A NODAL INVERSE PROBLEM: RECOVERING BOTH THE WEIGHT AND THE POTENTIAL OF A STURM-LIOUVILLE EQUATION WITHOUT USING A PRIORI ESTIMATES OF EIGENVALUES

MARTINA G. OVIEDO AND JUAN PABLO PINASCO

ABSTRACT. In this work we solve the nodal inverse problem when both the weight and the potential in a Sturm-Liouville problem are unknown. Our alogrithm does not use a priori information on the asymptotic behavior of the eigenvalues of the problem, and reconstruct the coefficients from the zeros of eigenfunctions.

1. Introduction

In this work we deal with the inverse nodal problem associated to the following weighted ordinary differential equation

$$-u'' + q(x)u = \lambda\rho(x)u \qquad x \in [0,1], \tag{1.1}$$

with zero Dirichlet boundary condition

$$u(0) = u(1) = 0. \tag{1.2}$$

Usually, there are two inverse problems for the previous equation: knowing the zeros of the eigenfunctions, one is to determine the weight ρ given q, and the other one is to determine the potential q when ρ is known. Both problems were introduced in the seminal work of Joyce McLaughlin [?], see also [?, ?, ?], and studied in several works since then, see for instance [?, ?, ?, ?, ?, ?, ?].

Both problems are solved in a similar way: it is possible to recover one of the coefficients from a two-term asymptotic expansion of the eigenvalues (plus a lower order error estimate) which depends on ρ and q, and also the nodal lengths are linked to this asymptotic expansion. However, this method requires precise estimates on the sequence $\{\lambda_n\}_{n\geq 1}$, which sometimes are far beyond our knowledge.

For instance, for $q \equiv 0$, the nodal inverse problem in unbounded intervals [?] or including sign changing weights [?] cannot be handled in this way since nodal domains can be large, and no two-term asymptotics were known, although the difficulties were overcome using only an expansion of the form $\lambda_n = C(\rho)n^2 + o(n^2)$, together with weak convergence results for families of measures supported on the nodes. Recently, in [?] a similar problem for measure geometric Laplacians on fractal sets was studied, where no asymptotic formulas were available, and only an estimate $\lambda_n = O(n^d)$ holds, and it is known that cannot be improved, see [?, ?].

Our main result in this work is the following theorem.

Theorem 1.1. *Let X be the nodal set of problem (??)-(??),*

$$X = \{x_k^n \ : \ n \geq 1 \text{ and } 0 \leq k \leq n\},$$

that is,

$$0 = x_0^n < \cdots < x_j^n < \cdots < x_n^n = 1$$

are the zeros of the n-th eigenfunction. Let q, $\rho \in C[0,1]$, and ρ a positive function satisfying $\|r\|_1 = 1$. Then, both q and r are uniquely determined and can be recovered from X.

Let us observe that this extend the classical Ambarzumyan theorem [?] on the uniqueness of the potential for $\rho \equiv 1$, to the pair q, ρ.

On the other hand, the proof gives an explicit algorithm to reconstruct both q and ρ from scratch, since we do not assume any a priori estimate of λ_n. The lack of precise information on the eigenvalue behavior appears in several problems, such as degenerate or singular operators [?], unbounded domains [?, ?, ?], pencils of operators [?, ?], nonlinear problems [?, ?], where non-Weyl asymptotics appears, and sometimes only the leading term is known.

Remark 1.2. Several improvements of Theorem ?? are possible:

(1) Different boundary conditions can be considered.
(2) It is possible to use only the zeros of a subsequence of eigenfunctions.
(3) The normalization condition $\|\rho\|_1 = 1$ can be changed to any other similar condition, using a different norm, or knowing the value of ρ at an arbitrary point in $[0,1]$.
(4) The regularity of q and ρ can be relaxed, by allowing finitely many discontinuity points.
(5) By using an estimate like $\lambda_n = C(\rho)n^2 + o(n^2)$, both q and ρ can be recovered at any point x using only a sequence of twin pairs of zeros $\{x_j^{n_k}, x_{j+1}^{n_k}\}_{k\geq 1}$, that is, a sequence of two consecutive zeros of the same eigenfunction, such that $x_j^{n_k} \to x$ when $k \to \infty$.

After few preliminary results on eigenvalues and the oscillation and comparison theory for Sturm-Liouville problems in Section §2, in Section §3 we show how to recover q knowing ρ / we prove Theorem ??. Finally, in Section §4 we present numerical computations and we deal with the generalizations in Remark ??.

2. Preliminaries

We consider a second order ODE of the form

$$u'' + q(x)u = 0 \tag{2.1}$$

where q is a continuous function.

Most of the times we can not have explicit solutions to ODE when the ODE has variable coefficients. However we can understand some of the qualitative properties of solutions of such equations, for example nature of zeroes of a solution. This is the goal of this section.

Theorem 2.1 (**Sturm-Liouville**). *The eigenvalues of the regular Sturm Liouville problem in* $[a, b]$ *given by:*

$$\begin{cases} u'' + (q(x) + \lambda\rho(x))u = 0 \\ u(a) = u(b) = 0 \end{cases} \tag{2.2}$$

are a sequence $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \ldots$ *such that* $\lim_{n\to\infty} \lambda_n = \infty$*. Besides, there exist one eigenfunction,* u_n*, associated to* λ_n *and this function has exactly* $n+1$ *zeros in* $[a, b]$.

When $q = 0$ and $\rho \equiv 1$, we have $\lambda_n = n^2\pi^2$ and the corresponding eigenfunction is $u_n(x) = \sin(n\pi x)$. Let us note that u_n has $n + 1$ zeros in $[0, 1]$.

We denote $\{x_j^n\}_{1\leq j\leq n+1}\}$ $n + 1$ zeros or nodal points of an eigenfunction which define n *nodal domains*, i.e., the interval between two consecutive zeros of u_n.

Theorem 2.2 (**Separation**). *Suppose that* u_1 *and* u_2 *be a fundamental pair of solutions (and hence are linearly independent) of*

$$u'' + q(x)u = 0. \tag{2.3}$$

Then:

i. *The zeros of nontrivial solutions of* **??** *are simple and isolated.*
ii. *Let* x_1 *and* x_2 *be two consecutive zeros of* u_1*, then* u_2 *has exactly one zero in* (x_1, x_2)

Theorem 2.3 (**Comparison**). *Let* u_1 *and* u_2 *be nontrivial solutions of equations*

$$\begin{aligned} u'' + q_1(x)u &= 0 \\ u'' + q_2(x)u &= 0 \end{aligned}$$

respectively, on an interval I; where q_1 *and* q_2 *are continuous functions such that* $q_1(x) \leq q_2(x)$ *on I. Then, between two consecutive zeros* x_1, x_2 *of* u_1*, there exists at least one zero of* u_2 *unless* $q_1 \equiv q_2$.

Remark 2.4. we will use this Theorem in the following way: given u_k and u_j, two eigenfunctions associated to $\lambda_k < \lambda_j$, between two zeros of u_k we have at least a zero of u_j.

We will based on this last remark no prove the following:

Lemma 2.5. *Let be* $x_j^n, x_{j+1}^n \in Z_n$ *two consecutive zeros from the* $n-th$ *eigenfunction,* u_n *associated to problem* **??** *and* **??**, *then* $|x_{j+1}^n - x_j^n| \to 0$

Proof. Considerer de problem:

$$-w'' = (\lambda\rho_m + q_m)w, \quad w(0) = w(1) = 0.$$

We know that $w_n = sin(\sqrt{\lambda_n\rho_m + q_m}x)$, thus if y_j^n and y_{j+1}^n are two consecutive zeros of w_n such that $|y_{j+1}^n - y_j^n| \to 0$, as $\lambda\rho_m + q_m \leq \lambda\rho + q$, then $x_j^n \in (y_j^n, y_{j+1}^n) \quad \forall j$ □

Let us define

$$N_n(x_0, \delta) = \#\Big\{\{x_j^n\}_{1\leq j\leq n+1} \cap (x_0 - \delta, x_0 + \delta)\Big\},$$

where $\{x_j^n\}_{1\leq j\leq n+1}$ are the zeros of the n-th. eigenfunction. We will show briefly that... , we have the following result: [[acumula ceros]]

Theorem 2.6. *Let* $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$ *be the eigenvalues of the following problems.*

$$\begin{aligned} u'' + \lambda\rho_1(x)u &= 0 \\ u'' + \mu\rho_2(x)u &= 0 \end{aligned}$$

with $\rho_1 \leq \rho_2$. *Then,* $\lambda_n \geq \mu_n$ *for any* $n \geq 1$, *and the inequalities are strict if* $\rho_1(x_0) < \rho_2(x_0)$ *in some* $x_0 \in [0, 1]$.

Theorem 2.7. *Let* $S = \{\{x_j^n\}_{1\leq j\leq n+1}\}_{n\geq 1}$ *be the set of nodal points of the eigenfunctions* $\{u_n\}_{n\geq 1}$ *of problem*

$$\begin{cases} u'' + \lambda u = 0 \\ u(a) = 0 \\ u(b) = 0 \end{cases} \tag{2.4}$$

. *Let* $x_0 \in [a, b]$. *Then, for any* $\delta > 0$,

$$\lim_{n\to\infty} N_n(x_0, \delta) = \infty.$$

Remark 2.8. If ρ changes signs, the previous result is not true, and we have

$$0 \le N_n(x_0, \delta) \le 1$$

for any δ sufficiently small, and any x_0 such that $\rho(x_0) < 0$.

3. Main results

3.1. **Characterization of the weight.** We characterize ρ, $\rho \in C([a,b])$. We will prove the result in $[0,1]$.

To start, let us write the problem **??** in the following way:

$$-u'' = \left(\frac{q(x)}{\lambda} + \rho(x)\right) u\lambda \tag{3.1}$$

Let us note that if $r(x) = q(x)/\lambda + \rho(x)$, then $\rho - \delta \le r(x) \le \rho + \delta$ with $\delta \to 0$ when $\lambda \to \infty$. Then we gonna find $r(x)$

Proof of Theorem. In order to characterize ρ, we will define a sequence of piecewise functions $\{\rho_m\}_{m\ge 1}$ and we will show that $\rho_m \to \rho$ as $m \to \infty$.

Lemma: for any m there exists $\lambda_{n(m)}$ such that the lengths of the nodal domains of $u_{n(m)}$ are less than $1/m$.

dem: partir en

$$\{[0, 1/2m], [1/2m, 2/2m], \dots, [1 - 1/2m, 1]\}.$$

existe autof con dos ceros en c/u y ahora sale que un cero de un intervalo y otro del intervalo siguiente estÃ¡n separados menos que $1/m$

For clarity, we divide the proof in three steps.

Step 1, a partition of $[0,1]$*:*

Let us consider the family of nodal intervals $\{J_j^n\}_{1\le j\le n+1}$ associated to the *nth* eigenfunction ,

$$\{[x_0^n, x_1^n], [x_1^n, x_2^m], \dots, [x_n^n, x_{n+1}^n]\}.$$

with n big enough so that $max_j|J_j^n| < 1/m$ where $|J_j^n| = |x_j^n - x_{j-1}^n|$.

Step 2, auxiliary problems: Let us consider a family of auxiliary eigenvalue problems in each interval

$$\begin{cases} -u'' = \Lambda u, \\ u(x_{k_j}^{n(m)}) = u(x_{k_j+1}^{n(m)}) = 0. \end{cases} \tag{3.2}$$

Let $\Lambda_1(J_j^m)$ be the first eigenvalue of problem **??** for each $1 \le j \le n+1$.

Step 3, an estimate for $\lambda_{n(m)}$*:*

Let us introduce the value

$$L_m = \sum_{j=1}^{m} \Lambda_1(J_j^m)|J_j^m|,$$

and let us show that $L_m = \lambda_{n(m)}(1 + O(\varepsilon_m))$, with $\varepsilon_m \to 0$ when $m \to \infty$.

Since ρ is a continuous function in $[a, b]$, there exists

$$c_j^m = \min\{\rho(x) : x \in J_j^m\}, \quad C_j^m = \max\{\rho(x) : x \in J_j^m\}$$

. Moreover, it is uniformly continuous, and there exists ε_m such that $|\rho(x) - \rho(y)| < \varepsilon_m$ whenever $|x - y| < 1/m$.

Hence, the first eigenvalue ζ_1 and ξ_1 of problems

$$\begin{cases} -u'' = \zeta c_j^m u, \\ u(x_{k_j}^{n(m)}) = u(x_{k_j+1}^{n(m)}) = 0, \end{cases} \quad \begin{cases} -v'' = \xi C_j^m v \\ v(x_{k_j}^{n(m)}) = v(x_{k_j+1}^{n(m)}) = 0, \end{cases} \tag{3.3}$$

satisfies $\xi_1 \le \lambda_{n(m)} \le \zeta_1$ since $c_j^m \le \rho \le C_j^m$ in J_j^m, due to the comparison result in Theorem **??**. On the other hand, $\zeta_1 c_j^m = \xi_1 C_j^m = \Lambda_1(J_j^m)$, being **??** and **??** constant coefficient problems.

Both inequalities imply

$$\lambda_{n(m)} c_j^m \le \Lambda_1(J_j^m) \le \lambda_{n(m)} C_j^m, \tag{3.4}$$

and

$$\lambda_{n(m)} \sum_{j=1}^{m} c_j^m |J_j^m| \le L^m \le \lambda_{n(m)} \sum_{j=1}^{m} C_j^m |J_j^m|. \tag{3.5}$$

Finally, since ρ is a continuous function, these are Riemann sums which converge to $\int \rho dx = 1$, and by using that $C_j^m - c_j^m \le \varepsilon_m \to 0$ when $m \to \infty$, due to the uniform continuity of ρ,

$$\Big| \sum_{j=1}^{m} C_j^m |J_j^m| - \sum_{j=1}^{m} c_j^m |J_j^m| \Big| \le \varepsilon_m,$$

so we have $L^m / \lambda_{n(m)} = 1 + O(\varepsilon_m)$.

Step 4, the sequence $\{\rho_m\}_{m \ge 1}$ and its convergence to ρ:

We define ρ_m by using L^m and $\mu(J_j^m)$ as follows:

$$\rho_j(x) = \frac{\Lambda_1(J_j^m)}{L^m} \quad x \in J_j^m$$

By using inequalities **??** and **??**, we get

$$\frac{c_j^m}{\sum_{j=1}^{m} C_j^m |J_j^m|} \le \rho_j(x) \le \frac{C_j^m}{\sum_{j=1}^{m} c_j^m |J_j^m|},$$

Let us observe that when m is big enough $\rho-\varepsilon < c_j^m$ and $C_j^m < \rho+\varepsilon$. Besides, $\sum_{j=1}^m C_j^m|J_j^m| = 1+\varepsilon$ and $\sum_{j=1}^m c_j^m|J_j^m| = 1-\varepsilon$ and

$$\frac{c_j^m}{\sum_{j=1}^m C_j^m|J_j^m|} - \frac{C_j^m}{\sum_{j=1}^m c_j^m|J_j^m|} = \frac{\rho-\varepsilon}{1+\varepsilon} - \frac{\rho+\varepsilon}{1-\varepsilon} \to 0$$

Then we get $\rho_m(x) \to \rho(x)$ in J_j^m and if we define $\rho_n = \sum_{j=1}^{n+1} p_j \mathcal{X}(J_j^n)$ we got the convergence in all the interval.

The proof is finished. □

3.2. **Characterization of λ.** Consider the problem **??** with boundary condition **??**, we'll assume that $q \in C([0,1])$ such that $\int_0^1 q = 0$ and $\rho > 0$ a known function. Suppose we also know the zeros $\{x_j^n\}$ of the eigenfunction u_n.

Lemma 3.1. *Consider the problem* **??** *with zero Dirichlet boundary conditions* **??** *and let us introduce the continuous function $f(x) = \lambda_k\rho(x) - q(x)$ with λ_k the $k-th$ eigenvalue and x_1 and x_2 two zeros corresponding to a eigenfunction u_k. Consider also the problem*

$$-u'' = \Lambda u, \qquad u(x_1^k) = u(x_2^k) = 0$$

and let be Λ_1 the first eigenvalue, then

$$\Lambda_1 = \lambda_k\rho(\tilde{x}) - q(\tilde{x}) \tag{3.6}$$

for $\tilde{x} \in [x_1, x_2]$

Proof. Begin noticed that u_k is a solution to

$$-u'' = f(x)u, \qquad u(x_1) = u(x_2) = 0.$$

Indeed, it is an eigenfunction corresponding to $\mu = 1$ of the problem

$$-u'' = \mu f(x)u, \qquad u(x_1) = u(x_2) = 0.$$

Let us introduce

$$f^m = \min\{f(x) : x_1 \le x \le x_2\} \le f^M = \max\{f(x) : x_1 \le x \le x_2\}.$$

Finally, we have the following eigenvalue problems in $[x_1, x_2]$:

$$-u'' = \Lambda u, \qquad u(x_1) = u(x_2) = 0,$$
$$-u'' = \Lambda^m f^m u, \qquad u(x_1) = u(x_2) = 0,$$
$$-u'' = \Lambda^M f^M u, \qquad u(x_1) = u(x_2) = 0.$$

Since they are constant coefficient problems, we have

$$\Lambda_1 = \pi^2/(x_2-x_1)^2 = \Lambda^m f^m = \Lambda^M f^M.$$

However, from

$$f^m \le f(x) \le f^M,$$

and the eigenvalue comparison theorem / Sturm Liouville theory, we deduce that

$$\Lambda^m \geq \mu_1 = 1 \geq \Lambda^M.$$

Therefore,

$$f^m \leq \Lambda_1 \leq f^M,$$

and the continuity of $f(x)$ implies that there exists some $\tilde{x} \in [x_1, x_2]$ such that

$$\lambda_k p(\tilde{x}) - q(\tilde{x}) = \Lambda_1.$$

□

Let be $\{x_j^n\}_{1 \leq j \leq n+1}$ the set of nodal points associated to the eigenfuntion u_k of the problem **??** with bounded condition **??**, let us define $L_j = x_j^n - x_{j-1}^n$. Now, consider the auxiliary problem

$$v'' + \Lambda v = 0, \qquad v(x_{j-1}^n) = v(x_j^n) = 0 \tag{3.7}$$

and let be Λ_1^j the first eigenvalue of this problem. We define now $\Lambda_M = \sum_{j=1}^{n+1} \Lambda_1^j L_j$. Then we can characterize the eigenvalue λ_n mediate the following theorem.

Theorem 3.2. *Let be λ_n the n-th eigenvalue of the problem* **??**, *then:*

$$\lambda_n = \frac{\Lambda_M + O(\varepsilon)}{\int_0^1 \rho(x)dx + O(\varepsilon)} \tag{3.8}$$

with $O(\varepsilon) \to 0$ as n goes to infinity.

Proof. By **??** we have

$$\Lambda_1^j = \lambda \rho(\tilde{x_j}) - q(\tilde{x_j}) \qquad \tilde{x_j} \in (x_{j-1}, x_j) \tag{3.9}$$

Being Λ_1^j and eigenvalue of **??**, moreover we know that $\Lambda_1^j = \pi^2/L_j^2$ being $L_j = x_j - x_{j-1}$.

We can replicate the same for the rest of the nodal intervals and then multiply **??** by L_j and sum over j we obtain:

$$\sum_{j=1}^{n+1} L_j \Lambda_1^j = \sum_{j=1}^{n+1} \lambda \rho(\tilde{x_j}) L_j - q(\tilde{x_j}) L_j \tag{3.10}$$

Observe that $\sum_j q(\tilde{x_j}) L_j = \int_0^1 q(x)dx + \epsilon = \epsilon$ and $\sum_j \rho(\tilde{x_j}) L_j = \int_0^1 \rho(x)dx + \epsilon$ Then, calling $\Lambda_M = \sum_{j=1}^{n+1} \Lambda_1^j L_j$

$$\lambda = \frac{\Lambda_M + O(\varepsilon)}{\int_0^1 \rho(x)dx + O(\varepsilon)} \tag{3.11}$$

and the proof is finished. □

3.3. **Characterization of the potential when $\rho \equiv 1$.** Let us focus now in the particular case of recover $q(x)$ when $\rho \equiv 1$

Theorem 3.3. *Given the equation $-u'' + q(x)u = \lambda u$ we can find $q(x)$ without using the fact $\lambda_n = cn^2 + o(n^2)$*

Proof. Let be $y \in [0,1]$, so there exist J_j^n such that $y \in J_j^n$. Besides there is an $\tilde{x}_j^n \in J_j^n$ such that $C_1 < q(\tilde{x}_j^n) = \frac{\pi^2}{L_j^2} - \Lambda_M < C_2$ and $\tilde{x}_j^n \to y$ when $n \to \infty$ and by continuity of q, $q(\tilde{x}_j^n) \to q(y)$ $\square$

3.4. **Characterization of the potential.** Now let us focus on recover $q(x)$. Once again, consider the problem

$$-u'' = \lambda\rho(x)u + q(x)u \textit{ with } u(0) = u(1) = 0$$

Take into account $\{(\lambda_n, u_n)\}$ such that $\lambda_n \to \infty$ as $n \to \infty$ and is well knowing that the eigenfunction u_n has $n+1$ zeros at [0,1]:

$$Z_n = \left\{x_j^n\right\}_{0 \le j \le n}$$

Theorem 3.4. *We can find q*

Proof. Let us choose $\{I_n\}_{n\ge1}$ each one defined as $I_n = [x_{j(n)}^n, x_{j(n)+1}^n]$ such that $x_0 \in I_n$ for all n. Now in order to simplify, set N so that

$$I_N = \left[x_{j(N)}^N, x_{j(N)+1}^N\right] := [a,b]$$

And we have:

$$-u'' = \lambda\rho(x)u + q(x)u, \quad u(a) = u(b) = 0$$

By lineality, u_N restricted to I_N is the solution and we can take $u = cu_N$ such that:

$$cu_N'(a) = 1. \tag{3.12}$$

Thanks to everything we do in the previous section, observes that we can find and approximation of ρ, ρ_k, in $[a,b]$ such that

$$\|\rho - \rho_k\|_\infty \le \varepsilon_k$$

Let us call

$$q_m = \min_{x\in[a,b]} q(x), \qquad q_M = \max_{x\in[a,b]} q(x)$$

Suppose now that we have v and w solutions of

$$-v'' = \lambda_N\rho(x)v + q_m(x)v$$
$$-w'' = \lambda_N\rho(x)w + q_M(x)w$$

Then, we can do:

$$\lambda_N(x) + q_m = \lambda_N(\rho(x) + \rho_k(x) - \rho_k(x)) + q_m = \lambda_N\rho_k + q_m - \lambda_N\varepsilon_k$$

and then we get

$$-v'' = \lambda_N \rho(x)_k v + q_m(x)v - \lambda_N \varepsilon_k v$$
$$-w'' = \lambda_N \rho(x)_k w + q_M(x)w + \lambda_N \varepsilon_k w$$

with (using **??**):

$$v(a) = 0, \quad v'(a) = 1$$
$$w(a) = 0, \quad w'(a) = 1$$

Besides,

$$q_m - \lambda_N \varepsilon_k \leq q + \lambda_N(\rho - \rho_k) \leq q_M + \lambda_N \varepsilon_k$$

Then, by Theorem **??**, we have $v(b) > 0$ and $w(b) < 0$, which implies, by Bolzano's theorem, that exist a constant $\tilde{q}$ such that:

$$-u'' = \lambda_N \rho_k u + \tilde{q} u$$

has a solution, u, with $u(a) = u(b) = 0$ with

$$q_m - \lambda_N \varepsilon_k < \tilde{q} < q_M + \lambda_N \varepsilon_k$$

and, as λ_N in fixed, we can take k such that $|\lambda_N \varepsilon_k| < \delta/2$, then:

$$\|q - \tilde{q}\|_\infty \leq q_M - q_m + \delta/2$$

Evermore, if N is big enough, $b - a \to 0$ and like q is continuous is uniformly continuous at $[0, 1]$, then

$$q_M - q_m < \delta/2$$

, and, finally:

$$\|q - \tilde{q}\| \leq \delta \tag{3.13}$$

□