Q3:

Because of $p(\tau_i)$ is gamma and $p(y_i|\tau_i)$ is normal (By question)

Because of $\int p(y_i|\tau_i)\,p(\tau_i)\,d\tau_i = \int \sqrt{\frac{\tau_i}{2\pi}}\, e^{-\frac{1}{2}\tau_i y_i^2} \frac{\frac{v}{2}^{\frac{v}{2}}}{\Gamma(\frac{v}{2})} \tau_i^{\frac{v}{2}-1} e^{-\frac{v}{2}\tau_i} d\tau_i$

$$p(y_i) = \int \sqrt{\frac{\tau_i}{\cancel{2\pi}}}\, \tau_i^{\frac{v}{2}-1}\, e^{-\tau_i(\frac{1}{2}y_i^2+\frac{v}{2})} \cdot \frac{\frac{v}{2}^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\, d\tau_i$$

c

- Only discuss: $\int \tau_i^{\frac{v}{2}-\frac{1}{2}} e^{-\tau_i(\frac{1}{2}y_i^2+\frac{v}{2})} d\tau_i$ ①

Because of the gamma distribution is:

$$\int x^{a-1} e^{-bx} dx = \frac{\Gamma(a)}{b^a}$$

$\Rightarrow$ we find $a-1 \Leftrightarrow \frac{v}{2}-\frac{1}{2} \Rightarrow a \Leftrightarrow \frac{v}{2}+\frac{1}{2}$

$\tau_i \Leftrightarrow x$

$b \Leftrightarrow \frac{1}{2}y_i^2+\frac{v}{2}$

$$\Rightarrow \int \tau_i^{\frac{v}{2}-\frac{1}{2}} e^{-\tau_i(\frac{1}{2}y_i^2+\frac{v}{2})} d\tau_i = \frac{\Gamma(\frac{v}{2}+\frac{1}{2})}{(\frac{1}{2}y_i^2+\frac{v}{2})^{(\frac{v+1}{2})}}$$

Then we plug in the results into the function $p(y_i)$, we have:

$$p(y_i) = \frac{\Gamma(\frac{v}{2}+\frac{1}{2})}{(\frac{1}{2}y_i^2+\frac{v}{2})^{(\frac{v+1}{2})}} \cdot \frac{\frac{v}{2}^{\frac{v}{2}}}{\Gamma(\frac{v}{2})} = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})} \cdot \frac{\cancel{\frac{v}{2}^{\frac{v}{2}}}}{(\frac{1}{2}y_i^2+\frac{v}{2})^{(\frac{v+1}{2})}}$$

$\propto$ c

Because of the function of t-distribution is: $f(t) = \frac{\Gamma(\frac{v+1}{2})}{\cancel{\sqrt{v\pi}}\,\Gamma(\frac{v}{2})} \left(1+\frac{t^2}{v}\right)^{-\frac{v+1}{2}}$

$\propto$ c

$\Rightarrow \sqrt{v\pi} \Leftrightarrow 2\pi$; $t \Leftrightarrow y_i$; $v=2$

we found that $p(y_i)$ is a t-distribution.