HW1:

Q1:

Let $P(pass)$ = the probability of passing the class

Let $P(quick \mid pass)$ = the probability of answering this question quickly if you'll pass the class)

Let $P(quick \mid not\ pass)$ = the probability of answering this question but not pass)

By question, we get: $P(pass) = 0.9$; $P(quick \mid pass) = 0.6$; $P(quick \mid not\ pass) = 0.3$

$\Rightarrow P(not\ pass) = 1 - P(pass) = 0.1$

Then $P(pass \mid quick) = \dfrac{P(quick \mid pass) \cdot P(pass)}{P(quick)}$ (By Bayes Theorem)

Because of $P(quick) = P(quick \mid pass) \cdot P(pass) + P(quick \mid not\ pass) \cdot P(not\ pass)$ (By law of total probability)

$= 0.6 \times 0.9 + 0.3 \times 0.1$

$$\Rightarrow P(pass \mid quick) = \frac{0.6 \times 0.9}{0.6 \times 0.9 + 0.3 \times 0.1} = \frac{0.54}{0.54 + 0.03} = 0.9474 \approx 0.947 = 94.7\%$$

Thus, 94.7% students who answer this question quickly will pass the class.

Q2:

Because of Likelihood $p(x|\theta, n)$ is multinomial distribution

$\Rightarrow$ the pmf of multinomial distribution is : $P(X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k) = \frac{n!}{x_1!\, x_2! \cdots x_k!} \prod_{i=1}^{k} \theta_i^{x_i}$

Because of Prior $p(\theta|\alpha)$ is dirichlet distribution

$\Rightarrow$ the pdf of Dirichlet distribution is $p(\theta|\alpha) = \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$

Because of posterior is always proportional to the joint $p(\theta|x) = \frac{1}{C} \times P(\theta, x) \propto p(\theta, x)$ (By question)

$\Rightarrow$ $P(\theta|x, n, \alpha) = \frac{\text{likelihood} \times \text{prior}}{P(x)} = \frac{\frac{n!}{x_1!\, x_2! \cdots x_k!} \cdot \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)}}{P(x)} \prod_{i=1}^{k} \theta_i^{x_i} \cdot \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$

Because of $P(x)$ is constant (By class notes) $\Rightarrow$ Let $p(x) = C''$

Also because of $\frac{n!}{x_1!\, x_2! \cdots x_k!}$ and $\frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)}$ are constant as well

$\Rightarrow$ Let $\frac{n!}{x_1!\, x_2! \cdots x_k!} = C$ and $\frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} = C'$

$\Rightarrow$ $P(\theta|x, n, \alpha) = \frac{CC'}{C''} \prod_{i=1}^{k} \theta_i^{x_i} \cdot \prod_{i=1}^{k} \theta_i^{\alpha_i - 1} = \frac{CC'}{C''} \prod_{i=1}^{k} \theta_i^{x_i + \alpha_i - 1}$

Let $x_i + \alpha_i = m_i$

$\Rightarrow$ $P(\theta|x, n, \alpha) = \frac{CC'}{C''} \prod_{i=1}^{k} \theta_i^{m_i - 1}$, with $\frac{CC'}{C''}$ is constant

we found the distribution of Posterior is Dirichlet and the parameteriza is $\alpha_{posterior} = (m_1, m_2 \cdots m_k) = (\alpha_1 + x_1, \cdots \alpha_k + x_k)$