Discrete Mathematics

Chapter 1 : Set Theory

Q. 1 Define set and subsets. Explain with examples.

Ans. :

Set :

A set is any well defined collection of objects, called the "elements" or "members" of the set. If there are only a few elements, they can be listed individually, by writing them between braces (curly brackets) and placing commas in between.

Example : the set of odd numbers between 0 and 100 can be written as,

$\{x \mid x \text{ is odd and } 0 < x < 100\}$.

And the definition reads the set of all x such that x is odd and $0 < x < 100$. The expression following the slash is a predicate containing the variable x. A set written in the form $\{x \mid P(x)\}$, where P(x) is a predicate is said to be written in predicate form or 'set builder form'.

Subset :

If every element of a set A is also an element of a set B, we say A is a **subset** of B or A is contained in B. This is denoted by writing,

$$A \subseteq B$$

If A is not a subset of B, this is indicated by writing

$$A \nsubseteq B$$

Examples :

(i) If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 5\}$, $C = \{1, 2, 3, 4, 5\}$ then $B \subseteq A$, $B \subseteq C$ and $C \subseteq A$.

(ii) If $A = \{1, 3, 6\}$, $B = \{-1, 1, 2, 3, 4, 6\}$, $C = \{1, 2, 3\}$ then $A \subseteq B$ but $A \nsubseteq C$.

It is clear from the definition that **Every set is a subset of itself.**

Q. 2 Let A = {a,b,c,d,e,f,g,h}. Consider the following subsets of A May 17

A1 = {a,b,c,d }

A2 = {a,c,e,g,h}

A3 = {a,c,e,g}

A4 = {b,d}

A5 ={ f,h}

Determine whether following is partition of A or not. Justify your answer.

(I) {A1, A2} (II) {A3, A4, A5}

Ans. :

(I) $\{A_1, A_2\}$

(a) $A_1 \cup A_2 = \{a, b, c, d\} \cup \{a, c, e, g, h\}$
$= \{a, b, c, d, e, g, h\} \neq A$

(b) $A_1 \cap A_2 = \{a\} \neq \phi$

Since $\{A_1, A_2\}$ is not partition of A. Since it is not satisfying for both condition.

(II) $\{A_3, A_4, A_5\}$

(a) $A_3 \cup A_4 \cup A_5 = \{a, c, e, g\} \cup \{b, d\} \cup \{f, h\}$
$= \{a, b, c, d, e, f, g, h\} = A$

(b) $A_3 \cap A_4 \cap A_5 = \phi$

$\therefore$ It is a partition of A.

Q. 3 In a class of students undergoing a computer course the following were observed. Out of a total of 50 students :

30 know Pascal,

18 know Fortran,

26 know COBOL,

9 know both Pascal and Fortran,

16 know both Pascal and COBOL,

8 know both Fortran and COBOL,

47 know at least one of the three Languages.

From this we have to determine

(a) How many students know none of these languages?

(b) How many students know all three languages?

(c) How many students know exactly one language? May 17

Ans. :

Let P, F and C denote the set of students who know Pascal, Fortran and COBOL respectively.

$|P| = 30$,

$|F| = 18$,

$|C| = 26$,

$|P \cap F| = 9$

$|P \cap C| = 16$,

$|F \cap C| = 8$

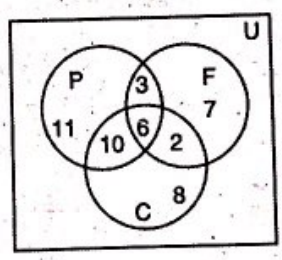

Fig. 1.1

(a) Students know none of these languages = 50 – 47 = 3

3 students know none of these languages

(b) $|P \cup F \cap C| = |P| + |F| + |C| - |P \cap F| - |P \cap C| - |F \cap C| + |P \cap F \cap C|$

$47 = 30 + 18 + 26 - 9 - 16 - 8 + |P \cap F \cap C|$

$|P \cap F \cap C| = 6$

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6 students know all three languages.

(c) The number of students who know only 'Pascal' (i.e. neither COBOL nor Fortran) is

$|P| - |P \cap C| - |P \cap F| + |P \cap F \cap C| = 30 - 16 - 9 + 6 = 11$

Similarly the number who know only 'Fortran' is,

$|F| - |F \cap C| - |P \cap F| + |P \cap F \cap C| = 18 - 8 - 9 + 6 = 7$

The number of students who know only 'COBOL'

$|C| - |F \cap C| - |P \cap C| + |P \cap F \cap C| = 26 - 8 - 16 + 6 = 8$

Hence number of students, who know exactly one language is $11 + 7 + 8 = 26$

Q. 4 Prove the following

$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. Dec. 16

Ans. :

$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

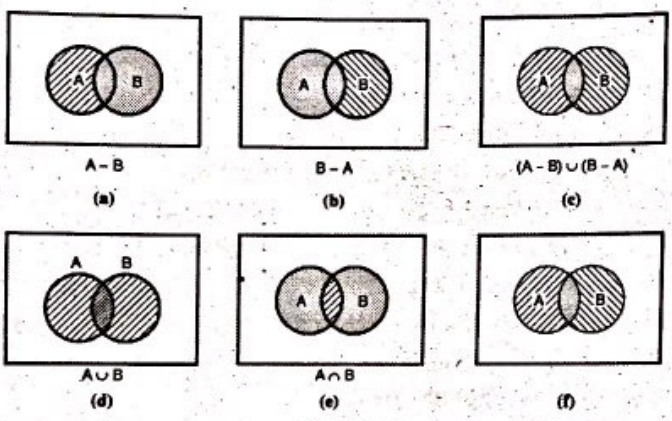

Fig. 1.2

Q. 5 Determine the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7. Dec. 16

Ans. :

Let the sets A_1, A_2, A_3, A_4 denote the set of integers between 1 and 250 that are divisible respectively by 2, 3, 5 and 7.

Then $|A_1|$ = number of elements in A_1

= largest number of elements in A_1 which are divisible by 2

$$= \left|\frac{250}{2}\right| = 125$$

Similarly, $|A_2| = \left|\frac{250}{3}\right| = 83$

$$|A_3| = \left|\frac{250}{5}\right| = 50$$

$$|A_4| = \left|\frac{250}{7}\right| = 35$$

Similarly, $|A_1 \cap A_2|$ = largest number of elements that are divisible by 2 and 3.

$$= \left|\frac{250}{2 \times 3}\right| = 41$$

and, $|A_1 \cap A_3| = \left|\frac{250}{2 \times 5}\right| = 25$

and $|A_2 \cap A_3| = \left|\frac{250}{3 \times 5}\right| = 16$

$$|A_1 \cap A_4| = \left|\frac{250}{2 \times 7}\right| = 17$$

$$|A_3 \cap A_4| = \left|\frac{250}{5 \times 7}\right| = 7$$

$$|A_2 \cap A_4| = \left|\frac{250}{3 \times 7}\right| = 11$$

$$|A_1 \cap A_2 \cap A_3| = \left|\frac{250}{2 \times 3 \times 5}\right| = 8$$

$$|A_1 \cap A_2 \cap A_4| = \left|\frac{250}{2 \times 3 \times 7}\right| = 5$$

$$|A_1 \cap A_3 \cap A_4| = \left|\frac{250}{2 \times 5 \times 7}\right| = 3$$

$$|A_2 \cap A_3 \cap A_4| = \left|\frac{250}{3 \times 5 \times 7}\right| = 2$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left|\frac{250}{2 \times 3 \times 5 \times 7}\right| = 1$$

Now we use the formula,

$$\begin{aligned}
|A_1 \cap A_2 \cap A_3 \cap A_4| &= \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| \\
&\quad + \Sigma |A_i \cap A_j \cap A_k| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| \\
&\quad - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| \\
&\quad - |A_2 \cap A_4| - |A_3 \cap A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\
&\quad + |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 125 + 83 + 50 + 35 - 41 - 25 - 11 - 16 \\
&\quad - 17 - 7 + 8 + 5 + 3 + 2 - 1 = 193
\end{aligned}$$

Q. 6 Find how many integers between 1 and 60 are not divisible by 2 nor by 3 and nor by 5 respectively. Dec. 14, May 16

Ans. :

Let A_1, A_2 and A_3 be the set of integers between 1 and 60 divisible by 2, 3 and 5 respectively.

$$\therefore \quad |A_1| = \left|\frac{60}{2}\right| = 30$$

$$|A_2| = \left|\frac{60}{3}\right| = 20$$

$$|A_3| = \left|\frac{60}{5}\right| = 12$$

and $|A_1 \cap A_2| = \left|\frac{60}{2 \times 3}\right| = 10$

$$|A_1 \cap A_3| = \left|\frac{60}{2 \times 5}\right| = 6$$

$$|A_2 \cap A_3| = \left|\frac{60}{3 \times 5}\right| = 4$$

and $|A_1 \cap A_2 \cap A_3| = \left|\frac{60}{2 \times 3 \times 5}\right| = 2$

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Number of integers between 1 and 60 which are divisible by 2, 3 or 5 are

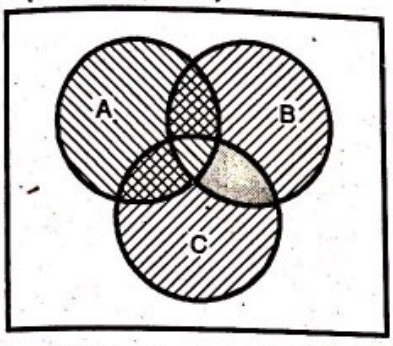
$$= |A_1 \cup A_2 \cup A_3|$$
$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$
$$= 30 + 20 + 12 - 10 - 6 - 4 + 2 = 44$$

Hence the number of integers between 1 and 60 are not divisible by 2, 3 or 5 = 60 – 44 = 16

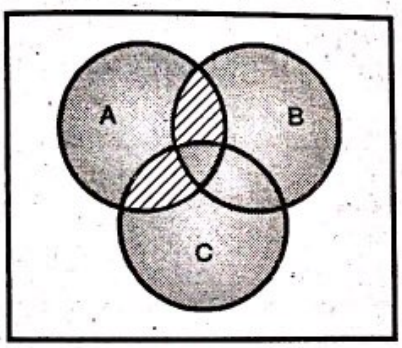
Q. 7 Show that $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$. Dec. 14

Ans. :

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

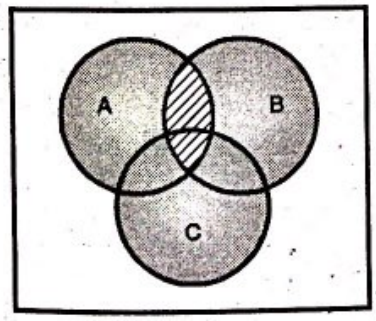


(a) $B \oplus C$ is shaded with ///



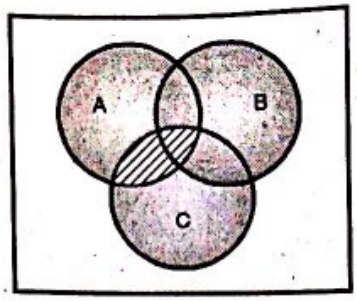
(b) $A \cap (B \oplus C)$ is shaded

**A is shaded with **



$A \cap B$

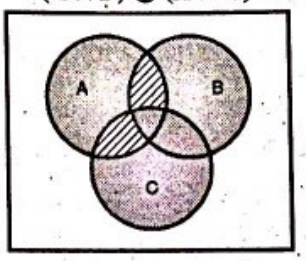
(c)



$A \cap C$

(d)

$(A \cap B) \oplus (A \cap C)$



(e)

Fig. 1.3

Fig. 1.3 (b) and (e) are same

$\therefore A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

Q. 8 Find how many integers between 1 to 60 are divisible by 2 not by 3 and nor by 5. Dec. 14

Ans. : Divisible by 2 not by 3 and nor by 5 :

Let A_1 denote the set of integers between 1 and 60 divisible by 2. Similarly A_2 and A_3 be the sets of integers divisible by 3 and 5 respectively.

$$\text{Then } |A_1| = \left|\frac{60}{2}\right| = 30,$$
$$|A_2| = \left|\frac{60}{3}\right| = 20,$$
$$|A_3| = \left|\frac{60}{5}\right| = 12$$

Number of integers between 1 and 60 which are divisible by 2 but not by 3 and nor by 5.

$$= |A_1| |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$
$$= 30 - 10 - 6 + 2 = 16$$

Q. 9 Find the number of integers between 1 and 1000 which are

(i) Divisible by 2, 3 or 5.

(ii) Divisible by 3 only but not by 2 nor by 5. Dec. 13

Ans. :

Let A_1 denote the set of integers between 1 and 1000 divisible by '2'. Similarly A_2 and A_3 be the sets of integers divisible by 3 and 5 respectively.

(i) Then $|A_1| = \left|\frac{1000}{2}\right| = 500$

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$$|A_2| = \left|\frac{1000}{3}\right| = 333$$

$$A_3 = \left|\frac{1000}{5}\right| = 200$$

$$\therefore |A_1 \cap A_2| = \left|\frac{1000}{2\times 3}\right| = 166,$$

$$|A_1 \cap A_3| = \left|\frac{1000}{2\times 5}\right| = 100$$

$$|A_2 \cap A_3| = \left|\frac{1000}{3\times 5}\right| = 66$$

and $$|A_1 \cap A_2 \cap A_3| = \left|\frac{1000}{2\times 3\times 5}\right| = 33$$

Hence using the principle of inclusion and exclusion we have number of integers which are divisible by 2, 3, or 5.

$$= |A_1 \cup A_2 \cup A_3|$$
$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$
$$= 500 + 333 + 200 - 166 - 100 - 66 + 33$$
$$= 734$$ ...Ans.

(ii) The set of integers divisible by 3 but not by 2 and 5 is the set $A_2 \cap \bar{A}_1 \cap \bar{A}_3$

$$A_2 \cap \bar{A}_1 \cap \bar{A}_3 = A_2 \cap (\overline{A_1 \cap A_3}) = A_2 - (A_1 \cup A_3)$$
$$|A_2 - (A_1 \cup A_3)| = |A_2| - |(A_2 \cap A_1) \cup (A_2 \cap A_3)|$$
$$= |A_2| - (|A_2 \cap A_1| + |A_2 \cap A_3|)$$
$$= 333 - (166 + 66) = 333 - 232 = 101$$ ...Ans.

Q. 10 In a survey of 60 people, it was found that 25 read Newsweek magazine, 26 read Time and 26 read Fortune. Also 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both Time and Fortune, 8 read no magazine at all.

(a) Find the number of people who read all three magazines.

(b) Determine the number of people who read exactly one magazine.

May 12, May 13

Ans. :

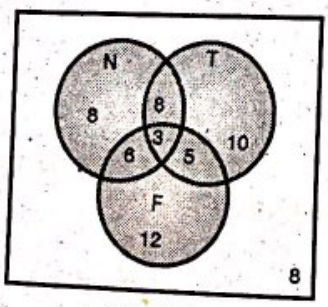

Fig. 1.4

(a) By the principle of inclusion and exclusion we have,

$$|N \cup T \cup F| = |N| + |T| + |F| - |N \cap T| - |T \cap F| + |N \cap T \cap F| \quad ...(1)$$

We have,

$$|N \cup T \cup F| = 60 - 8 = 52$$

$\because$ 8 read no magazine at all.

$$|N| = 25, \quad |T| = 26, \quad |F| 26,$$

and $|N \cap T| = 11$, $|N \cap F| = 9$, $|T \cap F| = 8$.

Hence from Equation (1),

$$52 = 25 + 26 + 26 - 11 - 9 - 8 + |N \cap T \cap F|$$
$$|N \cap T \cap F| = 3$$

Hence, 3 people read all three magazines.

(b) People who read only one magazine is $8 + 10 + 12 = 30$

Q. 11 Out of 250 candidates who failed in an examination, it was revealed that 128 failed in mathematics, 87 in physics and 134 in aggregate. 31 failed in mathematics and in Physics, 54 failed in the aggregate and in mathematics, 30 failed in the aggregate and in physics. Find how many candidates failed.

(i) in all the three subjects.

(ii) in mathematics but not in physics.

(iii) in the aggregate but not in mathematics.

(iv) in physics but not in aggregate or in mathematics.

Dec. 12

Ans. :

Let $|M| = 128$, $|P| = 87$, $|A| = 134$, $|M \cap P| = 31$, $|A \cap M| = 54$, $|A \cap P| = 30$

(i) By the principle of inclusion and exclusion we have,

$$|M \cup P \cup A| = |M| + |P| + |A| - |M \cap P| - |A \cap M| - |A \cap P| + |M \cap P \cap A|$$
$$250 = 128 + 87 + 134 - 31 - 54 - 30 + |M \cap P \cap A|$$
$$250 = 234 + |M \cap P \cap A|$$
$$250 - 234 = |M \cap P \cap A|$$
$$\therefore |M \cap P \cap A| = 16$$

16 candidates are failed in all three subjects.

(ii) Candidates failed in Mathematics but not in Physics

$$= |M| - |M \cap P| = 128 - 31 = 97$$

(iii) Candidates Failed in aggregate but not in Mathematics

$$= |A| - |A \cap M| = 134 - 54 = 80$$

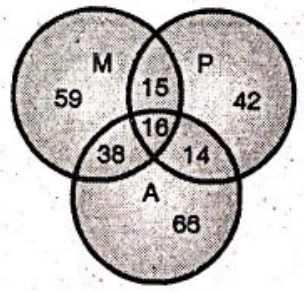

Fig. 1.5

(iv) Candidates Failed in Physics but not in aggregate or in Mathematics

$$|P| - |M \cap P| - |A \cap P| + |M \cap P \cap A|$$
$$= 87 - 31 - 30 + 16 = 42$$

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Q. 12 **A survey on a sample of 25 new cars being sold of a local auto dealer was conducted to see which of three popular options, air conditioning A, radio R, and power windows W, were already installed. The survey found,**

15 had air conditioning
12 had radio
11 had power windows
5 had air conditioning and power window
9 had air conditioning and radio
4 had radio and power windows
3 had all three options,
2 had no options.

Find the number of cars having,

(I) Only one of these options

(II) Radio and power windows but not air conditioning **Dec. 11**

Ans. :

We have,

$|A| = 15, |R| = 12, |W| = 11, |A \cap W| = 5,$

$|A \cap R| = 9,$

$|R \cap W| = 4;$ and $\therefore |A \cap R \cap W| = 3.$

3 had all three options.

We draw Venn diagram as shown in Fig. 1.6.

Using the Venn diagram, we have

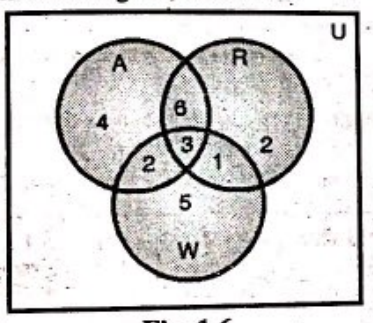

Fig. 1.6

(I) Only one of these options

$4 + 2 + 5 = 11$

$\therefore$ 11 cars had only one of the options.

(II) Radio and power windows but not air conditioning

$= |R \cap W| - |A \cap R \cap W| = 4 - 3 = 1$

Q.13 **A survey of 500 television watchers produced the following information :**

285 watch football games
195 watch hockey games
115 watch basketball games
45 watch football and basketball games
70 watch football and hockey games
50 watch hockey and basketball games
50 do not watch any of the 3 kinds of games.

(I) How many people in the survey watch all 3 kinds of games ?

(II) How many people watch exactly one of the sports ? **May 11**

Ans. : Let F be the set of people who watch football

Let H be the set of people who watch hockey.

Let B be the set of people who watch basketball.

Given :

$|F| = 285$

$|H| = 195$

$|B| = 115$

$|F \cap B| = 45$

$|F \cap H| = 70$

$|H \cap B| = 50$

50 people do not watch any of the 3 kinds of games.

(i) People in survey who watch all 3 kinds of games i.e. football or hockey or basketball,

$|F \cup H \cup B| = 500 - 50 = 450$

450 people watch atleast one of the 3 games.

Using the formula of inclusion and exclusion, we have,

$$|F \cup H \cup B| = |F| + |H| + |B| - |F \cap H| - |F \cap B| - |H \cap B| + |F \cap H \cap B|$$

$$\therefore \quad 450 = 285 + 195 + 115 - 45 - 70 - 50 + |F \cap H \cap B|$$

$$\therefore \quad |F \cap H \cap B| = 20$$

20 people in survey watch all 3 kinds of games.

(ii) People who watch only football.

$= |F| - |F \cap H| - |F \cap B| + |F \cap H \cap B|$

$= 285 - 70 - 45 + 20 = 190$

People who watch only hockey

$= |H| - |H \cap F| - |H \cap B| + |F \cap H \cap B|$

$= 195 - 70 - 50 + 20 = 95$

People who watch only basketball.

$= |B| - |F \cap B| - |B \cap H| + |F \cap H \cap B|$

$= 115 - 45 - 50 + 20 = 40$

Number of people watching exactly one kind of game

$= 190 + 95 + 40 = 325$

$\therefore$ 325 people watch exactly one kind of game.

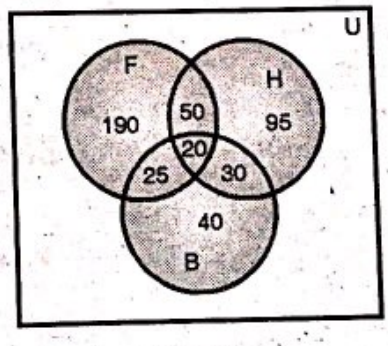

Fig. 1.7

Q. 14 **Define - Power Set.** **May 10**

Ans. : Power-set :

Let A be a set, then the set of all subsets of A is called the ***Power set*** of A and is denoted by **P(A).** A power set is an example of a set of sets, that is, a set whose elements are themselves sets.

Power set of A has 2^n elements where n is number of elements in A.

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Chapter 2 : Logic

Q. 1 Define Universal and Existential quantifiers? Explain with examples. Dec. 16

Ans. :

1. Universal quantifiers :

If P(x) is a predicate with the individual variable x as an argument, then the assertion "For all x, P(x)" which is interpreted as "For all values of x, the assertion P(x) is true," is a statement in which the variable x is said to be **universally quantified**.

We denote the phrase "For all" by ∀, called the **universal quantifier**. The meaning of ∀ is "for all" or "for every" or "for each". If P(x) is true for every possible value of x, then ∀ x P(x) is true; otherwise ∀ P(x) is false.

Example : Let P(x) be the predicate $x \geq 0$; where x is any positive integer. Then the proposition ∀ x P(x) is true. However, if x is any real number, then ∀ x P(x) is a false proposition.

2. Existential quantifiers :

Suppose for the predicate P(x), ∀x P(x) is false, but there exists atleast one value of x for which P(x) is true, then we say that in this proposition, x is bound by **existential quantification**. We denote the words "there exists" by the symbol ∃. Then the notation **∃ x P(x)** means "there exists a value of x (in the universe of discourse) for which P(x) is true".

Example : Let P(x) be the predicate "x + 3 = 5" and let the universe of discourse be the set of all integers. Then the proposition ∃ x P(x) is true (by setting x = 2) but ∀ x P(x) is false.

Let P(x, y) be a two-phase predicate, then **∃ x y P(x,y)** is the proposition "There exists a value of x such that for all values of y, P(x,y) is true".

∀ y ∃ x P(x, y) is the proposition "For each value of y, there exists an x such that P(x, y) is true."

∃ x ∃ y P(x, y) is the proposition "There exist a value of x and a value of y such that P(x,y) is true".

∀ x ∀ y P(x, y) is the proposition "For all values of x and y P(x, y) is true".

Q. 2 Prove that following by mathematical induction :

2 + 5 + 8 + ... + (3n − 1) = n(3n + 1)/2 Dec. 97, May 17

Ans. :

Let P(n) : 2 + 5 + 8 + ... (3n − 1) = n(3n + 1)/2

(i) Basis of induction :

For n = 1;

$$(3(1) - 1) = 1(3(1) + 1)/2$$
$$2 = \frac{4}{2}$$
$$2 = 2$$
$$\therefore \quad \text{L.H.S.} = \text{R.H.S.}$$

Hence, the given statement is true for n = 1

∴ P(1) is true.

(ii) Induction step :

Assume P(k) is true i.e.

$$2 + 5 + 8 + \ldots + (3k - 1) = k(3k + 1)/2 \quad \ldots(i)$$

To prove that the statement is true for P(k + 1)

i.e. $2 + 5 + 8 + \ldots + [3(k + 1) - 1] = (k + 1)[3(k + 1) + 1]/2$

$$\text{L.H.S} = 2 + 5 + 8 + \ldots + (3k - 1) + [3(k + 1) - 1]$$
$$= \frac{k(3k + 1)}{2} + 3(k + 1) - 1 \quad \ldots\text{from Equation (i)}$$
$$= \frac{(3k^2 + k + 6k + 6 - 2)}{2} = \frac{(3k^2 + 7k + 4)}{2}$$
$$= \frac{(3k^2 + 3k + 4k + 4)}{2} = \frac{(3k + 4)(k + 1)}{2}$$
$$= (k + 1)[3(k + 1) + 1]/2$$
$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Hence, given that the statement is true for n = k, it is true for n = k + 1.

Hence it has been proved by mathematical induction.

Q. 3 Show that (~ P∧ (~ Q ∧ R)) V (Q ∧ R) V (P ∧ R) ⇔ R May 16, May 17

Ans. :

Let S denote (Q ∧ R) ∨ (P ∧ R) and Z denote (~ P ∧ (~ Q ∧ R)) ∨ (Q ∧ R) ∨ (P ∧ R)

(In this example we have to show L.H.S. and R.H.S. are same.)

P	Q	R	Q∧R	P∧R	~P	~Q	(~Q∧R)	~P∧(~Q∧R)	S	Z
T	T	T	T	T	F	F	F	F	T	T
T	T	F	F	F	F	F	F	F	F	F
T	F	T	F	T	F	T	T	F	T	T
T	F	F	F	F	F	T	F	F	F	F
F	T	T	T	F	T	F	F	F	T	T
F	T	F	F	F	T	F	F	F	F	F
F	F	T	F	F	T	T	T	T	F	T
F	F	F	F	F	T	T	F	F	F	F

Hence, column Z and R are same.

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Q. 4 **Show that $n(n^2 - 1)$ is divisible by 24, where n is any odd positive integer.** **Dec. 16**

Ans. :

Let $P(n) : n(n^2 - 1)$ is divisible by 24.

(i) **Basis of induction :** For $n = 1$

$P(1) : 1((1)^2 - 1) = 0$ which is divisible by 24.

$\therefore$ P(1) is true.

(ii) **Induction step :** Assume P(k) is true i.e;

$P(k) : k(k^2 - 1)$ is divisible by 24,

where k is odd integer.

Now, the next odd integer is (k + 2); hence prove that P(k + 2) is also true.

$$\begin{aligned} P(k+2) : & (k+2)\left[(k+2)^2 - 1\right] \\ &= (k+2)\left[k^2 + 4k + 3\right] \\ &= (k+2)(k+1)(k+3) \\ &= k(k+1)(k+3) + 2(k+1)(k+3) \\ &= k(k+1)(k-1+4) + 2(k+1)(k+3) \\ &= k(k+1)(k-1) + 4k(k+1) + 2(k+1)(k+3) \\ &= k(k^2-1) + 2(k+1)[2k+k+3] \\ &= k(k^2-1) + 2(k+1)(3k+3) \\ &= k(k^2-1) + 6(k+1)^2 \end{aligned}$$

Now, $6(k+1)^2$ is divisible by 24 for all odd n and $k(k^2 - 1)$ is divisible by 24 by induction hypothesis. Hence the whole expression is divisible by 24.

Therefore P(n) is also true.

Q. 5 **By using mathematical induction prove that $1 + a + a^2 + \ldots a^n = \frac{1 - a^{n+1}}{1 - a}$ where $n \geq 0$.** **May 16**

Ans. : Let $P(n) : 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$

(i) **Basis of induction :**

For $n = 0, P(0) = \frac{1 - a}{1 - a} = 1$

$\therefore$ It is true for $n = 0$

For $n = 1, P(1) = \frac{1 - a^{1+1}}{1 - a} = \frac{1 - a^2}{1 - a} = 1 + a$

$\therefore$ True for $n = 1$.

(ii) **Induction step :**

Assume P(k) is true i.e.

$$P(k) : 1 + a + a^2 + \ldots + a^k = \frac{1 - a^{k+1}}{1 - a} \quad \ldots(i)$$

To prove that P(k + 1) is true.

$$P(k+1) : 1 + a + a^2 + \ldots + a^{k+1} = \frac{1 - a^{(k+1)+1}}{1 - a}$$

$$\begin{aligned} \text{L.H.S.} &= 1 + a + a^2 + \ldots + a^k + a^{k+1} \\ &= \frac{1 - a^{k+1}}{1 - a} + a^{k+1} \quad \ldots\text{from Equation (i)} \\ &= \frac{1 - a^{k+1} + a^{k+1} - a^{k+2}}{1 - a} = \frac{1 - a^{k+2}}{1 - a} \\ &= \frac{1 - a^{(k+1)+1}}{1 - a} \end{aligned}$$

$\therefore$ It is true for $n = k + 1$. Hence, P(n) is true for all n.

Q. 6 **Show that $(\neg q \wedge (p \Rightarrow q)) \Rightarrow \neg p$ is a tautology.** **Dec. 15**

Ans. : We need only to show that $p \rightarrow q$ and $\sim q$ both true imply $\sim p$ is true.

Since truth value of $\sim q$ is T, truth value of q is F. Since $p \rightarrow q$ is true, this means that p is false, i.e. truth value of $\sim p$ is T. Hence the proof.

Q. 7 **It was found that in a class, 80 students are passed in English, 60 in Science and 50 in Mathematics. If was also found that 30 students passed in both English and Science, 15 students passed in both English and Mathematics and 20 students passed in both Mathematics and Science, 10 students passed in all three subjects. If there are 150 students in the class, find.**

(i) How many students passed in at least one subject ?

(ii) How many students passed in English only ?

(iii) How many students failed in all three subjects ? **Dec. 15**

Ans. : 80 students are passed in English.

60 students are passed in Science.

50 students are passed in maths.

30 students are passed in both English and Science.

15 students are passed in both English and Mathematics.

20 students are passed in both Mathematics and Science.

10 students are passed in all 3 subjects.

Let 'E' denote students passed in English.

Let 'S' denote students passed in Science.

Let 'M' denote students passed in Mathematics.

$\therefore |E| = 80 \quad |S| = 60 \quad |M| = 50$

$|E \cap S| = 30 \quad |E \cap M| = 15 \quad |M \cap S| = 20$

$|E \cap S \cap M| = 10$

Using principle of inclusion and exclusion.

$$\begin{aligned} |E \cup S \cup M| &= |E| + |S| + |M| - |E \cap S| \\ &\quad - |E \cap M| - |M \cap S| \\ &\quad + |E \cap S \cap M| \\ &= 80 + 60 + 50 + 30 - 15 - 20 + 10 \\ &= 200 - 65 = 135 \end{aligned}$$

(i) $\therefore$ 135 students passed in atleast one subject.

(ii) Students passed in only English $= |E| - |E \cap S| - |E \cap M| + |E \cap S \cap M|$

$= 80 - 30 - 15 + 10 = 90 - 45 = 45$

(iii) Students failed in all three subjects.

$= 150 - |E \cup S \cup M| = 150 - 135 = 15$

Q. 8 **Prove by mathematical induction $x^n - y^n$ is divisible by $x - y$.** **Dec. 14**

Ans. : Let $P(n) : x^n - y^n$ is divisible by $x - y$

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(i) **Basis of induction : For n = 1**

$$P(1) = x - y \text{ is divisible by } x - y$$

(ii) **Induction step :**

Assume p(k) is true. i.e. $x^k - y^k$ is divisible by $x - y$ and prove P(k + 1) is also true.

$$P(k+1) = x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$$

$x^k - y^k$ is divisible by $x - y$ induction by hypothesis and $x - y$ is divisible by $x - y$ by Basis of induction, hence $x^n - y^n$ is divisible by $x - y$.

Q. 9 Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction, $n \geq 1$.

Dec. 13, May 14

Ans. : Let $P(n) : 8^n - 3^n$ is a multiple of 5

(i) **Basis of induction :** For n = 1

$P(1) : 8^1 - 3^1 = 5$ which is divisible by 5

$\therefore$ P(1) is true.

(ii) **Induction step :** Assume P(k) is true i.e. $(8^k - 3^k)$ is a multiple of 5, and prove P(k + 1) is also true.

$$\begin{aligned} P(k+1) &: 8^{k+1} - 3^{k+1} \\ &= 8^k \cdot 8 - 3^k \cdot 3 \\ &= 8^k \cdot 8 - 3^k \cdot 8 + 3^k \cdot 5 \\ &= 8(8^k - 3^k) + 3^k \cdot 5 \end{aligned}$$

Now $8(8^k - 3^k)$ is a multiple of 5 by induction hypothesis and $3^k \cdot 5$ is already a multiple of 5.

Hence $8^n - 3^n$ is a multiple of 5, for $n \geq 1$.

Q. 10 Prove by Mathematical induction method

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Dec. 09, May 13

Ans. : Let P (n) be the statement :

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(i) **Basis of induction : For n = 1**

$$P(1) : 1 = \frac{6}{6} = 1$$

Hence P (1) is true.

(ii) **Induction Step : Assume P (k) is true,**

$$P(k) : 1^2 + 2^2 + 3^2 + \ldots + k^2 = k(k+1)(2k+1) \quad \ldots(i)$$

(This assumption is called the induction hypothesis)

Prove P (k + 1) is also true.

$$P(k+1) : 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

$$= \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$$

$$= \frac{k+1(k+2)(2k+3)}{6} \quad \ldots(ii)$$

Using induction hypothesis (i)

$$\ldots\left[1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}\right]$$

Put Equation (i) in Equation (ii) we get

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k+1(k+2)(2k+3)}{6}$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] = \frac{k+1(k+2)(2k+3)}{6}$$

$$= (k+1)\left[\frac{2k2 + k + 6k + 6}{6}\right] = \frac{k+1(k+2)(2k+3)}{6}$$

$$= (k+1)\left[\frac{2k2 + 7k + 6}{6}\right] = \frac{k+1(k+2)(2k+3)}{6}$$

$$= \frac{k+1(k+2)(2k+3)}{6} = \frac{k+1(k+2)(2k+3)}{6}$$

Hence assuming P (k) is true, P (k + 1) is also true. Therefore P (n) is true by principle of mathematical induction for all n.

Q. 11 State the converse, inverse and contrapositive of the following.

(i) If it is cold, then he wears hat.

(ii) If integer is a multiple of 2, then it is even.

May 13

Ans. :

(i) If it is cold, then he wears hat.

Converse : If he wears hat then it is cold.

Contrapositive : If he do not wears hat, then it is not cold.

Inverse : If it is not cold then he do not wears hat.

(ii) If integer is multiple of 2, then it is even.

Converse : If integer is even, then it is multiple of 2.

Contrapositive

Q. 12 Use induction to prove that $7^n - 1$ is divisible by 6 for n = 1, 2, 3, ...

May 12

Ans. : Let $P(n) = 7^n - 1$

(i) **Basic induction :**

For n = 1, $7^1 - 1 = 6$ divisible by 6.

(ii) **Induction step :**

Assume that 7^{k-1} is dividible by 6.

Now put n = k + 1

$$P(k+1) = 7^{k+1} - 1$$

$$\text{We have, } 7^{k+1} = (7^k \cdot 7 - 7) + 6$$

$$= 7(7^k - 1) + 6$$

By induction hypothesis $7^k - 1$ is divisible by 6.

$\therefore$ Each term on the R.H.S. is divisible by 6.

$\therefore$ $7^{k-1} - 1$ divisible by 6.

Hence 7^{n-1} is divisible by 6.

Q. 13 Prove that (Use laws of logic)

$((P \vee Q) \wedge \neg (\neg P \wedge (\neg Q \vee \neg R)) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is tautology.

May 06, Dec. 06, May 07, Dec. 11, May 12

Ans. :

$$((P \vee Q) \wedge \sim(\sim P \wedge (\sim Q \vee \sim R))) \vee ((\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R))$$

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$\Rightarrow$ $(P \vee Q) \wedge \sim(\sim P \wedge \sim(Q \wedge R)) \vee (\sim(P \vee Q) \vee \sim(P \vee R))$

$\Rightarrow$ $(P \vee Q) \wedge \sim\sim(P \vee (Q \wedge R)) \vee (\sim(P \vee Q) \vee \sim(P \vee R))$

$\Rightarrow$ $(P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R)) \vee \sim((P \vee Q) \wedge (P \vee R)$

$\Rightarrow$ $((P \vee Q) \wedge (P \vee R)) \vee \sim(P \vee Q) \wedge (P \vee R))$

$\Rightarrow$ True : It is tautology.

Q. 14 Write English sentences corresponding to following :

(i) $\forall x \exists y R(x, y)$

(ii) $\exists x \forall y R(x, y)$

(iii) $\forall x (\sim Q(x))$

(iv) $\exists y (\sim P(y))$ Dec. 08

(v) $\forall x P(x)$

where P (x) : x is even

Q (x) : x is prime no.

R (x, y) : x + y is even

Dec. 05, Dec. 11

Ans. :

(i) $\forall x \exists y R(x, y)$
For all x there exists y, such that x + y is even integer.

(ii) $\exists x \forall y R(x, y)$
For all y there exists x, such that x + y is even integer.

(iii) $\forall x (\sim Q(x))$
For all x, x is not prime numbers.

(iv) $\exists y (\sim P(y))$
There exists y, such that y is odd.

(v) $\forall x P(x)$
For all x, x is even.

Q. 15 Translate the following statement into symbolic form :

If the utility cost goes up or the request for additional funding is desired, then a new company will be purchased if and only if we can show that the current computing facilities are indeed not adequate. May 11

Ans. :

p : The utility cost goes up

q : The request for additional funding is desired

r : A new computer will be purchased

s : We can show that the current computing facilities are indeed adquate.

Then $(p \vee q) \rightarrow (r \leftrightarrow \sim s)$ is the required symbolic form.

Q. 16 Using Mathematical Induction prove that $5^n - 1$ is divisible by 4 for n > = 1 Dec. 10

Ans. : Let $P(n) : 5^n - 1$

(i) **Basis of induction :**

For $n = 1$, $5^1 - 1 = 4$, divisible by 4

(ii) **Induction step :**

Assume that $5^k - 1$ is divisible by 4.

We have $5^{k+1} - 1 = (5^k \cdot 5 - 5) + 4 = 5(5^k - 1) + 4$

By induction hypothesis $5^k - 1$ is divisible by 4.

$\therefore$ Each term on the RHS is divisible by 4.

$\therefore$ $5^{k+1} - 1$ is divisible by 4.

Hence $5^n - 1$ is divisible by 4 for $n \geq 1$.

Q. 17 Prove that if x is a rational number and y is an irrational number, then x + y is an irrational number. May 10

Ans. :

$$x + y = \frac{p_1}{q_1} \quad \ldots(1)$$

$p, q \in I'$ I is integer

$$x = \frac{p_2}{q_2}$$ and y is irrational

Equation (1) is

$$\frac{p_2}{q_2} + y = \frac{p_1}{q_1}$$

$$y = \frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}$$

So $y = Q$ Q is contradiction

Q. 18 Converse of statement is given.

Write inverse and contra positive of statement "If I come early then I can get a car". May 10

Ans. :

Converse of the statement is given,

$\therefore$ p : I can get the car. q : I come early.

$\therefore$ Inverse of the statement is : " If I cannot get the car, then I shall not come early."

Contrapositive of the statement is :

"If I do not come early, then I cannot get the car."

Q. 19 Use mathematical induction to prove the following inequality $n < 2^n$ for all positive integers n. Dec. 06, May 10

Ans. : Basis step : P (1) is true since $1 < 2^1 = 2$ for n = 1

Inductive step : Assume P (k) is true for the positive integer k.

Assume P (k) i.e. $k < 2^k$ is true. We need to show that P (k + 1) is true i.e. $k + 1 < 2^{k+1}$ is true. Adding 1 to both sides of $k < 2^k$ we get,

$$k + 1 < 2^k + 1$$

We know that $1 \leq 2^k$, we get,

$$k + 1 < 2^k + 1 \leq 2^k + 2^k \leq 2^{k+1} = 2^{k+1}$$

So we have shown that P (k + 1) is true. The inductive step is complete.

Q. 20 Negate the statement.

For all real numbers x, if x > 3 then $x^2 > 9$. May 09

Ans. :

Given Statement : For all real numbers x, if x > 3 then $x^2 > 9$.

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Let P(x) : if $x > 3$ then $x^2 > 9$, since we can write given statement in symbolic form as follows :

$$\forall x, P(x)$$

The negation of $\forall x, P(x)$ is $\exists x \mid$ not P (x). Hence negation of given statement is given below :

There exist an x such that, if $x < 3$ then $x^2 < 9$.

Q. 21 **Let P denote Raju is rich, q denote Raju is happy. Write each of the following in symbolic form.**

1. **Raju is poor but happy**
2. **Raju is neither rich nor happy**
3. **Raju is either rich or unhappy**
4. **Raju is poor or else he is rich and unhappy** Dec. 96

Ans. :

(i) $\sim p \wedge q$ (ii) $p \vee \sim q$
(iii) $\sim p \wedge \sim q$ (iv) $\sim p \vee (p \wedge \sim q)$

Q. 22 **Using the following statements :**

p : I will study discrete structures.
q : I will go to a movie.
r : I am in a good mood.

Write the following statements in symbolic form :

(i) **If I am not in a good mood, then I will go to a movie.**
(ii) **I will not go to a movie and I will study discrete structures.**
(iii) **I will go to a movie only if I will not study discrete structures.**
(iv) **I will not study discrete structures, then I am not in a good mood.** Dec. 96

Ans. :

(i) $\sim r \rightarrow q$ (ii) $\sim q \wedge p$
(iii) $q \rightarrow \sim p$ (iv) $\sim p \rightarrow \sim r$

Q. 23 **Write the following statements in symbolic form :**

(i) **Indians will win the world cup if their fielding improves.**
(ii) **If I am not in a good mood or I am not busy, then I will go for a movie.**
(iii) **If you know object oriented programming and oracle, then you will get a job.**
(iv) **I will score good marks in the exam if and only if I study hard.** May 96

Ans. :

(i) Let p : Indians will win the world cup.
q : Their fielding improves.

Then $q \rightarrow p$ is the required form.

(ii) Let p : I am in a good mood.
q : I am busy.
r : I will go for a movie.

Then $(\sim p \vee \sim q) \rightarrow r$ is the required symbolic form.

(iii) Let p : You know object oriented programming.
q : You know oracle.
r : You will get a job.

Then $p \wedge q \rightarrow r$ is the required symbolic form.

(iv) Let p : I will score good marks in the exam.
q : I study hard.

Then $p \leftrightarrow q$ is the required symbolic form.

Q. 24 **Let p be 'He is tall', and Let q be 'He is handsome.' Write each of the following statements in symbolic form.**

1. **He is tall and handsome**
2. **He is tall but not handsome**
3. **It is false that he is short or handsome**
4. **He is neither tall nor handsome**
5. **He is tall, or he is short and handsome.**
6. **It is not true that he is short or not handsome.** Dec. 94

Ans. :

1. $p \wedge q$ 2. $p \wedge \sim q$
3. $\sim(\sim p \vee q)$ 4. $\sim p \wedge \sim q$
5. $p \vee (\sim p \wedge q)$ 6. $\sim(\sim p \vee \sim q)$

Q. 25 **Write English sentences corresponding to the following statement.**

(i) $((\sim p) \wedge q) \rightarrow r$.
(ii) $r \rightarrow (p \vee q)$.
(iii) $(\sim r) \rightarrow ((\sim q) \vee p)$.
(iv) $(q \vee (\sim p)) \leftrightarrow r$.

Where p : I will study discrete structures.
q : I will go to a movie.
r : I am in a good mood. Dec. 96

Ans. :

(i) $((\sim p) \wedge q) \rightarrow r$: If I do not study discrete structures and I go to a movie, then I am in a good mood.
(ii) $r \rightarrow (p \vee q)$: If I am in a good mood, then I will study discrete structure or I will go to a movie.
(iii) $(\sim r) \rightarrow ((\sim q) \vee p)$: If I am not in good mood, then I will not go to a movie or I will study discrete structure.
(iv) $(q \vee (\sim p)) \leftrightarrow r$: I will go to a movie or I will not study discrete structure if and only if I am in a good mood.

Q. 26 **To describe the various restaurants in the city, let p denote the statement "the food is good", q the statement "the service is good" and r the statement "the rating is three-star". Write the following statements in symbolic form :**

(i) **Either the food is good or service is good, or both.**

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(ii) Either the food is good or service is good, but not both.

(iii) The food is good while the service is poor.

(iv) It is not the case that both the food is good and the rating is three-star.

(v) If both the food and service are good, then the rating is three star.

(vi) It is not true that a three star rating always means good food and good service. **May 93**

Ans. :

(i) As "$\vee$" means "inclusive or", the statement required is $p \vee q$.

(ii) We have to use "exclusive or" i. e. "$\bar{\vee}$". Hence the statement is $p \bar{\vee} q$. Equivalently the statement is also $p \wedge \bar{q} \vee \bar{p} \wedge q$.

(iii) "While" is interpreted as "and". Hence the statement is $p \wedge \sim q$.

(iv) "It is not the case" implies negation. Hence the statement is $\sim (p \wedge r)$.

(v) The statement is $(p \wedge q) \rightarrow r$.

(vi) The statement is $\sim (r \rightarrow (p \wedge q))$.

Q. 27 **Using the following propositions :**

p : I am bored

q : I am waiting for one hour

r : There is no bus

translate the following into English.

(i) $(q \vee r) \rightarrow p$.

(ii) $\sim q \rightarrow \sim p$.

(iii) $(q \rightarrow p) \vee (r \rightarrow p)$. **May 94**

Ans. :

(i) If I am waiting for one hour or there is no bus, then I get bored.

(ii) If I am not waiting for one hour, then I am not bored.

(iii) If I am waiting for one hour then I am bored, or if there is no bus, then I am bored.

Q. 28 **Translate into symbolic form** **May 98**

If I play football, I cannot study.

Either I play football or I can pass DBMS.

I played football, therefore I passed DBMS.

Ans. :

Let p : I play football

q : I pass DBMS

r : I cannot study

(i) If I play football, I cannot study. In Symbolic form : $p \rightarrow r$

(ii) Either I play football, or I can pass DBMS. In symbolic form : $p \vee q$.

(iii) I played football, therefore I passed DBMS. In symbolic form : $p \leftrightarrow q$

Q. 29 **Determine the contrapositive of each statement.** **Dec. 2000**

(i) **If he has courage he will win.**

(ii) **It is necessary to be strong in order to be a sailor.**

(iii) **Only if he does not tire will he win.**

(iv) **It is sufficient for it to be a square in order to be a rectangle.**

Ans. :

(i) If he does not win, then he does not have courage.

(ii) If he is not strong, then he is not a sailor.

(iii) If he tires, then he will not win.

(iv) If it is not a rectangle, then it is not a square.

Q. 30 **Construct the truth table for the formula :** $\sim [p \wedge (p \vee \sim q)]$ **May 99**

Ans. :

p	q	~q	p∨~q	p∧(p∨~q)	~[p∧(p∨~q)]
T	T	F	T	T	F
T	F	T	T	T	F
F	T	F	F	F	T
F	F	T	T	F	T

Q. 31 **Compute the truth table of the statement, $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. Is this statement a tautology ?** **Dec. 96**

Ans. : We prepare the truth table as follows :

p	q	~p	~q	p→q	~q→~p	(p→q)↔(~q→~p)
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Q. 32 **Prove there is no rational number p/q whose square is 2.** **May 09**

Ans. :

A rational number is any number that can be expressed in the form p = q, where p and q are integers. Thus; what the theorem asserts is that no matter how p and q are chosen, it is never the case that $(p = q)^2 = 2$. The line of attack is indirect, using a type of argument referred to as a proof by contradiction. The idea is to assume that there is a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false. And so assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2 \qquad ...(1)$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, Equation (1) implies

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$$p^2 = 2q^2 \quad \ldots(2)$$

From this, we can see that the integer p^2 is an even number (it is divisible by 2), and hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute 2r for p in Equation (2), then a little algebra yields the relationship

$$2r^2 = q^2$$

But now the absurdity is at hand. This last equation implies that q^2 is even, and hence q must also be even. Thus, we have shown that p and q are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that Equation (1) cannot hold for any integers p and q, and thus the theorem is proved.

Q. 33 $A \vee B = (A \wedge \sim B) \vee (B \wedge \sim A) \vee (A \wedge B)$. **Dec. 02**

Ans. :

A	B	~B	A ∧ ~B	~A	B ∧ ~A	A ∧ ~B ∨ B ∧ ~A	A ∧ B	(A ∧ B) ∨ (B ∧ ~A) ∨ (A ∧ B)	A ∨ B
T	T	F	F	F	F	F	T	T	T
T	F	T	T	F	F	T	F	T	T
F	T	F	F	T	T	T	F	T	T
F	F	T	F	T	F	F	F	F	F

Q. 34 Prove the following : $[(p \vee q) \wedge (p \vee \sim q) \vee q \leftrightarrow p \vee q$ **Dec. 05**

Ans. :

p	q	p ∨ q	~q	p ∨ ~q	(p ∨ q) ∧ (p ∨ ~q)	(p ∨ q) ∧ (p ∨ ~q) ∨ q
F	F	F	T	T	F	F
F	T	T	F	F	F	T
T	F	T	T	T	T	T
T	T	T	F	T	T	T

Q. 35 **Given the truth value of P and Q as T and those of R and S as F, find the truth value of the following.**

$[P \wedge (Q \wedge R)] \vee \sim [(P \vee Q) \vee (R \vee S)]$

May 99, May 2000

Ans. : We substitute the given truth values in the expression and it reduces to

$$[T \wedge (T \wedge F)] \vee \sim [(T \vee T) \vee (F \vee F)]$$
$$\equiv [T \wedge F] \vee \sim [T \vee F]$$
$$\equiv F \vee \sim (T) = F \vee F \equiv F$$

Hence the truth value of the above expression is false.

Q. 36 **Show that :**

(i) $a \vee (a \wedge b) = a$

(ii) $a \wedge (a \vee b) = a$ **May 97**

Ans. :

(i) Using Identity law, we write

$a \vee (a \wedge b) = (a \wedge T) \vee (a \wedge b)$

where, $T \equiv$ True

Again, $a \vee (a \wedge b) \equiv a \wedge (T \vee b)$ - Distributive law

$\equiv a \wedge T$ - Identity law

$\equiv a$ - Identity law

$\therefore \quad a \vee (a \wedge b) \equiv a$

(ii) Using Identity law, we write

$a \wedge (a \vee b) \equiv (a \wedge T) \wedge (a \vee b)$

Where, $T \equiv$ true

$\therefore \quad a \wedge (a \vee b) \equiv a \vee (T \wedge b)$

- Distributive law

$\equiv a \wedge T$

- Identity law

$\equiv a$

- Identity law

$\therefore \quad a \wedge (a \vee b) \equiv a$

Q. 37 **Show that** **Dec. 96**

(I) $a \vee (\bar{a} \wedge b) = a \vee b$

(II) $a \wedge (\bar{a} \vee b) = a \wedge b$

Ans. :

(i) $a \vee (\bar{a} \wedge b) \equiv (a \vee \bar{a}) \wedge (a \vee b)$ - Distributive law

$\equiv 1 \wedge (a \vee b)$ - Complement law

$\equiv a \vee b$ - Identity law

Hence proved

(ii) $a \wedge (\bar{a} \vee b) \equiv (a \wedge \bar{a}) \vee (a \wedge b)$ - Distributive law

$\equiv 0 \vee (a \wedge b)$ - Complement law

$\equiv a \wedge b$ - Identity law

Hence proved.

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Q. 38 **Show the following equivalence without considering the truth table.**

$$(\bar{p} \wedge (\bar{q} \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \leftrightarrow r$$

Dec. 98, Dec. 02

Ans. :

$(q \wedge r) \vee (\bar{q} \wedge r \wedge \bar{p}) \vee (r \wedge p)$

$\equiv r \wedge (q \vee (\bar{q} \wedge \bar{p}) \vee p)$ - Distributive law

$\equiv r \wedge ([(\bar{q} \vee \bar{q}) \wedge (q \vee \bar{p})] \vee p$ - Distributive law

$\equiv r \wedge ([T \wedge (q \vee \bar{p})] \vee p)$ - Complement law

$\equiv r \wedge ([q \vee \bar{p}] \vee p)$ - Identity law

i.e. $p \wedge T \Leftrightarrow p$ $\equiv r \wedge (q \vee (\bar{p} \vee p))$ - Associative law

$\equiv r \wedge (q \vee T)$ - Complement law

$\equiv r \wedge T$ - Identity law

$\equiv r$ - Identity law

Q. 39 **Simplify the following Boolean expressions. Justify each step.** **Dec. 97**

(i) $((a \wedge \bar{b}) \vee c) \vee (a \vee \bar{b}) \wedge c$

(ii) $(a \wedge b) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (b \wedge c)$

Ans. :

(i) $((a \wedge \bar{b}) \vee c) \vee (a \vee \bar{b}) \wedge c$

$\equiv (a \vee c) \wedge (\bar{b} \vee c) \wedge (a \vee \bar{b}) \wedge c$ - Distributive law

$\equiv (a \vee c) \wedge c \wedge (\bar{b} \vee c) \wedge (a \vee \bar{b})$ - Associative law

$\equiv c \wedge (\bar{b} \vee c) \wedge (a \vee \bar{b})$ - Absorption law

i.e. $(a \vee c) \wedge c \Leftrightarrow c$ $\equiv c \wedge (a \vee \bar{b})$ - Absorption law

i.e. $c \wedge (\bar{b} \vee c) \Leftrightarrow c$

$\therefore$ $c \wedge (a \vee \bar{b})$ is Simplified form

(ii) $(a \wedge b) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (b \wedge c)$

$\equiv b \wedge (a \vee (\bar{a} \wedge \bar{c}) \vee c)$ - Distributive law

$\equiv b \wedge ([(a \vee \bar{a}) \wedge (a \vee \bar{c})] \vee c$ - Distributive law

$\equiv b \wedge ([T \wedge (a \vee \bar{c})] \vee c)$ - Complement law

$\equiv b \wedge ([a \vee \bar{c}] \vee c)$ - Identity law

i.e. $p \wedge T \Leftrightarrow p$ $\equiv b \wedge (a \vee (\bar{c} \vee c))$ - Associative law

$\equiv b \wedge (a \vee T)$ - Complement law

$\equiv b \wedge T$ - Identity law

$\equiv b$ - Identity law

Q. 40 **Using laws of logic proves the following.**

$$p \vee q \vee (\sim p \wedge \sim q \wedge r) \leftrightarrow p \vee q \vee r$$

May 04

Ans. :

L.H.S. $= p \vee q \vee (\sim p \wedge \sim q \wedge r)$

$\equiv p \vee q \vee (\sim p \wedge \sim q \vee \sim r)$

$\equiv p \vee q \vee [(\sim p \vee \sim r) \wedge (\sim q \vee \sim r)]$

$\equiv p \vee q \vee [\sim (p \wedge r) \wedge \sim (q \wedge r)]$

$\equiv p \vee q \vee [(p \wedge r) \vee (q \wedge r)]$

$\equiv p \vee q \vee [(p \vee q) \wedge r]$

$\equiv p \vee q \vee r$

Q. 41 **Prove the following logical equivalence (use laws of logic).**

(I) $[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \leftrightarrow p \wedge q$

(II) $(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)] \leftrightarrow \sim (q \vee p)$

Dec. 03

Ans. :

(i) L.H.S. $\equiv (\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)$

$\equiv \sim (\sim p \vee \sim q) \vee (p \wedge q \wedge r)$ - Implication law

$\equiv (p \wedge q) \vee (p \wedge q \wedge r)$ - De morgan's law

Let $p \wedge q$ be x

$\equiv x \vee (x \wedge r)$

$\equiv x$ - Absorption law

$\equiv p \wedge q$

(ii) $(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)] \leftrightarrow \sim (q \vee p)$

$\equiv (\sim p \vee q) \wedge [\sim q \wedge (r \vee \sim q)]$ - Implication law

$\equiv (\sim p \vee q) \wedge [\sim q \wedge (\sim q \vee r)]$

$\equiv (\sim p \vee q) \wedge \sim q$ - Absorption law

$\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q)$ - Distributive law

$\equiv (\sim p \wedge \sim q) \vee F$ - Complement law

$\equiv \sim (p \vee q) \vee F$ - De morgan's law

$\equiv \sim (q \vee p) \vee F$

$\equiv \sim (q \vee p)$ - Identity law

Q. 42 **Show that $(p \wedge q) \rightarrow (p \wedge q)$ is a tautology. Use laws of logic.** **May 05**

Ans. :

$(p \wedge q) \rightarrow (p \wedge q)$

$\equiv \sim (p \wedge q) \vee (p \wedge q)$ - Implication law

Let $p \wedge q$ be x.

$\equiv \sim x \vee x$

$\equiv T$ - Complement law

Q. 43 **Prove that**

$1.1! + 2.2! + 3.3! + \ldots n.n! = (n + 1)! - 1$,

where n is a positive integer. **Dec. 05**

Ans. :

Let $P(n) = 1.1! + 2.2! + 3.3! + \ldots n.n! = (n + 1)! - 1$

(i) **Basis of induction :**

For $n = 1$, $1.1! = 1$ and $(1 + 1)! - 1 = 2! - 1 = 1$

$\therefore$ P(1) is true.

(ii) **Induction step :**

Assume P(k) is true, i.e,

$1.1! + 2.2! + 3.3! + \ldots k.k! = (k + 1)! - 1$ ...(i)

To prove that P(k + 1) is true,

$P(k + 1) : 1.1! + 2.2! + 3.3! + \ldots (k + 1) \cdot (k + 1)! = (k + 2)! - 1$

L.H.S $= 1.1! + 2.2! + 3.3! + \ldots k\,k! + (k + 1) \cdot (k + 1)!$

$= (k + 1)! - 1 + (k + 1) \cdot (k + 1)!$

(by induction hypothesis).

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$= (k+1)! + (k+1)\cdot(k+1)! - 1$

$= (k+1)!\,(1+k+1) - 1$

$= (k+1)!\,(k+2) - 1 = (k+2)! - 1$

L.H.S. = R.H.S.

Hence the result is proved.

Q. 44 Prove by mathematical induction that, $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$. Dec. 99, Dec. 2000

Ans. :

Let P(n) : $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.

(i) **Basis of induction :**

For n = 2

$P(2) : 2^4 - 4(2)^2$ is divisible by 3

$\therefore$ P(2) is true.

(ii) **Induction step :**

Assume P(k) is true.

i.e. $(k^4 - 4k^2)$ is divisible by 3.

Assume P(k) is true, and prove P(k + 1) is also true.

For n = k + 1 we have,

$(k+1)^4 - 4(k+1)^2$

$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4k^2 - 8k - 4$

$= k^4 + 4k^3 + 2k^2 - 4k - 3$

$= k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3$

$= k^4 - 4k^2 + 3(2k^2 - 1) + 4k^3 - 4k$

$= k^4 - 4k^2 + 3(2k^2 - 1) + 4k(k^2 - 1)$

$= (k^4 - 4k^2) + 3(2k^2 - 1) + 4k(k-1)(k+1)$

Now $(k^4 - 4k^2)$ is divisible by 3 by induction hypothesis, as P(k) is true $3(2k^2 - 1)$ is divisible by 3 and $4k(k-1)(k+1)$ is divisible by 3, as it has a factor which is the product of three integers which are consecutive.

Hence P(n) is divisible by 3 for all $n \geq 2$.

Q. 45 Prove that $n! \geq 2^n$ for $n \geq 4$ Dec. 03

Ans. : Let P(n) : $n! \geq 2^n$

(i) **Basis of induction :**

For n = 4

P(4) : $4! \geq 2^4$

$\therefore 24 \geq 16$

$\therefore 4! \geq 2^4$

$\therefore$ P(4) is true.

(ii) **Induction step :**

Assume P(k) is true i.e.; $k! \geq 2^k$.

Prove that P(k + 1) is also true.

For n = k + 1 ...(i)

$P(k+1) : (k+1)! \geq 2^{k+1}$ ...(ii)

L.H.S. $= (k+1)\,k! \geq 2^k(k+1)$ ...from Equation (i)

$\geq 2^k(2) \geq 2^{k+1}$

$(k+1)! \geq 2^{k+1}$

i.e. $\geq$ R.H.S. of Equation (ii)

Hence $n! \geq 2^n$ for $n \geq 4$

Q. 46 Prove induction to show that $n! \geq 2^{n-1}$ for n = 1, 2, ... May 03

Ans. : Let P (n) be the given statement

(i) **Basis of induction :**

For n = 1, $1! \geq 2^{1-1}$

i.e. $1 \geq 1$ which is true

$\therefore$ P (n) is true for n = 1.

(ii) **Induction step :** We now assume that it is true for n = k.

i.e. $k! \geq 2^{k-1}$

To prove that statement is true for n = k + 1

Now $(k+1)! = (k+1)\cdot k!$

$\geq (k+1)2^{k-1} \geq 2\cdot 2^{k-1}$ Since $k+1 \geq 2$

$= 2^k$

Hence P (n) is true for n = m + 1

$\therefore$ P (n) is true for all $n \geq 1$..

Chapter 3 : Relations, Digraphs and Lattice

Q. 1 Explain the Equivalence Class. Dec. 08

Ans. : Equivalence Class :

Let R be an equivalence relation on a set A. For every $a \in A$, let $[a]_R$ (or R (a)) denote the set $\{x \in A \mid x\,R\,a\}$. Then $[a]_R$ is called as the **equivalence class of a with respect to R.**

$[a]_R \neq \phi$ since $a \in [a]_R$

The 'rank' of R is the number of distinct equivalence classes of R if the number of classes is finite; otherwise the rank is said to be infinite.

In what follows, we will drop the suffix R, and denote the equivalence class of a simply as [a].

$[a]_R$ is also denoted as R(a).

The following theorem gives an important characterisation of the equivalence classes.

Theorem 1 : Let R be an equivalence relation on a set A. The following hold :

(i) For all $a, b, \in A$, either [a] = [b] or $[a] \cap [b] = \phi$

(ii) $A = \approx_{a \in A} [a]$

Proof :

(i) If $A = \phi$, there is nothing to prove. Hence assume $A \neq \phi$. If A = {a}, singleton set, the result is trivially true. Therefore consider elements $a, b, \in A$. Suppose $[a] \neq [b]$. Then we have to show that $[a] \cap [b] = \phi$. Suppose this is not true. Let $c \in [a] \cap [b]$ then c R a and c R b. Since R is symmetric it follows that a R c and c R b. But R is transitive as well. Hence, we have a R b, i.e. $b \in [a]$ and $a \in [b]$, which mean that [a] = [b], a contradiction. Hence, $[a] \cap [b] = \phi$.

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(ii) Clearly for each $a \in A$, $[a] \subseteq A$.

Hence, $\bigcup_{a \in A} [a] \subseteq A$. Conversely, let $x \in A$. Then $x \in [a]$ for some $a \in A$. This implies that $x R a$, i.e. $a R x$. Hence $a \in [x]$ which means that $[a] = [x]$. Therefore $A \subseteq \bigcup_{a \in A} [a]$.

Hence it follows that $A = \bigcup_{a \in A} [a]$.

Q. 2 Let A = {1, 2, 3, 4, 6} and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find Matrix of R. Find inverse relation of R. Dec. 11

Ans. :

$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (6,6)\}$

$R^{-1} = \{(1,1), (2,1) (3,1) (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,6)\}$

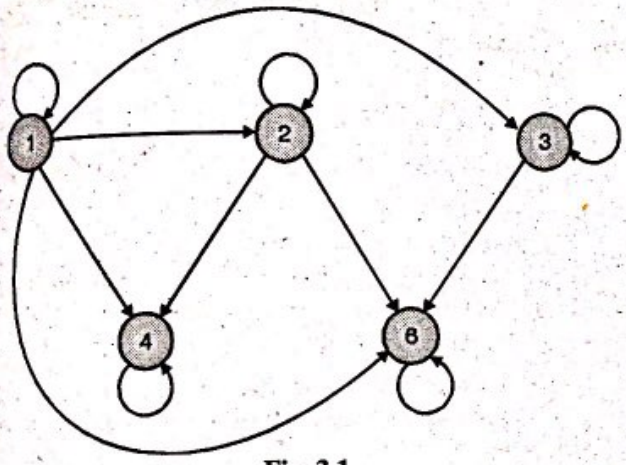


Fig. 3.1

Assume the rows and columns of M are each labelled 1,2,3,4,6, since R is relation from A to A, the matrix M_R is square, i.e. M_R has the same number of row as column

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Q. 3 Given A = {1, 2, 3, 4} and B {x, y, z}. Let R be the following relation from A to B :

R = {(1, y), (1, z), (3, y), (4, x), (4, z)}

(i) Determine the matrix of the relation.

(ii) Draw the arrow diagram of R.

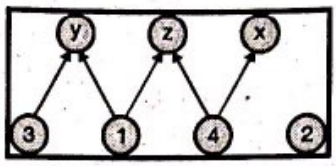
(iii) Find the inverse relation R^{-1} of R.

(iv) Determine the domain and range of R. Dec. 11

Ans. :

(i) Matrix =
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

(ii) **Diagraph :**



(iii) $R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$

(iv) Domain and range

DOM (R) = { 1, 3, 4 }

RAN (R) = { x, y, z }

Q. 4 Let R be the relation on set A. Prove that there is a path of length n from a to b if and only if $(a, b) \in R^n$. May 03

Ans. : Given R is relation on set A.

∴ The path of length n in R from A to B is a finite sequence.

$\pi_1 : a_1 x_1, x_2, x_3, \ldots\ldots x_{n-1} b$

beginning with a and ending with b such that $a R x_1, x_1, R x_2 \ldots x_{n-1} R b$

The path of length n involves n + 1 elements of A which may or may not be distinct.

For 'n' a positive integer we define relation R^n on A as $x R^n y$ which means that there is a path of length n from x to y in R.

It follows that path of length n from a to b.

if $a R^n b$ i.e. $(a, b) \in R^n$.

Q. 5 Determine whether the following relation are symmetric, asymmetric and antisymmetric.

(I) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (II) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Dec. 12

Ans. :

(i) Let $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

A = { 1, 2, 3 }

R = { (1, 1), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3) }

R is **irreflexive** because diagonal element (2, 2) is zero i.e. 2 R 2

R is **symmetric** because

1 R 3 and 3 R 1

2 R 3 and 3 R 2

Hence, R is **not asymmetric.**

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R is **not antisymmetric** since we have (1, 3) and (3, 1) ∈ R also (2, 3) and (3, 2) ∈ R.

(ii) Let $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Let A = { 1, 2, 3, 4 }

R = { (1, 1), (1, 4), (2, 2), (3, 3), (2, 3), (2, 4), (4, 4) }

R is reflexive since (1, 1), (2, 2), (3, 3), (4, 4) ∈ R i.e. all elements belonging to A are related to itself.

R is **assymetric**. Since (1, 4) ∈ R but (4, 1) ∉ R and (2, 3) ∈ R but (3, 2) ∉ R. Hence given relation R is **not symmetric**.

R is **antisymmetric**. Since whenever a R b and b R a, then a = b.

Q. 6 Give examples of relations R on A = {1, 2, 3} having the stated property.

(i) R is transitive but not symmetric.

(ii) R is symmetric but not transitive.

(iii) R is both symmetric and anti-symmetric.

(iv) R is neither symmetric nor anti-symmetric. May 98

Ans. :

(i) R = {(1, 2), (2, 3), (1, 3)}

(ii) R = {(1, 2), (2, 1)}

(iii) R = {(2, 3), (3, 2)}

(iv) R = {(1, 2), (2, 3)}

Q. 7 Define a relation on the set {a, b, c, d} that is

(i) transitive, reflexive and symmetric,

(ii) symmetric and transitive. Dec. 07

Ans. :

(i) **Transitive, reflexive and symmetric,**

A = { a, b, c, d }

R = { (a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (a, c), (c, a), (a, d), (d, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c) }

(ii) **Symmetric and transitive.**

A = { a, b, c, d }

R = { (a, b), (b, a), (c, d), (d, c), (a, a), (c, c)}

Q. 8 For each of these relations on the set {1, 2, 3, 4}, decide whether it is symmetric, whether it is reflexive, whether it is transitive and whether it is anti-symmetric. (One relation may satisfy more than one property.)

(i) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

(ii) {(1, 1), (2, 2), (3, 3), (4, 4)}

(iii) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)} May 03

Ans. :

Given A = {1, 2, 3, 4}

(i) Let R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R is reflexive since (1, 1), (2, 2), (3, 3), (4, 4) ∈ R

R is symmetric since (1, 2) ∈ R and (2, 1) ∈ R

R is transitive since if (1 R 2) and (2 R 1) then (1 R 1)

Hence R is reflexive, symmetric, transitive.

(ii) Let R = {(1, 1), (2, 2), (3, 3), (4, 4)}

Given relation R is only reflexive since (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

(iii) Let R = {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

R is not reflexive, since no element is related to itself.

R is not symmetric since

(1 R 3) and (3 R 1) but (1 $\not$R 1)

(2 R 3) and (3 R 1) but (2 $\not$R 1)

(3 R 1) and (1 R 3) but (3 $\not$R 3)

Hence relation is not reflexive, not symmetric, not transitive.

Q. 9 Show that if a relation on a set A is transitive and irreflexive, then it is asymmetric. Dec. 98, Dec. 08, Dec. 10, May 14

Ans. : Assume that a relation is transitive and irreflexive and let it be symmetric.

Let (a, b) and (b, c) belong to R and

Let (b, a) and (c, b) belong to R

∴ (a, c) ∈ R ... As R is transitive

Also (c, b) ∈ R

We have (b , c) ∈ R and (c, b) ∈ R

∴ (b, b) ∈ R ...As R is transitive.

But our relation is irreflexive → Assumption

∴ Our Assumption is false

∴ The relation is not symmetric i.e. it is asymmetric.

∴ If a relation on a set A is transitive and irreflexive, then it is asymmetric.

Q. 10 Define symmetric and anti-symmetric relation. Does there exist a relation which is symmetric and anti-symmetric. Justify your answer. Dec. 99, May 01

Ans. :

Let A be a set and R be a relation on A then :

(i) **Anti-symmetric relation :** A relation R on a set A is 'anti-symmetric' if whenever a R b and b R a, then a = b. The contrapositive of this definition is that R is anti-symmetric if whenever a ≠ b, then a $\not$R b or b $\not$R a. It follows that R is not anti-symmetric if we have a and b in A, a ≠ b, and both a R b and b R a.

(ii) **Symmetric relation :** A relation R on a set A is 'symmetric' if whenever a R b, then b R a. It then follows that R is not symmetric if we have some a and b ∈ A with a R b, but b $\not$R a. Consider the relation '=' defined on R certainly a = b implies b = a. So '=' is a symmetric relation on A. Surprisingly '=' is also anti-symmetric relation on R. This is due to the fact that the condition of the anti-symmetry property, a = b and a ≠ b is a contradiction.

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Q. 11 Show that if a relation on a set A is transitive and irreflexive, then it is symmetric. May 01

Ans. : Assume that a relation is transitive and irreflexive and let it be symmetric.

Let (a, b) and (b, c) ∈ R and let (b, a) and (c, b) ∈ R

∴ (a, c) ∈ R ... As R is transitive

Also as (c, b) ∈ R

we have (b, c) and (c, b) ∈ R

∴ (b, b) ∈ R ... As R is transitive.

But our relation is irreflexive → Assumption

∴ Our assumption is false.

∴ The relation is not symmetric, i.e. it is asymmetric.

∴ If a relation on a set A is transitive and irreflexive, then it is not symmetric (i.e. asymmetric)

Q. 12 Prove by induction that if a relation R on a set A is symmetric, then R^n is symmetric for $n \geq 1$. Dec. 07

Ans. :

(i) **Basis step :** For n = 1,
P (1) : If R is symmetric, then R' is symmetric is true.

(ii) **Induction step :** Use P(k) : If R is symmetric, then R^k is symmetric to show P (k + 1).
Suppose that a R^{k+1} b. Then there is a c ∈ A such that a R^kc and c R b. We have b R c and c R^k a. Hence b R^{k+1} a. Thus if a relation R on a set A is symmetric, then R^n is symmetirc for $n \geq 1$.

Q. 13 Define Equivalence relation with an example. May 2000, May 07, Dec. 10, May 11, May 15

Ans. : Equivalence relation : A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

The following are some of the common but important examples of equivalence relations.

Examples :

(i) Let A = IR and R be 'equality' of numbers.

(ii) Consider all subsets of a universal set and R be the relation, 'equality' of sets.

(iii) A is the set of triangles and R is 'similarity' of triangles.

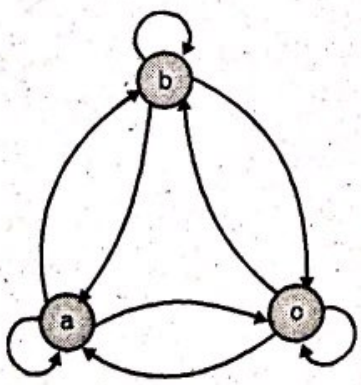

Fig. 3.2

(iv) A is a set of students and R is the relation of being in the same class or division.

(v) Let A be set of statement forms and R be the relation of 'Logical equivalence.'

(vi) A is set of lines in a plane and R is the relation of lines being 'parallel'.

The digraph of an equivalence relation will have the following characteristics.

(i) Every vertex will have a loop.

(ii) If there is an arc from a to b, there should be an arc from b to a.

(iii) If there is an arc from a to b and arc from b to c, there should be an arc from a to c

Q. 14 Let A = Z, the set of integers, and let R be defined by a R b if and only if a ≤ b. Is R an equivalence relation? Dec. 08

Ans. : Since a ≤ a, R is reflexive.

If a ≤ b, it need not follow that b ≤ a, so R is not symmetric. Incidentally, R is transitive, since a ≤ b and b ≤ c imply that a ≤ c. We see that R is not an equivalence relation.

Q. 15 Let R be the relation on set of real numbers such that aRb if and only if a – b is an integer. Prove that R is an equivalent relation. May 06, May 07, May 12

Ans. :

$$R = \{ (a, b) \mid (a - b \text{ is an integer} \}$$

For Reflexive relation :

$$a - a = 0 \text{ i.e. an integer.}$$

∴ a R a for all a in set of real numbers.

∴ Given relation R is **Reflexive** relation.

For Symmetric relation :

Let $a - b = c$, where c is an integer,

$b - a = -c$

∴ If a R b then b R a.

Hence R is a **Symmetric** relation.

For Transitive relation :

Let aRb ∴ a – b = integer value and

bRc ∴ b – c = integer value.

∴ a – c is also an integer value. Hence aRc

∴ If aRb and bRc then aRc.

Hence R is a **Transitive** relation.

∴ Given relation R is an **equivalence relation.**

Q. 16 Let R be a binary relation on the set of all positive integers such that,
R = {(a, b) | a – b is an odd positive integer}
Is R reflexive ? Symmetric ? Transitive ? An equivalence relation ? Dec. 98, May 01, Dec. 05

Ans. :

R is not Reflexive since :

$$a - a = 0 \neq \text{odd positive integer}$$

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$\therefore \quad a \not R a$

R is not symmetric also, since, as if a R b then

$a - b = 2n + 1$ where n = integer number

if b R a then $b - a = -2n - 1$ where n = integer number

$\therefore \quad b - a \neq$ odd positive integer

R is not transitive since,

Let a R b and b R c

i.e. $a - b = 2n_1 + 1$

$b - c = 2n_2 + 1$ ... i.e. odd positive integer

$a - c = (2n_1 + 1) + (2n_2 + 1) = 2(n_1 + n_2 + 1)$

$\neq$ odd positive integer

Hence R is not transitive.

Therefore R is not an equivalence relation.

Q. 17 Let S be the set of all points in a plane let R be the relation such that for any two points a and b (a, b) ∈ R if b is within one inch from a. Examine if R will be an equivalence relation. Dec. 96

Ans. : R is Reflexive since a is within one inch (i.e. 0 inch) from itself.

R is symmetric since if b is within 1 inch from a, a is also within 1 inch from b.

But R is not transitive since b is within 1 inch from a, c is within 1 inch from b need not imply that c is within 1 inch from a.

Hence R is not an equivalence relation.

Q. 18 Let A be a set of integers, let R be the relation on A × A defined by (a, b) R (c, d) if and only if a + d = b + c.

Prove that R is an equivalence relation. Dec. 02, May 17

Ans. : Let (a, b) R (a, b) $\therefore a + b = b + a$.

This expression is True. Hence (a, b) R (a, b). Thus given relation is reflexive. If (a, b) R (c, d) then check for (c, d) R (a, b).

Let (a, b) R (c, d) $\therefore a + d = b + c$ ...(i)

(c, d) R (a, b) $\therefore c + b = d + a$ ...(ii)

Above expressions (i) and (ii) are same. Hence if (a, b) R (c, d) then (c, d) R (a, b).

Thus given relation is symmetric. Let (a, b) R (c, d) and (c, d) R (e, f)

(a, b) R (c, d) $\therefore a + d = b + c$ ...(iii)

(c, d) R (e, f) $\rightarrow c + f = d + e$ ...(iv)

$\therefore$ (a, b) R (e, f) $\therefore$ expressions (iii) + (iv)

i.e. $(a + d)(c + f) = (b + c) + (d + e)$

$\therefore \quad a + f = b + \cancel{c} + \cancel{d} + e - \cancel{d} - \cancel{c}$

$\therefore \quad a + f = b + e$

$\therefore$ R is Transitive relation.

Hence given relation R is reflexive, symmetric, transitive. Thus 'R' is 'equivalence' relation.

Q. 19 Let S = {1, 2, 3} × {1, 2, 3, 4} and let a Relation R be defined on S as follows : For {(x, y), (u, v)} ∈ S (x, y) R (u, v) if and only if | x − y | = | u − v |. Show that R is an equivalence relation on S May 97

Ans. :

Every (x, y) R (x, y) as,

$|x - y| = |y - x|$

$\therefore$ R is Reflexive.

If $|x - y| = |u - v|$ hence (x, y) R (u, v)

$\therefore$ $|u - v| = |x - y|$ which is true. Hence if (x, y) R (u, v) then (u, v) R (x, y)

$\therefore$ R is symmetric relation.

Let (x, y) R (u, v) and (u, v) R (p, q)

$|x - y| = |u - v|$ ...(i)

$|u - v| = |p - q|$ ...(ii)

Substitute (ii) in (i)

$\therefore \quad |x - y| = |p - q|$ ...(iii)

From expressions (iii) and definition of R we have, (x, y) R (p, q)

$\therefore$ R is transitive relation.

$\therefore$ Hence R is equivalence relation.

Q. 20 Let R be a binary relation.

Let S = {(a, b) | (a, c) ∈ R and (c, b) ∈ R for some c}. Show that if R is an equivalence relation then S is also an equivalence relation. Dec. 98

Ans. : S = {(a, b) | (a, c) ∈ R and (c, b) ∈ R for some c}. As R is an equivalence relation → Given

Let (a, a) ∈ S as (a, a) ∈ R → Definition of S

$\therefore$ S is reflexive. Let (a, b) ∈ S

$\therefore$ (a, d) ∈ R and (c, b) ∈ R for some c

$\therefore$ (c, a) ∈ R and (b, c) ∈ R ...(i)

$\therefore$ As R is reflexive as it is an equivalance relation.

$\therefore$ (b, c) ∈ R and (c, a) ∈ R → from (i)

$\therefore$ (b, a) ∈ S → Definition of S.

$\therefore$ When (a, b) ∈ S then (b, a) ∈ S

$\therefore$ S is symmetric.

Let (a, b) and (b, c) ∈ S

$\therefore$ (a, d) ∈ R and (d, b) ∈ R

$\therefore$ and (b, e) ∈ R and (e, c) ∈ R

$\therefore$ for some d and e

$\therefore$ (d, e) ∈ R → Transitive property of equivalence relation

$\therefore$ And (d, c) ∈ R

$\therefore$ (a, d) ∈ R and (d, c) ∈ R

$\therefore$ (a, c) ∈ S → definition of S

$\therefore$ given that (a, b) and (b, c) ∈ S we have that (a, c) ∈ S

$\therefore$ Thus we have proved that S is reflexive, symmetric and transitive

$\therefore$ S is an equivalence relation.

Q. 21 Consider the relation "congruence modulo m" given by x R y iff x − y is divisible by m over the set of positive integers.

Show that R is an equivalence relation

Show that if x_1 R y_1 and x_2 R y_2 then $(x_1 + x_2)$ R $(y_1 + y_2)$. Dec. 99

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Ans. :

(i) We check for reflexivity of R. We have $x - x$ is divisible by m for all x.

$\therefore$ a R a for all $x \rightarrow$ Definition of R

$\therefore$ R is Reflexive. Now let x R y

$\therefore$ $x - y$ is divisible by m.

$\therefore$ $y - x$ is divisible by m. if a is divisible by m then $-a$ is also divisible by m (property of integers)

$\therefore$ Given x R y we have y R x for all x, y

$\therefore$ R is symmetric relation.

Let x R y and y R z

i.e. $(x - y)$ is divisible by m ...(1)

$(y - z)$ is divisible by m ...(2)

Equations (1) + (2) we have

$(x - y) + (y - z)$ which is divisible by m

i.e. $x - z$ is divisible by m

$\therefore$ x R z $\rightarrow$ Definition of R.

Thus given x R y and y R z we have x R z for all x, y, z. Thus R is transitive.

Thus, it can be seen that R is an equivalence relation.

(ii) $x_1 R y_1$ and $x_2 R y_2$

$\therefore$ $x_1 - y_1$ is divisible by m and $x_2 - y_2$ is divisible by m

$\therefore$ $x_1 - y_1 + x_2 - y_2$ is divisible by m

$\therefore$ $(x_1 + x_2) - (y_1 + y_2)$ is divisible by m

$\therefore$ $(x_1 + x_2) R (y_1 + y_2) \rightarrow$ Definition of R.

Q. 22 Let R be relation on set of all URLs (or web addresses) such that x R y if and only if the web page at x is same as web page at y. Show that R is equivalence relation. May 03

Ans. : Relation is equivalence relation if it is reflexive, symmetric and transitive. From given condition x belongs to set of URLs $\therefore$xRx $\therefore$ given relation is 'reflexive'. Also for x, y $\in$ URLs if x R y than y R x that is web page at x is same as web page at y and web page at y is same as web page at x.

$\therefore$ Relation R is symmetric.

And x, y, z $\in$ URLs.

If x R y, y R z then xRz also exists, that is, if web page at x is same as web page at y and web page at y is same as web page at z then web page at x is same as web page at z.

Hence given relation is transitive.

$\therefore$ R is an equivalence relation.

Q. 23 A relation R on A is called circular if a R b and b R c imply c R a. Show that R is reflexive and circular if and only if it is an equivalence relation.

Dec. 99, May 2000, Dec. 02, May 04, Dec. 14

Ans. :

Let R be reflexive and circular.

If a R b then a R b and b R b, so b R a. Hence R is symmetric.

If a R b and b R c, then c R a. But R is symmetric, so a R c and R is transitive. Let R be an equivalence relation.

Then R is reflexive

If a R b and b R c, then a R c (transitivity) and c R a (symmetry), so R is also circular.

Q. 24 Let R be the relation on the set of order pairs of positive integers such that ((a, b), (c, d)) $\in$ R if and only if ad = bc. Show that R is equivalence relation. May 05

Ans. : Let (a, b) R (a, b)

$\therefore$ ab = ba

This expression is true. Hence (a, b) R (a, b).

Thus given relation is reflexive.

If (a, b) R (c, d) $\therefore$ ad = bc,

then check for (c, d) R (a, b)

Let (a, b) R (c, d) $\therefore$ ad = bc

(c, d) R (a, b) $\therefore$ cb = da ...(i)

Above expressions (i) and (ii) are similar. ...(ii)

Hence if (a, b) R (c, d) then (c, d) R (a, b).

Thus the given relation is symmetric.

Let (a, b) R (c, d) and (c, d) R (e, f)

(a, b) R (c, d) $\therefore$ ad = bc ...(iii)

(c, d) R (e, f) $\therefore$ cf = de

(c, d) R (e, f) $\therefore$ cf = de ...(iv)

check for (a, b) R (e, f)

$$ad = bc$$

$$\therefore \quad a = \frac{bc}{d} \quad \text{...(v)}$$

$$\therefore \quad cf = de$$

$$\therefore \quad y = \frac{de}{c} \quad \text{...(vi)}$$

$$\therefore \quad af = \frac{bc}{d} \cdot \frac{de}{c} \quad \text{...(From Equations (v) and (vi))}$$

$$\therefore \quad af = be$$

$\therefore$ (a, b) R (e, f)

Hence given relation R is transitive.

Thus R is equivalence relation.

Q. 25 Let R be a reflexive relation on a set A. Show that R is an equivalence relation if and only if (a, b) and (a, c) are in R implies that (b, c) is in R. May 98, Dec. 96

Ans. :

Given that R is reflexive relation on set A i.e. a R a for all a $\in$ A. Assume that R is an equivalence relation.

i.e. R is reflexive.

R is symmetric and R is transitive.

and given that (a, b) and (a, c) are in R and (b, c) is not in R.

Now as R is equivalence relation it is symmetric

$\therefore$ as (b, a) is in R and (a, c) is in R, (b, c) is in R. However, this is contradictory to our assumption.

Therefore our assumption is false and hence we state that R is an equivalence relation if and only if (a, b) and (a, c) are in R implies that (b, c) is in R.

Q. 26 Let A = Z and let R {(a, b) $\in$ A $\times$ A' | a = r (mod 2) and b = r (mod 2)}. Determine A | R. May 01

Ans. : Now a R b if a mod 2 = r and b mod 2 = r

Here r = 0 or r = 1

$\therefore$ a R b if a mod 2 = 1 and b mod 2 = 1

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$\therefore$ or a R b if a mod 2 = 0 and b mod 2 = 0

$\therefore$ We can represent integers either as Z_n or Z_{n+1}

$\therefore$ partition of A i.e.

$\therefore$ A | R = {{2n}, {2n + 1}}

$\therefore$ where $n \in Z$.

Q. 27 Let m be the positive integer greater than 1. Show that the relation R = {(a, b) | a ≡ b (mod m)} i.e. aRb if and only if m divides a – b is equivalence relation on the set of integers. Dec. 06, Dec. 13, May 15

Ans. :

(a) **Reflexive :** We know a ≡ b (mod m) if and only if m divides a – b.

Now a – a = 0 is divisible by m.

So a ≡ a (mod m)

$\therefore$ Congruence module m is reflexive.

Hence given relation R is **reflexive.**

(b) **Symmetric :** Suppose a ≡ b (mod m) then (a – b) is divisible by m. So that a – b = km, whenever k is an integer.

If follows that(b – a) = (– km)

So that b ≡ a (mod m)

So congruence module m is symmetric. Hence given relation R is **symmertic.**

(c) **Transitive :** Suppose a ≡ b (mod m) and b ≡ c (mod m) then m divides both a – b and b – c.

Therefore there are integers k and *l* with

a – b = kmand b – c = *l*m

(a – b) + (b – c) = (k + *l*) m

a – c = (k + *l*) m

$\therefore$ a ≡ c (mod m)

$\therefore$ Congruence modulo m is transitive.

Hence given relation R is **transitive.**

$\therefore$ Give relation **R** is **Equivalence relation.**

Q. 28 Let R be the equivalence relation on set A then prove that the following statements are equivalent. (i) a R b (ii) [a] = [b] (iii) [a] ∩ [b] ≠ ϕ May 03

Ans. :

(i) Since R is equivalence relation on set A, $\therefore$ a, b ∈ A

Then [a] = [b] ⇒ a R b

i.e. (a, b) ∈ R

Since R is reflexive ⇒ a R a, also a R a ⇒ a ∈ [a] a ∈ [b] since [a] = [b] ⇒ a R b

(ii) If we assume a R b and let x ∈ [a] so that xRa but a R b hence R being Transitive x R a and a R b.

⇒ x R b ⇒ x ∈ [b]

⇒ [a] ⊆ [b] again for y ∈ [b] ⇒ y R b

$\therefore$ R being symmetric

a R b ⇒ b R a $\therefore$ y R b, b R a ⇒ y R a ⇒ yt [a]

⇒ [b] ⊆ [a] it follows that [a] = [b]

(iii) We assume [a] ∩ [b] ≠ ϕ

$\therefore \exists x \in A$

Such that x ∈ [a] ∩ [b] but x ∈ [a] ∩ [b] ⇒ x ∈ [a] and x ∈ [b]

⇒ x R a and a R b

⇒ aRx and a R b ⇒ a R b ⇒ [a] = [b]

$\therefore$ [a] ∩ [b] ≠ ϕ ⇒ [a] = [b]

Q. 29 List all partitions of A = {1, 2, 3}. Dec. 03

Ans. :

{{1}, {2}, {3}}, {{1}, {2, 3} }, {{1, 2}, {3}}, {{1, 3}, {2}} are partitions of set A = {1, 2, 3}

Q. 30 Let A = (1, 2, 3, 4, 5, 6} and let R be the equivalence relation on A defined by R = {(1, 1), (1, 5) (2, 2), (2, 3), (2, 6) (3, 2), (3, 3) (3, 6), (4, 4), (5, 1), (5, 5), (6, 2) (6, 3), (6, 6)}. Find equivalence classes of R and find the partition of A induced by R. Dec. 11

Ans. :

Equivalence classes :

R (1) = {1, 5} R (2) = {2, 3, 6}

R (3) = {2, 3, 6} R (4) = {4}

R (5) = {1, 5} R (6) = {2, 3, 6}

$\therefore$ R (1) = R (5)

R (2) = R (3) = R (6)

Accordingly, {{1, 5}, {2, 3, 6}, {4}} is the partition of A induced by R, i.e.

A | R = {{1, 5}, {2, 3, 6}, {4}}

Q. 31 Let I be the set of all integers. Let R_1 be a binary relation on I x I such that the ordered pair [(a, b), (c, d)] is in R_1 if and only if a – c = b – d.

(i) What is a geometric interpretation of the binary relation R_1 ?

(ii) Is R_1 an equivalence relation.

May 99

Ans. :

(i) If we plot the points of the relation, we observe that dx = dy i.e. we get a line of slope = 1 as

a – c = b – d → given

(ii) Lets check if R_1 is an equivalence relation

Let (a, b) ∈ R_1

$\therefore$ a – a = b – b

$\therefore$ 0 = 0 true

Therefore (a, b) R_1 (a, b) for any (a, b)

$\therefore$ R_1 is a reflexive relation.

Let (a, b) ∈ R_1 and (c, d) ∈ R_1

i.e. Let (a, b) R_1 (c, d)

Therefore (a – c) = (b – d) → definition of R_1 ...(1)

Now, multiply Equation (1) by –1 on both sides

$\therefore$ c – a = d – b

$\therefore$ (c, d) R_1 (a, b)

if (a, b) R_1 (c, d) then (c, d) R_1 (a, b).

Hence the relation R_1 is symmetric

Let (a, b) R_1 (c, d) and (c, d) R_1 (e, f)

a – c = b – d and

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$c - e = d - f$

$\therefore \quad a - b = c - d$

and $\quad c - d = e - f$

$\therefore \quad a - b = e - f$

$\therefore \quad a - e = b - f$

$\therefore \quad (a, b) R_1 (e, f)$

Hence $(a, b) R_1 (c, d)$ and $(c, d) R_1 (e, f)$ implies $(a, b) R_1 (e, f)$

Therefore R_1 is transitive.

Hence R_1 is an equivalence relation.

Q. 32 Let Z be the set of integers. Define a relation R on Z as a R b iff 6 | (a – b), show that R is an equivalence relation and find Z | R.

Dec. 08, Dec. 10, May 11

Ans. :

Since 6| (a – a), aRa. Hence R is reflexive. If 6 |(a – b), then 6 | (b – a) which shows that R is symmetric.

If 6 | (a – b) and 6 | (b – c) then obviously 6 | ((a – b) + (b – c)), i.e. 6 | (a – c). Hence R is also transitive. R is therefore an equivalence relation.

$Z | R = \{R(0), R(1), R(2), R(3), R(4), R(5)\}$

where

$R(0) = \{\ldots, -12, -6, 0, 6, 12, 18, \ldots\}$

$R(1) = \{\ldots, -11, -5, 1, 7, 13, \ldots\}$

$R(2) = \{\ldots, -10, -4, 2, 8, 14, \ldots\}$

$R(3) = \{\ldots, \ldots, -3, 3, 9, 15, \ldots\}$

$R(4) = \{\ldots, \ldots, -2, 4, 10, 16, \ldots\}$

$R(5) = \{\ldots, -7, -1, 5, 11, 17, \ldots\}$

The quotient set Z | R is denoted by Z_6 and is called as the set of congruence classes modulo 6. R is also called as a congruence relation.

Q. 33 Let R be the relation on the set A of integers, defined by :

x R y if x – y is divisible by 4 (i.e. x – y = 4 n for some integer n). Show that R is an equivalence relation, and describe the equivalence classes.

Dec. 2000, Dec. 09

Ans. :

If x is any integer, then x – x = 0, which is divisible by 4.

∴ R is reflexive.

Suppose that x – y is divisible by 4. This means that x – y = 4 n, n is integer.

Then y – x = – 4n, which is divisible by 4.

∴ R is symmetric.

Suppose x–y and y–z are divisible by 4. This means that x – y = 4 m

$y - z = 4n$, where m and n are integer

$\therefore \quad x - z = (x - y) + (y - z)$

$= 4m + 4n = 4(m + n)$

∴ x – z is divisible by 4.

∴ R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relations.

Now, we develop the equivalence classes. We choose an element of A and describe the equivalence class containing that element.

Suppose we choose 0.

The equivalence class is R (0);

i.e. $\quad R(0) = \{x \in A \mid x R 0\}$,

in word, it is the set of all integers x such that x – 0 is divisible by 4. This is the set of all multiples of 4 (positive, negative and zero);

$R(0) = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

We choose an element 1 in A. Then R (1) will contain all the integers x such that x – 1 is divisible by 4

$R(1) = \{\ldots, -11, -7, -3, 1, 5, 9, \ldots\}$

$R(2) = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$R(3) = \{\ldots, -9, -5, -1, 3, 7, 11, \ldots\}$

$R(4) = R(0)$

Q. 34 If R be a relation in the set of integers z defined by

R = {(x, y) : x ∈ z, y ∈ z, x – y is divisible by 3 }

Show that the relation R is an equivalence relation and describe the equivalence classes.

Dec. 12, Dec. 14

Ans. :

If x is any integer, then x – x = 0 which is divisible by 3.

∴ R is reflexive.

Suppose that x – y is divisible by 3. This means that x – y = 3n, n is integer. Then y – z = – 3n, which is divisible by 3.

∴ R is symmetric.

Suppose x – y and y – z are divisible by 3. This means that x – y = 3m

y – z = 3n, where m and n are integers.

$\therefore \quad x - z = (x - y) + (y - z) = 3m + 3n = 3(m + n)$

∴ x – z is divisible by 3.

R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Equivalence class : Now we develop the equivalence classes.

Suppose we choose θ. The equivalence class is R (θ).

i.e. $\quad R(\theta) = \{x \in A \mid x R 0\}$

It is the set of all integers x such that x – θ is divisible by 3. This is the set of all multiples of 3. (positive, negative, zero).

$R(\theta) = \{\ldots\ldots, -6, -3, 0, 3, 6, \ldots\ldots\}$

We choose an element 1 in A. Then R (1) will contain all the integers x such that x – 1 is divisible by 3.

$R(1) = \{\ldots\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\ldots\}$

$R(2) = \{\ldots\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\ldots\}$

$R(3) = R(0)$

Q. 35 Define equivalence relation on a set. Let R be a relation on the set of integers defined by aRb iff a – b is a multiple of 5. Prove that R is a equivalence relation.

May 13

Ans. : Let R on Z, (a, b) ∈ R iff a – b is multiple of 5.

For every a ∈ Z, a – a = 0, is clearly a multiple of 5, so that (a, a) ∈ R. It follows that R is reflexive.

Suppose now that a, b, c ∈ Z. If (a, b) ∈ R, then a – b is multiple of 5. In other wods, a – b = 5n for some n ∈ Z, so that b – a = 5 (– n). Hence, b – a is multiple of 5, so that (b , a) ∈ R. It follows that R is symmetric.

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Suppose that a, b, c, ∈ Z. If (a, b), (b, c) ∈ R the a − b and b − c are both multiples of 5. In other words a − b = 5n and b − c = 5m for some n, m ∈ Z. So that a − c = 5 (n + m). Hence a − c is multiple of 5, so that (a, c) ∈ R. It follows that R is transitive.

Hence, R is Equivalence relation.

Q. 36 Let s = {1, 2, 3, 4} and A= S x S. Define the following relation : R on A : (a, b) R (a', b') if and only if a + b = a' + b'. (i) Show that R is an equivalence relation. Dec. 13

Ans. : (a,b) R (a,b) because a + b = a + b Hence R is reflexive

If (a, b) R(a',b') then a + b = a' + b' and a' + b' = a + b

Hence R is symmetric.

Now suppose that (a, b) R(a', b') and (a', b') R(a'', b'') then

a + b = a' + b' and a' + b' = a'' + b'', so a + b = a'' + b''

Hence R is transitive.

Hence R is an equivalence relation.

Q. 37 Let R be a relation on the set of integers Z defined by a R b if and only if a ≡ m (mod 5). Prove that R is an equivalence relation. Find Z/R. May 14

Ans. :

Let A = Z and Let R = {(a, b)| a ≡ m(mod 5) and b ≡ m (mod 5)}

Equivalence Relation :

(a) **Reflexive :** We know a R b if a ≡ m (mod 5) and b = m (mod 5). Congruence module m is reflexive. This is reflexive relation.

(b) **Symmetric :** This relation is symmetric.

As a R b, a ≡ m (mod 5) and b ≡ m (mod 5)

Hence b R a, b ≡ m (mod 5) and a ≡ m (mod 5)

(c) **Transitive :** This relation is Transitive.

if a R c, a ≡ m(mod 5) and b ≡ m (mod 5)

b R c, b ≡ m (mod 5) and c ≡ m (mod 5)

then

a R c, a ≡ m (mod 5) and c = m (mod 5)

by transitivity a − m is divisible by 5 and c − m is divisible by 5

Z/R :

Now a R b if a mod 5 = m and b mod 5 = m

Here, m = 0 or m = 1

a R b if a mod 5 = 1 and b mod 5 = 1

a R b if a mod 5 = 0 and b mod 5 = 0

We can represent integers either as Z_n or Z_{n+1} partition of Z i.e.

$$Z|R = \{\{5n\}, \{5n+1\}\}$$

Q. 38 Let S = {1,2,3,4,5} and A− S × S. Define the following relation R on A : (a,b) R (c,d) if and only if ad = bc. Show that R is an equivalence relation and compute A/R. Dec. 15

Ans. :

Lets S = {1, 2, 3, 4, 5}, A = 5 × 5

R = (a, b) R (c, d) if ad = bc

Let (a, b) R (a, b) ∴ ab = ba

This expression is true, hence (a, b) R (a, b)

∴ Given relation is reflective.

If (a, b) R (c, d) ∴ ad = bc

Then check for (c, d) R (a, b) ∴ cb = da ...(i)

Above two expression are similar ...(ii)

∴ If (a, b) R (c, d) then (c, d) R (a, b)

∴ The given relation is symmetric

Let (a, b) R (c, d) and (c, d) R (e, f)

(a, b) R (c, d) ∴ ad = bc

(c, d) R (e, f) ∴ cf = ed

Check for (a, b) R (e, f)

$$ad = bc$$

$$\therefore a = \frac{bc}{d}$$

$$cf = ed$$

$$f = \frac{ed}{c}$$

$$af = \frac{bc}{d} + \frac{ed}{c}$$

$$\therefore af = be$$

∴ (a, b) R (e, f)

Hence R is transitive.

Thus R is an equivalence relation.

Q. 39 Suppose that A is non empty set and f is a function that has A as it's domain. Let R be the relation on A consisting of all ordered pairs (x, y) where f (x) = f(y). Show that R is an equivalence relation on A. Dec. 16

Ans. :

(x, x) ∈ R since f(x) = f(x). Hence R is reflexive + (x, y) ∈ R if and only if f(x) = f(y) which holds if and only if f(y) = f(x). Hence (x, y) ∈ R if and only if (y, x) ∈ R. Hence R is symmetric.

If (x, y) ∈R and (y, z) ∈v R then f(x) = f(y) and f(y) = f(z)

Hence f(x) = f(z). Thus (x, z) ∈ R

Hence R is transitive.

Q. 40 Let A = {a, b, c, d, e} and let R and S be two relations on A whose corresponding digraphs are shown in Fig. 3.3. Find R, R−1, R ∩ S, R ∪ S. Dec. 12

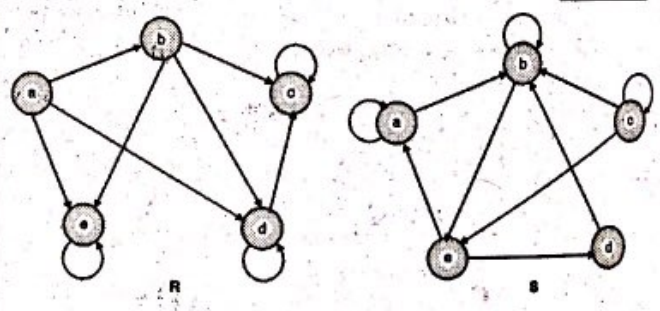

Fig. 3.3

Ans. :

R = {(a, b), (a, e), (a, d), (b, c), (b, d), (b, e), (c, c), (d, d), (d, c), (e, e)}

S = (a, a), (a, b), (b, b), (b, e), (c, c), (c, b), (c, e), (d, b), (e, d), (e, a)}

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First find $A \times A$

$R = \{(a, a), (b, b), (a, c), (b, a), (c, b), (c, d), (c, e), (c, a), (d, b), (d, a), (d, e), (e, b), (e, a), (e, d), (e, c)\}$

$R^{-1} = \{(b, a), (e, b), (c, c), (c, d), (d, d), (d, b), (c, b), (d, a), (e, e), (e, a)\}$

$R \cap S = \{(a, b), (b, e), (c, c)\}$

$R \cup S = \{(a, b), (a, a), (b, b), (c, c), (d, d), (a, e), (a, d), (b, c), (b, d), (b, e), (c, b), (c, e), (d, c), (c, b), (e, e), (e, d), (e, a)\}$

Q. 41 Let R = (1, 2), (4, 3), (2, 2), (2, 1), (3, 1} be a relation on s = { 1, 2, 3, 4}. Find the symmetric closure of R. May 13

Ans. :

Let $S = \{1, 2, 3, 4\}$,

$R = \{(1, 2), (4, 3), (2, 2), (2, 1), (3, 1)\}$

$R^* = R \cup R^{-1}$ = symmetric closure.

where $R^{-1} = \{(2, 1), (3, 4), (2, 2), (1, 2), (1, 3)\}$

$R \cup R^{-1} = \{(1, 2), (2, 1), (4, 3), (3, 4), (2, 2), (1, 2), (2, 1), (3, 1), (1, 3)\}$

= symmetric closure.

Q. 42 (i) Define reflexive closure, symmetric closure and transitive closure of a relation.

(ii) Find reflexive closure of the relation R

(iii) Find symmetric closure of the relation R where R is defined on set A = {1, 2, 3, 4}

R = {(1, 1), (1, 2), (1 ,4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4)} Dec. 96. May 15. May 17

Ans. :

(i) **Reflexive closure :** A relation R* is the reflexive closure of R if R* is the smallest relation containing R which is reflexive.

Symmetric closure : A relation R* is the symmetric closure of R if R* is the smallest relation containing R which is symmetric.

Transitive closure : A relation R* is the transitive closure of R if R* is the smallest relation containing R which is transitive.

(ii) Reflexive closure of R is

$R \cup \Delta = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3)\}$

(iii) Symmetric closure of R is

$R \cup R^{-1} = \{(1, 1), (1, 2), (2, 1), (1, 4), (4, 1), (2, 4), (4, 2), (3, 1), (1, 3), (3, 2), (2, 3), (4, 3), (3, 4), (4, 4)\}$

Q. 43 Define the terms : transitive relations and transitive closure . May 97

Ans. :

Transitive relation : We say that a relation on a set A is transitive if whenever a R b and b R c, then a R c.

Transitive closure : A relation R* is the transitive closure of R if R* is the smallest relation containing R which is transitive.

Q. 44 Let A = {a, b, c, d, e} and let R be the relation whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(i) Find the reflexive closure of R

(ii) Find the symmetric closure of R. Dec. 07

Ans. :

$R = \{(a, a), (a, d), (a, e), (b, c), (b, e), (c, a), (c, b), (c, c), (d, b), (d, c), (e, c), (e, e)\}$

(i) Reflexive closure of R is

$\{(a, a), (a, d), (a, e), (b, b), (b, c), (b, e), (c, a), (c, b), (c, c), (d, b), (d, c), (d, d), (e, c), (e, e)\}$

(ii) Symmetric closure of R is

$\{(a, a), (a, d), (d, a), (a, e), (e, a), (b, c), (c, b), (b, e), (e, b), (c, a), (a, c), (c, c), (d, b), (b, d), (d, c), (c, d), (e, c), (c, e), (e, e)\}$

Q. 45 Show that if set A has 3 elements, then we can find 8 relations on A that all have the same symmetric closure. Dec. 97. Dec. 02. May 09

Ans. : We know that relation R on a set A then R* is symmetric closure of R if R* is the smallest relation containing R which is symmetric and $R^* = R \cup R^{-1}$.

If R is symmetric.

For set of 'n' elements, each of ordered pair (a, a) for a $\in$ A must be in R and set of pairs of elements of form (a, b) and (b, a) for a, b $\in$ A must also be in R. There are $\frac{n(n-1)}{2}$ such pairs.

There are $2^n \frac{n(n-1)}{2}$ such pairs

$\therefore$ Set containing 3 elements, there are

$2^3 \frac{n(n-1)}{2} = 8$ elements

Q. 46 Prove that relation R on set A is transitive if and only if $R^n \subseteq R$ for n = 1, 2, 3 Dec. 03. Dec. 05

Ans. : Let a, b $\in$ R. Then since R is transitive closure of R, there exists a sequence of elements, $x_1, x_2, \ldots, x_R$ in A such that $a = x_1 = b = x_R$ and $(x_i, x_{i+1}) \in R$ for $1 \le i < R - 1$. This means $(a, b) \in R^{k-1}$.

Hence $R^* \subseteq R^2 \cup R^3 \cup \ldots \cup R^n$. Conversely $R^i \subseteq R$ for $1 \le i \le n$

Hence, $R^* = R \cup R^2 \cup R^3 \ldots \cup R^n$. Now we have only to show that R* is the smallest transitive relation containing R. Let S be a transitive relation containing R. Let $(a, c) \in R^2$. This implies that there exists an element b $\in$ A such that (a, b) $\in$ R and (b, c) $\in$ R

Since $R \subseteq S$ and S is transitive, if follows that (a, c) $\in$ S. Hence $R^2 \subseteq S$. Proceeding this way we can show that $R^3, R^4, \ldots R^n$ over all subsets of S.

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Hence $R \cup R^2 \cup \ldots \cup R^n \subseteq R^n \subseteq S$. $\therefore R^n \subseteq S$.

$\therefore \quad R^n \subseteq R$.

(a, e), (e, a), (b, c), (c, b), (b, e), (e, b), (c, a), (a, c), (c, c), (d, b), (b, d), (d, c), (c, d), (e, c), (c, e), (e, e)}

Q. 47 Let R be the relation represented by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$ **Find the matrix that represents R^4**

May 06

Ans. : By Warshall's algorithm :

$$MR^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

First we find MR^1, So that k = 1. $\therefore$ Check column 1 and row 1 of MR^0

$\therefore$ column : (3, 1) row : (1, 2)

To obtain MR^1, we must put it in position (3, 2). Thus,

$$\therefore MR^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now, we compute MR^2 so that k = 2. $\therefore$ Check column 2 and row 2 of MR^1.

$\therefore$ column : (1, 2) and (3, 2)

row : (2, 3)

To obtain MR^3, we must put it in position (1, 3) and (3,3).

$$\therefore \quad MR^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now, we compute MR^3 so that k = 3. $\therefore$ Check column 3 and row 3.

$\therefore$ column : (1, 3), (2, 1) and (3, 1)

Row : (3,1), (3, 2) and (3, 3)

To obtain MR^3, we must put is in position (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2) and (3, 3).

$$\therefore \quad MR^3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q. 48 Let A = {1, 2, 3, 4} and let R = {(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4)} Find transitive closure by using 'Warshall's algorithm'.

Dec. 96

Ans. :

Let $M_R = W_0$ and n = 4

$$M_R = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now compute W_1 so that k = 1 $\therefore$ check column 1 and row 1 of W_0

$p_1 : (1, 1), p_2 : (3, 1)$

$q_1 : (1, 1), q_2 : (1, 2), q_3 : (1, 4)$

To obtain W_1, we must put 1's in positions (1, 1), (1, 2), (1, 4), (3, 1), (3, 2) and (3, 4). Thus

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now compute W_2 so that k = 2 $\therefore$ check column 2 and row 2 of W_1

$p_1 : (1, 2), p_2 : (3, 2), p_3 : (4, 2)$

$q_1 : (2, 4)$

To obtain W_2, we must put 1's in positions (1, 4), (3, 4), (4, 4). Thus

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now compute W_3 so that k = 3 $\therefore$ check column 3 and row 3 of W_2

$p_1 : (4, 3)$

$q_1 : (3, 1), q_2 : (3, 2), q_3 : (3, 4)$

To obtain W_3, we must put 1's in positions (4, 1), (4, 2), and (4, 4). Thus

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now compute W_4 so that k = 4 $\therefore$ check column 4 and row 4 of W_3

$p_1 : (1, 4), p_2 : (2, 4), p_3 : (3, 4), p_4 : (4, 4)$

$q_1 : (4, 1), q_2 : (4, 2), q_3 : (4, 3), q_4 : (4, 4)$

To obtain W_4, we must put all 1's in positions (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3),

(2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3) and (4, 4). Thus

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore M_{R\infty} = W_4$ = Transitive closure

Q. 49 Let A = { 1, 2, 3, 4 } and R {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the transitive closure of R using Warshall's Algorithm.

May 13

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Ans. :

Let $A = \{1, 2, 3, 4\}$

and $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$

Let $M_R = W_0$ and $n = 4$.

$$M_R = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Check column 1 and row 1 of W_0

Column : (2, 1)

Row : (1, 2)

Hence insert (2, 2)

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Check column 2 and row 2 of W_1

Column : (1, 2), (2, 2)

Row : (2, 1), (2, 2), (2, 3)

Hence insert (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Check column 3 and row 3 of W_2

Column : (1, 3), (2, 3)

Row : (3, 4)

Hence insert (1, 4), (2, 4).

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Column : (1, 4), (2, 4), (3, 4)

Row : 6

$W_3 = W_4$

$$W_4 = H_R^{\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Transitive closure}$$

Q. 50 **Consider the following relations R defined on set S = {a, b, c, d, e}. R is given by**

R = {(a, a), (a, d), (b, b), (c, d), (c, e), (d, a), (e, b), (e, e)}

Find the transitive closure of R by using Warshall's algorithm. May 97, May 15

Ans. :

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Q. 51 **Given S = {1, 2, 3, 4} and a Relation R on S given by R = {(4, 3), (2, 2), (2, 1), (3, 1), (1, 2)}**

(i) Show that R is not transitive

(ii) Find a relation $R_1 \subseteq R$ such that R_1 is transitive.

(iii) Find transitive closure of R by Warshall's algorithm.

May 11, Dec. 12, Dec. 16

Ans. :

(i) $(4, 3) \in R$, $(3, 1) \in R$ but $(4, 1) \notin R$

$(1, 2) \in R$, $(2, 1) \in R$ but $(1, 1) \notin R$

$(3, 1) \in R$, $(1, 2) \in R$, but $(3, 2) \notin R$

Hence given relation R is not transitive

(ii) We find R_1 using Warshall's algorithm,

$$M_R = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now compute W_1, so that k = 1 ∴ check column 1 and row 1 of W_0

p_1 : (2, 1), p_2 : (3, 1)

q_1 : (1, 2)

To obtain W_1, we must put 1's in positions (2, 2) and (3, 2). Thus

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now compute W_2, so that k = 2 ∴ check column 2 and row 2 of W_1

p_1 : (1, 2), p_2 : (2, 2), p_3 : (3, 2)

q_1 : (2, 1), q_2 : (2, 2)

To obtain W_2, we must put 1's in positions (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), and (3, 2). Thus

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now compute W_3, so that k = 3 ∴ check column 3 and row 3 of W_2

p_1 : (4, 3),

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q_1 : (3, 1), q_2 : (3, 2)

To obtain W_3, we must put 1's in positions (4, 1) and (4, 2). Thus

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now compute W_4, so that k = 4 ∴ check column 4 and row 4 of W_3

p_1 : –

q_1 : (4, 1), (4, 2), (4, 3)

But no new additional ordered pairs. Thus $W_3 = W_4$ i.e.

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

∴ $MR_\infty = W_4$

∴ $R^\infty = R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

∴ $R_1 \supseteq R$ and R_1 is transitive

(iii) The transitive closure of R is R^∞ as defined in part (ii) of the Ans. above.

Q. 52 Consider the set A = {1, 4, 7, 13} and R = {(1, 4), (4, 7), (7, 4), (1, 13)}. Find out the transitive closure of R using Warshall's algorithm. May 2000. Dec. 04

Ans. : Let $M_R = W_0$ and n = 4

$$M_R = W_0 = \begin{array}{c} \\ 1 \\ 4 \\ 7 \\ 13 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now compute W_1, so that k = 1 ∴ check column 1 and row 1 of W_0

p_1 : –

q_1 : (1, 4), q_2 : (1, 3)

But no new additional ordered pair, hence $W_0 = W_1$. Thus

$$W_1 = \begin{array}{c} \\ 1 \\ 4 \\ 7 \\ 13 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now compute W_2, so that k = 2 ∴ check column 2 and row 2 of W_1

p_1 : (1, 4), p_2 : (7, 4)

q_1 : (4, 7)

To obtain W_2, we must put 1's in positions (1, 7) and (7, 7). Thus

$$W_1 = \begin{array}{c} \\ 1 \\ 4 \\ 7 \\ 13 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now compute W_3, so that k = 3 ∴ check column 3 and row 3 of W_2

p_1 : (1, 7), p_2 : (4, 7), p_3 : (7, 7)

q_1 : (7, 4), q_2 : (7, 7)

To obtain W_3, we must put 1's in positions (1, 4), (1, 7), (4, 4), (4, 7), (7, 4) and (7, 7). Thus

$$W_1 = \begin{array}{c} \\ 1 \\ 4 \\ 7 \\ 13 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 13 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now compute W_4, so that k = 4 ∴ check column 4 and row 4 of W_3

p_1 : (1, 13),

q_1 : –

No new additional ordered pairs, hence. $W_3 = W_4$. Thus

$$W_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ $MR_\infty = W_4$ = Transitive closure

∴ R^∞ = {(1, 4), (1, 7), (1, 13), (4, 4), (4, 7), (7, 4), (7, 7)}

Chapter 4 : Functions and Pigeon Hole Principle

Q. 1 Explain Primitive Recursive Function. Every primitive recursive function is a total function, Justify. May 09

Ans. :

Primitive Recursive Function : Let $g : N^n \to$ and $h : N^{n+2} \to N$ be functions. We say that $f : N^{n+1} \to N$ is defined from g and h by primitive recursion if f satisfies the conditions :

$f(x_1, x_2, \ldots\ldots, x_n, 0) = g(x_1, x_2, \ldots\ldots, x_n)$

and $f(x_1, x_2 \ldots\ldots, x_n, y+1) = h(x_1, x_2 \ldots\ldots\ldots, x_n, y, f(x_1, x_2, \ldots\ldots, x_n\, y)$

(where y is the inductive variable).

From the above definition it is clear that primitive recursion (or recursion) is the operation in the operation in which a function f of (n + 1) variables is defined by using two other functions g and h of n and (n + 2) variables.

Definition : A function f is said to be a primitive recursive if it can be obtained from the initial functions by a finite number of operations of composition and recursion (i.e. Primitive recursion).

Total Function :

Let X and Y be sets and A be a subset of X.

A function f from A to Y is called a partial function from X to Y. The set A is a called the domain of f. If the domain of f is the set X, then f is called a total function from X to Y.

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From the definition of the primitive recursive function, it is clear that, if f is a primitive recursive function from set X to set Y then the domain of f is the set X, it implies that f is also total function.

Example :

$f : R \to R$, defined by

$f(x) = 5 \ \forall \ x \in R$ is a constant mapping we have Ran $(f) = \{ 5 \}$

Q. 2 Explain surjective and injective function. May 09

Ans. : Surjective function :

A function from A to B is said to be an **onto** function if every element of B is the image of one or more elements of A. **Onto function** is also called **surjective** or 'f' is **ONTO** if Ran (f) = B.

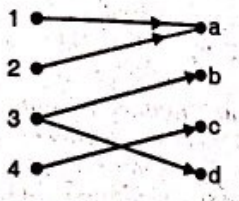

Fig. 4.1

Example : Let A = { 1, 2, 3, 4 } and B = { a, b, c, d }.

and f = { (1, a), (2, a), (3, d), (4, c), (3, b) }.

f (1) = a, f (2) = a, f (3) = d,

f (4) = c, f (3) = b.

Ran (*f*) = { a, b, c, d } = B.

So this function is onto or surjective function.

One to One or Injective Function :

A function from A to B is said to be a **one to one function** if no two elements of A have the same image. One to one function is also called as **injective function.**

Example : Let A = { 1, 2, 3, 4 }, and B = { a, b , c, d, e }.

and f = { (1, a), (2, e), (3, c), (4, d) }.

f (1) = a

f (2) = e

f (3) = c

f (4) = d

Given function *f* is one to one or injective function.

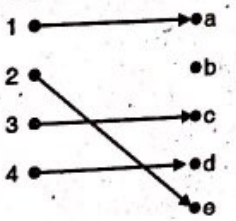

Fig. 4.1(a)

Q. 3 What is identity and inverse function ? Explain with example.

Ans. :

(1) Identity :

Definition :

Let A be a non-empty set. Then we can always define a function

$f : A \to A$ (i. e. B = A) as f (a) = a for all $a \in A$

f is called the **Identify** function on A and is denoted by I_A.

$\therefore \quad I_A = \{ (a,a) \mid a \in A \}$

Example

Let A = { 1, 2, 3 } and $f : A \to A$ is identify function since

f (1) = 1 f (2) = 2 f (3) = 3

(2) Inverse function :

The inverse of a function is a analogous to that of the converse of a relation.

Definition

Let $f : A \to B$, be function, then

$f^{-1} : B \to A$ is called the **inverse** mapping of f

f^{-1} is the set defined as

$f^{-1} = \{ (b, a) \mid (a, b) \in f \}$.

Example

Let A = { 1, 2, 3 } and *f* be the function defined on A such that f(1) = 2, f (2) = 3, f(3) = 1. Then $f^{-1} : A \to A$ is defined by

$$f^{-1}(1) = \{ 3 \}.$$
$$f^{-1}(3) = \{ 2 \}.$$
$$f^{-1}(2) = \{ 1 \}.$$
$$f^{-1} = \{ (1, 3) (3, 2), (2, 1) \}.$$

Q. 4 Define the term bijection.

Define $f : N \to N$ by f(n) = n + 1 if n is odd n − 1 if n is even. Prove the f is bijective function. May 97

Ans. : Refer Definition of bijection function.

Let n_1 and n_2 be 2 even natural numbers

n_3 and n_4 be 2 odd natural numbers

Such that $n_1 \neq n_2$ and $n_3 \neq n_4$

Let $f(n_1) = f(n_2)$

then $n_1 - 1 = n_2 - 1$

$\therefore \quad n_1 = n_2$

Also $f(n_3) = f(n_4)$

$n_3 + 1 = n_4 + 1$

$n_3 = n_4$

But this is contradictory to our assumption $f(n_1) \neq f(n_2)$ for all $n_1, n_2, \in N$

$\therefore$ f is one to one function ...(i)

Now consider any number $\in N$ we can represent the number as either n − 1 or n + 1. Thus every $n' \in N$ is the image of some $n \in N$.

$\therefore$ f is **onto** or **surjective function** ...(ii)

From Equations (i) and (ii) f is one to one and onto function. Hence it is bijective function.

Q. 5 Let A = { 0, − 1, 1 } and B = { 0,1 } Let $f : A \to B$ where f(a) = |a| . Is f onto? May 09

Ans. : The elements of B are 0 and 1. Hence f is onto if we can find $x, y \in A$ such that f(x) = 0 and f(y) = 1. Now, f(0) = |0| = 0 and f(− 1) = | − 1| = 1.

Hence to each element b of B there is an element $a \in A$.

$\therefore$ f(a) = b. Hence f is onto

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Q. 6 **Let A = B = Z and let f : A → B be defined by f(a) = a + 1, for a ∈ A. Which of the special properties does f possess?** May 09

Ans. :

Since the formula defining f makes sense for all integers,

Dom (f) = Z = A and so f is everywhere defined

Suppose that

$$f(a) = f(a') \text{ for } a, a' \in A \text{ then}$$
$$a + 1 = a' + 1$$

So $a = a'$ Hence f is one to one

To see if f is onto, Let b be an arbitrary element of B. Can we find an element a ∈ A such that f(a) = b?

Since $f(a) = a + 1$

we need an element a in A such that

$$a + 1 = b$$

of course, $a = b - 1$

will satisfy the desired equation. Since b – 1 is in A.

Hence Ran (f) = B; therefore f is onto,

Q. 7 **A function $f : R - \left\{\frac{7}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ is defined as : $f(x) = \frac{4x-5}{3x-7}$**

Prove that 'f' is bijective and find the rule for f^{-1}. Dec. 11

Ans. :

$$f(x) = \frac{4x-5}{3x-7}$$

for $f(x) = \frac{4x-5}{3x-7}$ to be one to one and onto

If $x, y \in A$ such that $f(x) = f(y)$

$$\frac{4x-5}{3x-7} = \frac{4y-5}{3y-7}$$
$$(4x-5)(3y-7) = (4y-5)(3x-7)$$
$$12xy - 28x - 15y + 35 = 12xy - 28y - 15x + 35$$
$$x = y$$

∴ f is one to one.

Now for $y = \frac{4x-5}{3x-7}$

$$3xy - 7y = 4x - 5$$
$$3xy - 4x = 7y - 5$$
$$x(3y-4) = 7y - 5$$
$$x = \frac{7y-5}{3y-4}$$

Set $y = f(x)$ and then

Interchange x and y as follows

$$y = \frac{4x-5}{3x-7}$$

By interchanging x and y

$$x = \frac{4y-5}{3y-7}$$
$$3xy - 7x = 4y - 5$$
$$3xy - 4y = 7x - 5$$
$$y(3x-4) = 7x - 5$$
$$y = \frac{7x-5}{3x-4}$$

$$f^{-1}(x) = \frac{7x-5}{3x-4}$$

Hence given function f is bijective.

Q. 8 **Determine whether the following function is bijective.**

f : I → I

f(j) = j/2 if j is even

= (j – 1) / 2 if j is odd

Dec. 98, May 2000, Dec. 04.

Ans. : Let a, b be any integers and a ≠ b

Now, if a and b are odd.

$$f(a) = (a-1)/2$$
$$f(b) = (b-1)/2$$

Let $f(a) = f(b)$

then $\frac{a-1}{2} = \frac{b-1}{2}$

∴ $a = b$

But this a contradution to our assumption a ≠ b

Therefore f(a) ≠ f(b) when a ≠ b

Therefore f is injective or one to one.

Let 'b' be any integer in I.

Then we can always find another integer 2b or 2b + 1

Therefore f(2b) = b or f(2b + 1) = b

This is true for all b ∈ I.

Thus the function is surjective or onto.

As the function is both injective and surjective we can say that f is bijective.

Q. 9 **Define g : N → N by g(n) = 1 if n is even**

= n + 1 if n is odd and n > 1

Prove that g is bijective function. May 97

Ans. : Let n_1 and n_2 be two even natural numbers and n_3, n_4 be odd natural numbers such that $n_1 \neq n_2$ and $n_3 \neq n_4$

(Let $n_1, n_2, n_3, n_4 \in N$)

Suppose $g(n_1) = g(n_2)$

∴ $n_1 + 1 = n_2 + 1$

∴ $n_1 = n_2$

But this contradicts our assumption

∴ $g(n_1) \neq g(n_2)$ where $n_1, n_2 \in N$ and n_1, n_2 even and $n_1 \neq n_2$.

Suppose $g(n_3) = g(n_4)$

∴ $n_3 - 1 = n_4 - 1$

∴ $n_3 = n_4$

But this contradicts our assumption.

∴ $g(n_3) \neq g(n_4)$ where $n_3, n_4 \in N$ and n_3, n_4 are odd and $n_3 \neq n_4$.

∴ Also $g(1) = g(1) = 1$

∴ g is an injective function ...(i)

Now any n' ∈ N and n' > 1 can be represented as n + 1 or n – 1 where n ∈ N

Thus every natural number is the image of some other natural number.

∴ g is surjective or anto ...(ii)

From Equations (i) and (ii) we conclude that g is bijective.

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Q. 10 **Let $f : A \to B$ be a function with finite domain and range. Suppose that $|Dom(f)| = n$ and $|Ran(f)| = m$. Prove that,**
(i) If f is one to one then $m = n$. Dec. 05
(ii) If f is not one to one then $m < n$

May 08

Ans. : A function from A to B is said to be a "one-to-one function", if no two elements of A have the same image.

Above definition is self explanatory $m = n$, i.e. $|Ran(f)| = |Dom(f)|$, then 'f' is one-to-one and if $m < n$ i.e. $|Ran(f)| < |Dom(f)|$ the f is not one-to-one.

Q. 11 **Let A = B = R, the set of real numbers. Let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$, and let $g : B \to A$ be given by $g(y) = \sqrt[3]{\frac{1}{2}Y + \frac{1}{2}}$ Show that f is bijection between. A and B and g is a bijection between B and A.**

May 03, Dec. 07

Ans. :

A function from A to B is Bijection if it is one to one and onto

$\therefore$ for $f(x) = 2x^3 - 1$ to be one to one and onto.

If $a, b, \in A$

such that $f(a) = f(b)$.

$\Rightarrow \quad 2a^3 - 1 = 2b^3 - 1.$

$a = b.$

$\therefore$ f is one to one

Now for $y = 2x^3 - 1.$

$1 + y = 2x^3.$

$x^3 = \frac{1}{2} + \frac{y}{2}.$

$\therefore \quad x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}.$

$\therefore$ for each $y \in B$. There is a unique x in A such that $f(x) = y$. $\therefore$ f is onto.

$\therefore$ f is bijective function between A and B.

Similarly for $g : B \to A$ to be one to one and onto

$$g(a) = g(b) = \sqrt[3]{\frac{1}{2} + \frac{a}{2}} = \sqrt[3]{\frac{1}{2} + \frac{b}{2}}$$

$\therefore \quad \frac{1}{2} + \frac{a}{2} = \frac{1}{2} + \frac{b}{a}$

$\Rightarrow \quad a = b$

$\therefore$ g is one to one.

Also for $x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}$

$x^3 = \frac{1}{2} + \frac{y}{2}$

$2x^3 = 1 + y$

$y = 2x^3 - 1$

for each x in A. There is a corresponding y in B. Such that $g(y) = x$

$\therefore$ g is onto function

so g is bijective function between B and A.

Q. 12 **Consider the above function $f(x) = 2x - 3$. Find a formula for the composition functions (i) $f^2 = f \circ f$ and (ii) $f^3 = f \circ f \circ f$.**

May 17

Ans. :

(i) $(f \circ f)(x) = f(f(x))$
$= f(2x - 3) = 2(2x - 3) - 3$
$= 4x - 6 - 3 = 4x - 9$

(ii) $(f \circ f \circ f)(x) = f(f(f(x)))$
$= f(f(2x - 3)) = f(4x - 9)$
$= 2(4x - 9) - 3 = 8x - 18 - 3 = 8x - 21$

Q. 13 **If A = B = C = R where R is set of real number and $f : A \to B$, $g : B \to C$ are functions defined by $f(x) = x + 1$, $g(x) = x^2 + 2$, then find $(g \circ f)(x)$ and $(f \circ g)(2)$.** Dec. 09

Ans. :

Given : $f(x) = x + 1, g(x) = x^2 + 2$

(i) $(g \circ f)(x) = g(f(x))$
$= g(x + 1) = (x + 1)^2 + 2 = x^2 + 2x + 3$

(ii) $(f \circ g)(2) = f(g(2))$
$= f(6) = 7$

Q. 14 **Let $f : R \to R$, $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$ find (i) $f \circ (g \circ f)$ (ii) $g \circ (f \circ g)$.** Dec. 12

Ans. :

Given $f : R \to R$, $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$

(i) $f \circ (g \circ f) = f(g(f(x)))$
$= f(g(x^2 - 1)) = f(4(x^2 - 1)^2 + 2)$
$= f(4(x^4 - 2x^2 + 1) + 2)$
$= f(4x^4 - 8x^2 + 4 + 2) = f(4x^4 - 8x^2 + 6)$
$= (4x^4 - 8x^2 + 6)^2 - 1$

(iii) $g \circ (f \circ g) = g(f(g(x)))$
$= g(f(4x^2 + 2)) = g((4x^2 + 2)^2 - 1)$
$= g(16x^4 + 16x^2 + 4)$
$= 4(16x^4 + 16x^2 + 4)^2 + 2$

Q. 15 **Let A = B = C = IR and Let $f : A \to B$, $g : B \to C$ be defined by $f(a) = a + 1$ and $g(b) = b^2 + 2$. Find.**

(i)	**$(g \circ f)(-2)$**	**(ii)**	**$(f \circ g)(-2)$**
(iii)	**$(g \circ f)(x)$**	**(iv)**	**$(f \circ g)(x)$**
(v)	**$(f \circ f)(y)$**	**(vi)**	**$(g \circ g)(y)$**

Ans. :

(i) $(g \circ f)(-2) = g(f(-2))$
$= g(-2 + 1) = g(-1)$
$= (-1)^2 + 2 = 1 + 2 = 3$

(ii) $(f \circ g)(-2) = f(g(-2))$
$= f((-2)^2 + 2)$
$= f(4 + 2) = f(6)$
$= 6 + 1 = 7$

(iii) $(g \circ f)(x) = g(f(x))$
$= g(x + 1) = (x + 1)^2 + 2$
$= x^2 + 2x + 3$

(iv) $(f \circ g)(x) = f(g(x))$
$= f(x^2 + 2) = (x^2 + 2) + 1$
$= x^2 + 3$

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(v) $(f \circ f)(y) = f(f(y))$
$= f(y+1)$
$= (y+1)+1 = y+2$

(vi) $(g \circ g)(y) = g(g(y))$
$= g(y^2+2)$
$= (y^2+2)^2+2$

Q. 16 **Define injective, surjective and bijective functions. If $f : R \to R$ and $g : R \to R$ are defined by the formulas $f(x) = x + 2$ and $g(x) = x^2$**
Find (i) f . g . f ; (ii) g . f. g. May 13, Dec. 15

Ans. :

Let $f(x) = x+2$ and $g(x) = x^2$

(i) $f \circ g \circ f = f(g(f(x)))$
$= f(g(x+2)) = f((x+2)^2)$
$= (x+2)^2+2 = x^2+4x+6$

(ii) $g(f(g(x)) = g(f(x^2))$
$= g(x^2+2) = (x^2+2)^2 = (x^2+2)(x^2+2)$
$= x^4+2x^2+2x^2+4 = x^4+4x^2+4$

Ans. : We may depict f, g, h graphically as shown in Fig. 4.2

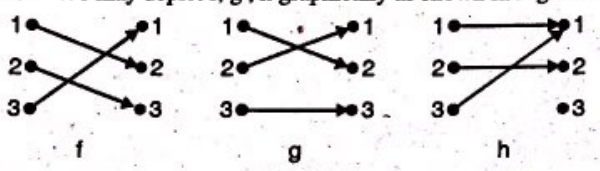

Fig. 4.2

(i) $f(1) = 2, \quad f(2) = 3, \quad f(3) = 1.$
$g(1) = 2, \quad g(2) = 1, \quad g(3) = -3.$
$f \circ g(1) = f(g(1)) = f(2) = 3.$
$f \circ g(2) = f(g(2)) = f(1) = 2.$
$f \circ g(3) = f(g(3)) = f(3) = 1.$
$\therefore \quad f \circ g = \{(1,3), (1,2), (1,1)\}.$

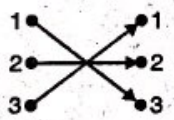

Fig. 4.2(a)

f o g is depicted as shown in Fig. 4.2(a)

$g \circ f(1) = g(f(1)) = g(2) = 1$
$g \circ f(2) = g(f(2)) = g(3) = 3$
$g \circ f(3) = g(f(3)) = g(1) = 2$
$\therefore g \circ f = \{(1,1), (2,3), (3,2)\}.$

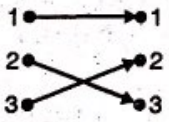

Fig. 4.2(b)

g o f is depicted as shown in Fig. 4.2 (b)

$\therefore f \circ g \neq g \circ f$

(ii) $h(1) = 1, \quad h(2) = 2, \quad h(3) = 1$
$f \circ g \circ h(1) = f(g(h(1))) = f(g(1)) = f(2) = 3.$
$f \circ g \circ h(2) = f(g(h(2))) = f(g(2)) = f(1) = 2.$
$f \circ g \circ h(3) = f(g(h(3))) = f(g(1)) = f(2) = 3.$
$\therefore \quad f \circ g \circ h = \{(1,3), (2,2), (3,3)\}.$

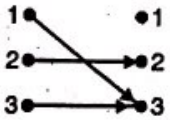

Fig. 4.2(c)

f o g o h can be depicted as shown in Fig. 4.2(c)

$f \circ h \circ g(1) = f(h(g(1))) = f(h(2)) = f(2) = 3.$
$f \circ h \circ g(2) = f(h(g(2))) = f(h(1)) = f(1) = 2.$
$f \circ h \circ g(3) = f(h(g(3))) = f(h(3)) = f(1) = 2.$
$\therefore \quad f \circ h \circ g = \{(1,3), (2,2), (3,2)\}.$

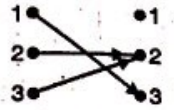

Fig. 4.2(d)

f o h o g can be depicted as shown in Fig. 4.2(d)

Q. 18 **Let $f(x) = 2x + 3$, $g(x) = 3x + 4$, $h(x) = 4x$ for $x \in R$, where R = set of real numbers.**
Find g o f, f o g, f o h, h o f, g o h. Dec. 14

Ans. :

(i) $g \circ f(x) = g(f(x))$
$= g(2x+3)$
$= 3(2x+3)+4$
$= 6x+13$

(ii) $f \circ g(x) = f(g(x))$
$= f(3x+4)$
$= 2(3x+4)+3$
$= 6x+11$

(iii) $f \circ h(x) = f(h(x))$
$= f(4x)$
$= 2(4x)+3$
$= 8x+3$

(iv) $h \circ f(x) = h(2x+3)$
$= 4(2x+3)$
$= 8x+12$

(v) $g \circ h(x) = g(4x)$
$= 3(4x)+4$
$= 12x+4$

Q. 19 **If $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from R to R, where R is the set of real numbers, find f o g and g o f.**

Ans. :

(i) $f \circ g(x) = f(g(x))$
$= f(x+2)$
$= (x+2)^2+1$
$= x^2+4x+5$

(ii) $g \circ f(x) = g(f(x))$
$= g(x^2+1)$
$= (x^2+1)+2$
$= x^2+3$

Q. 20 **Let $f(x) = ax + b$ and $g(x) = cx + d$ where a, b, c, d are constants. Determine for which constants a, b, c, d it is true that f o g = g o f**

Ans. :

$f \circ g(x) = f(g(x))$
$= f(cx+d)$
$= a(cx+d)+b.$
$= acx+ad+b$

$g \circ f(x) = g(f(x))$
$= g(ax+b)$
$= c(ax+b)+d$
$= acx+cb+d.$

$f \circ g = g \circ f \Rightarrow acx+ad+b = acx+cb+d$
$ad+b = cb+d$
$ad-d = cb-b$
$d(a-1) = b(c-1).$

i.e. $\frac{b}{d} = \frac{a-1}{c-1}$ is the relation between the constants if $f \circ g = g \circ f$.

Q. 21 **$f : R \to R$ is defined as $f(x) = x^3$.**
$g : R \to R$ is defined as $g(x) = 4x^2 + 1$.
$h : R \to R$ is defined as $h(x) = 7x - 2$.
Find the rule of defining as (h o g) of g o (h o f)
May 01, May 06, Dec. 10, May 11, Dec. 11

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Ans. :

We have $h \circ f = h[f(x)]$
$= h(x^3)$
$= 7(x^3) - 2$
$= 7x^3 - 2$

$\therefore \quad g \circ (h \circ f) = g[h \circ f]$
$= g[7x^3 - 2]$
$= 4(7x^3 - 2)^2 + 1$

Now, $h \circ g(x) = h[g(x)]$
$= h[4x^2 + 1]$
$= 7(4x^2 + 1) - 2 = 28x^2 + 5$

Hence
$(h \circ g)$ of $g \circ (h \circ f)(x) = 28[4(7x^3 - 2)^2 + 1]^2 + 5$

Q. 22 **If f and g be the functions from set of integers to set of integers defined by.**
$f(x) = 2x + 3$.
$g(x) = 3x + 2$.
Find (f o g) and (g o f). Dec. 02

Ans. :

We have $f(x) = 2x + 3$.
$g(x) = 3x + 2$.

(i) $(f \circ g)(x) = f(g(x))$
$= f(3x + 2)$
$= 2(3x + 2) + 3$
$= 6x + 4 + 3$
$= 6x + 7$.

(ii) $(g \circ f)(x) = g(f(x))$
$= g(2x + 3)$
$= 3(2x + 3) + 2$
$= 6x + 9 + 2$
$= 6x + 11$.

Q. 23 **Let A = B = C = R and Let $f : A \to B$ and $g : B \to C$ be defined by $f(a) = a - 1$, $g(b) = b^2$. Find**
(i) (f o g) (2) (ii) (g o f) (x) (iii) (f o f) (y) (iv) (g o f) (2) (v) (g o g) (y) Dec. 03

Ans. :

Given $f(a) = a - 1$ and $g(b) = b^2$

(i) $(f \circ g)(2) = f(g(2)) = f(4) = 4 - 1 = 3$.

(ii) $(g \circ f)(x) = g(f(x))$
$= g(x - 1)$
$= (x - 1)^2$.

(iii) $(f \circ f)(y) = f(f(y))$
$= f(y - 1)$
$= (y - 1) - 1$
$= y - 2$.

(iv) $(g \circ f)(2) = g(f(2))$
$= g(2 - 1)$
$= g(1)$
$= 1$.

(v) $(g \circ g)(y) = g(g(y))$
$= g(y^2)$
$= y^4$.

Q. 24 **Suppose that A is non empty set, and f is a function that has A as it's domain. Let R be the relation on A consisting of all ordered pairs (x, y) where f (x) = f (y). Show that R is an equivalence relation on A.** May 04

Ans. :

Here $R = \{(x, y) \mid f(x) = f(y)\}$ ... $x \cdot y \in A$
For $x \in A$
For ordered pair (x, x) each $f(x) = f(y)$
$\therefore$ $x R x$ for all $x \in A$
$\therefore$ R is Reflexive
Also for ordered pair (x, y)
as $f(x) = f(y)$
$\Rightarrow$ $f(y) = f(x)$
$\Rightarrow$ (y, x) $\therefore y R x$
$\therefore$ R is symmetric
Now for (x, y) and (y , z)
$f(x) = f(y)$ and $f(y) = f(z)$
$\Rightarrow$ $f(x) = f(z)$
$\Rightarrow$ (x, z) $\therefore x R z$
$\therefore$ R is Transitive
$\therefore$ R is an equivalence Relation.

Q. 25 **Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for all $x \in R$. (R is the set of real number)**
Find (i) $f \circ g \circ h$ (ii) $h \circ g \circ f$ (iii) $f \circ f \circ f$ May 14

Ans. :

(i) $f \circ g \circ h(x) = f(g \circ (h(x))) = f(g \circ (3x))$
$= f((3x) - 2) = (3x - 2) + 2 = 3x$

(ii) $h \circ g \circ f(x) = h(g(f(x))) = h(g(x + 2))$
$= h((x + 2) - 2) = h(x) = 3x$

(iii) $f \circ f \circ f = f(f(f(x))) = f(f(x + 2))$
$= f((x + 2) + 2)$
$= f(x + 4) = (x + 4) + 2 = x + 6$

Q. 26 **If $f : A \to B$ be both one-to-one and onto, then prove that $f^{-1} : B \to A$ is also both one-to-one and onto.** May 04, 05, Dec. 13, May 15

Ans. :

Let $f : A \to B$ be both one to one and onto then there exists elements.
$a_1, a_2 \in A$ and elements
$b_1, b_2 \in B$
Such that $f(a_1) = b_1$ and $f(a_2) = b_2$
$a_1 = f^{-1}(b_1)$ and $a_2 = f^{-1}(b_2)$
Now let $f^{-1}(b_1) = f^{-1}(b_2)$
$\Rightarrow$ $a_1 = a_2$
$\Rightarrow$ $f(a_1) = f(a_2)$
$b_1 = b_2$
$\therefore f^{-1}$ is one to one, again since f is onto, for $b \in B$, there is some element $a \in A$, such that
$f(a) = b$
$\Rightarrow a = f^{-1}(b)$
$\Rightarrow f^{-1}$ is onto
Hence f^{-1} is both one-one and onto

Q. 27 **If $f : A \to B$ and $g : B \to C$ are both one-one and onto, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$** Dec. 08, Dec. 15

Ans. : $f : A \to B$ is one to one and onto
$g : B \to C$ is one to one and onto
$\therefore$ $g \circ f : A \to C$ is one to one and onto

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$\Rightarrow \quad (g \circ f)^{-1} : C \to A$ is one to one and onto.

Let $a \in A$, then there exists an element $b \in B$ such that $f(a) = b$

$\Rightarrow \quad a = f^{-1}(b)$

Now $b \in B \Rightarrow$ there exists an element $c \in C$ such that $g(b) = C$

$\Rightarrow \quad b = g^{-1}(c)$

Then $(g \circ f)(a) = g[f(a)] = g[b] = c$

$\Rightarrow \quad a = (g \circ f)^{-1}(c)$ ...(i)

$(f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)]$

$= f^{-1}(b) = a$

$\Rightarrow \quad a = (f^{-1} \circ g^{-1})(c)$ ...(ii)

Combining (i) and (ii) we have,

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Q. 28 Show that function $f(x) = ax + b$ from R to R is inversible. Where a and b are constant with $a \neq 0$. Find the inverse of f. May 02

Ans. : For if $x, x_2 \in R$. $f(x_1) = f(x_2) \Rightarrow ax_1 + b = ax_2 + b$

$\Rightarrow \quad ax_1 = ax_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ f is one to one

Also if $y \in R$ Then

$y = f(x) = ax + b$

$\Rightarrow \quad x = \dfrac{y-b}{a}$

$\therefore$ for $x \in R$ there exists $\left(\dfrac{y-b}{a}\right) \in R$

Such that $f\left(\dfrac{y-b}{a}\right) = a\left(\dfrac{y-b}{a}\right) + b = y$

$\therefore$ f is onto

$\therefore f^{-1}$ exists and is defined by

$f^{-1}(y) = \dfrac{1}{a}(y-b)$

Q. 29 Show that the function $f : R - \{2\} \to R - \{0\}$ where R is set of real numbers defined by $f(x) = \dfrac{1}{x-2}$ is a bijection. Find its inverse. Dec. 09, May 11

Ans. : To find f^{-1},

Set $y = f(x)$ and then interchange x and y as follows.

$y = \dfrac{1}{x-2}$

One interchanging x and y

$x = \dfrac{1}{y-2}$

$x(y-2) = 1$

$xy - 2x = 1$

$xy = 1 + 2x$

$y = \dfrac{1+2x}{x}$

$f^{-1}(x) = \dfrac{1+2x}{x}$

Q. 30 Function $f(x) = \dfrac{(4x+3)}{(5x-2)}$. Find f^{-1} May 14

Ans. : $f(x) = \dfrac{(4x+3)}{(5x-2)}$

interchange x and y to get :

$x = \dfrac{(4y+3)}{(5y-2)}$

solve for "y"

$5xy - 2x = 4y + 3$

$5xy - 4y = 2x + 3$

$(5x-4)y = 2x + 3$

$y = \dfrac{(2x+3)}{(5x-4)}$

$f^{-1} = \dfrac{(2x+3)}{(5x-4)}$

Q. 31 Explain Pigeonhole principle and Extended Pigeonhole principle. Show that 7 colors are used to paint 50 bicycles, at least 8 bicycle will be of same color. Dec. 14, May 15, May 16

Ans. : Pigeonhole Principle :

If n pigenons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.

Proof :

Consider labeling the m plgeonholes with the numbers 1 through m and the n pigeons with the numbers 1 through n. Now, beginning with pigeon 1, assign each pigeon in order to the pigeonhole with the same number. This assigns as many pigeons as possible to individual pigeon holes, but because $m < n$, there are $n - m$ pigeons that have not yet been assigned to a pigeonhole. At least one pigeonhole will be assigned a second pigeon.

The Extended Pigeonhole Principle :

If there are m pigeonholes and more than 2m pigeons, then three or more pigeons will have to be assigned to at least. One of the pigeonholes. In general if the number at pigeons is much larger than the number at pigeonholes, previous theorm can be restated to give a stronger conclusion.

First , a word about notation. If n and m are positive integers, then $\lfloor n / m \rfloor$ stands for the largest integer less than or equal to the rational number n/m. Thus $\lfloor 3/2 \rfloor$ is 1, $\lfloor 9/4 \rfloor$ is 2 and $\lfloor 6/3 \rfloor$ is 2.

Theorem : (The extended pigeonhole principle)

If n pigeons are assigned to m, pigeonholes, then one of the pigeonholes must obtain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof : (by contradiction) :

If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$ pigeons, then there are at most $m. \lfloor (n-1)/m \rfloor \leq m. (n-1)/m = n - 1$ pigeons in all. This contradicts our assumption, so one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

7 colors are used to paint 50 bicycles, at least 8 bicycles will be of same color :

If n pigeons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.

By the extended pigeonhole principle at least.

$\lfloor (50-1)/7 \rfloor + 1 = 8$ will be of the same color.

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Q. 32 **Show that if any 11 numbers are chosen from the set {1, 2, ...20 }, then one of them will be a multiple of another.** May 05, Dec. 08, Dec. 15

Ans. :

Every positive integer n can be written as $n = 2^k$, where m is odd and $k \geq 0$. This can be seen by simply factoring all powers of 2 (if any) out of n. In this case let us call m the odd part of n. If 11 numbers are chosen from the set { 1, 2, ...20 }, then two of them must have the same odd parts. This follows from the pigeonhole principle since there are 11 numbers (pigeons), but only 10 odd numbers between 1 and 20 (pigeonholes) that can be odd parts of these numbers.

Let n_1 and n_2 be two chosen numbers with the same odd part we must have $n_1 = 2^{k1}$ m and $n_2 = 2^{k2}$ m, for some k_1 and k_2. If $k_1 \geq k_2$, then n_1 is a multiple at n_2; otherwise n_2 is a multiple of n_1.

Q. 33 **Prove that if any 14 integers from 1 to *25* are chosen, then one of them is a multiple of another.** Dec. 05, May 08, May 09, May 12

Ans. : Every positive integer n can be written as $n = 2^k$, where n is odd and $k \geq 0$. This can be seen by simply factoring all powers of 2 (if any) out of n. In this caselet us all m the odd part of n. If 14 numbers are chosen from the set {1, 2, ..., 25} then 2 of them must have the same odd parts. This follows from the pigeonhole principle since there are 14 numbers (pigeons), but only B odd numbers between 1 and 25 (pigeonholes) that can be odd parts of these numbers.

Let n_1 and n_2 be 2 chosen numbers with the same odd part we must have $n_1 = 2^{k1}$ m and $n_2 = 2^{k2}$ m. for some k and k_2. If $k_2 \geq k_2$ then n_1 is a multiple at n_2 ; otherwise n_2 is a multiple of n_1.

Q. 34 **Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any five points are chosen lying on or inside the triangle, then two of them must be no more than 1/2 unit apart.**

May 2000, Dec. 07, May 11, Dec. 13

Ans. :

m_1 , m_2 , m_3 are midpoints of sides AC, AB and BC, respectively. Let the four small triangles created be the pigeonholes. For any five points in or on triangle ABC, at least two must be in or on the small triangle and thus are no more than 1/2 unit apart.

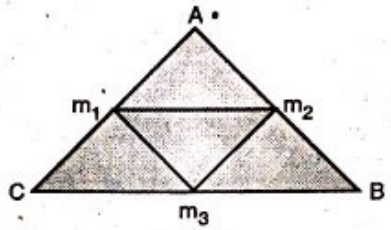

Fig. 4.3

Chapter 5 : Counting

Q. 1 **Let {1, 1, 1, ...} be a given sequence. Give the exponential generating function.**

Dec. 02, Dec.06, May 10

Ans. :

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \text{ is E.G.F.}$$

Q. 2 **Let {0, 1, 0, –1, 0, 1, 0, –1, ...} be a given sequence. Give the exponential generating function.** Dec. 02, 06, 09, May 10

Ans. :

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \text{ is E.G.F.}$$

Q. 3 **Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 214, 727, 2185, 65559, 19687, 59047.**

May 2000

Ans. : The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$$

$$= 1 + 7x + 25x^2 + 79x^3 + 241x^4 + 727x^5 + 2185x^6$$
$$6559x^7 + 19687x^8 + 59047x^9 + \ldots\ldots\ldots$$

Q. 4 **Determine the generating function of the numeric function a_r where**

(i) $a_r = 3^r + 4^{r+1}$, $r \geq 0$

(ii) $a_r = 5$, $r \geq 0$ Dec. 05, Dec. 06

Ans. :

(i) Let $a_r = b_r + c_r$

where $b_r = 3^r$

and $c_r = 4^{r+1}$

Let A(z), B(z) and C(z) be the generating functions of a, b and c.

For $b_r = 3^r$, the corresponding generating function is

$$B(z) = \frac{1}{1-3z}$$

For $c_r = 4^{r+1}$,

$$C(z) = \frac{4}{1-4z}$$

$$\therefore \quad A(z) = B(z) + C(z)$$

$$A(z) = \frac{1}{1-3z} + \frac{4}{1-4z}$$

$$A(z) = \frac{(5-16z)}{(1-3z)(1-4z)}$$

(ii) Given $a_r = 5$, $r \geq 0$

Therefore numeric function

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$$a = \{5, 5, 5, \ldots\}$$

$\therefore$ It generating function

$$A(z) = 5 + 5z + 5z^2 + 5z^3 + \ldots$$

$$= 5(1 + z + z^2 + z^3 + \ldots) = 5 \times \frac{1}{(1-z)}$$

$$\therefore \quad A(z) = \frac{5}{(1-z)}$$

Q. 5 Find the ordinary generating functions for the given sequences :

(i) $\{0, 1, 2, 3, 4, \ldots\}$ (ii) $\{1, 2, 3, 4, \ldots\}$

(iii) $\{0, 3, 3^2, 3^3, \ldots\}$ (iv) $\{2, 2, 2, 2, \ldots\}$

(v) $\{0, 0, 0, 1, 1, 1, 1, \ldots\ldots\ldots\ldots\ldots\ldots\}$

May 13, Dec. 13, May 14, May 15, May 16, Dec. 16, May 17

Ans. :

(i) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{0, 1, 2, 3, 4, \ldots\}$ is $0 + x + 2x^2 + 3x^3 + 4x^3 + \ldots$

(ii) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{1, 2, 3, 4, \ldots\}$ is $1 + 2x + 3x^2 + 4x^3 + \ldots$

(iii) The generation function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{0, 3, 3^2, 3^3, \ldots\}$ is $0 + 3x + 3^2x^2 + 3^3x^3 + \ldots$ i.e. $3x + 3^2x^2 + 3^3x^3 + \ldots$

(iv) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for the sequence $\{2, 2, 2, 2, \ldots\}$ is $2 + 2x + 2x^2 + 2x^3 + \ldots$

(v) $$\sum_{n=3}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots.$$

$\therefore$ The generating function for sequence

$$\sum_{n=3}^{\infty} (1)^n x^n = \{0, 0, 0, 1, 1, 1, 1, \ldots\ldots\ldots\ldots\ldots\} \text{ is } 0 + 0x + x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$$

Q. 6 Find formula for the sequence with the following first five terms

(i) 1, 3, 5, 7, 9 (ii) 1, –1, 1, –1, 1

(iii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

May 04

Ans. :

(i) 1, 3, 5, 7, 9 This is arithmetic progression

Here $a = 1; \quad d = 2$

$\therefore a + (n-1)d = 1 + (n-1)2 = 2n - 1$

$\therefore$ Generating function is $\sum_{n=1}^{5} (2n-1)\, x^n$.

(In this example question is for finding formula).

i.e. $$\sum_{n=1}^{5} (2n-1)$$

(ii) 1, –1, 1, –1, 1

Here alternative terms are positive and negative

$$\therefore \sum_{n=0}^{4} (-1)^n$$

(iii) $\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} = \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$

$\therefore$ Generating function is, $\sum_{n=0}^{4} \frac{1}{2^n}$

Q. 7 What is generating functions for the following sequences ?

(i) 1, 1, 1, 1, 1, 1 (ii) 2, 2, 2, 2, 2

(iii) 1, 1, 1, 1 .....

Dec. 03, Dec. 05, Dec. 09, May 15, May 16

Ans. :

(i) We know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

$$= \sum_{n=0}^{\infty} (1)^n x^n$$

$\therefore$ Generating function for sequence 1, 1, 1, 1, 1, 1 is

$$\frac{1}{1-x} = \sum_{n=0}^{5} (1)^n x^n$$

(ii) Generating function for sequence 2, 2, 2, 2, 2

$$\sum_{n=0}^{5} (2)^n x^n = 2^0 x^0 + 2^1 x^1 + 2^2 x^2 + 2^3 x^3 + 2^4 x^4 + 2^5 x^5$$

(iii) Similarly generating function for sequence 1, 1, 1, 1.....

$$\sum_{n=0}^{\infty} (1)^n x^n = \frac{1}{1-x}$$

Q. 8 Find the generating function for the following sequence.

(i) 1, 2, 3, 4, 5, 6....

(ii) 3, 3, 3, 3, 3...

Dec. 14

Ans. :

(i) 1, 2, 3, 4, 5, 6....

The generating function is $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_0 x^2 + \ldots$

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The generating function for sequence {1, 2, 3, 4, 5, 6} is $1+2x+3x^2+3x^2+4x^3+5x^4+6x^5$

(ii) {3, 3, 3, 3, 3}

The generating function for the sequence {3, 3, 3, 3, 3} is $3+3x+3x^2+3x^3+3x^4+\ldots$

Q. 9 Find the generating functions for the following sequence

(i) 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, ...

(ii) 6, – 6, 6, – 6, 6, – 6, 6 – ..... Dec. 15

Ans. : Find the generating function.

The generating function is,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

(i) The generating function for sequence {0, 0, 0, 1, 2, 3, 4, 5, 6, 7} is,

$0+0x+0x^2+1x^3+2x^4+3x^5+4x^6+5x^7+\ldots$

$x^3+2x^4+3x^5+4x^6+5x^7+\ldots$

(ii) The generating function for sequence {6, – 6, 6, – 6, 6, – 6,...}

$=6+-6x+6x^2-6x^3+6x^4-6x^5$

Q. 10 Find the solution of $a_{r+2}+2a_{r+1}-3a_r = 0$ that satisfies $a_0 = 1, a_1 = 2$. Dec. 02, Dec. 16, May 17

Ans. : The characteristic equation is

$$\alpha^2+2\alpha-3 = 0$$

$$\therefore \quad (\alpha+3)(\alpha-1) = 0$$

or $\alpha = -3, 1$

which are the characteristic roots of the equation.

Therefore, the solution of the given recurrence relation is

$$a_r = A_1(-3)^r + A_2(1)^r$$

$$= A_1(-3)^r + A_2 \quad \ldots(1)$$

where A_1, A_2 are constants

To find A_1 and A_2, putting r = 0, we get

$$a_0 = A_1 + A_2$$

$$1 = A_1 + A_2$$

Also putting r = 1 in Equation (1), we get

$$a_1 = -3A_1 + A_2$$

$$2 = -3A_1 + A_2$$

On solving, we get

$$A_1 = -\frac{1}{4} \text{ and}$$

$$A_2 = \frac{5}{4}$$

Hence the homogeneous solution of the given recurrence relation is

$$a_r = -\frac{1}{4}(-3)^r + \frac{5}{4}$$

Q. 11 Find the solution to the recurrence relation $a_n = 6a_{n-1} + 11a_{n-2} - 6a_{n-3}$ with condition $a_0 = 2, a_1 = 5$ and $a_2 = 15$. May 03, 05, May 15

Ans. :

Given $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

∴ The characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

$$\therefore (\alpha-1)(\alpha-2)(\alpha-3) = 0$$

$$\therefore \quad \alpha = 1, 2, 3$$

which are the characteristic roots of the equation.

Therefore the solution of the given recurrence relation is

$$a_n = A_1(1)^n + A_2(2)^n + A_3(3)^n \quad \ldots(1)$$

To find A_1, A_2 and A_3, putting r = 0, r = 1 and r = 2 we get

Given $a_0 = 2$

$\therefore \quad 2 = A_1 + A_2 + A_3$

$a_1 = 5$

$\therefore \quad 5 = A_1 + 2A_2 + 3A_3$

$a_2 = 15$

$\therefore \quad 15 = A_1 + 4A_2 + 9A_3$

On solving we get

$A_1 = 1, A_2 = -1$ and $A_3 = 2$

Hence the homogeneous solution of the given recurrence relation is

$$a_n = 1 - 2^n + 2.3^n$$

Q. 12 Determine the sequence whose R.R. is given by $C_n = 3C_{n-1} - 2C_{n-2}$ with initial conditions $C_1 = 5, C_2 = 3$. Dec. 08

Ans. :

Let $C_n - 3C_{n-1} + 2C_{n-2} = 0$

The characteristic equation is

$$\alpha^2 - 3\alpha + 2 = 0$$

$$(\alpha-2) \quad (\alpha-1)$$

$$\therefore \quad \alpha = 2, 1$$

which are the characteristic roots of the equation. Therefore, the solution of the given recurrence relation is

$$C_r = A_1(2)^r + A_2(1)^a$$

Where A_1, A_2 are constants ...(I)

To find A_1 and A_2 putting r = 1 we get

$$C_1 = 2A_1 + A_2 \qquad 5 = 2A_1 + A_2$$

Also putting r = 2 in Equation (I), we get

$$C_2 = 4A_1 + A_2 \qquad 3 = 4A_1 + A_2$$

On solving we get, $A_1 = -1$ and $A_2 = 7$

Hence the homogeneous solution of the given recurrence relation is

$$C_r = -2^r + 7^r$$

Q. 13 Solve the recurrence relation :

$d_n = 4(d_{n-1} - d_{n-2})$

Subject to initial conditions $d_0 = 1 = d_1$ Dec. 05, May 06, May 07, Dec. 07

Ans. :

$$d_n = 4d_{n-1} - 4d_{n-2}$$

$$d_n - 4d_{n-1} + 4d_{n-2} = 0$$

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The characteristic equation is,

$$\alpha^2 - 4\alpha + 4 = 0$$
$$\therefore \quad (\alpha - 2)(\alpha - 2) = 0$$
$$\therefore \quad \alpha = 2, 2$$

Which are the characteristic roots of the equation. Therefore solution of the given recurrence relation is

$$d_n = (A_1 n + A_2)(2)^n \ldots\ldots$$

Now, given $d_0 = 1$ and $d_1 = 1$. So by putting $n = 0$ and 1 in above equation we get,

$$1 = (A_1(0) + A_2(2))^0$$
$$1 = A_2$$
$$1 = (A_1(1) + A_2)(2)^1 = (A_1 + A_2)(2)$$
$$1 = 2A_1 + 2A_2$$
$$\text{put } A_2 = 1$$
$$1 = 2A_1 + 2$$
$$-1 = 2A_1$$
$$-\frac{1}{2} = A_1$$
$$\therefore \quad A_2 = 1 \text{ and } A_1 = -\frac{1}{2}$$

Hence the homogeneous solution of the given recurrence relation is,

$$d_n = \left[\left(-\frac{1}{2}\right) n + (1)\right](2)^n$$

Q. 14 Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$. Dec. 03

Ans. :

Given $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

$\therefore$ The characteristic equation is

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$$
$$\therefore \quad (\alpha + 1)^3 = 0$$
$$\therefore \quad \alpha = -1, -1, -1$$

Thus, the roots are $-1, -1, -1$ (tripple or 3–fold roots)

Therefore the solution of the given recurrence relation is

$$a_n = (A_1 n^2 + A_2 n + A_3)(-1)^n \quad \ldots(1)$$

To find A_1, A_2 and A_3 putting $n = 0, 1, 2$ in Equation (1) we get

For $n = 0$, $a_0 = 1$ $\quad \therefore 1 = A_3$

For $n = 1$, $a_1 = -2$

$$\therefore \quad -2 = (A_1 + A_2 + A_3)(-1)^1$$
$$= -A_1 - A_2 - A_3 = -A_1 - A_2 - 1$$
$$-2 + 1 = -A_1 - A_2$$
$$-1 = -A_1 - A_2$$

For $n = 2$, $a_2 = -1$

$$\therefore \quad -1 = (4A_1 + 2A_2 + A_3)(-1)^2$$
$$= 4A_1 + 2A_2 + 1$$
$$-2 = 4A_1 + 2A_2$$

Solving we get,

$$A_1 = -2, A_2 = 3, A_3 = 1$$

Hence the homogeneous solution of the given recurrence relation is

$$a_n = (-2n^2 - 3n + 1)(-1)^n$$

Q. 15 Determine whether the sequence $\{a_n\}$ is solution of recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ where $a_n = 3n$ for every non–negative integers n. Answer the same question for $a_n = 5$. May 03

Ans. :

Given $a_n - 2a_{n-1} + a_{n-2} = 0$

$\therefore$ The characteristic equation is

$$\alpha^2 - 2\alpha + 1 = 0$$
$$\therefore \quad (\alpha - 1)^2 = 0$$
$$\therefore \quad \alpha = 1, 1 \text{ which is repeated twice.}$$

Therefore the solution is,

$$a_n = (A_1 n + A_2)(1)^n \quad \ldots(1)$$

Now given $a_n = 3n$ for $n = 2, 3, 4, \ldots$, so by putting $n = 2$, and $n = 3$ in Equation (1), we get

For $n = 2$, $\quad 6 = 2A_1 + A_2$

$n = 3$, $\quad 9 = 3A_1 + A_2$

On solving, we get

$$A_1 = 3 \quad \text{and} \quad A_2 = 0$$

Hence the Homogeneous solution of the given recurrence relation is,

$$a_n = 3n$$

Now for $a_n = 5$ we get

For $n = 2$, $\quad 5 = 2A_1 + A_2$

$n = 3$, $\quad 5 = 3A_1 + A_2$

On solving we get

$$A_1 = 0 \quad \text{and} \quad A_2 = 5$$
$$\therefore \quad a_n = 5 \text{ is solution if } A_1 = 0$$

Q. 16 Solve the recurrence relation $a_n = -3a_{n-1}a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, $a_2 = 15$. Dec. 05, May 14

Ans. :

Given : $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

The characteristic equation is

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$$
$$(\alpha + 1)_3 = 0$$
$$\therefore \quad \alpha = -1, -1, -1$$

Thus, the roots are $-1, -1, -1$ (triple or 3-fold roots). Therefore the solution of the given recurrence relation is,

$$a_n = (A_1 n^2 + A_2 n + A_3)(-1)^n$$

To find A_1, A_2 and A_3 putting $n = 0, 1, 2$ in above equation we get,

For $n = 0$, $a_0 = 5$

$$\therefore \quad 5 = A_3$$

For $n = 1$, $a_1 = -9$

$$\therefore \quad -9 = (A_1 + A_2 + A_3)(-1)^1$$
$$= -A_1 - A_2 - A_3 = -A_1 - A_2 - 5$$
$$-9 + 5 = -A_1 - A_2$$
$$-4 = -A_1 - A_2$$

For $n = 2$, $a_2 = 15$

$$\therefore \quad 15 = (4A_1 + 2A_2 + A_3)(-1)^2$$
$$= 4A_1 + 2A_2 + 1$$

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$$14 = 4A_1 + 2A_2$$

Soling we get, $A_1 = 3$, $A_2 = 1$, $A_3 = 5$.

Hence the homogeneous solution of the given recurrence relation is,

$$a_n = (3n^2 + n + 5)\ (-1)^n$$

Q. 17 Solve $a_{r+2} - a_{r+1} - 6a_r = 4$ **Dec. 14**

Ans. : The corresponding homogeneous recurrence relation of the given recurrence relation is given by

$$a_{r+2} - a_{r+1} - 6\,a_r = 0$$

The characteristic equation is

$$\alpha^2 - \alpha - 6 = 0$$

$$\therefore \quad \alpha = 3, -2$$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A_1\,(3)^r + A_2\,(-2)^r$$

To find the particular solution, we consider the term f(r) (right hand side of the given equation). Since f (r) is a constant, the particular solution will also be a constant P.

i.e. $a_r = P$, for all r

$\Rightarrow \quad a_{r+1} = a_{r+2} = P$ (constant)

Substituting the values of a_r, a_{r+1}, and a_{r+2} in the given recurrence relation, we get,

$$P - P - 6P = 4$$

$$\therefore \quad P = -\frac{2}{3}$$

Hence, the particular solution is

$$a_r^{(p)} = -\frac{2}{3}$$

Thus the total solution or the general solution of the given recurrence relation is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\Rightarrow \quad a_r = A_1\,(3)^r + A_2\,(-2)^r - \frac{2}{3}$$

Q. 18 Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial conditions $a_0 = 1$, $a_1 = -1$. **Dec. 09**

Ans. :

The corresponding homogeneous recurrence relation of the given recurrence is given by

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

The characteristics equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\alpha = 3, 2$$

Therefore, the homogeneous solution of the given recurrence relation, we get

$$a_n^{(h)} = A_1(3)^n + A_2(2)^n$$

To find particular solution, we consider the term f(n). Since f(n) is a constant, the particular solution will also be a constant P.

i.e. $a_n = P$, for all n

$\Rightarrow \quad a_{n+1} = a_{n+2} = P$ (constant)

Substituting the values of a_n, a_{n+1}, a_{n+2} in the given recurrence relation, we get

$$P - 5P + 6P = 2$$

$$2P = 2$$

$$P = 1$$

Hence, the particular solution is,

$$a_n^{(p)} = 1$$

Thus the total solution of the given recurrence relation is,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A1(3)^n + A2(2)^n + 1$$

To determine A1 and A2, use given initial conditions, i.e. $a_0 = 1$, $a_1 = -1$.

Q. 19 Find the complete solution of the recurrence relation $a_n + 2\,a_{n-1} = n + 3$ for $n \geq 1$ and with $a_0 = 3$ **Dec. 07**

Ans. : The characteristic equation is

$$\alpha + 2 = 0$$

$$\therefore \quad \alpha = -2$$

Hence, Homogeneous solution is

$$a_n^{(h)} = A_1\,(-2)^n$$

For particular solution, since f (r) (Right hand side) is a linear polynomial, therefore, the particular solution will be of the form $(P_0 + P_{1n})$.

i.e. $a_n = P_0 + P_{1n}$

$a_{n-1} = P_0 + P_1\,(n-1)$

Substituting these values in the given recurrence relation, we get,

$$(P_0 + P_{1n}) + 2\,[P_0 + P_1\,(n-1)] = n + 3$$

$$P_0 + P_1 n + 2P_0 + 2P_1 n - 2P_1 = n + 3$$

$$(P_0 + 2P_0 - 2P_1) + n\,(P_1 + 2P_1) = n + 3$$

$$(3\,P_0 - 2P_1) + n\,(3P_1) = n + 3$$

On comparing the coefficients of polynomials, we get

and $3\,P_0 - 2P_1 = 3$

$3P_1 = 1$

Which on solving give $P_1 = 1/3$, $P_0 = 11/9$

Hence, the particular solution is

$$a_n^{(p)} = \frac{11}{9} + \frac{1}{3}\,n$$

Thus the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore \quad a_n = A_1\,(-2)^n + \frac{11}{9} + \frac{1}{3}$$

Using the initial conditions $a_0 = 3$ we get

$$a_0 = A_1 + \frac{11}{9}$$

Given $a_0 = 3$

$$3 = A_1 + \frac{11}{9}$$

$$3 - \frac{11}{9} = A_1$$

$$A_1 = 1.78$$

Therefore

$$a_n = 1.78\,(-2)^n + \frac{11}{9} + \frac{1}{3}\,n$$

Q. 20 Find the general solution of $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$ **May 06**

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Ans. :

The characteristic equation is

$$\alpha^2 + 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = -2, -3$$

Thus, the homogeneous solution is

$$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$$

For a particular solution, since f (r) (right hand side) of the given recurrence relation is a $42 \cdot 4^r$, the form of the particular solution will be, $P4^r$

Substituting in the given recurrence relation, we get,

$$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42 \cdot 4^r$$

Which simplifies to

$$\frac{21}{8} P4^r = 42 \cdot 4^r$$

$$\therefore \quad P = 16$$

Therefore, the particular solution is

$$a_r^{(p)} = 16 \cdot 4^r$$

Thus the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\therefore \quad a_r = A_1(-2)^r + A_2(-3)^r + 16 \cdot 4^r$$

Q. 21 Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

May 06, Dec. 06, Dec. 10, May 11, May 12, Dec. 13, Dec. 15

Ans. : This is nonhomogeneous recurrence relation. Solution of its associated homogeneous recurrence relation.

Given : $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$$\therefore \quad a_n - 5a_{n-1} + 6a_{n-2} = 7^n$$

The characteristic equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = 3, 2$$

The homogeneous solution is

$$a_n^{(h)} = A_1(3)^n + A_2(2)^n$$

Now, consider the form of particular solution.

Here f (n) is 7^n, the form of particular solution is, $P7^n$.

Subsituting in the given recurrence relation, we get,

$$P7^n - 5P7^{n-1} + 6P7^{n-2} = 7^n$$

which simplifies to

$$P = 2.44$$

Therefore, the particular solution is,

$$a_n^{(P)} = 2.447^n$$

Thus the general solution is

$$a_n = A_1(3)^n + A_2(2)^n + 2.447^n$$

Q. 22 Solve the following recurrence relation : $a_n - 5a_{n-1} + 6a_{n-2} = 2^n$ with initial conditions $a_0 = -1$ and $a_1 = 1$

Dec. 15

Ans. : This is nonhomogeneous recurrence relation. Solution of its associated homogeneous recurrence relation.

Given : $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$$\therefore \quad a_n - 5a_{n-1} + 6a_{n-2} = 7^n$$

The characteristic equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = 3, 2$$

The homogeneous solution is

$$a_n^{(h)} = A_1(3)^n + A_2(2)^n$$

Now, consider the form of particular solution. Here f (n) is 7^n, the form of particular solution is, $P7^n$.

Substituting in the given recurrence relation, we get,

$$P7^n - 5P7^{n-1} + 6P7^{n-2} = 7^n$$

which simplifies to

$$P = 2.44$$

Therefore, the particular solution is,

$$a_n^{(P)} = 2.447^n$$

Thus the general solution is

$$a_n = A_1(3)^n + A_2(2)^n + 2.447^n$$

Q. 23 Solve the following recurrence relation : $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with initial condition $a_0 = 1, a_1 = 6$

May 16

Ans. : **Given :** $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Initial condition : $a_0 = 1, \quad a_1 = 6$

The characteristic equation is,

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\therefore \quad (\alpha - 5)(\alpha - 2) = 0$$

Or $\quad \therefore \quad \alpha = 5, 2$

Which are the characteristic roots of the equation.

$\therefore$ The solution of the given recurrence relation is,

$$a_n = A_1(5)^n + A_2(2)^n$$

Where, A_1, A_2 are constants.

To find A_1 and A_2,

Putting n = 0, we get,

$$a_0 = A_1 + A_2$$

$$1 = A_1 + A_2 \quad \ldots(1)$$

Putting n = 1, we get,

$$a_1 = A_1 5^1 + A_2 2^1$$

$$6 = 5A_1 + 2A_2 \quad \ldots(2)$$

Solving two equations, we get,

$$A_1 = \frac{4}{3}$$

$$A_2 = \frac{-1}{3}$$

$\therefore$ Solution of given recurrence relation is,

$$a_n = \frac{4}{3}(5)^n - \frac{1}{3}(2)^n$$

Q. 24 Solve $a_r - 3a_{r-1} = 2, r \geq 1$ with $a_0 = 1$ using the generating functions.

Dec. 02, Dec. 13

Ans. : Given recurrence relation is

$$a_r - 3a_{r-1} = 2$$

Multiplying both sides by z^r, we obtain

$$a_r z^r - 3a_{r-1} z^r = 2z^r$$

Since $r \geq 1$, summing for all r, we get,

$$\sum_{r=1}^{\infty} a_r z^r - 3\sum_{r=1}^{\infty} a_{r-1} z^r = 2\sum_{r=1}^{\infty} z^r$$

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Consider the first term,

$$\sum_{r=1}^{\infty} a_r z^r = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

Since the generating function

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

$$\Rightarrow \sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$$

For the second term,

$$\sum_{r=1}^{\infty} a_{r-1} z^r = z \sum_{r=1}^{\infty} a_{r-1} z^{r-1}$$

$$= z A(z)$$

Also the third term gives

$$\sum_{r=1}^{\infty} z^r = \frac{z}{(1-z)}$$

Hence we obtain

$$[A(z) - a_0] - 3z A(z) = \frac{2z}{1-z}$$

or $$(1-3z) A(z) = \frac{2z}{1-z} + a_0$$

But, $a_0 = 1$

$$\therefore \quad (1-3z) A(z) = \frac{2z}{1-z} + 1$$

$$\therefore \quad (1-3z) A(z) = \frac{1+z}{1-z}$$

or $$A(z) = \frac{1+z}{(1-z)(1-3z)}$$

$$= \frac{2}{(1-3z)} - \frac{1}{(1-z)}$$

Consequently we have,

$$a_r = 2(3)^r - (1)^r, r \geq 0$$

which is the solution of the given recurrence relation.

Chapter 6 : Graphs

Q. 1. Explain graph with example.

Ans. : Graph :

A **graph** G consists of a finite set v of objects called **vertices**, a finite set E of objects called **edges** and a function γ that assigns to each edge a subset {v, w} where v and w are vertices (and may be the same). We will write G = (V, E, γ) when we need to name the parts of G. If e is an edge, and γ (e) = {v, w}, we say that e is an edge between v and w. The vertices v and w are the called **end points** of e.

Graphs are usually represented by pictures using a point for each vertex and a line for each edge. Graphs are often used to record information about relationships of connections.

Example :

Let V = {1, 2, 3, 4} and E = { e_1, e_2, e_3, e_4, e_5}. Let g be defined by

$\gamma(e_1) = \gamma(e_5) = \{1, 2\}, \gamma(e_2) = \{4, 3\}, \gamma(e_3) = \{1, 3\}, \gamma(e_4) = \{2, 4\}$.

Then G = {V, E, γ) is a graph as shown in Fig. 6.1.

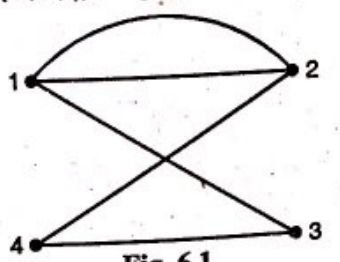

Fig. 6.1

Q. 2 Prove that a connected graph with n vertices must have atleast n–1 edges. May 98, Dec.16

Ans. :

Suppose there is a path from v_1 to v_2. Let ($v_1, \ldots v_i, \ldots v_2$) be the sequence of vertices that the path meets when it is traced from v_1 to v_2. If there *l* edges in the path, there are *l* + 1 vertices in the sequence. For *l* larger than n – 1, there must be a vertex v_k that appears more than one in the sequence, that is ($v_1, \ldots v_i, \ldots, v_k, \ldots v_k, \ldots, \ldots v_2$). Deleting the edges in path that leads v_k back to v_k, we have a path from v_1 to v_2 that has fewer edges than the original one. This argument can be repeated until we have a path that has n – 1 or fewer edges.

Q. 3 Show that the function f : R – {2} → R – {0} where R is set of real numbers defined by f(x) $= \frac{1}{x-2}$ is a bijection. Find its inverse. Dec. 09

Ans. : To find f^{-1},

Set y = f(x) and then interchange x and y as follows.

$$y = \frac{1}{x-2}$$

One interchanging x and y

$$x = \frac{1}{y-2}$$

$$x(y-2) = 1$$

$$xy - 2x = 1$$

$$xy = 1 + 2x$$

$$y = \frac{1+2x}{x}$$

$$f^{-1}(x) = \frac{1+2x}{x}$$

Q. 4 How many nodes are necessary to construct a graph with exactly 6 edges in which each node is of degree 2. Dec. 13, Dec. 14

Ans. : Suppose there are n vertices in the graph with 6 edges. Also, given the degree of each vertex is 2. Therefore by handshaking lemma,

$$\sum_{i=1}^{n} d(v_i) = 2e = 2 \times 6$$

$$\Rightarrow \quad d(v_1) + d(v_2) + \ldots + d(v_n) = 2 \times 6 = 12.$$

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$$\Rightarrow \quad 2+2+\underbrace{\ldots+2}_{n \text{ times}} = 12$$

$$\Rightarrow \quad 2n = 12$$

$$\Rightarrow \quad n = 6$$

Hence, 6 nodes are required to construct a graph with 6 edges in which each node is of degree 2.

Q. 5 Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs. Dec. 15

Ans. : Suppose the graph with 6 vertices has e number of edges. Therefore, by handshaking lemma.

$$\sum_{i=1}^{6} d(v_i) = 2e$$

$$\Rightarrow \quad d(v_1)+d(v_2)+d(v_3)+d(v_4)+d(v_5)+d(v_6) = 2e$$

Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence from the above equation

$$\Rightarrow (4+4)+(2+2+2+2) = 2e$$

$$\Rightarrow \quad 16 = 2e$$

$$\Rightarrow \quad e = 8$$

Hence the number of edges in a graph with 6 vertices with given conditions is 8.

Two such graphs are shown in Fig. 6.2 and 6.3.

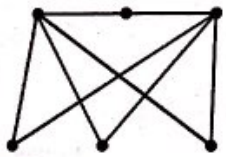

Fig. 6.2

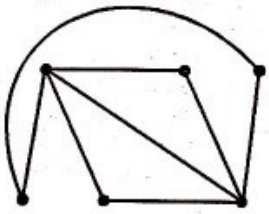

Fig. 6.3

Q. 6 Can a single undirected graph of 8 vertices have 40 edges excluding self loop ? Dec. 16

Ans. :

For a simple graph maximum edges are given by $\frac{n(n-1)}{2}$

($\because$ n = number of vertices)

$\therefore$ Number of edges $= \frac{8(8-1)}{2} = 28$ Maximum edges.

$\therefore$ A simple undirected graph of 8 vertices cannot have 40 edges excluding self loop.

Q. 7 How many paths of length 4 are there from a to d in simple graph shown in Fig. 6.4. May 06, May 07

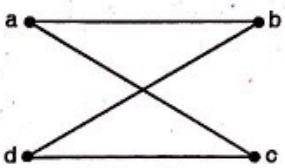

Fig. 6.4

Ans. :

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A^4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 8 & 8 & 8 \\ 8 & 0 & 0 & 8 \\ 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \end{bmatrix} \end{array}$$

path of length 4 is entry $(a, d)^{th}$

$\because$ Total 08 paths are there from a to d. These are,

ababd	abacd	abdbd	abdcd
acabd	acacd	acdbd	acdcd

Q. 8 Define Eular path and Hamilton path.

Ans. :

Eular graph :

A path in a graph G is called an **Euler path** if it includes every edge exactly once. An **Euler circuit** is an Euler path that is a circuit.

Hamilton path :

A **Hamiltonian path** is a path that contains each vertex exactly once. A **Hamiltonian circuit** is a circuit that contains each vertex exactly once except for the first vertex, which is also the last.

Q. 9 Give necessary and sufficient condition for an undirected graph to possess an Eulerian path and Eulerian circuit.

May 04, Dec 02, Dec 97, Dec 98, May 05, Dec. 06, May 06, May 07

Ans. : If a graph G has more than two vertices of odd degree, there can be no Euler path in G or in other words if G is connected and has exactly two vertices of odd degree, there is an euler path in G. Any Euler path in G must begin at one vertex of odd degree and end at the other. If G is a connected graph and every vertex has even degree, then there is an Eulerian circuit in G.

Q. 10 Determine Euler Cycle and path in graph shown below. May 15

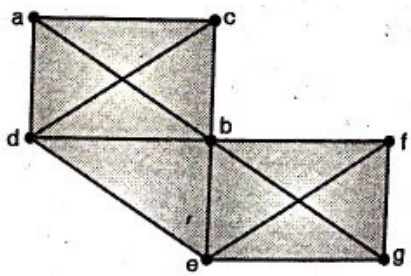

Fig. 6.5

Ans. : Vertex a – degree 3
Vertex b – degree 6
Vertex c – degree 3
Vertex d – degree 4
Vertex e – degree 4
Vertex f – degree 3

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Vertex g – degree 3

Number of vertices with odd degree = 4.

If a graph G has a vertex of odd degree, there can be no Euler circuit in G.

If a graph G has more than two vertices of odd degree then there can be no Euler path in G.

Hence given graph has no Euler path and no Euler circuit.

Q. 11 In Fig. 6.6 consider vertices 1 and 7 which are not adjacent to each other

Degree of 1 is 2.

Degree of 7 is 3. Dec. 07, May 11, May 16

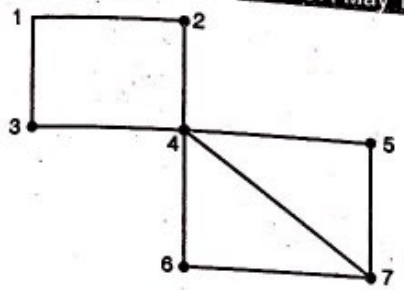


Fig. 6.6

Ans. :

∴ Degree of 1 + degree of 7 > number of vertices

$2 + 3 > 7$

$5 \not> 7$

∴ There is no Hamiltonian circuit.

But there is an Hamiltonian path π : 3, 1, 2, 4, 6, 7, 5.

Q. 12 Determine which of the following graph contain an Eulerian or Hamiltonian circuit. Dec. 12, May 13

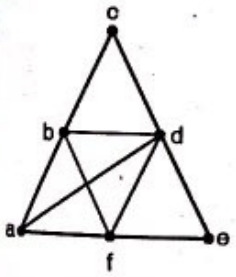


Fig. 6.7

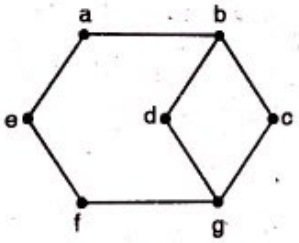


Fig. 6.7(a)

Ans. :

(i) **Eulerian Path :** π : a, b, c, d, b, f, d, a, f, e, d

G has 2 vertices of odd degree. Hence by Theorem 1 (a), no Eulerian circuit in given graph.

Hamiltonian Circuit : a, b, c, d, e, f, a.

Hamiltonian Path : a, b, c, d, e, f

(ii) **Eulerian Circuit :** This not exists in given graph.

Eulerian Path : g, d, b, a, e, f, g, c, b.

Hamiltonian Path : d, b, a, e, f, g, c

Q. 13 Determine whether the given graph has a Hamilton circuit or Eulerian circuit. If it does, find such a circuit. May 13

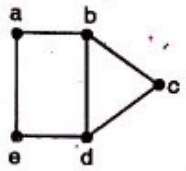


Fig. 6.8

Ans. :

Hamiltonian path = g, f, a, b, c, h, e, d

No Hamiltonian circuit

Hamiltonian path = e, a, b, d, c

Hamiltonian circuit = b, a, e, d, c, b.

Eulerian path = b, d, e, a, b, c, d

Eulerian circuit is not exist.

Q. 14 (i) Is Every Eulerian graph a Hamiltonian ?

(ii) Is every Hamiltonian graph a Eulerian ? Justify with the necessary graph. May 09

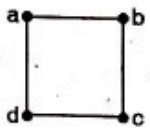
Ans. :

(i) Let G = (V, E) be a graph. A Eulerian Graph is a graph which passes through every edge exactly once.

Let G1 = (V1, E1) be a graph. A Hamiltonian circuit is a circuit which passes through every vertex exactly once (with only the first and last vertex being a repeat). A graph is called Hamiltonian if it possesses a Hamiltonian circuit.

In following Fig. 6.9(a) Graph G consists of both Eulerian and Hamiltonian Graph. But in Fig. 6.9(b) Graph G1 consist of Eulerain Graph but not Hamiltonian.

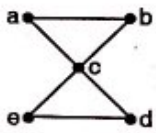
G



Eulerian Circuit : a, b, c, d, a (covers all edges of Graph G) Hamltonian Graph : a, b, c, d, a (covers all vertices of Graph G)

Fig. 6.9(a)

G1



Eulerian Circuit : c, a, b, c, e, d, c

No Hamiltonian Graph

Fig. 6.9(b)

Hence every Eulerian Graph is not necessarily Hamiltonian

(ii) In Hamiltonian Graph, we need to visit each and every vertex only once except the last vertex which is also the first vertex. In this type of graph we don't require to visit each and every edge of the graph (which is necessary in case of Eulerian Graph) i.e. need to cover all vertices only once.

Since every Hamiltonian graph not necessarily be Eulerian. Refer following Fig. 6.9(c) to better understand this statement. In following Graph G2 solid lines indicate Hamiltonian Graph, but it is not Eulerian Graph, since it is not possible to cover all edges exactly once.

G2

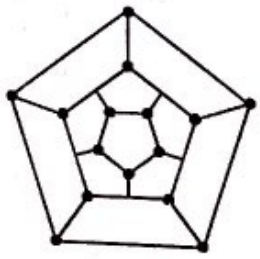


Fig. 6.9(c)

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Q. 15 . Determine the Eulerian and Hamiltonian path, if exists, in the following graphs. May 14

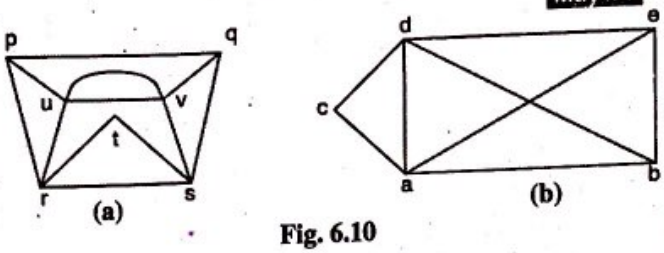


Fig. 6.10

Ans. :

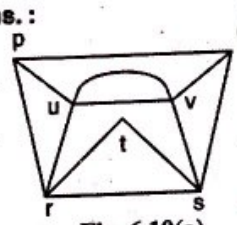


Fig. 6.10(a)

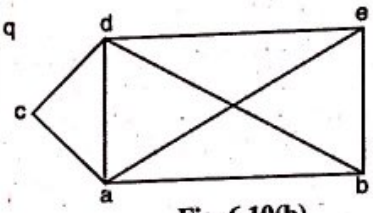


Fig. 6.10(b)

Hamiltonian path : p, u, v, q, s, t, r

Hamiltonian circuit : r, p, u, v, q, s, j, r

Eulerian path : (p, u, v, q, s, v, u, r, t, s, r, p, q)

Hamiltonian path : c, d, e, b, a

Hamiltonian circuit : c, d, b, a, c

Eulerian path : (e, d, b, a, d, c, a, e)

Q. 16 Determine Hamiltonian Cycle and path in graph shown in Fig. 6.11. May 17

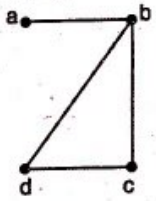


Fig. 6.11

Ans. : Hamiltonian Cycle :

Degree of each vertex :

a = 1 b = 3

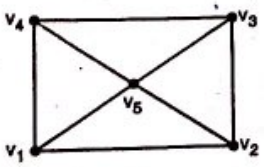
d = 2 c = 2

Total number of vertices n = 4

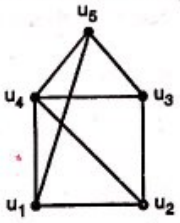
It has no Hamiltonian cycle consider 'a' and 'c' which are not adjacent to each other, degree (a) + degree = (c) = 3, which is not great, than unmber of vertices.

Hamiltonian path : a, b, c, d

Q. 17 Show that the following two graphs are isomorphic. May 98, May 2000, May 13, May 16



G_1
(a)



G_2
(b)

Fig. 6.12

Ans. : Both the graphs have 5 vertices and 8 edges.

Both the graphs have 4 vertices of degree 3 and 1 vertex of degree 4. And adjacency is also preserved.

The one-one correspondence between the vertices is given by, function $f : v \rightarrow v^*$,

where $v = \{v_1, v_2, v_3, v_4, v_5\}$,

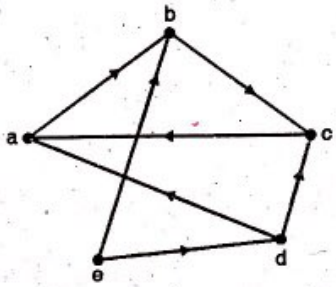
$v^* = \{u_1, u_2, u_3, u_4, u_5\}$

Such that {a, b} is an edge in G_1 if and only, if {f(a), f(b)} is an edge in G_2.

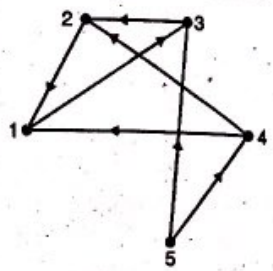
$\therefore \quad f = \{(v_1, u_1), (v_2, u_2), (v_3, u_3), (v_4, u_5), (v_5, u_4)\}$

Hence G_1 and G_2 are isomorphic graphs.

Q. 18 Show that the following two graphs are isomorphic. May 01, Dec. 13, Dec. 14



G_1
Fig. 6.13(a)



G_2
Fig. 6.13(b)

Ans. : Note that both the graphs have 5 vertices and 7 edges. Each graph has 4 vertices of degree 3 and 1 vertex of degree 2. Also the adjacency is preserved. The one–one correspondence between the vertices is given by, function $f : v \rightarrow v^*$

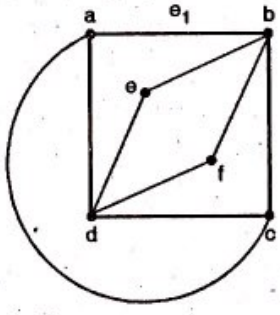
where $v = \{a, b, c, d, e\}$

$v^* = \{1, 2, 3, 4, 5\}$

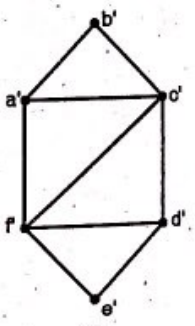
$\therefore \quad f = \{(a, 1), (b, 3), (c, 2), (d, 4), (e, 5)\}$

Hence G_1 and G_2 are isomorphic graphs.

Q. 19 Determine whether the following pair of graphs are isomorphic or not. Dec. 12



(a)



(b)

Fig. 6.14

Ans. : Here both graphs G_1 and G_2 have same number of vertices i.e. 6 and same number of edges that is 9.

Degree of each vertices in graph G and H is given below.

Graph G a = 3 c = 3 b = 4 d = 4 e = 2 f = 2

Graph H a' = 3 d' = 3 c' = 4 f' = 4 b' = 2 e' = 2

Each graph has two vertex of degree 4, two of degree 3 and two of degree 2. Now consider the graph G. In this graph, the vertex a of degree 3 is adjacent to 2 vertices of degree 3 and

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another of degree 4. Where in another graph H two vertices of degree 3 are adjacent to two vertices, they have degree 4 and 2 respectively. Hence G and H are not isomorphic graph.

$$G \not\cong H$$

Q. 20 Determine if following graphs g and G are isomorphic or not. **May 14**

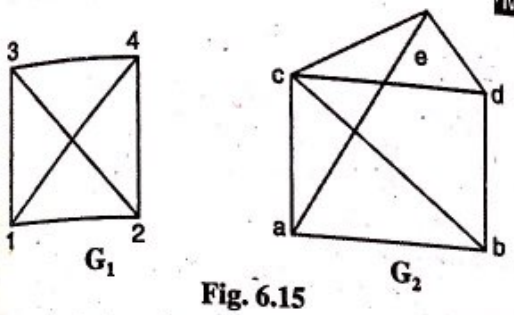

Fig. 6.15

Ans. : Graph G_1 have 4 vertices and 6 edges ; where Graph G_2 have 5 vertices and 8 edges. Adjacency is not available between G_1 and G_2.

Hence given two graphs are not isomorphic.

Q. 21 Determine if following graphs g and G are isomorphic or not. **May 15**

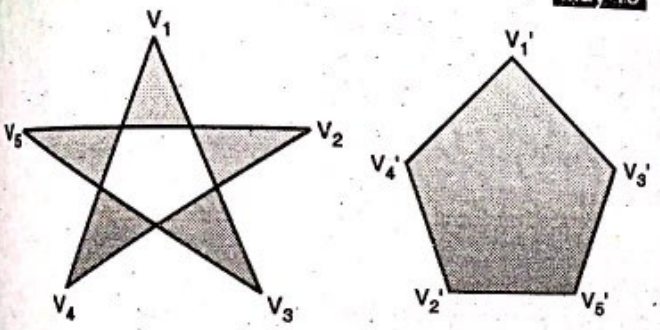

Fig. 6.16

Ans. : Number of vertices = both 5

Number of edges = both 5

Degree of corresponding vertices = all degree 2

connectedness : each is fully connected.

Number of connected components both 1

Pails of connected vertices : all correspond

Number of loops : 0

Number of parallel edges = 0

Everything is equal, so the graphs are isomorphic more formally,

$G = (V_1, E)$ where $V = \{V_1, V_2, V_3, V_4, V_5\}$ and

$E = \{(V_1, V_2), (V_2, V_3), (V_3, V_4), (V_4, V_5), (V_5, V_1)\}$

$= \{e_1, e_2, e_3, e_4, e_5\}$

$G' = \{V', E'\}$ where $V' = \{V_1', V_2', V_3', V_4', V_5'\}$ and

$E' = \{(V_1', V_2'), (V_2', V_3'), (V_3', V_4') (V_4', V_5'), (V_5', V_1')\}$

$= \{e_1', e_2', e_3', e_4', e_5'\}$

Construct 2 functions $f : V \to V'$ and $g : E \to E'$

$f : V \to V'$		$g : E \to E'$	
V	V'	E	E'
V_1	V_1'	e_1	e_1'
V_2	V_2'	e_2	e_2'
V_3	V_3'	e_3	e_3'
V_4	V_4'	e_4	e_4'
V_5	V_5'	e_5	e_5'

Q. 22 Determine if following graphs G1 and G2 are isomorphic or not. **May 17**

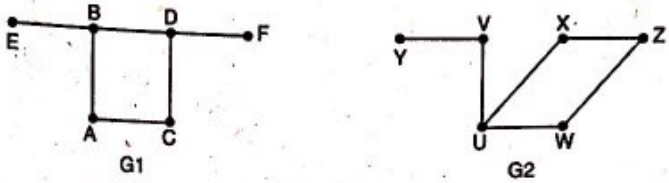

Fig. 6.17

Ans. : Both Graphs G1 and G2 contain 6 Vertices and 6 Edges

Let G1 = {A, B, C, D, E, F}

Let G2 = {V, U, W, X, Y, Z}

Degree of each node in G1	Degree of each node in G2
A = 2	V = 2
B = 3	U = 3
C = 2	W = 2
D = 3	X = 2
E = 1	Y = 1
F = 1	Z = 2

G2 is not holding any additional vertices of degree 3. Hence adjacency is not preserved. Hence given graphs are not isomorphic.

Q. 23 What are Planar graph ? **May 04, Dec. 06, May 07, Dec. 07, Dec. 12, May 13**

Ans. : Planar graph :

A graph is said to be **planar** if it can be drawn on a plane in such a way that no edges cross one another, except of course at common vertices.

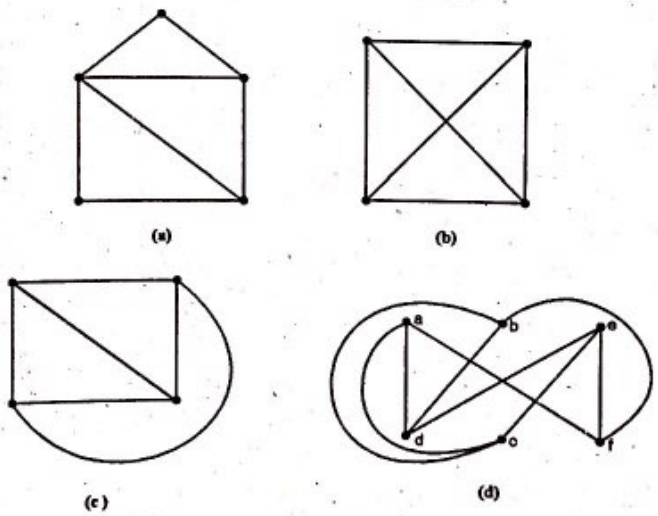

Fig. 6.18

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Fig. 6.18(a) shows a planar graph

Fig. 6.18(b) also shows a planar graph because it can be redrawn as that of figure shown in Fig. 6.18(c)

Fig. 6.18(d) shows a non-planar graph.

Suppose we draw a planar graph on the plane and take a sharp knife to cut along the edges then the plane will be divided into pieces that are called the regions of the graph. A region of a planar graph is defined to be an area of the plane that is bounded by edges and is not further divided into subareas.

Example :

The graph has five regions as shown in Fig. 6.19. Notice that cutting along the edge a does not further divide region 1 and cutting along the edges b, c, d does not further divide region 5.

A region is said to be finite if its area is finite and is said to be infinite if its area is infinite. Clearly a planer graph has exactly one infinite region.

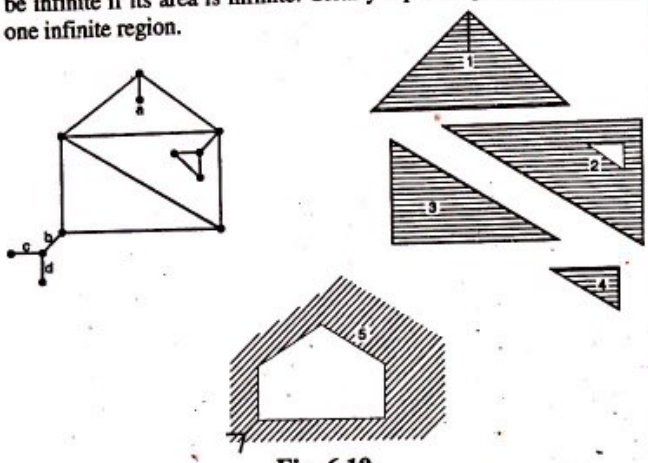

Fig. 6.19

Q. 24 A connected planar graph has 10 vertices each of degree 3. Into how many regions does a representation of this planar graph split the plane? Dec. 12

Ans. :

$|v| = 10 =$ number of vertices, degree of each vertex $= 3$.

By Handshaking Lemma

$$\sum_{i=1}^{n} v_i = 2.|e|, \quad [|e| = \text{number of edges}]$$

$$10 \times 3 = 2|e|$$

$$|e| = 15$$

By Euler's Theorem,

$$|v| - |E| + |R| = 2$$

$$10 - 15 + |R| = 2$$

$$|R| = 7$$

$\therefore$ Planer graph will split the plane in 7 regions.

Chapter 7 : Algebraic Structures

Q. 1 Define group and semi group. May 99, Dec. 2000, May 10, May 13

Ans. : Group :

Let (A, *) be an algebraic system, where * is a binary operation. (A, *) is called a **group** if the following conditions are satisfied.

1. * is a closed operation.
2. * is an associative operation
3. There is an identity.
4. Every element in A has a left inverse.

Because of associativity, a left inverse, of an element is also a right inverse of the element in a *group*.

Semi-group

Let (A, *) be an algebraic system, where * is a binary operation on A. (A, *) is called a **semigroup** if the following conditions are satisfied.

1. * is a closed operation.
2. * is an associative operation.

The semigroup (A, *) is said to be commutative if * is a commutative operation.

Q. 2 Show the relation o given by aob = a^b is a binary operation on set of natural numbers, is it associative ? Verify. Dec. 05, May 07

Ans. :

Given : aob $= a^b$, a and b are natural number

Let boa $= b^a$, a and b are natural number

$$aob \neq boa$$

$$a^b \neq b^a$$

$\therefore$ Given relation o is not an associative operation.

Q. 3 Let G be a group prove that the identity element e is unique May 98, Dec. 05

Ans. :

Suppose that $f \in G$, f is identity element of G and $f \neq e$.

We will now show that f = e, a contradiction which completes the proof.

$f = f * e$ since e is an identity

$= e$ since f is an identity

Q. 4 Let G be the group, prove that each element a in G has only one inverse in G. May 98, Dec. 02, May 04

Ans. : If possible consider $b, c \in G$ are two inverses of a such that $a * b = I$ and $a * c = I$

where I = identity element

$\Rightarrow \quad a * b = a * c$

$\Rightarrow \quad b = c$ ...left cancellation property

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$\therefore$ Inverse is unique.

$\therefore$ Each element 'a' in Group G has one inverse.

Q. 5 Show that if every element in a group is its own inverse, then the group must be Abelian. Dec. 98, Dec. 13, May 15

Ans. : The inverse exists for all $x \in A$ as $x * x = e$ i.e. every element is its own inverse

$\therefore$ Now, the identity is e

$\therefore x * e = e * x = x$ $\quad\because$ by definition of identity

$\therefore x * e = e * x$

$\therefore$ But for all $y \in A$. $y * y = e$

$\therefore x * y * y = y * y * x$

Now, we see that $y * y$ is another element of A say b

$$x * b = b * x$$

For all $x \in A$

$\therefore$ (A, *) is an Abelian group.

Q. 6 State and prove right or left cancellation property for a group. May 97, May 99, May 09

Ans. : Let G be a group and let a, b, c be elements of G then

(a) $ab = ac$ implies that $b = c$ **(left cancellation property)**

(b) $ba = ca$ implies that $b = c$ **(right cancellation property)**

Suppose that $ab = ac$

Multiplying both sides of this equation by a^{-1} on the left, we obtain

$$a^{-1}(ab) = a^{-1}(ac)$$

$$(a^{-1}a)b = (a^{-1}a)c \quad \because \text{associativity}$$

$$eb = ec \quad \because \text{by definition of inverse}$$

$$b = c \quad \text{by definition of an identity.}$$

Q. 7 Let (A, *) be an algebraic system where * is a binary operation. Such that for any a and b in A. a * b = a.

(i) Show that * is an associative operation.

(ii) Can * ever be a commutative operation. May 97

Ans. :

(i) $a * (b * c) = a * b = a$ ... by definition ...(1)

$(a * b) * c = a * c = a$ ... by definition ...(2)

From Equations (1) and (2)

$a * (b * c) = (a * b) * c$ $\quad\therefore$ * is associative operation

(ii) No, consider $a, b \in A$

Then $\left.\begin{array}{l} a * b = a \\ b * a = b \end{array}\right\}$ by definition of (A, *)

$\therefore a * b \neq b * a$ unless $a = b$

$\therefore$ * can never be a commutative operation

Q. 8 Determine whether the set together with the binary operation * is a semigroup, monoid or a group. Justify your answer.

(a) Set of real numbers with $a * b = a + b + 2$.

(b) The set of $m \times n$ matrices under the operation of multiplication. Dec. 96, May 2000

Ans. :

(a) (i) First check for associativity

$$a * (b * c) = (a * b) * c$$

$$a * (b + c + 2) = (a + b + 2) * c$$

$$a + b + c + 2 + 2 = a + b + 2 + c + 2$$

$$a + b + c + 4 = a + b + c + 4$$

(ii) Set of real number is closed under the binary operation *.

Therefore it is a semigroup.

(iii) It has an identity element i.e. -2,

$$a * (-2) = a + (-2) + 2 = a$$

Hence it is a monoid.

(iv) Every element in this set has an inverse.

For any element a in this set a^{-1} is

$$a * a^{-1} = -2$$

$$\therefore (a + a^{-1} + 2) = -2$$

$$\therefore a + a^{-1} = -2 - 2$$

$$\therefore a + a^{-1} = -4$$

$$\therefore a^{-1} = -4 - a$$

$$\therefore a^{-1} = -(4 + a)$$

$\therefore$ Hence it is a group.

(b) Multiplication of matrices is not associative operation

$\therefore$ It is not a semigroup, not a monoid, not a group

Q. 9 Let G be a group and let a and b elements of G then

(a) $(a^{-1})^{-1} = a$ (b) $(ab)^{-1} = b^{-1}a^{-1}$ May 98, May 03, May 04, Dec. 05, May 06, May 12

Ans. :

(a) We show that a acts as an inverse for a^{-1}

$$aa^{-1} = a^{-1}a = e$$

Since the inverse of an element is unique we conclude that $(a^{-1})^{-1} = a$

(b) We easily verify that

$$(ab)(b^{-1}a^{-1}) = a(b(b^{-1}a^{-1}))$$

$$= a((bb^{-1})a^{-1}) = a(ea^{-1}) = aa^{-1} = e$$

and similarly $(b^{-1}a^{-1})(ab) = e$

so $(ab)^{-1} = b^{-1}a^{-1}$

Q. 10 Show that a group G is Abelian if and only if $(ab)^2 = a^2b^2$ for all elements a and b in G. Dec. 96, Dec. 98, May 99, Dec. 99, Dec. 08, Dec. 10, May 14, May 16

Ans. :

Proof : Part I :

$$(ab)^2 = a^2b^2 \quad \text{...Given}$$

$$\therefore a * b * a * b = a * a * b * b$$

Premultiply by a^{-1} and post multiply by b^{-1} on both sides

$$\therefore a^{-1} * a * b * a * b * b^{-1} = a^{-1} * a * a * b * b * b^{-1}$$

$$\therefore b * a = a * b$$

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As $a * a^{-1} = b * b^{-1} = e$

and $e * x = x$

Now since $b * a = a * b$

G is an Abelian group.

Part II :

Now assume G is Abelian

Therefore $a * b = b * a$

$$(a * b) * (a * b) = (ab)^2$$
$$= a * (b * a) * b \quad \because \text{associative}$$
$$= a * (a * b) * b \quad \because \text{Abelian}$$
$$= (a * a) * (b * b) \quad \because \text{associative}$$
$$= a^2 b^2$$

Hence proved.

Q. 11 **Let G be the set of all non–zero real numbers and let $a * b = \frac{ab}{2}$. Show that (G, *) is an Abelian group.** May 03, May 09, May 14

Ans. :

(i) We first verify that * is a binary operation. If a and b are elements of G then ab/2 is a non zero real number and hence is in G.

(ii) We next verify associativity since

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)\,c}{4}$$

and since $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{(ab)\,c}{4}$

the operation * is associative.

(iii) The number 2 is the identity in G, for if $a \in G$, then

$$a * 2 = \frac{(a)\,(2)}{2} = a = \frac{(2)\,(a)}{2} = 2 * a$$

(iv) Finally, if $a \in G$, then $a' = 4/a$ is an inverse of a, since

$$a * a' = a * \frac{4}{a} = \frac{a(4/a)}{2} = 2$$
$$= \frac{(4/a)\,(a)}{2} = \frac{4}{a} * a = a' * a$$

(v) Since $a * b = b * a$ for all a and b in G, we conclude that G is an Abelian group.

Q. 12 **Let G be a group, and let a and b be elements of G. Then**

(a) The equation $ax = b$ has a unique solution in G.

(b) The equation $ya = b$ has a unique solution in G. May 98, Dec. 05

Ans. : Proof :

(a) The element $x = a^{-1}b$ is a solution of the equation $ax = b$ since

$$a(a^{-1}b) = (aa^{-1})\,b = eb = b$$

Suppose now that x_1 and x_2 are two solutions of the equations $ax = b$. Then $ax_1 = b$ and $ax_2 = b$.

Hence $ax_1 = ax_2$

$\therefore$ $x_1 = x_2$

(b) The proof is similar to that of part (a).

Q. 13 **Let (A, *) be a semigroup let a be an element of A. Consider a binary operation + on A such that for every x and y in A.**

$x + y = x * a * y$

Show that + is an associative operation. Dec. 97

Ans. : Here we have to prove that + is an associative operation.

Proof : i.e. To prove that

$$(x + y) + z = x + (y + z)$$

L.H.S. $= (x + y) + z = (x * a * y) + z$

$= x * a * y * a * z$ → definition of +

R.H.S. $= x + (y + z) = x + (y * a * z)$

$= x * a * y * a * z$ → definition of +

From above we see that LHS = RHS.

Hence we conclude that + is an associative operator.

Q. 14 **If (G, *) is an Abelian group, then for all a, b ∈ G show that $(a * b)^n = a^n * b^n$.** Dec. 97, May 01, Dec. 13, Dec. 15

Ans. :

(i) **Basis step :** For n = 1

$$(a * b)^1 = a^1 * b^1$$
$$\therefore a * b = a * b$$

$\therefore$ Statement is true for n = 1

(ii) **Induction step :**

Let the given statement be true for some n = k where $k \in N$.

$$\therefore (a * b)^k = a^k * b^k \quad \ldots(1)$$

To prove the statement for n = k + 1

$$(a * b)^{k+1} = (a * b)^k * (a * b) \quad \ldots \text{definition of } a^n$$
$$\therefore (a * b)^{k+1} = a^k * b^k * a * b \quad \ldots \text{from Equation (1)}$$
$$\therefore (a * b)^{k+1} = a^k * a * b^k * b$$

$\because$ (G, *) is an Abelian group

$$(a * b)^{k+1} = a^{k+1} * b^{k+1} \quad \because \text{definition of } a^n$$

Using the principle of mathematical induction we state that

$$(a * b)^n = a^n * b^n \text{ for all } n \in N$$

Q. 15 **Let G be the set of rational numbers other than 1. Let define an operation * on G by a * b = a + b - ab for all a, b ∈ G. Prove that (G, *) is a group.** Dec. 13

Ans. : Closure property :

Since the element $a * b \in Q$ for every $a, b \in Q$, hence, the set Q is closed under the operation *.

Associative property :

Let us assume $a, b, c \in Q$, then we have

$$(a * b) * c = (a + b - ab) * c$$
$$= (a + b - ab) + c - (a + b - ab)c$$
$$= a + b - ab + c - ac - bc + abc$$
$$= a + b + c - ab - ac - bc + abc$$

Similarly, $a * (b * c) = a + b + c - ab - ac - bc + abc$

Therefore, $(a * b) * c = a * (b * c)$

$\therefore$ * is associative.

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Identity :

Let e is an identity element. Then we have $a * e = a \; \forall \; a \in Q$.

$$\therefore \; a + e - ae = a$$

or $\quad e - ae = 0$

or $\quad e(1-a) = 0$

or $\quad e = 0$, if $1 - a \neq 0$

Similarly, for $e * a = a \; \forall \; a \in Q$, we have $e = 0$

Therefore, for $e = 0$, we have $a * e = e * a = a$

Thus, 0 is the identity element.

Inverse :

Let us assume an element $a \in Q$.

Let a^{-1} is an inverse of a. Then we have

$$a * a^{-1} = 0$$

$$\therefore \; a + a^{-1} - aa^{-1} = 0$$

or $\quad a^{-1}(1-a) = -a$ or $a^{-1} = \dfrac{a}{a-1}, a \neq 1.$

Now $\dfrac{a}{a-1} \in Q$, if $a \neq 1$

Therefore, every element has inverse such that $a \neq 1$.

Since, the algebraic system $(Q, *)$ satisfy all the properties of a group. Hence, $(Q, *)$ is a group.

Q. 16 Prove that the set G = {0, 1, 2, 3, 4, 5} is an Abelian group of order 6 with respect to addition modulo 6.

Dec. 04, May 07, Dec. 09, Dec. 11, May 13, May 16

Ans. :

$+_6$	0	1	2	3	4	5	
0	0	1	2	3	4	5	$0+0 \bmod 6 = 0$
1	1	2	3	4	5	0	$0+1 \bmod 6 = 1$
2	2	3	4	5	0	1	$0+2 \bmod 6 = 2$
3	3	4	5	0	1	2	$0+3 \bmod 6 = 3$
4	4	5	0	1	2	3	$0+4 \bmod 6 = 4$
5	5	0	1	2	3	4	$0+5 \bmod 6 = 5$

1st row

Similarly other rows are calculated.

(i) All the entries in the composition table are elements of the set G. Hence G is closed with respect to addition modulo 6 $(+_6)$.

(ii) The composition $+_6$ is associative. If a, b, c are any three elements of G, then

$$a +_6 (b +_6 c) = (a +_6 b) +_6 c$$

Let $\quad a = 1, \; b = 2, \; c = 3,$

$$1 +_6 (2 +_6 3) = (1 +_6 2) +_6 3$$

$$1 +_6 5 = 3 +_6 3$$

$$0 = 0$$

Let $\quad a = 3, \; b = 4, \; c = 5$

$$3 +_6 (4 +_6 5) = (3 +_6 4) +_6 5$$

$$3 +_6 3 = 1 +_6 5$$

$$0 = 0$$

Hence, $+_6$ is an associative operation. Since it is satisfying for all a, b, c, $\in$ G.

(iii) If a is any element of G, then from the composition table we see that

$$0 +_6 a = a = a +_6 0 = 0$$

that is,

$$0 +_6 0 = 0 +_6 0 = 0$$

$$0 +_6 1 = 1 +_6 0 = 1$$

$$0 +_6 2 = 2 +_6 0 = 2$$

$$0 +_6 3 = 3 +_6 0 = 3$$

$$0 +_6 4 = 4 +_6 0 = 4$$

$$0 +_6 5 = 5 +_6 0 = 5$$

$\therefore$ 0 is identity element.

(iv) From the composition table we can also see the left inverses of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively. Since,

$$0 +_6 0 = 0 \quad 1 +_6 5 = 0 \quad 2 +_6 4 = 0$$

$$3 +_6 3 = 0 \quad 4 +_6 2 = 0 \quad 5 +_6 1 = 0$$

e.g. $4 +_6 2 = 0 = 2 +_6 4$ implies 4 is the inverse of 2.

(v) The composition is commutative as the corresponding rows and columns in the position are *identical*.

(vi) The number of elements in the set G = 6

$\therefore$ $(G, +_6)$ is a finite Abelian group of order 6.

Q. 17 Prove that the set G = {1, 2, 3, 4, 5, 6} is a finite Abelian group of order 6 with respect to multiplication modulo 7.

May 06, May 10, Dec. 10, May 11, May 12, May 14, Dec. 15, May 17

Ans. :

$\times_7$	1	2	3	4	5	6	
1	1	2	3	4	5	6	$2 \times 1 \bmod 7 = 2$
2	2	4	6	1	3	5	$2 \times 2 \bmod 7 = 4$
3	3	6	2	5	1	4	$2 \times 3 \bmod 7 = 6$
4	4	1	5	2	6	3	$2 \times 4 \bmod 7 = 1$
5	5	3	1	6	4	2	$2 \times 5 \bmod 7 = 3$
6	6	5	4	3	2	1	$2 \times 6 \bmod 7 = 5$

2nd row

Similarly row 1, row 3, row 4, row 5, and row 6 are calculated.

(i) All the entries in the composition table are elements of G. Hence G is closed under multiplication modulo 7 $(\times_7)$

(ii) The composition of $\times_7$ is associative. Let a, b, c are any three elements of G, then

$$a \times_7 (b \times_7 c) = (a \times_7 b) \times_7 c$$

Let $a = 1, \; b = 2, \; c = 3$

$$1 \times_7 (2 \times_7 3) = (1 \times_7 2) \times_7 3$$

$$1 \times_7 6 = 2 \times_7 3$$

$$6 = 6$$

Let $a = 4, \; b = 5, \; c = 6$

$$4 \times_7 (5 \times_7 6) = (4 \times_7 5) \times_7 6$$

$$4 \times_7 2 = 6 \times_7 6$$

$$1 = 1$$

Hence, $\times_7$ is an associative operation. Since it is satisfying for all a, b, c $\in$ G.

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(iii) We have $1 \in G$

If a is any element of G, then from the composition table, we can see that

$$1 \times_7 a = a = a \times_7 1$$

that is,
$$1 \times_7 0 = 0 \times_7 0 = 0$$
$$1 \times_7 1 = 1 \times_7 1 = 1$$
$$1 \times_7 2 = 2 \times_7 1 = 2$$
$$1 \times_7 3 = 3 \times_7 1 = 3$$
$$1 \times_7 4 = 4 \times_7 1 = 4$$
$$1 \times_7 5 = 5 \times_7 1 = 5$$
$$1 \times_7 6 = 6 \times_7 1 = 6$$

$\therefore$ 1 is an identity element

(iv) From the composition table, we can see that the left inverses of 1, 2, 3, 4, 5, 6 are 1, 4, 5, 2, 3, 6 respectively. Since,

e.g. $3 \times_7 5 = 1 = 5 \times_7 3$ i.e. inverse of 3 is 5.

(v) The composition $\times_7$ is commutative as the corresponding rows and columns in the table are identical.

(vi) The set has 6 elements hence group $(G, \times_7)$ is a finite Abelian group of order 6.

Q. 18 Define cyclic group and subgroups with examples.

Dec. 97, Dec. 98, May 99, Dec. 2000, May 13

Ans. : Cyclic groups

A group $(G, *)$ is said to be a **cyclic group** if there exists an element $a \in G$ such that every element of G can be written as some power of a, viz a^k, for some integer k. By a^k, we mean $a * a * a \ldots a$ (k times). We then say that G is generated by a or a is a generator of G.

A cyclic group is Abelian, since for any two elements a^r, $a^s \in G$,

$$a^r * a^s = a^{r+s} = a^{s+r} = a^s * a^r.$$

Examples :

(i) The group $(Z_m, +)$ is cyclic group of order m, generated by 1.

(ii) Let S be the unit circle, and let P_0 be the rotation of the circle through an angle $2\pi/n$. Then the set of rotations $\{P_0, P_0 2, P_0 3, \ldots, P_n{}^n\}$ forms a cyclic group of order n, under the operation composition of functions.

Subgroups :

Let $(A, *)$ be a group and B be a subset of A, $(B, *)$ is said to be a **subgroup** of A if $(B, *)$ is also a group by itself. Suppose we want to check whether $(B, *)$ is a subgroup for a given subset B of A. We note that

1. We should test whether $*$ is a closed operation on B.
2. $*$ is known to be an associative operation.
3. Since there is only one element e in A such that $e * x = x * e = x$ for all x in A, we must check that e is in B. In other words the identity of $(A, *)$ must be in B as the identity of $(B, *)$
4. Since the inverse of every element in A is unique for every element b in B, we must check that its inverse is also in B.

Examples :

(i) For a positive integer n, let $H = nZ = \{nx \mid x \in Z\}$. Then $(H, +)$ is a subgroup of $(Z, +)$.

(ii) Let $H = \{0, 4)$ in $(Z_8, +)$ H is then a subgroup of Z_8.

(iii) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$ under matrix multiplication. Let $H = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad \neq 0\right\}$ H is then a subgroup of G.

(iv) Let G be the group of all non–zero complex numbers $a + ib$ (a, b real) under multiplication.

Let $H = \{a + ib \mid a^2 + b^2 = 1\}$

Then H is a subgroup of G.

(v) The subset $A = \{I, R_1, R_2, R_3\}$ of the group S of the symmetrics of the square, is closed under the operation $*$ because in this subset I is unity (0°), $R_1 = 90°$, $R_2 = 180°$, $R_3 = 270°$. The inverse of 90° is 270°, the inverse of 180° is 180° and the inverse of 270° is 90°.

The subset A contains an identity element 0° and it also contains the inverse of each element in the subset.

The operation $*$ is an associative operation on the large set S so it remains associative when restricted to the subset A. Therefore $(A, *)$ is itself a group and also a subgroup of S.

Q. 19 Prove that the set A = {0, 1, 2, 3, 4, 5} is a finite Abelian group under addition modulo 6.

May 04, Dec. 16

Ans. : Table addition modulo 6 for the set A is

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(i) From table we see that sum of any two elements of set belongs to set.

$\therefore$ Set A is closed under this binary operation.

(ii) For any element a in set A.

$0 +_6 a = a +_6 0 = a$ that is,

$0 +_6 0 = 0 +_6 0 = 0$ $\quad$ $1 +_6 0 = 0 +_6 1 = 1$

$2 +_6 0 = 0 +_6 2 = 2$ $\quad$ $3 +_6 0 = 0 +_6 3 = 3$

$4 +_6 0 = 0 +_6 4 = 4$ $\quad$ $5 +_6 0 = 0 +_6 5 = 5$

$\therefore$ '0' is identity element

(iii) Now check for associativity

For any a, b, c, $\in$ A

$$a +_6 (b +_6 c) = (a +_6 b) +_6 c$$

Let $a = 0, b = 1, c = 2$,

$$\therefore 0 +_6 (1 +_6 2) = (0 +_6 1) +_6 2$$
$$0 +_6 3 = 1 +_6 2$$
$$3 = 3$$

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Let $a = 3,\ b = 4,\ c = 5.$

$\therefore 3 +_6 (4 +_6 5) = (3 +_6 4) +_6 5$

$3 +_6 3 = 1 +_6 5$

$0 = 0$

$\therefore$ It is associative. Since it is satisfying for all $a, b, c \in A$.

(iv) Left Inverse of 0 is 0, since $0 +_6 0 = 0$

Left Inverse of 1 is 5, since $5 +_6 1 = 0$

Left Inverse of 2 is 4, since $4 +_6 2 = 0$

Left Inverse of 3 is 3, since $3 +_6 3 = 3$

Left Inverse of 4 is 2, since $2 +_6 4 = 0$

Left Inverse of 5 is 1, since $1 +_6 5 = 0$

(v) It is a group of finite elements

For any $a, b \in A$

$a +_6 b = b +_6 a$ that is,

Let $a = 2,\ b = 3$

$2 +_6 3 = 3 +_6 2$

$5 = 5$

Let $a = 4,\ b = 5,$

$4 +_6 5 = 5 +_6 4 = 3$

$\therefore +_6$ is commutative.

$\therefore$ It is a finite Abelian group.

Q. 20 Consider the set A = {1, 2, 3, 4, 5, 6} under the multiplication modulo 7.

(a) Find the multiplication table for the above

(b) Find the inverses of 2, 3 and 5, 6

(c) Prove that it is a cyclic group

(d) Find the orders and the subgroups generated by {3, 4} and {2, 3}

Dec. 99, May 2000, May 15, Dec. 16

Ans. :

(a) Multiplication modulo 7 table for set A is

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	1	3	2	1

(b) Note that 1 is an identity element of the algebraic system $(A, \times_7)$

Since for any $a \in A$,

$a \times_7 1 = a = 1 \times_7 a$

that is, $1 \times_7 1 = 1 \times_7 1 = 1$

$2 \times_7 1 = 1 \times_7 2 = 2$

$3 \times_7 1 = 1 \times_7 3 = 3$

$4 \times_7 1 = 1 \times_7 4 = 4$

$5 \times_7 1 = 1 \times_7 5 = 5$

$6 \times_7 1 = 1 \times_7 6 = 6$

Recall that a^{-1} is that element of G such that $a * a^{-1} = 1$

$2 \times_7 4 = 1 = 4 \times_7 2 = 1$

$3 \times_7 5 = 1 = 5 \times_7 3 = 1$

$6 \times_7 6 = 1 = 6 \times_7 6 = 1$

$\therefore$ Inverse of 2 is 4

Inverse of 4 is 2

Inverse of 3 is 5

Inverse of 5 is 3

Inverse of 6 is 6

(c) We have $2^1 = 2,$

$2^2 = 2 \times_7 2 = 4$

$2^3 = 2^2 \times_7 2 = 4 \times_7 2 = 1$

$2^4 = 2^3 \times_7 2 = 1 \times_7 2 = 2$

$\therefore$ Hence $|2| = 3$

$\therefore$ 2 is not generator of this group

We have $3^1 = 3$

$3^2 = 3 \times_7 3 = 2$

$3^3 = 3^2 \times_7 3 = 2 \times_7 3 = 6$

$3^4 = 3^3 \times_7 3 = 6 \times_7 3 = 4$

$3^5 = 3^4 \times_7 3 = 4 \times_7 3 = 5$

$3^6 = 3^5 \times_7 3 = 5 \times_7 3 = 1$

Hence $|3| = 6$

$\therefore$ 3 is generator of this group and this group is cyclic.

(d) Subgroup generated by {3, 4} is denoted by <{3, 4}> since 3, 4 are elements of this set they have to be there in <{3, 4}>

Inverse of 3 is 5, inverse of 4 is 2

$\therefore$ $3, 4, 5, 2, \in <\{3, 4\}>$

$3 \times_7 4 = 5$	$5 \times_7 4 = 6$
$3 \times_7 3 = 2$	$6 \times_7 6 = 1$
$3 \times_7 5 = 1$	$5 \times_7 1 = 5$
$4 \times_7 4 = 2$	$1 \times_7 1 = 1$
$3 \times_7 2 = 6$	$5 \times_7 2 = 3$
$5 \times_7 5 = 4$	$3 \times_7 6 = 4$
$5 \times_7 6 = 2$	$2 \times_7 2 = 4$

$\therefore$ $<\{3, 4\}> = <1, 2, 3, 4, 5, 6>$

$\therefore$ Subgroup generated by {3, 4} is the set A itself whose order is 6.

Subgroup generated by {2, 3} is denoted by <{2, 3}>.

Since 2, 3 are elements of this set they have to be there in <{2, 3}>.

Inverse of 2 is 4.

Inverse of 3 is 5

$\therefore$ $2, 3, 4, 5 \in <\{2, 3\}>$

$2 \times_7 3 = 6$	$3 \times_7 4 = 5$
$4 \times_7 4 = 2$	$5 \times_7 5 = 4$
$2 \times_7 4 = 1$	$3 \times_7 5 = 1$
$4 \times_7 1 = 4$	$5 \times_7 6 = 2$
$2 \times_7 5 = 3$	$3 \times_7 6 = 4$
$4 \times_7 5 = 6$	$5 \times_7 1 = 5$
$2 \times_7 6 = 5$	$3 \times_7 1 = 3$

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$6 \times_7 6 = 1$ $\quad$ $2 \times_7 1 = 2$

$3 \times_7 3 = 2$ $\quad$ $6 \times_7 1 = 6$

$2 \times_7 2 = 4$

$\therefore \langle \{2, 3\} \rangle = \langle 1, 2, 3, 4, 5, 6 \rangle$

$\therefore$ Subgroup generated by $\langle \{2, 3\} \rangle$ is the set A and is of order 6.

Q. 21 Show that the (2, 5) encoding function e : $B^2 \to B^5$ defined by

e (00) = 00000 $\quad$ e (10) = 10101

e (01) = 01110 $\quad$ e (11) = 11011

is a group code.

Dec. 97, May 2000, May 01, Dec. 02, May 04, Dec. 06, May 07, May 10

Ans. :

Let $N = \{00000, 01110, 10101, 11011\}$ be the set of all code words.

$\oplus$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101
10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

(i) For a, b $\in$ N, a $\oplus$ b $\in$ N

$\therefore$ N is closed under $\oplus$ operation.

(ii) Identity element of B^5 i.e. 00000 $\in$ N.

Since, $00000 \oplus 00000 = 00000 \oplus 00000 = 00000$

$01110 \oplus 00000 = 00000 \oplus 01110 = 01110$

$10101 \oplus 00000 = 00000 \oplus 10101 = 10101$

$11011 \oplus 00000 = 00000 \oplus 11011 = 11011$

(iii) $\oplus$ is an associative operation

for e.g. $01110 \oplus (00000 \oplus 10101) = (01110 \oplus 00000) \oplus 10101$

$01110 \oplus 10101 = 01110 \oplus 10101$

$11011 = 11011$

(iv) Every element is its own inverse.

$\therefore$ N is subgroup of B^5 and the given encoding function is a group code.

Q. 22 Show that the (2, 5) encoding function e : $B^2 \to B^5$ defined by

e (00) = 00000 $\quad$ e(01) = 01110

e (10) = 10101 $\quad$ e (11) = 11011

Is a group code. How many errors will it detect and correct?

Dec. 96, May 97, Dec. 09, Dec. 14, May 16, Dec. 16

Ans. :

Let $N = \{00000, 01110, 10101, 11011\}$ be the set of all code words.

$\oplus$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101
10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

(i) For any a, b $\in$ N, a $\oplus$ b $\in$ N

$\therefore$ Set N is closed under $\oplus$ operation.

(ii) Identity element of B^5 i.e. 00000 also belongs to N.

$00000 \oplus 00000 = 00000 \oplus 00000$

$01110 \oplus 00000 = 00000 \oplus 01110$

$10101 \oplus 00000 = 00000 \oplus 10101$

$11011 \oplus 00000 = 00000 \oplus 11011$

(iii) $\oplus$ is associative operation.

(iv) Each element of N is its own inverse.

$00000 \oplus 00000 = 00000 \oplus 00000 = 00000$

$01110 \oplus 01110 = 01110 \oplus 01110 = 00000$

$10101 \oplus 10101 = 10101 \oplus 10101 = 00000$

$11011 \oplus 11011 = 11011 \oplus 11011 = 00000$

$\therefore$ N is subgroup of B^5 and the given encoding function is a group code.

d (00000, 01110) = 3

d (00000, 10101) = 3

d (00000, 11011) = 4

d (01110, 10101) = 4

d (01110, 11011) = 3

d (10101, 11011) = 3

$\therefore$ Minimum distance is 3.

The code will detect k or fewer errors if and only if its minimum distance is atleast k + 1. Since the minimum distance is 3, we have $3 \geq k + 1$ or $k \leq 2$. The code will detect 2 or fewer errors.

The code will correct k or fewer errors if and only if its minimum distance is atleast 2 k + 1. Since the minimum distance is 3 we have $3 \geq 2k + 1$ or $k \leq 1$. So the code will correct 1 or fewer errors.

Q. 23 Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

be a parity check matrix. Determine the group code $e_H : B^3 \to B^6$.

Dec. 02, May 04, Dec. 04, May 07, Dec. 14, May 15, Dec. 15, May 17

Ans. :

We have $B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$.

Then

$e(000) = 000\, x_1 x_2 x_3$.

where $x_1 x_2 x_3$ are determined using equations.

$$x_1 = b_1 \cdot h_{11} + b_2 h_{21} + \ldots + b_m \cdot h_{m1}$$

$$x_2 = b_1 \cdot h_{12} + b_2 h_{22} + \ldots + b_m \cdot h_{m2}$$

$$\vdots$$

$$x_r = b_1 \cdot h_{1r} + b_2 h_{2r} + \ldots + b_m \cdot h_{mr}$$

$$\therefore \quad x_1 = 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = 0$$

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$x_2 = 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

$x_3 = 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

$\therefore$ e (000) = 000000

e (001) = 001 $x_1 x_2 x_3$

$x_1 = 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 = 1$

$x_2 = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$x_3 = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$\therefore$ e (001) = 001111

e(010) = 010 $x_1 x_2 x_3$

$x_1 = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 0$

$x_2 = 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

$x_3 = 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

$\therefore$ e (010) = 010011

e (011) = 011 $x_1 x_2 x_3$

$x_1 = 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 1$

$x_2 = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$

$x_3 = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$

$\therefore$ e (011) = 011100

e (100) = 100 $x_1 x_2 x_3$

$x_1 = 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = 1$

$x_2 = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

$x_3 = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

$\therefore$ e (100) = 100100

e (101) = 101 $x_1 x_2 x_3$

$x_1 = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 = 0$

$x_2 = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$x_3 = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$\therefore$ e (101) = 101011

e (110) = 110 $x_1 x_2 x_3$

$x_1 = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1$

$x_2 = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

$x_3 = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

$\therefore$ e (110) = 110111

e (111) = 111 $x_1 x_2 x_3$

$x_1 = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 0$

$x_2 = 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$

$x_3 = 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$

$\therefore$ e (111) = 111000

$\therefore$ e_H : $B^3 \rightarrow B^6$ is

e (000) = 000000	e (001) = 001111
e (010) = 010011	e (011) = 011100
e (100) = 100100	e (101) = 101011
e (110) = 110111	e (111) = 111000

Q. 24 **Consider the (3, 5) group encoding function e : $B^3 \rightarrow B^5$ defined by**

e (000) = 00000	**e (100) = 10011**
e (001) = 00110	**e (101) = 10101**
e (010) = 01001	**e (110) = 11010**
e (011) = 01111	**e (111) = 11100**

Decode the following words relative to a maximum likelihood decoding function.

(a) 11001 (b) 01010 (c) 00111

May 08, May 10, May 13, Dec. 13, May 15, May 17

Ans. : Prepare decoding table :

00000	00110	01001	01111	10011	10101	11010	11100
00001	<u>00111</u>	01000	01110	10010	10100	11011	11101
00010	00100	01011	01101	10001	10111	11000	11110
00100	00010	01101	01011	10111	10001	11110	11000
01000	01110	00001	00111	11011	11101	10010	10100
10000	10110	<u>11001</u>	11111	00011	00101	<u>01010</u>	01100
10001	10111	11000	11110	00010	00100	01011	01101
10010	10100	11011	11101	00001	00111	01000	01110

(i) If we receive the word 11001, we first locate it in the 3rd column of the decoding table. Where it is underlined once. The word at the top of the 3rd column is 01001. Since e (010) = 01001. We decode 11001 as 010.

(ii) If we receive the word 01010, we first locate it in the 7th column of the decoding table. Where it is underlined twice. The word at the top of the 7th column is 11010. Since e (110) = 11010. We decode 01010 as 110.

(iii) Similarly 00111 is located in 2nd column of the decoding. table where it is underlined thrice. The word at the top of the 2nd column is 00110. Since e (001) = 00110, we decode 00111 as 001.

Q. 25 **R = {0, 2, 4, 6, 8} show that R is commutative ring under addition and multiplication modulo 10.**

Dec. 99, Dec. 02, May 05, May 10, Dec. 10, Dec. 11

Ans. : We prepare composition Table

Table 7.1

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

Table 7.2

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

(i) Now we have to show $(R, +_{10})$ is abelian group

(a) $+_{10}$ is closed operation from Table 7.7, since all elements from Table 7.7 belong to set R.

(b) $+_{10}$ is associative operation i.e.,

$(a +_{10} b) +_{10} c = a +_{10} (b +_{10} c)$, for all a, b, c $\in$ R

For ex. : Let a = 0, b = 2, c = 4

$(0 +_{10} 2) +_{10} 4 = 0 +_{10} (2 +_{10} 4)$

$2 +_{10} 4 = 0 +_{10} 6$

$6 = 6$

Let a = 4, b = 6, c = 8

$(4 +_{10} 6) +_{10} 8 = 4 +_{10} (6 +_{10} 8)$

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$0 +_{10} 8 = 4 +_{10} 4$

$8 = 8$

(c) '0' is identity

$0 +_{10} 0 = 0$ $\quad$ $0 +_{10} 0 = 0$

$0 +_{10} 2 = 2$ $\quad$ $2 +_{10} 0 = 2$

$0 +_{10} 4 = 4$ $\quad$ $4 +_{10} 0 = 4$

$0 +_{10} 6 = 6$ $\quad$ $6 +_{10} 0 = 6$

$0 +_{10} 8 = 8$ $\quad$ $8 +_{10} 0 = 8$

'0' is left identity and right identity

'0' is identity.

(d) Every element has left inverse i.e.

$0 +_{10} 0 = 0$ $\therefore$ 0 is left inverse of 0

$2 +_{10} 8 = 0$ $\therefore$ 2 is left inverse of 8

$4 +_{10} 6 = 0$ $\therefore$ 4 is left inverse of 6

$6 +_{10} 4 = 0$ $\therefore$ 6 is left inverse of 4

$8 +_{10} 2 = 0$ $\therefore$ 8 is left inverse of 2

(e) $+_{10}$ is abelian i.e.

$a +_{10} b = b +_{10} a$, for all $a, b \in R$

For ex. Let $a = 2, b = 4$

$2 +_{10} 4 = 4 +_{10} 2$

$6 = 6$

Let $a = 6, b = 8$

$6 +_{10} 8 = 8 +_{10} 6$

$4 = 4$

$\therefore$ **$(R, +_{10})$ is an abelian group.**

(ii) We have to show $(R, \times_{10})$ is semigroup

(a) $\times_{10}$ is closed operation from Table 8.8, since all elements from Table 8.8 belongs to set R.

(b) $\times_{10}$ is associative operation i.e.

$(a \times_{10} b) \times_{10} c = a \times_{10} (b \times_{10} c)$, for all $a, b, c, \in R$

For ex. Let $a = 2, b = 4, c = 6$

$(2 \times_{10} 4) \times_{10} 6 = 2 \times_{10} (4 \times_{10} 6)$

$8 \times_{10} 6 = 2 \times_{10} 4$

$8 = 8$

Let $a = 4, b = 6, c = 8$

$(4 \times_{10} 6) \times_{10} 8 = 4 \times_{10} (6 \times_{10} 8)$

$4 \times_{10} 8 = 4 \times_{10} 8$

$2 = 2$

$\therefore$ **$(R, \times_{10})$ is semigroup.**

(iii) We have to show, the operation $\times_{10}$ is distributive over the operation $+_{10}$, i.e.

$a \times_{10} (b +_{10} c) = (a \times_{10} b) +_{10} (a \times_{10} c)$

For all $a, b, c \in R$

For ex. Let $a = 2, b = 4, c = 6$

$2 \times_{10} (4 +_{10} 6) = (2 \times_{10} 4) +_{10} (2 \times_{10} 6)$

$2 \times_{10} 0 = 8 +_{10} 2$

$0 = 0$

Let $a = 0, b = 2, c = 4$

$0 \times_{10} (2 +_{10} 4) = (0 \times_{10} 2) +_{10} (0 \times_{10} 4)$

$0 \times_{10} 6 = 0 +_{10} 0$

$0 = 0$

$\therefore$ **$\times_{10}$ is distributive over the operation $+_{10}$.**

(iv) $(R, +_{10}, \times_{10})$ is commutative ring, i.e.

$a \times_{10} b = b \times_{10} a$, for all $a, b, \in R. b = 4$

For ex. Let $a = 2, b = 4$.

$2 \times_{10} 4 = 4 \times_{10} 2$

$8 = 8$

Let $a = 6, b = 8$

$6 \times_{10} 8 = 8 \times_{10} 6$

$8 = 8$

Let $a = 4, b = 6$

$4 \times_{10} 6 = 6 \times_{10} 4$

$4 = 4$

$\therefore$ **$(R, +_{10}, \times_{10})$ is commutative ring.**

Q. 26 In any Ring (R + .) prove that : (i) The zero element z is unique. (ii) The additive inverse of each ring element is unique.

Dec. 05, Dec. 06, May 06, May 07, May 10, May 12

Ans. :

(i) The zero element z is unique. If R has more than one additive identity.

Let $z_1 z_2$ denote two such elements

$$z_1 = \underset{\substack{\uparrow \\ z_2 \text{ is identity}}}{z_1 + z_2} = \underset{\substack{\uparrow \\ \text{If } z_1 \text{ is identity}}}{z_2}$$

(ii) The additive inverse of each ring element is unique.

For $a \in R$. Let two elements $b, c \in R$ where

$a + b = b + a = z$

and $a + c = c + a = z$

then $b = b + z = b + (a + c)$

$= (b + a) + c = z + c = c$

Q. 27 Prove that if (F, +, ·) is a field then it is an integral domain.

Dec. 03, May 05, May 06, May 07, Dec. 08, May 09, Dec. 10, May 12, Dec. 12

Ans. : We have to show that every field is an integral domain.

Since a field is a commutative ring with unity. In order to show that it is integral domain. We have to show that field has no zero divisors.

Let a, b be elements of F with $a \neq 0$ such that $ab = 0$.

As $a \neq 0$, a^{-1} exists and we have,

$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1} 0$

$\Rightarrow (a^{-1} a) b = 0$

$\Rightarrow Ib = 0$

$\Rightarrow b = 0$

Similarly, let $ab = 0$ and $b \neq 0$

Since $b \neq 0$, b^{-1} exists.

$\therefore ab = 0 \Rightarrow (ab) b^{-1} = 0b^{-1}$

$\Rightarrow a(bb^{-1}) = 0$

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$$\Rightarrow \quad aI = 0$$
$$\Rightarrow \quad a = 0$$

Thus in the field $ab = 0 \Rightarrow a = 0$ or $b = 0$.

$\therefore$ Field has no zero divisor

Hence every field is an integral domain.

Q. 28 If $f : (R, +, \cdot) \to (S, \oplus, \odot)$ is a ring homomorphism then prove that, $f(-a) = -f(a)$ for all $a \in R$. **May 14**

Ans. :

Recall $f : R \to S$ is ring homomorphism

if $f(a+b) = f(a) \oplus f(b)$ and

$f(a \cdot b) = f(a) \odot f(b)$, for $\forall$ a, b, $\in R$

As $-a = (-1)(a)$

$\therefore \quad f(-a) = f[(-1)(a)] = f(-1) f(a)$

$= (-1) f(a) = -f(a)$

$\because (-1)$ is additive identity.

$\therefore$ the result.

□□□

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