Matrix Chain Multiplication —

Given a sequence of matrices, find the most efficient way to multiply these matrices together. The problem is not actually to perform multiplication but merely to decide in which order we should perform the multiplication.

We have many options to multiply a chain of matrices as the matrix multiplication is associative.

⇒ Two matrices can be multiplied if the no of columns in first is equal to no of rows of 2nd matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3 \times 2}$$

$$C = \begin{bmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31} & a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32} \\ a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31} & a_{21} \times b_{12} + a_{22} \times b_{22} + a_{23} \times b_{32} \end{bmatrix}_{2 \times 2}$$

$$\underset{2 \times 3}{A} \quad \underset{3 \times 2}{B} = \underset{2 \times 2}{C}$$

Total no of multiplication $= 2 \times 3 \times 2 = 12$

$$A_1 \times A_2 \times A_3$$

$$\underset{d_0 \; d_1}{2 \;\; 3} \qquad \underset{d_1 \; d_2}{3 \;\; 4} \qquad \underset{d_2 \; d_3}{4 \;\; 2}$$

$(A_1 A_2) A_3$

$\underbrace{2 \times 3 \;\; 3 \times 4} \;\; 4 \times 2$

$A_1 (A_2 A_3)$

$2 \times 3 \;\; \underbrace{3 \times 4 \;\; 4 \times 2}$

$0 + 24 + 12 = 36$

$$\boxed{C[i][j] = \min_{i \le k < j} \{ C[i,k] + C[k+1,j] + d_{i-1} \times d_k \times d_j \}}$$

$$\begin{array}{cccccccc} A_1 & \times & A_2 & \times & A_3 & \times & A_4 \\ d_0\ d_1 & & d_1\ d_2 & & d_2\ d_3 & & d_3\ d_4 \end{array}$$

1. $A_1(A_2(A_3A_4))$
2. $A_1((A_2A_3)A_4)$
3. $(A_1A_2)(A_3A_4)$
4. $(A_1(A_2A_3))A_4$
5. $((A_1A_2)A_3)A_4$

How many parenthesis way is possible for 4 matrices ??

$$\frac{{}^{2n}C_n}{n+1} \qquad (\text{Take } n = n-1)$$

$$= \frac{{}^{2(n-1)}C_{(n-1)}}{n}$$

If $\underline{n=4}$

$$\frac{{}^{2\times3}C_3}{4} = \frac{{}^6C_3}{4} = \frac{\frac{\cancel{6}\times5\times\cancel{4}}{\cancel{3\times2\times1}}}{\cancel{4}}$$

$$= 5$$

$$A_1 \times A_2 \times A_3 \times A_4$$

$$\begin{array}{cccccccc} 3 & 2 & 2 & 4 & 4 & 2 & 2 & 5 \\ d_0 & d_1 & d_1 & d_2 & d_2 & d_3 & d_3 & d_4 \end{array}$$

C

	1	2	3	4
1	0	24	28	58
2		0	16	36
3			0	40
4				0

K

	1	2	3	4
1		1	1	3
2			2	3
3				3
4				

$$C[1,2] = \min_{1 \leq K2} \overset{K=1}{\begin{Bmatrix} C[1,1] + C[2,2] + d_0 \times d_1 \times d_2 \\ 0 + 0 + 3 \times 2 \times 4 \\ = 24 \end{Bmatrix}}$$

$$C[2,3] = \min_{2 \leq K < 3} \overset{K=2}{\begin{Bmatrix} C[2,2] + C[3,3] + d_1 \times d_2 \times d_3 \\ 0 + 0 + 2 \times 4 \times 2 \\ = 16 \end{Bmatrix}}$$

$$C[3,4] = \min_{3 \leq K < 4} \overset{K=3}{\begin{Bmatrix} C[3,3] + C[4,4] + d_2 \times d_3 \times d_4 \\ 0 + 0 + 4 \times 2 \times 5 \\ = 40 \end{Bmatrix}}$$

$$C[1,3] = \min_{1 \leq K < 3} \begin{cases} K=1 & C[1,1] + C[2,3] + d_0 \times d_1 \times d_3 \\ & 0 + 16 + 3 \times 2 \times 2 = 28 \\ K=2 & C[1,2] + C[3,3] + d_0 \times d_2 \times d_3 \\ & 24 + 0 + 3 \times 4 \times 2 = 48 \end{cases}$$

$$C[2,4] = \min_{2 \leq K < 4} \begin{cases} K=2 & C[2,2] + C[3,4] + d_1 \times d_2 \times d_4 \\ & 0 + 40 + 2 \times 4 \times 5 = 80 \\ K=3 & C[2,3] + C[4,4] + d_1 \times d_3 \times d_4 \\ & 16 + 0 + 2 \times 2 \times 5 = 36 \end{cases}$$

$$C[1,4] = \min_{1 \leq K < 4} \begin{cases} K=1 & C[1,1] + C[2,4] + d_0 \times d_1 \times d_4 \\ & 0 + 36 + 3 \times 2 \times 5 = 86 \\ =2 & C[1,2] + C[3,4] + d_0 \times d_2 \times d_4 \\ & 24 + 40 + 3 \times 4 \times 5 = 124 \\ =3 & C[1,3] + C[4,4] + d_0 \times d_3 \times d_4 \\ & 28 + 0 + 3 \times 2 \times 5 = 58 \end{cases}$$

i.e cost of multiplying these A_1 to A_4 matrices is 58 number of multiplications.

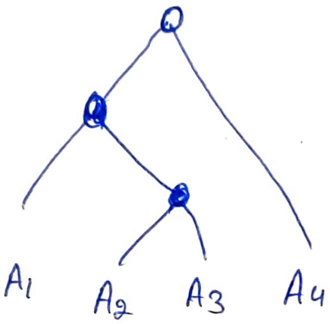
How the parenthasisation should be done

$\Rightarrow$ we will use K matrix

$A_1 \; A_2 \; A_3 \; A_4$

$(A_1 \; A_2 \; A_3) \; A_4$

$((A_1)(A_2 \; A_3))(A_4)$



Q for Practice

A_1	A_2	A_3	A
5×4	4×6	6×2	2×7

C

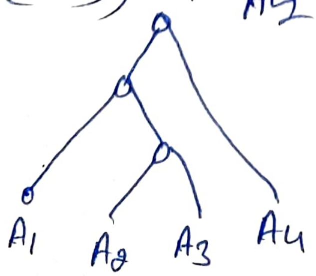
	1	2	3	4
1	0	120	88	158
2		0	48	104
3			0	84
4				0

Using K matrix

$A_1 \, A_2 \, A_3 \, A_4$

$(A_1 \, A_2 \, A_3) \, A_4$

$((A_1)(A_2 A_3)) \, A_4$. Ans



Ans

K

	1	2	3	4
1		1	1	3
2			2	3
3				3
4				

Time Complexity —

Number of elements generated $= \frac{n(n-1)}{2}$

$= n^2$

To find each element, we are calculating all and finding the min

$\therefore n^2 \times n$

$= n^3$

$\Rightarrow O(n^3)$

Program

↙ To find the min no of multiplication

```
main()
{
  int n=5;
  int P={5, 4, 6, 2, 7};
  int m[5][5]={0};
  int s[5][5]={0};
  int j, min, q;
  for(int d=1; d<n-1; d++)
  {
    for (int i=1; i<n-d; i++)
    {
      j=i+d;
      min=32767
      for(int k=i; k<=j-1; k++)
      {
        q=m[i][k]+m[k+1][j]+P[i-1]*P[k]*P[j];
        if (q<min)
        {
          min=q
          S[i][j]=k;
        }
      }
      m[i][j]=min;
    }
  }
  cout<<m[1][n-1];
}
```