Single Source Shortest Path

(1) Dijkstra's algorithm
(2) Bellman - Ford algorithm.

Dijkstra's algorithm

- Dijkstra's algorithm is a greedy algorithm that solve the single-source shortest path problem for a directed graph G with non-negative edge weight.
- Single-source shortest path algorithm are based on a technique known as relaxation.

INITIALIZE - SINGLE - SOURCE (G, s)

for each vertex $v \in V[G]$
    do $d[v] = \infty$
       $\pi[v] = NIL$
$d[s] = 0$

Here, $d[v]$: represents the shortest distance of v from the source.

$\pi[v]$: represents the predecessor of node v i.e. the node which precedes the given node in the shortest-path from source.

RELAX (u, v, w)

if $d[v] > d[u] + w(u, v)$
    then $d[v] = d[u] + w(u,v)$
         $\pi[v] = u$

Example

$\underset{u}{⑤} \xrightarrow{4} \underset{v}{⑪}$

$\Downarrow$ Relax

$\underset{u}{⑤} \xrightarrow{4} \underset{v}{⑨}$

$d[u] = 5$, $d[v] = 11$, $w(u,v) = 4$

$\therefore d[v] > d[u] + w(u,v)$

$11 > 5 + 4$

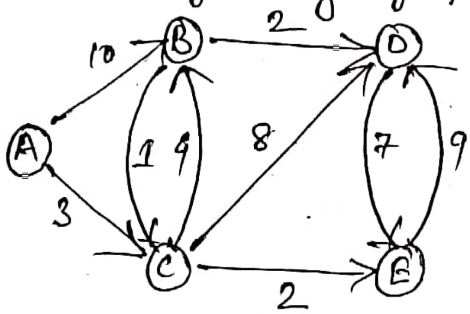
$11 > 9$

$\therefore$ do relax the edge

DIJKSTRA (G, w, s)

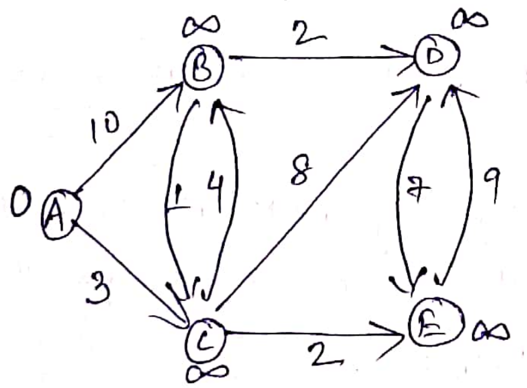
```
INITIALIZE-SINGLE-SOURCE (G, s)
S = φ
Q = V[G]
while Q ≠ φ
    do u = EXTRACT-MIN (Q)
        S = S ∪ {u}
        for each vertex v ∈ Adj[u]
        do RELAX (u, v, w)
```

Q Consider the following graph.



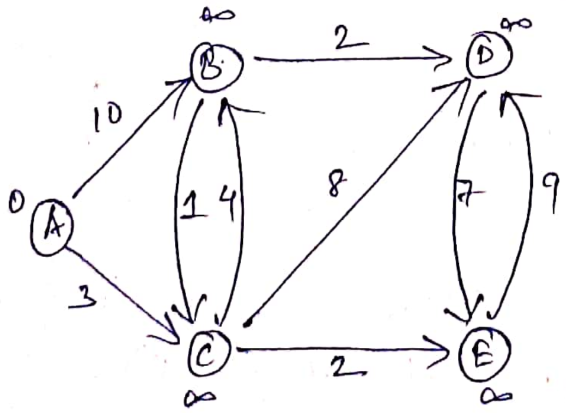
Let A be the source vertex.

<u>Soln</u>



Q:	A	B	C	D	E
	0	∞	∞	∞	∞

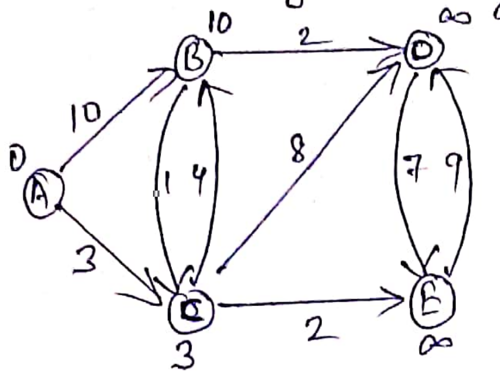
"A" ← EXTRACT-MIN (Q)



Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞

S: {A}

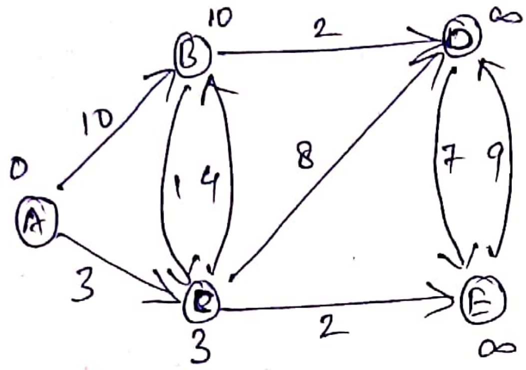
// : Relax all edges leaving A.



Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	3	–	–

S : {A}

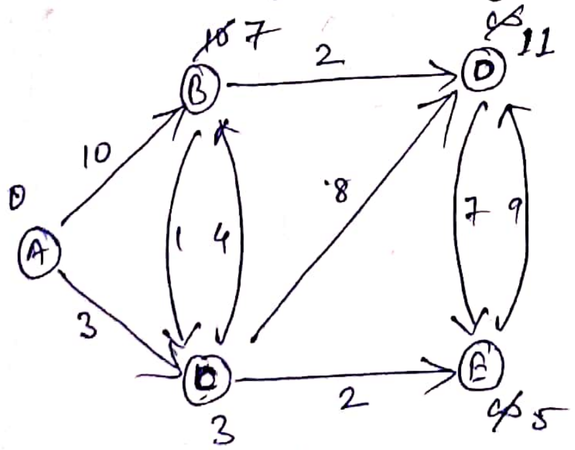
"C" ← EXTRACT – MIN (Q)



Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	–	–

S : {A, C}

Relax all edges leaving C.



Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	–	–
		7		11	5

S : {A, C}

"E" ← EXTRACT – MIN (Q)

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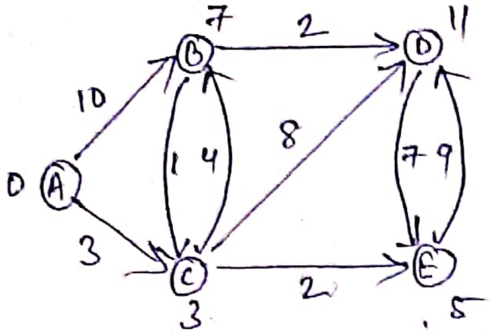
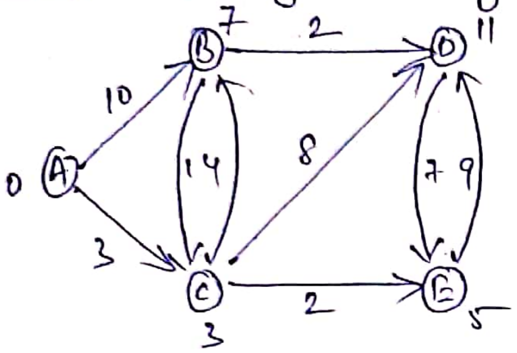


Q:

A	B	C	D	E
$\boxed{0}$	∞	∞	∞	∞
	10	$\boxed{3}$	—	—
	7		11	$\boxed{5}$

S: {A, C, E}

Relax all edges leaving E.

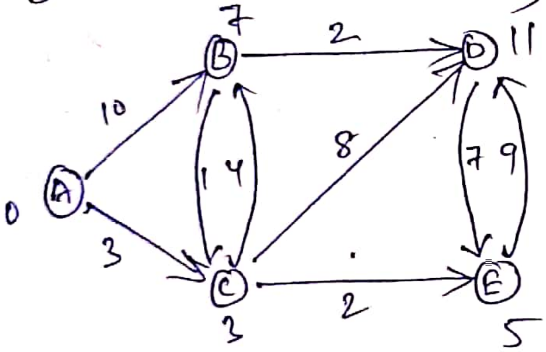


Q:

A	B	C	D	E
$\boxed{0}$	∞	∞	∞	∞
	10	$\boxed{3}$	—	—
	7		11	$\boxed{5}$
	7		11	

S: {A, C, E}

"B" ← EXTRACT – MIN(Q)

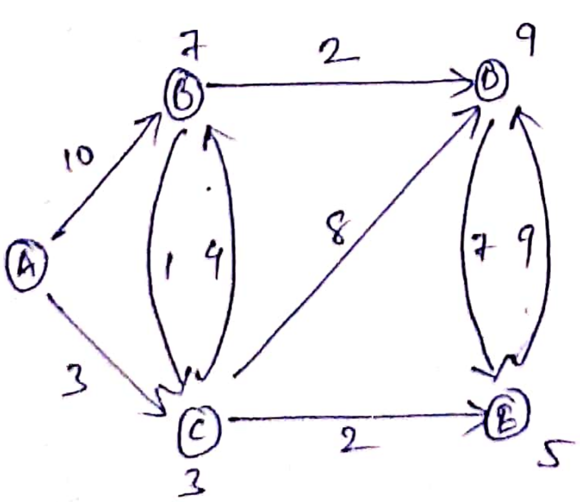


Q:

A	B	C	D	E
$\boxed{0}$	∞	∞	∞	∞
	10	$\boxed{3}$	—	—
	7		11	$\boxed{5}$
	$\boxed{7}$		11	

S: {A, C, E, B}

Relax all edges leaving B.

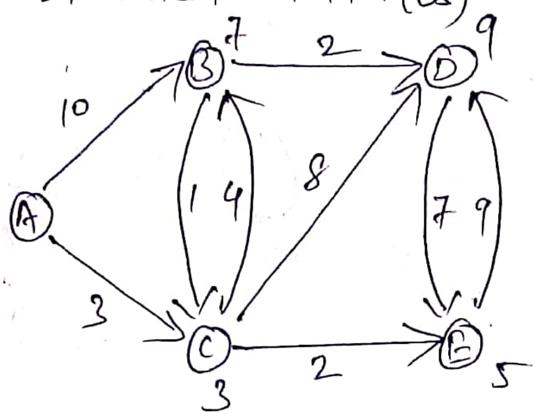


Q:

A	B	C	D	E
$\boxed{0}$	∞	∞	∞	∞
	10	$\boxed{3}$	—	—
	7		11	$\boxed{5}$
	$\boxed{7}$		11	
			9	

S: {A, C, E, B}

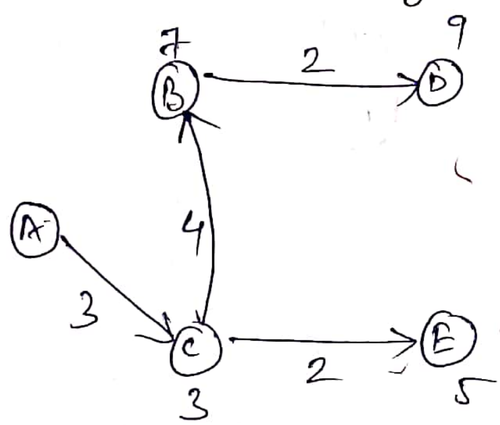
∴ "D" ← EXTRACT − MIN (Q)



	A	B	C	D	E
Q :	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	—	—
		7		11	$\boxed{5}$
		$\boxed{7}$		11	
				$\boxed{9}$	

S : {A, C, E, B, D}

∴ The minimum spanning tree is



∴ cost = 3 + 4 + 2 + 2

= 11

All - Pairs - Shortest - Path Algorithm

Floyd -Warshell Algorithm

- The algorithm considers the "intermediate" vertices of a shortest path where an intermediate vertex of a simple path $p = \langle v_1, v_2, \ldots, v_m \rangle$ is any vertex of p other than v_1 or v_m.

- The Floyd - Warshall algorithm is based on the following observation.

- Let the vertices of G be $V = \{1, 2, \ldots, n\}$ consider a sub-set of vertices $\{1, 2, \ldots, K\}$.

  For any pair of vertices $i, j \in V$, consider all paths from i to j whose intermediate vertices are all drawn from $\{1, 2, \ldots K\}$ and let p be the minimum - weight path among them.

- If K is not an intermediate vertex of path p, then all intermediate vertices of path p are in the set $\{1, 2, \ldots, K-1\}$.

- If K is an intermediate vertex of path p, then we break p down into $i \xrightarrow{p_1} K \xrightarrow{p_2} j$.

- Let $d_{ij}^{(K)}$ be the weight of a shortest path from vertex i to vertex j with all intermediate vertices in the set $\{1, 2, \ldots K\}$.

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

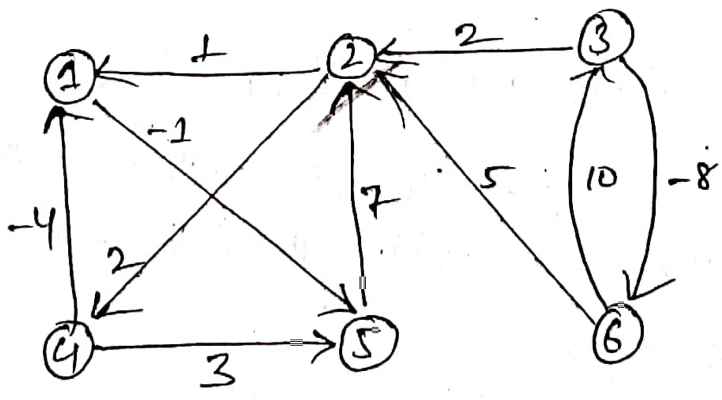
FLOYD-WARSHALL(W)

```
n ← rows[W]
D^(0) ← W
for k ← 1 to n
    do for i ← 1 to n
        do for j ← 1 to n
            do d_ij^(k) ← min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
return D^(n)
```

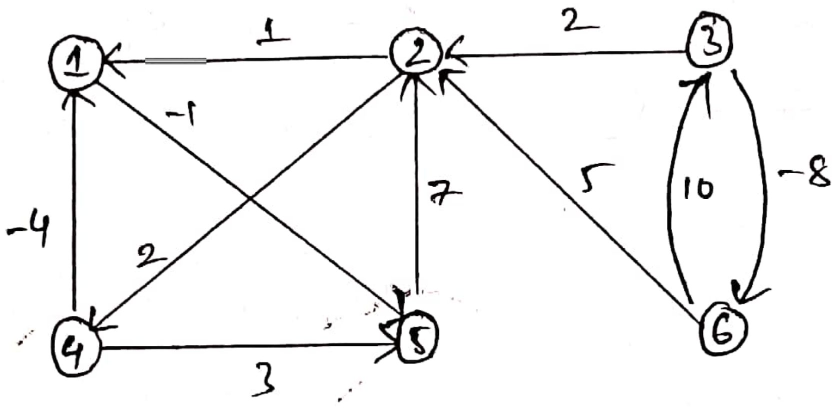
The strategy adopted by the Floyd-Warshall algorithm is dynamic programming.

The running time of Floyd-Warshall algorithm is $O(n^3)$.

Q. Apply Floyd-Warshall algorithm for constructing shortest path. Show the matrix $D^{(k)}$ that results each iteration.



Que Apply Floyd-Warshall Algorithm for constructing shortest path. Show the matrix $D^{(k)}$ that results each iteration. Also find the minimum cost from Node 3 to Node 1 and the corresponding shortest path also.



Soln

$$D^{(0)} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & \infty & \infty & \infty & -1 & \infty \\ 2 & 1 & 0 & \infty & 2 & \infty & \infty \\ 3 & \infty & 2 & 0 & \infty & \infty & -8 \\ 4 & -4 & \infty & \infty & 0 & 3 & \infty \\ 5 & \infty & 7 & \infty & \infty & 0 & \infty \\ 6 & \infty & 5 & 10 & \infty & \infty & 0 \end{array}$$

$$\pi^{(0)} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & NIL & NIL & NIL & NIL & 1 & NIL \\ 2 & 2 & NIL & NIL & 2 & NIL & NIL \\ 3 & NIL & 3 & NIL & NIL & NIL & 3 \\ 4 & 4 & NIL & NIL & NIL & 4 & NIL \\ 5 & NIL & 5 & NIL & NIL & NIL & NIL \\ 6 & NIL & 6 & 6 & NIL & NIL & NIL \end{array}$$

$$D^{(1)} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & \infty & \infty & \infty & -1 & \infty \\ 2 & 1 & 0 & \infty & 2 & \underline{0} & \infty \\ 3 & \infty & 2 & 0 & \infty & \infty & -8 \\ 4 & -4 & \infty & \infty & 0 & \underline{-5} & \infty \\ 5 & \infty & 7 & \infty & \infty & 0 & \infty \\ 6 & \infty & 5 & 10 & \infty & \infty & 0 \end{array}$$

$$\pi^{(1)} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & NIL & NIL & NIL & NIL & 1 & NIL \\ 2 & 2 & NIL & NIL & 2 & \underline{1} & NIL \\ 3 & NIL & 3 & NIL & NIL & NIL & 3 \\ 4 & 4 & NIL & NIL & NIL & \underline{1} & NIL \\ 5 & NIL & 5 & NIL & NIL & NIL & NIL \\ 6 & NIL & 6 & 6 & NIL & NIL & NIL \end{array}$$

$$D^{(2)} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & 0 & \infty \\ \underline{3} & 2 & 0 & \underline{4} & \underline{2} & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ \underline{8} & 7 & \infty & \underline{9} & \underline{0} & \infty \\ \underline{6} & 5 & 10 & \underline{7} & \underline{5} & 0 \end{bmatrix} \end{array}$$

$$\pi^{(2)} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} NIL & NIL & NIL & NIL & 1 & NIL \\ 2 & NIL & NIL & 2 & 1 & NIL \\ \underline{2} & 3 & NIL & \underline{2} & \underline{1} & 3 \\ 4 & NIL & NIL & NIL & 1 & NIL \\ \underline{2} & 5 & NIL & \underline{2} & \underline{NIL} & NIL \\ \underline{2} & 6 & 6 & \underline{2} & \underline{1} & NIL \end{bmatrix} \end{array}$$

$$D^{(3)} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & 0 & \infty \\ 3 & 2 & 0 & 4 & 2 & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ 8 & 7 & \infty & 9 & 0 & \infty \\ \underline{6} & \underline{5} & 10 & \underline{7} & \underline{5} & \underline{0} \end{bmatrix} \end{array}$$

$$\pi^{(3)} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} NIL & NIL & NIL & NIL & 1 & NIL \\ 2 & NIL & NIL & 2 & 1 & NIL \\ 2 & 3 & NIL & 2 & 1 & 3 \\ 4 & NIL & NIL & NIL & 1 & NIL \\ 2 & 5 & NIL & 2 & NIL & NIL \\ \underline{2} & \underline{6} & 6 & \underline{2} & \underline{1} & \underline{NIL} \end{bmatrix} \end{array}$$

$$D^{(4)} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ \underline{-2} & 0 & \infty & 2 & \underline{-3} & \infty \\ \underline{0} & 2 & 0 & 4 & \underline{-1} & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ \underline{5} & 7 & \infty & 9 & \underline{0} & \infty \\ \underline{3} & 5 & 10 & 7 & \underline{2} & 0 \end{bmatrix} \end{array}$$

$$\pi^{(4)} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} NIL & NIL & NIL & NIL & 1 & NIL \\ \underline{4} & NIL & NIL & 2 & \underline{1} & NIL \\ \underline{4} & 3 & NIL & 2 & \underline{1} & 3 \\ 4 & NIL & NIL & NIL & 1 & NIL \\ \underline{4} & 5 & NIL & 2 & \underline{NIL} & NIL \\ \underline{4} & 6 & 6 & 2 & \underline{1} & NIL \end{bmatrix} \end{array}$$

$$D^{(5)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 6 & \infty & 8 & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ 0 & 2 & 0 & 4 & -1 & -8 \\ -4 & 2 & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ 3 & 5 & 10 & 7 & 2 & 0 \end{array}\right] \end{array}$$

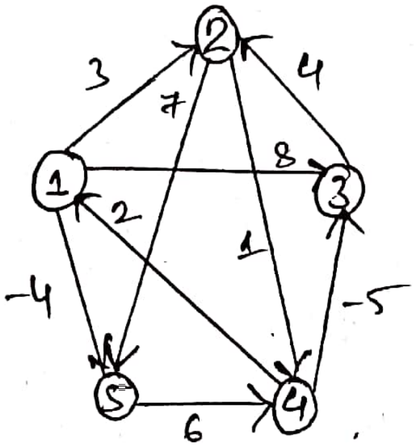
$$\pi^{(5)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} NIL & 5 & NIL & 2 & 1 & NIL \\ 4 & NIL & NIL & 2 & 1 & NIL \\ 4 & 3 & NIL & 2 & 1 & 3 \\ 4 & 5 & NIL & NIL & 1 & NIL \\ 4 & 5 & NIL & 2 & NIL & NIL \\ 4 & 6 & 6 & 2 & 1 & NIL \end{array}\right] \end{array}$$

$$D^{(6)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 6 & \infty & 8 & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ -5 & -3 & 0 & -1 & -6 & -8 \\ -4 & 2 & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ 3 & 5 & 10 & 7 & 2 & 0 \end{array}\right] \end{array}$$

$$\pi^{(6)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} NIL & 5 & NIL & 2 & 1 & NIL \\ 4 & NIL & NIL & 2 & 1 & NIL \\ 4 & 6 & NIL & 2 & 1 & 3 \\ 4 & 5 & NIL & NIL & 1 & NIL \\ 4 & 5 & NIL & 2 & NIL & NIL \\ 4 & 6 & 6 & 2 & 1 & NIL \end{array}\right] \end{array}$$

∴ The minimum cost from Node 3 to Node 1 is −5.
and the corresponding shortest path is $3 \to 6 \to 2 \to 4 \to 1$.

FLOYD - WARSHALL ALGORITHM



Solution

$$D^{(0)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \end{array}$$

$$\pi^{(0)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix} \end{array}$$

$$D^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \underline{5} & \underline{-5} & 0 & \underline{-2} \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \end{array}$$

$$\pi^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & \underline{1} & \underline{4} & NIL & \underline{1} \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix} \end{array}$$

$$D^{(2)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 8 & \underline{4} & \underline{-4} \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \underline{5} & \underline{11} \\ 2 & 5 & -5 & \underline{0} & \underline{-2} \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \end{array}$$

$$\pi^{(2)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} NIL & 1 & 1 & \underline{2} & \underline{1} \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & \underline{2} & \underline{2} \\ 4 & 1 & 4 & \underline{NIL} & \underline{1} \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix} \end{array}$$

$$D^{(3)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \end{array}$$

$$\pi^{(3)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{bmatrix} \end{array}$$

$$D^{(4)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \end{array}$$

$$\pi^{(4)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{bmatrix} \end{array}$$

$$D^{(5)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \end{array}$$

$$\pi^{(5)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{bmatrix} \end{array}$$

∴ The cost from Node 4 to Node 5 is -2 and the required path is $4 \rightarrow 1 \rightarrow 5$.