Strassen's Matrix multiplication →

$$A\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{\substack{2\times 2 \\ m\times n}} \times B\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2} = C\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2\times 2}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} * B_{kj}$$

```
For (i=0; i<n; i++)
{
    For (j=0; j<n; j++)
    {
        c[i][j]=0;
        For (k=0; k<n; k++)
        {
            c[i][j]+=A[i][k] * b[k][j];
        }
    }
}
```

$O(n^3)$

Time Complexity.

Above is simple algo to multiply two matrix using three loops. Let try to solve this problem with Divide and Conquer strategy where in we break the ~~the~~ problem into subproblems and solve them and finally combine the solutions. If problem is small we will solve it directly otherwise we will break it into further subproblems.

$$C_{11} = a_{11} * b_{11} + a_{12} * b_{21}$$
$$C_{12} = a_{11} * b_{12} + a_{12} * b_{22}$$
$$C_{21} = a_{21} * b_{11} + a_{22} * b_{21}$$
$$C_{22} = a_{21} * b_{12} + a_{22} * b_{22}$$

In case matrix is of size 2X2 only these 4 formula's be required to find their multiplication.

or

$$A[a_{11}] \times B[b_{11}] = C[a_{11} * b_{11}]$$

②

If the matrix size is greater than 2x2

* we assume that matrix have dimensions in power of 2 for e.g 2x2, 4x4, 8x8, 16x16 etc. If matrix is not in power of 2 then we can fill with 0's to make it power of 2

$$A = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ & A_{11} & A_{12} & \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ & A_{21} & A_{22} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \frac{4}{2} \times \frac{4}{2}$$

$$B = \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ & B_{11} & B_{12} & \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} \\ & B_{21} & B_{22} & \\ b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right] \quad \frac{4}{2} \times \frac{4}{2}$$

Algorithm Matrix_Multiplication (A, B, n)
mm

{
  If ($n \leq 2$)
  {

  }
  else
  {
    mid = n/2

    mm(A11, B11, n/2) + mm(A12, B21, n/2)

    mm(A11, B12, n/2) + mm(A12, B22, n/2)

    mm(A21, B11, n/2) + mm(A22, B21, n/2)

    mm(A21, B12, n/2) + mm(A22, B22, n/2)
  }

↑ Plus is a matrix addition.

$C_{11} = A_{11} * B_{11} + A_{12} * B_{21}$

$C_{12} = A_{11} * B_{12} + A_{12} * B_{22}$

$C_{21} = A_{21} * B_{11} + A_{22} * B_{21}$

$C_{22} = A_{21} * B_{12} + A_{22} * B_{22}$

Above algorithm is calling itself recursively 8 times
Lets write the Recurrence relation for it.

$$T(n) = \begin{cases} 1 & n <= 2 \\ 8T(n/2) + n^2 & n > 2 \end{cases}.$$

Apply master theorm

$a = 8$ $\quad \log_b a = \log_2 8 = 3$
$b = 2$
$f(n) = n^2$ $\quad n^k = n^2$

$$= \Theta(n^3)$$

Time complexity of divide and conquer is also n^3. Also as it is a recursive strategy it will require stack space to execute the recursive calls.

Can be do better than $O(n^3)$ — Yes

In the above divide and conquer method, the main component of complexity is 8 recursive calls. The idea of Strassen is to reduce the number of recursive calls to 7. Strassen's method is similar to the above method in the sense that this method also divide matrices to sub-matrices of size $n/2 \times n/2$ but in strassen's ~~matri~~ method, the four sub matrices of result are calculated using the following formulae:

P.TO

$P = (A_{11} + A_{22})(B_{11} + B_{22})$

$Q = (A_{21} + A_{22}) B_{11}$

$R = A_{11}(B_{12} - B_{22})$

$S = A_{22}(B_{21} - B_{11})$

$T = (A_{11} + A_{12}) B_{22}$

$U = (A_{21} - A_{11})(B_{11} + B_{12})$

$V = (A_{12} - A_{22})(B_{21} + B_{22})$

$C_{11} = P + S - T + V$

$C_{12} = R + T$

$C_{21} = Q + S$

$C_{22} = P + R - Q + U$

No of multiplication decreased but additions increased

$$T(n) = \begin{cases} 1 & n \leq 2 \\ 7T\left(\frac{n}{2}\right) + n^2 & n > 2 \end{cases}$$

$\log_2 7 = 2.81 \quad k = 2$

$\underline{O(n^{2.81})}$ complexity is reduced from $O(n^3$.

Q Multiply two matrices using strassens Algo.

$$A = \begin{bmatrix} 5 & 6 \\ -4 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -7 & 6 \\ 5 & 9 \end{bmatrix}$$

$P = (A_{11} + A_{22})(B_{11} + B_{22}) = (5+3)(-7+9) = 16$

$Q = B_{11}(A_{21} + A_{22}) = -7(-4+3) = 7$

$R = A_{11}(B_{12} - B_{22}) = 5(6-9) = -15$

$S = A_{22}(B_{21} - B_{11}) = 3(5-(-7)) = 36$

$T = (A_{11} + A_{12})B_{22} = (5+6)9 = 99$

$U = (A_{21} - A_{11})(B_{11} + B_{12}) = (-4-5)(-7+6) = 9$

$V = (A_{12} - A_{22})(B_{21} + B_{22}) = (6-3)(5+9) = 42$

$C_{11} = 16 + 36 - 99 + 42 = -5$

$C_{12} = -15 + 99 = 84$

$C_{21} = 7 + 36 = 43$

$C_{22} = 16 - 15 - 7 + 9 = 3$

$$= C \begin{bmatrix} -5 & 84 \\ 43 & 3 \end{bmatrix}$$ Ans.