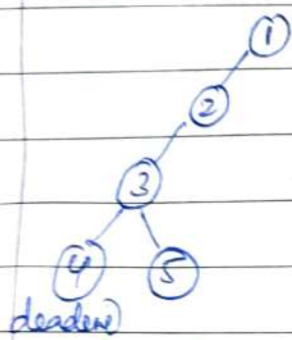
Branch and Bound Method

Branch and Bound method is similar to backtracking but it is used for solving optimization problems (minimi--zation)

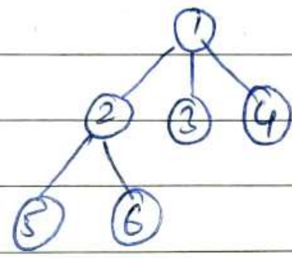
It performs a graph traversal on state space tree using BFS instead of DFS which is used by backtracking.

Backtracking



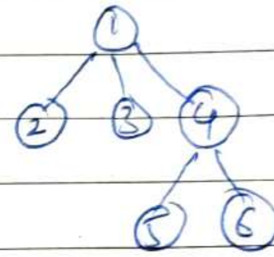
(DFS)

Branch & Bound



Queue | 2 | 3 | 4 |

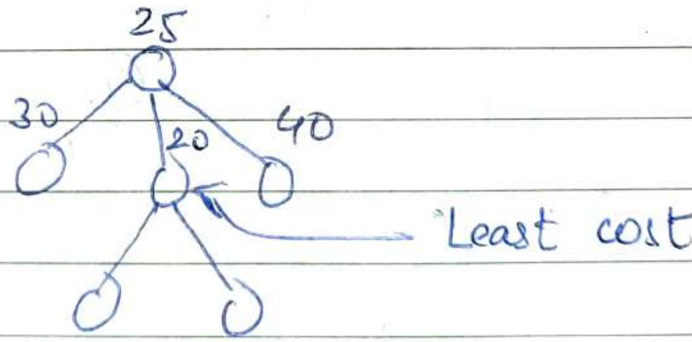
(FIFO Branch & Bound)



Stack

4
3
2

LIFO Branch & Bound.



Least-cost Branch and Bound.

Types of Nodes used in State-space Tree

(1) Live Node
(2) E-Node
(3) Dead Node

Live Node

- Node is generated but its children are not generated.

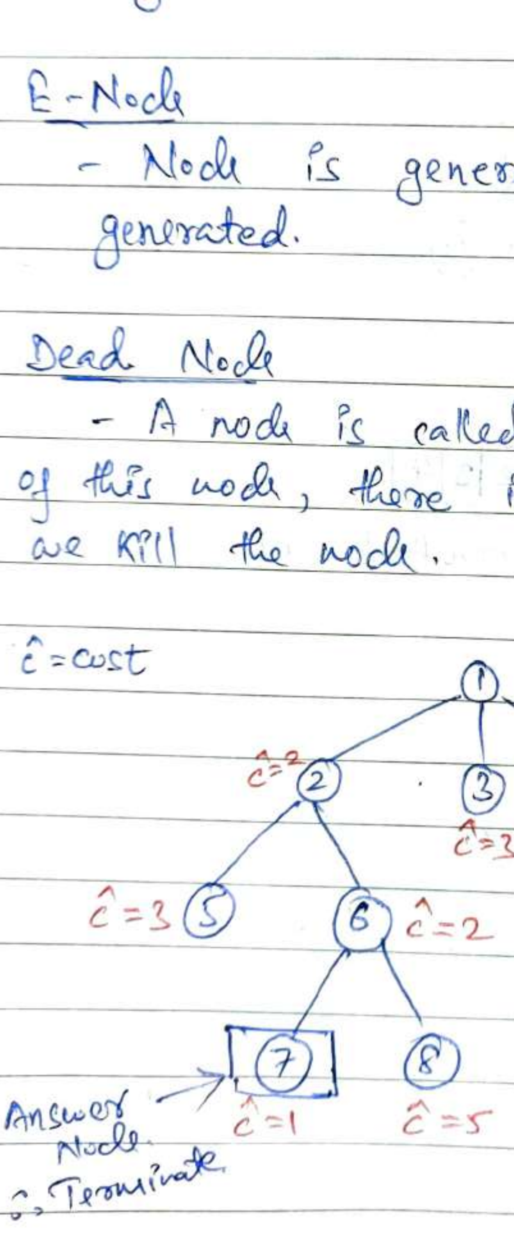
E-Node

- Node is generated and its children are also generated.

Dead Node

- A node is called dead node because of expansion of this node, there is no use in future. So we kill the node.

$\hat{c}$ = cost



Node 1 is E-node.

$\hat{c}(2) = 2$
$\hat{c}(3) = 3$
$\hat{c}(4) = 4$
using Ranking function.

Min cost is of Node 2
∴ Node 2 is expanded.
E-node is now Node 2.

E-Node : Node 1, Node 2, Node 6
Live Node : Node 3, Node 4, Node 5, Node 8

Branch and Bound method of algorithm design involves :

(1) Tree organization of Solution Space.
(2) Use of Bounding functions to limit the search ie to avoid the generation of sub-trees that do not contain an answer node.

Applications of Branch and Bound technique :

(1) 0/1 Knapsack problem.
(2) Travelling Salesperson problem (TSP)

Search techniques used in Branch and Bound :

(1) BFS
(2) DFS
(3) Least cost Search (LC)

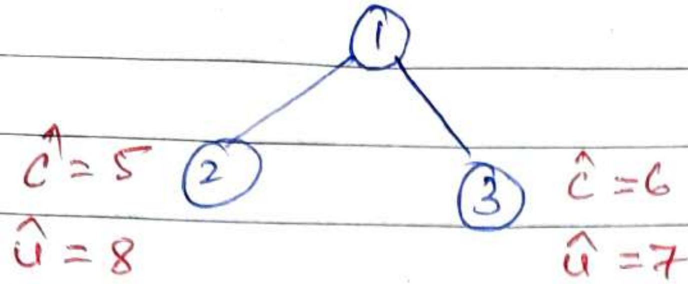
Two types of Bounds used in LCBB

(1) Lower Bound ($\hat{c}$)
(2) Upper Bound ($\hat{u}$)

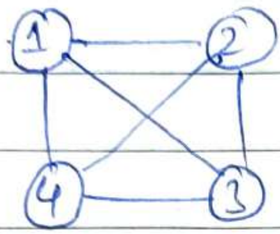
While calculating $\hat{c}$ for a node in state space tree, fractions are allowed.

While calculating $\hat{u}$ for a node in state space tree, fractions are not allowed.

which node $\hat{c}$ is minimum, that node is expanded, that node becomes E-node.



Travelling Salesperson Branch-and-Bound



Problem is we have to find out a minimum cost tour going through all the vertices once and returning back to starting vertex.

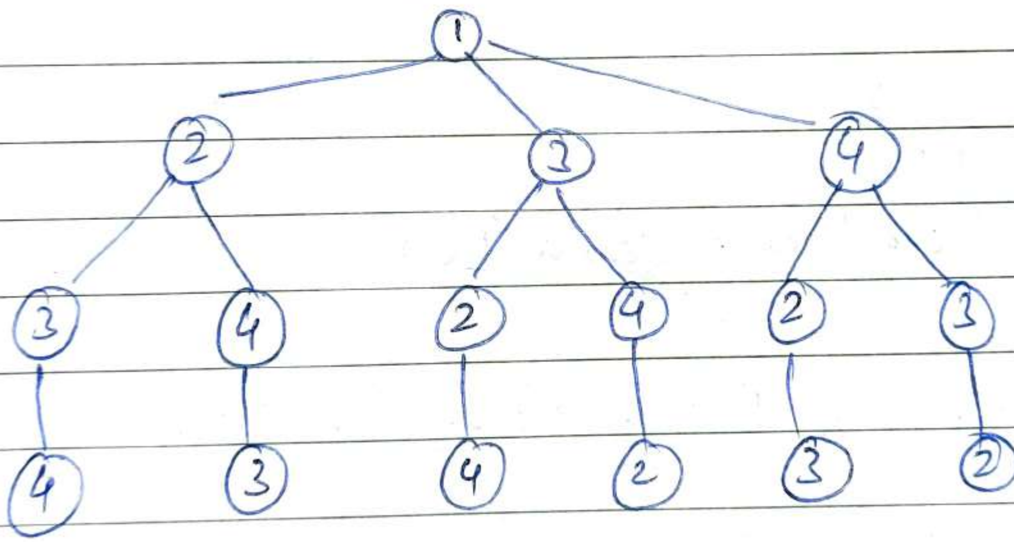
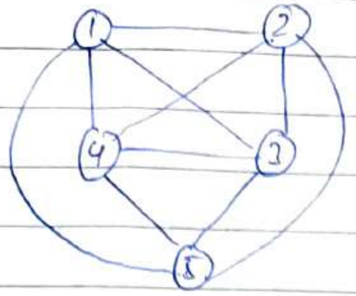


fig. state-space tree showing all possible tour.
(Nodes are generated using BFS)

Example



cost Adjacency matrix
⇓

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 20 & 30 & 10 & 11 \\
2 & 15 & \infty & 16 & 4 & 2 \\
3 & 3 & 5 & \infty & 2 & 4 \\
4 & 19 & 6 & 18 & \infty & 3 \\
5 & 16 & 4 & 7 & 16 & \infty
\end{array}$$

Step1 : Reduce the cost adjacency matrix (i.e Reduce row)

$$\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & 20 & 30 & 10 & 11 & 10 \\
2 & 15 & \infty & 16 & 4 & 2 & 2 \\
3 & 3 & 5 & \infty & 2 & 4 & 2 \\
4 & 19 & 6 & 18 & \infty & 3 & 3 \\
5 & 16 & 4 & 7 & 16 & \infty & 4 \\
\hline
 & & & & & & 21
\end{array}$$

21 ⇐ total cost of row reductions.

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 20 & 0 & 1 \\
2 & 13 & \infty & 14 & 2 & 0 \\
3 & 1 & 3 & \infty & 0 & 2 \\
4 & 16 & 3 & 15 & \infty & 0 \\
5 & 12 & 0 & 3 & 12 & \infty
\end{array}$$

Step2 : Reduce column.

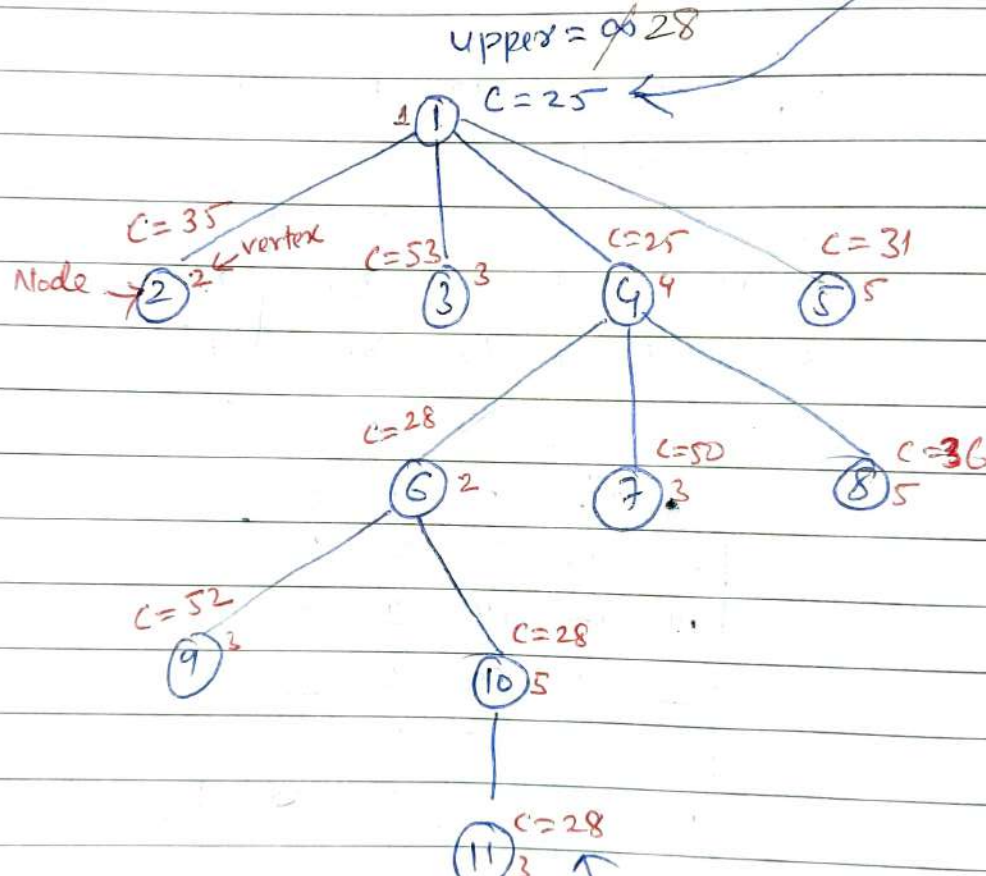
$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 20 & 0 & 1 \\
2 & 13 & \infty & 14 & 2 & 0 \\
3 & 1 & 3 & \infty & 0 & 2 \\
4 & 16 & 3 & 15 & \infty & 0 \\
5 & 12 & 0 & 3 & 12 & \infty \\
\hline
 & 1 & 0 & 3 & 0 & 0
\end{array} = 4$$

matrix after column reduction

$$\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}
\end{array} \Downarrow 21$$

$\Rightarrow 4$ $\quad \therefore 21 + 4 = 25$

reduced cost



cost of tour is

tour is 1-4-2-5-3-1

For Node 2 : 1^{st} Row & 2^{nd} col $\Rightarrow \infty$

$[2,1] \Rightarrow \infty$

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 17 & 0 & 1 \\
2 & 12 & \infty & 11 & 2 & 0 \\
3 & 0 & 3 & \infty & 0 & 2 \\
4 & 15 & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & 12 & \infty
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & 11 & 2 & 0 & 0 \\
3 & 0 & \infty & \infty & 0 & 2 & 0 \\
4 & 15 & \infty & 12 & \infty & 0 & 0 \\
5 & 11 & \infty & 0 & 12 & \infty & 0 \\
\hline
 & 0 & & 0 & 0 & 0 & \hat{r} = 0
\end{array}
$$

$c(1,2) + r + \hat{r}$

$= 10 + 25 + 0 = 35$

For Node 3 : 1^{st} Row & 3^{rd} col $\Rightarrow \infty$

$[3,1] \Rightarrow \infty$

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 17 & 0 & 1 \\
2 & 12 & \infty & 11 & 2 & 0 \\
3 & 0 & 3 & \infty & 0 & 2 \\
4 & 15 & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & 12 & \infty
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & 12 & \infty & \infty & 2 & 0 & 0 \\
3 & \infty & 3 & \infty & 0 & 2 & 0 \\
4 & 15 & 3 & \infty & \infty & 0 & 0 \\
5 & 11 & 0 & \infty & 12 & \infty & 0 \\
\hline
 & 11 & 0 & & 0 & 0 & \hat{r} = 11
\end{array}
$$

$\Downarrow$ reduce column

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 1 & \infty & \infty & 2 & 0 \\
3 & \infty & 3 & \infty & 0 & 2 \\
4 & 4 & 3 & \infty & \infty & 0 \\
5 & 0 & 0 & \infty & 12 & \infty
\end{array}
$$

$c(1,3) + r + \hat{r}$

$17 + 25 + 11 = 53$

For Node 4 : 1st row & 4th col $\Rightarrow \infty$

$[4,1] \Rightarrow \infty$

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 17 & 0 & 1 \\
2 & 12 & \infty & 11 & 2 & 0 \\
3 & 0 & 3 & \infty & 0 & 2 \\
4 & 15 & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & 12 & \infty \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & 12 & \infty & 11 & \infty & 0 & 0 \\
3 & 0 & 3 & \infty & \infty & 2 & 0 \\
4 & \infty & 3 & 12 & \infty & 0 & 0 \\
5 & 11 & 0 & 0 & \infty & \infty & 0 \\
\hline
 & 0 & 0 & 0 & & 0 &
\end{array}
$$

$\hat{r} = 0$

$c(1,4) + r + \hat{r}$

$0 + 25 + 0 = 25$

For Node 5 : 1st row & 5th Col $\Rightarrow \infty$

$[5,1] \Rightarrow \infty$

$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & 12 & \infty & 11 & 2 & \infty & 2 \\
3 & 0 & 3 & \infty & 0 & \infty & 0 \\
4 & 15 & 3 & 12 & \infty & \infty & 3 \\
5 & \infty & 0 & 0 & 12 & \infty & 0 \\
\hline
 & 0 & 0 & 0 & 0 & &
\end{array}
\xrightarrow{\text{reduce row}}
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 10 & \infty & 9 & 0 & \infty \\
3 & 0 & 3 & \infty & 0 & \infty \\
4 & 12 & 0 & 9 & \infty & \infty \\
5 & \infty & 0 & 0 & 12 & \infty \\
\hline
\end{array}
$$

$\hat{r} = 5$

$c(1,5) + r + \hat{r}$

$= 1 + 25 + 5 = 31$

Node 2 : $c = 35$

Node 3 : $c = 53$

Node 4 : $c = 25$ ← min, So explore it.

Node 5 : $c = 31$

For Node 6 : 4^{th} row & 2^{nd} col $\Rightarrow \infty$

$[2,1] \Rightarrow \infty$

Node 4

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 12 & \infty & 11 & \infty & 0 \\
3 & 0 & 3 & \infty & \infty & 2 \\
4 & \infty & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & \infty & \infty
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & 11 & \infty & 0 & 0 \\
3 & 0 & \infty & \infty & \infty & 2 & 0 \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & 11 & \infty & 0 & \infty & \infty & 0 \\
\hline
 & 0 & & 0 & & 0 &
\end{array}$$

$\hat{r} = 0$

$c(4,2) + r + \hat{r}$

$3 + 25 + 0 = 28$

For Node 7 : 4^{th} row & 3^{rd} col $\Rightarrow \infty$.

$[3,1] \Rightarrow \infty$

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 12 & \infty & 11 & \infty & 0 \\
3 & 0 & 3 & \infty & \infty & 2 \\
4 & \infty & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & \infty & \infty
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & 12 & \infty & \infty & \infty & 0 & 0 \\
3 & \infty & 3 & \infty & \infty & 2 & 2 \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & 11 & 0 & \infty & \infty & \infty & 0 \\
\hline
 & 11 & 0 & & & 0 &
\end{array}$$

$\hat{r} = 11 + 2 = 13$

$\Downarrow$ reduce row & col

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 1 & \infty & \infty & \infty & 0 \\
3 & \infty & 1 & \infty & \infty & 0 \\
4 & \infty & \infty & \infty & \infty & \infty \\
5 & 0 & 0 & \infty & \infty & \infty
\end{array}$$

$c(4,3) + r + \hat{r}$

$12 + 25 + 13 = 50$

For Node 8 : 4^{th} row & 5^{th} col $\Rightarrow \infty$

$[5,1] \Rightarrow \infty$

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 12 & \infty & 11 & \infty & 0 \\
3 & 0 & 3 & \infty & \infty & 2 \\
4 & \infty & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & \infty & \infty \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & 12 & \infty & 11 & \infty & \infty & 11 \\
3 & 0 & 3 & \infty & \infty & \infty & 0 \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & \infty & 0 & 0 & \infty & \infty & 0 \\
\hline
 & 0 & 0 & 0 & & & \hat{r} = 11
\end{array}$$

$\Downarrow$ reduce row

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & 1 & \infty & 0 & \infty & \infty \\
3 & 0 & 3 & \infty & \infty & \infty \\
4 & \infty & \infty & \infty & \infty & \infty \\
5 & \infty & 0 & 0 & \infty & \infty \\
\hline
\end{array}$$

$C(4,5) + r + \hat{r}$

$0 + 25 + 11 = 36$

Node 6 : $c = 28$ ← min. So explore it.

Node 7 : $c = 50$

Node 8 : $c = 36$

For Node 9 : 2^{nd} row & 3^{rd} col $\Rightarrow \infty$

$[3,1] \Rightarrow \infty$

Node 6

D_1

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & \infty & \infty & 11 & \infty & 0 \\
3 & 0 & \infty & \infty & \infty & 2 \\
4 & \infty & \infty & \infty & \infty & \infty \\
5 & 11 & \infty & 0 & \infty & \infty \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & \infty & \infty & \infty & \\
3 & \infty & \infty & \infty & \infty & 2 & 2 \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & 11 & \infty & \infty & \infty & \infty & 11 \\
\hline
\end{array}$$

$\hat{r} = 13$

Now, reduce row

$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & \infty & \infty & \infty & \\
3 & \infty & \infty & \infty & \infty & 0 & 0 \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & 0 & \infty & \infty & \infty & \infty & \\
\hline
 & 0 & & & & 0 &
\end{array}
$$

$\therefore\ c(2,3) + r + \hat{r}$

$11 + 28 + 13 = 52$

For Node 10 : 2^{nd} row & 5^{th} col $\Rightarrow \infty$

$[5,1] \Rightarrow \infty$

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & \infty & \infty & 11 & \infty & 0 \\
3 & 0 & \infty & \infty & \infty & 2 \\
4 & \infty & \infty & \infty & \infty & \infty \\
5 & 11 & \infty & 0 & \infty & \infty
\end{array}
\Longrightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & \infty & \infty & \infty & \\
3 & 0 & \infty & \infty & \infty & \infty & 0 \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & \infty & \infty & 0 & \infty & \infty & 0 \\
\hline
 & 0 & & 0 & & &
\end{array}
$$

$\hat{r} = 0$

$c(2,5) + r + \hat{r}$

$0 + 28 + 0 = 28$

Node 9 : $c = 52$

Node 10 : $c = 28$ ← min. So explore it

For Node 11 : 5^{th} row & 3^{rd} col $\Rightarrow \infty$

$[3,1] \Rightarrow \infty$

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & \infty & \infty & \infty & \infty \\
2 & \infty & \infty & \infty & \infty & \infty \\
3 & 0 & \infty & \infty & \infty & \infty \\
4 & \infty & \infty & \infty & \infty & \infty \\
5 & \infty & \infty & 0 & \infty & \infty
\end{array}
\Longrightarrow
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & \infty & \infty & \infty & 0 \\
3 & \infty & \infty & \infty & \infty & \infty & \\
4 & \infty & \infty & \infty & \infty & \infty & \\
5 & \infty & \infty & \infty & \infty & \infty &
\end{array}
$$

$\hat{r} = 0$

$$c(5,3) + r + r^{1}$$
$$0 + 28 + 0 = 28$$

Q Least cost Branch and Bound

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 5 & 2 & 3 \\ 2 & 4 & \infty & 2 & 3 \\ 3 & 4 & 2 & \infty & 3 \\ 4 & 7 & 6 & 8 & \infty \end{array}$$

Soln Subtract row minimum value from corresponding row.

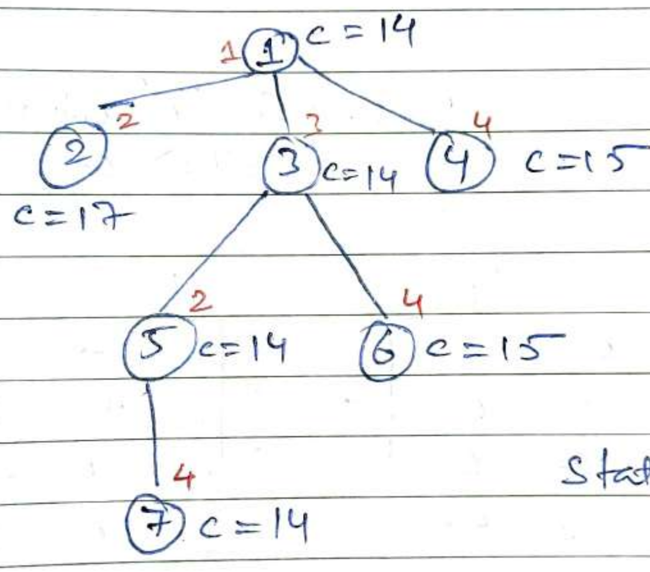
$$\begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline 1 & \infty & 5 & 2 & 3 & 2 \\ 2 & 4 & \infty & 2 & 3 & 2 \\ 3 & 4 & 2 & \infty & 3 & 2 \\ 4 & 7 & 6 & 8 & \infty & 6 \\ \hline & & & & & 12 \end{array} \Rightarrow \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 3 & 0 & 1 \\ 2 & 2 & \infty & 0 & 1 \\ 3 & 2 & 0 & \infty & 1 \\ 4 & 1 & 0 & \cancel{\infty}\ 2 & \infty \end{array}$$

Subtract column minimum value from corresponding column.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 3 & 0 & 1 \\ 2 & 2 & \infty & 0 & 1 \\ 3 & 2 & 0 & \infty & 1 \\ 4 & 1 & 0 & 2 & \infty \\ \hline & 1 & 0 & 0 & 1 = 2 \end{array} \Rightarrow \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 3 & 0 & 0 \\ 2 & 1 & \infty & 0 & 0 \\ 3 & 1 & 0 & \infty & 0 \\ 4 & 0 & 0 & 2 & \infty \end{array}$$

reduced matrix.

Total cost reduced $= 12 + 2 = 14$



State-space tree

For Node 2 : 1^{st} row & 2^{nd} column $\Rightarrow \infty$

$[2,1] \Rightarrow \infty$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 3 & 0 & 0 \\ 2 & 1 & \infty & 0 & 0 \\ 3 & 1 & 0 & \infty & 0 \\ 4 & 0 & 0 & 2 & \infty \end{array} \Rightarrow \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline 1 & \infty & \infty & \infty & \infty & \\ 2 & \infty & \infty & 0 & 0 & 0 \\ 3 & 1 & \infty & \infty & 0 & 0 \\ 4 & 0 & \infty & 2 & \infty & 0 \\ \hline & 0 & & 0 & 0 & \hat{r} = 0 \end{array}$$

$c(1,2) + r + \hat{r}$

$= 3 + 14 + 0 = 17$

For Node 3 :-> 1^{st} row & 3^{rd} column $\Rightarrow \infty$

$[3,1] \Rightarrow \infty$

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
1 & \infty & 3 & 0 & 0 \\
2 & 1 & \infty & 0 & 0 \\
3 & 1 & 0 & \infty & 0 \\
4 & 0 & 0 & 2 & \infty
\end{array}
\Rightarrow
\begin{array}{c|cccc|c}
 & 1 & 2 & 3 & 4 & \\
1 & \infty & \infty & \infty & \infty & \\
2 & 1 & \infty & \infty & 0 & 0 \\
3 & \infty & 0 & \infty & 0 & 0 \\
4 & 0 & 0 & \infty & \infty & 0 \\
 & 0 & 0 & & 0 &
\end{array}$$

$\hat{r} = 0$

$c(1,3) + r + \hat{r}$

$= 0 + 14 + 0 = 14$

For Node 4 :-> 1^{st} row & 4^{th} column $\Rightarrow \infty$

$[4,1] \Rightarrow \infty$

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
1 & \infty & 3 & 0 & 0 \\
2 & 1 & \infty & 0 & 0 \\
3 & 1 & 0 & \infty & 0 \\
4 & 0 & 0 & 2 & \infty
\end{array}
\Rightarrow
\begin{array}{c|cccc|c}
 & 1 & 2 & 3 & 4 & \\
1 & \infty & \infty & \infty & \infty & \\
2 & 1 & \infty & 0 & \infty & 0 \\
3 & 1 & 0 & \infty & \infty & 0 \\
4 & \infty & 0 & 2 & \infty & 0 \\
 & 1 & 0 & 0 & &
\end{array}$$

$\hat{r} = 1$

$\Downarrow$

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
1 & \infty & \infty & \infty & \infty \\
2 & 0 & \infty & 0 & \infty \\
3 & 0 & 0 & \infty & \infty \\
4 & \infty & 0 & 2 & \infty
\end{array}$$

$c(1,4) + r + \hat{r}$

$= 0 + 14 + 1 = 15$

For Node 5 :→ 3^{rd} row & 2^{nd} Column ⇒ ∞

[2,1] ⇒ ∞

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & \infty & \infty & \infty & \infty \\
2 & 1 & \infty & \infty & 0 \\
3 & \infty & 0 & \infty & 0 \\
4 & 0 & 0 & \infty & \infty \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|cccc|c}
 & 1 & 2 & 3 & 4 & \\
\hline
1 & \infty & \infty & \infty & \infty & \\
2 & \infty & \infty & \infty & 0 & 0 \\
3 & \infty & \infty & \infty & \infty & \\
4 & 0 & \infty & \infty & \infty & \\
\hline
 & 0 & & & 0 &
\end{array}$$

$\hat{r} = 0$

$c(3,2) + r + \hat{r}$

$= 0 + 14 + 0 = 14$

For Node 6 :⇒ 3^{rd} row & 4^{th} column ⇒ ∞

[4,1] ⇒ ∞

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & \infty & \infty & \infty & \infty \\
2 & 1 & \infty & \infty & 0 \\
3 & \infty & 0 & \infty & 0 \\
4 & 0 & 0 & \infty & \infty \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|cccc|c}
 & 1 & 2 & 3 & 4 & \\
\hline
1 & \infty & \infty & \infty & \infty & \\
2 & 1 & \infty & \infty & \infty & 1 \\
3 & \infty & \infty & \infty & \infty & \\
4 & \infty & 0 & \infty & \infty & 0 \\
\hline
\end{array}$$

$\hat{r} = 1$

⇓

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & \infty & \infty & \infty & \infty \\
2 & 0 & \infty & \infty & \infty \\
3 & \infty & \infty & \infty & \infty \\
4 & \infty & 0 & \infty & \infty \\
\hline
\end{array}$$

$c(3,4) + r + \hat{r}$

$= 0 + 14 + 1 = 15$

For Node 7 :→ 2^{nd} row & 4^{th} column ⇒ ∞

$[4,1] \Rightarrow \infty$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & \infty & 0 \\ 3 & \infty & \infty & \infty & \infty \\ 4 & 0 & \infty & \infty & \infty \end{array} \Rightarrow \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & \infty & \infty \\ 3 & \infty & \infty & \infty & \infty \\ 4 & \infty & \infty & \infty & \infty \end{array}$$

$\hat{r} = 0$

$c(2,4) + r + \hat{r}$

$= 0 + 14 + 0 = 14$

Path is (1-3-2-4-1) with min cost = 14