Scalar SCET

1. Applying SCET to the scalar field's Lagrangian

Scalar Field's Lagrangian: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{3!}\phi^3$ (massless, ϕ^3 theory)

Apply SCET introduce different regions as we talked before.

$$\phi = \phi_c + \phi_{\bar{c}} + \phi_s$$

—— Why no "hard" region? It was absorbed into the "Wilson coefficients".

How to understand?
The "hard" means high energy, but all we interested about is the low energy's "effective" theory. So we "integrate out" this region into a "Wilson coefficients" as a new coupling "constant".

So the original Lagrangian can be written as:

$$\mathcal{L} \equiv \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_s + \mathcal{L}_{c+s} = \mathcal{L}(\phi_c) + \mathcal{L}(\phi_{\bar{c}}) + \mathcal{L}(\phi_s) + \mathcal{L}_{c+s}(\phi_c, \phi_{\bar{c}}, \phi_s)$$

where $\mathcal{L}(\phi_x) = \frac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x - \frac{g}{3!}\phi_x^3 \qquad (x = c, \bar{c}, s)$

$$\mathcal{L}_{c+s} = -\frac{g}{2}\phi_c^2\phi_s - \frac{g}{2}\phi_{\bar{c}}^2\phi_s$$

* Other terms are "illegal", for they break momentum conservation.
e.g. $\phi_c\phi_{\bar{c}}\phi_s \Rightarrow P_c^\mu \sim (\lambda^2, 1, \lambda)Q$; $P_{\bar{c}}^\mu \sim (1, \lambda^2, \lambda)Q$; $P_s^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q \Rightarrow (P_c + P_{\bar{c}} + P_s)^2 \neq 0 + O(\lambda^2)$

so it's vertex could not be momentum conservative

Here we take $\phi_c^2\phi_s$ as an example

$$\text{Interaction vertex} = \int d^dx\, \phi_c^2(x)\phi_s(x) = \int d^dx \frac{d^dP_1}{(2\pi)^d}\frac{d^dP_2}{(2\pi)^d}\frac{d^dP_3}{(2\pi)^d}\tilde{\phi}_c(P_1)\tilde{\phi}_c(P_2)\tilde{\phi}_s(P_3)\, e^{-i(P_1+P_2+P_3)\cdot x}$$

$$P_1^\mu + P_2^\mu + P_s^\mu \sim (\lambda^2, 1, \lambda)Q$$

so the component of x must has the form of ($P_{total}\cdot x \sim 1$)

$$x^\mu \sim \left(1, \frac{1}{\lambda^2}, \frac{1}{\lambda}\right)\frac{1}{Q}$$

$$\Rightarrow P_s\cdot x = \underbrace{(P_s)_+\cdot x_-}_{O(1)} + \underbrace{(P_s)_-\cdot x_+}_{O(\lambda^2)} + \underbrace{(P_s)_\perp\cdot x_\perp}_{O(\lambda)} \sim (P_s)_+\cdot x_-$$

So we can expand the soft field at the large component "x_-" (x_- is too large to be a expanding parameter)

$$\phi_s(x) = \phi_s(x_-) + \underbrace{x_\perp\cdot\partial_\perp\phi_s(x_-)}_{O(\lambda)} + \underbrace{x_+\cdot\partial\phi_s(x_-)}_{O(\lambda^2)} + \underbrace{\frac{1}{2}\left(x_\perp^\mu x_\perp^\nu\partial_\mu\partial_\nu\phi_s(x_-)\right)}_{O(\lambda^2)} + \cdots$$

$$= \phi_s(x_-)(1 + O(\lambda))$$

So expand the interacting Lagrangian to the leading order, we can get:

$$\mathcal{L}_{c+s} = -\frac{g}{2}\phi_c^2(x)\phi_s(x_-) - \frac{g}{2}\phi_c^2(x)\phi_s(x_+)$$

3.2 Matching Procedure and Current Operator.

In effective theory, the hard contributions leads to matching corrections.

The procedure is the following:

(i) Write down the most general form of the Lagrangian (with symmetry or gauge), each term would be multiplied by an arbitary coefficient (Wilson coefficient).

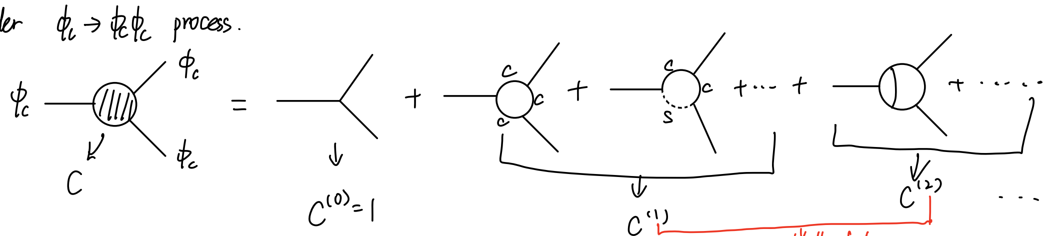
(ii) Calculate the given process both in the full theory and effective theory.

(iii) Compare the results of full theory and effective theory, change the Wilson coefficient to make it coincide. ⟶ Matching.

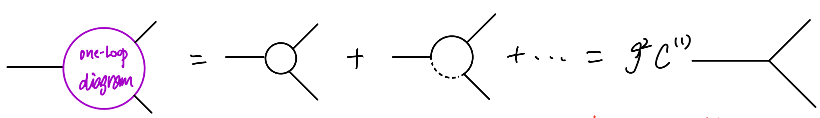
Eg. The $-\frac{g}{3!}\phi_c^3$ terms

$$-\frac{g}{3!}\phi_c^3 \longrightarrow -\frac{g}{3!}\,C\,\phi_c^3 = -\frac{g}{3}\big(\underset{\substack{\uparrow\\ C^{(0)}}}{1} + g^2C^{(1)} + g^4C^{(2)} + \cdots\big)\phi_c^3$$

Consider $\phi_c \to \phi_c\phi_c$ process.

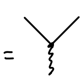


so if we want to fix $C^{(1)}$, we should take all one-loop diagrams into our calculation.



it's easy to see that when all external legs are on-shell (→ Kinetic condition.), all the diagrams are scaleless.

$\Rightarrow C = C^{(0)} = 1$ (Free Lagrangian don't have not-equal-to-1 "Wilson coefficients")

However, if we consider the "current operator" $J = \phi^2 =$ 

it means we insert a momentum to our system at the vertex.

$$J = \underbrace{J_2 + J_3}_{\#\text{ of involved fields}} + \cdots = C_2\phi_c\phi_c + \frac{C_3}{2!}(\phi_c^2\phi_c + \phi_c\phi_c^2) + \cdots$$

As we talked above, when we're expanding the fields at a particular position with a a small pertubation in other directions, the drivative term appears.

Consider the dirivatives: ($P \sim (\lambda^2, 1, \lambda)$ "Q", only consider the pertubation parameter)

$$\begin{cases} n\cdot\partial\phi_c(x) \sim n\cdot P_c\,\phi_c(x) = P_c^+\phi_c(x) \sim \lambda^2\phi_c(x) \\ \partial_\perp^\mu\phi_c(x) \sim P_\perp^\mu\phi_c(x) \sim \lambda\phi_c(x). \\ \bar{n}\cdot\phi_c(x) \sim n\cdot P_c\,\phi_c(x) = P_c^-\phi_c(x) \sim \lambda^0\phi_c(x) \end{cases}$$

$$\begin{cases} n\cdot\partial\phi_{\bar{c}}(x) \sim \lambda^0\phi_{\bar{c}}(x) \\ \partial_\perp^\mu\phi_{\bar{c}}(x) \sim \lambda\phi_{\bar{c}}(x) \\ \bar{n}\cdot\partial\phi_{\bar{c}}(x) \sim \lambda^2\phi_{\bar{c}}(x) \end{cases}$$

We can see, at leading order of "λ", just "$\bar{n}\cdot\partial\phi_c(x)$" and "$n\cdot\partial\phi_{\bar{c}}(x)$" contribute.

So we can expand an arbitary (anti-) collinear field in large momentum direction.

$$\Rightarrow \quad \phi_c(x+s\bar{n}) = \sum_{i=0}^{+\infty}\frac{s^i}{i!}(\bar{n}\cdot\partial)^i\phi_c(x)$$

$$\phi_{\bar{c}}(x+tn) = \sum_{i=0}^{+\infty}\frac{t^i}{i!}(n\cdot\partial)^i\phi_{\bar{c}}(x)$$

if we treat it inversely, that is all order of derivations can be described as the sum/integration of non-local (not at the same time/position, e.g. $x^\mu \neq y^\mu$) fields.

$$\Rightarrow \quad J_2(x) = \int ds\,dt\; C_2(s,t,\underset{\text{at an exact energy scale}}{\mu})\,\phi_c(x+s\bar{n})\,\phi_{\bar{c}}(x+tn)$$

in momentum space:

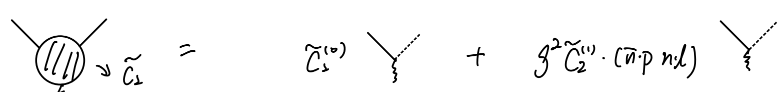
Why? It moves opposite direction.

$$\tilde{C}_2(\bar{n}\cdot p, n\cdot l, \mu) = \int ds\,dt\; e^{is\bar{n}\cdot p}\,e^{-itn\cdot l}\,C_2(s,t,\mu)$$

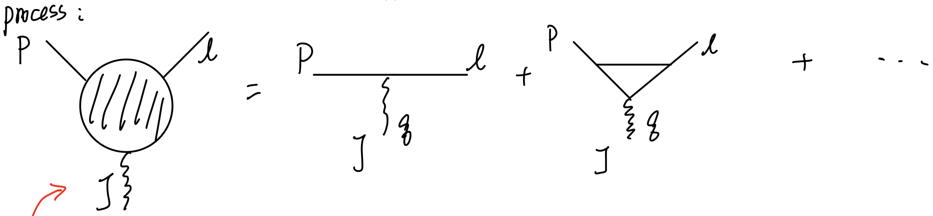
then we can expand $\tilde{C}_2$ by the parameter "g".

$$\tilde{C}_2 = \tilde{C}_2^{(0)} + g^2\tilde{C}_2^{(1)} + g^4\tilde{C}_2^{(2)} + \cdots$$

it means



Real process:

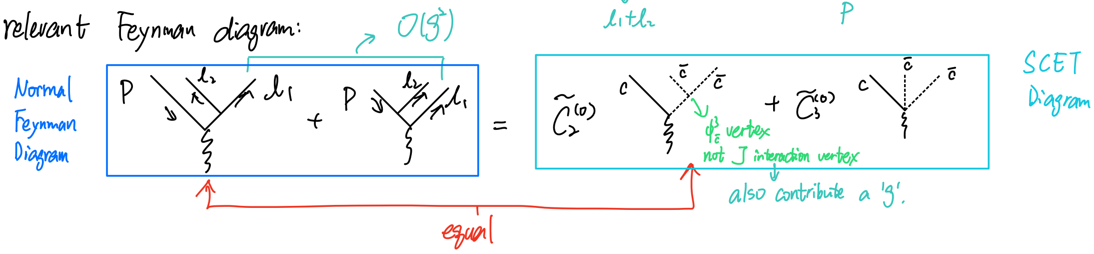


Momentum J (it means some terms of $Q\frac{n^\mu}{2}, Q\frac{\bar{n}^\mu}{2}$) insert, p & l out.

then we consider $J_{3\bar{c}} = \frac{1}{2}\phi_c \phi_{\bar{c}}^2$ → two momenta of $Q\frac{\bar{n}^\mu}{2}$, one momentum of $Q\frac{n^\mu}{2}$ insert.

$\downarrow$ $l_1 + l_2$ $\qquad \downarrow$ p

relevant Feynman diagram:



$$\text{L.H.S diagram 2} \propto \frac{g}{(p-l_2)^2 + i\epsilon} = \frac{g}{-p^- l_2^+ + i\epsilon}$$

$$\Rightarrow \quad \tilde{C}_3^{(0)}(n\cdot l_1, n\cdot l_2, \bar{n}\cdot p, \mu) = \frac{g}{-(\bar{n}\cdot p)(n\cdot l_2)}$$

with replace $\bar{n}\cdot p \Rightarrow (\bar{n}\cdot\partial)_{\phi_c}$ (acting on ϕ_c), $n\cdot l_2 \rightarrow (n\cdot\partial)\phi_{\bar{c}}$

so $\frac{g}{-(\bar{n}\cdot p)(n\cdot l_2)+i\epsilon} \longrightarrow -g\left(\frac{1}{\bar{n}\cdot\partial + i\epsilon}\phi_c\right)\left(\frac{1}{n\cdot\partial + i\epsilon}\phi_{\bar{c}}\right)\phi_{\bar{c}}$ (formal form)

to adapt the formal form to the practical form, we should drop the derivation in the denominator with this formula:

$$\frac{i}{i n\cdot\partial + i\epsilon}\phi(x) = \int_{-\infty}^{0} ds\, \phi(x+sn)$$

Proof: that is to prove: $n\cdot\partial\int_{-\infty}^{0} ds\,\phi(x+sn) = \phi(x)$

x is fixed, just s changes

$$\Rightarrow \quad n^\mu \int_{-\infty}^{0} ds\, \partial_\mu \phi(x+sn) = n^\mu \cdot \int_{-\infty}^{0} ds \frac{\partial}{\partial x^\mu + s n^\mu}\phi(x+sn)$$

$$= \cancel{n^\mu}\int_{-\infty}^{0} ds\, \frac{1}{\cancel{n^\mu}}\cdot\frac{\partial}{\partial s}\phi(x+sn)$$

$$= \phi(x+sn)\Big|_{-\infty}^{s=0} = \phi(x) \qquad \checkmark$$

Similarly: $\frac{-i}{i n\cdot\partial - i\epsilon}\phi(x) = \int_{0}^{\infty} ds\,\phi(x+sn)$

(To prove mathematically needs to do it in Fourier space, and use $\frac{1}{x+i\epsilon} = PV(\frac{1}{x}) - 2i\delta(x)$)

So $\quad J_3 = J_{3c} + J_{3\bar{c}} = \int_{-\infty}^{+\infty} ds\, dt_1 dt_2\, C_3(s, t_1, t_2, \mu)\, \phi_c(x+sn)\phi_{\bar{c}}(x+t_1\bar{n})\phi_{\bar{c}}(x+t_2\bar{n}) + (c \Leftrightarrow \bar{c})$

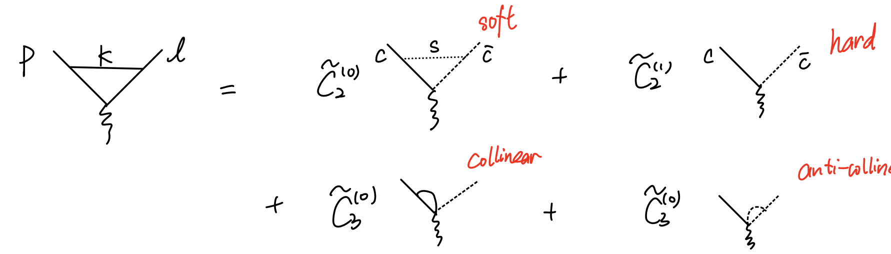
it's easy to probe the leading order : $C_2(s, t, \mu) = \delta(s)\delta(g) + O(g^2)$

$C_3(s, t_1, t_2, \mu) = g\,\theta(-s)\theta(t_1)\delta(t_2) + O(g^3)$

and give this replacement rule:

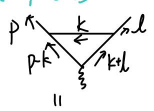
$$\delta(s) \Leftrightarrow 1 \quad ; \quad \theta(-s) \longleftrightarrow \frac{1}{iQ}$$

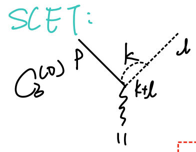
3. Sudakov Form Factor in SCET → Only (anti-) collinear fields has Wilson Coefficients.



Real Process: SCET:

e.g.





$$-i\int d^dk \frac{(-ig)^2}{(k^2+i\epsilon)[(k+l)^2+i\epsilon][(p-k)^2+i\epsilon]} \quad \frac{k\sim Q(1,\lambda^2,\lambda)}{(p+k)^2\sim 2k_+ p_-} \quad ig\int d^dk \frac{1}{(k^2+i\epsilon)((k+l)^2+i\epsilon)} \cdot \boxed{\frac{g}{2k_+\cdot p_- + i\epsilon}} \to C_3^{(0)}$$

same as anticollinear region.

4. Sudakov Problem in dimension 6.

$$S = \int d^dx \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - g\phi^3\right)$$

The dimension of ϕ? $[\phi]\neq 1$!

$[S]=0$, $[\partial_\mu]=1$, $[d^dx]=-d$

$\Rightarrow [\phi] = \frac{d}{2}-1$ $\Rightarrow$

$[g] = 3-\frac{d}{2}$

$[X]$ \ d	4	6
$[\phi]$	1	2
$[g]$	1	0

Similarly, we can introduce the expansion parameter 'λ' to do our POWER COUNTING.

e.g. We have a hard scale Q^2, and the collinear momentum P_c.

Where $P_c = Q(\lambda^2, 1, \lambda)$. Also, the anti-collinear momentum $P_{\bar{c}} = Q(1, \lambda^2, \lambda)$.

To emphasize, it is a little bit different from our former talk that the transverse momentum has dimension 4 ($[P_\perp]=4$, not 2).

The two-point correlator for collinear fields: $p^2\sim Q^2\lambda^2$

$$\langle 0|T\{\phi_c(x)\phi_c(0)\}|0\rangle \sim \underbrace{\int d^6p}_{\lambda^6}\cdot \underbrace{e^{-ip\cdot x}}_{\lambda^0}\underbrace{\frac{i}{p^2}}_{\lambda^{-2}} \sim \lambda^4 \quad \Rightarrow \phi_c \sim \lambda^2$$

For anti-collinear fields, it acts similarly.

For soft fields: $p^2 \sim Q^2\lambda^4$

$$\langle 0|T\{\phi_s(x)\phi_s(0)\}|0\rangle \sim \int \underset{\lambda^{12}}{d^6p}\ \underset{\lambda^{0}}{e^{-ip\cdot x}}\ \underset{\lambda^{-4}}{\frac{i}{p^2}} \sim \lambda^8 \quad \Rightarrow \quad \phi_s \sim \lambda^4$$

For component of x, it's the conjugate variable of p. it has $x\cdot p \sim 1$

So we can get every term in the effective Lagrangian:

$$\int d^6x \cdot \frac{1}{2}\partial_\mu\phi_{c,\bar c}\partial^\mu\phi_{c,\bar c} \sim \frac{1}{\lambda^6}\cdot(\lambda\lambda^2)^2 \sim \lambda^0$$

$$\int d^6x\ g\phi_{c,\bar c}^3 \sim \frac{1}{\lambda^6}\cdot(\lambda^2)^3 \sim \lambda^0$$

$$\int d^6x\ \frac{1}{2}\partial_\mu\phi_s\partial^\mu\phi_s \sim \frac{1}{\lambda^{12}}(\lambda^2\cdot\lambda^4)^2 \sim \lambda^0$$

$$\int d^6x\ g\phi_s^3 \sim \frac{1}{\lambda^{12}}(\lambda^4)^3 \sim \lambda^0$$

$$\int d^6x\ g\cdot\phi_{c,\bar c}^2\phi_s \sim \frac{1}{\lambda^6}(\lambda^2)^2\cdot\lambda^4 \sim \lambda^2 \longrightarrow \text{suppressed.}$$

And in the current operator J.

J_2 : $\int d^6x\, J_2(x) \sim \int d^6x\ \phi_c(x)\phi_{\bar c}(x) \sim \underset{\text{Why } \lambda^4?}{\frac{1}{\lambda^4}}\cdot\lambda^2\lambda^2 \sim \lambda^0$

$J_2(x)$ contains ϕ_c & $\phi_{\bar c}$, it means $P_J \sim p+l$

$P_J^- \sim p^- \sim \lambda^0$; $P_J^+ \sim l^+ \sim \lambda^0$; $P_J^\perp \sim p^\perp + l^\perp \sim \lambda$

$\Rightarrow x^+ \sim \lambda^0$; $x^- \sim \lambda^0$; $x^\perp \sim \frac{1}{\lambda}$

$\Rightarrow d^6x \sim \frac{1}{\lambda^4}$

J_3 :
$$\left.\begin{aligned}&\int d^6x\ \phi_c^2(x)\phi_{\bar c}(x) \sim \frac{1}{\lambda^4}\cdot(\lambda^2)^2\lambda^2 \sim \lambda^2\\ &\int d^6x\ \phi_c(x)\phi_{\bar c}^2(x) \sim \frac{1}{\lambda^4}\ \lambda^2(\lambda^2)^2 \sim \lambda^2\end{aligned}\right] \longrightarrow \text{Both suppressed}$$

In summary, the effective Lagrangian:

$$\int d^6x\ \mathcal{L}_{SCET} = \int d^6x\,[\mathcal{L}_c + \mathcal{L}_{\bar c} + \mathcal{L}_s] + O(\lambda^2)$$ → All mixed terms are power suppressed.

$$\int d^6x\ J(x) = \int d^6x \int ds\,dt\ C(s,t,\mu)\ \phi_c(x+s\bar n)\phi_{\bar c}(x+tn) + O(\lambda^2)$$ → All more-than-two-fields operators are power suppressed

* It is the base of factorization.

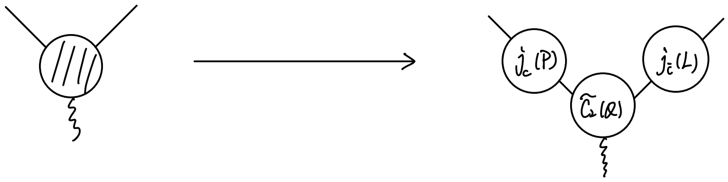
So the correlator:

$$G(p,l,\mu) = \int d^6x_1\int d^6x_2\ e^{-ip\cdot x_1 + il\cdot x_2}\langle 0|T\{\phi_c(x_1)J(0)\phi_{\bar c}(x_2)\}|0\rangle$$

$$= \int d^6x_1\int d^6x_2\ e^{-ip\cdot x_1 + il\cdot x_2}\int ds\int dt\ \underbrace{C_2(s,t,\mu)}_{\nearrow\ \tilde C_2(Q)}$$

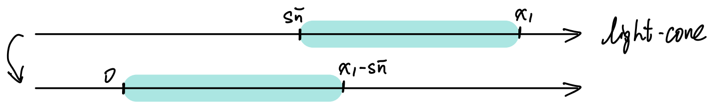
$$\times \underbrace{\langle 0|T\{\phi_c(x_1)\phi_c(s\bar n)\}|0\rangle}_{j_c(p)}\underbrace{\langle 0|T\{\phi_{\bar c}(tn)\phi_{\bar c}(x_2)\}|0\rangle}_{j_{\bar c}(l)}$$

In the correlator, we can see that the contribution of collinear & anti-collinear is separated.



$+ \; O(\lambda^2)$

Furthermore, we have translation invariance. (it's easy to understand redefine the "0" point).



$$\langle 0|T\{\phi_c(x_1)\,\phi_c(s\bar{n})\}|0\rangle = \langle 0|T\{\phi_c(x_1 - s\bar{n})\,\phi_c(0)\}|0\rangle$$

Similarly: $\langle 0|T\{\phi_{\bar{c}}(tn)\,\phi_{\bar{c}}(x_2)\}|0\rangle = \langle 0|T\{\phi_{\bar{c}}(0)\,\phi_{\bar{c}}(x_2 - tn)\}|0\rangle$

So we can get our correlator:

$$G(p,l,\mu) \underset{x_2 \to x_2 + tn}{\overset{x_1 \to x_1 + s\bar{n}}{=\!=\!=}} \int d^d x_1\, d^d x_2 \int ds\,dt\, e^{-is\,p\cdot\bar{n} + it\,l\cdot n}\, C(s,t,\mu)$$
$$\times\, e^{-ip\cdot x_1}\langle 0|T\{\phi_c(x_1)\phi_c(0)\}|0\rangle \cdot e^{il\cdot x_2}\langle 0|T\{\phi_{\bar{c}}(0)\,\phi_{\bar{c}}(x_2)\}|0\rangle$$

$$= \int ds\,dt\, e^{-is\,p\cdot\bar{n} + it\,l\cdot n}\, C(s,t,\mu)\; j(p^2,\mu)\; j(l^2,\mu)$$

where $j(p^2,\mu) = \int d^d x_1\, e^{-ip\cdot x_1}\langle 0|T\{\phi_c(x_1)\phi_c(0)\}|0\rangle$

$j(l^2,\mu) = \int d^d x_2\, e^{il\cdot x_2}\langle 0|T\{\phi_{\bar{c}}(0)\,\phi_{\bar{c}}(x_2)\}|0\rangle$

Since $j(p^2,\mu)$ & $j(l^2,\mu)$ are independent of s & t. so we can introduce the "Fourier Transformation" of $C(s,t,\mu)$

$$\tilde{C}_2(\bar{n}\cdot p, n\cdot l, \mu) \equiv \int ds\,dt\, e^{-is\,p\cdot\bar{n} + it\,l\cdot n}\, C(s,t,\mu)$$

$$\Rightarrow \quad G(p,l,\mu) = \underbrace{\tilde{C}_2(\bar{n}\cdot p, n\cdot l, \mu)}_{\text{Hard}}\; \underbrace{j(p^2,\mu)\; j(l^2,\mu)}_{\text{Jets.}} \;\Rightarrow\; \text{Factorize.}$$