Generization to QCD

Almost same as before talk, slight differences occurs.

(i) Different components of fermion field $\psi(x)$ & Gauge Boson $A_\mu(x)$ scale differently with expansion parameter.

(ii) Theory should be gauge invariance.

(iii) non-local operators involve WILSON LINES to ensure gauge invariance.

First we take collinear mode as an example. ($p^\mu \sim (\lambda^2, 1, \lambda) Q$

the gluon field & quark field:

$$A^\mu(x) \to A^\mu_c(x) + A^\mu_s(x) \quad ; \quad \psi(x) \Rightarrow \psi_c(x) + \psi_s(x)$$

Since quark fields always contain spinors, so we introduce two Projection Operator

$$P_+ = \frac{\not{n}\not{\bar{n}}}{4} \quad ; \quad P_- = \frac{\not{\bar{n}}\not{n}}{4} \qquad \bar{\xi}P_- = \overline{\xi P_-} = \overline{\tfrac{\not{n}\not{\bar{n}}}{4} \cdot P_+\psi_c} = \overline{P_+\psi_c} = \bar{\xi}$$

Obviously, we have $P_+ + P_- = \frac{n^\mu \bar{n}^\nu}{4}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \frac{n\cdot\bar{n}}{2} = 1$

And, $P_+^2 = \frac{\not{n}\not{\bar{n}}\not{n}\not{\bar{n}}}{16} = \frac{\not{n}\not{\bar{n}}}{4} = P_+ \,;\quad P_-^2 = P_-$.

So we can introduce two separated fields for collinear quark fields.

$$\xi = P_+\psi_c = \frac{\not{n}\not{\bar{n}}}{4}\psi_c \quad ; \quad \eta = P_-\psi_c = \frac{\not{\bar{n}}\not{n}}{4}\psi_c$$

$\Rightarrow \not{n}\xi = \bar{\xi}\not{n} = 0 \;;\; \not{\bar{n}}\eta = \bar{\eta}\not{\bar{n}} = 0$

1. Power Counting.

$$* \quad \bar{\xi} = \overline{P_+\psi_c} \propto \overline{(\not{n}\not{\bar{n}})\psi_c} = \psi_c^\dagger \not{\bar{n}}^\dagger \not{n}^\dagger \gamma_0 = \psi_c^\dagger \gamma_0 \cdot \underbrace{\gamma_0 \not{\bar{n}}^\dagger \not{n}^\dagger \gamma_0}_{\gamma_0\gamma^{\mu\dagger}\gamma_0 = \gamma^\mu} = \bar{\psi}_c \not{\bar{n}}\not{n}$$

Consider two-point correlators:

$$\langle 0|T\{\xi(x)\bar{\xi}(0)\}|0\rangle = \frac{\not{n}\not{\bar{n}}}{4}\langle 0|T\{\psi_c(x)\bar{\psi}_c(0)\}|0\rangle \frac{\not{\bar{n}}\not{n}}{4}$$
$$= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot x}\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4}$$

where $\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4} = \frac{\not{n}\not{\bar{n}}}{4}\left(\bar{n}\cdot p\frac{\not{n}}{2} + n\cdot p\frac{\not{\bar{n}}}{2} + \not{p}_\perp\right)\frac{\not{\bar{n}}\not{n}}{4} = \bar{n}\cdot p \cdot \frac{\not{n}}{2} \sim P_c \sim \lambda^0$

So that $\langle 0|T\{\xi(x)\bar{\xi}(0)\}|0\rangle \sim \lambda^4 \cdot \frac{1}{\lambda^2} \cdot \lambda^0 \sim \lambda^2 \;\Rightarrow\; \xi(x) \sim \lambda$

Similarly: $\langle 0|T\{\eta(x)\eta(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot x}\,\boxed{\frac{\bar{\not{n}}\not{n}}{4}\not{p}\frac{\not{n}\bar{\not{n}}}{4}} \sim \not{n}\cdot\frac{1}{\lambda^2}\cdot\lambda^2 \sim \lambda^4 \Rightarrow \eta(x)\sim\lambda^2$

$\mapsto p_c$

$\langle 0|T\{\psi_s(x)\psi_s(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\cdot\frac{i\not{p}}{p^2+i\epsilon}e^{-ip\cdot x} \sim (\lambda^2)^4\cdot\frac{1}{\lambda^4}\cdot\lambda^2 \sim \lambda^6 \quad \Rightarrow \psi_s(x)\sim\lambda^6$

for gluon fields, $\langle 0|T\{A^\mu(x)A^\nu(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot x}\left[-g^{\mu\nu}+\xi\frac{p^\mu p^\nu}{p^2}\right] \Rightarrow A^\mu(x)$ acts like p^μ

$\Rightarrow \quad \bar{n}\cdot A_c \sim p_c^+ \sim \lambda^0$; $n\cdot A_c \sim p_c^- \sim \lambda^2$; $A_c^\perp \sim p_c^\perp \sim \lambda$; $A_s^\mu \sim \lambda^2$

so, for gluon fields: $A^\mu = A_s^\mu + A_c^\mu$; two components of soft field vanishes ($A_s^\perp$ & $\bar{n}\cdot A_s$),

only $n\cdot A_s$ isn't power suppressed. $\rightarrow$ Gauge Invariance has broken.

$\downarrow$

May induce the unphysical longitudinal polarization.

$\downarrow$

Wilson Line should be introduced.

2. Effective Lagrangian.

Original Lagrangian: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\,i\not{D}\psi$ (QCD, not full SM)

where $\not{D} = \gamma_\mu\cdot D^\mu = \gamma_\mu\cdot(\partial^\mu - igA^\mu)$

In SCET: $iD^\mu = i\partial^\mu + gA^\mu = i\partial^\mu + g\,(A_s^{\mu,a}+A_c^{\mu,a})\boxed{t^a}$

$\mapsto$ generator of $SU(3)$, Also called "Gell-Mann Matrix".

First, we consider the collinear Lagrangian (of quarks)

$$\mathcal{L}_c = \bar{\psi}_c(i\not{D})\psi_c$$

$$= (\bar{\xi}+\bar{\eta})\left[\frac{\not{n}}{2}i\bar{n}\cdot D + \frac{\bar{\not{n}}}{2}in\cdot D + i\not{D}_\perp\right](\xi+\eta)$$

We have the relation: $\not{n}\cdot\xi = \bar{\xi}\cdot\not{n} = 0$; $\bar{n}\cdot\eta = \bar{\eta}\cdot\bar{n} = 0$

and: $\bar{\xi}\not{D}_\perp\xi = \bar{\xi}P_-\not{D}_\perp P_+\xi = \bar{\xi}\not{D}_\perp P_+P_+\xi = 0$ & $\bar{\eta}\not{D}_\perp\eta = 0$

$\uparrow$ $\{\not{n},\not{D}_\perp\} = \{\bar{\not{n}},\not{D}_\perp\} = 0$

$$\Rightarrow \quad \mathcal{L}_c = \bar{\xi}\frac{\bar{\not{n}}}{2}in\cdot D\xi + \bar{\eta}\frac{\not{n}}{2}i\bar{n}\cdot D\eta + \bar{\xi}\,i\not{D}_\perp\eta + \bar{\eta}\,i\not{D}_\perp\xi$$

This Lagrangian is quadratic and contains mixed term.

we can use Lagrange Equation to reduce one of these fields.

$$\begin{cases} \partial_\mu \dfrac{\partial \mathcal{L}_c}{\partial(\partial_\mu \bar{\xi})} - \dfrac{\partial \mathcal{L}_c}{\partial \bar{\xi}} = -\dfrac{\bar{\not{n}}}{2} i n\cdot D \xi - i \not{D}_\perp \eta = 0 \\ \partial_\mu \dfrac{\partial \mathcal{L}_c}{\partial(\partial_\mu \bar{\eta})} - \dfrac{\partial \mathcal{L}_c}{\partial \bar{\eta}} = -\dfrac{\not{n}}{2} i \bar{n}\cdot D \eta - i \not{D}_\perp \cdot \xi = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \not{D}_\perp \eta = -\dfrac{\bar{\not{n}}}{2} n\cdot D \xi \\ \not{D}_\perp \xi = -\dfrac{\not{n}}{2} \bar{n}\cdot D \eta \end{cases} \Rightarrow \begin{aligned} &\frac{\not{n}}{2} D_\perp \eta = -\frac{\not{n}\cdot\bar{\not{n}}}{4} n\cdot D \xi = -n\cdot D\xi \\ &\frac{\bar{\not{n}}}{2} D_\perp \xi = -\frac{\bar{\not{n}}\not{n}}{4} \bar{n}\cdot D \eta = -\bar{n}\cdot D \eta \end{aligned}$$

$$\Rightarrow \eta = -\frac{\bar{\not{n}}}{2\bar{n}\cdot D} \not{D}_\perp \xi \quad \& \quad \bar{\eta} = -\bar{\xi} \overleftarrow{\not{D}}_\perp \frac{\bar{\not{n}}}{2\bar{n}\cdot \overleftarrow{D}}$$ (here we integrate out η)

where the arrow indicates the derivate is acting to the left.

$$\Rightarrow \mathcal{L}_c = \bar{\xi} \frac{\bar{\not{n}}}{2} i n\cdot D \xi + \bar{\eta} \frac{\not{n}}{2} i \bar{n}\cdot D \eta + \bar{\xi} i \not{D}_\perp \eta + \bar{\eta} i \not{D}_\perp \xi$$

$$= \bar{\xi} \frac{\bar{\not{n}}}{2} i n\cdot D \xi + \bar{\xi} i \not{D}_\perp \frac{\bar{\not{n}}}{2\bar{n}\cdot D} \not{D}_\perp \xi + \cancel{\bar{\xi} \overleftarrow{\not{D}}_\perp \frac{\bar{\not{n}}}{2\bar{n}\cdot \overleftarrow{D}} \boxed{\frac{\not{n}}{2} i \bar{n}\cdot D \frac{\bar{\not{n}}}{2\bar{n}\cdot D}} \not{D}_\perp \xi} - \cancel{\bar{\xi} \overleftarrow{\not{D}}_\perp \frac{\bar{\not{n}}}{2\bar{n}\cdot \overleftarrow{D}} i \not{D}_\perp \xi}$$

$$i \frac{\not{n}\bar{\not{n}}}{4} = i P_+ \Rightarrow 1$$

$$= \bar{\xi} \left(\frac{\bar{\not{n}}}{2} i n\cdot D + i \not{D}_\perp \frac{\bar{\not{n}}}{2\bar{n}\cdot D} \not{D}_\perp \right) \xi = \bar{\xi} \frac{\bar{\not{n}}}{2} \left[i n\cdot D + i \not{D}_\perp \frac{1}{i\bar{n}\cdot D} i \not{D}_\perp \right] \xi$$

* It's the same when we do path integral to integrate out the field η.

$$\int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}] \exp\left\{ \int d^4x \, \bar{\eta} \frac{\not{n}}{2} i \bar{n}\cdot D \eta \right\} = \det\left(\frac{\not{n}}{2} i \bar{n}\cdot D \right)$$

it is invariant under SU(N) gauge transformation.

So we can choose a proper gauge to vanish its contribution (e.g. in the light cone gauge, where $\bar{n}\cdot D = \bar{n}\cdot\partial$, it's trivially independent of A^μ), it will become a "c-number" independent of $A^\mu \longrightarrow$ integrated out.

For gluon, this topic is trival, we can just do replacement $A^\mu \to A_c^\mu$

For soft: $\mathcal{L}_s = \bar{\psi}_s i \not{D}_s \psi - \frac{1}{4} F_s^{a,\mu\nu} \cdot F^a_{s,\mu\nu}$ also trival

$\longrightarrow$ No-interaction Lagrangian.

3. Soft-Collinear Interactions

As we've talked above, the scaling of different fields is different.

A_c: $\quad (n\cdot A_c, \bar n\cdot A_c, A_c^\perp) \sim (\lambda^2, 1, \lambda)$

A_s: $\quad (n\cdot A_s, \bar n\cdot A_s, A_s^\perp) \sim (\lambda^2, \lambda^2, \lambda^2)$

ψ: $\quad \xi \sim \lambda \;;\; \psi_s \sim \lambda^3$

for quarks, the soft-collinear interactions do not appear at leading order.

but for gluons are not $(n\cdot A_c \sim n\cdot A_s \sim \lambda^2)$

$$A^\mu(x) \longrightarrow \underline{(n\cdot A_c(x) + n\cdot A_s(x))}\,\frac{\bar n^\mu}{2} + \bar n\cdot A_c(x)\cdot\frac{n^\mu}{2} + A^\mu_{c\perp}(x)$$

$$iD^\mu = i\partial^\mu + gA^\mu \longrightarrow i\partial^\mu + gA_s^\mu + gA_c^\mu = in\cdot D\,\frac{\bar n^\mu}{2} + i\bar n\cdot D_c\,\frac{n^\mu}{2} + iD^\mu_{c\perp}$$

$$in\cdot D = in\cdot\partial + g\,n\cdot A \longrightarrow in\cdot\partial + g\,n\cdot A_c(x) + g\,n\cdot A_s(x_-)$$

where $D_c^\mu = \partial^\mu - igA_c^\mu$ (act on collinear quark fields)

& $D_s^\mu = \partial^\mu - igA_s^\mu$ (act on soft quark fields)

So the full Lagrangian of SCET is:

$$\mathcal{L}_{SCET} = \underbrace{\underset{\lambda^3}{\bar\psi_s}\, \underset{\boxed{\lambda^2}}{i\not{D}}\, \underset{\lambda^3}{\psi_s}}_{\lambda^8} + \underbrace{\underset{\lambda}{\bar\xi}\,\frac{\not{\bar n}}{2}\Big(\underset{\lambda^2}{in\cdot D} + \underset{\lambda}{i\not{D}_\perp}\, \underset{O(1)}{\frac{1}{i\bar n\cdot D}}\, \underset{\lambda}{i\not{D}_\perp}\Big)\underset{\lambda}{\xi}}_{\lambda^4} - \frac{1}{4}(F^{s,a}_{\mu\nu})^2 - \frac{1}{4}(F^{c,a}_{\mu\nu})^2$$

$\lambda^4 \to$ soft contribution vanishes except in $n\cdot D$ term.

$\hookrightarrow O(1)$

↳ Why there's no A_c^μ terms? → It invades momentum conservation.

the field strengths of gluon are given:

$$ig\,F^s_{\mu\nu} = [iD^s_\mu, iD^s_\nu]$$

$$ig\,F^c_{\mu\nu} = [iD_\mu, iD_\nu]$$

Why not $igF^c_{\mu\nu} = [iD^c_\mu, iD^c_\nu]$?

Let simplify this commutator:

$igF_{\mu\nu} = [iD_\mu, iD_\nu]$

$$= -\left[\underbrace{n\cdot D}_{\substack{\downarrow \\ n\cdot D\,(\lambda^2)}}\frac{\bar{n}^\mu}{2} + \underbrace{\bar{n}\cdot D}_{\substack{\downarrow \\ \bar{n}\cdot D_c\,(1)}}\cdot\frac{n^\mu}{2} + \underbrace{D_\perp^\mu}_{\substack{\downarrow \\ D_{c\perp}^\mu\,(\lambda)}},\ n\cdot D\frac{\bar{n}^\mu}{2} + \bar{n}\cdot D\cdot\frac{n^\mu}{2} + D_\perp^\mu\right]$$

contains soft field at LO.

But $igF_{\mu\nu}^S = [iD_\mu^s, iD_\nu^s]$ is purely independent of collinear field.

The dependence of soft field in collinear field strength seems to cause gauge invariance, but it's actually not, we will talk it later.

In practical, there will be another mode of collinear: $l \sim (1, \lambda^2, \lambda)$; $p \sim (\lambda^2, 1, \lambda)$, we should just replace $n^\mu \Leftrightarrow \bar{n}^\mu$ & $x_+ \Leftrightarrow x_-$ in our talk above.

4. Gauge Transformation and Reparameterization Invariance.

As we know, gauge symmetry is not a natural symmetry, it's just the redundancies in our theory. E.g. we use four components vector field A^μ to describe only two physical degrees of free, our theory should be free for whichever the gauge is chosen to be like. Simply we can say, gauge transformation is the different choices of gauge, and gauge invariance/symmetry is the physical result is independent from our chosen gauge.

Reparameter invariance is that the final result must be independend from the choice of these reference vectors, n_μ & $\bar{n}_\mu$, in the construction of EFT. It's straightforward to conclude that the final result should be free for our choice of coordinate.

Normal gauge transformation:

Abelian Theory (QED): $V(x) = e^{-i\alpha(x)}$ → non-local

$$\psi \to e^{-i\alpha(x)}\psi = V(x)\psi$$

$$A^\mu \to A^\mu + \frac{1}{g}\partial^\mu\alpha(x) = V(x)A^\mu V^\dagger(x) - \frac{i}{g}V(x)[\partial^\mu, V^\dagger(x)]$$

Non-Abelian Theory (QCD): $V(x) = e^{-i\alpha^a(x)t^a}$

$$\psi \to e^{-i\alpha^a(x)t^a}\psi = V(x)\psi$$

$$A^\mu \to V(x)A^\mu V^\dagger(x) - \frac{i}{g}V(x)[\partial^\mu, V^\dagger(x)]$$ → expansion is complicated.

Gauge transformation for SCET:

Soft Gauge Transformation: $V_s(x) = e^{-i\alpha^a(x)t^a}$

$$\psi_s \to V_s(x)\psi_s$$

$$A_s^\mu \to V_s(x)A_s^\mu V_s^\dagger(x) - \frac{i}{g}V_s(x)[\partial^\mu, V_s^\dagger(x)]$$

$$\xi \to V_s(x)\xi$$

→ Large momentum: "x_-"

$p_c \sim (\lambda^2, 1, \lambda) \Leftrightarrow x_c \sim (\frac{1}{\lambda^2}, 1, \frac{1}{\lambda})$

→ the only survived.

$$A_c^\mu \to \boxed{V_s(x_-)A_c^\mu V_s^\dagger(x_-)} - \frac{i}{g}V_s(x)\underbrace{[\partial^\mu, V_s^\dagger(x)]}_{O(\lambda^2)}$$

↳ $V_s(x) = \underbrace{V_s(x_-)}_{O(1)} + \underbrace{x_+\cdot\partial_- V_s(x_-)}_{O(\lambda^2)} + \underbrace{x_\perp\cdot\partial_\perp V_s(x_-)}_{O(\lambda)} + \cdots$

$$n\cdot A \longrightarrow V_s(x_-)\, n\cdot A\, V_s^\dagger(x_-) - \frac{i}{g}V_s(x_-)[n\cdot\partial, V_s^\dagger(x_-)]$$

$$i\,n\cdot D \longrightarrow V_s(x_-)\, i n\cdot D\, V_s^\dagger(x_-)$$ other terms are power suppressed.

Collinear Gauge Transformation: $V_c(x) = e^{-i\alpha^a(x)t^a}$

for momentum conservation, soft fields don't involve Collinear Gauge Transformation.

$$\psi_s \to \psi_s$$

$$A_s^\mu \to A_s^\mu$$

$$\xi \to V_c(x)\xi$$

$$A_c^\mu \to V_c(x)A_c^\mu V_c^\dagger(x) - \frac{i}{g}V_c(x)\left[i\partial^\mu + g\frac{\bar{n}^\mu}{2}n\cdot A_s(x_-), V_c^\dagger(x)\right]$$

it's clear that A_s come up in the anti collinear component of A_c.

$$in\cdot D \to V_c\, in\cdot D\, V_c^\dagger$$

5. Wilson Lines.

matching the full theory.

We've introduced the Wilson coefficient in ϕ^3 theory $\longrightarrow$ non-local

In gauge theory, the product of fields at different space-time points is gauge invariant only if these fields are connected by the Wilson Line. It's defined as:

$$[x+s\bar{n}, x] \equiv \mathbb{P}\exp\left[ig\int_0^s ds'\, \underbrace{\bar{n}\cdot A(x+s'\bar{n})}\right]$$

Along with the large component.

non-Abelian

the operator $\mathbb{P}$ indicates the "Path ordering" of color matrices:

e.g. $\mathbb{P}[A(x)\cdot A(x+s\bar{n})] = A(x+s\bar{n})\,A(x)$ for any $s>0$

Wilson Lines gauge transform's behavior is:

$$[x+s\bar{n}, x] = V(x+s\bar{n})\,[x+s\bar{n}, x]\,V^\dagger(x)$$

therefore the products like this

$$\bar{\psi}(x+s\bar{n})\,[x+s\bar{n}, x]\,\psi(x)$$

are all invariant.

In SCET, it's customary to work with Wilson Lines which go to infinity.

$$W(x) \equiv [x, \underbrace{-\infty\bar{n}}] = \mathbb{P}\exp\left[ig\int_{-\infty}^0 ds\, \bar{n}\cdot A(x+s\bar{n})\right]$$

Assuming fields vanish at infinity, $x-\infty\bar{n} \Rightarrow -\infty\bar{n}$

$$\Rightarrow W(x+s\bar{n})\,W^\dagger(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^0 dt\, \bar{n}\cdot A(x+s\bar{n}+t\bar{n})\right]\,\overline{\mathbb{P}}\exp\left[-ig\int_{-\infty}^0 dt\, \bar{n}\cdot A(x+t\bar{n})\right]$$

$$= \mathbb{P}\exp\left[ig\int_0^s dt\, \bar{n}\cdot A(x+t\bar{n})\right] = [x+s\bar{n}, x]$$

Gauge transformation of $W(x)$: $W(x) \Rightarrow V(x)\,W(x)\,V^\dagger(-\infty\bar{n})$

If one assume gauge functions vanishing at infinity, it's $V(-\infty\bar{n}) = 1$

$\Rightarrow$ All forms like these are gauge invariant:

$$\chi(x) \equiv W^\dagger(x)\,\psi(x) \qquad ; \qquad \bar{\chi}(x) \equiv \bar{\psi}(x)\,W(x)$$

$\Rightarrow$ Building blocks to construct non-local operators.

* In addition, the covariant derivative of Wilson Lines along the integration path is exactly zero. In our case is:

$$\bar{n}\cdot D\, W(x) = 0\,.$$

Collinear Wilson Line: $W_c(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^{0} ds\ \bar{n}\cdot A_c(x+s\bar{n})\right]$ $\rightarrow$ base rock of constructing operators.

Soft Wilson Line: $S_n(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^{0} ds\ \bar{n}\cdot A_s(x+s\bar{n})\right]$ $\rightarrow$ soft structure

6. Decoupling Transform.

$$\mathcal{L}_{SCET} = \bar{\psi}_s\, i\not{D}\,\psi_s + \bar{\xi}\frac{\not{\bar{n}}}{2}\left(in\cdot D + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c}\, i\not{D}_{c\perp}\right)\xi - \frac{1}{4}\left(F^{s,a}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{c,a}_{\mu\nu}\right)^2$$

Consider leading order, only $\bar{\xi}\frac{\not{\bar{n}}}{2}\, in\cdot D\,\xi$ term contributes to interaction.

Redefine these field:

$$\xi(x) \rightarrow S_n(x_-)\cdot\xi^{(0)}(x)$$

$$A_c^\mu(x) \rightarrow S_n(x_-)\,A_c^{(0)\mu}(x)\,S_n^\dagger(x_-)$$

$$\begin{aligned}
in\cdot D\,\xi \longrightarrow\ & in\cdot D'\cdot S_n(x_-)\,\xi^{(0)}(x)\\
=\ & \left(in\cdot\partial + g\,n\cdot S_n(x_-)A_c^{(0)}(x)S_n^\dagger(x_-) + g\,n\cdot A_s(x_-)\right) S_n(x_-)\,\xi^{(0)}(x)\\
=\ & i\left\{\left[n\cdot\partial S_n(x_-)\right]\xi^{(0)}(x) + S_n(x_-)\left[n\cdot\partial\,\xi^{(0)}(x)\right]\right\} + \left[S_n(x_-)\,g\,n\cdot A_c^{(0)}(x) + S_n(x_-)\,g\,n\cdot A_s(x_-)\right]\xi^{(0)}(x)\\
=\ & \left[\underbrace{in\cdot D_- S_n(x_-)}_{\text{Wilson Lines}\ \to 0} + S_n(x_-)\,in\cdot\partial + S_n(x_-)\,g\,n\cdot A_c^{(0)}(x)\right]\xi^{(0)}(x)\\
=\ & S_n(x_-)\left(in\cdot\partial + g\,n\cdot A_c^{(0)}(x)\right)\xi^{(0)}(x)\\
\equiv\ & S_n(x_-)\; n\cdot D_c^{(0)}\,\xi^{(0)}(x)
\end{aligned}$$

derivatives along the path (x_-).

So under this redefinition, the soft field no longer appears in collinear Lagrangian.

$\longrightarrow$ Decoupling Transformation. (Also work in collinear Lagrangian, but just works @ LO)

In real problems, we should deal with two collinear directions.

E.g. QED current: $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \to \bar{\psi}_c(x)\gamma^\mu\psi_{\bar{c}}(x)$

non-local operator: $J^\mu \longrightarrow \int ds\int dt\, C_V(s,t)\,\bar{\chi}_c(x+s\bar{n})\,\gamma_\perp^\mu\,\chi_{\bar{c}}(x+tn)$

where: $\begin{cases} \chi_c = W_c^\dagger \xi_c \\ \chi_{\bar{c}} = W_{\bar{c}}^\dagger \xi_{\bar{c}} \end{cases}$, s.t. $\begin{cases} \not{n}\chi_c = 0 \\ \not{\bar{n}}\chi_{\bar{c}} = 0 \end{cases} \Rightarrow$ cancelled $\not{n}\frac{\bar{n}^\mu}{2}$ & $\not{\bar{n}}\frac{n^\mu}{2}$

gauge inv.

$\gamma^\mu = \not{n}\frac{\bar{n}^\mu}{2} + \not{\bar{n}}\frac{n^\mu}{2} + \gamma_\perp^\mu$

Applying decoupling transformations:

$$\chi_c(x) \to S_n(x_-)\,\chi_c^{(0)}(x)$$

$$\chi_{\bar{c}}(x) \to S_{\bar{n}}(x_+)\,\chi_{\bar{c}}^{(0)}(x)$$

$$\Rightarrow \quad J^\mu = \int ds\int dt\, C_V(s,t)\,\bar{\chi}_c^{(0)}(x+s\bar{n})\,S_n^\dagger(x_-)\,S_{\bar{n}}(x_+)\,\gamma_\perp^\mu\,\chi_{\bar{c}}^{(0)}(x)$$

$p^\mu = p_c^\mu + p_{\bar{c}}^\mu \sim (1,1,\lambda) \Rightarrow x^\mu \sim (1,1,\frac{1}{\lambda}) \Rightarrow x_+^\mu$ & x_-^μ are both small.

$$\Rightarrow \bar{\chi}_c^{(0)}(x+s\bar{n})\,S_n^\dagger(x_-) = \bar{\chi}_c^{(0)}(x_+ + x_\perp + s\bar{n})\,S_n^\dagger(0) + x_-\cdot\frac{\partial}{\partial x_-}\bar{\chi}_c^{(0)}(x+s\bar{n})\,S_n^\dagger(x_-)\Big|_{x_-=0} + \cdots$$

$$S_{\bar{n}}(x_+)\,\chi_{\bar{c}}^{(0)}(x+tn) = S_{\bar{n}}(0)\,\chi_{\bar{c}}^{(0)}(x_- + x_\perp + tn) + x_+\cdot\frac{\partial}{\partial x_+}S_{\bar{n}}(x_+)\,\chi_{\bar{c}}^{(0)}(x+tn)\Big|_{x_+=0} + \cdots$$

$$J^\mu = \int ds\int dt\, C_V(s,t)\,\underline{\bar{\chi}_c^{(0)}(x_+ + x_\perp + s\bar{n})}\,\underline{S_n^\dagger(0)\,S_{\bar{n}}(0)}\,\underline{\chi_{\bar{c}}^{(0)}(x_- + x_\perp + tn)} + \cdots$$

Seems to be "Factorized".

* Two definitions of factorization.

(i) Scale Separation. $\to$ Accutually we've done this.

(ii) No energy interactions. $\to$ Unfortunately, collinear parts are still interacting with soft parts.

7. Factorization and Collinear Anomaly.

As we've talked in section 3, the correlator can be written as:

$$G(p,l,\mu) = \tilde{C}(\bar{n}\cdot p, n\cdot l,\mu)\cdot J_c(p^2,\mu)\, J_{\bar{c}}(l^2,\mu)$$

but in massive problem, we can find it acts like this:

$$G(p,l,\mu) = \tilde{C}(\bar{n}\cdot p, n\cdot l, m,\mu)\, J_c\left(p^2, m^2, \ln\frac{\nu^2}{m^2},\mu\right) J_{\bar{c}}\left(l^2, m^2, \ln\frac{\nu^2}{Q^2},\mu\right)$$

Similar to the answer in section 2, where ν is the 't Hooft scale.

We can see, the product of two jet functions are independent of ν. It is,

$$P = J_c\left(p^2, m^2, \ln\frac{\nu^2}{m^2},\mu\right) J_{\bar{c}}\left(l^2, m^2, \ln\frac{\nu^2}{Q^2},\mu\right)$$

$$\Rightarrow \frac{d}{d\ln\nu}\ln P = \frac{d}{d\ln\nu}\left[\ln J_c\left(p^2, m^2, \ln\frac{\nu^2}{m^2},\mu\right) + \ln J_{\bar{c}}\left(l^2, m^2, \ln\frac{\nu^2}{Q^2},\mu\right)\right] = 0$$

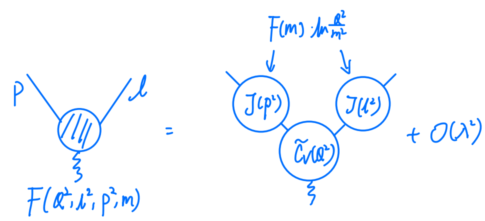
so, to exactly cancel the contribution of the scale ν, the coefficients of these logarithms must be opposite numbers. Therefore we can get:

→ because there only p or l in J_c and $J_{\bar{c}}$

$$\ln P = \ln J_c(p^2, m^2,\mu) + \ln J_{\bar{c}}(l^2, m^2,\mu) - F(m^2,\mu)\ln\frac{Q^2}{m^2}$$

so we can get the refactorize formula:

$$P = \left(\frac{Q^2}{m^2}\right)^{-F(m^2,\mu)} J_c(p^2, m^2,\mu)\, J_{\bar{c}}(l^2, m^2,\mu)$$



Why should we do this?

1. It extracts the divergence when $m \to 0$

2. It shows explicitly that the anomalous Q dependence exponentiates

8. Gauge Covariant Building Blocks.

In the previous section, we introduced the gauge invariant building blocks:

$$\chi(x) \equiv W^\dagger(x)\,\xi(x) = W^\dagger(x)\,\frac{\not{n}\not{\bar{n}}}{4}\,\psi(x)$$

But for the gauge field A^μ, we can only construct gauge COVARIANT terms, we note it as $\mathcal{A}^\mu$ for collinear gluon fields.

$$\mathcal{A}^\mu \equiv W^\dagger(x)\,(iD_c^\mu W(x))$$ ← we only take this into consider below.

or fields also contain soft part:

$$\mathcal{A}^\mu \equiv W^\dagger(x)\,(iD_c^\mu W(x)) + \frac{\bar{n}^\mu}{2}\left(W^\dagger(x)\, g\, n\cdot A_s(x_-)\,W(x) - g\, n\cdot A_s(x)\right)$$

How to understand? It adds the soft part to the collinear part ⟶ total gluon field.

$$A^\mu = (n\cdot A_c(x) + n\cdot A_s(x))\frac{\bar{n}^\mu}{2} + \frac{n^\mu}{2}\bar{n}\cdot A_c(x) + A_{c\perp}^\mu(x) = A_c^\mu + \boxed{\frac{\bar{n}^\mu}{2}\, n\cdot A_s}$$

From the definition above, we can easily get $\bar{n}\cdot\mathcal{A} = 0$ $(\bar{n}\cdot D\,W = 0)$

and $\quad n\cdot\mathcal{A} = W^\dagger(x)\,(in\cdot D_c W(x))$ $\quad \propto n\cdot P_c \sim \lambda^2 \to$ power-suppressed.

$$\mathcal{A}_\perp^\mu = W^\dagger(x)\,(iD_{c\perp}^\mu W(x)) \overset{\text{or}}{=\!=} W^\dagger(x)\,[iD_{c\perp}^\mu, W(x)]$$ these two are equivalent.

So, next step is to rewrite our collinear Lagrangian.

first we take a look at these expressions:

$$W^\dagger iD_c^\mu W = W^\dagger(i\partial^\mu + gA_c^\mu)W = W^\dagger[(i\partial^\mu + gA_c^\mu)W] + W^\dagger W\cdot i\partial^\mu$$

$$= W^\dagger(iD_c^\mu W) + i\partial^\mu = \mathcal{A}^\mu + i\partial^\mu \equiv i\mathcal{D}^\mu$$

and

$$W^\dagger i\bar{n}\cdot D_c W = W^\dagger(\overset{0}{\overbrace{i\bar{n}\cdot D_c W}}) + W^\dagger W\; i\bar{n}\cdot\partial$$

$$= i\bar{n}\cdot\partial$$

$$\Rightarrow \frac{1}{i\bar{n}\cdot D_c} = WW^\dagger(i\bar{n}\cdot D_c)^{-1}WW^\dagger = W\,\frac{1}{i\bar{n}\cdot\partial}\,W^\dagger$$

then

$$\mathcal{L}_c = \bar{\xi}\left(\frac{\not{n}}{2}\,in\cdot D_c + i\not{D}_{c\perp}\frac{\not{\bar{n}}}{2i\bar{n}\cdot D_c}\,i\not{D}_{c\perp}\right)\xi$$

$$= \bar{\chi}\frac{\not{n}}{2}\,in\cdot\mathcal{D}\,\chi + \bar{\chi}\,i\not{\mathcal{D}}_\perp\,\frac{1}{i\bar{n}\cdot\partial}\,i\not{\mathcal{D}}_\perp\frac{\not{\bar{n}}}{2}\,\chi$$

The gluon field:

$$W^\dagger F^{\mu\nu} W = \frac{1}{ig} W^\dagger [D_c^\mu, D_c^\nu] W = \frac{1}{g} (\partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu - i[\mathcal{A}^\mu, \mathcal{A}^\nu])$$

$\Rightarrow$ the Covariant field: $\mathcal{F}^{\mu\nu} = (\partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu - i[\mathcal{A}^\mu, \mathcal{A}^\nu])$

then the kenetic term: $-\frac{1}{4} F^{\mu\nu,a} F^a_{\mu\nu} = -\frac{1}{2} tr[F^{\mu\nu} F_{\mu\nu}] = -\frac{1}{2} tr[W^\dagger F^{\mu\nu} F_{\mu\nu} W]$

$$= -\frac{1}{2g^2} tr[\mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}]$$

where $F^{\mu\nu} = F^{\mu\nu,a} t^a \Rightarrow tr[F^{\mu\nu} F_{\mu\nu}] = F^{\mu\nu,a} F^b_{\mu\nu}\, tr[t^a t^b]$

$$= F^{\mu\nu,a} F^b_{\mu\nu}\, C_F \cdot \delta^{ab} \qquad (\text{for } SU(3),\ C_F = \tfrac{1}{2})$$

$$= \frac{1}{2} F^{\mu\nu,a} F^a_{\mu\nu}$$

6. Position Space Versus Label Formalism

Assuming a heavy quark in a meson, which moves as the velocity of v^μ. Note the mass of heavy quark as m_Q. (Heavy Quark ET)

its momentum should be $q^\mu = m_Q v^\mu + k^\mu$ — comes from interacting terms. (small $\sim \Lambda_{QCD}$)

then one can split off the large part of the momentum from the field:

$$Q(x) = e^{-i m_Q v\cdot x} h_v(x) = e^{-i q_Q \cdot x} h_{q_Q}(x)$$

Lable the removed momentum which is $m_Q v$

These are process independent, we can use them directly in SCET.

$$P_c^\mu = q^\mu + r^\mu \ ; \qquad q^\mu = q_-^\mu + q_\perp^\mu$$

collinear

$$\Rightarrow \quad A^\mu(x) = A_c^\mu(x) = \sum_q e^{-iq\cdot x} A_q^\mu(x) \qquad \text{— gluon}$$

$$\chi(x) = \sum_q e^{-iq\cdot x} \chi_q(x) \qquad \text{— quark.}$$

* Lable-operator: $\mathcal{P}$ —— exact the lable.

$$\mathcal{P}^\nu A_q^\mu(x) = q^\nu A_q^\mu(x)$$

we can do this replace in Lagrangian:

$$A^\mu \to A_q^\mu \ ; \quad \chi \to \chi_q \ ; \quad p^\mu \longrightarrow \mathcal{P}^\mu$$

$$\Rightarrow \mathcal{L}_c = \bar{\chi}_q \frac{\bar{n}\!\!\!/}{2} (i n\cdot\partial + n\cdot A_k)\chi_{q'} + \bar{\chi}_q (i \mathcal{P}\!\!\!/^\perp + A\!\!\!/_k^\perp) \frac{1}{i\bar{n}\cdot\mathcal{P}} (i\mathcal{P}\!\!\!/^\perp + A\!\!\!/_{k'}^\perp) \frac{\bar{n}\!\!\!/}{2} \chi_{q'}$$

where it's implied the sum over all momenta.

then the Wilson Line:

$$W = \sum \exp\left(-g \frac{1}{\bar{n}\cdot\mathcal{P}} A_{c,q}\right)$$