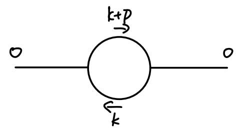
I : Effective Field Theory.

Why EFT?

For some integrals are hard to calculate directly.
Or itself contains divergence.

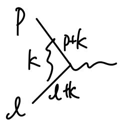
e.g. 

threshold.

$$= \int d^4k \cdot \frac{1}{(k^2-m^2)(k^2-m^2)}$$

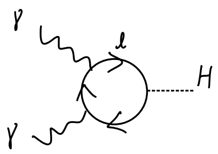
$$= 4\pi \cdot \int \frac{k^3 dk}{(k^2-m^2)^2}$$

$\rightarrow$ divergent.

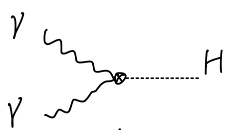
or. 

$$\propto \int d^4k \frac{1}{k^2 (k+l)^2 (k+p)^2} \rightarrow \text{hard to calculate.}$$

$\rightarrow$ contain angular analysis.

or. 

$\longrightarrow$



$\downarrow$

Effective Vertex

1. Regions Strategy.

Ex. 1. $I = \int_0^{+\infty} dk \frac{k}{(k^2+m^2)(k^2+M^2)}$ (Assuming $M \gg m$)

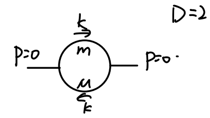
$= \frac{1}{2}\int_0^{+\infty} \frac{dk^2}{(k^2)^2+(m^2+M^2)k^2+m^2M^2}$

$= \frac{1}{2}\int_0^{+\infty} \frac{dk^2}{\left(k^2+\frac{m^2+M^2}{2}\right)^2-\left(\frac{m^2-M^2}{2}\right)^2}$

$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{m^2-M^2}\cdot\int_{\frac{m^2+M^2}{m^2-M^2}}^{+\infty}\frac{dx}{x^2-1}$

$= \frac{1}{m^2-M^2}\cdot\left(\frac{1}{2}\ln 1 - \frac{1}{2}\ln\frac{2M^2}{2m^2}\right)$

$= \frac{\ln M - \ln m}{M^2-m^2}$



order of N : 2
order of D : 4 $\rightarrow$ convergent.

$\int\frac{dx}{x^2-1} = \frac{1}{2}\int\frac{1}{x-1}-\frac{1}{x+1}\,dx$
$= \frac{1}{2}\ln\frac{x-1}{x+1} + C$

We can expand the result with $\frac{M}{m} \gg 1$

$I = \frac{\ln\frac{M}{m}}{M^2}\frac{1}{1-\frac{m^2}{M^2}} = \frac{\ln\frac{M}{m}}{M^2}\left(1+\frac{m^2}{M^2}+\frac{m^4}{M^4}+\cdots\right)$ → Large Logs.

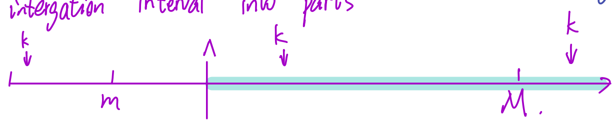
But, what if we just expand the intergrand? (Assuming m is quite small)

$\Rightarrow \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{k}{k^2(k^2+M^2)}\left(1-\frac{m^2}{k^2}+\frac{m^4}{k^4}+\cdots\right)$

then integrate it from 0 to $+\infty$, every term divergent.

Why?

Divide the integration interval into parts "Cut-off regularization"



The expansion above only works for $k \gg m$ (the Miku Blue region).

We can expand the integrands with $\frac{k}{M} \ll 1$, it is valid (since $m \ll M$).

$\Rightarrow I = \underbrace{\int_0^{\Lambda} dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{\text{low-energy / Infra-red}} + \underbrace{\int_{\Lambda}^{+\infty} dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{\text{high-energy / ultra-violet}}$

(with $m \sim k \ll M$) I_1 (with $m \ll k \sim M$) I_2

$\Rightarrow I_1 = \int_0^{\Lambda} dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^{\Lambda} dk \frac{k}{(k^2+m^2)\cdot M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\cdots\right)$ → All terms are covergent

$I_2 = \int_{\Lambda}^{+\infty} dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_{\Lambda}^{+\infty} dk \frac{k}{k^2(k^2+M^2)}\left(1-\frac{m^2}{k^2}+\frac{m^4}{k^4}+\cdots\right)$ → Also covergent

Infra-red Cut-off ($m \ll \Lambda \ll M$)

So, we can calculate I_1, I_2 independently. To emphasize, the final answer couldn't contain "Λ" terms.

$$I_1 = \frac{1}{2M^2}\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{1}{2M^4}\left(\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right) + O\left(\frac{\Lambda^4}{M^6},\ \frac{\Lambda^2 m^2}{M^6},\ \frac{m^4}{M^6}\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right)$$

$$= \frac{M^2+m^2}{2M^4}\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{2M^4}$$

$$I_2 = \frac{1}{2M^2}\ln\left(1+\frac{M^2}{\Lambda^2}\right) - \frac{m^2}{2\Lambda^2 M^4}\left(M^2 - \Lambda^2\ln\left(1+\frac{M^2}{\Lambda^2}\right)\right) + O\left(\frac{m^4}{M^4 L^2},\ \frac{m^4}{M^6}\ln\left(1+\frac{M^2}{\Lambda^2}\right)\right)$$

$$= \frac{M^2+m^2}{2M^4}\ln\left(1+\frac{M^2}{\Lambda^2}\right) - \frac{m^2}{2\Lambda^2 M^2} + O(\cdots)$$

Second expand I_1, I_2 with $m \ll \Lambda \ll M$; $\ln\left(1+\frac{\Lambda^2}{m^2}\right) = \ln\frac{\Lambda^2}{m^2} + \ln\left(1+\frac{m^2}{\Lambda^2}\right)$

$$I_1 = \frac{M^2+m^2}{2M^4}\ln\frac{\Lambda^2}{m^2} + \frac{M^2+m^2}{2M^4}\cdot\frac{m^2}{\Lambda^2} - \frac{\Lambda^2}{2M^4}$$

$$I_2 = \frac{M^2+m^2}{2M^4}\ln\frac{M^2}{\Lambda^2} + \frac{M^2+m^2}{2M^4}\cdot\frac{\Lambda^2}{M^2} - \frac{m^2}{2\Lambda^2 M^2}$$

$$\Rightarrow\ I_1 + I_2 = \frac{\ln\frac{M}{m}}{M^2} + \frac{m^2}{M^4}\ln\frac{M}{m} + \underline{\frac{m^2}{2M^4}\left(\frac{m^2}{\Lambda^2}+\frac{\Lambda^2}{M^2}\right)}$$

high order (can be cancelled by calculating high order terms)

leading order ; next-leading order.

↳ do not contain "Λ".

$\Rightarrow I = I_1 + I_2$ (equals in terms)

Goods : "Physical", easy to calculate, explicit.

Bads : needs to deal with more integrals, break some symmetries (e.g. Gauge symmetry)

"Dimensional regularization"

('t Hooft scale) mass dimension counter term; dimension

$$I = \int_0^{+\infty} dk\,\frac{k}{(k^2+m^2)(k^2+M^2)} \longrightarrow \mu^{\varepsilon}\int_0^{+\infty} d^{1-\varepsilon}k\,\frac{k}{(k^2+m^2)(k^2+M^2)}$$

using $d^N k = (N-1)\,dk\cdot d\Omega_N$; where Ω_N is the N dimension angular variables / sphere surface,

here we ignore the angular contributions (Actually, the contribution of $\Omega_{1-\varepsilon}$ is nearly "1")

$$\left(\Omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \Rightarrow \Omega_{1-\varepsilon} = \frac{2\pi^{\frac{1-\varepsilon}{2}}}{\Gamma(\frac{1-\varepsilon}{2})} \approx 2 \rightarrow \text{for } k>0 \Rightarrow \int d\Omega_{1-\varepsilon} = \frac{1}{2}\times\Omega_{1-\varepsilon} = 1\right)$$

$$\Rightarrow\ I = \mu^{\varepsilon}\int_0^{+\infty} dk\cdot k^{-\varepsilon}\frac{k}{(k^2+m^2)(k^2+M^2)}$$

(This ε is a hand-put and arbitrary parameter, just make integrals convergent.

$$I_{(I)} = \mu^{\varepsilon}\int_0^{+\infty} dk\, k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right)$$ → UV safe

At least the first term is covergent.

$$I_{(II)} = \mu^{\varepsilon}\int_0^{+\infty} dk\, k^{-\varepsilon}\frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right)$$ → IR safe.

At first term:

$$I_{(I)} = \frac{1}{M^2\varepsilon} - \frac{\ln\frac{m}{\mu}}{M^2} + O(\varepsilon)$$

$$I_{(II)} = -\frac{1}{M^2\varepsilon} + \frac{\ln\frac{M}{\mu}}{M^2} + O(\varepsilon)$$

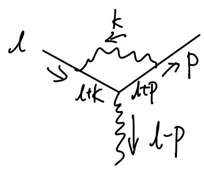
$$\Rightarrow\ I = I_{(I)} + I_{(II)} = \frac{1}{M^2}\ln\frac{M}{m}$$ → leading order in expansion of "ε".

Actually, we do the integral without considering the convergence radius of Maclaurin series.
↳ the "power" of regularization is not enough.
That is the reason why the other terms are divergent.
All also, we integrate the high-energy part over IR part and vice versa, but this part is trivial.

Since, $$I^{\Lambda}_{(1)} = \int_0^{\Lambda} dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\cdots\right)$$
$$= \left[\int_0^{+\infty} dk - \int_{\Lambda}^{+\infty} dk\right] k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\cdots\right)$$
$$= I_{(1)} - R_{(1)} \quad \to \text{Rest}$$
$$R_{(1)} = \int_{\Lambda}^{+\infty} dk\, k^{-\varepsilon} \frac{k}{k^2 M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\cdots\right) \qquad (k \gg m)$$
do the same, we get $$R_{(11)} = \int_0^{\Lambda} dk\, k^{-\varepsilon} \frac{k}{k^2 M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\cdots\right)$$
$$\Rightarrow R_{(11)}+R_{(1)} = \int_0^{+\infty} dk\, k^{-\varepsilon} \frac{k}{k^2 M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\cdots\right)$$ → do not depend on the cut-off.
↳ gives the scaleless integrals. ⇒ $R_{(1)}+R_{(11)}=0$
"Zero-bin" contribution

one can check by integrate out this integral: (1st term: $\int_0^{\Lambda} dk \frac{k^{-1-\varepsilon}}{M^2} + \int_{\Lambda}^{+\infty} dk \frac{k^{-1-\varepsilon}}{M^2} = -\frac{\Lambda^{-\varepsilon}}{\varepsilon M^2} + \frac{\Lambda^{-\varepsilon}}{\varepsilon M^2} = 0$ ✓)
It seems neither easy nor explicit, but it's useful for the intrinsic divergent integrals.

2. Sudakov Problem.



The general stratagy to obtain the expansion of a given Feynman diagram in a given kinematic limit is:

(i) Identify all regions of the integrand which leads to sigularities (UV/IR, hard/soft ⋯)
(ii) Expand the integrand in each region, then integrate the expansion over the full phase space. $(I_1, I_2 \cdots)$
(iii) Add all the results of different region's integrations to obtain the expansion of the original full integral. $(I = I_1 + I_2 + \cdots)$

The Feynman diagrams above can write as (with the dimension of "d")
$$I = i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^d k \frac{1}{(k^2+i\epsilon)[(k+l)^2+i\epsilon][(k+p)^2+i\epsilon]}$$

(it has IR divergence when $k \to 0$)

* Here we introduce the light-cone coordinate.

the four basis: $n_\mu=(1,0,0,1)$; $\bar{n}_\mu=(1,0,0,-1)$, $n^x_\mu=(0,1,0,0)$, $n^y_\mu=(0,0,1,0)$.

usually combine n^x and n^y together as $n_\perp$

we have $n\cdot\bar{n}=\bar{n}\cdot n=2$; $n\cdot n=\bar{n}\cdot\bar{n}=0$; $n\cdot n_\perp=\bar{n}\cdot n_\perp=0$

an arbitary vector, let: $k_\mu = A n_\mu + B\bar{n}_\mu + C n^\perp_\mu$

$$k\cdot n = 2B \Rightarrow B=\frac{k\cdot n}{2}$$

$$k\cdot\bar{n} = 2A \Rightarrow A=\frac{k\cdot\bar{n}}{2}$$

$$k\cdot n^\perp = C \Rightarrow C=(k_x,k_y)=k_\perp \quad \Rightarrow \text{Combine } k_\perp\cdot n^\perp_\mu = k^\perp_\mu = (0,k_x,k_y,0)$$

$$\Rightarrow k_\mu = k\cdot n\,\frac{\bar{n}_\mu}{2} + k\cdot\bar{n}\,\frac{n_\mu}{2} + k^\perp_\mu \equiv k_+\frac{\bar{n}_\mu}{2} + k_-\frac{n_\mu}{2} + k^\perp_\mu \Rightarrow (k_+,k_-,k_\perp)$$

Light-cone coordinate is usually used in large momentum or near light speed problems.

In the light-cone coordinate $p\cdot q=\frac{1}{2}(p_+q_-+p_-q_+)+p_\perp^2 = \frac{1}{2}\left[(p\cdot\bar{n})(q\cdot n)+(q\cdot\bar{n})(p\cdot n)\right]+p_\perp^2$

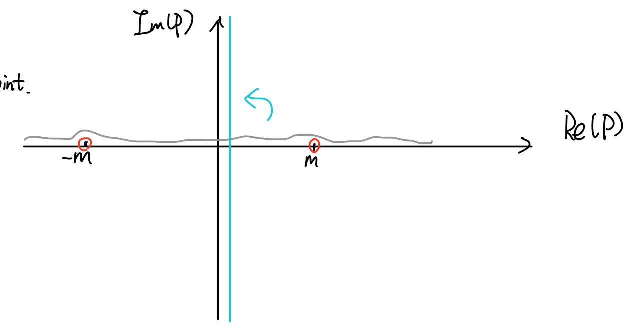
Introduce the following notation:

the inject momentum

$$L^2\equiv -l^2-i\epsilon \quad ; \quad P^2\equiv -p^2-i\epsilon \quad ; \quad Q^2\equiv -(l-p)^2-i\epsilon$$

Why "$-$"?

Avoid braching point.



Then we can introduce our expansion parameter λ

$$\lambda^2\sim\frac{P^2}{Q^2}\sim\frac{L^2}{Q^2} \quad \& \quad p^2\sim l^2\sim\lambda^2 Q^2$$

Choose the large component of momentum flow as the reference vector.

$$p^\mu\sim Q\frac{n^\mu}{2} \quad ; \quad l^\mu\sim Q\frac{\bar{n}^\mu}{2}$$

$\Rightarrow p^\mu\sim(\lambda^2,1,\lambda)Q$; $l^\mu\sim(1,\lambda^2,\lambda)Q$ $\rightarrow$ Arbitary, like $q\sim(\lambda^a,\lambda^b,\lambda^c)Q$

but it's necessary to make sure $q^2\sim\lambda^n Q^2$

$\Leftrightarrow \quad a+b=2c$

in this Sudakov problem, only four regions below contributes.

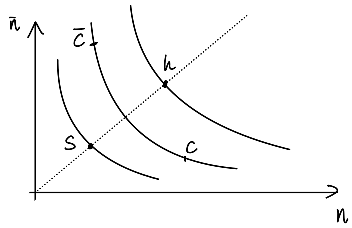
(i) **Hard region (h)** : $k_h^\mu \sim (1,1,1)Q$

(ii) **Region Collinear to p (c)** : $k_c^\mu \sim (\lambda^2, 1, \lambda)Q$

(iii) **Region Collinear to l ($\bar{c}$)** : $k_{\bar{c}}^\mu \sim (1, \lambda^2, \lambda)Q$ (in two-body problem, p & l are always back-to-back so it's also call anti-collinear)

(iv) **Soft region (s)** : $k_s^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$

Region	Theory	Scale
$\boxed{\text{Hard}}$	$\leftrightarrow$ QCD $\leftrightarrow$	Q
$\boxed{\text{Collinear}}$	$\leftrightarrow$ SCET $\leftrightarrow$	M
$\boxed{\text{Soft}}$	$\leftrightarrow$ SET $\leftrightarrow$	M^2/Q



$\boxed{\text{why not soft region looks like } k^\mu \sim (\lambda, \lambda, \lambda)Q}$ → semi-hard

consider $(k_s+k_c)^2 = (k_s+k_c)_+(k_s+k_c)_- + (k_s+k_c)_\perp^2$

$$= \underbrace{k_s^+ k_s^-}_{O(\lambda^2)} + \underbrace{k_s^+ k_c^-}_{O(\lambda)} + \underbrace{k_s^- k_c^+}_{O(\lambda^3)} + \underbrace{k_c^+ k_c^-}_{O(\lambda^2)} + \underbrace{(k_s+k_c)_\perp^2}_{O(\lambda^2)} \sim O(\lambda)$$

since the observables of collinear momentum is $L^2 \sim \lambda^2 Q^2$
there should be $(k_s+k_c)^2 \sim L^2 \sim O(\lambda^2)$
so it breaks this observation order. If $k_s^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$
then $(k_s+k_c)^2 \sim k_c^2 + k_s^+ \cdot k_c^- \sim O(\lambda^2)$ reserves order.

We have $k_h^2 \sim Q^2$; $\underbrace{k_c^2 \sim k_{\bar{c}}^2 \sim \lambda^2 Q^2 ; k_s^2 \sim \lambda^4}_{\text{order hierachy}} Q^2$ → ultra-soft $\Rightarrow$ SCET$_I$

SCET$_I$: $p_s^2 \sim \lambda^4$ (ultra-soft)

SCET$_{II}$: $p_s^2 \sim \lambda^2$ (semi-hard)

Only these four contributes, other contributions vanished.

take the semi-hard region as the example. $\left(l_+^\mu = (1,0,0)\frac{Q}{2} ;\ p_-^\mu = (0,1,0)\frac{Q}{2} \right.$
$\left. k_- l_+ = k \cdot l_+ \ ;\ k_+ p_- = k \cdot p_- \right)$

$$I_{sh} = i\pi^{\frac{d}{2}} \mu^{4-d} \int d^d k \frac{1}{(k^2+i0)(2k \cdot l_+ + i0)(2k \cdot p_- + i0)}$$

$$= i\pi^{\frac{d}{2}} \mu^{4-d} \int d^d k\, dx_1 dx_2 \frac{2}{(k^2 + 2k(x_1 l_+ + x_2 p_-))^3}$$

$$= \cancel{i\pi^{\frac{d}{2}}} \mu^{4-d} \frac{(-1)^3 \cancel{i\pi^{\frac{d}{2}}} \Gamma(3-\frac{d}{2})}{\Gamma(3)} \int dx_1 dx_2 \frac{2}{((x_1 l_+ + x_2 p_-)^2)^{3-\frac{d}{2}}}$$

$$= \mu^{2\varepsilon} \Gamma(1+\varepsilon) (l_+ p_-)^{-1-\varepsilon} \int_0^{+\infty} dx_1 dx_2 \, (x_1 x_2)^{-1-\varepsilon}$$

$= 0$

— Feynman Parameter 2

$$\frac{1}{ABC} = \int_0^{+\infty} dx_1 \int_0^{+\infty} dx_2 \frac{2}{(A + Bx_1 + Cx_2)^3}$$

(I) Hard Region:

$$(k+l)^2 = k^2 + 2l_+ k_- + O(\lambda)$$
$$(k+p)^2 = k^2 + 2p_- \cdot k_+ + O(\lambda)$$

$$I_h = i\pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{1}{k^2 (k^2 + 2k_- l_+)(k^2 + 2k_+ \cdot p_-)}$$

$$= i\pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k\, dx dy \frac{2}{[k^2 + 2k \cdot (x l_+ + y p_-) + (x l_+ + y p_-)^2 - xy\, l_+ p_-]^3}$$

$$= \cancel{i\pi^{-\frac{d}{2}}} \mu^{4-d} \int_0^1 dx dy \cdot \cancel{(-1)^3 \cdot i \pi^{\frac{d}{2}}} \Gamma(3-\tfrac{d}{2}) \left(\frac{1}{2xy\, l_+ p_-} \right)^{3-\frac{d}{2}}$$

$$= \frac{\mu^{2\varepsilon} \Gamma(1+\varepsilon)}{(2 l_+ p_-)^{1+\varepsilon}} \int_0^1 dx dy \, (xy)^{-1-\varepsilon}$$ → All from 0 to 1

$$= \frac{\Gamma(1+\varepsilon)}{2 l_+ p_-} \cdot \left(\frac{\mu^2}{2 l_+ p_-} \right)^{\varepsilon} \cdot \frac{1}{\varepsilon^2} = \frac{\Gamma(1+\varepsilon)}{Q^2} \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} + O(\varepsilon) \right)$$

— Feynman Parameter 1.

$$\frac{1}{ABC} = \int_0^1 dx_1 dx_2 dx_3 \frac{2\,\delta(1-x_1-x_2-x_3)}{(x_1 A + x_2 B + x_3 C)^3}$$
$$= \int_0^1 dx dy \frac{2}{((1-x-y)A + xB + yC)^3}$$

$\begin{cases} l_+ \sim Q \\ p_- \sim Q \end{cases} \Rightarrow 2 l_+ \cdot p_- = 2 \times \frac{1}{2} Q^2 = Q^2$

Or we can change our parameter to: $x_1=y;\ \underbrace{x_2=x-y}_{\text{gives } y\le x};\ x_3=1-x$

$$I_h=\mu^{2\varepsilon}\int_0^1 dx\,dy\,\Gamma(1+\varepsilon)\left(\frac{1}{2l_+P_-(1-x)y}\right)^{1+\varepsilon}$$

$$=\frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\cdot\boxed{\int_0^1 dx\int_0^x dy\,[(1-x)y]^{-1-\varepsilon}}=-\frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\varepsilon\,\Gamma(1-2\varepsilon)}=\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$=\frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}\boxed{-\frac{\pi^2}{6}}+O(\varepsilon)\right)$$

alternative form

$$=\frac{1}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{12}+O(\varepsilon)\right)$$

* So we can see that different parameter choice can derive diffent kind of constant term, but there scale-relying terms and dimensional divergence terms are all the same.

(II) Collinear Region. ($k^\mu\sim(\lambda^2,1,\lambda)Q$, or $k\parallel p$)

$(k+l)^2=2k_-l_++O(\lambda^2)$ → Only consider the leading order.

$(k+p)^2=k^2+2k\cdot p+p^2=O(\lambda^2)$

$$I_c=i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk\,\frac{1}{k^2(2k_-l_+)(k^2+2k\cdot p+p^2)}$$ — Feynman Parameter 2

$$=i\pi^{-\frac{d}{2}}\mu^{4-d}\int_0^{+\infty}dx\,dy\int d^dk\,\frac{2}{(k^2+2x\,k_-l_++yk^2+2yk\cdot p-yp^2)^3}$$

$$=i\pi^{-\frac{d}{2}}\mu^{4-d}\int_0^{+\infty}dx\,dy\int d^dk\,\frac{2}{\left\{(1+y)\left[k^2+2k\cdot\left(\frac{xl_++y\cdot p}{1+y}\right)-\frac{yp^2}{1+y}\right\}^3}$$

using space integration formula:

$$\int d^dk\,\frac{1}{(k^2+2k\cdot Q-M^2)^\alpha}=(-1)^\alpha\frac{i\pi^{\frac{d}{2}}}{(M^2+Q^2)^{\alpha-\frac{d}{2}}}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}$$

$$=\cancel{i}\cancel{\pi^{-\frac{d}{2}}}\mu^{4-d}\frac{\cancel{(-1)^3}\cancel{i}\cancel{\pi^{\frac{d}{2}}}\Gamma(3-\frac{d}{2})\cdot\cancel{2}}{\cancel{\Gamma(3)}}\int_0^{+\infty}dx\,dy\,\frac{1}{(1+y)^3\left(\frac{-yp^2}{(1+y)}+\frac{2xyl_+P_-+y^2p^2}{(1+y)^2}\right)^{3-\frac{d}{2}}}$$

$$=\mu^{2\varepsilon}\Gamma(1+\varepsilon)\int_0^{+\infty}dx\,dy\,\frac{1}{(1+y)^{1-2\varepsilon}\left[-y\,p^2+2xy\,l_+P_-\right]^{1+\varepsilon}}$$

$$=\mu^{2\varepsilon}\Gamma(1+\varepsilon)\int_0^{+\infty}dx\,dy\,\frac{1}{(1+y)^{1-2\varepsilon}\,y^{1+\varepsilon}\,[P^2+x\,Q^2]^{1+\varepsilon}}$$

Capital "P"

let $r=\frac{Q^2}{P^2}$

$$=\left(\frac{\mu^2}{P^2}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{P^2}\int_0^{+\infty}dy\,\frac{1}{(1+y)^{1-2\varepsilon}\,y^{1+\varepsilon}}\int_0^{+\infty}dx\,\frac{1}{(1+rx)^{1+\varepsilon}}$$

$$=\frac{\Gamma(1+\varepsilon)}{P^2}\left(\frac{\mu^2}{P^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot\frac{1}{\varepsilon\cdot r}$$

$$=\frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{\mu^2}{P^2}\right)^{\varepsilon}\left[-\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\right]$$

$$AF=\frac{1}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{12}\right)$$

$$=\frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\underbrace{\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}}_{\text{Collinear scale}}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}+O(\varepsilon)\right)$$

Similarly:

$$I_{\bar{c}}=\frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}+O(\varepsilon)\right)$$

$$AF=\frac{1}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{12}\right)$$

(III) Soft Region

$$(k+p)^2 = 2k_+ p_- + p^2 + O(\lambda^3)$$
$$(k+l)^2 = 2k_- l_+ + l^2 + O(\lambda^3)$$

$$I_s = i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk \frac{1}{k^2\,(2k_- l_+ + l^2)\,(2k_+ p_- + p^2)}$$

$$= i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk \int_0^{+\infty} dx\,dy \frac{2}{(k^2 + 2xk_- l_+ + 2yk_+ p_- + xl^2 + yp^2)^3}$$

$$= \mu^{2\varepsilon}\,\Gamma(1+\varepsilon)\int_0^{+\infty} dx\,dy \frac{1}{\left(-(xl^2+yp^2) + 2xy\,l_+ p_-\right)^{1+\varepsilon}}$$

$xL^2 \to x'$

$yP^2 \to y'$

$$= \mu^{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{L^2P^2}\int_0^{+\infty} dx\,dy\,(x + y + a\,xy)^{-1-\varepsilon}$$

let $a = \frac{Q^2}{L^2P^2}$

$$= \mu^{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{L^2P^2}\int_0^{+\infty} dx \int_0^{+\infty} du\; x^{-\varepsilon}(1+u+axu)^{-1-\varepsilon}$$

$\begin{cases} u = \frac{y}{x} \leftarrow \text{Parameter.} \\ dy = x\,du. \end{cases}$

$$= \mu^{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{L^2P^2}\int_0^{+\infty} dx \int_0^{+\infty} du\; x^{-\varepsilon}(1+u+axu)^{-1-\varepsilon}$$

$v = (1+ax)u.$

$dv = (1+ax)du.$

$$= \mu^{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{L^2P^2}\int_0^{+\infty}\frac{x^{-\varepsilon}}{1+ax}dx\int_0^{+\infty} dv\,(1+v)^{-1-\varepsilon}$$

$$= \mu^{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{L^2P^2}\,a^{-1+\varepsilon}\,\Gamma(1-\varepsilon)\Gamma(\varepsilon)\cdot\frac{1}{\varepsilon}$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{\mu^2Q^2}{L^2P^2}\right)^{\varepsilon}\left[-\Gamma(\varepsilon)\cdot\Gamma(-\varepsilon)\right]$$

$AF = \frac{1}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2} + \frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2} + \frac{\pi^2}{4}\right)$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2} + \frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2} + \frac{\pi^2}{6} + O(\varepsilon)\right)$$

A.F. just have differences in constants.

$$I_h = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\cancel{\frac{1}{\varepsilon^2}} + \cancel{\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \cancel{\frac{\pi^2}{6}}\right)$$

$$I_c = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\cancel{\frac{1}{\varepsilon^2}} - \cancel{\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}} - \frac{1}{2}\ln^2\frac{\mu^2}{P^2} + \cancel{\frac{\pi^2}{6}}\right)$$

$$I_{\bar{c}} = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\cancel{\frac{1}{\varepsilon^2}} - \cancel{\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}} - \frac{1}{2}\ln^2\frac{\mu^2}{L^2} + \frac{\pi^2}{6}\right)$$

$$I_s = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\cancel{\frac{1}{\varepsilon^2}} + \cancel{\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}} + \frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2} + \frac{\pi^2}{6}\right)$$

UV-divergence

IR-divergence

cancelled.

$$\Rightarrow I = I_h + I_c + I_{\bar{c}} + I_s \xlongequal{\text{Mathematically}} \frac{1}{Q^2}\left[\frac{\pi^2}{3} + \frac{1}{2}\left[(\ln\mu^2 - \ln Q^2)^2 - (\ln\mu^2 - \ln P^2)^2 - (\ln\mu^2 - \ln L^2)^2 + (\ln\mu^2 + \ln Q^2 - \ln P^2 - \ln L^2)^2\right]\right]$$

Imagine its expanding form.

$\ln^2\mu^2$, $\ln^2P^2$, $\ln^2Q^2$ terms vanished, $\ln\mu^2\cdot\ln X^2$ terms were also cancelled

$$= \frac{1}{Q^2}\left[\frac{\pi^2}{3} + \frac{1}{2}\left(2\ln^2Q^2 - 2\ln Q^2\ln P^2 - 2\ln Q^2\ln L^2 + 2\ln P^2\ln L^2\right)\right]$$

$$\xlongequal{\text{Factorise}} \frac{1}{Q^2}\left(\frac{\pi^2}{3} + \ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + O(\lambda)\right)$$

3. Massive Sudakov Problem (the virtual particle has mass of "m")

In this case, soft contribution vanishes.

That is:

$$m^2 \sim \lambda^2 Q^2$$

$$I_s = i\pi^{\frac{d}{2}} \mu^{4-d} \int d^d k \frac{1}{(k^2 - m^2 + i\epsilon)[(k+l)^2 + i\epsilon][(k+p)^2 + i\epsilon]} \quad (\text{where } k^2 \ll m^2)$$

$$\underline{\underline{LO}} \; - i\pi^{\frac{d}{2}} \mu^{4-d} \int d^d k \frac{1}{m^2 (2k_- l_+ + l^2)(2k_+ P_- + p^2)} \rightarrow \text{scaleless.}$$

Since soft region vanishes and hard $(k \gg m)$ region remains its former form, if the (anti-)collinear regions also remain unchanged, the divergence could no more be cancelled. So we should intoduce new regulators —— "Analytic regulator".

$$I = i\pi^{\frac{d}{2}} \mu^{4-d} \int d^d k \frac{\overbrace{(-\nu^2)}^{\text{"t Hooft scale"}}{}^{\overbrace{\alpha}^{\text{Analytic Regulator}}}}{(k^2 - m^2 + i\epsilon)[(k+l)^2 + i\epsilon][(k+p)^2 + i\epsilon]^{1+\alpha}}$$

For simplicity, we let the two outer legs be onshell and a mass of "0" $\Rightarrow l^2 = p^2 = 0$

Feynman Parameter 3.

$$\frac{1}{ABC^{1+\alpha}} = \int_0^\infty dx_1 \int_0^\infty dx_2 \frac{(2+\alpha)(1+\alpha)}{(Ax_1 + Bx_2 + C)^{3+\alpha}}$$

$$I_c = i\pi^{-\frac{d}{2}} \mu^{4-d} \int d^d k \frac{(-\nu^2)^\alpha}{(k^2 - m^2 + i\epsilon)[2k_- l_+ + i\epsilon][(k+p)^2 + i\epsilon]^{1+\alpha}}$$

$$= i\pi^{-\frac{d}{2}} \mu^{4-d} \int_0^\infty dx dy \int d^d k \frac{(2+\alpha)(1+\alpha)(-\nu^2)^\alpha}{\left[(1+x)\left(k^2 + 2k\cdot\left(\frac{y l_+ + P}{1+x}\right) - \frac{x m^2}{1+x}\right)\right]^{3+\alpha}}$$

$$= \cancel{i\pi^{-\frac{d}{2}}} \mu^{4-d} \frac{(-1)^{3+\alpha} \cdot \cancel{i\pi^{\frac{d}{2}}} \Gamma(3+\alpha-\frac{d}{2})(-\nu^2)^\alpha}{\Gamma(1+\alpha)} \int_0^\infty dx dy \left(\frac{x m^2}{1+x} + \frac{2y \cdot l_+ P_-}{(1+x)^2}\right)^{\frac{d}{2}-3-\alpha} \cdot \frac{1}{(1+x)^{3+\alpha}}$$

$$= \frac{(-1)^\alpha \mu^{2\varepsilon} \Gamma(1+\alpha+\varepsilon)(-\nu^2)^\alpha}{\Gamma(1+\alpha)} \int_0^\infty \frac{1}{(1+x)^{1-\alpha-2\varepsilon}} \left(m^2 x(x+1) + Q^2 \cdot y\right)^{-1-\alpha-\varepsilon} dx dy.$$

Integrate out y.

$$\int_0^\infty dy \frac{1}{(A+By)^{1+\alpha+\varepsilon}} = \frac{1}{(\alpha+\varepsilon)} \frac{1}{B \cdot A^{\alpha+\varepsilon}}$$

$$= \frac{\mu^{2\varepsilon} \Gamma(\alpha+\varepsilon) \nu^\alpha}{\Gamma(1+\alpha)} \int_0^\infty dx \frac{1}{(1+x)^{1-\alpha-2\varepsilon}} \frac{1}{Q^2 \cdot (m^2)^{\alpha+\varepsilon} \cdot x^{\alpha+\varepsilon} \cdot (x+1)^{\alpha+\varepsilon}}$$

$$= \frac{\Gamma(\alpha+\varepsilon)}{Q^2 \Gamma(1+\alpha)} \left(\frac{\mu^2}{m^2}\right)^\varepsilon \left(\frac{\nu^2}{m^2}\right)^\alpha \int_0^\infty \frac{1}{(1+x)^{1-\varepsilon} \cdot x^{\alpha+\varepsilon}}$$

$$= \frac{1}{Q^2} \left(\frac{\mu^2}{m^2}\right)^\varepsilon \left(\frac{\nu^2}{m^2}\right)^\alpha \cdot \frac{\Gamma(\alpha+\varepsilon) \cdot \Gamma(\alpha)\Gamma(1-\alpha-\varepsilon)}{\Gamma(1+\alpha) \quad \Gamma(1-\varepsilon)}$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2} \left(\frac{\mu^2}{m^2}\right)^\varepsilon \left(\frac{\nu^2}{m^2}\right)^\alpha \cdot \left(\frac{1}{\alpha\varepsilon} - \frac{1}{\varepsilon^2} + \frac{\pi^2}{3}\right)$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2} \left(-\frac{1}{\varepsilon^2} + \cancel{\frac{1}{\alpha\varepsilon}} + \cancel{\frac{1}{\alpha} \ln\frac{\mu^2}{m^2}} + \frac{1}{\varepsilon} \ln\frac{\nu^2}{\mu^2} + \ln\frac{\mu^2}{m^2} \ln\frac{\nu^2}{m^2} - \frac{1}{2}\ln^2\frac{\mu^2}{m^2} + \frac{\pi^2}{3}\right)$$

$$I_{\bar{c}} = i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk\,\frac{(-\nu^2)^\alpha}{(k^2-m^2+i\epsilon)[(k+l)^2+i\epsilon][k_+\cdot P_- + i\epsilon]^{1+\alpha}}$$

$$= \cancel{i\pi^{-\frac{d}{2}}}\mu^{2\varepsilon}\,\frac{\cancel{(-1)^{3+\alpha}\cdot i\cdot\pi^{\frac{d}{2}}}\,\Gamma(1+\alpha+\varepsilon)(\nu^2)^\alpha}{\Gamma(1+\alpha)}\int_0^\infty dx\,dy\,\frac{(x+y)^{2+2\alpha+2\varepsilon}}{(x+y)^{3+\alpha}\left[(2P_-\cdot l_+ + m^2x)y + m^2x^2\right]^{1+\alpha+\varepsilon}}$$

$$\underset{t=\frac{x}{y}}{=\!=} \frac{\mu^{2\varepsilon}\nu^{2\alpha}\Gamma(1+\alpha+\varepsilon)}{\Gamma(1+\alpha)}\int_0^\infty dt\,dy\,\frac{y^{-1+\varepsilon}}{(1+t)^{1-\alpha-2\varepsilon}\left[m^2t(1+t)y + Q^2\right]^{1+\alpha+\varepsilon}}$$

$$\int_0^\infty dy\,\frac{1}{y^{1-\varepsilon}(Ay+B)^{1+\alpha+\varepsilon}} = \frac{\Gamma(1+\alpha)\Gamma(\varepsilon)}{\Gamma(1+\alpha+\varepsilon)}\,\frac{1}{A^\varepsilon B^{1+\alpha}}$$

$$= \frac{1}{Q^2}\frac{\mu^{2\varepsilon}}{m^{2\varepsilon}}\frac{\nu^{2\alpha}}{Q^{2\alpha}}\Gamma(\varepsilon)\int_0^\infty dt\cdot\frac{1}{(1+t)^{1-\alpha-\varepsilon}\cdot t^\varepsilon}$$

$$= \frac{1}{Q^2}\left(\frac{\mu^2}{m^2}\right)^\varepsilon\left(\frac{\nu^2}{Q^2}\right)^\alpha\frac{\Gamma(\varepsilon)\cdot\Gamma(-\alpha)\Gamma(1-\varepsilon)}{\Gamma(1-\alpha-\varepsilon)}$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{\mu^2}{m^2}\right)^\varepsilon\left(\frac{\nu^2}{Q^2}\right)^\alpha\left(-\frac{1}{\alpha\varepsilon}+\frac{\pi^2}{6}\right)$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\cancel{\frac{1}{\alpha\varepsilon}} - \cancel{\frac{1}{\alpha}\ln\frac{\mu^2}{m^2}} - \frac{1}{\varepsilon}\ln\frac{\nu^2}{Q^2} - \ln\frac{\mu^2}{m^2}\ln\frac{\nu^2}{Q^2} + \frac{\pi^2}{6}\right)$$

$$I_{c+\bar{c}} = I_c + I_{\bar{c}} = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2} + \ln\frac{\mu^2}{m^2}\ln\frac{Q^2}{m^2} - \frac{1}{2}\ln^2\frac{\mu^2}{m^2} + \frac{\pi^2}{2}\right)$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\cancel{\frac{1}{\varepsilon^2}} - \cancel{\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}} - \ln\frac{\mu^2}{m^2}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{m^2} + \frac{\pi^2}{2}\right)$$

Our previous conclusion:

$$I_h = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\cancel{\frac{1}{\varepsilon^2}} + \cancel{\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6}\right)$$

$$\Rightarrow I = I_{c+\bar{c}} + I_h = \frac{1}{Q^2}\left(\frac{\pi^2}{3} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \ln\frac{\mu^2}{m^2}\ln\frac{m^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{m^2}\right)$$

$$= \frac{1}{Q^2}\left(\frac{\pi^2}{3} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2}\right)$$

—— Collinear Anomaly (Factorization Anomaly).

If we didn't introduce the "α"? We would have a divergence in the integral for the Feynman Parameter!

$*$ $$I_c = i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk\,\frac{1}{(k^2-m^2+i\epsilon)[(k+l)^2+i\epsilon][(k+p)^2+i\epsilon]}$$

$$= i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk\,\frac{1}{(k^2-m^2)(2k_-\cdot l_+)(k^2+2k\cdot p)}$$ — Feynman Parameter 2

$$= i\pi^{-\frac{d}{2}}\mu^{4-d}\int d^dk\int_0^\infty dx\,dy\,\frac{2}{[(1+y)k^2+2k\cdot(x\cdot l_+ + yP) - m^2]^3}$$

$$= \mu^{2\varepsilon}\Gamma(1+\varepsilon)\int_0^\infty dx\,dy\,\frac{1}{(1+y)^3\left[\frac{m^2}{1+y}+\frac{xyQ^2}{(1+y)^2}\right]^{1+\varepsilon}}$$

$$= \frac{1}{m^2}\left(\frac{\mu^2}{m^2}\right)^{\varepsilon}\Gamma(1+\varepsilon)\int_0^\infty dx\,dy\,\frac{1}{(1+y)^{1-2\varepsilon}\left((1+y)+xyr\right)^{1+\varepsilon}}$$

$$\int_0^\infty dx\,\frac{1}{(A+Bx)^{1+\varepsilon}} = \frac{1}{A^{\varepsilon}B\cdot\varepsilon}$$

$$= \frac{1}{m^2}\left(\frac{\mu^2}{m^2}\right)^{\varepsilon}\Gamma(1+\varepsilon)\int_0^\infty dy\,\frac{1}{(1+y)^{1-\varepsilon}\cdot yr\cdot\varepsilon} \longrightarrow \text{divergent when } y\to 0$$

so extra regulator is necessary!