Real Analysis
MT2223

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Contents

1 Real and complex number systems

The need for the real number system is replacing the rational number system is because the rational number system has "gaps".
Definition of a field: Set F with 2 operations.

- **Axioms of Addition:**
  (A1) If x ∈ F and y ∈ F, then their sum x + y is in F.
  (A2) Addition is commutative: x + y = y + x for all x, y ∈ F.
  (A3) Addition is associative: (x + y) + z = x + (y + z) for all x, y, z ∈ F.
  (A4) F contains an element O such that O + x = x for every x ∈ F.
  (A5) To every x ∈ F corresponds an element -x ∈ F such that
  $$x + (-x) = 0.$$

- **Axioms of Multiplication:**
  (M1) If x ∈ F and y ∈ F, then their product xy is in F.
  (M2) Multiplication is commutative:
  $$xy = yx$$
  for all x, y ∈ F.
  (M3) Multiplication is associative:
  $$(xy)z = x(yz)$$
  for all x, y, z ∈ F.
  (M4) F contains an element 1 ≠ 0 such that
  $$1x = x$$
  for every x ∈ F.
  (M5) If x ∈ F and x ≠ 0 then there exists an element 1/x ∈ F such that
  $$x \cdot (1/x) = 1$$

- **Distributive Law:**
  $$x(y + z) = xy + xz$$
  holds for all x, y, z e F.

Definition 1.1. An ordered relation(<) is:

- If $x \in S$ and $y \in S$ then one and only of the following statements are true:
  $$x < y, \ x = y, \ y < x$$
  is true

- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Definition 1.2. If S has an ordered relation we call it an ordered set.

Definition 1.3. Start with an ordered set S. Let $E \subset S$ be nonempty. If $\exists \beta \in S$ such that $\forall x \in E$ we have $x \leq \beta$ then we say β is an **upper bound** of E.

Definition 1.4. Start with an ordered set S. Let $E \subset S$ be nonempty. Supposed E bounded above. Suppose that $\exists \alpha \in S$ with the following:

- α is an upper bound of E

- $\gamma < \alpha$ then γ is not an upper bound

then α is called **least upper bound** or supremum.

$$sup(E) = \alpha$$

Definition 1.5. An ordered set has least upper bound property if for a non empty $E \subset S$, E bounded above then

$$sup(E) \in S$$

Remark. Rational numbers do not satisfy this property

Same property for infimum.

Theorem 1.1. Suppose S is an ordered set with the least-upper-bound property, B $\in$ S, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then α = sup L exists in S, and β = inf B

Definition 1.6. An ordered field is a field F which is also an ordered set, such that
(i) x + y < x + z if x, y, z $\in$ F and y < z,
(ii) xy > 0 if x $\in$ F, y $\in$ F, x > 0, and y > 0.
If x > 0, we call x positive; if x < 0, x is negative.

Theorem 1.2. There exists an ordered field R which has the least-upper-bound property.
Moreover, R contains Q as a subfield.
This is called **existence theorem**

2 Series

Theorem 2.1. If a_n and b_n are two sequences:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} = \sum_{n=p}^{q-1} (A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Theorem 2.2. Suppose:

- $\sum_{n=0}^{\infty} a_n$ *converges absolutely*
- $\sum_{n=0}^{\infty} a_n = A$
- $\sum_{n=0}^{\infty} b_n = B$
- $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then:

$$\sum_{n=0}^{\infty} c_n = AB$$