Advanced Linear Algebra
MT2123

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Lecture # 1

Determinants : (2×2 & 3×3.)

- Definition of det(A), A - 2×2 or 3×3 matrix.
- Basic Properties
- Examples.

§1.1 Definition of Determinant :

- F denotes a field. Most of the time $F = \mathbb{R}$ (real) or $F = \mathbb{C}$ (complex).
  (Sometimes, $F = \mathbb{Q}$ (rationals) or $\mathbb{F}_q$ - finite field).

- $M_{n\times n}(F)$ = all $n\times n$-matrices with entries in F.

  If $A \in M_{n\times n}(F)$, then we write $A = [a_{ij}]$, $1 \le i,j \le n$.
  a_{ij} = entry in the i^{th}-row and j^{th} column.

- $\det : M_{n\times n}(F) \longrightarrow F$

  Determinant is a function, denoted "det", that takes as input an $n\times n$-matrix A & gives us a scalar output denoted det (A).

- We will cover the general theory of determinants next week. Today we remind ourselves the definition when $n=2$ and $n=3$.

- Determinant of a 2×2-matrix.

$$\boxed{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc}$$

- ($n=3$) The definition is a little more complicated. We can expand w.r.t. (with respect to) any row or column. Let's see this expanded w.r.t the first row.

- **Determinant of a 3×3-matrix.**

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\cdot\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\cdot\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Think of this formula as:

$$\det(A) = \sum_{j=1}^{3} a_{1j}\, C_{1j}(A)$$

$C_{ij}(A)$ = $(i,j)^{th}$-cofactor = $(-1)^{i+j}\det(M_{ij}(A))$

$M_{ij}(A)$ = $(i,j)^{th}$- minor = matrix obtained by deleting the i^{th}-row & j^{th} column from A

Example: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1.\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2\cdot\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3\cdot\det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$= 1\cdot(5\cdot9 - 8\cdot6) - 2\cdot(4\cdot9 - 7\cdot6) + 3(4\cdot8 - 7\cdot5)$$

$$= 45 - 48 - 2(36 - 42) + 3(32 - 35)$$

$$= -3 - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9 = 0$$

1.2 Basic Properties of determinant function:

Proposition:

Let $A, B \in M_{n\times n}(F)$.

(i) $$\det(AB) = \det(A)\cdot\det(B).$$

(ii) If A' is obtained from A by interchanging two rows then
$$\det(A') = -\det(A)$$

(iii) If A' is obtained from A by adding a constant times a row to another row then:
$$\det(A') = \det(A)$$

(iv) If A' is obtained from A by replacing the i^{th}-row by a nonzero constant c times the i^{th} row then:
$$\det(A') = c\cdot\det(A).$$

(v) If ${}^{T}A$ = transpose of A then
$$\det({}^{T}A) = \det(A).$$

(vi) $\det(I_n) = 1$. $\quad I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1\end{bmatrix}$ $n\times n$-idetity matrix

(vi) If A is invertible then $\det(A) \neq 0$ and furthermore: $\det(A^{-1}) = \det(A)^{-1}$.

- (ii), (iii), (iv) also hold for elementary column operations.
- We will prove this proposition after discussing the general theory of determinants.

1.3 Examples / Exercises:

① $\det(I_n) = 1$ $\qquad \det\begin{bmatrix}1&0\\0&1\end{bmatrix} = 1, \quad \det\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = 1\cdot\det\begin{bmatrix}1&0\\0&1\end{bmatrix} = 1.$

② Determinant of a diagonal matrix is the product of the diagonal entries

$$\det\begin{bmatrix}a&0\\0&b\end{bmatrix} = ab \qquad \det\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix} = a\cdot\det\begin{bmatrix}b&0\\0&c\end{bmatrix} = abc.$$

③ Determinant of an upper-triangular matrix is the product of the diagonal entries:

$$\det\begin{bmatrix}a&b\\&c\end{bmatrix} = ac$$

$$\det\begin{bmatrix}a&b&c\\&d&e\\&&f\end{bmatrix} = a\cdot\det\begin{bmatrix}d&e\\0&f\end{bmatrix} - b\cdot\det\begin{bmatrix}0&e\\0&f\end{bmatrix} + c\cdot\det\begin{bmatrix}0&d\\0&0\end{bmatrix}$$

$$= a(df) - b(0) + c(0) = adf.$$

(It's simpler to expand via the 1^{st} column:)

④ **Exercise**: Prove the proposition for 2×2-matrices.

For e.g. if $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, $\quad {}^TA = \begin{bmatrix}a&c\\b&d\end{bmatrix}$

$$\det({}^TA) = \det\begin{bmatrix}a&c\\b&d\end{bmatrix} = ad - cb = \det(A).$$

⑤ **Exercise: (True / False?)**

(i) Determinant of a lower triangular matrix is the product of diagonal entries.

(ii) If P is an invertible matrix then

$$\det(PAP^{-1}) = \det(A).$$

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Lecture 2 | Theory of Determinants – I.

- Multilinear functions.
- Alternating functions
- Determinant function on $M_{n\times n}(F)$.
- **Theorem:** Determinant function exists.

§2.1 Multilinear functions

Let V be a vector space over a field F. (Just take $F = \mathbb{R}$)

Let $n \geq 1$ be a positive integer.

- A multilinear function on V is a function:

$$T : \underbrace{V \times \cdots \times V}_{n} \longrightarrow F$$

(domain = n-fold product of V, range = F - base field)

such that: for each $1 \leq i \leq n$, T is a linear function w.r.t. the i^{th} variable keeping the other $(n-1)$-variables fixed.

- For a given n, we also call this function n-linear.

- $n = 1$: just a linear fn. $n = 2$: bilinear function (we will study this in great detail later in the course.)

- Linearity in the i^{th} variable means:

$$T(v_1, \ldots, av_i + bv_i', \ldots, v_n) = aT(v_1, \ldots, v_i, \ldots, v_n) + b \cdot T(v_1, \ldots, v_i', \ldots, v_n)$$

($v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ are fixed.
a, b any scalars, v_i, v_i' any vectors)

- Example: $V = \mathbb{R}^3$ $v \in \mathbb{R}^3$ we write as $v = (x, y, z)$

(i) Bilinear fn. on $\mathbb{R}^3$: $T(v, v') = T((x,y,z), (x',y',z'))$

$$= xy' - yx' + zz'$$

(ii) $A((x,y,z),(x',y',z')) = (xx')^2 + (yy')^2 + (zz')^2$ is not bilinear

2.2 Alternating functions:

Let V - vector space $/F$, $n \geq 1$.

- an alternating n-linear function is an n-linear function

$$A : \underbrace{V \times \cdots \times V}_{n} \longrightarrow F \quad \text{such that}$$

$A(v_1, \ldots, v_n) = 0$ whenever any two of the variables v_i, v_j are equal.

- Lemma:

If A is alternating n-linear fn. on V then
$A(v_1', \ldots, v_n') = -A(v_1, \ldots, v_n)$ if $(v_1', \ldots, v_n')$ is obtained from $(v_1, \ldots, v_n)$ by interchanging two of the variables.

Pf: Only two variables matter and so it is best to see the proof for bilinear alternating fn. $A : V \times V \longrightarrow F$. Let $v, w \in V$.

$$0 = A(v+w, v+w) = \underbrace{A(v,v)}_{0} + A(v,w) + A(w,v) + \underbrace{A(w,w)}_{0}$$

$$\Rightarrow A(v,w) + A(w,v) = 0 \Rightarrow A(v,w) = -A(w,v). \qquad \blacksquare$$

Examples: $V = \mathbb{R}^3$

(i) $A((x,y,z),(x',y',z')) = xz' - x'z$ is alternating.

(ii) $A((x,y,z),(x',y',z')) = xx' + yy' + zz'$ is bilinear but not alternating. (It is "symmetric": $A(v,w) = A(w,v)$)

§2.3 Determinant function on $M_{n\times n}(F)$:

- Now we take V = n-dimensional vector space/F.

  Fix a basis; using which we identify $V = F^n$.

  Any $v \in V$ is of the form $v = (a_1, \ldots, a_n)$ $\quad a_i \in F$.

  For n-linear functions on F^n we look at functions on $\underbrace{F^n \times \cdots \times F^n}_{n}$. But this is the same as $M_{n\times n}(F)$, by thinking of a matrix $A \in M_{n\times n}(F)$ as an n-tuple $(R_1, \ldots, R_n)$ of its rows, with each $R_i \in F^n$.

$$M_{n\times n}(F) \cong F^n \times \cdots \times F^n \qquad R_i = (a_{i1}, \ldots, a_{in})$$

$$A \longleftrightarrow (R_1, \ldots, R_n) \qquad A = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

- **Definition**:

  A determinant function on $M_{n\times n}(F)$

$$D : M_{n\times n}(F) \longrightarrow F$$

  alternating n-linear function of the rows, normalised by $D(I_n) = 1$. ; I_n = $n\times n$ - identity matrix.

- Why ask for $D(I_n) = 1$?

- Example: $(n=2)$: $D : M_{2\times 2}(F) \to F$. $\qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\left.\begin{aligned} R_1 &= (a_{11}, a_{12}) = a_{11}\varepsilon_1 + a_{12}\varepsilon_2 \\ R_2 &= (a_{21}, a_{22}) = a_{21}\varepsilon_1 + a_{22}\varepsilon_2 \end{aligned}\right\} \quad \{\varepsilon_1 = (1,0),\ \varepsilon_2 = (0,1)\} \text{ Standard basis for } F^2.$$

$$\begin{aligned} D(A) = D(R_1, R_2) &= D(a_{11}\varepsilon_1 + a_{12}\varepsilon_2,\ a_{21}\varepsilon_1 + a_{22}\varepsilon_2) \\ &= a_{11}a_{21}\underbrace{D(\varepsilon_1,\varepsilon_1)}_{0} + a_{11}a_{22}D(\varepsilon_1,\varepsilon_2) + a_{12}a_{21}\underbrace{D(\varepsilon_2,\varepsilon_1)}_{-1\cdot D(\varepsilon_1,\varepsilon_2)} + a_{12}a_{22}\underbrace{D(\varepsilon_2,\varepsilon_2)}_{0} \\ &= (a_{11}a_{22} - a_{12}a_{21})\, D(\varepsilon_1, \varepsilon_2) \\ &= \det(A)\cdot D(I_2) = \det(A) \end{aligned}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

§2.4 Existence of Determinant Function.

Theorem:

(i) Let D be an $(n-1)$-linear alternating function $M_{(n-1)\times(n-1)}(F)$.
Fix any $1 \leq j \leq n$. Define $\Delta_j : M_{n\times n}(F) \longrightarrow F$ by:

$$\Delta_j(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot D(M_{ij}(A))$$

Then Δ_j is an n-linear alternating fn. on $M_{n\times n}(F)$.

(ii) If D is a determinant fn. on $M_{(n-1)\times(n-1)}(F)$, then Δ_j is a determinant fn. on $M_{n\times n}(F)$.

Remarks:

- $A = [a_{ij}]_{1\leq i,j \leq n}$
- $M_{ij}(A) = (i,j)^{th}$-minor of A = $(n-1)\times(n-1)$-matrix obtained by deleting the i^{th}-row & j^{th}-column.
- The formula for Δ_j is "expanding along the j^{th} column."
- The theorem is constructive!
- It's an existence theorem: each of the $\Delta_1, \ldots, \Delta_n$ can possibly be different functions. (We will see later that they are all equal! There is a <u>unique</u> determinant function.)
- The proof: Read it up from the book. (Theorem 1, pp. 146–147)
  There are no surprises in the proof. You need to carefully check that Δ_j is n-linear & alternating.
  - n-linear is "easy".
  - for the proof of <u>alternating</u> you will see the necessity of the <u>alternating</u> signs: $(-1)^{i+j}$.

Lecture 3

Theory of Determinants – II

- S_n = Permutations of $\{1, 2, \ldots, n\}$
- Sign of a permutation.
- Formula for $\det(A)$, $A \in M_{n\times n}(F)$
- <u>Theorem</u>: There is a unique determinant function on $M_{n\times n}(F)$.
- 3×3 - example.

§3.1 Permutations of $\{1, 2, \dots, n\}$

- A **permutation** of the set $\{1, 2, \dots, n\}$ is a bijection

$$\sigma : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$$

- $S_n = Bij(\{1, 2, \dots, n\})$ = set of all permutations of $\{1, 2, \dots, n\}$

Fact $|S_n| = n!$ (n-factorial)

The total number of permutations of $\{1, 2, \dots, n\}$ = size of $S_n = n!$

Let $\sigma \in S_n$. There are n possibilities for $\sigma(1)$. Once $\sigma(1)$ is fixed, there are $(n-1)$ possibilities for $\sigma(2)$ ... etc.

Total # of possibilities for σ = $n \cdot (n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$

- A **transposition** is a permutation that switches only two elements in $\{1, \dots, n\}$ and keeps others fixed.

We denote by (i, j) the permutation that interchanges i and j, and fixes k for all $k \neq i$, $k \neq j$.

- Examples / Notations:

(i) $(k_1, k_2, \dots, k_r)$ denotes the **cyclic permutation**.

$k_1 \to k_2$, $k_2 \to k_3$, ..., $k_{r-1} \to k_r$, $k_r \to k_1$; all other i's are fixed

(ii) $(1\ 2\ 3) = (1\ 3)(1\ 2)$ Verify this!

Note: $(13)(12)$ is the composition of the permutations; it means first apply (12), then apply (13).

Just like $(f \circ g)(x) = f(g(x))$

(iii) $(1, 2, 3, 4) = (1\ 4)(1\ 3)(1\ 2)$

(iv) $(k_1, k_2, \dots, k_r) = (k_1\ k_r)(k_1\ k_{r-1}) \cdots (k_1\ k_2)$

(v) $(1\ 3) = (1\ 2)(2\ 3)(1\ 2)$

§3.2 Sign of a permutation:

The basic facts we will need about permutations are:

Proposition:

1. Every permutation is a product of transpositions.
2. If $\sigma \in S_n$ is expressed as a product of transpositions $\sigma = s_1 \cdots s_m$, then m need not be unique, but the parity of m is uniquely determined by σ.

(See Theorem on next page.)

Remarks:

- Proof of 1. is in two steps: $\sigma \in S_n$. Then σ can be written as a product of cyclic permutations. Now use Example (iv) of §3.1: A cyclic perm. is a product of transpositions
- Example (v): $(1\,3) = (1\,2)(2\,3)(1\,2)$ already illustrates the statement in 2.

Definition

(i) Let $\sigma \in S_n$. Suppose $\sigma = s_1 \cdots s_m$ is a product of transpositions. Then the sign of σ, also called signature of σ, is

$$\operatorname{sgn}(\sigma) = (-1)^m.$$

(ii) σ is an even permutation : $\operatorname{sgn}(\sigma) = 1$

odd " : $\operatorname{sgn}(\sigma) = -1$

Examples:

(i) $(1\,2\,3) = (1\,3)(1\,2) \Rightarrow (1\,2\,3)$ is an even permutation.

(ii) $(1\,3) = (1\,2)(2\,3)(1\,2)$ is an odd permutation.

(iii) $(1\,4\,7)(2\,5)$ – even or odd?

Theorem

The function $sgn : S_n \longrightarrow \{\pm 1\}$ is a "homomorphism"

i.e. : $sgn(\sigma\tau) = sgn(\sigma)\cdot sgn(\tau)$

- This is a deep theorem! Why?
- We denote $sgn(\sigma)$ also as $\varepsilon(\sigma)$. ; $\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau)$.

§ 3.3 Formula for det (A) :

Let $A \in M_{n\times n}(F)$; $A = [a_{ij}] = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$

$R_i = (a_{i1}, \dots, a_{in}) = \sum_{j=1}^{n} a_{ij}\,\varepsilon_j$; $\{\varepsilon_1, \dots, \varepsilon_n\}$ = standard basis for F^n.

- The example in §2.3, p.3, Lecture_2 generalizes to $n\times n$-matrices.

i) If $D : M_{n\times n}(F) \longrightarrow F$ is an n-linear function of $R_1, \dots, R_n$

then $\boxed{D(A) = \sum_{1\le k_1,\dots,k_n\le n} a_{1k_1}\cdots a_{nk_n}\cdot D(\varepsilon_{k_1}, \dots, \varepsilon_{k_n})}$ (n^n-terms)

(ii) If $D : M_{n\times n}(F) \longrightarrow F$ is alternating then,

$\boxed{\text{if } k_i = k_j \text{ then } D(\varepsilon_{k_1}, \dots, \varepsilon_{k_n}) = 0}$. For nonzero terms : $(k_1, \dots, k_n) = (\sigma(1), \dots, \sigma(n))$

for a permutation σ. Also : $\boxed{D(\varepsilon_{\sigma(1)}, \dots, \varepsilon_{\sigma(n)}) = sgn(\sigma)\cdot D(\varepsilon_1, \dots, \varepsilon_n).}$

(iii) If $D : M_{n\times n}(F) \longrightarrow F$ is a determinant function, then

$\boxed{D(I_n) = D(\varepsilon_1, \dots, \varepsilon_n) = 1.}$

(i), (ii) & (iii) together proves the following fundamental theorem:

Theorem :

If $D : M_{n\times n}(F) \longrightarrow F$ is a determinant function, then for any $A = [a_{ij}]$, $D(A)$ is "the" determinant function:

$$\det(A) = \sum_{\sigma\in S_n} sgn(\sigma)\cdot a_{1\sigma(1)}\cdot a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

§ 3.4 Existence & Uniqueness:

- Existence : $\begin{cases} \text{We constructed determinant function(s) on } M_{n\times n}(F) \\ \text{using determinant function on } M_{(n-1)\times(n-1)}(F). \end{cases}$
- Uniqueness : $\begin{cases} \text{Any determinant function on } M_{n\times n}(F) \text{ is forced} \\ \text{to be } \det(A). \end{cases}$

Let's put both together in the following theorem:

Theorem

The determinant function on $M_{n\times n}(F) \longrightarrow F$ is given by :

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} \cdots a_{n\sigma(n)} .$$
$$= \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(M_{ij}(A))$$

Remarks:

- $\det(A)$ is a homogeneous polynomial in $\{a_{ij}\}$ of degree n.
- The coefficients of this polynomial are $\{\pm 1\}$.
- $a_{1\sigma(1)} \cdots a_{n\sigma(n)}$: is a product of n entries; exactly one from every row and every column.
- In the constructive proof, the expansions for determinant along two different columns give us the same answer. (See next page.)

§3.5 **Example:** $(n=3)$. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

- Permutations of S_3; and their signatures: $|S_3| = 3! = 6$

$\sigma \in S_3$	1	(12)	(13)	(23)	$(123$	(132)
$sgn(\sigma)$	1	-1	-1	-1	1	1

- $\det(A) = \sum_{\sigma} sgn(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$

$$= a_{11} a_{22} a_{33} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{23}$$

$$+ a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$

- Expansion along 1^{st} column: $\sum_{i=1}^{3} (-1)^{i+1} \cdot a_{i1} \cdot \det(M_{i1}(A))$

$$= a_{11} \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21} \det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31} \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

$$= a_{11}(a_{22} a_{33} - a_{32} a_{23}) - a_{21}(a_{12} a_{33} - a_{32} a_{13}) + a_{31}(a_{12} a_{23} - a_{22} a_{13})$$

$$= \det(A). \quad (\text{check}).$$

- Verify for yourselves that expanding via 2^{nd} column / 3^{rd} column also gives you $\det(A)$.

- (The determinant function feels like magic!!!)

X

Lecture 4

Properties of Determinants - I

- $\det(AB) = \det(A)\det(B)$
- $\det({}^tA) = \det(A)$
- $\det(\text{Row-Op}(A)) = \det(E)\det(A)$; $E = \text{Same-Row-Op}(I_n)$.
- A is invertible $\iff \det(A) \neq 0$.

§6.1 Homomorphism Property of Determinants:

Theorem:

$$\boxed{\det(AB) = \det(A)\det(B), \quad \forall A, B \in M_{n\times n}(F).}$$

Pf Fix B, and consider the function $D_B : M_{n\times n}(F) \longrightarrow F$

$$D_B(A) = \det(AB)$$

Then, we have to become aware of the fact that D_B is a n-linear alternating function of the rows of A. By the exact same calculation as in Lecture 3, § 3.3, we see that:

$$D_B(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)}\cdots a_{n,\sigma(n)} \cdot D_B(\varepsilon_1,\dots,\varepsilon_n)$$

$$= \det(A)\cdot D_B(I_n) = \det(A)\cdot \det(I_n B)$$

$$= \det(A)\det(B).$$

Recall: $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ $\quad (A = [a_{ij}],\ B = [b_{ij}])$

i^{th} row of AB = $R_i(AB) = \left(\sum_{k=1}^{n} a_{ik} b_{k1}, \sum_{k=1}^{n} a_{ik} b_{k2}, \dots, \sum_{k=1}^{n} a_{ik} b_{kn}\right)$

$R_i(A) = (a_{i1}, \dots, a_{in})$ $\longrightarrow$ $R_i(AB) = R_i(A)\cdot B$

$\therefore D_B(A) = D_B(R_1(A), \dots, R_n(A))$

$= \det(R_1(A)B, \dots, R_n(A)B)$ $\qquad \mathcal{R}_i = R_i \cdot B$

Since $\det(\mathcal{R}_1,\dots,\mathcal{R}_n)$ is an alternating fn. of $\mathcal{R}_1,\dots,\mathcal{R}_n$, we can see that $\det(R_1B,\dots,R_nB)$ is an alternating fn. of $R_1,\dots,R_n$. ■

§4.2 **Invariance of determinant under transpose:**

$A \in M_{n\times n}(F)$, then ${}^tA \in M_{n\times n}(F)$, called the transpose of A, is defined as: ${}^tA(i,j) = A(j,i)$.

Theorem: $\boxed{\det({}^tA) = \det(A)}$

Pf:
$$\det({}^tA) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cdot {}^tA(1,\sigma(1)) \cdots {}^tA(n,\sigma(n))$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cdot A(\sigma(1),1) \cdots\cdots A(\sigma(n),n)$$
Think!!

$\left(A(\sigma(i),i) : \text{ put } \sigma(i)=k \iff i = \sigma^{-1}(k).\right)$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cdot A(1,\sigma^{-1}(1)) \cdots\cdots A(n,\sigma^{-1}(n))$$

(put $\sigma^{-1} = \tau$)
$$= \sum_{\tau\in S_n} \operatorname{sgn}(\tau^{-1})\cdot A(1,\tau(1)) \cdots A(n,\tau(n))$$

$\sigma = s_1 \cdots\cdots s_m$, s_j - transposition, then $\sigma^{-1} = s_m \cdots\cdots s_1$

(Check: $\sigma\cdot\sigma^{-1} = s_1\cdots s_m\cdot s_m\cdots s_1 = 1$ (cancels out from the inside; $s_j^2 = 1$.))

$\Rightarrow \quad \operatorname{sgn}(\tau^{-1}) = \operatorname{sgn}(\tau)$

$$\therefore \quad \det({}^tA) = \cdots = \sum_{\tau\in S_n} \operatorname{sgn}(\tau)\cdot A(1,\tau(1)) \cdots\cdots A(n,\tau(n))$$
$$= \det(A). \qquad \blacksquare$$

§4.3 Behaviors of determinants under row operations:

- The **elementary row operations** are:

  (i) $R_i \longrightarrow R_i + cR_j$; add c times the j^{th} row to the i^{th} row

  (ii) $R_i \longrightarrow cR_i$; multiply the i^{th} row by c $(c \neq 0)$.

  (iii) $R_i \longleftrightarrow R_j$; interchange the i^{th} & j^{th} row.

- An **elementary matrix** is a matrix that you get by doing one elementary row operation to I_n.

  i) $E = \begin{bmatrix} 1 & & & c & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$; c in the $(i,j)^{th}$-position
  1's on the diagonal
  0 everywhere else.

  (ii) $E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$; the $(i,i)^{th}$ diagonal entry is c, etc

  (iii) $E = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & & 1 & \\ & & & & 1 \ddots 1 & & \\ & & & 1 & & 0 & \\ & & & & & & 1 \ddots 1 \end{bmatrix}$; $(i,j)^{th}$ & $(j,i)^{th}$ entries are 1,
  $(k,k)^{th}$ entry $= 1$ $k \neq i, j$
  $(i,i)^{th}$ & $(j,j)^{th}$ entry $= 0$.

- **Theorem:**

  $A \in M_{n\times n}(F)$. Let e be an elementary row operation.
  Suppose $E = e(I_n)$ is the corresponding elementary matrix.
  Then $$e(A) = E \cdot A$$

  Pf: See Theorem 9, Section 1.5, of the textbook.

- Elementary row operations are realized by left multiplying by elementary matrices.

- **Theorem:**

(i) $e: R_i \to R_i + cR_j$	then	$\det(e(A)) = \det(A)$
(ii) $e: R_i \to cR_i$	then	$\det(e(A)) = c \cdot \det(A)$
(iii) $e: R_i \leftrightarrow R_j$	then	$\det(e(A)) = -\det(A)$

Pf: In each case, $\det(e(A)) = \det(EA) = \det(E)\cdot\det(A)$,

where $E = e(I_n)$. Now the proof follows by computing $\det(E)$:

$$\det\left(\begin{bmatrix} 1 & & & c \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\right) = 1, \quad \det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & 1 & \\ & & & & \ddots \; 1 \end{bmatrix}\right) = c$$

$$\det\left(\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & 1 \ddots 1 & & \\ & & 1 & & 0 & \\ & & & & & 1 \ddots 1 \end{bmatrix}\right) = -1 \quad \text{(since } (i,j) \text{ is an odd permutation.)}.$$

$\blacksquare$

- The above theorem is computationally important!
- This theorem is the basis of generalizations of determinant; for example: "DIEUDONNE DETERMINANT" for matrices over division rings = non-commutative fields.
- Example: Row operations & determinant:-

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \xrightarrow{(R_2 - 4R_1) \to R_1} \underset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}} \xrightarrow{(R_3 - 7R_1) \mapsto R_1} \underset{C}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}} \xrightarrow{R_3 - 2R_2 \mapsto R_3} \underset{D}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}}$$

$\det(A) = \det(B) = \det(C) = \det(D) = 0$.

(since D has a zero row).

§4.4 Invertibility = Nonvanishing of determinants.

- Let's recall a theorem about elementary row operations/matrices and invertibility of a matrix. (Recall: A is invertible if $\exists B$ s.t. $AB = BA = I_n$; we call B the inverse of A & write $B = A^{-1}$.)

Theorem:

$\boxed{\begin{array}{l} \text{Let } A \in M_{n\times n}(F). \text{ ; } F\text{ - field.} \\ A \text{ is invertible} \iff A = E_1 \cdots E_k \text{ a product of elementary matrices.} \end{array}}$

See Theorem 12, Section 1.6 of the Textbook

(The point is: you start row-reducing A by row operations, and A is invertible if and only if you can reduce it all the way I_n, i.e., A - invertible $\iff$ the row-reduced echelon matrix of $A = I_n$)

- **Theorem.** $\boxed{A \text{ is invertible} \iff \det(A) \neq 0.}$

Pf: "$\Rightarrow$"

A - invertible $\iff$ $A = E_1 \cdots E_k$. $\Rightarrow$ $\det(A) = \det(E_1) \cdots \det(E_k) \neq 0$,

as we have seen in §4.3 that $\det(E) \neq 0$ for an elementary matrix.

"$\Leftarrow$" We can prove this by proving:

"A is not-invertible $\Rightarrow$ $\det(A) = 0$".

Suppose A is not-invertible then its row-reduced echelon matrix R has to have a row full of 0's. Hence $\exists$ elementary matrices $E_1, \dots, E_k$ s.t. $E_k \cdots E_1 \cdot A = R$ & furthermore,

$\det(R) = 0 \Rightarrow \det(E_k) \cdots \det(E_1) \cdot \det(A) = 0$

$\Rightarrow \det(A) = 0$. ∎

- The above proof needs the theory of row-operations on matrices. If you do not understand these, then go back to Chapter 1 of the textbook, and review Section 1.3, 1.4, 1.5 and 1.6.

- We will see another proof in the next lecture.

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Lecture 5

Properties of Determinants – II

- $\mathrm{adj}(A)$ = The classical adjoint of A
- $A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) \cdot I_n$
- A - invertible $\Longleftrightarrow \det(A) \neq 0$; Formula for A^{-1}
- Cramer's Rule to solve the system $AX = Y$.
- Determinant of a linear operator $T: V \longrightarrow V$.

5.1 The classical adjoint of a matrix:

Let $A \in M_{n\times n}(F)$; F- field.

- $M_{ij}(A)$ = the (i,j)-th minor of A = $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row & j^{th} column.
- $C_{ij}(A) = (-1)^{i+j}\det(M_{ij}(A))$ = $(i,j)^{th}$- cofactor of A.
- $\operatorname{adj}(A)$ = $n\times n$ -matrix called the classical adjoint of A = $[C_{ji}(A)]_{1\le i,j\le n}$ (note the interchange of i & j) = transpose of the matrix of cofactors of A
- $$\boxed{\operatorname{adj}(A)(i,j) = C_{ji}(A) = (-1)^{i+j}\det(M_{ji}(A)).}$$

5.2 The main theorem about adjoint of a matrix:

Theorem:

$$\boxed{A\cdot\operatorname{adj}(A) = \operatorname{adj}(A)\cdot A = \det(A)\, I_n = \begin{bmatrix} \det(A) & & & \\ & \det(A) & & \\ & & \ddots & \\ & & & \det(A)\end{bmatrix}}$$

Pf: We have already seen the expansion of $\det(A)$ along the j^{th} column of A as:

$$\det(A) = \sum_{i=1}^{n}(-1)^{i+j} a_{ij}\cdot\det(M_{ij}(A))$$

$$= \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} a_{ij}\operatorname{adj}(A)_{ji} \begin{cases} = (A\cdot\operatorname{adj}(A))_{ii}. \\ = (\operatorname{adj}(A)\cdot A)_{jj}. \end{cases}$$

$\Rightarrow (A\cdot\operatorname{adj}(A))_{ii} = \det(A)$ & $(\operatorname{adj}(A)\cdot A)_{jj} = \det(A)$

This proves the theorem by checking equality for diagonal entries.

For non-diagonal entries we need to prove: if $i \neq j$ that:

$$(A \cdot \operatorname{adj}(A))_{ij} = 0 \quad \text{or} \quad \sum_{k=1}^{n} a_{ik} \operatorname{adj}(A)_{kj} = 0$$

or

$$\underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{ik} \det(M_{jk}(A))}_{(*)} = 0$$

We note that the expression $(*)$ is nothing but the expansion for determinant along the j^{th} row of the matrix $\tilde{A}$ obtained by replacing the j^{th} row of A by the i^{th} row; in particular $\tilde{A}$ has two equal rows, and hence $\det(\tilde{A}) = 0$.

Similarly,

$$(\operatorname{adj}(A) \cdot A)_{ij} = \sum_{k=1}^{n} \operatorname{adj}(A)_{ik} \cdot a_{kj}$$

$$= \sum_{k=1}^{n} \underbrace{(-1)^{i+k} a_{kj} \det(M_{ki}(A))}_{(**)} = 0,$$

since $(**)$ is the determinant expansion along the i^{th} column of the matrix $\tilde{\tilde{A}}$ obtained by replacing the i^{th} column by the j^{th} column; in particular $\tilde{\tilde{A}}$ has two equal columns & so $\det(\tilde{\tilde{A}}) = 0$. ∎

- This proof might seem tricky or slick. The magic is really in various expansions of det along any row or any column.

- The importance of the theorem is that it gives an explicit formula for inverse of a matrix.

§5.3 Invertibility & formula for inverse:

Theorem.

(i) A is invertible $\iff$ $\det(A) \neq 0$

(ii) If A is invertible then $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$

Pf: (i) "$\Rightarrow$" If A is invertible then $\exists\, B$ s.t. $AB = BA = I_n$

Then $\det(AB) = \det(I_n) \Rightarrow \det(A)\cdot\det(B) = 1 \Rightarrow \det(A) \neq 0$

(i) "$\Leftarrow$" If $\det(A) \neq 0$. then we have seen:

$$A \cdot \operatorname{adj}(A) = \operatorname{adj}(A)\cdot A = \begin{bmatrix} \det(A) & & \\ & \ddots & \\ & & \det(A) \end{bmatrix}$$

$$\Rightarrow A \cdot \left(\det(A)^{-1}\operatorname{adj}(A)\right) = \left(\det(A)^{-1}\operatorname{adj}(A)\right)\cdot A = \det(A)^{-1}\cdot \begin{bmatrix} \det(A) & & \\ & \ddots & \\ & & \det(A) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I_n$$

(ii) Follows from "(i), $\Leftarrow$". , since the inverse of a matrix if it exists is unique: if B & C are such that $AB = BA = I_n = AC = CA$ then $B = B(I_n) = B(AC) = (BA)C = I_n \cdot C = C$.

Therefore, if A is invertible then: its inverse has to be:

$$A^{-1} = \frac{1}{\det(A)}\cdot \operatorname{adj}(A).$$

- Let's note:

$$\boxed{(A^{-1})_{ij} = \frac{(-1)^{i+j}\det(M_{ji}(A))}{\det(A)}.}$$

The entries of A^{-1} are rational functions of the entries of A

§5.4 Cramer's Rule:

Consider a system of n linear equations in n-variables:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}\right\}$$

$AX = b$

$A = [a_{ij}]$ - matrix of coefficients

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

The system $AX=b$ has a unique solution when A is invertible, because:

$$AX = b \Rightarrow \left.\underbrace{A^{-1}(AX)}_{(A^{-1}A)\cdot X} = A^{-1}b\right\} \Rightarrow X = A^{-1}b$$

Now we use the formula for A^{-1} as in §5.3, to get:

$$x_i = X_{i1} = (A^{-1}b)_{i1} = \sum_{k=1}^{n} (A^{-1})_{ik}\, b_{k1}$$

$$= \sum_{k=1}^{n} (-1)^{i+k} \frac{\det[M_{ki}(A)]}{\det(A)} \cdot b_k$$

$$x_i = \frac{\left(\sum_{k=1}^{n} (-1)^{i+k} \det(M_{ki}(A)) \cdot b_k\right)}{\det(A)} \longleftarrow \text{Cramer's Rule.}$$

Theorem (Cramer's Rule)

If A is invertible, the unique solution of $AX=b$ is given by the rule:

$$x_i = \frac{\det(\text{Replace } i^{th} \text{ column of } A \text{ by } b)}{\det(A)}.$$

§5.5 Determinant of an operator:

- Let $T: V \longrightarrow V$ be a linear operator

(Linear Operator = linear transformation from a vector space to itself.)

Suppose $\dim(V) = n$.

Fix a basis $\mathcal{B} = \{v_1, \dots, v_n\}$. Let $A = [T]_{\mathcal{B}}$ be the matrix of T relative to $\mathcal{B}$:

$$\boxed{T v_j = \sum_{i=1}^{n} a_{ij} v_i}$$

Memorize this formula!

Now, let $\mathcal{B}' = \{w_1, \dots, w_n\}$ be another basis of V, and let $B = [T]_{\mathcal{B}'}$. Then $T w_j = \sum_{i=1}^{n} b_{ij} w_i$.

Consider the change of basis matrix:

$$P : \mathcal{B} \rightsquigarrow \mathcal{B}' \quad \text{or} \quad P : \{v_1, \dots, v_n\} \rightsquigarrow \{w_1, \dots, w_n\}$$

which we may write as: $\boxed{w_j = \sum_{i=1}^{n} P_{ij} v_i}$ Memorize this formula!

Let $Q : \mathcal{B}' \rightsquigarrow \mathcal{B}$ be the change of basis in the other direction; $Q = P^{-1}$; $v_j = \sum_{i=1}^{n} Q_{ij} w_i$.

Then:

$$T(w_j) = T\left(\sum_{i=1}^{n} P_{ij} v_i\right) = \sum_{i=1}^{n} P_{ij} T(v_i) = \sum_{i=1}^{n} P_{ij} \cdot \sum_{k=1}^{n} a_{ki} v_k$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} P_{ij} a_{ki} v_k = \sum_{i=1}^{n} \sum_{k=1}^{n} P_{ij} a_{ki} \sum_{\ell=1}^{n} Q_{\ell k} w_\ell$$

$$= \left(\sum_{i,k,\ell} P_{ij} a_{ki} Q_{\ell k}\right) w_\ell = \sum_{\ell=1}^{n} \left(\sum_i \sum_k Q_{\ell k} a_{ki} P_{ij}\right) w_\ell$$

$$= \sum_{\ell=1}^{n} (QAP)_{\ell j} w_\ell \implies B = QAP \quad \text{or}$$

$$\boxed{B = P^{-1} A P.}$$

- **Defn:** If $T : V \to V$ is a linear operator; $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis for V; $A = [T]_{\mathcal{B}}$ matrix of T relative to $\mathcal{B}$; then we define the determinant of the operator T as:

$$\det(T) = \det([T]_{\mathcal{B}}) = \det(A).$$

This definition is independent of choice of $\mathcal{B}$, because if $\mathcal{B}'$ is another basis, and $P : \mathcal{B} \rightsquigarrow \mathcal{B}'$ the change of basis matrix, then $B = P^{-1}AP$ and

$$\det(B) = \det(P^{-1}AP) = \cancel{\det(P)^{-1}}\det(A)\cancel{\det(P)} = \det(A).$$

- To compute the determinant of an operator, it is convenient to have a basis relative to which the matrix of this operator is particularly convenient. This idea leads us to the next main topic: Eigenvalues & Eigenvectors.

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Lecture 6 | Chapter-1 of Textbook | Review-1

- Fields
- Linear Systems of Equations
- Row operations on matrices
- RREF : Row-reduced echelon form.
- Matrix Multiplication ; elementary matrices
- Invertible matrices
- characterizing invertibility.

§1 A list of concepts whose definitions you need to know:

- Field. Some Examples:

  $\mathbb{Q}$ = rational numbers

  $\mathbb{R}$ = real numbers

  $\mathbb{C}$ = complex numbers

  $\mathbb{Q}(i)$ = Gaussian numbers

  $\mathbb{F}_p$ = finite field with p elements (p-prime)

  Some non-examples:

  $\mathbb{Z}$ = integers (not a field)

  $\mathbb{Z}[i]$ = Gaussian integers (not field)

- The characteristic of a field. ($\mathrm{char}(\mathbb{Q}) = 0$, $\mathrm{char}(\mathbb{F}_p) = p$.)
- Linear system of m-equations in n-unknowns. ($AX = b$)
- Homogeneous system of equations. ($AX = 0$)
  - Trivial solution of a homogeneous system.
  - Nontrivial " " " " .
- Equivalent systems of equations: $(AX = b) \sim (BX = c)$
- Coefficient matrix of a linear system of equations.
- Augmented matrix of a " " " "
- Solving a linear system by eliminating variables $\Big\} = \Big\{$ Doing row operations on the coefficient / augmented matrix
- Elementary row operations:
$$\begin{cases} R_i + cR_j \longrightarrow R_i \\ cR_i \longrightarrow R_i \quad (c \neq 0) \\ R_i \longleftrightarrow R_j \end{cases}$$

- $A, B \in M_{m \times n}(F)$, A is row-equivalent to B if ___

- RREF = row-reduced echelon form. ②

  (a matrix is in RREF if ———)

- The leading 1's of a matrix in RREF.
- When can we multiply matrices AB

  $(m \times n)\ (n \times p) \rightsquigarrow (n \times p)$

  $(AB)_{ij} = \sum a_{ik} \cdot b_{kj}$

- A - $n\times n$ matrix ; A is invertible if ———

  A^{-1} = inverse of A (if it exists).

- $M_{n\times m}(F)$ = all $n \times m$ matrices over a field F
- $M_{n\times n}(F) = M_n(F)$ = "$n \times n$" or square matrices of size n
- I_n = $n\times n$ identity matrix = $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

- Scalar matrix $(n\times n)$: $cI_n = \begin{bmatrix} c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{bmatrix}$

- Diagonal matrix $(n \times n)$: $\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$

- Kronecker delta: $\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$; $\delta_{ij} = \delta(i,j)$

§2 a compendium of theorems / results:

Theorem 1

Equivalent systems of linear equations have same solutions, i.e., if $AX=b$ & $BX=c$ are equivalent then any solution of $AX=b$ is a solⁿ of $BX=c$, and conversely, any solⁿ of $BX=c$ is a solⁿ of $AX=b$.

Theorem 2

An elementary row operation e has an inverse row operation e' which is also of the same kind; $e'(e(A)) = A$.

e	$R_i \longrightarrow R_i + cR_j$	$R_i \longrightarrow cR_i$	$R_i \longleftrightarrow R_j$
e'	$R_i \longrightarrow R_i - cR_j$	$R_i \longrightarrow \frac{1}{c} \cdot R_i$	$R_i \longleftrightarrow R_j$

Theorem 3.

$A, B \in M_{m\times n}(F)$ are row-equivalent $\Rightarrow$ $AX=0$ & $BX=0$ have the same set of solutions

Basically Theorem 1 restated for matrices & for homogeneous systems.

Theorem 4

Every $m\times n$-matrix A is row-equivalent to a unique matrix in RREF.

- work out an example.
- Uniqueness is not so obvious!

Theorem 5

$\boxed{\text{If } m<n \text{ then a homogeneous system of } m\text{-equations in } n\text{-unknowns has a nontrivial solution.}}$

This is a very important theorem. Given $AX=0$, $A \in M_{m\times n}(F)$, let R be in RREF & $A \sim R$. Suffices to prove $RX=0$ has nontrivial solutions.

$\left\{\begin{array}{l} \text{Leading 1's of } R \longleftrightarrow \text{Leading Variables} \longleftrightarrow \text{Dependent variables.} \\ \text{All other variables are independent variables.} \end{array}\right.$

Theorem 6

- $A \in M_{m\times n}(F) \Longrightarrow I_m \cdot A = A$, $A \cdot I_n = A$.
- $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$, $C \in M_{p\times r}(F)$ then

$$(AB)C = A(BC) \qquad \text{(associative law)}$$

Proof is an exercise! Do it!

Theorem 8

- A-invertible $\Rightarrow$ A^{-1} is invertible & $(A^{-1})^{-1} = A$.
- A, B invertible $\Rightarrow$ AB is invertible & $(AB)^{-1} = B^{-1}A^{-1}$
- A-invertible $\Rightarrow$ ${}^tA$ is invertible & $({}^tA)^{-1} = {}^t(A^{-1})$
- an elementary matrix is invertible; its inverse is also an elementary matrix (of the same kind).

Since $({}^tA)^{-1} = {}^t(A^{-1})$, we are justified in writing: ${}^tA^{-1}$.

Theorem 9

The following are equivalent for $A \in M_{n\times n}(F)$.

(i) A is invertible.

(ii) A is row-equivalent to I_n.

(iii) A is a product of elementary matrices.

(iv) $AX = 0$ has only the trivial solution.

(v) $AX = b$ has a unique solution for any $b \in M_{n\times 1}(F)$.

If you can find a proof for theorem 9, then you have mastered the concepts of this chapter. As a mental exercise try to prove each of the $\binom{5}{2} = 10$ implications.

Theorem 10

i) If A has a left-inverse then A is invertible.
$(\exists B \text{ s.t. } BA = I_n)$

ii) If A has a right-inverse then A is invertible.
$(\exists C \text{ s.t. } AC = I_n)$

(iii) $A = A_1 \cdots A_k$ is invertible $\Longleftrightarrow$ each A_i is invertible.

$$(A_1 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}$$

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Lecture 7 — Chapter 2 (2.1–2.4) of Textbook — Review-2

- Vector space : Definition & Examples
- Linear combination; Subspace; Span; Linear independence; Basis; Dimension;
- Ordered basis; coordinates of a vector; change of basis.

§1 A list of concepts / Definitions:

- Vector space V over a field F
  - Scalars (elements of F)
  - vectors (elements of V)
  - vector addition $+$ $\quad (V, +, 0)$ - abelian group
  - scalar multiplication.

- Examples of vector spaces:
  - $\mathbb{R}^n$: "euclidean space" $/\mathbb{R}$ $\qquad$ - $\mathbb{Z}^n$ is not a vector space
  - $\mathbb{C}^n$: "unitary space" $/\mathbb{C}$
  - $\mathbb{Q}^n$ is a vector sp $/\mathbb{Q}$.
  - F^n is a vector space $/F$. $\qquad$ (Ex: $F^n \cong M_{n\times 1}(F) \cong M_{1\times n}(F)$)
  - $M_{m\times n}(F)$: space of $m\times n$ - matrices $/F$
  - $\mathrm{Funct}(S, F) = \{ f : S \to F \}$ set of all functions from S to F.
  - $F[x]$: all polynomials in one variable x over F
    (Ex: $F[x] = \mathrm{Funct}(\mathbb{Z}_{\geq 0}, F)$; $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots, n, \dots\}$.)
  - $\mathcal{C}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$

    $\mathcal{C}([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \}$

    $\mathcal{C}^\infty(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable} \}$

- Linear combination of vectors $v_1, \dots, v_n$ in V.
- Subspace W of a vector space V.

  e.g. – $A \in M_{m \times n}(F)$, $N_A = \{ X \in F^{n \times 1} \mid AX = 0 \}$ is a subspace of F^n.
- Subspace of V spanned by a set $S = \{v_1, \dots, v_n\}$ of vectors.
- $S \subset V$ is linearly independent. (LI)
- $B \subset V$ is a basis $\begin{cases} B \text{ spans } V \\ B \text{ is L.I.} \end{cases}$
- Standard basis for F^n

  $\mathcal{B} = \{ \varepsilon_1 = (1, 0, \dots, 0), \ \varepsilon_2 = (0, 1, 0, \dots, 0), \ \dots, \ \varepsilon_n = (0, 0, \dots, 0, 1) \}$
- Standard basis for $M_{m \times n}(F)$, $\{ E_{ij} \mid 1 \leq i, j \leq n \}$.
- Dimension of a vector space V/F ; $\dim_F(V)$.
- Sum of subspaces: $W_1 + W_2 = \{ \omega_1 + \omega_2 \mid \omega_1 \in W_1, \ \omega_2 \in W_2 \}$
- Ordered basis B of a vector space V/F.
- The coordinates of a vector $v \in V$ w.r.t. an ordered basis $\mathcal{B} = \{v_1, \dots, v_n\}$

  $v = \sum_{i=1}^{n} a_i v_i \implies [v]_{\mathcal{B}} = (a_1, \dots, a_n)$
- $\mathcal{B} = \{v_1, \dots, v_n\}$, $\mathcal{B}' = \{\omega_1, \dots, \omega_n\}$ ordered bases for V.

  Change of basis matrix $P : \mathcal{B} \rightsquigarrow \mathcal{B}'$ $P = [P_{ij}]$

  $\omega_j = \sum_{i=1}^{n} P_{ij} v_i$

§ A Compendium of Theorems :

Theorem 1

Let S be a subset of a vector space $V/_F$.

The subspace spanned by S is the set of all linear combination of vectors in S :

$$\langle S \rangle = \{ c_1 v_1 + \cdots + c_k v_k : c_1, \ldots, c_k \in F; \; v_1, \ldots, v_k \in S \}$$

Theorem 2

Let V be a vector space over a field F.

Suppose a finite set $\{v_1, \ldots, v_m\}$ spans V. Then any linearly independent set in S has at most m vectors.

- an important technical theorem towards "dimension of a vector sp."
- Uses the fact that a homogeneous system of equations, with #(equations) < #(variables), has a nontrivial solution.

Theorem 3 (Dimension of a vector space)

Let V be a vector space$/F$.

Any two bases have the same number of elements.

- This common number is called dimension of the vector space
- You should know a proof for finite-dimensional spaces.

(It is also true for infinite-dimensional spaces.)

Theorem 4

Suppose $\dim_F(V) = n$.

Let S be a subset of V, and $|S|$ = # of vectors in S.

(i) $|S| > n \implies S$ is linearly dependent.

(ii) $|S| < n \implies S$ cannot span V.

<u>Note</u>: $|S| = n$ does not mean S is a basis.

Lemma 5 (Building a basis of V)

Let S be a linearly independent set in V.

Suppose $v \in V$ is not in the $\text{Span}(S)$ = subspace spanned by S.

$v \notin \text{Span}(S) \implies S \cup \{v\}$ is also linearly independent.

Corollaries to Theorem 4 & Lemma 5

- W is a subspace of $V \implies \dim(W) \leq \dim(V)$.
- W is a proper subspace of $V \implies \dim(W) < \dim(V)$
- Any nonzero vector of V is part of a basis for V.
- Any linearly independent set in V is part of a basis for V

Theorem 6

Let W_1, W_2 be finite-dimensional subspaces of a vector space V.

Then $W_1 + W_2$ and $W_1 \cap W_2$ are finite-dimensional.

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

- Vector space analogue of the counting principle of inclusion-exclusion

Theorem 7 (Coordinates & change of basis)

Let V be an n-dimensional vector space.

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ & $\mathcal{B}' = \{w_1, \dots, w_n\}$ be ordered bases of V.

Let $P : \mathcal{B} \rightsquigarrow \mathcal{B}'$ be the change of basis matrix: $\boxed{w_j = \sum_{i=1}^{n} P_{ij} v_i}$

(i) P is invertible

(ii) For $v \in V$, $v = \sum a_i v_i$, $[v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $v = \sum c_i w_i$, $[v]_{\mathcal{B}'} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Then $\boxed{[v]_{\mathcal{B}} = P \cdot [v]_{\mathcal{B}'}.}$ $\quad (n \times 1) = (n \times n)(n \times 1)$

- Memorize the recipe for the change of basis matrix.
- To remember $[v]_{\mathcal{B}} = P \cdot [v]_{\mathcal{B}'}$,

  put $v = w_j$, $[w_j]_{\mathcal{B}'} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow j^{\text{th}}. = \varepsilon_j$, then

  $P [w_j]_{\mathcal{B}'} = P \cdot \varepsilon_j = j^{\text{th}}$ column of P

  $[w_j]_{\mathcal{B}} = \sum P_{ij} v_i$ $\Bigg\}$ Definition of P.

- If $Q : \mathcal{B}' \rightsquigarrow \mathcal{B}$ is the change of basis matrix in the other direction, then $PQ = QP = I_n$.

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Lecture 8 | **Linear Transformations** | Review 3

- Linear Transf., Kernel (T), Image (T), nullity (T), rank (T).
- $\mathrm{Hom}_F(V, W) \cong M_{m\times n}(F)$
- $\mathrm{End}_F(V) \cong M_n(F)$
- Invertible linear transformation / Non-singular
- $V^* = \mathrm{Hom}_F(V, F)$ = dual of V; V^{**} = double-dual of V
- $\dim(V^*) = \dim(V)$, Dual basis.

§1 A list of concepts / Definitions :

- Linear Transformation $T : V \longrightarrow W$; V, W vect. spaces over F.

- Kernel of T = Null space of T = $\{ v \in V \mid T(v) = 0 \} = \mathrm{Ker}(T)$

  Image of T = $\{ T(v) \mid v \in V \}$ = $\mathrm{Im}(T)$

  Nullity(T) = dimension (Kernel (T)).

  rank (T) = dimension (Image(T)).

  $T : V \longrightarrow W$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \cup & & \cup \\ \mathrm{Ker}(T) & & \mathrm{Im}(T) \end{array}$$

- $\mathrm{Hom}_F(V, W)$ = set of all F-linear transformations from V to W.

  Linear Transformation = "homomorphism" or "morphism" of vector spaces.

- Linear Operator = $T : V \longrightarrow V$, also called "endomorphism".

  $\mathrm{End}_F(V) = \mathrm{Hom}_F(V, V)$

- Invertible linear transformation: $T : V \longrightarrow W$ is invertible, if $\exists$ linear transformation $U : W \longrightarrow V$ such that

$$TU = 1_W , \quad UT = 1_V .$$

  Invertible $\Leftrightarrow$ bijection. T : isomorphism.

- Non-singular linear transformation:

  Non-singular $\Leftrightarrow$ T is 1:1 $\Leftrightarrow$ T is injective $\Leftrightarrow$ $\mathrm{Ker}(T) = 0$.

- Matrix of a linear transformation: $T : V \longrightarrow W$

  $\mathcal{B} - \{v_1, \dots, v_n\}$ basis for V.

  $\mathcal{B}' = \{w_1, \dots, w_m\}$ " " W.

  $[T] = [T]_{\mathcal{B},\mathcal{B}'} =$ $m \times n$ matrix $[a_{ij}]$

  $$\boxed{T(v_j) = \sum_{i=1}^{m} a_{ij} w_i}$$

  MEMORIZE !!

- $A, B \in M_n(F)$ are similar if $\exists$ invertible matrix P such that $B = P^{-1}AP$. (or $A = PBP^{-1}$).

- A linear functional on a vector space V: $f : V \longrightarrow F$, i.e., f is a linear transf. from V to the base field F.

  $V^* = \text{Hom}_F(V, F) =$ set of all linear functionals on V

  $=$ dual vector space.

  $V^{**} = \text{Hom}_F(V^*, F) =$ double - dual of V

§2. A Compendium of Theorems :

Theorem 1

A linear transformation $T : V \longrightarrow W$ is completely determined by its values on basis vectors for V. , i.e.,
if $\{v_1, \dots, v_n\}$ is a basis for V, (assuming $\dim_F(V) = n$)
$\{w_1, \dots, w_n\}$ any set of n vectors in W, then
there is a unique linear transformation $T : V \longrightarrow W$ such that
$$T(v_i) = w_i .$$

Theorem 2

Let $T : V \longrightarrow W$ be a linear transformation. Suppose $\dim(V) < \infty$.
$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$$

$\{v_1, \dots, v_k\}$ - basis for $\operatorname{Ker}(T)$
build it up to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V.
Then $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $\operatorname{Im}(T)$.

Theorem 3

If V, W are vector spaces$/_F$ then $\operatorname{Hom}_F(V, W)$ is a vector space$/_F$.
If V, W are finite-dimensional, then so is $\operatorname{Hom}_F(V, W)$,
$$\dim\left(\operatorname{Hom}_F(V, W)\right) = \dim(V) \cdot \dim(W)$$

$\{v_1, \dots, v_n\}$ - basis for V, $\{w_1, \dots, w_m\}$ - basis for W.

Define T_{ij}, $\begin{matrix} 1 \le i \le n \\ 1 \le j \le m \end{matrix}$: $T_{ij}(v_k) = \delta_{ik} \cdot w_j$

Theorem 4.

Let V be a vector space$/_F$ then $\mathrm{End}_F(V)$ is an F-algebra:

- $\mathrm{End}_F(V)$ is a vector space$/_F$
- $\exists$ multiplication: $T, U \in \mathrm{End}_F(V) \Rightarrow TU : V \longrightarrow V$ is given by composition $(TU)(v) = T(U(v))$; $TU = T \circ U$.
- $\exists$ identity element: $I \in \mathrm{End}_F(V)$: $IT = TI = T$, $\forall\, T \in \mathrm{End}_F(V)$

(See Lemma on p. 77 of the textbook)

Theorem 5

$T : V \longrightarrow W$ is non-singular if T preserves the property of linear independence, i.e., if

$\{v_1, \dots, v_n\}$ is LI in $V \implies \{T(v_1), \dots, T(v_n)\}$ is LI in W.

Theorem 6

Let V, W be finite-dimensional vector spaces$/_F$.

Suppose $\dim(V) = \dim(W)$. Then, the following are equivalent:

(i) T is invertible. (T is a bijection)

(ii) T is non-singular (T is an injection)

(iii) T is onto (T is a surjection)

Theorem 7

Let V be an n-dimensional vector space.

Let W be an m-dimensional vector space.

(i) A choice of basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for V gives an isomorphism $V \xrightarrow{\sim} F^n$. $V \ni v = \sum a_i v_i \longmapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n$.

Similarly, $\mathcal{B}' = \{w_1, \ldots, w_m\}$ gives an isomorphism $W \xrightarrow{\sim} F^m$

(ii) choice of bases $\mathcal{B}$ & $\mathcal{B}'$ as in (i) gives an isomorphism: of vector spaces.

$$\mathrm{Hom}_F(V, W) \xrightarrow{\sim} M_{m\times n}(F).$$
$$T \longmapsto [T].$$

(iii) choice of basis $\mathcal{B}$, gives an isomorphism of F-algebras:

$$\mathrm{End}_F(V) \xrightarrow{\sim} M_n(F),$$

in particular, $[TU] = [T]\cdot[U]$

$$V \simeq F^n, \quad \mathrm{Hom}(V, W) \simeq M_{m\times n}(F), \quad \mathrm{End}_F(V) \simeq M_n(F).$$

Theorem 8

V is an n-dimensional vector space

$\mathcal{B} = \{v_1, \ldots, v_n\}$ & $\mathcal{B}' = \{v_1', \ldots, v_n'\}$ bases for V.

$P : \mathcal{B} \rightsquigarrow \mathcal{B}'$ change of basis matrix: $v_j' = \sum P_{ij} v_i$

$T \in \mathrm{End}_F(V)$. Then

$$[T]_{\mathcal{B}'} = P^{-1} \cdot [T]_{\mathcal{B}} \cdot P$$

Theorem 9 (The dual basis).

Let V be an n-dimensional vector space$_{\mathbb{F}}$.

Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V.

Define vectors in V^*: $\mathcal{B}^* = \{f_1, \ldots, f_n\}$ as:

$$f_i(v_j) = \delta_{ij}$$

Then $\{f_1, \ldots, f_n\}$ is a basis for V^*; in particular

$$\dim(V^*) = \dim(V)$$

Theorem 10 (The double-dual)

Let V be an n-dimensional vector space$_{\mathbb{F}}$.

Consider the map $\mathcal{E} : V \longrightarrow V^{**}$, $\quad v \longmapsto \mathcal{E}_v$,

defined via evaluation: $\quad \mathcal{E}_v(f) = f(v) \quad , \ \forall f \in V^*$.

Then $\mathcal{E}$ is an isomorphism.

$\mathcal{E}$ is a "natural isomorphism".

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Lecture 9 | Matrix Operations. | Review 4

- $A \in M_{m\times n}(F)$.
  - Row-Space(A) $\subset$ F^n
  - Column-Space (A) $\subset$ F^m
  - Null-space(A) $\subset$ F^n
- Row-Rank(A) = Column-Rank(A).
- computational questions on vector spaces via matrix operations.

§1 Row-Rank = Column-Rank

Let $A \in M_{m\times n}(F)$. m-rows & n-columns.

- Row-Space(A) = subspace of F^n spanned by the rows of A.

  Row-Rank(A) = dim(Row-Space(A)).

- Column-Space(A) = subspace of F^m spanned by the columns of A

  Column-Rank(A) = dim(Column-Space(A)).

- Null-space(A) = space of solutions of the homogeneous system $AX=0$ $\quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

  subspace of F^n.

- $T_A : F^n \longrightarrow F^m \qquad T_A(X) = AX.$

  if $A = [C_1, \dots, C_n]$; $C_j = j^{th}$ column $\in F^m$.

  $T_A(X) = AX = x_1 C_1 + \dots + x_n C_n.$

  Kernel of T_A = Null-space of A

  Image of T_A = $\{ b \in F^m \mid AX = b \text{ has a solution} \}$

- R = row-reduced echelon form of A

  r = number of nonzero rows of R.

  = # leading 1's.

  = # dependent variables of $AX=0$

Theorem: 1

- Row Space (A) = Row Space (R).
- Row-Rank $(A) = r$
- Nullity $(T_A) = n-r$ (# free variables of $AX=0$).
- Image (T_A) = Column-Space (A)
- Rank (T_A) = Column-Rank (A).
- Column Rank (A) = Row-Rank (A).

Theorem 2 (Invertibility)

Let $A \in M_{n\times n}(F)$. The following are equivalent:

1. A is invertible.
2. Row Space $(A) = F^n$
3. Row Rank $(A) = n$
4. Column Space $(A) = F^n$
5. Column Rank $(A) = n$.
6. Row-Reduced Echelon form of $A = I_n$.

§2. Vector Space Computations Using Matrices:

($V = F^n$, n-dimensional vector space, $\{\varepsilon_1, \dots, \varepsilon_n\}$ - Standard basis)

1. Given $w_1, \dots, w_m \in F^n$,
   Determine $\dim(\mathrm{Span}\{w_1, \dots, w_m\})$

2. Given $w_1, \dots, w_m \in F^n$, given $v \in F^n$,
   Determine if $v \in \mathrm{Span}\{w_1, \dots, w_m\}$.

3. Given $A \in M_n(F)$,
   Determine if A is invertible, and if yes, find its inverse.

(Reading assignment: Read Example 22 on p.63 of the text book.)

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Lecture 10 | Eigenvalues & Diagonalizable Operators

- Eigenvalues & Eigenvectors
- characteristic Polynomial
- Diagonalizable Operator
- Linear Independence of "distinct" eigenspaces
- Diagonalizability.

§10.1 Eigenvalues & Eigenvectors & Eigenspace

- V = n-dimensional vector space $/F$

  $T : V \longrightarrow V$ linear operator on V.

GOAL: If possible find a basis $\{v_1, \dots, v_n\}$ for V such that $T(v_i) = c_i v_i$, $c_i \in F$. Then $[T] = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}$

When this is possible, it will be very easy to answer any kind of question about T.

- Defn: (i) An eigenvalue for T is a scaler $c \in F$, for which there is a nonzero vector v such that $T(v) = cv$.

(ii) If $c \in F$ is an eigenvalue then, any vector v (including 0) for which $Tv = cv$ is called an eigenvector for that eigenvalue.

(iii) If $c \in F$ is an eigenvalue, then

$$V(c) = \{ v \in V \mid Tv = cv \} = \text{set of all eigenvectors for } c$$

is called the eigenspace of c.

- $\left.\begin{array}{l} \text{Eigenvalue} = \text{Characteristic value} \\ \text{Eigenvector} = \text{characteristic vector} \\ \text{Eigenspace} = \text{characteristic space} \end{array}\right\}$ Terminology in the textbook.

- **Theorem**

$$\boxed{\begin{array}{l} \text{The following are equivalent:} \\ \text{(i) } c \text{ is an eigenvalue of } T \\ \text{(ii) } T - cI \text{ is singular, i.e., nontrivial kernel.} \\ \text{(iii) } \det(T - cI) = 0. \end{array}}$$

Pf:

$Tv = cv$

$\iff (T - cI)(v) = 0$

$\iff v \in \operatorname{Ker}(T - cI)$. $\blacksquare$

§10.2 The characteristic polynomial:

- We can transfer our attention to matrices $A \in M_n(F)$.

Defn: An eigenvalue of A is a scalar $c \in F$ such that

$$\det(cI - A) = 0.$$

Suppose $\mathcal{B}, \mathcal{B}'$ are ordered bases for V, and $A = [T]_{\mathcal{B}}$ & $B = [T]_{\mathcal{B}'}$, then $B = P^{-1}AP$, for the change of basis matrix $P: \mathcal{B} \rightsquigarrow \mathcal{B}'$. Then

(*)
$$\det(cI - B) = \det(cI - P^{-1}AP) = \det(cP^{-1}P - P^{-1}AP)$$
$$= \det(P^{-1}(cI - A)P) = \det(cI - A).$$

$\therefore$ c is an eigenvalue of T $\iff$ c is an eigenvalue for $A = [T]_{\mathcal{B}}$

$\iff$ c " " " $B = [T]_{\mathcal{B}'}$.

- **Defn:** $A \in M_{n\times n}(F)$. The characteristic polynomial $f_A(x)$ is a monic polynomial of degree n with coefficients in F defined as:

$$f_A(x) = \det(xI - A).$$

- **Defn:** $T: V \to V$ linear operator & $A = [T]_{\mathcal{B}}$ for some ordered basis $\mathcal{B}$, then the characteristic polynomial of T is

$$f_T(x) = f_A(x).$$

The above calculation (*) shows that the definition of $f_T(x)$ is independent of the choice of $\mathcal{B}$.

- **Proposition:**

$$\boxed{\text{The eigenvalues of } T = \text{roots of } f_T(x), \text{ i.e., solutions of } f_T(x) = 0,}$$

§10.3 Diagonalizable operator:

- **Defn:** A linear operator $T : V \longrightarrow V$ is said to be diagonalizable if V has a basis consisting of eigenvectors of T, i.e., $\exists$ basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ and scalars $c_1, \ldots, c_n$ (need not be distinct) such that $T(v_i) = c_i v_i$; hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}.$$

- Let's draw some consequences of T being diagonalizable. Let's collect together the scalars $c_1, \ldots, c_n$ by grouping those that are equal; relabel them; suppose

c_1 appears d_1 many times, ..., c_k appears d_k many times; then

$$[T]_{\mathcal{B}} = \begin{bmatrix} c_1 I_{d_1} & & & \\ & c_2 I_{d_2} & & \\ & & \ddots & \\ & & & c_k I_{d_k} \end{bmatrix}; \qquad d_1 + d_2 + \cdots + d_k = n$$

Then
$$f_T(x) = \det\left(\begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix} - \begin{bmatrix} c_1 I_{d_1} & & \\ & \ddots & \\ & & c_k I_{d_k} \end{bmatrix} \right) = \det \begin{bmatrix} (x-c_1) I_{d_1} & & \\ & \ddots & \\ & & (x-c_k) I_{d_k} \end{bmatrix}$$

$$= (x-c_1)^{d_1} (x-c_2)^{d_2} \cdots (x-c_k)^{d_k}$$

- $V(c_j)$ = eigenspace for c_j. $\quad V(c_j) = \operatorname{Ker}(c_j I - T)$

e.g. $V(c_1) = \text{Null Space} \begin{bmatrix} 0_{d_1} & & & \\ & (c_1-c_2) I_{d_2} & & \\ & & \ddots & \\ & & & (c_1-c_k) I_{d_k} \end{bmatrix}$; $\dim(V(c_1)) = d_1$

$\therefore \dim(V(c_1)) + \dim(V(c_2)) + \cdots + \dim(V(c_k)) = d_1 + \cdots + d_k = n$,

the sum of the dimensions of the distinct eigenspaces is the dimension of the ambient vector space V.

Theorem

Let $T: V \longrightarrow V$ be a linear operator. ; $\dim(V) = n$.

Let $c_1, \dots, c_k$ be distinct eigenvalues of T;

$V(c_j) = \ker(c_j I - T)$ = eigenspace for c_j. TFAE:

(i) T is diagonalizable

(ii) $f_T(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$, $d_i = \dim(V(c_i))$

(iii) $\dim(V(c_1)) + \cdots + \dim(V(c_k)) = \dim(V)$.

Pf: We have seen (i) $\Rightarrow$ (ii) & (iii).

(ii) $\Rightarrow$ (iii) is obvious because $\deg(f_T(x)) = n = \dim(V)$.

(iii) $\Rightarrow$ (i) needs a proof.

The key point in the proof of (iii) $\Rightarrow$ (i) is that

Lemma

eigenvectors for distinct eigenvalues are linearly independent.

Say $w_1 \in V(c_1), \dots, w_k \in V(c_k)$ & $w_1 + w_2 + \cdots + w_k = 0$.

apply T: $c_1 w_1 + \cdots + c_k w_k = 0$

apply T^2: $c_1^2 w_1 + \cdots + c_k^2 w_k = 0$. ... etc ... take linear combinations

For any polynomial $g(x) \in F[x]$, we have:

$$g(c_1) w_1 + g(c_2) w_2 + \cdots + g(c_k) w_k = 0.$$

For $j=1$, take $g_1(x) = \dfrac{(x-c_2)\cdots(x-c_k)}{(c_1-c_2)\cdots(c_1-c_k)}$. Then $g_1(c_1) = 1$

$g_1(c_2) = \cdots = g_1(c_k) = 0$

For $g = g_1$, we get: $w_1 = 0$. Similarly $w_2 = \cdots = w_k = 0$.

Hence: If $\mathcal{B}_1$ = basis for $V(c_1)$, ..., $\mathcal{B}_k$ = basis for $V(c_k)$

then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is a basis for V.

$\Rightarrow$ T is diagonalizable ———— $\blacksquare$.

§10.4 <u>Computations:</u> $A \in M_n(F)$ (A matrix of an op. T)

- Eigenvalues: compute $f_A(a) = \det(aI_n - A)$.

  solve $f_A(a) = 0$.

  Suppose distinct solutions are $c_1, \ldots, c_k$.

- Eigenspaces: For each c_j, find

  $V(c_j) = \{ x \in F^n \mid Ax = c_j x \}$.

- If $\dim(V(c_1)) + \cdots + \dim(V(c_k)) = n$ then A is diagonalizable.

  If " " $\neq n$ then A is not " .

- When diagonalizable:-

  $\mathcal{B}_1 = \{x_{11}, \ldots, x_{1d_1}\}$ - basis for $V(c_1)$ $\quad x_{ij} \in F^n$.

  $\vdots \qquad\qquad\qquad \vdots$

  $\mathcal{B}_k = \{x_{k1}, \ldots, x_{kd_k}\}$ - " " $V(c_k)$.

- Put P = $n \times n$-matrix whose columns are built out $\mathcal{B}_1, \ldots, \mathcal{B}_k$.

  $P = [x_{11}\ x_{12} \cdots x_{1d_1}\ \ x_{21} \cdots x_{2d_2} \cdots x_{k1} \cdots x_{kd_k}]$

  Put $D = \begin{bmatrix} c_1 & & & & & & \\ & \ddots & & & & & \\ & & c_1 & & & & \\ & & & c_2 \ddots c_2 & & & \\ & & & & \ddots & & \\ & & & & & c_k & \\ & & & & & & \ddots\ c_k \end{bmatrix} = \begin{bmatrix} c_1 I_{d_1} & & & \\ & c_2 I_{d_2} & & \\ & & \ddots & \\ & & & c_k I_{d_k} \end{bmatrix}$

EX Show that $A \cdot P = P \cdot D$

Conclude $P^{-1} A P = D$

Read EXAMPLE 3, p. 187–189 of the textbook.

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