

Real Analysis

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Chirayu Gupta

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1 Real and complex number systems

The need for the real number system is replacing the rational number system is because the rational number system has "gaps".

Definition of a field: Set F with 2 operations.

- **Axioms of Addition:**

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F .

(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.

(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.

(A4) F contains an element O such that $O + x = x$ for every $x \in F$.

(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = O.$$

- **Axioms of Multiplication:**

(M1) If $x \in F$ and $y \in F$, then their product xy is in F .

(M2) Multiplication is commutative:

$$xy = yx$$

for all $x, y \in F$.

(M3) Multiplication is associative:

$$(xy)z = x(yz)$$

for all $x, y, z \in F$.

(M4) F contains an element $1 \neq O$ such that

$$1x = x$$

for every $x \in F$.

(M5) If $x \in F$ and $x \neq O$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1$$

- **Distributive Law:**

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

Definition 1.1. An ordered relation ($<$) is:

- If $x \in S$ and $y \in S$ then one and only of the following statements are true:

$$x < y, \quad x = y, \quad y < x$$

is true

- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Definition 1.2. If S has an ordered relation we call it an ordered set.

Definition 1.3. Start with an ordered set S . Let $E \subset S$ be nonempty. If $\exists \beta \in S$ such that $\forall x \in E$ we have $x \leq \beta$ then we say β is an **upper bound** of E .

Definition 1.4. Start with an ordered set S . Let $E \subset S$ be nonempty. Supposed E bounded above. Suppose that $\exists \alpha \in S$ with the following:

- α is an upper bound of E
- $\gamma < \alpha$ then γ is not an upper bound

then α is called **least upper bound** or supremum.

$$\sup(E) = \alpha$$

Definition 1.5. An ordered set has least upper bound property if for a non empty $E \subset S$, E bounded above then

$$\sup(E) \in S$$

Remark. Rational numbers do not satisfy this property

Same property for infimum.

Theorem 1.1. Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B . Then $\alpha = \sup L$ exists in S , and $\beta = \inf B$

Definition 1.6. An ordered field is a field F which is also an ordered set, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,

(ii) $xy > 0$ if $x \in F, y \in F, x > 0$, and $y > 0$.

If $x > 0$, we call x positive; if $x < 0$, x is negative.

Theorem 1.2. There exists an ordered field R which has the least-upper-bound property.

Moreover, R contains Q as a subfield.

This is called **existence theorem**