Quantum Field Theory
PH

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1 Lecture 1

All states contribute to relativistic corrections:

$$H \to H + \partial V$$

Where the corrections in energy is:

$$E_{corr} = \langle 0| \partial V |0\rangle + \sum_n \frac{|\langle 0| \partial V |n\rangle|^2}{E_0 - E_n} + ...$$

As you can see it depends on n.
Since for relativistic cases processes like

$$p + p \to p + p + \pi_0$$

are possible. That is new particles can be created, the number of such states $|n\rangle$ are infinite. Typically, the relativistic corrections are af the order $\mathcal{O}(v^2)$ because, typically the denominator is mc^2 and since

$$\frac{E}{mc^2} = \frac{\gamma c^2}{c^2} = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{v^2}{c^2}$$

2 Lecture 2

2.1 Natural Units

- $h_{cross} = 1$
- $c = 1$

Above 2 conditions are for natural units. There are implications on dimentions because of this which are:

- $[m] = [E] = [L^{-1}] = [T^{-1}]$

2.2 Theory of Single free spinless particle of mass μ[From coleman notes]

Momentum operator:

$$\vec{\mathbf{P}}\,|k\rangle = k\,|k\rangle$$

$$\langle k|k'\rangle = \partial(k-k')$$

Completeness: $\int \langle k|k\rangle\, d^3k = 1$
We know:

$$\psi(k) = \langle k|\psi\rangle$$

Also classically we know:

$$H\,|k\rangle = |k|^2/2\mu\,|k\rangle$$

To introduce relativity we introduce:

$$H\,|k\rangle = \sqrt{|k|^2+\mu^2}\cdot|k\rangle = \omega\,|k\rangle$$

How do we know this makes the theory relativistic(or Lorentz invariant)?
First we define Translational and Rotational Invariance:

2.3 Translational Invariance

For a linear operator$U$ specifying a translation: we have

$$U(a)U(a)^\dagger = 1 \tag{1}$$

$$U(0) = 1 \tag{2}$$

$$U(a)U(b) = U(a+b) \tag{3}$$

The U satisfying these is $U(a) = e^{\iota\mathbf{p}a}$ where $\mathbf{p} = (H,\vec{\mathbf{p}})$

$$U(a)\,|0\rangle = |a\rangle$$

$$\langle a|\,O(x+a)\,|a\rangle = \langle 0|\,O(x)\,|0\rangle$$

2.4 Rotational Invariance

Similar to [Equation 1 - 3], we have equations for rotation

$$U(R)U(R)^\dagger = 1 \tag{4}$$

$$U(1) = 1 \tag{5}$$

$$U(R_1)U(R_2) = U(R_1R_2) \tag{6}$$

A U satisfying all these is given by

$$U(R)\left|\vec{\mathbf{k}}\right\rangle = \left|R\vec{\mathbf{k}}\right\rangle$$

TODO: Put Lorentz invariance and the relativistic contour integral calculations from coleman

3 Lecture 3

3.1 Klien-Gordan equation

A heuristic way to derive SC equation:

$$E = \frac{P^2}{2m}$$

$$E \to \iota \frac{\partial}{\partial t}$$

$$p_j \to -\iota \frac{\partial}{\partial x_j}$$

Action is defined as:

$$\mathbf{S}[\phi] = \int (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2(x))$$

Similarly we derive KG eqn:

$$E^2 = p^2 + m^2$$

$$(\frac{\partial}{\partial t} - \nabla^2)\phi + m^2 \phi(x,t) = 0$$

A solution to this equation is $e^{\iota(k \cdot x + \omega t)}$. When you apply $\hat{H}$ we get energies like $-\omega\phi$.**Negative Energy solution**. If you perturb this system slightly it keeps going down the potential well(since energy not bounded from below). In QM continuity eqn:

$$\nabla J + \frac{\partial \rho}{\partial t} = 0$$

$$J_i = \frac{1}{2im}(\phi^* \partial \phi - (\partial \phi^*)\phi)$$

$$\rho = \frac{\iota}{2m}(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t})$$

With KG its same but ρ is not positive definite.
Hence, KG equation failed!.
TODO: Dont really understand why KG failed

3.2 Rotation

Rotation is like a gauge transformation: we apply some transformation on $\hat{A}, \hat{\phi}$ which does not change E and B. The choice of coordinations/description has nothing to do with the system!!
Quantites that don't change under a transformation are **scalar** under that transformation.
Eg: $x^2 + y^2$ does not depend on choice of axes given a origin.
A scalar is always associated with a transformation.

3.3 Scalar Field, Vector Field

Given a point and density of air(ρ) we take two coordinate systems or 2 different description:

$$\rho(\vec{\mathbf{x}}) \to \rho(R\vec{\mathbf{x}})$$

Rho is a scalar field.
For a vector field under rotation:

$$E_x(\vec{\mathbf{X}}, \vec{\mathbf{Y}}), E_y(\vec{\mathbf{X}}, \vec{\mathbf{Y}}) \to RE(R\vec{\mathbf{X}})$$

Both field components and arguments changed.
Question: I do a rotation on the coordinates then what is the effent on components of $\psi_n(\vec{\mathbf{X}})$

$$\mathbf{R}\psi_{ij}(R\vec{\mathbf{x}})$$

$\mathbf{R}$ is representation of the Rotation group $SO(3)$

3.4 Lorentz Group

Group is

$$SO(1,3)$$

not $SO(4)$ in which

$$(-x_0^2 + \sum x_i^2)$$

is invariant.
Transformation is represented by:

$$\mathbf{R}_{ij}\psi_j(\Lambda\vec{\mathbf{x}})$$

$$\eta = Diag(-1, 1, 1, 1) \tag{7}$$

Lambda satisfies:

$$\Lambda^t \eta \Lambda = \eta$$

$$\Lambda_0^0 \geq 1 \to Orthochoronum$$

$$\Lambda_0^0 \leq 1 \to Not\ Considered\ as\ impllies\ time\ reversal$$

Formally Lorentz group defined as $SO^+(1,3)$ to denote orthochronum.
For $SO(1,3)$ $Det\Lambda = 1$ For parity: $\Lambda = diag(-1, 1, 1, 1)$
For time reversal: $\Lambda = diag(1, -1, -1, -1)$
Parity and time reversal are outside the $SO(1,3)$ group.
For 2-D parity is inside $SO(1,3)$ (TODO:Why?)

4 Lecture 4

4.1 Classical Field Theory

$$\partial \mathbf{S} = \partial \int dt L(x, \dot{x}) = 0$$

Subject to constraint:

$$\partial x|_{t_1, t_2} = 0$$

For QFT:

$$\mathbf{S} = \int d^4x \mathbf{L}(\phi(x), \partial_\mu \phi(x))$$

$$\partial \mathbf{S} = 0$$

Boundary condition:

$$\partial \phi(x) = 0$$

$$\delta S = \int d^4x [\frac{\partial L}{\partial \phi(x)}]\delta\phi(x) + \frac{\partial L}{\partial \partial_\mu \phi(x)} \partial(\delta\phi(x))$$

$$\delta S = \int d^4x [\frac{\partial L}{\partial \phi(x)}]\delta\phi(x) + -\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x)} \delta\phi(x) + \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x)} \delta\phi$$

On solving we get:

$$\frac{\partial L}{\partial \phi(x)} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi(x))} = 0$$

- In a Boundary condition: we set ϕ at t_1 and t_2. For a double derivative equation: we can have only 2 boundary condition hence cant set velocity at t_1 and t_2 as well
- Lagrangian is taken to be real
- If Lagrangian not real Hamiltonian also not real$\rightarrow$ problem in time evolution
- Lagrangian is Lorentz invariant

$$L = (\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2$$

$$\frac{\partial L}{\partial \phi} = -2m^2\phi$$

$$\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} = 4\partial_\mu(\partial^\mu \phi)$$

This gives KG equation(TODO:How?)

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

If we write a new Lagrangian:

$$L' = -\phi \partial_\mu \partial^\mu - m^2\phi^2$$

$$= L - \partial_\mu(\phi \partial^\mu \phi)$$

On putting in action principle we get an extra $\partial \phi$ (which can be cancelled as we can change that in BT) and $\partial_\mu(\phi \partial^\mu \phi)$ which need to be cancelled **hence in defining Lagrangian like this we might need to add an extra boundary terms** which can be important in many theories.

$$S = \int d^4 L + BT$$

4.2 Global Symeetries(Reference Wienberg: Section 7.3)

$$\psi(x) \to \psi(x) + \iota\epsilon F(x)$$

$$\delta S = \iota\epsilon \int^{F(x)} \frac{\delta S[\psi]}{\delta\psi(x)} = 0$$

Then this transformation is the symmetry of the transformation.
Now, if epsilon is function of x:

$$\delta S = \int -J^{\mu}(x)\frac{\partial\epsilon(x)}{\partial X^{\mu}}$$

Integrating by parts,

$$-\int \frac{\partial J^{\mu}\epsilon(x)}{\partial X^{\mu}} + \frac{\partial J^{\mu}}{\partial X^{\mu}}\epsilon(x)$$

Implies,

$$\frac{\mathrm{d}J^{\mu}}{\mathrm{d}X^{\mu}} = 0$$

$$\partial_{\mu}J^{\mu}(x) = 0$$

This is known as **Noether's Theorem**

5 Lecture 5

5.1 Recap

Now consider the Lagrangian density is also invariant: Assuming(Global symmetry):

$$\frac{\partial L}{\partial\phi}F(x) + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\mu}F(x) = 0$$

$$S = \int d^4x L(\phi, \partial_{\mu}\phi)$$

take $\phi \to (x) + \iota\epsilon(x)F(x)$

$$\delta S = \int \frac{\partial L}{\partial\phi}\iota\epsilon F(X) + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\mu}(\iota\epsilon F(x))$$

$$= \int d^4x \frac{\partial L}{\partial\phi}\iota\epsilon F(x) + \frac{\partial L}{\partial\partial_{\mu}}[\iota\partial_{\mu}\epsilon(x)F(x)] + (\iota\epsilon(x)\partial_{\mu}F(x))$$

$$= \int d^4x \frac{\partial L}{\partial\partial_{\mu}\phi}\iota F(x)\partial_{\mu}\epsilon(x)$$

$$J^{\mu} = -\frac{\partial L}{\partial\partial_{\mu}\phi}\iota F(x)$$

5.2 New

$$\phi(x) \to \phi(x+\epsilon(x))$$
$$= \phi(x) + \epsilon^{\mu}(x)\partial_{\mu}\phi$$
$$\delta S = \int d^4x[\frac{\partial L}{\partial \phi}\epsilon^{\mu}(x)\partial_{\mu}\phi + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\mu}(\epsilon(x)\partial_{\mu}\phi(x))]$$
$$= ... + \frac{\partial L}{\partial\partial_{\mu}\phi}\epsilon^{\nu}(x)\partial_{\mu}\partial_{\nu}\phi(x) + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\mu}\epsilon^{\nu}(x)\partial_{\nu}\phi(x)$$
$$\frac{\partial L}{\partial X^{\mu}} = \frac{\partial L}{\partial \phi}\partial_{\mu}\phi + \frac{\partial L}{\partial\partial_{\nu}\phi}\partial_{\mu}\partial_{\nu}\phi$$

Going back to previous eqn:

$$= \int d^4x(\epsilon^{\nu}(x)\partial_{\nu}L + \frac{\partial L}{\partial\partial_{\nu}\phi}\partial_{\mu}\epsilon_{\nu})(\partial_{\nu}\phi(x))$$

Integrating by parts

$$-\frac{\partial L}{\partial\partial_{\mu}\phi}\epsilon^{\nu}\partial_{\mu}\partial_{\nu}\phi(x) + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\mu}(\epsilon^{\nu}`6_{\nu}\phi(x))$$
$$\delta S = \int d^4x\delta^{\mu}_{\nu}\partial_{\mu}\epsilon^{\nu}(x)L - \partial_{\nu}\epsilon^{\nu}(x)L + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\nu}\phi(\partial_{\mu}\epsilon^{\nu})$$
$$= d^4x[(\partial_{\mu}\epsilon^{\nu})[-\delta^{\mu}_{\nu}L + \frac{\partial L}{\partial\partial_{\mu}\phi}\partial_{\nu}\phi]]$$
$$\delta S = -\int d^4x\partial_{\mu}\epsilon^{\nu}(x)T^{\mu}_{\nu}$$

Finally we get using $\epsilon \to 0 at\infty$ and using equation of motion(Use the idea that for that equation of motion is derived from these kinds of variations) by applying by parts

$$\partial_{\mu}T^{\mu}_{\nu} = 0$$

Where:

$$T^{\mu}_{\nu} = -\delta^{\mu}_{\nu}...$$

This means energy momentum four Vector is conserved. **This happens ONLY because Lagrangian is NOT an explicit function of x and t.**. This is not related to Noether's Theorem.