

# Quantum Field Theory

PH

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## 1 Lecture 1

All states contribute to relativistic corrections:

$$H \rightarrow H + \partial V$$

Where the corrections in energy is:

$$E_{corr} = \langle 0 | \partial V | 0 \rangle + \sum_n \frac{|\langle 0 | \partial V | n \rangle|^2}{E_0 - E_n} + \dots$$

As you can see it depends on  $n$ .

Since for relativistic cases processes like

$$p + p \rightarrow p + p + \pi_0$$

are possible. That is new particles can be created, the number of such states  $|n\rangle$  are infinite. Typically, the relativistic corrections are of the order  $\mathcal{O}(v^2)$  because, typically the denominator is  $mc^2$  and since

$$\frac{E}{mc^2} = \frac{\gamma c^2}{c^2} = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{v^2}{c^2}$$

## 2 Lecture 2

### 2.1 Natural Units

- $\hbar_{cross} = 1$
- $c = 1$

Above 2 conditions are for natural units. There are implications on dimensions because of this which are:

- $[m] = [E] = [L^{-1}] = [T^{-1}]$

## 2.2 Theory of Single free spinless particle of mass $\mu$ [From coleman notes]

Momentum operator:

$$\begin{aligned}\vec{\mathbf{P}}|k\rangle &= k|k\rangle \\ \langle k|k'\rangle &= \delta(k - k')\end{aligned}$$

Completeness:  $\int \langle k|k\rangle d^3k = 1$

We know:

$$\psi(k) = \langle k|\psi\rangle$$

Also classically we know:

$$H|k\rangle = |k|^2/2\mu|k\rangle$$

To introduce relativity we introduce:

$$H|k\rangle = \sqrt{|k|^2 + \mu^2} \cdot |k\rangle = \omega|k\rangle$$

How do we know this makes the theory relativistic (or Lorentz invariant)?

First we define Translational and Rotational Invariance:

## 2.3 Translational Invariance

For a linear operator  $U$  specifying a translation: we have

$$U(a)U(a)^\dagger = 1 \tag{1}$$

$$U(0) = 1 \tag{2}$$

$$U(a)U(b) = U(a+b) \tag{3}$$

The  $U$  satisfying these is  $U(a) = e^{i\mathbf{p}a}$  where  $\mathbf{p} = (H, \vec{\mathbf{p}})$

$$U(a)|0\rangle = |a\rangle$$

$$\langle a|O(x+a)|a\rangle = \langle 0|O(x)|0\rangle$$

## 2.4 Rotational Invariance

Similar to [Equation 1 - 3], we have equations for rotation

$$U(R)U(R)^\dagger = 1 \tag{4}$$

$$U(1) = 1 \tag{5}$$

$$U(R_1)U(R_2) = U(R_1R_2) \tag{6}$$

A  $U$  satisfying all these is given by

$$U(R)|\vec{\mathbf{k}}\rangle = |R\vec{\mathbf{k}}\rangle$$

TODO: Put Lorentz invariance and the relativistic contour integral calculations from coleman

### 3 Lecture 3

#### 3.1 Klien-Gordan equation

A heuristic way to derive SC equation:

$$E = \frac{P^2}{2m}$$

$$E \rightarrow \iota \frac{\partial}{\partial t}$$

$$p_j \rightarrow -\iota \frac{\partial}{\partial x_j}$$

Action is defined as:

$$\mathbf{S}[\phi] = \int (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2(x))$$

Similarly we derive KG eqn:

$$E^2 = p^2 + m^2$$

$$\left( \frac{\partial}{\partial 2t - \nabla^2} \right) \phi + m^2 \phi(x, t) = 0$$

A solution to this equation is  $e^{\iota(k+\omega t)}$ . When you apply  $\hat{H}$  we get energies like  $-\omega\phi$ . **Negative Energy solution.** If you perturb this system slightly it keeps going down the potential well (since energy not bounded from below). In QM continuity eqn:

$$\nabla J + \frac{\partial \rho}{\partial t} = 0$$

$$J_i = \frac{1}{2im} (\phi^* \partial \phi - (\partial \phi^*) \phi)$$

$$\rho = \frac{\iota}{2m} (\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t})$$

With KG its same but  $\rho$  is not positive definite.

Hence, KG equation failed!.

TODO: Dont really understand why KG failed

#### 3.2 Rotation

Rotation is like a gauge transformation: we apply some transformation on  $\hat{A}, \hat{\phi}$  which does not change E and B. The choice of coordinations/description has nothing to do with the system!!

Quantites that don't change under a transformation are **scalar** under that transformation.

Eg:  $x^2 + y^2$  does not depend on choice of axes given a origin.

A scalar is always associated with a transformation.

### 3.3 Scalar Field, Vector Field

Given a point and density of air( $\rho$ ) we take two coordinate systems or 2 different description:

$$\rho(\vec{x}) \rightarrow \rho(R\vec{x})$$

Rho is a scalar field.

For a vector field under rotation:

$$E_x(\vec{X}, \vec{Y}), E_y(\vec{X}, \vec{Y}) \rightarrow RE(R\vec{X})$$

Both field components and arguments changed.

**Question:** I do a rotation on the coordinates then what is the effect on components of  $\psi_n(\vec{X})$

$$\mathbf{R}\psi_{ij}(R\vec{x})$$

$\mathbf{R}$  is representation of the Rotation group  $SO(3)$

### 3.4 Lorentz Group

Group is

$$SO(1, 3)$$

not  $SO(4)$  in which

$$(-x_0^2 + \sum x_i^2)$$

is invariant.

Transformation is represented by:

$$\begin{aligned} \mathbf{R}_{ij}\psi_j(\Lambda\vec{x}) \\ \eta = \text{Diag}(-1, 1, 1, 1) \end{aligned} \tag{7}$$

Lambda satisfies:

$$\Lambda^t \eta \Lambda = \eta$$

$$\Lambda_0^0 \geq 1 \rightarrow \text{Orthochronum}$$

$$\Lambda_0^0 \leq 1 \rightarrow \text{Not Considered as implies time reversal}$$

Formally Lorentz group defined as  $SO^+(1, 3)$  to denote orthochronum.

For  $SO(1, 3)$   $\text{Det}\Lambda = 1$  For parity:  $\Lambda = \text{diag}(-1, 1, 1, 1)$

For time reversal:  $\Lambda = \text{diag}(1, -1, -1, -1)$

Parity and time reversal are outside the  $SO(1, 3)$  group.

For 2-D parity is inside  $SO(1, 3)$  (TODO:Why?)

## 4 Lecture 4

### 4.1 Classical Field Theory

$$\partial \mathbf{S} = \partial \int dt L(x, \dot{x}) = 0$$

Subject to constraint:

$$\partial x|_{t_1, t_2} = 0$$

For QFT:

$$\mathbf{S} = \int d^4x \mathbf{L}(\phi(x), \partial_\mu \phi(x))$$
$$\partial \mathbf{S} = 0$$

Boundary condition:

$$\partial \phi(x) = 0$$
$$\delta S = \int d^4x \left[ \frac{\partial L}{\partial \phi(x)} \delta \phi(x) + \frac{\partial L}{\partial \partial_\mu \phi(x)} \partial(\delta \phi(x)) \right]$$
$$\delta S = \int d^4x \left[ \frac{\partial L}{\partial \phi(x)} \delta \phi(x) + -\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x)} \delta \phi(x) + \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x)} \delta \phi \right]$$

On solving we get:

$$\frac{\partial L}{\partial \phi(x)} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi(x))} = 0$$

- In a Boundary condition: we set  $\phi$  at  $t_1$  and  $t_2$ . For a double derivative equation: we can have only 2 boundary condition hence cant set velocity at  $t_1$  and  $t_2$  as well
- Lagrangian is taken to be real
- If Lagrangian not real Hamiltonian also not real  $\rightarrow$  problem in time evolution
- Lagrangian is Lorentz invariant

$$L = (\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2$$
$$\frac{\partial L}{\partial \phi} = -2m^2 \phi$$
$$\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} = 4\partial_\mu (\partial^\mu \phi)$$

This gives KG equation(TODO:How?)

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

If we write a new Lagrangian:

$$L' = -\phi \partial_\mu \partial^\mu \phi - m^2 \phi^2$$
$$= L - \partial_\mu (\phi \partial^\mu \phi)$$

On putting in action principle we get an extra  $\partial \phi$  (which can be cancelled as we can change that in BT) and  $\partial_\mu (\phi \partial^\mu \phi)$  which need to be cancelled **hence in defining Lagrangian like this we might need to add an extra boundary terms** which can be important in many theories.

$$S = \int d^4x L + BT$$

## 4.2 Global Symmetries (Reference: Weinberg: Section 7.3)

$$\psi(x) \rightarrow \psi(x) + \epsilon F(x)$$

$$\delta S = \epsilon \int \frac{\delta S[\psi]}{\delta \psi(x)} = 0$$

Then this transformation is the symmetry of the transformation.  
Now, if epsilon is function of x:

$$\delta S = \int -J^\mu(x) \frac{\partial \epsilon(x)}{\partial X^\mu}$$

Integrating by parts,

$$- \int \frac{\partial J^\mu \epsilon(x)}{\partial X^\mu} + \frac{\partial J^\mu}{\partial X^\mu} \epsilon(x)$$

Implies,

$$\begin{aligned} \frac{dJ^\mu}{dX^\mu} &= 0 \\ \partial_\mu J^\mu(x) &= 0 \end{aligned}$$

This is known as **Noether's Theorem**