Quantum Field Theory
PH

Chirayu Gupta

1 Lecture 1

All states contribute to relativistic corrections:

$$H \to H + \partial V$$

Where the corrections in energy is:

$$E_{corr} = \langle 0| \, \partial V \, |0\rangle + \sum_n \frac{|\langle 0| \, \partial V \, |n\rangle|^2}{E_0 - E_n} + ...$$

As you can see it depends on n.
Since for relativistic cases processes like

$$p + p \to p + p + \pi_0$$

are possible. That is new particles can be created, the number of such states $|n\rangle$ are infinite. Typically, the relativistic corrections are af the order $\mathcal{O}(v^2)$ because, typically the denominator is mc^2 and since

$$\frac{E}{mc^2} = \frac{\gamma c^2}{c^2} = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{v^2}{c^2}$$

2 Lecture 2

2.1 Natural Units

- $h_{cross} = 1$

- $c = 1$

Above 2 conditions are for natural units. There are implications on dimentions because of this which are:

- $[m] = [E] = [L^{-1}] = [T^{-1}]$

2.2 Theory of Single free spinless particle of mass μ[From coleman notes]

Momentum operator:
$$\vec{\mathbf{P}} |k\rangle = k |k\rangle$$

$$\langle k|k'\rangle = \partial(k - k')$$

Completeness: $\int \langle k|k\rangle \, d^3k = 1$
We know:
$$\psi(k) = \langle k|\psi\rangle$$

Also classically we know:
$$H |k\rangle = |k|^2/2\mu |k\rangle$$

To introduce relativity we introduce:

$$H |k\rangle = \sqrt{|k|^2 + \mu^2} \cdot |k\rangle = \omega |k\rangle$$

How do we know this makes the theory relativistic(or Lorentz invariant)?
First we define Translational and Rotational Invariance:

2.3 Translational Invariance

For a linear operator$U$ specifying a translation: we have

$$U(a)U(a)^\dagger = 1 \tag{1}$$

$$U(0) = 1 \tag{2}$$

$$U(a)U(b) = U(a + b) \tag{3}$$

The U satisfying these is $U(a) = e^{\iota \mathbf{P} a}$ where $\mathbf{p} = (H, \vec{\mathbf{p}})$

$$U(a) |0\rangle = |a\rangle$$

$$\langle a| O(x + a) |a\rangle = \langle 0| O(x) |0\rangle$$

2.4 Rotational Invariance

Similar to [Equation 1 - 3], we have equations for rotation

$$U(R)U(R)^\dagger = 1 \tag{4}$$

$$U(1) = 1 \tag{5}$$

$$U(R_1)U(R_2) = U(R_1 R_2) \tag{6}$$

A U satisfying all these is given by

$$U(R) \left|\vec{\mathbf{k}}\right\rangle = \left|R\vec{\mathbf{k}}\right\rangle$$

TODO: Put Lorentz invariance and the relativistic contour integral calculations from coleman

3 Lecture 3

3.1 Klien-Gordan equation

A heuristic way to derive SC equation:

$$E = \frac{P^2}{2m}$$

$$E \rightarrow \iota\frac{\partial}{\partial t}$$

$$p_j \rightarrow -\iota\frac{\partial}{\partial x_j}$$

Action is defined as:
$$\mathbf{S}[\phi] = \int (\partial^\mu\phi\partial_\mu\phi - m^2\phi^2(x))$$

Similarly we derive KG eqn:
$$E^2 = p^2 + m^2$$

$$(\frac{\partial}{\partial t} - \nabla^2)\phi + m^2\phi(x,t) = 0$$

A solution to this equation is $e^{\iota(k \cdot x + \omega t)}$. When you apply $\hat{H}$ we get energies like $-\omega\phi$.**Negative Energy solution**. If you perturb this system slightly it keeps going down the potential well(since energy not bounded from below). In QM continuity eqn:

$$\nabla J + \frac{\partial \rho}{\partial t} = 0$$

$$J_i = \frac{1}{2im}(\phi^*\partial\phi - (\partial\phi^*)\phi)$$

$$\rho = \frac{\iota}{2m}(\phi^*\frac{\partial\phi}{\partial t} - \frac{\partial\phi^*}{\partial t})$$

With KG its same but ρ is not positive definite.
Hence, KG equation failed!.
TODO: Dont really understand why KG failed

3.2 Rotation

Rotation is like a gauge transformation: we apply some transformation on $\hat{A}, \hat{\phi}$ which does not change E and B. The choice of coordinations/description has nothing to do with the system!!
Quantites that don't change under a transformation are **scalar** under that transformation.
Eg: $x^2 + y^2$ does not depend on choice of axes given a origin.
A scalar is always associated with a transformation.

3.3 Scalar Field, Vector Field

Given a point and density of air(ρ) we take two coordinate systems or 2 different description:

$$\rho(\vec{\mathbf{x}}) \rightarrow \rho(R\vec{\mathbf{x}})$$

Rho is a scalar field.
For a vector field under rotation:

$$E_x(\vec{\mathbf{X}}, \vec{\mathbf{Y}}), E_y(\vec{\mathbf{X}}, \vec{\mathbf{Y}}) \rightarrow RE(R\vec{\mathbf{X}})$$

Both field components and arguments changed.
Question: I do a rotation on the coordinates then what is the effent on components of $\psi_n(\vec{\mathbf{X}})$

$$\mathbf{R}\psi_{ij}(R\vec{\mathbf{x}})$$

$\mathbf{R}$ is representation of the Rotation group $SO(3)$

3.4 Lorentz Group

Group is
$$SO(1,3)$$

not $SO(4)$ in which
$$(-x_0^2 + \sum x_i^2)$$

is invariant.
Transformation is represented by:
$$\mathbf{R}_{ij}\psi_j(\Lambda\vec{\mathbf{x}})$$
$$\eta = Diag(-1,1,1,1) \tag{7}$$

Lambda satisfies:
$$\Lambda^t \eta \Lambda = \eta$$

$$\Lambda_0^0 \geq 1 \rightarrow Orthochoronum$$

$$\Lambda_0^0 \leq 1 \rightarrow Not\ Considered\ as\ impllies\ time\ reversal$$

Formally Lorentz group defined as $SO^+(1,3)$ to denote orthochronum.
For $SO(1,3)$ $Det\Lambda = 1$ For parity: $\Lambda = diag(-1,1,1,1)$
For time reversal: $\Lambda = diag(1,-1,-1,-1)$
Parity and time reversal are outside the $SO(1,3)$ group.
For 2-D parity is inside $SO(1,3)$ (TODO:Why?)

4 Lecture 4

4.1 Classical Field Theory

$$\partial \mathbf{S} = \partial \int dt L(x, \dot{x}) = 0$$

Subject to constraint:

$$\partial x|_{t_1, t_2} = 0$$

For QFT:

$$\mathbf{S} = \int d^4 x \mathbf{L}(\phi(x), \partial_\mu \phi(x))$$

$$\partial \mathbf{S} = 0$$

Boundary condition:

$$\partial \phi(x) = 0$$

$$\delta S = \int d^4 x [\frac{\partial L}{\partial \phi(x)}] \delta \phi(x) + \frac{\partial L}{\partial \partial_\mu \phi(x)} \partial(\delta \phi(x))$$

$$\delta S = \int d^4 x [\frac{\partial L}{\partial \phi(x)}] \delta \phi(x) + -\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x)} \delta \phi(x) + \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x)} \delta \phi$$

On solving we get:

$$\frac{\partial L}{\partial \phi(x)} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi(x))} = 0$$

- In a Boundary condition: we set ϕ at t_1 and t_2. For a double derivative equation: we can have only 2 boundary condition hence cant set velocity at t_1 and t_2 as well

- Lagrangian is taken to be real

- If Lagrangian not real Hamiltonian also not real$\rightarrow$ problem in time evolution

- Lagrangian is Lorentz invariant

$$L = (\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2$$

$$\frac{\partial L}{\partial \phi} = -2m^2 \phi$$

$$\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} = 4 \partial_\mu(\partial^\mu \phi)$$

This gives KG equation(TODO:How?)

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

If we write a new Lagrangian:

$$L' = -\phi \partial_\mu \partial^\mu - m^2 \phi^2$$

$$= L - \partial_\mu(\phi \partial^\mu \phi)$$

On putting in action principle we get an extra $\partial \phi$ (which can be cancelled as we can change that in BT) and $\partial_\mu(\phi \partial^\mu \phi)$ which need to be cancelled **hence in defining Lagrangian like this we might need to add an extra boundary terms** which can be important in many theories.

$$S = \int d^4 L + BT$$

4.2 Global Symeetries(Reference Wienberg: Section 7.3)

$$\psi(x) \rightarrow \psi(x) + \iota\epsilon F(x)$$

$$\delta S = \iota\epsilon \int^{F(x)} \frac{\delta S[\psi]}{\delta\psi(x)} = 0$$

Then this transformation is the symmetry of the transformation.
Now, if epsilon is function of x:

$$\delta S = \int -J^\mu(x) \frac{\partial\epsilon(x)}{\partial X^\mu}$$

Integrating by parts,

$$-\int \frac{\partial J^\mu \epsilon(x)}{\partial X^\mu} + \frac{\partial J^\mu}{\partial X^\mu}\epsilon(x)$$

Implies,

$$\frac{\mathrm{d}J^\mu}{\mathrm{d}X^\mu} = 0$$

$$\partial_\mu J^\mu(x) = 0$$

This is known as **Noether's Theorem**

5 Lecture 5

5.1 Recap

Now consider the Lagrangian density is also invariant: Assuming(Global symmetry):

$$\frac{\partial L}{\partial\phi}F(x) + \frac{\partial L}{\partial\partial_\mu\phi}\partial_\mu F(x) = 0$$

$$S = \int d^4x L(\phi, \partial_\mu\phi)$$

take $\phi \rightarrow (x) + \iota\epsilon(x)F(x)$

$$\delta S = \int \frac{\partial L}{\partial\phi}\iota\epsilon F(X) + \frac{\partial L}{\partial\partial_\mu\phi}\partial_\mu(\iota\epsilon F(x))$$

$$= \int d^4x \frac{\partial L}{\partial\phi}\iota\epsilon F(x) + \frac{\partial L}{\partial\partial_\mu}[\iota\partial_\mu\epsilon(x)F(x)] + (\iota\epsilon(x)\partial_\mu F(x))$$

$$= \int d^4x \frac{\partial L}{\partial\partial_\mu\phi}\iota F(x)\partial_\mu\epsilon(x)$$

$$J^\mu = -\frac{\partial L}{\partial\partial_\mu\phi}\iota F(x)$$

5.2 New

$$\phi(x) \to \phi(x + \epsilon(x))$$

$$= \phi(x) + \epsilon^\mu(x)\partial_\mu\phi$$

$$\delta S = \int d^4x [\frac{\partial L}{\partial \phi}\epsilon^\mu(x)\partial_\mu\phi + \frac{\partial L}{\partial \partial_\mu \phi}\partial_\mu(\epsilon(x)\partial_\mu\phi(x))]$$

$$= ... + \frac{\partial L}{\partial \partial_\mu \phi}\epsilon^\nu(x)\partial_\mu\partial_\nu\phi(x) + \frac{\partial L}{\partial \partial_\mu \phi}\partial_\mu\epsilon^\nu(x)\partial_\nu\phi(x)$$

$$\frac{\partial L}{\partial X^\mu} = \frac{\partial L}{\partial \phi}\partial_\mu\phi + \frac{\partial L}{\partial \partial_\nu \phi}\partial_\mu\partial_\nu\phi$$

Going back to previous eqn:

$$= \int d^4x (\epsilon^\nu(x)\partial_\nu L + \frac{\partial L}{\partial \partial_\nu \phi}\partial_\mu\epsilon_\nu)(\partial_\nu\phi(x))$$

Integrating by parts

$$-\frac{\partial L}{\partial \partial_\mu \phi}\epsilon^\nu\partial_\mu\partial_\nu\phi(x) + \frac{\partial L}{\partial \partial_\mu \phi}\partial_\mu(\epsilon^\nu `6_\nu\phi(x))$$

$$\delta S = \int d^4x \delta^\mu_\nu \partial_\mu\epsilon^\nu(x)L - \partial_\nu\epsilon^\nu(x)L + \frac{\partial L}{\partial \partial_\mu \phi}\partial_\nu\phi(\partial_\mu\epsilon^\nu)$$

$$= d^4x[(\partial_\mu\epsilon^\nu)[-\delta^\mu_\nu L + \frac{\partial L}{\partial \partial_\mu \phi}\partial_\nu\phi]]$$

$$\delta S = -\int d^4x \partial_\mu\epsilon^\nu(x)T^\mu_\nu$$

Finally we get using $\epsilon \to 0 at \infty$ and using equation of motion(Use the idea that for that equation of motion is derived from these kinds of variations) by applying by parts

$$\partial_\mu T^\mu_\nu = 0$$

Where:

$$T^\mu_\nu = -\delta^\mu_\nu ...$$

This means energy momentum four Vector is conserved. **This happens ONLY because Lagrangian is NOT an explicit function of x and t.**. This is not related to Noether's Theorem.

6 Lecture 5

$$S = \frac{1}{2}\int d^3x dt[\dot\phi^2 - [\nabla\phi]^2 - m^2\phi^2]$$

See raghav Notes.

$$\ddot\phi + \frac{R^2 + m^2}{\omega_k^2}\phi_k$$

See solution from raghav notes.

$$[a_k^-, a_{k'}^+] = \delta(k - k')$$

TODO: Prove $\dot{V}_k V_k^* - V_k \dot{V}_k^* = 2\iota$ using phi form from raghav notes and above eqn
Above quantity is called Wrongskian Choose V_K

$$\dot{V}_k = \alpha_k \dot{U}_k(t) + \beta_k U_k(t)$$

This will satisfy the diff eqn as long as

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

This is known as Bogolyubov transformation
TODO: Why for above eqn.
Choice of mode function will not change $\phi(x,t)$. It will just be in different "Basis"

$$N = a^+ a^-$$

$\left|\partial_b^0\right\rangle$ is a vaccum state(0 partice state) in b basis.

$$\left\langle \partial_b^0 \right| N^{(}b) \left|\partial_b^0\right\rangle = 0$$

Now we are interested in:
$$\left\langle \partial_b^0 \right| N^{(}a) \left|\partial_b^0\right\rangle = |\beta_k|^2 \delta^3(0)$$

which is non 0
Based on choice of mode function, one person may or may not see particles.
A particle is defined as a excitation with respect to a ground state.
In adiabatic approx time dependence is neglected
Choose a mode function:
$$V_k^* = \frac{1}{\sqrt{\omega_k}} e^{-\iota \omega_k t}$$

Show that ϕ is Lorentz invariant after putting this mode function. Now the theory is Lorentz invariant!!

7 Homework

TODO: Homework

$$1.\ S = \int d^4 x[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}], F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Find the EOM. Write the EOM in standard form identifying $E^i = -F^d$ and $\epsilon^{isk} B^k = -F^{ij}$

- Compute $T^{\mu\nu}$. Notice that it is symmetric in $\mu_{i\nu}$

- Define $T^{\mu\nu}_{tilde} = T^{\mu\nu}$... Show that $T^{\mu\nu}_{tilde}$ is symmtric and satisfies $\partial_\mu T^{\mu\nu}_{tilde} = 0$

  ...

  2. Consider the two complex Klein Gorden Scalar Fields

$$L = \sum$$

8 Lecture 8

$$L \ (\partial_\mu \phi)^2 - m^2 \phi^2$$

To include interaction(scattering) term.

$$L \ (\partial_\mu \phi)^2 - m^2 \phi^2 - \lambda \phi^3$$

ϕ^3 means one phi interacts with 2 more phis
ϕ^2 means 2 phis just propagated
See diagram from photos.
COnstruct a scattering problem with two foch spaces. One for incpming particles, one for outgoing. To calculate $\langle \beta, out | \alpha, in \rangle$ we get the $\alpha\beta$ elements of the S matrix. S is diagonal if there are no interactions. Elements of S matrix are $\langle out | S | in \rangle$ Similar to QM cross section:

$$d\sigma = \frac{1}{2E_1 2E_2 |\nu_1 \nu_2|} |S|^2 d\pi$$

Where $d\pi$ is the density of states(Final available states subject to energy momnetum conservation) which is given by

$$\pi_j \frac{d^3 p_j}{(2\pi)^3 2E_{p_j}} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

Formula for S matrix LSZ reduction formula:

$$\langle p_3 p_4 | S | p_1 p_2 \rangle$$

= Given in pics