Running Lecture Outline: 707

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1.1 Order, Linear, and PDE vs ODE

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2.1 Miscellaneous

I'm doing this to not fall asleep in class. Prof is Phillip Hutton. There are several ways to take the quiz. Quiz, office hours, etc. you can also take it in class. All quiz get one page cheat sheet.

2.2 More miscellany

Prof is doing some incredible projector gymnastics. He should record balance beam events.

2.3 Verifying solutions using initial conditions

To verify potential solutions, plug into the original diffeq. Use algebra (ha!) to make LHS = RHS. On the other hand, we can simply plug in various numbers for x and check for equivalency. Obviously, check for domains.

Example 2.1.

$$y' = xy^{\frac{1}{2}}$$

Potential Solution:

$$y = \frac{1}{16}x^4$$

Then,

$$\frac{dy}{dx} = \frac{1}{4}x^3$$

Then, plug into the original diffeq. We have:

$$\frac{1}{4}x^3 = x(\dot{\frac{1}{16}}x^4)^{\frac{1}{2}}$$

or that

$$\frac{1}{4}x^3 = \frac{1}{4}x^3$$

as desired.

Example 2.2.

$$(y-x)\frac{dy}{dx} = y - x + 8$$

Potential Solution 1:

$$y = 2x + 4\sqrt{x+2}$$

Potential Solution 2:

$$y = x + 4\sqrt{x+2}$$

Case 1:

$$\frac{dy}{dx} = 2 + 2(x+2)^{\frac{-1}{2}}$$

Then, plug into the original diffeq. We have:

$$(2x + 4\sqrt{x+2} - x)(2 + 2(x+2)^{\frac{-1}{2}}) = 2x + 4\sqrt{x+2} - x + 8$$

Simplifying,

$$(x + 4\sqrt{x+2})(2 + 2(x+2)^{\frac{-1}{2}}) = x + 8 + 4\sqrt{x+2}$$

Consider the case that $x = 0$. Then,

$$(4\sqrt{2})(2 + 2(2)^{\frac{-1}{2}}) = 8 + 4\sqrt{2}$$
$$8\sqrt{2} + 8 = 8 + 4\sqrt{2}$$

or that

$$8\sqrt{2} = 4\sqrt{2}$$

which is clearly false.

Solution 2 works. You plug it in like above, but end with a true statement.

2.4 IVPs

Solving a diffeq yields a *general* solution with unknowns. Using initial values we can then solve for said unknowns. For n unknowns, we need n initial values.

Let's do an example!

Example 2.3. $y' = y$, where $y(0) = 3$. We know our general solution is

$$y = Ce^x$$

but what is C? clearly, since $y(0) = 3$, and at $x = 0$, $y = c$, $3 = C$. Thus, the equation is really

$$y = 3e^x$$

Example 2.4. $y' + 2xy^2 = 0$, where $y(0) = 1$. The general solution to this is $y = \frac{1}{x+C}$. Plugging in at $x = 0$, we have $C = -1$. Final solution is $y = \frac{1}{x-1}$.

Example 2.5. $x'' + 16x = 0$, where $x(\frac{\pi}{2}) = -2$, $x'(\frac{\pi}{2}) = 1$. The general solution is

$$x = C_1 \cos 4t + C_2 \sin 4t$$

You plug in twice, you get $C_1 = -2$ and after the second step you get $C_2 = \frac{1}{4}$. Then plug in to general equation. Yay.

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3.1 Integrating Factor

This is the most fun McIlwain review. When do we use this?

Theorem 1. *If you can write a differential equation to be of the form* $\frac{dy}{dx} + p(x)y = f(x)$*, you are eligible to use Integrating Factor.*

The algorithm for solving goes something like this:

$$\frac{dy}{dx} + p(x)y = f(x)$$

Multiply both sides by:

$$I_f = e^{\int p(x)\,dx}$$

You get:

$$e^{\int p(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}p(x)y = e^{\int p(x)\,dx}f(x)$$

Using reverse chain rule:

$$\frac{d}{dx}[e^{\int p(x)\,dx}y] = e^{\int p(x)\,dx}f(x)$$

Integrating both sides, we get:

$$e^{\int p(x)\,dx}y = \int e^{\int p(x)\,dx}f(x)$$

This gets us:

$$y(x) = \frac{\int e^{\int p(x)\,dx}f(x)}{e^{\int p(x)\,dx}}$$

Example 3.1.

$$\frac{dy}{dx} = 5y$$

This seems separable, and it is. But if you were to use integrating factor, it goes like this:

$$\frac{dy}{dx} - 5y = 0$$

Note that this makes $p(t) = -5$

$$y = Ce^{5x}$$

Example 3.2.

$$\frac{dy}{dx} + y = e^{3x}$$

Note that this makes $p(x) = 1,\ f(x) = e^{3x}$

$$e^x\frac{dy}{dx} = e^{3x}e^x = e^{4x}$$

$$e^x y = \frac{1}{4}e^{4x} + C$$

$$y = \frac{\frac{1}{4}e^{4x} + C}{e^x}$$

or:

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We have a quiz lol. Three questions, about 20-25 minutes.

4.1 Separation of Variables

Definition 1 (separable). *A differential equation is separable if you can put it into the form* $\frac{dy}{dx} = f(x)h(y)$, *where* f h *are functions.*

The steps to solving them are as follows:

$$\frac{dy}{dx} = f(x)h(y)$$

$$\frac{dy}{h(y)} = f(x)dx$$

$$\int \frac{dy}{h(y)} = \int f(x)dx$$

then, solve. Let's check for separability of some cases:

Example 4.1.

$$\frac{dy}{dx} - y = \cos x$$

$$\frac{dy}{dx} = \cos x + y$$

This does not satisfy the form specified in the definition, so it is *not separable.*

Example 4.2.

$$\frac{dy}{dx} = x^2y^4e^{5x-3y}$$

We can rewrite as:

$$\frac{dy}{dx} = (x^2e^{5x})(y^4e^{-3y})$$

This is separable, as it satisfies the above form.

Now, let's solve one:

Example 4.3.

$$(1-x)dy = -ydx$$

where $y(0) = 5$

Rewriting, we get:

$$\frac{1}{1-x}dx = \frac{-1}{y}dy$$

Integrating, we get:

$$-\ln 1 - x = -lny + C$$

When $y(0) = 5$, we have $0 = -\ln 5 + C$, so $C = \ln 5$ So,

$$-\ln 1 - x = -lny + \ln 5$$

and then:

$$lny = \ln 1 - x + \ln 5$$

Exponentiating,

$$e^{\ln y} = e^{\ln 1 - x + \ln 5}$$

$$y = 5(1-x)$$

Example 4.4.

$$\frac{1}{y}\frac{dy}{dx} = 1 - x$$

where $y(0) = 2$ The joke is you don't actually need to use separation to do this. But you can.

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5.1 Exact Equations

Exact equations are a specific form of differential equation.

Definition 2. *For $f(x,y) = k$, where k is a constant, then $df = \frac{df}{dx}dx + \frac{df}{dy}dy = 0$.*

Then, let us define $M(x,y) = \frac{df}{dx}$ and $N(x,y) = \frac{df}{dy}$. So, $M(x,y)dx + N(x,y)dy = 0$. Differentiating, we get:

$$\frac{dm}{dy} = \frac{d^2 f}{dxdy} = \frac{dN}{dx}$$

Definition 3. *If $M(x,y)dx + N(x,y)dy = 0$ and $\frac{dm}{dy} = \frac{d^2 f}{dxdy} = \frac{dN}{dx}$, then $\frac{df}{dx} = M$ and $\frac{df}{dy} = N$.*

This gives us a general algorithm for solving differential equations of some classes:

1. Put the DE into form $M(x,y)dx + N(x,y)dy = 0$
2. Then, identify $M(x,y)$ and $N(x,y)$
3. Then, test for exactness: that is, $\frac{dM}{dy} = \frac{dN}{dx}$. If true, we have an exact equation.
4. From $\frac{df}{dx} = M$, we have $df = Mdx$, which, when integrating, gets $f(x,y) = g(x,y) + h(y)$.
5. Similarly, from $\frac{df}{dy} = N$, we have $\frac{df}{dy} = \frac{dg}{dy} + \frac{dh}{dy} = N$, which, when integrating, gets $f(x,y) = g(x,y) + h(y)$. Think of $h(y)$ as a constant term.
6. This gets us $h(y) = \int N - \frac{dg}{dy}dy$. Then we substitute $h(y)$ into $f(x,y)$ and set $f(x,y) = C$.

I think an example may help more.

Example 5.1. $-2xydx = (x^2 - 1)dy$. Set up involves rewriting into the form, which gives us:

$$2xydx + (x^2 - 1)dy = 0$$

where $M = 2xy$, $N = x^2 - 1$. Then, we check if the equation is exact, which requires us to take the partials of both sides, that is $\frac{dM}{dy}$ and $\frac{dN}{dx}$. Since they are both $2x$, we're good to continue. Now, $\frac{df}{dx} = 2xy$, and then $f(x,y) = \int 2xydx$. We get $f(x,y) = x^2 y + h(y)$. We now do the same thing for $\frac{df}{dy} = N$. That is, $\frac{df}{dy} = N$. That is,

$$x^2 + \frac{dh}{dy} = x^2 - 1$$

, and so $\frac{dh}{dy} = -1$. Then, $\int dh = -\int dy$, and so $h(y) = -y + C$. Setting $f(x,y) = C$, we have $x^2 y - y = C$.

Another example.

Example 5.2. $(x^2 + 2xy + y^2)dx + (2xy + x^2 - 1)dy = 0$ There's a cheeky sum of squares method for this. $M = (x^2 + 2xy + y^2)$ and $N = (2xy + x^2 - 1)$. Doing the test, we get $\frac{dM}{dy} = 2x + 2y = \frac{dN}{dx}$. This is true, by the wonders of the commutative property. Then, $\frac{df}{dx} = M = x^2 + 2xy + y^2$. Integrating, we get some silly little equation:

$$f(x,y) = \frac{1}{3}x^3 + x^2 y + xy^2 + h(y)$$

. Then, if $\frac{df}{dy} = N$, we can solve for $h(y)$, as then $\frac{d}{dy}f(x,y) = x^2 y + 2xy + -1$, and so $\frac{dh}{dy} = -1$, and then $h(y) = -y + C$. And then we set $f(x,y) = C$.

Another one. Cue DJ Khaled.

Example 5.3. $(x^3 + \cos y + \frac{1}{x})dy = (\frac{y}{x^2} - 3x^2 y)dx$. Initially, N is on the left, M is on the right. More accurately,

$$(\frac{y}{x^2} - 3x^2 y)dx - (x^3 + \cos y + \frac{1}{x})dy = 0$$

$\frac{dM}{dy} = \frac{1}{x^2} - 3x^2$. This is the same as $\frac{dN}{dx}$. Practice these differentiations, kids. Therefore, this is an exact equation.

$\frac{df}{dx} = M \implies f(x,y) = \frac{-y}{x} - x^3 y + h(y)$. Similarly, $\frac{df}{dy} = N$ and $\frac{df}{dy} = \frac{-1}{x} - x^3 + \frac{dh}{dy} = -(x^3 + \cos y + \frac{1}{x})$. Then, solve for $h(y)$ and continue, and you're done.