

Running Lecture Outline: 707

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1 26-AUG-24

1.1 Order, Linear, and PDE vs ODE

Not notated. Will add when I have time.

2 28-AUG-24

2.1 Miscellaneous

Prof is Phillip Hutton. There are several ways to take the quiz. Quiz, office hours, etc. you can also take it in class. All quiz get one page cheat sheet.

2.2 Verifying solutions using initial conditions

To verify potential solutions, plug into the original diffeq. Use algebra (ha!) to make LHS = RHS. On the other hand, we can simply plug in various numbers for x and check for equivalency. Obviously, check for domains.

Example 2.1.

$$y' = xy^{\frac{1}{2}}$$

Potential Solution:

$$y = \frac{1}{16}x^4$$

Then,

$$\frac{dy}{dx} = \frac{1}{4}x^3$$

Then, plug into the original diffeq. We have:

$$\frac{1}{4}x^3 = x \cdot \left(\frac{1}{16}x^4\right)^{\frac{1}{2}}$$

or that

$$\frac{1}{4}x^3 = \frac{1}{4}x^3$$

as desired.

Example 2.2.

$$(y-x)\frac{dy}{dx} = y-x+8$$

Potential Solution 1:

$$y = 2x + 4\sqrt{x+2}$$

Potential Solution 2:

$$y = x + 4\sqrt{x+2}$$

Case 1:

$$\frac{dy}{dx} = 2 + 2(x+2)^{-\frac{1}{2}}$$

Then, plug into the original diffeq. We have:

$$(2x + 4\sqrt{x+2} - x)(2 + 2(x+2)^{-\frac{1}{2}}) = 2x + 4\sqrt{x+2} - x + 8$$

Simplifying,

$$(x + 4\sqrt{x+2})(2 + 2(x+2)^{-\frac{1}{2}}) = x + 8 + 4\sqrt{x+2}$$

Consider the case that $x = 0$. Then,

$$(4\sqrt{2})(2 + 2(2)^{-\frac{1}{2}}) = 8 + 4\sqrt{2}$$

$$8\sqrt{2} + 8 = 8 + 4\sqrt{2}$$

or that

$$8\sqrt{2} = 4\sqrt{2}$$

which is clearly false.

Solution 2 works. You plug it in like above, but end with a true statement.

2.3 IVPs

Solving a diffeq yields a *general* solution with unknowns. Using initial values we can then solve for said unknowns. For n unknowns, we need n initial values.

Let's do an example!

Example 2.3. $y' = y$, where $y(0) = 3$. We know our general solution is

$$y = Ce^x$$

but what is C ? clearly, since $y(0) = 3$, and at $x = 0$, $y = c$, $3 = C$. Thus, the equation is really

$$y = 3e^x$$

Example 2.4. $y' + 2xy^2 = 0$, where $y(0) = 1$. The general solution to this is $y = \frac{1}{x+C}$. Plugging in at $x = 0$, we have $C = -1$. Final solution is $y = \frac{1}{x-1}$.

Example 2.5. $x'' + 16x = 0$, where $x(\frac{\pi}{2}) = -2$, $x'(\frac{\pi}{2}) = 1$. The general solution is

$$x = C_1 \cos 4t + C_2 \sin 4t$$

You plug in twice, you get $C_1 = -2$ and after the second step you get $C_2 = \frac{1}{4}$. Then plug in to general equation. Yay.

3 30-AUG-24

3.1 Integrating Factor

This is the most fun McIlwain review. When do we use this?

Theorem 1. *If you can write a differential equation to be of the form $\frac{dy}{dx} + p(x)y = f(x)$, you are eligible to use Integrating Factor.*

The algorithm for solving goes something like this:

$$\frac{dy}{dx} + p(x)y = f(x)$$

Multiply both sides by:

$$I_f = e^{\int p(x) dx}$$

You get:

$$e^{\int p(x) dx} \frac{dy}{dx} + e^{\int p(x) dx} p(x)y = e^{\int p(x) dx} f(x)$$

Using reverse chain rule:

$$\frac{d}{dx} [e^{\int p(x) dx} y] = e^{\int p(x) dx} f(x)$$

Integrating both sides, we get:

$$e^{\int p(x) dx} y = \int e^{\int p(x) dx} f(x)$$

This gets us:

$$y(x) = \frac{\int e^{\int p(x) dx} f(x)}{e^{\int p(x) dx}}$$

Example 3.1.

$$\frac{dy}{dx} = 5y$$

This seems separable, and it is. But if you were to use integrating factor, it goes like this:

$$\frac{dy}{dx} - 5y = 0$$

Note that this makes $p(t) = -5$

$$y = Ce^{5x}$$

Example 3.2.

$$\frac{dy}{dx} + y = e^{3x}$$

Note that this makes $p(x) = 1$, $f(x) = e^{3x}$

$$e^x \frac{dy}{dx} = e^{3x} e^x = e^{4x}$$

$$e^x y = \frac{1}{4} e^{4x} + C$$

$$y = \frac{\frac{1}{4} e^{4x} + C}{e^x}$$

or:

4 04-SEP-24

We have a quiz lol. Three questions, about 20-25 minutes.

4.1 Separation of Variables

Definition 1 (separable). *A differential equation is separable if you can put it into the form $\frac{dy}{dx} = f(x)h(y)$, where f h are functions.*

The steps to solving them are as follows:

$$\frac{dy}{dx} = f(x)h(y)$$

$$\frac{dy}{h(y)} = f(x)dx$$

$$\int \frac{dy}{h(y)} = \int f(x)dx$$

then, solve. Let's check for separability of some cases:

Example 4.1.

$$\frac{dy}{dx} - y = \cos x$$

$$\frac{dy}{dx} = \cos x + y$$

This does not satisfy the form specified in the definition, so it is *not separable*.

Example 4.2.

$$\frac{dy}{dx} = x^2 y^4 e^{5x-3y}$$

We can rewrite as:

$$\frac{dy}{dx} = (x^2 e^{5x})(y^4 e^{-3y})$$

This is separable, as it satisfies the above form.

Now, let's solve one:

Example 4.3.

$$(1 - x)dy = -ydx$$

where $y(0) = 5$

Rewriting, we get:

$$\frac{1}{1 - x}dx = \frac{-1}{y}dy$$

Integrating, we get:

$$-\ln 1 - x = -\ln y + C$$

When $y(0) = 5$, we have $0 = -\ln 5 + C$, so $C = \ln 5$ So,

$$-\ln 1 - x = -\ln y + \ln 5$$

and then:

$$\ln y = \ln 1 - x + \ln 5$$

Exponentiating,

$$e^{\ln y} = e^{\ln 1 - x + \ln 5}$$

$$y = 5(1 - x)$$

Example 4.4.

$$\frac{1}{y} \frac{dy}{dx} = 1 - x$$

where $y(0) = 2$ The joke is you don't actually need to use separation to do this. But you can.

5 06-SEP-24

5.1 Exact Equations

Exact equations are a specific form of differential equation.

Definition 2. For $f(x, y) = k$, where k is a constant, then $df = \frac{df}{dx}dx + \frac{df}{dy}dy = 0$.

Then, let us define $M(x, y) = \frac{df}{dx}$ and $N(x, y) = \frac{df}{dy}$. So, $M(x, y)dx + N(x, y)dy = 0$. Differentiating, we get:

$$\frac{dm}{dy} = \frac{d^2f}{dx dy} = \frac{dN}{dx}$$

Definition 3. If $M(x, y)dx + N(x, y)dy = 0$ and $\frac{dm}{dy} = \frac{d^2f}{dx dy} = \frac{dN}{dx}$, then $\frac{df}{dx} = M$ and $\frac{df}{dy} = N$.

This gives us a general algorithm for solving differential equations of some classes:

1. Put the DE into form $M(x, y)dx + N(x, y)dy = 0$
2. Then, identify $M(x, y)$ and $N(x, y)$
3. Then, test for exactness: that is, $\frac{dM}{dy} = \frac{dN}{dx}$. If true, we have an exact equation.
4. From $\frac{df}{dx} = M$, we have $df = Mdx$, which, when integrating, gets $f(x, y) = g(x, y) + h(y)$.
5. Similarly, from $\frac{df}{dy} = N$, we have $\frac{df}{dy} = \frac{dg}{dy} + \frac{dh}{dy} = N$, which, when integrating, gets $f(x, y) = g(x, y) + h(y)$.
Think of $h(y)$ as a constant term.
6. This gets us $h(y) = \int N - \frac{dg}{dy}dy$. Then we substitute $h(y)$ into $f(x, y)$ and set $f(x, y) = C$.

I think an example may help more.

Example 5.1. $-2xydx = (x^2 - 1)dy$. Set up involves rewriting into the form, which gives us:

$$2xydx + (x^2 - 1)dy = 0$$

where $M = 2xy$, $N = x^2 - 1$. Then, we check if the equation is exact, which requires us to take the partials of both sides, that is $\frac{dM}{dy}$ and $\frac{dN}{dx}$. Since they are both $2x$, we're good to continue. Now, $\frac{df}{dx} = 2xy$, and then $f(x, y) = \int 2xydx$. We get $f(x, y) = x^2y + h(y)$. We now do the same thing for $\frac{df}{dy} = N$. That is, $\frac{df}{dy} = N$. That is,

$$x^2 + \frac{dh}{dy} = x^2 - 1$$

, and so $\frac{dh}{dy} = -1$. Then, $\int dh = -\int dy$, and so $h(y) = -y + C$. Setting $f(x, y) = C$, we have $x^2y - y = C$.

Another example.

Example 5.2. $(x^2 + 2xy + y^2)dx + (2xy + x^2 - 1)dy = 0$ There's a cheeky sum of squares method for this. $M = (x^2 + 2xy + y^2)$ and $N = (2xy + x^2 - 1)$. Doing the test, we get $\frac{dM}{dy} = 2x + 2y = \frac{dN}{dx}$. This is true, by the wonders of the commutative property. Then, $\frac{df}{dx} = M = x^2 + 2xy + y^2$. Integrating, we get some silly little equation:

$$f(x, y) = \frac{1}{3}x^3 + x^2y + xy^2 + h(y)$$

. Then, if $\frac{df}{dy} = N$, we can solve for $h(y)$, as then $\frac{d}{dy}f(x, y) = x^2y + 2xy + -1$, and so $\frac{dh}{dy} = -1$, and then $h(y) = -y + C$. And then we set $f(x, y) = C$.

Another one. Cue DJ Khaled.

Example 5.3. $(x^3 + \cos y + \frac{1}{x})dy = (\frac{y}{x^2} - 3x^2y)dx$. Initially, N is on the left, M is on the right. More accurately,

$$(\frac{y}{x^2} - 3x^2y)dx - (x^3 + \cos y + \frac{1}{x})dy = 0$$

$\frac{dM}{dy} = \frac{1}{x^2} - 3x^2$. This is the same as $\frac{dN}{dx}$. Practice these differentiations, kids. Therefore, this is an exact equation.

$\frac{df}{dx} = M \implies f(x, y) = \frac{-y}{x} - x^3y + h(y)$. Similarly, $\frac{df}{dy} = N$ and $\frac{df}{dy} = \frac{-1}{x} - x^3 + \frac{dh}{dy} = -(x^3 + \cos y + \frac{1}{x})$. Then, solve for $h(y)$ and continue, and you're done.

6 09-SEP-24

6.1 Separation of Variables, but more

Example 6.1. Find the general solution to the differential equation:

$$e^{-2y}dy = e^{3x}dx$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C$$

Multiplying by -2 , we have:

$$e^{-2y} = \frac{-2}{3}e^{3x} + C$$

Taking natural logs:

$$\ln e^{-2y} = \ln \frac{-2}{3}e^{3x} + C$$

$$-2y = \ln \frac{-2}{3}e^{3x} + C$$

6.2 Exact equations, but more

Example 6.2. Find the general solution of:

$$(\sin y - y \sin x)dx = -(\cos x + x \cos y)dy$$

First, get it into the form:

$$(\sin y - y \sin x)dx + (\cos x + x \cos y)dy = 0$$

Note that $M_y = \cos y - \sin x$ and $N_x = -\sin x + \cos y$. It is exact.

$$\int M dx = x \sin y + \cos xy$$

Therefore, $\psi(x, y) = x \sin y + \cos xy + h(y)$. Then, we take the partial again:

$$\psi(x, y)_y = x \cos y + \cos x + h'(y)$$

Note that $N = \cos x + x \cos y$ and so $h'(y) = 0$. We get: $x \sin y + \cos xy = C$.

7 11-SEP-24

7.1 Bernoulli Equations

So what is Bernoulli's equation?

Definition 4. Bernoulli equations are of the form:

$$\frac{dy}{dx} + p(x)y = f(x)y^n$$

Notice that this is nonlinear. So we have to make it linear, to make our life easier.

Here's the steps:

1. Put the differential equation in the form $\frac{dy}{dx} + p(x)y = f(x)y^n$, and then identify n .
2. Then, substitute $y = u^{\frac{1}{1-n}}$, which implies that $\frac{dy}{dx} = \frac{1}{1-n} u^{\frac{1}{1-n}-1} \frac{du}{dx}$.
3. This will be an ODE. We then solve for u using either separation of variables, integrating factor, or exact equations. It's usually integrating factor.
4. Then, once we solve for u , substitute back $y = u^{1-n}$, and then solve for $y(x)$.

Example 7.1. Let's solve:

$$x \frac{dy}{dx} + y = x^2 y^2$$

The first step is easy. To put it in the form $\frac{dy}{dx} + p(x)y = f(x)y^n$, we divide by x , which gives us:

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

Note that this makes $n = 2$. Then, we substitute for u :

$$y = u^{-1} \implies \frac{dy}{dx} = -u^{-2} \frac{du}{dx}$$

Continuing our evaluation, we get:

$$-u^{-2} \frac{du}{dx} + \frac{1}{x} \frac{1}{u} = x(u^{-1})^2$$

which gets us

$$\frac{du}{dx} - \frac{1}{x}u = -x$$

This is easy to solve using integrating factor. Note that $p(x) = \frac{-1}{x}$. We end up with $u(x) = -x^2 - Cx$. (Note that I would always write $+Cx$ here, but I'm following the professor's directive). Plugging it back in, we have $y = u^{-1}$. We end up with $y = (-x^2 - Cx)^{-1}$.

Another one.

Example 7.2. Let's solve:

$$x^2 \frac{dy}{dx} - 2xy = 3y^4$$

We divide by x^2 now, which gives us:

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4$$

So, $n = 4$. Then, we substitute for u :

$$y = u^{-\frac{1}{3}} \implies \frac{dy}{dx} = \frac{-1}{3}u^{-\frac{4}{3}} \frac{du}{dx}$$

This gives us

$$\frac{-1}{3}u^{-\frac{4}{3}} \frac{du}{dx} - \frac{2}{x}u^{-\frac{1}{3}} = \frac{3}{x^2}u^{-\frac{4}{3}}$$

which gets us

$$\frac{du}{dx} + \frac{6}{x}u = -\frac{9}{x^2}$$

Integrating factor gets us $u(x) = C/x^6 - \frac{9}{5x}$. Since $u(x) = y^{-3}$, $y^{-3} = C/x^6 - \frac{9}{5x}$, which gets us:

$$y = (C/x^6 - \frac{9}{5x})^3$$

Example 7.3. Let's solve:

$$x^2 \frac{dy}{dx} - 2xy = 3y^4$$

We divide by x^2 now, which gives us:

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4$$

So, $n = 4$. Then, we substitute for u :

$$y = u^{-\frac{1}{3}} \implies \frac{dy}{dx} = \frac{-1}{3}u^{-\frac{4}{3}} \frac{du}{dx}$$

This gives us

$$\frac{-1}{3}u^{-\frac{4}{3}} \frac{du}{dx} - \frac{2}{x}u^{-\frac{1}{3}} = \frac{3}{x^2}u^{-\frac{4}{3}}$$

which gets us

$$\frac{du}{dx} + \frac{6}{x}u = -\frac{9}{x^2}$$

Integrating factor gets us $u(x) = C/x^6 - \frac{9}{5x}$. Since $u(x) = y^{-3}$, $y^{-3} = C/x^6 - \frac{9}{5x}$, which gets us:

$$y = (C/x^6 - \frac{9}{5x})^{-\frac{1}{3}}$$

Another one.

Example 7.4. Let's solve:

$$x \frac{dy}{dx} + y - y^{-2} = 0$$

Step 1:

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}$$

So, $n = -2$. Substitute for u :

$$y = u^{\frac{1}{3}} \implies \frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}} \frac{du}{dx}$$

Then,

$$\frac{1}{3}u^{-\frac{2}{3}} \frac{du}{dx} + \frac{1}{x}u^{\frac{1}{3}} = \frac{1}{x}u^{-\frac{2}{3}}$$

which gets us

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x}$$

Integrating factor gets us $u(x) = \frac{C}{x^3} + 1$. So $y^3 = \frac{C}{x^3} + 1$, which gets us:

$$y = (\frac{C}{x^3} + 1)^{\frac{1}{3}}$$

8 13-SEP-24

8.1 Second order ODEs with constant coefficients

The form of this equation is:

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = 0$$

where a, b, c are constants.

Here are the general steps:

1. Put the DE into the form $ay'' + by' + c = 0$, and then identify a, b, c
2. Then, find the roots to the characteristic equation $am^2 + bm + c = 0$, using the quadratic equation.
3. Your solution depends on your solutions to this characteristic equation m_1, m_2 .

Theorem 2. If $m_1, m_2 \in \mathbb{R}$ and $m_1 \neq m_2$, then the solution becomes

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Theorem 3. If $m_1, m_2 \in \mathbb{R}$ and $m_1 = m_2$, then the solution becomes

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

Theorem 4. If $m_1, m_2 = \alpha \pm i\beta$, then the solution becomes

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

Here's an example:

Example 8.1.

$$2y'' + 12y = -10y' = 0$$

This is the same as

$$y'' + 5y' + 6 = 0$$

Solving the quadratic we get $m_1 = 2$ and $m_2 = 3$. By Theorem 2 we have $C_1 e^{-2x} + C_2 e^{-3x}$

Example 8.2.

$$y'' - 10y' + 25y = 0$$

Some work gets us $m_1 = m_2 = 5$. Using Theorem 3, we get $C_1 e^{5x} + C_2 x e^{5x}$

Example 8.3.

$$4y'' + 4y' + 17y = 0$$

where $y(0) = -1$ and $y'(0) = 2$ Using the silly quadratic formula, we get $m_{1,2} = \frac{-1}{2} \pm j2$ where $\alpha = \frac{-1}{2}$ and $\beta = 2$. Using Theorem 4, our general solution is

$$y = e^{\frac{-1}{2}x} (C_1 \cos 2x + C_2 \sin 2x)$$

Then, we solve for initial conditions. Using product rule, we get:

$$\frac{dy}{dx} = \frac{-1}{2} e^{\frac{-1}{2}x} (C_1 \cos 2x + C_2 \sin 2x)$$

We can then apply initial conditions. Using $y(0) = -1$ on the general solution, we have $-1 = C_1$. Plugging the second initial condition and $-1 = C_1$ into the equation for $\frac{dy}{dx}$ we get that $C_2 = \frac{3}{4}$.

We end up with:

$$y = e^{\frac{-1}{2}x} (-1 \cos 2x + \frac{3}{4} \sin 2x)$$

Example 8.4.

$$y'' - y' - 6y = 0$$

where $y(0) = 4$ $y'(0) = -3$ Using math, $m_1 = 3$, $m_2 = -2$. Using 2, we get the general solution is:

$$y = C_1 e^{3x} + C_2 e^{-2x}$$

We do some similar malarkey to find C_1 and C_2 . We end up with a systems of equations:

$$\begin{cases} C_1 + C_2 = 4 \\ 3C_1 - 2C_2 = -3 \end{cases}$$

We end up with $C_1 = 1$, $C_2 = 3$. I end up with:

$$y = e^{3x} - 3e^{-2x}$$

9 16-SEP-2024

9.1 Examples, again

Here's a Bernoulli's equation example:

Example 9.1. Give the general solution to $y' = e^x y^2 + y$. The form of Bernoulli is

$$y' - y = e^x y^2$$

So, $n = 2$. We then substitute, $y = u^{-1}$, and so $\frac{dy}{dx} = -u^{-2} \frac{du}{dx}$. Then,

$$u^{-2} \frac{du}{dx} - u^{-1} = e^x u^{-2}$$

This is solvable using integrating factor. We get $u = \frac{-1}{2}e^x + Ce^{-x}$. Since $y = u^{-1}$, $u = y^{-1}$. So, $y = (\frac{-1}{2}e^x + Ce^{-x})^{-1}$.

Let's do a second order ODE example.

Example 9.2.

$$y'' = 36y$$

We get that $m^2 - 36 = 0$, or that $m = \pm 6$. This gets us $C_1 e^{-6x} + C_2 e^{6x}$.

Another Bernoulli's.

Example 9.3.

$$3(1 - x^2)y' = 2xy(y^3 - 1)$$

In standard Bernoulli Form, we have:

$$y' + \frac{2x}{3(1 - x^2)}y = \frac{2x}{3(1 - x^2)}y^4$$

$n = 4$, and so $y = u^{-\frac{1}{3}}$. $\frac{dy}{dx} = -\frac{1}{3}u^{-\frac{4}{3}} \frac{du}{dx}$. Then, we rewrite and some small mental gymnastics. We have:

$$\frac{du}{dx} - \frac{2x}{1 - x^2}u = \frac{-2x}{1 - x^2}$$

Using integrating factor again, we get:

$$u = 1 + C(1 - x^2)$$

So, $y^{-3} = 1 + C(1 - x^2)$

10 18-SEP-24

10.1 Non-homogenous second order ODE with constant coefficients

Recall the form of a homogenous equation:

Definition 5. A homogeneous equation has the form $ay'' + by' + cy = 0$.

On the other hand, a non-homogeneous equation has the form $ay'' + by' + cy = g(x)$. This has the solution as follows:

$$y = y_h + y_p$$

where y_h is the solution to the homogeneous equation and y_p is the solution to the non-homogeneous equation (I think it's called the *particular solution*).

Here's some basic steps:

1. Find solution to y_h .
2. Based on $g(x)$, take an educated guess using Table 3.4.1 (page 131).
3. Plug in the solution from Table 3.4.1. Solve for A, B, C, etc.
4. $y = y_h + y_p$

$g(x)$	Trial solution y_p
k (constant)	A
x	Ax
x^n	$Ax^n + Bx^{n-1} + Cx^{n-2} + \dots$
$e^{\alpha x}$	$Ae^{\alpha x}$
$\sin(\beta x)$ or $\cos(\beta x)$	$A \sin(\beta x) + B \cos(\beta x)$
$x \sin(\beta x)$ or $x \cos(\beta x)$	$x(A \sin(\beta x) + B \cos(\beta x))$
$e^{\alpha x} \sin(\beta x)$ or $e^{\alpha x} \cos(\beta x)$	$e^{\alpha x}(A \sin(\beta x) + B \cos(\beta x))$
$x^n e^{\alpha x}$	$(Ax^n + Bx^{n-1} + Cx^{n-2} + \dots)e^{\alpha x}$

Table 1: Common forms of $g(x)$ and their corresponding trial solutions y_p

Additional Rules:

- For $g(x) = g_1(x) + g_2(x)$, use $y_p = y_{p1} + y_{p2}$
- For $g(x) = g_1(x) \cdot g_2(x)$, use $y_p = y_{p1} \cdot y_{p2}$

where y_{p1} and y_{p2} are trial solutions for $g_1(x)$ and $g_2(x)$ respectively. Here's some more stuff.

Example 10.1. $y'' + 3y' + 2y = 6$. Roots of this is -1 and -2 . That means that:

$$y_h = C_1 e^{-x} + C_2 e^{-2x}$$

Let's try $y_p = k$. Substituting, we have: $y_p = a$, $y_p' = 0$ and $y_p'' = 0$, and so $2A = 6 \implies A = 3$. So $y_p = 3$. We end with $y = C_1 e^{-x} + C_2 e^{-2x} + 3$.

Example 10.2. Solve the non-homogeneous 2nd order ODE: $y'' + y' - 6y = 2x$

Step 1: Find the complementary solution y_h

The characteristic equation is:

$$m^2 + m - 6 = 0$$

Factoring this equation:

$$(m + 3)(m - 2) = 0$$

Solving for m :

$$m_1 = -3, \quad m_2 = 2$$

Therefore, the complementary solution is:

$$y_h = C_1 e^{-3x} + C_2 e^{2x}$$

Step 2: Find the particular solution y_p

From the given table, since $g(x) = 2x$, we use the trial solution:

$$y_p = Ax + B$$

Calculating derivatives:

$$\begin{aligned}y_p' &= A \\y_p'' &= 0\end{aligned}$$

Substituting into the original ODE:

$$0 + A - 6(Ax + B) = 2x$$

Simplifying:

$$A - 6Ax - 6B = 2x$$

Equating coefficients:

$$\begin{aligned}x : \quad -6A &= 2 \quad \Rightarrow \quad A = -\frac{1}{3} \\ \text{constant} : \quad A - 6B &= 0 \quad \Rightarrow \quad -\frac{1}{3} - 6B = 0 \quad \Rightarrow \quad B = -\frac{1}{18}\end{aligned}$$

Therefore, the particular solution is:

$$y_p = -\frac{1}{3}x - \frac{1}{18}$$

Step 3: Combine y_h and y_p for the general solution

$$y = y_h + y_p = C_1e^{-3x} + C_2e^{2x} - \frac{1}{3}x - \frac{1}{18}$$

This is the complete general solution to the given ODE.

Example 10.3. Solve the non-homogeneous 2nd order ODE: $y'' + 4y' + 2y = 2x^2 - 3x + 6$

Step 1: Find the complementary solution y_h

The characteristic equation is:

$$m^2 + 4m + 2 = 0$$

Using the quadratic formula:

$$m = \frac{-4 \pm \sqrt{16 - 8}}{2} = -2 \pm \sqrt{2}$$

Therefore:

$$m_1 = -2 + \sqrt{2}, \quad m_2 = -2 - \sqrt{2}$$

The complementary solution is:

$$y_h = C_1e^{(-2+\sqrt{2})x} + C_2e^{(-2-\sqrt{2})x}$$

Step 2: Find the particular solution y_p

From the given table, since $g(x) = 2x^2 - 3x + 6$, we use the trial solution:

$$y_p = Ax^2 + Bx + C$$

Calculating derivatives:

$$\begin{aligned}y_p' &= 2Ax + B \\y_p'' &= 2A\end{aligned}$$

Substituting into the original ODE:

$$2A + 4(2Ax + B) + 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

Expanding and grouping terms:

$$2Ax^2 + (8A + 2B)x + (2A + 4B + 2C) = 2x^2 - 3x + 6$$

Equating coefficients:

$$x^2 : 2A = 2 \Rightarrow A = 1$$

$$x : 8A + 2B = -3 \Rightarrow 8 + 2B = -3 \Rightarrow B = -\frac{11}{2}$$

$$\text{constant} : 2A + 4B + 2C = 6 \Rightarrow 2 + 4\left(-\frac{11}{2}\right) + 2C = 6 \Rightarrow C = 13$$

Therefore, the particular solution is:

$$y_p = x^2 - \frac{11}{2}x + 13$$

Step 3: Combine y_h and y_p for the general solution

$$y = y_h + y_p = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x} + x^2 - \frac{11}{2}x + 13$$

This is the complete general solution to the given ODE.

11 20-SEP-2024

11.1 Homogeneous Cauchy-Euler Equation

Recall the form of a homogeneous second order ODE solutions with constant coefficients, as discussed last week. This is basically the same thing.

The Cauchy-Euler equation is of the form:

$$ax^2y'' + bxy' + cy = 0$$

. Therefore, the characteristic equation is:

$$am^2 + (b-a)m + c = 0$$

The solution then depends on the roots.

Theorem 5. If $m_1, m_2 \in \mathbb{R}$ and $m_1 \neq m_2$, then the solution becomes

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

Theorem 6. If $m_1, m_2 \in \mathbb{R}$ and $m_1 = m_2$, then the solution becomes

$$y = C_1 x^{m_1} + C_2 x^{m_2} \ln |x|$$

Theorem 7. If $m_1, m_2 = \alpha \pm i\beta$, then the solution becomes

$$y = x^\alpha (C_1 \cos \beta \ln |x| + C_2 \sin \beta \ln |x|)$$

Here are the general steps:

1. Put the DE in the form $ax^2y'' + bxy' + cy = 0$.
2. Find a, b, c .
3. Then, find the roots of the characteristic equation $am^2 + (b-a)m + c = 0$.
4. Find solution based on roots.

Example 11.1. Solve:

$$x^2y'' - 3xy' + 3y = 0$$

It's already in the form, which is nice. $a = 1, b = -3, c = 3$. This gives us:

$$m^2 - 4m + 3 = 0$$

$$(m-3)(m-1) = 0 \implies m = 1, 3$$

By Theorem 5, $y = C_1 x + C_2 x^3$.

Example 11.2. Solve:

$$4x^2y'' + y = -8xy'$$

This is just:

$$4x^2y'' + 8xy' + y = 0'$$

Then, $a = 4, b = 8, c = 1$. So, the form of the characteristic equation is

$$4m^2 + 4m + 1 = 0$$

$$(2m + 1)(2m + 1) = 0 \implies m = \frac{-1}{2}$$

By Theorem 6, $y = C_1x^{\frac{-1}{2}} + C_2x^{\frac{-1}{2}} \ln |x|$.

Example 11.3. Solve:

$$4x^2y'' + 17y = 0'$$

Then, $a = 4, b = 0, c = 17$. So, the form of the characteristic equation is

$$4m^2 - 4m + 17 = 0$$

By the quadratic formula, $m = \frac{1}{2} \pm i2$ By Theorem 7, $y = x^{\frac{1}{2}}(C_1 \cos 2 \ln |x| + C_2 \sin 2 \ln |x|)$.

Example 11.4. Solve:

$$x^2y'' + xy' + 4y = 0'$$

Then, $a = 1, b = 1, c = 4$. So, the form of the characteristic equation is

$$m^2 + 4 = 0$$

By the quadratic formula, $m = 0 \pm i2$ By Theorem 7, $y = (C_1 \cos 2 \ln |x| + C_2 \sin 2 \ln |x|)$.

Example 11.5. Solve:

$$x^2y'' - 2xy' - 4y = 0'$$

Then, $a = 1, b = -2, c = -4$. So, the form of the characteristic equation is

$$m^2 - 3m - 4 = 0$$

$(m - 4)(m + 1) = 0 \implies m = -1, 4$. By Theorem 5, $y = C_1x^{-1} + C_2x^4$.

Example 11.6. Solve:

$$x^2y'' + 5xy' + 4y = 0'$$

Then, $a = 1, b = 5, c = 4$. So, the form of the characteristic equation is

$$m^2 + 4m + 4 = 0$$

$(m + 2)(m + 2) = 0 \implies m = -2$ By Theorem 6, $y = C_1x^{-2} + C_2x^{-2} \ln |x|$.

Example 11.7. Solve:

$$4x^2y'' + y = 0'$$

Then, $a = 4, b = 0, c = 1$. So, the form of the characteristic equation is

$$m^2 - 4m + 1 = 0$$

Using the quadratic formula, $m = \frac{16 \pm \sqrt{16-4}}{8}$ or $m = \frac{1}{2}$. By Theorem 6, $y = C_1x^{\frac{1}{2}} + C_2x^{\frac{1}{2}} \ln |x|$.

12 23-SEP-24

12.1 Non-homogeneous Cauchy-Euler differential equation

Recall the definition of a homogeneous Cauchy-Euler differential equation.

Definition 6. *The homogeneous Cauchy-Euler equation is of the form:*

$$ax^2y'' + bxy' + cy = 0$$

.

Then, non-homogeneity just adds the $g(x)$.

Definition 7. *The non-homogeneous Cauchy-Euler equation is of the form:*

$$ax^2y'' + bxy' + cy = g(x)$$

.

We start by finding the homogeneous solution of the following equation:

$$ax^2y'' + bxy' + cy = 0$$

This is explained in the notes for 20-SEP-2024. We call the homogeneous solution Y_h . Next, we rewrite Y_h as:

$$Y_h = C_1y_1 + C_2y_2$$

In the case of Theorem 5, the functions y_1 and y_2 are given by:

$$y_1 = x^{m_1}, \quad y_2 = x^{m_2}$$

In the case of Theorem 6, they are:

$$y_1 = x^{m_1}, \quad y_2 = x^{m_2} \ln x$$

In the case of Theorem 7, the solutions are:

$$y_1 = x^\alpha \cos \beta \ln x, \quad y_2 = x^\alpha \sin \beta \ln x$$

Next, let $f(x) = \frac{g(x)}{ax^2}$. Now, we calculate the Wronskian of y_1 and y_2 .
The Wronskian of y_1 and y_2 is given by:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

We then solve for u_1 and u_2 using the formulas:

$$u_1 = \int \frac{-y_2f(x)}{W} dx$$
$$u_2 = \int \frac{y_1f(x)}{W} dx$$

After dropping the constants, we find the particular solution y_p , which is:

$$y_p = u_1y_1 + u_2y_2$$

Thus, the general solution is:

$$y = C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2$$

Now, consider the following example:

Example 12.1. Solve the equation:

$$y'' - \frac{4}{x}y' = x^2$$

The homogeneous solution is obtained by solving the associated equation:

$$y'' - \frac{4}{x}y' = 0$$

The solution is:

$$Y_h = C_1x^0 + C_2x^5$$

Now, we compute the Wronskian of $y_1 = 1$ and $y_2 = x^5$:

$$W(y_1, y_2) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4$$

Next, we define $f(x) = \frac{x^4}{ax^2} = x^2$, and solve for u_1 and u_2 :

$$u_1 = \int \frac{-x^5x^2}{5x^4} dx = -\frac{1}{20}x^4$$

$$u_2 = \int \frac{x^2}{5x^4} dx = -\frac{1}{5}x^{-1}$$

Thus, the particular solution is:

$$-\frac{1}{20}x^4(1) - \frac{1}{5}x^{-1}(x^5)$$

And then it is easy to finish.

13 27-SEP-24

13.1 Variation of Parameters

Recall the definition of a homogeneous Cauchy-Euler differential equation.

Definition 8. *The homogeneous Cauchy-Euler equation is of the form:*

$$ax^2y'' + bxy' + cy = 0$$

.

Then, non-homogeneity just adds the $g(x)$.

Definition 9. *The non-homogeneous Cauchy-Euler equation is of the form:*

$$ax^2y'' + bxy' + cy = g(x)$$

.

For Variation of parameters for 2nd order ODE, there is only one change: $f(x) = \frac{g(x)}{a}$, not $\frac{g(x)}{ax^2}$. We start by finding the homogeneous solution of the following equation:

$$ax^2y'' + bxy' + cy = 0$$

This is explained in the notes for 20-SEP-2024. We call the homogeneous solution Y_h . Next, we rewrite Y_h as:

$$Y_h = C_1y_1 + C_2y_2$$

In the case of Theorem 5, the functions y_1 and y_2 are given by:

$$y_1 = x^{m_1}, \quad y_2 = x^{m_2}$$

In the case of Theorem 6, they are:

$$y_1 = x^{m_1}, \quad y_2 = x^{m_2} \ln x$$

In the case of Theorem 7, the solutions are:

$$y_1 = x^\alpha \cos \beta \ln x, \quad y_2 = x^\alpha \sin \beta \ln x$$

Next, let $f(x) = \frac{g(x)}{ax^2}$. Now, we calculate the Wronskian of y_1 and y_2 .

The Wronskian of y_1 and y_2 is given by:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

We then solve for u_1 and u_2 using the formulas:

$$u_1 = \int \frac{-y_2 f(x)}{W} dx$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx$$

After dropping the constants, we find the particular solution y_p , which is:

$$y_p = u_1 y_1 + u_2 y_2$$

Thus, the general solution is:

$$y = C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2$$

14 30-SEP-2024

14.1 Laplace Transform Fundamentals

We have a differential equation. We then transform the differential equation into Laplace Space, as follows:

$$\mathcal{L}(f(t)) \rightarrow F(s)$$

We then use algebra to solve for $Y(s)$. We then take the inverse transform: that is:

$$\mathcal{L}(Y(s))^{-1} \rightarrow y(t)$$

The transform is defined as follows:

Definition 10. $F(s) = \int_0^\infty e^{-st} f(t) dt$

Example 14.1. If $f(t) = 1$, find $F(s)$.

Clearly, $F(s) = \int_0^\infty e^{-st} (1) dt$. This evaluates to $\frac{1}{s}$, with some introductory calculus.

Example 14.2. If $f(t) = e^{at}$, find $F(s)$.

Clearly, $F(s) = \int_0^\infty e^{-st} (e^{at}) dt$. This evaluates to $\int_0^\infty e^{-(s-a)t} dt$. We get $\frac{1}{s-a}$, for $s > a$.

People have done this work for us for a lot of functions, so we're going to steal their efforts, in the form of a nice little table.

Let's do some more examples.

Example 14.3. Find the Laplace transform of $f(t) = t^3 + 3t^2 + 2$. Using 2, we can split up the three terms into:

$$\frac{3!}{s^{3+1}} + 3 \frac{2!}{s^{2+1}} + \frac{2}{s}$$

So, $F(s) = \frac{6}{s^4} + \frac{6}{s^3} + \frac{2}{s}$.

Example 14.4. Find the Laplace transform of $f(t) = 4t^2 - 5 \sin 3t$ Using 2, we can split up the two terms into:

$$4 \frac{2!}{s^{2+1}} + -5 \frac{3}{s^2 + 3^2}$$

So, $F(s) = \frac{8}{s^3} - \frac{15}{s^2 + 9}$.

Function	Laplace Transform
$f(t) = 1$	$F(s) = \frac{1}{s}$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{n+1}}$
$f(t) = e^{at}$	$F(s) = \frac{1}{s-a}$
$f(t) = \sin(bt)$	$F(s) = \frac{b}{s^2+b^2}$
$f(t) = \cos(bt)$	$F(s) = \frac{s}{s^2+b^2}$
$f(t) = \sinh(bt)$	$F(s) = \frac{b}{s^2-b^2}$
$f(t) = \cosh(bt)$	$F(s) = \frac{s}{s^2-b^2}$
$f(t) = te^{at}$	$F(s) = \frac{1}{(s-a)^2}$
$f(t) = u(t-a)$	$F(s) = \frac{e^{-as}}{s}$
$f(t) = \delta(t-a)$	$F(s) = e^{-as}$
$y(t)$	$Y(s)$
$\frac{dy}{dt}$	$sY(s) - y(0)$
$\frac{d^2y}{dt^2}$	$s^2Y(s) - sy(0) - \frac{dy}{dt}(0)$
$\int_0^t y(\tau) d\tau$	$\frac{Y(s)}{s}$
$f(t) = e^{at} \sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$
$f(t) = e^{at} \cos(bt)$	$F(s) = \frac{s-a}{(s-a)^2+b^2}$

Table 2: Laplace Transform Table

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15.1 Partial Fraction Expansion

Recall the Laplace Table 2.

We begin with an example.

Example 15.1.

$$f(t) = 4 \sin 2t + 3 \cos 6t + 4e^{2t}$$

Using 2, we have

$$F(s) = 4 \cdot \frac{2}{s^2 + 4} + 3 \cdot \frac{s}{s^2 + 36} + 4 \cdot \frac{1}{s - 2}$$

.

Another example:

Example 15.2.

$$f(t) = 10 + 2t^4$$

Using 2, we have

$$F(s) = 10 \cdot \frac{1}{s} + 2 \cdot \frac{4!}{s^5} = \frac{10}{s} + \frac{48}{s^5}$$

.

Now, let's take the inverse!

Example 15.3.

$$F(s) = \frac{5s}{s^2 + 25} + \frac{8}{s^2 + 4}$$

This is the same as:

$$5 \frac{s}{s^2 + 25} + 4 \frac{2}{s^2 + 4}$$

which evaluates to $f(t) = 5 \cos 5t + 4 \sin 2t$.

Example 15.4.

$$F(s) = \frac{2}{s^3} + \frac{1}{s+9} + \frac{5}{s}$$

This is $f(t) = t^2 - e^{-9t} + 5$.

Now, let's talk about examples with multiple terms multiplied by each other in the denominator. This requires Partial Fraction Expansion. You may remember similar things in Calculus. Note that fractions of the form:

$$F(s) = \frac{N(s)}{(s+a)(s+b)^n(s^2+c)}$$

evaluate to:

$$\frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \cdots + \frac{B_n}{(s+b)^n} + \frac{D_1s + D_2}{s^2+c}$$

So, we have a somewhat decent method to evaluate Partial Fraction Expansion.

1. Multiply both sides by the denominator
2. Next, solve by either substitution or equating coefficients
3. Then, use the Laplace Transform to find the inverse of components.

Example 15.5.

$$F(s) = \frac{6s^2 + 50}{(s+3)(s^2+4)}$$

Using Partial Fraction Expansion, this is of the form:

$$\frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

Next, we need to find A, B, C ! Let's use substitution.

$$6s^2 + 50 = A(s^2 + 4) + (Bs + C)(s + 3)$$

Evaluating at $s = -3$, which is one of the zeros, we find that $A = \frac{104}{13} = 8$. You can verify it for yourself!

How can we find B and C ? Let's just use equating coefficients, what we should have done at the start.

$$6s^2 + 50 = As^2 + 4A + Bs^2 + Cs + 3Bs + 3C$$

Note that this means:

$$6 = A + B$$

$$0 = 3B + C$$

$$5 = 4A + 3C$$

This can just be reducing the matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 3 & 1 & 0 \\ 4 & 0 & 3 & 5 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{array} \right)$$

Or $A = 8, B = -2, C = 6$. So,

$$\frac{8}{s+3} + \frac{-2s}{s^2+4} + \frac{6}{s^2+4}$$

Taking the inverse Laplace Transform,

$$f(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

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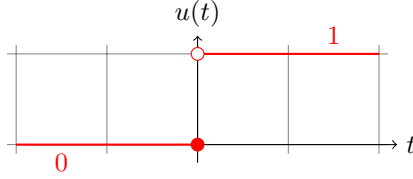
17 09-OCT-2024

17.1 Translation on the t-axis

Recall the *unit step function*.

Definition 11 (Unit step function). *The unit step function $u(t)$ is defined as:*

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$



Now consider the case where $f(t) = g(t)u(t - a)$, where $u(t - a)$ is the shifted unit step function. The Laplace transform of $f(t)$ is:

$$F(s) = e^{-as} + \mathcal{L}\{g(t - a)\}.$$

The steps to evaluate this are as follows:

1. Identify $g(t)$ and a
2. Replace t with $t - a$ to get $g(t - a)$.
3. Expand $g(t - a)$
4. Take the Laplace transform. That is, $\mathcal{L}\{g(t - a)\}$
5. $F(s) = e^{-as} \mathcal{L}\{g(t - a)\}$

Example 17.1. Evaluate $f(t) = u(t - 2)$.

Note that $g(t) = 1$, and $a = 2$. Then, $g(t - 2) = 1$. We get $\mathcal{L}\{1\} = \frac{1}{s}$, or $F(s) = e^{-2s} \frac{1}{s}$.

Example 17.2. Evaluate $f(t) = t^2 u(t - 2)$.

Note that $g(t) = t^2$, and $a = 2$. Then, $g(t - 2) = (t - 2)^2$. Evaluating, $g(t - 2) = t^2 - 4t + 4$. So, Taking the Laplace Transform, we have $\mathcal{L}\{t^2 - 4t + 4\} = \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}$, or $F(s) = e^{-2s} [\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}]$.

Example 17.3. Evaluate $f(t) = 2 - 3u(t - 2) + u(t - 3)$.

We get $F(s) = 2(\frac{1}{s}) - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}$.

Example 17.4. Evaluate $f(t) = 4 \cos t u(t - \pi)$.

Recall that $\cos t - \pi = -\cos t$. This will help us a lot in the future.

Now, we have $a = \pi$, $g(t) = 4 \cos t$. Then, $g(t - \pi) = 4 \cos(t - \pi) = -4 \cos t$. Knowing that $\mathcal{L}\{-4 \cos t\} = \frac{4s}{s^2 + 1}$, we get $F(s) = -e^{-\pi s} \frac{4s}{s^2 + 1}$.

Example 17.5. Evaluate for $Y(s)$: $y' + y = 4 \cos t u(t - \pi)$, given initial conditions $y(0) = 0$.

$$sY - 0 + Y = e^{-\pi s} \frac{4s}{s^2 + 1}$$

$$Y(s + 1) = -e^{-\pi s} \frac{4s}{s^2 + 1}$$

$$Y(s) = \frac{-e^{-\pi s} \frac{4s}{s^2 + 1}}{s + 1}$$

But how do we find $Y(t)$? We use the *Second Translation Theorem*.

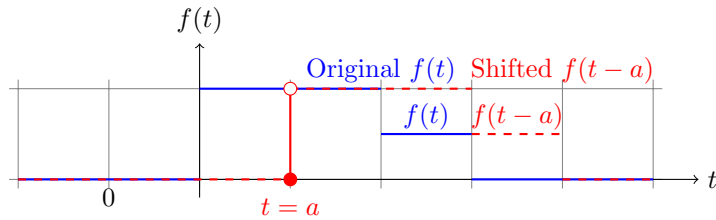
Definition 12. The **Second Translation Theorem** in Laplace Transforms is as follows:
If

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s),$$

where $u(t-a)$ is the unit step function that shifts the function by a units in time.



Example 17.6. $f(t) = (t-1)u(t-1)$. Find $F(s)$. Through some standard malarkey, $F(s) = e^{-s} \frac{1}{s^2}$.