Summary of Heuristic-based Kinematics of Containment, 2D case

May 21, 2017

1. **General settings**
   Let $\mathbf{a} = [a_1, a_2]^T$ denotes the semi-axis of the smaller ellipse E_a, and so the semi-axis of bigger ellipse E_b can be defined as $\mathbf{b} = [b_1, b_2]^T = (1+\varepsilon)\mathbf{a}$, where ε is the "explode factor". We further define the aspect ratio of the ellipses as $\alpha = a_1/a_2 = b_1/b_2$. Therefore, there are only 3 parameters to be considered in this problem, say "a_2, α, ε".

2. **Largest angle that the smaller ellipse can rotate**
   The maximum angle that the smaller ellipse can rotate comes when the center point is fixed in the origin. And the angle can be found in closed-form as follows.

   We use the parametric forms for the two ellipses,

$$\mathbf{u}_a = \begin{pmatrix} a_1 \cos\theta_a \\ a_2 \cos\theta_a \end{pmatrix} \quad \mathbf{u}_b = \begin{pmatrix} b_1 \cos\theta_b \\ b_2 \cos\theta_b \end{pmatrix} \tag{1}$$

   and the rotation of E_a can be written as,

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{2}$$

   where ϕ is the rotational angle of E_a.

   Assume that the bigger ellipse E_b is fixed with the world frame, with the center in the origin, and the larger semi-axis is aligned with x-axis. Then the points on the boundary of E_b as seen in the world frame is exactly $\mathbf{u}_b$. Since E_a only rotates around the origin, the points on its boundary as seen in the world frame should be $R(\phi)\mathbf{u}_a$.

   The intersecting point of the two ellipses can be obtained by enforcing the two parametric equations as seen in the world frame to be equal as,

$$\begin{aligned} &\mathbf{u}_b = R(\phi)\mathbf{u}_a \\ \iff &\begin{cases} b_1 \cos\theta_b = a_1 \cos\phi\cos\theta_a - a_2 \sin\phi\sin\theta_a \\ b_2 \sin\theta_b = a_1 \sin\phi\cos\theta_a + a_2 \cos\phi\sin\theta_a \end{cases} \end{aligned} \tag{3}$$

   Substituting $a_1 = \alpha a_2, b_1 = (1+\varepsilon)a_1, b_2 = (1+\varepsilon)a_2$, and observing the fact that $\theta_b = \phi + \theta_a$ (rotational angle transformation from local frame to world frame of E_a) gives,

$$\begin{aligned} &\begin{cases} \alpha a_2(1+\varepsilon)\cos(\theta_a+\phi) = \alpha a_2 \cos\phi\cos\theta_a - a_2\sin\phi\sin\theta_a \\ a_2(1+\varepsilon)\sin(\theta_a+\phi) = \alpha a_2 \sin\phi\cos\theta_a + a_2\cos\phi\sin\theta_a \end{cases} \\ \iff &\begin{cases} \alpha\varepsilon\cos\theta_a\cos\phi = (\alpha(1+\varepsilon)-1)\sin\theta_a\sin\phi \\ \varepsilon\sin\theta_a\cos\phi = (\alpha-\varepsilon-1)\cos\theta_a\sin\phi \end{cases} \end{aligned} \tag{4}$$

   Since always $a_2 < b_2$, the points on smaller semi-axis of the smaller ellipse will be always inside the bigger ellipse, so when we consider the case of intersecting, $\theta_a \neq \pi/2$, or $3\pi/2$, or $\cos\theta_a \neq 0$. Further, One sufficient condition when two ellipses intersect should be $a_1 \geq b_2$, or $\alpha \geq 1+\varepsilon$, and the equal sign occurs only when $\phi = \pi/2, 3\pi/2$ and $\theta_a = 0, \pi$, or $\cos\phi = 0$ and $\sin\theta_a = 0$.

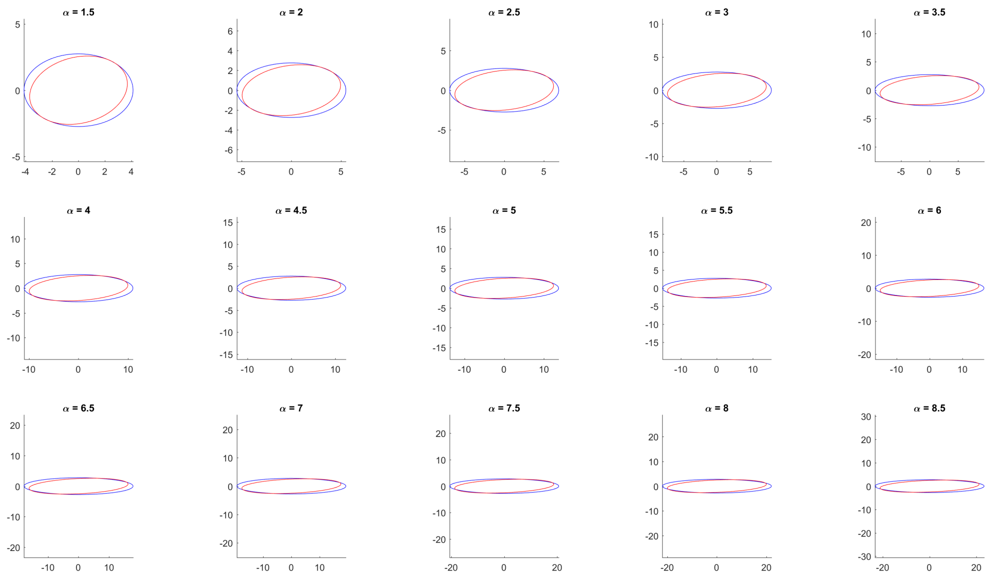

Figure 1: Test with different values of aspect ratio α, fixed $\varepsilon = 0.1$

So, for general cases, we could rearrange the above system of equations as,

$$\begin{cases} \dfrac{\alpha\varepsilon}{(\alpha(1+\varepsilon)-1)} = \tan\theta_a \tan\phi \\ \dfrac{\varepsilon}{(\alpha-\varepsilon-1)} \tan\theta_a = \tan\phi \end{cases} \tag{5}$$

Since $b_1 \geq a_1$, $\phi \neq 0, \pi$ ($\tan\phi \neq 0$) when two ellipses intersect, then we could get,

$$\begin{aligned} \frac{\alpha\varepsilon}{(\alpha(1+\varepsilon)-1)\tan\phi} &= \frac{(\alpha-\varepsilon-1)\tan\phi}{\varepsilon} \\ \Rightarrow \tan^2\phi &= \frac{\alpha\varepsilon^2}{(\alpha(1+\varepsilon)-1)(\alpha-\varepsilon-1)} \end{aligned} \tag{6}$$

Solving for ϕ gives,

$$\phi = \arctan(\pm\varepsilon\sqrt{\frac{\alpha}{(\alpha(1+\varepsilon)-1)(\alpha-\varepsilon-1)}}) \tag{7}$$

Figures 1 and 2 show the numerical experiments to verify the result, with different α and ε values respectively.

3. **Heuristic KC with twisted ellipsoid fit**
It is observed that the convex hull of the sample-based c-space of KC is twisted along x-axis. So we try to untwist the space along x-axis, fit a convex shape inside the sample-based convex hull, and twist back to get a better fit.

At first, figure 3 is a comparison of twisted and untwisted C-Space. Note that the untwisting angle β is set manually, which is proportional to the x-coordinate, i.e $\beta = kx$, where k is set by observation, so the next step is to find an automatic way to set the angle. Also, β might also be proportional to a higher order of x so that the untwisted version of the c-space might look better.

For the current experiment, I fit an ellipsoid into the untwisted c-space, with the equation $\frac{x^2}{(a_1\varepsilon)^2} + \frac{y^2}{(a_2\varepsilon)^2} + \frac{\theta^2}{\phi^2} = 1$, where each denominator $(a_1\varepsilon, a_2\varepsilon, \phi)$ is the maximum displacement in the corresponding axis in c-space.

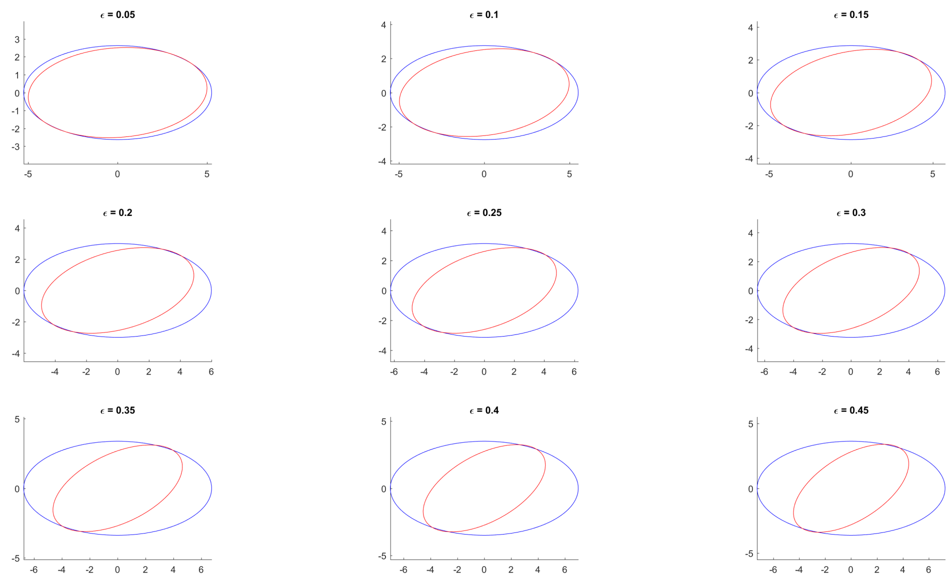

Figure 2: Test with different values of explode factor ε, fixed $\alpha = 2$

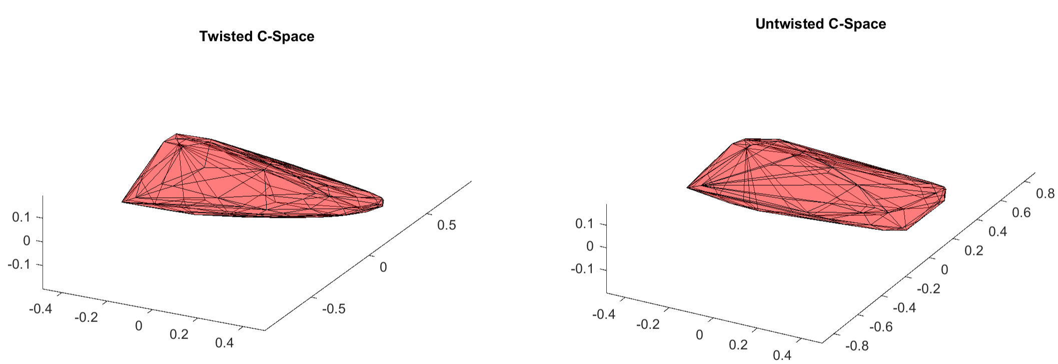

Figure 3: Comparison of twisted and untwisted c-space

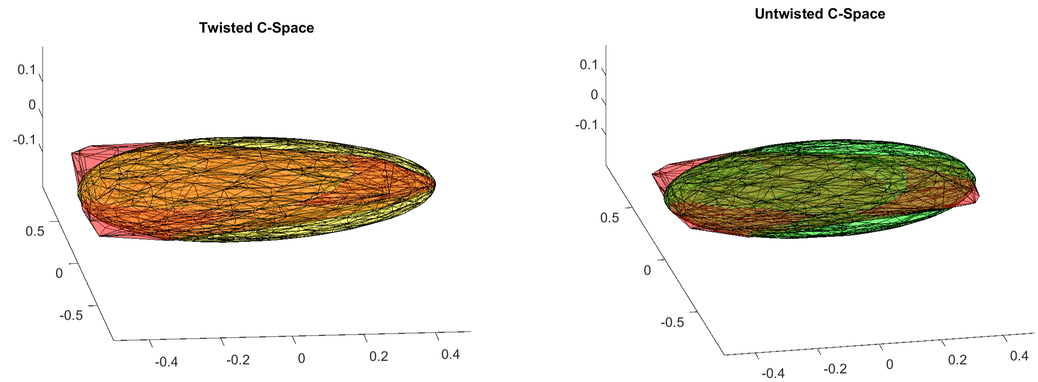

Figure 4: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 2$, $\varepsilon = 0.1$, $\beta = 0.6x$

Figure 4 is one example of the heuristic fit. The fitted ellipsoid exceeds the range of the convex hull, which is infeasible. To solve this, we could define a scale factor to the ellipsoid parameters, factor can be different for each parameter, but this still needs manually observation. Figure 5 shows the scaled heuristic fit, with θ scaled only.

4. **Heuristic fit with different aspect ratios** Each time we change the aspect ratio α, we need to manually change the factor k of twisting function $\beta = kx$ correspondingly. Table 4 lists the aspect ratios and the corresponding twisting factors.

Aspect ratio α	1.5	1.8	2	2.2	2.5	3
Twisting factor k	1.75	1	0.6	0.5	0.35	0.25

Figures 6, 7, 8, 9 and 10 show the heuristic fit with different aspect ratios.

5. **TODO list**
   (a) Find an automatic way to define the twisting function along x-axis, e.g. calculate the maximum ratio of θ/y
   (b) Find a way to determine the scale factor of the heuristic fit inside the untwisted convex hull
   (c) Fit the untwisted convex hull with other shapes, e.g. superquadrics
   (d) Try the heuristic fit with different aspect ratios and explode factors

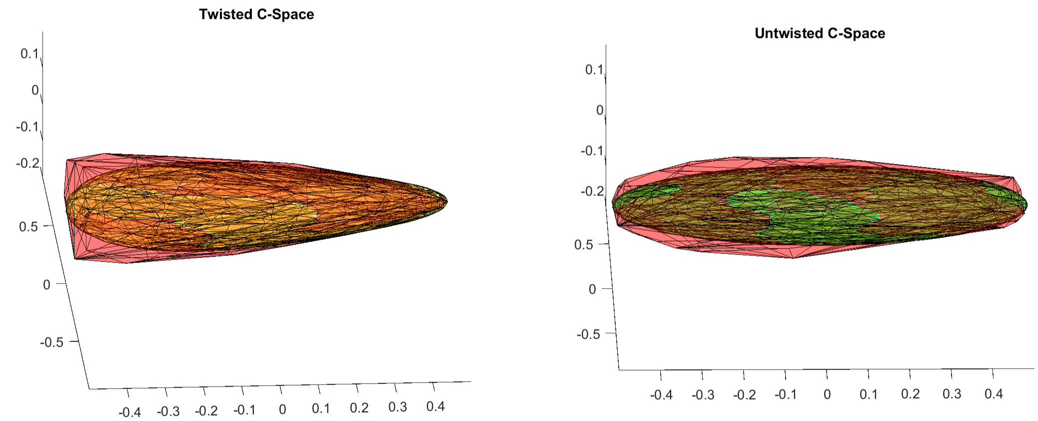

Figure 5: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 2$, $\varepsilon = 0.1$, $\beta = 0.6x$, scale factor: 0.6

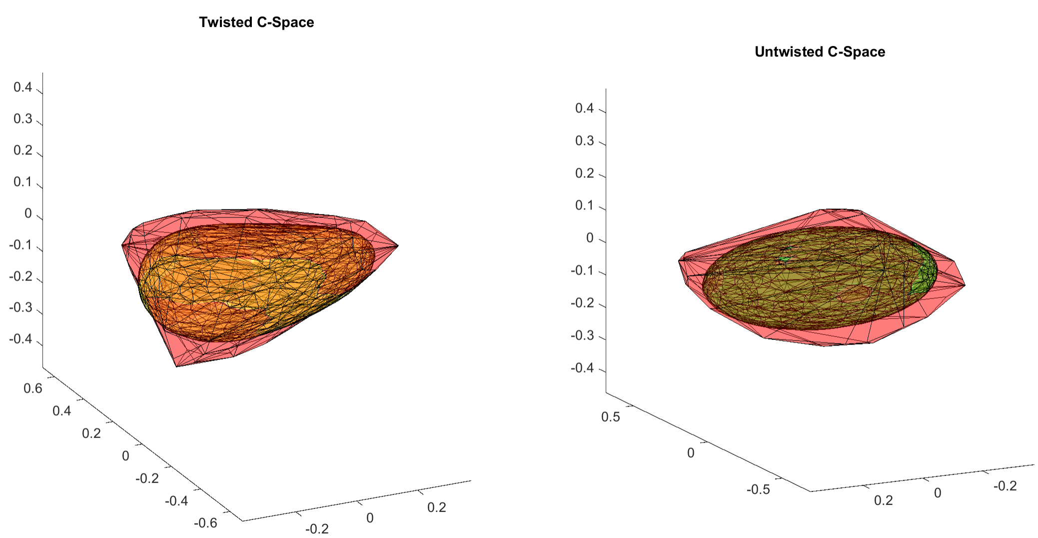

Figure 6: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 1.5$, $\varepsilon = 0.1$, $\beta = 0.75x$, scale factor: 0.6

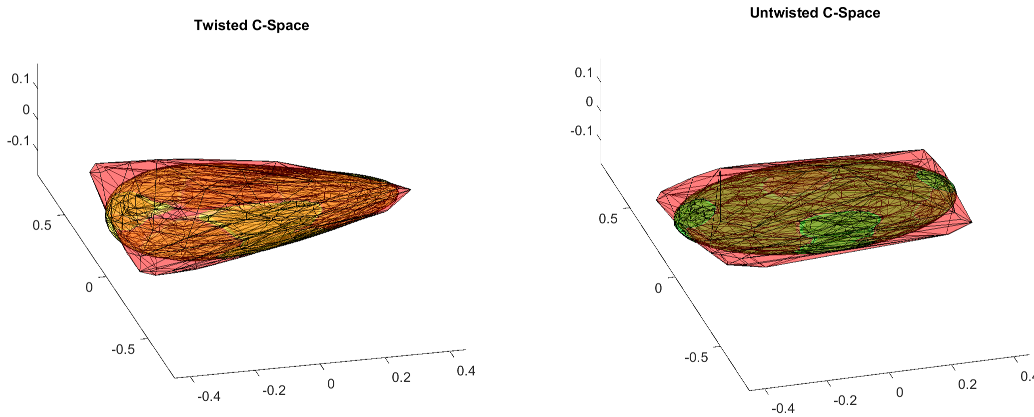

Figure 7: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 1.8$, $\varepsilon = 0.1$, $\beta = x$, scale factor: 0.6

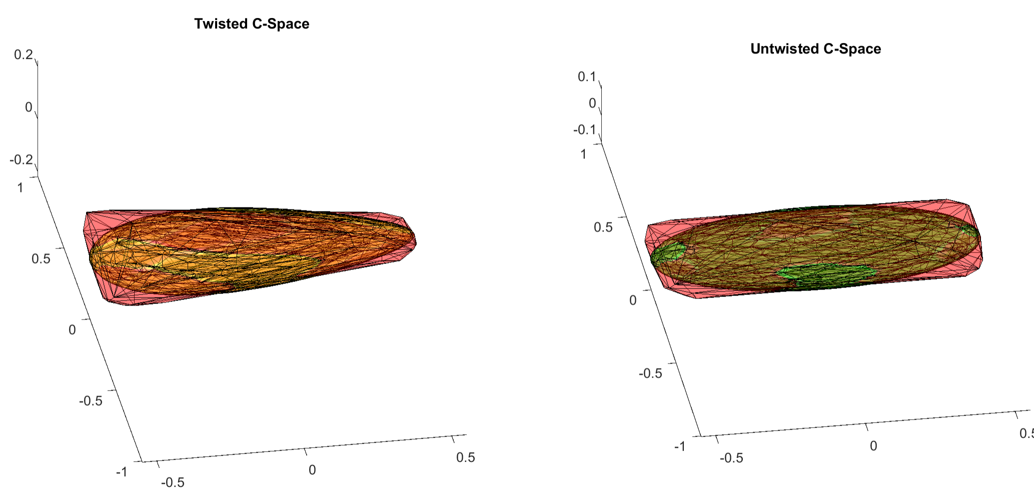

Figure 8: heuristic fit, with parameters: $a_2 = 1.5$, $\alpha = 2.2$, $\varepsilon = 0.1$, $\beta = 0.5x$, scale factor: 0.6

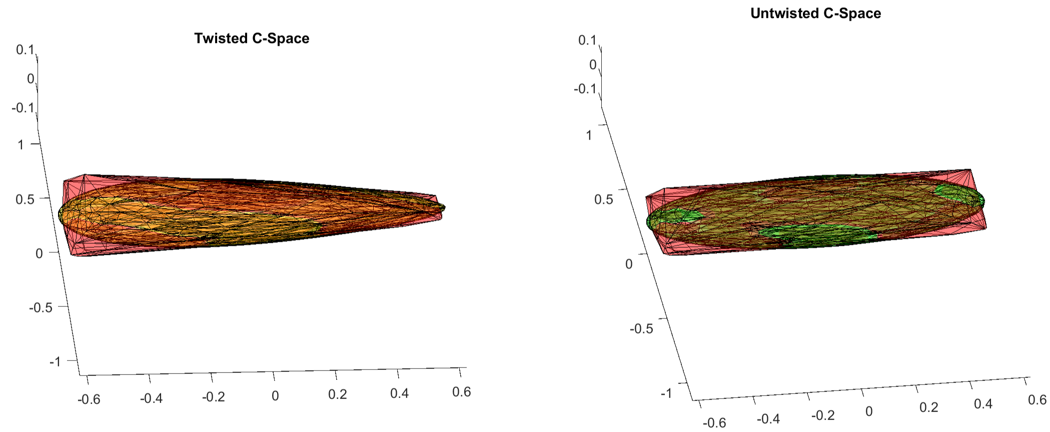

Figure 9: heuristic fit, with parameters: $a_2 = 1.5$, $\alpha = 2.5$, $\varepsilon = 0.1$, $\beta = 0.35x$, scale factor: 0.6

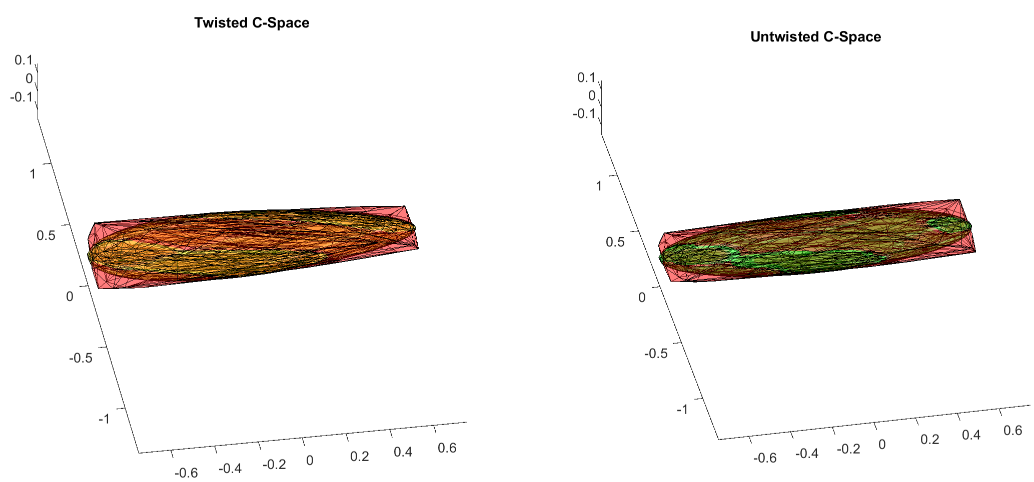

Figure 10: heuristic fit, with parameters: $a_2 = 1.5$, $\alpha = 3$, $\varepsilon = 0.1$, $\beta = 0.25x$, scale factor: 0.6