Summary of Heuristic-based Kinematics of Containment, 2D case

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1. **General settings**
   Let $\mathbf{a} = [a_1, a_2]^T$ denotes the semi-axis of the smaller ellipse E_a, and so the semi-axis of bigger ellipse E_b can be defined as $\mathbf{b} = [b_1, b_2]^T = (1 + \varepsilon)\mathbf{a}$, where ε is the "explode factor". We further define the aspect ratio of the ellipses as $\alpha = a_1/a_2 = b_1/b_2$. Therefore, there are only 3 parameters to be considered in this problem, say "a_2, α, ε".

2. **Largest angle that the smaller ellipse can rotate**
   The maximum angle that the smaller ellipse can rotate comes when the center point is fixed in the origin. And the angle can be found in closed-form as follows.

   We use the parametric forms for the two ellipses,

   $$\mathbf{u}_a = \begin{pmatrix} a_1 \cos \theta_a \\ a_2 \cos \theta_a \end{pmatrix} \quad \mathbf{u}_b = \begin{pmatrix} b_1 \cos \theta_b \\ b_2 \cos \theta_b \end{pmatrix} \tag{1}$$

   and the rotation of E_a can be written as,

   $$R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \tag{2}$$

   where ϕ is the rotational angle of E_a.

   Assume that the bigger ellipse E_b is fixed with the world frame, with the center in the origin, and the larger semi-axis is aligned with x-axis. Then the points on the boundary of E_b as seen in the world frame is exactly $\mathbf{u}_b$. Since E_a only rotates around the origin, the points on its boundary as seen in the world frame should be $R(\phi)\mathbf{u}_a$.

   The intersecting point of the two ellipses can be obtained by enforcing the two parametric equations as seen in the world frame to be equal as,

   $$\mathbf{u}_b = R(\phi)\mathbf{u}_a$$
   $$\iff \begin{cases} b_1 \cos \theta_b = a_1 \cos \phi \cos \theta_a - a_2 \sin \phi \sin \theta_a \\ b_2 \sin \theta_b = a_1 \sin \phi \cos \theta_a + a_2 \cos \phi \sin \theta_a \end{cases} \tag{3}$$

   Substituting $a_1 = \alpha a_2, b_1 = (1+\varepsilon)a_1, b_2 = (1+\varepsilon)a_2$, and observing the fact that $\theta_b = \phi + \theta_a$ (rotational angle transformation from local frame to world frame of E_a) gives,

   $$\begin{cases} \alpha a_2(1+\varepsilon)\cos(\theta_a + \phi) = \alpha a_2 \cos \phi \cos \theta_a - a_2 \sin \phi \sin \theta_a \\ a_2(1+\varepsilon)\sin(\theta_a + \phi) = \alpha a_2 \sin \phi \cos \theta_a + a_2 \cos \phi \sin \theta_a \end{cases}$$
   $$\iff \begin{cases} \alpha \varepsilon \cos \theta_a \cos \phi = (\alpha(1+\varepsilon) - 1)\sin \theta_a \sin \phi \\ \varepsilon \sin \theta_a \cos \phi = (\alpha - \varepsilon - 1)\cos \theta_a \sin \phi \end{cases} \tag{4}$$

   Since always $a_2 < b_2$, the points on smaller semi-axis of the smaller ellipse will be always inside the bigger ellipse, so when we consider the case of intersecting, $\theta_a \neq \pi/2$, or $3\pi/2$, or $\cos \theta_a \neq 0$. Further, One sufficient condition when two ellipses intersect should be $a_1 \geq b_2$, or $\alpha \geq 1 + \varepsilon$, and the equal sign occurs only when $\phi = \pi/2, 3\pi/2$ and $\theta_a = 0, \pi$, or $\cos \phi = 0$ and $\sin \theta_a = 0$.
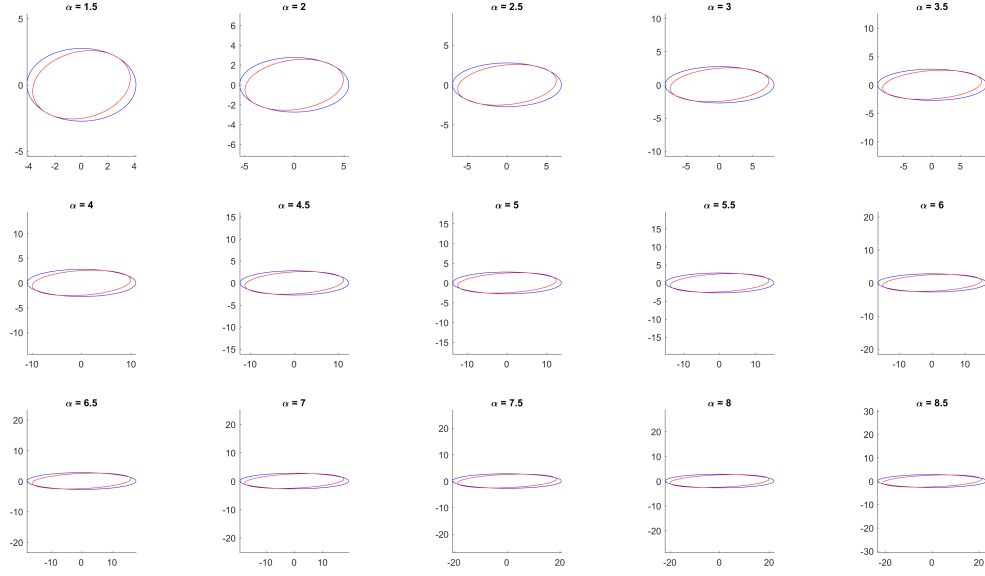
Figure 1: Test with different values of aspect ratio α, fixed $\varepsilon = 0.1$

So, for general cases, we could rearrange the above system of equations as,

$$\begin{cases} \dfrac{\alpha\varepsilon}{(\alpha(1+\varepsilon)-1)} = \tan\theta_a \tan\phi \\ \dfrac{\varepsilon}{(\alpha-\varepsilon-1)}\tan\theta_a = \tan\phi \end{cases} \tag{5}$$

Since $b_1 \geq a_1$, $\phi \neq 0, \pi$ ($\tan\phi \neq 0$) when two ellipses intersect, then we could get,

$$\frac{\alpha\varepsilon}{(\alpha(1+\varepsilon)-1)\tan\phi} = \frac{(\alpha-\varepsilon-1)\tan\phi}{\varepsilon}$$
$$\Rightarrow \tan^2\phi = \frac{\alpha\varepsilon^2}{(\alpha(1+\varepsilon)-1)(\alpha-\varepsilon-1)} \tag{6}$$

Solving for ϕ gives,

$$\phi = \arctan(\pm\varepsilon\sqrt{\frac{\alpha}{(\alpha(1+\varepsilon)-1)(\alpha-\varepsilon-1)}}) \tag{7}$$

Figures 1 and 2 show the numerical experiments to verify the result, with different α and ε values respectively.

3. **Heuristic KC with twisted ellipsoid fit**
It is observed that the convex hull of the sample-based c-space of KC is twisted along x-axis. So we try to untwist the space along x-axis, fit a convex shape inside the sample-based convex hull, and twist back to get a better fit.

At first, figure 3 is a comparison of twisted and untwisted C-Space. Note that the untwisting angle β is set manually, which is proportional to the x-coordinate, i.e $\beta = kx$, where k is set by observation, so the next step is to find an automatic way to set the angle. Also, β might also be proportional to a higher order of x so that the untwisted version of the c-space might look better.

For the current experiment, I fit an ellipsoid into the untwisted c-space, with the equation $\frac{x^2}{(a_1\varepsilon)^2} + \frac{y^2}{(a_2\varepsilon)^2} + \frac{\theta^2}{\phi^2} = 1$, where each denominator $(a_1\varepsilon, a_2\varepsilon, \phi)$ is the maximum displacement in the corresponding axis in c-space.
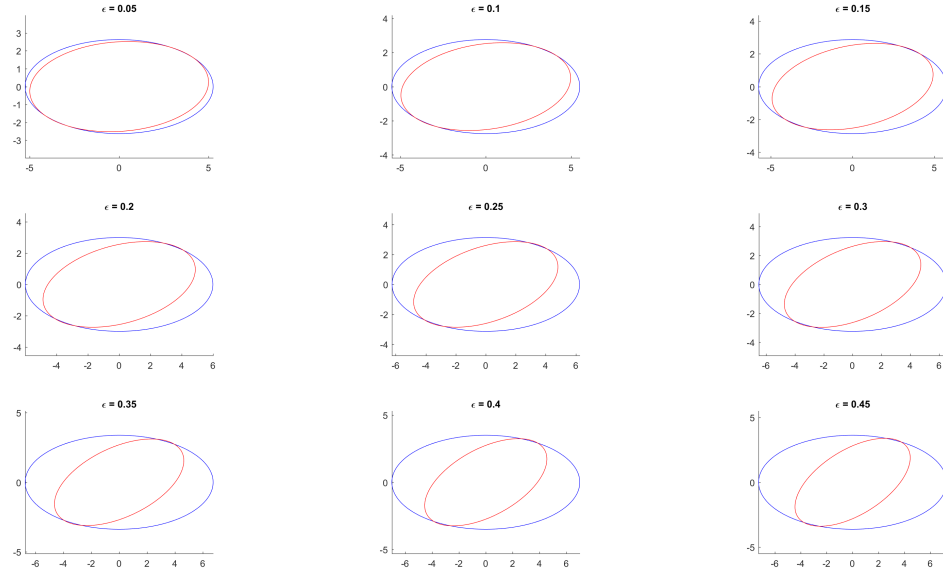
Figure 2: Test with different values of explode factor ε, fixed $\alpha = 2$
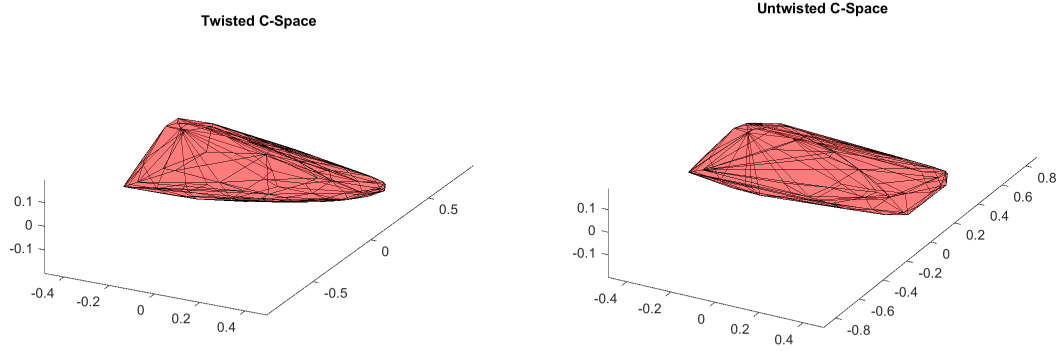

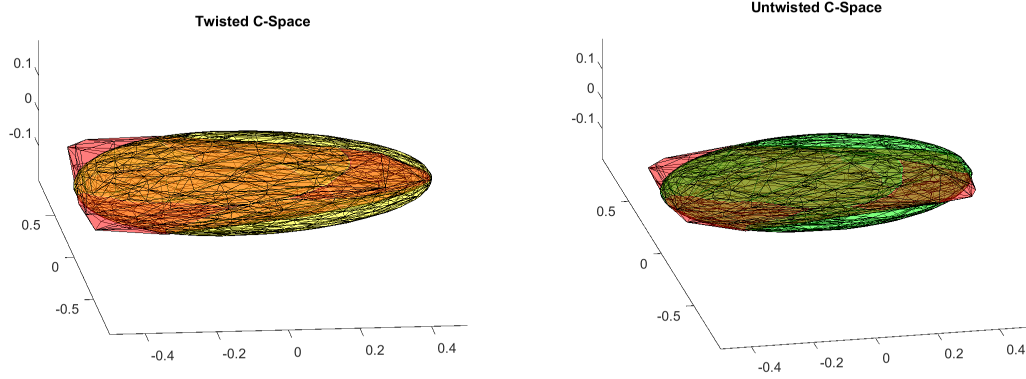Figure 3: Comparison of twisted and untwisted c-space

Figure 4: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 2$, $\varepsilon = 0.1$, $\beta = 0.6x$

Figure 4 is one example of the heuristic fit. The fitted ellipsoid exceeds the range of the convex hull, which is infeasible. To solve this, we could define a scale factor to the ellipsoid parameters, factor can be different for each parameter, but this still needs manually observation. Figure 5 shows the scaled heuristic fit, with θ scaled only.

4. **Heuristic fit with different aspect ratios** Each time we change the aspect ratio α, we need to manually change the factor k of twisting function $\beta = kx$ correspondingly. Table 4 lists the aspect ratios and the corresponding twisting factors.

Aspect ratio α	1.5	1.8	2	2.2	2.5	3
Twisting factor k	1.75	1	0.6	0.5	0.35	0.25

Figures 6, 7, 8, 9 and 10 show the heuristic fit with different aspect ratios.

5. **TODO list**

   (a) Find an automatic way to define the twisting function along x-axis, e.g. calculate the maximum ratio of θ/y

   (b) Find a way to determine the scale factor of the heuristic fit inside the untwisted convex hull

   (c) Fit the untwisted convex hull with other shapes, e.g. superquadrics

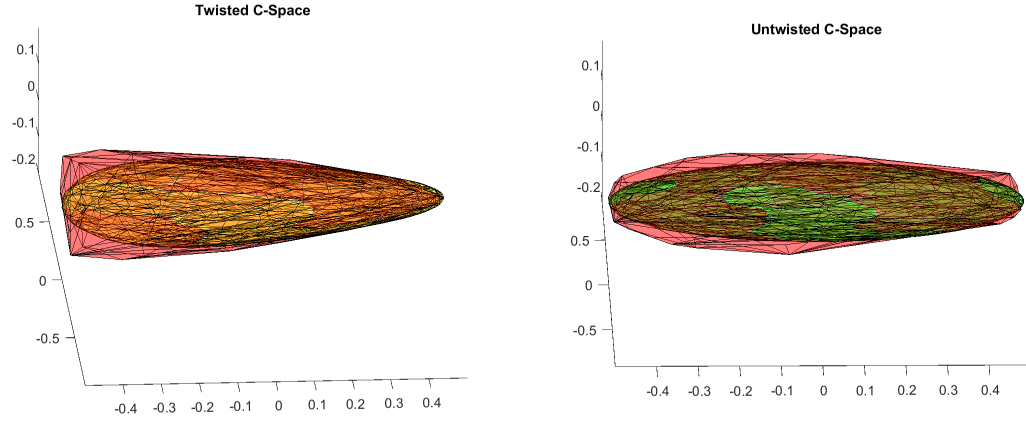   (d) Try the heuristic fit with different aspect ratios and explode factors

Figure 5: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 2$, $\varepsilon = 0.1$, $\beta = 0.6x$, scale factor: 0.6
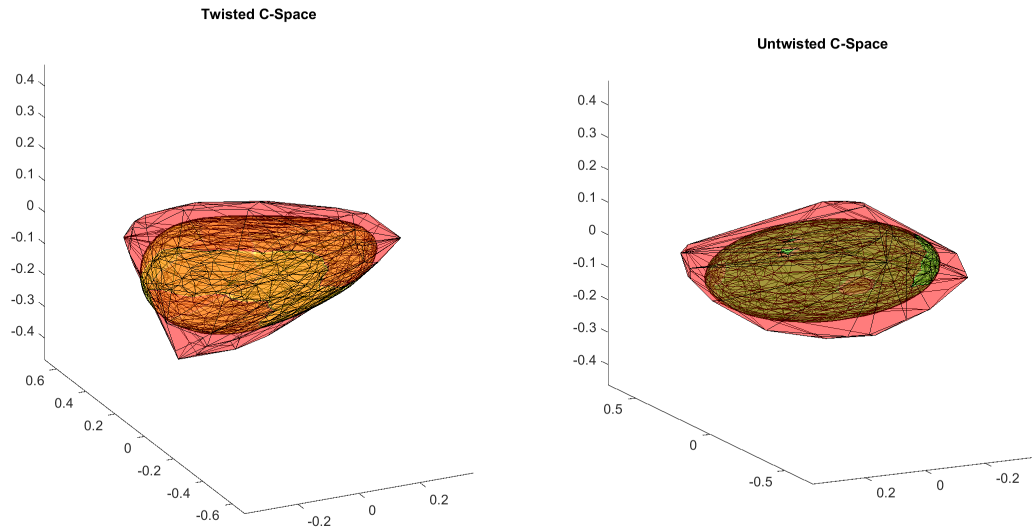

Figure 6: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 1.5$, $\varepsilon = 0.1$, $\beta = 0.75x$, scale factor: 0.6
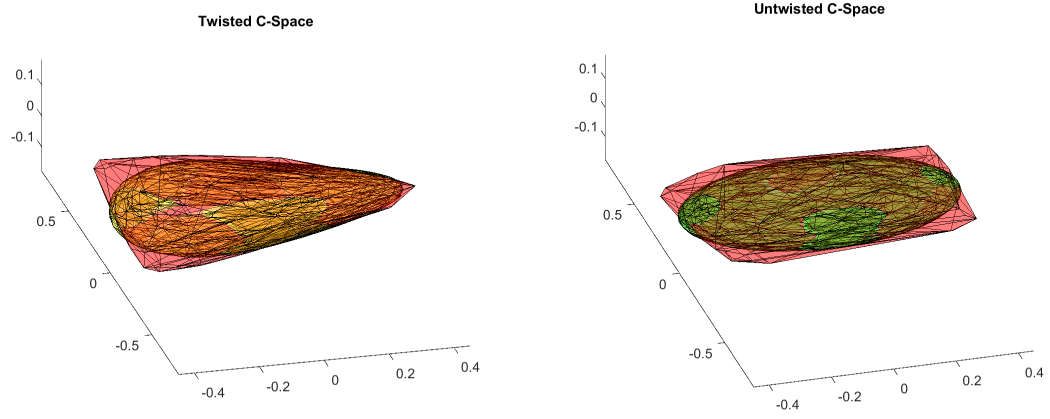
Twisted C-Space

Untwisted C-Space

Figure 7: heuristic fit, with parameters: $a_2 = 2.5$, $\alpha = 1.8$, $\varepsilon = 0.1$, $\beta = x$, scale factor: 0.6
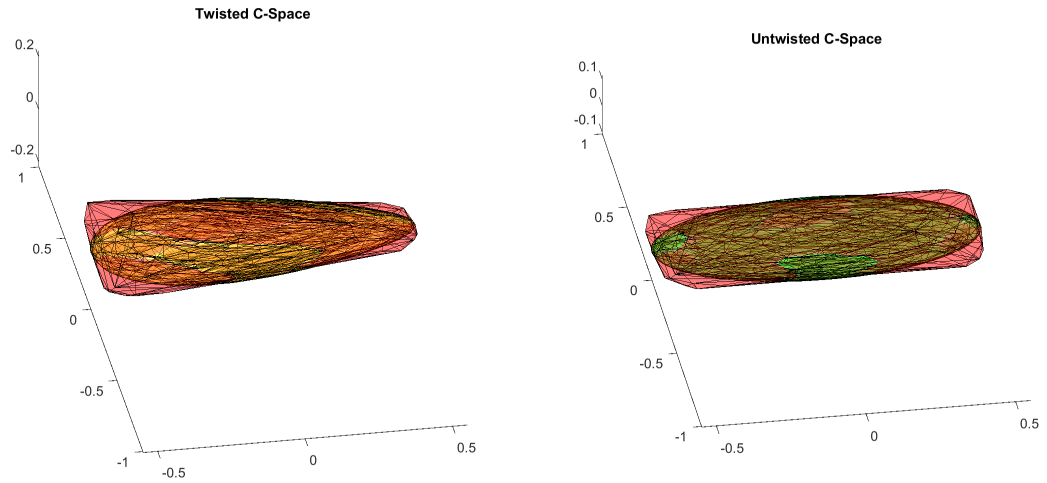
Twisted C-Space

Untwisted C-Space

Figure 8: heuristic fit, with parameters: $a_2 = 1.5$, $\alpha = 2.2$, $\varepsilon = 0.1$, $\beta = 0.5x$, scale factor: 0.6
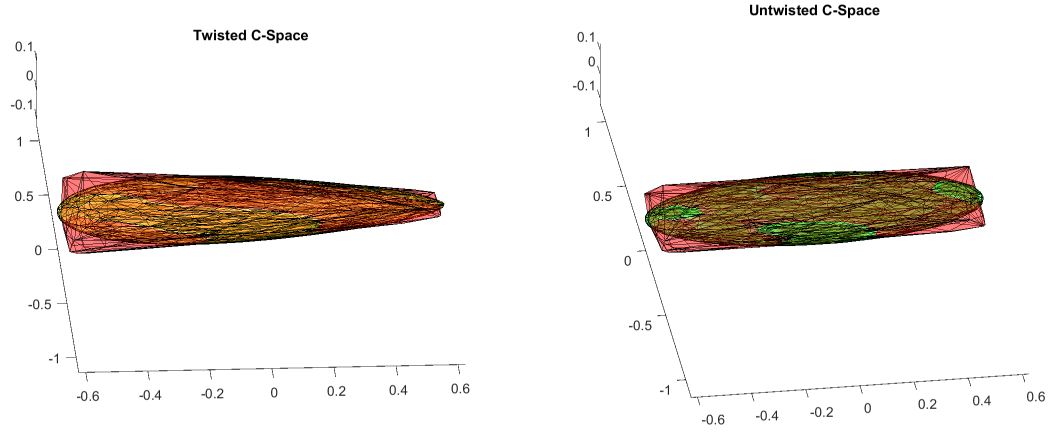
Figure 9: heuristic fit, with parameters: $a_2 = 1.5$, $\alpha = 2.5$, $\varepsilon = 0.1$, $\beta = 0.35x$, scale factor: 0.6
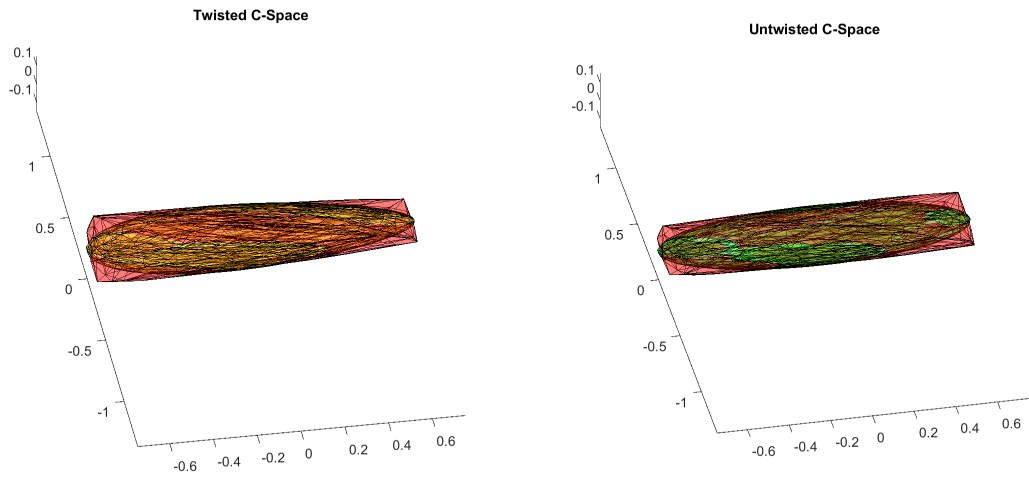

Figure 10: heuristic fit, with parameters: $a_2 = 1.5$, $\alpha = 3$, $\varepsilon = 0.1$, $\beta = 0.25x$, scale factor: 0.6

6. Heuristic KC with Polyhedron fit in C-Space, 2D

Another observation from the sample-based KC is that surfaces between 3 different extreme points which seems flat forms a tetrahedron-shaped c-space. This gives an intuition to construct the motion c-space as a tetrahedron or generally a polyhedron. From this point, we first seek to find the extreme points that represent the polyhedron as follows.

(a) **Finding extreme vertices that represent the polyhedron**

Extreme points in x, y and θ axes can be simply found by fixing the other two values to zero. For example, if we set $y, \theta = 0$, then the extreme distance in x-axis is $x_{extreme} = \varepsilon a_1$; similarly $y_{extreme} = \varepsilon a_2$ if fixing $x, \theta = 0$. The extreme distance in θ-axis is obtained above in closed-form as $\theta_{extreme} = \arctan(\pm \varepsilon \sqrt{\frac{\alpha}{(\alpha(1+\varepsilon)-1)(\alpha-\varepsilon-1)}})$. Since in each axis, there are 2 extreme points (positive and negative), here we get 6 vertices for constructing the polyhedron.

Now we seek to find the vertices that are farthest to the origin. Let $\mathbf{z} = [\theta, x, y]^T$, then our goal is to maximize its distance function to origin, $f = \mathbf{z}^T \mathbf{z}$. Further, the constraint for $\mathbf{z}$ is that $\mathbf{t} = [x, y]^T$ and θ should satisfy the algebraic condition that the smaller ellipse E_1 must move inside the larger one E_2 without collision.

$$(R(\theta)\Lambda(\mathbf{a})\mathbf{u} + \mathbf{t})^T \Lambda^{-2}(\mathbf{b})(R(\theta)\Lambda(\mathbf{a})\mathbf{u} + \mathbf{t}) \leq 1 \tag{8}$$

Regrouping the terms of the first-order approximation of the above inequality gives the constraint for $\mathbf{z}$:

$$C(\mathbf{z}) = \mathbf{z}^T H(\mathbf{u})\mathbf{z} + \mathbf{h}(\mathbf{u})^T \mathbf{z} + c(\mathbf{u}) \leq 1 \tag{9}$$

Such constraint is a family of inequality constraints where $\mathbf{u}_i = [\cos(\phi)_i, \sin(\phi)_i]^T$ represents the i-th point on the boundary of a unit circle. If we treat $\mathbf{z}$ as unknown variable and $\mathbf{u}$ as a group of known parameters, finding the maximum distance can be formed as an optimization problem as

$$\mathbf{z}_{extreme} = \arg\max \mathbf{z}^T \mathbf{z}$$
$$\text{s.t. } C_i(\mathbf{z}) = \mathbf{z}^T H(\mathbf{u}_i)\mathbf{z} + \mathbf{h}(\mathbf{u}_i)^T \mathbf{z} + c(\mathbf{u}_i) \leq 1 \quad (i = 0, ..., n) \tag{10}$$

where, n is the number of discretized points on the boundary of the unit circle.

Since the objective function is quadratic, the solutions for each variable have 2 possibilities, so the total number of solutions can be up to 8. However, not all of those possibilities are feasible solutions, meaning we have to validate them by substituting back to the constraint inequalities. Interestingly, after some numerical tests, there are only 4 of them that are feasible, i.e. for each fixed x, y pair, there is only one corresponding θ that the smaller ellipse can stay inside the large one without collision. Fig 11 and 12 show the resulting c-space polyhedron and the corresponding scenarios in euclidean space.
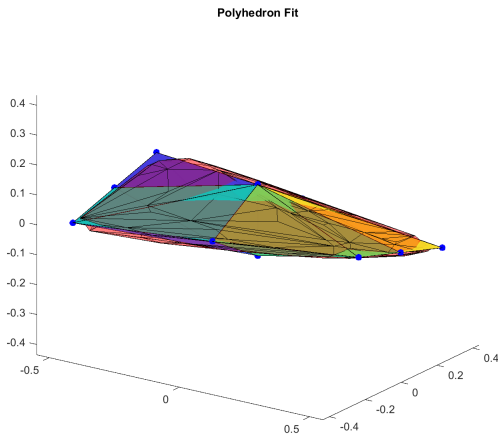

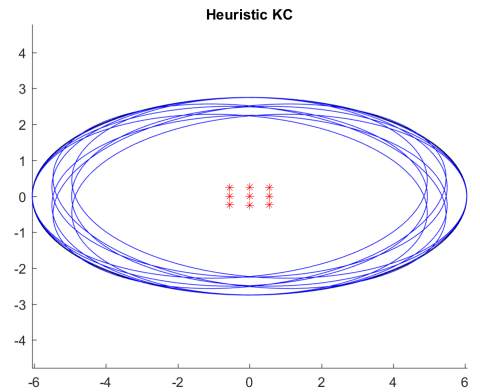Figure 11: Heuristic fit as polyhedron represented by 10 extreme vertices, C-space

Figure 12: Heuristic fit as polyhedron represented by 10 extreme vertices

At this time, we get totally 10 extreme vertices to construct the "c-space motion polyhedron". By saying "polyhedron", we should prove that the surfaces we observed is truly flat, in other word, points on the straight line connecting two vertices all represent safe configurations. This observation becomes obvious if we could firstly show that the constraint functions are convex.

(b) **Show that the c-space points inside the convex hull of the extreme vertices are safe configurations**
We have the following equivalent claim: the c-space points that lies on the line segment connecting any two extreme points also satisfy the conditions $C(\mathbf{z}) = \mathbf{z}^T H(\mathbf{u})\mathbf{z} + \mathbf{h}(\mathbf{u})^T \mathbf{z} + c(\mathbf{u}) \leq 1$ for all $\mathbf{u} \in \mathscr{B}(1)$.

Proof. From the condition we could also claim that among all $\mathbf{u}$,

$$\max C_i(\mathbf{z}) = \max \mathbf{z}^T H(\mathbf{u}_i)\mathbf{z} + \mathbf{h}(\mathbf{u}_i)^T \mathbf{z} + c(\mathbf{u}_i) \leq 1 \tag{11}$$

Then we could further prove that "given any two extreme points $\mathbf{z}_1, \mathbf{z}_2$ satisfying (11), the points on line segment between these two points also satisfy (11)". The proof can be achieved by showing $\max C_i(\mathbf{z})$ is convex.

For any fixed $\mathbf{u}_i$, we seek to prove, at first, that the function $C_i(\mathbf{z})$ is convex, which is to show that, given $\mathbf{z}_1, \mathbf{z}_2$

$$C_i(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) \leq \alpha C_i(\mathbf{z}_1) + (1-\alpha)C_i(\mathbf{z}_2), \forall \alpha \in [0,1] \tag{12}$$

Expanding both sides and moving right-hand side to the left gives

$$
\begin{aligned}
&[LHS] - [RHS] \\
=&[(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2)^T H(\mathbf{u}_i)(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) + \mathbf{h}(\mathbf{u}_i)^T(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) + c(\mathbf{u}_i)] \\
&- [\alpha(\mathbf{z}_1^T H(\mathbf{u}_i)\mathbf{z}_1 + \mathbf{h}(\mathbf{u}_i)^T\mathbf{z}_1 + c(\mathbf{u}_i)) + (1-\alpha)(\mathbf{z}_2^T H(\mathbf{u}_i)\mathbf{z}_2 + \mathbf{h}(\mathbf{u}_i)^T\mathbf{z}_2 + c(\mathbf{u}_i))] \\
=&-\alpha(1-\alpha)[(\mathbf{z}_1 - \mathbf{z}_2)^T H(\mathbf{u}_i)(\mathbf{z}_1 - \mathbf{z}_2)]
\end{aligned}
\tag{13}
$$

The above expression satisfies the inequality $[LHS] - [RHS] \leq 0$ if and only if $(\mathbf{z}_1 - \mathbf{z}_2)^T H(\mathbf{u}_i)(\mathbf{z}_1 - \mathbf{z}_2) \geq 0$, or equivalently, $H(\mathbf{u}_i)$ is symmetric positive semi-definite, which can be showed by expanding the original expression of the condition to the 1st-order approximation.

Since $\mathbf{z} = [\mathbf{r}^T, \mathbf{t}^T]^T$, only the quadratic terms with respect to both $\mathbf{r}$ and $\mathbf{t}$ from the 1st-order approximation will contribute to the calculation of $\mathbf{z}^T H(\mathbf{u}_i)\mathbf{z}$ for any $\mathbf{z}$. Thus, we have, $\forall \mathbf{z} \in \mathfrak{R}^{\frac{n(n+1)}{2}}$

$$\mathbf{z}^T H(\mathbf{u}_i)\mathbf{z} = (S(\mathbf{r})\Lambda(\mathbf{a})\mathbf{u} + \mathbf{t})^T \Lambda^{-2}(\mathbf{b})(S(\mathbf{r})\Lambda(\mathbf{a})\mathbf{u} + \mathbf{t}) \tag{14}$$

Since $\Lambda^{-2}(\mathbf{b})$ is diagonal with non-negative entries on diagonal, $(14) \geq 0$, which means that $H(\mathbf{u}_i)$ is symmetric positive semi-definite. Hence we showed the inequality condition for (13), from which we conclude that each condition function $C_i(\mathbf{z})$ is convex.

Next, we continue the proof by showing that the maximum of a family of convex functions is also convex. Since $C_i(\mathbf{z})$ is convex, then taking maximum for both sides of (12) gives

$$
\begin{aligned}
\max C_i(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) &\leq \max[\alpha C_i(\mathbf{z}_1) + (1-\alpha)C_i(\mathbf{z}_2)], \forall \alpha \in [0,1] \\
&\leq \max \alpha C_i(\mathbf{z}_1) + \max(1-\alpha)C_i(\mathbf{z}_2), \forall \alpha \in [0,1]
\end{aligned}
\tag{15}
$$

Thus $\max C_i(\mathbf{z})$ is convex. And we can conclude that if for two extreme points $\mathbf{z}_1, \mathbf{z}_2$, $\max C_i(\mathbf{z}_j) \leq 1, j = 1, 2$ are hold, then for points on the line segment between them, $\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2, \forall \alpha \in [0,1]$, $\max C_i(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) \leq \alpha \max C_i(\mathbf{z}_1) + (1-\alpha)\max C_i(\mathbf{z}_2) \leq 1$ is also satisfied. $\square$

Figs 13 and 14 show the numerical results of randomly sampled configurations of smaller ellipses that verify the above claim. The ellipse whose configuration point is inside the c-space polyhedron (convex hull of the 10 extreme vertices) lies safely inside the larger one without collision.
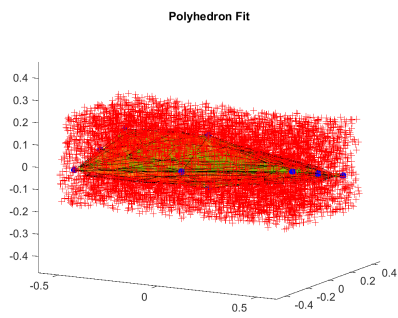
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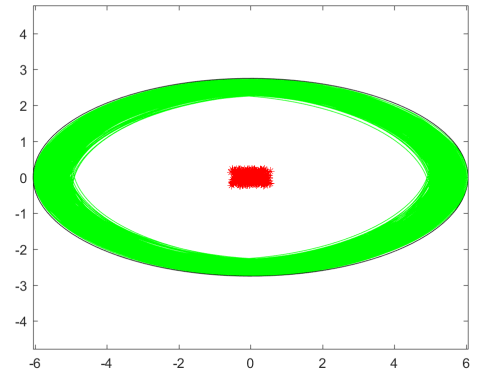Figure 13: C-space polyhedron fit with samples. Red: outside the convex hull; green: inside the convex hull

Figure 14: Smaller ellipses that lies safely inside the larger ellipse.