EDDP – Seminar

21 nov – partial

25.10.2011

3) Să se det. mult. sol. ec.

$$x' = \frac{tx^2 + t^3 - t}{2x(t^2-1)}, \quad t \in (1, +\infty). \text{ Să se det. sol. care verif. } x(2) = 1$$

$$x' = \frac{tx^2}{2x(t^2-1)} + \frac{t\cancel{(t^2-1)}}{2x\cancel{(t^2-1)}} \Rightarrow x' = \underbrace{\frac{t}{2(t^2-1)}}_{a(t)} x + \underbrace{\frac{t}{2}}_{b(t)} \cdot x^{-1} \quad (1)$$

$\alpha = -1$

$$\frac{d\bar{x}}{dt} = a(t)\bar{x} \Rightarrow \bar{x} = c \cdot e^{A(t)} = c \cdot e^{\ln(t^2-1)^{\frac{1}{4}}} = c \cdot (t^2-1)^{\frac{1}{4}}$$

$$\frac{dx}{dt} = a(t)x \Rightarrow x(t) = c \cdot e^{\int a(t)dt}$$

$$A(t) = \int \frac{t}{2(t^2-1)}\,dt = \frac{1}{4}\ln|t^2-1| = \ln(t^2-1)^{\frac{1}{4}}$$

Metoda var. const.: det. funcția $c(t)$ a.î. $x(t) = c(t)(t^2-1)^{\frac{1}{4}}$ să verif. ec. Bernoulli (1)

$$\frac{d}{dt}\left[c(t)(t^2-1)^{\frac{1}{4}}\right] = \frac{t}{2(t^2-1)}\left[c(t)(t^2-1)^{\frac{1}{4}}\right] + \frac{t}{2} \cdot \frac{1}{c(t)(t^2-1)^{\frac{1}{4}}} \Rightarrow$$

$$\Rightarrow c'(t)\cdot(t^2-1)^{\frac{1}{4}} + \cancel{c(t)\cdot\frac{1}{4}\cdot(t^2-1)^{-\frac{3}{4}}\cdot 2t} = \cancel{\frac{t}{2}\cdot c(t)\cdot(t^2-1)^{-\frac{3}{4}}} + \frac{t}{2}\cdot\frac{1}{c(t)\cdot(t^2-1)^{\frac{1}{4}}}$$

(totdeauna se simplifică termenul cu $c(t)$)

$$\frac{dc}{dt}(t^2-1)^{\frac{1}{4}} = \frac{t}{2(t^2-1)}\cdot\frac{1}{c} \Rightarrow$$

$$\Rightarrow \frac{dc}{dt} = \underbrace{\frac{t}{2(t^2-1)^{\frac{1}{2}}}}_{a_1(t)} \cdot \underbrace{\frac{1}{c}}_{b_1(c)}$$

$b_1(c) = 0 \Rightarrow \frac{1}{c} = 0 \Rightarrow$ nu are sol. staționare

Separarea variabilelor

$$\frac{dc}{\frac{1}{c}} = \frac{t}{2\sqrt{t^2-1}}\,dt$$

$$\int c\,dc = \frac{c^2}{2}$$

$$\int \frac{t}{2\sqrt{t^2-1}}\,dt = \frac{1}{2}\int \underbrace{\left(\sqrt{t^2-1}\right)'}_{=\frac{1}{2\sqrt{t^2-1}}\cdot 2t} dt = \frac{1}{2}\sqrt{t^2-1}$$

$c_1 \in \mathbb{R}$

Sol. implicită $\Rightarrow$ $\frac{c^2}{\cancel{2}} = \frac{1}{2}\sqrt{t^2-1} + \frac{c_1}{2}$, $\Rightarrow$

$\Rightarrow c^2 = \cancel{4}\sqrt{t^2-1} + c_1 \Rightarrow c = \pm\sqrt{\sqrt{t^2-1}+c_1}$, $c_1 \in \mathbb{R}$

$$x(t) = \pm\sqrt{\sqrt{t^2-1}+c_1}\cdot\left(t^2-1\right)^{\frac{1}{4}} \quad c_1 \in \mathbb{R}$$

$x(2) = 1$
$t = 2$
$x = 1$

$\cancel{\pm}\sqrt{\sqrt{3}+c_1}\cdot\sqrt[4]{3} = 1$ (doar partea cu +)

$\sqrt{\sqrt{3}+c_1}\cdot\sqrt[4]{3} = 1 \;\Big|^2$

$(\sqrt{3}+c_1)\cdot\sqrt{3} = 1$

$3 + \sqrt{3}c_1 = 1 \Rightarrow c_1 = \frac{-2}{\sqrt{3}}$

PC: $\begin{cases} (1) \\ x(2) = 1 \end{cases} \Rightarrow x(t) = \sqrt{\sqrt{t^2-1} - \frac{2}{\sqrt{3}}}\,\sqrt[4]{t^2-1}$

Ec. Riccati

$$\frac{dx}{dt} = a(t)x^2 + b(t) + c(t)$$

$a, b, c : I \subset \mathbb{R} \to \mathbb{R}$ funcții continue

Rezolvarea presupune cunoașterea unei sol. particulare φ_0.

Se efectuează schimbarea de var. $x(t) = y(t) + \varphi_0(t)$, care

$(x = y + \varphi_0)$

conduce la ec. Bernoulli

Ex: Fie ecuația $x' = -x^2 + 2tx + 5 - t^2$ (2) $t \in \mathbb{R}$

~~a) Stabiliți tipul ecuației~~

~~b) Determinați $m, n \in \mathbb{R}$ a.î. $\varphi_0(t) = mt + n$ să~~ fie sol. a ec (2)

c) Det. mulț. sol. ec(2)

d) Det. soluția care verifică condiția $x(1) = 3$

a) Ec Riccati

$a(t) = -1$

$b(t) = 2t$

$c(t) = 5 - t^2$

b) Înlocuim φ_0 în ec. (pe x cu φ_0)

$$\underset{(mt+n)'}{m} = -(mt+n)^2 + 2t(mt+n) + 5 - t^2 \quad \cancel{\forall t \in \mathbb{R}}$$

$$m = -(m^2t^2 + 2mnt + n^2) + 2mt^2 + 2nt + 5 - t^2$$

$$m = -m^2t^2 - 2mnt - n^2 + 2mt^2 + 2nt + 5 - t^2$$

$$m = t^2(-m^2 + 2m - 1) + t(-2mn + 2n) + 5 - n^2$$

$$\begin{cases} -m^2 + 2m - 1 = 0 \\ -2mn + 2n = 0 \\ 5 - n^2 = m \end{cases} \Leftrightarrow \begin{cases} -(m-1)^2 = 0 \Rightarrow m = 1 \\ -2mn + 2n = 0 \quad (A) \\ 5 - n^2 = m \Rightarrow n^2 = 4 \Rightarrow n = \pm 2 \end{cases}$$

$\varphi_0(t) = t + 2$ sau $\varphi_0(t) = t - 2$

c) S.v.:

$x = y + \varphi_0 \Rightarrow x = y + t + 2$

$$\frac{d}{dt}(y + t + 2)' = -(y+t+2)^2 + 2t(y+t+2) + 5 - t^2$$

$$(y' + 1) = -(y^2 + t^2 + 4 + 2yt + 4t + 4y) + 2ty + 2t^2 + 4t + 5 - t^2$$

$$y' + 1 = -y^2 \cancel{- t^2} - 4 \cancel{- 2yt} \cancel{- 4t} - 4y \cancel{+ 2ty} \cancel{+ 2t^2} \cancel{+ 4t} + 5 \cancel{- t^2}$$

$$y' = -y^2 - 4y \cancel{+1}$$

$\cancel{y^2+5y=0 \Rightarrow y(y+5)=0 \Rightarrow y=0}$

$\cancel{y=-5}$

$y' = -y^2 - 4y$ ec. bernoulli

$$\frac{dy}{dt} = 1(-4y - y^2)$$

$$a_1(t) = 1$$

$$b_1(y) = -4y - y^2$$

I) $b_1(y) = 0 \Rightarrow -4y - y^2 = 0 \Rightarrow 4y + y^2 = 0 \Rightarrow$

$\Rightarrow y(y+4) = 0 \Rightarrow y = 0$

$y = -4$

Sol lui x: $\boxed{x_1 = t+2}$ $x(1) = 3$

$x_2 = t-2$

II $b_1(y) \neq 0 \Rightarrow y^2 + 4y \neq 0 \Rightarrow y \in \mathbb{R} \setminus \{-4, 0\}$

$$dy \frac{1}{-4y-y^2} = dt \Rightarrow \frac{dy}{y^2+4y} = -dt$$

$$\int \frac{dy}{y^2+4y} \cancel{= \int \frac{1}{y(y+4)} dy =}$$

$$= \int \frac{(y+2)'}{(y+2)^2 - 2^2} = \frac{1}{2\cdot 2} \ln\left|\frac{y+2-2}{y+2+2}\right| = \frac{1}{4} \ln\left|\frac{y}{y+4}\right|$$

Sol implicită: $\frac{1}{4} \ln\left|\frac{y}{y+4}\right| = -t + \frac{C_1}{4} \quad |\cdot 4$

$$\ln\left|\frac{y}{y+4}\right| = -4t + C_1$$

$$\left|\frac{y}{y+4}\right| = e^{-4t} \cdot \underbrace{e^{C_1}}_{C_2 \in \mathbb{R}, \; C_2 > 0}$$

$$\frac{y}{y+4} = C_2 e^{-4t}, \quad C_2 \in \mathbb{R}^*$$

$$y = (y+4)\cdot c_2 e^{-4t}$$

$$y(1 - \cancel{4}\, c_2 e^{-4t}) = 4 c_2 e^{-4t}$$

$$y = \frac{4 c_2 e^{-4t}}{1 - \cancel{4}\, c_2 e^{-4t}}$$

$$x(t) = \frac{4 c_2 e^{-4t}}{1 - \cancel{4}\, c_2 e^{-4t}} + t + 2 \qquad c_2 \in \mathbb{R}^*$$

$x(1) = 3 \Rightarrow$
$t = 1$

$$\frac{4 c_2 e^{-4}}{1 - c_2 e^{-4}} \cancel{+3} = \cancel{3}$$

$$\frac{4 c_2 e^{-4}}{1 - c_2 e^{-4}} = 0 \Rightarrow c_2 = 0$$

Temă

Det. mult. sol. ec. urm.:

1) $\frac{dx}{dt} = x^2 - \frac{x}{t} - \frac{4}{t^2}$, $t \in (-\infty, 0)$

{ dacă are sol. particulară $\varphi_0(t) = \frac{\alpha}{t}$ cu $\alpha \in \mathbb{R}$ convenabil determin.
(ca ex. ant.) $\rightarrow \alpha$ tb. det

Găsiți soluțiile care verif. $x(-1) = 0$

2) $\frac{dx}{dt} = x^2 - \frac{3}{t} x + \frac{1}{t^2}$, $t \in (0, \infty)$

(ac. cerințe)

3) $\frac{dx}{dt} = x^2 + x - 4e^{2t}$; $\varphi_0 = \alpha e^t$, $\alpha \in \mathbb{R}$