EAAP - Seminar 3

17.10.2011

din temă

1)
$$\frac{dx}{dt} = \frac{xt}{x^2+t^2}$$

$$\frac{dx}{dt} = \frac{x\cancel{t}}{t^{\cancel{2}}\left(\left(\frac{x}{t}\right)^2+1\right)} \Rightarrow \frac{dx}{dt} = \frac{x}{t\left[\left(\frac{x}{t}\right)^2+1\right]} \Rightarrow \frac{dx}{dt} = \frac{\frac{x}{t}}{\left(\frac{x}{t}\right)^2+1}$$

$$\frac{x}{t} = y \Rightarrow x = ty \qquad x(t) = t\,y(t)$$

$$\frac{d}{dt}(ty) = \frac{y}{y^2+1}$$

$$y + t\frac{dy}{dt} = \frac{y}{y^2+1}$$

$$\frac{dy}{dt} = \frac{\frac{y}{y^2+1} - y}{t} \Rightarrow \frac{dy}{dt} = \frac{\cancel{y} - y^3 - \cancel{y}}{t(y^2+1)} \Rightarrow$$

$$\Rightarrow \frac{dy}{dt} = \underbrace{\frac{1}{t}}_{a(t)} \cdot \underbrace{\left(\frac{-y^3}{y^2+1}\right)}_{b(y)}$$ ec. cu var separabile

i) $b(y) = 0 \Rightarrow \frac{-y^3}{y^2+1} = 0 \Rightarrow y = 0$ sol staționară $\Rightarrow$

$\Rightarrow x(t) = 0$

ii) $b(y) \neq 0 \Rightarrow$ separăm variabilele

$$\frac{y^2+1}{-y^3}\,dy = \frac{1}{t}\,dt$$

$$B(y) = -\int \frac{y^2+1}{y^3}\,dy = -\int \frac{y^2}{y^3}\,dy - \int \frac{1}{y^3}\,dy = -\ln|y| - \frac{y^{-2}}{-2} =$$

$$= -\ln|y| + \frac{1}{2y^2} + c$$

$$A(t) = \int \frac{1}{t}\,dt = \ln\underbrace{|t|}_{>0} = \ln t + c$$

$-\ln|y| + \frac{1}{2y^2} = \ln|t| + c$ sol în formă implicită pt $y$

$-\ln\left|\frac{x}{t}\right| + \frac{1}{2\left(\frac{x}{t}\right)^2} = \ln|t| + c$ formă implicită pt ec în $x$

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2) $\frac{dx}{dt} = \frac{x-t-1}{t-x+1}$ soluția generală

Ec. de tipul $\frac{dx}{dt} = g\left(\frac{at+bx+c}{\alpha t+\beta x+\gamma}\right)$

$$\begin{cases} |a|+|\alpha|>0 \\ |\beta|+|b|>0 \end{cases}$$

$$\left.\begin{array}{l} a=-1 \\ b=1 \\ \alpha=1 \\ \beta=-1 \end{array}\right| \Rightarrow \Delta = 1-1=0$$

Avem $b \neq 0 \Rightarrow$ se face schimbarea de var $y = x-t$

$x = y+t$

$x(t) = y(t)+t$

$$\frac{d}{dt}(y+t) = \frac{-t+(y+t)-1}{t-(y+t)+2}$$

$$\frac{dy}{dt}+1 = \frac{y-1}{-y+2}$$

$$\frac{dy}{dt} = \frac{y-1}{-y+2} - 1 \Rightarrow \frac{dy}{dt} = \frac{y-1+y-2}{-y+2} \Rightarrow \frac{dy}{dt} = \underbrace{\frac{2y-3}{-y+2}}_{b(y)} \cdot \underbrace{1}_{a(t)}$$

i) $b(y)=0 \Rightarrow 2y-3=0 \Rightarrow y=\frac{3}{2}$ $\Rightarrow x = \frac{3}{2}+t$

$y=\frac{3}{2}$: sol staționară pt ec în $y$

$x = \frac{3}{2}+t$: sol. particulară

II) $b(y) \neq 0 \Rightarrow \dfrac{-y+2}{2y-3}\, dy = 1\, dt$

$$B(y) = \int \frac{-y+2}{2y-3}\, dy = -\frac{1}{2} \int \frac{2y-3+1}{2y-3}\, dy =$$

$$= -\frac{1}{2}\left( y - \int \frac{1}{2y-3}\, dy \right) =$$

$$= -\frac{1}{2} y + \frac{1}{2} \cdot \frac{\ln|2y-3|}{2}$$

$A(t) = t$

$B(y) = A(t) + c$

$-\frac{1}{2} y + \frac{1}{4} \ln|2y-3| = t + c$ sol implicită a ec în y

$-\frac{1}{2}(x-t) + \frac{1}{4} \ln|2x-2t-3| = t + c$

) $\dfrac{dx}{dt} = \dfrac{x+t-3}{t-x+1}$

| | |
|---|---|
| $a = 1$<br>$b = 1$<br>$c = -3$<br>$\alpha = 1$<br>$\beta = -1$<br>$\gamma = 1$ | $\Delta = -1 - 1 = -2 \neq 0$ |

$$\begin{cases} t + x - 3 = 0 \\ t - x + 1 = 0 \end{cases}$$

$2t - 2 = 0 \Rightarrow t = 1 \Rightarrow t_0 = 1$

$x = 2 \Rightarrow x_0 = 2$

Schimbare de var

$$\left| \begin{array}{l} y = x - x_0 \\ s = t - t_0 \end{array} \right| \begin{array}{l} = x - 2 \Rightarrow x = y + \\ = t - 1 = s(t) \end{array}$$

$y(s(t)) = x(t) - 2$  $y(s) = x(t+2)$

$x = y + 2 \Rightarrow$

$\Rightarrow x(t) = y(s(t)) + 2$

$$\frac{d}{dt}\left( y(s(t)) + 2 \right) = \frac{y + \cancel{2} + s + \cancel{1} - \cancel{3}}{s + 1 - y - \cancel{2} + \cancel{1}}$$

$\| \; y(s(t))' \cdot (s(t))'$

(termenii liberi trebuie să se reducă!)

$$\frac{dy}{ds} \cdot \overset{1}{\frac{ds}{dt}} = \frac{y+s}{s-y}$$

$$\frac{dy}{ds} = \frac{s+y}{s-y}$$

$$\frac{dy}{ds} = \frac{1+\frac{y}{s}}{1-\frac{y}{s}}$$

$$\frac{y}{s} = w \Rightarrow \frac{y(s)}{s} = w(s) \Rightarrow y(s) = s \cdot w(s)$$

Ex.: $t = e^s$, $t > 0$

$x(t) = y(s(t))$

$s = \ln t \Rightarrow \frac{ds}{dt} = \frac{1}{t}$

$$\frac{dx}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt} = \frac{1}{t} \frac{dy}{ds}$$

$$x(t) \xrightarrow[s=t-1]{} y(s) \xrightarrow{y(s) = s w(s)} w(s)$$

$y = x - 2$ $(y(s(t)) = x(t) - 2)$

$$\frac{d}{ds}(s \cdot w) = \frac{1+w}{1-w}$$

$$w + s \cdot \frac{dw}{ds} = \frac{1+w}{1-w}$$

$$\frac{dw}{ds} = \frac{1}{s} \cdot \left( \frac{1+w}{1-w} - w \right)$$

$$\frac{dw}{ds} = \underbrace{\frac{1}{s}}_{a(s)} \cdot \underbrace{\frac{1 + \cancel{w} - \cancel{w} + w^2}{1-w}}_{b(w)}$$

I) $b(w) = 0 \Rightarrow \frac{1+w^2}{1-w} = 0 \Rightarrow$ nu are soluții

II $b(w) \neq 0$

$B(w) = A(s) + C$

$$B(w) = \int \frac{1-w}{1+w^2}\,dw = -\frac{1}{2} \int \frac{2w-2}{w^2+1}\,dw = -\frac{1}{2}\left( \int \frac{(w^2+1)'}{w^2+1}\,dw - 2\int \frac{1}{w^2+1}\,dw \right) =$$

$$= -\frac{1}{2}\left(\ln(w^2+1) \mp 2\ \mathrm{arctg}\, w\right)$$

$$A(s) = \ln|s|$$

$$-\frac{1}{2}\left\{\ln(w^2+1) + 2\ \mathrm{arctg}\, w\right\} = \ln|s| + c$$

$$-\frac{1}{2}\ln\left(\left(\frac{y}{s}\right)^2+1\right) + 2\ \mathrm{arctg}\,\frac{y}{s} = \ln|s| + c$$

$$-\frac{1}{2}\ln\left(\left(\frac{x-2}{t-1}\right)^2+1\right) + 2\ \mathrm{arctg}\left(\frac{x-2}{t-1}\right) = \ln|t-1| + C$$

Ec. Bernoulli

$$\frac{dx}{dt} = a(t)x + b(t)x^{\alpha}$$

sau $\alpha \in \mathbb{R}\setminus\{0,1\}$

$\rightarrow$ o ecuație răsturnată de tip Bernoulli

$$\frac{dt}{dx} = a_1(x)t + b_1(x)t^{\alpha}$$

(4) $$\frac{dx}{dt} = \frac{1}{t^2 e^x - 2t}$$

(*) $$\frac{dt}{dx} = -2t + e^x \cdot t^{2} \quad \alpha = 2$$

$$a_1(x) = -2$$

$$b_1(x) = e^x$$

$$\alpha = 2$$

$$\frac{d\bar{t}}{dx} = a_1(x)\cdot \bar{t}$$

$\bar{t} = c\cdot e^{A_1(x)}$ unde $A_1(x)$ primitivă a lui $a_1(x)$

$$A_1(x) = \int a_1(x)\,dx = -\int 2\,dx = -2x$$

$$\bar{t} = c\cdot e^{-2x}$$

Aflăm $c(x)$ a.î. $\underline{t(x) = c(x)\cdot e^{-2x}}$ să verif. ec inițială (*)

$$\frac{d}{dx}\left(c(x)e^{-2x}\right) = -2\left(c(x)e^{-2x}\right) + e^x\left(c(x)\cdot e^{-2x}\right)^2$$

$$\frac{dc}{dx}\cdot e^{-2x} + \cancel{c(x)\cdot\left(-2\cdot e^{-2x}\right)} = \cancel{-2\left(c(x)\cdot e^{-2x}\right)} + e^x\left\{c(x)\cdot e^{-2x}\right\}^2$$

$$e^{-2x}\frac{dc}{dx} = \underbrace{e^x\cdot c^2(x)\cdot e^{-4x}}_{e^{-3x}}$$

$$\frac{dc(x)}{dx} = e^{-x} \cdot c^2(x)$$

$$\frac{dc}{dx} = \underbrace{e^{-x}}_{a_1(x)} \cdot \underbrace{c^2}_{b_1(c)}$$

$b_1(c) = 0 \Rightarrow c^2 = 0 \Rightarrow c = 0$

$t(x) = 0$ sol. stat.

$b_1(c) \neq 0$

$B_1(c) = A_1(x) + C$

$B_1(c) = \int \frac{1}{c^2} dc = \frac{c^{-1}}{-1} + k = \frac{-1}{c}$

$A_1(x) = \int e^{-x} dx = -e^{-x} = -\frac{1}{e^x}$

$-\frac{1}{c} = -\frac{1}{e^x} + k \Rightarrow \cancel{c(x) =}$

$\frac{1}{c} = \frac{1}{e^x} - \cancel{\frac{e^x}{k}} k \Rightarrow c = \frac{e^x}{1 - e^x \cdot k} \Rightarrow$

$\Rightarrow t(x) = \frac{e^x}{1 - e^x \cdot k} \cdot \left(e^{-2x}\right) \quad k \in \mathbb{R}$

Să se det. sol ec urm: (ec. Bernoulli)

<u>Temă</u> I. 1) $x' = \frac{2tx - x^2}{t^2}$

$\frac{dx}{dt}$

2) $x' = x\,tg\,t + x^4 \cos t$

3) $x' = \frac{tx^2 + t^3 - t}{2x(t^2 - 1)}$

II. Se dă ec $x' = at^\alpha + bx^\beta$ unde $\alpha, \beta, a, b$ sunt constante reale. Să se det. $\alpha$ și $\beta$ a.î. schimbarea de var $x = y^m$ să conducă la ec. omogenă în $y$.