Eddp - Seminar 1

6.10.2011

① Să se determine primitivele funcției

a) $f(x) = \dfrac{1}{\cos^2 x}$, $x \in \left(0, \dfrac{\pi}{2}\right)$

b) $f(x) = \dfrac{1}{\sin^2 x \cos^2 x}$ $x \in \left(0, \dfrac{\pi}{2}\right)$

c) $f(x) = (x^2 + 2x) \cdot e^{3x}$ $x \in \mathbb{R}$

d) $f(x) = (x^3 + 1) \log_2 x$ $x \in (0, \infty)$

e) $f(x) = \sqrt{4 - x^2}$ $x \in (-2, 2)$

f) $f(x) = \dfrac{e^{\operatorname{arctg} x}}{x^2 + 1}$ $x \in \mathbb{R}$

② Să se determine mulțimea sol. ec. dif. $\dfrac{dx}{dt} = f(t)$ unde

) $f(t) = t\sqrt{t^2 + 1}$, $t \in \mathbb{R}$

) $f(t) = e^{t} \sin t$, $t \in \mathbb{R}$

① a) $F(x) = \int f(x)\,dx = \int \dfrac{1}{\cos^2 x}\,dx = \operatorname{tg} x + \mathcal{C}$

b) $F(x) = \int f(x)\,dx = \int \dfrac{1}{\sin^2 x \cos^2 x}\,dx =$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx = \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x}\,dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x \cancel{\cos^2 x}}\,dx =$$

$$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx = \operatorname{tg} x - \operatorname{ctg} x + \mathcal{C}$$

c) $F(x) = \int f(x)\,dx = \int (x^2 + 2x)\, e^{3x}\,dx = \dfrac{1}{3} \int (x^2 + 2x)(e^{3x})'\,dx$

$$= \frac{1}{3}\left[(x^2 + 2x) \cdot e^{3x} - \int (x^2 + 2x)' \cdot e^{3x}\,dx\right] =$$

$$= \frac{1}{3}\left[(x + 2x) \cdot e^{3x} - \int (2x + 2)\, e^{3x}\,dx\right] =$$

$$= \frac{1}{3}\left[(x^2 + 2x) \cdot e^{3x} - \frac{1}{3}\int (2x+2)(e^{3x})'\,dx\right] = \frac{1}{3}\left[(x^2 + 2x)\, e^{3x} - \frac{1}{3}\left[(2x+2) \cdot\right.\right.$$

$$\left.\left. \cdot e^{3x} - 2\int e^{3x}\,dx\right]\right] = \frac{1}{3}(x^2 + 2x)\, e^{3x} - \frac{1}{9}(2x+2)\, e^{3x} + \frac{2}{27} e^{3x} + \mathcal{C}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

d) $F(x) = \int f(x)\,dx = \int (x^3+1)\log_2 x \,dx =$

$$= \int \left(\frac{x^4}{4} + x\right)' \log_2 x \,dx =$$

$$= \left(\frac{x^4}{4} + x\right)\log_2 x - \int \cancel{x}\left(\frac{x^3}{4} + 1\right)\underbrace{(\log_2 x)'}_{\frac{1}{\cancel{x}\ln 2}}\,dx =$$

$$= \left(\frac{x^4}{4} + x\right)\log_2 x - \frac{1}{\ln 2}\int \left(\frac{x^3}{4} + 1\right) dx =$$

$$= \left(\frac{x^4}{4} + x\right)\log_2 x - \frac{1}{\ln 2}\left(\frac{1}{4}\,\frac{x^4}{4} + x\right) + \mathcal{C}$$

$$= \left(\frac{x^4}{4} + x\right)\log_2 x - \frac{1}{\ln 2}\left(\frac{x^4}{16} + x\right) + \mathcal{C}$$

e) $F(x) = \int f(x)\,dx = \int \sqrt{4-x^2}\,dx =$

$$= \int x'\sqrt{4-x^2}\,dx = x\sqrt{4-x^2} - \int x\left(\sqrt{4-x^2}\right)' dx =$$

$$= x\sqrt{4-x^2} - \int x\,\frac{1}{2\sqrt{4-x^2}}\cdot(4-x^2)'\,dx =$$

$$= x\sqrt{4-x^2} - \int x\,\frac{1}{\cancel{2}\sqrt{4-x^2}}\cdot(-\cancel{2}x)\,dx =$$

$$= x\sqrt{4-x^2} - \int \frac{-x^2}{\sqrt{4-x^2}}\,dx = x\sqrt{4-x^2} - \int \frac{(4-x^2)-4}{\sqrt{4-x^2}}\,dx =$$

$$= x\sqrt{4-x^2} - \left(\int \sqrt{4-x^2}\,dx - \int \frac{4}{\sqrt{4-x^2}}\,dx\right) =$$

$$= x\sqrt{4-x^2} - \int \sqrt{4-x^2}\,dx + \int \frac{4}{\sqrt{4-x^2}}\,dx =$$

$$= x\sqrt{4-x^2} - \underbrace{\int \sqrt{4-x^2}\,dx}_{F(x)} + 4\int \frac{1}{\sqrt{4-x^2}}\,dx \Rightarrow$$

$\Rightarrow F(x) = x\sqrt{4-x^2} - F(x) + 4\arcsin\frac{x}{2} \Rightarrow$

$\Rightarrow F(x) = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C$

f) $F(x) = \int f(x)\,dx = \int \frac{e^{\operatorname{arctg} x}}{x^2+1}\,dx = e^{\operatorname{arctg} x} + C$

$\operatorname{arctg} x = t \Rightarrow \frac{1}{x^2+1}\,dx = dt$

$\Rightarrow \int e^t\,dt = e^t$

② a) $\frac{dx}{dt} = t\sqrt{t^2+1}$

$\Rightarrow x(t) = \frac{1}{2}\int 2t\sqrt{t^2+1}\,dt = \frac{1}{2} = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}(t^2+1)^{3/2} + C$

$t^2+1 = u \Rightarrow 2t\,dt = du$

$\int\sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$

b) $\frac{dx}{dt} = e^t \sin t$

$x(t) = \underbrace{\int e^t \sin t\,dt}_{} = \int (e^t)' \sin t\,dt = e^t \sin t - \int e^t \underset{\cos t}{(\sin t)'}\,dt =$

$= e^t \sin t - \int (e^t)' \cos t\,dt = e^t \sin t - e^t \cos t + \int e^t \underset{-\sin t}{(\cos t)'}\,dt =$

$= e^t \sin t - e^t \cos t - \underbrace{\int e^t \sin t\,dt}_{} \Rightarrow$

$\Rightarrow x(t) = \frac{e^t \sin - e^t \cos t}{2} = \frac{1}{2}e^t(\sin t - \cos t) + C$

$\frac{dx}{dt} = f(t, x)$ nu se scrie $x(t) = \dots$ ⚠

③ Să se det. mulț. sol ec. dif.

$$\frac{dx}{dt} = \frac{1}{t}(x^2-9) \quad x\in\mathbb{R},\ t>0$$

$f:(0,\infty)\times\mathbb{R}\to\mathbb{R}$

$$f(t,x) = \underbrace{\frac{1}{t}}_{a(t)}\underbrace{(x^2-9)}_{b(x)}$$ $\Rightarrow$ ec. este cu variabile separabile

Sol. staționare

I) $b(x)=0 \Rightarrow x^2-9=0 \Rightarrow x=\pm 3 \Rightarrow \begin{cases}\varphi_1(t)=3 & \forall t>0\\ \varphi_2(t)=-3 & \forall t>0\end{cases}$ (1)

II) $b(x)\neq 0 \Rightarrow x\in(-\infty,-3)\cup(-3,3)\cup(3,+\infty)$

$$\frac{dx}{x^2-9} = \frac{1}{t}\,dt$$

$$B(x) = \int\frac{dx}{x^2-9} = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|$$

$$A(t) = \int\frac{1}{t}\,dt = \ln|t|$$

Sol. în formă implicită: $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| = \ln|t| + \ln c,\ c>0$

$$\ln\left|\frac{x-3}{x+3}\right| = 6\ln|t|c \Rightarrow$$

$$\Rightarrow \ln\left|\frac{x-3}{x+3}\right| = \ln(c|t|)^6$$

$$\left|\frac{x-3}{x+3}\right| = \underbrace{c^6}_{>0}|t|^6 \Rightarrow$$

$$\Rightarrow \frac{x-3}{x+3} = \pm c^6 t^6 \Rightarrow \frac{x-3}{x+3} = c_1 t^6 \Rightarrow$$

$c_1\neq 0$

$$\Rightarrow x-3 = xc_1t^6 + 3c_1t^6 \Rightarrow x(1-c_1t^6) = 3c_1t^6+3 \Rightarrow$$

$$\Rightarrow x = \frac{3(c_1t^6+1)}{1-c_1t^6} \quad c_1\neq 0 \ (2)$$

Mult. sol = (1) ∪ (2)

④ Să se det mult. sol ec dif.

a) $\frac{dx}{dt} = \frac{\cos^2 x}{t^2+3}$ $\quad t \in \mathbb{R}$, $x \in \mathbb{R}$

b) $\frac{dx}{dt} = \frac{1}{t}(x^2-4x+3)$ $\quad t>0$, $x \in \mathbb{R}$

c) $\frac{dx}{dt} = (tg^2 t+1)\frac{1}{\cos^2 x}$, $\quad t \in \mathbb{R}$, $x \in (0, \frac{\pi}{2})$

d) $\frac{dx}{dt} = \frac{e^{arctg\, t}}{(t^2+1)(2x+1)e^{x^2+x}}$ $\quad x \in (-\frac{1}{2}, \infty)$, $t \in \mathbb{R}$

a) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$f(t,x) = \underbrace{\frac{1}{t^2+3}}_{a(t)} \cdot \underbrace{\cos^2 x}_{b(x)}$$

I) $b(x)=0 \Rightarrow \cos^2 x = 0 \Rightarrow x \in \{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}$

$\varphi_n(t) = (2n+1)\frac{\pi}{2} \quad \forall n \in \mathbb{Z}, t \in \mathbb{R}$

II $b(x) \neq 0 \Rightarrow \frac{dx}{\cos^2 x} = \frac{1}{t^2+3}dt$

$B(x) = \int \frac{1}{\cos^2 x}dx = tg\, x$

$A(t) = \int \frac{1}{t^2+(\sqrt{3})^2}dt = \frac{1}{\sqrt{3}} arctg \frac{t}{\sqrt{3}}$

$\Rightarrow tg\, x = \frac{1}{\sqrt{3}} arctg \frac{t}{\sqrt{3}} + C$

$x = arctg\left(\frac{1}{\sqrt{3}} arctg \frac{t}{\sqrt{3}}\right) + c$

$$\int \frac{1}{a^2-t^2}\,dt = \cancel{\operatorname{arctg}}\ \operatorname{arcth} \frac{t}{a} + \mathcal{C}$$

$$\int \frac{1}{\sqrt{t^2-a^2}}\,dt = \ln\left|t+\sqrt{t^2-a^2}\right| + \mathcal{C}$$

$$\int \frac{1}{t^2-a^2}\,dt = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \mathcal{C}$$

$$\int \frac{1}{\sqrt{t^2+a^2}}\,dt = \ln\left(t+\sqrt{t^2+a^2}\right) + \mathcal{C}$$

$$\int \frac{1}{t^2+a^2}\,dt = \frac{1}{a}\operatorname{arctg}\frac{t}{a} + \mathcal{C}$$