Assignment-1

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2) For every $b > 1$ and every constant $x > 0$,

$\log_b n = O(n^x)$

Solution:- Let us assume this to be true.

$\therefore f(n) = O(g(n))$

$O(g(n)) = \{f(n)$: for positive constant c,

$f(n) \le c g(n) \ \forall \ n \ge n_0\}$

~~$f(n) = \log_b n$; $g(n) = n^x$~~.

And,

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

$f(n) = \log_b n$

$g(n) = n^x$

$\forall \ b > 1$

$x > 1$

$$\lim_{n \to \infty} \frac{\log_b n}{n^x}$$

$$f'(n) = \frac{1}{n(\log b)} \qquad g'(n) = x n^{x-1}$$

$$\lim_{n \to \infty} \frac{\left(\frac{1}{n \log b}\right)}{x n^{x-1}} \Rightarrow \left(\frac{1}{x \log b}\right) \lim_{n \to \infty} \left(\frac{1}{n^{x-1}}\right)\left(\frac{1}{n}\right)$$

$$\Rightarrow \left(\frac{1}{x \log b}\right) \lim_{n\to\infty} \frac{1}{n^{x-1+1}} = \frac{1}{x \log b} \lim_{n\to\infty} \left(\frac{1}{n^{x}}\right)$$

$$\Rightarrow \frac{1}{x \log b}(0) \Rightarrow \underline{\underline{0}}$$

Hence $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \underline{\underline{0}}$

$\therefore$ Our assumption is true.

3) For every $r > 1$ and every constant $d > 0$

$$n^d = \Omega(r^n).$$

Solution- Let us assume A is true.

$f(n) = \Omega(g(n))$

$\Omega(g(n)) = \{ f(n)$: There exists positive constants c & n_0 such that

$$0 \le c\,g(n) \le f(n) \ \forall\ n \ge n_0 \}$$

And $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \underline{\underline{\infty}}$.

$f(n) = n^d$, $g(n) = r^n$

$$\lim_{n\to\infty} \frac{n^d}{r^n} \Rightarrow \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$ [Applying L'Hospital's Rule]

$f'(n) = dn^{d-1}$

$g'(n) = r^n(\ln r)$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{dn^{d-1}}{r^n(\ln r)}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{f''(n)}{g''(n)}, \ldots$$

$$\Rightarrow \lim_{n\to\infty} \frac{dn^{d-1}}{r^n(\ln r)} = \lim_{n\to\infty} \frac{d(d-1)n^{d-2}}{r^n(\ln r)^2}.$$

We can observe that as we derivate the numerator decreases and the denominator increases.

Hence $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

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But, this is a contradiction to our assumption

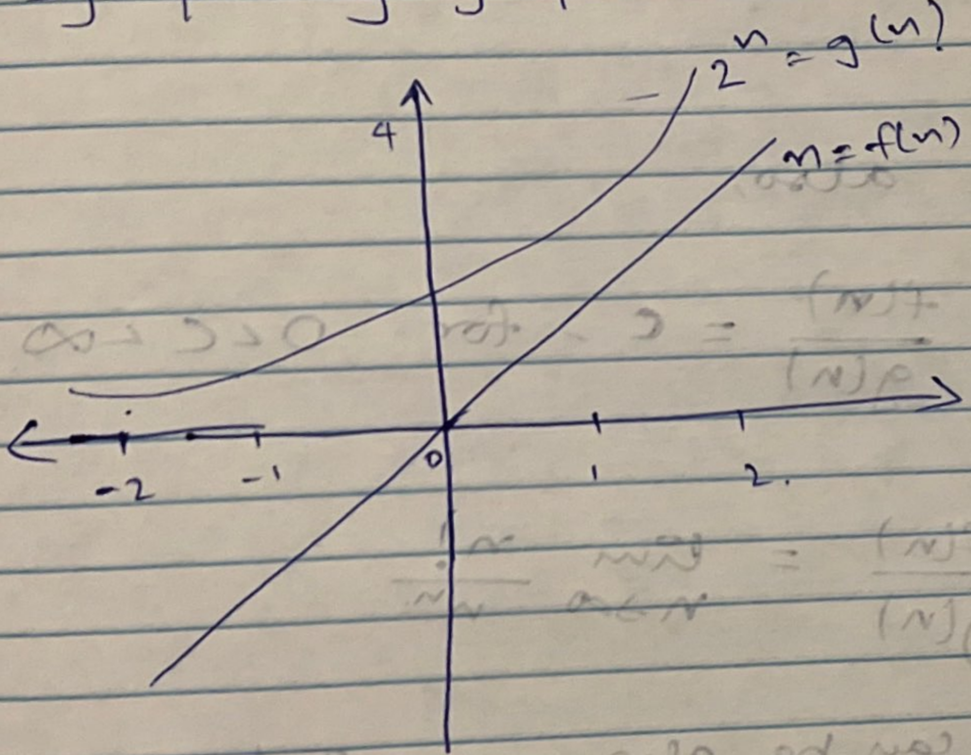
Hence, it is false.

$n^d = \Omega(r^n)$ $r > 1$; $d > 0$

$r = 2$; $d = 1$

$n = \Omega(2^n)$

By plotting graphs: for better understanding



; $\underline{f(n) \le c g(n)}$

6) $n! = \Theta(n^n)$

Solution:- Let us assume it is true.

For $f(n) = \Theta(g(n))$

$\Theta(g(n)) = \{ f(n)$: there exists positive constants c_1, c_2 & n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \forall n \geq n_0 \}$$

$f(n) = n!$

$= n \times (n-1) \times \dots \times 1$

$g(n) = n^n$

$\Rightarrow n \times n \times \dots \times n$

And also,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \text{ for } 0 < c < \infty$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n!}{n^n}$$

$\Rightarrow$ It can be also represented as

$$\lim_{n\to\infty} \left[\frac{n \times (n-1) \times (n-2) \dots 1}{n \times n \times \dots \times n} \right]$$

$$\Rightarrow \lim_{n\to\infty} \left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots \left(\frac{1}{n}\right) \right];$$

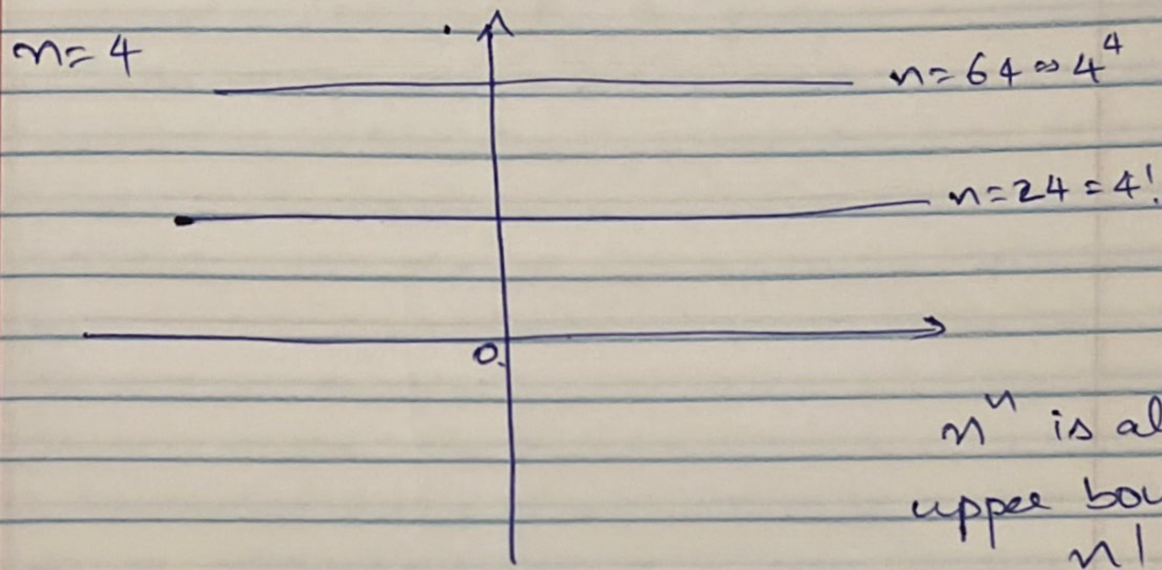
$$\lim_{n\to\infty} (1/n) = \underline{\underline{0}}$$

c should be greater than 0.

Hence it is a contradiction to our assumption.

Hence, it is false.

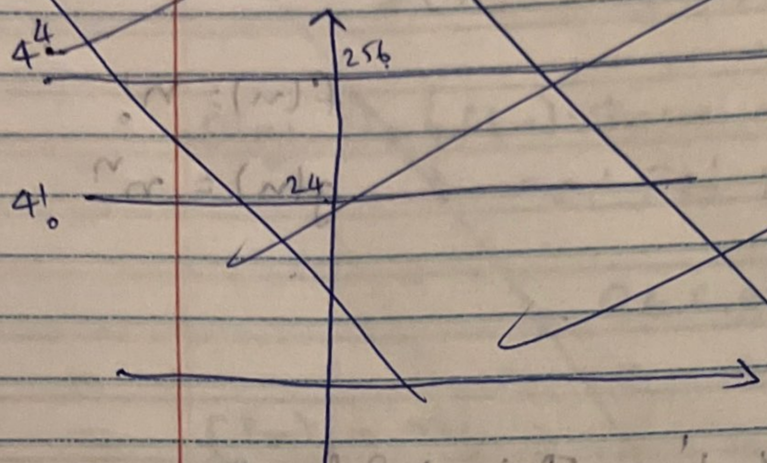
for example,



n^n is always upper bound of $n!$.

~~Let us take~~ ~~$n = 4$.~~

~~$f(n) = 4!$~~ ~~$g(n) = 4^4$~~



~~4^4 is always upper bounded of $4!$~~

7). $2^n = \Theta(4^n)$

lets assume it is true.

Solution:- $f(n) = \Theta(g(n))$

$\Theta(g(n)) = \{ f(n)$: there exists positive constants 'c_1', 'c_2' & 'n_0' such that

$0 \le c_1 g(n) \le f(n) \le c_2 g(n) \; \forall \; n \ge n_0 \}$

Here $f(n) = 2^n$; $g(n) = 4^n = 2^{2n}$.

And $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$; $0 < c < \infty$

$\lim_{n\to\infty} \frac{2^n}{2^{2n}} \Rightarrow \lim_{n\to\infty} 2^{n-2n} \Rightarrow \lim_{n\to\infty} 2^{-n}$

$\Rightarrow$ Hence. $\lim_{n\to\infty} 2^{-n} = 0.$

Again, 'c' has to be greater than '0'.

Hence, $f(n) \neq \Theta(g(n))$. Hence, our assumption is false.

But by the theorem,

As $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) = O(g(n))$ &

$f(n) \neq \Omega(g(n))$

$\Rightarrow$ ~~$2^n = \Theta g$~~

$$\Rightarrow \boxed{2^n = O(4^n) \text{ \& } 2^n \neq \Omega(4^n)}$$

For values of 'n':

$\Rightarrow 2^n \leq c \cdot 4^n.$

$\Rightarrow 2^n \leq c \cdot 2^{2n}$

$\Rightarrow 1 \leq c \cdot 2^n$ Holds true for all values of 'n'.

5) $n^{1.01} = O(n \log^2 n)$

Let's assume it is true

Solution: $f(n) = O(g(n))$

$O(g(n)) = \{ f(n) ;$ for some positive constants 'c',

$f(n) \leq c g(n) \; \forall \; n \geq n_0 \}$

And.

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

$f(n) = n^{1.01}$

$g(n) = n \log^2 n$

$$\Rightarrow \lim_{n \to \infty} \frac{n^{1.01}}{n \log^2 n} \Rightarrow \text{Applying L' Hospital Rule}$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}$$

$f(n) = n^{1.01}$

$f'(n) = (1.01)(n^{1.01-1}) \Rightarrow \underline{\underline{(1.01)(n^{0.01})}}$

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$g(n) = n\log^2 n$

$g'(n) = \frac{d}{dn}(n\log^2 n)$ Now, using product rule,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Here, $u = n$

$v = \log^2 n$

$\therefore \frac{d(n\log^2 n)}{dn} = n\frac{d}{dn}(\log^2 n) + \log^2 n\left(\frac{dn}{dn}\right)$

$\Rightarrow \left(\cancel{n} \times 2\log n \times \frac{1}{\cancel{n}}\right) + (\log^2 n \times 1)$

$\Rightarrow \cancel{2\log n[1+\log n]}$ $\log n[2+\log n]$

$g'(n) = g'(n\log^2 n) = \log n(2+\log n)$

$\therefore \lim_{n\to\infty} \frac{(1.01)n^{0.01}}{\log n(2+\log n)}$ $\Rightarrow$ Here as we observe, the numerator grows asymptotically faster than the denominator as '$n \to \infty$'.

$\therefore \lim_{n\to\infty} \frac{(1.01)n^{0.01}}{\log n(2+\log n)} = \infty.$

Hence, our assumption is contradicting.

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Hence,

$$n^{1.01} \neq O(n\log^2 n) \text{ and}$$

$$\boxed{n^{1.01} = \Omega(n\log^2 n) \ \& \ n^{1.01} \neq \Omega(n\log^2 n)}$$

i) Let $f(n)$ be a polynomial running time-function of 'n' and 'g' be an exponential running-time function of n. Then $f(n) = O(g(n))$.

__Solution:__- Let us assume it is true..

$f(n)$ = polynomial running-time function

$\Rightarrow a_0 x^n + a_1 x^{n-1} + \dots + a_n$

$g(n)$ = exponential running-time function

$\Rightarrow b^n$

Now,

By definition of $f(n) = O(g(n))$

$O(g(n)) = \{ f(n)$: for a positive constant 'c',

$f(n) \leq c\,g(n) \ \forall \ n \geq n_0$

And

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

$$\therefore \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

$$f'(n) = n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + \dots +$$

$$g'(n) = b^n \ln b$$

Hence,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \cancel{\lim_{n\to\infty} [n a_0 x^{n-1} + \dots +}$$

$$= 0$$

let us consider a monomial

$f(n) = n$

$g(n) = b^n$

$$\lim_{n\to\infty} \frac{n}{b^n} = \lim_{n\to\infty} \frac{1}{b^n (\ln b)} = 0$$

for a binomial.

$f(n) = an^2 + bn + c$

$g(n) = b^n$

$$\lim_{n\to\infty} \frac{an^2+bn+c}{b^n} \Rightarrow \lim_{n\to\infty} \frac{[2an+b]}{b^n(\ln b)} \Rightarrow \lim_{n\to\infty} \frac{2a}{b^n(\ln b)^2}$$

Therefore, even for a ~~quadri~~ binomial.

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. [As the denominator increases enormously than the numerator as $n\to\infty$]

It holds true for all polynomials.

Hence, $f(n) = O(g(n))$

Where, $f(n)$ = polynomial function

$g(n)$ = exponential function.

Hence, it ~~our base~~ assumption is true.

8) $\sum_{i=1}^{n} i^k = O(n^{k+1})$, where 'k' is a constant

Solution:- Let us assume it is true.

$f(n) = O(g(n))$

$O(g(n)) = \{f(n)$: for positive constant c, $f(n) \le c g(n) \ \forall\ n \ge n_0\}$

~~Here,~~ And $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Here, $f(n) = \sum_{i=1}^{n} i^k$ where 'k' is a constant.

$g(n) = n^{k+1}$

$f(n) = 1^k + 2^k + \dots + n^k$

$g(n) = n^{k+1} \Rightarrow (n^k)n = n^k \times n^k \times \dots \times n^k$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(1^k + 2^k + \dots + n^k)}{n(n^k)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\left[\frac{1^k}{n^k} + \frac{2^k}{n^k} + \dots + \frac{n^k}{n^k}\right]}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\left[\left(\frac{1}{n}\right)^k + \left(\frac{2}{n}\right)^k + \dots + \left(\frac{n}{n}\right)^k\right]}{n}$$

$$\text{let } \left(\frac{k}{n}\right)^k = \underline{\underline{0}}$$

Hence, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \underline{0}$

Hence, our assumption is $\underline{\underline{\text{true}}}$.