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Assignment - 12

Problem: You are given a set of 'n' objects, where the size s_i of the i^{th} object satisfies $0 < s_i < 1$. Your goal is to pack all the objects into the minimum number of unit-size bins. Each bin can hold any subset of the objects whose total size does not exceed 1. The first-fit heuristic takes each other object in turn and places it into the first bin that can accomodate it, as follows. It maintains an ordered list of bins. Let 'b' denote the number of bins in the list, where b increases over the course of the algorithm, and let $\langle B_1, B_2, \ldots B_b \rangle$ be the list of bins. Initially $b=0$ and the list is empty. The algorithm take each object 'i' in turn and places it in the lowest-numbered bin that can still accomodate it. If no bin can accomodate object i, then b is incremented and a new bin B_b is opened, containing object i. Let $S = \sum_{i=1}^{n} s_i$

Q1) Argue that the optimal number of bins required is at least $\lceil S \rceil$.

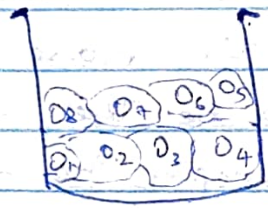
Solution: Number of given objects $= n$
the sizes of the objects would be respectively s_i for Object O_i

Size of the bin $= 1$.

Let there be 'b' bins as $\langle B_1, B_2, \ldots B_b \rangle$.

Now, we understand that the bin can be either filled with objects or can be empty.

So, we can calculate the size of the bin by calculating the size of the objects in the bin and any leftover space if any.



B_i

Let us say, this bin B_i is filled with 8 objects $\langle O_1, O_2, \ldots O_8 \rangle$ of sizes $\langle S_1, S_2, \ldots S_8 \rangle$ respectively and $\sum_{i=1}^{8} S_i < 1$.

This signifies that there is some empty space left.

So, the total bin size = total objects size + space left.

total objects space $\Big\} \Rightarrow S = \sum_{i=1}^{n} S_i$

Space left $\Big\} \Rightarrow L = \sum_{i=1}^{n} L_i$

$\therefore$ Therefore, total bins size would be as follow

$$\Rightarrow \sum_{i=1}^{n} S_i + \sum_{i=1}^{n} L_i \Rightarrow S + L$$

As, we know that size of each bin is '1'.

Total bins size = number of bins × size of each bin

$\sum_{i=1}^{n} S_i + \sum_{i=1}^{n} L_i$ = number of bins × 1

$\Rightarrow$ $\boxed{\text{number of bins} = \sum_{i=1}^{n} S_i + \sum_{i=1}^{n} L_i}$

$\therefore$ Therefore, number of bins = $\sum_{i=1}^{n} S_i + \sum_{i=1}^{n} L_i$

$\Rightarrow$ $S+L$

So, there might be a case where a bin can be completely filled with objects of sizes 'S_i' then we don't have any left over space. Hence, in that case $L_i = 0$

Therefore the optimal case would be $L_i = 0$.

Now, generalizing it left over space $\boxed{L \geq 0}$

Now,

Number of bins = $S + L$

$[L \geq 0]$

$\therefore$ Therefore,

number of bins $\geq S$ [As we substitute $L \geq 0$, $\geq$ comes]

As, each bin size is 1, and the size of the objects should be less than 1 in a bin B,

The best optimal way of accomodating in least number of bins of total number of objects size 'S' would be $\lceil S \rceil$.

Proof:- Lets consider 100 Objects.

each Object size = 0.727

Total Size, $S = \sum_{i=1}^{n} S_i \Rightarrow \sum_{i=1}^{100} S_i \Rightarrow (0.727) \times 100$

$\Rightarrow \underline{\underline{72.7}}$

$\therefore$ Therefore, number of bins $= \lceil S \rceil$

$= \lceil 72.7 \rceil = \underline{\underline{73}}$

As no two objects cannot be accomodated in a single bin.
as each Object size is 0.727

and 2 objects size will be greater than size of bin i.e. 1.

Q2) Argue that the first-fit heuristic leaves at most one bin at most half full.

Solution:

The first-fit algorithm uses the following heuristic. It keeps a list of open bins, which is initially empty. When an item arrives, find the first bin into which the item can fit, if any. If such a bin is found, then the new item is placed inside it.

As the question says first-fit heuristic leaves at most one bin less than half full.

Lets try to consider a case for 2 bins. 'B_1' & 'B_2'.
Lets consider 'B_1' & 'B_2' are half filled.

Now, when the objects are placed according to the first-fit heuristic, the objects would be placed in 'B_1' only but not in 'B_2'.

Therefore, there would not be a case where there would be 2 bins less than half full.
Hence, the first-fit heuristic leaves at most one bin less than half full.

Q3) Prove that the number of bins used by the first-fit heuristic never exceeds $\lceil 2S \rceil$.

Solution:- According to the first-fit heuristic method.

We place the objects in the empty bins and go on until it can fit into.

Hence, we end up with the case where bins are filled more than half.

Hence,

Total bin size = Total Objects size + Space left Over,

$$Bin~~Size~~ = \sum_{i=1}^{n} S_i + \sum_{i=1}^{n} L_i$$

$$Bin~~Size~~ = S + L$$

As we consider where bins are half full or more than half filled with the objects.

We can conclude that the left over space would be half of the Bin Size.

$\therefore$ Therefore, $L \leq \frac{Bin~Size}{2}$

$\Rightarrow$ $Bin~Size \leq S + \frac{Bin~Size}{2}$

$\Rightarrow$ $Bin~Size - \frac{Bin~Size}{2} \leq S$

$\Rightarrow$ $Bin~Size \leq 2S$

Which in turn proves

$$\boxed{Bin~Size \leq \lceil 2S \rceil}$$

Q4) Prove an approximation ratio of 2 for the first-fit heuristic.

Solution

From the definition of approximation ratio,

$$\text{Approximation Ratio} = \frac{\text{Worst Case}}{\text{Optimal Case}}$$

Now the number of bins used in Worst Case would be $\lceil 2S \rceil$

And the number of bins used in the optimal case would be $\lceil S \rceil$.

$\therefore$ Therefore Approximation Ratio $= \dfrac{\lceil 2S \rceil}{\lceil S \rceil} \Rightarrow \underline{\underline{2}}$