

# Fourier Descriptors for Plane Closed Curves

CHARLES T. ZAHN AND RALPH Z. ROSKIES

**Abstract**—A method for the analysis and synthesis of closed curves in the plane is developed using the Fourier descriptors FD's of Cosgriff [1]. A curve is represented parametrically as a function of arc length by the accumulated change in direction of the curve since the starting point. This function is expanded in a Fourier series and the coefficients are arranged in the amplitude/phase-angle form. It is shown that the amplitudes are pure form invariants as well as are certain simple functions of phase angles. Rotational and axial symmetry are related directly to simple properties of the Fourier descriptors. An analysis of shape similarity or symmetry can be based on these relationships; also closed symmetric curves can be synthesized from almost arbitrary Fourier descriptors. It is established that the Fourier series expansion is optimal and unique with respect to obtaining coefficients insensitive to starting point. Several examples are provided to indicate the usefulness of Fourier descriptors as features for shape discrimination and a number of interesting symmetric curves are generated by computer and plotted out.

**Index Terms**—Boundary functions, curvature functions, form-invariant features, Fourier descriptors, image processing, intrinsic functions, pattern analysis, pattern recognition, pattern symmetry, pattern synthesis, planar curves, shape description.

## INTRODUCTION

THE problem of discriminating planar shapes is one of the most familiar and fundamental problems in pattern recognition. It entails the assignment of an unknown shape to one of several classes of shapes based on a finite set of measurements (features) made on the shape. The digital representation of the shape usually is initially a digital picture [14]—that is, a rectangular matrix whose elements are from some finite range  $(0, B)$  of nonnegative integers. When  $B=1$  we have a black/white digital picture and each connected subset of elements having the value 1 (black points) constitutes a two-dimensional shape. Fig. 5(a) exhibits such a shape. There are many different ways to obtain numerical features from digital shapes but we find features based on the boundary of the shape to be most compelling intuitively. There is a substantial body of psychological and psychophysical evidence suggesting the importance of edges in visual scene analysis. We cite in particular the work of Attneave and Arnoult [15], Attneave [16], Uhr [17], Lettvin *et al.* [18], and Hubel and Wiesel [19].

The transformation from binary matrix to polygonal boundary can be effected by the method of Ledley [20] or Zahn [8], [9]. When  $B>1$  (so-called grey-scale pictures) more sophisticated methods are sometimes required to extract boundary curves but whatever method is employed the theory developed in this paper is appropriate. The topic is analysis and synthesis of closed curves in the plane and parts

of the theory could be applied to nonclosed curves. Although the boundary of a shape has as many separate closed curves as its degree of connectivity we discuss single curves only. Full comparison of multiply connected shapes should involve analysis of all boundary curves but the treatment of each curve will be similar.

At this point our problem is reduced to one of extracting a finite set of numerical features from a closed curve, features that will tend to separate the shapes of different classes relative to the intraclass dispersion. For this purpose, we shall use the Fourier descriptors, first suggested by Cosgriff [1]. A starting point on the boundary is selected and a function  $\theta(l)$  is defined which measures angular direction of the curve as a function of arc length. After appropriate normalization this periodic function is expanded in a Fourier series and the coefficients of a truncated expansion are used as shape features called Fourier descriptors. Higher order terms represent changes in direction of the curve over very small arc lengths and their elimination will probably reduce noise and serve to accentuate lower order terms that contain more macroscopic information on the shape.

The report by Cosgriff [1] was followed by a sequence of reports by Fritzsche [2], Raudseps [3], Borel [4], and Brill [5]–[7]. Material from these reports tends to support the idea that look-alike shapes are usually near each other in a space of Fourier descriptor features endowed with the Euclidean metric. The last two papers in the sequence Brill [6], [7] are especially noteworthy because they report on the results of recognition experiments which had remarkable success considering the context in which the recognition was performed. A more recent use of Fourier descriptors is reported by Barrow and Popplestone [25, pp. 395–396].

In this paper we define a normalized cumulative angular function  $\phi^*(t)$  for a simple closed curve  $\gamma$  and expand  $\phi^*$  in a Fourier series to obtain descriptive coefficients that we call Fourier descriptors (FD's), although  $\phi^*$  is a slightly different function than was used in the earlier work [1]–[7]. There follows a formula for reconstructing a curve from its Fourier descriptors and the explicit formulas for calculating the Fourier descriptors of a polygonal curve. Next we state several important relationships between the geometry of shapes and algebraic properties of the Fourier descriptors. Using these results we are able to define form-invariant measures based on the phase angle Fourier descriptors, that is functions of these phase angles which are not dependent on the starting point used to generate the function  $\phi^*$ . Also using the basic relationships, we suggest methods for shape similarity analysis and the testing of symmetry with several concrete examples. A short discussion of Fourier descriptors

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C. T. Zahn is with the Computation Group, Stanford Linear Acceleration Center, Stanford, Calif.

R. Z. Roskies is with the Department of Physics, Yale University, New Haven, Conn.

based on the curvature function of  $\gamma$  shows that they differ only slightly from those based on  $\phi^*$ . Next we consider the synthetic generation of curves from arbitrary Fourier descriptors and establish some simple sufficient conditions for such curves to be closed. It is then proved that the Fourier series expansion is the only expansion for which the coefficients are largely independent of starting point. Several examples are shown to indicate the usefulness of Fourier descriptors as features for shape discrimination and the theoretical material on curve reconstruction is used to generate some interesting curves with axial as well as rotational symmetry.

#### FOURIER DESCRIPTORS OF A CURVE AND THE RECONSTRUCTION THEOREM

We assume  $\gamma$  is a clockwise-oriented simple closed curve with parametric representation  $(x(l), y(l)) = Z(l)$  where  $l$  is arc length and  $0 \leq l \leq L$ . Denote the angular direction of  $\gamma$  at point  $l$  by the function  $\theta(l)$  assuming that  $\gamma$  is smooth in the sense of Buck [21] and let  $\delta_0 = \theta(0)$  be the absolute angular direction at the starting point  $Z(0)$ . We now define the cumulative angular function  $\phi(l)$  as the net amount of angular bend between starting point and point  $l$  [see Fig. 1(a)]. With this definition  $\phi(0) = 0$  and  $\phi(l) + \delta_0$  is identical to  $\theta(l)$  except for a possible multiple of  $2\pi$ . If the curve  $\gamma$  winds in a spiral then  $|\phi(l)|$  achieves values larger than  $2\pi$ . Fig. 2(b) shows  $\phi(l)$  for the shape depicted in Fig. 2(a).

It is not hard to see that  $\phi(L) = -2\pi$  because all smooth simple closed curves with clockwise orientation have a net angular bend of  $-2\pi$ . As a result,  $\phi(L)$  does not convey any shape information. The domain of definition  $[0, L]$  of  $\phi(l)$  simply contains absolute size information and we would like to normalize to the interval  $[0, 2\pi]$  which is standard for periodic functions. Hence we define a normalized variant  $\phi^*(t)$  [see Fig. 2(c)] whose domain is  $[0, 2\pi]$  and such that  $\phi^*(0) = \phi^*(2\pi) = 0$ . The formal definition is

$$\phi^*(t) = \phi\left(\frac{Lt}{2\pi}\right) + t \quad (1)$$

and  $\phi^*$  is invariant under translations, rotations, and changes of perimeter  $L$ . As intuitive justification for the definition of  $\phi^*$  we note that  $\phi^* \equiv 0$  for a circle which is in some sense the most *shapeless* closed curve! Viewed in this light the function  $\phi^*(t)$  measures the way in which the shape in question differs from a circular shape.

What we have done so far is to map all plane simple closed curves with starting point into the class of periodic functions on  $[0, 2\pi]$  in such a way that all curves of identical shape and starting point go into the same function  $\phi^*$ . Two plane curves have *identical shape* if they differ only by a combination of translation, rotation, and change in size.

We now expand  $\phi^*$  as a Fourier series

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt). \quad (2)$$

In polar form the expansion is

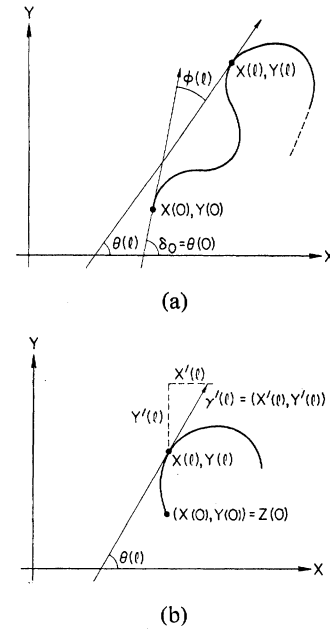


Fig. 1. (a) Parametric representation of a plane curve with tangential direction  $\theta(l)$  and cumulative angular bend function  $\phi(l)$ . (b) Tangent or velocity vector of plane curve.

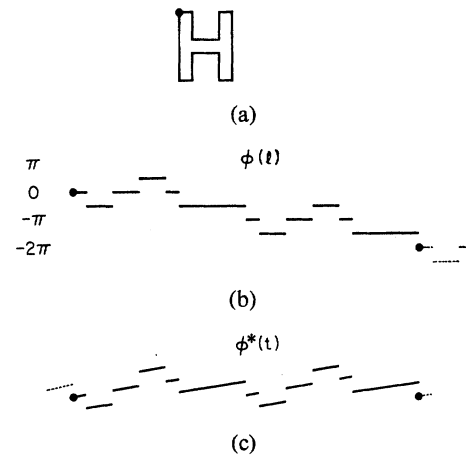


Fig. 2. (a) Plane closed curve with starting point from boundary of character "H". (b) Cumulative angular bend function  $\phi(l)$  for curve of Fig. 2(a). (c) Normalized cumulative angular bend function  $\phi^*(l)$  for curve of Fig. 2(a).

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} A_k \cos(kt - \alpha_k) \quad (3)$$

where  $(A_k, \alpha_k)$  are polar coordinates of  $(a_k, b_k)$ . These numbers  $A_k$  and  $\alpha_k$  are the Fourier descriptors for curve  $\gamma$  and are known respectively as the  $k$ th *harmonic amplitude* and *phase angle*.

Having developed this description  $\{A_k, \alpha_k\}_{1^{\infty}}$  it is of interest to know how easily the curve might be reconstructed from these numbers and how well a truncated description  $\{A_k, \alpha_k\}_{1^N}$  would reconstruct the curve. The theorem that follows answers the first question and a later section provides some insight into the second. The first question reduces quickly to one of reconstructing  $\gamma$  easily from its angle versus length function  $\theta(l)$ .

### Reconstruction Theorem

If curve  $\gamma$  is described by  $\theta(l)$  and starting point  $Z(0)$  the position of point  $Z(l)$  can be gotten from the expression

$$Z(l) = Z(0) + \int_0^l e^{i\theta(\lambda)} d\lambda \quad (4)$$

which is equivalent to

$$\begin{aligned} x(l) &= x(0) + \int_0^l \cos \theta(\lambda) d\lambda \\ y(l) &= y(0) + \int_0^l \sin \theta(\lambda) d\lambda. \end{aligned}$$

*Proof:* By definition  $\theta(l)$  measures the direction of the velocity vector  $\gamma'(l) = (x'(l), y'(l))$  which is tangent to  $\gamma$  at  $l$  [see Fig. 1(b)]. Whenever a curve is parametrized by its own arc length the speed  $|\gamma'(l)|$  is always 1 (see Buck [22]) and we obtain immediately  $x'(l) = \cos \theta(l)$  and  $y'(l) = \sin \theta(l)$ . The result follows by substituting these values into the fundamental theorem of integral calculus

$$x(l) - x(0) = \int_0^l x'(\lambda) d\lambda$$

and

$$y(l) - y(0) = \int_0^l y'(\lambda) d\lambda.$$

*Corollary:* Any function  $\theta^*(l) \equiv \theta(l)$  modulo  $2\pi$  works as well in the above reconstruction formula.

Using the reconstruction theorem and a truncated Fourier series expansion of  $\phi^*$  we shall give a practical formula for curve reconstruction based on Fourier descriptors  $\{A_k, \alpha_k\}_{1^N}$ . Invoking the corollary we can use  $\theta^*(l) = \phi(l) + \delta_0$  in place of  $\theta(l)$ ; then using (4) and (1), the transformation of variable  $l = Lt/2\pi$  and the truncated version of (3) we obtain the formula

$$\begin{aligned} Z(l) = Z(0) + \frac{L}{2\pi} \int_0^{2\pi l/L} \exp \left\{ i \left[ -t + \delta_0 + \mu_0 \right. \right. \\ \left. \left. + \sum_{k=1}^N A_k \cos (kt - \alpha_k) \right] \right\} dt. \quad (5) \end{aligned}$$

If we desire a curve of similar shape but starting point  $Z^*$ , initial direction  $\delta^*$ , and total arc length  $L^*$  we use the same formula but with these values instead of  $Z(0)$ ,  $\delta_0$ , and  $L$ .

### CALCULATING FOURIER DESCRIPTORS FROM A POLYGONAL CURVE

In this section we shall derive formulas for the Fourier coefficients  $\{a_k, b_k\}$  and  $\mu_0$  when  $\gamma$  is a polygonal curve. We assume the curve  $\gamma$  has  $m$  vertices  $V_0, \dots, V_{m-1}$  and that the edge  $(V_{i-1}, V_i)$  has length  $\Delta l_i$ . The change in angular direction at vertex  $V_i$  is  $\Delta \phi_i$  and  $L = \sum_{i=1}^m \Delta l_i$ . With these definitions (see Fig. 3) it is not hard to verify that

$$\phi(l) = \sum_{i=1}^k \Delta \phi_i \quad \text{for} \quad \sum_{i=1}^k \Delta l_i \leq l < \sum_{i=1}^{k+1} \Delta l_i$$

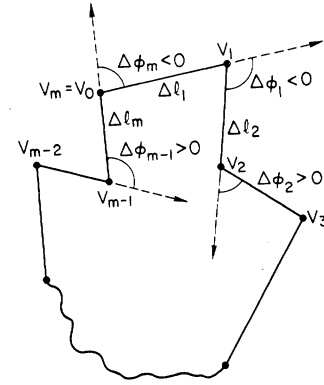


Fig. 3. Description of simple closed planar polygon in terms of edge lengths  $\Delta l_i$  and vertex bends  $\Delta \phi_i$ .

and

$$\phi(l) = 0 \quad \text{for} \quad 0 \leq l < \Delta l_1.$$

Expanding  $\phi^*$  we get

$$\phi^*(t) = \mu_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$\mu_0 = \frac{1}{2\pi} \int_0^{2\pi} \phi^*(t) dt$$

and

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \phi^*(t) \cos nt dt$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \phi^*(t) \sin nt dt.$$

Remembering that  $\phi^*(t) = \phi(Lt/2\pi) + t$  and changing variable  $\lambda = Lt/2\pi$  we get

$$\mu_0 = \frac{1}{L} \int_0^L \phi(\lambda) d\lambda + \pi$$

$$a_n = \frac{2}{L} \int_0^L \left( \phi(\lambda) + \frac{2\pi\lambda}{L} \right) \cos \frac{2\pi n\lambda}{L} d\lambda$$

$$b_n = \frac{2}{L} \int_0^L \left( \phi(\lambda) + \frac{2\pi\lambda}{L} \right) \sin \frac{2\pi n\lambda}{L} d\lambda.$$

Exploiting the fact that  $\phi(l)$  is a step function we obtain (after reasonably straightforward but somewhat lengthy calculations [3])

$$\mu_0 = -\pi - \frac{1}{L} \sum_{k=1}^m l_k \Delta \phi_k \quad (6)$$

$$a_n = \frac{-1}{n\pi} \sum_{k=1}^m \Delta \phi_k \sin \frac{2\pi n l_k}{L} \quad (7)$$

and

$$b_n = \frac{1}{n\pi} \sum_{k=1}^m \Delta \phi_k \cos \frac{2\pi n l_k}{L} \quad (8)$$

where

$$l_k = \sum_{i=1}^k \Delta l_i.$$

The final forms of the expressions for  $a_n, b_n$  are especially appealing because of their similarity and also because  $\Delta\phi_k$  represents the angular change (bend) in the curve's direction at the  $k$ th polygonal vertex, and  $l_k$  is the arc length from starting vertex to  $k$ th vertex. The dependence on  $\Delta\phi_k$  and  $l_k$  is nice for computation of the coefficients since a very natural data structure for representing polygons [8] uses  $\Delta\phi_k$  and  $\Delta l_k$ .

It is clear from these expressions alone that the Fourier coefficients ( $a_n, b_n$ ) contain *no* information relating to absolute position or rotational orientation of the curve.

In the amplitude/phase angle form of the Fourier series

$$\phi^*(t) = \mu_0 + \sum_{n=1}^{\infty} A_n \cos(nt - \alpha_n)$$

the  $(A_n, \alpha_n)$  are polar coordinates for the point  $(a_n, b_n)$ .

$A_n$  is called the  $n$ th *harmonic amplitude* and  $\alpha_n$  the  $n$ th *harmonic phase angle*. Of course when  $A_n=0$  the  $n$ th term vanishes and  $\alpha_n$  is undefined.

#### RELATIONSHIPS BETWEEN FOURIER DESCRIPTORS AND GEOMETRY OF SHAPES

A number of useful relationships link algebraic properties of the Fourier descriptors of a simple closed curve with geometric properties of the shape bounded by that curve. Probably the most powerful justification for the use of FD's is the invariance of harmonic amplitudes  $\{A_n\}$  and certain simple functions of pairs of phase angles  $\{\alpha_n\}$  under translations, rotations, changes in size, and shifts in the starting point. It could be said that the  $\{A_n\}$  are pure shape features although pure shape information resides in the  $\{\alpha_n\}$  also. Intuitively speaking, if we cut from cardboard a two-dimensional shape bounded by a simple closed curve, lay it flat on either side in any position and orientation, and select an arbitrary starting point—then respective values of  $\{A_n\}$  will be the same each time. When an algorithm computes the FD's for a polygonal curve the information about position, orientation, and size is so to speak factored out in the values  $Z_0, \delta_0$ , and  $L$  and is therefore still available for discrimination tests. Mirror images and shifts of the starting point can be detected by an appropriate examination of the phase angles  $\{\alpha_n\}$  as described in what follows.

We now describe informally several important algebraic-geometric properties of the Fourier descriptors of a simple closed curve. They are stated formally and proved as Theorems 1, 2, and 3 in the Appendix. First we remind the reader that two curves which differ only in position, orientation, and size (i.e.,  $Z_0, \delta_0, L$ ) but have analogous starting points are transformed into the same waveform  $\phi^*(t)$  and hence have identical Fourier descriptors  $\{A_n, \alpha_n\}$ . If  $\gamma$  and  $\gamma'$  represent the same curve with different starting points  $Z_0$

and  $Z_0'$  then the FD's  $\{A_n, \alpha_n\}$  and  $\{A_n', \alpha_n'\}$  for  $\gamma$  and  $\gamma'$  satisfy

$$A_n' = A_n, \quad \alpha_n' = \alpha_n + n\Delta\alpha, \quad \Delta\alpha = \frac{-2\pi\Delta l}{L} \quad (9)$$

where  $\Delta l$  is the clockwise arc length along the curve from  $Z_0$  to  $Z_0'$ .

Two curves  $\gamma$  and  $\gamma'$  which are reflections of one another (mirror images) but have identical starting points satisfy

$$A_n' = A_n \quad \alpha_n' + \alpha_n \equiv \pi \pmod{2\pi}. \quad (10)$$

It should be understood that the relations involving  $\alpha_n, \alpha_n'$  hold only when  $\alpha_n, \alpha_n'$  exist (i.e.,  $A_n, A_n' \neq 0$ ).

In addition to relationships that exist between two different curves we can investigate the effect on FD's of the axial and rotational symmetries of a single curve. We find that a curve  $\gamma$  with axial symmetry satisfies

$$2\alpha_n = \pi - n\Delta\alpha. \quad (11a)$$

After a shift of starting point by  $\beta = \Delta\alpha/2$  the resulting curve  $\gamma'$  has its starting point on an axis of symmetry and the  $\{\alpha_n'\}$  have the following simple property:

$$\alpha_n' \equiv \pi/2 \pmod{\pi}. \quad (11b)$$

Turning our attention to rotational symmetry we find that a curve having  $k$ -fold rotational symmetry (i.e., goes into itself under a rotation of  $2\pi/k$ ) has zero harmonic amplitudes for all indices that are not integral multiples of  $k$ , that is

$$A_n = 0 \quad \text{for } n \not\equiv 0 \pmod{k}. \quad (12)$$

The general form of the propositions above was to give necessary conditions on the Fourier descriptors for certain geometric properties to exist. They are sufficient conditions as well so that we obtain a one to one correspondence between a class of geometric properties of two-dimensional shapes and a class of algebraic properties of the Fourier descriptors extracted from the boundary curves of those shapes. These statements are proved in the Appendix.

In practice we shall want to infer geometric properties from the properties of a truncated sequence of Fourier descriptors  $\{A_n, \alpha_n\}_{1^N}$  with the knowledge that similarity of shapes is likely to be only approximate. The experience we have gathered using FD's suggests that shapes which "look similar" to people are near each other in the space of low-order FD's. This assumes that the phase angles have been normalized so different starting points do not affect the comparison.

The omission of  $\mu_0$  from this discussion warrants some comment. Because  $\phi^*(0)=0$  we find that

$$\mu_0 = - \sum_{n=1}^{\infty} a_n$$

and therefore  $\mu_0$  carries information related to the particular starting point used. This makes it undesirable as a shape discrimination feature.

## FORM-INVARIANT MEASURES BASED ON PHASE ANGLES

Since the phase angles  $\{\alpha_n\}$  contain shape information it is of interest to have measures based on  $\{\alpha_n\}$  that are not affected by change of starting point. In other words, we desire form-invariant measures which are functions of  $\{\alpha_n\}$ . Furthermore, because the harmonic amplitudes  $\{A_n\}$  do not discriminate reflected images we would hope the measures based on  $\{\alpha_n\}$  would discriminate such reflections.

Define  $F_{kj}[\gamma] = j^* \alpha_k - k^* \alpha_j$  where  $j^* = j/\gcd(j, k)$ ,  $k^* = k/\gcd(j, k)$ , and  $\gcd$  denotes greatest common divisor. In other words,  $k^*/j^*$  is  $k/j$  reduced to lowest terms.

Now if  $\gamma = \{A_n, \alpha_n\}$  and  $\gamma' = \{A_n', \alpha_n'\}$  is  $\gamma$  with a shifted starting point then  $\alpha_n' = \alpha_n + n\Delta\alpha$  and we get

$$\begin{aligned} F_{kj}[\gamma] - F_{kj}[\gamma'] &= j^* \alpha_k - k^* \alpha_j - (j^* \alpha_k' - k^* \alpha_j') \\ &= j^* \alpha_k - k^* \alpha_j - j^* (\alpha_k + k\Delta\alpha) \\ &\quad + k^* (\alpha_j + j\Delta\alpha) \\ &= (k^* j - j^* k) \Delta\alpha \\ &= \frac{(kj - jk)}{\gcd(j, k)} \cdot \Delta\alpha \\ &= 0. \end{aligned}$$

Hence

$$F_{kj}[\gamma] \equiv F_{kj}[\gamma'] \pmod{2\pi} \quad (13)$$

when  $\gamma' = \text{shift}(\gamma)$ .

On the other hand, if  $\gamma' = \{A_n', \alpha_n'\}$  is a reflected and shifted version of  $\gamma$  then  $\alpha_n' = (\pi - \alpha_n) + n\Delta\alpha$  and we obtain

$$\begin{aligned} F_{kj}[\gamma] + F_{kj}[\gamma'] &= j^* \alpha_k - k^* \alpha_j + j^* (\pi - \alpha_k + k\Delta\alpha) \\ &\quad - k^* (\pi - \alpha_j + j\Delta\alpha) \\ &= (j^* - k^*) \pi. \end{aligned}$$

Hence

$$F_{kj}[\gamma] + F_{kj}[\gamma'] \equiv (j^* - k^*) \pi \pmod{2\pi} \quad (14)$$

when  $\gamma' = \text{shift}(\text{reflect}(\gamma))$ .

Since axial symmetry means invariance under a reflection and the appropriate shift we can let  $\gamma' = \gamma$  in the above condition and obtain

$$F_{kj}[\gamma] \equiv (j^* - k^*) \frac{\pi}{2} \pmod{\pi} \quad (15)$$

when  $\gamma$  is axially symmetric.

[We get mod  $\pi$  because  $2\alpha \equiv \beta \pmod{2\pi}$  means  $2\alpha - \beta = 2\pi n$  so that  $\alpha - \beta/2 = \pi n$  which is equivalent to  $\alpha \equiv \beta/2 \pmod{\pi}$ .]

The choice of  $j^*$  and  $k^*$  as multipliers was made because they are the smallest pair of positive integral values whose ratio is  $j/k$ . The ratio condition is all that is really required to prove the above relationships and nonintegral values are avoided because every time we divide our values  $\{\alpha_n\}$  we divide the modulus a corresponding amount [e.g., if  $\alpha_1 \equiv \alpha_3 \pmod{2\pi}$  then  $\alpha_1/3 \equiv \alpha_3/3 \pmod{2\pi/3}$ ].

A simple example helps to explain why we choose  $(j^*, k^*)$

over  $(j, k)$  as multipliers. Suppose that  $j=2, k=4$  so that  $j^*=1, k^*=2$ . If we calculate values of  $\alpha_4 - 2\alpha_2$  over two classes representing distinct shapes it is possible that the values would fall in disjoint intervals  $[0, 2\pi/3]$  and  $[4\pi/3, 2\pi]$  with a significant gap between. Unfortunately, the intervals for the values of  $2\alpha_4 - 4\alpha_2$  would be  $[0, 4\pi/3]$  and  $[2\pi/3, 2\pi]$  which overlap considerably. The change in scale involves loss of discrimination because the phase angles are only known mod  $2\pi$ .

We must add a warning about applying these properties to real data where the geometric properties are not exact. Whenever a harmonic amplitude  $A_k$  is small then the significance of  $\alpha_k$  is greatly diminished and therefore one cannot expect the measure  $F_{kj}$  based on  $\alpha_k$  to reveal anything significant about the geometry of the shape.

## ANALYSIS OF SHAPE SYMMETRIES AND SIMILARITIES

If we are given the FD's  $\{A_n, \alpha_n\}$  for a closed curve  $\gamma$  we might like to know how nearly symmetric  $\gamma$  is in the rotational or axial sense. For  $k$  rotational symmetry we propose the following measure

$$R_k[\gamma] = \sum_{j \neq 0 \pmod{k}} A_j \quad (16)$$

where the sum is finite depending on what frequency has been chosen to truncate the sequence of FD's. A curve with perfect  $k$  rotational symmetry has  $R_k[\gamma] = 0$  and we expect that  $R_k[\gamma]$  near zero will generally indicate a shape that would be judged nearly symmetric.

For axial symmetry we propose

$$X[\gamma] = \sum \left| F_{kj} - \frac{\pi}{2} (j^* - k^*) \right| \quad (17)$$

where the sum is taken over some subset of pairs  $(k, j)$  preferably avoiding those pairs where either or both of  $A_k, A_j$  are small. Another possibility is to weight each term with a weight  $w_{kj} = (A_k + A_j)/2$  or  $w_{kj} = \min\{A_k, A_j\}$ . The latter seems to be what is required since *one* small harmonic amplitude suffices to destroy the measure  $F_{jk}$ .

Given curves  $\gamma$  and  $\gamma'$  with Fourier descriptors  $\{A_n, \alpha_n\}$  and  $\{A_n', \alpha_n'\}$  how similar are they? Or how close are they to being mirror images? For similarity the following measures are suggested by the previous sections

$$S_A[\gamma, \gamma'] = \sum |A_k - A_k'| \quad (18)$$

$$S_\alpha[\gamma, \gamma'] = \sum |F_{kj} - F_{kj}'| \quad (19)$$

where, as before, weights can be inserted in the formula for  $S_\alpha$ .

To measure the extent to which  $\gamma'$  is a mirror image of  $\gamma$  we use  $S_A$  and the following measure based on the phase angles

$$M[\gamma, \gamma'] = \sum |F_{kj} + F_{kj}' - (j^* - k^*) \pi|. \quad (20)$$

In computing these absolute differences care must be taken

TABLE I

Shape	$X$	$Y$	$Z$	$W$	$R$	$S$	$T$	$U$
$A_1$	0.366	0.223	0.440	0.363	0.024	0.468	0	1.558
$A_2$	0.866	0.729	0.872	0.717	0.116	0.840	0.004	1.081
$A_3$	0.333	0.289	0.500	0.513	0.272	0.384	0	0.592
$A_4$	0.250	0.217	0.108	0.243	0.025	0.229	0.023	0.122
$A_5$	0.273	0.105	0.135	0	0.139	0.244	0	0.161
$\alpha_1$	2.099	1.043	4.361	2.509	4.414	6.269	—	3.867
$\alpha_2$	2.099	1.156	0.274	3.878	1.252	4.702	5.473	0.033
$\alpha_3$	3.142	0	3.670	4.397	3.844	6.237	—	2.583
$\alpha_4$	1.571	1.113	4.222	0.827	1.969	4.723	6.237	6.057
$\alpha_5$	1.043	2.099	6.108	—	3.467	6.181	—	4.562
$F_{21}$	4.184	5.353	4.118	5.143	4.990	4.730	—	4.865
$F_{32}$	-0.013	-3.468	0.235	-2.840	3.932	-1.638	—	5.067
$F_{42}$	—	—	—	—	—	—	-4.709	—

Note:  $\pi/2=1.571$ ;  $3\pi/2=4.713$ .

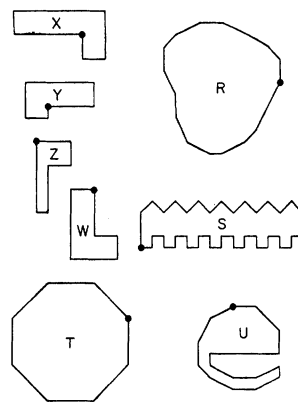


Fig. 4. Simple examples of plane closed polygonal curves.

to get the smallest magnitude change modulo the appropriate uncertainty. The modulus is  $2\pi$  for all  $\alpha$  based measures except  $X[\gamma]$  which uses  $\pi$ .

If we have found two curves to be almost identical in shape by virtue of  $S_A[\gamma, \gamma']$  and  $S_\alpha[\gamma, \gamma']$  being nearly zero then what is a good estimate for the shift  $\Delta\alpha$  between their respective starting points? A least squares fit of  $f(n)=n\Delta\alpha$  to  $(\alpha_n' - \alpha_n)$  suggests itself but there is a serious difficulty. The quantities  $(\alpha_n' - \alpha_n)$  are not single valued because phase angles are mod  $2\pi$ . In practice we have found that for quite similar shapes one can fix the  $(\alpha_n' - \alpha_n)$  at what seem clearly to be the best choices mod  $2\pi$  and then use least squares to get the finely tuned estimate of  $\Delta\alpha$ . It is helpful that  $|\Delta\alpha| < \pi$  can be assumed!

This pseudo least squares can be used (with analogous caution) to find the approximate axis of best axial symmetry and the shift between analogous starting points for mirror images.

To get a feel for how well the values  $\{A_n, \alpha_n\}$  reflect shape properties we have calculated in Table I some FD's for the eight shapes in Fig. 4. In particular, the four shapes  $X, Y, Z, W$  are in form quite similar except that  $Y, W$  are reflected versions of  $X, Z$ . More precisely, the mirror image of  $X$  is more like  $Y$  than  $X$  itself is like  $Y$ . We expect to find that

$$|F_{21}[X] + F_{21}[Y] - \pi| \ll |F_{21}[X] - F_{21}[Y]|.$$

In actuality we get  $0.112 \ll 1.169$  so that  $X$  is very nearly the same as  $Y$  without reflection. Performing the corresponding test on  $F_{32}$  we get  $0.339 \ll 2.828$  so both  $F_{21}$  and  $F_{32}$  strongly support the hypothesis that  $X$  and  $Y$  are nearly mirror images rather than similar without reflection.

Turning our attention to  $Y$  and  $W$  we find that

$$|F_{21}[Y] - F_{21}[W]| = 0.210 \ll 1.071$$

$$= |F_{21}[Y] + F_{21}[W] - \pi|$$

and correspondingly for  $F_{32}$  we get  $0.628 \ll 3.117$  so  $Y$  and  $W$  are more nearly identical than they are mirror images.

According to formulas of the previous section axial symmetry is revealed by  $F_{21}$  and  $F_{32}$  being near  $\pi/2$  or  $3\pi/2$  (i.e., nearly  $\pi/2 \bmod \pi$ ). The reader is invited to check for himself the extent to which Table I reflects such geometric properties of the shapes in Fig. 4.

#### FOURIER DESCRIPTORS BASED ON THE CURVATURE FUNCTION

In the original report on Fourier descriptors Cosgriff [1] suggests two alternative ways of defining the descriptors. One way is to expand the angle versus length function  $\theta(l)$  in a Fourier series as we have done in this paper. The second way is to expand the curvature function  $k(l)=d\theta/dl$  instead. As is well known,  $k(l)=-1/R_l$  where  $R_l$  is the radius of

curvature of curve  $\theta$  at the point  $l$ . The minus sign is not normal but simply reflects our convention that clockwise is negative.

One might have expected  $k(l)$  to be the prime choice for basing Fourier descriptors but there is a problem. For a polygonal curve,  $\theta(l)$  is a step function so  $k(l) = d\theta/dl$  is zero almost everywhere and infinite at the discrete jumps of  $\theta(l)$ . This makes  $k(l)$  a poor candidate for polygonal curves and therefore of little use for practical shape discrimination. We can however investigate what such Fourier descriptors would be like if the curve were smooth.

If we take polygonal curve  $\gamma$  and compute the Fourier descriptors  $\{A_n, \alpha_n\}$  based on the angle versus length function  $\theta(l)$  then [see integrand in (5)]

$$\hat{\theta}(l) = \frac{-2\pi l}{L} + \mu_0 + \sum_{k=1}^N A_k \cos\left(\frac{2\pi k l}{L} - \alpha_k\right) + \delta_0$$

is a reasonable approximation to  $\theta$  and what is more it is smooth so  $d\hat{\theta}/dl$  is well defined and easily calculated!

If we define  $k(l) = d\hat{\theta}/dl$  and  $k^*(t) = k(Lt/2\pi)$  for  $t \in [0, 2\pi]$  we find

$$k^*(t) = B_0 + \sum_{k=1}^N B_k \cos(kt - \beta_k)$$

where

$$B_0 = -\frac{2\pi}{L}$$

$$B_k = \frac{2\pi k A_k}{L}$$

and

$$\beta_k = \left(\alpha_k - \frac{\pi}{2}\right).$$

In other words,  $\{B_k, \beta_k\}$  are the curvature based Fourier descriptors for a smooth curve which approximates the curve for which  $\{A_k, \alpha_k\}$  are the angle versus length based Fourier descriptors. The fact that  $B_k$  depends on  $L$  shows that  $k(l)$  is not a size-invariant function the way  $\theta(l)$  is. To fix this we should probably define  $k^*(t) = Lk(Lt/2\pi)$ . Note that the constant term  $B_0 = -2\pi/L$  is correct for the average curvature. In particular, for a circle of radius  $R$  we get constant curvature  $k^*(t) = -1/R$  so the average curvature is also  $-1/R$ . But  $L = 2\pi R$  so  $B_0 = -2\pi/L = -1/R$  making  $B_0$  the average curvature.

The most important thing we find from calculating the  $\{B_k, \beta_k\}$  is their extremely simple relationship to the angle versus length based Fourier descriptors  $\{A_k, \alpha_k\}$ . The new harmonic amplitudes  $\{B_k\}$  differ from  $\{A_k\}$  only in the sense that the  $\{B_k\}$  give greater weight to higher frequencies. The descriptors based on curvature are therefore essentially equivalent to those based on the angle versus length function and there is no need to further consider these alternative descriptors.

## SYNTHETIC GENERATION OF CURVES FROM FOURIER DESCRIPTORS

According to the reconstruction theorem proved in an earlier section a curve can be reconstructed from a knowledge of its Fourier descriptors  $\{A_k, \alpha_k, \mu_0\}$  and the triple  $(L, \delta_0, Z_0)$  giving length, initial tangential direction, and position of starting point. We repeat the formula here for ease of reference.

$$Z(l) = Z_0 + \frac{L}{2\pi} \int_0^{2\pi l/L} \exp \left[ i \left\{ -t + \delta_0 + \mu_0 + \sum_{j=1}^N A_j \cos(jt - \alpha_j) \right\} \right] dt. \quad (21)$$

This formula suggests the possibility of making up a fictitious set of Fourier descriptors  $\{A_k, \alpha_k\}$  and seeing what sort of curve  $Z(l)$  results. We should mention that  $\mu_0$  is not really a free variable but is given by  $\{A_k, \alpha_k\}$  as

$$\mu_0 = \sum_{k=1}^N A_k \cos(-\alpha_k) = \sum_{k=1}^N A_k \cos \alpha_k.$$

Now given any  $\{A_k, \alpha_k\}_{k=1}^N$  and a triple  $(L, \delta_0, Z_0)$ ,  $Z(l)$  becomes a well-defined continuous function on  $[0, L]$  and hence traces out a curve in the plane. Almost at once we would like to know which  $\{A_k, \alpha_k\}$  generate closed curves.

We have obtained some simple conditions under which  $Z(L) = Z_0$  and they are presented in the following theorem (see Appendix for proof).

### Closure Theorem

The function  $Z(l)$  represents a closed curve if one of the following conditions holds.

*Condition 1:*  $A_1$  is a zero of the first Bessel function  $J_1(x)$  and  $A_n = 0$  for  $n \geq 2$ .

*Condition 2:*  $A_n = 0$  for all  $n \not\equiv 0 \pmod{k}$  where  $k \geq 2$ .

As a special case of Condition 2 in the closure theorem we see that if only one  $A_k (k \geq 2)$  is nonzero then  $Z(l)$  is a closed curve with  $k$  rotational symmetry. After seeing a number of curves reconstructed from single terms we can hazard the following interpretation of the resulting curve. Starting with a circle you push in toward the center at  $k$  equally spaced points on the circle and how hard you push is determined by the magnitude of  $A_k$ .

Using relationships stated earlier [see (11a)] one can generate curves with axial symmetry simply by putting  $\alpha_n = \pi/2$  for all  $n$ . This will not, however, insure a closed curve but an axially symmetric closed curve could be made by adding a straight line between  $Z(0)$  and  $Z(L)$ .

It should be pointed out that our closure theorem does not insure that you get a *simple* closed curve. The conditions for simplicity are probably very complicated.

### OPTIMALITY OF FOURIER BASIS FOR GENERATING CURVE DESCRIPTORS

In earlier sections we introduced the angle versus length function  $\theta(l)$  and its normalized cumulative variant  $\phi^*(t)$ . We then expanded  $\phi^*(t)$  as a Fourier series and used its co-

efficients as descriptors of the shape of the curve. The familiar real-variable forms of the Fourier series are

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

and

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} A_k \cos (kt - \alpha_k).$$

We can, however, express the Fourier series in a complex form [10], where the family of functions generating the expansion is  $\{e^{int}\}_{-\infty}^{\infty}$  and the expansion is

$$\phi^*(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

with coefficients given by

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \phi^*(t) e^{-ikt} dt.$$

These coefficients  $c_k$  are complex numbers related to the real  $a_k$ ,  $b_k$  and  $A_k$ ,  $\alpha_k$  by the equations

$$c_0 = \mu_0, \quad 2c_k = a_k - ib_k = A_k e^{-i\alpha_k}$$

and

$$2c_{-k} = a_k + ib_k = A_k e^{i\alpha_k}.$$

Since there are other families  $\{f_n(t)\}$  of functions upon which such an expansion could be based we are led to question whether or not the Fourier basis  $\{e^{int}\}_{-\infty}^{\infty}$  has any advantage over other families. One feature of the Fourier series which we have already found to be extremely valuable is the invariance of harmonic amplitudes under shifts of starting point and the simple relationship  $\alpha_m' = \alpha_m + m\Delta\alpha$  which holds under shifts of starting point. We shall soon show that among all possible expansions  $\phi^*(t) = \sum c_n f_n(t)$  only the Fourier series allows a simple functional relationship between  $c_n(\phi_{\Delta\alpha}^*)$ ,  $c_n(\phi^*)$  and  $\Delta\alpha$  where  $\phi_{\Delta\alpha}^*$ ,  $\phi^*$  are the functions derived from a single curve with starting points differing by  $\Delta\alpha$ . For the Fourier series expressed in complex form we obtain the simple relation

$$c_n(\phi_{\Delta\alpha}^*) = c_n(\phi^*) e^{in\Delta\alpha}, \quad n \neq 0$$

embodying the relations  $A_n(\phi_{\Delta\alpha}^*) = A_n(\phi^*)$  and  $\alpha_n(\phi_{\Delta\alpha}^*) = \alpha_n(\phi^*) - n\Delta\alpha$  which we have already discovered in (9). The earlier work of Raudseps [3] contains some empirical evidence favoring the Fourier series but no theoretical results on its uniqueness with respect to a desirable property.

Let  $\phi^*(t)$  and  $\phi_{\Delta\alpha}^*(t)$  be the normalized cumulative variants of the angle versus length functions for two curves differing only in starting point  $\Delta\alpha$ . Using the definition of  $\phi^*$ , (1), and the relationships  $\theta_{\Delta\alpha}(l) = \theta(l - \Delta\alpha)$  and  $\theta(l) = \phi(l) + \delta_0$  we find that

$$\phi_{\Delta\alpha}^*(t) = \phi^*(t - \Delta\alpha) + \Delta\alpha + \delta_0 - \delta_0^{\Delta\alpha}. \quad (22)$$

Now suppose  $\{f_n(t)\}$  is a basis of linearly independent functions in the Hilbert space  $\mathcal{L}^2[0, 2\pi]$  of complex-valued

square-summable periodic functions on the interval  $[0, 2\pi]$  employing Lebesgue measure on the real line. Let  $\phi^*$  and  $\phi_{\Delta\alpha}^*$  be expanded using  $\{f_n(t)\}$  as

$$\begin{aligned} \phi^*(t) &= \sum c_n f_n(t) \\ \phi_{\Delta\alpha}^* &= \sum c_n^{\Delta\alpha} f_n(t). \end{aligned}$$

Although  $\{f_n(t)\}$  is not necessarily an orthogonal basis we can still compute the coefficients  $c_n$  and  $c_n^{\Delta\alpha}$  by selecting a family  $\{g_m(t)\}$  in such a way that each  $g_m$  is orthogonal to the entire subspace spanned by all functions  $\{f_n\}$  save the one  $f_m$ . The existence of each  $g_m$  is assured by well-known facts about Hilbert space [11]. We can in fact choose  $g_m(t)$  so that

$$\int_0^{2\pi} f_n(t) \overline{g_m(t)} dt = \delta_{nm}$$

where the overbar means complex conjugate. Furthermore,  $g_m(t)$  can be chosen periodic in  $[0, 2\pi]$ .

With this family  $\{g_m(t)\}$  we can compute coefficients

$$\begin{aligned} c_m &= \int_0^{2\pi} \phi^*(t) \overline{g_m(t)} dt \\ c_m^{\Delta\alpha} &= \int_0^{2\pi} \phi_{\Delta\alpha}^*(t) \overline{g_m(t)} dt. \end{aligned}$$

Now we ask the question, "Is there any family  $\{f_n(t)\}$  such that  $c_m^{\Delta\alpha} = c_m$  for all save one value of  $m$  (e.g.,  $m=0$ ), for all closed curves  $\phi^*$ ? If not then is there some  $\{f_n(t)\}$  such that  $c_m^{\Delta\alpha}$  depends only on  $c_m$  and  $\Delta\alpha$ , (i.e., is independent of  $c_k$ ,  $k \neq m$ ) for all  $\phi^*$ ?"

The answer is provided by the following theorem whose proof is in the Appendix.

#### Optimality Theorem

- 1) There is no family  $\{f_n(t)\}$  such that  $c_m^{\Delta\alpha} = c_m$  for all save one value of  $m$ , for all closed curves  $\phi^*$ .
- 2) The Fourier family  $\{e^{int}\}$  is the unique family for which  $c_m^{\Delta\alpha}$  depends only on  $c_m$  and  $\Delta\alpha$  independent of curve  $\phi^*$  and other coefficients  $c_k$ ,  $k \neq m$ .

We must point out that the criterion of optimality used here concerns the problem of different starting points for identically shaped curves and that if a canonical preferred starting point could be determined in some other fashion then expansion of  $\phi^*(t)$  with respect to other orthogonal functions may well be quite useful.

#### USEFULNESS OF FOURIER DESCRIPTORS FOR SHAPE DISCRIMINATION

We present in this section a sampling of the evidence indicating that Fourier descriptors are very good features for use in shape discrimination. Our own experiments involve numerals from the hand-printed character sets collected by Munson [24]. Each character of this set is represented by a  $24 \times 24$  binary matrix or black and white digital picture. Fig. 5(a) is one such picture of a hand-printed numeral "4." The curvature point method [8], [9] was used to obtain a polygonal boundary description as shown in Fig. 5(b). It



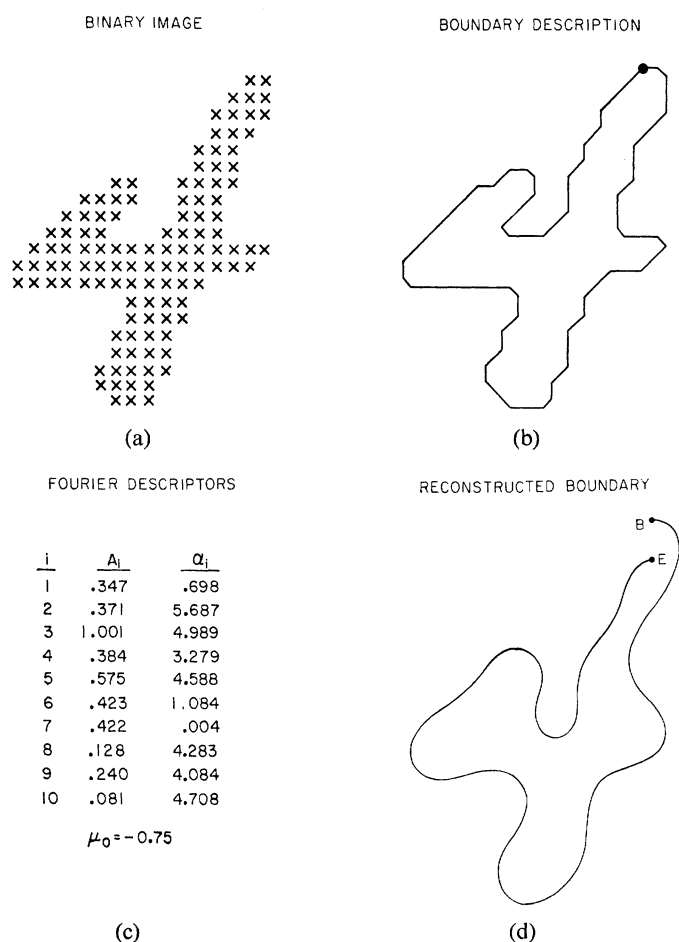


Fig. 5. Fourier descriptor analysis and reconstruction of boundary of a binary image.

was then an easy matter to calculate the Fourier descriptors  $\{A_k, \alpha_k\}$  using formulas derived earlier. Fig. 5(c) shows the first ten pairs of harmonic amplitude and phase angle. Fig. 5(d) depicts a reconstruction of the "4" based on the ten terms of Fig. 5(c). This reconstruction was generated by the reconstruction formula of an earlier section in conjunction with numerical integration. The shape of Fig. 5(d) is visual proof that the "fourness" of the digital picture is contained in the twenty numbers  $\{A_k, \alpha_k\}_{1^{10}}$ . Brill [6] contains examples of five numerals reconstructed from 5, 10, and 15 FD pairs indicating that approximately seven pairs should be sufficient to discriminate the five numerals. Barrow and Popplestone [25] use  $\{A_k\}_{1^6}$  for shape discrimination.

Using 39 samples each of numerals 1, 2, and 3 from the Munson data [24] representing the work of 13 separate authors we plotted these 117 numerals in the two-dimensional space of features  $(A_1, A_2)$  to observe how well the numeral classes separate, if at all. Fig. 6 shows the degree to which these three classes cluster in  $(A_1, A_2)$  space. Knowing that harmonic amplitudes alone cannot discriminate shapes that are nearly reflections of one another we tested the feature  $F_{42} = \alpha_4 - 2\alpha_2$  on similar samples of the numerals 2 and 5. The results were that with few exceptions  $F_{42}$  was in the interval  $[4.7, 5.6]$  for numeral 2 and in  $[3.1, 4.4]$  for numeral 5. Of the three exceptions out of 80, two 5's were outside both intervals and one 2 was in the interval populated by 5's.

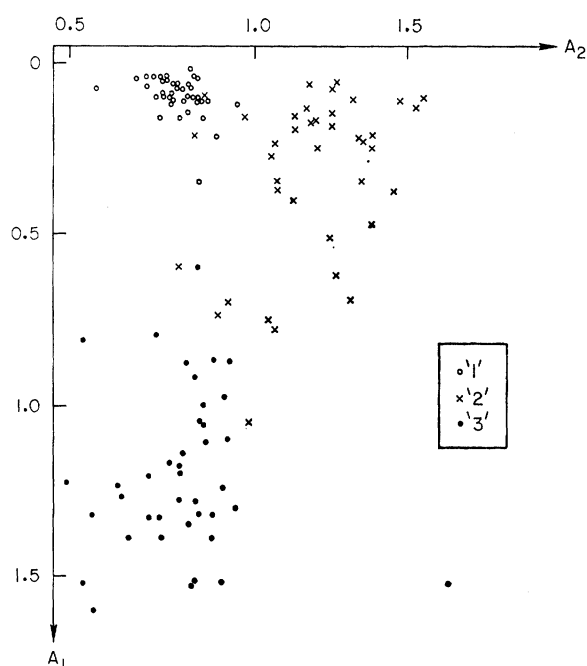


Fig. 6. Clustering of hand-printed characters in the two-dimensional space of the lowest order harmonic amplitudes  $A_1$  and  $A_2$ .

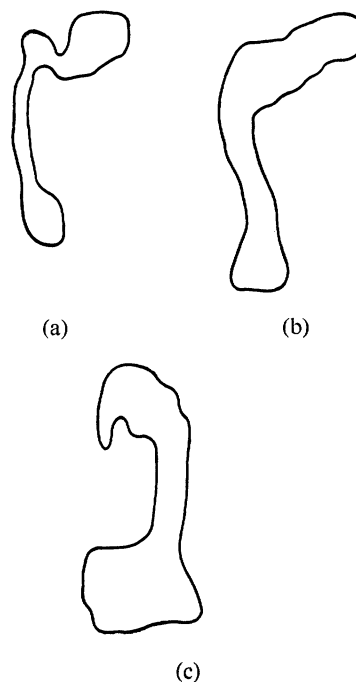


Fig. 7. Figure boundaries of two abnormal "5's" and one abnormal "2".

As can be seen from Fig. 7 these three samples are not very standard looking numerals.

It would be convenient if we could see visually what the numeral classes of Fig. 6 looked like in the five-dimensional space  $(A_1, A_2, A_3, A_4, A_5)$  but there is no way to directly display such higher dimensional spaces. An indirect method has been recently reported [23] for characterizing the degree to which classes of points in a general metric space cluster internally compared to the separation between classes. A tree is constructed with the points as nodes so that the total

of all edge lengths in the tree is a minimum among all possible trees spanning the point set. In case the class membership of each point is known this *minimal spanning tree* (MST) can be used to calculate the relative compactness of the classes with respect to each other—that is, the tendency for points of one class to be near points of the same class rather than other classes. The number of MST edges with end nodes in different classes measures this tendency rather well. We have performed such an analysis on the 117 points used earlier to see how well the classes would separate in the space  $(A_1, \dots, A_8)$ . The results are that the MST contains one edge linking a 1 and 2 and three edges linking 2's and 3's. With three classes two of these edges were absolutely necessary and the extra two (out of 114) are a reflection of the tendency toward intermixture of classes. The tendency is quite minimal. Looking closely at the nodes causing the crossover edges we were able to determine that they were due entirely to one author whose 2's curled downward making their shapes approximate 3's more closely.

The recognition experiments carried out by Brill [6] give impressive indications of the power of Fourier descriptors. We briefly summarize the results. In one experiment 600 machine-printed numerals from 50 different fonts were recognized with an error rate of 1.5 percent. In another, 400 hand-printed numerals from 40 different styles were recognized with an error rate of 9.5 percent. These are very good if not spectacular error rates. What is rather surprising is that the recognition was accomplished using lower order Fourier descriptors and a reference set containing about *two samples* per numeral class. In other words, numerals from 50 fonts were recognized by an algorithm whose training set consisted of at most two fonts—similarly, for the hand-printed numerals. Brill *et al.* [7] report on further recognition experiments with machine-printed characters which are known to make trouble.

#### EXPERIMENTS IN SHAPE GENERATION USING FOURIER DESCRIPTORS

Programs have been written in PL/1 to reconstruct and plot (on-line printer or digital Calcomp plotter) curves from arbitrary Fourier descriptors. We have experimented very slightly with the generation of curves using these programs and Fig. 8 contains a small sample of some of the more interesting results. In line with the closure theorem of an earlier section we reconstructed the presumably closed curves from a single FD  $A_1$  where  $A_1$  is one of the zeros of the first Bessel function. Fig. 8(a) and (b) are from  $A_1 = 3.83171$  and  $7.01559$ , respectively. Most of the remaining curves generated have both  $k$  rotational and axial symmetries which can be forced by putting  $A_j = 0$  for  $j \neq 0 \pmod{k}$  and  $\alpha_{nk} \equiv \pi/2 \pmod{\pi}$ . Rather attractive curves are generated by a very small set of FD's; for example, Fig. 8(c) is generated by  $A_5 = 4$ ,  $\alpha_5 = 0$  and Fig. 8(d) is generated by  $A_4 = 3$ ,  $\alpha_4 = \pi/2$ ,  $A_8 = 2$ , and  $\alpha_8 = 3\pi/2$ . The very interesting shape of Fig. 8(e) involves only two nonzero harmonic amplitudes  $A_6$  and  $A_9$ . Table II lists the FD input to generate each of the curves in Fig. 8.

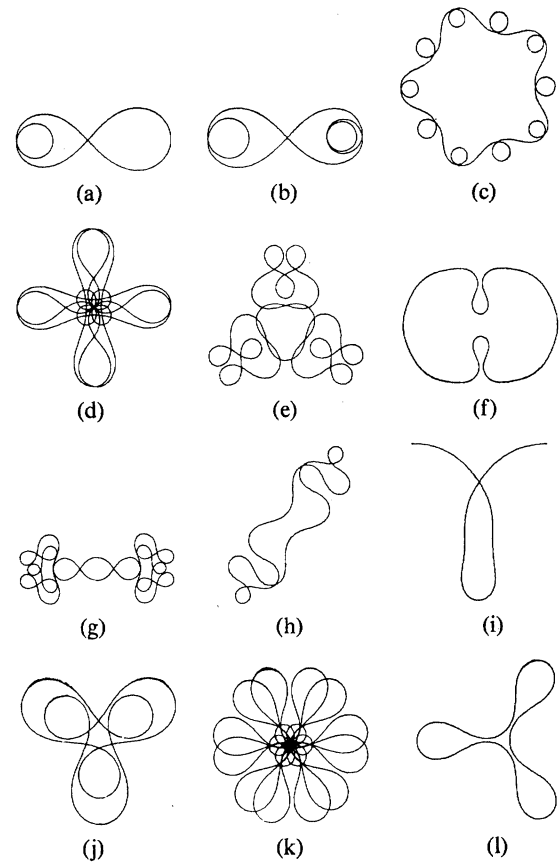


Fig. 8. Examples of synthetically generated curves from simple Fourier descriptor representations, many exhibiting axial and rotational symmetries under control of simple algebraic properties of the Fourier descriptors.

TABLE II

Figure	Generating Fourier Descriptors
8(a)	$A_1 = 3.83171$
8(b)	$A_1 = 7.01559$
8(c)	$A_5 = 4.0$
8(d)	$A_4 = 3.0, \alpha_4 = \pi/2, A_8 = 2.0, \alpha_8 = 3\pi/2$
8(e)	$A_6 = 1.0, \alpha_6 = \pi/2, A_9 = 2.0, \alpha_9 = \pi/2$
8(f)	$A_{2k} = 1.2 - (0.2)k, \alpha_{2k} = \pi/2$ for $k = 1, 2, 3, 4, 5$
8(g)	$A_8 = 1.0, \alpha_8 = 3\pi/2, A_{10} = 2.0, \alpha_{10} = \pi/2$
8(h)	$A_{2k} = (0.5)k, \alpha_{2k} = 1.0(k-1)$ for $k = 1, 2, 3$
8(i)	$A_k = 0.6 - (0.1)k$ for $k = 1$ to $5$ and $\alpha_1 = \alpha_3 = \alpha_5 = \pi/2$ , $\alpha_2 = \alpha_4 = 3\pi/2$
8(j)	$A_3 = 2.5, \alpha_3 = \pi/2, A_6 = 1.0, \alpha_6 = 3\pi/2, A_9 = 0.5, \alpha_9 = \pi/2$
8(k)	$A_6 = 3.0, \alpha_6 = \pi/2, A_{12} = 2.0, \alpha_{12} = 3\pi/2$
8(l)	$A_3 = 1.5$

#### APPENDIX

**Theorem 1:** If two closed curves  $\gamma$  and  $\gamma'$  differ only in starting point by  $\Delta l$  in units of arc length (clockwise from  $Z_0$  to  $Z_0'$ ) then we have the following.

**Condition 1:**  $A_k' = A_k$ .

**Condition 2:**  $\alpha_k' = \alpha_k + k\Delta\alpha$  where  $\Delta\alpha = -2\pi\Delta l/L$ .

**Condition 3:**  $\mu_0' = \mu_0 + \delta_0 - \delta_0' + \Delta\alpha$ .

Conversely if these three conditions hold then  $\gamma$  and  $\gamma'$  differ only in a shift of starting point by  $\Delta l$ .

**Proof:** If  $\gamma$  and  $\gamma'$  differ by  $\Delta l$  in starting point then  $\theta'(l) = \theta(l + \Delta l)$  and so

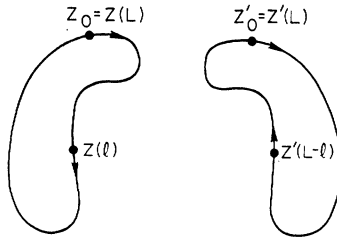


Fig. 9. Relationship between parametric descriptions of reflected curves with identical starting point.

$$\phi'(l) + \delta_0' = \phi(l + \Delta l) + \delta_0$$

or employing (1)

$$\begin{aligned} \phi^{*'}\left(\frac{2\pi l}{L}\right) - \frac{2\pi l}{L} + \delta_0' \\ = \phi^*\left(\frac{2\pi l}{L} + \frac{2\pi \Delta l}{L}\right) - \frac{2\pi l}{L} - \frac{2\pi \Delta l}{L} + \delta_0. \end{aligned}$$

Letting  $t = 2\pi l/L$  and  $\Delta t = 2\pi \Delta l/L$  we get

$$\phi^{*'}(t) - \phi^*(t + \Delta t) = \delta_0 - \delta_0' - \Delta t.$$

The two functions  $F(t) = \phi^{*'}(t) = \mu_0' + \sum A_k' \cos(kt - \alpha_k')$  and  $G(t) = \phi^*(t + \Delta t) = \mu_0 + \sum A_k \cos(k(t + \Delta t) - \alpha_k)$  differ by a constant and therefore their Fourier coefficients must match term by term except that  $\mu_0' - \mu_0 = \delta_0 - \delta_0' - \Delta t$ . The term by term equality of harmonic amplitudes and phase angles assures that

$$A_k' = A_k$$

and

$$\alpha_k' = \alpha_k - k\Delta t.$$

Letting  $\Delta\alpha = -\Delta t$  we obtain Condition 2. Reading the sequence of equations backwards we obtain the converse.

**Theorem 2:** If  $\gamma$  and  $\gamma'$  differ only in the sense that  $\gamma'$  is a mirror image of  $\gamma$  with the same starting point then we have the following.

**Condition 1:**  $A_k' = A_k$ .

**Condition 2:**  $\alpha_k' + \alpha_k = \pi$ .

**Condition 3:**  $\mu_0' + \mu_0 = -\Delta\phi_0$  where  $\Delta\phi_0$  is the bend at  $Z_0$ .

Conversely if these three conditions hold then  $\gamma$  and  $\gamma'$  are mirror images of each other with the same starting point.

**Proof:** Referring to Fig. 9 we see that the curvature of  $\gamma'$  at  $Z'(L-l)$  is identical to that of  $\gamma$  at  $Z(l)$ . This is because reflection of the curve changes the sign of curvature but the change of orientation also changes the sign resulting in no net change. As a result of this the cumulative change of direction  $\phi'$  between  $(L-l)$  and  $L$  is the same as the change in  $\phi$  between 0 and  $l$ .

Hence we can write

$$\phi'(L-l) + \phi(l) + \Delta\phi_0 = -2\pi$$

where  $\Delta\phi_0$  is the change in direction at  $Z_0$  to take care of situations in which  $\gamma$  is polygonal and  $Z_0$  is a vertex.

Transforming to the normalized functions  $\phi^*$  and  $\phi^{*'} [see (1)]$  and variable  $t = 2\pi l/L$  we get

$$\begin{aligned} \phi^{*'}\left(\frac{2\pi(L-l)}{L}\right) - \frac{2\pi(L-l)}{L} + \phi^*\left(\frac{2\pi l}{L}\right) - \frac{2\pi l}{L} \\ + \Delta\phi_0 = -2\pi \end{aligned}$$

or

$$\phi^{*'}(2\pi - t) - (2\pi - t) + \phi^*(t) - t + \Delta\phi_0 = -2\pi$$

or

$$\phi^{*'}(-t) + \phi^*(t) + \Delta\phi_0 = 0$$

since  $\phi^{*'}$  is periodic  $[2\pi]$ .

If  $f(t)$  and  $g(t)$  satisfy

$$f(t) + g(-t) = c$$

and if

$$g(t) = \mu_0' + \sum_{k=1}^{\infty} (a_k' \cos kt + b_k' \sin kt)$$

while  $a_k, b_k$ , and  $\mu_0$  are the analogous coefficients for  $f(t)$ , a straightforward calculation yields

$$a_k' = -a_k \quad b_k' = b_k \quad \mu_0' = -\mu_0 + c.$$

For  $g = \phi^{*'}, f = \phi^*$ , and  $c = -\Delta\phi_0$  we get

$$a_k' = -a_k \quad b_k' = b_k \quad \mu_0' = -\mu_0 - \Delta\phi_0.$$

Since  $A_k, \alpha_k$  are polar coordinates for  $a_k, b_k$  we obtain Conditions 1 and 2 of Theorem 2 immediately.

The converse follows from observing that if the Fourier coefficients of  $\phi^{*'}$  and  $\phi^*$  are related as above

$$\phi^{*'}(-t) + \phi^*(t) + \Delta\phi_0 = 0$$

and this implies

$$\phi'(L-l) + \phi(l) + \Delta\phi_0 = -2\pi.$$

**Theorem 3:** A curve  $\gamma$  has  $n$ -fold rotational symmetry if and only if  $A_k = 0$  for  $k \not\equiv 0 \pmod{n}$ .

**Proof:** First assume  $n$ -fold rotational symmetry. This implies immediately that  $\phi^*(t)$  has period  $2\pi/n$  and we must show that Fourier coefficients  $a_k, b_k$  are zero except when  $k \equiv 0 \pmod{n}$  for a function  $f$  which is periodic  $[2\pi/n]$ . It is easy to show this by direct calculations.

The converse is almost immediate because a function of the form

$$f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} A_{nj} \cos (njt - \alpha_{nj})$$

is periodic with a period  $2\pi/n$ .

#### Closure Theorem

Let

$$Z(l) = Z_0 + \frac{L}{2\pi} \int_0^{2\pi l/L} \exp \left[ i \left\{ -t + \delta_0 + \mu_0 + \sum_{j=1}^N A_j \cos (jt - \alpha_j) \right\} \right] dt.$$

Then  $Z(l)$  represents a closed curve if one of the following conditions holds.

*Condition 1:*  $A_1$  is a zero of the first Bessel function  $J_1(x)$  and  $A_n=0$  for  $n>1$ .

*Condition 2:*  $A_n=0$  for all  $n \neq 0 \pmod k$  where  $k \geq 2$ .

*Proof:* If Condition 2 holds then  $f(t) = \sum_{j=1}^N A_j \cos (jt - \alpha_j)$  is periodic with period  $2\pi/k$  and we must show that

$$\int_0^{2\pi} e^{-it + if(t)} dt = 0.$$

A simple calculation shows that when  $f(t)$  is periodic  $(2\pi/k)$  this expression has as a factor the sum of all the  $k$ th roots of unity and so it is zero.

If Condition 1 of the closure theorem holds then  $A_1$  is the only nonzero  $A_j$  and we need to show

$$\int_0^{2\pi} \exp [-it + iA_1 \cos (t - \alpha_1)] dt = 0.$$

A simple change of variable  $t' = t - \alpha_1$  gets rid of  $\alpha_1$  and our requirement on  $A_1$  becomes

$$\int_0^{2\pi} e^{-it'} e^{iA_1 \cos t'} dt' = 0.$$

This equation holds for  $A_1$  such that  $J_1(A_1)=0$  where  $J_1$  is the first Bessel function [13] and the theorem is proved.

#### Optimality Theorem

*Condition 1:* There is no family  $\{f_n(t)\}$  such that  $c_m^{\Delta\alpha} = c_m$  for all save one value of  $m$ , for all closed curves  $\phi^*$ .

*Condition 2:* The Fourier family  $\{e^{int}\}$  is the unique family for which  $c_m^{\Delta\alpha}$  depends only on  $c_m$  and  $\Delta\alpha$  independent of curve  $\phi^*$  and the other coefficients  $c_k$ ,  $k \neq m$ .

*Proof:* Suppose a family  $\{f_n(t)\}$  as described in Condition 1 of the optimality theorem did exist. If such a situation did hold we would have

$$\int_0^{2\pi} [\phi^*(t) - \phi_{\Delta\alpha}^*(t)] \overline{g_m(t)} dt = 0, \quad \text{for } m \neq 0.$$

The function  $F(t) = \phi^*(t) - \phi_{\Delta\alpha}^*(t)$  in square brackets can also be expanded as

$$F(t) = \sum b_m f_m(t)$$

where

$$b_m = \int_0^{2\pi} F(t) \overline{g_m(t)} dt.$$

From the expression above  $b_m=0$  for  $m \neq 0$  and so

$$F(t) = b_0 f_0(t)$$

or

$$\phi^*(t) - \phi_{\Delta\alpha}^*(t) = b_0(\Delta\alpha) f_0(t)$$

which expresses the possible dependence of  $b_0$  on  $\Delta\alpha$ .

Substituting for  $\phi_{\Delta\alpha}^*(t)$  from (22), dividing by  $\Delta\alpha$ , and taking the limit as  $\Delta\alpha \rightarrow 0$  we obtain

$$\lim_{\Delta\alpha \rightarrow 0} \frac{\phi^*(t) - \phi^*(t - \Delta\alpha)}{\Delta\alpha} - 1 + \frac{\delta_0 - \delta_0^{\Delta\alpha}}{\Delta\alpha} - \frac{b_0(\Delta\alpha)}{\Delta\alpha} f_0(t) = 0.$$

The first quotient approaches the derivative of  $\phi^*$  at  $t$ , the second approaches the derivative of  $\theta$  at its starting point, and the third quotient approaches  $b_0'(0)$  since  $b_0(0)=0$ . We finally obtain

$$f_0(t) = Dk(t) + E$$

where

$$k(t) = \frac{d\phi^*}{dt}$$

is the curvature of the curve being expanded and  $D, E$  are constants not depending on  $t$ . Hence  $f_0(t)$  depends on the curvature of the curve whose function  $\phi^*$  is being expanded and therefore the expansion could not work for arbitrary curves. We conclude that no family  $\{f_n(t)\}$  can be chosen to obtain  $c_n^{\Delta\alpha} = c_n$  for all but  $n=0$ , for all closed curves  $\phi^*$ .

Since we cannot arrange to have  $c_n^{\Delta\alpha} = c_n$  the next best thing would be to have  $c_n^{\Delta\alpha}$  depend in some simple way on  $c_n$  and  $\Delta\alpha$ . We shall now see what happens when we assume the basis  $\{f_n(t)\}$  is such that  $c_n^{\Delta\alpha} = h_n(c_n, \Delta\alpha)$  for all  $n$  and  $h_n$  independent of  $\phi^*(t)$ . In general,  $c_n^{\Delta\alpha}$  would depend on all the  $c_i$  and not just on  $c_n$ . This condition becomes

$$c_n^{\Delta\alpha} = \int_0^{2\pi} \phi_{\Delta\alpha}^*(t) \overline{g_n(t)} dt = h_n \left( \int_0^{2\pi} \phi^*(t) \overline{g_n(t)} dt, \Delta\alpha \right).$$

Since the preceding equation must hold for arbitrary  $\phi^*$  we can let  $\phi^*$  approach a delta function  $\phi^*(t) \approx \lambda \delta(t - \mu)$ . Substituting for  $\phi_{\Delta\alpha}^*$  and then  $\lambda \delta(t - \mu)$  for  $\phi^*$  we get

$$\lambda \overline{g_m(\Delta\alpha + \mu)} + (\Delta\alpha + \delta_0 - \delta_0^{\Delta\alpha}) \int_0^{2\pi} \overline{g_m(t)} dt = h_m(\lambda \overline{g_m(\mu)}, \Delta\alpha)$$

Putting

$$y = \lambda \overline{g_m(\mu)}$$

and

$$D = (\Delta\alpha + \delta_0 - \delta_0^{\Delta\alpha}) \int_0^{2\pi} \frac{1}{g_m(t)} dt$$

we get

$$y \frac{\overline{g_m(\Delta\alpha + \mu)}}{g_m(\mu)} + D = h_m(y, \Delta\alpha).$$

Treating  $y$  and  $\Delta\alpha$  as variables, we see that the right side does not depend on  $\mu$ , and therefore neither does the left side, so that

$$G_m(\Delta\alpha) = \frac{\overline{g_m(\Delta\alpha + \mu)}}{g_m(\mu)}$$

is independent of  $\mu$  for all  $\Delta\alpha$ . The only reasonable (e.g., anywhere continuous) solution [12] of such a functional equation is

$$g_m(x) = A_m e^{\sigma_m x}.$$

Since  $g_m(x)$  is periodic on  $[0, 2\pi]$ ,

$$\sigma_m = in(m)$$

where  $n(m)$  is an integer, and since  $\{g_m\}$  is a complete set, the set  $\{n(m)\}$  must include all integers. Therefore the family  $\{g_m(t)\}$  is in fact  $\{e^{int}\}_{-\infty}^{\infty}$  and aside from appropriate multiplicative constants the family  $\{f_n(t)\}$  is precisely the Fourier basis  $\{e^{-int}\}_{-\infty}^{\infty}$ .

There is a slight flaw in the above argument where we approximated  $\phi^*$  to a delta function but the problem is simple enough to repair. We were wrong in saying that  $\phi^*$  was arbitrary because  $\phi^*$  comes from a closed curve and must therefore satisfy the integral formula [see reconstruction theorem, (4)]

$$\int_0^{2\pi} e^{i(\phi^*(t)-t)} dt = 0.$$

If we define an infinite sequence of functions  $\delta^*_{\lambda, m, \pm}$  by

$$\delta^*_{\lambda, m, \pm}(t) = \text{if } |t - \mu| \leq \frac{\lambda}{4\pi m} \text{ then } \pm 2\pi m \text{ else } 0$$

where  $\lambda > 0$  and  $m$  is a positive integer then each  $\delta^*_{\lambda, m, \pm}$  satisfies the integral formula for closure and for fixed  $\lambda$ ,  $\pm$  it is easily verified that

$$\lim_{m \rightarrow \infty} \delta^*_{\lambda, m, \pm}(t) = \pm \lambda \delta(t - \mu).$$

The proof of the optimality of Fourier series is now complete.

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#### REFERENCES

- [1] R. L. Cosgriff, "Identification of shape," Ohio State Univ. Res. Foundation, Columbus, Rep. 820-11, ASTIA AD 254 792, Dec. 1960.
- [2] D. L. Fritzsche, "A systematic method for character recognition," Ohio State Univ. Res. Foundation, Columbus, Rep. 1222-4, ASTIA AD 268 360, Nov. 1961.
- [3] J. G. Raudseps, "Some aspects of the tangent-angle vs arc length representation of contours," Ohio State Univ. Res. Foundation, Columbus, Rep. 1801-6, ASTIA AD 462 877, Mar. 1965.
- [4] R. J. Borel, "A mathematical pattern recognition technique based on contour shape properties," Ohio State Univ. Res. Foundation, Columbus, Rep. 1801-11, ASTIA AD 476 113, Dec. 1965.
- [5] E. L. Brill, "The development and evaluation of three recognition criteria for pattern recognition," M.S. thesis, Ohio State Univ., Columbus, 1967.
- [6] —, "Character recognition via Fourier descriptors," presented at WESCON, Session 25, Qualitative Pattern Recognition Through Image Shaping, Los Angeles, Calif., Aug. 1968.
- [7] E. L. Brill *et al.*, "Some approaches to character recognition for postal address reader applications," presented at the Conf. Automatic Pattern Recognition, U. S. Post Office, May 1969.
- [8] C. T. Zahn, "Two-dimensional pattern description and recognition via curvature points," Stanford Linear Accelerator Cent. (USAEC), Stanford Univ., Stanford, Calif., Tech. Rep. SLAC-70, Dec. 1966.
- [9] —, "A formal description for two-dimensional patterns," in *Proc. Int. Joint Conf. Artificial Intelligence*, May 1969, pp. 621-628; available also as Rep. SLAC-PUB-538, Stanford Linear Accelerator Cent., Stanford Univ., Stanford, Calif.
- [10] W. Kaplan, *Advanced Calculus*. Reading, Mass.: Addison-Wesley, 1968, p. 434.
- [11] P. R. Halmos, *Introduction to Hilbert Space*. New York: Chelsea, p. 23.
- [12] J. Aczel, *Lectures on Functional Equations and Their Applications*. New York: Academic Press, Inc., 1966, p. 143.
- [13] Bateman Manuscript Project, *Higher Transcendental Functions*, vol. II, p. 14.
- [14] A. Rosenfeld, *Picture Processing by Computer*. New York: Academic Press, Inc.,
- [15] F. Attneave and M. D. Arnoult, "The quantitative study of shape and pattern perception," in *Pattern Recognition*, L. Uhr, Ed. New York: Wiley.
- [16] F. Attneave, "Some informational aspects of visual perception," *Psychol. Rev.*, pp. 183-193, 1954.
- [17] L. Uhr, "Pattern recognition," in *Pattern Recognition*, L. Uhr, Ed. New York: Wiley.
- [18] J. Y. Lettvin *et al.*, "What the frog's eye tells the frog's brain," *Proc. IRE*, vol. 47, pp. 1940-1951, Nov. 1959.
- [19] D. H. Hubel and T. N. Wiesel, "Receptive fields, binocular interaction, and functional architecture in the cat's visual cortex," *J. Physiol.*, vol. 160, pp. 106ff, 1962.
- [20] R. S. Ledley, *Use of Computers in Biology and Medicine*. New York: McGraw-Hill, 1965, p. 340.
- [21] R. C. Buck, *Advanced Calculus*. New York: McGraw-Hill, 1962, p. 252.
- [22] —, *Advanced Calculus*. New York: McGraw-Hill, 1962, p. 261.
- [23] C. T. Zahn, "Graph-theoretical methods for detecting and describing Gestalt clusters," *IEEE Trans. Comput.*, vol. C-20, pp. 68-86, Jan. 1971.
- [24] J. H. Munson, "Experiments in the recognition of handprinted text: Part I—character recognition," in *1968 Fall Joint Comput. Conf., AFIPS Conf. Proc.*, vol. 33. Washington, D. C.: Thompson, 1968, pp. 1125-1138.
- [25] H. G. Barrow and R. J. Popplestone, "Relational descriptions in picture processing," in *Machine Intelligence*, vol. 6, B. Meltzer and D. Michie, Eds. New York: Elsevier, 1971.