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FLCD SEMINAR 3 HW

1. Given grammar $G = (\{S, H\}, \{b, c, d, e\}, \{S \to \overset{(1)}{b^2 S e} \mid \overset{(2)}{H}, H \to \overset{(3)}{c H d^2} \mid \overset{(4)}{cd}\}, S)$, find the generated language (+ proof). - 25 p

$$(S) \underset{(2)}{\Rightarrow} H \underset{(4)}{\Rightarrow} cd$$

$$H \underset{(3)}{\Rightarrow} cHd^2 \underset{(4)}{\Rightarrow} c^2 d^3$$

$$\underset{(3)}{\Rightarrow} c^3 H d^6 \underset{(4)}{\Rightarrow} c^4 d^7$$

$$\underset{(3)}{\Rightarrow} c^4 H d^8 \underset{(4)}{\Rightarrow} c^5 d^9$$

$$\underset{(3)}{\Rightarrow} \dots$$

$$? \ \underbrace{H \overset{*}{\Rightarrow} c^n d^{2n-1}, \forall n \in \mathbb{N}^*}_{P(n)}$$

I $\ ?\ P(1)$ is true

$$H \underset{(4)}{\Rightarrow} cd = c^1 d^{2 \cdot 1 - 1} \Rightarrow P(1) \text{ is true}$$

II $P(k) \overset{?}{\Rightarrow} P(k+1), \forall k \in \mathbb{N}^*$

$$P(k) \text{ is true} \Rightarrow H \overset{*}{\Rightarrow} c^k d^{2k-1} \underset{(3)}{\Rightarrow}$$

$$\Rightarrow H \overset{*}{\Rightarrow} c c^k d^{2k-1} d^2 = c^{k+1} d^{2(k+1)-1} \Rightarrow P(k+1) \text{ is true}$$

$\mathrm{I}, \mathrm{II} \Rightarrow P(n)$ is true, $\forall n \in \mathbb{N}^* \Rightarrow$

$\Rightarrow H \overset{*}{\Rightarrow} c^n d^{2n-1}, \forall n \in \mathbb{N}^*$

$$S \underset{(2)}{\Rightarrow} H$$

$$S \underset{(1)}{\Rightarrow} b^2 S e \underset{(2)}{\Rightarrow} b^2 H e$$

$$b^2 S e \underset{(1)}{\Rightarrow} b^4 S e^2 \underset{(2)}{\Rightarrow} b^4 H e^2$$

$$b^4 S e^2 \underset{(1)}{\Rightarrow} b^6 S e^4 \underset{(2)}{\Rightarrow} b^6 H e^4$$

$$b^6 S e^4 \underset{(1)}{\Rightarrow} b^8 S e^6 \ldots$$

$$\cancel{? \ \underbrace{S \overset{*}{\Rightarrow} b^{2m} H e^m, \forall m \in \mathbb{N}}_{P(m)}}$$

$$? \ \underbrace{S \overset{*}{\Rightarrow} b^{2m} c^n d^{2n-1} e^m, \forall m \in \mathbb{N}, \forall n \in \mathbb{N}^*}_{P(n, m)}$$

I ? $P(1,0)$ is true $\Leftrightarrow S \overset{*?}{\Rightarrow} cd$

$$S \underset{(2)}{\Rightarrow} H \underset{(4)}{\Rightarrow} cd \Rightarrow P(1,0) \text{ is true}$$

$k \geq 1,\ l \geq 0$

$k \in \mathbb{N}^*, l \in \mathbb{N}$

II $P(k, l) \overset{?}{\Longrightarrow} P(k+1, l), P(k, l+1), P(k+1, l+1)$

$P(k, l)$ is true $\Longrightarrow S \overset{*}{\Longrightarrow} b^{2l} c^{k} d^{2k-1} e^{l}$

~~$S \overset{*}{\underset{\text{ind. hyp.}}{\Longrightarrow}} c^{k} d^{2k-1} \underset{(3)}{\Longrightarrow} S \overset{*}{\Longrightarrow} c^{k+1} d^{2(k+1)-1}$~~

$H \underset{(4)}{\Longrightarrow} cd \overset{k}{\underset{(3)}{\Longrightarrow}} c^{k+1} d^{2(k+1)-1} \underset{(2)}{\Longrightarrow} S \overset{*}{\Longrightarrow} c^{k+1} d^{2(k+1)}$

....

$\overset{l}{\underset{(1)}{\Longrightarrow}} S \overset{*}{\Longrightarrow} b^{2l} c^{k+1} d^{2(k+1)-1} \Rightarrow P(k+1, l)$ is true

$H \underset{(4)}{\Longrightarrow} cd \overset{k-1}{\underset{(3)}{\Longrightarrow}} c^{k} d^{2k-1} \underset{(2)}{\Longrightarrow} S \overset{*}{\Longrightarrow} c^{k} d^{2k+1}$

$\overset{l+1}{\underset{(1)}{\Longrightarrow}} S \overset{*}{\Longrightarrow} b^{2(l+1)} c^{k} d^{2k+1} e^{l} \Rightarrow P(k, l+1)$ is true

by applying the same logic, we can deduce that

$P(k, l) \Rightarrow P(k+1, l+1)$

.....

$$\Rightarrow \underbrace{\{ b^{2m} c^{n} d^{2n-1} e^{m} \mid n \in \mathbb{N}^*, m \in \mathbb{N} \}}_{L} \subseteq L(G)$$

$? \; L(G) \subseteq L$

$S \underset{(2)}{\Rightarrow} H \underset{(4)}{\Rightarrow} cd \in L$

$H \underset{(3)}{\Rightarrow} cHd^2 \underset{(4)}{\Rightarrow} c^2d^3 \in L$

$cHd^2 \underset{(3)}{\Rightarrow} c^2Hd^4 \underset{(4)}{\Rightarrow} c^3d^5 \in L$

$c^2Hd^4 \underset{(3)}{\Rightarrow} c^3Hd^6 \underset{(4)}{\Rightarrow} c^4d^7 \in L$

$c^3Hd^6 \underset{(4)}{\Rightarrow} \ldots$

$S \underset{(1)}{\Rightarrow} b^2Se \underset{(2)}{\Rightarrow} b^2He \underset{(4)}{\Rightarrow} b^2cde \in L$

$b^2He \underset{(3)}{\Rightarrow} b^2cHd^2e \underset{(4)}{\Rightarrow} b^2c^2d^3e \in L$

$b^2cHd^2e \underset{(3)}{\Rightarrow} b^2c^2Hd^4e \underset{(4)}{\Rightarrow} b^2c^3d^5e \in L$

$b^2c^2Hd^4e \underset{(3)}{\Rightarrow} b^2c^3Hd^6e \ldots$

$b^2Se \underset{(1)}{\Rightarrow} b^4Se^2 \underset{(2)}{\Rightarrow} b^4He^2 \underset{(4)}{\Rightarrow} b^4cde^2 \in L$

$b^4He^2 \underset{(3)}{\Rightarrow} b^4cHd^2e^2 \underset{(4)}{\Rightarrow} b^4c^2d^3e^2 \in L$

$b^4cHd^2e^2 \underset{(3)}{\Rightarrow} b^4c^2Hd^4e^2 \underset{(4)}{\Rightarrow} b^4c^3d^5e^2 \in L$

$b^4c^2Hd^4e^2 \underset{(3)}{\Rightarrow} \ldots$

$b^4Se^2 \underset{(1)}{\Rightarrow} b^6Se^4 \underset{(2)}{\Rightarrow} b^6He^4 \underset{(4)}{\Rightarrow} b^6cde^4 \in L$

$b^6He^4 \underset{(3)}{\Rightarrow} b^6cHd^2e^4 \underset{(4)}{\Rightarrow} b^6c^2d^3e^4 \in L$

$b^6cHd^2e^4 \underset{(3)}{\Rightarrow} b^6c^2Hd^4e^4 \underset{(4)}{\Rightarrow} b^6c^3d^7e^4 \in L$

$b^6c^2Hd^4e^4 \underset{(3)}{\Rightarrow} \ldots$

$b^6Se^4 \underset{(1)}{\Rightarrow} b^8Se^6 \underset{(2)}{\Rightarrow} \ldots$

$b^8Se^6 \underset{(1)}{\Rightarrow} b^{10}Se^8 \underset{(2)}{\Rightarrow} \ldots$

$b^{10}Se^8 \underset{(1)}{\Rightarrow} \ldots$

All grammar productions have been used in all possible combinations $\Rightarrow$ valid justif.

$L \subseteq L(G)$, $L(G) \subseteq L \Rightarrow L(G) = L$

2. Find grammars that generate the following languages

A. $L_1 = \{ x^n y^n \mid n \in \mathbb{N} \}$ + proof

It is enough to find G_1 s.t. $L_1 \subseteq L(G_1)$ and $L(G_1) \subseteq L_1$

Let $G_1 = (N, \Sigma, P, S)$

$N = \{S\}$

$\Sigma = \{x, y\}$

$P: \ S \longrightarrow \overset{(1)}{xSy} \mid \overset{(2)}{\varepsilon}$

I $L_1 \overset{?}{\subseteq} L(G_1)$

$L_1 \subseteq L(G_1) \Leftrightarrow \forall n \in \mathbb{N}: x^n y^n \in L(G_1)$

Let $n \in \mathbb{N}$ - fixed

$S \underset{(2)}{\Rightarrow} \varepsilon = x^0 y^0 \underset{(1)}{\Rightarrow} xy \underset{(1)}{\overset{n-1}{\Rightarrow}} x^n y^n \Rightarrow$

$\Rightarrow S \overset{*}{\Rightarrow} x^n y^n \Rightarrow \left.\begin{array}{l} x^n y^n \in L(G_1) \\ L_1 = \{x^n y^n \mid n \in \mathbb{N}\} \end{array}\right\} \Rightarrow L_1 \subseteq L(G_1)$

II $L(G_1) \overset{?}{\subseteq} L_1 \iff$? S generates $x^n y^n, n \in \mathbb{N}$

$$S \underset{(2)}{\Rightarrow} \varepsilon = x^0 y^0 \in L_1$$

$$\Downarrow_{(1)}$$

$$xSy \underset{(2)}{\Rightarrow} xy \in L_1$$

$$\Downarrow_{(1)}$$

$$x^2 S y^2 \underset{(2)}{\Rightarrow} x^2 y^2 \in L_1$$

$$\Downarrow_{(1)}$$

$$x^3 S y^3 \underset{(2)}{\Rightarrow} x^3 y^3 \in L_1$$

$$\Downarrow \quad \dots$$

$\Rightarrow$ S, only generates $x^n y^n$, $\forall n \in \mathbb{N}$

$\Rightarrow$ $L(G_1) \subseteq L_1$

$\text{I}, \text{II} \Rightarrow$ $L(G_1) = L_1$

B. $L_2 = \{ a^n b^{2n} \mid n \in \mathbb{N}^* \}$ + proof

Let $G_2 = (N, \Sigma, P, S)$

$N = \{S\}$

$\Sigma = \{a, b\}$

$P:$ $S \to a S b^2 \overset{(1)}{} \mid a b^2 \overset{(2)}{}$

I $L_2 \overset{?}{\subseteq} L(G_2) \iff \forall n \in \mathbb{N}$?: $a^n b^{2n} \in L(G_2)$

Let $n \in \mathbb{N}^*$ - fixed

$$S \underset{(2)}{\Rightarrow} ab^2 \underset{(1)}{\Rightarrow} a^2 b^4 \overset{n-2}{\underset{(1)}{\Rightarrow}} a^n b^{2n}$$

$$S \overset{*}{\Rightarrow} a^n b^{2n} \Rightarrow a^n b^{2n} \in L(G_2) \Rightarrow L_2 \subseteq L(G_2)$$

$\underline{II}$ $L(G_2) \overset{?}{\subseteq} L_2$

$S \underset{(2)}{\Rightarrow} a b^2 \in L_2$

$\overset{(1)}{\Downarrow}$ $a S b^2 \underset{(2)}{\Rightarrow} a^2 b^4 \in L_2$

$\overset{(1)}{\Downarrow}$ $a^2 S b^4 \underset{(2)}{\Rightarrow} a^3 b^6 \in L_2$

$\overset{(1)}{\Downarrow}$ $a^3 S b^6 \underset{(2)}{\Rightarrow} a^4 b^8 \in L_2$

$\Downarrow$ ....

$\Rightarrow$ S only generates $a^n b^{2n}$, $\forall n \in \mathbb{N}^*$

$\Rightarrow L(G_2) \subseteq L_2$

$I, II \Rightarrow L(G_2) = L_2$

C. $L_3 = \{a^n b^m \mid n, m \in \mathbb{N}^*\}$ - regular grammar required + proof

Let $G_3 = (N, \Sigma, P, S)$

$N = \{S, H\}$

$\Sigma = \{a, b\}$

$P: \left.\begin{array}{l} S \Rightarrow \overset{(1)}{a S} \mid \overset{(2)}{a H} \\ H \Rightarrow \overset{(3)}{b H} \mid \overset{(4)}{b} \end{array}\right\}$ ← right-linear

$\underline{I}$ $L_3 \overset{?}{\subseteq} L(G_3) \Leftrightarrow \forall n, m \in \mathbb{N}^* ?: a^n b^m \in L(G_3)$

Let $n, m \in \mathbb{N}^*$ fixed

$H \underset{(4)}{\Rightarrow} b \overset{m-1}{\underset{(3)}{\Longrightarrow}} b^m$ (i)

$$S \underset{(2)}{\Rightarrow} aH \overset{n-1}{\underset{(1)}{\Rightarrow}} a^n H \underset{(i)}{\Rightarrow} a^n b^m \Rightarrow$$

$$\Rightarrow \quad S \overset{*}{\Rightarrow} a^n b^m \Rightarrow a^n b^m \in L(G_3)$$

$$\Rightarrow L_3 \subseteq L(G_3)$$

II $L(G_3) \overset{?}{\subseteq} L_3 \Leftrightarrow$? S only generates $a^n b^m$, $n, m \in \mathbb{N}^*$

a) ? H only generates b^m, $m \in \mathbb{N}^*$

$$H \underset{(4)}{\Rightarrow} b$$

$$\Downarrow (3)$$

$$bH \underset{(4)}{\Rightarrow} b^2$$

$$\Downarrow (3)$$

$$b^2 H \underset{(4)}{\Rightarrow} b^3$$

$$\Downarrow (3)$$

$$b^3 H \underset{(4)}{\Rightarrow} b^4$$

$$\Downarrow \; \ldots$$

$\Rightarrow$ H only generates b^m, $m \in \mathbb{N}^*$

b) ? S only generates $a^n H$, $n \in \mathbb{N}^*$

$$S \underset{(2)}{\Rightarrow} aH$$

$$\Downarrow (1)$$

$$aS \underset{(2)}{\Rightarrow} a^2 H$$

$$\Downarrow (1)$$

$$a^2 S \underset{(2)}{\Rightarrow} a^3 H$$

$$\Downarrow (1)$$

$$a^3 S \underset{(2)}{\Rightarrow} a^4 H$$

$$\Downarrow \; \ldots$$

$\Rightarrow$ S only generates $a^n H$, $n \in \mathbb{N}^*$

a), b), (2) $\Rightarrow$ G_3 only generates $a^n b^m$, $n, m \in \mathbb{N}^*$

(Considering that the other prod. rule of S besides (2) is (1), and that applying (1) still generates only results in L_3, then this should be sound.) ~~I realized late that I could have used $N = \{S, A, B\}$, $\Sigma = \{a, b\}$, $P = \{S \to AB$, (not regular)~~

$\Rightarrow L(G_3) \subseteq L_3$

$I, II \Rightarrow L(G_3) = L_3$