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A Nonconforming Rectangle Element for Planar Elasticity with Pure Traction Boundary Condition *

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Abstract: A new nonconforming rectangle element for the problem of planar elasticity with pure traction boundary condition is considered. The discrete Korn's second inequality for the element is valid. Error estimates in the energy norm and L^2-norm are $O(h^2)$ and $O(h^3)$, respectively. The convergence rate is uniformly optimal with respect to λ.

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1. Introduction

It is well known that the standard finite element method[1-3] deteriorates as the Lamé constant $\lambda \to \infty$, i. e., when the elastic material is incompressible. Various methods have been proposed on overcoming this problem, for example the mixed method[4,5]. the p-version method and hp-version[6-7], and the nonconforming method[3,8-10].

However some methods work well in the case of displacement boundary conditions, but must be modified for pure traction or mixed boundary conditions. The main difficulty stems from the proof of appropriate discrete versions of Korn's second inequality which can be avoided in the pure displacement case. For instance, Falk[8] introduced a local projection and modified the variational equations, so that the modified discrete versions of Korn's second inequality hold for the standard Crouziex-Raviart finite element space. Meanwhile using mixed methods, Falk obtained optimal-order error estimates for nonconforming piecewise polynomials of degree $\leqslant 3$. Brenner and Sung[3] considered a linear conforming triangular element by the method of reduced integration. Kouhia and Stenberg[11] introduced a new triangle element in which linear conforming element is used for one of the displacement components and linear nonconforming element for the other component, and obtained optimal-order error estimates.

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This idea has been extended to NRQ_1 element[12]. ZHANG[10] also analyzed some nonconforming elements on arbitrary quadrilateral meshes by modifying the variational form. In addition, Korn's inequalities for piecewise H^1 vector fields are established in [13], which can be applied to classical nonconforming finite element methods, mortar methods and discontinuous Galerkin methods.

In this paper, we construct a new nonconforming rectangle element for the pure traction planar elasticity. We don't need any modification on the variational form and can directly prove that the discrete version of Korn's second inequality is valid. The convergence rate of the finite element is uniformly optimal with respect to λ. Error estimates in the energy norm and L^2 - norm are $O(h^2)$ and $O(h^3)$, respectively.

For simplicity, the letter C denotes a generic constant, not necessarily the same in each occurrence. We denote by $P_k(\Omega)$ the space of all polynomials on Ω of degree $\leqslant k$, and by $Q_{kj}(\Omega)$ the space of all polynomials on Ω of degree $\leqslant k$ in variable x and $\leqslant j$ in variable y. We will use the same notations for Sobolev as in [2].

The pure traction boundary value problem for planar linear isotropic elasticity is given by

$$\begin{cases} -\operatorname{div}\sigma(\boldsymbol{u}) = \boldsymbol{f}, & \text{in } \Omega, \\ \sigma(\boldsymbol{u})\boldsymbol{v} = \boldsymbol{g}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\sigma(\boldsymbol{u}) = (2\mu\varepsilon(\boldsymbol{u}) + \lambda \operatorname{tr}(\varepsilon(\boldsymbol{u}))I)$ is the stress tensor, $\varepsilon(\boldsymbol{u}) = [\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^{\mathrm{T}}]/2$ is the strain tensor, I is the 2×2 identity matrix, $\boldsymbol{u}$ is the displacement, $\boldsymbol{f}$ is the body force, $\boldsymbol{g}$ is the boundary traction, $\boldsymbol{v}$ is the unit outer normal, $(\mu,\lambda) \in [\mu_1, \mu_2] \times (0,\infty)$ are the Lamé constants and $0 < \mu_1 < \mu_2$. For simplicity, we only consider the case where $\Omega \subseteq \mathbf{R}^2$ is a bounded convex polygonal domain. We introduce the spaces

$$\hat{V} = \{\boldsymbol{v} \in (H^1(\Omega))^2 : \int_\Omega \boldsymbol{v}\,\mathrm{d}x\mathrm{d}y = \boldsymbol{0}, \int_\Omega \operatorname{rot} \boldsymbol{v}\,\mathrm{d}x\mathrm{d}y = 0\},$$

$$RM = \{\boldsymbol{v} : \boldsymbol{v} = (a+by, c-bx)^{\mathrm{T}}, a, b, c, \in \mathbf{R}\},$$

where $\operatorname{rot} \boldsymbol{v} = -\partial v_1/\partial y + \partial v_2/\partial x$. If and only if $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the following compatibility condition[2-3]

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v}\,\mathrm{d}s = 0, \forall \boldsymbol{v} \in RM, \tag{1.2}$$

the problem (1.1) is solvable, and there exists a unique solution in $\hat{V}$.

The variational problem for (1.1) is: find $\boldsymbol{u} \in \hat{V}$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v}\,\mathrm{d}s, \forall \boldsymbol{v} \in \hat{V}, \tag{1.3}$$

where $a(\boldsymbol{u},\boldsymbol{v}) = 2\mu\int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x\mathrm{d}y + \lambda\int_\Omega (\operatorname{div} \boldsymbol{u})(\operatorname{div} \boldsymbol{v})\,\mathrm{d}x\mathrm{d}y$.

The following Korn's second inequality[2] is well known: there exists a positive constant C such that

$$\| \varepsilon(\boldsymbol{v}) \|_{0,\Omega} \geqslant C \mid \boldsymbol{v} \mid_{1,\Omega}, \forall \boldsymbol{v} \in \hat{V}. \tag{1.4}$$

When compatibility condition (1.2) holds, problem (1.3) has a unique solution[2].

2. A Nonconforming Rectangle Element

In this section we present a locking-free rectangle finite element with 18 degrees of freedom.

Suppose reference element $\hat{T}$ is a square on (ξ,η) plane with vertices $\hat{A}_1(-1,-1)$, $\hat{A}_2(1,$

$-1),\hat{A}_3(1,1),\hat{A}_4(-1,1)$ and sides $\hat{e}_i=\hat{A}_i\hat{A}_{i+1}$ (the index on A_i modulo 4), $i=1,2,3,4$. Let

$$W_1(\hat{T})=(Q_{32}(\hat{T})/\{\xi\eta^2,\xi^2\eta\})\oplus\{\eta^3\},W_2(\hat{T})=(Q_{23}(\hat{T})/\{\eta^3,\xi\eta^2\})\oplus\{\xi^3\}.$$

We define finite element $(\hat{T},\hat{P},\hat{\Sigma})$ as follows: The shape function space is defined by

$$\hat{P}=\{\hat{\boldsymbol{p}}:\hat{\boldsymbol{p}}\in W_1(\hat{T})\times W_2(\hat{T}),\operatorname{div}\hat{\boldsymbol{p}}\in P_1(\hat{T})\}. \tag{2.1}$$

Let $\hat{\boldsymbol{v}}=(\hat{v}_1,\hat{v}_2)^{\mathrm{T}}$, the degree of freedom is defined by

$$\begin{aligned}\hat{\Sigma}=\Big\{&\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{v}_j\mathrm{d}\hat{s},i=1,2,3,4,j=1,2;\\ &\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{v}_j\xi\mathrm{d}\hat{s},i=1,3,j=1,2;\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{v}_j\eta\mathrm{d}\hat{s},i=2,4,j=1,2;\\ &\frac{1}{|\hat{T}|}\int_{\hat{T}}\operatorname{div}\hat{\boldsymbol{v}}\cdot\xi\mathrm{d}\xi\mathrm{d}\eta,\frac{1}{|\hat{T}|}\int_{\hat{T}}\operatorname{div}\hat{\boldsymbol{v}}\cdot\eta\mathrm{d}\xi\mathrm{d}\eta\Big\}.\end{aligned} \tag{2.2}$$

The above finite element is well defined, and $\forall\hat{\boldsymbol{p}}=(\hat{p}_1,\hat{p}_2)^{\mathrm{T}}\in\hat{P}$ can be represented as:

$$\begin{aligned}\hat{\boldsymbol{p}}=&\sum_{i=1}^{4}\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{p}_1\mathrm{d}\hat{s}\cdot\hat{\boldsymbol{q}}_{1,i}+\sum_{i=1}^{4}\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{p}_2\mathrm{d}\hat{s}\cdot\hat{\boldsymbol{q}}_{2,i}\\ &+\sum_{i=1,3}\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{p}_1\xi\mathrm{d}\hat{s}\cdot\hat{\boldsymbol{q}}_{1,i+4}+\sum_{i=2,4}\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{p}_1\eta\mathrm{d}\hat{s}\cdot\hat{\boldsymbol{q}}_{1,i+4}\\ &+\sum_{i=1,3}\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{p}_2\xi\mathrm{d}\hat{s}\cdot\hat{\boldsymbol{q}}_{2,i+4}+\sum_{i=2,4}\frac{1}{|\hat{e}_i|}\int_{\hat{e}_i}\hat{p}_2\eta\mathrm{d}\hat{s}\cdot\hat{\boldsymbol{q}}_{2,i+4}\\ &+\frac{1}{|\hat{T}|}\int_{\hat{T}}\operatorname{div}\hat{\boldsymbol{p}}\cdot\xi\mathrm{d}\xi\mathrm{d}\eta\,\hat{\boldsymbol{\varphi}}_1+\frac{1}{|\hat{T}|}\int_{\hat{T}}\operatorname{div}\hat{\boldsymbol{p}}\cdot\eta\mathrm{d}\xi\mathrm{d}\eta\,\hat{\boldsymbol{\varphi}}_2,\end{aligned} \tag{2.3}$$

where

$$\hat{\boldsymbol{q}}_{1,1}=\left(-\frac{1}{4}+\frac{3}{4}\eta+\frac{3}{4}\eta^2-\frac{5}{4}\eta^3,0\right)^{\mathrm{T}},\hat{\boldsymbol{q}}_{1,3}=\left(-\frac{1}{4}-\frac{3}{4}\eta+\frac{3}{4}\eta^2+\frac{5}{4}\eta^3,0\right)^{\mathrm{T}},$$

$$\hat{\boldsymbol{q}}_{1,2}=\left(\frac{3}{4}+\frac{13}{16}\xi-\frac{3}{4}\eta^2+\frac{5}{24}\xi^3-\frac{25}{16}\xi^3\eta^2,-\frac{5}{16}\eta-\frac{5}{8}\xi^2\eta+\frac{25}{16}\xi^2\eta^3\right)^{\mathrm{T}},$$

$$\hat{\boldsymbol{q}}_{1,4}=\left(\frac{3}{4}-\frac{13}{16}\xi-\frac{3}{4}\eta^2-\frac{5}{24}\xi^3+\frac{25}{16}\xi^3\eta^2,\frac{5}{16}\eta+\frac{5}{8}\xi^2\eta-\frac{25}{16}\xi^2\eta^3\right)^{\mathrm{T}},$$

$$\begin{aligned}\hat{\boldsymbol{q}}_{1,5}=\Big(&\frac{15}{16}\xi-\frac{15}{4}\xi\eta-\frac{5}{8}\xi^3+\frac{15}{4}\xi^3\eta+\frac{75}{16}\xi^3\eta^2,\\ &-\frac{5}{8}+\frac{15}{16}\eta+\frac{15}{8}\xi^2+\frac{15}{8}\eta^2+\frac{15}{8}\xi^2\eta-\frac{45}{8}\xi^2\eta^2-\frac{75}{16}\xi^2\eta^3\Big)^{\mathrm{T}},\end{aligned}$$

$$\hat{\boldsymbol{q}}_{1,6}=\left(\frac{15}{4}\eta-\frac{9}{4}\xi\eta-\frac{15}{4}\eta^3+\frac{15}{4}\xi^3\eta,\frac{3}{8}+\frac{9}{8}\xi^2+\frac{9}{8}\eta^2-\frac{45}{8}\xi^2\eta^2\right)^{\mathrm{T}},$$

$$\begin{aligned}\hat{\boldsymbol{q}}_{1,7}=\Big(&\frac{15}{16}\xi+\frac{15}{4}\xi\eta-\frac{5}{8}\xi^3-\frac{15}{4}\xi^3\eta+\frac{75}{16}\xi^3\eta^2,\\ &\frac{5}{8}+\frac{15}{16}\eta-\frac{15}{8}\xi^2-\frac{15}{8}\eta^2+\frac{15}{8}\xi^2\eta+\frac{45}{8}\xi^2\eta^2-\frac{75}{16}\xi^2\eta^3\Big)^{\mathrm{T}},\end{aligned}$$

$$\hat{\boldsymbol{q}}_{1,8}=\left(\frac{15}{4}\eta+\frac{9}{4}\xi\eta-\frac{15}{4}\eta^3-\frac{15}{4}\xi^3\eta,-\frac{3}{8}-\frac{9}{8}\xi^2-\frac{9}{8}\eta^2+\frac{45}{8}\xi^2\eta^2\right)^{\mathrm{T}},$$

$$\hat{\boldsymbol{q}}_{2,1}=\left(\frac{5}{16}\xi-\frac{5}{24}\xi^3-\frac{5}{16}\xi^3\eta^2,\frac{3}{4}-\frac{13}{16}\eta-\frac{3}{4}\xi^2+\frac{5}{8}\xi^2\eta+\frac{5}{16}\xi^2\eta^3\right)^{\mathrm{T}},$$

$$\hat{\boldsymbol{q}}_{2,2}=\left(0,-\frac{1}{4}-\frac{3}{4}\xi+\frac{3}{4}\xi^2+\frac{5}{4}\xi^3\right)^{\mathrm{T}},\hat{\boldsymbol{q}}_{2,4}=\left(0,-\frac{1}{4}+\frac{3}{4}\xi+\frac{3}{4}\xi^2-\frac{5}{4}\xi^3\right)^{\mathrm{T}},$$

$$\hat{\boldsymbol{q}}_{2,3}=\left(-\frac{5}{16}\xi+\frac{5}{24}\xi^3+\frac{5}{16}\xi^3\eta^2,\frac{3}{4}+\frac{13}{16}\eta-\frac{3}{4}\xi^2-\frac{5}{8}\xi^2\eta-\frac{5}{16}\xi^2\eta^3\right)^{\mathrm{T}},$$

$$\hat{q}_{2,5} = \left(-\frac{3}{8} - \frac{9}{8}\xi^2 - \frac{9}{8}\eta^2 + \frac{45}{8}\xi^2\eta^2, \frac{15}{4}\xi + \frac{9}{4}\xi\eta - \frac{15}{4}\xi^3 - \frac{15}{4}\xi\eta^3\right)^{\mathrm{T}},$$

$$\hat{q}_{2,6} = \Big(\frac{5}{8} + \frac{15}{16}\xi - \frac{15}{8}\xi^2 - \frac{15}{8}\eta^2 - \frac{5}{8}\xi^3 + \frac{45}{8}\xi^2\eta^2 - \frac{15}{16}\xi^3\eta^2,$$
$$-\frac{15}{16}\eta + \frac{15}{4}\xi\eta + \frac{15}{8}\xi^2\eta - \frac{15}{4}\xi\eta^3 + \frac{15}{16}\xi^2\eta^3\Big)^{\mathrm{T}},$$

$$\hat{q}_{2,7} = \left(\frac{3}{8} + \frac{9}{8}\xi^2 + \frac{9}{8}\eta^2 - \frac{45}{8}\xi^2\eta^2, \frac{15}{4}\xi - \frac{9}{4}\xi\eta - \frac{15}{4}\xi^3 + \frac{15}{4}\xi\eta^3\right)^{\mathrm{T}},$$

$$\hat{q}_{2,8} = \Big(-\frac{5}{8} + \frac{15}{16}\xi + \frac{15}{8}\xi^2 + \frac{15}{8}\eta^2 - \frac{5}{8}\xi^3 - \frac{45}{8}\xi^2\eta^2 - \frac{15}{16}\xi^3\eta^2,$$
$$-\frac{15}{16}\eta - \frac{15}{4}\xi\eta + \frac{15}{8}\xi^2\eta + \frac{15}{4}\xi\eta^3 + \frac{15}{16}\xi^2\eta^3\Big)^{\mathrm{T}},$$

$$\hat{\varphi}_1 = \left(-2 + \frac{3}{2}\xi^2 + \frac{3}{2}\eta^2, 0\right)^{\mathrm{T}}, \hat{\varphi}_2 = \left(0, -2 + \frac{3}{2}\xi^2 + \frac{3}{2}\eta^2\right)^{\mathrm{T}}.$$

Let $\mathscr{T}_h$ be the quasi-uniform subdivision of Ω. For a general rectangle $T \in \mathscr{T}_h$, we denote $A_1(x_0 - h_1, y_0 - h_2)$, $A_2(x_0 + h_1, y_0 - h_2)$, $A_3(x_0 + h_1, y_0 + h_2)$, $A_4(x_0 - h_1, y_0 + h_2)$ and $e_i = A_iA_{i+1}$ $(i = 1,2,3,4)$ by the vertices and sides of T, respectively. Let $h_T = \max(h_1, h_2)$, $h = \max\limits_{T\in\mathscr{T}_h} h_T$. Then there exists an affine mapping $F_T : \hat{T} \to T$ given by

$$(x,y)^{\mathrm{T}} = F_T(\xi,\eta)^{\mathrm{T}} = B(\xi,\eta)^{\mathrm{T}} + (x_0, y_0)^{\mathrm{T}}, \text{where } B = \operatorname{diag}(h_1, h_2). \tag{2.4}$$

For any vector function $\hat{\boldsymbol{v}}$ defined on $\hat{T}$, by Piola's transformation, we associate $\boldsymbol{v}$ on T by

$$\boldsymbol{v} = B\hat{\boldsymbol{v}} = (h_1\hat{v}_1 \circ F_T^{-1}, h_2\hat{v}_2 \circ F_T^{-1})^{\mathrm{T}}. \tag{2.5}$$

The shape function space on general element T is defined by

$$P(T) = \{\boldsymbol{p} : \boldsymbol{p} = B\hat{\boldsymbol{p}} \circ F_T^{-1}, \hat{\boldsymbol{p}} \in \hat{P} \text{ is given by (2.3)}\}. \tag{2.6}$$

Suppose $\hat{\Pi}$ and Π_T are the finite-element interpolation operators introduced by $\hat{P}$ and $P(T)$, respectively. Since under the Piola's transformation (2.5), $\operatorname{div} \boldsymbol{v} = \operatorname{div} \hat{\boldsymbol{v}} \circ F_T^{-1}$ for any $\boldsymbol{v} \in (H^1(T))^2$. It can be shown that the interpolation operators are affine equivalent, i. e.

$$\widehat{\Pi_T\boldsymbol{v}} = B^{-1}\Pi_T\boldsymbol{v} \circ F_T^{-1} = \hat{\Pi}\hat{\boldsymbol{v}}, \tag{2.7}$$

and the following properties hold for Π_T:

$$\frac{1}{|e_i|}\int_{e_i} p\boldsymbol{v}\,\mathrm{d}s = \frac{1}{|e_i|}\int_{e_i} p\Pi_T\boldsymbol{v}\,\mathrm{d}s, p \in P_1(e_i), i = 1,2,3,4; \tag{2.8}$$

$$\frac{1}{|T|}\int_T \operatorname{div}\boldsymbol{v} \cdot p\,\mathrm{d}x\mathrm{d}y = \frac{1}{|T|}\int_T \operatorname{div}\Pi_T\boldsymbol{v} \cdot p\,\mathrm{d}x\mathrm{d}y, p \in P_1(T). \tag{2.9}$$

(2.8) and (2.9) are the key properties which can ensure the element has two order of convergence.

Denote Π_h by the global finite element interpolation operator. For a piecewise function $\boldsymbol{u}$, we define $\operatorname{grad}_h \boldsymbol{u}$ to be the $(L^2(\Omega))^4$ function whose restriction to each triangle $T \in \mathscr{T}_h$ is given by $\operatorname{grad} \boldsymbol{u}|_T$. Analogous definitions hold for rot_h and ε_h. By Green's formula and (2.8), we can derive the following lemma, which is the key fact for the proof of the discrete Korn's second inequality.

Lemma 2.1 For any $\boldsymbol{v} \in \hat{V}$, $\int_\Omega \operatorname{rot}_h \Pi_h\boldsymbol{v}\,\mathrm{d}x\mathrm{d}y = 0$.

Define finite element spaces as follows:

$$V_h = \{\boldsymbol{v}_h : \boldsymbol{v}_h|_T \in P(T)\},$$

$$\hat{V}_h = \left\{ \boldsymbol{v}_h \in V_h : \int_\Omega \mathrm{rot}_h\, \boldsymbol{v}_h \mathrm{d}x\mathrm{d}y = 0, \int_\Omega \boldsymbol{v}_h \mathrm{d}x\mathrm{d}y = \boldsymbol{0} \right\}. \tag{2.10}$$

We deduce from (2. 8) that for any edge $e \subset \partial T$,

$$\int_e [\boldsymbol{v}_h \cdot \boldsymbol{q}] \mathrm{d}s = 0 \text{ for all } \boldsymbol{v}_h \in V_h, \boldsymbol{q} \in (P_1(e))^2, \tag{2.11}$$

where $[v]$ is the jump of v across the edge e, and $[v] = v$ if $e \subset \partial\Omega$. It is another key fact in the proof of the Korn's second inequality.

3. Error Estimate

The discrete bilinear form is defined by

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = 2\mu \sum_T \int_T \varepsilon(\boldsymbol{u}_h) : \varepsilon(\boldsymbol{v}_h) \mathrm{d}x\mathrm{d}y + \lambda \sum_T \int_T (\mathrm{div}\ \boldsymbol{u}_h)(\mathrm{div}\ \boldsymbol{v}_h) \mathrm{d}x\mathrm{d}y.$$

Then the approximation form to (1. 3) is:

$$\begin{cases} \text{Find } \boldsymbol{u}_h \in \hat{V}_h \text{ such that} \\ a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v}_h \mathrm{d}s,\ \forall\, \boldsymbol{v}_h \in \hat{V}_h. \end{cases} \tag{3.1}$$

The following lemma shows the relationship between V_h and $\hat{V}_h$.

Lemma 3. 1 The finite element space V_h can be decomposed into $V_h = \hat{V}_h \oplus RM$.

Proof Obviously, $\hat{V}_h \oplus RM \subset V_h$. Conversely, given any $\boldsymbol{v} \in V_h$, there exists a unique pair $(\boldsymbol{z}, \boldsymbol{w}) \in \hat{V}_h \times RM$ such that $\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{w}$. In particular, $\boldsymbol{w} = \boldsymbol{c} + b(y, -x)^{\mathrm{T}}$, where

$$b = -\frac{1}{2|\Omega|} \sum_T \int_T \mathrm{rot}\ \boldsymbol{v} \mathrm{d}x\mathrm{d}y, \boldsymbol{c} = \frac{1}{|\Omega|} \int_\Omega [\boldsymbol{v} - b(y, -x)^{\mathrm{T}}] \mathrm{d}x\mathrm{d}y.$$

Lemma 3. 2 Define $\| \boldsymbol{v} \|_h = a_h(\boldsymbol{v}, \boldsymbol{v})^{1/2}$, then $\| \cdot \|_h$ is actually a norm on $\hat{V}_h$.

Proof We only need to show that $\| \boldsymbol{v} \|_h = 0$ yields $\boldsymbol{v} = \boldsymbol{0}$ for all $\boldsymbol{v} \in \hat{V}_h$. By the definition of $a_h(\cdot, \cdot)$, $\| \boldsymbol{v} \|_h = 0$ results in $\| \varepsilon_h(\boldsymbol{v}) \|_{0,\Omega} = 0$. Thus $\boldsymbol{v}|_T = (a_T + b_T y, c_T - b_T x)^{\mathrm{T}}$, where a_T, b_T, c_T are constants dependent on T. Suppose T^+ and T^- are adjacent elements with a common edge e, $\bar{a} = a_{T^+} - a_{T^-}$, $\bar{b} = b_{T^+} - b_{T^-}$, $\bar{c} = c_{T^+} - c_{T^-}$. By virtue of (2. 18), we have

$$\int_e (\bar{a} + \bar{b}y, \bar{c} - \bar{b}x)^{\mathrm{T}} \cdot \boldsymbol{q} \mathrm{d}s = 0, \forall\, \boldsymbol{q} \in (P_1(e))^2.$$

Take $\boldsymbol{q} = (\bar{a} + \bar{b}y, \bar{c} - \bar{b}x)^{\mathrm{T}}$, then $\bar{a} + \bar{b}y|_e = 0$ and $\bar{c} - \bar{b}x|_e = 0$. Since the equation of e can not be $\bar{a} + \bar{b}y = 0$ and $\bar{c} - \bar{b}x = 0$ at the same time, we conclude $\bar{a} = \bar{b} = \bar{c} = 0$, which implies that a_T, b_T, c_T are independent of T and $\boldsymbol{v} \in RM$. By Lemma 3. 1 which implies $RM \cap \hat{V}_h = \{\boldsymbol{0}\}$, we have $\boldsymbol{v} = \boldsymbol{0}$.

By virtue of property (2. 11) and the property that $\int_\Omega \mathrm{rot}_h\, \boldsymbol{v}_h \mathrm{d}x\mathrm{d}y = 0$ for all $\boldsymbol{v}_h \in \hat{V}_h$, the following discrete Korn's second inequality is a direct consequence of the estimate (1. 16) in [13].

Lemma 3. 3 For all $\boldsymbol{u} \in \hat{V}_h$, there exists a positive constant C independent of h such that

$$C \| \mathrm{grad}_h\, \boldsymbol{u} \|_{0,\Omega} \leqslant \| \varepsilon_h(\boldsymbol{u}) \|_{0,\Omega}. \tag{3.2}$$

Since $\hat{\Pi}$ is exact for the space $(P_2(T))^2$, the following interpolation error estimate holds.

Lemma 3. 4 There exists a positive constant C, independent of h_T and T, such that

$$\| \boldsymbol{v} - \Pi_T \boldsymbol{v} \|_{0,T} + h_T | \boldsymbol{v} - \Pi_T \boldsymbol{v} |_{1,T} \leqslant C h_T^3 | \boldsymbol{V} |_{3,T}, \forall\, \boldsymbol{v} \in (H^3(T))^2. \tag{3.3}$$

Define γ_T, the L^2 projection operator from $L^2(T)$ onto $P_1(T)$. By the interpolation theorem and the technique of affine transformation, we can obtain the following results.

Lemma 3. 5 There exists a positive constant C independent of h_T and T, such that

$$\operatorname{div} \Pi_T \boldsymbol{v} = \gamma_T \operatorname{div} \boldsymbol{v}, \forall \boldsymbol{v} \in (H^1(T))^2, \forall T \in \mathcal{T}_h, \tag{3.4}$$

$$\| w - \gamma_T w \|_{L^2(T)} \leqslant C h_T^2 \mid w \mid_{2,T}, \forall w \in H^2(T), \forall T \in \mathcal{T}_h. \tag{3.5}$$

Theorem 3.1 Suppose $\boldsymbol{u} \in (H^3(\Omega))^2 \cap \hat{V}$ and $\boldsymbol{u}_h$ are the solutions of problem (1.3) and (3.1), respectively. Then there exists a positive constant C independent of h such that

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_h \leqslant C h^2 (\mid \boldsymbol{u} \mid_{3,\Omega} + \lambda \mid \operatorname{div} \boldsymbol{u} \mid_{2,\Omega}), \tag{3.6}$$

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_{0,\Omega} \leqslant C h^3 (\mid \boldsymbol{u} \mid_{3,\Omega} + \lambda \mid \operatorname{div} \boldsymbol{u} \mid_{2,\Omega}). \tag{3.7}$$

Proof Using Strang second lemma[2], we have

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_h \leqslant \inf_{\boldsymbol{v}_h \in V_h} \| \boldsymbol{u} - \boldsymbol{v}_h \|_h + \sup_{0 \neq \boldsymbol{w}_h \in V_h} \frac{\mid a_h(\boldsymbol{u}, \boldsymbol{w}_h) - \langle \boldsymbol{l}, \boldsymbol{w}_h \rangle \mid}{\| \boldsymbol{w}_h \|_h}, \tag{3.8}$$

where

$$\langle \boldsymbol{l}, \boldsymbol{w}_h \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_h \, dx dy + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{w}_h \, ds.$$

By Lemma 3.4 and (2.11), using Cauchy-Schwarz inequality, we deduce the approximation error as follows:

$$\begin{aligned}
\inf_{\boldsymbol{v}_h \in V_h} \| \boldsymbol{u} - \boldsymbol{v}_h \|_h &\leqslant \| \boldsymbol{u} - \Pi_h \boldsymbol{u} + \frac{1}{\mid \Omega \mid} \int_\Omega \Pi_h \boldsymbol{u} \, dx dy \|_h = \| \boldsymbol{u} - \Pi_h \boldsymbol{u} \|_h \\
&\leqslant (2\mu C \sum_T \mid \boldsymbol{u} - \Pi_T \boldsymbol{u} \mid_{1,T}^2 + \lambda \sum_T \| \operatorname{div} \boldsymbol{u} - \gamma_T \operatorname{div} \boldsymbol{u} \|_{0,T}^2)^{1/2} \\
&\leqslant C h^2 (\mid \boldsymbol{u} \mid_{3,\Omega} + \lambda \mid \operatorname{div} \boldsymbol{u} \mid_{2,\Omega}).
\end{aligned} \tag{3.9}$$

Since $\varepsilon(\boldsymbol{u}) : \operatorname{grad} \boldsymbol{w}_h = \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{w}_h)$ and $(\operatorname{tr} \varepsilon(\boldsymbol{u}))\delta : \operatorname{grad} \boldsymbol{w}_h = (\operatorname{div} \boldsymbol{u})(\operatorname{div} \boldsymbol{w}_h)$, by Green's formula, the nonconforming error can be written as

$$\begin{aligned}
E_h(\boldsymbol{u}, \boldsymbol{w}_h) = a_h(\boldsymbol{u}, \boldsymbol{w}_h) - \langle \boldsymbol{l}, \boldsymbol{w}_h \rangle &= \sum_T \int_T \sigma(\boldsymbol{u}) : \operatorname{grad} \boldsymbol{w}_h \, dx dy - \langle \boldsymbol{l}, \boldsymbol{w}_h \rangle \\
&= \sum_T \sum_{\substack{e \subset \partial T \\ e \not\subset \partial\Omega}} \int_e \sigma(\boldsymbol{u}) \boldsymbol{\nu} \cdot \boldsymbol{w}_h \, ds.
\end{aligned} \tag{3.10}$$

Let $P_0^e(w) = \frac{1}{\mid e \mid} \int_e w \, ds$. For the rectangle subdivision $\mathcal{T}_h$ on Ω, suppose $I_h(w)$ is the continuous piecewise bilinear interpolation function defined by the value of $w \in H^2(\Omega)$ on the element vertices. Using (2.11), we may write

$$\begin{aligned}
\mid E_h(\boldsymbol{u}, \boldsymbol{w}_h) \mid &= \left| \sum_T \sum_{\substack{e \subset \partial T \\ e \not\subset \partial\Omega}} \int_e [\sigma(\boldsymbol{u}) - I_h(\sigma(\boldsymbol{u}))] \boldsymbol{\nu} \cdot [\boldsymbol{w}_h - P_0^e(\boldsymbol{w}_h)] ds \right| \\
&\leqslant \sum_T \sum_{\substack{e \subset \partial T \\ e \not\subset \partial\Omega}} \| \sigma(\boldsymbol{u}) - I_h(\sigma(\boldsymbol{u})) \|_{0,\partial T} \| \boldsymbol{w}_h - P_0^e(\boldsymbol{w}_h) \|_{0,\partial T}.
\end{aligned} \tag{3.11}$$

It follows from trace theorem, interpolation theorem[2] and Lemma 3.3 that

$$\begin{aligned}
\mid E_h(\boldsymbol{u}, \boldsymbol{w}_h) \mid &\leqslant C h^2 \mid \sigma(\boldsymbol{u}) \mid_{2,\Omega} \| \boldsymbol{w}_h \|_{1,h} \\
&\leqslant C h^2 (\mid \boldsymbol{u} \mid_{3,\Omega} + \lambda \mid \operatorname{div} \boldsymbol{u} \mid_{2,\Omega}) \| \boldsymbol{w}_h \|_h.
\end{aligned} \tag{3.12}$$

(3.6) now follows by combing (3.8), (3.9) and (3.12).

Remark 3.1 The uniform priori error estimates depend on a proper regularity result of problem (1.1) in the following form

$$\| \boldsymbol{u} \|_{k,\Omega} + \lambda \| \operatorname{div} \boldsymbol{u} \|_{k-1,\Omega} \leqslant C \| \boldsymbol{f} \|_{k,\Omega}. \tag{3.13}$$

This result has been proved for all $k \geqslant 1$ by Vogelius[7] for a smooth domain with C^∞ - boundary.

The regularity (3.13) with $k = 2$ for the case of a convex polygon was proved by Brenner and Sung[3]. The regularity is valid for $k = 3$ when the boundary is of class C^3 [14]. However, the finite element method for such domains containing curved boundaries is not the object of this paper.

References:

[1] BABUSKA I,SURI M. Locking effects in the finite element approximation of elasticity problems[J]. Numer. Math. ,1992,62:439-463.

[2] BRENNER S C,SCOTT L R. The Mathematical Theory of Finite Element Methods[M]. New York: Springer-Verlag,1994.

[3] BRENNER S C,SUNG L Y. Linear finite element methods for planar linear elasticity[J]. Math. Comp. , 1992,59(220):321-330.

[4] ARNOLD D N,BREZZI F,DOUGLAS J P. A new mixed finite element for plane elasticity[J]. Japan J. Appl. Math. ,1984,1:347-367.

[5] ARNOLD D N,DOUGLAS J P,GUPTA C P. A family of higher order mixed finite element methods for plane elasticity[J]. Numer. Math. ,1984,45:1-22.

[6] STENBERG R,SURI M. Mixed hp finite element methods for problems in elasticity and Stokes flow[J]. Numer. Math. ,1996,72:367-390.

[7] VOGELIUS M. An analysis of the p-version of finite element method for nearly incompressible materials,uniformly valid,optimal order estimates[J]. Numer. Math. ,1983,41:39-53.

[8] FALK R S. Nonconforming finite element methods for the equations of linear elasticity[J]. Math. Comp. ,1991,57:529-550.

[9] WANG Lieheng,QI He. On locking-free finite element for the planar elasticity[J]. J. Comp. Math. ,2005, 23:101-112.

[10] ZHANG Zhimin. Analysis of some quadrilateral nonconforming elements for incompressible elasticity [J]. SIAM J. Numer. Anal. ,1997,34:640-663.

[11] KOUHIA R,STENBERG R. A linear nonconforming finite element method for nearly incompressible elasticity and Stokes flow[J]. Comput. Meths. Appl. Mech. Engrg. ,1995,124:195-212.

[12] MING Pingbing,SHI Zhongci. Nonconforming rotated Q_1 element for Reissner-Mindlin plate[J]. Math. Model Meth. Appl. Scie. ,2001,11:1311-1342.

[13] BRENNER S C. Korn's inequalities for piecewise H^1 vector fields[J]. Math. Comp. ,2003,73(247): 1067-1087.

[14] GIRAULT V,RAVIART P A. Finite Element Methods for Navier-Stokes Equations,Theory and Algorithms[M]. Berlin:Springer-Verlag,1986.

纯应力平面弹性问题的一个非协调矩形元

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摘要:针对纯应力平面弹性问题构造了一个非协调矩形元. 该单元满足离散的第二 Korn 不等式,并且关于 λ 有一致最优收敛阶,其误差的能量模和 L^2-模分别为 $O(h^2)$ 和 $O(h^3)$.

关键词:平面弹性;Locking-free;第二 Korn 不等式