 密级：公开

南京师范大学

硕士学位论文

An extended finite element method for the elasticity interface problem

一种求解弹性界面问题的扩展有限元方法

研究生：曹培
指导教师：陈金如 教授
培养单位：数学科学学院
一级学科：数学
二级学科：计算数学
完成时间：2018年3月1日
答辩时间：2018年5月23日

学位论文独创性声明

本人郑重声明：所提交的学位论文是本人在导师指导下进行的研究工作和取得的研究成果。本论文中除引文外，所有实验、数据和有关材料均是真实的。本论文中除引文和致谢的内容外，不包含其他人或其它机构已经发表或撰写过的研究成果。其他同志对本研究所做的贡献均已在论文中作了声明并表示了谢意。

学位论文作者签名：曹培　　　日　　期：2018.5.24

学位论文使用授权声明

研究生在校攻读学位期间论文工作的知识产权单位属南京师范大学。学校有权保存本学位论文的电子和纸质文档，可以借阅或上网公布本学位论文的部分或全部内容，可以采用影印、复印等手段保存、汇编本学位论文。学校可以向国家有关机关或机构送交论文的电子和纸质文档，允许论文被查阅和借阅。（保密论文在解密后遵守此规定）

学位论文作者签名：曹培　　　指导教师签名：

日　　期：2018.5.24　　　日　　期：2018.5.24

Contents

摘要

本文提出了一种求解弹性界面问题的扩展有限元方法, 给出了最优先验误差估计和条件数估计, 通过数值试验验证了理论结果.

我们首先给出了混合形式下的弹性界面问题, 并介绍了弹性界面问题的扩展有限元方法, 通过引入鬼罚项, 保证了该方法的适定性与界面位置无关; 其次，给出了该方法的有界性, 强制性分析, 并由此证明了解的存在唯一性, 同时也给出了最优先验误差估计; 接着给出了条件数估计, 并证明了刚度矩阵的条件数是 $\mathcal{O}(h^{-2})$ 的, 且条件数估计的系数与界面位置无关; 最后, 通过数值试验验证了理论结果的正确性.

关键词: 弹性界面问题, 扩展有限元方法, 最优先验误差估计, 条件数估计.

Abstract

In the thesis, we propose an extended finite element method for elasticity interface problems, and present optimal a priori error estimate and the estimates of the condition number of the discrete system of equation . The method proposed is verified by numerical experiments.

We start with elasticity interface problems with a mixed form, and introduce the extended finite element method for the problem. By introducing a ghost penalty stabilization, we proved the well-posedness of the method which is independent of the location of the interface; Next, we present the analysis of boundedness and coercivity, which is necessary in the proof of the existence and uniqueness of the solution. At the same time, we also get the optimal a priori error estimate. Then, we show that the condition number of stiffness matrix is $\mathcal{O}(h^{-2})$ independent of the location of the interface.

Key words: elasticity interface problem, extended finite element method, optimal a priori error estimates, estimates of condition number.

Introduction

Interface problems have many applications in material sciences, solid mechanics and fluid dynamics, such as the stationary and non-stationary heat conduction problems with different conduction coefficients, the elasticity interface problems describing various material behaviors and the two-phase flows involving with different viscosities, etc. Interface problems usually lead to differential equations with discontinuous or non-smooth solutions across interfaces.

Due to the discontinuity of the coefficients across the interface, if using standard finite element method to solve interface problems, one usually enforces mesh lines along the interface in order to get optimal a priori error estimates. However, for many problems in which the interface evolves with time, it's very costly that one repeats remeshing of the domain to obtain fitted meshes. Therefore, developing numerical methods based on unfitted mesh of the domain is attractive. There are mainly two classes of interface-unfitted methods: the extended finite element methods(XFEM) and the immersed finite element methods(IFEM).

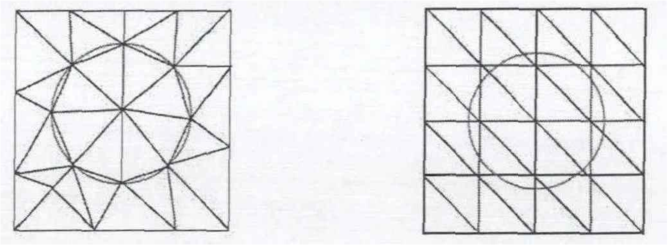

Figure 1: The skeletons of interface-fitted meshes (left) and interface-unfitted meshes (right)

Z. Li firstly proposed the immersed finite element method in 1998 ([24]). Later there are many authors studied this method (see [25], [23], [1], [26]). However, there are no error analysis for IFEM for the elasticity interface problem.

Recently, the extended finite element methods have become a popular approach. Based on a variant of Nitsche's method [27], A. Hansbo and P. Hansbo ([19]) firstly proposed an unfitted finite element method allowing for discontinuous or weakly discontinuous situations in 2002. Whereafter, the extended finite element methods represent a vivid subject of research in the field of the computational mechanics and flow problems such as [16], [29], [18], [30] and so on.

Particularly, A. Hansbo and P. Hansbo applied this method to solve the elasticity interface problem in 2004 ([20]), but excluding the incompressible case. In order to deal with the incompressible case, R. Becker, E. Burman and P. Hansbo [4] used the extended finite element spaces for both the displacement and the pressure in 2009. They obtained the optimal a priori estimates in the incompressible case, but the coefficient is independent of the mesh size h not of the ratio $\left(\frac{\mu_{max}}{\mu_{min}}\right)$. What's more, they avoided the case when the interface cuts the mesh in a way that very small sub-elements are created, which will lead to the instability of the method.

In this thesis, we consider the mixed form of the elasticity interface problem, use XFEM $P_1^b - P_1$ pair to approximate the displacement and the pressure. As for the pressure jump between adjacent elements, stabilization techniques are used to enhance the stability. In order to allow both displacement and pressure to be discontinuous across the interface, we enrich the finite element space. Meantime, in case of the instabilities because of "small cuts", we apply the ghost penalty stabilization near the interface. And we also derive an inf-sup stability result for the discrete bilinear form uniform with respect to h and the quotient $\frac{\mu_1}{\mu_2}$. Based on this, we obtain the optimal a priori estimates in energy and L^2 norms.

The outline of the thesis is as follows. In chapter 1, we introduce the extended finite element method. In chapter 2, the analysis of the extended finite element method are presented. In chapter 3, we prove that the condition number of the stiffness matrix is independent of the relation between the interface and the mesh. In chapter 4, we provide some numerical examples to verify our theoretical analysis.

Chapter 1

The extended finite element method

In this chapter, we introduce the elasticity interface problem and give the extended finite element method.

§1.1 The elasticity interface problem

Let Ω be a bounded domain in $\mathbf{R}^2$, with convex polygonal boundary $\partial\Omega$. A smooth interface defined by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ divides Ω into two open set Ω_1 and Ω_2 such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Consider the following elasticity interface problems: find a displacement $\mathbf{u}$ and a pressure p such that

$$-\mathrm{div}(2\mu\epsilon(\mathbf{u})) + \mathrm{grad}p = \mathbf{f} \quad \text{in } \Omega_1 \cup \Omega_2, \tag{1.1}$$

$$\mathrm{div}\mathbf{u} + \frac{1}{\lambda}p = 0 \quad \text{in } \Omega_1 \cup \Omega_2, \tag{1.2}$$

$$[\mathbf{u}] = 0 \quad \text{on } \Gamma, \tag{1.3}$$

$$[p\mathbf{n} - 2\mu\epsilon(\mathbf{u})\mathbf{n}] = -\sigma\kappa\mathbf{n} \quad \text{on } \Gamma, \tag{1.4}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

where, on the basis of Young's modulus E_i and Poisson's ratio ν_i, we have $\mu_i = E_i/(2(1+\nu_i))$ and $\lambda_i = E_i\nu_i/((1+\nu_i)(1-2\nu_i))$. Here, $\epsilon(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ is the strain rate tensor, σ is the surface tension coefficient, κ is the curvature of the interface, $\mathbf{n}$ is the outward pointing normal to Ω_1 on Γ, μ is a piecewise constant Lamé parameter

$$\mu = \begin{cases} \mu_1 & \text{in } \Omega_1, \\ \mu_2 & \text{in } \Omega_2, \end{cases}$$

$\mathbf{f} \in (L^2(\Omega))^2$ is given function, and $[v]|_\Gamma = (v_1 - v_2)|_\Gamma$ is the jump on the interface Γ, where $v_i = v\,|_{\Omega_i}, i = 1, 2$. Introduce the following L^2 inner product

$$(\mathbf{u}, \mathbf{v})_\Omega = \int_\Omega \mathbf{u} \cdot \mathbf{v} dx.$$

For the weak formulation, we first introduce the space

$$L^2_\mu(\Omega) := \{p \in L^2(\Omega) | \int_\Omega \mu^{-1} p(x) dx = 0\}.$$

The weak formulation of the problem can be read as follows: given $\mathbf{f} \in \mathbf{V}'$, find $(\mathbf{u}, p) \in \mathbf{V} \times Q = (H^1_0(\Omega))^2 \times L^2_\mu(\Omega)$ such that

$$B[(\mathbf{u}, p), (\mathbf{v}, q)] = L(\mathbf{v}), \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q, \tag{1.6}$$

where

$$B[(\mathbf{u}, p), (\mathbf{v}, q)] = \sum_{i=1}^{2} ((2\mu_i \epsilon(\mathbf{u}), \epsilon(\mathbf{v}))_{\Omega_i} - (\mathrm{div}\mathbf{v}, p)_{\Omega_i} + (\mathrm{div}\mathbf{u}, q)_{\Omega_i} + (\lambda_i^{-1} p, q)_{\Omega_i}),$$

$$L(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + \int_\Gamma \sigma \kappa \mathbf{v} \cdot \mathbf{n} ds.$$

§1.2 The extended finite element space

Let $\mathcal{T}_h$ be the triangulation of Ω. For any element $K \in \mathcal{T}_h$, denote the diameter of K by h_K, define $h := \underset{K \in \mathcal{T}_h}{max} h_K$. For any element $K \in \mathcal{T}_h$, let $K_i := K \cap \Omega_i$ denote the part of K in Ω_i.

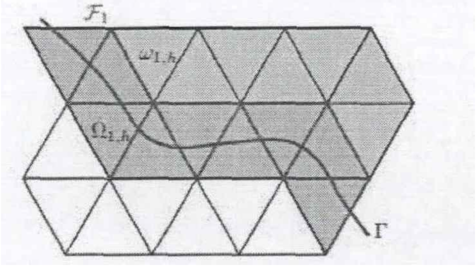

Figure 1: Set of faces $\mathcal{F}_1$ (in red lines) and subdomains $\omega_{1,h}$(in light blue triangles) and $\Omega_{1,h}$ (light and darker blue triangles)

After these preliminary considerations we are now ready to propose a finite element discretization of the problem, with this aim we introduce the following notations. By $G_h := \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$, we denote the set of elements that are intersected with the interface. For an element $K \in G_h$, let $\Gamma_K := \Gamma \cap K$.

Let the subdomains $\Omega_{i,h} := \{K \in \mathcal{T}_h \mid K \subset \Omega_i \ or \ meas(K \cap \Gamma) > 0\}, i = 1, 2$. Let $\mathcal{T}_{h,i}$ be the conforming triangulations of $\Omega_{i,h}$ such that the union of $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$ gives $\mathcal{T}_h$ and for every triangle $K \in \mathcal{T}_{h,1} \cap \mathcal{T}_{h,2}$, we have $K \cap \Gamma \neq \emptyset$. Define $\omega_{i,h} := \{K \in \Omega_{i,h} \mid meas(K \cap \Gamma) = 0\}, i = 1, 2$, Note that $\mathcal{T}_h = G_h \cup \omega_{1,h} \cup \omega_{2,h}$. Let $\mathcal{F}_i := \{e \subset \partial K \mid K \in G_h, e \subsetneq \partial\Omega_{i,h}\}, i = 1, 2$.

We make the following assumptions with respect to the mesh and the interface from [19].

- The triangulation is non-degenerate, i.e. there exists a constant $C > 0$, such that $h_K/\rho_K \leq C$, $\forall K \in \mathcal{T}_h$, where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K.
- Γ intersects with the boundary ∂K of an element K in G_h exactly twice and each (open)at most one.
- Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between Γ and ∂K. We assume that Γ_K is a function of length on $\Gamma_{K,h}$, in local coordinates:

$$\Gamma_{K,h} := \{(\xi, \eta) \mid 0 < \xi < |\Gamma_{K,h}|, \eta = 0\},$$

and

$$\Gamma_K := \{(\xi, \eta) \mid 0 < \xi < |\Gamma_{K,h}|, \eta = \delta(\xi)\}.$$

Assume the curvature of Γ is bounded, then the second assumption and the third assumption are always fulfilled on sufficiently fine meshes. Denote by $q_{h,i}$ the restriction of q_h on $\Omega_{i,h}$. Define

$$V_{h,i} := [\{v_h \in H^1(\Omega_{i,h}) \mid v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_{h,i},\ v_h|_{\partial\Omega} = 0,\ i = 1, 2\} \oplus B]^2,$$

$$Q_{h,i} := \{q_h \in H^1(\Omega_{i,h}) \mid q_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_{h,i},\ i = 1, 2\},$$

where

$$B := \{b \in C^0(\Omega_{i,h}) \mid b|_K \in P_3(K) \cap H_0^1(K),\ \forall K \in \mathcal{T}_{h,i},\ i = 1, 2\}.$$

Note that for $x \in G_h$, a given $q_h = (q_{h,1}, q_{h,2}) \in Q_{h,1} \times Q_{h,2}$ has two values: $q_{h,1}(x)$ and $q_{h,2}(x)$. We define a uni-valued function $q_h^{\Gamma} \in \mathcal{C}(\Omega_1 \cup \Omega_2)$ by

$$q_h^\Gamma = q_{h,i}(x), \text{ for } x \in \Omega_i.$$

We use a norm denoted by $\|q_h\|_{\Omega_{1,h}\cup\Omega_{2,h}} = (\|q_{h,1}\|^2_{\Omega_{1,h}} + \|q_{h,2}\|^2_{\Omega_{2,h}})^{\frac{1}{2}}$ on $Q_{h,1} \times Q_{h,2}$.

Thus, our extended mini finite element spaces are

$$\begin{aligned}
&\mathbf{V}_h := \mathbf{V}_{h,1} \times \mathbf{V}_{h,2},\\
&Q_h := (Q_{h,1} \times Q_{h,2})/R = \{q_h \in (Q_{h,1} \times Q_{h,2}), (\mu^{-1} q_h^\Gamma, 1)_\Omega = 0\}.
\end{aligned}$$

§1.3 The finite element formulation

To define the stabilization terms, we first decompose the interface zone of the triangulation $\mathcal{T}_{h,i}$ in $\mathcal{N}_{l,i}$ patches $\mathcal{P}_{l,i}$ with diameter $h_{\mathcal{P}_{l,i}} = \mathcal{O}(h)$ consisting of a moderate number of elements, in such a way that each interface element is involved in one patches $\mathcal{P}_{l,i}$. We assume that there exist constants $c_{\mathcal{P}_{l,i}}$ and $c_{h,h_{\mathcal{P}_{l,i}}}$ such that for every l, i, $c_{\mathcal{P}_{l,i}} \leq \dfrac{meas(\mathcal{P}_{l,i} \cap \Omega_i)}{meas(\mathcal{P}_{l,i})}$ and $c_{h,h_{\mathcal{P}_{l,i}}} \leq \min\limits_{K\in\mathcal{P}_{l,i}} \dfrac{h_K}{h_{\mathcal{P}_{l,i}}}$.

Under the first condition, the patches always have sufficient overlap with the physical domain Ω to ensure stability and under the second their sizes remain of the same order as the mesh size asymptotically for optimal accuracy. We then introduce the set of polynomials of order 1 on each $\mathcal{P}_{l,i}$. Denote $\Pi_{l,i} : L^2(\mathcal{P}_{l,i}) \mapsto P_1(\mathcal{P}_{l,i})$ be the L^2-projection onto $P_1(\mathcal{P}_{l,i})$.

For any function φ defined on $K \in G_h$, we define the weight average $\{\varphi\} := \omega_1\varphi_1 + \omega_2\varphi_2$ and $\{\varphi\}_* := \omega_2\varphi_1 + \omega_1\varphi_2$ with $\omega_i = \dfrac{|K_i|}{|K|}$, where $|K| := measK$. Clearly, $0 \leq \omega_i \leq 1$ and $\omega_1 + \omega_2 = 1$. Recalling the definition of $[\varphi]$, we have $[\varphi\psi] = \{\varphi\}[\psi] + [\varphi]\{\psi\}_*$.

The discrete approximation of the elasticity interface problem is to find $(\mathbf{u_h}, p_h) \in \mathbf{V}_h \times Q_h$, such that for any $(\mathbf{v_h}, q_h) \in \mathbf{V}_h \times Q_h$,

$$B_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)] + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{v_h}) + \varepsilon_p J_p(p_h, q_h) = L_h(\mathbf{v_h}), \qquad (1.7)$$

where

$$\begin{aligned}
&B_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)] = a_h(\mathbf{u_h}, \mathbf{v_h}) + b_h(\mathbf{v_h}, p_h) - b_h(\mathbf{u_h}, q_h) + c_h(p_h, q_h),\\
&L_h(\mathbf{v_h}) = (\mathbf{f}, \mathbf{v_h})_\Omega + \int_\Gamma \sigma\kappa\{\mathbf{v_h} \cdot \mathbf{n}\}_* ds,
\end{aligned}$$

with

$$\begin{aligned} a_h(\mathbf{u_h}, \mathbf{v_h}) \;&= \int_{\Omega_1 \cup \Omega_2} 2\mu\epsilon(\mathbf{u_h}) : \epsilon(\mathbf{v_h}) dx - \int_\Gamma \{2\mu\epsilon(\mathbf{u_h}) \cdot \mathbf{n}\}[\mathbf{v_h}] ds \\ &- \int_\Gamma \{2\mu\epsilon(\mathbf{v_h}) \cdot \mathbf{n}\}[\mathbf{u_h}] ds + \sum_{K \in G_h} \int_{\Gamma_K} \gamma_u h_K^{-1} \mu_{max} [\mathbf{u_h}][\mathbf{v_h}] ds, \end{aligned}$$

$$\begin{aligned} b_h(\mathbf{v_h}, p_h) &= -\int_{\Omega_1 \cup \Omega_2} p_h \mathrm{div} \mathbf{v_h} dx + \int_\Gamma \{p_h\}[\mathbf{v_h} \cdot \mathbf{n}] ds, \\ c_h(p_h, q_h) &= \int_{\Omega_1 \cap \Omega_2} \lambda^{-1} p_h q_h dx, \end{aligned}$$

$$\begin{aligned} J_u(\mathbf{u_h}, \mathbf{v_h}) \;&= \sum_{i=1}^{2} \sum_{l=1}^{\mathcal{N}_{l,i}} \mu h_{\mathcal{P}_{l,i}}^{-2} \int_{\mathcal{P}_{l,i}} (\mathbf{u_{h,i}} - \Pi_{l,i} \mathbf{u_{h,i}})(\mathbf{v_{h,i}} - \Pi_{l.i} \mathbf{v_{h,i}}) dx, \\ J_p(p_h, q_h) \;&= \sum_{i=1}^{2} j_i(p_{h,i}, q_{h,i}) = \sum_{i=1}^{2} \mu_i^{-1} h^3 \sum_{e \in \mathcal{F}_i} ([\nabla p_{h,i}], [\nabla q_{h,i}])_e. \end{aligned}$$

In Eq.(1.7) ε_u and ε_p are positive constants.

Chapter 2

Analysis of the extended finite element method

Since the stabilization terms $J_u(\mathbf{u_h}, \mathbf{v_h})$, $J_p(p_h, q_h)$ are not the residual of the equations, the finite element formulation (1.7) is not consistent. We have the following weak consistent relation.

Lemma 2.1. Assume $(\mathbf{u}, p) \in (H^2(\Omega_1 \cap \Omega_2) \cap H_0^1(\Omega))^2 \times (H^1(\Omega_1 \cup \Omega_2) \cap L_\mu^2(\Omega))$ be the solution of the problem (1.6) and $(\mathbf{u_h}, \mathbf{p_h}) \in \mathbf{V}_h \times Q_h$ be the solution of the discrete approximate problem (1.7). Then for any $(\mathbf{v_h}, q_h) \in \mathbf{V}_h \times Q_h$,

$$B_h[(\mathbf{u} - \mathbf{u_h}, p - p_h), (\mathbf{v_h}, q_h)] = \varepsilon_u J_u(\mathbf{u_h}, \mathbf{v_h}) + \varepsilon_p J_p(p_h, q_h). \tag{2.1}$$

Proof. Multiplying (1.1) and (1.2) by testing functions $\mathbf{v_h}$ and q_h, respectively, using integration by parts and noting that the interface conditions (1.3) and (1.4), we obtain

$$a_h(\mathbf{u}, \mathbf{v_h}) + b_h(\mathbf{v_h}, p) - b_h(\mathbf{u}, q_h) + c_h(p, q_h) = L_h(\mathbf{v_h}). \tag{2.2}$$

Subtracting (1.7) from (2.2), we get (2.1). ∎

In the following analysis, we need to introduce some mesh dependent norms, i.e.,

$$
\begin{aligned}
\|\mathbf{v}\|_{\frac{1}{2},h,\Gamma} :=& (\sum_{K \in G_h} h_K^{-1} \|\mathbf{v}\|_{0,\Gamma_K}^2)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{V_h}, \\
\|\mathbf{v}\|_{-\frac{1}{2},h,\Gamma} :=& (\sum_{K \in G_h} h_K \|\mathbf{v}\|_{0,\Gamma_K}^2)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{V_h}, \\
|||\mathbf{v}||| :=& (\|\mu^{\frac{1}{2}} \epsilon(\mathbf{v})\|_{0,\Omega_1 \cup \Omega_2}^2 + \|\mu_{max}^{\frac{1}{2}} [\mathbf{v}]\|_{\frac{1}{2},h,\Gamma}^2 \\
&+ \|\mu_{max}^{-\frac{1}{2}} \{2\mu\epsilon(\mathbf{v}) \cdot \mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}^2)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{V_h}, \\
|||\mathbf{v_h}|||_h :=& (\|\mu^{\frac{1}{2}} \epsilon(\mathbf{v_h})\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}^2 + \|\mu_{max}^{\frac{1}{2}} [\mathbf{v_h}]\|_{\frac{1}{2},h,\Gamma}^2 \\
&+ \|\mu_{max}^{-\frac{1}{2}} \{2\mu\epsilon(\mathbf{v_h}) \cdot \mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}^2 + J_u(\mathbf{u_h}, \mathbf{v_h}))^{\frac{1}{2}}, \quad \forall \mathbf{v_h} \in \mathbf{V_h}, \\
|||(\mathbf{v}, q)||| :=& (|||\mathbf{v}|||^2 + \|\mu^{-\frac{1}{2}} q\|_{0,\Omega_1 \cup \Omega_2}^2 + \|\mu_{max}^{-\frac{1}{2}} \{p\}\|_{-\frac{1}{2},h,\Gamma}^2)^{\frac{1}{2}}, \quad \forall (\mathbf{v}, q) \in (\mathbf{V_h}, Q_h), \\
|||(\mathbf{v_h}, q_h)|||_h :=& (|||\mathbf{v_h}|||_h^2 + \|\mu^{-\frac{1}{2}} q_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}^2 + J_p(q_h, q_h))^{\frac{1}{2}}, \quad \forall (\mathbf{v_h}, q_h) \in (\mathbf{V_h}, Q_h).
\end{aligned}
$$

We denote the inner products in $L^2(\Omega)$ by $(\cdot,\cdot)_\Omega$. It's obvious

$$(\mathbf{u},\mathbf{v})_\Gamma \le \|\mathbf{u}\|_{\frac{1}{2},h,\Gamma}\|\mathbf{v}\|_{-\frac{1}{2},h,\Gamma}. \tag{2.3}$$

§2.1 Continuity analysis

Lemma 2.2. For all $\mathbf{u_h}\in\mathbf{V_h}$, we have

$$a_h(\mathbf{u_h},\mathbf{v_h}) \le C_1|||\mathbf{u_h}|||\,|||\mathbf{v_h}|||, \quad \forall\mathbf{v_h}\in\mathbf{V_h}, \tag{2.4}$$

where C_1 is a positive constant independent of μ_1, μ_2 and the mesh size.

Proof. By the definition of $a_h(\mathbf{u_h},\mathbf{v_h})$ and (2.3), we get that

$$\begin{aligned}
a_h(\mathbf{u_h},\mathbf{v_h}) =& \int_{\Omega_1\cup\Omega_2} 2\mu\epsilon(\mathbf{u_h}):\epsilon(\mathbf{v_h})dx - \int_\Gamma \{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}[\mathbf{v_h}]ds \\
&- \int_\Gamma \{2\mu\epsilon(\mathbf{v_h})\cdot\mathbf{n}\}[\mathbf{u_h}]ds + \sum_{K\in G_h}\int_{\Gamma_K}\gamma_u h_K^{-1}\mu_{max}[\mathbf{u_h}][\mathbf{v_h}]ds \\
\le& 2\|\mu^{\frac{1}{2}}\epsilon(\mathbf{u_h})\|_{0,\Omega_1\cup\Omega_2}\|\mu^{\frac{1}{2}}\epsilon(\mathbf{v_h})\|_{0,\Omega_1\cup\Omega_2} \\
&+ \|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}\cdot\|\mu_{max}^{\frac{1}{2}}[\mathbf{v_h}]\|_{\frac{1}{2},h,\Gamma} \\
&+ \|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{v_h})\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}\cdot\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|_{\frac{1}{2},h,\Gamma} \\
&+ \gamma_u\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|_{\frac{1}{2},h,\Gamma}\cdot\|\mu_{max}^{\frac{1}{2}}[\mathbf{v_h}]\|_{\frac{1}{2},h,\Gamma} \\
\le& C|||\mathbf{u_h}|||\,|||\mathbf{v_h}|||,
\end{aligned}$$

Hence (2.4) follows. ∎

Lemma 2.3. For any $q_h\in Q_h$, we have

$$\|\mu_{max}^{-\frac{1}{2}}\{q_h\}\|_{-\frac{1}{2},h,\Gamma} \le C_2\|\mu_{max}^{-\frac{1}{2}}q_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}},$$

where C_2 is a positive constant independent of μ_1, μ_2 and the mesh size.

Proof. By the definition, the trace theorems and the inverse inequality,

$$\begin{aligned}
\|\mu_{max}^{-\frac{1}{2}}\{q_h\}\|^2_{-\frac{1}{2},h,\Gamma} &\le C\sum_{K\in G_h} h_K(h_K^{-1}\|\mu_{max}^{-\frac{1}{2}}q_h\|^2_{0,K} + h_K|\mu_{max}^{-\frac{1}{2}}q_h|^2_{1,K}) \\
&\le C\sum_{K\in G_h} h_K\cdot h_K^{-1}\|\mu_{max}^{-\frac{1}{2}}q_h\|^2_{0,K} \\
&\le C_2\|\mu_{max}^{-\frac{1}{2}}q_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}.
\end{aligned}$$

Therefore, we complete the proof. ∎

Lemma 2.4. For any $\mathbf{u_h} \in \mathbf{V_h}, \mathbf{v_h} \in \mathbf{V_h}, p_h \in Q_h, q_h \in Q_h$, we have

$$B_h(\mathbf{u_h}, p_h; \mathbf{v_h}, q_h) + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{v_h}) + \varepsilon_p J_p(p_h, q_h) \leq C_B |||(\mathbf{u_h}, p_h)|||_h \cdot |||(\mathbf{v_h}, q_h)|||_h,$$

where C_B is a positive constant independent of μ_1, μ_2 and the mesh size.

Proof. By lemma 2.2, we know that

$$a_h(\mathbf{u_h}, \mathbf{v_h}) \leq C_1 |||\mathbf{u_h}||| |||\mathbf{v_h}|||, \quad \forall \mathbf{v_h} \in \mathbf{V_h}. \tag{2.5}$$

Using Cauchy-Schwarz inequality and lemma 2.3,

$$\begin{aligned} b_h(\mathbf{v_h}, p_h) &= -\int_{\Omega_1 \cup \Omega_2} p_h \mathrm{div}\mathbf{v_h} dx + \int_\Gamma \{p_h\}[\mathbf{v_h} \cdot \mathbf{n}] ds \\ &\leq \|\mu^{\frac{1}{2}} \mathrm{div}\mathbf{v_h}\|_{0,\Omega_1 \cup \Omega_2} \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_1 \cup \Omega_2} + \|\mu_{max}^{\frac{1}{2}} [\mathbf{v_h}]\|_{\frac{1}{2},h,\Gamma} \|\mu_{max}^{-\frac{1}{2}} \{p_h\}\|_{-\frac{1}{2},h,\Gamma} \\ &\lesssim |||\mathbf{v_h}||| \cdot (\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_1 \cup \Omega_2} + \|\mu_{max}^{-\frac{1}{2}} \{p_h\}\|_{-\frac{1}{2},h,\Gamma}) \\ &\lesssim |||\mathbf{v_h}||| \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}. \end{aligned}$$

Similarly,

$$b_h(\mathbf{u_h}, q_h) \lesssim |||\mathbf{u_h}||| \|\mu^{-\frac{1}{2}} q_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} c_h(p_h, q_h) &= \int_{\Omega_1 \cup \Omega_2} \lambda^{-1} p_h q_h dx \lesssim \lambda^{-1} \|p_h\|_{0,\Omega_1 \cup \Omega_2} \|q_h\|_{0,\Omega_1 \cup \Omega_2} \\ &\lesssim \|p_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}} \|q_h\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}. \end{aligned}$$

Noting

$$B_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)] = a_h(\mathbf{u_h}, \mathbf{v_h}) + b_h(\mathbf{v_h}, p_h) - b_h(\mathbf{u_h}, q_h) + c_h(p_h, q_h).$$

Combining above estimates, we obtain

$$B_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)] \leq C_B |||(\mathbf{u_h}, p_h)||| \cdot |||(\mathbf{v_h}, q_h)|||, \tag{2.6}$$

and thus we finish the proof. ∎

§2.2 Stability analysis

Lemma 2.5. There exists a positive constant C_I independent of μ_1 and μ_2 such that

$$\|\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} \le C_I\mu_{max}\|\mu^{\frac{1}{2}}\epsilon(\mathbf{u_h})\|^2_{0,\Omega_1\cup\Omega_2}, \quad \forall \mathbf{u_h}\in\mathbf{V_h}.$$

Proof. By the definition of $\|\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}$ and the scaling argument

$$\begin{aligned}\|\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} &\le \sum_{K\in G_h}\sum_{i=1}^{2} 2h\frac{|K_i|^2}{|K|^2}\int_{\Gamma_K}|2\mu\epsilon(\mathbf{u_{h,i}})|^2dx\\ &\le \sum_{K\in G_h}\sum_{i=1}^{2} 2h\frac{|K_i|^2}{|K|^2}|\Gamma_K|\cdot\|2\mu\epsilon(\mathbf{u_{h,i}})\|^2_{L^\infty(K_i)}\\ &\le \sum_{K\in G_h}\sum_{i=1}^{2} 2h\frac{|K_i|}{|K|^2}|\Gamma_K|\cdot\|2\mu\epsilon(\mathbf{u_{h,i}})\|^2_{L^2(K_i)}\\ &\le C_I\mu_{max}\|\mu^{\frac{1}{2}}\epsilon(\mathbf{u_h})\|^2_{0,\Omega_1\cup\Omega_2}.\end{aligned}$$

The result follows. ∎

Lemma 2.6. There exists a positive constant C_3 independent of μ_1 and μ_2 such that

$$a_h(\mathbf{u_h},\mathbf{u_h}) + \varepsilon_u J_u(\mathbf{u_h},\mathbf{u_h}) \ge C_3|||\mathbf{u_h}|||_h^2, \quad \forall \mathbf{u_h}\in\mathbf{V_h}.$$

Proof. By theorem 2.12 in [9], we know

$$\|\epsilon(\mathbf{v})\|^2_{0,\Omega_1\cup\Omega_2} + \|[\mathbf{v}]\|^2_{\frac{1}{2},h,\Gamma} \ge C|\mathbf{v}|^2_{1,\Omega_1\cup\Omega_2}, \quad \forall \mathbf{v}\in\mathbf{V}_h. \tag{2.7}$$

Next, using the definition of $a_h(\mathbf{u_h},\mathbf{u_h})$ and Cauchy-Schwarz inequality, it follows

$$\begin{aligned}a_h(\mathbf{u_h},\mathbf{u_h}) \ge& 2\|\mu^{\frac{1}{2}}\epsilon(\mathbf{u_h})\|^2_{0,\Omega_1\cup\Omega_2}\\ &- \|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|_{\frac{1}{2},h,\Gamma} + \gamma_u\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|^2_{\frac{1}{2},h,\Gamma}\\ \ge& 2\|\mu^{\frac{1}{2}}\epsilon(\mathbf{u_h})\|^2_{0,\Omega_1\cup\Omega_2} - \frac{1}{\varepsilon}\|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma}\\ &- \varepsilon\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|^2_{\frac{1}{2},h,\Gamma} + \gamma_u\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|^2_{\frac{1}{2},h,\Gamma}\\ \ge& C_1|\mu^{\frac{1}{2}}\mathbf{u_h}|^2_{1,\Omega_1\cup\Omega_2} - \frac{1}{\varepsilon}\|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} + (C_2-\varepsilon)\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|^2_{\frac{1}{2},h,\Gamma}.\end{aligned}$$

According to lemma 4.2 in [12],

$$\alpha_s\|\nabla \mathbf{u_h}\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq \|\nabla \mathbf{u_h}\|^2_{0,\Omega_1\cup\Omega_2} + J_u(\mathbf{u_h},\mathbf{u_h}), \quad \forall \mathbf{u_h}\in \mathbf{V}_h.$$

Therefore, by lemma 2.5 and the inequality above, we get

$$\begin{aligned}
a_h(\mathbf{u_h},\mathbf{u_h}) + \varepsilon_u J_u(\mathbf{u_h},\mathbf{u_h}) \geq & C_1|\mu^{\frac{1}{2}}\mathbf{u_h}|^2_{1,\Omega_{1,h}\cup\Omega_{2,h}} + \alpha_s(\varepsilon_u - C_1)J_u(\mathbf{u_h},\mathbf{u_h}) \\
& + \frac{1}{\varepsilon}\|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} \\
& - \frac{2}{\varepsilon}\|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} + (C_2-\varepsilon)\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|^2_{\frac{1}{2},h,\Gamma} \\
\geq & (C_1 - \frac{2C_I}{\varepsilon})\|\mu^{\frac{1}{2}}\epsilon(\mathbf{u_h})\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + (C_2-\varepsilon)\|\mu_{max}^{\frac{1}{2}}[\mathbf{u_h}]\|^2_{\frac{1}{2},h,\Gamma} \\
& + \frac{1}{\varepsilon}\|\mu_{max}^{-\frac{1}{2}}\{2\mu\epsilon(\mathbf{u_h})\cdot\mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} + \alpha_s(\varepsilon_u - C_1)J_u(\mathbf{u_h},\mathbf{u_h}) \\
\geq & C_3|||\mathbf{u_h}|||^2_h.
\end{aligned}$$

Finally, the proof is finished. ∎

Lemma 2.7. Let $p_h = (p_{h,1}, p_{h,2}) \in Q_h$, there exists a constant $C > 0$ such that

$$\|\mu_i^{-\frac{1}{2}}p_{h,i}\|^2_{0,\Omega_{i,h}} \leq C(\|\mu_i^{-\frac{1}{2}}p_{h,i}\|^2_{0,\omega_{i,h}} + j_i(p_{h,i},p_{h,i})). \tag{2.8}$$

Proof. Note that

$$\|p_{h,i}\|^2_{0,\Omega_{i,h}} = \|p_{h,i}\|^2_{0,\omega_{i,h}} + \sum_{K\in G_h}\|p_{h,i}\|^2_{0,K}. \tag{2.9}$$

For the simplicity, we write $p = p_{h,i}$, which is a piecewise linear function on $\Omega_{i,h}$. Take $K_0 = K \in G_h$ and $x \in K_0$. There is a sequence of $\{K_j\}_{j=1}^s$ with faces $e_j = \bar{K}_j \cap \bar{K}_{j-1}, j = 1, ..., s$, and $K_s \in \omega_{i,h}$. The number s is usually bounded. We denote m_j be the barycenter of e_j. With an appropriate orientation of the jump operator $[\cdot]_e$, we have

$$\sum_{j=1}^s [\nabla p]_{e_j} = \nabla p|_{T_s} - \nabla p|_{T_0},$$

and

$$\sum_{j=1}^s [\nabla p]_{e_j}\cdot m_j = p(m_1) - p(m_s) + \nabla p|_{K_s}\cdot m_s - \nabla p|_{K_0}\cdot m_1.$$

Using above equalities, for $x \in K_0$, expressing p Taylor expansion around m_1, we obtain

$$\begin{aligned} p(x) &= p(m_1) + \nabla p|_{K_0} \cdot (x - m_1) \\ &= p(m_s) + \nabla p|_{K_s} \cdot (x - m_s) + \sum_{j=1}^{s} [\nabla p]_{e_j} \cdot (m_j - x). \end{aligned}$$

Let ξ, η be the barycenter of K_0, K_1 respectively, then by Taylor expansion, we get

$$\begin{aligned} p(\xi) &= p(m_1) + \nabla p|_{K_0} \cdot (\xi - m_1) \\ &= p(\eta) + \nabla p|_{K_0} \cdot (\xi - m_1) + \nabla p|_{K_1} \cdot (m_1 - \eta). \end{aligned} \tag{2.10}$$

Taking the square and multiplying $|K_0|$ on the two sides of (2.10),

$$|p(\xi)|^2 |K_0| \leq \frac{|K_0|}{|K_1|} |K_1| |p(\eta)|^2 + |K_0| (|\nabla p|_{K_0} (\xi - m_1) + |\nabla p|_{K_1} (m_1 - \eta))^2.$$

Thus, we have

$$\|p\|_{0,K_0}^2 \leq \frac{|K_0|}{|K_1|} \|p\|_{0,K_1}^2 + h^3 \|[\nabla p \cdot n_{e_j}]\|_{0,e_1}^2.$$

Using the same technique as above, the estimates $\|x - m_j\| \leq Ch_{e_j}$ and $|K_0| = |K_j|, j = 1, ..., s$, we obtain

$$\|p\|_{0,K_0}^2 \leq c(\|p\|_{0,K_s}^2 + \sum_{j=1}^{s} h_{e_j}^3 \|[\nabla p \cdot n_{e_j}]\|_{0,e_j}^2).$$

Summing over $K_0 = K \in G_h$, it follows

$$\begin{aligned} \sum_{K \in G_h} \|p_{h,i}\|_{0,K}^2 &\leq c(\|p_{h,i}\|_{0,\omega_{i,h}}^2 + \sum_{e \in \mathcal{F}_i} h_e^3 \|[\nabla p_{h,i} \cdot n_e]\|_{0,e}^2) \\ &\leq c(\|p_{h,i}\|_{0,\omega_{i,h}}^2 + \mu_i j_i(p_{h,i}, p_{h,i})), \end{aligned}$$

which completes the proof. ∎

To prove the inf-sup stability of $b_h(\cdot, \cdot)$, we need the following decomposition of our extended finite element space. Define

$$\overline{p} = \begin{cases} \mu_1 |\Omega_1|^{-1} & \text{on } \Omega_1, \\ \mu_2 |\Omega_2|^{-1} & \text{on } \Omega_2. \end{cases}$$

Let M_0=span$\{\bar{p}\}$. For each $p_h \in Q_h$ we can write

$$p_h = p_0 + p_0^{\perp}, \quad p_0 \in M_0, \quad p_0^{\perp} \in M_0^{\perp}.$$

where $M_0^{\perp} = \{p_0^{\perp} \in Q_h \mid (p_0^{\perp}, 1)_{\Omega_i} = 0, i = 1, 2\}$,

Lemma 2.8. For any $p_0 \in M_0$ and sufficiently small h, there exist $\mathbf{v_0} \in \mathbf{V_h}$ such that

$$b_h(\mathbf{v_0}, p_0) \geq C_{1,p_0} \|\mu^{-\frac{1}{2}} p_0\|^2_{0,\Omega_1 \cup \Omega_2},$$

and

$$|||\mathbf{v_0}|||_h \leq C_{2,p_0} \|\mu^{-\frac{1}{2}} p_0\|_{0,\Omega_1 \cup \Omega_2},$$

where C_{1,p_0}, C_{2,p_0} are positive constants independent of μ_1 and μ_2.

Proof. Let $\tilde{p}_0 = \mu^{-1} p_0 = (|\Omega_1|^{-1}, -|\Omega_2|^{-1}) \in Q_{1,h} \times Q_{2,h}$, thus we know that

$$(\mu^{-1} p_0, 1)_\Omega = (\tilde{p}_0, 1)_\Omega = 0.$$

Denote by $I(\tilde{p}_0)$ the continuous piecewise linear nodal interpolation of $\tilde{p}_0$ which differs from $\tilde{p}_0$ only in elements $K \in \mathcal{T}_h^{\Gamma}$. Let $q_h = I(\tilde{p}_0) - \alpha$, where

$$\alpha = \frac{(I(\tilde{p}_0), 1)_\Omega}{\|1\|^2_{0,\Omega_1 \cup \Omega_2}},$$

It's easy to see that $(q_h, 1)_{\Omega_1 \cup \Omega_2} = 0$, and q_h is continuous in $\Omega_1 \cup \Omega_2$. Since the continuous finite element spaces are inf-sup stable, there exist $\mathbf{v_0} \in \mathbf{V_h} \cap C(\Omega_1 \cup \Omega_2)$, such that

$$\frac{b_h(\mathbf{v_0}, q_h)}{\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}} \geq C \|q_h\|_{0,\Omega_1 \cup \Omega_2}. \tag{2.11}$$

Therefore, by the continuity of $b_h(\mathbf{v_0}, \tilde{p}_0 - q_h)$ and the triangle inequality,

$$\begin{aligned}
\frac{b_h(\mathbf{v_0}, \tilde{p}_0)}{\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}} &= \frac{b_h(\mathbf{v_0}, q_h) + b_h(\mathbf{v_0}, \tilde{p}_0 - q_h)}{\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h} \cup \Omega_{2,h}}} \\
&\geq C \|q_h\|_{0,\Omega_1 \cup \Omega_2} - \sqrt{2} \|\tilde{p}_0 - q_h\|_{0,\Omega_1 \cup \Omega_2} \\
&\geq C \|\tilde{p}_0\|_{0,\Omega_1 \cup \Omega_2} - (C + \sqrt{2}) \|\tilde{p}_0 - q_h\|_{0,\Omega_1 \cup \Omega_2}.
\end{aligned}$$

Using the definitions of $\tilde{p}_0$, q_h, α and the triangle inequality, we get

$$\|\tilde{p}_0 - q_h\|_{0,\Omega_1 \cup \Omega_2} = \|\tilde{p}_0 - I(\tilde{p}_0) + \alpha\|_{0,\Omega_1 \cup \Omega_2} \leq \|I(\tilde{p}_0) - \tilde{p}_0)\|_{0,\Omega_1 \cup \Omega_2} + \|\alpha\|_{0,\Omega_1 \cup \Omega_2}.$$

and

$$|\alpha| = \frac{|(I(\tilde{p}_0), 1)_\Omega|}{\|1\|^2_{0,\Omega_1\cup\Omega_2}} = \frac{|(I(\tilde{p}_0) - \tilde{p}_0, 1)_\Omega|}{\|1\|^2_{0,\Omega_1\cup\Omega_2}} \le \frac{\|I(\tilde{p}_0) - \tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}}{\|1\|_{0,\Omega_1\cup\Omega_2}},$$

then

$$\frac{\|\tilde{p}_0 - q_h\|_{0,\Omega_1\cup\Omega_2}}{\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}} \le 2\frac{\|I(\tilde{p}_0) - \tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}}{\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}} \le ch^{\frac{1}{2}},$$

and hence

$$\begin{aligned}\frac{b_h(\mathbf{v_0}, \tilde{p}_0)}{\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}} &\ge (C - (C + \sqrt{2})\frac{\|\tilde{p}_0 - q_h\|_{0,\Omega_1\cup\Omega_2}}{\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}})\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2} \\ &\ge (C - ch^{\frac{1}{2}})\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2} \\ &= C_1\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}.\end{aligned}$$

From the definition of M_0, we can see

$$\|\mu^{-\frac{1}{2}}p_0\|^2_{0,\Omega_1\cup\Omega_2} = C(\mu,\Omega)\|\tilde{p}_0\|^2_{0,\Omega_1\cup\Omega_2}, \text{ with } C(\mu,\Omega) = \frac{\mu_1|\Omega_1|^{-1} + \mu_2|\Omega_2|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}}.$$

For any $\mathbf{v_0} \in \mathbf{V_h} \cap \mathcal{C}(\Omega_1 \cup \Omega_2)$, by Green formula,

$$\int_{\Omega_1} \mathrm{div}\mathbf{v_0} dx + \int_{\Omega_2} \mathrm{div}\mathbf{v_0} dx = 0, \tag{2.12}$$

we have

$$b_h(\mathbf{v_0}, p_0) = C(\mu,\Omega) b_h(\mathbf{v_0}, \tilde{p}_0).$$

Choosing $\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} = \|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}$, we get

$$\begin{aligned} b_h(\mathbf{v_0}, p_0) &= C(\mu,\Omega) b_h(\mathbf{v_0}, \tilde{p}_0) \ge C_1 C(\mu,\Omega)\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2} \\ &= C_1 C(\mu,\Omega)\|\tilde{p}_0\|^2_{0,\Omega_1\cup\Omega_2} \\ &= C_{1,p_0}\|\mu^{-\frac{1}{2}}p_0\|^2_{0,\Omega_1\cup\Omega_2}, \end{aligned} \tag{2.13}$$

and

$$|||\mathbf{v_0}|||_h = \|\mu^{\frac{1}{2}}\epsilon(\mathbf{v_0})\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \le C\mu^{\frac{1}{2}}_{max}\|\nabla \mathbf{v_0}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} = C\mu^{\frac{1}{2}}_{max}\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2}.$$

Since

$$C(\mu,\Omega) = \frac{\mu_1|\Omega_1|^{-1} + \mu_2|\Omega_2|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} \ge \mu_{max}\min_{i=1,2}\frac{|\Omega_1|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} = \tilde{c}\mu_{max},$$

we have

$$|||\mathbf{v_0}|||_h \le C\mu_{max}^{\frac{1}{2}}\|\tilde{p}_0\|_{0,\Omega_1\cup\Omega_2} \le C\tilde{c}^{-\frac{1}{2}}\|\mu^{-\frac{1}{2}}p_0\|_{0,\Omega_1\cup\Omega_2}.$$

Finally, note that for $\mathbf{v_0} = (\mathbf{v_{h,1}}, \mathbf{v_{h,2}}) \in \mathbf{V_h} \cap \mathcal{C}(\Omega_1 \cup \Omega_2)$, we can choose $\mathbf{v_{h,i}} = \mathbf{v_{h,j}}$ in $\Omega_{i,h} \cap \Omega_j, j \ne i$, so that

$$|||\mathbf{v_0}|||_h \le C|||\mathbf{v_0}||| \le C_{2,p_0}\|\mu^{-\frac{1}{2}}p_0\|_{0,\Omega_1\cup\Omega_2}.$$

The proof is completed. ∎

Lemma 2.9. For any $p_0^\perp \in M_0^\perp$ and sufficiently small h, there exist $\mathbf{v_0^\perp} \in M_0^\perp$, such that

$$b_h(\mathbf{v_0^\perp}, p_0^\perp) \ge C_{1,p_0^\perp}\|\mu^{-\frac{1}{2}}p_0^\perp\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - C_{2,p_0^\perp}J_p(p_0^\perp, p_0^\perp),$$

and

$$|||\mathbf{v_0^\perp}|||_h \le C_{3,p_0^\perp}\|\mu^{-\frac{1}{2}}p_0^\perp\|_{0,\Omega_{1,h}\cup\Omega_{2,h}},$$

where $C_{1,p_0^\perp}, C_{2,p_0^\perp}, C_{3,p_0^\perp}$ are positive constants independent of μ_1 and μ_2.

Proof. For $p_0^\perp = (p_{h,1}^\perp, p_{h,2}^\perp) \in M_0^\perp$, let $q_{h,i} = p_{h,i}^\perp - \alpha_i$, where

$$\alpha_i = \frac{(p_{h,i}^\perp, 1)_{\omega_{i,h}}}{|\omega_{i,h}|}.$$

We can see $(q_{h,i}, 1)_{\omega_{i,h}} = 0$. Since the underlying finite element space is inf-sup stable, we have that for any $q_{h,i} \in Q_{h,i}$, there exist $\mathbf{v_{h,i}^\perp} \in \mathbf{V_{h,i}}$ with $supp(\mathbf{v_{h,i}^\perp}) \subset \bar{\omega}_{i,h}$ and $\|\nabla \mathbf{v_{h,i}^\perp}\|_{0,\omega_{i,h}} = \|q_{h,i}\|_{0,\omega_{i,h}}$, such that

$$b_h(\mathbf{v_{h,i}^\perp}, q_{h,i}) \ge C\|q_{h,i}\|^2_{0,\omega_{i,h}}. \tag{2.14}$$

Noting $\mathbf{v_{h,i}^\perp} = 0$ on $\partial\Omega$ and $p_{h,i}^\perp - q_{h,i} = \alpha_i$ is a constant, by Green formula, we have

$$b_h(\mathbf{v_{h,i}^\perp}, q_{h,i}) = b_h(\mathbf{v_{h,i}}, p_{h,i}^\perp).$$

Thus, by (2.14) and lemma 2.7,

$$\begin{aligned} b_h(\mu_i^{-1}\mathbf{v_{h,i}^\perp}, q_{h,i}) &= b_h(\mu_i^{-1}\mathbf{v_{h,i}^\perp}, p_{h,i}^\perp) \\ &\ge C\mu_i^{-1}\|q_{h,i}\|^2_{0,\omega_{i,h}} = C\|\mu_i^{-\frac{1}{2}}q_{h,i}\|^2_{0,\omega_{i,h}} \\ &\ge C_1\|\mu_i^{-\frac{1}{2}}q_{h,i}\|^2_{0,\Omega_{i,h}} - j_i(q_{h,i}, q_{h,i}). \end{aligned}$$

Since $j_i(p_{h,i}^\perp, p_{h,i}^\perp)$ depends only on $\nabla q_{h,i}^\perp$, we have $j_i(p_{h,i}^\perp, p_{h,i}^\perp) = j_i(q_{h,i}, q_{h,i})$, therefore

$$b_h(\mu_i^{-1}\mathbf{v}_{\mathbf{h},\mathbf{i}}^\perp, p_{h,i}^\perp) \geq C_1\|\mu_i^{-\frac{1}{2}}q_{h,i}\|_{0,\Omega_{i,h}}^2 - j_i(p_{h,i}^\perp, p_{h,i}^\perp). \tag{2.15}$$

For $p_0^\perp \in M_0^\perp$, noting $(p_{h,i}^\perp, 1)_{\Omega_i} = 0$, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\alpha_i| &= \frac{|(p_{h,i}^\perp, 1)_{\omega_{i,h}}|}{|\omega_{i,h}|} = \frac{1}{|\omega_{i,h}|} \mid \int_{\Omega_i\setminus\omega_{i,h}} p_{h,i}^\perp dx \mid \\ &\leq \frac{|\Omega_i \setminus \omega_{i,h}|^{\frac{1}{2}}}{|\omega_{i,h}|}\|p_{h,i}^\perp\|_{0,\Omega_{i,h}} \leq ch^{\frac{1}{2}}\|p_{h,i}^\perp\|_{0,\Omega_{i,h}}. \end{aligned} \tag{2.16}$$

Thus, for sufficiently small h, there exists a constant $c > 0$ such that

$$\|q_{h,i}\|_{0,\Omega_{i,h}} \geq \|p_{h,i}^\perp\|_{0,\Omega_{i,h}} - c|\alpha_i| \geq \|p_{h,i}^\perp\|_{0,\Omega_{i,h}}(1 - ch^{\frac{1}{2}}) \geq c\|p_{h,i}^\perp\|_{0,\Omega_{i,h}}. \tag{2.17}$$

From (2.16) and (2.17),

$$b_h(\mu_i^{-1}\mathbf{v}_{\mathbf{h},\mathbf{i}}^\perp, p_{h,i}^\perp) \geq C_1\|\mu_i^{-\frac{1}{2}}p_{h,i}^\perp\|_{0,\Omega_{i,h}}^2 - j_i(p_{h,i}^\perp, p_{h,i}^\perp). \tag{2.18}$$

Taking $\mathbf{v}_\mathbf{0}^\perp = \mu_1^{-1}\mathbf{v}_{\mathbf{h},\mathbf{1}}^\perp + \mu_2^{-1}\mathbf{v}_{\mathbf{h},\mathbf{2}}^\perp \in \mathbf{V}_\mathbf{h}(\omega_{1,h} \cup \omega_{2,h})$, we get

$$\begin{aligned} b_h(\mathbf{v}_\mathbf{0}^\perp, p_0^\perp) &= b_h(\mu_1^{-1}\mathbf{v}_{\mathbf{h},\mathbf{1}}^\perp, p_{h,1}^\perp) + b_h(\mu_2^{-1}\mathbf{v}_{\mathbf{h},\mathbf{2}}^\perp, p_{h,2}^\perp) \\ &\geq C_{1,p_0^\perp}\|\mu^{-\frac{1}{2}}p_0^\perp\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 - C_{2,p_0^\perp}J_p(p_0^\perp, p_0^\perp). \end{aligned}$$

By the triangular inequality and (2.16),

$$\|\nabla\mathbf{v}_{\mathbf{h},\mathbf{i}}^\perp\|_{0,\omega_{i,h}} = \|q_{h,i}\|_{0,\omega_{i,h}} \leq \|p_{h,i}^\perp\|_{0,\omega_{i,h}} + c|\alpha_i| \leq c\|p_{h,i}^\perp\|_{0,\Omega_{i,h}},$$

and thus,

$$\begin{aligned} |||\mathbf{v}_\mathbf{0}^\perp|||_h^2 &= \sum_{i=1}^2 \|\mu^{-\frac{1}{2}}\epsilon(\mathbf{v}_{\mathbf{h},\mathbf{i}}^\perp)\|_{0,\omega_{i,h}}^2 \leq \sum_{i=1}^2 \mu^{-1}\|\nabla\mathbf{v}_{\mathbf{h},\mathbf{i}}^\perp\|_{0,\omega_{i,h}}^2 \\ &\leq c\sum_{i=1}^2 \mu^{-1}\|p_{h,i}^\perp\|_{0,\Omega_{i,h}} = c\|\mu^{-\frac{1}{2}}p_0^\perp\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2. \end{aligned}$$

The desired result follows. ∎

Lemma 2.10. For any $p_h \in Q_h$ and sufficiently small h, there exist $\mathbf{v_h} \in \mathbf{V_h}$ such that

$$b_h(\mathbf{v_h}, p_h) \geq C_{1,h}\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 - C_{2,h}J_p(p_h, p_h),$$

and

$$|||\mathbf{v_h}|||_h \leq C_{3,h}\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}},$$

where $C_{1,h}, C_{2,h}, C_{3,h}$ are positive constants independent of μ_1 and μ_2.

Proof. If p_h is a piecewise constant, i.e. $p_h \in M_0$, then the result follows from lemma 2.8 with $C_{2,h} = 0$. Otherwise, we have $p_h = p_0 + p_0^{\perp}$, where $p_0 \in M_0$, $p_0^{\perp} \in M_0^{\perp}$. Let $\mathbf{v_0}$ be such that lemma 2.8 is satisfied and $\mathbf{v_0^{\perp}}$ be such that lemma 2.9 is satisfied.

For $\alpha > 0$, define $\mathbf{v_h} = \mathbf{v_0} + \alpha\mathbf{v_0^{\perp}}$. Since $\mathbf{v_0^{\perp}}$ vanishes on $\Gamma\cup\partial\Omega$ and p_0 is a constant on each subdomain $\Omega_i, i = 1, 2$, we have

$$b_h(\mathbf{v_0^{\perp}}, p_0) = 0.$$

Noting $\mathbf{v_0}$ is continuous on $\Omega_1 \cup \Omega_2$ and vanishes on $\partial\Omega$, we obtain

$$|b_h(\mathbf{v_0}, p_0^{\perp})| = |(\nabla \cdot \mathbf{v_0}, p_0^{\perp})_{0,\Omega_1\cup\Omega_2}| \leq C|||\mathbf{v_0}|||_h\|\mu^{-\frac{1}{2}}p_0^{\perp}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}.$$

Since p_0 is a constant on $\Omega_{i,h}$, we get

$$J_p(p_h, p_h) = J_p(p_0^{\perp}, p_0^{\perp}).$$

By the definition of $b_h(\mathbf{v_h}, p_h)$, lemmas 2.8-2.9, and the arithmetic-geometric inequality, it follows

$$\begin{aligned}
b_h(\mathbf{v_h}, p_h) =& b_h(\mathbf{v_0}, p_0) + b_h(\mathbf{v_0}, p_0^{\perp}) + \alpha b_h(\mathbf{v_0^{\perp}}, p_0) + \alpha b_h(\mathbf{v_0^{\perp}}, p_0^{\perp}) \\
\geq& C_{1,p_0}\|\mu^{-\frac{1}{2}}p_0\|^2_{0,\Omega_1\cup\Omega_2} - C|||\mathbf{v_0}|||_h\|\mu^{-\frac{1}{2}}p_0^{\perp}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\
&+ \alpha(C_{1,p_0^{\perp}}\|\mu^{-\frac{1}{2}}p_0^{\perp}\|^2_{0,\Omega_1\cup\Omega_2} - C_{2,p_0^{\perp}}J_p(p_0^{\perp}, p_0^{\perp})) \\
\geq& C_{1,p_0}\|\mu^{-\frac{1}{2}}p_0\|^2_{0,\Omega_1\cup\Omega_2} - CC_{2,p_0}\|\mu^{-\frac{1}{2}}p_0\|_{0,\Omega_1\cup\Omega_2}\|\mu^{-\frac{1}{2}}p_0^{\perp}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\
&+ \alpha(C_{1,p_0^{\perp}}\|\mu^{-\frac{1}{2}}p_0^{\perp}\|^2_{0,\Omega_1\cup\Omega_2} - C_{2,p_0^{\perp}}J_p(p_0^{\perp}, p_0^{\perp})) \\
\geq& (C_{1,p_0} - \frac{\varepsilon CC_{2,p_0}}{2})\|\mu^{-\frac{1}{2}}p_0\|^2_{0,\Omega_1\cup\Omega_2} \\
&+ (\alpha C_{1,p_0^{\perp}} - \frac{CC_{2,p_0}}{2\varepsilon})\|\mu^{-\frac{1}{2}}p_0^{\perp}\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - \alpha C_{2,p_0^{\perp}}J_p(p_0^{\perp}, p_0^{\perp}) \\
\geq& C_{1,h}\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - C_{2,h}J_p(p_h, p_h),
\end{aligned}$$

for sufficiently large α.

Since p_0 is a constant on each subdomain $\Omega_i, i = 1, 2$,

$$\|p_0\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq (1 + ch^{\frac{1}{2}})\|p_0\|_{0,\Omega_1\cup\Omega_2}.$$

By the triangular inequality, lemmas 2.8-2.9,

$$\begin{aligned}
|||\mathbf{v_h}|||_h &\leq |||\mathbf{v_0}|||_h + \alpha|||\mathbf{v_0}^{\perp}|||_h \\
&\leq C_{2,p_0}\|\mu^{-\frac{1}{2}}p_0\|_{0,\Omega_1\cup\Omega_2} + \alpha C_{3,p_0^{\perp}}\|\mu^{-\frac{1}{2}}p_0^{\perp}\|_{0,\Omega_1\cup\Omega_2} \\
&\leq C_{3,h}\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}.
\end{aligned}$$

The proof is completed. ∎

Theorem 2.1. For $(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h) \in \mathbf{V_h} \times Q_h$, Let

$$A_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)] = B_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)] + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{v_h}) + \varepsilon_p J_p(p_h, q_h).$$

For sufficiently small h, there exists a constant $C_s > 0$, such that for any pair $(\mathbf{u_h}, p_h) \in V_h \times Q_h$,

$$\sup_{(\mathbf{v_h}, q_h)\in\mathbf{V_h}\times Q_h} \frac{A_h[(\mathbf{u_h}, p_h), (\mathbf{v_h}, q_h)]}{|||(\mathbf{v_h}, q_h)|||_h} \geq C_s|||(\mathbf{u_h}, p_h)|||_h. \tag{2.19}$$

Proof. Let $\mathbf{v_h} = \mathbf{u_h}$, $q_h = p_h$, by lemma 2.10, we obtain

$$\begin{aligned}
&B_h[(\mathbf{u_h}, p_h), (\mathbf{u_h}, p_h)] + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{u_h}) + \varepsilon_p J_p(p_h, p_h) \\
=&a_h(\mathbf{u_h}, \mathbf{u_h}) + b_h(\mathbf{u_h}, p_h) - b_h(\mathbf{u_h}, p_h) + c_h(p_h, p_h) \\
&+ \varepsilon_u J_u(\mathbf{u_h}, \mathbf{u_h}) + \varepsilon_p J_p(p_h, p_h) \\
=&a_h(\mathbf{u_h}, \mathbf{u_h}) + c_h(p_h, p_h) + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{u_h}) + \varepsilon_p J_p(p_h, p_h) \\
\geq& \, min(1, \varepsilon_u)|||\mathbf{u_h}|||_h^2 + \varepsilon_p J_p(p_h, p_h) + c_h(p_h, p_h).
\end{aligned}$$

Let $\mathbf{v_h} = -\mathbf{w}, q_h = 0$, where $\mathbf{w}$ be such that lemma 2.9 is satisfied, with $|||\mathbf{w}||| \leq \|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$, we can choose $|||\mathbf{w}|||_h = \|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$. By lemma 2.2, lemma

2.9, using the arithmetic-geometric inequality, then

$$
\begin{aligned}
&B_h[(\mathbf{u_h},p_h),(-\mathbf{w},0)]+\varepsilon_u J_u(\mathbf{u_h},-\mathbf{w})\\
=&a_h(\mathbf{u_h},-\mathbf{w})+b_h(-\mathbf{w},p_h)-\varepsilon_u J_u(\mathbf{u_h},\mathbf{w})\\
=&-a_h(\mathbf{u_h},\mathbf{w})-b_h(\mathbf{w},p_h)-\varepsilon_u J_u(\mathbf{u_h},\mathbf{w})\\
\geq&-C_1|||\mathbf{u_h}|||\,|||\mathbf{w}|||_h-b_h(\mathbf{w},p_h)\\
\geq&C_{1,h}\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}-C_{2,h}J_p(p_h,p_h)-C_1|||\mathbf{u_h}|||_h\|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}\\
\geq&C_1\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}-C_2J_p(p_h,p_h)-C_3|||\mathbf{u_h}|||^2_h.
\end{aligned}
$$

Finally, taking $\mathbf{v_h}=\mathbf{u_h}-\alpha\mathbf{w}, q_h=p_h$, then

$$
\begin{aligned}
&B_h[(\mathbf{u_h},p_h),(\mathbf{u_h}-\alpha\mathbf{w},p_h)]+\varepsilon_u J_u(\mathbf{u_h},\mathbf{u_h}-\alpha\mathbf{w})+\varepsilon_p J_p(p_h,p_h)\\
=&B_h[(\mathbf{u_h},p_h),(\mathbf{u_h},p_h)]+\varepsilon_u J_u(\mathbf{u_h},\mathbf{u_h})+\varepsilon_p J_p(p_h,p_h)\\
&+B_h[(\mathbf{u_h},p_h),(-\alpha\mathbf{w},0)]+\varepsilon_u J_u(\mathbf{u_h},-\alpha\mathbf{w})\\
\geq&\ min(1,\varepsilon_u)|||\mathbf{u_h}|||^2_h+\varepsilon_p J_p(p_h,p_h)+c_h(p_h,p_h)\\
&+\alpha(C_1\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}-C_2J_p(p_h,p_h)-C_3|||\mathbf{u_h}|||^2_h)\\
\geq&\ (min(1,\varepsilon_u)-\alpha C_3)|||\mathbf{u_h}|||^2_h+\alpha C_1\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}\\
&-c_h(p_h,p_h)+(\varepsilon_p-\alpha C_2)J_p(p_h,p_h)\\
\geq&C(|||\mathbf{u_h}|||^2_h+J_p(p_h,p_h)+\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}})\\
\geq&\ C|||(\mathbf{u_h},p_h)|||^2_h.
\end{aligned}
\tag{2.20}
$$

We also have

$$
|||(\mathbf{u_h}-\alpha\mathbf{w},p_h)|||_h\leq|||(\mathbf{u_h},p_h)|||_h+\alpha|||(\mathbf{w}|||_h\leq(1+\alpha)|||(\mathbf{u_h},p_h)|||_h. \tag{2.21}
$$

From (2.20)-(2.21), we know (2.19) holds . ∎

Remark 2.1. *Combing lemma* 2.4 *with theorem* 2.1 *and applying Babuska theorem, we can know that the solution is exist and unique.*

§2.3 Approximate properties

Define extension operators $E_i^k: H^k(\Omega_i)\to H^k(\Omega)$, such that

$$
(E_i^2\mathbf{v_i},E_i^1q_i)|_{\Omega_i}=(\mathbf{v_i},q_i),
$$

and

$$\|E_i^k \mathbf{v_i}\|_{H^s(\Omega)} \le \|\mathbf{v_i}\|_{H^s(\Omega_i)} \quad \forall \mathbf{v_i} \in H^s(\Omega_i), i=1,2, s=0,...,k. \tag{2.22}$$

cf. Dautray and Lions[14].

Let $I_h : H^2(\Omega) \cap H_0^1(\Omega) \to \mathbf{V_h}$ be the nodal interpolation operation, $\pi_h : H^1(\Omega) \to Q_h$ be the local L^2 projection. Define

$$(I_h^* \mathbf{v}, \pi_h^* q) = ((I_{h,1}^* \mathbf{v_1}, I_{h,2}^* \mathbf{v_2}), (\pi_{h,1}^* q_1, \pi_{h,2}^* q_2)),$$

where $I_{h,i}^* \mathbf{v_i} = (I_h E_i^2 \mathbf{v_i})|_{\Omega_i}, \pi_{h,i}^* \mathbf{v_i} = (\pi_h E_i^1 \mathbf{v_i})|_{\Omega_i}$.

We restate the following lemma without proof (see [19]).

Lemma 2.11. Map a triangle K onto the unit reference triangle $\tilde{K}$ by an affine map and denote by $\tilde{\Gamma}_{\tilde{K}}$ the corresponding image of Γ_K. Under the assumptions $A1 - A3$ in chapter 1, there exists a constant C, depending on Γ but independent of the mesh, such that

$$\|\mathbf{w}\|_{0,\tilde{\Gamma}_{\tilde{K}}} \le C\|\mathbf{w}\|_{0,\tilde{K}} \|\mathbf{w}\|_{1,\tilde{K}}, \quad \forall \mathbf{w} \in H^1(\tilde{K}). \tag{2.23}$$

Theorem 2.2. There exist a positive constant independent of μ_1, μ_2 and the relation between the interface Γ and the triangulation, such that

$$|||(\mathbf{v} - I_h^* \mathbf{v}, q - \pi_h^* q)||| \le Ch(\|\mu_{max}^{\frac{1}{2}} \mathbf{v}\|_{2,\Omega_1 \cup \Omega_2} + \|\mu^{-\frac{1}{2}} q\|_{1,\Omega_1 \cup \Omega_2}). \tag{2.24}$$

Proof. Let $\mathbf{v_i} = \mathbf{v}|_{\Omega_i}, K_i = K \cap \Omega_i, i = 1,2$, we have

$$\begin{aligned} \|\nabla(\mathbf{v_i} - I_{h,i}^* \mathbf{v_i})\|_{0,K_i}^2 &= \|\nabla(E_i^2 \mathbf{v_i} - I_h E_i^2 \mathbf{v_i})\|_{0,K_i}^2 \\ &\le \|\nabla(E_i^2 \mathbf{v_i} - I_h E_i^2 \mathbf{v_i})\|_{0,K}^2 \le Ch^2 \|E_i^2 \mathbf{v_i}\|_{2,K}^2, \end{aligned} \tag{2.25}$$

similarly,

$$\|q_i - \pi_{h,i}^* q_i\|_{0,K_i}^2 \le Ch_K^2 \|E_i^1 q_i\|_{1,K}^2. \tag{2.26}$$

Summing over all the triangles that intersect with Ω_i, it follows from (2.25) and (2.26) that

$$\begin{aligned} \sum_{i=1}^{2} \sum_{K \in \Omega_i} \|\nabla(\mathbf{v_i} - I_{h,i}^* \mathbf{v_i})\|_{0,K_i}^2 &\le \sum_{i=1}^{2} \sum_{K \in \Omega_i} Ch^2 \|I_h E_i^2 \mathbf{v_i}\|_{2,K}^2 \le C \sum_{i=1}^{2} h^2 \|E_i^2 \mathbf{v_i}\|_{2,\Omega}^2 \\ &\le C \sum_{i=1}^{2} h^2 \|\mathbf{v_i}\|_{2,\Omega_i}^2, \end{aligned} \tag{2.27}$$

and

$$\sum_{i=1}^{2}\sum_{K\in\Omega_i}\|q_i-\pi_{h,i}^*q_i\|_{0,K_i}^2\le\sum_{i=1}^{2}\sum_{K\in\Omega_i}Ch_K^2\|E_i^1q_i\|_{1,K}^2\le\sum_{i=1}^{2}Ch^2\|q_i\|_{1,\Omega_i}^2. \tag{2.28}$$

Next, we consider the jumps on the interface. By the scaling argument, it follows from lemma 2.11,

$$\|\mathbf{w}\|_{0,\Gamma_K}^2\le C(h_K^{-1}\|\mathbf{w}\|_{0,K}^2+h_k|\mathbf{w}|_{1,K}^2),\quad \forall\mathbf{w}\in H^1(K).$$

Using the inverse inequality, we obtain

$$\begin{aligned}\|[\mathbf{v}-I_h^*\mathbf{v}]\|_{\frac{1}{2},h,\Gamma}^2&\le\sum_{i=1}^{2}\sum_{K\in G_h}h_K^{-1}\|\mathbf{v}_i-I_{h,i}^*\mathbf{v}_i\|_{0,\Gamma_K}^2\\&=\sum_{i=1}^{2}\sum_{K\in G_h}h_K^{-1}\|E_i^2\mathbf{v}_i-I_hE_i^2\mathbf{v}_i\|_{0,\Gamma_K}^2\\&\le\sum_{i=1}^{2}\sum_{K\in G_h}(h_K^{-2}\|E_i^2\mathbf{v}_i-I_hE_i^2\mathbf{v}_i\|_{0,K}^2+|E_i^2\mathbf{v}_i-I_hE_i^2\mathbf{v}_i|_{1,K}^2)\\&\le\sum_{i=1}^{2}\sum_{K\in G_h}Ch_K^2\|E_i^2\mathbf{v}_i\|_{2,K}^2\le\sum_{i=1}^{2}Ch^2\|E_i^2\mathbf{v}_i\|_{2,\Omega}^2\\&\le\sum_{i=1}^{2}Ch^2\|\mathbf{v}_i\|_{2,\Omega_i}^2,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\|\{q-\pi_h^*q\}\|_{-\frac{1}{2},h,\Gamma}^2&\le C\sum_{i=1}^{2}\sum_{K\in G_h}(\|E_i^1q_i-\pi_{h,i}E_i^1q_i\|_{0,K}^2+h_K^2|E_i^1q_i-\pi_{h,i}E_i^1q_i|_{1,K}^2)\\&\le C\sum_{i=1}^{2}\sum_{K\in G_h}Ch_K^2\|E_i^1q_i\|_{1,K}^2\le C\sum_{i=1}^{2}Ch^2\|q_i\|_{1,\Omega_i}^2.\end{aligned} \tag{2.30}$$

According to lemma 2.11, it holds

$$\|\nabla\mathbf{w}\cdot\mathbf{n}\|_{0,\Gamma_K}^2\le C(h_K^{-1}|\mathbf{w}|_{1,K}^2+h_k|\mathbf{w}|_{2,K}^2),\quad\forall\mathbf{w}\in H^2(K). \tag{2.31}$$

Therefore

$$\|\{\nabla(\mathbf{v}-I_h^*\mathbf{v})\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma}^2 \le C\sum_{i=1}^{2}\sum_{K\in G_h} h_K\|\nabla(E_i^2\mathbf{v}_i - I_hE_i^2\mathbf{v}_i)\cdot\mathbf{n}\|_{0,\Gamma_K}^2$$
$$\le C\sum_{i=1}^{2}\sum_{K\in G_h}(|E_i^2\mathbf{v}_i - I_hE_i^2\mathbf{v}_i|_{1,K}^2 + h_K^2|E_i^2\mathbf{v}_i - I_hE_i^2\mathbf{v}_i|_{2,K}^2)$$
$$\le C\sum_{i=1}^{2}\sum_{K\in G_h} h_K^2\|E_i^2\mathbf{v}_i\|_{2,K}^2 \le C\sum_{i=1}^{2} h^2\|\mathbf{v}_i\|_{2,\Omega_i}^2. \tag{2.32}$$

From (2.27), (2.29) and (2.32), it follows

$$|||\mathbf{v}-I_h^*\mathbf{v}||| \le Ch\|\mu_{max}^{\frac{1}{2}}\mathbf{v}\|_{2,\Omega_1\cup\Omega_2}, \tag{2.33}$$

Combing (2.28), (2.30) and (2.33), the proof is completed. ▮

Theorem 2.3. Assume $(\mathbf{u},p)\in(H^2(\Omega_1\cap\Omega_2)\cap H_0^1(\Omega))^2\times(H^1(\Omega_1\cup\Omega_2)\cap L_\mu^2(\Omega))$ be the solution of the elasticity interface problem (1.6), and $(\mathbf{u_h},p_h)\in\mathbf{V_h}\times Q_h$ be the solution of the discrete problem (1.7),

$$|||(\mathbf{u}-\mathbf{u_h},p-p_h)||| \le Ch(\|\mu_{max}^{\frac{1}{2}}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2} + \|\mu^{-\frac{1}{2}}p\|_{1,\Omega_1\cup\Omega_2}), \tag{2.34}$$

where C is a positive constant independent of μ_1, μ_2 and the relation between the interface Γ and the triangulation.

Proof. Note that

$$|||(\mathbf{u}-\mathbf{u_h},p-p_h)||| \le |||(\mathbf{u}-I_h^*\mathbf{u},p-\pi_h^*p)||| + |||(I_h^*\mathbf{u}-\mathbf{u_h},\pi_h^*p-p_h)|||_h, \tag{2.35}$$

by theorem 2.2, we only need to estimate $|||(I_h^*\mathbf{u}-\mathbf{u_h},\pi_h^*p-p_h)|||_h$. Applying theorem 2.1 and lemma 2.1, we have

$$|||(I_h^*\mathbf{u}-\mathbf{u_h},\pi_h^*p-p_h)|||_h$$
$$\le C\sup_{(\mathbf{u_h},p_h)\in\mathbf{V_h}\times Q_h}\frac{I+\varepsilon_uJ_u(I_h^*\mathbf{u}-\mathbf{u_h},\mathbf{v_h})+\varepsilon_pJ_p(\pi_h^*p-p_h,q_h)}{|||(\mathbf{v_h},q_h)|||_h}$$
$$\le C\sup_{(\mathbf{u_h},p_h)\in\mathbf{V_h}\times Q_h}\frac{I+\varepsilon_uJ_u(I_h^*\mathbf{u},\mathbf{v_h})+\varepsilon_pJ_p(\pi_h^*p,q_h)}{|||(\mathbf{v_h},q_h)|||_h}. \tag{2.36}$$

where $I = B_h[(I_h^*\mathbf{u} - \mathbf{u_h}, \pi_h^* p - p_h), (\mathbf{v_h}, q_h)]$. Using the continuity of $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&B_h[(I_h^*\mathbf{u} - \mathbf{u}, \pi_h^* p - p), (\mathbf{v_h}, q_h)] \\
=&a_h(I_h^*\mathbf{u} - \mathbf{u}, \mathbf{v_h}) + b_h(\mathbf{v_h}, \pi_h^* p - p) - b_h(I_h^*\mathbf{u} - \mathbf{u}, q_h) + c_h(\pi_h^* p - p, q_h) \\
\leq&|||I_h^*\mathbf{u} - \mathbf{u}||| \cdot |||\mathbf{v_h}||| + |||\mathbf{v_h}||| \cdot \left|\left|\left|\mu^{-\frac{1}{2}}(\pi_h^* p - p)\right|\right|\right| + |||I_h^*\mathbf{u} - \mathbf{u}||| \cdot \left|\left|\left|\mu^{-\frac{1}{2}} q_h\right|\right|\right| \\
&+ \|\pi_h^* p - p\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \cdot \|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\
\leq&|||(I_h^*\mathbf{u} - \mathbf{u}, \pi_h^* p - p)||| \cdot |||(\mathbf{v_h}, q_h)|||_h. \qquad (2.37)
\end{aligned}$$

For $\mathbf{u} \in H^2(\Omega_i)$ and $E^2\mathbf{u} = (E_i^2\mathbf{u}_1, E_i^2\mathbf{u}_2)$, by Cauchy-Schwarz inequality, the properties of L^2 projection, it holds

$$\begin{aligned}
J_u(I_h^*\mathbf{u}, \mathbf{v_h}) \leq&\sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}} \mu h_{\mathcal{P}_{l,i}}^{-2} (ch|E_i^2\mathbf{u_i} - I_{h,i}^*\mathbf{u_i}|_{1,\mathcal{P}_{l,i}} + ch^2|E_i^2\mathbf{u_i}|_{1,\mathcal{P}_{l,i}}) \times ch|\mathbf{v_{h,i}}|_{1,\mathcal{P}_{l,i}} \\
\leq&\sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}} \mu h_{\mathcal{P}_{l,i}}^{-2} ch^2\|E_i^2\mathbf{u_i}\|_{2,\mathcal{P}_{l,i}} \times ch\|\nabla\mathbf{v_{h,i}}\|_{0,\mathcal{P}_{l,i}} \\
\leq&\sum_{i=1}^{2} ch\|E_i^2\mathbf{u_i}\|_{2,\Omega} \cdot \|\nabla\mathbf{v_{h,i}}\|_{0,\mathcal{P}_{l,i}} \\
\leq&\sum_{i=1}^{2} ch\|\mathbf{u_i}\|_{2,\Omega_i}\|\nabla\mathbf{v_{h,i}}\|_{0,\Omega_{i,h}} \\
\leq&ch\|\mu_{max}^{\frac{1}{2}}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2}|||(\mathbf{v_h}, q_h)|||_h. \qquad (2.38)
\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
J_p(\pi_h^* p, q_h) &= \sum_{i=1}^{2} \mu_i^{-1} h^3 \sum_{e\in\mathcal{F}_i} ([\nabla\pi_{h,i}^* p_i], [\nabla q_{h,i}])_e \\
&\leq \sum_{i=1}^{2} \mu_i^{-1} h^3 \sum_{e\in\mathcal{F}_i} \|[\nabla\pi_{h,i}^* p_i]\|_{0,e}\|[\nabla q_{h,i}]\|_{0,e} \\
&= J_p(\pi_h^* p, \pi_h^* p)^{\frac{1}{2}} J_p(q_h, q_h)^{\frac{1}{2}} \\
&\leq J_p(\pi_h^* p, \pi_h^* p)^{\frac{1}{2}} |||(\mathbf{v_h}, q_h)|||_h. \qquad (2.39)
\end{aligned}$$

By the trace theorem, the inverse inequality and stability of π_h, it follows

$$
\begin{aligned}
J_p(\pi_h^* p, \pi_h^* p) &= \sum_{i=1}^{2}\sum_{e\in\mathcal{F}_i}\mu_i^{-1}h^3\|[\nabla\pi_{h,i}^* p_i]\|_{0,e}^2 \le ch^2\sum_{i=1}^{2}\mu_i^{-1}\sum_{e\in\mathcal{F}_i}\|\nabla\pi_{h,i}^* p_i\|_{0,K_F^+\cup K_F^-}^2 \\
&\le ch^2\sum_{i=1}^{2}\mu_i^{-1}\sum_{K\in\mathcal{T}_h}\|\pi_h E_i^1 p_i\|_{1,K}^2 \le ch^2\sum_{i=1}^{2}\mu_i^{-1}\sum_{K\in\mathcal{T}_h}\|E_i^1 p_i\|_{1,K}^2 \\
&\le ch^2\mu_i^{-1}\|p\|_{1,\Omega_1\cup\Omega_2}^2,
\end{aligned}
\tag{2.40}
$$

where K_F^+, K_F^- are two elements sharing an edge F.

Therefore, combining (2.37), (2.38), (2.39), (2.40), (2.36) with (2.35), the desired result follows. ∎

Chapter 3

Estimate of the condition number

Let $\{\phi_i\}_{i=1}^{N_1}$, $\{\varphi_i\}_{i=1}^{N_2}$ be a standard finite element basis in V_h and Q_h, respectively, and G be the stiffness matrix associated with the weak formulation (1.7). We denote by $|V|_N = (\sum_{i=1}^{N} V_i^2)^{\frac{1}{2}}$ the Euclidean norm of a vector $V = (V_1, V_2, ..., V_N)^T \in \mathbb{R}^N$, here $N = N_1 + N_2$. Define the condition number $\kappa(G)$ by

$$\kappa(G) = |G|_N \cdot |G^{-1}|_N,$$

where

$$|G|_N = \sup_{U \in \mathbb{R}} \frac{|GU|_N}{|U|_N}.$$

Let $V = \{V_i\}_{i=1}^{N}$ denote the expansion coefficients of $(\mathbf{v_h}, q_h)$ in the basis $\{\phi_i\}_{i=1}^{N_1}$ and $\{\varphi_i\}_{i=1}^{N_2}$. Since the triangulation is quasi-uniform, we have the following estimates

$$|V|_N \geq c_1 h^{-1}(\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}) \tag{3.1}$$

and

$$|V|_N \leq c_2 h^{-1}(\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}). \tag{3.2}$$

In order to estimate the condition number, we firstly give the following two lemmas.

Lemma 3.1. For any $(\mathbf{v_h}, q_h) \in \mathbf{V_h} \times Q_h$, we have

$$\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq C max(\mu_{max}^{\frac{1}{2}}, \mu_{min}^{-\frac{1}{2}}) |||(\mathbf{v_h}, q_h)|||_h, \tag{3.3}$$

where C is a positive constant.

Proof. Noting that $\|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq \mu_{max}^{\frac{1}{2}} \|\mu^{-\frac{1}{2}} q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$, we only need to prove that $\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq C|||(\mathbf{v_h}, q_h)|||_h$. We claim

$$\|\mu^{\frac{1}{2}} \mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq C(\|\mu^{\frac{1}{2}} \nabla \mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|\mu_{max}^{\frac{1}{2}} \mathbf{v_h}\|_{\frac{1}{2},h,\Gamma}). \tag{3.4}$$

In fact, we assume the right side of (3.4) equals to 0, then $\mathbf{v_h} = c$, this means that $\mathbf{v_h}$ is a piecewise constant in $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$. Since $[\mathbf{v_h}] = 0$, it follows that $\mathbf{v_h} \equiv 0$. Thus, if the right is zero, the left side is also zero. By the equivalence of norm in finite dimensional space, (3.4) holds. By lemma 4.2 in [12] and discrete korn's inequality,

$$
\begin{aligned}
\|\mu^{\frac{1}{2}}\nabla\mathbf{v_h}\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} &\leq \|\mu^{\frac{1}{2}}\nabla\mathbf{v_h}\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + J_u(\mathbf{v_h},\mathbf{v_h}) \\
&\leq \|\mu^{\frac{1}{2}}\varepsilon(\mathbf{v_h})\|^2_{0,\Omega_1\cup\Omega_2} + \|\mu_{max}^{\frac{1}{2}}[\mathbf{v_h}]\|^2_{\frac{1}{2},h,\Gamma} + J_u(\mathbf{v_h},\mathbf{v_h}) \\
&\leq |||\mathbf{v_h}|||^2_h,
\end{aligned}
$$

thus $\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq C\mu_{min}^{-\frac{1}{2}}|||\mathbf{v_h}|||_h$ and the proof is completed. ∎

Lemma 3.2. For all $(\mathbf{v_h}, q_h) \in \mathbf{V_h}\times Q_h$, we have

$$
|||(\mathbf{v_h},q_h)|||_h \leq Cmax(\mu_{max}^{\frac{1}{2}},\mu_{min}^{-\frac{1}{2}})h^{-1}(\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}), \tag{3.5}
$$

where C is a positive constant.

Proof. For any $(\mathbf{v_h}, q_h) \in \mathbf{V_h}\times Q_h$, it's obvious,

$$
\|\mu^{-\frac{1}{2}}q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq \mu_{min}^{-\frac{1}{2}}\|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}},
$$

$$
\|\mu^{\frac{1}{2}}\varepsilon(\mathbf{v_h})\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq \|\mu^{\frac{1}{2}}\nabla\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq ch^{-1}\mu_{max}^{\frac{1}{2}}\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}.
$$

Using the properties of L^2 projection, we get

$$
\begin{aligned}
J_u(\mathbf{v_h},\mathbf{v_h}) &\leq \sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}}\mu_i h_{\mathcal{P}_{l,i}}^{-2}\|\mathbf{v_{h,i}} - \Pi_{l,i}\mathbf{v_{h,i}}\|^2_{0,\mathcal{P}_{l,i}} \\
&\lesssim \sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}}\mu_i h_{\mathcal{P}_{l,i}}^{-2}\cdot h_{\mathcal{P}_{l,i}}^{2}|\mathbf{v_{h,i}}|^2_{1,\mathcal{P}_{l,i}} \\
&\lesssim \mu_{max}\|\nabla\mathbf{v_h}\|^2_{1,\Omega_{1,h}\cup\Omega_{2,h}} \\
&\lesssim \mu_{max}h^{-2}\|\mathbf{v_h}\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}.
\end{aligned}
$$

By the trace theorem and inverse inequality, we obtain

$$
\begin{aligned}
J_p(q_h,q_h) &\leq \sum_{i=1}^{2}\mu_i^{-1}h^3\sum_{e\in\mathcal{F}_i}\|[\nabla q_{h,i}]\|^2_{0,e} \leq \sum_{i=1}^{2}\mu_i^{-1}h^2\sum_{K\in\mathcal{T}_{h,i}}\|\nabla q_{h,i}\|^2_{0,K} \\
&\leq \sum_{i=1}^{2}\mu_i^{-1}\sum_{K\in\mathcal{T}_{h,i}}\|q_{h,i}\|^2_{0,K} \leq C\mu_{min}^{-1}\|q_h\|^2_{0,\Omega_{1,h}\cup\Omega2,h},
\end{aligned}
$$

combing the above estimates, we get the result (3.5). ∎

Theorem 3.1. $\kappa(G) \leq Cmax(\mu_{max}^2, \mu_{min}^{-2})h^{-2}$, where C is a positive constant.

Proof. Let $U \in \mathbb{R}^N$ and $V \in \mathbb{R}^N$ be the expansion coefficients of $(\mathbf{u_h}, p_h)$ and $(\mathbf{v_h}, q_h)$ in the basis $\{\phi_i\}_{i=1}^{N_1}$, $\{\varphi_i\}_{i=1}^{N_2}$, respectively. By the definition of $|G|_N$, the continuity of $B_h[(\cdot,\cdot),(\cdot,\cdot)]$, Cauchy-Schwarz inequality, lemma 3.2 and the inequality (3.1), we have

$$\begin{aligned}
|GU|_N &= \sup_{U\in\mathbb{R}} \frac{(GU,V)_N}{|V|_N} \\
&= \sup_{U\in\mathbb{R}} \frac{B_h[(\mathbf{u_h}, p_h),(v_h, q_h)] + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{v_h}) + \varepsilon_p J_p(p_h, q_h)}{|V|_N} \\
&\leq C\frac{|||(\mathbf{u_h}, p_h)|||_h |||(\mathbf{v_h}, q_h)|||_h}{|V|_N} \\
&\leq Cmax(\mu_{max}, \mu_{min}^{-1})\frac{|U|_N|V|_N}{|V|_N} \leq Cmax(\mu_{max}, \mu_{min}^{-1})|U|_N.
\end{aligned}$$

Thus, we obtain

$$|G|_N = \sup_{U\in\mathbb{R}} \frac{|GU|_N}{|U|_N} \leq Cmax(\mu_{max}, \mu_{min}^{-1}). \tag{3.6}$$

Next, we turn to estimate $|G^{-1}|_N$, Using the inequality (3.2) and lemma 3.1 and theorem 2.1,

$$\begin{aligned}
|U|_N &\leq Ch^{-1}(\|\mathbf{u_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}) \\
&\leq Ch^{-1}max(\mu_{max}^{\frac{1}{2}}, \mu_{min}^{-\frac{1}{2}})|||(\mathbf{u_h}, p_h)|||_h \\
&\leq Ch^{-1}max(\mu_{max}^{\frac{1}{2}}, \mu_{min}^{-\frac{1}{2}})\sup_{U\in\mathbb{R}} \frac{B_h[(\mathbf{u_h}, p_h),(v_h, q_h)] + \varepsilon_u J_u(\mathbf{u_h}, \mathbf{v_h}) + \varepsilon_p J_p(p_h, q_h)}{|||(\mathbf{v_h}, q_h)|||_h} \\
&\leq Ch^{-1}max(\mu_{max}^{\frac{1}{2}}, \mu_{min}^{-\frac{1}{2}})\sup_{V\in\mathbb{R}} \frac{(GU,V)_N}{|V|_N}\cdot\frac{|V|_N}{|||(\mathbf{v_h}, q_h)|||_h} \\
&\leq Ch^{-2}max(\mu_{max}^{\frac{1}{2}}, \mu_{min}^{-\frac{1}{2}})|GU|_N\frac{\|\mathbf{v_h}\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}}{|||(\mathbf{v_h}, q_h)|||_h} \\
&\leq Cmax(\mu_{max}, \mu_{min}^{-1})|GU|_N,
\end{aligned}$$

Setting $U = G^{-1}V$, we obtain

$$|G^{-1}|_N \leq Cmax(\mu_{max}, \mu_{min}^{-1})h^{-2}. \tag{3.7}$$

Combining (3.6) with (3.7), the desired result holds. ∎

Chapter 4

Numerical examples

In this section, we are going to present a test to demonstrate our method.

Example 1. In our experiment, the domain $\Omega = (0,1) \times (0,1)$, the interface Γ is a line with equation $\Phi(x) = x - \pi/8$. The domain is divided into two subdomains $\Omega_1 = \{(x,y)|x < \pi/8\}$ and $\Omega_1 = \{(x,y)|x > \pi/8\}$, the exact solution u is

$$u(x,y) = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}$$

$$= \begin{cases} \begin{pmatrix} u_1^-(x,y) \\ u_2^-(x,y) \end{pmatrix} \\ \begin{pmatrix} u_1^+(x,y) \\ u_2^+(x,y) \end{pmatrix} \end{cases} = \begin{cases} \begin{pmatrix} \frac{1}{\mu_1}x(x-1)y(y-1)(x-\pi/8)^2 \\ \frac{1}{\mu_1}x(x-1)y(y-1)(x-\pi/8)^2 \end{pmatrix} & in\ \Omega_1, \\ \begin{pmatrix} \frac{1}{\mu_2}x(x-1)y(y-1)(x-\pi/8)^2 \\ \frac{1}{\mu_2}x(x-1)y(y-1)(x-\pi/8)^2 \end{pmatrix} & in\ \Omega_2, \end{cases}$$

We transform the Lamé parameters λ_1, μ_1 and λ_2, μ_2 so that we can test problems with different discontinuities in Lamé parameters. In the first two tables, the material property is compressible. The first table has small jumps in Lamé parameters: $\mu_1 = 40, \mu_2 = 4, \lambda_1 = 80, \lambda_2 = 8$. The second table has larger jumps in Lamé parameters:$\mu_1 = 400, \mu_2 = 4, \lambda_1 = 800, \lambda_2 = 8$. The third table has the jumps in Lamé parameters:$\mu_1 = 4000, \mu_2 = 4, \lambda_1 = 8000, \lambda_2 = 8$.

Table 4.1: The finite element errors with $\mu_1 = 40, \mu_2 = 4, \lambda_1 = 80, \lambda_2 = 8$

n	$\lvert\mathbf{u}-\mathbf{u_h}\rvert_{H^1}$	order	$\lVert\mathbf{u}-\mathbf{u_h}\rVert_{L^2}$	order	$\lVert p-p_h\rVert_{L^2}$	order
8	2.6190e-03		8.8870e-05		6.1961e-03	
16	1.3311e-03	9.7636e-01	2.1896e-05	2.0210e+00	2.0403e-03	1.6026e+00
32	6.6785e-04	9.9504e-01	5.3448e-06	2.0345e+00	6.9453e-04	1.5547e+00
64	3.3421e-04	9.9878e-01	1.3748e-06	1.9589e+00	2.4402e-04	1.5090e+00

Table 4.2: The finite element errors with $\mu_1 = 400, \mu_2 = 4, \lambda_1 = 800, \lambda_2 = 8$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	2.6308e-03		9.3428e-05		6.6212e-03	
16	1.3318e-03	9.8206e-01	2.2311e-05	2.0661e+00	2.1151e-03	1.6464e+00
32	6.6715e-04	9.9734e-01	5.3365e-06	2.0638e+00	6.9768e-04	1.6001e+00
64	3.3385e-04	9.9883e-01	1.3825e-06	1.9486e+00	2.4484e-04	1.5107e+00

Table 4.3: The finite element errors with $\mu_1 = 4000, \mu_2 = 4, \lambda_1 = 8000, \lambda_2 = 8$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	2.6504e-03		9.8701e-05		7.5144e-03	
16	1.3341e-03	9.9032e-01	2.2849e-05	2.1109e+00	2.2377e-03	1.7476e+00
32	6.6723e-04	9.9962e-01	5.3403e-06	2.0971e+00	7.0191e-04	1.6727e+00
64	3.3384e-04	9.9901e-01	1.3813e-06	1.9509e+00	2.4431e-04	1.5226e+00

We transform the jump ratio of $\frac{\mu_1}{\mu_2}$ through the next three tables. From the numerical results, we observe that both the XFEM interpolation errors and the corresponding XFEM solution errors in L^2 norm and semi H^1 norm are optimal independent of the quotient $\frac{\mu_1}{\mu_2}$.

Table 4.4: The finite element errors with $\mu_1 = 4, \mu_2 = 4000, \lambda_1 = 8, \lambda_2 = 8000$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	1.2055e-03		4.1110e-05		6.0754e-03	
16	6.2273e-04	9.5295e-01	1.0467e-05	1.9736e+00	2.0592e-03	1.5609e+00
32	3.1511e-04	9.8272e-01	2.8950e-06	1.8543e+00	8.2469e-04	1.3202e+00
64	1.5735e-04	1.0019e+00	6.4564e-07	2.1647e+00	2.5308e-04	1.7042e+00

Table 4.5: The finite element errors with $\mu_1 = 4, \mu_2 = 400, \lambda_1 = 8, \lambda_2 = 800$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	1.2059e-03		4.1114e-05		6.0709e-03	
16	6.2263e-04	9.5367e-01	1.0454e-05	1.9756e+00	2.0484e-03	1.5674e+00
32	3.1401e-04	9.8757e-01	2.7131e-06	1.9460e+00	6.9982e-04	1.5494e+00
64	1.5739e-04	9.9646e-01	6.4646e-07	2.0693e+00	2.4545e-04	1.5116e+00

Table 4.6: The finite element errors with $\mu_1 = 4, \mu_2 = 40, \lambda_1 = 8, \lambda_2 = 80$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	1.2791e-03		4.2218e-05		6.0436e-03	
16	6.3771e-04	1.0041e+00	1.0651e-05	1.9869e+00	2.0372e-03	1.5688e+00
32	3.2067e-04	9.9181e-01	2.7102e-06	1.9745e+00	6.9444e-04	1.5527e+00
64	1.6167e-04	9.8801e-01	6.6027e-07	2.0373e+00	2.4433e-04	1.5070e+00

Example 2. In our experiment, the domain $\Omega = (-1,1) \times (-1,1)$, the interface Γ is a circle with $r_0 = \pi/8$. The domain is divided into two domains $\Omega_1 = \{(x,y)|x^2 + y^2 > r_0^2\}$ and $\Omega_1 = \{(x,y)|x^2 + y^2 < r_0^2\}$, the exact solution u is

$$
\begin{aligned}
u(x,y) &= \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} u_1^-(x,y) \\ u_2^-(x,y) \end{pmatrix} \\ \begin{pmatrix} u_1^+(x,y) \\ u_2^+(x,y) \end{pmatrix} \end{cases}
= \begin{cases} \begin{pmatrix} \frac{1}{\lambda^-}(x^2+y^2)^{\frac{\alpha_1}{2}} \\ \frac{1}{\lambda^-}(x^2+y^2)^{\frac{\alpha_2}{2}} \end{pmatrix} & in\ \Omega_1, \\ \begin{pmatrix} \frac{1}{\lambda^+}(x^2+y^2)^{\frac{\alpha_1}{2}} + (\frac{1}{\lambda^-} - \frac{1}{\lambda^+})^{r_0^{\alpha_1}} \\ \frac{1}{\lambda^+}(x^2+y^2)^{\frac{\alpha_2}{2}} + (\frac{1}{\lambda^-} - \frac{1}{\lambda^+})^{r_0^{\alpha_2}} \end{pmatrix} & in\ \Omega_2, \end{cases}
\end{aligned}
$$

with $\alpha_1 = 5, \alpha_2 = 7$. We vary the Lamé parameters λ_1, μ_1 and λ_2, μ_2 so that we can test problems with different discontinuities in Lamé parameters.

Table 4.7: The finite element errors with $\mu_1 = 40, \mu_2 = 4, \lambda_1 = 80, \lambda_2 = 8$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	9.5374e-01		7.7226e-02		1.6847e+00	
16	4.9385e-01	9.4953e-01	2.0001e-02	1.9490e+00	6.5002e-01	1.3740e+00
32	2.4657e-01	1.0020e+00	4.7555e-03	2.0724e+00	2.2391e-01	1.5376e+00
64	1.2338e-01	9.9887e-01	1.1841e-03	2.0058e+00	8.1079e-02	1.4655e+00

Table 4.8: The finite element errors with $\mu_1 = 400, \mu_2 = 4, \lambda_1 = 800, \lambda_2 = 8$

n	$\|\mathbf{u}-\mathbf{u_h}\|_{H^1}$	order	$\|\mathbf{u}-\mathbf{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	9.5356e-01		7.7166e-02		1.7114e+00	
16	4.9030e-01	9.5967e-01	1.8937e-02	2.0268e+00	6.5731e-01	1.3805e+00
32	2.4653e-01	9.9189e-01	4.7613e-03	1.9918e+00	2.2555e-01	1.5432e+00
64	1.2338e-01	9.9865e-01	1.1762e-03	2.0172e+00	8.1107e-02	1.4755e+00

Table 4.9: The finite element errors with $\mu_1 = 4000, \mu_2 = 4, \lambda_1 = 8000, \lambda_2 = 8$

n	$\lvert \mathbf{u} - \mathbf{u_h} \rvert_{H^1}$	order	$\lVert \mathbf{u} - \mathbf{u_h} \rVert_{L^2}$	order	$\lVert p - p_h \rVert_{L^2}$	order
8	9.5401e-01		7.7154e-02		1.9592e+00	
16	4.9004e-01	9.6109e-01	1.8955e-02	2.0251e+00	6.3211e-01	1.6320e+00
32	2.4653e-01	9.9114e-01	4.7383e-03	2.0002e+00	3.2082e-01	9.7843e-01
64	1.2338e-01	9.9863e-01	1.1677e-03	2.0207e+00	8.3995e-02	1.9334e+00

Chapter 5

Conclusions and future work

In this thesis, we give the elasticity interface problem on mixed form, and introduce the extended finite element method for the problem. We introduce a ghost penalty term in order to keep the well-posedness of the method, and prove the existence and uniqueness of the solution. At the same time, we get the optimal a priori error estimates and the estimate of the condition number. Finally, we test our method by the numerical examples which verify that the convergence rates are optimal for the displacement in H^1 norm and the pressure in L^2 norm independent of the jump ratio of the constant Lamé parameter.

As far as I know, there is mainly two ways to solve the "locking" of the elasticity problem: the nonconforming finite element method and the mixed finite element method. As we have tried to solve the problem with the mixed finite element method, we shall try to use the nonconforming finite element method to solve the elasticity interface problem in the future.

Bibliography

[1] N. An, H. Chen. *A partially penalty immersed interface finite element method for anisotropic elliptic interface problems*, Numer. Methods PDE., 30: 1984-2028, 2014.

[2] C. Annavarapu, M. Hautefeuille, J. Dolbow. *A robust Nitsche's formulation for interface problems*, Comput. Methods Appl. Mech. Eng., 225-228: 44-54, 2012.

[3] I. Babuška. *The finite element method for elliptic equations with discontinuous coefficients*, Computing, 5: 207-213, 1970.

[4] R. Becker, E. Burman, P. Hansbo. *A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity*, Comput. Methods Appl. Mech. Engrg., 198: 3352-3360, 2009.

[5] R. Becker, P. Hansbo. *A simple pressure stabilization method for the Stokes equation*, Commun. Numer. Methods Engrg., 24: 1421-1430, 2008.

[6] M. Bendsoe, O. Sigmund. *Topology Optimization: Theory, Methods, and Applications*, Springer Science & Business Media, 2013.

[7] P. Bochev, C. Dohrmann, M. Gunzburgers. *Stabilization of low-order mixed finite elements for the Stokes equations*, SIAM J. Numer. Anal., 44: 82-101, 2006.

[8] S. Brenner. *Poincaré-Friedrichs inequalities for piecewise H^1 functions*, SIAM J. Numer. Anal., 41: 306-324, 2003.

[9] S. Brenner. *Korn's inequalities for piecewise H^1 vector field*, Math. Comp., 73: 1067-1087, 2003.

[10] F. Brezzi, M. Fortin. *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[11] F. Brezzi and J. Pitkaranta. *On the stabilization of finite element approximations of the Stokes equations*, Efficient Solutions of Elliptic Systems, W. Hackbusch, ed., Notes Numer. Fluid Mech. 10, Vieweg Braunschweig, Wiesbaden, 11-19, 1984.

[12] E. Burman. *Ghost penalty*, C. R. Acad. Sci. Paris, 348: 1217-1220, 2010.

[13] L. Cattaneo, L. Formaggia, G. Iori, A. Scotti, P. Zunino. *Stabilized extended finite elements for the approximation of saddle point problems with unfitted interfaces*, Calcolo, 52: 123-152, 2015.

[14] R. Dautray, J. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology, Functional and Variational Methods, vol. 2*, SpringerVerlag, Berlin, 1988.

[15] A. Ern, J. Guermond. *Theory and Practice of Finite Elements*, Appl. Math. Sci. 159, Springer, New York, 2004.

[16] T. Fries, T. Belytschko. *The extended/generalized finite element method: an overview of the method and its applications*, Int. J. Numer. Methods Eng., 84: 253-304, 2010.

[17] H. Gao, Y. Huang, F. Abraham. *Continuum and atomistic studies of intersonic crack propagation*, Journal of the Mechanics and Physics of Solids, 49: 2113-2132, 2001.

[18] S. Gross, A. Reusken. *Numerical methods for two-phase incompressible flows*, Springer Series in Computational Mathematics, vol. 40, Springer, Berlin, 2011.

[19] A. Hansbo, P. Hansbo. *An unfitted finite element method, based on Nitsche's method, for elliptic interface problems*, Comput. Methods Appl. Mech. Engrg., 191: 5537-5552, 2002.

[20] A. Hansbo, P. Hansbo. *A finite element method for the simulation of strong and weak discontinuities in solid mechanics*, Comput. Methods Appl. Mech. Engrg., 193: 3523-3540, 2004.

[21] P. Hansbo, M. Larson, S. Zahedi. *A cut finite element method for a Stokes interface problem*, Appl. Numer. Math., 8: 90-114, 2014.

[22] M. Kirchhrat, S. Gross, A. Reusken. *Analysis of an XFEM discretization for Stokes interface problems*, SIAM J. Sci. Comput., 38: 1019-1043, 2014.

[23] D. Kwak, K. Wee, K. Chang. *An analysis of a broken P1-nonconforming finite element method for interface problems*, SIAM J. Numer. Anal., 48: 2117-2134, 2010.

[24] Z. Li. *The immersed interface method using a finite element formulation*, Appl. Numer. Math., 27: 253-267, 1998.

[25] Z. Li, T. Lin, X. Wu. *New Cartesian grid methods for interface problems using the finite element formulation*, Numer. Math., 96: 61-98, 2003.

[26] T. Lin, Y. Lin, X. Zhang. *Partially penalized immersed finite element methods for elliptic interface problems*, SIAM J. Numer. Anal., 53: 1121-1144, 2015.

[27] J. Nitsche. *Über ein Variationsprinzip zur Lösungvon Dirichlet-Problemen bei Verwendungvon Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Univ. Hamburg, 36: 9-15, 1971.

[28] M. Olshanskii, A. Reusken. *Analysis of a Stokes interface problem*, Numer. Math., 103: 129-149, 2006.

[29] A. Reusken. *Analysis of extended pressure finite element space for two-phase incompressible flows*, Comput. Vis. Sci., 11: 293-305, 2008.

[30] H. Sauerland, T. Fries. *The extended finite element method for two-phase and free-surface flows: a systematic study*, J. Comput. Phys., 230: 3369-3390, 2011.

[31] L. Scott, S. Zhang. *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54: 483-493, 1990.

[32] Q. Wang, J. Chen. *A new unfitted stabilized Nitsche's finite element method for Stokes interface problems*, Comput. Math. Appl., 70: 820-834, 2015.

[33] Z. Zhong, S. Meguid. *On the imperfectly bonded spherical inclusion problem*, J. Appl. Mech., 66: 839-846, 1999.

致 谢

首先, 非常感谢我的导师-陈金如教授, 进入师门以来, 陈老师便在学习上给予了我极大的关心和鼓励. 刚进入研究生学习时, 因为基础不好, 在学习中, 只要遇到自己不清楚的点就一步一步深究下去, 有时读着一本书, 慢慢桌上就堆了一堆书: 数学分析, 高等代数, 泛函分析, 实变函数等. 有一次, 碰到了自己实在没法解决的问题, 便去请教了我的导师陈老师, 他讲的很清楚, 明白之后才发现其实并非难题. 我懊恼自己基础差思考不连贯, 但陈老师却鼓励说:"基础不好没关系, 你喜欢提问喜欢思考, 这就是一个很好的习惯, 坚持下去, 肯定可以收获不少的". 非常感谢陈老师一开始的鼓励, 让我对专业学习也有了更浓厚的兴趣.

与此同时, 我还要感谢王锋老师. 在完成论文理论部分开展程序书写的过程中, 王锋老师给了我极大的帮助, 理解算法并逐步写出算法. 关于程序上的帮助, 我还要感谢我的师姐秦芳芳, 那时为了让我学会写界面单刚并合成总刚, 师姐陪我熬了一整晚, 期间不断讨论以帮助我理解算法在编程中的运用.我还要感谢我的师兄王华, 在我程序报错的时候, 耐心的给我分析算法可能出现的问题, 并指导做出相应的修改, 才使得后期程序慢慢有了进展. 我还要感谢我的其他同门以及数科院的朋友们. 感谢他们在学习上的鼓励生活中的照顾.

其次, 我要感谢南京师范大学, 是她让我拥有了良好的学习环境以及良师益友, 让我收获满满.

最后, 我要感谢我的家人, 他们是我最坚强的后盾. 无论我做出怎样的决定, 只要我不后悔, 他们便全力支持, 感谢他们的理解, 感谢他们给予我的无私的爱.