弹性问题

2022 年 11 月 24 日

1 模型

令

$$
\begin{gathered}
\sigma(u) = 2\mu\epsilon(u) + \lambda tr(\epsilon(u))\delta \\
\epsilon(u) = \tfrac{1}{2}(gradu + (gradu)^t) \\
tr(\tau) = \tau_{11} + \tau_{22} \\
grad(u) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} \\
\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
divu = \tfrac{\partial u_1}{\partial x} + \tfrac{\partial u_2}{\partial y} \\
div\tau = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \\ \frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} \end{pmatrix}
\end{gathered}
$$

考察模型

$$
\begin{gathered}
-div\sigma(u) = f \quad \in \Omega \\
u|_\Gamma = 0
\end{gathered}
$$

其中 $u = (u_1, u_2)^t$ 为求解向量，$f = (f_1, f_2)^t$ 为右端向量，$\Omega = [0,1] \times [0,1]$

$$
\begin{gathered}
u_1 = (x-1)(y-1)ysin(x) \\
u_2 = (x-1)(y-1)xsin(y) \\
f_1 = -((2\mu+\lambda)y(y-1)(2cos(x) - (x-1)sin(x)) \\
+(\mu+\lambda)(2x-1)(sin(y) + (y-1)cos(y)) \\
+2\mu(x-1)sin(x)) \\
f_2 = -((2\mu+\lambda)x(x-1)(2cos(y) - (y-1)sin(y)) \\
+(\mu+\lambda)(2y-1)(sin(x) + (x-1)cos(x)) \\
+2\mu(y-1)sin(y))
\end{gathered}
$$

2 变分

该问题的变分问题为，求 $u \in H^1(\Omega)$ 使得 $u|_{\Gamma_1} = g$，并且

$$
a(u,\nu) = \int_\Omega f \cdot \nu dxdy \quad \forall \nu \in V
$$

其中

$$
\begin{aligned}
a(u,\nu) :&= \int_\Omega \sigma(u) : grad\nu dxdy \\
&= \int_\Omega 2\mu\epsilon(u) : grad\nu + \lambda divudiv\nu dxdy \\
V :&= \{\nu \in H^1(\Omega) \quad | \quad \nu|_\Gamma = 0\}
\end{aligned}
$$

证其与原问题的等价性

1. 若 u 为原问题的解
   设 $\nu = (\nu_1, \nu_2)^t, \quad \nu_1, \nu_2 \in C_0^\infty(\Omega)$，方程两边同乘 ν 并积分得

$$
-\int_\Omega div\sigma(u)\nu dxdy = \int_\Omega f\nu dxdy
$$

   由

$$
\begin{gathered}
fdiva = div(fa) - a : gradf \\
\textstyle\int_\Omega divadV = \int_{\partial\Omega} adS
\end{gathered}
$$

   得

$$
\begin{aligned}
&-\textstyle\int_\Omega div\sigma(u)\nu dxdy \\
&= -\int_\Omega div(\sigma(u)\nu)dxdy - \int_\Omega \sigma(u) : grad\nu dxdy \\
&= -\int_\Gamma \sigma(u)\nu dxdy + \int_\Omega \sigma(u) : grad\nu dxdy \\
&= \int_\Omega \sigma(u) : grad\nu dxdy
\end{aligned}
$$

   所以

$$
\int_\Omega \sigma(u) : grad\nu dxdy = \int_\Omega f\nu dxdy
$$

2. 若 u 为变分问题的解
   由

$$
\begin{aligned}
&\textstyle\int_\Omega \sigma(u) : grad\nu dxdy \\
&= -\int_\Gamma \sigma(u)\nu dxdy + \int_\Omega \sigma(u) : grad\nu dxdy \\
&= -\int_\Omega div\sigma(u)\nu dxdy
\end{aligned}
$$

得

$$-\int_\Omega div\sigma(u)\nu dxdy = \int_\Omega f\nu dxdy$$

由变分法基本引理得

$$-div\sigma(u) = f$$

3 剖分与基函数

3.1 剖分

对区间 Ω 按图 1 方式剖分，并对节点和单元进行编号，各节点坐标为 (x_i, y_i)，i=0, ... , n，

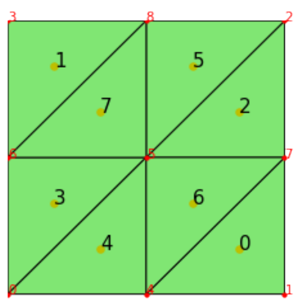

图 1

设基函数为

$(\varphi_0, 0)^t, (0, \varphi_0)^t, (\varphi_1, 0)^t, (0, \varphi_1)^t, ..., (\varphi_n, 0)^t, (0, \varphi_n)^t$

φ_i 为线性元，以下得到其在各单元上表达式。

3.2 线性元

如图 2，设 $\triangle(p_0, p_1, p_2)$ 是以 p_0, p_1, p_2 为顶点的任意三角型元，面积为 S。在 $\triangle(p_0, p_1, p_2)$ 内任取一点 p_3，坐标为 (x, y)。过 p_3 点作与三个顶点的连线，将 $\triangle(p_0, p_1, p_2)$ 分成三个三角形:

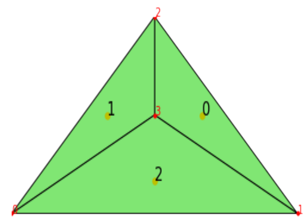

图 2

$\triangle(p_1, p_2, p_3), \triangle(p_0, p_3, p_2), \triangle(p_0, p_1, p_3)$，其面积分别为 S_0, S_1, S_2

显然 $S_0 + S_1 + S_2 = S$，令

$$L_0 = \frac{S_0}{S}, \quad L_1 = \frac{S_1}{S}, \quad L_2 = \frac{S_2}{S}$$

$$\begin{cases} L_0 = \frac{1}{2S}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\ L_1 = \frac{1}{2S}[(x_3y_0 - x_0y_3) + (y_3 - y_0)x + (x_0 - x_3)y] \\ L_2 = \frac{1}{2S}[(x_0y_1 - x_1y_0) + (y_0 - y_1)x + (x_1 - x_0)y] \end{cases}$$

因为

$$\begin{cases} L_0 = \begin{cases} 1, & x = x_0, y = y_0 \\ 0, & x = x_1, y = y_1 \\ 0, & x = x_2, y = y_2 \end{cases} \\ L_1 = \begin{cases} 0, & x = x_0, y = y_0 \\ 1, & x = x_1, y = y_1 \\ 0, & x = x_2, y = y_2 \end{cases} \\ L_2 = \begin{cases} 0, & x = x_0, y = y_0 \\ 0, & x = x_1, y = y_1 \\ 1, & x = x_2, y = y_2 \end{cases} \end{cases}$$

所以在此区间上 $\varphi_i = L_i$。

4 形成线性方程组

4.1 总刚度矩阵

设 $\varphi_{xi} = (\varphi_i, 0)^t, \varphi_{yi} = (0, \varphi_i)^t$，i = 0, ... , n 为试探函数空间 U_h 的基函数，则任一 $u_h \in U_h$ 可表成

$$u_h = \sum_{i=1}^{n} u_1^i \varphi_{xi} + \sum_{i=1}^{n} u_2^i \varphi_{yi}, \quad u^i = u_h(x_i, y_i)$$

令 $\phi_{2i} = \varphi_{xi}$ ，$\phi_{2i+1} = \varphi_{yi}$, $c_{2i} = u_1^i$ ，$c_{2i+1} = u_2^i$ 带入变分形式得

$$\sum_{j=0}^{2n+1} a(\phi_j, \phi_i) c_i = (f, \phi_i) \quad i = 0, ..., 2n+1$$

矩阵形式为

$$Ac = F$$
$$A = (a(\phi_i, \phi_j))_{(2n+1)\times(2n+1)}$$
$$F = ((f, \phi_i))_{(2n+1)\times 1}$$
$$c = (c_i)_{(2n+1)\times 1}$$

4.2 边界条件

模型为齐次边界条件，若 (x_i, y_i) 为边界点，则 A 第 2i 行第 2i 列，第 2i+1 行第 2i+1 列元素为 1，其他元素及 F(2i)，F(2i+1) 都为 0。