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# Weak Galerkin finite element method for linear elasticity interface problems

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ABSTRACT

In this paper, we apply a weak Galerkin finite element method to a linear elasticity interface model. Since the solution may become discontinuous while crossing the interface, we first discretize the model by double-valued weak functions on the interface. Then, in order to facilitate theoretical analysis and algorithm implementation, we substitute interface conditions into the weak Galerkin formulation and construct a weak Galerkin method with single-valued functions on the interface. Furthermore, we prove the well-posedness of the weak Galerkin scheme and derive a priori error estimates in energy norm and $L^2$ norm. Finally, we present some numerical experiments to demonstrate the efficiency and the locking-free property of our method.



## 1. Introduction

Linear elasticity equations often occur in modeling solid mechanics. Such equations describe the relationship between displacement, stress and strain of an object under external force. Usually, the elastic body is made of multiple materials, and the corresponding material coefficients, displacement, stress and strain may appear jumps while crossing the interface. This is a typical linear elasticity interface problem. Linear elasticity interface models are also applied to continuum mechanics [6,31], microstructural evolution [11,12] and so on.

Due to the important applications of linear elasticity interface models, the corresponding numerical methods have received increasing attention. In order to design efficient numerical methods, some key points need to be concerned. Firstly, when the material is nearly incompressible (One of Lamé parameters tends to infinite), the numerical solution obtained by the lowest-order conforming finite element method shows poor convergence rate, which is called locking phenomenon [2,3]. Secondly, the Crouzeix-Raviart P1-nonconforming element is not applicable to linear elasticity model due to its violating the Korn's inequality [7]. Thirdly, the unknowns involved in interface model may become discontinuous while crossing the interface, then some numerical methods based on continuity assumption are no longer appropriate. Finally, the geometric complexity of interface sometimes increases the difficulty for mesh generation. Therefore, it is a challenge to construct efficient numerical methods to approximate linear elasticity interface models.

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In the past two decades, various numerical methods have been developed for linear elasticity interface problems. Xie et al. [30] and Zhang et al. [33] respectively apply the conforming finite element method and the discontinuous Galerkin method to linear elasticity interface models on fitted meshes, where they present some numerical experiments to demonstrate the effectiveness of their approximation. Sometimes, the generation of fitted meshes consumes much time and energy, then the methods based on unfitted meshes are proposed and developed. One kind of the unfitted methods is the Nitsche extended finite element method [1,10], which constructs a solution on each subdomain, and then implements the interface conditions by Nitsche's technique. Another popular method is the immersed finite element method [8,14,15], whose fundamental idea is to modify the basis functions along the interface to satisfy interface conditions. In addition to above finite element methods, some numerical approaches based on finite difference method have been successfully applied to linear elasticity interface models, such as the immersed interface method [32], the matched interface and boundary method [24,25].

Weak Galerkin (WG) finite element method is an innovative numerical method developed in recent years, which was first proposed by Wang and Ye [27] in 2013 for a second order elliptic equation. The key idea of WG method is to replace the traditional differential operators by weak differential operators in numerical scheme. Later on, in order to ensure the uniqueness of numerical solutions, a stabilizer is introduced to the numerical formulation [28]. Then this method has been widely applied to various models, such as Brinkman equations [18,29], linear elasticity problems [4,26], eigenvalue problems [34,35], Biharmonic equations [19,36], stochastic parabolic equations [37,38] and so on.

People already investigate the WG discretizations for interface models. Mu et al. [17] first apply the original WG method [27] adopting RT elements for elliptic interface problems, and their numerical experiments show the convergence order is between 1.75 and 2 in $L^\infty$ norm. Mu et al. [20] reconsider the same model by a WG method equipped with a stabilizer, where they use piecewise high order polynomials on polygon meshes. Mu [16] also present a posteriori error estimate of WG method for elliptic interface problems, and investigate the reliability and efficiency. Song et al. [22] introduce an over-penalized WG method with double-valued edge functions for elliptic interface problems, and they point out that this method is applicable to problems with local low regularity. Mu and Zhang [21] develop an immersed weak Galerkin finite element method on unfitted meshes for elliptic interface problems, and then Dehghan and Gharibi [5] apply this method to a second-order hyperbolic interface problem.

The purpose of this paper is aimed to develop a WG method to approximate an elasticity interface model. For any given partition, we discretize the displacement variable by linear elements inside the cell and by the trace of rigid body motions on its boundary. This WG element is proposed in Wang et al. [26] for linear elasticity problems. Obviously, the solution may become discontinuous while crossing the interface, then it is natural to describe the numerical solution on the interface by double-valued weak functions, which is not convenient for theoretical analysis and numerical implementation. Thus, we use the same technique in Mu et al. [20] to incorporate the interface conditions into the numerical scheme. Then, we get a new WG scheme with single-valued elements on the interface. Furthermore, we obtain the error estimates and optimal convergence order of the numerical solution, which also implies the locking-free property of the WG method.

This paper is organized as follows. In Section 2, we introduce a linear elasticity interface model. And in Section 3, we construct WG methods for this model. Then, in Section 4, we prove the well-posedness of the WG solution. Furthermore, in Section 5, we derive a priori error estimates in energy norm and $L^2$ norm. Finally, in Section 6, we present some numerical experiments to demonstrate our theoretical analysis.

## 2. Model problem

Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$. We suppose that $\Omega$ is divided into two disjoint subdomains $\Omega^+$ and $\Omega^-$ by a $C^2$ curve $\Gamma = \partial\Omega^+ \cap \partial\Omega^-$. Then we consider a linear elasticity interface problem on $\Omega$: find displacement $\mathbf{u}$ such that

$$-\nabla\cdot\sigma(\mathbf{u}) = \mathbf{f}, \qquad \text{in } \Omega^\pm, \tag{2.1}$$

$$\mathbf{u} = \mathbf{0}, \qquad \text{on } \partial\Omega, \tag{2.2}$$

$$[\![\mathbf{u}]\!] = \mathbf{u}|_{\Omega^+} - \mathbf{u}|_{\Omega^-} = \boldsymbol{\psi}, \qquad \text{on } \Gamma, \tag{2.3}$$

$$[\![\sigma(\mathbf{u})\mathbf{n}]\!] = \sigma(\mathbf{u})|_{\Omega^+}\mathbf{n}^+ + \sigma(\mathbf{u})|_{\Omega^-}\mathbf{n}^- = \boldsymbol{\phi}, \qquad \text{on } \Gamma, \tag{2.4}$$

where $\mathbf{f}\in[L^2(\Omega)]^2$ denotes an external force, $\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda\nabla\cdot\mathbf{u}I$ is a linear strain tensor, and $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is a symmetric Cauchy stress tensor, $I$ is a identity matrix. The Lamé parameters $\lambda$ and $\mu$ are piecewise constants

$$(\lambda(\mathbf{x}), \mu(\mathbf{x})) = \begin{cases} (\lambda^+, \mu^+), & \mathbf{x}\in\Omega^+, \\ (\lambda^-, \mu^-), & \mathbf{x}\in\Omega^-. \end{cases}$$

They are respectively determined by the modulus of elasticity $E > 0$ and the Poisson ratio $0 < \nu < \frac{1}{2}$,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

From [10,13], it follows that there is a unique solution $\mathbf{u} \in [H^2(\Omega^+ \cup \Omega^-)]^2$ of (2.1)–(2.4). We assume that the solution $\mathbf{u}$ satisfies a regularity property [2,23].

$$||\mathbf{u}||_{[\mathbf{H}^2(\Omega^+\cup\Omega^-)]^2} + |\lambda| \cdot ||\nabla \cdot \mathbf{u}||_{[\mathbf{H}^1(\Omega^+\cup\Omega^-)]^2} \le \mathbf{C}\Big(||\mathbf{f}||_{[\mathbf{L}^2(\Omega)]^2} + ||\boldsymbol{\psi}||_{[\mathbf{H}^{1/2}(\Gamma)]^2} + ||\boldsymbol{\phi}||_{[\mathbf{H}^{1/2}(\Gamma)]^2}\Big). \tag{2.5}$$

## 3. Weak Galerkin discretization

In this section, we first introduce weak functions and weak function spaces. Then, we define weak differential operators acting on weak functions. Based upon these settings, we construct a WG method to approximate the solution of the linear elasticity interface model (2.1)–(2.4).

### 3.1. Weak Galerkin discrete space

In this part, we will introduce a basic concept of weak function and some weak function spaces.

Let $\mathcal{T}_h^i$ be a regular partition [28] of $\Omega^i$, respectively, for $i \in \{+,-\}$. We assume that $\mathcal{T}_h^+$ and $\mathcal{T}_h^-$ are aligned along $\Gamma$. Let $\mathcal{T}_h = \mathcal{T}_h^+ \cup \mathcal{T}_h^-$, then it is a fitted partition of $\Omega$. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$. The set of the edges whose vertices all lie on the interface is denoted by $\Gamma_h$. For each $T \in \mathcal{T}_h$, we denote by $h_T$ the diameter of $T$, then $h = \max_{T\in\mathcal{T}_h} h_T$ characterizes the size of the partition $\mathcal{T}_h$.

Define a weak function on $T$

$$\mathbf{v} = \begin{cases} \mathbf{v}_0 \in \left[L^2(T)\right]^2, & \textit{for} \ \ \mathbf{x} \in T, \\ \mathbf{v}_b \in \left[L^2(\partial T)\right]^2, & \textit{for} \ \ \mathbf{x} \in \partial T. \end{cases}$$

A weak function is formed by an internal function $\mathbf{v}_0$ and a boundary function $\mathbf{v}_b$, where $\mathbf{v}_b$ may not necessarily be related to the trace of $\mathbf{v}_0$ on $\partial T$. For convenience, we write $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$.

Next, define a local weak function space on $T$

$$W(T) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \ \mathbf{v}_0 \in [L^2(T)]^2, \mathbf{v}_b \in [L^2(\partial T)]^2\},$$

and a global weak function space on $\Omega$

$$W(\Omega) = \{\mathbf{v} : \ \Omega \to \mathbb{R}^2 | \ \mathbf{v}|_T \in W(T)\}.$$

Define a space of rigid body motions on $T$

$$RM(T) = \{\mathbf{v} : \ T \to \mathbb{R}^2 | \ \mathbf{v}(\mathbf{x}) = \mathbf{a} + \eta\mathbf{x}, \ \mathbf{a} \in \mathbb{R}^2, \ \eta \in so(2)\},$$

where $so(2)$ is the set of skew-symmetric $2 \times 2$ matrices. On each edge $e \in \mathcal{E}_h \cap \partial T$, the trace of the rigid body motions forms a finite dimensional space

$$P_{RM}(e) = \{\mathbf{v} \in [L^2(e)]^2 : \mathbf{v} = \tilde{\mathbf{v}}|_e, \ \text{for some a } \tilde{\mathbf{v}} \in RM(T)\}.$$

For any $T \in \mathcal{T}_h$, define a local WG finite element space

$$V(T) = \{\mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} : \ \mathbf{v}_{h0} \in [P_1(T)]^2, \ \mathbf{v}_{hb} \in P_{RM}(e), \ e \in \mathcal{E}_h \cap \partial T\}.$$

From now on, we shall simply write $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\}$ and no confusion should occur.

Define a global WG finite element space on $\Omega$

$$V_h = \{\mathbf{v}_h \in W(\Omega) : \ \mathbf{v}_h|_T \in V(T), \ T \in \mathcal{T}_h\},$$

and its subspace

$$V_h^0 = \{\mathbf{v}_h \in V_h : \ \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega\}.$$

Similarly, we define finite element spaces $V_h^{\pm}$ on $\Omega^{\pm}$

$$V_h^{\pm} = \{\mathbf{v}_h \in W(\Omega^{\pm}) : \ \mathbf{v}_h|_T \in V(T), \ T \in \mathcal{T}_h^{\pm}\},$$

and their subspaces

$$V_h^{\pm,0} = \{\mathbf{v}_h \in V_h^{\pm} : \ \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega^{\pm}/\Gamma\}.$$

Notice that any weak function $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h^0$ only has a single-valued $\mathbf{v}_b$ on each edge $e \in \mathcal{E}_h$, which is hard to describe the jump phenomenon on $\Gamma_h$. To overcome this difficulty, we define a global finite element space

$$\tilde{V}_h^0 = V_h^{+,0} \times V_h^{-,0}.$$

Note that functions in $\tilde{V}_h^0$ are double-valued on $e \in \Gamma_h$, which are flexible to characterize the jump properties on $\Gamma_h$.

### 3.2. Discrete weak differential operators

In this subsection, we extend the concept of differential operators to a weak version acting on WG finite element spaces.

**Definition 3.1.** For any $T \in \mathcal{T}_h$, define a local discrete weak gradient operator $\nabla_{w,T}$: $V(T) \to [P_0(T)]^{2\times 2}$, such that for any $\mathbf{v}_h \in V(T)$, $\nabla_{w,T}\mathbf{v}_h$ is the unique polynomial satisfying

$$(\nabla_{w,T}\mathbf{v}_h, \tau_h)_T = -(\mathbf{v}_0, \nabla\tau_h)_T + \langle \mathbf{v}_b, \tau_h\mathbf{n}\rangle_{\partial T}, \quad \forall\ \tau_h \in [P_0(T)]^{2\times 2}, \tag{3.1}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial T$.

Analogously, we define a local discrete weak divergence operator.

**Definition 3.2.** For any $T \in \mathcal{T}_h$, define a local discrete weak divergence operator $\nabla_{w,T}\cdot$ : $V(T) \to P_0(T)$, such that for any $\mathbf{v}_h \in V(T)$, $\nabla_{w,T}\cdot\mathbf{v}_h$ is the unique polynomial satisfying

$$(\nabla_{w,T}\cdot\mathbf{v}_h, q_h)_T = -(\mathbf{v}_0, \nabla q_h)_T + \langle \mathbf{v}_b, q_h\mathbf{n}\rangle_{\partial T}, \quad \forall\ q_h \in P_0(T). \tag{3.2}$$

Now, we define the global weak differential operators $\nabla_w$ and $\nabla_w\cdot$ by assembling the local counterparts, i.e. for any $\mathbf{v}_h \in V_h^0$,

$$(\nabla_w\mathbf{v}_h)|_T = \nabla_{w,T}(\mathbf{v}_h|_T), \quad (\nabla_w\cdot\mathbf{v}_h)|_T = \nabla_{w,T}\cdot(\mathbf{v}_h|_T).$$

For simplicity of notations, we respectively use $\nabla_w$ and $\nabla_w\cdot$ to denote both the local and global weak gradient operators and weak divergence operators.

To end this subsection, we define a weak strain tensor and a weak stress tensor, respectively, for $\mathbf{v}_h \in V_h^0$,

$$\varepsilon_w(\mathbf{v}_h) = \frac{1}{2}(\nabla_w\mathbf{v}_h + \nabla_w^T\mathbf{v}_h), \tag{3.3}$$

$$\sigma_w(\mathbf{v}_h) = 2\mu\varepsilon_w(\mathbf{v}_h) + \lambda\nabla_w\cdot\mathbf{v}_h I. \tag{3.4}$$

### 3.3. Numerical scheme

Now, we start to construct a WG method for linear elasticity interface model (2.1)–(2.4).

For $i \in \{+,-\}$ and $\mathbf{v}_h,\ \mathbf{w}_h \in V_h^{i,0}$, define bilinear forms

$$s^i(\mathbf{v}_h, \mathbf{w}_h) = \textstyle\sum_{T\in\mathcal{T}_h^i} h_T^{-1}\langle Q_b\mathbf{v}_0 - \mathbf{v}_b, Q_b\mathbf{w}_0 - \mathbf{w}_b\rangle_{\partial T},$$
$$a_h^i(\mathbf{v}_h, \mathbf{w}_h) = \textstyle\sum_{T\in\mathcal{T}_h^i}\left(\left(2\mu^i\varepsilon_w(\mathbf{v}_h), \varepsilon_w(\mathbf{w}_h)\right)_T + \left(\lambda^i\nabla_w\cdot\mathbf{v}_h, \nabla_w\cdot\mathbf{w}_h\right)_T\right) + s^i(\mathbf{v}_h, \mathbf{w}_h).$$

In a similar way, we define two bilinear forms $s(\mathbf{v}_h, \mathbf{w}_h)$ and $a_h(\mathbf{v}_h, \mathbf{w}_h)$ on $V_h^0 \times V_h^0$.

$$s(\mathbf{v}_h, \mathbf{w}_h) = \textstyle\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b\mathbf{v}_0 - \mathbf{v}_b, Q_b\mathbf{w}_0 - \mathbf{w}_b\rangle_{\partial T},$$
$$a_h(\mathbf{v}_h, \mathbf{w}_h) = \textstyle\sum_{T\in\mathcal{T}_h}((2\mu\varepsilon_w(\mathbf{v}_h), \varepsilon_w(\mathbf{w}_h))_T + (\lambda\nabla_w\cdot\mathbf{v}_h, \nabla_w\cdot\mathbf{w}_h)_T) + s(\mathbf{v}_h, \mathbf{w}_h).$$

For each $e \in \mathcal{E}_h$, denote by $Q_b$ the orthogonal $L^2$-projection from $[L^2(e)]^2$ onto $P_{RM}(e)$.

The WG finite element method for (2.1)–(2.4) is to seek a function $\mathbf{u}_h \in \tilde{V}_h^0$ such that

$$a_h^i(\mathbf{u}_h, \mathbf{v}_h^i) - \langle\sigma_w(\mathbf{u}_h|_{T^i})\mathbf{n}^i, \mathbf{v}_b^i\rangle_{\Gamma_h} = (\mathbf{f}, \mathbf{v}_0^i)_{\Omega^i}, \qquad \forall\ \mathbf{v}_h^i \in V_h^{i,0},\ i \in \{+,-\}, \tag{3.5}$$

$$\mathbf{u}_b|_{(\partial T^-\cap e)} = \mathbf{u}_b|_{(\partial T^+\cap e)} - Q_b\boldsymbol{\psi}|_e, \forall\ e \in \Gamma_h,\ e = \partial T^+ \cap \partial T^-, \tag{3.6}$$

$$(\sigma_w(\mathbf{u}_h|_{T^+})\mathbf{n}^+)|_e + (\sigma_w(\mathbf{u}_h|_{T^-})\mathbf{n}^-)|_e = Q_b\boldsymbol{\phi}|_e, \qquad \forall\ e \in \Gamma_h,\ e = \partial T^+ \cap \partial T^-. \tag{3.7}$$

Here, for any $e \in \Gamma_h$, $\mathbf{n}^+$ denotes the unit normal vector of $e$ pointing from $\Omega^+$ to $\Omega^-$, and $\mathbf{n}^- = -\mathbf{n}^+$. In the rest of this paper, for any $e \in \Gamma_h$, we will fix a unit normal vector of $e$ by $\mathbf{n}_e = \mathbf{n}^+$ in further computation.

In order to reduce the computation complexity of above algorithm, we remove $\mathbf{u}_b|_{\partial T^-\cap e}$ by substitute (3.6) and (3.7) into (3.5) and then we obtain a simplified WG finite element method: find $\mathbf{u}_h \in V_h^0$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_0) + \langle\boldsymbol{\phi}, \mathbf{v}_b\rangle_{\Gamma_h} + \alpha_h(\mathbf{v}_h) - \beta_h(\mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in V_h^0, \tag{3.8}$$

where

$$\alpha_h(\mathbf{v}_h) = \textstyle\sum_{e\in\Gamma_h}\left\langle\boldsymbol{\psi}, \sigma_w\left(\mathbf{v}_h|_{T_e^-}\right)\mathbf{n}_e\right\rangle_e,$$
$$\beta_h(\mathbf{v}_h) = \textstyle\sum_{e\in\Gamma_h} h_{T_e^-}^{-1}\left\langle\boldsymbol{\psi}, Q_b\left(\mathbf{v}_0|_{T_e^-}\right) - \mathbf{v}_b\right\rangle_e.$$

Here, $T_e^- \in \mathcal{T}_h^-$ is the cell with $e$ as its one edge.

Once having solved (3.8) in $V_h^0$, we obtain the solution of (3.5) in $\tilde{V}_h^0$ by (3.6).

## 4. Existence and uniqueness

In this section, our main purpose is to study the well-posedness of WG method (3.8).

Define a functional on $V_h^0$

$$|||\mathbf{v}_h||| = \sqrt{a_h(\mathbf{v}_h, \mathbf{v}_h)}.$$

**Lemma 1.** *$|||\cdot|||$ is a norm in $V_h^0$.*

**Proof.** It is obvious that $|||\cdot|||$ is a semi-norm in $V_h^0$. We only need to verify that $|||\mathbf{v}_h||| = 0$ implies $\mathbf{v}_h = \mathbf{0}$. Assume $|||\mathbf{v_h}||| = 0$ for a $\mathbf{v}_h \in V_h^0$, then we have

$$\begin{aligned} &\varepsilon_w(\mathbf{v}_h) = 0, \text{ and } \nabla_w \cdot \mathbf{v}_h = 0, && \text{in each } T \in \mathcal{T}_h, \\ &Q_b\mathbf{v}_0 - \mathbf{v}_b = \mathbf{0}, && \text{on each } e \in \mathcal{E}_h. \end{aligned} \tag{4.1}$$

For any $\tau_h \in [P_0(T)]^{2\times 2}$,

$$\begin{aligned} (\nabla_w \mathbf{v}_h, \tau_h)_T &= -(\mathbf{v}_0, \nabla\cdot\tau_h)_T + \langle \mathbf{v}_b, \tau_h\mathbf{n}\rangle_{\partial T} \\ &= (\nabla\mathbf{v}_0, \tau_h)_T - \langle \mathbf{v}_0, \tau_h\mathbf{n}\rangle_{\partial T} + \langle \mathbf{v}_b, \tau_h\mathbf{n}\rangle_{\partial T} \\ &= (\nabla\mathbf{v}_0, \tau_h)_T - \langle Q_b\mathbf{v}_0 - \mathbf{v}_b, \tau_h\mathbf{n}\rangle_{\partial T} \\ &= (\nabla\mathbf{v}_0, \tau_h)_T. \end{aligned}$$

This implies that $\nabla_w\mathbf{v}_h = \nabla\mathbf{v}_0$. Then we obtain

$$\varepsilon(\mathbf{v}_0) = \frac{1}{2}(\nabla\mathbf{v}_0 + \nabla^T\mathbf{v}_0) = \frac{1}{2}(\nabla_w\mathbf{v}_h + \nabla_w^T\mathbf{v}_h) = \varepsilon_w(\mathbf{v}_h) = 0.$$

Thus, $\nabla\mathbf{v}_0$ is a skew-symmetric matrix. Combining with $\mathbf{v}_0 \in [P_1(T)]^2$, we know that $\mathbf{v}_0|_e \in P_{RM}(e)$, which means $Q_b\mathbf{v}_0 = \mathbf{v}_0$ on $e \in \mathcal{E}_h$. By virtue of Eq. (4.1) and the fact $\mathbf{v}_h = \mathbf{0}$ on $\partial\Omega$, we have $\mathbf{v}_h \in [H_0^1(\Omega)]^2$. According to the estimate (A.7), we get $\|\mathbf{v}_h\|_1 = 0$, which implies $\mathbf{v}_h = \mathbf{0}$ in $\Omega$. □

**Lemma 2.** *The WG method (3.8) has a unique solution.*

**Proof.** The Eq. (3.8) is actually a finite dimensional linear system and then it suffices to show that the solution is unique. Set $\mathbf{f} = \mathbf{0}$, $\boldsymbol{\phi} = \mathbf{0}$, $\boldsymbol{\psi} = \mathbf{0}$. Then taking $\mathbf{v}_h = \mathbf{u}_h$, we get

$$|||\mathbf{u}_h|||^2 = a_h(\mathbf{u}_h, \mathbf{u}_h) = 0,$$

which implies that $\mathbf{u}_h = \mathbf{0}$. □

Throughout this paper, we will use $\mathbf{u} \in [H^2(\Omega^+ \cup \Omega^-)]^2$ to denote the exact solution of (2.1)–(2.4) and $\mathbf{u}_h \in V_h^0$ to denote the numerical solution of (3.8).

At the end of this part, we present some estimates used in later error analysis.

**Lemma 3.** *([29]) There exists a constant $C$ independent of $h$ such that*

$$||\mathbf{v}_0 - Q_b\mathbf{v}_0||_{\partial T}^2 \le Ch_T||\varepsilon(\mathbf{v}_0)||_T^2, \quad \forall\ \mathbf{v}_0 \in [P_1(T)]^2.$$

**Lemma 4.** *There exists a constant $C$ independent of $h$ such that*

$$\sum_{T\in\mathcal{T}_h} h_T^{-1/2}||\mathbf{v}_b - Q_b\mathbf{v}_0||_{\partial T} \le C|||\mathbf{v}_h|||, \quad \forall\ \mathbf{v}_h \in \mathrm{V}_\mathrm{h}^0.$$

**Proof.** According to Lemma 3, we only need to prove that $||\varepsilon(\mathbf{v}_0)|| \le C|||\mathbf{v}_h|||$. Notice that

$$\begin{aligned} (\varepsilon_w(\mathbf{v}_h), \tau_h)_T &= \tfrac{1}{2}(\nabla_w\mathbf{v}_h, \tau_h)_T + \tfrac{1}{2}\left(\nabla_w^T\mathbf{v}_h, \tau_h\right)_T \\ &= \tfrac{1}{2}\left(\nabla_w\mathbf{v}_h, \tau_h + \tau_h^T\right)_T \\ &= -\tfrac{1}{2}\left(\mathbf{v}_0, \nabla\cdot\left(\tau_h + \tau_h^T\right)\right)_T + \tfrac{1}{2}\langle \mathbf{v}_b, \left(\tau_h + \tau_h^T\right)\mathbf{n}\rangle_{\partial T} \\ &= \tfrac{1}{2}\left(\nabla\mathbf{v}_0, \left(\tau_h + \tau_h^T\right)\right)_T - \tfrac{1}{2}\left(\mathbf{v}_0 - \mathbf{v}_b, \left(\tau_h + \tau_h^T\right)\mathbf{n}\right)_{\partial T} \\ &= (\varepsilon(\mathbf{v}_0), \tau_h)_T - \tfrac{1}{2}\langle Q_b\mathbf{v}_0 - \mathbf{v}_b, \left(\tau_h + \tau_h^T\right)\mathbf{n}\rangle_{\partial T}. \end{aligned}$$

By virtue of the trace inequality (A.4) and the inverse inequality (A.5), we get

$$\begin{aligned} |(\varepsilon(\mathbf{v_0}), \tau_h)_T| &\le ||\varepsilon_w(\mathbf{v}_h)||_T||\tau_h||_T + ||Q_b\mathbf{v}_0 - \mathbf{v}_\mathrm{b}||_{\partial T}\left|\left|\left(\tau_h + \tau_h^T\right)\mathbf{n}\right|\right|_{\partial T} \\ &\le \left(||\varepsilon_w(\mathbf{v}_h)||_T + Ch_T^{-1/2}||Q_b\mathbf{v}_0 - \mathbf{v}_\mathrm{b}||_{\partial T}\right)||\tau_h||_T. \end{aligned}$$

Taking $\tau_h = \varepsilon(\mathbf{v}_0)$ in above equation gives

$$||\varepsilon(\mathbf{v}_0)||_T \le ||\varepsilon_w(\mathbf{v}_h)||_T + Ch_h^{-1/2}||Q_b\mathbf{v}_0 - \mathbf{v}_b||_{\partial T} \le C\ |||\mathbf{v}_h|||.$$

This completes the proof of the lemma. □

## 5. Error analysis

In this section, we first derive an error equation and then investigate a priori error estimates in energy norm and $L^2$ norm.

For each $T \in \mathcal{T}_h$, define an orthogonal $L^2$-projection $Q_0$: $[L^2(T)]^2 \to [P_1(T)]^2$. Then we define two global projections, also denoted by $Q_0$ and $Q_b$, assembled peicewisely on the interior and boundary of each $T \in \mathcal{T}_h$. Combining the projection $Q_0$ and $Q_b$, we have the projection $Q_h\mathbf{v} = \{Q_0\mathbf{v}, Q_b\mathbf{v}\}$. By $\mathbf{Q}_h$, we denote an orthogonal $L^2$-projection from $[L^2(T)]^{2\times 2}$ onto $[P_0(T)]^{2\times 2}$, and by $\mathbb{Q}_h$ the orthogonal $L^2$-projection from $L^2(T)$ onto $P_0(T)$.

In order to ensure that the exact solution $\mathbf{u}$ is well projected onto $V_h^0$, we define $(Q_b\mathbf{u})|_e = Q_b(\mathbf{u}^+|_e)$ for $e \in \Gamma_h$, and hence $Q_b(\mathbf{u}^-|_e) = Q_b\mathbf{u} - Q_b\boldsymbol{\psi}$.

**Lemma 5.** [21] *On each $T \in \mathcal{T}_h$, there holds for any $\tau_h \in [P_0(T)]^{2\times 2}$ and $q_h \in P_0(T)$*

- *if $T \notin \mathcal{T}_h^-$ or $\partial T \cap \Gamma_h = \emptyset$,*

$$(\nabla_w(Q_h\mathbf{u}), \tau_h)_T = (\mathbf{Q}_h(\nabla\mathbf{u}), \tau_h)_T, \tag{5.1}$$

$$(\nabla_w \cdot (Q_h\mathbf{u}), q_h)_T = (\mathbb{Q}_h(\nabla\cdot\mathbf{u}), q_h)_T, \tag{5.2}$$

- *if $T \in \mathcal{T}_h^-$ and $\partial T \cap \Gamma_h \neq \emptyset$,*

$$(\nabla_w(Q_h\mathbf{u}), \tau_h)_T = (\mathbf{Q}_h(\nabla\mathbf{u}), \tau_h)_T + \langle\boldsymbol{\psi}, \tau_h\mathbf{n}\rangle_{\partial T\cap\Gamma_h}, \tag{5.3}$$

$$(\nabla_w \cdot (Q_h\mathbf{u}), q_h)_T = (\mathbb{Q}_h(\nabla\cdot\mathbf{u}), q_h)_T + \langle\boldsymbol{\psi}, q_h\mathbf{n}\rangle_{\partial T\cap\Gamma_h}. \tag{5.4}$$

Define an error function $\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = \mathbf{u}_h - Q_h\mathbf{u} \in V_h^0$.

**Lemma 6.** *The error function $\mathbf{e}_h$ satisfies*

$$a_h(\mathbf{e}_h, \mathbf{v}_h) = \ell_1(\mathbf{u}, \mathbf{v}_h) - \ell_2(\mathbf{u}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in V_h^0, \tag{5.5}$$

*where*

$$\begin{aligned}
\ell_1(\mathbf{u}, \mathbf{v}_h) &= \textstyle\sum_{T\in\mathcal{T}_h} \langle 2\mu(\mathbf{Q}_h\varepsilon(\mathbf{u}) - \varepsilon(\mathbf{u}))\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} \\
&\qquad\qquad + \textstyle\sum_{T\in\mathcal{T}_h} \langle\lambda(\mathbb{Q}_h(\nabla\cdot\mathbf{u}) - \nabla\cdot\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}, \\
\ell_2(\mathbf{u}, \mathbf{v}_h) &= \textstyle\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0\mathbf{u} - \mathbf{u}, Q_b\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}.
\end{aligned}$$

**Proof.** For any $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h^0$, testing (2.1) with $\mathbf{v}_0$ and using the interface condition (2.4), we have

$$\begin{aligned}
(\mathbf{f}, \mathbf{v}_0) &= -\textstyle\sum_{T\in\mathcal{T}_h} (\nabla\cdot\sigma(\mathbf{u}), \mathbf{v}_0)_T \\
&= \textstyle\sum_{T\in\mathcal{T}_h} (\sigma(\mathbf{u}), \nabla\mathbf{v}_0)_T - \sum_{T\in\mathcal{T}_h} \langle\sigma(\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle\sigma(\mathbf{u})\mathbf{n}, \mathbf{v}_b\rangle_{\partial T} \\
&= \textstyle\sum_{T\in\mathcal{T}_h} (\sigma(\mathbf{u}), \nabla\mathbf{v}_0)_T - \sum_{T\in\mathcal{T}_h} \langle\sigma(\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} - \langle\boldsymbol{\phi}, \mathbf{v}_b\rangle_{\Gamma_h} \\
&= \textstyle\sum_{T\in\mathcal{T}_h} ((2\mu\varepsilon(\mathbf{u}), \nabla\mathbf{v}_0)_T + (\lambda(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}_0)_T) \\
&\quad - \textstyle\sum_{T\in\mathcal{T}_h} \langle 2\mu\varepsilon(\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle\lambda(\nabla\cdot\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} - \langle\boldsymbol{\phi}, \mathbf{v}_b\rangle_{\Gamma_h}.
\end{aligned} \tag{5.6}$$

Direct computation gives

$$\begin{aligned}
&\textstyle\sum_{T\in\mathcal{T}_h} ((2\mu\varepsilon(\mathbf{u}), \nabla\mathbf{v}_0)_T + (\lambda(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}_0)_T) \\
&= \textstyle\sum_{T\in\mathcal{T}_h} ((2\mu\mathbf{Q}_h\varepsilon(\mathbf{u}), \nabla\mathbf{v}_0)_T + (\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}_0)_T) \\
&= \textstyle\sum_{T\in\mathcal{T}_h} (-(\nabla\cdot 2\mu\mathbf{Q}_h\varepsilon(\mathbf{u}), \mathbf{v}_0)_T - (\nabla\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u}), \mathbf{v}_0)_T) \\
&\quad + \textstyle\sum_{T\in\mathcal{T}_h} (\langle 2\mu\mathbf{Q}_h\varepsilon(\mathbf{u})\mathbf{n}, \mathbf{v}_0\rangle_{\partial T} + \langle\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u})\mathbf{n}, \mathbf{v}_0\rangle_{\partial T}) \\
&= \textstyle\sum_{T\in\mathcal{T}_h} (2\mu\mathbf{Q}_h\varepsilon(\mathbf{u}), \nabla_w\mathbf{v}_h)_T + \sum_{T\in\mathcal{T}_h} (\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u}), \nabla_w\cdot\mathbf{v}_h)_T \\
&\quad + \textstyle\sum_{T\in\mathcal{T}_h} \langle 2\mu\mathbf{Q}_h\varepsilon(\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h} \langle\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} \\
&= \textstyle\sum_{T\in\mathcal{T}_h} (2\mu\mathbf{Q}_h\varepsilon(\mathbf{u}), \varepsilon_w(\mathbf{v}_h))_T + \sum_{T\in\mathcal{T}_h} (\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u}), \nabla_w\cdot\mathbf{v}_h)_T \\
&\quad + \textstyle\sum_{T\in\mathcal{T}_h} \langle 2\mu\mathbf{Q}_h\varepsilon(\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h} \langle\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u})\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}.
\end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.6) and applying Lemma 5, we have

$$\begin{aligned}(\mathbf{f},\mathbf{v}_0) &= \sum_{T\in\mathcal{T}_h}(2\mu\mathbf{Q}_h\varepsilon(\mathbf{u}),\varepsilon_w(\mathbf{v}_h))_T+\sum_{T\in\mathcal{T}_h}(\lambda\mathbb{Q}_h(\nabla\cdot\mathbf{u}),\nabla_w\cdot\mathbf{v}_h)_T-\langle\boldsymbol{\phi},\mathbf{v}_b\rangle_{\Gamma_h}\\ &\quad+\sum_{T\in\mathcal{T}_h}\langle 2\mu(\mathbf{Q}_h\varepsilon(\mathbf{u})-\varepsilon(\mathbf{u}))\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}\\ &\quad+\sum_{T\in\mathcal{T}_h}\langle\lambda(\mathbb{Q}_h(\nabla\cdot\mathbf{u})-\nabla\cdot\mathbf{u})\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}\\ &=\sum_{T\in\mathcal{T}_h}(2\mu\varepsilon_w(Q_h\mathbf{u}),\varepsilon_w(\mathbf{v}_h))_T+\sum_{T\in\mathcal{T}_h}(\lambda\nabla_w\cdot Q_h\mathbf{u},\nabla_w\cdot\mathbf{v}_h)_T\\ &\quad-\alpha_h(\mathbf{v}_h)+\ell_1(\mathbf{u},\mathbf{v}_h)-\langle\boldsymbol{\phi},\mathbf{v}_b\rangle_{\Gamma_h}.\end{aligned} \tag{5.8}$$

Adding $s(Q_h\mathbf{u},\mathbf{v}_h)$ to both sides of above equation, we get

$$a_h(Q_h\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_0)+s(Q_h\mathbf{u},\mathbf{v}_h)-\ell_1(\mathbf{u},\mathbf{v}_h)+\alpha_h(\mathbf{v}_h)+\langle\boldsymbol{\phi},\mathbf{v}_b\rangle_{\Gamma_h}.$$

From the definition of $s(\cdot,\cdot)$, it follows that

$$\begin{aligned}s(Q_h\mathbf{u},\mathbf{v}_h)&=\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle Q_b(Q_0\mathbf{u})-Q_b\mathbf{u},Q_b\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}-\beta_h(\mathbf{v}_h)\\ &=\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle Q_b(Q_0\mathbf{u}-\mathbf{u}),Q_b\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}-\beta_h(\mathbf{v}_h)\\ &=\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle Q_0\mathbf{u}-\mathbf{u},Q_b\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}-\beta_h(\mathbf{v}_h)\\ &=\ell_2(\mathbf{u},\mathbf{v}_h)-\beta_h(\mathbf{v}_h).\end{aligned}$$

Thus, we get

$$a_h(Q_h\mathbf{u},\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_0)+\langle\boldsymbol{\phi},\mathbf{v}_b\rangle_{\Gamma_h}+\alpha_h(\mathbf{v}_h)-\beta_h(\mathbf{v}_h)-\ell_1(\mathbf{u},\mathbf{v}_h)+\ell_2(\mathbf{u},\mathbf{v}_h).$$

Combing with (3.8), we obtain the error equation (5.5). □

**Lemma 7.** *There hold estimates*

$$|\ell_1(\mathbf{u},\mathbf{v}_h)|\le Ch\Big(||\mathbf{u}||_{2,\Omega^+}+||\mathbf{u}||_{2,\Omega^-}+|\lambda^+|\cdot||\nabla\cdot\mathbf{u}||_{1,\Omega^+}+|\lambda^-|\cdot||\nabla\cdot\mathbf{u}||_{1,\Omega^-}\Big)|||\mathbf{v}_h|||,$$

$$|\ell_2(\mathbf{u},\mathbf{v}_h)|\le Ch\Big(||\mathbf{u}||_{2,\Omega^+}+||\mathbf{u}||_{2,\Omega^-}\Big)|||\mathbf{v}_h|||.$$

**Proof.** By virtue of the trace inequality (A.4), the projection inequality (A.2) and Lemma 4, we have

$$\begin{aligned}|\ell_1(\mathbf{u},\mathbf{v}_h)|^2&\le C\Big(\sum_{T\in\mathcal{T}_h^+\cup\mathcal{T}_h^-}\big(h_T||\mathbf{Q}_h\varepsilon(\mathbf{u})-\varepsilon(\mathbf{u})||_{\partial T}^2+h_T||\lambda(\mathbb{Q}_h(\nabla\cdot\mathbf{u})-\nabla\cdot\mathbf{u})||_{\partial T}^2\big)\Big)\\ &\quad\cdot\Big(\sum_{T\in\mathcal{T}_h}\big(h_T^{-1}||\mathbf{v}_0-Q_b\mathbf{v}_0||_{\partial T}^2+h_T^{-1}||Q_b\mathbf{v}_0-\mathbf{v}_b||_{\partial T}^2\big)\Big)\\ &\le Ch^2\Big(||\mathbf{u}||_{2,\Omega^+}^2+||\mathbf{u}||_{2,\Omega^-}^2+|\lambda^+|^2\cdot||\nabla(\nabla\cdot\mathbf{u})||_{\Omega^+}^2+|\lambda^-|^2\cdot||\nabla(\nabla\cdot\mathbf{u})||_{\Omega^-}^2\Big)|||\mathbf{v}_h|||^2\\ &\le Ch^2\Big(||\mathbf{u}||_{2,\Omega^+}^2+||\mathbf{u}||_{2,\Omega^-}^2+|\lambda^+|^2\cdot||\nabla\cdot\mathbf{u}||_{1,\Omega^+}^2+|\lambda^-|^2\cdot||\nabla\cdot\mathbf{u}||_{1,\Omega^-}^2\Big)|||\mathbf{v}_h|||^2.\end{aligned}$$

Therefore, we have proved the estimate for $\ell_1(\mathbf{u},\mathbf{v}_h)$. In a similar way, we get the estimate for $\ell_2(\mathbf{u},\mathbf{v}_h)$. □

**Theorem 1.** *There exists a constant $C$ independent of $h$ and $\lambda$, such that*

$$|||\mathbf{e}_h|||\le Ch\left(||\mathbf{f}||+\big|\big|\boldsymbol{\psi}\big|\big|_{[H^{1/2}(\Gamma)]^2}+||\boldsymbol{\phi}||_{[H^{1/2}(\Gamma)]^2}\right).$$

**Proof.** Taking $\mathbf{v}_h=\mathbf{e}_h$ in (5.5), it yields

$$|||\mathbf{e}_h|||^2=\ell_1(\mathrm{u},\mathrm{e}_\mathrm{h})-\ell_2(\mathrm{u},\mathrm{e}_\mathrm{h}).$$

According to Lemma 7 and the regularity property (2.5), we get

$$|||\mathbf{e}_h|||\le Ch\left(||\mathbf{f}||+\big|\big|\boldsymbol{\psi}\big|\big|_{[H^{1/2}(\Gamma)]^2}+||\boldsymbol{\phi}||_{[H^{1/2}(\Gamma)]^2}\right).$$

Hence the proof is completed. □

In order to derive an optimal estimate in $L^2$ norm, we consider a dual problem:

$$-\nabla\cdot\sigma(\mathbf{w})=\mathbf{e}_0,\quad\text{in }\Omega^\pm, \tag{5.9}$$

$$\mathbf{u}=\mathbf{0},\quad\text{on }\partial\Omega, \tag{5.10}$$

$$[\![\mathbf{w}]\!]=\mathbf{0},\quad\text{on }\Gamma, \tag{5.11}$$

$$[\![\sigma(\mathbf{w})\mathbf{n}]\!]=\mathbf{0},\quad\text{on }\Gamma. \tag{5.12}$$

According to Hansbo and Hansbo [10], Leguillon and Sánchez-Palencia [13], these equations possess a unique solution $\mathbf{w} \in [H_0^1(\Omega)]^2 \cap [H^2(\Omega^+ \cup \Omega^-)]^2$. We assume that the dual problem has a piecewise $H^2$-regular property

$$||\mathbf{w}||_{[H^2(\Omega^+\cup\Omega^-)]^2} + |\lambda| \cdot ||\nabla \cdot \mathbf{w}||_{[H^1(\Omega^+\cup\Omega^-)]^2} \le C||\mathbf{e}_0||, \tag{5.13}$$

where $C$ is a constant independent of $h$ and $\lambda$.

**Theorem 2.** *There exists a constant $C$ independent of $h$ and $\lambda$, such that*

$$\|\mathbf{e}_0\| \le Ch^2(\|\mathbf{f}\| + \|\boldsymbol{\psi}\|_{[H^{1/2}(\Gamma)]^2} + \|\boldsymbol{\phi}\|_{[H^{1/2}(\Gamma)]^2}). \tag{5.14}$$

**Proof.** Similar to the argument of Eq. (5.8), we obtain an equation that the solution $\mathbf{w}$ of (5.9)–(5.12) satisfies

$$\sum_{T\in\mathcal{T}_h} 2(\mu\varepsilon_w(Q_h\mathbf{w}), \varepsilon_w(\mathbf{v}_h))_T + \sum_{T\in\mathcal{T}_h} (\lambda\nabla_w \cdot (Q_h\mathbf{w}), \nabla_w \cdot \mathbf{v}_h)_T = (\mathbf{e}_0, \mathbf{v}_0) - \ell_1(\mathbf{w}, \mathbf{v}_h).$$

Adding $s(Q_h\mathbf{w}, \mathbf{v}_h)$ to both sides of above equation and taking $\mathbf{v}_h = \mathbf{e}_h$, we get

$$\|\mathbf{e}_0\|^2 = a_h(Q_h\mathbf{w}, \mathbf{e}_h) + \ell_1(\mathbf{w}, \mathbf{e}_h) - \ell_2(\mathbf{w}, \mathbf{e}_h).$$

Combining with the error Eq. (5.5), we have

$$\|\mathbf{e}_0\|^2 = \ell_1(\mathbf{u}, Q_h\mathbf{w}) - \ell_2(\mathbf{u}, Q_h\mathbf{w}) + \ell_1(\mathbf{w}, \mathbf{e}_h) - \ell_2(\mathbf{w}, \mathbf{e}_h).$$

It is obvious that Lemma 7 is also applicable to the solution $\mathbf{w}$ of (5.9)–(5.12), then we have

$$\begin{aligned}
&|\ell_1(\mathbf{w}, \mathbf{e}_h)| + |\ell_2(\mathbf{w}, \mathbf{e}_h)| \\
&\le Ch\Big(||\mathbf{w}||_{2,\Omega^+} + ||\mathbf{w}||_{2,\Omega^-} + |\lambda^+| \cdot ||\mathbf{w}||_{1,\Omega^+} + |\lambda^-| \cdot ||\mathbf{w}||_{1,\Omega^-}\Big)|||\mathbf{e}_h||| \\
&\le Ch|||\mathbf{e}_h||| \cdot ||\mathbf{e}_0||.
\end{aligned}$$

Using the trace inequality (A.4), projection inequalities (A.1), (A.2) and (A.3), estimates (2.5) and (5.13), we obtain

$$\begin{aligned}
&|\ell_1(\mathbf{u}, Q_h\mathbf{w})|^2 \\
&\le C\big(\textstyle\sum_{T\in\mathcal{T}_h} \big(||2\mu(\mathbf{Q}_h\varepsilon(\mathbf{u}) - \varepsilon(\mathbf{u}))||_T^2 + ||\lambda(\mathbb{Q}_h(\nabla\cdot\mathbf{u}) - \nabla\cdot\mathbf{u})||_T^2\big)\big) \\
&\cdot\big(\textstyle\sum_{T\in\mathcal{T}_h} ||Q_0\mathbf{w} - \mathbf{w}||_{\partial T}^2\big) \\
&\le Ch^2\big(||\mathbf{u}||_{2,\Omega^+}^2 + ||\mathbf{u}||_{2,\Omega^-}^2 + |\lambda^+|^2 \cdot ||\nabla(\nabla\cdot\mathbf{u})||_{\Omega^+}^2 + |\lambda^-|^2 \cdot ||\nabla(\nabla\cdot\mathbf{u})||_{\Omega^-}^2\big) \\
&\cdot Ch^2\big(||\mathbf{w}||_{2,\Omega^+}^2 + ||\mathbf{w}||_{2,\Omega^-}^2\big) \\
&\le Ch^4\Big(||\mathbf{f}|| + \big|\big|\boldsymbol{\psi}\big|\big|_{[H^{1/2}(\Gamma)]^2} + ||\boldsymbol{\phi}||_{[H^{1/2}(\Gamma)]^2}\Big)^2 ||\mathbf{e}_0||^2.
\end{aligned}$$

In a similar way, we have

$$|\ell_2(\mathbf{u}, Q_h\mathbf{w})|^2 \le Ch^4\Bigg(||\mathbf{f}|| + \big|\big|\boldsymbol{\psi}\big|\big|_{[H^{1/2}(\Gamma)]^2} + ||\boldsymbol{\phi}||_{[H^{1/2}(\Gamma)]^2}\Bigg)^2 ||\mathbf{e}_0||^2.$$

The proof follows from these estimates. □

## 6. Numerical experiments

In this section, we present some examples to demonstrate our theoretical analysis.

**Example 3.** Consider an interface model (2.1)–(2.4) in $\Omega = [-1, 1] \times [-1, 1]$. The interface $\Gamma$ is parameterized with the polar angle $\theta$

$$r = \frac{1}{2} + \frac{\sin(6\theta)}{7}.$$

Let $\Omega^+$ be the subdomain inside $\Gamma$, and $\Omega^-$ be the subdomain outside $\Gamma$. The analytic solution is

$$\mathbf{u} = \begin{cases} \mathbf{u}^+ = \begin{bmatrix} e^{x^2+y^2} \\ e^x\big(y^2 + x^2\sin(y)\big) \end{bmatrix}, & \text{in } \Omega^+, \\ \mathbf{u}^- = \begin{bmatrix} 0.1\big(x^2+y^2\big)^2 - 0.01\ln(2\sqrt{x^2+y^2}) \\ -x^2 - y^2 \end{bmatrix}, & \text{in } \Omega^-. \end{cases}$$

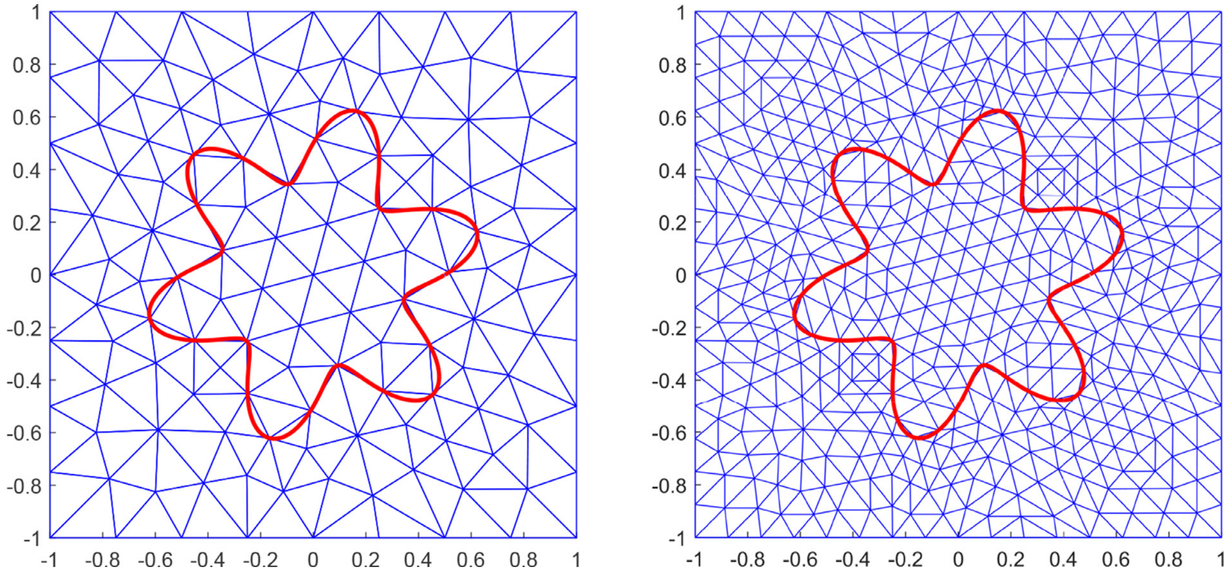

**Fig. 1.** Left: the first level grid, right: the second level grid.

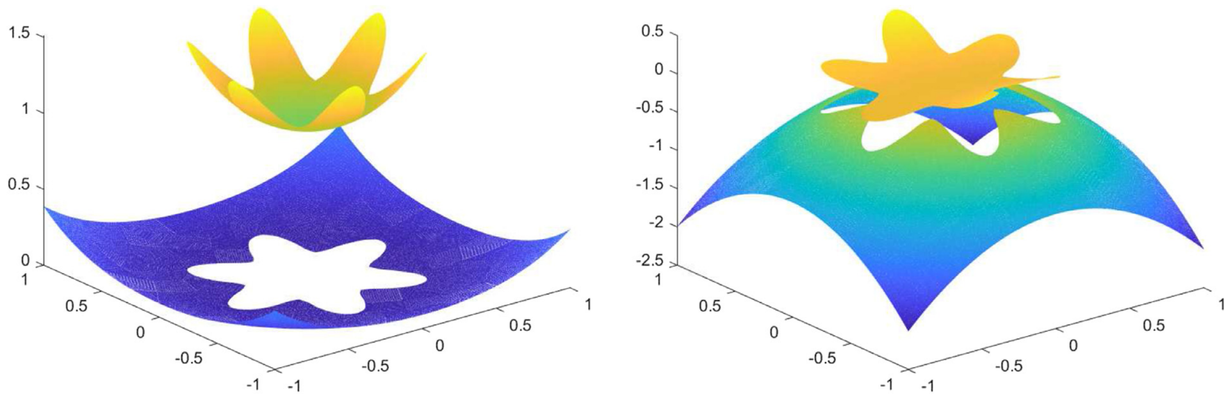

**Fig. 2.** The solution of Example (3) on level 5, left: the first component $(\mathbf{u}^+;\mathbf{u}^-)_1$, right: the second component $(\mathbf{u}^+;\mathbf{u}^-)_2$.

**Table 1**
Errors and convergence orders for WG solution of Example 3.

| Level | $\|\|\|Q_h\mathbf{u}-\mathbf{u}_h\|\|\|$ | order | $\|Q_0\mathbf{u}-\mathbf{u}_0\|_0$ | order |
|---|---|---|---|---|
| 1 | 4.9039 | – | 0.6543 | – |
| 2 | 2.4897 | 0.9849 | 0.1726 | 1.9361 |
| 3 | 1.2497 | 0.9979 | 0.0439 | 1.9821 |
| 4 | 0.6250 | 1.0014 | 0.0110 | 2.0003 |
| 5 | 0.3123 | 1.0018 | 0.0028 | 1.9758 |

We adopt 5 different levels of partitions to carry out the numerical calculation. In our calculations, meshes are generated by Gmsh generator [9]. The first level and the second level partitions are depicted in Fig. 1. We take $\mu^+ = 0.5,\ \lambda^+ = 1,\ \mu^- = 0.1\,\lambda^- = 10$ for numerical calculating. A numerical solution computed on level 5 partition is exhibited in Fig. 2. The errors and convergence orders are presented in Table 1. We observe that the numerical solution converges to the exact one in energy norm and $L^2$ norm at the optimal order, respectively, which is consistence with Theorems 1 and 2.

**Example 4.** [14] Consider an interface model (2.1)–(2.4) within domain $\Omega = [-1,1]\times[-1,1]$ and an interface $\Gamma = \{x_0\}\times[-1,1]$, $-1 < x_0 < 1$. The interface divides the domain $\Omega$ into two subdomains: $\Omega^- = [-1,x_0]\times[-1,1]$ and $\Omega^+ = [x_0,1]\times$

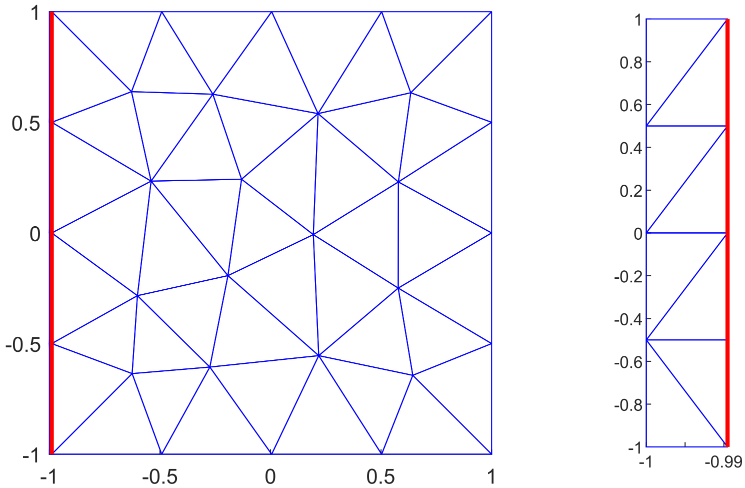


**Fig. 3.** Left: a partition of Ω, right: a zoomed in partition of $\Omega^-$.

**Table 2**
Errors and convergence orders for WG solution of Example 4.

| Level | $\|\|\|Q_h\mathbf{u}-\mathbf{u}_h\|\|\|$ | order | $\|\|Q_0\mathbf{u}-\mathbf{u}_0\|\|$ | order |
|---|---|---|---|---|
| 1 | 2.4288 | – | 0.3331 | – |
| 2 | 1.2131 | 1.0191 | 0.0855 | 1.9963 |
| 3 | 0.5990 | 1.0271 | 0.0214 | 2.0160 |
| 4 | 0.2965 | 1.0190 | 0.0053 | 2.0225 |
| 5 | 0.1476 | 1.0086 | 0.0013 | 2.0320 |

$[-1,1]$. The analytic solution is

$$\mathbf{u}=\begin{cases}\mathbf{u}^+=\begin{bmatrix}\frac{1}{2\mu^++\lambda^+}(x-x_0)\cos((x+x_0)y)\\ \frac{1}{\mu^+}(x-x_0)\cos((x+x_0)y)\end{bmatrix}, & \text{in } \Omega^+,\\ \mathbf{u}^-=\begin{bmatrix}\frac{1}{2\mu^-+\lambda^-}(x-x_0)\cos(2xy)\\ \frac{1}{\mu^-}(x-x_0)\cos(2xy)\end{bmatrix}, & \text{in } \Omega^-.\end{cases}$$

We take $\mu^+=0.5,\ \lambda^+=1,\ \mu^-=0.05\,\lambda^-=10$ and $x_0=-1+\pi/300$ to carry out numerical computation. Since the subdomain $\Omega^-$ is very narrow, the normal regular triangular partition will lead to a great amount of partition cells. Therefore, our strategy is to employ a regular partition in subdomain $\Omega^+$ and use irregular partition in $\Omega^-$ where the ratio of the longest edge to the shortest is about 48. The first level partition together with a zoomed in partition of $\Omega^-$ is shown in Fig. 3. We carry out numerical calculation on such partitions and also obtain a desired convergence order, cf. Table 2.

**Example 5.** [26] In this example, we will illustrate the locking-free property of the WG method. Consider an interface model (2.1)–(2.4) within domain $\Omega=\Omega^+\cup\Omega^-$, where $\Omega^+=[0,1]\times[-1,0]$ and $\Omega^-=[0,1]\times[0,1]$. The interface is $\Gamma=[0,1]\times\{0\}$. The analytic solution is

$$\mathbf{u}=\begin{cases}\mathbf{u}^+=\begin{bmatrix}\sin(x)\sin(y)\\ \cos(x)\cos(y)\end{bmatrix}+\frac{1}{\lambda^+}\begin{bmatrix}1\\ y\end{bmatrix}, & in\Omega^+,\\ \mathbf{u}^-=\begin{bmatrix}\sin(x)\sin(y)\\ \cos(x)\cos(y)\end{bmatrix}+\frac{1}{\lambda^-}\begin{bmatrix}1\\ y\end{bmatrix}, & in\Omega^-.\end{cases}$$

We fix $\mu^+=2,\ \lambda^+=1,\ \mu^-=2$ and take $\lambda^-=100,\ 10,000,\ 1,000,000$ to implement numerical experiments. The errors and convergence orders in energy norm and $L^2$ norm for different sufficiently large $\lambda^-$ are presented in Tables 3 and 4, respectively. We see that the error and convergence orders are independent of $\lambda^-$, which demonstrates the locking-free convergence for the WG method.

**Table 3**
Errors and convergence orders for WG solution of Example 5 in energy norm with different $\lambda^-$.

| | $\lambda^- = 100$ | | $\lambda^- = 10,000$ | | $\lambda^- = 1,000,000$ | |
|---|---|---|---|---|---|---|
| Level | $\|\|\|Q_h\mathbf{u} - \mathbf{u}_h\|\|\|$ | order | $\|\|\|Q_h\mathbf{u} - \mathbf{u}_h\|\|\|$ | order | $\|\|\|Q_h\mathbf{u} - \mathbf{u}_h\|\|\|$ | order |
| 1 | 0.9945 | – | 0.9920 | – | 0.9919 | – |
| 2 | 0.5277 | 0.91 | 0.5255 | 0.92 | 0.5255 | 0.92 |
| 3 | 0.2800 | 0.91 | 0.2790 | 0.91 | 0.2790 | 0.91 |
| 4 | 0.1448 | 0.95 | 0.1445 | 0.95 | 0.1445 | 0.95 |
| 5 | 0.0733 | 0.98 | 0.0733 | 0.98 | 0.0733 | 0.98 |
| 6 | 0.0368 | 0.99 | 0.0368 | 0.99 | 0.0368 | 0.99 |

**Table 4**
Errors and convergence orders for WG solution of Example 5 in $L^2$ norm with different $\lambda^-$.

| | $\lambda^- = 100$ | | $\lambda^- = 10,000$ | | $\lambda^- = 1,000,000$ | |
|---|---|---|---|---|---|---|
| Level | $\|Q_0\mathbf{u} - \mathbf{u}_0\|$ | order | $\|Q_0\mathbf{u} - \mathbf{u}_0\|$ | order | $\|Q_0\mathbf{u} - \mathbf{u}_0\|$ | order |
| 1 | 0.2239 | – | 0.2272 | – | 0.2272 | – |
| 2 | 0.0605 | 1.89 | 0.0628 | 1.86 | 0.0629 | 1.85 |
| 3 | 0.0172 | 1.81 | 0.0184 | 1.77 | 0.0184 | 1.77 |
| 4 | 0.0048 | 1.84 | 0.0052 | 1.82 | 0.0052 | 1.82 |
| 5 | 0.0012 | 2.00 | 0.0014 | 1.89 | 0.0014 | 1.89 |
| 6 | 3.1616e-04 | 1.92 | 3.4768e-04 | 2.01 | 3.4808e-04 | 2.01 |

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## Appendix A. Some known estimates

**Lemma 8.** *Let $\mathcal{T}_h$ be a regular partition of domain $\Omega$ satisfying the assumptions specified in Wang and Ye [29], we assume $\mathbf{w}$ and $\rho$ are sufficiently smooth. Then, for $0 \le m \le 1$, $1 \le r \le k$ we have*

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}||\mathbf{w} - Q_0\mathbf{w}||_{T,m}^2 \le Ch^{2(r+1)}||\mathbf{w}||_{r+1}^2, \tag{A.1}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}||\nabla\mathbf{w} - \mathbf{Q}_h(\nabla\mathbf{w})||_{T,m}^2 \le Ch^{2r}||\mathbf{w}||_{r+1}^2, \tag{A.2}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}||\rho - \mathbb{Q}_h\rho||_{T,m}^2 \le Ch^{2r}||\rho||_r^2. \tag{A.3}$$

Let $T$ be an element satisfying the assumption verified in Wang and Ye [28] with $e$ as a edge. For any function $g \in H^1(T)$, there holds the trace inequality [28]

$$||g||_e^2 \le \left(h_T^{-1}||g||_T^2 + h_T||\nabla g||_T^2\right). \tag{A.4}$$

Particularly, if $g$ is a polynomial in $T$ we have the inverse inequality [28]

$$||\nabla g||_T^2 \le Ch_T^{-2}||g||_{T,}^2 \tag{A.5}$$

where $C$ is a constant only related to the degree of polynomial.

Therefore, we get

$$||g||_e^2 \le Ch_T^{-1}||g||_T^2. \tag{A.6}$$

**Lemma 9.** *[27] Let $\Omega$ be a connected, open bounded domain, and the boundary $\partial\Omega$ is Lipschitz continuous. $\Gamma_1 \subset \partial\Omega$ is a non-trivial portion of $\partial\Omega$. Then for $1 \le p < \infty$, there exists a constant $C$ such that*

$$||\mathbf{v}||_1 \le C\left(||\varepsilon(\mathbf{v})|| + ||\mathbf{v}||_{L^p(\Gamma_1)}\right), \tag{A.7}$$

*for all $\mathbf{v} \in [H^1(\Omega)]^2$.*

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## Further reading


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