```
(*
This Mathematica notebook verifies the CHSH-breaking threshold for maximally
entangled states and the two-sided amplitude damping channel.
*)
```

In[1]:=

```
(* Clearing variables and declaring as real values*)
γ1=.
Im[γ1] ^:= 0
Conjugate[γ1] ^:= γ1

γ2=.
Im[γ2] ^:= 0
Conjugate[γ2] ^:= γ2

θ=.
Im[θ] ^:= 0
Conjugate[θ] ^:= θ

ϕ=.
Im[ϕ] ^:= 0
Conjugate[ϕ] ^:= ϕ

ω=.
Im[ω] ^:= 0
Conjugate[ω] ^:= ω

(* Pauli Operators *)
σx = {{0, 1}, {1, 0}};
σy = {{0, -ⅈ}, {ⅈ, 0}};
σz = {{1, 0}, {0, -1}};

(*Amplitude damping channel Kraus operators*)
K1A = {{1, 0}, {0, Sqrt[1 - γ1]}};
K1B = {{1, 0}, {0, Sqrt[1 - γ2]}};
K2A = {{0, Sqrt[γ1]}, {0, 0}};
K2B = {{0, Sqrt[γ2]}, {0, 0}};

Print["Two-Sided Amplituded Damping Kraus Operators"]
Kraus = {
KroneckerProduct[K1A, K1B], KroneckerProduct[K1A, K2B],
   KroneckerProduct[K2A, K1B], KroneckerProduct[K2A, K2B]
};
MatrixForm /@ Kraus
```

Two-Sided Amplituded Damping Kraus Operators

Out[25]=

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma 2} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma 1} & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{\gamma 2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma 1}\ \sqrt{\gamma 2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \sqrt{\gamma 1} & 0 \\ 0 & 0 & 0 & \sqrt{\gamma 1}\ \sqrt{1-\gamma 2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{\gamma 1}\ \sqrt{\gamma 2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

In[26]:=

```
(*Setting up unitaries to prepare maximally entangled states*)
Print["Arbitrary Qubit Unitary"]
arbQubitUnitary = {
{Exp[-ⅈ * (ϕ + ω) / 2] * Cos[θ / 2], -Exp[ⅈ * (ϕ - ω) / 2] * Sin[θ / 2]},
{Exp[-ⅈ * (ϕ - ω) / 2] * Sin[θ / 2], Exp[ⅈ * (ϕ + ω) / 2] * Cos[θ / 2]}
};
arbQubitUnitary // MatrixForm

Print["Real Qubit Unitary"]
realQubitUnitary = {
{Cos[θ / 2], -Sin[θ / 2]},
{Sin[θ / 2], Cos[θ / 2]}
};
realQubitUnitary // MatrixForm

Print["Bell State"]
Φplus = {{1, 0, 0, 1}, {0, 0, 0, 0}, {0, 0, 0, 0}, {1, 0, 0, 1}} / 2;
Φplus // MatrixForm
```

Arbitrary Qubit Unitary

Out[28]//MatrixForm=

$$\begin{pmatrix} e^{-\frac{1}{2} i (\phi+\omega)} \mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{1}{2} i (\phi-\omega)} \mathrm{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{1}{2} i (\phi-\omega)} \mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{1}{2} i (\phi+\omega)} \mathrm{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

Real Qubit Unitary

Out[31]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}\left[\frac{\theta}{2}\right] & -\mathrm{Sin}\left[\frac{\theta}{2}\right] \\ \mathrm{Sin}\left[\frac{\theta}{2}\right] & \mathrm{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

Bell State

Out[34]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

In[35]:=

```
(*
Constructing two-qubit unitary to prepare general maximally
entangled states from Bell state
*)
Print["Unitary for Arbitrary Entangled State Preparation"]
arbU = KroneckerProduct[arbQubitUnitary, {{1, 0}, {0, 1}}];
arbU // MatrixForm

Print["Unitary for Real Entangled State Preparation"]
realU =  KroneckerProduct[realQubitUnitary, {{1, 0}, {0, 1}}];
realU // MatrixForm
```

Unitary for Arbitrary Entangled State Preparation

Out[37]//MatrixForm=

$$\begin{pmatrix} e^{-\frac{1}{2} i (\phi+\omega)} \mathrm{Cos}\left[\frac{\theta}{2}\right] & 0 & -e^{\frac{1}{2} i (\phi-\omega)} \mathrm{Sin}\left[\frac{\theta}{2}\right] & 0 \\ 0 & e^{-\frac{1}{2} i (\phi+\omega)} \mathrm{Cos}\left[\frac{\theta}{2}\right] & 0 & -e^{\frac{1}{2} i (\phi-\omega)} \mathrm{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{1}{2} i (\phi-\omega)} \mathrm{Sin}\left[\frac{\theta}{2}\right] & 0 & e^{\frac{1}{2} i (\phi+\omega)} \mathrm{Cos}\left[\frac{\theta}{2}\right] & 0 \\ 0 & e^{-\frac{1}{2} i (\phi-\omega)} \mathrm{Sin}\left[\frac{\theta}{2}\right] & 0 & e^{\frac{1}{2} i (\phi+\omega)} \mathrm{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

Unitary for Real Entangled State Preparation

Out[40]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}\left[\frac{\theta}{2}\right] & 0 & -\mathrm{Sin}\left[\frac{\theta}{2}\right] & 0 \\ 0 & \mathrm{Cos}\left[\frac{\theta}{2}\right] & 0 & -\mathrm{Sin}\left[\frac{\theta}{2}\right] \\ \mathrm{Sin}\left[\frac{\theta}{2}\right] & 0 & \mathrm{Cos}\left[\frac{\theta}{2}\right] & 0 \\ 0 & \mathrm{Sin}\left[\frac{\theta}{2}\right] & 0 & \mathrm{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

In[41]:=

```
(*Constructing parameterized maximally entangled states.*)
Print["Arbitrary Maximally Entangled State"]
ρ = arbU.Φplus.ConjugateTranspose[arbU] // FullSimplify;
ρ // MatrixForm

Print["Real Maximally Entangled State"]
realρ = realU.Φplus.ConjugateTranspose[realU] // FullSimplify;
realρ // MatrixForm
```

Arbitrary Maximally Entangled State

Out[43]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}(1+\mathrm{Cos}[\theta]) & -\frac{1}{4}e^{-i\phi}\mathrm{Sin}[\theta] & \frac{1}{4}e^{-i\omega}\mathrm{Sin}[\theta] & \frac{1}{4}e^{-i(\phi+\omega)}(1+\mathrm{Cos}[\theta]) \\ -\frac{1}{4}e^{i\phi}\mathrm{Sin}[\theta] & \frac{1}{4}(1-\mathrm{Cos}[\theta]) & \frac{1}{4}e^{i(\phi-\omega)}(-1+\mathrm{Cos}[\theta]) & -\frac{1}{4}e^{-i\omega}\mathrm{Sin}[\theta] \\ \frac{1}{4}e^{i\omega}\mathrm{Sin}[\theta] & \frac{1}{4}e^{-i(\phi-\omega)}(-1+\mathrm{Cos}[\theta]) & \frac{1}{4}(1-\mathrm{Cos}[\theta]) & \frac{1}{4}e^{-i\phi}\mathrm{Sin}[\theta] \\ \frac{1}{4}e^{i(\phi+\omega)}(1+\mathrm{Cos}[\theta]) & -\frac{1}{4}e^{i\omega}\mathrm{Sin}[\theta] & \frac{1}{4}e^{i\phi}\mathrm{Sin}[\theta] & \frac{1}{4}(1+\mathrm{Cos}[\theta]) \end{pmatrix}$$

Real Maximally Entangled State

Out[46]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}(1+\mathrm{Cos}[\theta]) & -\frac{\mathrm{Sin}[\theta]}{4} & \frac{\mathrm{Sin}[\theta]}{4} & \frac{1}{4}(1+\mathrm{Cos}[\theta]) \\ -\frac{\mathrm{Sin}[\theta]}{4} & \frac{1}{4}(1-\mathrm{Cos}[\theta]) & \frac{1}{4}(-1+\mathrm{Cos}[\theta]) & -\frac{\mathrm{Sin}[\theta]}{4} \\ \frac{\mathrm{Sin}[\theta]}{4} & \frac{1}{4}(-1+\mathrm{Cos}[\theta]) & \frac{1}{4}(1-\mathrm{Cos}[\theta]) & \frac{\mathrm{Sin}[\theta]}{4} \\ \frac{1}{4}(1+\mathrm{Cos}[\theta]) & -\frac{\mathrm{Sin}[\theta]}{4} & \frac{\mathrm{Sin}[\theta]}{4} & \frac{1}{4}(1+\mathrm{Cos}[\theta]) \end{pmatrix}$$

In[47]:=

```
(*
Applying two-sided amplitude damping Kraus operators to
parameterized maximally entangled states
*)
Print["Noisy State Preparation"]
KrausApply[krausOp_] = krausOp.ρ.krausOpᵀ;
ρNoise = Total[KrausApply /@ Kraus] // FullSimplify;
ρNoise // MatrixForm

Print["Noisy State Preparation"]
realKrausApply[krausOp_] = krausOp.realρ.krausOpᵀ;
realρNoise = Total[realKrausApply /@ Kraus] // FullSimplify;
realρNoise // MatrixForm
```

Noisy State Preparation

Out[50]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}((1+\gamma1)(1+\gamma2)+(-1+\gamma1)(-1+\gamma2)\,\mathrm{Cos}[\theta]) & \frac{1}{4}e^{-i\phi}(-1+\gamma1)\sqrt{1-\gamma2}\,\mathrm{Sin}[\theta] & -\frac{1}{4}e^{-i\omega}\sqrt{1-\gamma1} \\ \frac{1}{4}e^{i\phi}(-1+\gamma1)\sqrt{1-\gamma2}\,\mathrm{Sin}[\theta] & -\frac{1}{4}(-1+\gamma2)(1+\gamma1+(-1+\gamma1)\,\mathrm{Cos}[\theta]) & \frac{1}{4}e^{i(\phi-\omega)}\sqrt{1-\gamma1} \\ -\frac{1}{4}e^{i\omega}\sqrt{1-\gamma1}(-1+\gamma2)\,\mathrm{Sin}[\theta] & \frac{1}{4}e^{-i(\phi-\omega)}\sqrt{1-\gamma1}\sqrt{1-\gamma2}(-1+\mathrm{Cos}[\theta]) & -\frac{1}{4}(-1+\gamma1)(1+\gamma \\ \frac{1}{4}e^{i(\phi+\omega)}\sqrt{1-\gamma1}\sqrt{1-\gamma2}(1+\mathrm{Cos}[\theta]) & \frac{1}{4}e^{i\omega}\sqrt{1-\gamma1}(-1+\gamma2)\,\mathrm{Sin}[\theta] & -\frac{1}{4}e^{i\phi}(-1+\gamma1) \end{pmatrix}$$

Noisy State Preparation

Out[54]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}((1+\gamma1)(1+\gamma2)+(-1+\gamma1)(-1+\gamma2)\,\mathrm{Cos}[\theta]) & \frac{1}{4}(-1+\gamma1)\sqrt{1-\gamma2}\,\mathrm{Sin}[\theta] & -\frac{1}{4}\sqrt{1-\gamma1}(-1+ \\ \frac{1}{4}(-1+\gamma1)\sqrt{1-\gamma2}\,\mathrm{Sin}[\theta] & -\frac{1}{4}(-1+\gamma2)(1+\gamma1+(-1+\gamma1)\,\mathrm{Cos}[\theta]) & \frac{1}{4}\sqrt{1-\gamma1}\sqrt{1-\gamma2} \\ -\frac{1}{4}\sqrt{1-\gamma1}(-1+\gamma2)\,\mathrm{Sin}[\theta] & \frac{1}{4}\sqrt{1-\gamma1}\sqrt{1-\gamma2}(-1+\mathrm{Cos}[\theta]) & -\frac{1}{4}(-1+\gamma1)(1+\gamma2+(\\ \frac{1}{4}\sqrt{1-\gamma1}\sqrt{1-\gamma2}(1+\mathrm{Cos}[\theta]) & \frac{1}{4}\sqrt{1-\gamma1}(-1+\gamma2)\,\mathrm{Sin}[\theta] & -\frac{1}{4}(-1+\gamma1)\sqrt{1- \end{pmatrix}$$

In[55]:=

```
(*Constructing the correlation matrix
 from amplitude damped maximally entangled states*)
Print["Correlation Matrix for Noisy arbitrary Two-Qubit State"]
Tρ = {
{Tr[ρNoise.KroneckerProduct[σx, σx]],
      Tr[ρNoise.KroneckerProduct[σx, σy]], Tr[ρNoise.KroneckerProduct[σx, σz]]},
{Tr[ρNoise.KroneckerProduct[σy, σx]],
      Tr[ρNoise.KroneckerProduct[σy, σy]], Tr[ρNoise.KroneckerProduct[σy, σz]]},
{Tr[ρNoise.KroneckerProduct[σz, σx]],
      Tr[ρNoise.KroneckerProduct[σz, σy]], Tr[ρNoise.KroneckerProduct[σz, σz]]}
} // TrigToExp // FullSimplify;
Tρ // MatrixForm

Print["Correlation Matrix for Noisy real Two-Qubit State"]
realTρ = {
{Tr[realρNoise.KroneckerProduct[σx, σx]], Tr[realρNoise.KroneckerProduct[σx, σy]],
      Tr[realρNoise.KroneckerProduct[σx, σz]]},
{Tr[realρNoise.KroneckerProduct[σy, σx]], Tr[realρNoise.KroneckerProduct[σy, σy]],
      Tr[realρNoise.KroneckerProduct[σy, σz]]},
{Tr[realρNoise.KroneckerProduct[σz, σx]], Tr[realρNoise.KroneckerProduct[σz, σy]],
      Tr[realρNoise.KroneckerProduct[σz, σz]]}
} // TrigToExp // FullSimplify;
realTρ // MatrixForm
```

Correlation Matrix for Noisy arbitrary Two-Qubit State

Out[57]//MatrixForm=

$$\left(\begin{array}{ll} \sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2}\ (\text{Cos}[\theta]\times\text{Cos}[\phi]\times\text{Cos}[\omega]-\text{Sin}[\phi]\times\text{Sin}[\omega]) & \sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2}\ (\text{Cos}[\theta]\times\text{Cos}[\omega]\times\text{Sin}[\phi]+ \\ \sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2}\ (\text{Cos}[\omega]\times\text{Sin}[\phi]+\text{Cos}[\theta]\times\text{Cos}[\phi]\times\text{Sin}[\omega]) & \sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2}\ (-\text{Cos}[\phi]\times\text{Cos}[\omega]+\text{Cos}[\theta]\ \rangle \\ (-1+\gamma 1)\ \sqrt{1-\gamma 2}\ \text{Cos}[\phi]\times\text{Sin}[\theta] & (-1+\gamma 1)\ \sqrt{1-\gamma 2}\ \text{Sin}[\theta]\times\text{Sin} \end{array}\right.$$

Correlation Matrix for Noisy real Two-Qubit State

Out[60]//MatrixForm=

$$\left(\begin{array}{ccc} \sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2}\ \text{Cos}[\theta] & 0 & -\sqrt{1-\gamma 1}\ (-1+\gamma 2)\ \text{Sin}[\theta] \\ 0 & -\sqrt{1-\gamma 1}\ \sqrt{1-\gamma 2} & 0 \\ (-1+\gamma 1)\ \sqrt{1-\gamma 2}\ \text{Sin}[\theta] & 0 & \gamma 1\ \gamma 2+(-1+\gamma 1)\ (-1+\gamma 2)\ \text{Cos}[\theta] \end{array}\right)$$

In[67]:=
```
(*Creating symmetric R matrix from correlation matrices *)
Print["arb R matrix"]
Rρ = Tρᵀ.Tρ // FullSimplify;
Rρ // MatrixForm

Print["real R Matrix"]
realRρ = realTρᵀ.realTρ // FullSimplify;
realRρ // MatrixForm
```

arb R matrix

Out[69]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}(-1+\gamma 1)(-1+\gamma 2)\left(4-\gamma 1+\gamma 1\,\mathrm{Cos}[2\,\theta]-2\,\gamma 1\,\mathrm{Cos}[2\,\phi]\,\mathrm{Sin}[\theta]^2\right) & -\left((-1+\gamma 1)\,\gamma 1\,(-1+\gamma 2)\,\mathrm{Cos}[\phi]\,\mathrm{Si} \right. \\ -\left((-1+\gamma 1)\,\gamma 1\,(-1+\gamma 2)\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\phi]\right) & \frac{1}{4}(-1+\gamma 1)(-1+\gamma 2)\left(4-\gamma 1+\gamma 1\,\mathrm{Cos}[2\,\theta]+\right. \\ (-1+\gamma 1)\,\gamma 1\,\sqrt{1-\gamma 2}\,(\gamma 2+(-1+\gamma 2)\,\mathrm{Cos}[\theta])\,\mathrm{Cos}[\phi]\times\mathrm{Sin}[\theta] & (-1+\gamma 1)\,\gamma 1\,\sqrt{1-\gamma 2}\,(\gamma 2+(-1+\gamma 2)\,\mathrm{Cos} \end{pmatrix}$$

real R Matrix

Out[72]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}(-1+\gamma 1)(-1+\gamma 2)(2-\gamma 1+\gamma 1\,\mathrm{Cos}[2\,\theta]) & 0 & (-1+\gamma 1)\,\gamma 1\,\sqrt{1-\gamma 2}\,(\gamma 2+(-1 \\ 0 & (-1+\gamma 1)(-1+\gamma 2) & 0 \\ (-1+\gamma 1)\,\gamma 1\,\sqrt{1-\gamma 2}\,(\gamma 2+(-1+\gamma 2)\,\mathrm{Cos}[\theta])\,\mathrm{Sin}[\theta] & 0 & (\gamma 1\,\gamma 2+(-1+\gamma 1)(-1+\gamma 2)\,\mathrm{Cos}[\theta])^2-(\end{pmatrix}$$

In[73]:=
```
Print["Eigenvalues of Rρ"]
evalsRρ = Eigenvalues[Rρ] // FullSimplify;
evalsRρ[[1]]
evalsRρ[[2]]
evalsRρ[[3]]

Print["Eigenvalues of realRρ"]
evalsrealRρ = Eigenvalues[realRρ] // FullSimplify;
evalsrealRρ[[1]]
evalsrealRρ[[2]]
evalsrealRρ[[3]]
```

Eigenvalues of Rρ

Out[75]= $(-1+\gamma 1)(-1+\gamma 2)$

Out[76]=

$$\frac{1}{4}\Big(2(-2+\gamma 2)(-1+\gamma 2)-\gamma 1(-1+\gamma 2)(-6+3\,\gamma 2+4\,\gamma 2\,\mathrm{Cos}[\theta]+\gamma 2\,\mathrm{Cos}[2\,\theta])+ \gamma 1^2(2+3(-1+\gamma 2)\gamma 2+(-1+\gamma 2)\gamma 2(4\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\,\theta]))- \frac{1}{2}\sqrt{\Big(-64(-1+\gamma 1)(-1+\gamma 2)((-1+\gamma 1)(-1+\gamma 2)+\gamma 1\,\gamma 2\,\mathrm{Cos}[\theta])^2+4\big(4+2(-3+\gamma 1)\gamma 1-6\,\gamma 2- 3(-3+\gamma 1)\gamma 1\,\gamma 2+(2+3(-1+\gamma 1)\gamma 1)\gamma 2^2+(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2(4\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\,\theta])\big)^2\Big)}\Big)$$

Out[77]= $$\frac{1}{4}\Big(2(-2+\gamma 2)(-1+\gamma 2)-\gamma 1(-1+\gamma 2)(-6+3\gamma 2+4\gamma 2\,\mathrm{Cos}[\theta]+\gamma 2\,\mathrm{Cos}[2\theta])+\gamma 1^2(2+3(-1+\gamma 2)\gamma 2+(-1+\gamma 2)\gamma 2(4\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\theta]))+\frac{1}{2}\sqrt{\Big(-64(-1+\gamma 1)(-1+\gamma 2)((-1+\gamma 1)(-1+\gamma 2)+\gamma 1\,\gamma 2\,\mathrm{Cos}[\theta])^2+4\big(4+2(-3+\gamma 1)\gamma 1-6\gamma 2-3(-3+\gamma 1)\gamma 1\,\gamma 2+(2+3(-1+\gamma 1)\gamma 1)\gamma 2^2+(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2(4\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\theta])\big)^2\Big)}\Big)$$

```
Eigenvalues of realRρ
```

Out[80]= $(-1+\gamma 1)(-1+\gamma 2)$

Out[81]= $$\frac{1}{8}\Big(4(-2+\gamma 2)(-1+\gamma 2)-6\gamma 1(-2+\gamma 2)(-1+\gamma 2)+2\gamma 1^2(2+3(-1+\gamma 2)\gamma 2)+2(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2(4\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\theta])-\sqrt{2}\sqrt{\Big(8(-1+\gamma 2)^2\gamma 2^2-24\gamma 1(-1+\gamma 2)^2\gamma 2^2+\gamma 1^2(-1+\gamma 2)^2(8+\gamma 2(-8+59\gamma 2))+\gamma 1^4(8+(-1+\gamma 2)\gamma 2(24+35(-1+\gamma 2)\gamma 2))-2\gamma 1^3(-1+\gamma 2)(-8+\gamma 2(16+\gamma 2(-47+35\gamma 2)))+(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2\big(8\big(4(-1+\gamma 2)\gamma 2-\gamma 1(-1+\gamma 2)(-4+7\gamma 2)+\gamma 1^2(4+7(-1+\gamma 2)\gamma 2)\big)\mathrm{Cos}[\theta]+4\big(4+2(-3+\gamma 1)\gamma 1-6\gamma 2+(9-7\gamma 1)\gamma 1\,\gamma 2+(2+7(-1+\gamma 1)\gamma 1)\gamma 2^2\big)\mathrm{Cos}[2\theta]+(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2(8\,\mathrm{Cos}[3\theta]+\mathrm{Cos}[4\theta])\big)\Big)}\Big)$$

Out[82]= $$\frac{1}{8}\Big(4(-2+\gamma 2)(-1+\gamma 2)-6\gamma 1(-2+\gamma 2)(-1+\gamma 2)+2\gamma 1^2(2+3(-1+\gamma 2)\gamma 2)+2(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2(4\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\theta])+\sqrt{2}\sqrt{\Big(8(-1+\gamma 2)^2\gamma 2^2-24\gamma 1(-1+\gamma 2)^2\gamma 2^2+\gamma 1^2(-1+\gamma 2)^2(8+\gamma 2(-8+59\gamma 2))+\gamma 1^4(8+(-1+\gamma 2)\gamma 2(24+35(-1+\gamma 2)\gamma 2))-2\gamma 1^3(-1+\gamma 2)(-8+\gamma 2(16+\gamma 2(-47+35\gamma 2)))+(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2\big(8\big(4(-1+\gamma 2)\gamma 2-\gamma 1(-1+\gamma 2)(-4+7\gamma 2)+\gamma 1^2(4+7(-1+\gamma 2)\gamma 2)\big)\mathrm{Cos}[\theta]+4\big(4+2(-3+\gamma 1)\gamma 1-6\gamma 2+(9-7\gamma 1)\gamma 1\,\gamma 2+(2+7(-1+\gamma 1)\gamma 1)\gamma 2^2\big)\mathrm{Cos}[2\theta]+(-1+\gamma 1)\gamma 1(-1+\gamma 2)\gamma 2(8\,\mathrm{Cos}[3\theta]+\mathrm{Cos}[4\theta])\big)\Big)}\Big)$$

```
In[83]:= (*No CHSH violations are found beyond the critical noise threshold*)
Print["arb entangled state optimizations"]
Maximize[{evalsRρ[[3]] + evalsRρ[[1]],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[2]] + evalsRρ[[1]],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[3]] + evalsRρ[[2]],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]

Print["real entangled state optimizations"]
Maximize[{evalsrealRρ[[3]] + evalsrealRρ[[1]],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsrealRρ[[2]] + evalsrealRρ[[1]],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsrealRρ[[3]] + evalsrealRρ[[2]],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]

arb entangled state optimizations

Out[84]= {1., {γ1 → 1., γ2 → 1., θ → 0.}}

Out[85]= {0.87868, {γ1 → 0.292893, γ2 → 0.292893, θ → 5.13947}}

Out[86]= {1., {γ1 → 1., γ2 → 1., θ → 0.}}

real entangled state optimizations

Out[88]= {1., {γ1 → 1., γ2 → 1., θ → 0.}}

Out[89]= {0.87868, {γ1 → 0.292893, γ2 → 0.292893, θ → 5.13947}}

Out[90]= {1., {γ1 → 1., γ2 → 1., θ → 0.}}
```

In[91]:=

```
(*Besides the γ1=γ2=1 point,
M(ρNoise)=1 is maximum found along the critical noise threshold*)
Print["arb entangled state optimizations"]
Maximize[{evalsRρ[[3]] + evalsRρ[[1]],
  0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[2]] + evalsRρ[[1]],
  0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[3]] + evalsRρ[[2]],
  0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]

Print["real entangled state optimizations"]
Maximize[{evalsrealRρ[[3]] + evalsrealRρ[[1]],
  0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsrealRρ[[2]] + evalsrealRρ[[1]],
  0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsrealRρ[[3]] + evalsrealRρ[[2]],
  0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.5, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
```

arb entangled state optimizations

Out[92]= $\{1., \{\gamma 1 \to 0.060587, \gamma 2 \to 0.467753, \theta \to 2.95066 \times 10^{-7}\}\}$

Out[93]= $\{0.87868, \{\gamma 1 \to 0.292893, \gamma 2 \to 0.292893, \theta \to 1.14372\}\}$

Out[94]= $\{0.843146, \{\gamma 1 \to 0.292893, \gamma 2 \to 0.292893, \theta \to 3.66142 \times 10^{-8}\}\}$

real entangled state optimizations

Out[96]= $\{1., \{\gamma 1 \to 0.06059, \gamma 2 \to 0.467751, \theta \to 1.57196 \times 10^{-8}\}\}$

Out[97]= $\{0.87868, \{\gamma 1 \to 0.292893, \gamma 2 \to 0.292893, \theta \to 1.14372\}\}$

Out[98]= $\{0.843146, \{\gamma 1 \to 0.292893, \gamma 2 \to 0.292893, \theta \to 2.5895 \times 10^{-8}\}\}$

In[99]:=
```
(*Extending slightly beyond the critical boundary allows for nonlocality*)
Print["arb entangled state optimizations"]
Maximize[{evalsRρ[[3]] + evalsRρ[[1]],
   0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.51, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[2]] + evalsRρ[[1]],
   0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.51, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[3]] + evalsRρ[[2]],
   0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.51, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]

Print["real entangled state optimizations"]
Maximize[{evalsrealRρ[[3]] + evalsrealRρ[[1]],
   0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.51, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsrealRρ[[2]] + evalsrealRρ[[1]],
   0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.51, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsrealRρ[[3]] + evalsrealRρ[[2]],
   0.1 ≤ (1 - γ1) * (1 - γ2) ≤ 0.51, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
```

arb entangled state optimizations

Out[100]=
$\{1.02, \{\gamma 1 \to 0.195035, \gamma 2 \to 0.366432, \theta \to 5.10788 \times 10^{-8}\}\}$

Out[101]=
$\{0.897571, \{\gamma 1 \to 0.285857, \gamma 2 \to 0.285857, \theta \to 1.15897\}\}$

Out[102]=
$\{0.860126, \{\gamma 1 \to 0.285857, \gamma 2 \to 0.285857, \theta \to 1.00403 \times 10^{-7}\}\}$

real entangled state optimizations

Out[104]=
$\{1.02, \{\gamma 1 \to 0.195035, \gamma 2 \to 0.366432, \theta \to 1.90484 \times 10^{-8}\}\}$

Out[105]=
$\{0.897571, \{\gamma 1 \to 0.285857, \gamma 2 \to 0.285857, \theta \to 1.15897\}\}$

Out[106]=
$\{0.860126, \{\gamma 1 \to 0.285857, \gamma 2 \to 0.285857, \theta \to 1.45481 \times 10^{-7}\}\}$

In[107]:=

```
(*In the range γ1,γ2 in [0,0.5] no arbitrary maximally
 entangled state performs better than the Bell state.
In this range, the Bell state's sum of maximal eigenvalues is 2*(1-γ1)*(1-γ2)*)
Maximize[{evalsRρ[[3]] + evalsRρ[[1]] - 2 * (1 - γ1) * (1 - γ2),
   0.5 ≤ (1 - γ1) * (1 - γ2) ≤ 1, 0 ≤ γ1 ≤ 0.5, 0 ≤ γ2 ≤ 0.5, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[2]] + evalsRρ[[1]] - 2 * (1 - γ1) * (1 - γ2),
   0.5 ≤ (1 - γ1) * (1 - γ2) ≤ 1, 0 ≤ γ1 ≤ 0.5, 0 ≤ γ2 ≤ 0.5, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[3]] + evalsRρ[[2]] - 2 * (1 - γ1) * (1 - γ2),
   0.5 ≤ (1 - γ1) * (1 - γ2) ≤ 1, 0 ≤ γ1 ≤ 0.5, 0 ≤ γ2 ≤ 0.5, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
```

Out[107]=

$\{1.88738 \times 10^{-15}, \{\gamma 1 \to 0.229651, \gamma 2 \to 0.0919844, \theta \to 8.34177 \times 10^{-8}\}\}$

Out[108]=

$\{0., \{\gamma 1 \to 0., \gamma 2 \to 0., \theta \to 4.70973\}\}$

Out[109]=

$\{0., \{\gamma 1 \to 0., \gamma 2 \to 0., \theta \to 1.51654 \times 10^{-6}\}\}$

In[110]:=

```
(*In the range γ1,γ2 in [0,1] there exists arbitrary maximally
 entangled states that perform better than the Bell state,
but these stated do not violate the CHSH inequality.*)
Maximize[
 {evalsRρ[[3]] + evalsRρ[[1]] - (1 - γ1) * (1 - γ2) - Max[(1 - γ1) * (1 - γ2), (1 - γ1 - γ2 + 2 * γ1 * γ2) ^ 2],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 1.0, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[
 {evalsRρ[[2]] + evalsRρ[[1]] - (1 - γ1) * (1 - γ2) - Max[(1 - γ1) * (1 - γ2), (1 - γ1 - γ2 + 2 * γ1 * γ2) ^ 2],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 1.0, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[
 {evalsRρ[[3]] + evalsRρ[[2]] - (1 - γ1) * (1 - γ2) - Max[(1 - γ1) * (1 - γ2), (1 - γ1 - γ2 + 2 * γ1 * γ2) ^ 2],
   0 ≤ (1 - γ1) * (1 - γ2) ≤ 1.0, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
```

Out[110]=

$\{0.0274935, \{\gamma 1 \to 0.267255, \gamma 2 \to 0.841128, \theta \to 4.77277\}\}$

NMaximize: The function value $1.97397 \times 10^{-10} + 6.53379 \times 10^{-10}\, i$ is not a real number at $\{\gamma 1, \gamma 2, \theta\} = \{0.994678, 0.0658896, 0.\}$.

Out[111]=

$\{-7.0157 \times 10^{-9}, \{\gamma 1 \to 0.994686, \gamma 2 \to 0.0658438, \theta \to 1.21651 \times 10^{-6}\}\}$

Out[112]=

$\{3.19189 \times 10^{-16}, \{\gamma 1 \to 0.990223, \gamma 2 \to 0.208649, \theta \to 7.7204 \times 10^{-10}\}\}$

In[113]:=

```
(*Along the CHSH-Breaking boundary the maximimum value is M(ρNoise)=1.
That is, arbitrary entangled states do not improve the violation
 of the CHSH inequality along the critical noise threshold.*)
Maximize[{evalsRρ[[3]] + evalsRρ[[1]],
  (1 - γ1) * (1 - γ2) == 1 / 2, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[2]] + evalsRρ[[1]],
  (1 - γ1) * (1 - γ2) == 1 / 2, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
Maximize[{evalsRρ[[3]] + evalsRρ[[2]],
  (1 - γ1) * (1 - γ2) == 1 / 2, 0 ≤ γ1 ≤ 1, 0 ≤ γ2 ≤ 1, 0 ≤ θ ≤ 2 * π}, {γ1, γ2, θ}]
```

Out[113]=

$$\left\{1, \left\{\gamma 1 \to \frac{11}{27}, \gamma 2 \to \frac{5}{32}, \theta \to 0\right\}\right\}$$

Out[114]=

$$\left\{-\frac{3}{2}\left(-2+\sqrt{2}\right), \left\{\gamma 1 \to \frac{-\frac{1}{2}+\frac{1}{2}\left(2-\sqrt{2}\right)}{-1+\frac{1}{2}\left(2-\sqrt{2}\right)}, \gamma 2 \to \frac{1}{2}\left(2-\sqrt{2}\right), \theta \to 2\,\pi + 2\,\mathrm{ArcTan}\left[-0.644\ldots\right]\right\}\right\}$$

Out[115]=

$$\left\{\frac{1}{2}\left(13-8\sqrt{2}\right), \left\{\gamma 1 \to \frac{-\frac{1}{2}+\frac{1}{2}\left(2-\sqrt{2}\right)}{-1+\frac{1}{2}\left(2-\sqrt{2}\right)}, \gamma 2 \to \frac{1}{2}\left(2-\sqrt{2}\right), \theta \to 0\right\}\right\}$$