粒子物理学

习题课1

作业1

1. 自然单位制

Table 2.1 Relationship between S.I. and natural units.

Quantity	[kg, m, s]	[$\hbar$, c, GeV]	$\hbar = c = 1$
Energy	kg m^2 s^{-2}	GeV	GeV
Momentum	kg m s^{-1}	GeV/c	GeV
Mass	kg	GeV/c^2	GeV
Time	s	(GeV/$\hbar$)$^{-1}$	GeV^{-1}
Length	m	(GeV/$\hbar c$)$^{-1}$	GeV^{-1}
Area	m^2	(GeV/$\hbar c$)$^{-2}$	GeV^{-2}

$$\tau = \frac{1}{\Gamma}$$

- $\hbar c$ = 0.197 GeV fm = 1 Energy ←→ Length
- $\hbar$ = 6.6 × 10^{-25} GeV s = 1 Energy ←→ Time
- c = 3.0 × 10^8 m s^{-1} = 1 Length ←→ Time

作业1

2. 画出s, t, u三个channel的费曼图，并且证明

$$s+t+u=m_1^2+m_2^2+m_3^2+m_4^2$$

$$s=(p_1+p_2)^2, t=(p_1-p_3)^2, u=(p_1-p_4)^2, \left(p_\mu\right)^2=p_\mu p^\mu=m^2$$

$$\begin{aligned} s+t+u &= (p_1+p_2)^2+(p_1-p_3)^2+(p_1-p_4)^2 \\ &= m_1^2+m_2^2+m_3^2+m_4^2+2p_1^2+2p_1p_2-2p_1p_3-2p_1p_4 \\ &= m_1^2+m_2^2+m_3^2+m_4^2+2\mathrm{p}_1(\mathrm{p}_1+\mathrm{p}_2-\mathrm{p}_3-\mathrm{p}_4) \\ &= m_1^2+m_2^2+m_3^2+m_4^2 \end{aligned}$$

作业1

3. 理解decay rate的推导过程，并从$f(p_1) = 0$开始解出p^*的表达式

$$f(p_1) = m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2} = 0, \qquad \text{with } p_1 = p^*$$

$$m_1^2 + p^{*2} = m_i^2 + m_2^2 + p^{*2} - 2m_i\sqrt{m_2^2 + p^{*2}}$$

$$\frac{m_i^2 + m_2^2 - m_1^2}{2m_i} = \sqrt{m_2^2 + p^{*2}}$$

$$p^* = \frac{1}{2m_i}\sqrt{\left(m_i^2 + m_2^2 - m_1^2\right)^2 - 4m_2^2 m_i^2}$$

$$= \frac{1}{2m_i}\sqrt{\left(m_i^2 - (m_1^2 + m_2^2)\right)\left(m_i^2 - (m_1^2 - m_2^2)\right)}$$

作业1

4. 证明:

a) 四维矢量的乘积$p_1 \cdot p_2$在洛伦兹变换下保持不变

b) $\mathrm{F} = 2\mathrm{E}_1 2\mathrm{E}_2(\mathrm{v}_1 + \mathrm{v}_2)$在洛伦兹变换下保持不变

$$\Lambda^{\sigma}{}_{\mu} g_{\sigma\rho} \Lambda^{\rho}{}_{\nu} = g_{\mu\nu}$$

$$\left(\Lambda^T\right)_{\mu}{}^{\sigma} g_{\sigma\rho} \Lambda^{\rho}{}_{\nu} = g_{\mu\nu}$$

$$\Lambda^T g \Lambda = g$$

a) $p_1 \cdot p_2 = p_1^{\mu} p_{2\mu} = g_{\mu\nu} p_1^{\mu} p_2^{\nu}$

洛伦兹变换: $p_1'^{\sigma} = \Lambda^{\sigma}{}_{\mu} p_1^{\mu},\ p_2'^{\rho} = \Lambda^{\rho}{}_{\nu} p_2^{\nu}$

$$p_1' \cdot p_2' = p_1'^{\sigma} p_{2\sigma}' = g_{\sigma\rho} p_1'^{\sigma} p_2'^{\rho} = \underline{g_{\sigma\rho} \Lambda^{\sigma}{}_{\mu} \Lambda^{\rho}{}_{\nu}} p_1^{\mu} p_2^{\nu} = g_{\mu\nu} p_1^{\mu} p_2^{\nu} = p_1 \cdot p_2$$

b)

$$\begin{aligned} F = 2E_a 2E_b(v_a + v_b) &= 4E_a E_b \left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b} \right) \\ &= 4(|\vec{p}_a| E_b + |\vec{p}_b| E_a) \end{aligned}$$

To show this is Lorentz invariant, first consider

$$p_a . p_b = p_a^{\mu} p_{b\mu} = E_a E_b - \vec{p}_a . \vec{p}_b = E_a E_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned} F^2/16 - (p_a^{\mu} p_{b\mu})^2 &= (|\vec{p}_a| E_b + |\vec{p}_b| E_a)^2 - (E_a E_b + |\vec{p}_a||\vec{p}_b|)^2 \\ &= |\vec{p}_a|^2 (E_b^2 - |\vec{p}_b|^2) + E_a^2 (|\vec{p}_b|^2 - E_b^2) \\ &= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\ &= -m_a^2 m_b^2 \\ F &= 4 \left[(p_a^{\mu} p_{b\mu})^2 - m_a^2 m_b^2 \right]^{1/2} \end{aligned}$$

作业1

6. 在LHC上测得Higgs在$\sqrt{s}=13$TeV时，截面约为57pb，试计算，LHC运行一年可以得到多少个Higgs。（LHC目前的亮度为$10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$）

$$N=\mathcal{L}\cdot\sigma\cdot t=17975520\sim10^{7}$$

作业2

1. Bethe-Bloch公式

$$-\frac{dE}{dx} = 2\pi N_A r_e^2 m_e c^2 \rho \frac{Z}{A}\frac{z^2}{\beta^2}\left[ln\left(\frac{2m_e\gamma^2 v^2 W_{max}}{I^2}\right) - 2\beta^2 - \delta - \frac{2C}{Z}\right]$$

$$-\frac{dE}{dx} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}ln\left(\frac{2m_e c^2\beta^2\gamma^2 T_{max}}{I^2}\right) - \beta^2 - \frac{\delta(\beta\gamma)}{2}\right]$$

with

$$2\pi N_a r_e^2 m_e c^2 = 0.1535 \text{ MeVcm}^2/\text{g}$$

r_e: classical electron radius $= 2.817\times10^{-13}$ cm
m_e: electron mass
N_a: Avogadro's number $= 6.022\times10^{23}\ \text{mol}^{-1}$
I: mean excitation potential
Z: atomic number of absorbing material
A: atomic weight of absorbing material

ρ: density of absorbing material
z: charge of incident particle in units of e
$\beta =$ v/c of the incident particle
$\gamma =$ $1/\sqrt{1-\beta^2}$
δ: density correction
C: shell correction
W_{max}: maximum energy transfer in a single collision.

A : atomic mass of absorber

$\frac{K}{A} = 4\pi N_A r_e^2 m_e c^2/A = 0.307075$ MeV g^{-1}cm^2, for A = 1g mol^{-1}

z: atomic number of incident particle

Z: atomic number of absorber

T_{max} : Maximum energy transfer in a single collision

$$T_{max} = \frac{2m_e c^2\beta^2\gamma^2}{1+2\gamma m_e/M+(m_e/M)^2}$$

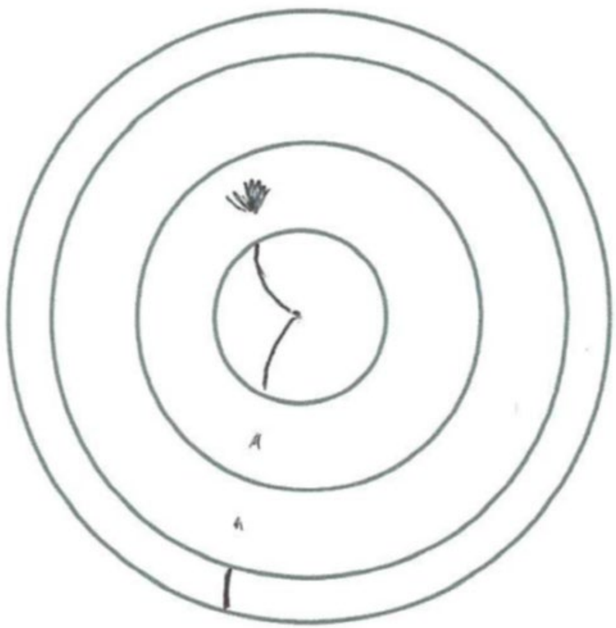
δ(βγ) : density effect correction to ionisation loss.

x = ρ s , surface density or mass thickness, with unit g/cm^2, where s is the length.

dE/dx has the units MeV cm^2/g

作业2

3. 观察下图，写出所发生的过程是$e^-e^+ \to Z \to ? \to ?$，并画出对应的费曼图



$$e^-e^+ \to Z \to \tau^-\tau^+$$

$$\tau^- \to \nu_\tau e^- \bar{\nu}_e$$

$$\tau^+ \to \bar{\nu}_\tau \mu^+ \nu_\mu$$

作业2

4. 尝试证明以下几个γ矩阵的性质：

$a)\ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix};\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- 从 α_j和 β 矩阵的性质 (D2)-(D4)，可得：

$$(\gamma^0)^2 = \beta^2 = 1 \quad \text{和} \quad (\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -1$$

完整关系式

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \quad (j \neq k) \end{aligned}$$

表达为

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

(defines the algebra)

$\boldsymbol{b})\ (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$\boldsymbol{b})\ \boldsymbol{\mu = 0}: (\boldsymbol{\gamma^0})^\dagger = \boldsymbol{\gamma^0}, (\boldsymbol{\gamma^0})^2 = \boldsymbol{1}, (\boldsymbol{\gamma^0})^\dagger = \boldsymbol{\gamma^0\ \gamma^0\ \gamma^0}$

$\boldsymbol{\mu \neq 0}: (\boldsymbol{\gamma^\mu})^\dagger = -\boldsymbol{\gamma^\mu}, \{\boldsymbol{\gamma^\mu}, \boldsymbol{\gamma^0}\} = \boldsymbol{0}, (\boldsymbol{\gamma^0})^\dagger = -\boldsymbol{\gamma^\mu\gamma^0\gamma^0} = \boldsymbol{\gamma^0\gamma^\mu\gamma^0}$

$\boldsymbol{c})\ \overline{\gamma^\mu} = \gamma^\mu$

$\boldsymbol{c})\ \overline{\boldsymbol{\gamma^\mu}} = \boldsymbol{\gamma^0}(\boldsymbol{\gamma^\mu})^\dagger\boldsymbol{\gamma^0} = \boldsymbol{\gamma^0\gamma^0\gamma^\mu\gamma^0\gamma^0} = \boldsymbol{\gamma^\mu}$

作业2

d) 定义: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$，证明:

i. $(\gamma^5)^2 = I$

ii. $\{\gamma^\mu, \gamma^5\} = 0$

iii. $(\gamma^5)^\dagger = \gamma^5$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix};\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$d/i)\ \left(\gamma^5\right)^2 = -1 \times \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -1 \times 1 \times -1 \times -1 \times -1 = 1$$

$$d/ii)\ \{\gamma^0, \gamma^5\} = i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = i(\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3) = 0$$

$$\{\gamma^1, \gamma^5\} = i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = i(\gamma^0\gamma^2\gamma^3 - \gamma^0\gamma^2\gamma^3) = 0$$

$$\{\gamma^2, \gamma^5\} = i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = i(-\gamma^0\gamma^1\gamma^3 + \gamma^0\gamma^1\gamma^3) = 0$$

$$\{\gamma^3, \gamma^5\} = i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i(\gamma^0\gamma^1\gamma^2 - \gamma^0\gamma^1\gamma^2) = 0$$

$$d/iii)\left(\gamma^5\right)^\dagger = -i\left(\gamma^3\right)^\dagger\left(\gamma^2\right)^\dagger\left(\gamma^1\right)^\dagger\left(\gamma^0\right)^\dagger = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

作业2

5. 从动量空间的Dirac方程出发：

$$(\gamma^\mu p_\mu - m)u = 0$$

a) 证明相应的伴随旋量符合方程：

$$\bar{u}(\gamma^\mu p_\mu - m) = 0$$

$a)$ $(\gamma^\mu p_\mu - m)u = 0$ 取厄米共轭：

$$u^\dagger\left((\gamma^\mu)^\dagger p_\mu - m\right) = 0$$

$$u^\dagger(\gamma^0\gamma^\mu\gamma^0 p_\mu - m)\gamma^0 = 0$$

$$u^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0 p_\mu - mu^\dagger\gamma^0 = 0$$

$$\bar{u}\gamma^\mu p_\mu - m\bar{u} = 0$$

$$\bar{u}(\gamma^\mu p_\mu - m) = 0$$

b) 由a)的结果，不写出$\bar{u}$的具体形式，使用归一化条件：$u^\dagger u = 2E$，证明：

i. $\bar{u}u = 2m$

ii. $\bar{u}\gamma^\mu u = 2p^\mu$

$b)$ $\bar{u}\gamma^\nu(\gamma^\mu p_\mu - m)u = 0$

$$\bar{u}(\gamma^\mu p_\mu - m)\gamma^\nu u = 0$$

两式相加：

$$\bar{u}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu u - 2m\bar{u}\gamma^\nu u = 0$$

$$2\bar{u}g^{\mu\nu}p_\mu u - 2m\bar{u}\gamma^\nu u = 0$$

$$\bar{u}up^\nu - m\bar{u}\gamma^\nu u = 0 \ (*)$$

取$\nu = 0$：

$$\bar{u}uE - m\bar{u}\gamma^0 u = 0$$

$$\bar{u}uE - mu^\dagger u = 0$$

$$\bar{u}u = m2E/E = 2m$$

代回(*)：

$$\bar{u}\gamma^\mu u = 2p^\mu$$

作业3

1. 证明螺旋度算符和狄拉克哈密顿量对易.

$\hat{H}_D = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m$，$\beta$中包含单位矩阵，只需要考虑 $\vec{\alpha}\cdot\hat{\vec{p}}$

$$\left[h, \vec{\alpha}\cdot\hat{\vec{p}}\right] \propto [\Sigma\cdot p, \alpha\cdot p] = \left[\Sigma_x p_x + \Sigma_y p_y + \Sigma_z p_z, \alpha_x p_x + \alpha_y p_y + \alpha_z p_z\right]$$

$$= p_x p_x[\Sigma_x, \alpha_x] + p_x p_y\left[\Sigma_x, \alpha_y\right] + p_x p_z[\Sigma_x, \alpha_z] + y \ldots + z \ldots$$

$$[\Sigma_x, \alpha_x] = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} = 0$$

$$\left[\Sigma_x, \alpha_y\right] = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_x\sigma_y - \sigma_y\sigma_x & 0 \\ 0 & \sigma_x\sigma_y - \sigma_y\sigma_x \end{pmatrix} = 2i\alpha_z$$

$$\left[\Sigma_y, \alpha_x\right] = -2i\alpha_z$$

$$\left[h, \vec{\alpha}\cdot\hat{\vec{p}}\right] = 0 \rightarrow [h, H_D] = 0$$

作业3

2. 从狄拉克方程开始推导：

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) + \frac{i}{m}\bar{u}(p')\Sigma^{\mu\nu}q_\nu u(p)$$

其中$q = p' - p, \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$\bar{u}(p')\gamma^\nu(\gamma^\mu p_\mu - m)u(p) = 0$ $\bar{u}(p')(\gamma^\mu p'_\mu - m)\gamma^\nu u(p) = 0$

两式相加：

$$\bar{u}(p')\gamma^\nu\gamma^\mu p_\mu u(p) + \bar{u}(p')\gamma^\mu\gamma^\nu p'_\mu u(p) - 2m\bar{u}(p')\gamma^\nu u(p) = 0$$

$$\bar{u}(p')\gamma^\nu u(p) = \frac{1}{2m}(\bar{u}(p')\gamma^\nu\gamma^\mu p_\mu u(p) + \bar{u}(p')\gamma^\mu\gamma^\nu p'_\mu u(p))$$

$$= \frac{1}{4m}(\bar{u}(p')\gamma^\nu\gamma^\mu p_\mu u(p) + \bar{u}(p')\gamma^\mu\gamma^\nu p'_\mu u(p) + \bar{u}(p')\gamma^\mu\gamma^\nu p_\mu u(p) + \bar{u}(p')\gamma^\nu p'_\mu p_\mu u(p))$$

$$+ \frac{1}{4m}(\bar{u}(p')\gamma^\nu\gamma^\mu p_\mu u(p) + \bar{u}(p')\gamma^\mu\gamma^\nu p'_\mu u(p) - \bar{u}(p')\gamma^\mu\gamma^\nu p_\mu u(p) - \bar{u}(p')\gamma^\nu p'_\mu p_\mu u(p))$$

$$= \frac{1}{4m}\left(\bar{u}(p')(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)(p_\mu + p'_\mu)u(p)\right) + \frac{1}{4m}\left(\bar{u}(p')(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'_\mu - p_\mu)u(p)\right)$$

$$= \frac{1}{2m}\bar{u}(p')(p+p')^\nu u(p) + \frac{i}{m}\bar{u}(p')\frac{-i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'_\mu - p_\mu)u(p)$$

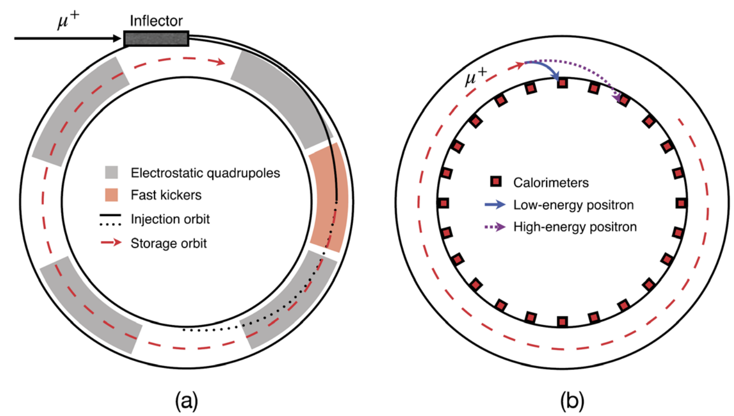
$$= \frac{1}{2m}\bar{u}(p')(p+p')^\nu u(p) + \frac{i}{m}\bar{u}(p')\Sigma^{\nu\mu}q_\mu u(p)$$

作业3

3. b) 缪子的自旋磁矩满足以下公式：

$$\vec{\mu} = g_\mu \frac{e}{2m_\mu}\vec{S}$$

定义缪子反常磁矩$a_\mu = (g_\mu - 2)/2$，
出反常磁矩如何影响缪子在储存环中的
式，并进一步学习实验中通过观测那些



(a) (b)

$$\vec{\omega}_a \equiv \vec{\omega}_s - \vec{\omega}_c = -a_\mu \frac{q\vec{B}}{m_\mu}$$

$$\vec{\omega}_a \equiv \vec{\omega}_s - \vec{\omega}_c = -\frac{q}{m_\mu}\left[a_\mu \vec{B} - a_\mu\left(\frac{\gamma}{\gamma+1}\right)(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(a_\mu \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

$$a_\mu = \frac{R}{\lambda - R}, R = \omega_a/\omega_p$$

作业3

- 展示狄拉克方程在洛伦兹变换下的协变性（协同变换）

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi \quad \text{(A.5)} \qquad \text{变换到} \qquad i\gamma^{\mu}\partial'_{\mu}\psi' = m\psi' \quad \text{(A.6)}$$

其中 $\partial'_{\mu} \equiv \frac{\partial}{\partial x'^{\mu}} = \left(\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$ 且 $\psi'(x') = S\psi(x)$ 是变换后的旋量

- 如果存在 4x4 矩阵 S，狄拉克方程有协变性
- Consider a Lorentz transformation with the primed frame moving with velocity v along the x axis

$$\partial'_{\mu} = \Lambda_{\mu}^{\nu}\partial_{\nu} \quad \text{其中} \quad \Lambda_{\nu}^{\mu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With this transformation equation (A.6)

$$i\gamma^{\nu}\partial'_{\nu}\psi' = m\psi' \Rightarrow i\gamma^{\nu}\Lambda_{\nu}^{\mu}\partial_{\mu}S\psi = mS\psi$$

which should be compared to the matrix S multiplying (A.5)

$$iS\gamma^{\mu}\partial_{\mu}\psi = mS\psi$$

★ **Therefore the covariance of the Dirac equation will be demonstrated if we can find a matrix S such that**

$$\begin{aligned} i\gamma^\nu \Lambda^\mu_\nu \partial_\mu S \psi &= iS\gamma^\mu \partial_\mu \psi \\ \Rightarrow \quad \gamma^\nu \Lambda^\mu_\nu S \partial_\mu \psi &= S\gamma^\mu \partial_\mu \psi \\ \Rightarrow \quad \boxed{S\gamma^\mu = \gamma^\nu S \Lambda^\mu_\nu} \end{aligned} \qquad \textbf{(A.7)}$$

• **Considering each value of** $\mu = 0,\ 1,\ 2,\ 3$

$$\left.\begin{aligned} S\gamma^0 &= \gamma\gamma^0 S - \beta\gamma\gamma^1 S \\ S\gamma^1 &= -\beta\gamma\gamma^0 S + \gamma\gamma^1 S \\ S\gamma^2 &= \gamma^2 S \\ S\gamma^3 &= \gamma^3 S. \end{aligned}\right\}$$

where $\gamma = (1-\beta^2)^{-1/2}$
and $\beta = v/c$

- It is easy (although tedious) to demonstrate that the matrix:

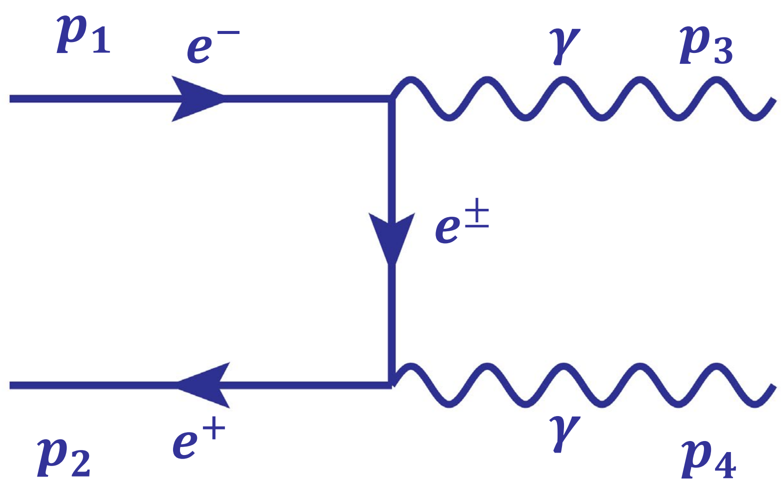
$$\boxed{S = aI + b\gamma^0\gamma^1}$$ **with** $$\boxed{a = \sqrt{\tfrac{1}{2}(\gamma+1)}, \quad b = \sqrt{\tfrac{1}{2}(\gamma-1)}}$$

satisfies the above simultaneous equations

$$S = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}$$

作业3

5. 画出最低阶的$e^+e^- \to \gamma\gamma$过程的t-channel和u-channel的费曼图，并通过QED费曼规则写出对应的矩阵元。



$$-iM_t = [\epsilon_\mu^*(p_3)ie\gamma^\mu u(p_1)] \cdot \left[\frac{i(\gamma^\rho q_\rho + m_e)}{q^2 - m_e}\right] \cdot [\bar{v}(p_2)ie\gamma^\nu \epsilon_\mu^*(p_4)]$$

$$-iM_u = [\epsilon_\mu^*(p_4)ie\gamma^\mu u(p_1)] \cdot \left[\frac{i(\gamma^\rho q_\rho + m_e)}{q^2 - m_e}\right] \cdot [\bar{v}(p_2)ie\gamma^\nu \epsilon_\mu^*(p_3)]$$

- **Overview:**

- **S matrix in QM and QFT**
- **2 particle scattering in QM**
- **Fermion-fermion scattering in QFT**
- **Obtention of the Yukawa potential**

S matrix in QM

- Consider a scattering process in QM, hamiltonian $\hat{H}$, free hamiltonian $\hat{H}_0$ and potential $\hat{V}$:

$$\hat{H} = \hat{H}_0 + \hat{V}$$

- Write down eigenstate of $\hat{H}$ and $\hat{H}_0$ with the same eigenvalue:

$$\hat{H}_0|\phi_a\rangle = E_a|\phi_a\rangle \quad (1)$$

$$\hat{H}\,|\psi_a\rangle = E_a|\psi_a\rangle \quad (2)$$

- Define a S matrix for initial and final state:

$$S_{fi} = \langle\psi_f|\psi_i\rangle$$

- Then we want to work out how $|\psi_i\rangle$ and $|\psi_f\rangle$ relate to their free solutions $|\phi_i\rangle$ and $|\phi_f\rangle$, manipulating format (1) and (2) can get(need knowledge in Green's function):

$$\left|\psi_a^{\pm}\right\rangle = |\phi_a\rangle + \frac{1}{E_a - \hat{H}_0 \pm i\epsilon}\hat{V}\left|\psi_a^{\pm}\right\rangle \quad (3)$$

$$\left|\psi_a^{\pm}\right\rangle = |\phi_a\rangle + \frac{1}{E_a - \hat{H} \pm i\epsilon}\hat{V}|\phi_a\rangle \quad (4)$$

S matrix in QM

$$|\psi_a^\pm\rangle = |\phi_a\rangle + \frac{1}{E_a - \hat{H}_0 \pm i\epsilon}\hat{V}|\psi_a^\pm\rangle$$

$$|\psi_a^\pm\rangle = |\phi_a\rangle + \frac{1}{E_a - \hat{H} \pm i\epsilon}\hat{V}|\phi_a\rangle$$

- Where $\pm i\epsilon$ are used to avoid the singularities, $\frac{1}{E_a - \hat{H}_0 \pm i\epsilon}$ are actually time-retarded and time-advanced Green's function. Then write S matrix with $\pm$ sign:

$$S_{fi} = \langle\psi_f^-|\psi_i^+\rangle$$

- Take the hermitian conjugate of (4) and multiply back to (3), we can get:

$$\langle\psi_f^-|\psi_i^+\rangle = \langle\phi_f|\phi_i\rangle - \frac{2i\epsilon}{\left(E_f - E_i\right)^2 + \epsilon^2}\langle\phi_f|\hat{V}|\psi_i^+\rangle \qquad \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} = \delta(x)$$

$$= \delta_{fi} - 2\pi i\delta(E_f - E_i)\langle\phi_f|\hat{V}|\psi_i^+\rangle \quad (5)$$

- Actually we can do this many times to get higher orders.

S matrix in QFT

- Given a interacting hamiltonian, $\widehat{H}_0$ describes the particle fields and $\widehat{H}_{int}$ the interaction between them:

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_{int}$$

- We will use interaction picture:

$$|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S$$
$$\hat{O}_I = e^{iH_0 t}\hat{O}_S e^{-iH_0 t}$$

- States evolve according to the following equation:

$$i\frac{d|\psi(t)\rangle_I}{dt} = \widehat{H}_I|\psi(t)\rangle_I$$

- Integrate this function:

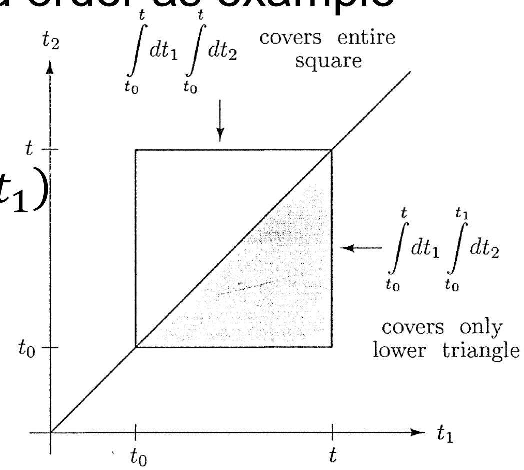
$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle_I$$

- We can expand the last term multiple times:

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)|\psi(t_0)\rangle_I + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)|\psi(t_0)\rangle_I + \cdots \quad (6)$$

S matrix in QFT

- Take time ordering operator T into consideration, use the second order as example and exchange t_1 and t_2:



$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I(t_2) H_I(t_1)$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 \left(\int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \int_{t_1}^{t} dt_2 H_I(t_2) H_I(t_1) \right)$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T[H_I(t_1) H_I(t_2)]$$

- With T we can write (6) as:

$$|\psi(t)\rangle_I = U_I(t, t_0)|\psi(t_0)\rangle_I$$

$$U_I(t, t_0) = T\left\{e^{-i\int_{t_0}^{t} \hat{H}_I(t')dt'}\right\}$$

- Then S matrix in QFT is the defined as:

$$\hat{S} = \lim_{t_\pm \to \pm\infty} U(t_-, t_+)$$

2 particle scattering in QM

$$S_{fi} = \delta_{fi} - 2\pi i \delta(E_f - E_i)\langle \phi_f | \hat{V} | \psi_i^+ \rangle$$

- Take the first order of V, the second term in (5) becomes:

$$2\pi i \delta(E_f - E_i)\langle \phi_f | \hat{V} | \phi_i \rangle$$

- In order to compare the result with QFT, we take $|\phi_i\rangle = |\boldsymbol{p_1}, \boldsymbol{p_2}\rangle$, $|\phi_f\rangle = |\boldsymbol{p'_1}, \boldsymbol{p'_2}\rangle$:

$$\langle \phi_f | \hat{V} | \phi_i \rangle = \int d^3x_1 \int d^3x_2\, V(\boldsymbol{x_1} - \boldsymbol{x_2}) e^{i(\boldsymbol{p_1} - \boldsymbol{p'_1})\boldsymbol{x_1}} e^{i(\boldsymbol{p_2} - \boldsymbol{p'_2})\boldsymbol{x_2}}$$

- If the masses of the two particles are m_1, m_2, we can change $\{\boldsymbol{x_1}, \boldsymbol{x_2}\}$ to $\{\boldsymbol{x_{CM}}, \boldsymbol{x_R}\}$:

$$\int d^3x_{CM} e^{i(\boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p'_1} - \boldsymbol{p'_2})\boldsymbol{x_{CM}}} \int d^3x_R\, V(\boldsymbol{x_R}) e^{i\left(\frac{m_2(\boldsymbol{p_1} - \boldsymbol{p'_1})}{m_1 + m_2} - \frac{m_1(\boldsymbol{p_2} - \boldsymbol{p'_2})}{m_1 + m_2}\right)\boldsymbol{x_R}}$$

$$\boldsymbol{x}_{CM} = \frac{m_1 \boldsymbol{x}_1 + m_2 \boldsymbol{x}_2}{m_1 + m_2}$$

$$\boldsymbol{x}_R = \boldsymbol{x}_1 - \boldsymbol{x}_2$$

- Define $\boldsymbol{q} = \boldsymbol{p_1} - \boldsymbol{p'_1}$, the first term is actually a delta function and the second term is actually a Fourier transform for $\boldsymbol{q}$. With this, the S matrix becomes:

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta\left(E_f - E_i\right) \delta(\boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p'_1} - \boldsymbol{p'_2}) \tilde{V}(\boldsymbol{q}) \qquad (7)$$

Fermion-fermion scattering in QFT

- Consider Yukawa interaction for fermion-fermion scattering:

$$H_I = \int d^3x g\phi\bar{\psi}\psi$$

- Where g is the coupling constant in the interaction, and $\phi, \bar{\psi}, \psi$ are fields for scalar boson and fermions, they are:

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\left(\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}\right) \quad (7)$$

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_\lambda\left(e^{-ipx}u_\lambda(p)\hat{b}_p^\lambda + e^{ipx}v_\lambda(p)\hat{c}_p^{\lambda\dagger}\right) \quad (8)$$

$$\hat{\bar{\psi}}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_\lambda\left(e^{ipx}\bar{u}_\lambda(p)\hat{b}_p^{\lambda\dagger} + e^{-ipx}\bar{v}_\lambda(p)\hat{c}_p^{\lambda}\right) \quad (9)$$

$$\left[\hat{a}_p, \hat{a}_q^\dagger\right] = (2\pi)^3\delta(\vec{p}-\vec{q}) \quad (10)$$

$$\left\{\hat{b}_p^\lambda, \hat{b}_q^{\lambda'\dagger}\right\}\left\{\hat{c}_p^\lambda, \hat{c}_q^{\lambda'\dagger}\right\} = (2\pi)^3\delta(\vec{p}-\vec{q})\delta^{\lambda\lambda'} \quad (11)$$

Fermion-fermion scattering in QFT

- Then write down $|i\rangle$ and $|f\rangle$:

$$|i\rangle \equiv |p_1, p_2\rangle = \sqrt{2E_1}\sqrt{2E_2}\, b_{p_1}^{r\dagger} b_{p_2}^{s\dagger}|0\rangle$$
$$\langle f| \equiv \langle p_1', p_2'| = \langle 0| b_{p_2'}^{s'} b_{p_1'}^{r'}\sqrt{2E_1'}\sqrt{2E_2'} \tag{12}$$

- The corresponding S matrix element is:

$$\langle p_1', p_2'|T\left\{e^{-ig\int d^4x\phi\bar{\psi}\psi}\right\}|p_1, p_2\rangle$$

- The zero order contributes to $\mathbb{1}$, the first order only have one $\phi\sim a^\dagger + a$, and a's commute with b's and c's as they create different particles, so the first order will end up with either $a|0\rangle$ or $\langle 0|a^\dagger$, giving zero contribution.
- The second order will give the main contribution:

$$\langle p_1', p_2'|\frac{(-ig)^2}{2!}\int d^4x\int d^4y\, T\{(\phi\bar{\psi}\psi)_x(\phi\bar{\psi}\psi)_y\}|p_1, p_2\rangle \tag{13}$$

- The next step is integrating over all the possible time orders, but we can find that the only contribution from the product is the one that annihilates two fermions on the right and two fermions on the left.

Fermion-fermion scattering in QFT

- Before integrating, we can firstly show how a fermion act on the initial state, using (8), (11) and (12):

$$
\begin{aligned}
\psi(x)|p_1\rangle &= \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{\lambda}\left(e^{-ip'x}u_\lambda(p')\hat{b}^{\lambda}_{p'} + e^{ip'x}\cancel{v_\lambda(p')\hat{c}^{\lambda\dagger}_{p'}}\right)\sqrt{2E_{p_1}}b^{r\dagger}_{p_1}|0\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3p' \sqrt{\frac{E_{p_1}}{E_{p'}}}e^{-ip'x}u_\lambda(p')\hat{b}^{\lambda}_{p'}\, b^{r\dagger}_{p_1}|0\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3p' \sqrt{\frac{E_{p_1}}{E_{p'}}}e^{-ip'x}u_\lambda(p')\left\{\hat{b}^{\lambda}_{p'}, b^{r\dagger}_{p_1}\right\}|0\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3p' \sqrt{\frac{E_{p_1}}{E_{p'}}}e^{-ip'x}u_\lambda(p')\left\{\hat{b}^{\lambda}_{p'}, b^{r\dagger}_{p_1}\right\}|0\rangle \\
&= \int d^3p' \sqrt{\frac{E_{p_1}}{E_{p'}}}e^{-ip'x}u_\lambda(p')\delta^3(p'-p_1)\delta^{\lambda r}|0\rangle \\
&= e^{-ip_1x}u_r(p_1)|0\rangle \qquad (14)
\end{aligned}
$$

Fermion-fermion scattering in QFT

- Another thing need to clarify is the propagator:

$$\contraction{}{\phi}{_x}{\phi}\phi_x\phi_y = D_F(x-y)$$

- Try to calculate $\langle 0|\phi_x\phi_y|0\rangle$, but not succeeded, then calculate $\langle 0|[\phi_x,\phi_y]|0\rangle$

$\langle 0|[\phi_x,\phi_y]|0\rangle$

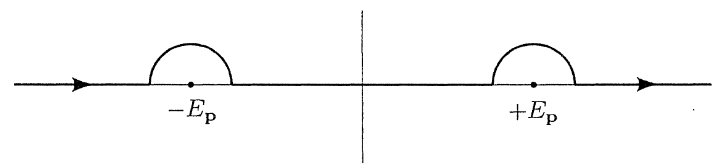
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\left[\left(\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}\right), \left(\hat{a}_q e^{-iqy} + \hat{a}_q^\dagger e^{iqy}\right)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$

- Assume $x^0 > y^0$:

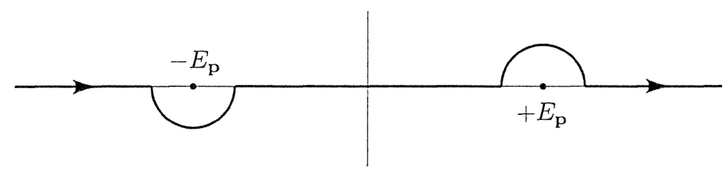
$$= \int \frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2p^0}e^{-ip\cdot(x-y)}\Big|_{p^0=E_p} + \frac{1}{-2p^0}e^{-ip\cdot(x-y)}\Big|_{p^0=-E_p}\right\}$$

$$\underset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip\cdot(x-y)}$$



- Actually we use time ordering operator T, so we can consider $x^0 > y^0$ at one time:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2 - m_\phi^2 + i\epsilon}$$



Fermion-fermion scattering in QFT

$$\phi_x \phi_y = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m_\phi^2 + i\epsilon} \qquad (15)$$

- With (14) and (15) we can start to integrate(13) over x and y:

$$\frac{(-ig)^2}{2!} \langle p_1', p_2' | \int d^4x \int d^4y \, (\phi \bar{\psi} \psi)_x (\phi \bar{\psi} \psi)_y | p_1, p_2 \rangle \qquad x \leftrightarrow y$$

$$= (-ig)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} \int d^4x \int d^4y \, e^{-ip_1x} e^{ip_1'x} e^{-ip_2y} e^{ip_2'y} e^{-ip(x-y)} u(p_1) u(p_2) \bar{u}(p_1') \bar{u}(p_2')$$

$$= (-ig)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} \int d^4x \int d^4y \, e^{-i(p_1 - p_1' + p)x} e^{-i(p_2 - p_2' - p)y} u(p_1) u(p_2) \bar{u}(p_1') \bar{u}(p_2')$$

$$= (-ig)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} \int d^4x \int d^4y \, e^{-i(p_1 - p_1' + p)x} e^{-i(p_2 - p_2' - p)y} u(p_1) u(p_2) \bar{u}(p_1') \bar{u}(p_2')$$

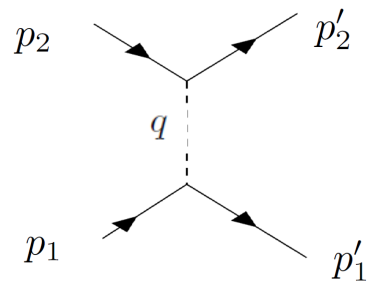
$$= (-ig)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} (2\pi)^4 \delta(p_1 - p_1' + p) (2\pi)^4 \delta(p_2 - p_2' - p) u(p_1) u(p_2) \bar{u}(p_1') \bar{u}(p_2') \qquad (16)$$

Fermion-fermion scattering in QFT

- Then integrate over p in (16):

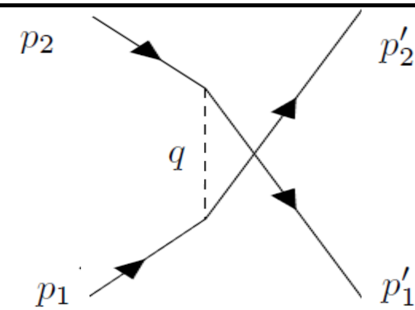
$$= (2\pi)^4 \delta(p_1 - p_1' + p_2 - p_2')(-ig^2)\frac{1}{q^2 - m_\phi^2} u(p_1)u(p_2)\bar{u}(p_1')\bar{u}(p_2')$$
$$= (2\pi)^4 \delta(p_1 - p_1' + p_2 - p_2')(-iM)$$

- Where $q = p_2' - p_2 = p_1 - p_1'$, thus the matrix element describes this Feynman diagram:

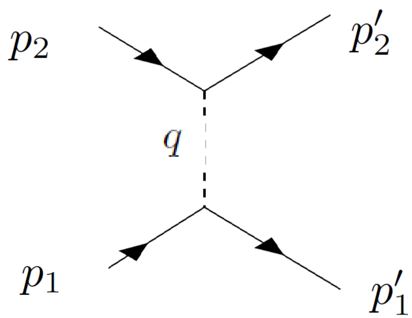


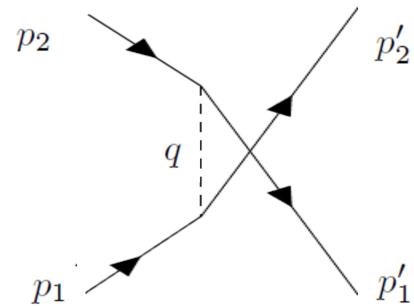
- Similarly, if we cross x and y when integrating, we can get another result:

$$\frac{(-ig)^2}{2!}\langle p_1', p_2' | \int d^4x \int d^4y \, (\phi\bar{\psi}\psi)_x (\phi\bar{\psi}\psi)_y | p_1, p_2 \rangle$$



Fermion-fermion scattering in QFT





- Now we can write down the total matrix element:

$$-iM = (-ig^2)$$

$$\left(\bar{u}(p_2')u(p_2)\frac{1}{(p_1 - p_1')^2 - m_\phi^2}\bar{u}(p_1')u(p_1) - \bar{u}(p_2')u(p_1)\frac{1}{(p_1 - p_2')^2 - m_\phi^2}\bar{u}(p_1')u(p_2)\right) \quad (17)$$

Obtention of the Yukawa potential

- One more thing is we can find in QFT, there're two terms but only one term in QM, this comes from we didn't force two particle distinguishable in initial and final state when calculating QFT, if we do this, the second term in (17) will vanish.
- Now we should try to combine (7) and (17), take non-relativistic limit in (17):

$$p^0 = m + \frac{\boldsymbol{p}^2}{2m} \to m$$

$$u_\lambda(p) \to \sqrt{2m}\begin{pmatrix} \xi_\lambda \\ \xi_\lambda \end{pmatrix}$$

$$(p_1 - p_1')^2 = 0 - |\boldsymbol{p}_1 - \boldsymbol{p}_1'|^2 \equiv -\boldsymbol{q}^2$$

- Where ξ is two component spinor with $\xi_\lambda^\dagger \xi_{\lambda'} = \delta_{\lambda\lambda'}$, then S matrix in QFT reads:

$$\mathbb{1} - ig^2\,(2\pi)^4\delta(p_1 - p_1' + p_2 - p_2')(2m)^2\left(\frac{\delta_{rr'}\delta_{ss'}}{-\boldsymbol{q}^2 - m_\phi^2}\right) \quad (18)$$

- The factor $(2m)^2$ comes from the relativistic normalization in QFT: $\langle p|p'\rangle_{QFT} = 2E\delta(\boldsymbol{p} - \boldsymbol{p}')$ with $E = m$. Now we are in non-relativistic limit, so let's drop it.

Obtention of the Yukawa potential

- Now we can compare (7) and (18) straightforwardly:

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta\left(E_f - E_i\right)\delta(\boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p}'_1 - \boldsymbol{p}'_2)\tilde{V}(\boldsymbol{q})$$

$$S_{fi} = \mathbb{1} - ig^2\,(2\pi)^4\delta(p_1 - p'_1 + p_2 - p'_2)\left(\frac{\delta_{rr'}\delta_{ss'}}{-\boldsymbol{q}^2 - m_\phi^2}\right)$$

- Imposing spin conservation in this last expression, $\tilde{V}(\boldsymbol{q})$ can be written as:

$$\tilde{V}(\boldsymbol{q}) = \frac{-g^2}{\boldsymbol{q}^2 + m_\phi^2}$$

- Then apply Fourier transform back:

$$V(r) = -g^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{iqr}}{\boldsymbol{q}^2 + m_\phi^2}$$

- Do the integrate and get the final format:

$$V(r) = -\frac{g^2}{4\pi r} e^{-m_\phi r} \qquad (19)$$

- This is the Yukawa potential