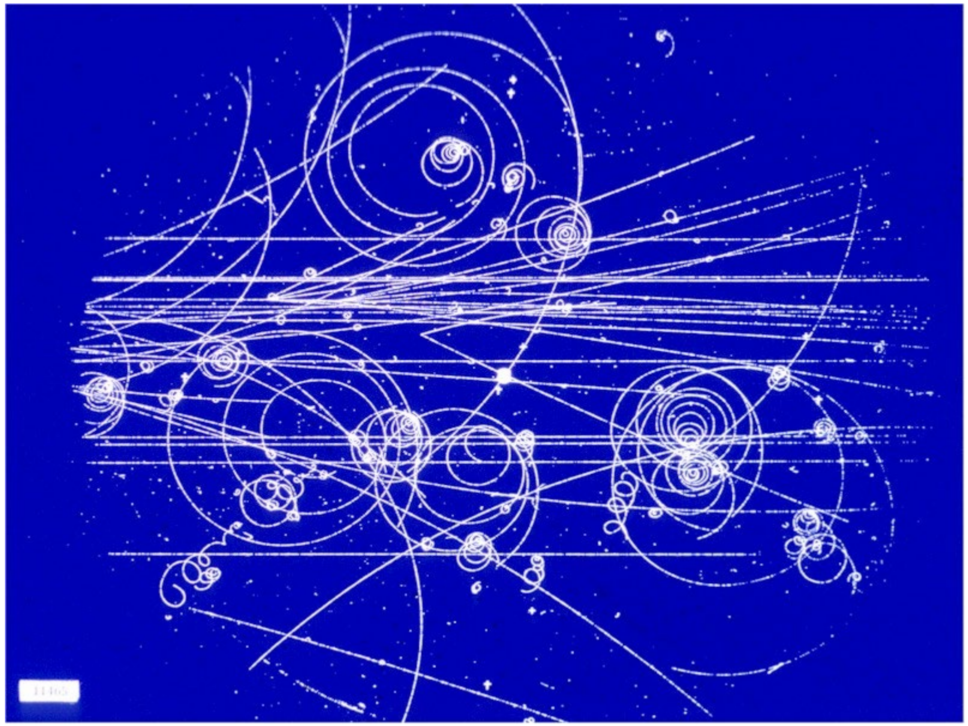
# 粒子物理学

## 第 4 章： 正负电子湮灭



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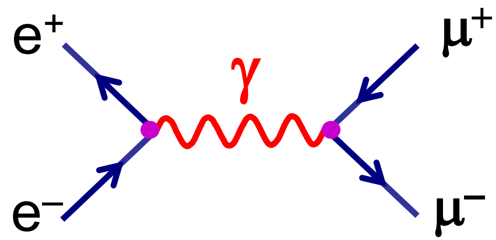
Based on M. Thomson’s notes

# QED Calculations

➢ 如何利用QED计算截面（如：$e^+e^- \rightarrow \mu^+\mu^-$）：

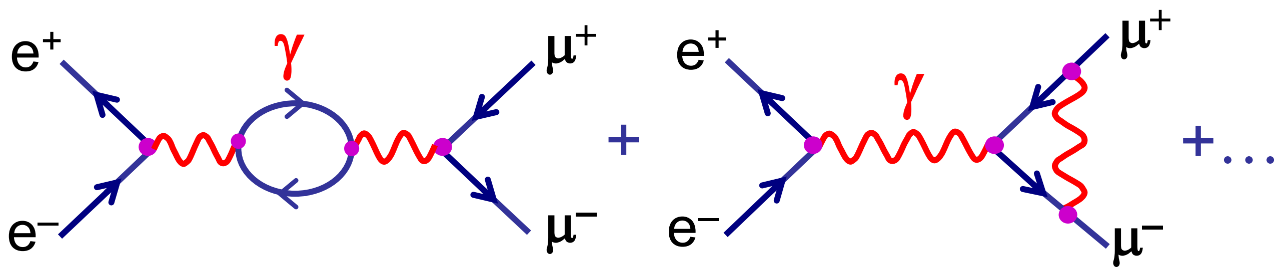
1. 画出所有可能得费曼图

   • 对于 $e^+e^- \rightarrow \mu^+\mu^-$，只有一张**最低阶**图




$$M \propto e^2 \propto \alpha_{em}$$

+ 很多 **二阶**图 + ···




$$M \propto e^4 \propto \alpha_{em}^2$$

2. 对每张图，利用（上节课的）费曼规则计算ME

3. 对各个ME求和（即，振幅求和）　$M_{fi} = M_1 + M_2 + M_3 + \ldots.$

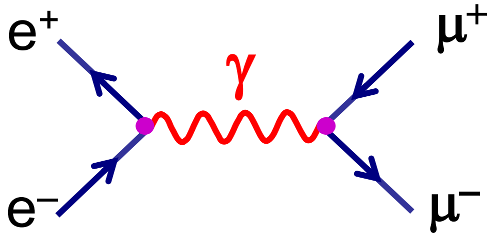
注意：振幅求和导致相同末态的不同图会产生或者正或负的干涉！

# QED Calculations

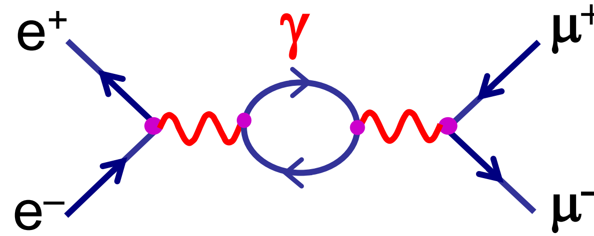
接着平方 $|M_{fi}|^2 = (M_1 + M_2 + M_3 + ....)(M_1^* + M_2^* + M_3^* + ....)$

➡ 给出完整的微扰展开$\alpha_{em}$

- 对于QED $\alpha_{em} = 1/137$，最低阶图占主导，大多数情况可以忽略高阶图




$$M^2 \propto \alpha_{em}^2$$




$$M^2 \propto \alpha_{em}^4$$

4. 计算衰变率/截面（利用课程2的公式）

- 例如，对于衰变：

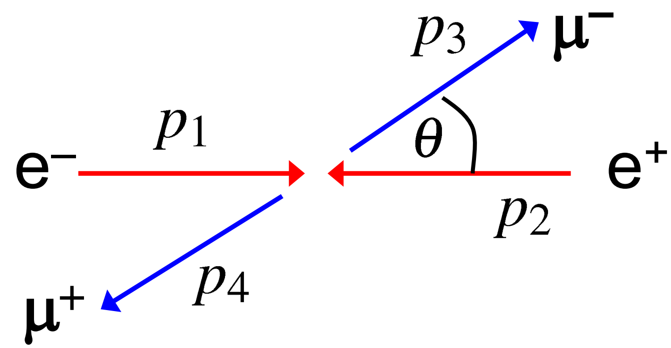
$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 d\Omega$$

- 对于质心系的散射

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \qquad (1)$$

# Electron Positron Annihilation

➢ 计算过程 $e^+e^- \to \mu^+\mu^-$

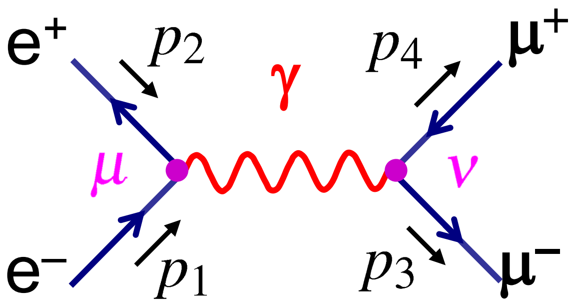



- 选质心系（对于大多数 $e^+e^-$对撞机适用）

$$p_1 = (E,0,0,p) \qquad p_2 = (E,0,0,-p)$$

$$p_3 = (E,\vec{p}_f) \qquad p_4 = (E,-\vec{p}_f)$$

- 考虑最低阶费曼图：




- 费曼规则给出：

$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

**NOTE:**
- **Incoming anti-particle $\overline{v}$**
- **Incoming particle $u$**
- **Adjoint spinor written first**

- 质心系内：

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}|M_{fi}|^2 \qquad \text{其中} \quad s = (p_1+p_2)^2 = (E+E)^2 = 4E^2$$

# Electron and Muon Currents

- 这里 $q^2 = (p_1 + p_2)^2 = s$

  且矩阵元 $$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

  $$\Rightarrow \quad M = -\frac{e^2}{s}g_{\mu\nu}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma^\nu v(p_4)]$$

- 第二节课中引入了四矢量流 $j^\mu = \bar{\psi}\gamma^\mu\psi$ 　其在矩阵元中的[ ]中的两项具有同样的形式

- 矩阵元可以用电子和缪子流来表示

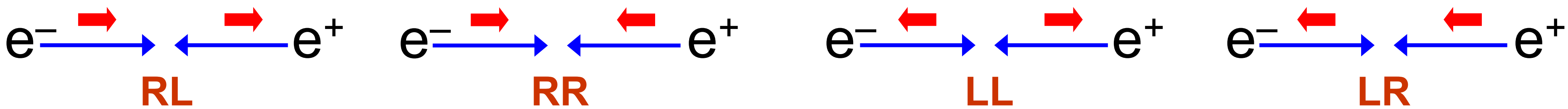
$$(j_e)^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad (j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$$

$$\Rightarrow \quad M = -\frac{e^2}{s}g_{\mu\nu}(j_e)^\mu(j_\mu)^\nu \qquad \boxed{M = -\frac{e^2}{s}j_e \cdot j_\mu}$$

➢ 矩阵元是四矢量的标量积 – 体现洛伦兹不变性

# Spin in e⁺e⁻ Annihilation

- 一般而言，电子和反电子没有极化，即正负螺旋度态的数量相同
  - 初态有四种可能组合




- 类似，末态也有四种螺旋度态组合。总共16种组合，如 RL→RR, RL→RL, ...
  - 对所有16个可能螺旋度态组合求和，然后对初态螺旋度数目求平均：

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} \left( |M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots \right)$$

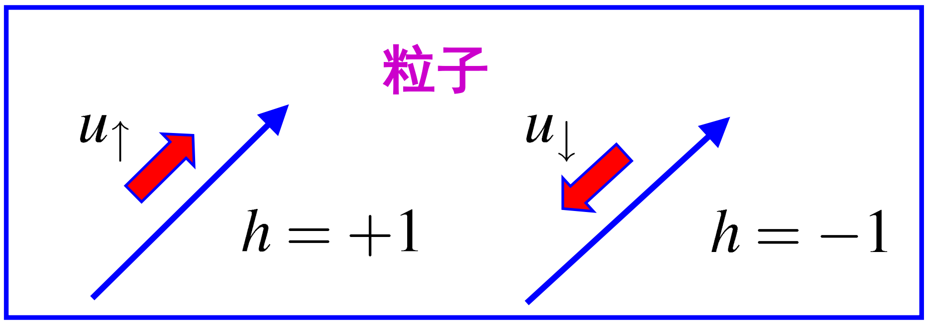
需要对16个全部的螺旋度组合计算： $M = -\frac{e^2}{s} j_e \cdot j_\mu$

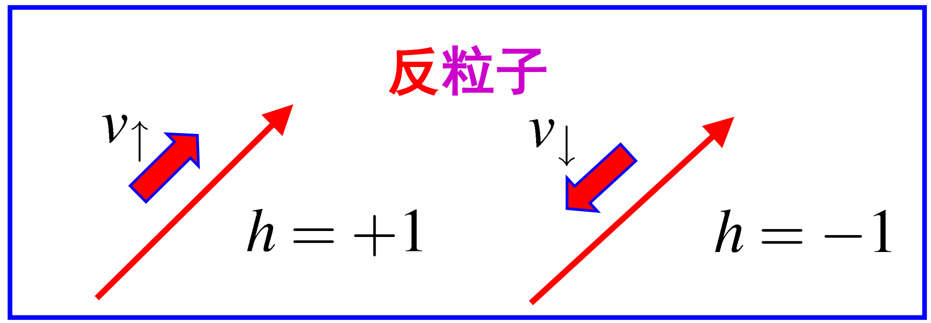
➢ 在$E \gg m_\mu$的极限下，其实只有4个螺旋度组合有非零的矩阵元
  - 这是QED/QCD一个重要的特征

# （回顾第二节课）粒子和反粒子的螺旋度本征态：

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$



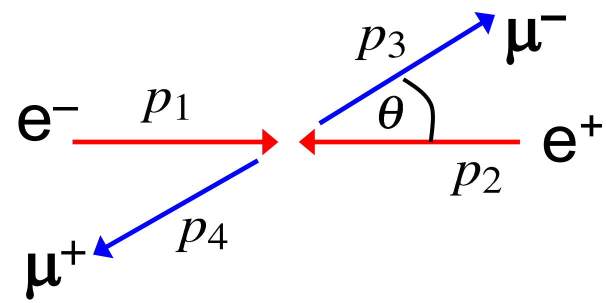




- 全部四个态归一化到单位体积 $2E$ 个粒子，得到 $\boxed{N=\sqrt{E+m}}$

# Spin in e⁺e⁻ Annihilation

- 质心系 $E \gg m_\mu$的极限下

$$p_1 = (E,0,0,E); \qquad p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_2 = (E,0,0,-E) \qquad p_4 = (E, -\sin\theta, 0, -E\cos\theta)$$




- 正反粒子左手 和 右手 螺旋度旋量

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

其中 $s = \sin\frac{\theta}{2}$; $c = \cos\frac{\theta}{2}$ 和 $N = \sqrt{E+m}$

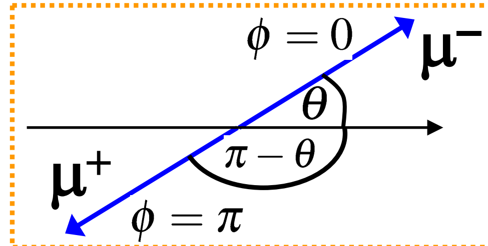
- 考虑$E \gg m_\mu$极限，得：

$$\boxed{u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}}$$

# Spin in e⁺e⁻ Annihilation

- 初态电子e⁻是左手或者右手螺旋度态

$$u_\uparrow(p_1)=\sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\ u_\downarrow(p_1)=\sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix};$$




- 初态正电子e⁺ (θ=π):

$$v_\uparrow(p_2)=\sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\ v_\downarrow(p_2)=\sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- 设末态$\mu^-$ 极化角$\theta$，选 ϕ=0

$$u_\uparrow(p_3)=\sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix};\ u_\downarrow(p_3)=\sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix};$$

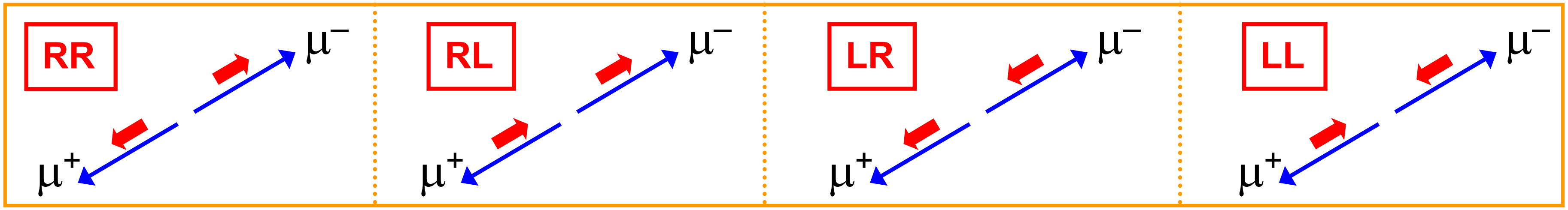
- 并且对末态的$\mu^+$ 做替换：$\theta\to\pi-\theta;\quad \phi\to\pi$ 得到

$$v_\uparrow(p_4)=\sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix};\ v_\downarrow(p_4)=\sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix};\ \left\{\begin{array}{l}\text{利用}\ \sin\left(\frac{\pi-\theta}{2}\right)=\cos\frac{\theta}{2}\\ \cos\left(\frac{\pi-\theta}{2}\right)=\sin\frac{\theta}{2}\\ e^{i\pi}=-1\end{array}\right.$$

# Spin in e$^+$e$^-$ Annihilation

**目标：计算矩阵元** $M = -\dfrac{e^2}{s} j_e \cdot j_\mu$

- **首先考虑缪子流 $j_\mu$ 的四种可能得螺旋度组合**




## 复习：γ-matrices (Dirac–Pauli representation)

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

## 复习：Helicity spinors

$$u_\uparrow(p) = N\begin{pmatrix} c \\ se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix}, u_\downarrow(p) = N\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \end{pmatrix}, v_\uparrow(p) = N\begin{pmatrix} \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}, v_\downarrow(p) = N\begin{pmatrix} \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

# Muon Current

- 对4种螺旋度组合估算 $(j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$
- 对任意的旋量 $\psi,\ \phi$ 很容易知道 $\overline{\psi}\gamma^\mu\phi$ 的各个成分是

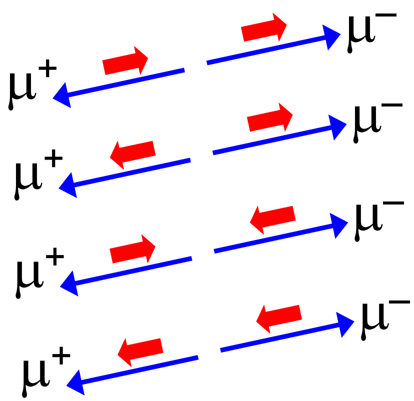
$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 && (3)\\
\overline{\psi}\gamma^1\phi &= \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 && (4)\\
\overline{\psi}\gamma^2\phi &= \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) && (5)\\
\overline{\psi}\gamma^3\phi &= \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 && (6)
\end{aligned}$$

- 考虑组合 $\mu_R^-\mu_L^+$ 并利用 $\psi = u_\uparrow \quad \phi = v_\downarrow$ 其中 $v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$; $u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}$;

$$\begin{aligned}
\overline{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) &= E(cs - sc + cs - sc) = 0\\
\overline{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) &= E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta\\
\overline{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) &= -iE(-c^2 - s^2 - c^2 - s^2) = 2iE\\
\overline{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) &= E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta
\end{aligned}$$

# Muon Current

- 因此，**RL**组合的缪子四矢量流为：$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta)$
- （同样方法得到）4种螺旋度组合的结果：




$$
\begin{aligned}
\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) &= 2E(0,-\cos\theta,i,\sin\theta) && \text{RL}\\
\bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) &= (0,0,0,0) && \text{RR}\\
\bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) &= (0,0,0,0) && \text{LL}\\
\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) &= 2E(0,-\cos\theta,-i,\sin\theta) && \text{LR}
\end{aligned}
$$

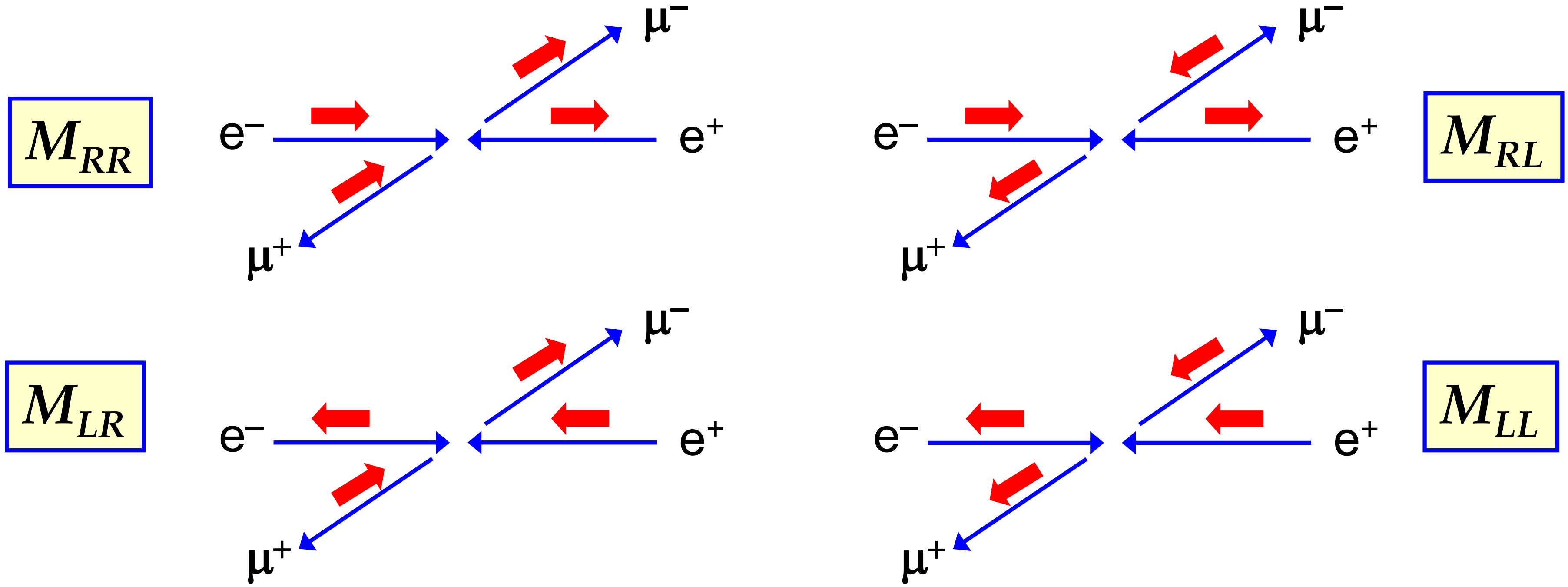
> ➢ $E \gg m$ 的极限下，只有2种螺旋度组合非零！

  - QED重要的特征，同样适用于QCD。
  - 弱相互作用中，只有1个螺旋度组合有贡献。后面会讨论手征起源

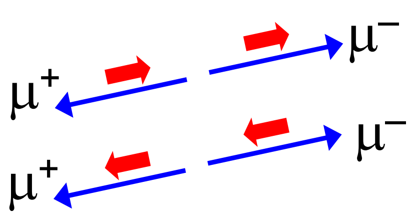
- 因此，16个螺旋度组合中，只有4个有非零矩阵元

# Electron Positron Annihilation cont.

➢ 对于ee⁻→ μ⁺μ⁻ 现在只需要考虑4个矩阵元（R,L代表e- mu-的helicity）




• 之前，推导出了允许螺旋度组合的缪子流：




$$
\boxed{\begin{array}{lcll} \mu_R^- \mu_L^+ & : & \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) & = & 2E(0,-\cos\theta, i, \sin\theta) \\ \mu_L^- \mu_R^+ & : & \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) & = & 2E(0,-\cos\theta, -i, \sin\theta) \end{array}}
$$

➢ 现在需要考虑电子流

# Electron Current

- 入射电子和正电子旋量 (L 和 R 螺旋度):

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix};\quad v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- 电子流要么以前一样通过方程 (3)-(6) 得到，要么直接从缪子流的表达式得到

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \Longleftrightarrow (j_\mu)^\mu = \overline{u}(p_3)\gamma^\mu v(p_4)$$

- 提醒：矩阵元为 $-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\dfrac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$

➢ 缪子流做厄米共轭得到

$$\begin{aligned}
[\overline{u}(p_3)\gamma^\mu v(p_4)]^\dagger &= [u(p_3)^\dagger\gamma^0\gamma^\mu v(p_4)]^\dagger \\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3) \\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_3) \\
&= v(p_4)^\dagger\gamma^0\gamma^\mu u(p_3) \\
&= \overline{v}(p_4)\gamma^\mu u(p_3)
\end{aligned}$$

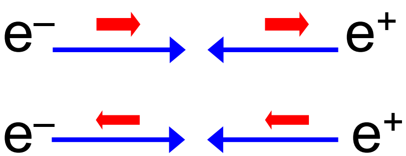
$$(AB)^\dagger = B^\dagger A^\dagger$$
$$\gamma^{0\dagger} = \gamma^0$$
$$\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$$

# Electron Current

- 缪子流的两个非零螺旋度组合的复共轭（dagger等于*，3对1，theta=0）

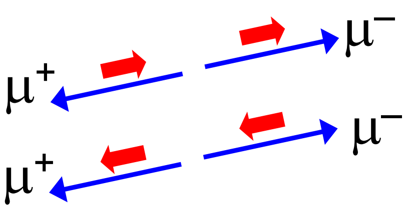
$$\begin{aligned}\bar{v}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_3) &= \left[\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)\right]^* = 2E(0,-\cos\theta,-i,\sin\theta)\\ \bar{v}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_3) &= \left[\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4)\right]^* = 2E(0,-\cos\theta,i,\sin\theta)\end{aligned}$$

为得到电子流，我们设$\theta$=0




$$\boxed{\begin{aligned} e_R^- e_L^+ &: \quad \bar{v}_\downarrow(p_2)\gamma^\nu u_\uparrow(p_1) &= \quad 2E(0,-1,-i,0)\\ e_L^- e_R^+ &: \quad \bar{v}_\uparrow(p_2)\gamma^\nu u_\downarrow(p_1) &= \quad 2E(0,-1,i,0)\end{aligned}}$$

把所有流都写在这里！



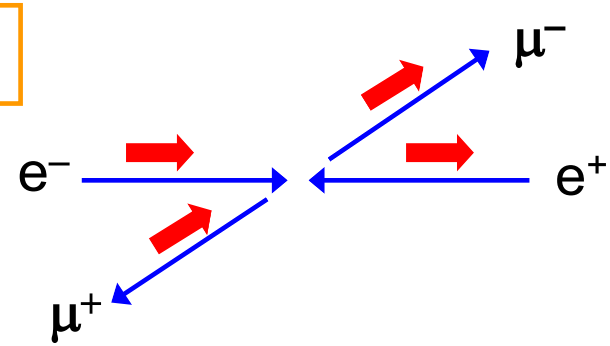

$$\boxed{\begin{aligned} \mu_R^- \mu_L^+ &: \quad \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) &= \quad 2E(0,-\cos\theta,i,\sin\theta)\\ \mu_L^- \mu_R^+ &: \quad \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) &= \quad 2E(0,-\cos\theta,-i,\sin\theta)\end{aligned}}$$

# Matrix Element Calculation

- **对四种螺旋度组合，计算** $M = -\frac{e^2}{s} j_e \cdot j_\mu$

例如 $e_R^- e_L^+ \to \mu_R^- \mu_L^+$ 的矩阵元被记作：$\boxed{M_{RR}}$

- 第一下标为 $e^-$ 螺旋度，第二为 $\mu^-$的螺旋度
- 由于"螺旋度守恒"不需要指明另外一个
- 仅有特定的手征(Chiral)组合才非零




**利用：**

$$e_R^- e_L^+ : \quad (j_e)^\mu = \bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0,-1,-i,0)$$

$$\mu_R^- \mu_L^+ : \quad (j_\mu)^\nu = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta, i, \sin\theta)$$

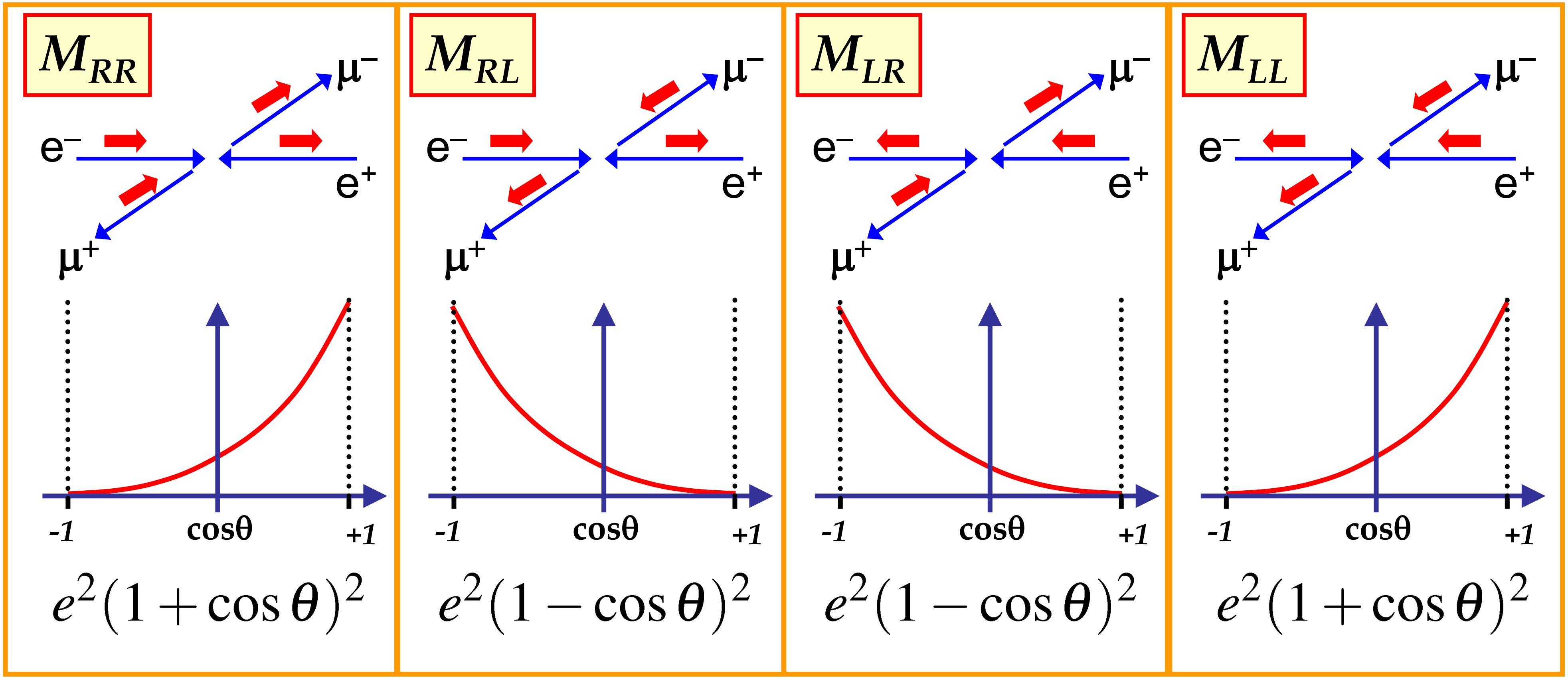
**得**

$$\begin{aligned} M_{RR} &= -\frac{e^2}{s}[2E(0,-1,-i,0)].[2E(0,-\cos\theta,i,\sin\theta)] \\ &= -e^2(1+\cos\theta) \\ &= -4\pi\alpha(1+\cos\theta) \end{aligned}$$

**where** $\alpha = e^2/4\pi \approx 1/137$

# Matrix Element Calculation

- 类似得 $|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$
  $|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$




- 下一步：假设入射电子和正电子没有极化，初态四种可能的螺旋度态几率相同

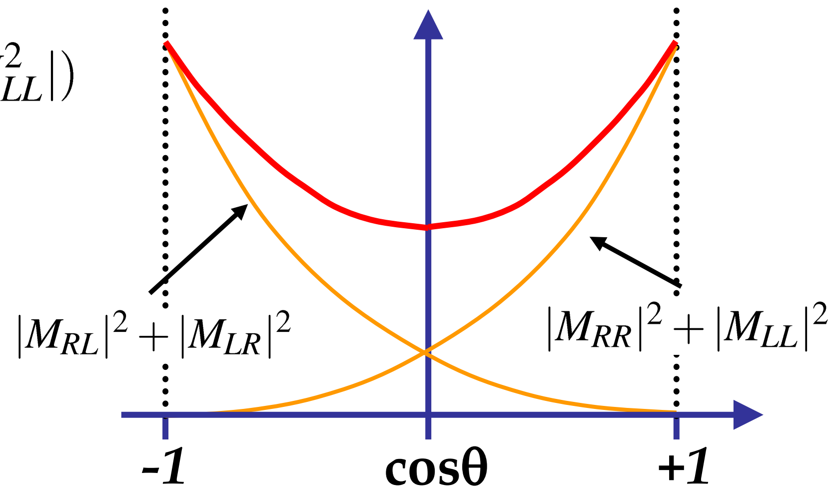
# Differential Cross Section

- **初态自旋求平均，末态自旋求和，从而得到截面：**

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}^2|)$$

$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$



# Differential Cross Section

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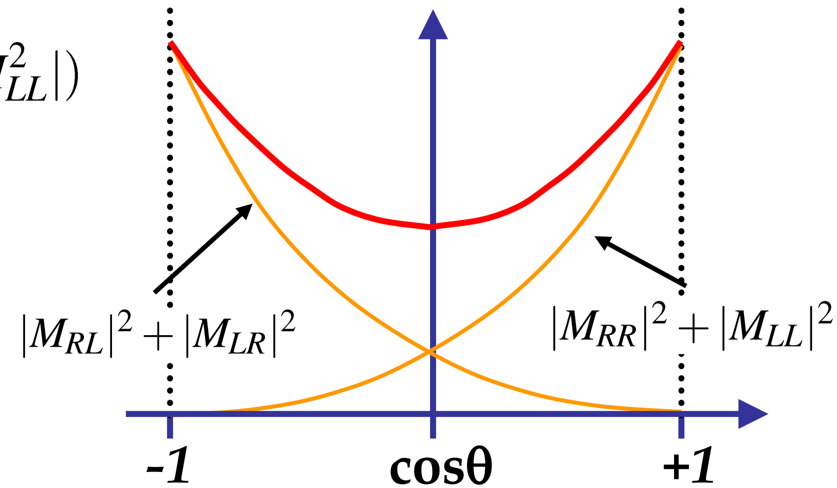


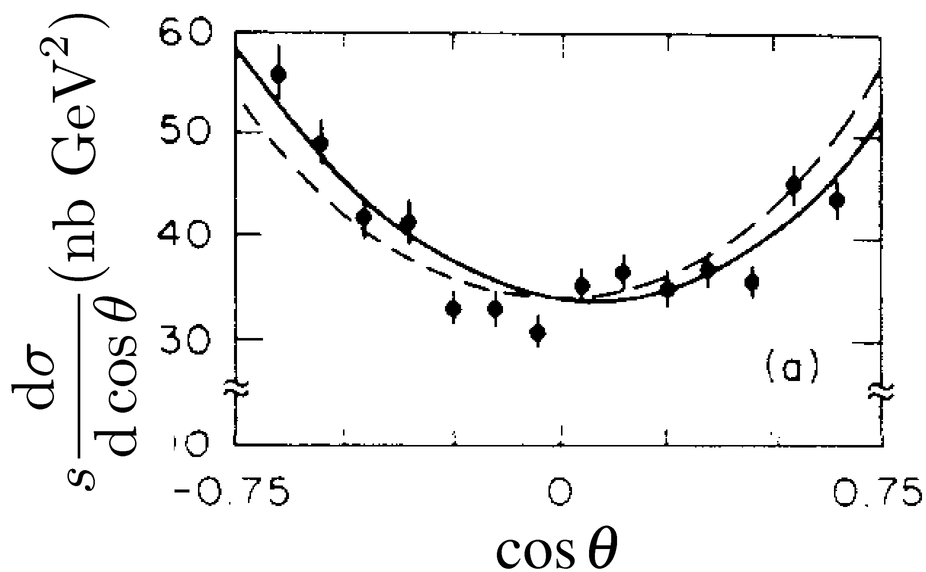

**Example:**

$e^+e^- \to \mu^+\mu^-$ $\sqrt{s} = 29$ GeV

- - - - - pure QED, $O(\alpha^3)$

——— QED plus Z contribution

- 在高阶QED中或者引入Z玻色子的贡献时，角分布变得稍微不对称




Mark II Expt., M.E.Levi et al., PRL 51 (1983) 1941

# Total Cross Section

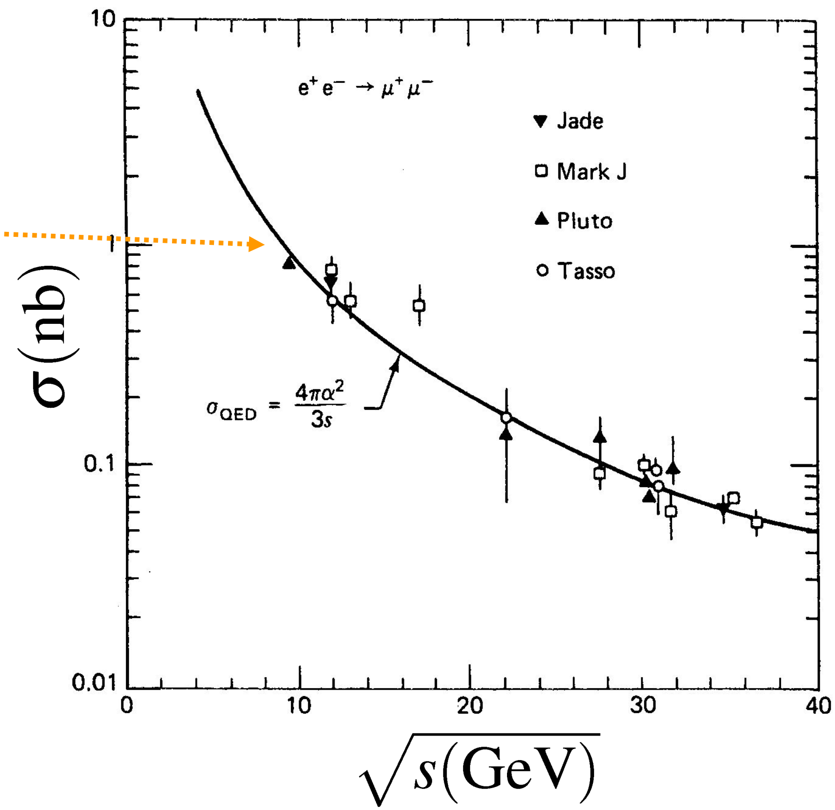
- 总截面通过对 $\theta$, $\phi$ 积分得到

$$\int (1+\cos^2\theta)\mathrm{d}\Omega = 2\pi \int_{-1}^{+1} (1+\cos^2\theta)\mathrm{d}\cos\theta = \frac{16\pi}{3}$$

给出$e^+e^- \to \mu^+\mu^-$ 过程的 **QED** 总截面

$$\boxed{\sigma = \frac{4\pi\alpha^2}{3s}}$$

➢ 最低阶截面就可以很好地描述实验数据

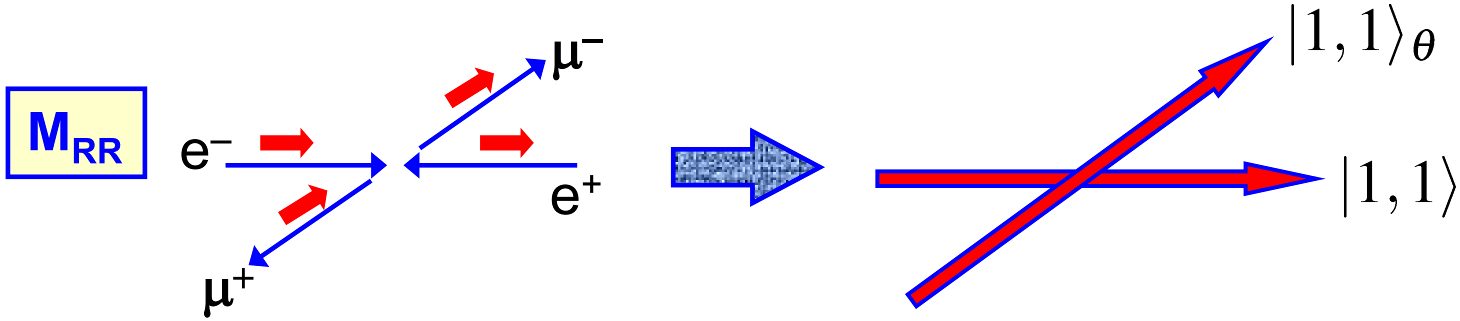



我们从第一性原理出发得到**1%**精度的正反电子湮灭的截面

# (optional) Spin Considerations $(E \gg m)$

- QED 正反电子矩阵元的角度依赖可以通过角动量来理解
  - 由于允许的螺旋度态，正反电子相互作用的自旋态为 $S_z = \pm 1$，
    - 即，总自旋为 1 态延z轴：$|1,+1\rangle$ 或 $|1,-1\rangle$
  - 类似地，缪子和反缪子产自一个总自旋为1且延极化角θ 的态

例如




- 因此 $M_{RR} \propto \langle \psi|1,1\rangle$，这里 $\psi$ 对于缪子对的自旋态 $|1,1\rangle_\theta$

$|1,1\rangle_\theta$ 被表达成$S_z$的本征态 详见附录 (和 QM)

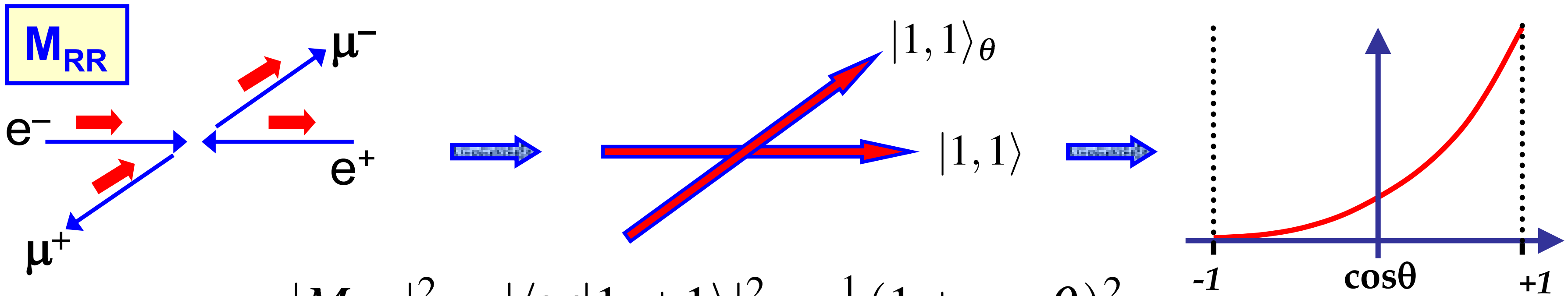
$$\boxed{|1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle}$$

# (optional) Spin Considerations $(E \gg m)$

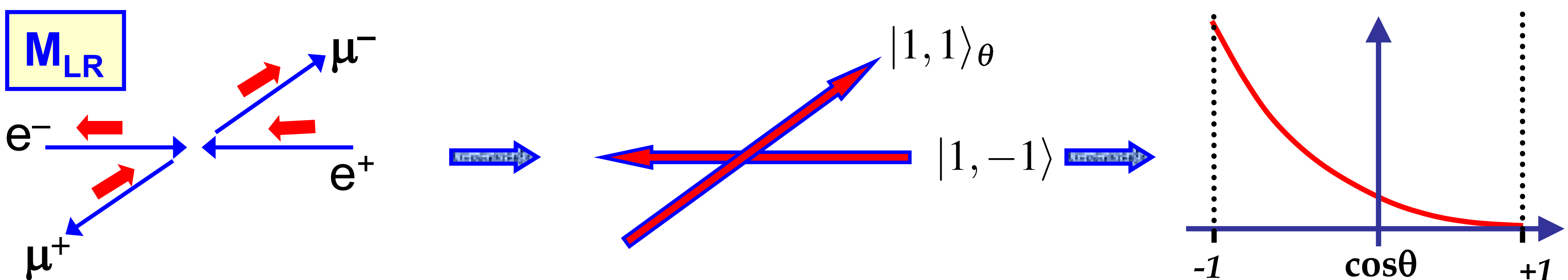
- **利用自旋为1且与轴夹角θ的态的波函数**

$$\psi = |1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

**能立即理解角度依赖**




$$|M_{\mathrm{RR}}|^2 \propto |\langle\psi|1,+1\rangle|^2 = \tfrac{1}{4}(1+\cos\theta)^2$$




$$|M_{\mathrm{LR}}|^2 \propto |\langle\psi|1,-1\rangle|^2 = \tfrac{1}{4}(1-\cos\theta)^2$$

# Lorentz Invariant form of Matrix Element

- 自旋平均的矩阵元表达为缪子在质心系的角度 $\langle |M_{fi}|^2 \rangle \quad = \quad e^4 (1+\cos^2\theta)$

- 矩阵元 洛伦兹不变（四矢量流的标量积）即，坐标系无关的表达形式

  - 在质心系 $p_1 = (E, 0, 0, E) \quad p_2 = (E, 0, 0, -E)$

    $p_3 = (E, E\sin\theta, 0, E\cos\theta) \quad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$

    得到： $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

- 可以写出

  （证明略去）

$$\boxed{\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2}} \quad \equiv 2e^4 \left( \frac{t^2+u^2}{s^2} \right)$$

★Valid in any frame !

# CHIRALITY

- $E \gg m$时，正反粒子的螺旋度本征态：　　其中 $s = \sin\frac{\theta}{2}$; $c = \cos\frac{\theta}{2}$

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

定义矩阵

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}$$

- $E \gg m$ 极限，螺旋度态也是 $\gamma^5$ 的本征态

$$\gamma^5 u_\uparrow = +u_\uparrow;\ \ \gamma^5 u_\downarrow = -u_\downarrow;\ \ \gamma^5 v_\uparrow = -v_\uparrow;\ \ \gamma^5 v_\downarrow = +v_\downarrow$$

➢ 定义$\gamma^5$本征态为左手和右手的手征态（CHIRAL）：$u_R$, $u_L$, $v_R$, $v_L$

i.e. $$\gamma^5 u_R = +u_R;\ \ \gamma^5 u_L = -u_L;\ \ \gamma^5 v_R = -v_R;\ \ \gamma^5 v_L = +v_L$$

- 在$E \gg m$ 极限下（且仅在此极限下）：

$$u_R \equiv u_\uparrow; \qquad u_L \equiv u_\downarrow;$$
$$v_R \equiv v_\uparrow; \qquad v_L \equiv v_\downarrow$$

# CHIRALITY

## 注意

- 手征是QED以及任何$\bar{u}\gamma^\nu u$形式相互作用的一个重要概念
- 一般而言，螺旋度和手征本征态并不相同，只在相对论极限下手征本征态对应螺旋度的
- **手征算符本征态的普遍定义为：**

$$\gamma^5 u_R = +u_R;\ \ \gamma^5 u_L = -u_L;\ \ \gamma^5 v_R = -v_R;\ \ \gamma^5 v_L = +v_L$$

- 定义投影算符：

$$\boxed{P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)}$$

投影出手征本征态

$$\boxed{\begin{array}{llll} P_R u_R = u_R; & P_R u_L = 0; & P_L u_R = 0; & P_L u_L = u_L \\ P_R v_R = 0; & P_R v_L = v_L; & P_L v_R = v_R; & P_L v_L = 0 \end{array}}$$

- **注意 $P_R$ 投影出 右手粒子态 和 左手反粒子态**
  - 任意旋量都可以写成左手和右手的手征分量

$$\psi = \psi_R + \psi_L = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi$$

# Chirality in QED

- **QED中费米子和光子的基本相互作用：**

**旋量分解为左手和右手的手征分量**

$$
\begin{aligned}
ie\overline{\psi}\gamma^\mu\phi &= ie(\overline{\psi}_L+\overline{\psi}_R)\gamma^\mu(\phi_R+\phi_L) \\
&= ie(\overline{\psi}_R\gamma^\mu\phi_R+\overline{\psi}_R\gamma^\mu\phi_L+\overline{\psi}_L\gamma^\mu\phi_R+\overline{\psi}_L\gamma^\mu\phi_L)
\end{aligned}
$$

- **利用$\gamma^5$的性质**

$$\boxed{(\gamma^5)^2=1; \quad \gamma^{5\dagger}=\gamma^5; \quad \gamma^5\gamma^\mu=-\gamma^\mu\gamma^5}$$

**直接得到** $\overline{\psi}_R\gamma^\mu\phi_L=0; \quad \overline{\psi}_L\gamma^\mu\phi_R=0$

  - **因此只有特定组合的手征态对相互作用有贡献。此结论永远成立！**

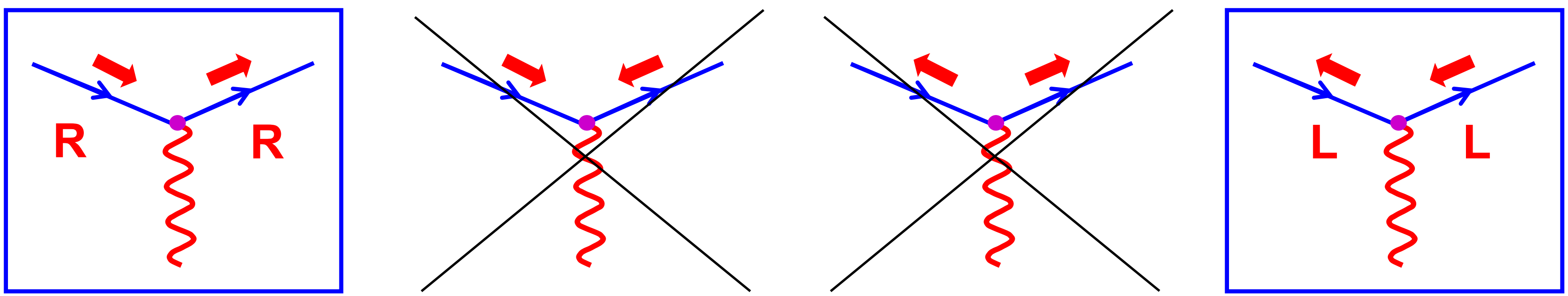
- **$E \gg m$ 极限，手征与螺旋度的本征态等价**
  - 这说明$E \gg m$ 时，只有特定螺旋度组合对 QED 顶角有贡献！
  - This is why previously two of four helicity combinations for muon current were zero

# Allowed QED Helicity Combinations

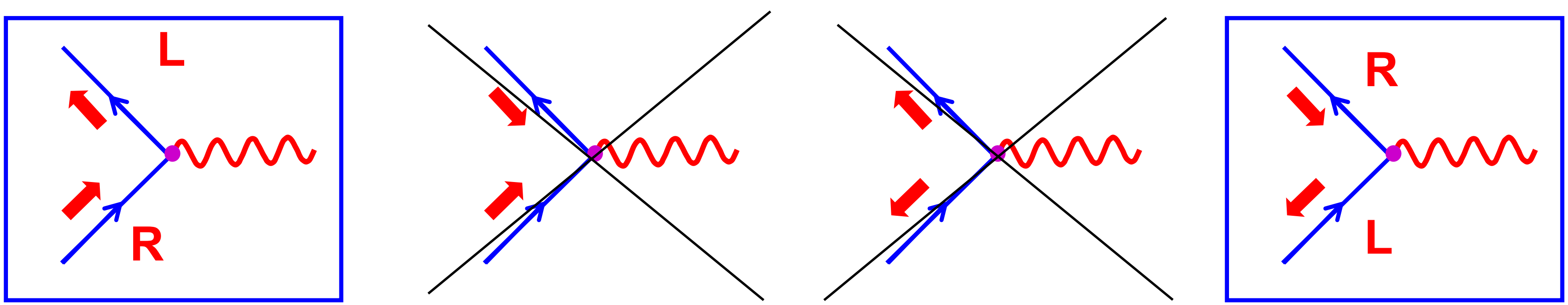
- In ultra-relativistic limit the helicity eigenstates ≡ chiral eigenstates
- In this limit, the only non-zero helicity combinations in QED are:

**Scattering:**

"Helicity conservation"




**Annihilation:**



# Summary

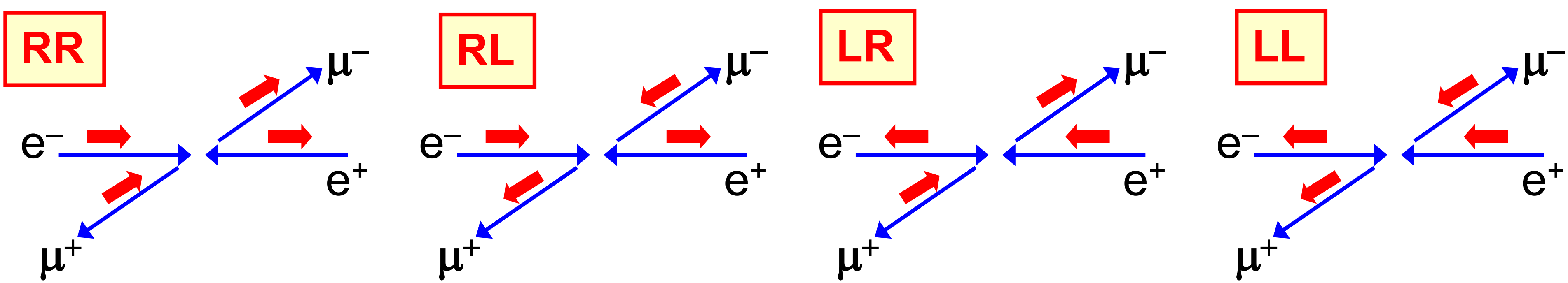
- 在质心系 $e^+e^- \to \mu^+\mu^-$ 的微分截面：$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

  注意：忽略缪子质量，即假设 $E \gg m_\mu$

- 在 QED 中，只有特定左手 和 右手 手征态组合会给出非零的矩阵元

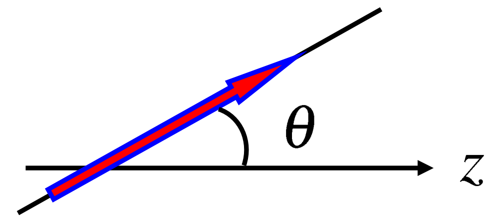
- 通过手征投影算符定义手征态 $$P_R = \frac{1}{2}(1+\gamma^5); \qquad P_L = \frac{1}{2}(1-\gamma^5)$$

- $E \gg m$ 极限，手征本征态与螺旋度本征态对应
  - 只有特定螺旋度组合会给出非零的矩阵元



# Appendix : Spin 1 Rotation Matrices

- 考虑自旋为1态，
  其自旋+1方向延单位矢量：$\vec{n} = (\sin\theta, 0, \cos\theta)$




- 自旋态是$\vec{n}\cdot\vec{S}$的本征态，本征值为 +1 $\quad (\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle$ **(A1)**

- 表达成自旋为1态的
  $S_z$ 本征态的线性组合 $\quad |\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$ 其中 $\alpha^2+\beta^2+\gamma^2=1$

- **(A1)** 变成 $\quad (\sin\theta S_x + \cos\theta S_z)(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle)$
  $= \alpha|1,1\rangle + \beta|1,0\rangle\gamma|1,-1\rangle$ **(A2)**

- $S_x$ 写成梯形算符 $S_x = \frac{1}{2}(S_+ + S_-)$

其中 $\quad S_+|1,1\rangle = 0 \qquad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \qquad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$

$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \qquad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \qquad S_-|1,-1\rangle = 0$

- from which we find
$$S_x|1,1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$
$$S_x|1,0\rangle = \tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$
$$S_x|1,-1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$

- (A2) becomes
$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] +$$
$$\alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle\gamma|1,-1\rangle$$

- which gives
$$\left.\begin{aligned} \beta\frac{\sin\theta}{\sqrt{2}} + \alpha\cos\theta &= \alpha \\ (\alpha+\gamma)\frac{\sin\theta}{\sqrt{2}} &= \beta \\ \beta\frac{\sin\theta}{\sqrt{2}} - \gamma\cos\theta &= \gamma \end{aligned}\right\}$$

- using $\alpha^2 + \beta^2 + \gamma^2 = 1$ the above equations yield
$$\alpha = \frac{1}{\sqrt{2}}(1+\cos\theta) \qquad \beta = \frac{1}{\sqrt{2}}\sin\theta \qquad \gamma = \frac{1}{\sqrt{2}}(1-\cos\theta)$$

- hence $\psi = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle$

- The coefficients $\alpha, \beta, \gamma$ are examples of what are known as quantum mechanical **rotation matrices**. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction

$$\boxed{d^{j}_{m',m}(\theta)}$$

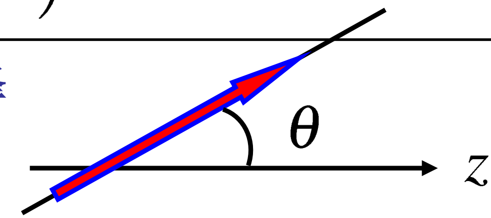
- For spin-1 $(j=1)$ we have just shown that

$$d^{1}_{1,1}(\theta) = \tfrac{1}{2}(1+\cos\theta) \qquad d^{1}_{0,1}(\theta) = \frac{1}{\sqrt{2}}\sin\theta \qquad d^{1}_{-1,1}(\theta) = \tfrac{1}{2}(1-\cos\theta)$$

- For spin-1/2 it is straightforward to show

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \qquad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}$$

# Spin Considerations $(E \gg m)$




- 正负电子湮灭的**LL** 和**LR**过程截面的差异可用**QM**角动量守恒解释
- 碰撞前系统的自旋态记作$|j, j_z\rangle$，碰撞后系统绕**y**轴转动**θ**角
- 此时系统的自旋态记作$|j, j_z'\rangle$。碰撞后的系统可以由态系统通过转动得到，转动矩阵为

$$d^j_{j_z, j_z'} = \langle j, j_z' | e^{-iJ_y\theta} | j, j_z\rangle$$

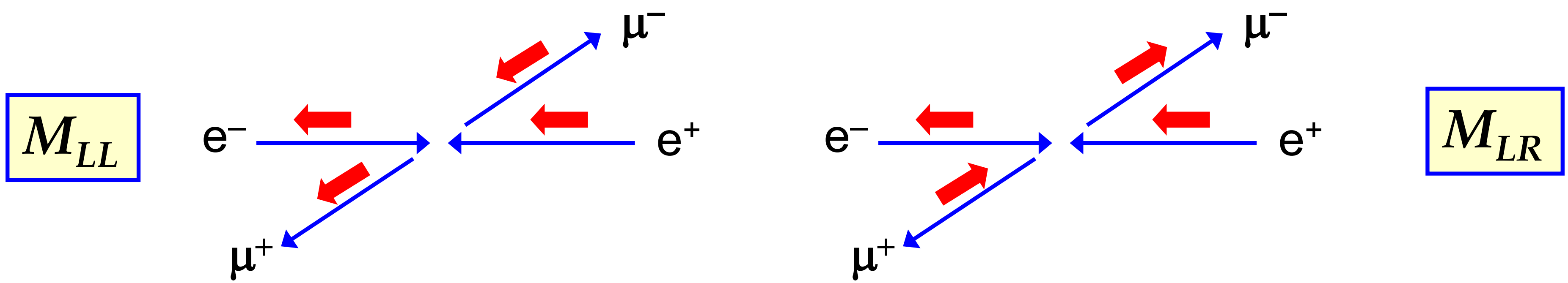
- 矩阵元的一般计算公式为**(Wigner's expression)**：

$$d^j_{j_z, j_z'} = \sum_s \frac{(-1)^s\left[(j+j_z)!(j-j_z)!(j+j_z')!(j-j_z')!\right]^{1/2}}{s!(j-s-j_z')!(j+j_z-s)!(j_z'+s-j_z)!} \times \left(\cos\frac{\theta}{2}\right)^{2j+j_z-j_z'-2s} \left(-\sin\frac{\theta}{2}\right)^{j_z'-j_z+2s}$$

- 其中对所有满足阶乘内$\geq 0$的**s**求和
- 这里列出几个：

$$d^0_{0,0} = 1, \qquad \begin{pmatrix} d^1_{1,1} & d^1_{1,0} & d^1_{1,-1} \\ d^1_{0,1} & d^1_{0,0} & d^1_{0,-1} \\ d^1_{-1,1} & d^1_{-1,0} & d^1_{-1,-1} \end{pmatrix} = \begin{pmatrix} \dfrac{1+\cos\theta}{2} & -\dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1-\cos\theta}{2} \\ \dfrac{\sin\theta}{\sqrt{2}} & \cos\theta & -\dfrac{\sin\theta}{\sqrt{2}} \\ \dfrac{1-\cos\theta}{2} & \dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1+\cos\theta}{2} \end{pmatrix}$$

- 不变矩阵元和转动矩阵有关：$\left|M_{fi}\right|^2 \propto \left|d^j_{j_z, j_z'}\right|^2$




- **LL** 过程：碰撞前系统总自旋为**1**，自旋态为$|j,j_z\rangle = |1,-1\rangle$，碰撞后系统总自旋仍为**1**，自旋态为$|j,j_z'\rangle = |1,-1\rangle$，于是

$$\left.\left|M_{fi}\right|^2\right|_{LL} \propto \left|d^1_{-1,-1}\right|^2 = \left(\frac{1+\cos\theta}{2}\right)^2$$

- **LR** 过程：碰撞前系统总自旋为**1**，自旋态为$|j,j_z\rangle = |1,-1\rangle$，碰撞后系统为$|j,j_z'\rangle = |1,1\rangle$，于是

$$\left.\left|M_{fi}\right|^2\right|_{LR} \propto \left|d^1_{-1,1}\right|^2 = \left(\frac{1-\cos\theta}{2}\right)^2$$

# Clebsh-Gordan coefficients

- 将两个不同的角动量相加，可以得到

$$\boldsymbol{J} = \boldsymbol{J_1} + \boldsymbol{J_2}$$

- 有两组可以互相转换的本征值：

$$|j_1, j_2; j, j_z\rangle$$

$$|j_1, j_2; j_{z1}, j_{z2}\rangle \equiv |j_1, j_{z1}\rangle \otimes |j_2, j_{z2}\rangle$$

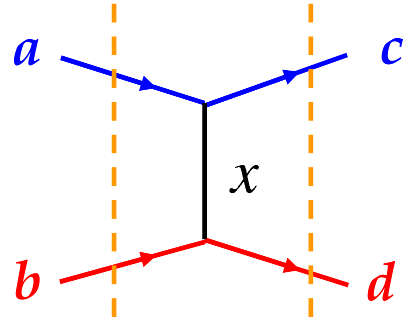
- 其满足：

$$|j_1, j_2; j, j_z\rangle = \sum_{j_{z1}, j_{z2}} C(j_1, j_2, j; j_{z1}, j_{z2}, j_z) |j_1, j_2; j_{z1}, j_{z2}\rangle$$

- $C(j_1, j_2, j; j_{z1}, j_{z2}, j_z)$就是**CG**系数

# 传播子与共振态

**Recap:**



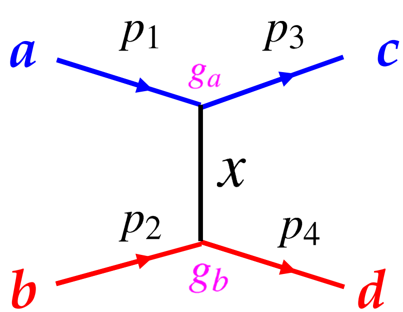

$$\boxed{M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}}$$

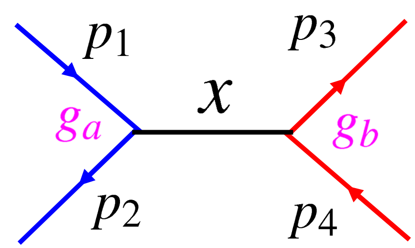
- 量子场论中，将费曼图中的内线称呼为传播子，传播子往往不满足质壳关系，即

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

因此传播子也被称为虚粒子。

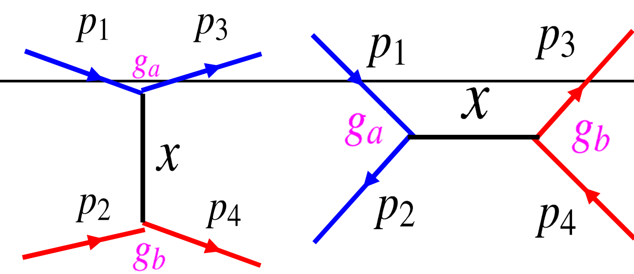
- 但是如果在特殊条件下满足了$q^2 = m_X^2$，共振态就出现了。




- 这两种过程哪个可能产生共振态？

# 传播子与共振态




- 写出两张图中传播子对应的不变矩阵元：

$$M = g_a g_b\left(\frac{1}{(p_1 - p_3)^2 - m_X^2} + \frac{1}{(p_1 + p_2)^2 - m_X^2}\right)$$

- 在质心系中计算，假设外线粒子质量$m$，内线粒子质量$m_X$
- 写出四动量：

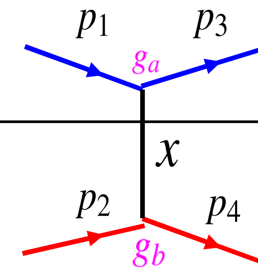
$$p_1: (E, \vec{p}), \qquad p_2: (E, -\vec{p}), \qquad p_3: \left(E, \overrightarrow{p'}\right), \qquad p_4: \left(E, -\overrightarrow{p'}\right)$$

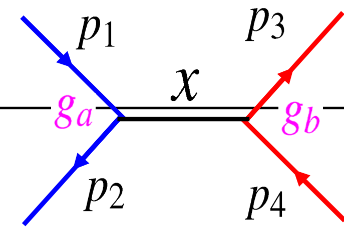
- 第一项$\frac{1}{(p_1 - p_3)^2 - m_X^2}$是**t(u)-channel**的贡献：

$$t = \mathrm{q}^2 = (p_1 - p_3)^2 = (E - E)^2 - \left(\vec{p} - \overrightarrow{p'}\right)^2 < 0$$

- 因此，第一项分母$t - M_X^2$总不为零，第一项是有限的，不会导致共振态的出现

# 传播子与共振态




- 第二项$\frac{1}{(p_1+p_2)^2-m_X^2}$是**s-channel**的贡献：

$$s = q^2 = (p_1 + p_2)^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2 > 4m^2$$

  - ➢ $m_X^2 < 4m^2 < s$
  - $s - m_X^2$永远不为零，第二项有限。此时的传播子永远是虚粒子。
  - ➢ $m_X^2 > 4m^2$
  - 此时，在某种动量下，$s = m_X^2$可以满足，分母等于**0**，第二项发散！此时传播子在壳，不是虚粒子，可以产生共振态。

- 第二项可能发散说明式子中的写法有错误。实际上，此时的传播子不稳定，可以衰变到两个粒子**3**和**4**，**s-channel**的矩阵元必须考虑到传播子的衰变。

- **Recap:**

  **粒子衰变的量子力学描述：**设 $t = 0$时形成能量为 $E_0$、平均寿命为 $\tau$ 的态，

$$\psi(t) = \psi(0)e^{-iE_0t}e^{-t/2\tau} \qquad |\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau}$$

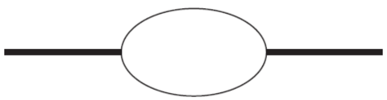
  i.e. the probability density decays exponentially (as required)

- $\Gamma = \frac{1}{\tau}$，相当于对能量进行修正：$E = E - i\frac{\Gamma}{2}$

# 传播子与共振态

- 类似的，在量子场论中对传播子场作如图所示圈图修正时，正是对传播子矩阵元中的质量项进行修正：

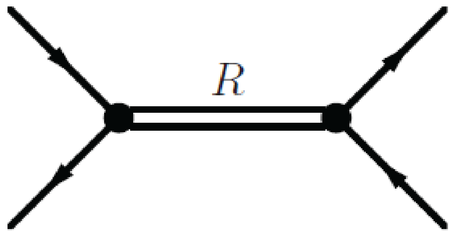
$$m_X = m_R - i\frac{\Gamma}{2}$$



- 于是**s-channel**的矩阵元被修正为：

$$M_s = \frac{g^2}{s - \left(m_R - i\frac{\Gamma}{2}\right)^2}$$

- 为了计算反应截面，假设$\Gamma \ll m_R$，得到$\tau = \frac{1}{\Gamma} \gg \frac{1}{m_R} = \lambda$，这意味着传播子的寿命$\tau$远大于光通过$\lambda$距离（传播子的**Compton** 波长）所需要的时间。因此可以认为，此时**s-channel**中产生了一个真实的中间态粒子**R**，它通过初态粒子的碰撞产生，存在一段时间后衰变到末态粒子，这个中间态粒子成为共振态。



# 传播子与共振态

- 此时有：

$$\left(m_R - i\frac{\Gamma}{2}\right)^2 \simeq m_R^2 - im_R\Gamma$$

$$M_s = \frac{g^2}{s - m_R^2 + im_R\Gamma}$$

- 代入质心系弹性散射的微分截面公式：$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|M|^2$：

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{g^4}{[E^2 - m_R^2 + im_R\Gamma]^2}$$

- 上式中E为质心系总能量。对于共振态，可以用可观测量Γ替代掉式中的g：
- 二体衰变：

$$\Gamma = \frac{|\overrightarrow{p^*}|}{32\pi^2 E_i m_i}\int |M|^2 d\Omega = \frac{\frac{1}{2m_R}\sqrt{(m_R^2 - 4m^2)}}{32\pi^2 E m_R}\int g^2 d\Omega = \frac{g^2}{16\pi E m_R}\sqrt{1 - \frac{4m^2}{m_R^2}}$$

- 代入得到：

$$\frac{d\sigma}{d\Omega} = \frac{4m_R^2}{m_R^2 - 4m^2}\frac{\Gamma^2}{(E^2 - m_R^2)^2 + m_R^2\Gamma^2}$$

# 传播子与共振态

- 根据$R \to 3 + 4$能量守恒：$m_R = 2\sqrt{m^2 + \overrightarrow{p'}^2}$：

$$m_R^2 - 4m^2 = 4\left|\overrightarrow{p'}\right|^2 = 4|\vec{p}|^2$$

- 代入微分截面表达式：

$$\frac{d\sigma}{d\Omega} = \frac{m_R^2}{|\vec{p}|^2}\frac{\Gamma^2}{(E^2 - m_R^2)^2 + m_R^2\Gamma^2}$$

- 积分得到：

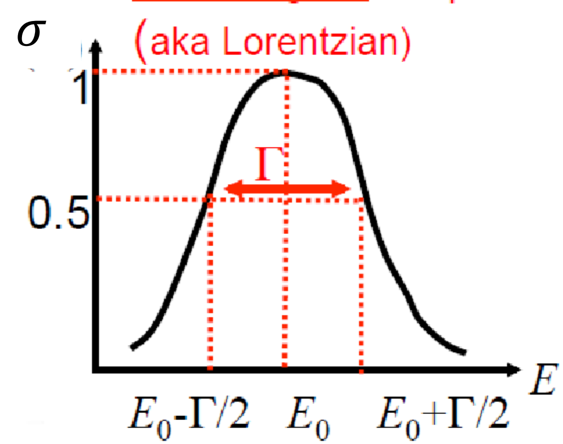
$$\sigma \simeq \frac{4\pi m_R^2}{|\vec{p}|^2}\frac{\Gamma^2}{(E^2 - m_R^2)^2 + m_R^2\Gamma^2}$$

- 此时被称为相对论**Breit-Wigner**共振态公式。

# 传播子与共振态

- 考虑特殊情况，共振态很窄： $E \simeq m_R$， $E^2 - m_R^2 \simeq 2m_R(E - m_R)$

$$\sigma \simeq \frac{4\pi}{|\vec{p}|^2} \frac{(\frac{\Gamma}{2})^2}{(E - m_R)^2 + (\frac{\Gamma}{2})^2}$$




- 此时被称为非相对论**Breit-Wigner**共振态公式。
- 当$E = m_R$时总截面达到最大$\sigma_{max} = \frac{4\pi}{|\vec{p}|^2}$
- 当$E = m_R \pm \frac{\Gamma}{2}$时总截面为：

$$\sigma = \frac{4\pi}{|\vec{p}|^2} \frac{(\frac{\Gamma}{2})^2}{\left(m_R \pm \frac{\Gamma}{2} - m_R\right)^2 + (\frac{\Gamma}{2})^2} = \frac{1}{2}\sigma_{max}$$

- 可见Γ是共振峰半高上的全宽度（**full width at half maximum**），称为共振峰的宽度。