第14章 变分法及应用

14.1 变分的基本问题

速降线问题，Euler方程，自然边界条件

14.2 泛函的条件极值问题

测地线问题,等周问题, 归一化问题

14.3 Hamilton原理与最小位能原理

弦的横向振动,膜的横振动,静电场方程

14.4 本征值问题中的应用

本征值与泛函的关系,完备性定理, 近似方法

14.5 边值问题中的应用

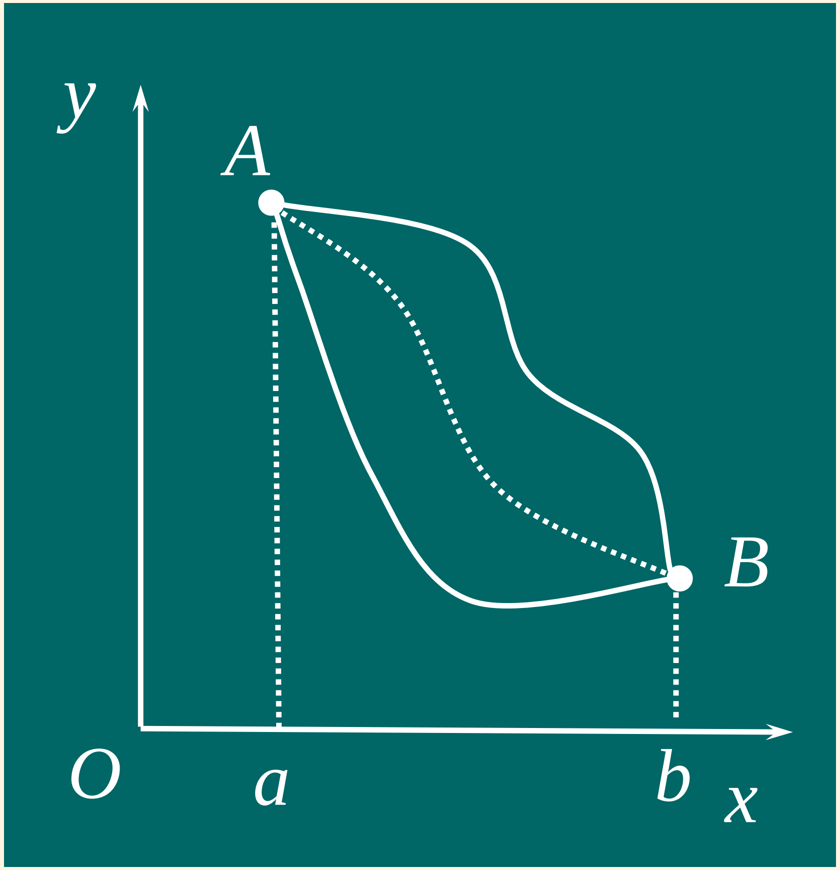
正算子, Ritz−Galerkin法, 有限元方法(FEM)

14.1变分的基本问题

■ 固定边界问题

最速降线问题: A点到B点所需总时间为积分

$$T(y)=\frac{1}{\sqrt{2g}}\int_a^b\sqrt{\frac{1+y'^2}{y}}\mathrm{d}x$$



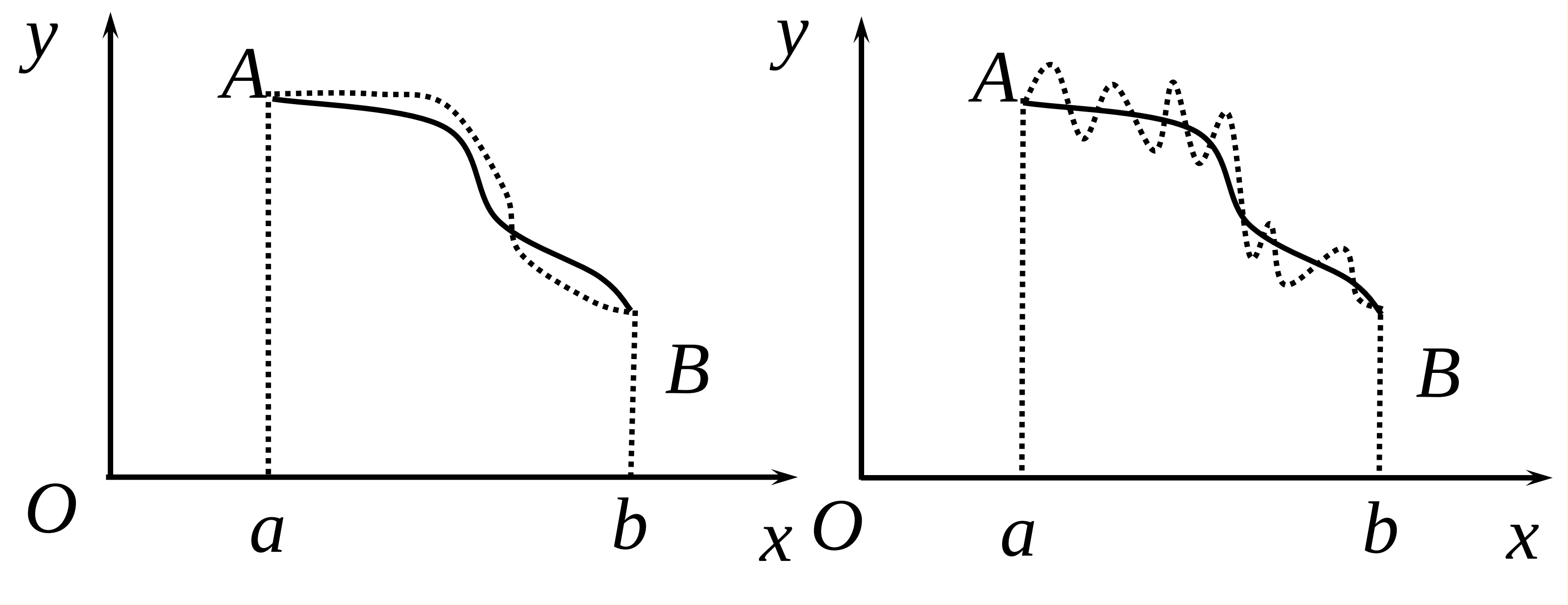
——泛函：随给定函数取确定值的对应关系

- **变分：求泛函的极值问题，称为变分问题**
- **邻域：泛函T在$y=\xi(x)$处达到极值，满足**

$$|y(x)-\xi(x)|<\varepsilon,\ \ x\in(a,b)$$

$$|y'(x)-\xi'(x)|<\varepsilon,\ \ x\in(a,b)$$

的函数$y(x)$称为属于$\xi(x)$的一阶ε邻域.



■ **极值函数:对某一阶ε邻域中所有$y(x)$都使**

$$T(\xi) \le T(y); \quad T(\xi) \ge T(y)$$

$\xi(x)$称为极值函数

■ **必要条件** **设$y(x)$使泛函**

$$J(y) = \int_a^b f(x, y, y')\mathrm{d}x$$

取极值, 则对$y(x)$邻域内的函数$y^*(x)$, 应有

$$J(y) \ge J(y^*) \quad \text{或者} \quad J(y) \ge J(y^*)$$

取

$$y^*(x) = y(x) + \alpha\eta(x)$$

——当α足够小， $y^*(x)$属于$y(x)$的邻域

泛函J在$y^*(x)$的值

$$J(y^*)=\int_a^b f\left(x,y^*,\frac{\mathrm{d}y^*}{\mathrm{d}x}\right)\mathrm{d}x$$

变量α的函数

$$=\int_a^b f\left[x,y+\alpha\eta,y'(x)+\alpha\eta'(x)\right]\mathrm{d}x\equiv J(\alpha)$$

■ **函数极值的必要条件** $J'(\alpha)\big|_{\alpha=0}=0$

$$\left.\frac{\mathrm{d}J(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0}=\int_a^b\left[\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)\right]\mathrm{d}x=0$$

第二项分部积分

$$\left.\frac{\mathrm{d}J(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0}=\left[\frac{\partial f}{\partial y'}\eta(x)\right]\Bigg|_a^b+\int_a^b\eta(x)\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\mathrm{d}x=0$$

■ 固定端点问题

$$y^*(a)=y(a)+\alpha\eta(a)=0\Rightarrow\eta(a)=0$$

$$y^*(b)=y(b)+\alpha\eta(b)=0\Rightarrow\eta(b)=0$$

$$\left.\frac{\mathrm{d}J(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0}=\int_a^b\eta(x)\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\mathrm{d}x=0$$

■ Euler方程，必要条件 对任意$\eta(x)$成立

$$\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}=0$$

——$y(x)$满足的二阶常微分方程

■ 泛函J的一阶变分

$$\delta J \equiv \left.\frac{\mathrm{d}J(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} \cdot \alpha = \left[\frac{\partial f}{\partial y'}\delta y\right]_a^b + \int_a^b \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y \mathrm{d}x$$

——当函数由$y \to y+\delta y$, 泛函J的一阶变化: $J \to J+\delta y$

函数的一阶微分
极值点为零

→

泛函的一阶变分
极值函数为零

■ 泛函J的二阶变分

$$J(y+\delta y) = \int_a^b f(x, y+\delta y, y'+\delta y')\mathrm{d}x$$

——看作二个独立变量的展开，首先对y展开

$$J(y+\delta y)=\int_a^b\left[f_1+\frac{\partial f_1}{\partial y}\delta y+\frac{1}{2}\frac{\partial^2 f_1}{\partial y^2}(\delta y)^2\right]\mathrm{d}x$$

$$\Uparrow$$

$$f_1\equiv f(x,y,y'+\delta y')$$

再对*y*′展开，并且仅保留二阶项

$$J(y+\delta y)-J(y)\equiv\delta J+\delta^2 J$$

$$\Uparrow$$

$$\delta^2 J=\frac{1}{2}\int_a^b\left[\frac{\partial^2 f}{\partial y'^2}(\delta y')^2+\frac{\partial^2 f}{\partial y\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}(\delta y)^2+\frac{\partial^2 f}{\partial y^2}(\delta y)^2\right]\mathrm{d}x$$

$$=\left[\frac{1}{2}\frac{\partial^2 f}{\partial y\partial y'}(\delta y)^2\right]_a^b+\frac{1}{2}\int_a^b\left[\frac{\partial^2 f}{\partial y'^2}(\delta y')^2+\left(\frac{\partial^2 f}{\partial y^2}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial^2 f}{\partial y\partial y'}\right)(\delta y)^2\right]\mathrm{d}x$$

■ 特殊情况：$f(y, y')$—不显含x

$$J(y)=\int_a^b f(y,y')\mathrm{d}x \Rightarrow y'\frac{\partial f}{\partial y'}-f=C(\text{常数})$$

证明 不难计算

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}-f\right)=y''\frac{\partial f}{\partial y'}+y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}-\frac{\mathrm{d}f}{\mathrm{d}x}$$

$$=y''\frac{\partial f}{\partial y'}+y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}-y'\frac{\partial f}{\partial y}-y''\frac{\partial f}{\partial y'}=y'\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}\right)=0$$

$$\Downarrow$$

$$y'\frac{\partial f}{\partial y'}-f=C(\text{常数}) \Leftarrow$$

物理上相当于存在守恒量

例1 最速降线问题

$$f(x,y,y')=\sqrt{(1+y'^2)/y}$$

$$y'\frac{\partial f}{\partial y'}-f=-\frac{1}{\sqrt{y(1+y'^2)}}=\text{常数} \Rightarrow y(1+y'^2)=C$$

$$y(1+y'^2)=C \Leftarrow y'=\cot(t)$$

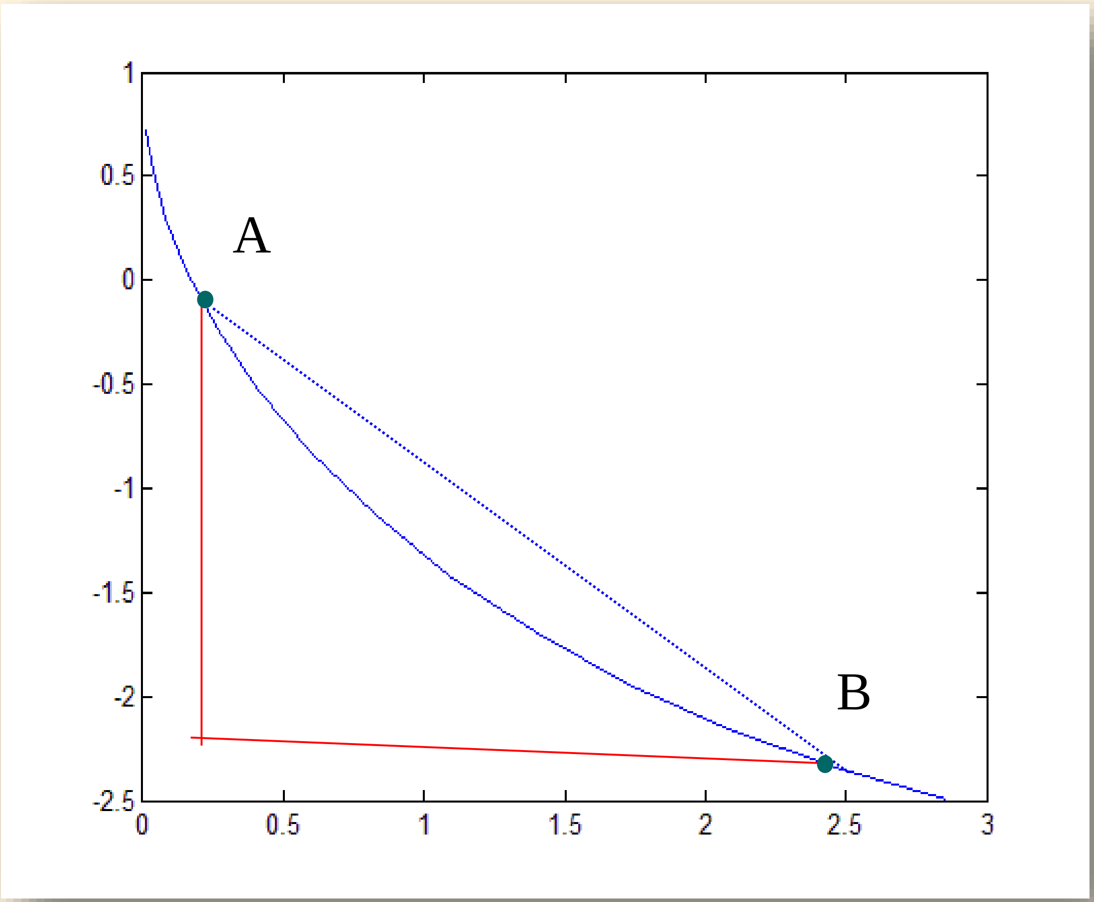
$$\Downarrow$$

$$y=C\sin^2 t=\frac{1}{2}C(1-\cos 2t)$$

$$\Downarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos t}{\sin t}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\sin t}{\cos t}\frac{\mathrm{d}y}{\mathrm{d}t}=C(1-\cos 2t)$$

$$x(t)=C\left(t-\frac{1}{2}\sin 2t\right)+C_1;\ y(t)=\frac{C}{2}(1-\cos 2t)$$



① 常数C和C_1由A和B二点决定;

② A~B是摆线(cycloid)的一部分;

③ 也是等时线。

例2 求下列泛函的Euler方程, 并要求通过两定点

$$J(y)=\frac{1}{2}\int_a^b\left[p(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+q(x)y^2\right]\mathrm{d}x$$

- **一阶变分**

$$\delta J(y)=\int_a^b\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\delta\frac{\mathrm{d}y}{\mathrm{d}x}+q(x)y\delta y\right]\mathrm{d}x$$

$$=\int_a^b\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}\delta y}{\mathrm{d}x}+q(x)y\delta y\right]\mathrm{d}x$$

第一项分部积分

$$\delta J(y)=\left[p(x)y'(x)\delta y\right]\Big|_a^b+\int_a^b\left\{-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right]+q(x)y\right\}\delta y\mathrm{d}x$$

因端点固定, 故

$$\delta J(y)=\int_a^b\left\{-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right]+q(x)y\right\}\delta y\mathrm{d}x$$

Euler方程

$$\Downarrow$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right]+q(x)y(x)=0$$

——第一类边界条件下的Sturm−Liouville边值问题

■ 二阶变分

$$J(y+\delta y)=\frac{1}{2}\int_a^b\left[p(x)\left(\frac{\mathrm{d}(y+\delta y)}{\mathrm{d}x}\right)^2+q(x)(y+\delta y)^2\right]\mathrm{d}x$$

$$\equiv J(y)+\delta J+\delta^2 J$$

$$\delta^2 J\equiv\frac{1}{2}\int_a^b\left[p(x)\left(\frac{\mathrm{d}\delta y}{\mathrm{d}x}\right)^2+q(x)(\delta y)^2\right]\mathrm{d}x$$

如果*p*(*x*)>0, *q*(*x*)>0, 则 $\delta^2 J>0$ ➡ **泛函极小**

如果*p*(*x*)<0, *q*(*x*)<0, 则 $\delta^2 J<0$ ➡ **泛函极大**

❑多个变量的变分问题

$$J(u)=\iint_G F(x,y,u,u_x,u_y)\mathrm{d}x\mathrm{d}y$$

$$u(x,y)|_{\partial G}=u_0(x,y)$$

在*u*(*x*,*y*)的邻域内取比较函数

$$u^*(x,y)=u(x,y)+\alpha\eta(x,y);\eta(x,y)|_{\partial G}=0$$

$$\Downarrow$$

$$\begin{aligned}J(u^*)&=\iint_G F(x,y,u^*,u_x^*,u_y^*)\mathrm{d}x\mathrm{d}y\\&=\iint_G F(x,y,u+\alpha\eta,u_x+\alpha\eta_x,u_y+\alpha\eta_y)\mathrm{d}x\mathrm{d}y\\&\equiv J(\alpha)\end{aligned}$$

$$\left.\frac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} = \iint_G (F_u\eta + F_{u_x}\eta_x + F_{u_y}\eta_y)\mathrm{dxdy}$$

$$= \iint_G \left(F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} \right)\eta(x,y)\mathrm{dxdy}$$

$$+\iint_G \left[\frac{\partial (F_{u_x}\eta)}{\partial x} + \frac{\partial (F_{u_y}\eta)}{\partial y} \right]\mathrm{dxdy}$$

第二项可用平面Green公式化成∂G上的积分

$$\iint_G \left[\frac{\partial (F_{u_x}\eta)}{\partial x} + \frac{\partial (F_{u_y}\eta)}{\partial y} \right]\mathrm{dxdy} = \int_{\partial G} \eta(F_{u_y}\,\mathrm{dy} - F_{u_x}\,\mathrm{dx})$$

于是，对固定边界问题

$$\iint_G \left(F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} \right) \eta(x, y) \mathrm{d}x\mathrm{d}y \equiv 0$$

Euler方程

$$F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} = 0$$

❑*n*个变量的情况

$$J(u) = \iint_G F(x_1, \cdots, x_n; u, u_{x_1}, \cdots, u_{x_n}) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

***n*维空间的Green公式**

$$\iint_G \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (F_{u_{x_i}} \delta u) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n = \sum_{i=1}^{n} \int_{\partial G} F_{u_{x_i}} \cos \vartheta_i \delta u \mathrm{d}S$$

一阶变分

$$\delta J(u)=\iint_G\left[\frac{\partial F}{\partial u}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial u_{x_i}}\right)\right]\delta u\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n=0$$

Euler方程

$$\frac{\partial F}{\partial u}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial u_{x_i}}\right)=0$$

例1 求泛函的Euler方程

$$J(u)=\int_G\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}\tau$$

边界固定

$$u(x,y,z)\big|_{\partial G}=u_0(x,y,z)$$

Euler方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

——泛函的极值问题化为求Laplace方程的第一类边值问题

二阶变分

$$J(u+\delta u)=\int_G\left[\left(\frac{\partial u}{\partial x}+\frac{\partial \delta u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial \delta u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial \delta u}{\partial z}\right)^2\right]\mathrm{d}\tau$$

$$J(u+\delta u)-J(u)=\delta J+\delta^2 J$$

$$\delta J=2\int_G\left(\frac{\partial u}{\partial x}\frac{\partial\delta u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial\delta u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial\delta u}{\partial z}\right)\mathrm{d}\tau$$

$$=2\iint_{\partial G}\delta u(\nabla u)\cdot\mathrm{d}\boldsymbol{S}-2\int_G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\delta u\,\mathrm{d}\tau$$

$$\delta^2 J=\int_G\left[\left(\frac{\partial\delta u}{\partial x}\right)^2+\left(\frac{\partial\delta u}{\partial y}\right)^2+\left(\frac{\partial\delta u}{\partial y}\right)^2\right]\mathrm{d}\tau>0$$

例2 求泛函的Euler方程，边界值固定为零

$$J(u)=\int_G\left[p(\nabla u)^2+(q-\lambda\rho)u^2\right]\mathrm{d}\tau$$

Euler方程

$$-\nabla\cdot(p\nabla u)+qu=\lambda\rho u$$

——转化为第一类边界条件的本征值问题

❑变端点问题和自然边界条件

$$J(y)=\int_a^b f(x,y,y')\mathrm{d}x$$

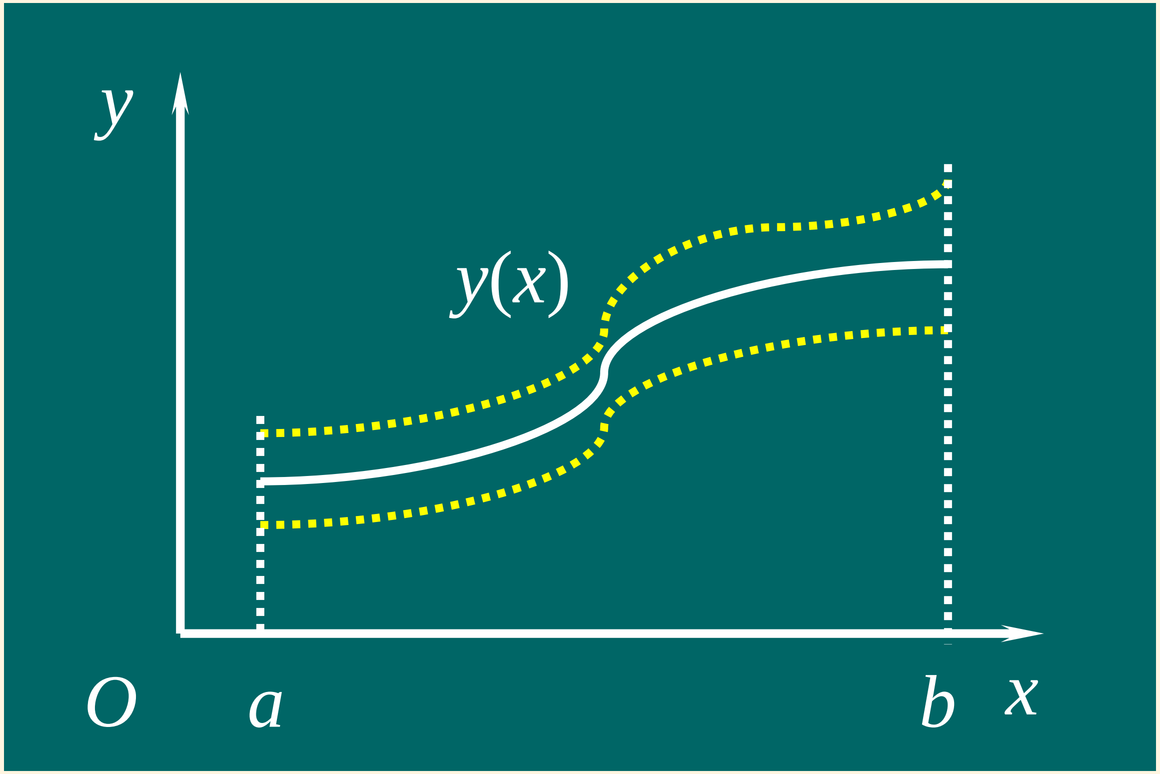
一阶变分

$$\delta J=\left[\frac{\partial f}{\partial y'}\delta y\right]\Bigg|_a^b+\int_a^b\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y\mathrm{d}x$$

独立变分：$\delta y(a)$、 $\delta y(b)$和$\delta y(x)$, 从$\delta J=0$得到

Euler方程和自然边界条件

$$\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}=0;\qquad \left.\frac{\partial f}{\partial y'}\right|_a=0,\ \left.\frac{\partial f}{\partial y'}\right|_b=0$$



■ 二个变数的泛函

$$J(u)=\iint_G F(x,y,u,u_x,u_y)\mathrm{d}x\mathrm{d}y$$

一阶变分

$$\delta J(u)=\iint_G\left(\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}\right)\delta u\mathrm{d}x\mathrm{d}y$$

$$+\int_{\partial G}\left[\frac{\partial F}{\partial u_x}\cos(n,x)+\frac{\partial F}{\partial u_y}\cos(n,y)\right]\delta u\mathrm{d}S$$

Euler方程

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}=0$$

自然边界条件

$$\frac{\partial F}{\partial u_x}\cos(n,x)+\frac{\partial F}{\partial u_y}\cos(n,y)=0$$

例3 求泛函的Euler方程

$$J(u)=\int_G\left[(\nabla u)^2+2fu\right]\mathrm{d}\tau$$

Euler方程为Possion方程;自然边界条件为第二类边界条件

$$\nabla^2 u=f;\quad \left.\frac{\partial u}{\partial n}\right|_{\partial G}=0$$

例4 求下面泛函的Euler方程及自然边界条件

$$J(u)=\int_G\left[p(\nabla u)^2+qu^2-2fu\right]\mathrm{d}\tau$$
$$-\iint_{\partial G}p(2gu-hu^2)\mathrm{d}S$$

一阶变分

$$\delta J(u)=2\int_G\left[-\nabla\cdot(p\nabla u)+qu-f\right]\delta u\mathrm{d}\tau$$
$$+2\iint_{\partial G}\left[p\left(\frac{\partial u}{\partial n}+hu-g\right)\right]\delta u\mathrm{d}S$$

Euler方程和自然边界条件

$$-\nabla\cdot(p\nabla u)+qu=f;\left.\left(hu+\frac{\partial u}{\partial n}\right)\right|_{\partial G}=g$$

14.2 泛函的条件极值问题

■约束条件为函数方程：测地线问题

$$J(y,z)=\int_a^b F(x,y,y',z,z')\mathrm{d}x$$

$$G(x,y,z)=0$$

在端点固定条件下, J的一阶变分为

$$\delta J(y,z)=\int_a^b \left\{\left[\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y+\left[\frac{\partial F}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial z'}\right)\right]\delta z\right\}\mathrm{d}x$$

——δy和δz不是独立变分, 不能直接推出Euler方程

$$G(x,y,z)=0 \Rightarrow G_y\delta y+G_z\delta z=0$$

$$\delta J(y,z)=\int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}+\left(\frac{\partial F}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial z'}\right)\left(-\frac{G_y}{G_z}\right)\right]\delta y\,\mathrm{d}x$$

$$\Downarrow$$

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}+\left(\frac{\partial F}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial z'}\right)\left(-\frac{G_y}{G_z}\right)=0$$

- **Lagrange乘子法**

$$\frac{\dfrac{\partial F}{\partial y}-\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\partial F}{\partial y'}}{G_y}=\frac{\dfrac{\partial F}{\partial z}-\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\partial F}{\partial z'}}{G_z}\equiv\lambda(x)$$

□ Euler方程

$$\frac{\partial F}{\partial y}+\lambda(x)\frac{\partial G}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}=0$$

$$\frac{\partial F}{\partial z}+\lambda(x)\frac{\partial G}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial z'}=0$$

等价泛函

$$\tilde{J}(y,z)=J(y,z)+\int_a^b \lambda(x)G(x,y,z)\mathrm{d}x$$

——新泛函, δy和δz都是独立变分

例 1 求圆柱面上两点A及B之间长度最短的曲线.

约束条件 $G(x,y,z)=x^2+y^2-R^2=0$

参数方程 $x = x(t),\ \ y = y(t),\ z = z(t)$

泛函关系

$$J(x,y,z) = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t$$

构成新泛函

$$\tilde{J}(x,y,z) = J(x,y,z) + \int_{t_1}^{t_2} \lambda(x)(x^2 + y^2 - R^2)\,\mathrm{d}t$$

柱面上

$$x = R\cos t;\ \ y = R\sin t,\ \ 0 \le t \le 2\pi$$

$$\Downarrow$$

$$\tilde{J}(x,y,z) = \int_{t_1}^{t_2} \sqrt{R^2 + \dot{z}^2}\,\mathrm{d}t$$

$$\delta\tilde{J}(x,y,z)=\delta\int_{t_1}^{t_2}\sqrt{R^2+\dot{z}^2}\,\mathrm{d}t$$

$$=\int_{t_1}^{t_2}\frac{\dot{z}\delta\dot{z}}{\sqrt{R^2+\dot{z}^2}}\mathrm{d}t=\int_{t_1}^{t_2}\frac{\dot{z}}{\sqrt{R^2+\dot{z}^2}}\mathrm{d}\delta z$$

$$=\frac{\dot{z}}{\sqrt{R^2+\dot{z}^2}}\delta z\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\delta z\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{z}}{\sqrt{R^2+\dot{z}^2}}\mathrm{d}t=0$$

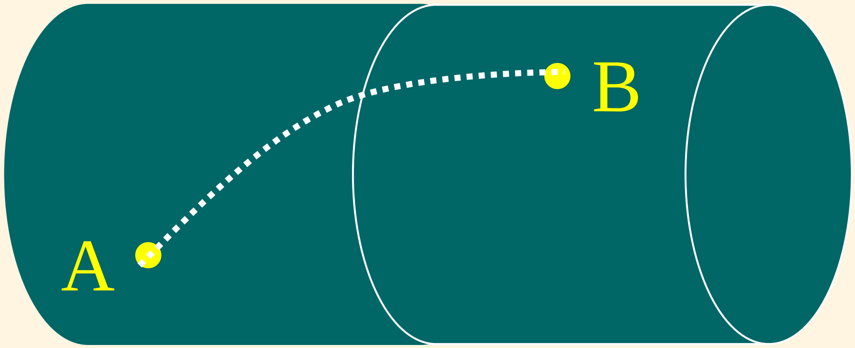
Euler 方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{z}}{\sqrt{R^2+\dot{z}^2}}\right)=0\Rightarrow\dot{z}=C_1\Rightarrow z=C_1t+C_2$$

$$\Downarrow$$

$$x=R\cos t;\quad y=R\sin t;\quad z=C_1t+C_2\ \ (0\le t\le 2\pi)$$

圆柱面上的螺旋线



- **约束条件是积分形式: 等周问题**

$$J(y)=\int_{x_0}^{x_1}F(x,y,y')\mathrm{d}x$$

约束条件

$$J_1(y)=\int_{x_0}^{x_1}G(x,y,y')\mathrm{d}x=l$$

在$y(x)$的邻域内, 可取比较函数

$$y^*(x) = y(x) + \alpha\eta_1(x) + \beta\eta_2(x)$$

$$J(\alpha,\beta) \equiv J(y^*) = \int_{x_0}^{x_1} F(x, y+\alpha\eta_1+\beta\eta_2, y'+\alpha\eta_1'+\beta\eta_2')\mathrm{d}x$$

$$J_1(\alpha,\beta) \equiv J_1(y^*) = \int_{x_0}^{x_1} G(x, y+\alpha\eta_1+\beta\eta_2, y'+\alpha\eta_1'+\beta\eta_2')\mathrm{d}x = l$$

约束条件泛函极值 → 约束条件函数极值

Lagrange乘子法

$$\left[\frac{\partial J(\alpha,\beta)}{\partial \alpha}+\lambda\frac{\partial J_1(\alpha,\beta)}{\partial \alpha}\right]\Bigg|_{\substack{\alpha=0\\ \beta=0}}=0$$

$$\left[\frac{\partial J(\alpha,\beta)}{\partial \beta}+\lambda\frac{\partial J_1(\alpha,\beta)}{\partial \beta}\right]\Bigg|_{\substack{\alpha=0\\ \beta=0}}=0$$

注意: 这里 λ 是常数

$$\int_{x_0}^{x_1}\left[(F_y+\lambda G_y)-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}+\lambda\frac{\partial G}{\partial y'}\right)\right]\eta_1\mathrm{d}x=0$$

$$\int_{x_0}^{x_1}\left[(F_y+\lambda G_y)-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}+\lambda\frac{\partial G}{\partial y'}\right)\right]\eta_2\mathrm{d}x=0$$

Euler方程

$$F_y + \lambda G_y - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'} + \lambda \frac{\partial G}{\partial y'}\right) = 0$$

等价泛函

$$\tilde{J}(y) = J(y) + \lambda \int_{x_0}^{x_1} G\mathrm{d}x = \int_{x_0}^{x_1} (F + \lambda G)\mathrm{d}x$$

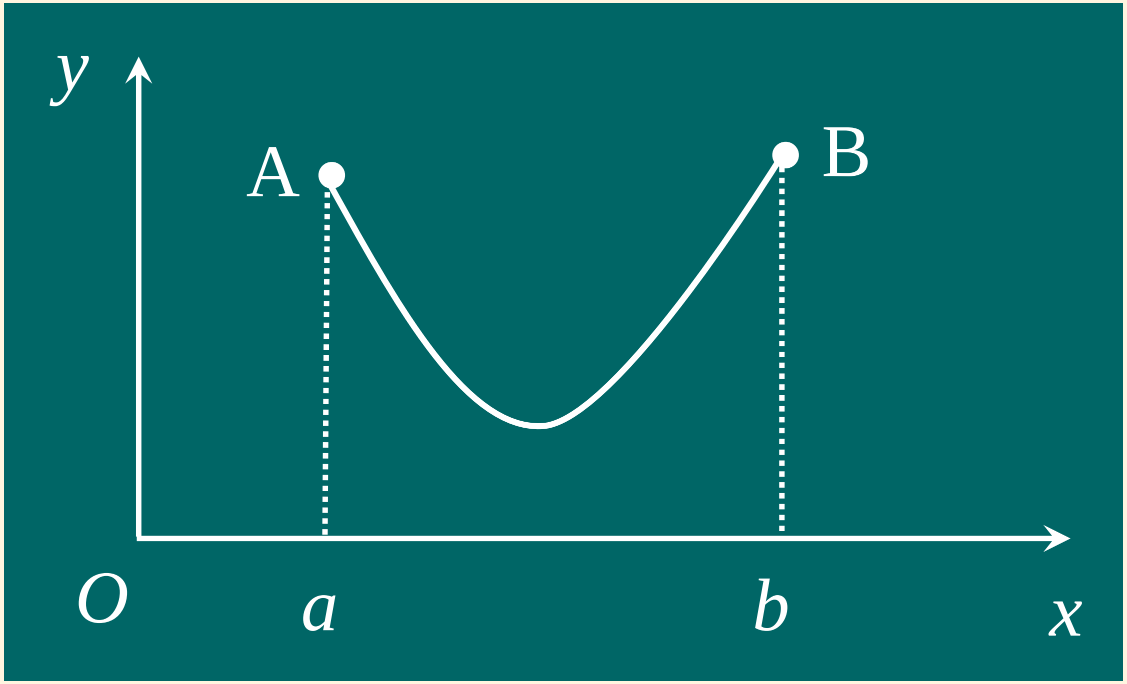
例2 固定两点悬挂长度为l的铁链, 求在重力作用下铁链的形状.

势能 $$J(y) = \int_a^b y\sqrt{1 + y'^2}\mathrm{d}x$$

约束条件 $$\int_a^b \sqrt{1 + y'^2}\mathrm{d}x = l$$

在重力作用下势能极小，故问题变成在约束条件下，求$J(y)$的极小

$$\tilde{J}(y)=\int_a^b (y+\lambda)\sqrt{1+y'^2}\,\mathrm{d}x$$



悬链线问题

一阶变分

$$\delta \tilde{J}(y) = \int_a^b \left[\delta y \sqrt{1+y'^2} + (y+\lambda) \frac{y' \delta y'}{\sqrt{1+y'^2}} \right] \mathrm{d}x$$

$$= (y+\lambda) \frac{y' \delta y}{\sqrt{1+y'^2}} \Bigg|_a^b + \int_a^b \left\{ \sqrt{1+y'^2} - \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{(y+\lambda) y'}{\sqrt{1+y'^2}} \right] \right\} \delta y \mathrm{d}x$$

Euler方程

$$\sqrt{1+y'^2} - \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{(y+\lambda) y'}{\sqrt{1+y'^2}} \right] = 0$$

- ***f*(*y*, *y*′)不显含*x***

$$f - y' \frac{\partial f}{\partial y'} = C_1; \; f \equiv (y+\lambda)\sqrt{1+y'^2}$$

$$(y+\lambda)\sqrt{1+y'^2}-(y+\lambda)\frac{(y')^2}{\sqrt{1+y'^2}}=C_1$$

$$\Downarrow$$

$$y+\lambda=C_1\cosh\left(\frac{x-C_2}{C_1}\right)$$

例3 在约束条件下, 使泛函取极值

$$J(u)=\int_G F(x,y,z,u,u_x,u_y,u_z)\mathrm{d}\tau$$

$$\int_G \rho u^2\mathrm{d}\tau=1;\quad u|_{\partial G}=0$$

一阶变分

$$\delta J(u)=\int_G\left(\frac{\partial F}{\partial u}\delta u+\sum_{j=x,y,z}\frac{\partial F}{\partial u_j}\frac{\partial \delta u}{\partial x_j}\right)\tau$$

$$=\int_G\left(\frac{\partial F}{\partial u}-\sum_{j=x,y,z}\frac{\partial}{\partial x_j}\frac{\partial F}{\partial u_j}\right)\delta u\mathrm{d}\tau+\iint_{\partial G}\frac{\partial F}{\partial u_j}\delta u\mathrm{d}S$$

$$=\int_G\left(\frac{\partial F}{\partial u}-\sum_{j=x,y,z}\frac{\partial}{\partial x_j}\frac{\partial F}{\partial u_j}\right)\delta u\mathrm{d}\tau$$

$$\Downarrow$$

$$\delta J(u)=\int_G\left(\frac{\partial F}{\partial u}-\sum_{j=x,y,z}\frac{\partial}{\partial x_j}\frac{\partial F}{\partial u_j}\right)\delta u\mathrm{d}\tau$$

$$\delta\int_G \rho u^2\mathrm{d}\tau=0\Rightarrow\int_G\rho u\delta u\mathrm{d}\tau=0$$

Lagrange乘子法

$$\delta\tilde{J}(u)=\int_G\left(\frac{\partial F}{\partial u}-\sum_{j=x,y,z}\frac{\partial}{\partial x_j}\frac{\partial F}{\partial u_j}-\lambda\rho u\right)\delta u\,\mathrm{d}\tau$$

$$\frac{\partial F}{\partial u}-\frac{\partial F_{u_x}}{\partial x}-\frac{\partial F_{u_y}}{\partial y}-\frac{\partial F_{u_z}}{\partial z}=\lambda\rho u$$

当取

$$F=\frac{1}{2}[p(\nabla u)^2+qu^2]$$

$$-\nabla\cdot(p\nabla u)+qu=\lambda\rho u$$

——本征值问题与泛函的条件极值问题等价。

Lagrange乘子：**本征值**；约束条件：**归一化**

14.3Hamilton原理与最小位能原理

Hamilton 原理：任何力学系统，若给定初始状态和终结状态，则从一切可能的运动状态中，真实运动使作用量泛函的变分δJ=0

$$J=\int_{t_1}^{t_2} L\mathrm{d}t$$

Lagrange 函数：$L=T-U$ **（动能：*T*；位能：*U*）**

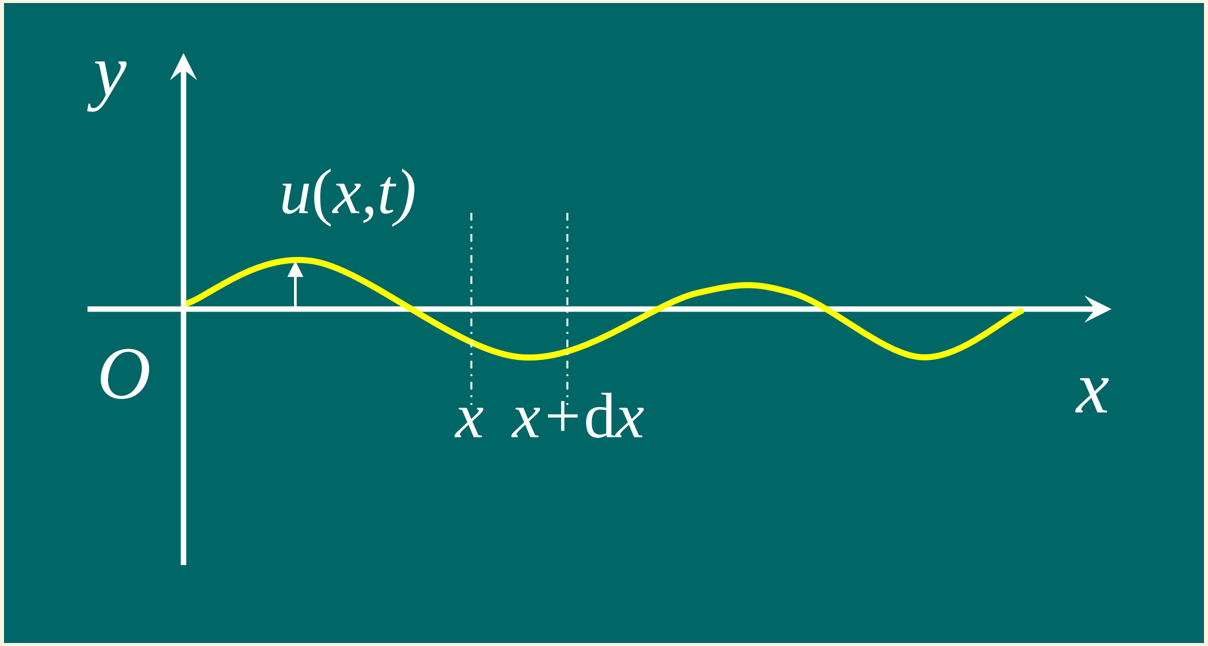
- **连续分布的力学系统**

$$J=\int_{t_1}^{t_2}\int_G \ell \mathrm{d}\tau \mathrm{d}t$$

例1 两端固定的弦, 设弦长为l, 密度分布为$\rho(x)$, 内部张力分布为$\tau(x)$, 在外力作用下作横向振动.

- **动能 dT**

$$\mathrm{d}T = \frac{1}{2}\rho(x)u_t^2\mathrm{d}x$$



弦的横向振动

- **位能 dU**

$$\mathrm{d}U=\tau(x)\left(\sqrt{1+u_x^2}\,\mathrm{d}x-\mathrm{d}x\right)\approx\frac{1}{2}\tau(x)u_x^2\mathrm{d}x$$

- **系统的Lagrange函数**

$$L(t)=\frac{1}{2}\int_0^l\left[\rho(x)u_t^2-\tau(x)u_x^2\right]\mathrm{d}x$$

- **作用量泛函**

$$J(u)=\int_{t_0}^{t_1}L(t)\mathrm{d}t=\frac{1}{2}\int_{t_0}^{t_1}\int_0^l\left[\rho(x)u_t^2-\tau(x)u_x^2\right]\mathrm{d}x\mathrm{d}t$$

- **Hamilton 原理**

$$\delta J(u)=0$$

一阶变分

$$\delta J(u)=\int_{t_0}^{t_1}\int_0^l\left[\rho(x)u_t\delta u_t-\tau(x)u_x\delta u_x\right]\mathrm{d}x$$

$$=\int_{t_0}^{t_1}\int_0^l\rho(x)u_t\frac{\partial\delta u}{\partial t}\mathrm{d}x\mathrm{d}t-\int_{t_0}^{t_1}\int_0^l\tau(x)u_x\frac{\partial\delta u}{\partial x}\mathrm{d}x\mathrm{d}t$$

$$=\int_0^l\rho(x)u_t\,\delta u\Big|_{t_0}^{t_1}\mathrm{d}x-\int_{t_0}^{t_1}\tau(x)u_x\,\delta u\Big|_0^l\mathrm{d}t$$

$$-\int_{t_0}^{t_1}\int_0^l\left\{\rho(x)\frac{\partial^2u}{\partial t^2}-\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right]\right\}\delta u\mathrm{d}x\mathrm{d}t$$

- **二端固定** $\delta u\big|_0^l=0$
- **二端自由** $u_x\big|_0^l=0$
- **等时变分** $\delta u\big|_{t_0}^{t_1}=0$

$$\delta J(u)=\int_{t_0}^{t_1}\int_0^l\left\{\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right]-\rho(x)\frac{\partial^2u}{\partial t^2}\right\}\delta u\mathrm{d}x\mathrm{d}t$$

■ 横向振动方程

$$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right]-\rho(x)\frac{\partial^2 u}{\partial t^2}=0$$

例2 膜的横振动

$$L_1=\int_{t_1}^{t_2}\int_G\left[\frac{\rho}{2}u_t^2-\frac{\tau}{2}(u_x^2+u_y^2)+fu\right]\mathrm{d}x\mathrm{d}y\mathrm{d}t$$

$$L_2=\int_{t_1}^{t_2}\int_{\partial G}\left[p(S)u-\frac{1}{2}\sigma(S)u^2\right]\mathrm{d}S\mathrm{d}t$$

作用量泛函

$$J(u)\equiv L_1+L_2$$

一阶变分

$$\delta J(u)\equiv \delta L_1+\delta L_2$$

$$\delta J=\int_{t_1}^{t_2}\int_G\left[\rho u_t\frac{\partial\delta u}{\partial t}-\tau\left(u_x\frac{\partial\delta u}{\partial x}+u_y\frac{\partial\delta u}{\partial y}\right)+f\delta u\right]\mathrm{d}x\mathrm{d}y\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_{\partial G}\left[p(S)\delta u-\sigma(S)u\delta u\right]\mathrm{d}S\mathrm{d}t$$

$\Downarrow$

对等时变分 $\delta u\Big|_{t_1}^{t_2}=0$

$$\delta J=-\int_{t_1}^{t_2}\int_G\left[\rho\frac{\partial^2 u}{\partial t^2}-\left[\frac{\partial}{\partial x}(\tau u_x)+\frac{\partial}{\partial y}(\tau u_y)\right]-f\right]\delta u\mathrm{d}x\mathrm{d}y\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_{\partial G}\left[p(S)-\sigma(S)u-\tau\frac{\partial u}{\partial n}\right]\delta u\mathrm{d}S\mathrm{d}t$$

Euler方程

$$\rho\frac{\partial^2 u}{\partial t^2}-\left[\frac{\partial}{\partial x}(\tau u_x)+\frac{\partial}{\partial y}(\tau u_y)\right]=f$$

自然边界条件

$$\sigma(S)u+\tau\frac{\partial u}{\partial n}=p(S)$$

例2 电势方程 空间电场分布$E=-\nabla U$的静电场能量为

$$J(U)=\frac{1}{2}\int_G\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2\right]\mathrm{d}\tau$$

■ 能量变分

$$\Delta J(U)=J(U+\delta U)-J(U)=\delta J(U)+\delta^2 J(U)$$

■ 一阶变分

$$\delta J(U)=\int_G\left(\frac{\partial U}{\partial x}\frac{\partial \delta U}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial \delta U}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial \delta U}{\partial z}\right)\mathrm{d}\tau$$

$$=0$$

■ 二阶变分

$$\delta^2 J(U)=\int_G\left[\left(\frac{\partial \delta U}{\partial x}\right)^2+\left(\frac{\partial \delta U}{\partial y}\right)^2+\left(\frac{\partial \delta U}{\partial z}\right)^2\right]\mathrm{d}\tau$$

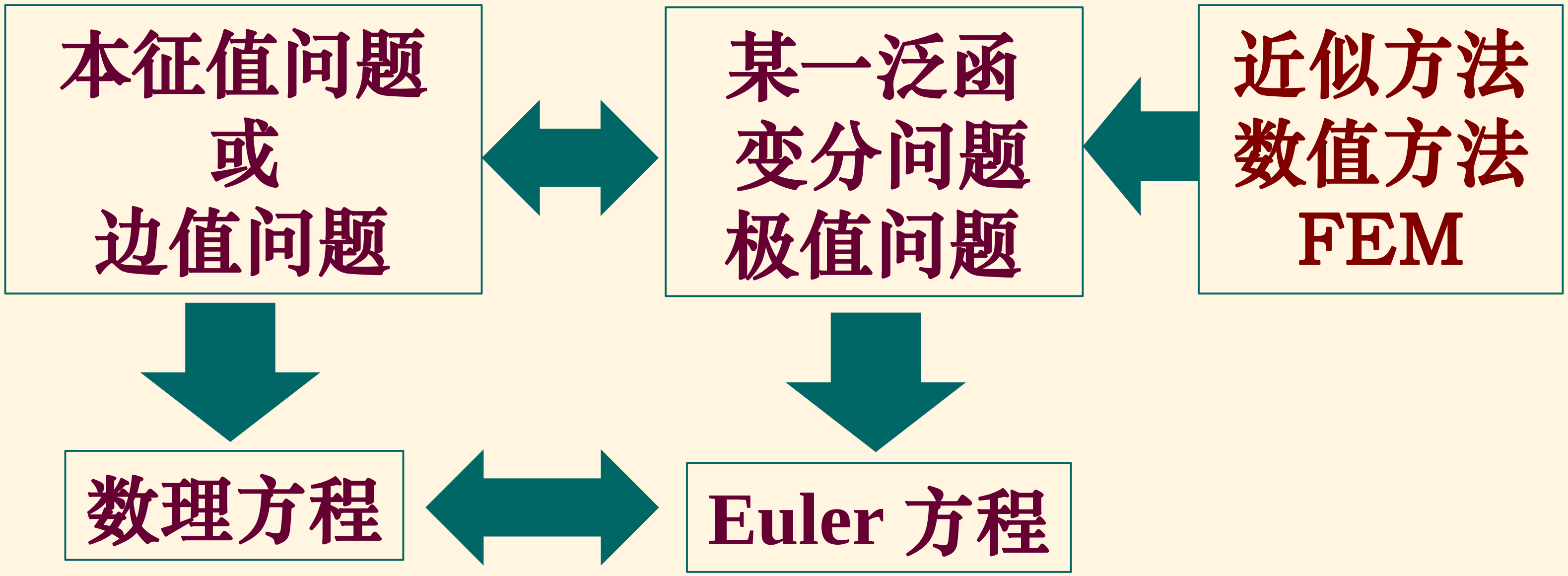
$$\delta J(U) = -\int_G \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial z^2} \right) \delta U \mathrm{d}\tau + \int_{\partial G} \frac{\partial U}{\partial n} \delta U \mathrm{d}S$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

$$\delta^2 J(U) = \int_G \left[\left(\frac{\partial \delta U}{\partial x} \right)^2 + \left(\frac{\partial \delta U}{\partial y} \right)^2 + \left(\frac{\partial \delta U}{\partial z} \right)^2 \right] \mathrm{d}\tau > 0$$

故满足Laplace方程的电势使静电场能量分布达到极小

14.4 本征值问题中的应用



如果用其他方法解决了泛函的变分问题，求出了极值函数——就是原本征值问题或边值问题的解

❑本征值问题与泛函极值问题的等价

- **Hermite算子本征值问题**

$$\boldsymbol{L}\varphi = \lambda\rho\varphi$$

- **定义泛函**

$$\lambda(\varphi) = \frac{\int_G \varphi^* \boldsymbol{L}\varphi \,\mathrm{d}\tau}{\int_G \rho\varphi^* \varphi \,\mathrm{d}\tau} \equiv \frac{A}{B}$$

一阶变分

$$\delta\lambda(\varphi) = \frac{B\delta A - A\delta B}{B^2} = \frac{1}{B}(\delta A - \lambda\delta B)$$

$$\delta A = \delta\int_G \varphi^* \boldsymbol{L}\varphi\,\mathrm{d}\tau = \int_G \delta\varphi^* \boldsymbol{L}\varphi\,\mathrm{d}\tau + \int_G \varphi^* \boldsymbol{L}\delta\varphi\,\mathrm{d}\tau$$

$$\delta B = \delta\int_G \rho\varphi^*\varphi\,\mathrm{d}\tau = \int_G \rho\varphi^*\delta\varphi\,\mathrm{d}\tau \int_G \rho\delta\varphi^*\varphi\,\mathrm{d}\tau$$

$$\int_G (\boldsymbol{L}\varphi_1)^* \varphi_2\,\mathrm{d}\tau = \int_G \varphi_1^* (\boldsymbol{L}\varphi_2)\,\mathrm{d}\tau$$

Hermite
对称算子

$$\delta A = 2\,\mathrm{Re}\int_G \delta\varphi^* \boldsymbol{L}\varphi\,\mathrm{d}\tau;\ \ \delta B = 2\,\mathrm{Re}\int_G \rho\varphi\delta\varphi^*\,\mathrm{d}\tau$$

$$\delta\lambda(\varphi) = \frac{2}{B}\mathrm{Re}\left[\int_G (\boldsymbol{L} - \lambda\rho)\varphi\delta\varphi^*\,\mathrm{d}\tau\right]$$

Euler方程

$$\boldsymbol{L}\varphi = \lambda\rho\varphi \Rightarrow \{0 < \lambda_1 < \lambda_2, \ldots, \lambda_j, \ldots, \infty; \varphi_1, \varphi_2, \ldots, \varphi_j, \ldots.\}$$

二阶变分

$$\lambda(\varphi+\delta\varphi) = \frac{\int_G (\varphi+\delta\varphi)^* \boldsymbol{L}(\varphi+\delta\varphi)\mathrm{d}\tau}{\int_G \rho(\varphi+\delta\varphi)^*(\varphi+\delta\varphi)\mathrm{d}\tau} = \frac{A+\delta A+\delta^2 A}{B+\delta B+\delta^2 B}$$

$$\delta^2 A \equiv \int_G \delta\varphi^* \boldsymbol{L}\delta\varphi\mathrm{d}\tau; \delta^2 B = \int_G \rho\delta\varphi^*\delta\varphi\mathrm{d}\tau$$

$$\Downarrow$$

$$\Delta\lambda = \lambda(\varphi+\delta\varphi) - \lambda(\varphi) = \delta\lambda + \delta^2\lambda$$

$$\delta\lambda = \frac{1}{B}(\delta A - \lambda\delta B) = \frac{2\,\mathrm{Re}}{B}\left[\int_G \delta\varphi^*(\boldsymbol{L}-\lambda\rho)\varphi\mathrm{d}\tau\right]$$

$$\delta^2\lambda = \frac{1}{B}(\delta^2 A - \lambda\delta^2 B) - (\delta A - \lambda\delta B)\frac{\delta B}{B^2}$$

$$= \frac{1}{B}\left[\int_G \delta\varphi^*(\boldsymbol{L} - \lambda\rho)\delta\varphi\,\mathrm{d}\tau\right] - \frac{2\,\mathrm{Re}}{B}\left[\int_G \delta\varphi^*(\boldsymbol{L} - \lambda\rho)\varphi\,\mathrm{d}\tau\right]\frac{\delta B}{B}$$

$$(\boldsymbol{L} - \lambda_j\rho)\varphi_j = 0$$

$$\delta^2\lambda = \frac{1}{B}\left[\int_G \delta\varphi^*(\boldsymbol{L} - \lambda_j\rho)\delta\varphi\,\mathrm{d}\tau\right]$$

设

$$\delta\varphi = \sum_{k=1}^{\infty} a_k\varphi_k; a_k = \int_G \delta\varphi\varphi_k^*\mathrm{d}\tau$$

$$\delta^2\lambda = \frac{1}{B}\left[\sum_{k=1}^{\infty}|a_k|^2(\lambda_k - \lambda_j)\right]$$

(1) $\lambda_j = \lambda_1, (j=1)$

$$\delta^2\lambda = \frac{1}{B}\left[\sum_{k=2}^{\infty}|a_k|^2(\lambda_k - \lambda_1)\right] > 0$$

故泛函$\lambda(\varphi)$在φ_1达到极小值，对任意的函数φ

$$\lambda_1 \le \lambda(\varphi)$$ ——**估计基态能级**

(2) $\lambda_j = \lambda_p, (j=p)$

$$\delta^2\lambda = \frac{1}{B}\left[\sum_{k=1}^{p-1}|a_k|^2(\lambda_k - \lambda_p)\right] + \frac{1}{B}\left[\sum_{k=p+1}^{\infty}|a_k|^2(\lambda_k - \lambda_p)\right]$$

如果 if $a_k = \int_G \delta\varphi\varphi_k^* \mathrm{d}\tau = 0, (k=1,\ldots,p-1)$

$$\delta^2\lambda = \frac{1}{B}\left[\sum_{k=p+1}^{\infty}|a_k|^2(\lambda_k - \lambda_p)\right] > 0$$

故泛函$\lambda(\varphi)$在φ_p达到极小值，对任意的函数φ

$$\lambda_p \le \lambda(\varphi)$$

其中， 函数φ与$\{\varphi_1, \varphi_2, \dots, \varphi_{p-1}\}$**，正交**

$$\int_G \varphi \varphi_k^* \mathrm{d}\tau = 0, \ (k = 1, \dots, p-1)$$

例1 一般形式的Sturm−Liouville本征值问题

$$\boldsymbol{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x) \frac{\mathrm{d}}{\mathrm{d}x} \right] + q(x); \ p(x) > 0, \ q(x) \ge 0$$

任意函数作广义Fourier展开 $\varphi(x) \cong \sum_{i=1}^{\infty} a_i \varphi_i(x)$

$$\lambda(\varphi)=\frac{A}{B}=\frac{\sum_{i=1}^{\infty}|a_i|^2\lambda_i}{\sum_{i=1}^{\infty}|a_i|^2}\geq\frac{\lambda_1\sum_{i=1}^{\infty}|a_i|^2}{\sum_{i=1}^{\infty}|a_i|^2}=\lambda_1$$

(1)如果φ正交于φ_1, 则$a_1=0$, 于是

$$\lambda(\varphi)=\frac{A}{B}=\frac{\sum_{i=2}^{\infty}|a_i|^2\lambda_i}{\sum_{i=2}^{\infty}|a_i|^2}\geq\frac{\lambda_2\sum_{i=2}^{\infty}|a_i|^2}{\sum_{i=2}^{\infty}|a_i|^2}=\lambda_2$$

(2)如果φ正交于$\varphi_1, \varphi_2,\dots,\varphi_{n-1}$ 则

$$a_1=a_2=\cdots=a_{n-1}=0$$

$$\lambda(\varphi)=\frac{A}{B}=\frac{\sum_{i=n}^{\infty}|a_i|^2\lambda_i}{\sum_{i=n}^{\infty}|a_i|^2}\le\frac{\lambda_n\sum_{i=n}^{\infty}|a_i|^2}{\sum_{i=n}^{\infty}|a_i|^2}=\lambda_n$$

注意：如果零是本征值 $L\varphi_0(x)=0$

$$\varphi(x)\cong\sum_{i=0}^{\infty}a_i\varphi_i(x)=a_0\varphi_0(x)+\sum_{i=1}^{\infty}a_i\varphi_i(x)$$

$$\Downarrow$$

$$\lambda(\varphi)=\frac{A}{B}=\frac{\sum_{i=0}^{\infty}|a_i|^2\lambda_i}{\sum_{i=0}^{\infty}|a_i|^2}=\frac{\sum_{i=1}^{\infty}|a_i|^2\lambda_i}{\sum_{i=0}^{\infty}|a_i|^2}\ge\lambda_1\frac{\sum_{i=1}^{\infty}|a_i|^2}{\sum_{i=0}^{\infty}|a_i|^2}$$

如果$\varphi(x)$与零本征值相应的本征函数$\varphi_0(x)$正交

$$a_0 = \int_0^l \varphi(x)\varphi_0^*(x)\mathrm{d}x = 0$$

$$\lambda(\varphi) = \frac{A}{B} = \frac{\sum_{i=0}^{\infty} |a_i|^2 \lambda_i}{\sum_{i=0}^{\infty} |a_i|^2} = \frac{\sum_{i=1}^{\infty} |a_i|^2 \lambda_i}{\sum_{i=1}^{\infty} |a_i|^2} \geq \lambda_1$$

例2　一维情形, 第一类齐次边界条件

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \lambda\varphi, \quad x \in (0, l); \quad \varphi|_{x=0} = \varphi|_{x=l} = 0$$

严格解

$$\varphi_n(x)=\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right);\quad \lambda_n=\frac{n^2\pi^2}{l^2}$$

最小本征值

$$\lambda_1=\frac{\pi^2}{l^2}=\frac{9.8596}{l^2}$$

泛函

$$\lambda(\varphi)=\frac{1}{\int_0^l\varphi^*\varphi\mathrm{d}x}\left[-\int_0^l\varphi^*\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\mathrm{d}x\right]$$

取φ如下 $\varphi(x)=x(l-x)$ ⬅ **二阶导数，边界条件**

$$\lambda(\varphi)=\frac{2\int_0^l x(l-x)\mathrm{d}x}{\int_0^l x^2(l-x)^2\mathrm{d}x}=\frac{10}{l^2}>\lambda_1 \Rightarrow \frac{\lambda-\lambda_1}{\lambda_1}\approx 1.4\%$$

❑ 完备性定理的证明

完备性定理 设Hermite算子L的本征值可数

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \cdots$$

相应的本征函数系正交归一. 如果

(1)最小本征值λ_1有限

$$\lim_{n\to\infty} \lambda_n = +\infty$$

(2)本征函数系$\{\varphi_i\}$在均方收敛的意义下构成$L^2(G)$中的完备系

$$\lim_{n\to\infty} \int_G |\varphi(\boldsymbol{r}) - \sum_{i=1}^{n} a_i \varphi_i(\boldsymbol{r})|^2 \rho(\boldsymbol{r}) \mathrm{d}\tau = 0$$

$$\varphi(\mathbf{r}) \approx \sum_{i=1}^{\infty} a_i \varphi_i(\mathbf{r}); \quad a_i = \int_G \rho \varphi_i^*(\mathbf{r}) \varphi(\mathbf{r}) \mathrm{d}\tau = (\varphi_i, \varphi)$$

证明：令

$$R_n = \varphi(\mathbf{r}) - \sum_{i=1}^{n} a_i \varphi_i(\mathbf{r})$$

只要证明

$$\lim_{n\to\infty} \int_G \rho \,|\, R_n \,|^2 \, \mathrm{d}\tau = 0$$

从积分

$$\int_G \varphi_i^*(\mathbf{r}) R_n \rho(\mathbf{r}) \mathrm{d}\tau = \int_G \varphi_i^*(\mathbf{r}) \varphi(\mathbf{r}) \rho \mathrm{d}\tau - a_i = 0$$

$$\Downarrow$$

$$R_n \perp (\varphi_1, \varphi_2, \cdots, \varphi_n)$$

$$\lambda(R_n)=\frac{\int_G R_n^* \boldsymbol{L} R_n \mathrm{d}\tau}{\int_G \rho R_n^* R_n \mathrm{d}\tau} \geq \lambda_{n+1}$$

$$\int_G \rho\,|R_n|^2\,\mathrm{d}\tau \leq \frac{1}{\lambda_{n+1}}\left[\int_G \varphi^* \boldsymbol{L}\varphi \mathrm{d}\tau - \sum_{i=1}^{n}|a_i|^2\,\lambda_i\right]$$

最小本征值有限，对不等式右边放大，去掉所有正本征值的项

$$\int_G \rho\,|R_n|^2\,\mathrm{d}\tau \leq \frac{1}{\lambda_{n+1}}\left[\int_G \varphi^* \boldsymbol{L}\varphi \mathrm{d}\tau - \sum_{i=1}^{m}|a_i|^2\,\lambda_i\right]$$

当$n\to\infty$时，由条件$\lambda_{n+1}\to\infty$得

$$\lim_{n\to\infty}\int_G \rho\,|R_n|^2\,\mathrm{d}\tau = 0$$

❑ Ritz和Galerkin法解本征值问题

Ritz法求泛函问题的基本原理

$$\min[J(\varphi)] = m$$

寻找函数序列

$$\{\overline{\varphi}_n\} \Rightarrow \min[J(\overline{\varphi}_n)] \approx m$$

$$\lim_{n\to\infty} \overline{\varphi}_n = \varphi \Rightarrow$$ 关键：寻找函数序列

- 函数族

$$\varphi_n = \Phi_n(\boldsymbol{r}, a_1, a_2, \dots, a_n)$$

1. 代入泛函 $\Rightarrow J(\varphi_n) \equiv J(a_1, a_2, \dots, a_n)$

2.求极小

$$\frac{\partial J}{\partial a_k}=0,\ (k=1,2,....,n)$$

3.求代数方程

$$a=(\bar{a}_1,\bar{a}_2,...,\bar{a}_n)$$

4.近似解

$$\bar{\varphi}_n=\Phi_n(\mathbf{r},\bar{a}_1,\bar{a}_2,...,\bar{a}_n)$$

问题：精确解的较好近似?

n增加 → $\Phi_{n+1}\supset\Phi_n$ → $J(\bar{\varphi}_n)\geq J(\bar{\varphi}_{n+1})$

$$\lim_{n\to\infty}J(\bar{\varphi}_n)=\min[J(\varphi)]=m$$

充分条件 l 阶ε邻域

$$|\varphi_n-\varphi|<\varepsilon;\quad \left|\frac{\partial^k\varphi_n}{\partial x_i^k}-\frac{\partial^k\varphi}{\partial x_i^k}\right|<\varepsilon,\ (k=1,2,\dots,l)$$

- **Ritz法求泛函问题的具体方法**

取已知完备系：$\left\{\phi_i^R,(i=1,2,\dots\right\}$

取逼近函数族

$$\varphi_n^R=\sum_{i=1}^{n}a_i\phi_i^R$$

典型可取
① 多项式系
② 三角函数系;
③ 特殊函数系;
④ ……

- **Hermite算子L的本征值问题**

$$\mathbf{L}\varphi=\lambda\rho\varphi$$

■ **等价泛函** $J(\varphi)=\int_G(\varphi^* \mathbf{L}\varphi-\lambda\rho\varphi\varphi^*)\mathrm{d}\tau$

$$J(\varphi_n^R)=\sum_{ij=1}^{n}\left[\alpha_{ij}^R a_i a_j-\lambda\gamma_{ij}^R a_i a_j\right]$$

$$\alpha_{ij}^R\equiv\int_G\phi_i^{R*}\mathbf{L}\phi_j^R\mathrm{d}\tau;\gamma_{ij}^R\equiv\int_G\rho\phi_i^{R*}\phi_j^R\mathrm{d}\tau$$

注意：已知完备系不一定正交、归一

■ **极值条件**

$$\frac{1}{2}\frac{\partial J}{\partial a_j}=\sum_{i=1}\left(\alpha_{ij}^R-\lambda\gamma_{ij}^R\right)a_i=0,(j=1,2,\dots,n)$$

■ 解存在条件

$$\det|\alpha_{ij}^{R}-\lambda\gamma_{ij}^{R}|=0$$

决定前n个近似本征值的方程

■ 特别情况

$$\varphi_1^R=a_1\phi_1^R \Rightarrow \bar{\lambda}_1=\frac{\int_G \phi_1^{*R}\boldsymbol{L}\phi_1^R\mathrm{d}\tau}{\rho\int_G \phi_1^R\phi_1^{*R}\mathrm{d}\tau}>\lambda_1$$

——关键：基函数的选择！量子力学中估计基态能级

Galerkin法的基本原理

■ 取已知完备系：$\left\{\phi_i^G,(i=1,2,\ldots\right\}$

■ **近似解取成形式**

$$\varphi_n^G = \sum_{i=1}^{n} c_i \phi_i^G$$

Galerkin解是一种弱解

■ **求解方程**

$$\boldsymbol{L}(\varphi) = 0 \Longleftrightarrow \int_G \boldsymbol{L}(\varphi_n^G)\phi_j^G \mathrm{d}\tau = 0, (j = 1, 2, \ldots, n)$$

只能满足*n*个正交条件

$$\int_G \phi_j^{*G} \boldsymbol{L}(\varphi_n^G) \mathrm{d}\tau = \int_G \boldsymbol{L}\left(\sum_{i=1}^{n} c_i \phi_i^G\right) \phi_j^{*G} \mathrm{d}\tau = 0$$

$$(j = 1, 2, \ldots, n)$$

关键：没有用到方程与变分问题的等价关系！

■ 本征值问题

$$\boldsymbol{L}(\varphi)-\lambda\rho\varphi=0$$

$$\int_G\left[\boldsymbol{L}(\varphi_n^G)-\lambda\rho\varphi_n^G\right]\phi_j^{*G}\mathrm{d}\tau=0,(j=1,2,\ldots,n)$$

$$\sum_{i=1}^{\infty}\left(\alpha_{ij}^G-\lambda\gamma_{ij}^G\right)c_i=0,(j=1,2,\ldots,n)$$

$$\alpha_{ij}^G=\int_G\boldsymbol{L}(\phi_i^G)\phi_j^{*G}\mathrm{d}\tau;\quad \gamma_{ij}^G=\int_G\rho\phi_i^{*G}\phi_j^G\mathrm{d}\tau$$

$$\det\left(\alpha_{ij}^G-\lambda\gamma_{ij}^G\right)=0$$

决定前*n*个近似本征值的方程

例1 零阶Bessel方程的本征值问题

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)=\lambda r\varphi, r\in(0,a);\varphi|_{r=a}=0$$

满足边界条件的完备系

$$\phi_1=\cos\left(\frac{\pi r}{2a}\right);\ \phi_2=\cos\left(\frac{3\pi r}{2a}\right);\ \phi_3=\cos\left(\frac{5\pi r}{2a}\right);\cdots$$

- **第一次近似** $\varphi_1=a_1\phi_1$

$$\Downarrow$$

$$\int_0^a\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\varphi_1}{\mathrm{d}r}\right)+\lambda r\varphi_1\right]\cos\left(\frac{\pi}{2a}r\right)\mathrm{d}r=0$$

积分得 $$\frac{\pi^2}{4}\left(\frac{1}{2}+\frac{2}{\pi^2}\right)-\lambda a^2\left(\frac{1}{2}-\frac{2}{\pi^2}\right)=0$$

$$\lambda_1 \approx 5.83/a^2$$

相对误差 1.2%

精确解 $$\lambda_1 \approx 5.76/a^2$$

■ 第二次近似 $\varphi_2(r)=a_1\phi_1+a_2\phi_2$

$$\begin{cases}(1.7337-0.29736\lambda a^2)a_1+(0.20264\lambda a^2-1.5)a_2=0\\(0.20264\lambda a^2-1.5)a_1+(11.603-0.47748\lambda a^2)a_2=0\end{cases}$$

$$\lambda_1 = 5.792 / a^2$$

相对误差0.5%

Ritz法和Galerkin法 的比较

$$\sum_{i=1} \left(\alpha_{ij}^{R} - \lambda \gamma_{ij}^{R} \right) a_i = 0 \Longleftrightarrow \sum_{i=1} \left(\alpha_{ij}^{G} - \lambda \gamma_{ij}^{G} \right) a_i = 0$$

$$\alpha_{ij}^{R} = \int_G \phi_i^{*R} \boldsymbol{L} \phi_j^{R} \mathrm{d}\tau; \quad \gamma_{ij}^{R} = \int_G \rho \phi_i^{*R} \phi_j^{R} \mathrm{d}\tau$$

$$\Updownarrow$$

$$\alpha_{ij}^{G} = \int_G \phi_i^{*G} \boldsymbol{L} \phi_j^{G} \mathrm{d}\tau; \quad \gamma_{ij}^{G} = \int_G \rho \phi_i^{*G} \phi_j^{G} \mathrm{d}\tau$$

$$\{\phi_i^{R}\} \Leftrightarrow \{\phi_i^{G}\}$$

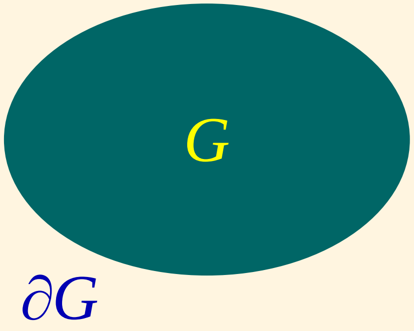
Ritz-Galerkin法

14.5 边值问题中的应用

Hilbert空间上的Hermite对称算子*L*

$$\boldsymbol{L}\varphi = f, \boldsymbol{r} \in G; \left.\left(\alpha\varphi + \beta\frac{\partial\varphi}{\partial n}\right)\right|_{\partial G} = 0 \qquad (1)$$

假定边界条件齐次，否则，进行齐次化处理。困难！



❑与泛函极值问题的等价

唯一性定理：如果*L*是正算子, 即对所有属于允许函数类的函数φ, 内积 $(\boldsymbol{L}\varphi, \varphi) > 0$

则边值问题

$$\boldsymbol{L}\varphi = f, \boldsymbol{r} \in G; \left(\alpha\varphi + \beta \frac{\partial \varphi}{\partial n} \right) \Big|_{\partial G} = 0$$

至多只有一个解.

证明 设存在二个解φ_1和φ_2，则$\varphi=\varphi_1-\varphi_2$应满足齐次方程$L\varphi=0$，即$(L\varphi,\varphi)=0$，而$L$是正算子，根据假定，如果$\varphi\neq 0$，应有$(L\varphi,\varphi)>0$，故$\varphi\equiv 0$(几乎处处)，即$\varphi_1=\varphi_2$.

例1 一维S-L算子是正算子

$$\boldsymbol{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x} \right] + q(x), x \in (0, l)$$

$$\left(\alpha_1\varphi - \beta_1 \frac{\mathrm{d}\varphi}{\mathrm{d}x} \right) \Big|_{x=0} = 0; \left(\alpha_2\varphi + \beta_2 \frac{\mathrm{d}\varphi}{\mathrm{d}x} \right) \Big|_{x=l} = 0$$

证明 $$(\boldsymbol{L}\varphi,\varphi)=\int_0^l\left[p\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2+q\varphi^2\right]\mathrm{d}x+\frac{\alpha_1}{\beta_1}p(0)\varphi^2(0)$$

$$+\frac{\alpha_2}{\beta_2}p(l)\varphi^2(l)>0$$

例2 三维S-L算子是正算子

$$\boldsymbol{L}=-\nabla\cdot[p(\boldsymbol{r})\nabla]+q(\boldsymbol{r});\quad\left.\left(\alpha\varphi+\beta\frac{\partial\varphi}{\partial n}\right)\right|_{\partial G}=0$$

$$\Downarrow$$

$$(\boldsymbol{L}\varphi,\varphi)=\int_G[-\nabla\cdot(p\nabla\varphi^*)+q\varphi^*]\varphi\mathrm{d}\tau$$

$$=\int_G(p|\nabla\varphi|^2+q|\varphi|^2)\mathrm{d}\tau-\iint_{\partial G}p\varphi\left(\frac{\partial\varphi^*}{\partial n}\right)\mathrm{d}S$$

$$=\int_G(p|\nabla\varphi|^2+q|\varphi|^2)\mathrm{d}\tau+\iint_{\partial G}\frac{\alpha}{\beta}p|\varphi|^2\mathrm{d}S$$

□等价定理

设L是正的Hermite算子, 若

$$\boldsymbol{L}\varphi \equiv -\nabla\cdot[p(\boldsymbol{r})\nabla\varphi]+q(\boldsymbol{r})\varphi = f;\quad \left.\left(\alpha\varphi+\beta\frac{\partial\varphi}{\partial n}\right)\right|_{\partial G}=0$$

有解, 则此解必使泛函

$$J(\varphi)=(\boldsymbol{L}\varphi,\varphi)-[(\varphi,f)+(f,\varphi)]$$

取极小值; 反之, 若允许函数类中函数使上式取极小, 则必定是问题的解.

证明 $\delta J(\varphi)=(\boldsymbol{L}\delta\varphi,\varphi)+(\boldsymbol{L}\varphi,\delta\varphi)-[(\delta\varphi,f)+(f,\delta\varphi)]$

$$=(\delta\varphi,\boldsymbol{L}\varphi)+(\boldsymbol{L}\varphi,\delta\varphi)-[(\delta\varphi,f)+(f,\delta\varphi)]$$

$$=(\boldsymbol{L}\varphi,\delta\varphi)^*+(\boldsymbol{L}\varphi,\delta\varphi)-[(f,\delta\varphi)^*+(f,\delta\varphi)]$$

$$=2\,\mathrm{Re}[(\boldsymbol{L}\varphi-f,\delta\varphi)]$$

(1)如果 $\boldsymbol{L}\varphi = f$ ➡ $\delta J(\varphi) = 2\,\mathrm{Re}[(\boldsymbol{L}\varphi - f, \delta\varphi)] = 0$

对正算子，二阶变分 $\delta^2 J(\varphi) = (\boldsymbol{L}\delta\varphi, \delta\varphi) > 0$

故泛函取极小值

(2)如果 $\delta J(\varphi) = 2\,\mathrm{Re}[(\boldsymbol{L}\varphi - f, \delta\varphi)] = 0$，但是

$\boldsymbol{L}\varphi - f = \phi \neq 0$，可取 $\delta\varphi = \varepsilon\phi$，于是

$$\delta J(\varphi) = 2\varepsilon\,\mathrm{Re}[(\phi, \phi)] > 0$$

除非 $\phi = 0$ (几乎处处)，因此 $\boldsymbol{L}\varphi - f = 0$

例3 一维S-L算子

$$J(\varphi) = \int_0^l \left[p\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 + q\varphi^2 \right]\mathrm{d}x + \frac{\alpha_1}{\beta_1} p(0)\varphi^2(0)$$

$$+\frac{\alpha_2}{\beta_2} p(l)\varphi^2(l) - 2\int_0^l f\varphi\mathrm{d}x$$

如果 $\beta_1 = \beta_2 = 0$

$$J(\varphi) = \int_0^l \left[p\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 + q\varphi^2 \right]\mathrm{d}x - 2\int_0^l f\varphi \mathrm{d}x$$

例4 三维S-L算子

$$J(\varphi) = \int_G (p|\nabla\varphi|^2 + q|\varphi|^2)\mathrm{d}\tau$$

$$-\int_G (\varphi^* f + f^* \varphi)\mathrm{d}\tau + \iint_{\partial G} \frac{\alpha}{\beta} p|\varphi|^2 \,\mathrm{d}S$$

如果 $\beta = 0$

$$J(\varphi) = \int_G (p|\nabla\varphi|^2 + q|\varphi|^2)\mathrm{d}\tau$$

$$-\int_G (\varphi^* f + f^* \varphi)\mathrm{d}\tau$$

注意：泛函仅要求一阶偏导存在，而原方程要求二阶偏导存在，故变分解是一类广义解

❑ Ritz法解边值问题

Hermite算子方程

$$\boldsymbol{L}\varphi = f$$

■ 等价泛函

$$J(\varphi) = (\boldsymbol{L}\varphi, \varphi) - \left[(f, \varphi) + (\varphi, f)\right]$$

■ 取完备系 $\{\phi_i^R\}$ (不一定是L的本征函数系)

$$\varphi_n = \sum_{k=1}^{n} a_k \phi_k^R$$

$$J(\varphi_n) = \sum_{j,k=1}^{n} a_j a_k (\boldsymbol{L}\phi_j^R, \phi_k^R) - \sum_{j=1}^{n} a_j \left[(f, \phi_j^R) + (\phi_j^R, f)\right]$$

■ 极小的条件

$$\frac{\partial J(\varphi_n)}{\partial a_j}=0,(j=1,2,\ldots,n)$$

$$\sum_{k=1}^{n} a_k(\boldsymbol{L}\phi_k^R,\phi_j^R)=\frac{1}{2}\left[(f,\phi_j^R)+(\phi_j^R,f)\right]$$

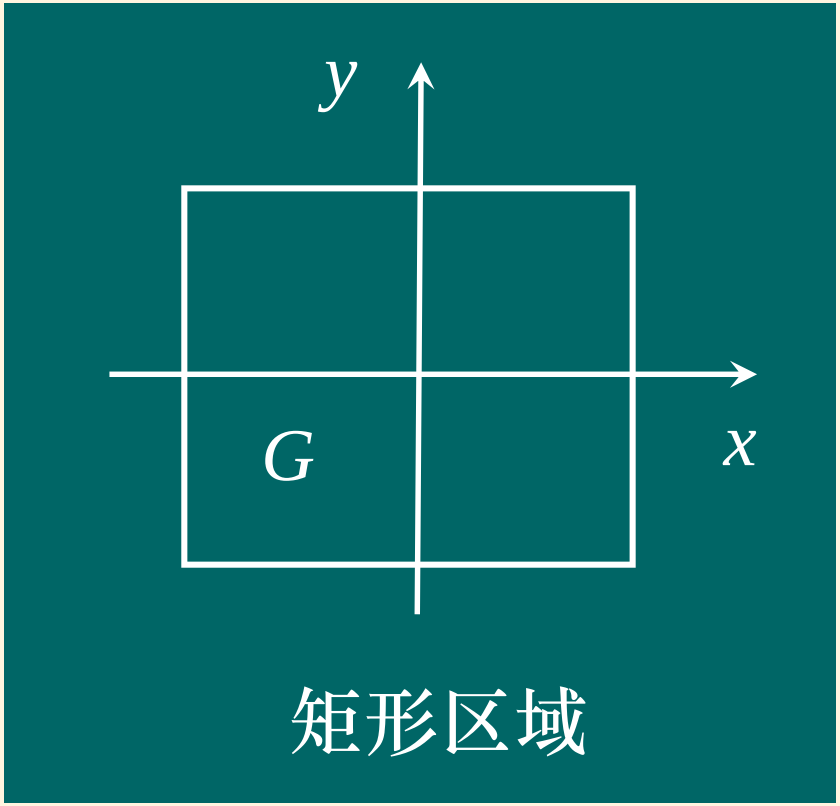
$$(j=1,2,\ldots,n)$$

有唯一解

例5 考虑边值问题

$$-\nabla^2 u(x,y)=2,\quad (x,y)\in G;\quad u|_{\partial G}=0$$

矩形区域 $[-a \le x \le a;\ -b \le y \le b]$



令

$$w(x,y)=(a^2-x^2)(b^2-y^2)$$

多项式完备系

$$\phi_1(x,y)=w(x,y);\ \ \phi_2(x,y)=w(x,y)\cdot x^2$$

$$\phi_3(x,y)=w(x,y)\cdot y^2;\ \ \phi_4(x,y)=w(x,y)\cdot x^4$$

$$\cdots\cdots\cdots\cdots$$

当取 $n=1$

$$(\boldsymbol{L}\phi_1,\phi_1)=\iint_G (w_x^2+w_y^2)\mathrm{d}x\mathrm{d}y=\frac{128}{45}a^3b^3(a^2+b^2)$$

$$(\phi_1, f)=2\iint_G w(x,y)\mathrm{d}x\mathrm{d}y=\frac{32}{9}a^3b^3$$

一级近似解

$$u(x,y)\approx\frac{5}{4}\frac{(a^2-x^2)(b^2-y^2)}{a^2+b^2}$$

❑ Galerkin法

算子方程(不一定要求Hermite对称性)

$$\boldsymbol{L}\varphi = f$$

取完备系 $\{\phi_i^G\}$

$$\varphi_n = \sum_{k=1}^{n} a_k \phi_k^G;\ \sum_{k=1}^{n} a_k \boldsymbol{L}\phi_k^G = f$$

根据Galerkin法应有

$$\sum_{k=1}^{n} a_k (\boldsymbol{L}\phi_k^G, \phi_j^G) = (f, \phi_j^G), (j = 1, 2, \ldots, n)$$

——没有用到泛函与方程等价条件：①不要求Hermite对称性; ② 不要求是正算子。

Ritz法

$$\sum_{k=1}^{n} a_k(\boldsymbol{L}\phi_k^R,\phi_j^R)=\frac{1}{2}\left[(f,\phi_j^R)+(\phi_j^R,f)\right]$$

$$(j=1,2,\ldots,n)$$

如果 $$(f,\phi_j^R)=(\phi_j^R,f)$$

Galerkin法

$$\sum_{k=1}^{n} a_k(\boldsymbol{L}\phi_k^G,\phi_j^G)=(f,\phi_j^G)$$

$$(j=1,2,\ldots,n)$$

如果不苛求函数系的完备性, Ritz法和Galerkin法能给出同样的结果——Ritz–Galerkin法

❑有限元方法

(1)当G的边界非常复杂时，寻找这样的完备系相当困难，甚至不可能;(2)非齐次问题.

■ 一维边值问题

$$\boldsymbol{L}\psi \equiv -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right]+q(x)\psi = f(x),\ \ x\in(a,b)$$

$$\psi(a)=\psi_0;\ \left(\alpha\psi+\beta\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)\Bigg|_{x=b}=g$$

$$J(\psi)=(\psi,\boldsymbol{L}\psi)-2(f,\psi)-\frac{2p(b)}{\beta}\psi(b)g$$

$$=\int_a^b\left[p\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2+q\psi^2-2f\psi\right]\mathrm{d}x+\frac{\alpha p(b)}{\beta}\psi^2(b)-\frac{2p(b)}{\beta}\psi(b)g$$

(1)划分网格节点

$$a = x_0 < x_1 < \ldots < x_{i-1} < x_i < x_{i+1} \ldots < x_{M-1} < x_M = b$$

$$\psi_0,\ \psi_1, \cdots, \psi_{i-1},\ \psi_i,\ \psi_{i+1}, \cdots, \psi_{M-1},\ \psi_M$$

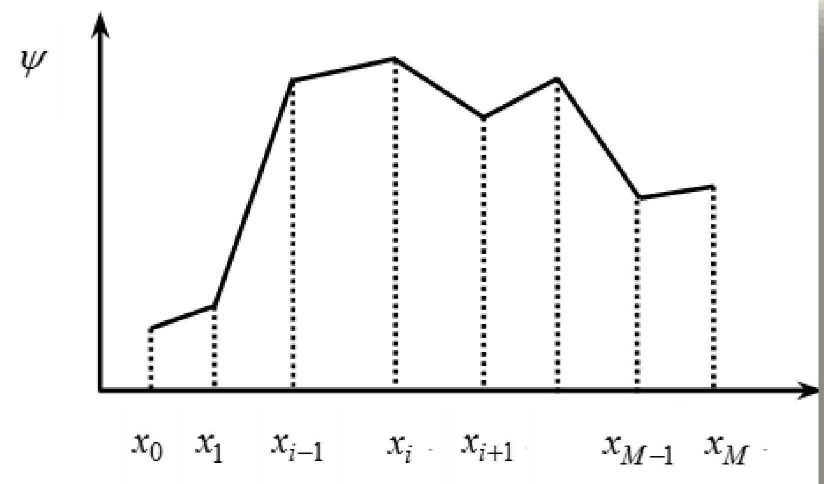
(2)在单元e用线性插值(一次单元)

$$\psi^e(x) \approx \psi_{i-1} \frac{x_i - x}{h_i} + \psi_i \frac{x - x_{i-1}}{h_i}$$

$$x \in e, \quad (e = 1, 2, \ldots, M)$$

(3)在整个区间[a,b]上

$$\psi(x) \approx \psi_0 N_0(x) + \sum_{i=1}^{M} \psi_i N_i(x)$$



区间 $[a,b]$ 分成 M 个

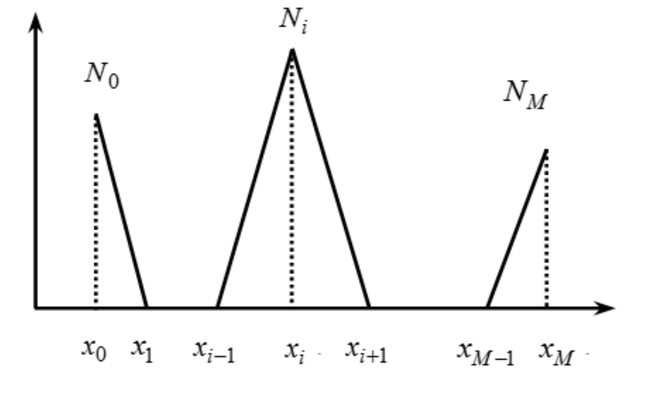
相邻的子区间 $e_i = [x_{i-1}, x_i]$

□ 基函数

$$N_i(x)=\begin{cases}(x-x_{i-1})/h_i, & x\in[x_{i-1},x_i]\\ (x_{i+1}-x)/h_{i+1}, & x\in[x_i,x_{i+1}],\quad (i=1,2,\ldots,M-1)\\ 0, & \text{其它}\end{cases}$$

$$N_0(x)=\begin{cases}(x_1-x)/h_1, & x\in[x_0,x_1]\\ 0, & x\notin[x_0,x_1]\end{cases}$$

$$N_M(x)=\begin{cases}(x-x_{M-1})/h_M, & x\in[x_{M-1},x_M]\\ 0, & x\notin[x_{M-1},x_M]\end{cases}$$



□ 基函数的性质

$$N_i(x_j)=\delta_{ij},\ (1\le i,j\le M)$$

$$N_i(x)N_j(x)=0,\ |i-j|\ge 2$$

$$N_i'(x)N_j'(x)=0,\ |i-j|\ge 2$$

(4)泛函离散化

$$J(\psi)=\sum_{i,j=0}^{M}\chi_{ij}\psi_i\psi_j-2\sum_{i=1}^{M}\beta_i\psi_i+\frac{\alpha p(b)}{\beta}\psi_M^2-\frac{2p(b)}{\beta}\psi_M g$$

- 系数矩阵——刚度矩阵

$$\chi_{ij}\equiv\int_a^b\left[p(x)\frac{\mathrm{d}N_i(x)}{\mathrm{d}x}\cdot\frac{\mathrm{d}N_j(x)}{\mathrm{d}x}+q(x)N_i(x)N_j(x)\right]\mathrm{d}x$$

- 泛函后二项可合并到刚度矩阵

$$J(\psi)=\sum_{i,j=0}^{M}\chi_{ij}\psi_i\psi_j-2\sum_{i=1}^{M}\beta_i\psi_i;\beta_i\equiv\int_a^b f(x)N_i(x)\mathrm{d}x$$

(5)极值条件——决定结点值的线性代数方程组

$$\partial J(\psi)/\partial\psi_i=0 \Rightarrow \sum_{i=1}^{M}\chi_{ij}\psi_i=\beta_j,\ (j=1,2,\ldots,M)$$

(6)二阶变分

$$\delta^2 J(\psi)=\sum_{i,j=0}^{M}\chi_{ij}\delta\psi_i\delta\psi_j>0$$

——*L*是Hermite对称的正算子，刚度矩阵正定

(7)定解问题的解

$$\psi(x)\approx\psi_0 N_0(x)+\sum_{i=1}^{M}\psi_i N_i(x),\ \ x\in[a,b]$$

■ 一维本征值问题

$$\boldsymbol{L}u\equiv-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right]+q(x)\psi=\rho(x)\lambda\psi,\ \ x\in(a,b)$$

$$\psi(a)=0;\ \left.\left(\alpha\psi+\beta\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)\right|_{x=b}=0$$

$$\Rightarrow\ f(x)=\rho(x)\lambda\psi$$

$$\beta_j = \int_a^b f(x) N_j(x) \mathrm{d}x = \lambda \int_a^b \rho(x) \psi(x) N_j(x) \mathrm{d}x$$

$$= \lambda \int_a^b \rho(x) \sum_{i=1}^{M} \psi_i N_i(x) N_j(x) \mathrm{d}x = \lambda \sum_{i=1}^{M} (N_i, N_j) \psi_i$$

$$\Uparrow$$

$$\psi(x) \approx \psi_0 N_0(x) + \sum_{i=1}^{M} \psi_i N_i(x) = \sum_{i=1}^{M} \psi_i N_i(x), (\psi_0 = 0)$$

$$(N_i, N_j) \equiv \int_a^b \rho(x) N_i(x) N_j(x) \mathrm{d}x$$

$$\Downarrow$$

$$\sum_{i=1}^{M} \chi_{ij} \psi_i = \lambda \sum_{i=1}^{M} (N_i, N_j) \psi_i, \quad (j = 1, 2, \ldots, M)$$

$$\Downarrow$$

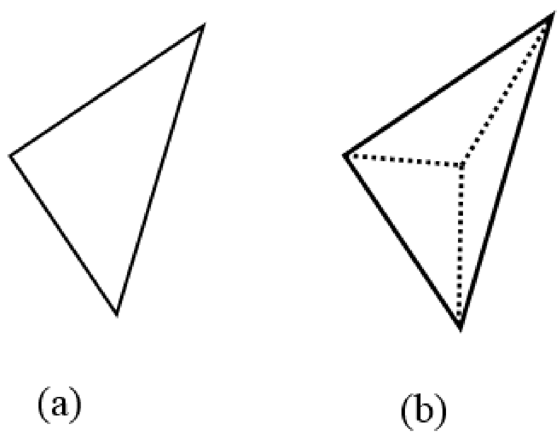
$$\sum_{i=1}^{M} [\chi_{ij} - \lambda (N_i, N_j)] \psi_i = 0, \quad (j = 1, 2, \ldots, M)$$

■ 二维边值问题

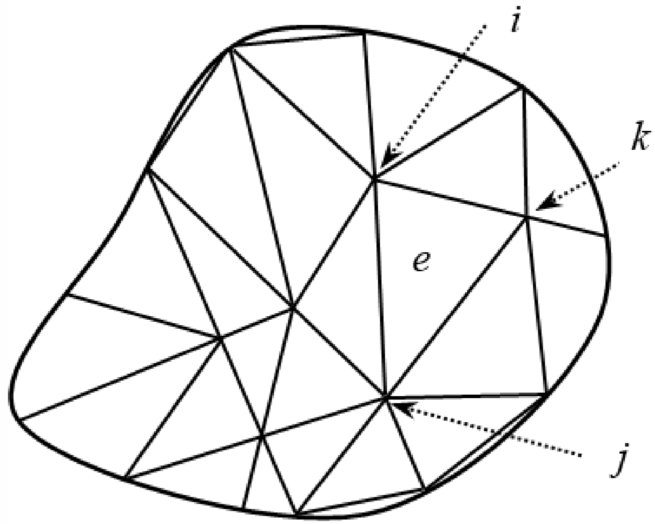
$$\boldsymbol{L}\psi(x,y)=f,\ (x,y)\in G$$

$$\psi(x,y)|_{\partial G_1}=\psi_0(x,y),(x,y)\in\partial G_1$$

$$\left(\alpha\psi+\beta\frac{\partial\psi}{\partial n}\right)_{\partial G_2}=b(x,y),\ (x,y)\in\partial G_2$$



子区域为：(a)三角形(二维)
(b)四面体(三维)



区域 G 用 N 个三角形来划分