第13章 Green 函数理论

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13.1 常微分方程边值问题的Green函数

点源的响应: (1)非齐次方程的解; (2)微分方程化为积分方程。

❑Green函数的定义

考虑二阶常微分方程的边值问题

$$\begin{cases} \boldsymbol{L}[u] \equiv -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u = f(x),\ (a < x < b) \\ (\alpha_1 u - \beta_1 u')\big|_{x=a} = 0;\ (\alpha_2 u + \beta_2 u')\big|_{x=b} = 0 \end{cases}$$

——必须写成S-L形式

利用 Dirac Delta 函数的抽样特性

$$f(x) = \int_a^b f(\xi)\delta(x-\xi)\mathrm{d}\xi$$

如果求得下列问题的解

$$\begin{cases} \boldsymbol{L}[G(x,\xi)] = \delta(x-\xi),\ (a < x < b) \\ (\alpha_1 G - \beta_1 G')|_{x=a} = 0, \quad (\alpha_2 G + \beta_2 G')|_{x=b} = 0 \end{cases}$$

则利用叠加原理， $u(x)$ 可表示为

$$u(x) = \int_a^b G(x,\xi) f(\xi)\mathrm{d}\xi$$

——$G(x,\xi)$称为L的Green函数(注意与边界条件一起).

构造法求 Green 函数：设 $u_1(x)$ 和 $u_2(x)$ 分别是齐次方程 $L[u]=0$ 的解，且分别满足边界条件

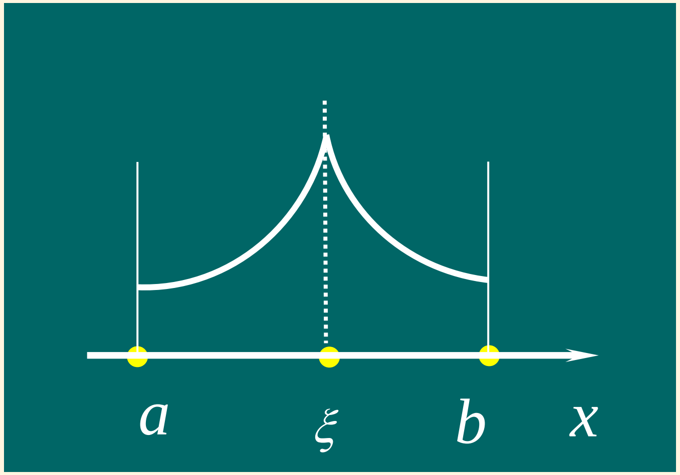
$$(\alpha_1 u_1 - \beta_1 u_1')\big|_{x=a} = 0, \quad (\alpha_2 u_2 + \beta_2 u_2')\big|_{x=b} = 0$$

取下列形式的 Green

$$G(x,\xi) = \begin{cases} C_1(\xi)u_1(x), \ (a \le x \le \xi) \\ C_2(\xi)u_2(x), (\xi \le x \le b) \end{cases}$$

显然，这样定义的 Green 函数除 $x=\xi$ 点外满足方程和边界条件。$C_1(\xi)$和$C_2(\xi)$由$G(x,\xi)$的连续性和导数在$x=\xi$的跃变决定

$$\begin{cases} G(x,\xi)\big|_{x=\xi+0} = G(x,\xi)\big|_{x=\xi-0} \\ \left.\dfrac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=\xi+0} - \left.\dfrac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=\xi-0} = -\dfrac{1}{p(\xi)} \end{cases}$$



①连续性：如果G在$x=\xi$点不连续，一阶导数出现δ函数，二阶导数将出现δ函数的导数;

②一阶导数的跃变：对方程在区间$[\xi-\varepsilon, \xi+\varepsilon]$ 积分

$$-\int_{\xi-\varepsilon}^{\xi+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}G}{\mathrm{d}x}\right]\mathrm{d}x + \int_{\xi-\varepsilon}^{\xi+\varepsilon} q(x)G\mathrm{d}x = 1$$

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\mathrm{d}x = 1$$

连续函数 $\Rightarrow$ $$\lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon} q(x)G(x,\xi)\mathrm{d}x = 0$$

$$-\int_{\xi-\varepsilon}^{\xi+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}G}{\mathrm{d}x}\right]dx = -p(\xi)\left(\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{\xi+0} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{\xi-0}\right)$$

$$\Downarrow$$

$$\left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=\xi+0} - \left.\frac{\mathrm{d}G}{\mathrm{d}x}\right|_{x=\xi-0} = -\frac{1}{p(\xi)}$$

■ **系数方程**

$$C_2(\xi)u_2(\xi) - C_1(\xi)u_1(\xi) = 0$$

$$C_2(\xi)\frac{\mathrm{d}u_2(\xi)}{\mathrm{d}\xi} - C_1(\xi)\frac{\mathrm{d}u_1(\xi)}{\mathrm{d}\xi} = -\frac{1}{p(\xi)}$$

$$C_1(\xi) = -\frac{u_2(\xi)}{p(\xi)W(u_1,u_2)};\quad C_2(\xi) = -\frac{u_1(\xi)}{p(\xi)W(u_1,u_2)}$$

$$G(x,\xi) = \begin{cases} -\dfrac{u_2(\xi)u_1(x)}{p(\xi)W(u_1,u_2)}, & (a \le x \le \xi) \\ -\dfrac{u_1(\xi)u_2(x)}{p(\xi)W(u_1,u_2)}, & (\xi \le x \le b) \end{cases}$$

- **Wronski行列式**

$$W(u_1,u_2) = u_1(\xi)u_2'(\xi) - u_1'(\xi)u_2(\xi)$$

$$-p(\xi)W(u_1,u_2) = \text{常数}C$$

$$G(x,\xi)=\frac{1}{C}\begin{cases}u_1(x)u_2(\xi), & (a\le x\le \xi)\\ u_2(x)u_1(\xi), & (\xi\le x\le b)\end{cases}$$

例1 求算子 $L=-\mathrm{d}^2/\mathrm{d}x^2$ 在第一类边界条件下的 Green 函数

$$\begin{cases}-\dfrac{\mathrm{d}^2G(x,\xi)}{\mathrm{d}x^2}=\delta(x-\xi),\ (0<x<l)\\ G(x,\xi)|_{x=0}=0;\quad G(x,\xi)|_{x=l}=0\end{cases}$$

解：先求齐次方程的解：显然可取

$$u_1(x)=x;\quad u_2(x)=(l-x)$$

因此, 所求Green 函数为 $(-p(\xi)W(u_1,u_2)=l)$

$$G(x,\xi)=\frac{1}{l}\begin{cases} x(l-\xi), & (0\le x\le\xi) \\ \xi(l-x), & (\xi\le x\le l)\end{cases}$$

- **边值问题**

$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=f(x);\quad u(0)=u(l)=0$$

$$\Downarrow$$

$$u(x)=\int_0^l f(\xi)G(x,\xi)\mathrm{d}\xi$$

例2 求 $L=-\mathrm{d}^2/\mathrm{d}x^2+\lambda$ 在第一类边界条件的Green 函数

$$\begin{cases} -\dfrac{\mathrm{d}^2 G(x,\xi)}{\mathrm{d}x^2} + \lambda G(x,\xi) = \delta(x-\xi),\ (0 < x < l) \\ G(x,\xi)\,|_{x=0} = 0;\quad G(x,\xi)\,|_{x=l} = 0 \end{cases}$$

解：先求齐次方程的解

$$\boldsymbol{L}(u) \equiv -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \lambda u = 0$$

①如果λ>0，显然可取

$$u_1(x) = \sinh\left(\sqrt{\lambda}x\right);\quad u_2(x) = \sinh\left[\sqrt{\lambda}(l-x)\right]$$

$$-p(\xi)W(u_1, u_2) = -\sqrt{\lambda}\sinh\left(\sqrt{\lambda}l\right)$$

因此所求的 Green 函数

$$G(x,\xi)=\begin{cases}\dfrac{\sinh\left(\sqrt{\lambda}x\right)\sinh\left[\sqrt{\lambda}(l-\xi)\right]}{\sqrt{\lambda}\sinh\left(\sqrt{\lambda}l\right)}, & (0\le x\le\xi)\\[2ex] \dfrac{\sinh\left(\sqrt{\lambda}\xi\right)\sinh\left[\sqrt{\lambda}(l-x)\right]}{\sqrt{\lambda}\sinh\left(\sqrt{\lambda}l\right)}, & (\xi\le x\le l)\end{cases}$$

②如果λ<0，显然可取

$$u_1(x)=\sin\left(\sqrt{|\lambda|}x\right);\quad u_2(x)=\sin\left[\sqrt{|\lambda|}(l-x)\right]$$

$$\Downarrow$$

$$-p(\xi)W(u_1,u_2)=\sqrt{|\lambda|}\sin\left(\sqrt{|\lambda|}l\right)$$

$$G(x,\xi)=\begin{cases}\dfrac{\sin\left(\sqrt{|\lambda|}x\right)\sin\left[\sqrt{|\lambda|}(l-\xi)\right]}{\sqrt{|\lambda|}\sin\left(\sqrt{|\lambda|}l\right)}, & (0\le x\le\xi)\\[2ex] \dfrac{\sin\left(\sqrt{|\lambda|}\xi\right)\sin\left[\sqrt{|\lambda|}(l-x)\right]}{\sqrt{|\lambda|}\sin\left(\sqrt{|\lambda|}l\right)}, & (\xi\le x\le l)\end{cases}$$

- **本征函数展开法求 Green 函数：令**

$$G(x,\xi)=\sum_{m=0}^{\infty}C_m\psi_m(x)$$

其中：$\psi_m(x)$ 是 Sturm-Liouville 方程的本征函数，相应的本征值为λ_m

$$\begin{cases} \boldsymbol{L}(\psi_m) = \lambda_m \rho(x)\psi_m \\ \left.\left(\alpha_1 y - \beta_1 \dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\right|_{x=a} = 0; \quad \left.\left(\alpha_2 y + \beta_2 \dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\right|_{x=b} = 0 \end{cases}$$

代入 Green 函数的定义方程

$$\boldsymbol{L}[G(x,\xi)] = \sum_{m=0}^{\infty} C_m \boldsymbol{L}[\psi_m(x)] = \sum_{m=0}^{\infty} C_m \lambda_m \rho(x)\psi_m(x) = \delta(x-\xi)$$

两边乘$[\psi_m(x)]^*$

$$\lambda_m C_m = \int_a^b \psi_m^*(x)\delta(x-\xi)\mathrm{d}x = \psi_m^*(\xi)$$

首先假定零不是本征值：$\lambda_m \neq 0$ $\quad C_m = \psi_m^*(\xi)/\lambda_m$

$$G(x,\xi)=\sum_{m=0}^{\infty}\frac{1}{\lambda_m}\psi_m^*(\xi)\psi_m(x)$$

例3 求算子 *L*=–d²/dx² 在第一类边界条件下的 Green 函数

$$-\frac{\mathrm{d}^2\psi_m}{\mathrm{d}x^2}=\lambda_m\psi_m;\psi_m\,|_{x=0}=0,\ \ \psi_m\,|_{x=l}=0$$

$$\psi_m(x)=\sqrt{\frac{2}{l}}\sin\left(\frac{m\pi x}{l}\right),\quad \lambda_m=\frac{m^2\pi^2}{l^2}$$

$$G(x,\xi)=\sum_{m=1}^{\infty}\frac{4l}{m^2\pi^2}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{m\pi\xi}{l}\right)$$

❑ 广义 Green 函数

问题：当L存在零本征值(λ_0=0)时

$$L[\psi_0(x)] = \lambda_0\psi_0(x) = 0$$

$$\boxed{G(x,\xi) = \frac{1}{\lambda_0}\psi_0^*(\xi)\psi_0(x) + \sum_{m=1}^{\infty}\frac{1}{\lambda_m}\psi_m^*(\xi)\psi_m(x)}$$

——Green函数发散！能否定义Green函数?

唯一性和存在性问题

$$\begin{cases} \boldsymbol{L}[u] \equiv -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u = f(x),\ (a<x<b) \\ (\alpha_1 u - \beta_1 u')\,|_{x=a} = 0,\ (\alpha_2 u + \beta_2 u')\,|_{x=b} = 0 \end{cases}$$

■唯一性：设u是边值问题的解，则$u+A\psi_0(x)$也是解，不唯一；

■存在性：设u是边值问题的解，则

$$\int_a^b (u\boldsymbol{L}\psi_0^* - \psi_0^* \boldsymbol{L}u)\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[p\left(u\frac{\mathrm{d}\psi_0^*}{\mathrm{d}x} - \psi_0^*\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right]\mathrm{d}x = 0$$

$$\Downarrow$$

$$\int_a^b \psi_0^*(x) f(x)\mathrm{d}x = 0$$

即：要求$f(x)$与$\psi_0(x)$正交——存在u的相容性条件，否则方程无解

■ 广义 Green 函数

$$\begin{cases} \boldsymbol{L}[G(x,\xi)] = \delta(x-\xi), \ \ (a < x < b) \\ \alpha_1 G - \beta_1 G' |_{x=a} = 0, \quad \alpha_2 G + \beta_2 G' |_{x=b} = 0 \end{cases}$$

——解不存在

分析

$$G(x,\xi) = \sum_{m=0}^{\infty} C_m \psi_m(x); \quad \delta(x-\xi) = \sum_{m=0}^{\infty} \rho(x)\psi_m^*(\xi)\psi_m(x)$$

⬇

$$\boldsymbol{L}\left[\sum_{m=0}^{\infty} C_m \psi_m(x)\right] = \sum_{m=0}^{\infty} \rho(x)\psi_m^*(\xi)\psi_m(x)$$

⬆

$$\boldsymbol{L}[\psi_m(x)] = \lambda_m \rho(x)\psi_m(x)$$

$$C_0\lambda_0\rho(x)\psi_0(x)+\sum_{m=1}^{\infty}C_m\lambda_m\rho(x)\psi_m(x)$$

不等于0

等于0

$$=\rho(x)\psi_0^*(\xi)\psi_0(x)+\sum_{m=1}^{\infty}\rho(x)\psi_m^*(\xi)\psi_m(x)$$

——不可能成立

① 问题出在:δ函数包含所有“基函数”上的“投影”,而左边经L作用后,不包含“基函数”$\psi_0(x)$上的投影!

② 解决方法非常简单:右边减去“基函数”$\psi_0(x)$上的投影即开。

定义广义Green函数

$$\begin{cases} L[\tilde{G}(x,\xi)] = \delta(x-\xi) - \rho(x)\psi_0^*(\xi)\psi_0(x), \quad (a<x<b) \\ \alpha_1\tilde{G} - \beta_1\tilde{G}'|_{x=a} = 0; \ \alpha_2\tilde{G} + \beta_2\tilde{G}'|_{x=b} = 0 \end{cases}$$

$$\tilde{G}(x,\xi) = \sum_{m=0}^{\infty} C_m\psi_m(x)$$ ——**完备性要求**

$$\sum_{m=1}^{\infty} C_m\lambda_m\rho(x)\psi_m(x) = \sum_{m=1}^{\infty}\rho(x)\psi_m^*(\xi)\psi_m(x)$$

$$\tilde{G}(x,\xi) = C_0\psi_0(x) + \sum_{m=1}^{\infty}\frac{1}{\lambda_m}\psi_m^*(\xi)\psi_m(x)$$

考虑对称性后

量纲常数，最后不出现

$$\tilde{G}(x,\xi) = L\psi_0^*(\xi)\psi_0(x) + \sum_{m=1}^{\infty} \frac{1}{\lambda_m} \psi_m^*(\xi)\psi_m(x)$$

非齐次方程的边值问题

$$\begin{cases} \boldsymbol{L}(u) \equiv -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u = f(x), \; (a < x < b) \\ (\alpha_1 u - \beta_1 u')\big|_{x=a} = 0; \; (\alpha_2 u + \beta_2 u')\big|_{x=b} = 0 \end{cases}$$

$$\Downarrow$$

$$\int_a^b \left[u\boldsymbol{L}(\tilde{G}^*) - \tilde{G}^*\boldsymbol{L}(u)\right]\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[p\left(u\frac{\mathrm{d}\tilde{G}^*}{\mathrm{d}x} - \tilde{G}^*\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right]\mathrm{d}x = 0$$

$$\int_a^b \left\{ u(x)\left[\delta(x-\xi)-\psi_0(\xi)\psi_0^*(x)\right] - f(x)\tilde{G}^*(x,\xi)\right\}\mathrm{d}x = 0$$

$$u(\xi)=\int_a^b f(x)\tilde{G}^*(x,\xi)\mathrm{d}x + \psi_0(\xi)\int_a^b u(x)\psi_0^*(x)\mathrm{d}x$$

$$u(x)=\int_a^b f(\xi)\tilde{G}^*(\xi,x)\mathrm{d}\xi + \psi_0(x)\int_a^b u(\xi)\psi_0^*(\xi)\mathrm{d}\xi$$

Green函数
共轭对称性

$$\tilde{G}^*(\xi,x)=\tilde{G}(x,\xi)$$

$$u(x)=\int_a^b f(\xi)\tilde{G}(x,\xi)\mathrm{d}\xi + A\psi_0(x)$$

$$u(x)=A\psi_0(x)+\int_a^b f(\xi)G_0(x,\xi)\mathrm{d}\xi$$

$$\int_a^b f(\xi)\psi_0^*(\xi)\mathrm{d}\xi=0;\ G_0(x,\xi)=\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\psi_m^*(\xi)\psi_m(x)$$

❑**Green 函数的对称性 直接从表达式**

$$G(x,\xi)=\sum_{m=0}^{\infty}\frac{1}{\lambda_m}\psi_m^*(\xi)\psi_m(x)\ \Rightarrow\ G^*(\xi,x)=G(x,\xi)$$

■**一般性证明**

$$u=G(x,\xi_1);\ \ \boldsymbol{L}(u)=\delta(x-\xi_1)$$

$$v=G(x,\xi_2);\ \ \boldsymbol{L}(v)=\delta(x-\xi_2)$$

$$\int_a^b\left[u^*\boldsymbol{L}(v)-v\boldsymbol{L}(u^*)\right]\mathrm{d}x=G^*(\xi_2,\xi_1)-G(\xi_1,\xi_2)$$

左边直接计算

$$\int_a^b (u^* \boldsymbol{L}v - v\boldsymbol{L}u^*)\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[p\left(u^*\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u^*}{\mathrm{d}x}\right)\right]dx = \int_a^{\xi_1} + \int_{\xi_1}^{\xi_2} + \int_{\xi_2}^{b}$$

$$= p(x)\left[G^*(x,\xi_1)\frac{\mathrm{d}G(x,\xi_2)}{\mathrm{d}x} - G(x,\xi_2)\frac{\mathrm{d}G^*(x,\xi_2)}{\mathrm{d}x}\right]\Bigg|_{x=a}^{x=b} = 0$$

$$G^*(\xi_2,\xi_1) = G(\xi_1,\xi_2)$$

❑相容性条件的意义

$$u(x) = \int_a^b G(x,\xi) f(\xi)\mathrm{d}\xi; G(x,\xi) = \sum_{m=0}^{\infty}\frac{1}{\lambda_m}\psi_m^*(\xi)\psi_m(x)$$

$$u(x)=\frac{\int_a^b \psi_0^*(\xi)f(\xi)\mathrm{d}\xi}{\lambda_0}\psi_0(x)+\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\left[\int_a^b \psi_m^*(\xi)f(\xi)\mathrm{d}\xi\right]\psi_m(x)$$

如果λ_0=0，为了保证第一项有限

$$\frac{\int_a^b \psi_0^*(\xi)f(\xi)\mathrm{d}\xi}{\lambda_0}=C \quad \Rightarrow \quad \int_a^b \psi_0^*(\xi)f(\xi)\mathrm{d}\xi=0$$

① 数学意义：源的分布*f*(*x*)与零本征值“基函数”正交；

② 物理意义：在*x*=*a*,*b*绝热条件下，体源的空间平均为零，即热源和热汇平均为零。

例1 求$L=-\mathrm{d}^2/\mathrm{d}x^2$在第二类边界条件下的Green 函数

$$\begin{cases} -\dfrac{\mathrm{d}^2 g}{\mathrm{d}x^2} = \delta(x-\xi)-1, \ (0<x<1) \\ g'|_{x=0}=0, \quad g'|_{x=1}=0 \end{cases}$$

当 $x\neq\xi$时, 方程有解

$$g(x,\xi)=\begin{cases} A+Bx+\dfrac{1}{2}x^2, \quad (0<x<\xi) \\ C+Dx+\dfrac{1}{2}x^2, \ (\xi<x<1) \end{cases}$$

(1)由边界条件： $B=0,\ D=-1$

(2)由连续性条件： $g(x,\xi)|_{x=\xi-0}=g(x,\xi)|_{x=\xi+0}$

得到 $A = C - \xi$

$$g(x,\xi) = \begin{cases} C - \xi + \dfrac{1}{2}x^2, & (0 < x < \xi) \\ C - x + \dfrac{1}{2}x^2, & (\xi < x < 1 \end{cases}$$

注意：上式自动满足在$x=\xi$的跃变条件。

(3)对称性条件： $C = \xi^2 / 2$

$$g(x,\xi) = \begin{cases} -\xi + \dfrac{1}{2}(x^2 + \xi^2), & (0 < x < \xi) \\ -x + \dfrac{1}{2}(\xi^2 + x^2), & (\xi < x < 1) \end{cases}$$

□ 本征函数展开法

$$\begin{cases} -\dfrac{\mathrm{d}^2\psi_m(x)}{\mathrm{d}x^2} = \lambda_m \psi_m(x) \\ \psi_m'(x)\big|_{x=0} = 0, \quad \psi_m'(x)\big|_{x=1} = 0 \end{cases}$$

$$\Downarrow$$

$$\psi_0(x) = 1; \lambda_0 = 0$$

$$\psi_m(x) = \sqrt{2}\cos(m\pi x); \lambda_m = (m\pi)^2, \ (m = 1, 2, \ldots)$$

$$\Downarrow$$

$$g(x, \xi) = 1 + \frac{2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos(m\pi x)\cos(m\pi\xi)$$

❑ 非齐次边界问题

$$\begin{cases} \boldsymbol{L}(u) \equiv -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u = f(x), \ (a < x < b) \\ (\alpha_1 u - \beta_1 u')|_{x=a} = g_1; \ (\alpha_2 u + \beta_2 u')|_{x=b} = g_2 \end{cases}$$

⬇

$$\begin{cases} \boldsymbol{L}[G(x,\xi)] = \delta(x-\xi), \quad (a < x < b) \\ (\alpha_1 G - \beta_1 G')|_{x=a} = 0, \quad (\alpha_2 G + \beta_2 G')|_{x=b} = 0 \end{cases}$$

如何用Green函数表示*u*?

$$u(x) = \int_a^b G(x,\xi) f(\xi)\mathrm{d}\xi$$

⇦ **不满足边界条件**

$$\int_a^b [u^* \boldsymbol{L}(v) - v\boldsymbol{L}(u^*)]\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[p\left(u^* \frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u^*}{\mathrm{d}x} \right)\right]\mathrm{d}x$$

$$\Uparrow$$

$$v(x) = G(x,\xi)$$

$$\int_a^b [u^* \delta(x,\xi) - G(x,\xi)\, f^*]\mathrm{d}x = p\left(u^* \frac{\mathrm{d}G}{\mathrm{d}x} - G\frac{\mathrm{d}u^*}{\mathrm{d}x} \right)\Bigg|_a^b$$

$$\Downarrow$$

$$u^*(\xi) = \int_a^b f^*(x)G(x,\xi)\mathrm{d}x + p(b)\left(u^* \frac{\mathrm{d}G}{\mathrm{d}x} - G\frac{\mathrm{d}u^*}{\mathrm{d}x} \right)\Bigg|_{x=b}$$

$$- p(a)\left(u^* \frac{\mathrm{d}G}{\mathrm{d}x} - G\frac{\mathrm{d}u^*}{\mathrm{d}x} \right)\Bigg|_{x=a}$$

□ 边界处理

实系数

$$(\alpha_1 G - \beta_1 G')|_{x=a} = 0, \quad (\alpha_2 G + \beta_2 G')|_{x=b} = 0$$

$$(\alpha_1 u - \beta_1 u')|_{x=a} = g_1, \quad (\alpha_2 u + \beta_2 u')|_{x=b} = g_2$$

$$(\alpha_1 G - \beta_1 G')|_{x=a} = 0, \quad (\alpha_2 G + \beta_2 G')|_{x=b} = 0$$

$$(\alpha_1 u^* - \beta_1 u'^*)|_{x=a} = g_1^*, \quad (\alpha_2 u^* + \beta_2 u'^*)|_{x=b} = g_2^*$$

$$(u^* G' - G u'^*)|_{x=a} = \frac{g_1^*}{\alpha_1} G'(a,\xi)$$

$$(u^* G' - G u'^*)|_{x=b} = \frac{g_2^*}{\alpha_2} G'(b,\xi)$$

$$(G' u^* - G u'^*)|_{x=a} = \frac{G g_1^*}{\beta_1}$$

$$(G' u^* - G u'^*)|_{x=b} = -\frac{G g_1^*}{\beta_2}$$

$$u^*(\xi) = \int_a^b f^*(x)G(x,\xi)\mathrm{d}x + \frac{p(b)}{\alpha_2} g_1^* \left.\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}x}\right|_{x=b}$$

$$-\frac{p(a)}{\alpha_1} g_1^* \left.\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}x}\right|_{x=a}$$

$$\Downarrow$$

$$u(x) = \int_a^b f(\xi)G^*(\xi,x)\mathrm{d}\xi + \frac{p(b)}{\alpha_2} g_2 \left.\frac{\mathrm{d}G^*(\xi,x)}{\mathrm{d}\xi}\right|_{\xi=b}$$

$$-\frac{p(a)}{\alpha_1} g_1 \left.\frac{\mathrm{d}G^*(\xi,x)}{\mathrm{d}\xi}\right|_{\xi=a}$$

$$G^{*}(\xi,x)=G(x,\xi)$$

$$u(x)=\int_a^b f(\xi)G(x,\xi,)\mathrm{d}\xi+g_2\frac{p(b)}{\alpha_2}\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi}\bigg|_{\xi=b}$$

$$-g_1\frac{p(a)}{\alpha_1}\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi}\bigg|_{\xi=a}$$

——如果α_1或者α_2为零，解如何变化——不能同时为零，否则必须引进广义Green函数，解如何变化?

□ 问题

$$\begin{cases} L(u) \equiv a(x)\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + c(x)u = f(x), \ (a < x < b) \\ (\alpha_1 u - \beta_1 u')|_{x=a} = g_1; \ (\alpha_2 u + \beta_2 u')|_{x=b} = g_2 \end{cases}$$

如何定义Green函数？—共轭算子？Hermite对称算子——非Hermite对称算子

$$\int_a^b \left[v^* \boldsymbol{L}(u) - u\boldsymbol{L}^+(v^*) \right] \mathrm{d}x = 0$$

由共轭算子定义Green函数。对常微分方程，一般首先转化场S-L类型(不改变边界条件的形式)，但对偏微分方程，则必须引进共轭算子.

13.2 高维边值问题的Green函数

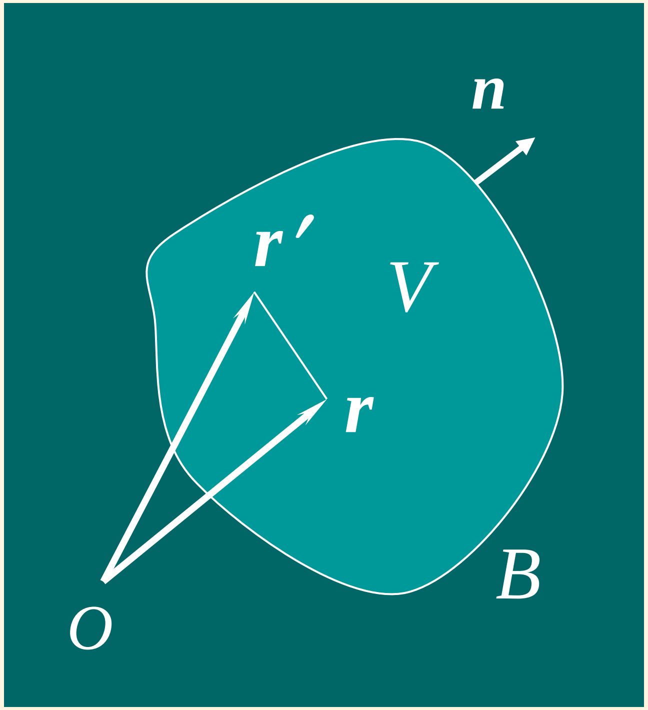
■**Poisson方程**

考虑非齐次边值问题

$$\begin{cases} -\nabla^2 u = f,\ (\boldsymbol{r} \in V) \\ \left.\left(\alpha u + \beta \dfrac{\partial u}{\partial n}\right)\right|_{\boldsymbol{r}\in B} = b(\boldsymbol{r}), (\boldsymbol{r} \in B) \end{cases}$$

定义 Green 函数

$$\begin{cases} -\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r},\boldsymbol{r}'),\ \ (\boldsymbol{r},\boldsymbol{r}') \in V \\ \left.\left(\alpha G + \beta \dfrac{\partial G}{\partial n}\right)\right|_{\boldsymbol{r}\in B} = 0, (\boldsymbol{r}' \in G + B) \end{cases}$$



利用 Green 公式

$$\int_V (u^* \nabla^2 v - v \nabla^2 u^*) \mathrm{d}\tau = \iint_B \left(u^* \frac{\partial v}{\partial n} - v \frac{\partial u^*}{\partial n} \right) \mathrm{d}S$$

取 $v = G(\boldsymbol{r}, \boldsymbol{r}')$

$$\int_V [-u^* \delta(\boldsymbol{r}, \boldsymbol{r}') + f^* G] \mathrm{d}\tau = \iint_B \left(u^* \frac{\partial G}{\partial n} - G \frac{\partial u^*}{\partial n} \right) \mathrm{d}S$$

$$u^*(\boldsymbol{r}') = \int_V f^*(\boldsymbol{r}) G(\boldsymbol{r}, \boldsymbol{r}') \mathrm{d}\tau - \iint_B \left(u^* \frac{\partial G}{\partial n} - G \frac{\partial u^*}{\partial n} \right) \mathrm{d}S$$

(1)第一类边界条件：在边界上

$$u(\boldsymbol{r})\big|_{\boldsymbol{r}\in B}=b(\boldsymbol{r}),\quad G(\boldsymbol{r},\boldsymbol{r}')\big|_{\boldsymbol{r}\in B}=0$$

$$u^*(\boldsymbol{r}')=\int_V f^*(\boldsymbol{r})G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}\tau-\iint_B b^*(\boldsymbol{r})\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n}\mathrm{d}S$$

(2)第二类边界条件：在边界上

$$\left.\frac{\partial u}{\partial n}\right|_{\boldsymbol{r}\in B}=b(\boldsymbol{r}),\quad \left.\frac{\partial G}{\partial n}\right|_{\boldsymbol{r}\in B}=0$$

$$u^*(\boldsymbol{r}')=\int_V f^*(\boldsymbol{r})G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}\tau+\iint_B G(\boldsymbol{r},\boldsymbol{r}')b^*(\boldsymbol{r})\mathrm{d}S$$

(3)第三类边界条件：在边界上

$$\left.\left(\alpha G+\beta\frac{\partial G}{\partial n}\right)\right|_{\mathbf{r}\in B}=0;\left.\left(\alpha u^{*}+\beta\frac{\partial u^{*}}{\partial n}\right)\right|_{\mathbf{r}\in B}=b^{*}(\mathbf{r})$$

$$\left.\left(u^{*}\frac{\partial G}{\partial n}-G\frac{\partial u^{*}}{\partial n}\right)\right|_{\mathbf{r}\in B_1}=\frac{b^{*}(\mathbf{r})}{\alpha}\frac{\partial G}{\partial n}$$

$$\left.\left(u^{*}\frac{\partial G}{\partial n}-G\frac{\partial u^{*}}{\partial n}\right)\right|_{\mathbf{r}\in B_2}=-\frac{b^{*}(\mathbf{r})}{\beta}G$$

——注意： $\alpha(r)$和$\beta(r)$都是r的函数．设在部分边界B_1上 $\beta(r)=0$；在部分边界B_2上$\alpha(r)=0$，且$B=B_1+B_2$

$$u^*(\mathbf{r}')=\int_V f^*(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}\tau+\iint_{B_2}\frac{b^*(\mathbf{r})}{\beta(\mathbf{r})}G(\mathbf{r},\mathbf{r}')\mathrm{d}S$$

$$-\iint_{B_1}\frac{b^*(\mathbf{r})}{\alpha(\mathbf{r})}\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\mathrm{d}S$$

——第一、二类边界条件都可以由上式表示。

操作：①交换变量$r'\leftrightarrow r$；②二边求复共轭

$$u(\mathbf{r})=\int_V f(\mathbf{r}')G^*(\mathbf{r}',\mathbf{r})\mathrm{d}\tau'+\iint_{B_2}\frac{b(\mathbf{r}')}{\beta(\mathbf{r}')}G^*(\mathbf{r}',\mathbf{r})\mathrm{d}S'$$

$$-\iint_{B_1}\frac{b(\mathbf{r}')}{\alpha(\mathbf{r}')}\frac{\partial G^*(\mathbf{r}',\mathbf{r})}{\partial n'}\mathrm{d}S'$$

存在问题

$$\begin{cases} -\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}'), \quad (\mathbf{r},\mathbf{r}') \in G \\ \left. \left(\alpha G + \beta \dfrac{\partial G}{\partial n} \right) \right|_{\mathbf{r}\in B} = 0 \end{cases} \Rightarrow G(\mathbf{r},\mathbf{r}')$$

——r'是常量，作为解函数的变量不适合。

Green函数的共轭对称性: $G^*(\mathbf{r}',\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$

$$u(\mathbf{r}) = \int_V f(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\mathrm{d}\tau' + \iint_{B_2} \frac{b(\mathbf{r}')}{\beta(\mathbf{r}')} G(\mathbf{r},\mathbf{r}')\mathrm{d}S'$$

$$-\iint_{B_1} \frac{b(\mathbf{r}')}{\alpha(\mathbf{r}')} \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} \mathrm{d}S'$$

——结果与第10章一致

❑Green 函数的对称性质

$$G^*(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$$

证明：利用Green 公式

$$\int_V (u\nabla^2 v - v\nabla^2 u)\mathrm{d}\tau = \iint_B \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right)\mathrm{d}S$$

$$\Uparrow$$

$$u = G(\mathbf{r},\mathbf{r}'),\quad v = G^*(\mathbf{r},\mathbf{r}'')$$

$$\Downarrow$$

$$-\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}');\qquad \left[\alpha G(\mathbf{r},\mathbf{r}') + \beta\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\right]\Bigg|_B = 0$$

$$-\nabla^2 G^*(\mathbf{r},\mathbf{r}'') = \delta(\mathbf{r},\mathbf{r}'');\quad \left[\alpha G^*(\mathbf{r},\mathbf{r}'') + \beta\frac{\partial G^*(\mathbf{r},\mathbf{r}'')}{\partial n}\right]\Bigg|_B = 0$$

Green 公式左边

$$\int_V (u\nabla^2 v - v\nabla^2 u)\mathrm{d}\tau = -G(\mathbf{r}'',\mathbf{r}') + G^*(\mathbf{r}',\mathbf{r}'')$$

Green 公式右边

$$\left[\alpha G(\mathbf{r},\mathbf{r}') + \beta \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\right]\Bigg|_B = 0; \quad \left[\alpha G^*(\mathbf{r},\mathbf{r}'') + \beta \frac{\partial G^*(\mathbf{r},\mathbf{r}'')}{\partial n}\right]\Bigg|_B = 0$$

$$\Downarrow$$

$$\left[G(\mathbf{r},\mathbf{r}')\frac{\partial G^*(\mathbf{r},\mathbf{r}'')}{\partial n} - \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} G^*(\mathbf{r},\mathbf{r}'')\right]\Bigg|_{\mathbf{r}\in B} = 0$$

$$\iint_B \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S = 0 \quad \Rightarrow \quad G(\mathbf{r}'',\mathbf{r}') = G^*(\mathbf{r}',\mathbf{r}'')$$

❑广义Green 函数

与一维情况类似，对第二类边界条件，齐次方程存在非零解，或者存在零本征值，必须定义广义Green函数

$$
\begin{cases}
-\nabla^2 \tilde{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}') - \psi_0(\mathbf{r}')\psi_0(\mathbf{r}), \quad (\mathbf{r},\mathbf{r}') \in G \\
\left.\dfrac{\partial G}{\partial n}\right|_{\mathbf{r}\in B} = 0
\end{cases}
$$

$$\psi_0(\mathbf{r}) = 1/\sqrt{V}$$

其中V是区域的体积。由Green 公式

$$
u(\mathbf{r}) = \int_V \left[\frac{u(\mathbf{r}')}{V} + \tilde{G}(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \right] \mathrm{d}\tau' + \iint_B \tilde{G}(\mathbf{r},\mathbf{r}') b(\mathbf{r}') \mathrm{d}S'
$$

第一个积分为常数，故积分解

$$u(\boldsymbol{r})=\frac{A}{V}+\int_V \tilde{G}(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\mathrm{d}\tau'+\iint_B \tilde{G}(\boldsymbol{r},\boldsymbol{r}')b(\boldsymbol{r}')\mathrm{d}S'$$

■ 相容条件

由Green公式

$$\int_V (u\nabla^2 v-v\nabla^2 u)\mathrm{d}\tau=\iint_B\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)\mathrm{d}S$$

令 $v=\psi_0(\boldsymbol{r})=1/\sqrt{V}$ ，则有相容条件

这里提出一个小思考：从三维情况入手好像理解更容易

$$\int_V f(\boldsymbol{r})\mathrm{d}\tau=\iint_B b(\boldsymbol{r})\mathrm{d}S$$

对齐次边界： ***b*=0**

$$\int_V f(\boldsymbol{r})\mathrm{d}\tau=0$$

物理上，要求热源的分布：源和汇抵消，这样才能在边界绝热的情况下，温度稳定地分布。

❑Helmholtz方程的Green函数

$$\begin{cases} \left(-\nabla^2+\lambda\right)u(\boldsymbol{r}) = f(\boldsymbol{r}), \quad \boldsymbol{r}\in G \\ \left.\left(\alpha u+\beta\dfrac{\partial u}{\partial n}\right)\right|_B = b(\boldsymbol{r}), \ \boldsymbol{r}\in B \end{cases}$$

定义 Green 函数

$$\begin{cases} \left(-\nabla^2+\lambda\right)G(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r},\boldsymbol{r}') \\ \left.\left(\alpha G+\beta\dfrac{\partial G}{\partial n}\right)\right|_{\boldsymbol{r}\in B} = 0 \end{cases}$$

满足齐次边界条件

可以证明积分解仍然成立

$$u(\boldsymbol{r})=\int_V f(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}\tau'+\iint_{B_2}\frac{b(\boldsymbol{r}')}{\beta(\boldsymbol{r}')}G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S'$$

$$-\iint_{B_1}\frac{b(\boldsymbol{r}')}{\alpha(\boldsymbol{r}')}\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}\mathrm{d}S'$$

——注意：尽管积分解的形式一样，但是Green函数完全不同

正交展开法求Green 函数

设Laplace 算子的本征函数集完备正交且归一

$$\begin{cases} -\nabla^2\psi_m=\lambda_m\psi_m \\ \left.\left(\alpha\psi_m+\beta\dfrac{\partial\psi_m}{\partial n}\right)\right|_B=0 \end{cases}$$

——注意：本征函数与Green函数一样满足齐次边界条件

令

$$G(\mathbf{r},\mathbf{r}')=\sum_{m=1}^{\infty}C_m\psi_m(\mathbf{r}) \quad \left(-\nabla^2+\lambda\right)G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r},\mathbf{r}')$$

$$\sum_{m=1}^{\infty}C_m(\lambda_m+\lambda)\psi_m(\mathbf{r})=\delta(\mathbf{r},\mathbf{r}')$$

$$C_m(\lambda_m+\lambda)=\int_G\delta(\mathbf{r},\mathbf{r}')\psi_m^*(\mathbf{r})\mathrm{d}\tau=\psi_m^*(\mathbf{r}')$$

① $\lambda_m+\lambda\neq 0$

$$C_m=\frac{\psi_m^*(\mathbf{r}')}{\lambda_m+\lambda} \quad \boxed{G(\mathbf{r},\mathbf{r}')=\sum_{m=1}^{\infty}\frac{\psi_m^*(\mathbf{r}')\psi_m(\mathbf{r})}{\lambda_m+\lambda}}$$

②第 M 个本征值刚好等于$-\lambda$： $\lambda_M + \lambda = 0$

$$C_m = \frac{\psi_m^*(\mathbf{r}')}{\lambda_m + \lambda},\ (m \neq M); C_M = C_M$$

系数 C_M 为任意常数, 但考虑到 Green 函数的对称性质，取

$$G(\mathbf{r},\mathbf{r}') = A\psi_M(\mathbf{r})\psi_M^*(\mathbf{r}') + \sum_{m\neq M}^{\infty} \frac{\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')}{\lambda_m + \lambda}$$

A为量纲常数。

■定义广义Green函数

$$\begin{cases} \left(-\nabla^2+\lambda\right)\tilde{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}') - \psi_M(\mathbf{r})\psi_M^*(\mathbf{r}') \\ \left.\left(\alpha\tilde{G}+\beta\dfrac{\partial\tilde{G}}{\partial n}\right)\right|_B = 0 \end{cases}$$

$$\tilde{G}(\mathbf{r},\mathbf{r}') = A\psi_M(\mathbf{r})\psi_M^*(\mathbf{r}') + \sum_{m\neq M}^{\infty}\frac{\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')}{\lambda_m+\lambda}$$

- **非齐次问题的积分解**

$$\begin{cases} \left(-\nabla^2+\lambda\right)u(\mathbf{r}) = f(\mathbf{r}), \quad \mathbf{r}\in G \\ \left.\left(\alpha u+\beta\dfrac{\partial u}{\partial n}\right)\right|_B = b(\mathbf{r}), \ \mathbf{r}\in B \end{cases}$$

$$\int_V (u^*\nabla^2 v - v\nabla^2 u^*)\mathrm{d}\tau = \iint_B \left(u^* \frac{\partial v}{\partial n} - v\frac{\partial u^*}{\partial n} \right)\mathrm{d}S$$

取 $v = \tilde{G}(\boldsymbol{r}, \boldsymbol{r}')$

$$\int_V \left[u^*\nabla^2 \tilde{G}(\boldsymbol{r}, \boldsymbol{r}') - \tilde{G}(\boldsymbol{r}, \boldsymbol{r}')\nabla^2 u^* \right]\mathrm{d}\tau$$

$$= \iint_B \left[u^* \frac{\partial \tilde{G}(\boldsymbol{r}, \boldsymbol{r}')}{\partial n} - \tilde{G}(\boldsymbol{r}, \boldsymbol{r}')\frac{\partial u^*}{\partial n} \right]\mathrm{d}S$$

$$u(\boldsymbol{r}) = \int_V \tilde{G}(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}')\mathrm{d}\tau' + \left[\int_V u(\boldsymbol{r}')\psi_M^*(\boldsymbol{r}')\mathrm{d}\tau' \right] \cdot \psi_M(\boldsymbol{r})$$

$$+ \iint_{B_2} \frac{b(\boldsymbol{r}')}{\beta(\boldsymbol{r}')} \tilde{G}(\boldsymbol{r}, \boldsymbol{r}')\mathrm{d}S' - \iint_{B_1} \frac{b(\boldsymbol{r}')}{\alpha(\boldsymbol{r}')} \frac{\partial \tilde{G}(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'}\mathrm{d}S'$$

$$\tilde{G}(\boldsymbol{r},\boldsymbol{r}') = A\psi_M^*(\boldsymbol{r}')\psi_M(\boldsymbol{r}) + G_M(\boldsymbol{r},\boldsymbol{r}')$$

$$G_M(\boldsymbol{r},\boldsymbol{r}') \equiv \sum_{m\neq M}^{\infty} \frac{\psi_m(\boldsymbol{r})\psi_m^*(\boldsymbol{r}')}{\lambda_m + \lambda}$$

$$u(\boldsymbol{r}) = \left[\int_V \psi_M^*(\boldsymbol{r}') f(\boldsymbol{r}')\mathrm{d}\tau' + \iint_{B_2} \frac{b(\boldsymbol{r}')}{\beta(\boldsymbol{r}')}\psi_M^*(\boldsymbol{r}')\mathrm{d}S'\right]\psi_M(\boldsymbol{r})$$

$$-\left[\iint_{B_1} \frac{b(\boldsymbol{r}')}{\alpha(\boldsymbol{r}')}\frac{\partial \psi_M^*(\boldsymbol{r}')}{\partial n'}\mathrm{d}S'\right]\psi_M(\boldsymbol{r}) + C\psi_M(\boldsymbol{r})$$

$$+\int_V G_M(\boldsymbol{r},\boldsymbol{r}') f(\boldsymbol{r}')\mathrm{d}\tau'$$

$$+\iint_{B_2} \frac{b(\boldsymbol{r}')}{\beta(\boldsymbol{r}')} G_M(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S' - \iint_{B_1} \frac{b(\boldsymbol{r}')}{\alpha(\boldsymbol{r}')}\frac{\partial G_M(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}\mathrm{d}S'$$

■ 相容条件

$$\int_V (u\nabla^2 v - v\nabla^2 u)\mathrm{d}\tau = \iint_B \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right)\mathrm{d}S \Leftarrow v = \psi_M(\mathbf{r})$$

$$\int_V \psi_M(\mathbf{r}) f(\mathbf{r})\mathrm{d}\tau + \iint_{B_2} \frac{b(\mathbf{r}')}{\beta(\mathbf{r}')}\psi_M(\mathbf{r}')\mathrm{d}S' - \iint_{B_1} \frac{b(\mathbf{r}')}{\alpha(\mathbf{r}')}\frac{\partial \psi_M(\mathbf{r}')}{\partial n'}\mathrm{d}S' = 0$$

$$u(\mathbf{r}) = C\psi_M(\mathbf{r}) + \int_V G_M(\mathbf{r},\mathbf{r}') f(\mathbf{r}')\mathrm{d}\tau' + \iint_{B_2} \frac{b(\mathbf{r}')}{\beta(\mathbf{r}')} G_M(\mathbf{r},\mathbf{r}')\mathrm{d}S' - \iint_{B_1} \frac{b(\mathbf{r}')}{\alpha(\mathbf{r}')}\frac{\partial G_M(\mathbf{r},\mathbf{r}')}{\partial n'}\mathrm{d}S'$$

注意：

① 在实际物理问题中，相当于共振激发，当激发频率与系统的某一个本征频率相等时，该模式激发无限大，但由于衰减的存在，物理上总是有限大小；

② 在计算过程中，往往取近似计算方法，如波动中$\lambda=-k^2$(k为波数)，在波数中引进衰减因子(虚部，表示波的吸收)

$$G(\boldsymbol{r},\boldsymbol{r}')=\lim_{\varepsilon\to 0}\sum_{m=1}^{\infty}\frac{\psi_m^*(\boldsymbol{r}')\psi_m(\boldsymbol{r})}{\lambda_m-(k+\mathrm{i}\varepsilon)^2}$$

——从复变函数的角度，相当于把实轴上的奇点平移到上、下半平面(后面讨论)。

例1 二维 Helmholtz方程的Green 函数

$$\begin{cases} -(\nabla^2 + k^2)G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}') \\ G|_{\rho=0} < \infty, \quad G|_{\rho=a} = 0 \end{cases}$$

□ 本征函数展开法

$$\begin{cases} -\nabla^2 \psi_m = \lambda_m \psi_m \\ \psi_m|_{\rho=0} < \infty, \ \psi_m|_{\rho=a} = 0 \end{cases}$$

$$\{\psi_{mn}, \lambda_{mn}\} = \left\{ \frac{\sqrt{2}}{aJ_m'(\mu_{mn}a)} J_m\left(\mu_{mn}\frac{\rho}{a}\right) e^{im\varphi}, \left(\frac{\mu_{mn}}{a}\right)^2 \right\}$$

其中 μ_{mn}为 $J_m(x) = 0$ 的第n个正根。

Green 函数为

$$G(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{2}{(\mu_{mn}/a)^2-k^2}\times\frac{J_m(\mu_{mn}\rho'/a)J_m(\mu_{mn}\rho/a)}{a^2\left[J_m'(\mu_{mn})\right]^2}e^{\mathrm{i}m(\varphi-\varphi')}$$

如果$k=\mu_{MN}/a$, 广义Green函数为

$$G(\mathbf{r},\mathbf{r}')=\frac{2J_M(\mu_{MN}\rho'/a)J_M(\mu_{MN}\rho/a)}{a^2\left[J_M'(\mu_{MN})\right]^2}e^{\mathrm{i}M(\varphi-\varphi')}+\sum_{n\neq N}^{\infty}\sum_{m\neq M}^{\infty}\frac{2e^{\mathrm{i}m(\varphi-\varphi')}}{(\mu_{MN}/a)^2-k^2}\frac{J_m(\mu_{mn}\rho'/a)J_m(\mu_{mn}\rho/a)}{a^2\left[J_m'(\mu_{mn})\right]^2}$$

□ 构造方法

$$G(\mathbf{r},\mathbf{r}') = \sum_{m=-\infty}^{\infty} g_m(\rho) e^{\mathrm{i}m(\varphi-\varphi')}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\varphi^2} + k^2 G = -\frac{\delta(\rho,\rho')\delta(\varphi,\varphi')}{\rho}$$

$$\sum_{m=-\infty}^{\infty} e^{\mathrm{i}m(\varphi-\varphi')}\left\{\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}g_m(\rho)}{\mathrm{d}\rho}\right] + \left(k^2 - \frac{m^2}{\rho^2}\right) g_m(\rho)\right\}$$

$$= -\delta(\rho,\rho')\delta(\varphi,\varphi')/\rho$$

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}g_m(\rho)}{\mathrm{d}\rho}\right] + \left(k^2 - \frac{m^2}{\rho^2}\right) g_m(\rho) = -\frac{\delta(\rho,\rho')}{2\pi\rho}$$

■ 齐次方程的通解

$$g_m(\rho) = A_m J_m(k\rho) + B_m N_m(k\rho)$$

■ 构造非齐次方程的解

$$g_m(\rho) = \begin{cases} A_m J_m(k\rho) + B_m N_m(k\rho), & (\rho < \rho' < a) \\ C_m J_m(k\rho) + D_m N_m(k\rho), & (\rho' < \rho < a) \end{cases}$$

$$g_m(\rho)\big|_{\rho=0} < \infty, \quad g_m(\rho)\big|_{\rho=a} = 0$$

$$g_m(\rho)\big|_{\rho=\rho'-0} = g_m(\rho)\big|_{\rho=\rho'+0}$$

$$\left.\frac{\mathrm{d}g_m(\rho)}{\mathrm{d}\rho}\right|_{\rho=\rho'+0} - \left.\frac{\mathrm{d}g_m(\rho)}{\mathrm{d}\rho}\right|_{\rho=\rho'-0} = -\frac{1}{2\pi\rho'}$$

利用关系 $k\rho'[N_m(k\rho')J'_m(k\rho')-J_m(k\rho')N'_m(k\rho')]=-1$

$$g_m(\rho)=\begin{cases}\dfrac{1}{2\pi}\dfrac{\Im(k\rho')}{J_m(ka)}J_m(k\rho), & (\rho<\rho'<a)\\[2ex] \dfrac{1}{2\pi}\dfrac{J_m(k\rho')}{J_m(ka)}\Im(k\rho), & (\rho'<\rho<a)\end{cases}$$

$$\Im(k\rho)\equiv[N_m(ka)J_m(k\rho)-J_m(ka)N_m(k\rho)]$$

$$\Downarrow$$

$$G(\mathbf{r},\mathbf{r}')=\sum_{m=-\infty}^{\infty}g_m(\rho)e^{\mathrm{i}m(\varphi-\varphi')}$$

注意：当波数k满足$J_m(ka)=0$时，发生共振。本征函数展开法能够给出较为明显的结果。

13.3 无限空间的Green函数, 基本解

- **无界空间的 Green 函数：称为方程的基本解**

对有限空间，一般令

$$G = g + G_1;\quad \boldsymbol{L}[g] = \delta(\mathbf{r},\mathbf{r}') \Rightarrow \boldsymbol{L}[G_1] = 0$$

其中：g 为方程的基本解，含有奇点，但是不满足边界条件；G_1 在区域内正则，无奇点，使 G 满足边界条件

$$\left.\left(\alpha G_1 + \beta\frac{\partial G_1}{\partial n}\right)\right|_B = -\left.\left(\alpha g + \beta\frac{\partial g}{\partial n}\right)\right|_B$$

g 容易求得，而 G_1一般用级数法求得，因不包含奇点，级数有比较好的收敛性质

❑三维Laplace 算子的基本解

$$-\nabla^2 g = \delta(\boldsymbol{r}, \boldsymbol{r}')$$

首先求无限空间的本征函数

$$-\nabla^2 \psi_k(\boldsymbol{r}) = k^2 \psi_k(\boldsymbol{r})$$

本征值构成连续谱 $\boldsymbol{k}^2$ 相应的本征函数为

$$\psi_k(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

故g为(连续谱，求和变化成积分)

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{(2\pi)^3}\sum_k\frac{e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2}$$

三维k空间积分

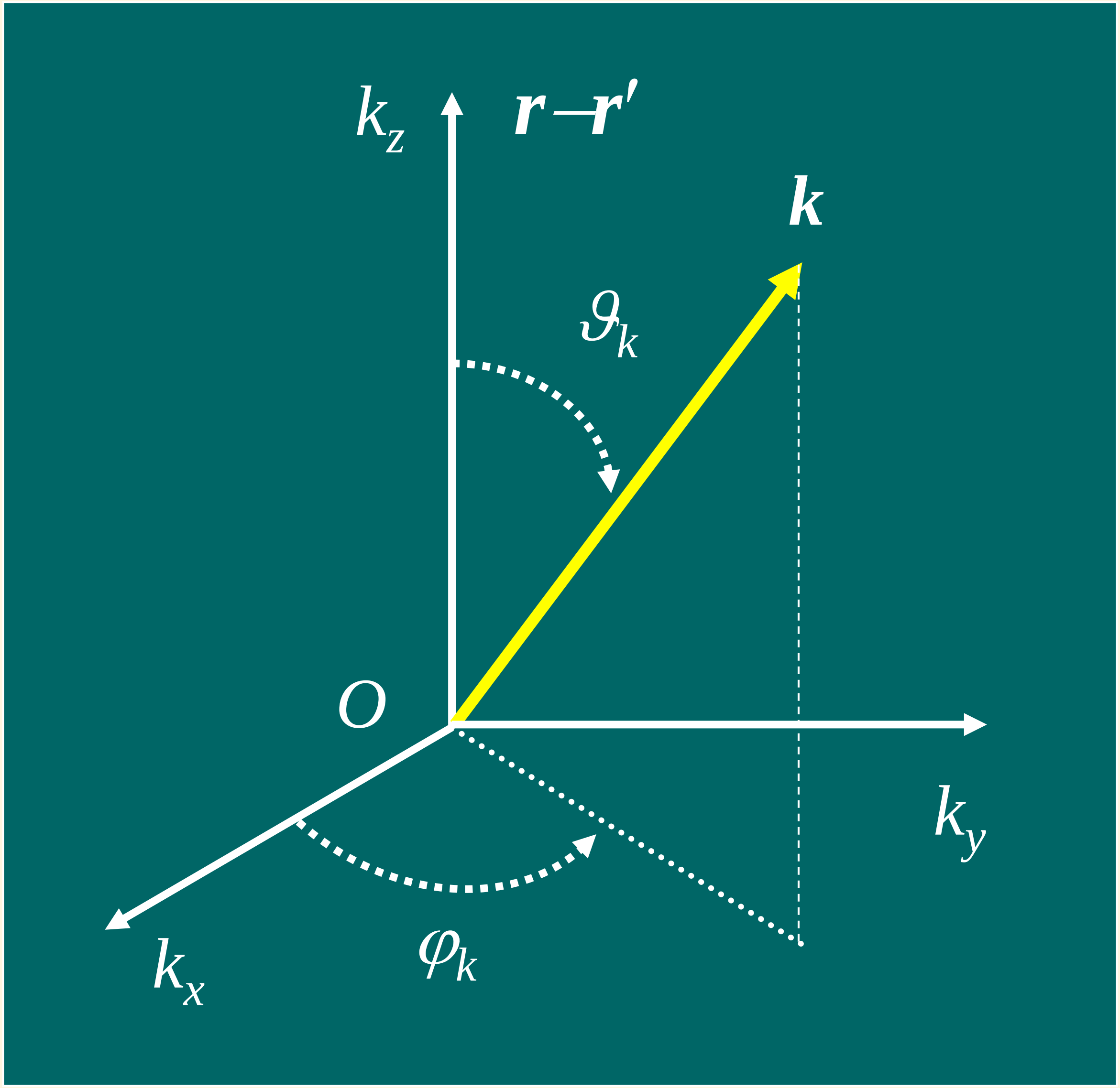
$$=\frac{1}{(2\pi)^3}\int_0^{\pi}\int_0^{\infty}\frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta_k}}{k^2}k^2 2\pi\sin\vartheta_k\mathrm{d}\vartheta_k\mathrm{d}k$$

$$=\frac{1}{2\pi^2}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\int_0^{\infty}\frac{\sin k|\boldsymbol{r}-\boldsymbol{r}'|}{k}\mathrm{d}k=\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')=k|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta_k$$

即Laplace算子的基本解为

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$



k_z
$\boldsymbol{r}-\boldsymbol{r}'$
$\boldsymbol{k}$
ϑ_k
O
k_y
φ_k
k_x

❑Helmholtz 算子的基本解

$$-(\nabla^2+\lambda)g=\delta(\boldsymbol{r},\boldsymbol{r}')$$

用 Fourier 积分法求之：令

$$g(\boldsymbol{r},\boldsymbol{r}')=\int g(\boldsymbol{k},\boldsymbol{r}')e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

代入方程

$$\int(k^2-\lambda)g(\boldsymbol{k},\boldsymbol{r}')e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}=\delta(\boldsymbol{r},\boldsymbol{r}')$$

$$\Downarrow$$

$$g(\boldsymbol{k},\boldsymbol{r}')=\frac{1}{(2\pi)^3(k^2-\lambda)}\int\delta(\boldsymbol{r},\boldsymbol{r}')e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}=\frac{e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}}{(2\pi)^3(k^2-\lambda)}$$

$$\Downarrow$$

$$\boxed{g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{(2\pi)^3}\int\frac{e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2-\lambda}\mathrm{d}^3\boldsymbol{k}}$$

①λ=0，Laplace算子情况，已讨论；

②$\lambda<0$，令$\lambda=-\kappa^2$

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{(2\pi)^3}\int\frac{e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2-\lambda}\mathrm{d}^3\boldsymbol{k}$$

$$=\frac{1}{(2\pi)^3}\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta_k)}{k^2+\kappa^2}k^2\sin\vartheta_k\mathrm{d}k\mathrm{d}\vartheta_k\mathrm{d}\varphi_k$$

$$=-\frac{1}{(2\pi)^2}\int_0^{\infty}k^2\left[\int_0^{\pi}\frac{\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta_k)}{k^2+\kappa^2}\mathrm{d}\cos\vartheta_k\right]\mathrm{d}k$$

$$=\frac{1}{(2\pi)^2\mathrm{i}\,|\boldsymbol{r}-\boldsymbol{r}'|}\int_0^{\infty}k\left[\frac{\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)-\exp(-\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2+\kappa^2}\right]\mathrm{d}k$$

$$=\frac{1}{(2\pi)^2 2\mathrm{i}\,|\boldsymbol{r}-\boldsymbol{r}'\}}\int_{-\infty}^{\infty}k\left[\frac{\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)-\exp(-\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2+\kappa^2}\right]\mathrm{d}k$$

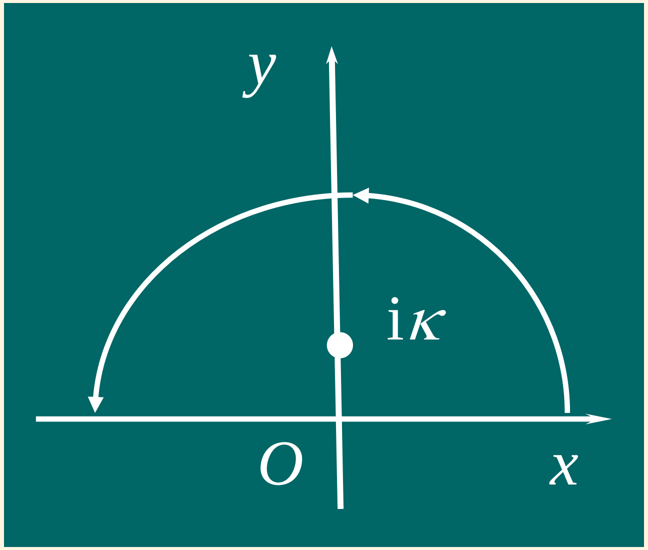
$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{(2\pi)^2 2\mathrm{i}\,|\boldsymbol{r}-\boldsymbol{r}'\}}(I_+-I_-)$$

$$I_\pm\equiv\int_{-\infty}^{\infty}\frac{k\exp(\pm\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'\,|)}{k^2+\kappa^2}\mathrm{d}k$$

奇点 $k=\pm\mathrm{i}\kappa$ 在虚轴上：①对积分I_+作上半平面围道，仅包含$k=+\mathrm{i}\kappa$，因此

$$I_+=2\pi\mathrm{i}\mathrm{Res}\left\{\frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}k}{(k^2+\kappa^2)},\mathrm{i}\kappa\right\}$$

$$=\pi\mathrm{i}e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}$$



②对积分I_-作下半平面围道，仅包含$k=-\mathrm{i}\kappa$，因此

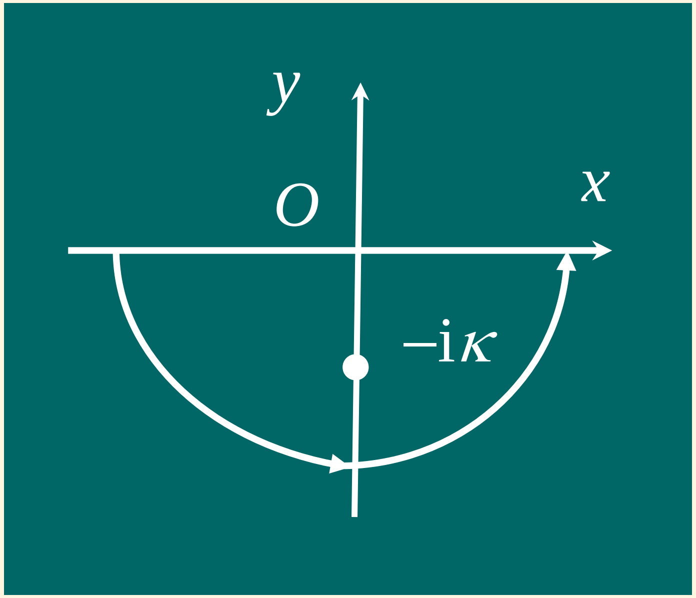
$$I_- = -2\pi\mathrm{i}\,\mathrm{Res}\left\{\frac{e^{-\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}k}{(k^2+\kappa^2)}, -\mathrm{i}\kappa\right\} = -\pi\mathrm{i}e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}$$

——下半平面围道积分反向，故加负号

因此，Green 函数为

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{(I_+ - I_-)}{(2\pi)^2 2\mathrm{i}\,|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$= \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}$$



——该Green函数描述短程相互作用势，在量子力学的散射理论中有重要应用

③$\lambda>0$，令$\lambda=q^2$

角度部分积分后

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{2(2\pi)^2\mathrm{i}\,|\boldsymbol{r}-\boldsymbol{r}'|}(I_+-I_-)$$

$$I_\pm\equiv\int_{-\infty}^{\infty}\frac{k\exp(\pm\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-q^2}\mathrm{d}k$$

奇点 $k_{1,2}=\pm q$ 在实轴上。为了避免奇点在实轴上，我们引进小的虚部：

$$I_\pm=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{k\exp(\pm\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-(q+\mathrm{i}\varepsilon)^2}\mathrm{d}k$$

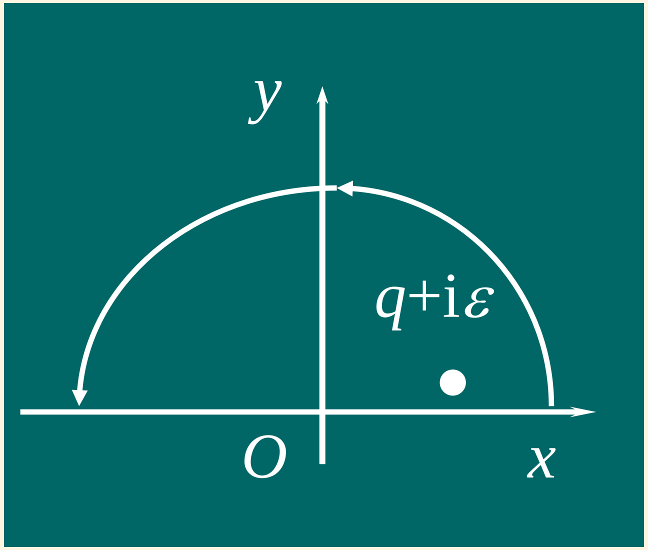
于是，二个奇点分别为$k_1=q+\mathrm{i}\varepsilon$和$k_2=-q-\mathrm{i}\varepsilon$

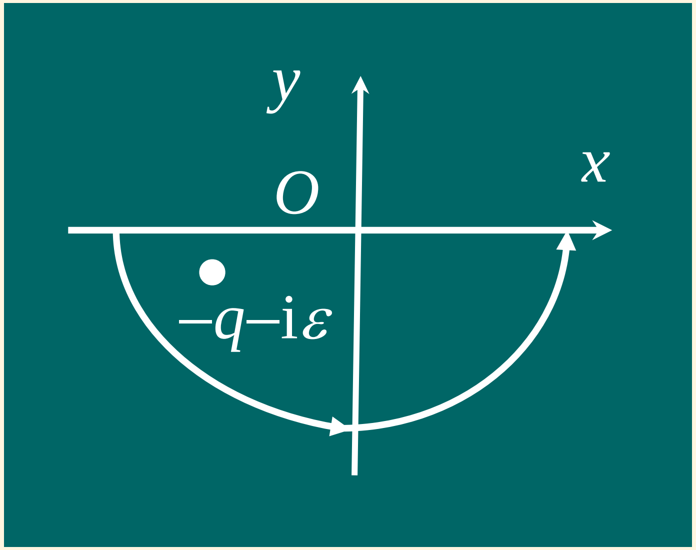
对积分 I_+, 取上半平面的围道,积分为

$$I_+ = 2\pi\mathrm{i}\lim_{\varepsilon\to 0}\mathrm{Res}\left[\frac{k\exp(+\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-(q+\mathrm{i}\varepsilon)^2}, q+\mathrm{i}\varepsilon\right] = \pi\mathrm{i}\exp(\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|)$$

对积分 I_-, 取下半平面的围道,积分为

$$I_- = -2\pi\mathrm{i}\lim_{\varepsilon\to 0}\mathrm{Res}\left[\frac{k\exp(-\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-(q+\mathrm{i}\varepsilon)^2}, -(q+\mathrm{i}\varepsilon)\right] = -\pi\mathrm{i}\exp(\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|)$$





于是，Green函数为

$$g^{+}(\boldsymbol{r},\boldsymbol{r}') \equiv \frac{(I_{+}-I_{-})}{2(2\pi)^{2}\mathrm{i}\,|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\exp(\mathrm{i}q\,|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}$$

- 也可以这样引进小虚部

$$I_{\pm} = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{k\exp(\pm\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^{2}-(q-\mathrm{i}\varepsilon)^{2}}\mathrm{d}k$$

于是，二个奇点分别为$k_1=q-\mathrm{i}\varepsilon$和$k_2=-q+\mathrm{i}\varepsilon$

对积分 I_+, 取上半平面的围道,积分为

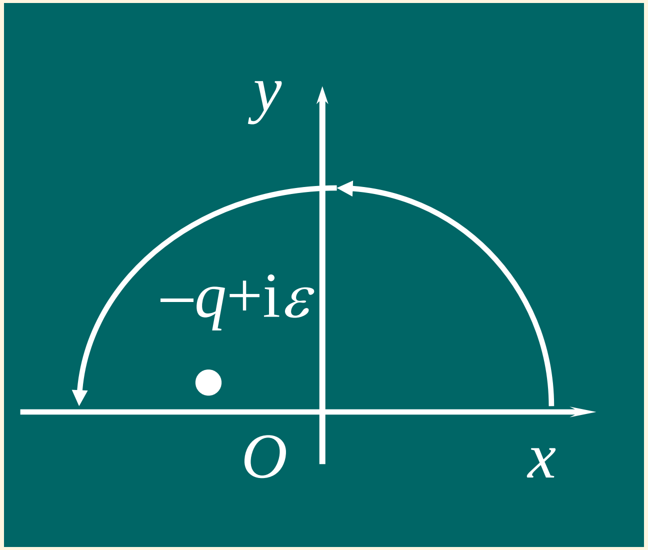
$$I_{+} = 2\pi\mathrm{i}\lim_{\varepsilon\to 0}\mathrm{Res}\left[\frac{k\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^{2}-(q-\mathrm{i}\varepsilon)^{2}},-q+\mathrm{i}\varepsilon\right] = \pi\mathrm{i}\exp(-\mathrm{i}q\,|\boldsymbol{r}-\boldsymbol{r}'|)$$

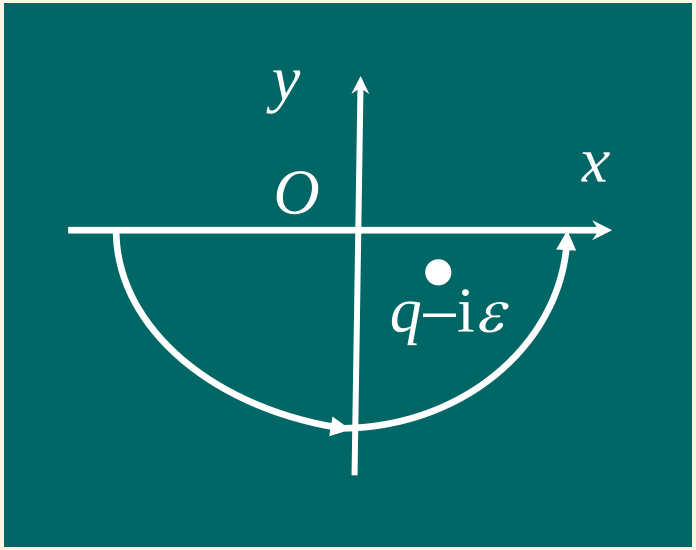
对积分 I_-, 取下半平面的围道,积分为

$$I_- = -2\pi \mathrm{i} \lim_{\varepsilon\to 0} \mathrm{Res}\left[\frac{k\exp(-\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-(q-\mathrm{i}\varepsilon)^2}, (q-\mathrm{i}\varepsilon)\right] = -\pi\mathrm{i}\exp(-\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|)$$

于是，Green函数为

$$g^-(\boldsymbol{r},\boldsymbol{r}') \equiv \frac{(I_+ - I_-)}{2(2\pi)^2\mathrm{i}|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\exp(-\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$





$$g^{+}(\boldsymbol{r},\boldsymbol{r}')=\frac{\exp(+\mathrm{i}q\,|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}$$

——辐射Green函数，波由r'点发出，以球面波的形式向外传播

$$g^{-}(\boldsymbol{r},\boldsymbol{r}')=\frac{\exp(-\mathrm{i}q\,|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}$$

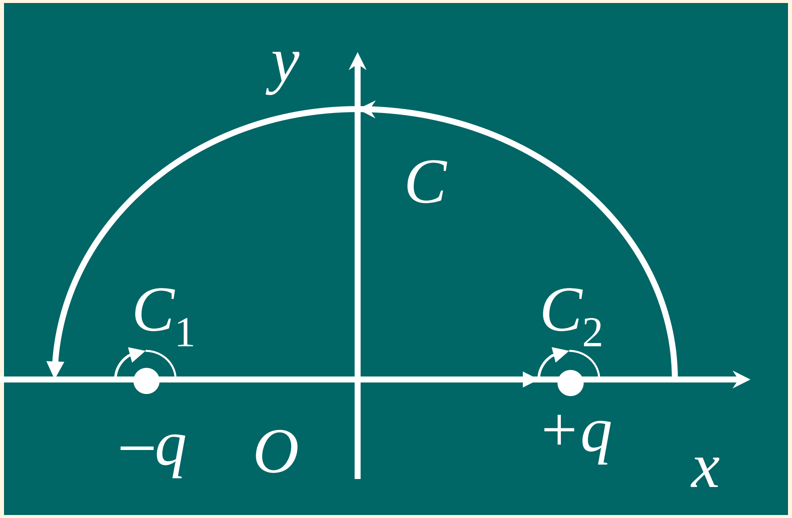
——接收Green函数，波由无限远处发出，向r'点以球面波的形式汇聚

出现 “$+q$” 和 “$-q$” 是因为 Helmholtz 方程中 q 以 q^2 出现。

□ 如果不引进小虚部会怎么样?

$$I_{\pm} \equiv \int_{-\infty}^{\infty} \frac{k\exp(\pm \mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-q^2}\mathrm{d}k$$

二个奇点 $k_{1,2}=\pm q$ 在实轴上。对积分 I_+，取上半平面的围道，积分为



$$I_+ = \pi\mathrm{i}\mathrm{Res}\left[\frac{k\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-q^2}, q\right] + \pi\mathrm{i}\mathrm{Res}\left[\frac{k\exp(\mathrm{i}k\,|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-q^2}, -q\right]$$

$$= \frac{\pi\mathrm{i}}{2}[\exp(\mathrm{i}q\,|\boldsymbol{r}-\boldsymbol{r}'|) + \exp(-\mathrm{i}q\,|\boldsymbol{r}-\boldsymbol{r}'|)]$$

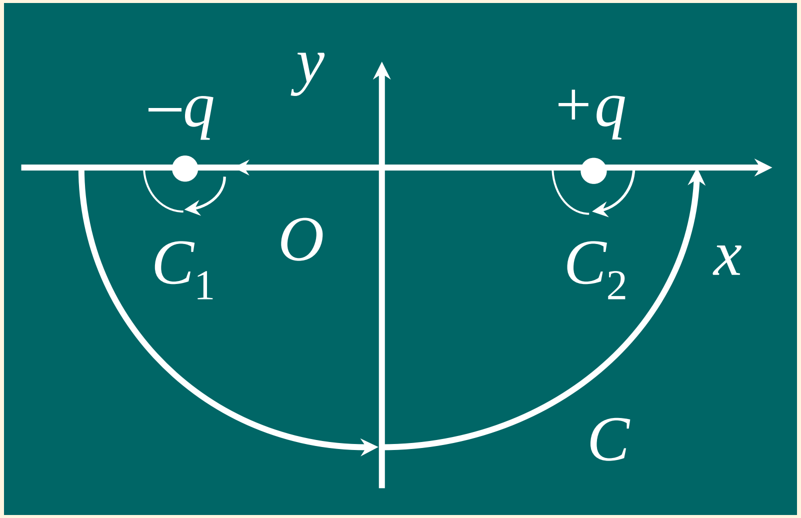
对积分 I_-, 取 下 ~~上半平面~~的围道, 积分为

$$I_- = -\pi \mathrm{i}\,\mathrm{Res}\left[\frac{k\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-q^2}, q\right] - \pi \mathrm{i}\,\mathrm{Res}\left[\frac{k\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2-q^2}, -q\right]$$

$$= -\frac{\pi \mathrm{i}}{2}\left[\exp(\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|)+\exp(-\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|)\right]$$

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{\mathrm{e}^{\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|}+\mathrm{e}^{-\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|}}{8\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$= \frac{1}{2}\left[g^{+}(\boldsymbol{r},\boldsymbol{r}')+g^{-}(\boldsymbol{r},\boldsymbol{r}')\right]$$



——从数学的角度，满足Green函数方程，但不满足Sommerfeld辐射条件.

□一维Helmholtz方程

$$-\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q^2\right)g(x,x')=\delta(x,x')$$

$$g(x,x')=\int_{-\infty}^{\infty}g(k)e^{\mathrm{i}kx}\,\mathrm{d}k$$

$$\int_{-\infty}^{\infty}g(k)\left(-k^2+q^2\right)e^{\mathrm{i}kx}\,\mathrm{d}k=-\delta(x,x')$$

$$g(k)=\frac{e^{-\mathrm{i}kx'}}{2\pi(k^2-q^2)};g(x,x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}k(x-x')}}{k^2-q^2}\mathrm{d}k$$

——奇点在实轴上：仿三维情况讨论

- **首先考虑$q^2<0$: $q=\mathrm{i}\sigma$**

(1) $x > x'$ **(上半平面围道)**

$$g(x,x') = \frac{1}{2\pi} 2\pi \mathrm{i}\,\mathrm{Res}\left[\frac{e^{\mathrm{i}k(x-x')}}{k^2+\sigma^2}, \mathrm{i}\sigma\right] = \frac{1}{2\sigma} e^{-\sigma(x-x')}$$

(2) $x' > x$ **(下半平面围道)**

$$g(x,x') = -\frac{1}{2\pi} 2\pi \mathrm{i}\,\mathrm{Res}\left[\frac{e^{\mathrm{i}k(x-x')}}{k^2+\sigma^2}, -\mathrm{i}\sigma\right] = \frac{1}{2\sigma} e^{-\sigma(x'-x)}$$

$$\boxed{g(x,x') = \frac{1}{2\sigma} e^{-\sigma|x-x'|}}$$

- **其次考虑$q^2>0$的情况**

(1) $q \Rightarrow q + \mathrm{i}\varepsilon;\ x > x'$ **(上半平面围道)**

$$g(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\mathrm{i}k(x-x')}}{k^2 - (q+\mathrm{i}\varepsilon)^2}\mathrm{d}k$$

$$= \frac{1}{2\pi} 2\mathrm{i}\pi \mathrm{Res}\left[\frac{e^{\mathrm{i}k(x-x')}}{k^2-(q+\mathrm{i}\varepsilon)^2}, q+\mathrm{i}\varepsilon\right] = \mathrm{i}\frac{e^{\mathrm{i}q(x-x')}}{2q}$$

(2) $q \Rightarrow q + \mathrm{i}\varepsilon;\ x' > x$ **(下半平面围道)**

$$g(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}k(x'-x)}}{k^2 - (q+\mathrm{i}\varepsilon)^2}\mathrm{d}k$$

$$= -\frac{1}{2\pi} 2\mathrm{i}\pi \mathrm{Res}\left[\frac{e^{-\mathrm{i}k(x'-x)}}{k^2-(q+\mathrm{i}\varepsilon)^2}, -q-\mathrm{i}\varepsilon\right] = \mathrm{i}\frac{e^{\mathrm{i}q(x'-x)}}{2q}$$

所以，辐射Green函数为

$$g^{+}(x,x') = \mathrm{i}\frac{e^{\mathrm{i}q|x-x'|}}{2q}$$

(3) $q \Rightarrow q-\mathrm{i}\varepsilon;\ x > x'$ **(上半平面围道)**

$$g(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}k(x-x')}}{k^2-(q-\mathrm{i}\varepsilon)^2}\mathrm{d}k$$

$$= \frac{1}{2\pi}2\mathrm{i}\pi\mathrm{Res}\left[\frac{e^{\mathrm{i}k(x-x')}}{k^2-(q-\mathrm{i}\varepsilon)^2}, -q+\mathrm{i}\varepsilon\right] = -\mathrm{i}\frac{e^{-\mathrm{i}q(x-x')}}{2q}$$

(4) $q \Rightarrow q-\mathrm{i}\varepsilon;\ x' > x$ **(下半平面围道)**

$$g(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}k(x'-x)}}{k^2-(q-\mathrm{i}\varepsilon)^2}\mathrm{d}k$$

$$= -\frac{1}{2\pi}2\mathrm{i}\pi\mathrm{Res}\left[\frac{e^{-\mathrm{i}k(x'-x)}}{k^2-(q-\mathrm{i}\varepsilon)^2}, q-\mathrm{i}\varepsilon\right] = -\mathrm{i}\frac{e^{-\mathrm{i}q(x'-x)}}{2q}$$

所以，接收Green函数为

$$g^{-}(x,x') = -\mathrm{i}\frac{e^{-\mathrm{i}q|x-x'|}}{2q}$$

❑二维Helmholtz方程

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+q^2\right)g(x,y;x',y') = \delta(x,x')\delta(y,y')$$

$$\Downarrow$$

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{(2\pi)^2}\int\frac{e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2-q^2}\mathrm{d}^2\boldsymbol{k}$$

$$\Downarrow$$

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{(2\pi)^2}\int_0^{\infty}\frac{k}{k^2-q^2}\left[\int_0^{2\pi}e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|\cos\varphi_k}\mathrm{d}\varphi_k\right]\mathrm{d}k$$

①辐射Green函数

$$J_0(k|\boldsymbol{r}-\boldsymbol{r}'|)=\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|\cos\varphi}\mathrm{d}\varphi$$

$$2J_0(x)=H_0^{(1)}(x)-H_0^{(1)}(xe^{\mathrm{i}\pi})$$

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{k}{k^2-q^2}H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}k$$

$$H_0^{(1)}(x)\approx\frac{2\mathrm{i}}{\pi}\ln\frac{x}{2},(x\to 0)$$

$$H_0^{(1)}(x)\approx\sqrt{\frac{2}{\pi x}}\exp\left[\mathrm{i}\left(x-\frac{\pi}{4}\right)\right],(x\to\infty)$$

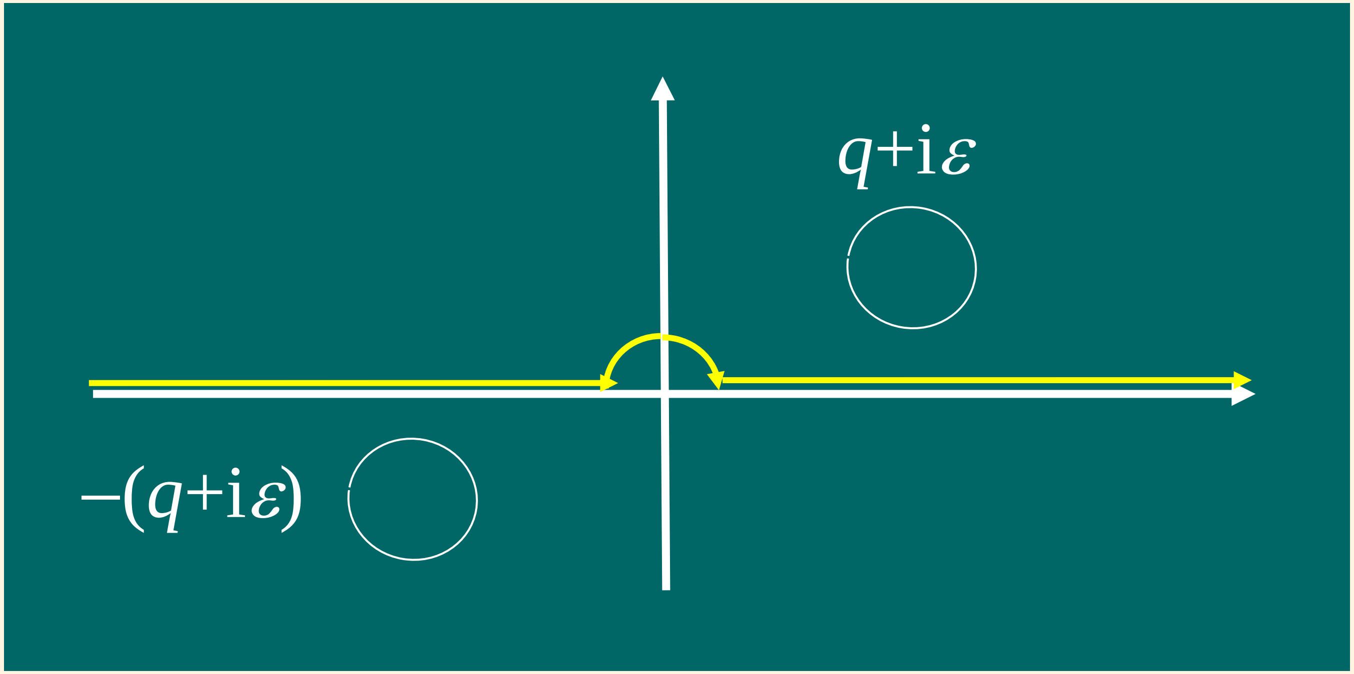
分析：*k*=0, 支点； $k_{1,2}=\pm q$, 实轴上的一阶极点

引进小的虚部: $k_{1,2}=\pm(q+\mathrm{i}\varepsilon)$, 积分围道取上半平面, 用半径为$\delta$的小半圆包围原点积分

- **支点贡献：在小圆上**

支点贡献为零

$$k\mathrm{d}k \sim \delta^2 \ln\frac{\delta}{2}\mathrm{d}\varphi_\delta \to 0,(\delta \to 0)$$



■ 极点贡献

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi}2\pi\mathrm{i}\mathrm{Res}\left[\frac{k}{k^2-q^2}H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|),q+\mathrm{i}\varepsilon\right]$$

$$=\frac{\mathrm{i}}{4}H_0^{(1)}(q|\boldsymbol{r}-\boldsymbol{r}'|)$$

$$\Downarrow$$

$$g^+(\boldsymbol{r},\boldsymbol{r}')=\frac{\mathrm{i}}{4}H_0^{(1)}(q|\boldsymbol{r}-\boldsymbol{r}'|)$$

②接收Green函数

$$2J_0(x)=H_0^{(2)}(xe^{\mathrm{i}\pi})-H_0^{(2)}(x)$$

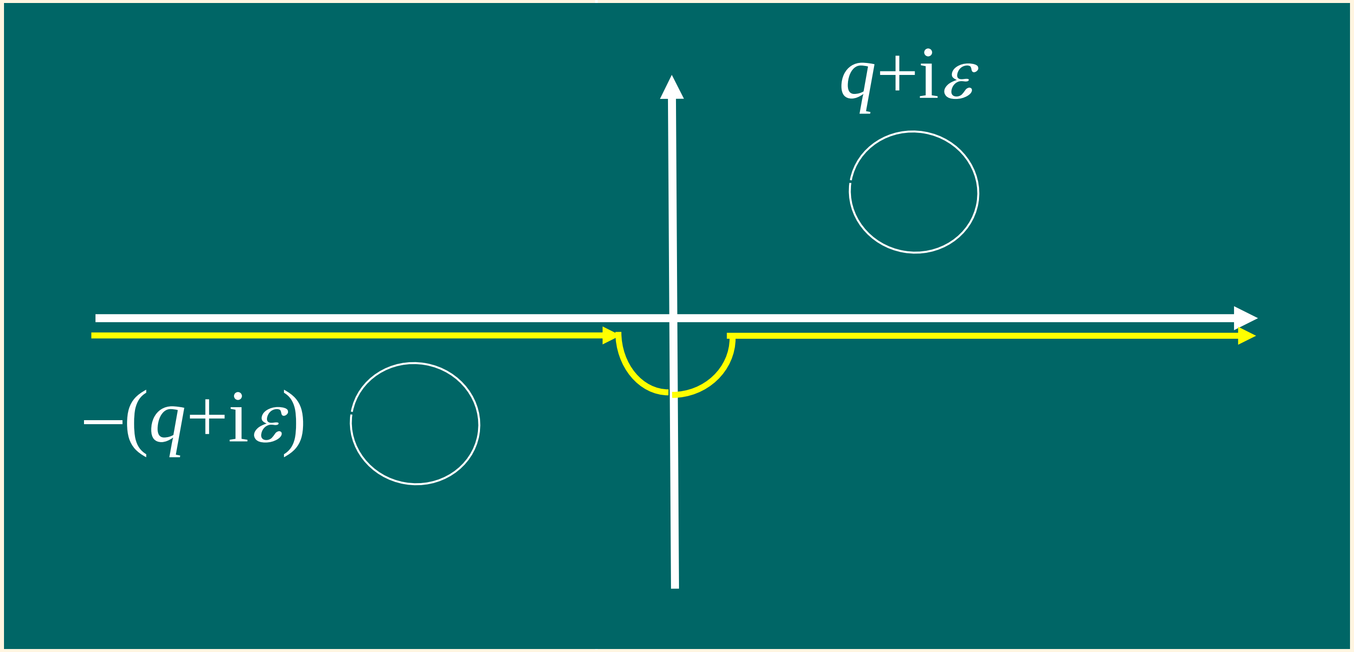
$$\Downarrow$$

$$g^-(\boldsymbol{r},\boldsymbol{r}')=-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{k}{k^2-q^2}H_0^{(2)}(k|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}k$$

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} \exp\left[-\mathrm{i}\left(x - \frac{\pi}{4}\right)\right], (x \to \infty)$$

根据以上展开，积分围道取下半平面

$$g^{-}(\boldsymbol{r}, \boldsymbol{r}') = \frac{\mathrm{i}}{4} H_0^{(2)}(q\,|\boldsymbol{r} - \boldsymbol{r}'\,|)$$



③如果$q^2<0$: $q=\mathrm{i}\sigma$

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{k}{k^2+\sigma^2}H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}k$$

$k_{1,2}=\pm\mathrm{i}\sigma$, 二个一阶极点位于虚轴上的，取上半平面围道

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi}2\pi\mathrm{i}\mathrm{Res}\left[\frac{k}{k^2+\sigma^2}H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|),\mathrm{i}\sigma\right]$$

$$=\frac{\mathrm{i}}{4}H_0^{(1)}(\mathrm{i}\sigma|\boldsymbol{r}-\boldsymbol{r}'|)=\frac{1}{2\pi}K_0(\sigma|\boldsymbol{r}-\boldsymbol{r}'|)$$

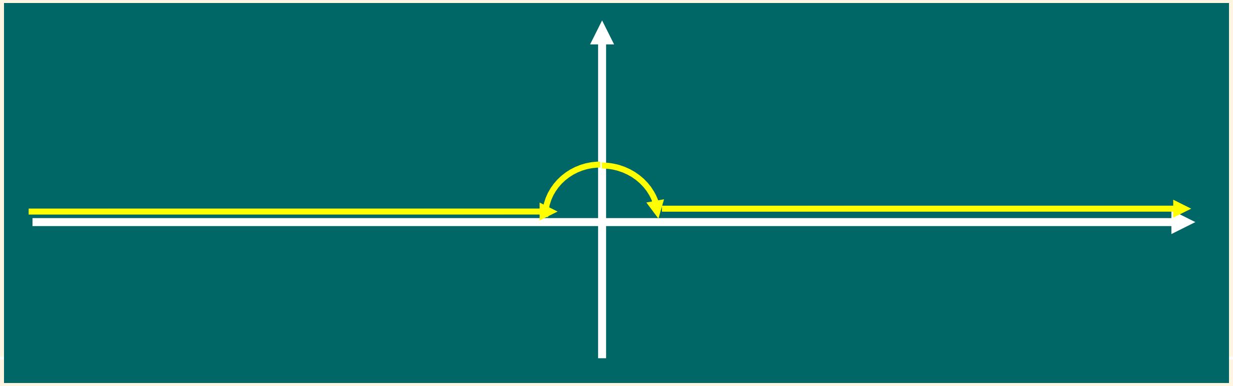
$$K_0(x)=\frac{\mathrm{i}\pi}{2}H_0^{(1)}(\mathrm{i}x)$$

■二维Laplace方程的基本解

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{(2\pi)^2}\int\frac{e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2}\mathrm{d}^2\boldsymbol{k}$$

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{k}H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}k$$

积分围道取上半平面，用半径为ε的小半圆包围原点



$$g(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{4\pi}\int_{\varepsilon}\frac{1}{\varepsilon e^{\mathrm{i}\phi}}H_0^{(1)}(\varepsilon e^{\mathrm{i}\phi}\,|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{i}\varepsilon e^{\mathrm{i}\phi}\mathrm{d}\phi$$

$$= -\frac{\mathrm{i}}{4\pi}\int_{\pi}^{0}H_0^{(1)}(\varepsilon e^{\mathrm{i}\phi}\,|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}\phi \Leftarrow H_0^{(1)}(x)\approx \mathrm{i}\frac{2}{\pi}\ln x$$

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{\mathrm{i}}{4\pi}\int_{0}^{\pi}H_0^{(1)}(\varepsilon e^{\mathrm{i}\phi}\,|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}\phi$$

$$= \boxed{-\frac{1}{2\pi^2}\int_0^{\pi}\ln\left(\varepsilon e^{\mathrm{i}\phi}\right)\mathrm{d}\phi} + \frac{1}{2\pi}\ln\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

常数
(为什么)

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2\pi}\ln\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

■ 镜像法求 Green 函数

有限空间

$$\boldsymbol{L}[G(\boldsymbol{r},\boldsymbol{r}')] = \delta(\boldsymbol{r},\boldsymbol{r}'),(\boldsymbol{r},\boldsymbol{r}' \in V)$$

$$\boldsymbol{B}[G(\boldsymbol{r},\boldsymbol{r}')] = 0,(\boldsymbol{r} \in B,\boldsymbol{r}' \in V + B)$$

$$\Uparrow$$

$$G(\boldsymbol{r},\boldsymbol{r}') = g(\boldsymbol{r},\boldsymbol{r}') + G_1(\boldsymbol{r},\boldsymbol{r}')$$

$$\Downarrow$$

$$\boldsymbol{L}[g(\boldsymbol{r},\boldsymbol{r}')] = \delta(\boldsymbol{r},\boldsymbol{r}')$$

$$\boldsymbol{L}[G_1(\boldsymbol{r},\boldsymbol{r}')] = 0; \boldsymbol{B}[G_1(\boldsymbol{r},\boldsymbol{r}')] = -\boldsymbol{B}[g(\boldsymbol{r},\boldsymbol{r}')]$$

下面以具体例子，用镜像法来求 G_1

例1 上半空间Laplace方程第一、二 边值问题的 Green 函数

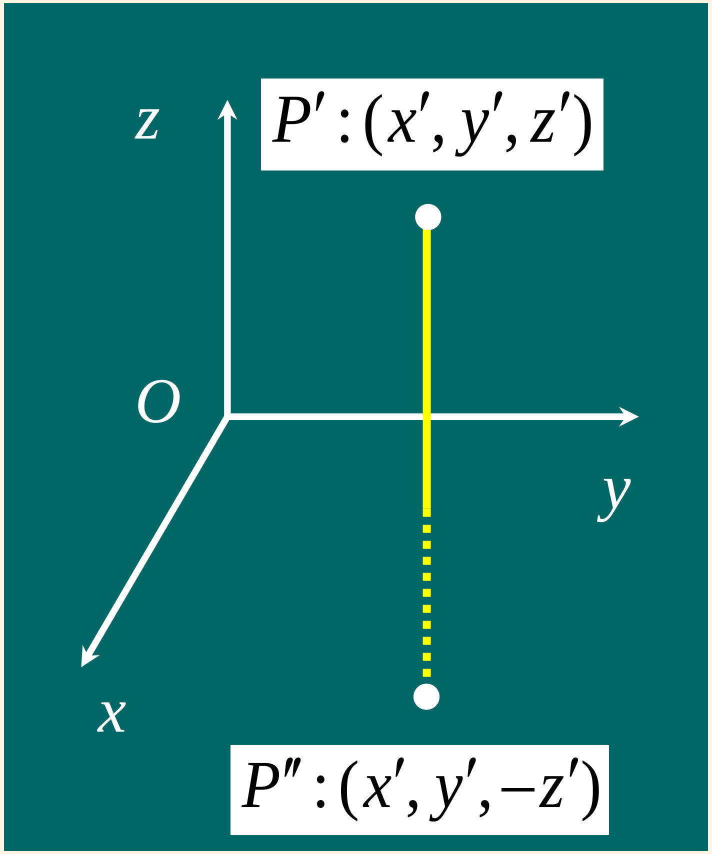
$$\begin{cases} -\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'), \quad z>0 \\ G|_{z=0} = 0 \end{cases}$$

解：(1)Laplace方程的基本解为

$$g = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

(2)在边界 z=0 上的值

$$g|_{z=0} = \frac{1}{4\pi\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + z'^2}}$$



(3)为了满足边界条件，必须选择 G_1 使

$$G_1|_{z=0} = -g|_{z=0} = -\frac{1}{4\pi\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + z'^2}}$$

并且在上半空间满足 Laplace 方程。于是取下半空间的镜像点 $P'':(x',y',-z')$，在镜像点放负点源产生的场

$$G_1 = -\frac{1}{4\pi\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + (z+z')^2}}$$

因此， Green 函数为

$$G = \frac{1}{4\pi}\left[\frac{1}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + (z-z')^2}} - \frac{1}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + (z+z')^2}}\right]$$

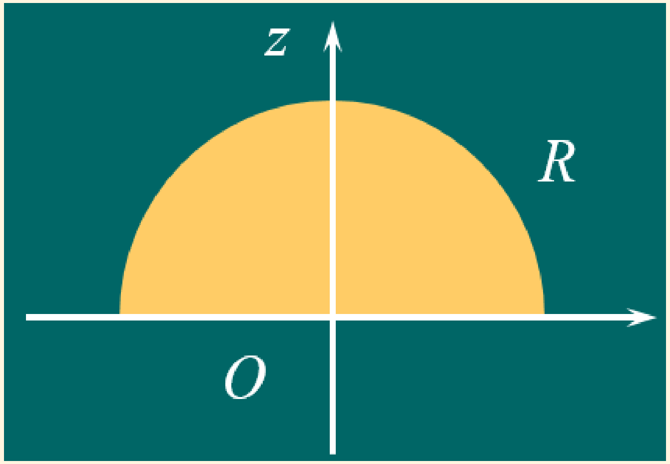
第一类边值问题

$$\begin{cases}\nabla^2 u(\boldsymbol{r}) = f(\boldsymbol{r}), & z>0 \\ u|_{z=0} = u_0(x,y) \end{cases} ;\ f(\boldsymbol{r})=0$$

对有限空间 V，第一类边值问题的积分解为

$$u(\boldsymbol{r}) = \int_V f(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}\tau' - \iint_B \left(u\frac{\partial G}{\partial n'} - G\frac{\partial u}{\partial n'} \right)\mathrm{d}S'$$

对上半平面问题，上式如何处理？作半径为 $R\to\infty$ 的大半球，底面 D 在 xOy 平面上，且覆盖所有 $u_0(x,y)\neq 0$ 的区域.



设B:半球球面S_R+D，V为S_R与D包围的有限空间，于是

$$u(\mathbf{r}) = -\iint_D \left(u\frac{\partial G}{\partial n'} - G\frac{\partial u}{\partial n'} \right) \mathrm{d}S' - \iint_{S_R} \left(u\frac{\partial G}{\partial n'} - G\frac{\partial u}{\partial n'} \right) \mathrm{d}S'$$

$$= -\iint_D u_0(x', y') \frac{\partial G}{\partial n'}\bigg|_{z'=0} \mathrm{d}x'\mathrm{d}y' - \iint_{S_R} \left(u\frac{\partial G}{\partial n'} - G\frac{\partial u}{\partial n'} \right) \mathrm{d}S'$$

注意：大球面上的u仍然是未知的，但如果

$$I_R \equiv \lim_{R\to\infty} \iint_{S_R} \left(u\frac{\partial G}{\partial n'} - G\frac{\partial u}{\partial n'} \right) \mathrm{d}S' \to 0$$

$$\Downarrow$$

$$u(\mathbf{r}) = -\iint_D u_0(x', y') \frac{\partial G}{\partial n'}\bigg|_{z'=0} \mathrm{d}x'\mathrm{d}y'$$

分析：在大球面上，如果 $\lim\limits_{R\to\infty} u \to \lim\limits_{R\to\infty} 1/R \to 0$

$$\iint_{S_R} G\frac{\partial u}{\partial n'}\mathrm{d}S' \sim G\frac{\partial u}{\partial R}4\pi R^2 \sim G \to 0$$

$$\iint_{S_R} u\frac{\partial G}{\partial n'}\mathrm{d}S' \sim u\frac{\partial G}{\partial R}4\pi R^2 \sim u \to 0$$

故只要至少

$$\lim_{R\to\infty} u \to \lim_{R\to\infty} 1/R \to 0 \quad\Longrightarrow\quad I_R \to 0$$

于是

$$u(x,y,z) = -\iint_D u_0(x',y')\left.\frac{\partial G}{\partial n'}\right|_{z'=0}\mathrm{d}x'\mathrm{d}y'$$

注意到 $\boldsymbol{n}' = (0, 0, -1)$

$$\left.\frac{\partial G}{\partial n'}\right|_{z'=0} = \boldsymbol{n}' \cdot \nabla G\,|_{z'=0} = -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = -\frac{z}{2\pi[|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + z^2]^{3/2}}$$

$$u(x, y, z) = \frac{z}{2\pi}\iint_D \frac{u_0(x', y')\mathrm{d}x'\mathrm{d}y'}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}$$

注意：上式在验证边界条件时，不能直接把z=0代入，而应该利用Dirac Delta函数序列关系

$$\lim_{z\to 0}\frac{1}{2\pi}\frac{z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} = \delta(x-x')\delta(y-y')$$

$$\delta_z(x,y;x',y') \equiv \frac{1}{2\pi}\frac{z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

(1)证明积分为1 用极坐标积分

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_z(x,y,0,0)\mathrm{d}x\mathrm{d}y = \frac{z}{2\pi}\int_0^{\infty}\int_0^{2\pi}\frac{\rho\mathrm{d}\rho\mathrm{d}\varphi}{(\rho^2+z^2)^{3/2}} = 1$$

(2)与连续函数的积分

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_z(x,y;x',y')f(x,y)\mathrm{d}x\mathrm{d}y = f(x',y')$$

$$\frac{z}{2\pi}\lim_{z\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x,y)}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\mathrm{d}x\mathrm{d}y$$

$$= \frac{z}{2\pi}\lim_{z\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\alpha+x',\beta+y')}{(\alpha^2+\beta^2+z^2)^{3/2}}\mathrm{d}x\mathrm{d}y$$

在极坐标下

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_z(x,y;x',y')f(x,y)\mathrm{d}x\mathrm{d}y$$

$$=\frac{z}{2\pi}\lim_{z\to 0}\int_0^{2\pi}\int_0^{\infty}\frac{f(\rho\cos\varphi+x',\rho\cos\varphi+y')}{(\rho^2+z^2)^{3/2}}\rho\mathrm{d}\rho\mathrm{d}\varphi$$

$$=\frac{z}{2\pi}\lim_{z\to 0}\int_0^{2\pi}\int_z^{\infty}\frac{f(\rho'\cos\varphi+x',\rho'\cos\varphi+y')}{\rho'^3}\rho'\mathrm{d}\rho'\mathrm{d}\varphi$$

$$\rho^2+z^2=\rho'^2 \Rightarrow \rho\approx\rho',(z\to 0)$$

积分主要贡献是零点

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_z(x,y;x',y')f(x,y)\mathrm{d}x\mathrm{d}y\approx f(x',y')\frac{z}{2\pi}\lim_{z\to 0}\int_0^{2\pi}\int_z^{\infty}\frac{1}{\rho'^3}\rho'\mathrm{d}\rho'\mathrm{d}\varphi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_z(x,y;x',y')f(x,y)\mathrm{d}x\mathrm{d}y=f(x',y')$$

■ Fourier积分方法?

$$u(x,y,z)=\iint u(k_x,k_y,z)\exp[\mathrm{i}(k_x x+k_y y)]\mathrm{d}k_x\mathrm{d}k_y$$

$$\Downarrow$$

$$-(k_x^2+k_y^2)u(k_x,k_y,z)+\frac{\mathrm{d}^2u(k_x,k_y,z)}{\mathrm{d}z^2}=0$$

$$\Downarrow$$

$$u(k_x,k_y,z)=A(k_x,k_y)\exp\left(-\sqrt{k_x^2+k_y^2}\,z\right)$$

$$\Downarrow$$

角谱

传播子

$$u(x,y,z)=\iint A(k_x,k_y)\exp\left(-\sqrt{k_x^2+k_y^2}\,z\right)\exp[\mathrm{i}(k_x x+k_y y)]\mathrm{d}k_x\mathrm{d}k_y$$

边界条件

$$u(x,y,0)=\iint A(k_x,k_y)\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y=u_0(x,y)$$

$$A(k_x,k_y)=\frac{1}{(2\pi)^2}\iint u_0(x',y')\exp[-\mathrm{i}(k_x x'+k_y y')]\mathrm{d}x'\mathrm{d}y'$$

$$u(x,y,z)=\frac{1}{(2\pi)^2}\iint u_0(x',y')g(x-x';y-y',z)\mathrm{d}x'\mathrm{d}y'$$

$$g(x-x',y-y',z)\equiv\iint \mathrm{e}^{-\sqrt{k_x^2+k_y^2}z}\exp\{\mathrm{i}[k_x(x-x')+k_y(y-y')]\}\mathrm{d}k_x\mathrm{d}k_y$$

K-空间: $k_x=k\cos\phi_k;k_y=k\sin\phi_k$

实-空间: $x-x'=\rho\cos\varphi;y-y'=\rho\sin\varphi$

$$g(x-x',y-y',z)\equiv\int_0^\infty \mathrm{e}^{-kz}k\mathrm{d}k\int_0^{2\pi}\exp[\mathrm{i}k\rho\cos(\phi_k-\varphi)]\mathrm{d}\phi_k$$

$$=2\pi\int_0^\infty \mathrm{e}^{-kz}J_0(k\rho)k\mathrm{d}k$$

$$J_0(k\rho)=\frac{1}{2\pi}\int_0^{2\pi}\exp[\mathrm{i}k\rho\cos(\phi_k-\varphi)]\mathrm{d}\phi_k$$

$$\int_0^{\infty}k^{m+1}\mathrm{e}^{-kz}J_\nu(k\rho)\mathrm{d}k=(-1)^{m+1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}\alpha^{m+1}}\left[\frac{\left(\sqrt{z^2+\rho^2}-z\right)^{\nu}}{\sqrt{z^2+\rho^2}}\right](\rho>0,\mathrm{Re}\,\nu>-m-2)$$

$$\int_0^{\infty}\mathrm{e}^{-kz}J_0(k\rho)k\mathrm{d}k=-\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\sqrt{\rho^2+z^2}}\right]=\frac{z}{(\rho^2+z^2)^{3/2}}$$

$$g(x-x';y-y',z)=2\pi\frac{z}{(\rho^2+z^2)^{3/2}}$$

$$u(x,y,z)=\frac{z}{2\pi}\iint_D\frac{u_0(x',y')}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\mathrm{d}x'\mathrm{d}y'$$

■ 特殊情况：无限均匀（调和函数为常数）

$$\begin{cases} \nabla^2 u(\mathbf{r}) = 0, \quad z > 0 \\ u\big|_{z=0} = u_0(\text{常数}) \end{cases}$$

$$u(x,y,z) = \frac{zu_0}{2\pi}\iint_D \frac{1}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\mathrm{d}x'\mathrm{d}y'$$

$$= \frac{1}{2} zu_0 \int_0^{\infty} \frac{\mathrm{d}\rho'^2}{(\rho'^2+z^2)^{3/2}} = -zu_0 \frac{1}{(\rho'^2+z^2)^{1/2}}\Bigg|_0^{\infty} = u_0$$

■ 直接从边界条件?

$$u(x,y,0) = \iint A(k_x,k_y)\exp[\mathrm{i}(k_x x + k_y y)]\mathrm{d}k_x \mathrm{d}k_y = u_0$$

$$A(k_x,k_y) = \frac{u_0}{(2\pi)^2}\iint \exp[-\mathrm{i}(k_x x' + k_y y')]\mathrm{d}x'\mathrm{d}y' = u_0\delta(k_x)\delta(k_y)$$

$$u(x,y,z) = \iint A(k_x,k_y)\mathrm{e}^{-\sqrt{k_x^2+k_y^2}z}\exp[\mathrm{i}(k_x x + k_y y]\mathrm{d}k_x \mathrm{d}k_y = u_0$$

第二类边值问题：显然只要取

$$G=\frac{1}{4\pi}\left[\frac{1}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}}+\frac{1}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z+z')^2}}\right]$$

$$\begin{cases}\nabla^2 u(\mathbf{r})=f(\mathbf{r}),\quad z>0\\ \left.\dfrac{\partial u}{\partial n}\right|_{z=0}=-\left.\dfrac{\partial u}{\partial z}\right|_{z=0}=b(x,y)\end{cases};\ f(\mathbf{r})=0$$

$$u(x,y,z)=\iint_D b(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\mathrm{d}S'$$

$$=\frac{1}{2\pi}\iint_D\frac{b(x',y')\mathrm{d}x'\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

作业验证满足边界条件

■ Fourier积分方法?

$$u(x,y,z)=\iint u(k_x,k_y,z)\exp[\mathrm{i}(k_x x+k_y y)]\mathrm{d}k_x\mathrm{d}k_y$$

$$u(x,y,z)=\iint A(k_x,k_y)\exp\left(-\sqrt{k_x^2+k_y^2}\,z\right)\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y$$

边界条件

$$-\frac{\partial u}{\partial z}\bigg|_{z=0}=b(x,y)$$

$$\iint \sqrt{k_x^2+k_y^2}A(k_x,k_y)\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y=b(x,y)$$

$$A(k_x,k_y)=\frac{1}{(2\pi)^2\sqrt{k_x^2+k_y^2}}\iint_D b(x',y')\exp[-\mathrm{i}(k_x x'+k_y y']\mathrm{d}x'\mathrm{d}y'$$

$$u(x,y,z)=\iint A(k_x,k_y)\exp\left(-\sqrt{k_x^2+k_y^2}\,z\right)\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y$$

$$u(x,y,z)=\frac{1}{(2\pi)^2}\iint_D b(x',y')g(x-x',y-y',z)\mathrm{d}x'\mathrm{d}y'$$

$$g(x-x',y-y',z)\equiv\iint\frac{1}{\sqrt{k_x^2+k_y^2}}\mathrm{e}^{-\sqrt{k_x^2+k_y^2}\,z}\,\mathrm{e}^{\mathrm{i}[k_x(x-x')+k_y(y-y')]}\,\mathrm{d}k_x\mathrm{d}k_y$$

$$g(x-x',y-y',z)=\int_0^\infty \mathrm{e}^{-kz}\,J_0(k\rho)\mathrm{d}k=\frac{2\pi}{\sqrt{z^2+\rho^2}}$$

$$u(x,y,z)=\frac{1}{2\pi}\iint_D\frac{b(x',y')\mathrm{d}x'\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

■ 特殊情况：无限金属板（匀强电场—容易求解）

$$u(x,y,z)=\frac{1}{2\pi}\iint_D \frac{b(x',y')\mathrm{d}x'\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

$$=\frac{1}{2\pi}b_0\int_0^{\infty}\int_0^{2\pi}\frac{\rho'\mathrm{d}\rho'\mathrm{d}\varphi'}{\sqrt{\rho'^2+z^2}}=b_0\int_z^{\infty}\mathrm{d}\eta \quad （令：\rho'^2+z^2=\eta^2）$$

$$=-b_0 z+\text{无限大常数}$$

——求解基本失败

■ 直接从边界条件？ $-\left.\frac{\partial u}{\partial z}\right|_{z=0}=b_0$

$$\iint\sqrt{k_x^2+k_y^2}A(k_x,k_y)\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y=b_0$$

⇓

$$A(k_x,k_y)=\frac{b_0\delta(k_x)\delta(k_y)}{\sqrt{k_x^2+k_y^2}} \Rightarrow u(x,y,z)\to\infty$$

——求解失败

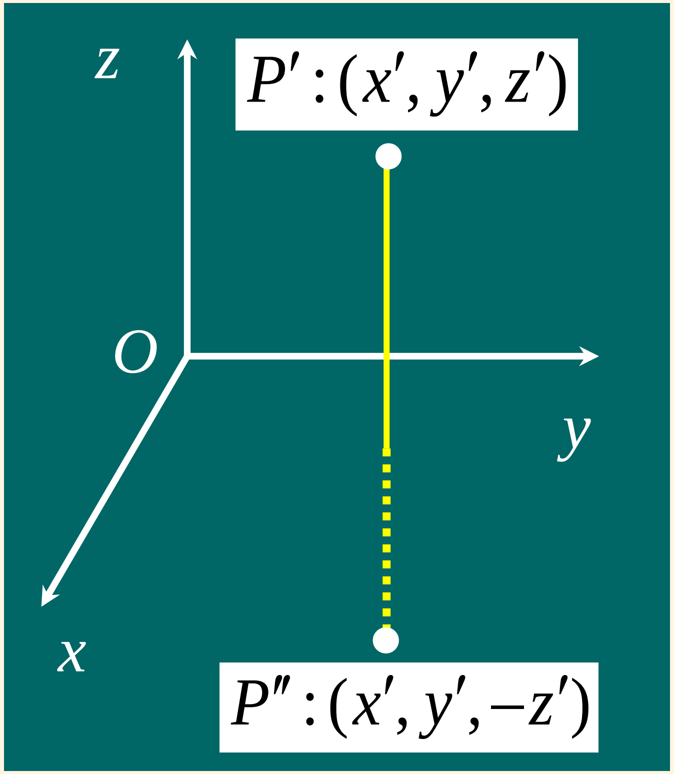
例2 上半空间 Helmholtz方程第一、二 类边值问题的Green 函数。

$$g(\boldsymbol{r},\boldsymbol{r}')=\frac{e^{\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|};G_1(\boldsymbol{r},\boldsymbol{r}'')=-\frac{e^{\mathrm{i}q|\boldsymbol{r}-\boldsymbol{r}''|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}''|}$$

其中

$$|\boldsymbol{r}-\boldsymbol{r}'|=\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}$$

$$|\boldsymbol{r}-\boldsymbol{r}''|=\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z+z')^2}$$



上半空间 Helmholtz 方程第一、二 边值问题的Green函数分别为

$$G=g+G_1;\quad G=g-G_1$$

第二类边值问题

$$\begin{cases}(\nabla^2+q^2)u(\boldsymbol{r})=0, \quad z>0 \\ \left.\dfrac{\partial u}{\partial n}\right|_{z=0}=-\left.\dfrac{\partial u}{\partial z}\right|_{z=0}=b(x,y)\end{cases}$$

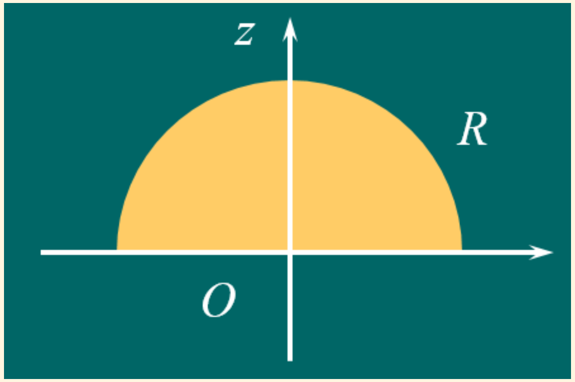
第二类边值问题的解(有限空间*V*)

$$u(\boldsymbol{r})=\int_V f(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'-\iint_B\left(u\frac{\partial G}{\partial n'}-G\frac{\partial u}{\partial n'}\right)\mathrm{d}S'$$

取*V*为半径*R*的大球

$$u(\boldsymbol{r})=\iint_D G(\boldsymbol{r};x',y',0)b(x',y')\mathrm{d}x'\mathrm{d}y'$$

$$-\iint_{S_R}\left(u\frac{\partial G}{\partial n'}-G\frac{\partial u}{\partial n'}\right)\mathrm{d}S'$$



$$\left.\frac{\partial u}{\partial z'}\right|_{z'=0} = b(x', y'); \quad \left.\frac{\partial G}{\partial z'}\right|_{z'=0} = 0; \quad \frac{\partial}{\partial n'} = -\frac{\partial}{\partial z'}$$

$$I_R \equiv \iint_{S_R} \left(u\frac{\partial G}{\partial n'} - G\frac{\partial u}{\partial n'} \right) \mathrm{d}S'$$

$$= \lim_{R\to\infty} \iint \left(u\frac{\partial G}{\partial r'} - G\frac{\partial u}{\partial r'} \right) r'^2 \mathrm{d}\Omega \to 0$$

$$\lim_{R\to\infty} \frac{\partial u}{\partial r'} \sim \mathrm{i}qu; \quad \lim_{R\to\infty} \frac{\partial G}{\partial r'} \sim \mathrm{i}qG$$

——Sommerfeld 辐射条件

$$u(\mathbf{r}) = \iint G(\mathbf{r}; x', y', 0) b(x', y') \mathrm{d}x' \mathrm{d}y'$$

$$u(\boldsymbol{r}) = \frac{1}{2\pi}\iint_D \frac{b(x',y')\exp(\mathrm{i}qR)}{R}\mathrm{d}x'\mathrm{d}y'$$

$$R \equiv [(x-x')^2 + (y-y')^2 + z^2]^{1/2}$$

二维Fourier变换

□ 远场

$$u(\boldsymbol{r}) \approx \frac{\exp(\mathrm{i}q|\boldsymbol{r}|)}{2\pi|\boldsymbol{r}|}\iint_D b(x',y')\exp[-\mathrm{i}q(xx'+yy')/r]\mathrm{d}x'\mathrm{d}y'$$

$$R \approx \sqrt{x^2 - 2xx' + y^2 - 2yy' + z^2} = r - (xx'+yy')/r$$

本例的 Green 函数在声学中声波经海面的反射有重要应用:①如果声源在海面上，海平面的反射相当于第一类边界条件(硬边界)；②反之,如果声源在水下，海平面的反射相当于第二类边界条件(软边界).

■ 满足边界条件?

$$-\frac{\partial u(\boldsymbol{r})}{\partial z}=-\frac{1}{2\pi}\iint_D b(x',y')\frac{\partial}{\partial R}\left[\frac{\exp(\mathrm{i}qR)}{R}\right]\frac{\partial R}{\partial z}\mathrm{d}x'\mathrm{d}y'$$

$$=\frac{z}{2\pi}\iint_D b(x',y')\left(\frac{1}{R^3}-\mathrm{i}q\frac{1}{R^2}\right)\exp(\mathrm{i}qR)\mathrm{d}x'\mathrm{d}y'$$

$$R\equiv[(x-x')^2+(y-y')^2+z^2]^{1/2}$$

极坐标下: $x'-x=\rho'\cos\varphi';\quad y'-y=\rho'\sin\varphi'$

$$-\frac{\partial u(\boldsymbol{r})}{\partial z}\bigg|_{z=0}=\lim_{z\to 0}\frac{z}{2\pi}\int_0^{2\pi}\int_0^{\infty}b(x+\rho'\cos\varphi',y+\rho'\sin\varphi')$$

$$\times\left[\frac{1}{(\rho'^2+z^2)^{3/2}}-\frac{\mathrm{i}q}{\rho'^2+z^2}\right]\exp(\mathrm{i}qR)\rho'\mathrm{d}\rho'\mathrm{d}\varphi'$$

变量变换： $\rho'^2 + z^2 = \eta^2$

$$-\left.\frac{\partial u(\boldsymbol{r})}{\partial z}\right|_{z=0} = \lim_{z\to 0}\frac{z}{2\pi}\int_0^{2\pi}\int_z^{\infty} b(x+\rho'\cos\varphi', y+\rho'\sin\varphi')$$

$$\times[(1-\mathrm{i}q\eta)\exp(\mathrm{i}q\eta)]\frac{1}{\eta^2}\mathrm{d}\eta\mathrm{d}\varphi'$$

当z→0时： $\rho'\to 0;\eta\to 0$ **，积分主要来自** $\eta\sim 0$

$$-\left.\frac{\partial u(\boldsymbol{r})}{\partial z}\right|_{z=0} = \lim_{z\to 0} b(x+\overline{\rho'\cos\varphi'}, y+\overline{\rho'\sin\varphi'})[(1-\mathrm{i}q\bar{\eta})\exp(\mathrm{i}q\bar{\eta})]$$

$$\times\frac{z}{2\pi}\int_0^{2\pi}\int_z^{\infty}\frac{1}{\eta^2}\mathrm{d}\eta\mathrm{d}\varphi'$$

$$-\left.\frac{\partial u(\boldsymbol{r})}{\partial z}\right|_{z=0} = b(x,y)\lim_{z\to 0} z\int_z^{\infty}\frac{1}{\eta^2}\mathrm{d}\eta = b(x,y)$$

■ Fourier积分方法?

传播子

$$u(x,y,z)=\iint A(k_x,k_y)\exp\left[\mathrm{i}\sqrt{q^2-(k_x^2+k_y^2)}z\right]\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y$$

角谱

——倏逝波的存在性

$$u(x,y,z)=\frac{1}{(2\pi)^2}\iint_D b(x',y')g(x-x',y-y',z)\mathrm{d}x'\mathrm{d}y'$$

$$g(x-x',y-y',z)\equiv \mathrm{i}\iint \frac{\mathrm{e}^{\mathrm{i}\sqrt{q^2-(k_x^2+k_y^2)}z}}{\sqrt{q^2-(k_x^2+k_y^2)}}\mathrm{e}^{\mathrm{i}[k_x(x-x')+k_y(y-y')]}\mathrm{d}k_x\mathrm{d}k_y$$

$$g(x-x',y-y',z)=2\pi\mathrm{i}\int_0^\infty \frac{1}{\sqrt{q^2-k^2}}\mathrm{e}^{\mathrm{i}\sqrt{q^2-k^2}z}J_0(k\rho)k\mathrm{d}k$$

球面波的柱面波展开公式

$$\frac{e^{\mathrm{i}q\sqrt{\rho^2+z^2}}}{4\pi\sqrt{\rho^2+z^2}}=\frac{\mathrm{i}}{4\pi}\int_0^{\infty}\frac{1}{\sqrt{q^2-k^2}}J_0(k\rho)e^{\mathrm{i}\sqrt{q^2-k^2}z}k\mathrm{d}k$$

——后面证明

$$g(x-x',y-y',z)=2\pi\frac{\mathrm{e}^{\mathrm{i}q\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}$$

$$u(x,y,z)=\frac{1}{2\pi}\iint_D\frac{b(x',y')\,\mathrm{e}^{\mathrm{i}q\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\mathrm{d}x'\mathrm{d}y'$$

——瑞利积分

■ 特殊情况：无限大平板

$$\begin{cases} (\nabla^2 + q^2)u(\boldsymbol{r}) = 0, \quad z > 0 \\ \left.\dfrac{\partial u}{\partial n}\right|_{z=0} = -\left.\dfrac{\partial u}{\partial z}\right|_{z=0} = b_0 \end{cases}$$

$$u(x,y,z) = \frac{b_0}{2\pi}\iint_\infty \frac{\mathrm{e}^{\mathrm{i}q\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\mathrm{d}x'\mathrm{d}y'$$

$$= b_0\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}q\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}\rho\mathrm{d}\rho$$

$$\frac{e^{\mathrm{i}q\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} = \mathrm{i}\int_0^\infty \frac{1}{\sqrt{q^2-k^2}} J_0(k\rho)e^{\mathrm{i}\sqrt{q^2-k^2}z}k\mathrm{d}k$$

$$u(x,y,z)=\mathrm{i}b_0\int_0^{\infty}\frac{1}{\sqrt{q^2-k^2}}\left[\int_0^{\infty}J_0(k\rho)\rho\,\mathrm{d}\rho\right]e^{\mathrm{i}\sqrt{q^2-k^2}z}k\mathrm{d}k$$

$$\int_0^{\infty}J_0(k\rho)\rho\,\mathrm{d}\rho=\frac{1}{k}\delta(k)$$

$$u(x,y,z)=\mathrm{i}\frac{b_0}{q}e^{\mathrm{i}qz}$$ ——z方向传播的平面波

■ 直接从边界条件方程出发

$$-\left.\frac{\partial u}{\partial z}\right|_{z=0}=-\iint \mathrm{i}\sqrt{q^2-(k_x^2+k_y^2)}A(k_x,k_y)\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y=b_0$$

角谱

$$A(k_x,k_y)=\frac{\mathrm{i}b_0}{(2\pi)^2\sqrt{q^2-(k_x^2+k_y^2)}}\iint \mathrm{e}^{-\mathrm{i}(k_x x'+k_y y')}\mathrm{d}x'\mathrm{d}y'=\frac{\mathrm{i}b_0}{q}\delta(k_x)\delta(k_y)$$

$$A(k_x,k_y)=\frac{\mathrm{i}b_0}{q}\delta(k_x)\delta(k_y)$$

——只有z方向传播的分量

$$u(x,y,z)=\iint A(k_x,k_y)\exp\left[\mathrm{i}\sqrt{q^2-(k_x^2+k_y^2)}z\right]\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y$$

$$=\frac{\mathrm{i}}{q}b_0\iint \delta(k_x)\delta(k_y)\exp\left[\mathrm{i}\sqrt{q^2-(k_x^2+k_y^2)}z\right]\exp[\mathrm{i}(k_x x+k_y y]\mathrm{d}k_x\mathrm{d}k_y$$

$$=\frac{\mathrm{i}}{q}b_0\exp(\mathrm{i}qz)$$

$$u(x,y,z)=\mathrm{i}\frac{b_0}{q}\exp(\mathrm{i}qz)$$

——z方向传播的平面波

■球面波用平面波展开

直角坐标中满足非齐次波动方程

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k_0^2\right)u(x,y,z)=\delta(x)\delta(y)\delta(z)$$

$$\Downarrow$$

$$u(\boldsymbol{r},\omega)=\frac{1}{4\pi|\boldsymbol{r}|}\exp(\mathrm{i}k_0|\boldsymbol{r}|)$$

把球面波用平面波展开, 平面波展开形式

$$u(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(k_x,k_y,z)e^{\mathrm{i}(k_x x+k_y y)}\mathrm{d}k_x\mathrm{d}k_y$$

——物理本质：用无限多个平面逼近球面

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{[k_0^2-(k_x^2+k_y^2)]g+\frac{\mathrm{d}^2 g}{\mathrm{d}z^2}\right\}\mathrm{e}^{\mathrm{i}(k_x x+k_y y)}\,\mathrm{d}k_x\mathrm{d}k_y$$

$$=-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_x x+k_y y)}\mathrm{d}k_x\mathrm{d}k_y\delta(z)$$

$$\Downarrow$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2}+\xi^2 g=-\frac{1}{(2\pi)^2}\delta(z);\;\; \xi\equiv\sqrt{k_0^2-(k_x^2+k_y^2)}$$

$$\Downarrow$$

$$g(k_x,k_y,z)=\begin{cases}A\exp(\mathrm{i}\xi z), & (z>0)\\ B\exp(-\mathrm{i}\xi z), & (z<0)\end{cases}$$

$$g(k_x,k_y,z)\big|_{z=0-0}=g(k_x,k_y,z)\big|_{z=0+0}$$

$$\left.\frac{\mathrm{d}g(k_x,k_y,z)}{\mathrm{d}z}\right|_{z=0+0}-\left.\frac{\mathrm{d}g(k_x,k_y,z)}{\mathrm{d}z}\right|_{z=0-0}=-\frac{1}{(2\pi)^2}$$

$$A=B=\mathrm{i}/(8\pi^2\xi)$$

$$g(k_x,k_y,z)=\frac{\mathrm{i}}{8\pi^2\xi}\exp(\mathrm{i}\xi\,|\,z\,|)$$

$$u(x,y,z)=\frac{\mathrm{i}}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\xi}\exp[\mathrm{i}(k_x x+k_y y+\xi\,|\,z\,|)]\mathrm{d}k_x\mathrm{d}k_y$$

$$\frac{\exp\left(\mathrm{i}k_0\,|\boldsymbol{r}-\boldsymbol{r}'|\right)}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{\mathrm{i}}{8\pi^2}\iint\frac{1}{\xi}\exp[\mathrm{i}\boldsymbol{k}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')+\mathrm{i}\xi\,|\mathrm{z}-\mathrm{z}'|]\mathrm{d}^2\boldsymbol{k}_\rho$$

$$\boldsymbol{k}_\rho=(k_x,k_y);\boldsymbol{\rho}=(x,y);\boldsymbol{\rho}'=(x',y')$$

——Weyl公式

$$\frac{\exp\left(\mathrm{i}k_0\,|\boldsymbol{r}-\boldsymbol{r}'|\right)}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{\mathrm{i}}{8\pi^2}\iint_{|k|<k_0}\exp\left\{\mathrm{i}[\boldsymbol{k}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')+\xi\,|\mathrm{z}-\mathrm{z}'|]\right\}\mathrm{d}^2\boldsymbol{k}_\rho$$

$$+\frac{\mathrm{i}}{8\pi^2}\iint_{|k|>k_0}e^{-\nu|z-z'|}\exp[\mathrm{i}\boldsymbol{k}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')]\mathrm{d}^2\boldsymbol{k}_\rho$$

$$\xi=\sqrt{k_0^2-(k_x^2+k_y^2)};\mu\equiv\sqrt{(k_x^2+k_y^2)-k_0^2}$$

——圆内—平面波；圆外—倏逝波

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k_0^2\right)u(x,y,z)=\delta(x)\delta(y)\delta(z)$$

$$\Downarrow$$

$$u(x,y,z)=\int_{-\infty}^{\infty} g(k_x,k_y,k_z)e^{\mathrm{i}(k_x x+k_y y+k_z z)}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z$$

$$\Downarrow$$

$$\left[k_0^2-(k_x^2+k_y^2+k_z^2)\right]g=-\frac{1}{(2\pi)^3}$$

$$\Downarrow$$

$$u(x,y,z)=-\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}(k_x x+k_y y+k_z z)}}{k_0^2-(k_x^2+k_y^2+k_z^2)}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z$$

$$=-\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(k_x,k_y)e^{\mathrm{i}(k_x x+k_y y)}\mathrm{d}k_x\mathrm{d}k_y$$

$$I(k_x,k_y)\equiv\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}k_z z}}{k_0^2-(k_x^2+k_y^2)-k_z^2}\mathrm{d}k_z$$

仿照前面求一维Helmholtz方程Green的方法，可以得到

$$I(k_x,k_y)=-\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}k_z z}}{k_z^2-\xi^2}\mathrm{d}k_z=-\frac{\mathrm{i}\pi}{\xi}e^{\mathrm{i}\xi|z|}$$

$$\xi=\sqrt{k_0^2-(k_x^2+k_y^2)}$$

$$\Downarrow$$

$$\frac{\exp\left(\mathrm{i}k_0\,|\boldsymbol{r}|\right)}{4\pi\,|\boldsymbol{r}|}=\frac{\mathrm{i}}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\xi}e^{\mathrm{i}(k_x x+k_y y+\xi|z|)}\mathrm{d}k_x\mathrm{d}k_y$$

——上式一般用于z方向具有分层介质的散射问题

球面波用柱函数展开

柱坐标中Green函数方程为

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right]g+k_0^2 g=-\delta(\mathbf{r},\mathbf{r}')$$

$$\delta(\mathbf{r},\mathbf{r}')=\frac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$$

①利用 $\Phi_m(\varphi)=\Phi_0\exp(\mathrm{i}m\varphi)$ **的完备性，作展开**

$$g(\mathbf{r},\mathbf{r}')=\sum_{m=-\infty}^{\infty}g_m^1(\rho,z)\exp(\mathrm{i}m\varphi)$$

$$\sum_{m=-\infty}^{\infty}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial z^2}+\left(k_0^2-\frac{m^2}{\rho^2}\right)\right]g_m^1(\rho,z)\exp(\mathrm{i}m\varphi)=-\delta(\mathbf{r},\mathbf{r}')$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial z^2}+\left(k_0^2-\frac{m^2}{\rho^2}\right)\right]g_m^1(\rho,z)$$

$$=-\frac{1}{2\pi\rho}\delta(\rho-\rho')\delta(z-z')\exp(-im\varphi')$$

②作Hankel变换

$$g_m^1(\rho,z)=\int_0^\infty g_m^2(\lambda,z)J_m(\lambda\rho)\lambda\mathrm{d}\lambda$$

$$\Downarrow$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(k_0^2-\lambda^2)\right]g_m^2(\lambda,z)=-\frac{1}{2\pi}\delta(z-z')J_m(\lambda\rho')\exp(-im\varphi')$$

$$\Uparrow$$

$$g_m^2(\lambda,z)\big|_{z=z'-\varepsilon}=g_m^2(\lambda,z)\big|_{z=z'+\varepsilon}$$

$$\left.\frac{\mathrm{d}g_m^2(\lambda,z)}{\mathrm{d}z}\right|_{z=z'+\varepsilon}-\left.\frac{\mathrm{d}g_m^2(\lambda,z)}{\mathrm{d}z}\right|_{z=z'-\varepsilon}=-\frac{1}{2\pi}J_m(\lambda\rho')\exp(-im\varphi')$$

$$g_m^2(\lambda, z) = \begin{cases} A\exp\left[\mathrm{i}\sigma(z-z')\right], z > z' \\ B\exp\left[\mathrm{i}\sigma(z'-z)\right], z < z' \end{cases}$$

其中： $\sigma = \sqrt{k_0^2 - \lambda^2}, (k_0 > \lambda); \sigma = \mathrm{i}\sqrt{\lambda^2 - k_0^2 -}, (k_0 < \lambda)$

$$g_m^2(\lambda, z) = -\frac{1}{4\pi\mathrm{i}\sigma} J_m(\lambda\rho')\exp(-\mathrm{i}m\varphi')\exp\left[\mathrm{i}\sigma|z-z'|\right]$$

$$\frac{e^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\mathrm{i}}{4\pi}\sum_{m=-\infty}^{\infty}\left[\int_0^\infty \frac{1}{\sigma} J_m(\lambda\rho')J_m(\lambda\rho)e^{\mathrm{i}\sigma|z-z'|}\lambda\mathrm{d}\lambda\right] \times e^{\mathrm{i}m(\varphi-\varphi')}$$

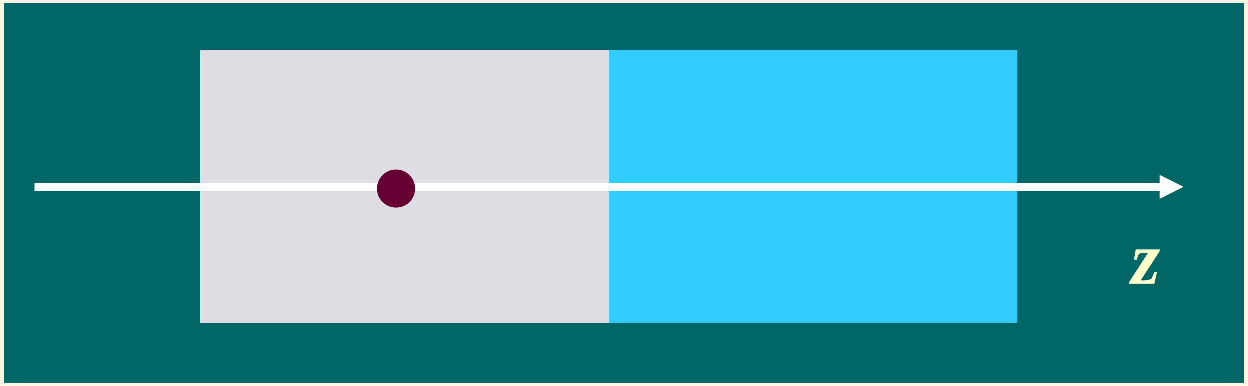
——物理本质：用无限多个柱面逼近球面

特殊情况：点源位于z轴上，即 $\rho'=0$

$$\frac{e^{\mathrm{i}k_0 R}}{4\pi R}=\frac{\mathrm{i}}{4\pi}\int_0^{\infty}\frac{1}{\sqrt{k_0^2-\lambda^2}}J_0(\lambda\rho)e^{\mathrm{i}\sqrt{k_0^2-\lambda^2}|z-z'|}\lambda\mathrm{d}\lambda$$

$$R\equiv\sqrt{\rho^2+(z-z')^2}$$

——积分部分形成z方向的倏逝波



③作z方向的Fourier变换

$$g_m^1(\rho, z) = \int_{-\infty}^{\infty} g_m^3(\sigma, \rho)\exp(\mathrm{i}\sigma z)\mathrm{d}\sigma$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m^3}{\partial\rho}\right)+\left(k_\rho^2-\frac{m^2}{\rho^2}\right)g_m^3\right] \Longleftarrow k_\rho \equiv \sqrt{k_0^2-\sigma^2}$$

$$=-\frac{1}{(2\pi)^2\rho}\delta(\rho-\rho')e^{-\mathrm{i}(m\varphi'+\sigma z')}$$

$$g_m^3(\rho,\sigma) = \frac{\mathrm{i}}{8\pi}\exp[-\mathrm{i}(m\varphi'+\sigma z')]$$

$$\times\begin{cases} J_m(k_\rho\rho')H_m^{(1)}(k_\rho\rho) & (\rho'<\rho) \\ H_m^{(1)}(k_\rho\rho')J_m(k_\rho\rho) & (\rho<\rho') \end{cases}$$

$$\frac{e^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{\mathrm{i}}{8\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[\mathrm{i}\sigma(z-z')]\overset{\mathrm{d}\sigma}{\cancel{\mathrm{d}k_z}}\,\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')}$$

$$\times\begin{cases}J_m(k_\rho\rho')H_m^{(1)}(k_\rho\rho) & (\rho'<\rho)\\ H_m^{(1)}(k_\rho\rho')J_m(k_\rho\rho) & (\rho<\rho')\end{cases}$$

圆柱体对点源的散射问题.

特殊情况：点源位于z轴上，即 $\rho'=0$

$$\frac{e^{\mathrm{i}k_0 R}}{4\pi R}=\frac{\mathrm{i}}{8\pi}\int_{-\infty}^{\infty}H_0^{(1)}\left(\sqrt{k_0^2-\sigma^2}\,\rho\right)\exp[\mathrm{i}\sigma(z-z')]\mathrm{d}\sigma$$

$$H_0^{(1)}(\mathrm{i}\kappa_\rho\rho)\sim K_0(\kappa_\rho\rho)$$

——积分部分形成ρ方向的倏逝波

球面波用球函数展开

三维Laplace方程

$$-\nabla^2 g(\mathbf{r},\mathbf{r}') = \frac{1}{r^2\sin\vartheta}\delta(r-r')\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')$$

利用球谐函数的完备性，作展开

$$g(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{lm}(r)Y_{lm}(\vartheta,\varphi)$$

$$\Downarrow$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{l(l+1)}{r^2}\right]g_{lm}(r)Y_{lm}(\vartheta,\varphi)$$

$$= -\frac{1}{r^2\sin\vartheta}\delta(r-r')\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')$$

$$\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)-\frac{l(l+1)}{r^2}\right]g_{lm}(r)=-\frac{1}{r^2}\delta(r-r')Y_{lm}^*(\vartheta',\varphi')$$

$$g_{lm}(r)\big|_{r=r'-\varepsilon}=g_{lm}(r)\big|_{r=r'+\varepsilon}$$

$$\left.\frac{\mathrm{d}g_{lm}(r)}{\mathrm{d}r}\right|_{r=r'+\varepsilon}-\left.\frac{\mathrm{d}g_{lm}(r)}{\mathrm{d}r}\right|_{r=r'-\varepsilon}=-\frac{1}{r^2}Y_{lm}^*(\vartheta',\varphi')$$

零点有限

$$g_{lm}(r,r')=\frac{Y_{lm}^*(\vartheta',\varphi')}{2l+1}\begin{cases}\dfrac{1}{r'}\left(\dfrac{r}{r'}\right)^l, & r\le r'\\ \dfrac{1}{r}\left(\dfrac{r'}{r}\right)^{l+1}, & r\ge r'\end{cases}\equiv g_l(r,r')Y_{lm}^*(\vartheta',\varphi')$$

无限为零

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_l(r,r')Y_{lm}(\vartheta,\varphi)Y_{lm}^{*}(\vartheta',\varphi')$$

- **加法公式** **取** $t=r/r'<1$

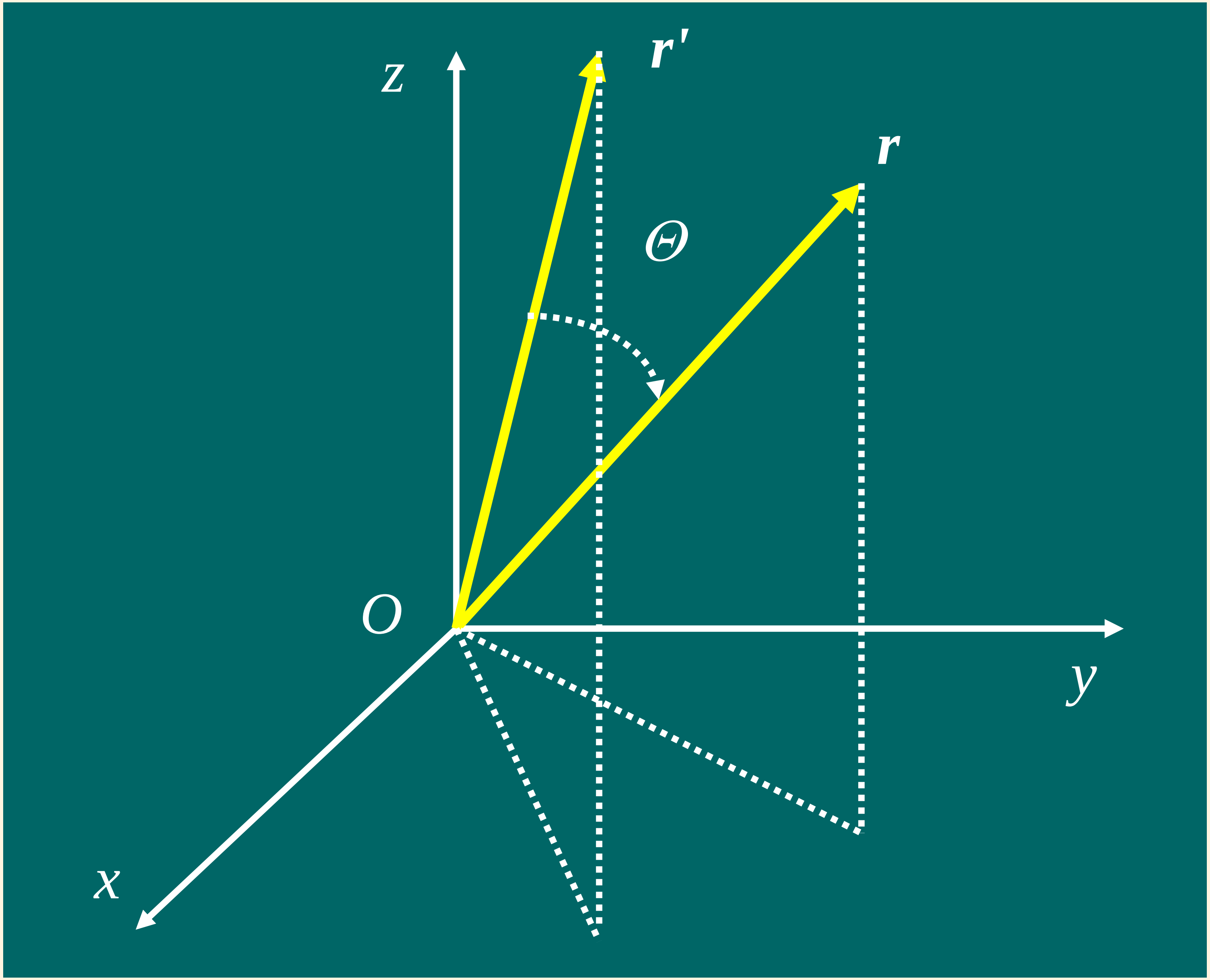
$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{4\pi r'\sqrt{1-2t\cos\Theta+t^2}}=\frac{1}{4\pi r'}\sum_{l=0}^{\infty}t^l P_l(\cos\Theta)$$

$$\Downarrow$$

$$\frac{1}{r'}\sum_{l=0}^{\infty}t^l P_l(\cos\Theta)=\frac{1}{r'}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}t^l\sum_{m=-l}^{l}Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta_2,\varphi_2)$$

$$\Downarrow$$

$$P_l(\cos\Theta)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta_2,\varphi_2)$$


z
r'
r
Θ
O
y
x

■三维Helmholtz方程

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]g+k_0^2 g$$

$$=-\frac{1}{r^2\sin\vartheta}\delta(r-r')\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')$$

■ 利用球谐函数的完备性，作展开

$$g(\mathbf{r},\mathbf{r}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_{lm}(r)Y_{lm}(\vartheta,\varphi)$$

$$\Downarrow$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+k_0^2-\frac{l(l+1)}{r^2}\right]g_{lm}(r)Y_{lm}(\vartheta,\varphi)$$

$$=-\frac{1}{r^2\sin\vartheta}\delta(r-r')\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')$$

$$\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)+k_0^2-\frac{l(l+1)}{r^2}\right]g_{lm}(r)=-\frac{1}{r^2}\delta(r-r')Y_{lm}^*(\vartheta',\varphi')$$

$$g_{lm}(r)\Big|_{r=r'-\varepsilon}=g_{lm}(r)\Big|_{r=r'+\varepsilon}$$

$$\left.\frac{\mathrm{d}g_{lm}(r)}{\mathrm{d}r}\right|_{r=r'+\varepsilon}-\left.\frac{\mathrm{d}g_{lm}(r)}{\mathrm{d}r}\right|_{r=r'-\varepsilon}=-\frac{1}{r^2}Y_{lm}^*(\vartheta',\varphi')$$

$$g_{lm}(r)=\begin{cases}A_l h_l^{(1)}(k_0 r), & (r>r')\\ B_l j_l(k_0 r), & (r<r')\end{cases}$$

向外辐射形式的解

原点必须有限，驻波形式的解

$$A_l h_l^{(1)}(k_0 r') = B_l\, j_l(k_0 r'); \qquad h_l'^{(1)}(k_0 r') = \frac{\mathrm{d}h_l^{(1)}(k_0 r')}{\mathrm{d}(k_0 r')}$$

$$A_l h_l'^{(1)}(k_0 r') - B_l\, j_l'(k_0 r') = -\frac{1}{k_0 r'^2} Y_{lm}^*(\vartheta', \varphi')$$

$$\Downarrow$$

$$A_l = \frac{j_l(k_0 r')}{k_0 r'^2 \left[j_l(k_0 r') h_l'^{(1)}(k_0 r') - h_l^{(1)}(k_0 r') j_l'(k_0 r') \right]} Y_{lm}^*(\vartheta', \varphi')$$

$$B_l = \frac{h_l^{(1)}(k_0 r')}{k_0 r'^2 \left[j_l(k_0 r') h_l'^{(1)}(k_0 r') - h_l^{(1)}(k_0 r') j_l'(k_0 r') \right]} Y_{lm}^*(\vartheta', \varphi')$$

$$\Uparrow$$

$$(k_0 r')^2 \left[j_l(k_0 r') h_l'^{(1)}(k_0 r') - h_l^{(1)}(k_0 r') j_l'(k_0 r') \right] = -\mathrm{i}$$

$$\Downarrow$$

$$g_{lm}(r) = \mathrm{i}k_0 \begin{cases} h_l^{(1)}(k_0 r)\, j_l(k_0 r') Y_{lm}^*(\vartheta', \varphi'), & (r > r') \\ j_l(k_0 r) h_l^{(1)}(k_0 r') Y_{lm}^*(\vartheta', \varphi'), & (r < r') \end{cases}$$

$$g(\boldsymbol{r},\boldsymbol{r}')=\mathrm{i}k_0\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}Y_{lm}(\vartheta,\varphi)Y_{lm}^*(\vartheta',\varphi')\begin{cases}h_l^{(1)}(k_0r)j_l(k_0r'), & (r>r')\\ j_l(k_0r)h_l^{(1)}(k_0r'), & (r<r')\end{cases}$$

$$\frac{\exp(\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}=\mathrm{i}k_0\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}Y_{lm}(\vartheta,\varphi)Y_{lm}^*(\vartheta',\varphi')\begin{cases}h_l^{(1)}(k_0r)j_l(k_0r'), & r>r'\\ j_l(k_0r)h_l^{(1)}(k_0r'), & r<r'\end{cases}$$

——物理本质：用无限个球心在原点的球面逼近偏心球面

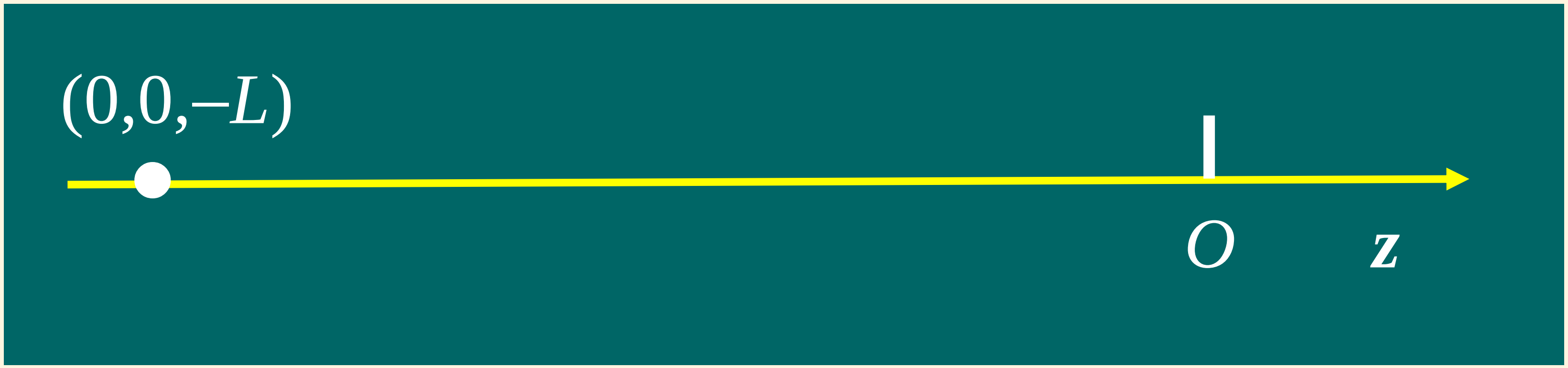
- **特殊情况：点源位于原点**

$$r'=0\Rightarrow j_0(k_0r')=j_0(0)\neq 0\Rightarrow l=0\Rightarrow m=0;r>r'=0$$

$$\frac{\exp(\mathrm{i}k_0|\boldsymbol{r}|)}{4\pi|\boldsymbol{r}|}=\mathrm{i}k_0Y_{00}(\vartheta,\varphi)Y_{00}^*(\vartheta',\varphi')h_0^{(1)}(k_0r)=\frac{1}{4\pi r}e^{\mathrm{i}k_0r}$$

■ 平面波展开公式

$$\boldsymbol{r}' = (x', y', z') = (0, 0, -L)(L > 0);\quad r' >> r;\quad \vartheta' = \pi$$



$$\frac{e^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = \mathrm{i}k_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\vartheta, \varphi) Y_{lm}^{*}(\vartheta', \varphi') j_l(k_0 r) h_l^{(1)}(k_0 r')$$

$$(r << r')$$

$$\frac{\mathrm{e}^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{\exp(\mathrm{i}k_0 L)}{4\pi|\boldsymbol{r}'|} \exp(\mathrm{i}k_0 z)$$

$$h_l^{(1)}(k_0 r') \sim -\frac{\mathrm{i}}{k_0 r'}\exp\left[\mathrm{i}\left(k_0 r' - \frac{l\pi}{2}\right)\right] = -\frac{\mathrm{i}}{k_0 r'}\exp(\mathrm{i}k_0 r')(-\mathrm{i})^l$$

$$\frac{\exp(\mathrm{i}k_0 L)}{4\pi|\boldsymbol{r}'|}\exp(\mathrm{i}k_0 z) \approx \frac{1}{4\pi r'}\exp(\mathrm{i}k_0 r')\sum_{l=0}^{\infty}\mathrm{i}^l(2l+1)j_l(k_0 r)P_l(\cos\vartheta)$$

$$\exp(\mathrm{i}k_0 r\cos\vartheta) = \sum_{l=0}^{\infty}\mathrm{i}^l(2l+1)j_l(k_0 r)P_l(\cos\vartheta)$$

——平面波用球面波展开，可以看作为用Legedre函数展开

$$\exp(\mathrm{i}k_0 r\cos\vartheta) = \sum_{l=0}^{\infty}A_l(r)P_l(\cos\vartheta)$$

$$A_l(r) = \frac{2l+1}{2}\int_0^{\pi}\exp(\mathrm{i}k_0 r\cos\vartheta)P_l(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta$$

计算复杂

□波动方程的基本解

与纯空间的 Green 函数不同点：时间变量t与t'不能简单对调。 如何定义含时 Green函数?

无限空间的初值问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0, \ (t>0) \\ u\big|_{t=0} = \psi_1(\boldsymbol{r}); \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi_2(\boldsymbol{r}) \end{cases}$$

如何定义Green函数?

首先用Fourier方法求解，令

$$u(\boldsymbol{r},t) = \int g(\boldsymbol{k},t)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

$$\frac{\mathrm{d}^2 g(\boldsymbol{k},t)}{\mathrm{d}t^2}+c^2k^2g(\boldsymbol{k},t)=0,\left(k=\sqrt{k_x^2+k_y^2+k_z^2}\right)$$

$$\Downarrow$$

$$g(\boldsymbol{k},t)=A(k)\sin(ckt)+B(k)\cos(ckt)$$

$$\Downarrow$$

$$u(\boldsymbol{r},t)=\int\left[A(k)\sin(ckt)+B(k)\cos(ckt)\right]e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

初始条件

$$u(\boldsymbol{r},0)=\int B(k)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}=\psi_1(\boldsymbol{r})$$

$$u_t(\boldsymbol{r},0)=\int ckA(k)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}=\psi_2(\boldsymbol{r})$$

$$\Downarrow$$

$$B(k)=\frac{1}{(2\pi)^3}\int\psi_1(\boldsymbol{r})e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}$$

$$A(k)=\frac{1}{(2\pi)^3}\int\frac{1}{ck}\psi_2(\boldsymbol{r})e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}$$

用Green函数表示的积分形式解

$$u(\boldsymbol{r},t)=\int\left[A(k)\sin(ckt)+B(k)\cos(ckt)\right]e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

$$=\int\frac{\partial g(\boldsymbol{r},\boldsymbol{r}',t)}{\partial t}\psi_1(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'+\int g(\boldsymbol{r},\boldsymbol{r}',t)\psi_2(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

$$\Uparrow$$

$$g(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{(2\pi)^3}\int\frac{1}{ck}\sin(ckt)e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\mathrm{d}^3\boldsymbol{k}$$

如果取初值

$$\psi_1(\boldsymbol{r})=0;\quad \psi_2(\boldsymbol{r})=\delta(\boldsymbol{r},\boldsymbol{r}'')$$

$$\Downarrow$$

$$u(\boldsymbol{r},t)=\int g(\boldsymbol{r},\boldsymbol{r}',t)\delta(\boldsymbol{r},\boldsymbol{r}'')\mathrm{d}^3\boldsymbol{r}'=g(\boldsymbol{r},\boldsymbol{r}'',t)$$

于是，无限空间初值问题的Green函数定义为

$$\begin{cases} \dfrac{\partial^2 g}{\partial t^2} - c^2\nabla^2 g = 0, (t > 0) \\ g\,|_{t=0} = 0;\ g_t\,|_{t=0} = \delta(\boldsymbol{r}, \boldsymbol{r}') \end{cases}$$

方程的解为

$$u(\boldsymbol{r}, t) = \int \frac{\partial g(\boldsymbol{r}, \boldsymbol{r}', t)}{\partial t} \psi_1(\boldsymbol{r}') \mathrm{d}^3\boldsymbol{r}' + \int g(\boldsymbol{r}, \boldsymbol{r}', t) \psi_2(\boldsymbol{r}') \mathrm{d}^3\boldsymbol{r}'$$

$$\Uparrow$$

$$g(\boldsymbol{r}, \boldsymbol{r}', t) = \frac{1}{(2\pi)^3} \int \frac{1}{ck} \sin(ckt) e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \mathrm{d}^3\boldsymbol{k}$$

因此

$$g(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{(2\pi)^2 c}\int_0^\infty k\sin(ckt)\int_0^\pi e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta}\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}k$$

$$=\frac{1}{4\pi^2 c}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\int_{-\infty}^\infty \sin(ckt)\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}k$$

$$=\frac{1}{4\pi c}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta(ct-|\boldsymbol{r}-\boldsymbol{r}'|)-\delta(ct+|\boldsymbol{r}-\boldsymbol{r}'|)$$

$$g(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{4\pi c}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left[\delta(ct-|\boldsymbol{r}-\boldsymbol{r}'|)-\delta(ct+|\boldsymbol{r}-\boldsymbol{r}'|)\right]$$

$$g(\boldsymbol{r},\boldsymbol{r}',t)=\frac{\delta(|\boldsymbol{r}-\boldsymbol{r}'|-ct)}{4\pi c|\boldsymbol{r}-\boldsymbol{r}'|}$$

推迟势

无限空间的非齐次问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = f(\boldsymbol{r}, t) \\ u\big|_{t=0} = 0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

如何定义Green函数?

首先也用Fourier方法求解，令

$$u(\boldsymbol{r}, t) = \int g(\boldsymbol{k}, t) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3\boldsymbol{k}$$

$$\int \left[\frac{\mathrm{d}^2 g(\boldsymbol{k}, t)}{\mathrm{d}t^2} + c^2 k^2 g(\boldsymbol{k}, t) \right] e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3\boldsymbol{k} = f(\boldsymbol{r}, t)$$

$$\frac{\mathrm{d}^2 g(\boldsymbol{k}, t)}{\mathrm{d}t^2} + c^2 k^2 g(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3} \int f(\boldsymbol{r}, t) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3\boldsymbol{r} \equiv F(\boldsymbol{k}, t)$$

零初始条件的特解

$$g(\mathbf{k},t)=\frac{1}{ck}\int_0^t F(\mathbf{k},\tau)\sin[ck(t-\tau)]\mathrm{d}\tau$$

$$\Downarrow$$

$$u(\mathbf{r},t)=\int_0^t\int f(\mathbf{r}',\tau)g(\mathbf{r},\mathbf{r}',t-\tau)\mathrm{d}^3\mathbf{r}'\mathrm{d}\tau$$

$$g(\mathbf{r},\mathbf{r}',t-\tau)=\frac{\delta[|\mathbf{r}-\mathbf{r}'|-c(t-\tau)]}{4\pi c|\mathbf{r}-\mathbf{r}'|}$$

$$\Downarrow$$

$$u(\mathbf{r},t)=\int_0^\infty\int f(\mathbf{r}',\tau)\tilde{g}(\mathbf{r},\mathbf{r}',t-\tau)\mathrm{d}^3\mathbf{r}'\mathrm{d}\tau$$

$$\Uparrow$$

$$\tilde{g}(\mathbf{r},\mathbf{r}',t-\tau)\equiv H(t-\tau)g(\mathbf{r},\mathbf{r}',t-\tau)$$

如果非齐次项为

$$f(\mathbf{r},t)=\delta(\mathbf{r}-\mathbf{r}'')\delta(t-t'')$$

$$\Downarrow$$

$$u(\mathbf{r},t)=\int_0^{\infty}\int\delta(\mathbf{r}'-\mathbf{r}'')\delta(\tau-t'')\tilde{g}(\mathbf{r},\mathbf{r}',t-\tau)\mathrm{d}^3\mathbf{r}'\mathrm{d}\tau$$

$$=\tilde{g}(\mathbf{r},\mathbf{r}'',t-t'')$$

因此，可以定义Green函数

$$\begin{cases}\dfrac{\partial^2\tilde{g}}{\partial t^2}-c^2\nabla^2\tilde{g}=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \\ \tilde{g}\,|_{t=0}=0;\tilde{g}_t\,|_{t=0}=0\end{cases}$$

无限空间的非齐次、非零初值问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = f(\mathbf{r}, t) \\ u|_{t=0} = \psi_1(\mathbf{r}), u_t|_{t=0} = \psi_2(\mathbf{r}) \end{cases}$$

二部分解叠加

$$u(\mathbf{r}, t) = \int \frac{\partial g(\mathbf{r}, \mathbf{r}', t)}{\partial t} \psi_1(\mathbf{r}') \mathrm{d}^3 \mathbf{r}' + \int g(\mathbf{r}, \mathbf{r}', t) \psi_2(\mathbf{r}') \mathrm{d}^3 \mathbf{r}'$$

$$+ \int_0^\infty \int f(\mathbf{r}', \tau) \tilde{g}(\mathbf{r}, \mathbf{r}', t - \tau) \mathrm{d}^3 \mathbf{r}' \mathrm{d}\tau$$

二维波动方程

三维推迟势，明显的波前和波后

$$g(\mathbf{r}, \mathbf{r}', t) = \frac{\delta(|\mathbf{r} - \mathbf{r}'| - ct)}{4\pi c |\mathbf{r} - \mathbf{r}'|}$$

二维: 相当于在(x,y)存在线源产生的场，可由三维Green函数通过降维方法得到

$$g_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}',t)=\int_{-\infty}^{\infty}g(\mathbf{r},\mathbf{r}',t)\mathrm{d}z'=\int_{-\infty}^{\infty}\frac{\delta(|\mathbf{r}-\mathbf{r}'|-ct)}{4\pi c|\mathbf{r}-\mathbf{r}'|}\mathrm{d}z'$$

$$\Downarrow$$

$$g_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}',t)=\int_{-\infty}^{\infty}\frac{\delta\left(\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z'-z)^2}-ct\right)}{4\pi c\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z'-z)^2}}\mathrm{d}(z'-z)$$

$$=\int_{-\infty}^{\infty}\frac{\delta\left(\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta^2}-ct\right)}{4\pi c\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta^2}}\mathrm{d}\eta$$

Dirac Delta 函数的零点分析:

$$g(\eta)\equiv\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta^2}-ct=0 \Rightarrow \eta^2=(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2$$

①当满足下式时，Dirac Delta函数无零点，积分为零

$$(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2<0\Rightarrow G_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}',t)=0$$

②当$(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2>0$ **时， Dirac Delta函数存在**

二个零点,表示关于z′=0对称的二个点源产生的波达到场点

$$\eta_{\pm}=\pm\sqrt{(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}$$

因此， Dirac Delta函数可表示为

$$\delta\left(\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta^2}-ct\right)=\frac{1}{|g'(\eta_+)|}\delta(\eta-\eta_+)+\frac{1}{|g'(\eta_-)|}\delta(\eta-\eta_-)$$

$$=\frac{ct}{\sqrt{(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}}[\delta(\eta-\eta_+)+\delta(\eta-\eta_-)]$$

$$\Downarrow$$

$$g_{\mathrm{2D}}(\boldsymbol{\rho},\boldsymbol{\rho}',t)=\frac{ct}{\sqrt{(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}}\int_{-\infty}^{\infty}\frac{[\delta(\eta-\eta_+)+\delta(\eta-\eta_-)]}{4\pi c\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta^2}}\mathrm{d}\eta$$

$$=\frac{ct}{\sqrt{(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}}\left[\frac{1}{4\pi c\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta_+^2}}+\frac{1}{4\pi c\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+\eta_-^2}}\right]$$

$$=\frac{1}{2\pi c\sqrt{(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}}$$

因此，二维波动方程的Green函数为

$$g_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}',t)=\begin{cases}\dfrac{1}{2\pi c\sqrt{(ct)^2-|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}}, & ct>|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2\\ 0, & ct<|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2\end{cases}$$

❑扩散方程的 Green 函数

无限空间的初值问题

$$\begin{cases}\dfrac{\partial u}{\partial t}-c^2\nabla^2 u=0\\ u|_{t=0}=\psi(\boldsymbol{r})\end{cases}\Rightarrow\begin{cases}\dfrac{\partial g}{\partial t}-c^2\nabla^2 g=0\\ g|_{t=0}=\delta(\boldsymbol{r},\boldsymbol{r}')\end{cases}$$

于是，方程的解为

$$u(\boldsymbol{r},t)=\int G(\boldsymbol{r},\boldsymbol{r}',t)\psi(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

用 Fourier 积分法求 Green 函数：令

$$g(\boldsymbol{r},\boldsymbol{r}',t)=\int g(\boldsymbol{k},\boldsymbol{r}',t)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

$$\Downarrow$$

$$\frac{\mathrm{d}g}{\mathrm{d}t}+c^2k^2g=0;\quad g\,|_{t=0}=\frac{1}{(2\pi)^3}\int\delta(\boldsymbol{r},\boldsymbol{r}')e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}=\frac{e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}}{(2\pi)^3}$$

$$g=\frac{e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'-c^2k^2t}}{(2\pi)^3}\Rightarrow g(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{(2\pi)^3}\int e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')-k^2c^2t}\mathrm{d}^3\boldsymbol{k}$$

$$=g_1g_2g_3$$

其中积分

$$g_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathrm{i}(x-x')k_x-k_x^2c^2t}\mathrm{d}k_x=\frac{1}{\sqrt{4\pi c^2 t}}e^{-\frac{(x-x')^2}{4c^2t}}$$

$$g_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathrm{i}(y-y')k_y-k_y^2c^2t}\mathrm{d}k_y=\frac{1}{\sqrt{4\pi c^2 t}}e^{-\frac{(y-y')^2}{4c^2t}}$$

$$g_3=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathrm{i}(z-z')k_z-k_z^2c^2t}\mathrm{d}k_x=\frac{1}{\sqrt{4\pi c^2 t}}e^{-\frac{(z-z')^2}{4c^2t}}$$

扩散方程的基本解

$$g(\mathbf{r},\mathbf{r}',t)=\frac{1}{(4\pi c^2 t)^{n/2}}e^{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4c^2t}},(n=3,2,1)$$

无限空间的非齐次问题

$$\begin{cases} \dfrac{\partial u}{\partial t} - c^2\nabla^2 u = f(\mathbf{r},t) \\ u\big|_{t=0} = 0 \end{cases}$$

Green函数定义为

$$\begin{cases} \dfrac{\partial \tilde{g}}{\partial t} - c^2\nabla^2 \tilde{g} = \delta(\mathbf{r},\mathbf{r}')\delta(t,t') \\ \tilde{g}(\mathbf{r},\mathbf{r}';t,t')\big|_{t=0} = 0 \end{cases}$$

$$\Downarrow$$

$$u(\mathbf{r},t) = \int_0^\infty \int f(\mathbf{r}',t')\tilde{g}(\mathbf{r},\mathbf{r}';t,t')\mathrm{d}^3\mathbf{r}'\mathrm{d}t'$$

用 Fourier 积分法求 Green 函数：令

$$\tilde{g}(\boldsymbol{r},\boldsymbol{r}',t,t')=\int g(\boldsymbol{k},\boldsymbol{r}',t,t')e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

$$\Downarrow$$

$$\int\left(\frac{\mathrm{d}g}{\mathrm{d}t}+c^2k^2g\right)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}=\delta(\boldsymbol{r},\boldsymbol{r}')\delta(t,t')$$

$$g(\boldsymbol{r},\boldsymbol{r}',0)=\int g(\boldsymbol{k},\boldsymbol{r}',0,t')e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}=0$$

$$\Downarrow$$

$$\frac{\mathrm{d}g}{\mathrm{d}t}+c^2k^2g=\frac{1}{(2\pi)^3}e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}\delta(t,t')$$

常数变易法可求解

$$g(\boldsymbol{k},\boldsymbol{r}',0,t')=0$$

$$g(\boldsymbol{k},\boldsymbol{r}',t,t')=\frac{1}{(2\pi)^3}e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}\int_0^t\delta(\tau,t')\exp[-c^2k^2(t-\tau)]\mathrm{d}\tau$$

$$=\frac{1}{(2\pi)^3}e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}\begin{cases}\exp[-c^2k^2(t-t')],&(t>t')\\0,&(t<t')\end{cases}$$

$$\Downarrow$$

$$\tilde{g}(\boldsymbol{r},\boldsymbol{r}',t,t')=\frac{1}{(2\pi)^3}H(t-t')\int\exp[-c^2k^2(t-t')]e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\mathrm{d}^3\boldsymbol{k}$$

$$=\frac{1}{[4\pi c^2(t-t')]^{3/2}}e^{-\frac{|\boldsymbol{r}-\boldsymbol{r}'|^2}{4c^2t}}H(t-t')$$

$$\boxed{\tilde{g}(\boldsymbol{r},\boldsymbol{r}',t,t')=\frac{1}{[4\pi c^2(t-t')]^{n/2}}e^{-\frac{|\boldsymbol{r}-\boldsymbol{r}'|^2}{4c^2t}}H(t-t'),(n=3,2,1)}$$

因此，无限空间非齐次问题的解

$$u(\mathbf{r},t)=\int_0^{\infty}\int f(\mathbf{r}',t')\frac{H(t-t')}{\left[4\pi c^2(t-t')\right]^{3/2}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4c^2(t-t')}\right]\mathrm{d}^3\mathbf{r}'\mathrm{d}t'$$

$$=\int_0^{t}\int f(\mathbf{r}',t')g(\mathbf{r},\mathbf{r}',t-t')\mathrm{d}^3\mathbf{r}'\mathrm{d}t'$$

无限空间的非齐次、初值问题

$$\frac{\partial u}{\partial t}-c^2\nabla^2 u=f(\mathbf{r},t),(t>0);u|_{t=0}=\psi(\mathbf{r})$$

$$\Downarrow$$

$$u(\mathbf{r},t)=\int g(\mathbf{r},\mathbf{r}',t)\psi(\mathbf{r}')\mathrm{d}^3\mathbf{r}'$$

$$+\int_0^{t}\int f(\mathbf{r}',t')g(\mathbf{r},\mathbf{r}',t-t')\mathrm{d}^3\mathbf{r}'\mathrm{d}t'$$

13.4 广义Green公式和积分解

一般形式的二阶线性偏微分算子为

$$\boldsymbol{L}\equiv\sum_{\mu,\nu=1}^{n}a_{\mu\nu}(\boldsymbol{r})\frac{\partial^2}{\partial x_\mu\partial x_\nu}+\sum_{\mu=1}^{n}b_\mu(\boldsymbol{r})\frac{\partial}{\partial x_\mu}+c(\boldsymbol{r})$$

□ 共轭算子 $\boldsymbol{L}^+$ 其定义为：使下列等式成立

$$(\boldsymbol{L}\psi_i)^*\psi_j-\psi_i^*\boldsymbol{L}^+\psi_j=\sum_{\mu=1}^{n}\frac{\partial R_\mu(\psi_i^*,\psi_j)}{\partial x_\mu}$$

——注意：右边具有散度形式

$$\boldsymbol{L}^+\psi_j\equiv\sum_{\mu,\nu=1}^{n}\frac{\partial^2(a_{\mu\nu}^*\psi_j)}{\partial x_\mu\partial x_\nu}-\sum_{\mu=1}^{n}\frac{\partial(b_\mu^*\psi_j)}{\partial x_\mu}+c^*\psi_j$$

$$R_\mu \equiv \sum_{\nu=1}^{n}\left[a_{\mu\nu}^{*}\psi_j\frac{\partial\psi_i^{*}}{\partial x_\nu}-\psi_i^{*}\frac{\partial(a_{\mu\nu}^{*}\psi_j)}{\partial x_\nu}\right]+b_\mu^{*}\psi_j\psi_i^{*}$$

◻ 广义Green公式

$$\int_G[(\boldsymbol{L}\psi_i)^{*}\psi_j-\psi_i^{*}\boldsymbol{L}^{+}\psi_j]\mathrm{d}\tau=\iint_{\partial G}[(\boldsymbol{P}\psi_i)^{*}\psi_j-\psi_i^{*}\boldsymbol{P}^{+}\psi_j]\mathrm{d}S$$

$$\boldsymbol{P}\psi_i \equiv \sum_{\mu,\nu=1}^{n}a_{\mu\nu}\frac{\partial\psi_i}{\partial x_\nu}\cos(n_\nu,x_\nu)+\beta\psi_i$$

$$\boldsymbol{P}^{+}\psi_j \equiv \sum_{\mu,\nu=1}^{n}a_{\mu\nu}^{*}\frac{\partial\psi_j}{\partial x_\nu}\cos(n_\nu,x_\nu)+(\beta^{*}-b)\psi_j$$

$$b \equiv \sum_{\mu=1}^{n}\left(b_\mu^{*}-\sum_{\nu=1}^{n}\frac{\partial a_{\mu\nu}^{*}}{\partial x_\nu}\right)\cos(n_\nu,x_\nu)$$

□ 自共轭算子：$\boldsymbol{L}=\boldsymbol{L}^{+}$

注意：与Hermite对称的区别，与边界的关系

例 下列3个典型的微分算子与它们的共轭算子

①实系数三维S-L算子——自共轭算子

$$\boldsymbol{L}=-\nabla\cdot[p(\boldsymbol{r})\nabla]+q(\boldsymbol{r});\boldsymbol{L}^{+}=-\nabla\cdot(p\nabla\psi)+q\psi$$

②实系数三维波动算子——自共轭算子

$$\boldsymbol{\Pi}=\frac{\partial^2}{\partial t^2}-\nabla\cdot[p(\boldsymbol{r})\nabla]+q(\boldsymbol{r});\boldsymbol{\Pi}^{+}=\frac{\partial^2}{\partial t^2}-\nabla\cdot[p(\boldsymbol{r})\nabla]+q(\boldsymbol{r})$$

③实系数三维热扩散算子——非自共轭算子

$$\boldsymbol{\Pi}=\frac{\partial}{\partial t}-\nabla\cdot[p(\boldsymbol{r})\nabla]+q(\boldsymbol{r});\boldsymbol{\Pi}^{+}=-\frac{\partial}{\partial t}-\nabla\cdot[p(\boldsymbol{r})\nabla]+q(\boldsymbol{r})$$

□ 三维Laplace算子或者Helmholtz算子

$$\boldsymbol{L}=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+k_0^2;\ \boldsymbol{L}^{+}=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+k_0^2$$

实数,如果有吸收,就不自共轭

$$\boldsymbol{P}u\equiv\sum_{i=1}^{3}\cos(n_i,x_i)\frac{\partial u}{\partial x_i}+\beta u=\frac{\partial u}{\partial n}+\beta u$$

$$\boldsymbol{P}^{+}v\equiv\sum_{i=1}^{3}\cos(n_i,x_i)\frac{\partial v}{\partial x_i}+\beta v=\frac{\partial v}{\partial n}+\beta v$$

$$\int_G(v\boldsymbol{L}u-u\boldsymbol{L}^{+}v)\mathrm{d}\tau=\iint_{\partial G}\left[v\left(\frac{\partial u}{\partial n}+\beta u\right)-u\left(\frac{\partial v}{\partial n}+\beta v\right)\right]\mathrm{d}S$$

$$=\iint_{\partial G}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)\mathrm{d}S$$

□ 边值问题的积分解

$$\boldsymbol{L}\psi = f(\boldsymbol{r}),\ \ \boldsymbol{r}\in\Omega;\ \ \boldsymbol{P}\psi\,|_{\partial\Omega} = B(\boldsymbol{r}),\ \ \boldsymbol{r}\in\partial\Omega$$

分别定义*L*和*L*⁺的Green函数

$$\boldsymbol{L}G(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'),\ \ \boldsymbol{r},\boldsymbol{r}'\in\Omega$$

$$\boldsymbol{P}G(\boldsymbol{r},\boldsymbol{r}')\Big|_{\partial\Omega} = 0,\ \ \boldsymbol{r}'\in\Omega+\partial\Omega$$

$$\boldsymbol{L}^{+}G^{+}(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'),\ \ \boldsymbol{r},\boldsymbol{r}'\in\Omega$$

$$\boldsymbol{P}^{+}G^{+}(\boldsymbol{r},\boldsymbol{r}')\Big|_{\partial\Omega} = 0,\ \ \boldsymbol{r}'\in\Omega+\partial\Omega$$

$$\Downarrow$$

$$\int_{\Omega}[(\boldsymbol{L}\psi)^{*}\psi^{+} - \psi^{*}\boldsymbol{L}^{+}\psi^{+}]\mathrm{d}\Omega = \iint_{\partial\Omega}[(\boldsymbol{P}\psi)^{*}\psi^{+} - \psi^{*}\boldsymbol{P}^{+}\psi^{+}]\mathrm{d}\Sigma$$

$$\Uparrow$$

$$\psi^{+} = G^{+}(\boldsymbol{r},\boldsymbol{r}')$$

$$\psi^*(\boldsymbol{r}') = \int_\Omega G^+(\boldsymbol{r},\boldsymbol{r}') f^*(\boldsymbol{r})\mathrm{d}\Omega - \iint_{\partial\Omega} B^*(\boldsymbol{r}) G^+(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}\Sigma$$

$$\Downarrow$$

$$\psi(\boldsymbol{r}) = \int_\Omega [G^+(\boldsymbol{r}',\boldsymbol{r})]^* f(\boldsymbol{r}')\mathrm{d}\Omega' - \iint_{\partial\Omega} B(\boldsymbol{r}')[G^+(\boldsymbol{r}',\boldsymbol{r})]^*\mathrm{d}\Sigma'$$

$$\Uparrow$$

$$G^+(\boldsymbol{r}',\boldsymbol{r}) = G^*(\boldsymbol{r},\boldsymbol{r}')$$

$$\Downarrow$$

$$\psi(\boldsymbol{r}) = \int_\Omega G(\boldsymbol{r},\boldsymbol{r}') f(\boldsymbol{r}')\mathrm{d}\Omega' - \iint_{\partial\Omega} B(\boldsymbol{r}') G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}\Sigma'$$

——对含时间的波动方程或者热扩散方程,由于时间变量t的特殊性，问题较为复杂,下面把时间变量和空间变量分开处理.

有限空间波动方程的混合问题

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = f(\boldsymbol{r}, t), \quad (t > 0, \ \boldsymbol{r} \in V)$$

$$u|_{t=0} = \psi_1(\boldsymbol{r}), u_t|_{t=0} = \psi_2(\boldsymbol{r}), (\boldsymbol{r} \in V + \partial V)$$

$$\left.\left(\alpha u + \beta \frac{\partial u}{\partial n}\right)\right|_{\partial V} = b(\boldsymbol{r}, t), \quad (t > 0, \ \boldsymbol{r} \in \partial V)$$

定义波动算子和共轭算子的Green函数分别满足

$$\frac{\partial^2 G(\boldsymbol{r}, \boldsymbol{r}', t)}{\partial t^2} - c^2 \nabla^2 G(\boldsymbol{r}, \boldsymbol{r}', t) = 0, \quad t > 0$$

$$G|_{t=0} = 0, \quad G_t\big|_{t=0} = \delta(\boldsymbol{r}, \boldsymbol{r}')$$

$$\left.\left(\alpha G + \beta \frac{\partial G}{\partial n}\right)\right|_{\partial V} = 0, \ \boldsymbol{r} \in \partial V$$

$$\frac{\partial^2 G^+(\boldsymbol{r},\boldsymbol{r}',t)}{\partial t^2} - c^2\nabla^2 G^+(\boldsymbol{r},\boldsymbol{r}',t) = 0, \quad 0 < t < T$$

$$G^+ |_{t=T} = 0;\ G^+ |_{t=T} = \delta(\boldsymbol{r},\boldsymbol{r}')$$

$$\left.\left(\alpha G^+ + \beta\frac{\partial G^+}{\partial n}\right)\right|_{\partial V} = 0,\ \ \boldsymbol{r} \in \partial V$$

注意：①波动算子是自共轭算子；②对共轭算子，给定*T*时刻的初始条件，求*t*<*T*的分布

对空间变量应用Green公式

$$\int_V (u^*\nabla^2 G^+ - G^+\nabla^2 u^*)\mathrm{d}V = \iint_B \left(u^*\frac{\partial G^+}{\partial n} - G^+\frac{\partial u^*}{\partial n}\right)\mathrm{d}S$$

$$\int_V \frac{\partial}{\partial t}\left(u^* \frac{\partial G^+}{\partial t} - G^+ \frac{\partial u^*}{\partial t}\right)\mathrm{d}V + \int_V G^+(\mathbf{r},\mathbf{r}',t) f^*(\mathbf{r},t)\mathrm{d}V$$

$$= c^2 \iint_B \left(u^* \frac{\partial G^+}{\partial n} - G^+ \frac{\partial u^*}{\partial n}\right)\mathrm{d}S$$

上式两边对时间*t*积分，并且注意到

$$\int_0^T \int_V \frac{\partial}{\partial t}\left(u^* \frac{\partial G^+}{\partial t} - G^+ \frac{\partial u^*}{\partial t}\right)\mathrm{d}V\mathrm{d}t = \int_V \left(u^* \frac{\partial G^+}{\partial t} - G^+ \frac{\partial u^*}{\partial t}\right)_{t=0}^{t=T} \mathrm{d}V$$

$$= \int_V u^* G_t^+ \big|_{t=T} \,\mathrm{d}V - \int_V \left(\psi_1^* G_t^+ - G^+ \psi_2^*\right)\big|_{t=0} \,\mathrm{d}V$$

$$= u^*(\mathbf{r}',T) - \int_V \left(\psi_1^* G_t^+ - G^+ \psi_2^*\right)\big|_{t=0} \,\mathrm{d}V$$

$$u^*(\mathbf{r}',T)=\int_V\left(\psi_1^* G_t^+ - G^+\psi_2^*\right)\big|_{t=0}\,\mathrm{d}V-\int_0^T\int_V G^+(\mathbf{r},\mathbf{r}',t)f^*(\mathbf{r},t)\mathrm{d}V\mathrm{d}t$$

$$+c^2\int_0^T\iint_B\left(u^*\frac{\partial G^+}{\partial n}-G^+\frac{\partial u^*}{\partial n}\right)\mathrm{d}S\mathrm{d}t$$

右边面积分项

$$\left(\alpha G^+ +\beta\frac{\partial G^+}{\partial n}\right)\Bigg|_{\partial V}=0 \qquad \left(u^*\frac{\partial G^+}{\partial n}-G^+\frac{\partial u^*}{\partial n}\right)\Bigg|_{B_1}=-\frac{1}{\beta}G^+ b^*(\mathbf{r},t)$$

$$\left(\alpha u^* +\beta\frac{\partial u^*}{\partial n}\right)\Bigg|_{\partial V}=b^*(\mathbf{r},t) \qquad \left(u^*\frac{\partial G^+}{\partial n}-G^+\frac{\partial u^*}{\partial n}\right)\Bigg|_{B_2}=\frac{1}{\alpha}\frac{\partial G^+}{\partial n}b^*(\mathbf{r},t)$$

两边取复共轭并且交换变量

$$\mathbf{r}'\leftrightarrow\mathbf{r}$$

$$u(\mathbf{r},T)=\int_V\left(\psi_1[G_t^+]^*-[G^+]^*\psi_2\right)\big|_{t=0}\,\mathrm{d}V'$$

$$-\int_0^T\int_V[G^+(\mathbf{r}',\mathbf{r},\tau)]^*f(\mathbf{r},\tau)\mathrm{d}V'\mathrm{d}\tau$$

$$-c^2\int_0^T\iint_{B_1}\frac{1}{\beta}[G^+]^*b(\mathbf{r}',\tau)\mathrm{d}S'\mathrm{d}\tau$$

$$+c^2\int_0^T\iint_{B_2}\frac{1}{\alpha}\frac{\partial[G^+]^*}{\partial n'}b(\mathbf{r}',\tau)\mathrm{d}S'\mathrm{d}\tau$$

Green函数的对称性

$$G(\mathbf{r},\mathbf{r}',t)\equiv\sum_{m=0}^{\infty}\frac{1}{c\sqrt{\lambda_m}}\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')\sin\left(c\sqrt{\lambda_m}t\right)$$

$$G^+(\mathbf{r},\mathbf{r}',t)\equiv\sum_{m=0}^{\infty}\frac{1}{c\sqrt{\lambda_m}}\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')\sin\left[c\sqrt{\lambda_m}(t-T)\right]$$

$$[G^+(\mathbf{r}',\mathbf{r},t)]^* = G(\mathbf{r},\mathbf{r}',t-T) = -G(\mathbf{r},\mathbf{r}',T-t)$$

$$u(\mathbf{r},T) = \frac{\partial}{\partial T}\int_V \psi_1(\mathbf{r}')G(\mathbf{r},\mathbf{r}',T)\mathrm{d}V' + \int_V G(\mathbf{r},\mathbf{r}',T)\psi_2(\mathbf{r}')\mathrm{d}V'$$

$$+\int_0^T\int_V G(\mathbf{r},\mathbf{r}',T-\tau)f(\mathbf{r},\tau)\mathrm{d}V'\mathrm{d}\tau$$

$$+c^2\int_0^T\left[\begin{array}{l}\displaystyle\iint_{B_1}\frac{1}{\beta}G(\mathbf{r},\mathbf{r}',T-\tau)b(\mathbf{r}',\tau)\mathrm{d}S' \\ \displaystyle-\iint_{B_2}\frac{1}{\alpha}\frac{\partial G(\mathbf{r},\mathbf{r}',T-\tau)}{\partial n'}b(\mathbf{r}',\tau)\mathrm{d}S'\end{array}\right]\mathrm{d}\tau$$

——由*T*的任意性，上式就是有限区域波动方程的积分解——与10章得到的结果一致。

有限空间扩散方程的混合问题

$$\frac{\partial u}{\partial t}-c^2\nabla^2 u = f(\boldsymbol{r},t), \quad (t>0,\ \boldsymbol{r}\in V)$$

$$u\,|_{t=0}=\psi(\boldsymbol{r})$$

$$\left.\left(\alpha u+\beta\frac{\partial u}{\partial n}\right)\right|_{\partial V}=b(\boldsymbol{r},t), \quad (t>0,\ \boldsymbol{r}\in\partial V)$$

定义扩散算子和共轭算子的Green函数分别满足

$$\frac{\partial G(\boldsymbol{r},\boldsymbol{r}',t)}{\partial t}-c^2\nabla^2 G(\boldsymbol{r},\boldsymbol{r}',t)=0, \quad t>0$$

$$G\,|_{t=0}=\delta(\boldsymbol{r},\boldsymbol{r}')$$

$$\left.\left(\alpha G+\beta\frac{\partial G}{\partial n}\right)\right|_{\partial V}=0,\ \ \boldsymbol{r}\in\partial V$$

$$-\frac{\partial G^{+}(\boldsymbol{r},\boldsymbol{r}',t)}{\partial t}-c^{2}\nabla^{2}G^{+}(\boldsymbol{r},\boldsymbol{r}',t)=0,\quad 0<t<T$$

$$G^{+}\big|_{t=T}=\delta(\boldsymbol{r},\boldsymbol{r}')$$

$$\left.\left(\alpha G^{+}+\beta\frac{\partial G^{+}}{\partial n}\right)\right|_{\partial V}=0,\ \ \boldsymbol{r}\in\partial V$$

注意：①扩散算子不是自共轭算子；②对共轭算子，给定*T*时刻的初始条件，求*t*<*T*的分布，解是稳定的

对空间变量应用Green公式

$$\int_{V}(u^{*}\nabla^{2}G^{+}-G^{+}\nabla^{2}u^{*})\mathrm{d}V=\iint_{B}\left(u^{*}\frac{\partial G^{+}}{\partial n}-G^{+}\frac{\partial u^{*}}{\partial n}\right)\mathrm{d}S$$

$$\int_V \frac{\partial}{\partial t}(u^* G^+)\mathrm{d}V = \int_V G^+ f^*(\boldsymbol{r},t)\mathrm{d}V - c^2 \iint_B \left(u^* \frac{\partial G^+}{\partial n} - G^+ \frac{\partial u^*}{\partial n} \right) \mathrm{d}S$$

上式两边对时间*t*积分,并且注意到面积分与前类似

$$u^*(\boldsymbol{r}',T) = \int_V \psi^*(\boldsymbol{r}) G^+|_{t=0} \mathrm{d}V + \int_0^T \int_V G^+ f^*(\boldsymbol{r},t)\mathrm{d}V\mathrm{d}t$$

$$+c^2 \int_0^T \left[\begin{array}{l} \displaystyle\iint_{B_1} \frac{1}{\beta} G^+ b^*(\boldsymbol{r},t)\mathrm{d}S \\ \displaystyle -\iint_{B_2} \frac{1}{\alpha} \frac{\partial G^+}{\partial n} b^*(\boldsymbol{r},t)\mathrm{d}S \end{array} \right] \mathrm{d}t$$

两边取复共轭并且交换变量

$$\boldsymbol{r}' \leftrightarrow \boldsymbol{r}$$

$$u(\mathbf{r},T)=\int_V \psi(\mathbf{r}')[G^+(\mathbf{r}',\mathbf{r},t)]^*|_{t=0}\mathrm{d}V'+\int_0^T\int_V [G^+(\mathbf{r}',\mathbf{r},t)]^* f(\mathbf{r}',t)\mathrm{d}V'\mathrm{d}t$$

$$+c^2\int_0^T\left[\begin{array}{l}\displaystyle\iint_{B_1}\frac{1}{\beta}[G^+(\mathbf{r}',\mathbf{r},t)]^* b(\mathbf{r}',t)\mathrm{d}S' \\ \displaystyle-\iint_{B_2}\frac{1}{\alpha}\frac{\partial[G^+(\mathbf{r}',\mathbf{r},t)]^*}{\partial n'}b(\mathbf{r},t)\mathrm{d}S'\end{array}\right]\mathrm{d}t$$

Green函数的对称性

$$G(\mathbf{r},\mathbf{r}',t)=\sum_{m=1}^{\infty}\exp(-c^2\lambda_m t)\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')$$

$$G^+(\mathbf{r},\mathbf{r}',t)=\sum_{m=1}^{\infty}\exp[c^2\lambda_m(t-T)]\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')$$

$$\Downarrow$$

$$[G^+(\mathbf{r}',\mathbf{r},t)]^*=G(\mathbf{r},\mathbf{r}',T-t)$$

$$u(\mathbf{r},T)=\int_V \psi(\mathbf{r}')G(\mathbf{r},\mathbf{r}',T)\mathrm{d}V'+\int_0^T\int_V G(\mathbf{r},\mathbf{r}',T-\tau)f(\mathbf{r}',\tau)\mathrm{d}V'\mathrm{d}\tau$$

$$+c^2\int_0^T\left[\begin{array}{l}\displaystyle\iint_{B_1}\frac{1}{\beta}G(\mathbf{r},\mathbf{r}',T-\tau)b(\mathbf{r}',\tau)\mathrm{d}S' \\ \displaystyle-\iint_{B_2}\frac{1}{\alpha}\frac{\partial G(\mathbf{r},\mathbf{r}',T-\tau)}{\partial n'}b(\mathbf{r},\tau)\mathrm{d}S'\end{array}\right]\mathrm{d}\tau$$

$$\Downarrow$$

$$u(\mathbf{r},t)=\int_V \psi(\mathbf{r}')G(\mathbf{r},\mathbf{r}',t)\mathrm{d}V'+\int_0^t\int_V G(\mathbf{r},\mathbf{r}',t-\tau)f(\mathbf{r}',\tau)\mathrm{d}V'\mathrm{d}\tau$$

$$+c^2\int_0^T\left[\begin{array}{l}\displaystyle\iint_{B_1}\frac{1}{\beta}G(\mathbf{r},\mathbf{r}',t-\tau)b(\mathbf{r}',\tau)\mathrm{d}S' \\ \displaystyle-\iint_{B_2}\frac{1}{\alpha}\frac{\partial G(\mathbf{r},\mathbf{r}',t-\tau)}{\partial n'}b(\mathbf{r},\tau)\mathrm{d}S'\end{array}\right]\mathrm{d}\tau$$

13.5 把微分方程化成积分方程

Green函数的目的：(1)解方程；(2)把微分方程转化为积分方程——更有意义.

□ 一般形式

$$\boldsymbol{L}[u(\boldsymbol{r})]=\varepsilon \boldsymbol{L}_1[u(\boldsymbol{r}),\boldsymbol{r}]\equiv f(\boldsymbol{r})$$

定义Green函数 (基本解)——已经求得

$$\boldsymbol{L}[g(\boldsymbol{r},\boldsymbol{r}')]=\delta(\boldsymbol{r},\boldsymbol{r}')$$

$$\boldsymbol{L}[u_0(\boldsymbol{r})]=0$$

$$u(\boldsymbol{r})=\boldsymbol{L}^{-1}f(\boldsymbol{r})=u_0(\boldsymbol{r})+\int f(\boldsymbol{r}')g(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

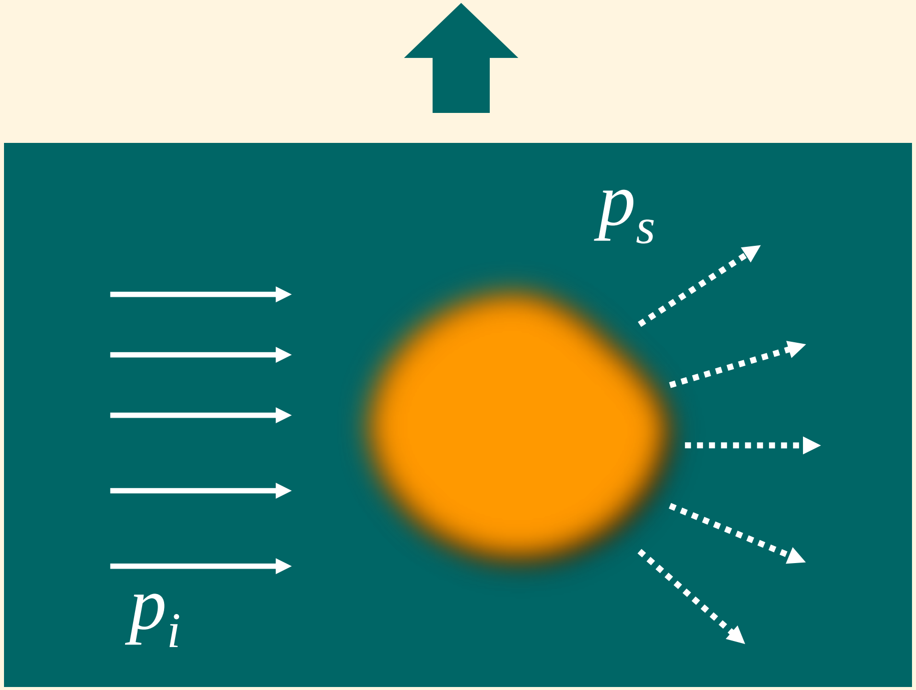
$$=u_0(\boldsymbol{r})+\varepsilon\int \boldsymbol{L}_1[u(\boldsymbol{r}'),\boldsymbol{r}']g(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

形式解

意义：①微分方程转化成积分方程，有利于数值计算；②当ε足够小，容易得到迭代公式 。

- 非均匀区的声散射

$$-(\nabla^2+k_0^2)p(\mathbf{r},\omega)=k_0^2\gamma_\kappa(\mathbf{r})p(\mathbf{r},\omega)$$



利用无界空间的Green函数

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp(\mathrm{i}k_0 |\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$p(\boldsymbol{r},\omega) = p_i(\boldsymbol{r},\omega) + k_0^2\int_V \gamma_\kappa(\boldsymbol{r}')p(\boldsymbol{r}',\omega)g(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

——第二类Fredholm积分方程

Born近似

当散射比较“弱”可以用迭代法求解方程

第一次近似(Born近似)

$$p(\boldsymbol{r},\omega) \approx p_i(\boldsymbol{r},\omega) + p_1(\boldsymbol{r},\omega)$$

$$p_1(\boldsymbol{r},\omega) \equiv k_0^2\int_V \gamma_\kappa(\boldsymbol{r}')p_i(\boldsymbol{r}',\omega)g(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

第二次近似

$$p(\mathbf{r},\omega) \approx p_i(\mathbf{r},\omega) + p_1(\mathbf{r},\omega) + p_2(\mathbf{r},\omega)$$

$$p_2(\mathbf{r},\omega) \equiv k_0^2 \int_V \gamma_\kappa(\mathbf{r}') p_1(\mathbf{r},\omega) g(\mathbf{r},\mathbf{r}') \mathrm{d}^3\mathbf{r}'$$

第*N*次近似

$$p(\mathbf{r},\omega) = p_i(\mathbf{r},\omega) + \sum_{j=1}^{N} p_j(\mathbf{r},\omega)$$

$$\Uparrow$$

$$p_N(\mathbf{r},\omega) \equiv k_0^2 \int_V \gamma_\kappa(\mathbf{r}') p_{N-1}(\mathbf{r}',\omega) g(\mathbf{r},\mathbf{r}') \mathrm{d}^3\mathbf{r}'$$

当$N\to\infty$时，由$(p_1,p_2,\ldots,p_N,\ldots)$形成的级数称为Born级数. Born级数的收敛性讨论在理论上讲非常困难. 充分条件：①低频($k_0a<<1$); ②非均匀度较小($\|\gamma_\kappa(r)\|<<1$).

量子力学散射

Lippman-Schwinger积分方程

把Schrödinger方程改写成形式

$$-(\nabla^2 + k^2)\psi = -\frac{2m}{\hbar^2}U(\mathbf{r})\psi$$

入射粒子波数

$k^2 = 2mE/\hbar^2$

定义Green 函数

$$-(\nabla^2 + k^2)g(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}') \Rightarrow g(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

Schrödinger方程化成积分方程

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{2m}{\hbar^2}\int g(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}')\mathrm{d}^3\mathbf{r}'$$

入射波

$$(\nabla^2+k^2)\psi_0=0 \Rightarrow \psi_0(\boldsymbol{r})=\psi_i(\boldsymbol{r})=\psi_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$$

Lippman-Schwinger积分方程

$$\psi(\boldsymbol{r})=\psi_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})-\frac{2m}{\hbar^2}\int g(\boldsymbol{r},\boldsymbol{r}')U(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

$$=\psi_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})-\frac{m}{2\pi\hbar^2}\int\frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}U(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

叠代求解

0级近似，即为入射波

$$\psi^{(0)}(\boldsymbol{r})\approx\psi_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$$

1级近似，Born近似

$$\psi^{(1)}(\boldsymbol{r}) \approx \psi_0 \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) - \frac{m}{2\pi\hbar^2}\int \frac{\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}U(\boldsymbol{r}')\psi^{(0)}(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

$$\psi^{(0)}(\boldsymbol{r}) = \psi_0 \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$$

❑远场特性

空间Fourier变换

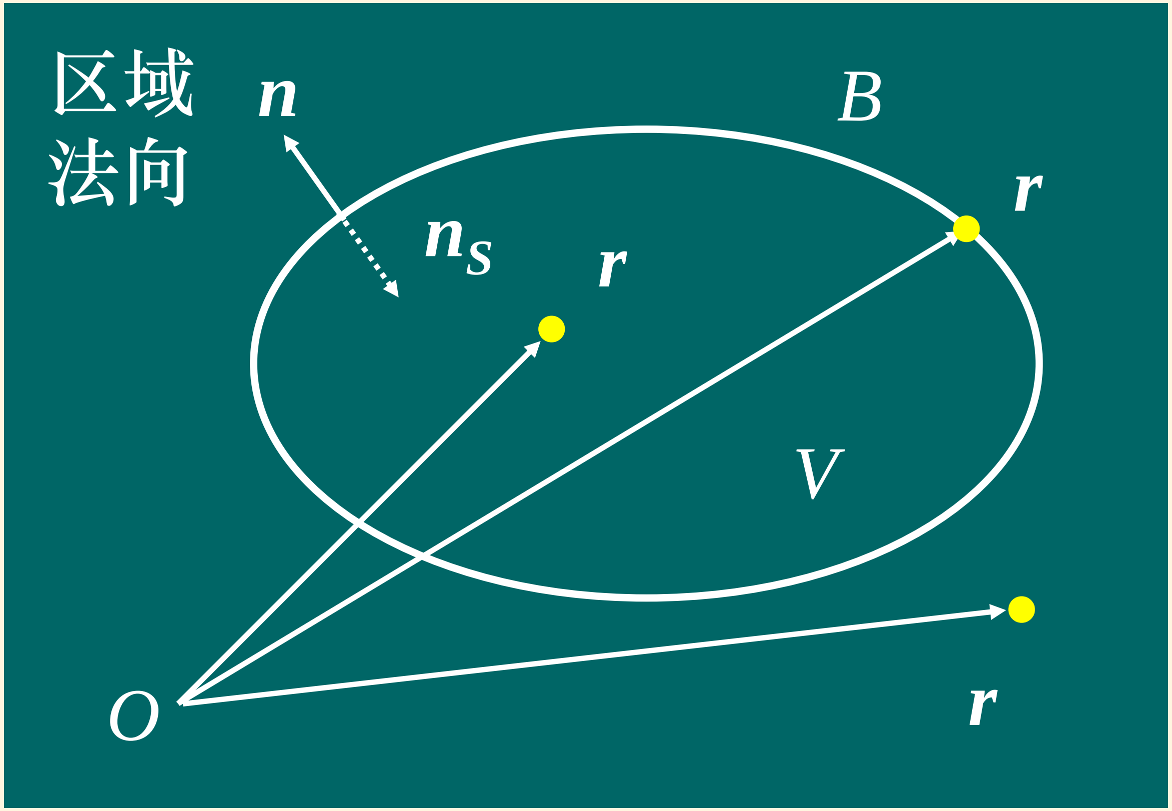
$$\lim_{|\boldsymbol{r}|\to\infty}\psi_s(\boldsymbol{r}) = -\psi_0 \frac{2m}{\hbar^2}\frac{e^{\mathrm{i}k|\boldsymbol{r}|}}{4\pi|\boldsymbol{r}|}\int e^{\mathrm{i}k(\boldsymbol{e}_i-\boldsymbol{e}_r)\cdot\boldsymbol{r}'}U(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

$$\boldsymbol{e}_r \equiv \boldsymbol{r}/|\boldsymbol{r}|;\boldsymbol{e}_i \equiv \boldsymbol{k}/|\boldsymbol{k}|$$

$$|\boldsymbol{r}-\boldsymbol{r}'| \approx |\boldsymbol{r}|\left(1-\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{|\boldsymbol{r}|^2}\right);\quad \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\approx\frac{1}{|\boldsymbol{r}|}$$

Kirchhoff积分公式——边界元方法

场中的任意曲面*B*，包围的区域*V*，如果区域*V*内不存在源



$$\int_V (u\nabla'^2 g - g\nabla'^2 u)\mathrm{d}V' = \iint_B \left(u\frac{\partial g}{\partial n'} - g\frac{\partial u}{\partial n'} \right)\mathrm{d}S'$$

$$-(\nabla'^2 + k_0^2)g = \delta(\boldsymbol{r}', \boldsymbol{r}); \quad -(\nabla'^2 + k_0^2)u = f(\boldsymbol{r}')$$

① 点*r*在区域*V*外： $\boldsymbol{r} \neq \boldsymbol{r}' \Rightarrow -(\nabla'^2 + k_0^2)g = 0$

$$\int_V g(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}')\mathrm{d}V' = \iint_B \left(u\frac{\partial g}{\partial n'} - g\frac{\partial u}{\partial n'} \right)\mathrm{d}S'$$

$$f(\boldsymbol{r}') = 0$$

$$\iint_B \left(u\frac{\partial g}{\partial n'_S} - g\frac{\partial u}{\partial n'_S} \right)\mathrm{d}S' = 0$$

②点$\boldsymbol{r}$在区域V内：当$\boldsymbol{r}'$在V上作体积分时，总有可能$\boldsymbol{r}'=\boldsymbol{r}$, 于是

$$-(\nabla'^2+k_0^2)g=\delta(\boldsymbol{r}',\boldsymbol{r})$$

$$\Downarrow$$

$$u(\boldsymbol{r})=\int_V g(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\mathrm{d}V'+\iint_B\left(u\frac{\partial g}{\partial n_S'}-g\frac{\partial u}{\partial n_S'}\right)\mathrm{d}S'$$

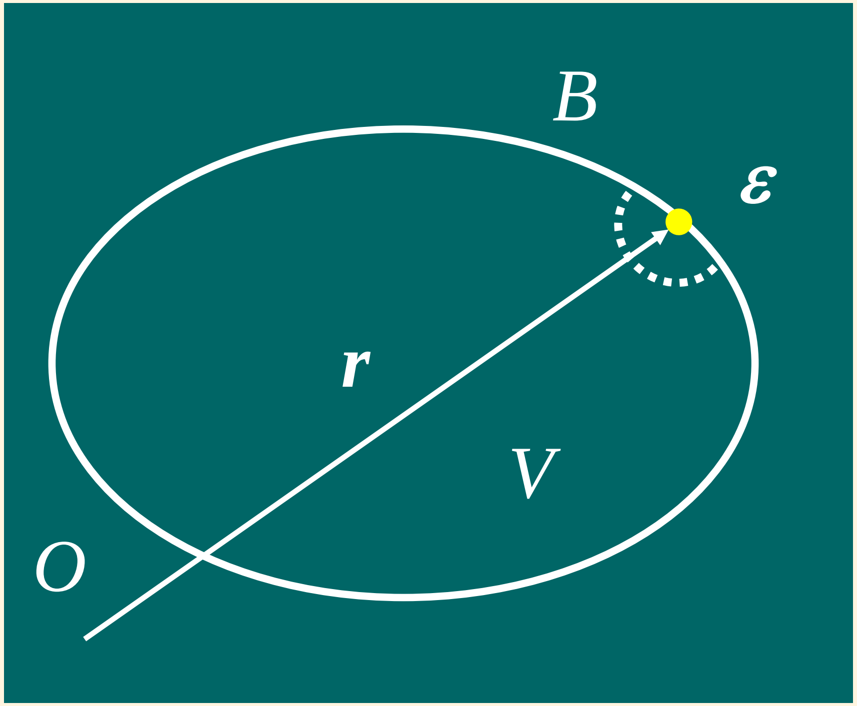
$$\Downarrow$$

$$u(\boldsymbol{r})=\iint_B\left(u\frac{\partial g}{\partial n_S'}-g\frac{\partial u}{\partial n_S'}\right)\mathrm{d}S'$$

③点$\boldsymbol{r}$恰好在区域V的边界B：当$\boldsymbol{r}'$在B上作面积分时，总有可能$\boldsymbol{r}'=\boldsymbol{r}$, 于是上式是一个反常积分. 在$\boldsymbol{r}$周围去掉半径为$\varepsilon$的半球，形成新的区域$V-\varepsilon$, 则$\boldsymbol{r}$在新区域外，于是

$$\iint_{B-\varepsilon}\left(u\frac{\partial g}{\partial n'}-g\frac{\partial u}{\partial n'}\right)\mathrm{d}S'=0$$

$$P\iint_{B}\left(u\frac{\partial g}{\partial n'}-g\frac{\partial u}{\partial n'}\right)\mathrm{d}S'+\lim_{\varepsilon\to 0}\iint_{\varepsilon}\left(u\frac{\partial g}{\partial n'}-g\frac{\partial u}{\partial n'}\right)\mathrm{d}S'=0$$



作运算

$$\frac{\partial g(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n'_\varepsilon}=-\boldsymbol{e}_R\cdot\nabla' g(R)=-\frac{\mathrm{d}g(R)}{\mathrm{d}R}(\boldsymbol{e}_R\cdot\nabla' R)$$

$$=-\boldsymbol{e}_R\cdot\boldsymbol{e}_R\frac{\mathrm{i}k_0R-1}{R}g(R)=-\frac{\mathrm{i}k_0\varepsilon-1}{4\pi\varepsilon^2}\exp(\mathrm{i}\varepsilon)$$

$$R=|\boldsymbol{r}-\boldsymbol{r}'|=\varepsilon;\boldsymbol{e}_R=(\boldsymbol{r}'-\boldsymbol{r})/R$$

$$\lim_{\varepsilon\to 0}\iint_\varepsilon\left[u(\boldsymbol{r}')\frac{\partial g(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n'_\varepsilon}-g(|\boldsymbol{r}-\boldsymbol{r}'|)\frac{\partial u(\boldsymbol{r}')}{\partial n'_\varepsilon}\right]\mathrm{d}S'$$

$$=\lim_{\varepsilon\to 0}\left[-u(\boldsymbol{r}+\boldsymbol{\varepsilon})\frac{\mathrm{i}k_0\varepsilon-1}{4\pi\varepsilon^2}\exp(\mathrm{i}\varepsilon)-\frac{1}{4\pi\varepsilon}\exp(\mathrm{i}\varepsilon)\frac{\partial u(\boldsymbol{r}+\boldsymbol{\varepsilon})}{\partial R}\right]2\pi\varepsilon^2$$

$$=\frac{1}{2}u(\boldsymbol{r})-\lim_{\varepsilon\to 0}\varepsilon\frac{\partial u(\boldsymbol{r})}{\partial R}=\frac{1}{2}u(\boldsymbol{r}),(\boldsymbol{\varepsilon}\equiv\boldsymbol{r}'-\boldsymbol{r})$$

因此，点*r*恰好在区域*V*的边界*B*时

$$P\iint_B\left(u\frac{\partial g}{\partial n_S'}-g\frac{\partial u}{\partial n_S'}\right)\mathrm{d}S'=\lim_{\varepsilon\to 0}\iint_\varepsilon\left(u\frac{\partial g}{\partial n'}-g\frac{\partial u}{\partial n'}\right)\mathrm{d}S'=\frac{1}{2}u(\boldsymbol{r})$$

于是，合成为

$$\int_V g(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\mathrm{d}V'+P\iint_B\left(u\frac{\partial g}{\partial n_S'}-g\frac{\partial u}{\partial n_S'}\right)\mathrm{d}S'=\begin{cases}u(\boldsymbol{r}), & (\boldsymbol{r}\in V)\\ \dfrac{1}{2}u(\boldsymbol{r}), & (\boldsymbol{r}\in B)\\ 0, & (\boldsymbol{r}\notin V+B)\end{cases}$$

注意：*g*是自由空间的Green函数，即基本解，故上式仍然是关于*u*的积分方程.

$$g=\frac{1}{4\pi}\frac{\exp(\mathrm{i}k_0\,|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|};g=\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

■ 边界元方法

①数值求解下列积分方程，得到边界上的*u*和法向导数

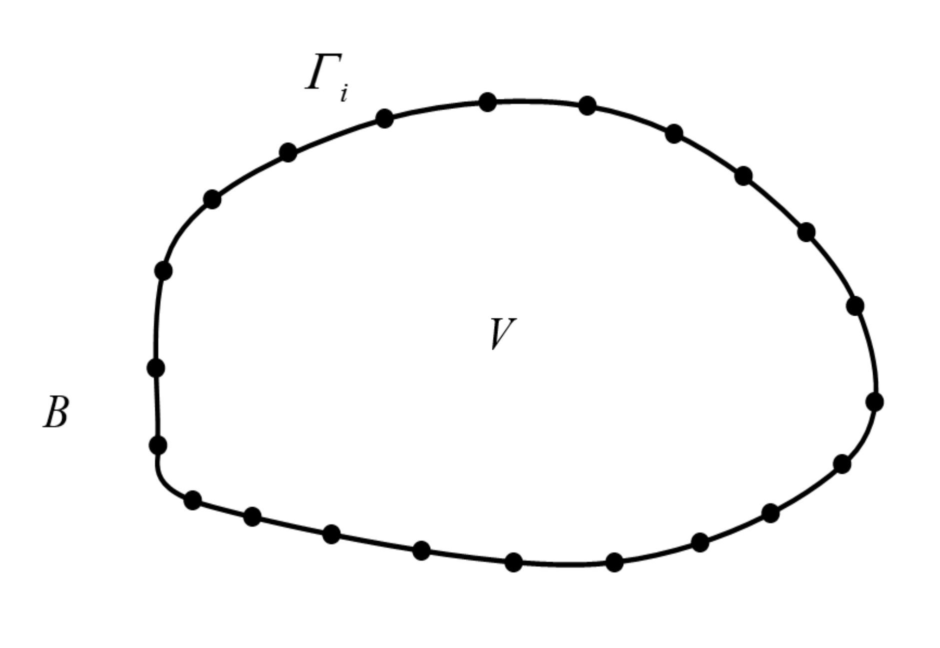
$$\int_V g(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\mathrm{d}V' + P\iint_B \left(u\frac{\partial g}{\partial n_S'} - g\frac{\partial u}{\partial n_S'}\right)\mathrm{d}S' = \frac{1}{2}u(\mathbf{r}),\ (\mathbf{r}\in B)$$

②然后计算体内场的分布

$$u(\mathbf{r}) = \int_V g(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\mathrm{d}V'$$

$$+P\iint_B \left(u\frac{\partial g}{\partial n_S'} - g\frac{\partial u}{\partial n_S'}\right)\mathrm{d}S'$$

$$(\mathbf{r}\in V)$$



■ **外区域问题(散射问题):*V*以外的无限大空间*G*(无限远处边界+*B*形成闭合曲面)**