第11章 球函数及其应用

11.1 Legendre多项式及其应用

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Legendre函数，径向和极角方向边界条件，半球问题

11.1 Legendre多项式及其应用

球坐标中： Laplace方程或Helmholtz方程

$$\nabla^2 u(r,\vartheta,\varphi)=0$$

$$\nabla^2 u(r,\vartheta,\varphi)+k^2 u(r,\vartheta,\varphi)=0$$

分离变量

■**径向：满足径向方程(Euler方程或球Bessel方程)**

■**角度方向：满足球函数方程**

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial Y(\vartheta,\varphi)}{\partial\vartheta}\right]+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y(\vartheta,\varphi)}{\partial\varphi^2}+\nu(\nu+1)Y(\vartheta,\varphi)=0$$

❑Legendre多项式

进一步分离变量

$$Y(\vartheta,\varphi)=\Theta(\vartheta)\Phi(\varphi)$$

方位角部分：本征值问题

$$\begin{cases} \Phi''+\lambda\Phi=0 \\ \Phi(\varphi)=\Phi(2\pi+\varphi) \end{cases}$$

解为

$$\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi} \quad (m=0,\pm1,\pm2,\ldots.)$$

$$\Phi_m(\varphi)=A_m\sin(m\varphi)+B_m\cos(m\varphi)$$

$$(m=0,1,2,\ldots)$$

极角部分：本征值问题

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\sin\vartheta\frac{\mathrm{d}\Theta(\vartheta)}{\mathrm{d}\vartheta}\right]+\left[\nu(\nu+1)-\frac{m^2}{\sin^2\vartheta}\right]\Theta(\vartheta)=0$$

北极和南极: $\vartheta=0,\pi$是方程的奇点，存在自然边界条件

$$\Theta(\vartheta)\,|_{\vartheta=0,\pi}<\infty$$

令 $x=\cos\vartheta$，连带Legendre方程本征值问题

$$\begin{cases}-\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]+\dfrac{m^2}{1-x^2}\Theta=\nu(\nu+1)\Theta\\[2ex] \Theta(x)\,|_{x=\pm1}<\infty\end{cases}$$

当 m=0 时，为 Legendre 方程的本征值问题

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]=\nu(\nu+1)\Theta \\ \Theta(x)\big|_{x=\pm 1}<\infty \end{cases}$$

当 $\nu=l$ 是零和正整数时 Legendre 方程存在 $x=\pm 1$ 有限的解—Legendre多项式，表示为 $P_l(x)$。级数形式为

$$P_l(x)=\sum_{0}^{[l/2]}\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k}$$

*(-1)^k
原式少了这个

$$[l/2]=\begin{cases} l/2 & (l=\text{even}) \\ (l-1)/2 & (l=\text{odd}) \end{cases}$$

对称性关系

$$P_l(-x)=(-1)^l P_l(x)$$

前8个Legendre多项式

$$P_0(x)=1;\quad P_1(x)=x;\quad P_2(x)=\frac{1}{2}(3x^2-1);$$

$$P_3(x)=\frac{1}{2}(5x^3-3x);\quad P_4(x)=\frac{1}{8}(35x^4-30x^2+3);$$

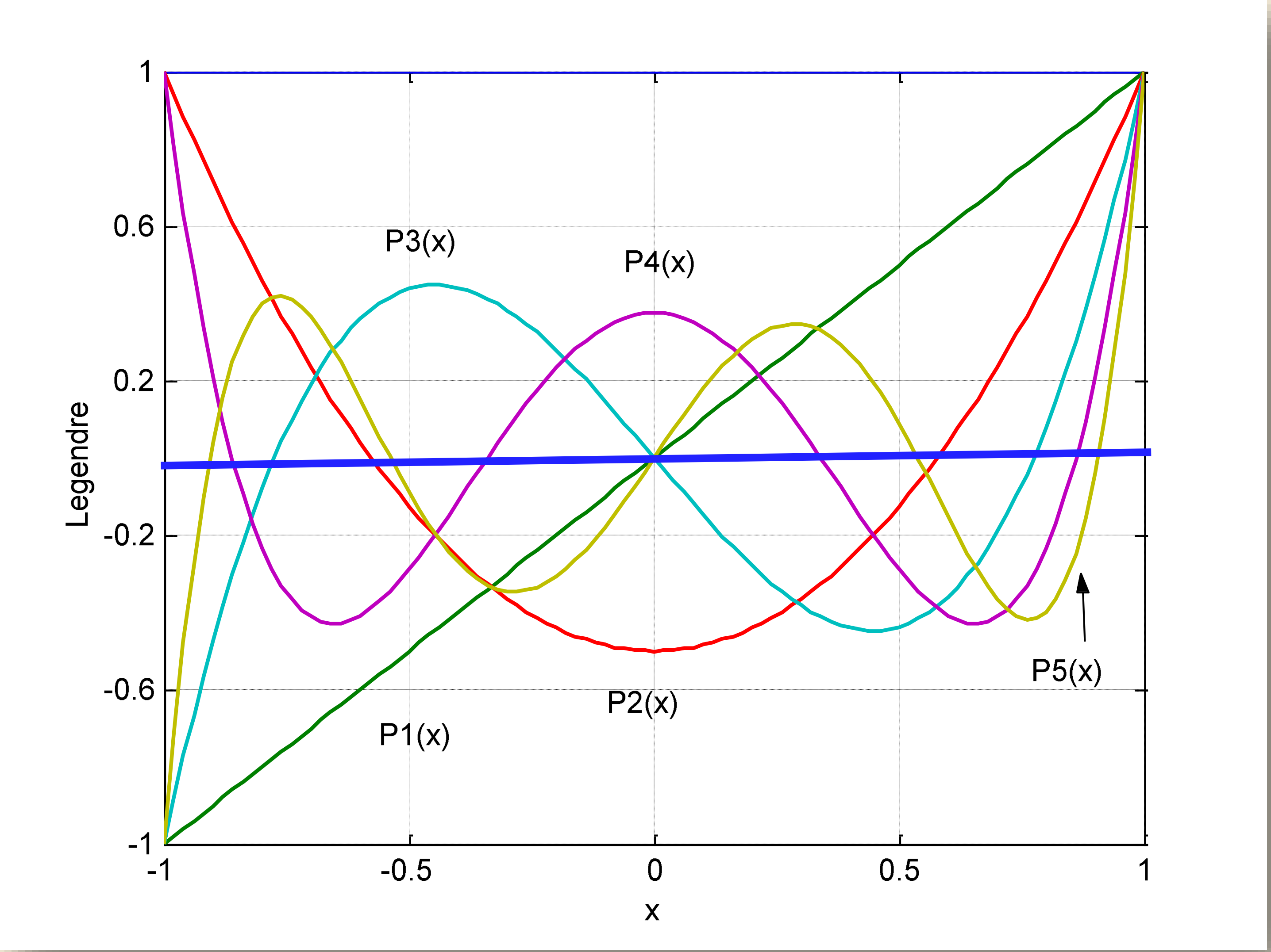
$$P_5(x)=\frac{1}{8}(63x^5-70x^3+15x);$$

$$P_6(x)=\frac{1}{16}(231x^6-315x^4+105x^2-5);$$

$$P_7(x)=\frac{1}{16}(429x^7-693x^5+315x^3-35x).$$

Legendre多项式曲线

$P_{2k+1}(0)=0,\ P'_{2k}(0)=0$



Legendre多项式的微分和积分形式

①Rodrigues公式

$$P_l(x)=\frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l$$

常用于求积分

②Schlafli 积分

$$P_l(x)=\frac{1}{2\pi\mathrm{i}}\frac{1}{2^l}\oint_C\frac{(z^2-1)^l}{(z-x)^{l+1}}\mathrm{d}z$$

$$f^{(n)}(z)=\frac{n!}{2\pi\mathrm{i}}\oint_C\frac{f(\xi)}{(\xi-z)^{n+1}}\mathrm{d}\xi$$

C 为 z 平面上包含 $z=x$ 的任一闭合曲线

③Laplace 积分(证明见10页)

$$P_l(x)=\frac{1}{\pi}\int_0^{\pi}\left(x+\mathrm{i}\sqrt{1-x^2}\cos\psi\right)^l\mathrm{d}\psi$$

常用于求特殊点的值

$$P_l(\pm 1)=\frac{1}{\pi}\int_0^{\pi}(\pm 1)^l \mathrm{d}\psi=(\pm 1)^l$$

$$P_l(0)=\frac{\mathrm{i}^l}{\pi}\int_0^{\pi}\cos^l\psi \mathrm{d}\psi$$

$$P_{2n-1}(0)=0;\quad P_{2n}(0)=\frac{(-1)^n(2n-1)!!}{(2n)!}$$

$$\begin{cases}\cos^{2n-1}\psi=\dfrac{1}{2^{2n-2}}\left\{\displaystyle\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos[(2n-2k-1)\psi]\right\}\\ \cos^{2n}\psi=\dfrac{1}{2^{2n}}\left\{\displaystyle\sum_{k=0}^{n-1}2\binom{2n}{k}\cos[2(n-k)\psi]+\binom{2n}{n}\right\}\end{cases}$$

$$\binom{p}{n}=\frac{p!}{n!(p-n)!}$$

证明：**利用如图的围道: 圆心在实轴的x, 半径为$\sqrt{x^2-1}$**

则C上任意一点的坐标为

$$z=x+\sqrt{x^2-1}e^{i\psi};\mathrm{d}z=i\sqrt{x^2-1}e^{i\psi}\mathrm{d}\psi=i(z-x)\mathrm{d}\psi$$

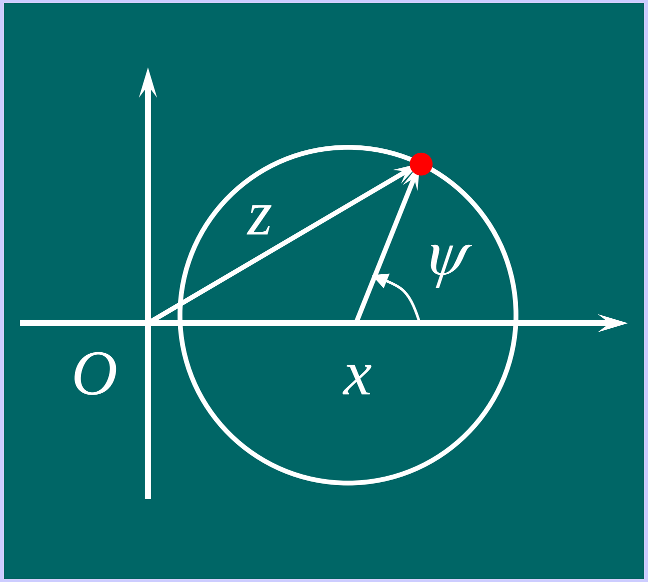
$$z^2-1=\left(x+\sqrt{x^2-1}e^{i\psi}\right)^2-1=2(z-x)\left(x+\sqrt{x^2-1}\cos\psi\right)$$

$$P_l(x)=\frac{1}{2\pi}\int_0^{2\pi}\left(x+\sqrt{x^2-1}\cos\psi\right)^l\mathrm{d}\psi$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\cos\psi\right)^l\mathrm{d}\psi$$

$$|P_l(x)|\le\frac{1}{\pi}\int_0^{\pi}\left|x+i\sqrt{1-x^2}\cos\psi\right|^l\mathrm{d}\psi$$

$$=\frac{1}{\pi}\int_0^{\pi}(x^2\sin^2\psi+\cos^2\psi)\mathrm{d}\psi\le\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\psi\le 1$$



第二类 Legendre 函数

当 l 为零或正整数，Legendre 方程的另一个线性独立解

$$Q_l(x) = P_l(x)\int \frac{1}{(1-x^2)[P_l(x)]^2}\mathrm{d}x$$

——称为第二类Legendre函数

可见，当 $x=\pm 1$ 时, $Q_l(x)$ 对数发散

$$Q_l(x) = \frac{1}{2}P_l(x)\ln\frac{1+x}{1-x}$$

$$+\frac{1}{2^l}\sum_{k=0}^{[\frac{l-1}{2}]}\left[\sum_{n=0}^{k}\frac{(-1)^{n+1}}{(2k-2n+1)}\frac{(2l-2n)!}{n!(l-n)!(l-2n)!}\right]x^{l-1-2k}$$

$$(-1<x<1, \quad l\geq 1)$$

前三个函数形式

$$Q_0(x)=\int\frac{\mathrm{d}x}{1-x^2}=\frac{1}{2}\ln\frac{1+x}{1-x};$$

$$Q_1(x)=x\int\frac{\mathrm{d}x}{(1-x^2)x^2}=\frac{1}{2}P_1(x)\ln\frac{1+x}{1-x}-1;$$

$$Q_2(x)=\frac{1}{2}P_2(x)\ln\frac{1+x}{1-x}-\frac{3}{2}x.$$

- **Legendre方程的通解可表示为**

$$y(x)=C_1P_l(x)+C_2Q_l(x)$$

如果物理问题包含ϑ=0和π，就构成自然边界条件. 因此:

① l是零或正整数；②常数必须取零, C_2=0。

但如果物理问题不包含ϑ=0和π，就构不成自然边界条件，对于 $l=\mu$一般情形, P_μ不是多项式(后面讨论)。

Legendre 多项式的正交性

不同阶的Legendre 多项式在区间[–1,+1]上正交

$$\int_{-1}^{1} P_k(x)P_l(x)\mathrm{d}x = 0 \ (k \neq l)$$

$$\int_{0}^{\pi} P_k(\cos\vartheta)P_l(\cos\vartheta)\sin\vartheta \mathrm{d}\vartheta = 0 \ (k \neq l)$$

当 $l=k$, 得到 Legendre 函数的模 N_l

$$N_l = \sqrt{\int_{-1}^{1}[P_l(x)]^2\mathrm{d}x} = \sqrt{\frac{2}{2l+1}}, \quad (l = 0,1,2,\ldots)$$

Legendre 多项式的完备性

函数系$\{P_l(x)\}$是完备的. 因此, 定义在[−1,+1]上的平方可积函数$f(x)$可展成广义 Fourier 级数

$$\begin{cases} \displaystyle\sum_{l=0}^{\infty} f_l P_l(x) = \frac{1}{2}[f(x^+) + f(x^-)] \\ \displaystyle f_l = \frac{2l+1}{2}\int_{-1}^{1} f(x) P_l(x)\,\mathrm{d}x \end{cases}$$

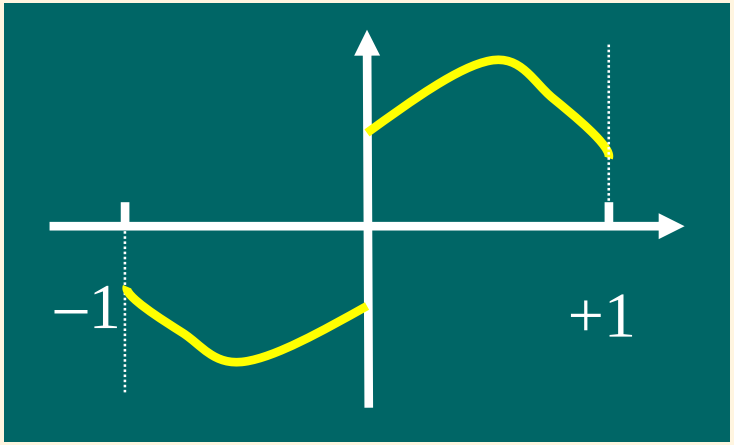
$$\begin{cases} \displaystyle f(\vartheta) = \sum_{l=0}^{\infty} f_l P_l(\cos\vartheta) \\ \displaystyle f_l = \frac{2l+1}{2}\int_{0}^{\pi} f(\vartheta) P_l(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta \end{cases}$$

注意：以ϑ为变量时,区间为[0,π] ,并且带权：$\sin\vartheta$

■ 奇函数 $f(x)=-f(-x)$

$$f_l=\frac{2l+1}{2}\left[-\int_{-1}^{0}f(-x)P_l(x)\mathrm{d}x+\int_0^1 f(x)P_l(x)\mathrm{d}x\right]$$

$$=\frac{2l+1}{2}\int_0^1 f(x)[P_l(x)-P_l(-x)]\mathrm{d}x$$



$$\boxed{P_l(-x)=(-1)^l P_l(x)}$$

$$f_l=0\quad(l=2k)$$

$$\frac{1}{2}[f(x^+)+f(x^-)]=\sum_{k=0}^{\infty}f_{2k+1}P_{2k+1}(x);\quad f_{2k+1}=(4k+3)\int_0^1 f(x)P_{2k+1}(x)\mathrm{d}x$$

原点的函数值收敛到零

$$\frac{1}{2}[f(0^+)+f(0^-)]=\sum_{k=0}^{\infty}f_{2k+1}P_{2k+1}(0)=0$$

■ 偶函数 $f(x)=f(-x)$

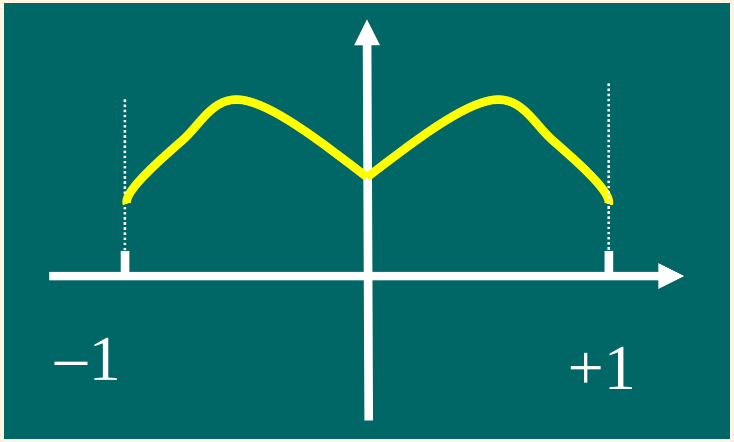
$$f_l=\frac{2l+1}{2}\left[\int_{-1}^{0}f(-x)P_l(x)\mathrm{d}x+\int_{0}^{1}f(x)P_l(x)\mathrm{d}x\right]$$

$$=\frac{2l+1}{2}\int_{0}^{1}f(x)[P_l(x)+P_l(-x)]\mathrm{d}x$$

$$f_l=0\quad(l=2k+1)$$

$$\frac{1}{2}[f(x^+)+f(x^-)]=\sum_{k=0}^{\infty}f_{2k}P_{2k}(x)$$

$$f_{2k}=(4k+1)\int_{0}^{1}f(x)P_{2k}(x)\mathrm{d}x$$



原点导数值收敛到零

$$\frac{1}{2}[f'(0^-)+f'(0^+)]\simeq\sum_{k=0}^{\infty}f_{2k}P'_{2k}(0)=0$$

例1 在[–1,+1]上把

$$f(x) = 2x^3 + 3x + 4$$

展开成 Legendre 多项式。

$$f(x) = 2x^3 + 3x + 4 = \sum_{l=0}^{3} f_l P_l(x)$$

$$= f_0 P_0(x) + f_1 P_1(x) + f_2 P_2(x) + f_3 P_3(x)$$

$$= f_0 \cdot 1 + f_1 \cdot x + f_2 \cdot \frac{1}{2}(3x^2 - 1) + f_3 \cdot \frac{1}{2}(5x^3 - 3x)$$

$$= \left(f_0 - \frac{1}{2} f_2 \right) + \left(f_1 - \frac{3}{2} f_3 \right) x + \frac{3}{2} f_2 x^2 + \frac{5}{2} f_3 x^3$$

上式对任意的x成立，因此

$$f_0 - \frac{1}{2} f_2 = 4;\ \ f_1 - \frac{3}{2} f_3 = 3;\ \ \frac{3}{2} f_2 = 0;\ \ \frac{5}{2} f_3 = 2$$

$$f_0 = 4;\ f_1 = \frac{21}{5};\ f_0 = 0;\ f_3 = \frac{4}{5}$$

因此

$$2x^3 + 3x + 4 = 4P_0(x) + \frac{21}{5} P_1(x) + \frac{4}{5} P_3(x)$$

例2 在[–1,+1]上把函数

$$f(x) = |x|$$

展开成 Legendre 多项式。

解

$$|x|=\sum_{l=0}^{\infty} f_l P_l(x)$$

$$f_l=\frac{2l+1}{2}\left[\int_{-1}^{1}|x|P_l(x)\mathrm{d}x\right]=\frac{2l+1}{2}\left[\int_{0}^{1}x[P_l(-x)+P_l(x)]\mathrm{d}x\right]$$

$$\boxed{P_l(-x)=(-1)^l P_l(x)}$$

$$f_{2n+1}=0;\quad f_{2n}=(4n+1)\left[\int_0^1 \xi P_{2n}(\xi)\mathrm{d}\xi\right]$$

$$|x|=\frac{1}{2}P_0(x)+\sum_{l=1}^{\infty}(-1)^{l+1}\frac{(4l+1)(2l-1)!!}{(2l-1)(2l+2)!!}P_{2l}(x)$$

Legendre多项式应用

例1 在球内部 *r<a* 求解 Laplace 方程使满足边界条件

$$u\big|_{r=a} = f(\vartheta)$$

解：显然问题与φ无关，相应的 Laplace 方程为

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial u(r,\vartheta)}{\partial\vartheta}\right]=0$$

$$r<a,\quad 0\le\vartheta\le\pi$$

①通解为

$$u(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}][C_l P_l(\cos\vartheta)+D_l Q_l(\cos\vartheta)]$$

②球内问题: $r=0$, $u<\infty$, $B_l\equiv 0$; 包含$\vartheta=0$和π: $D_l\equiv 0$

$$u(r,\vartheta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\vartheta)$$

③待定系数决定： 由边界条件

$$u(r,\vartheta)|_{r=a}=\sum_{l=0}^{\infty}A_l a^l P_l(\cos\vartheta)=f(\vartheta)$$

由Legendre 多项式的正交性质

$$f(\vartheta)=\sum_{l=0}^{\infty}f_l P_l(\cos\vartheta)$$

$$f_l=\frac{2l+1}{2}\int_0^{\pi}f(\vartheta)P_l(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta \quad\Rightarrow\quad A_l=\frac{f_l}{a^l}$$

④解的级数和积分形式

$$u(r,\vartheta)=\sum_{l=0}^{\infty} f_l\left(\frac{r}{a}\right)^l P_l(\cos\vartheta)=\int_0^{\pi} g(r,\vartheta,\vartheta')f(\vartheta')\sin\vartheta'\mathrm{d}\vartheta'$$

$$g(r,\vartheta,\vartheta')\equiv\sum_{l=0}^{\infty}\frac{2l+1}{2}\left(\frac{r}{a}\right)^l P_l(\cos\vartheta')P_l(\cos\vartheta)$$

■**简单例子** $f(\vartheta)=\cos^2\vartheta$

$$f_l=\frac{2l+1}{2}\int_0^{\pi} f(\vartheta)P_l(\cos\vartheta)\sin\vartheta\mathrm{d}\vartheta$$

$$=\frac{2l+1}{2}\int_{-1}^{1} x^2P_l(x)\mathrm{d}x$$

$$P_2(x)=\frac{1}{2}(3x^2-1)\Rightarrow x^2=\frac{1}{3}[1+2P_2(x)]=\frac{1}{3}P_0(x)+\frac{2}{3}P_2(x)$$

$$f_l=\frac{2l+1}{2}\int_{-1}^{1}\left\{\frac{1}{3}P_0(x)+\frac{2}{3}P_2(x)\right\}P_l(x)\mathrm{d}x$$

利用正交性质

$$\int_{-1}^{1}P_0(x)P_0(x)\mathrm{d}x=2;\ \int_{-1}^{1}P_2(x)P_2(x)\mathrm{d}x=\frac{2}{5}$$

$$\int_{-1}^{1}P_0(x)P_l(x)\mathrm{d}x=0\ (l\neq 0);\ \int_{-1}^{1}P_2(x)P_l(x)dx=0\ (l\neq 2).$$

$$f_0=\frac{1}{3};\quad f_1=0;\ f_2=\frac{2}{3};\ f_l=0\ (l>2)$$

$$u(r,\vartheta)=\frac{1}{3}+\frac{2}{3}\left(\frac{r}{a}\right)^2P_2(\cos\vartheta)$$

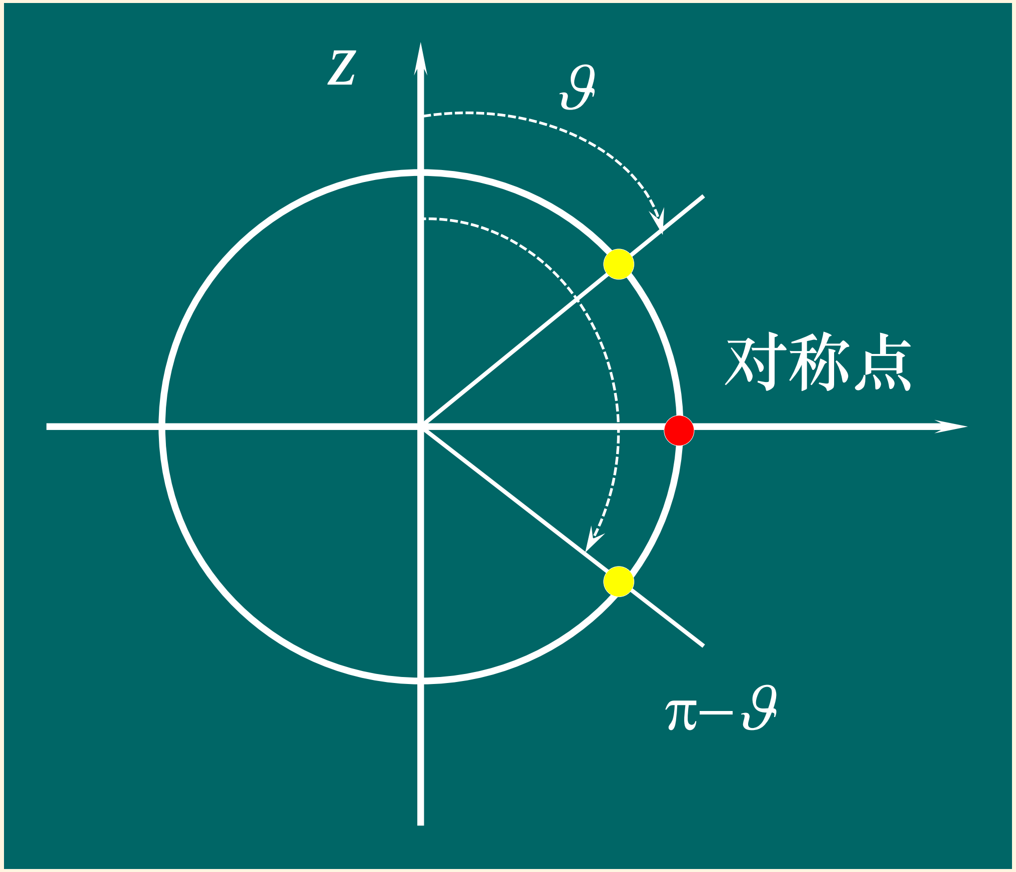
例2 在半球内部 $r<a$， $0<\vartheta<\pi/2$ 求解 Laplace方程使满足边界条件：(1) 半球面 $f(\vartheta)$; (2) 底面绝热。解：

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial u(r,\vartheta)}{\partial\vartheta}\right]=0$$

$$(r<a,\ \ 0\le\vartheta\le\pi/2)$$

$$u\big|_{r=a}=f(\vartheta)\ (0<\vartheta<\pi/2);\quad \left.\frac{\partial u}{\partial\vartheta}\right|_{\vartheta=\pi/2}=0\ (r<a)$$

分析： $0<\vartheta<\pi/2$ 即 $0<x=\cos\vartheta<1$, 而 Legendre多项式定义在 $-1<x<+1$, 因此必须把问题延拓到整个球内, 为了满足底面 $\vartheta=\pi/2$的边界条件, 作关于 $\vartheta=\pi/2$(即$x=0$)的偶延拓



球坐标中关于ϑ=0的延拓

①偶延拓

$$u(r,\vartheta)|_{r=a}=F(\vartheta)=\begin{cases} f(\vartheta) & (0\le\vartheta\le\pi/2) \\ f(\pi-\vartheta) & (\pi/2\le\vartheta\le\pi) \end{cases}$$

②级数解

$$u(r,\vartheta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\vartheta)$$

③求系数

$$u(r,\vartheta)|_{r=a}=\sum_{l=0}^{\infty}A_l a^l P_l(\cos\vartheta)=F(\vartheta)$$

$$\Downarrow$$

$$A_l=\frac{2l+1}{2a^l}\int_0^{\pi}F(\vartheta)P_l(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta$$

$$=\frac{2l+1}{2a^l}\left[\int_0^{\pi/2}f(\vartheta)P_l(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta+\int_{\pi/2}^{\pi}f(\pi-\vartheta)P_l(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta\right]$$

第二个积分变量变化

$$A_l=\frac{2l+1}{2a^l}\left[\int_0^{\pi/2}f(\vartheta)P_l(\cos\vartheta)\sin\vartheta\mathrm{d}\vartheta+\int_0^{\pi/2}f(\psi)P_l(-\cos\psi)\sin\psi\mathrm{d}\psi\right]$$

$$=\frac{2l+1}{2a^l}\int_0^{\pi/2}f(\vartheta)[P_l(\cos\vartheta)+P_l(-\cos\vartheta)]\sin\vartheta\mathrm{d}\vartheta$$

$$=\frac{2l+1}{2a^l}\int_0^{\pi/2}f(\vartheta)P_l(\cos\vartheta)[1+(-1)^l]\sin\vartheta\mathrm{d}\vartheta$$

$$A_{2k+1}=0;\quad A_{2k}=\frac{4k+1}{a^{2k}}\int_0^{\pi/2}f(\vartheta)P_{2k}(\cos\vartheta)\sin\vartheta\mathrm{d}\vartheta$$

$$u(r,\vartheta)=\sum_{k=0}^{\infty}A_{2k}r^{2k}P_{2k}(\cos\vartheta)$$

④验证是否满足底面边界条件

$$\left.\frac{\partial u(r,\vartheta)}{\partial \vartheta}\right|_{\vartheta=\pi/2} = \sum_{k=0}^{\infty} A_{2k} r^{2k} \left.\frac{\partial P_{2k}(\cos\vartheta)}{\partial \vartheta}\right|_{\vartheta=\pi/2} = -\sum_{k=0}^{\infty} A_{2k} r^{2k} \left.\frac{\mathrm{d}P_{2k}(x)}{\mathrm{d}x}\right|_{x=0}$$

$$P_l(x) = \frac{1}{\pi}\int_0^{\pi}\left(x + \mathrm{i}\sqrt{1-x^2}\cos\psi\right)^l \mathrm{d}\psi \Rightarrow \left.\frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\right|_{x=0} = \frac{l}{\pi}\mathrm{i}^{l-1}\int_0^{\pi}\cos^{l-1}\psi\,\mathrm{d}\psi$$

$$\left.\frac{\mathrm{d}P_{2k}(x)}{\mathrm{d}x}\right|_{x=0} = \frac{2k}{\pi}\mathrm{i}^{2k-1}\int_0^{\pi}\cos^{2k-1}\psi\,\mathrm{d}\psi = 0$$

$$\left.\frac{\mathrm{d}P_{2k+1}(x)}{\mathrm{d}x}\right|_{x=0} = \frac{2k+1}{\pi}\mathrm{i}^{2k}\int_0^{\pi}\cos^{2k}\psi\,\mathrm{d}\psi \neq 0$$

$$\left.\frac{\partial u(r,\vartheta)}{\partial \vartheta}\right|_{\vartheta=\pi/2} = 0$$

$$\begin{cases} \cos^{2n-1}\psi = \dfrac{1}{2^{2n-2}}\left\{\displaystyle\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos[(2n-2k-1)\psi]\right\} \\ \cos^{2n}\psi = \dfrac{1}{2^{2n}}\left\{\displaystyle\sum_{k=0}^{n-1}2\binom{2n}{k}\cos[2(n-k)\psi] + \binom{2n}{n}\right\} \end{cases} ;\binom{p}{n} = \frac{p!}{n!(p-n)!}$$

❑如果底面边界条件：底面保持零度

$$u(r,\vartheta)|_{\vartheta=\pi/2}=0$$

则作关于$\vartheta=\pi/2$(即$x=0$)的奇延拓

$$u(r,\vartheta)|_{r=a}=F(\vartheta)=\begin{cases} f(\vartheta) & (0\le\vartheta\le\pi/2) \\ -f(\pi-\vartheta) & (\pi/2\le\vartheta\le\pi) \end{cases}$$

$$A_l=\frac{2l+1}{2a^l}\int_0^{\pi/2} f(\vartheta)P_l(\cos\vartheta)[1-(-1)^l]\sin\vartheta\,\mathrm{d}\vartheta$$

$$A_{2k}=0;\quad A_{2k+1}=\frac{2(2k+1)+1}{a^{2k+1}}\int_0^{\pi/2} f(\vartheta)P_{2k+1}(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta$$

$$u(r,\vartheta)=\sum_{k=0}^{\infty}A_{2k+1}r^{2k+1}P_{2k+1}(\cos\vartheta)$$

❑如果底面边界条件：底面保持常数

$$u(r,\vartheta)|_{\vartheta=\pi/2}=u_0 \Rightarrow u(r,\vartheta)=u_0+v(r,\vartheta)$$

❑如果底面边界条件：底面温度分布

$$u(r,\vartheta)|_{\vartheta=\pi/2}=u_0(r)$$

——无法用简单的延拓方法：Green函数法，本征函数开展法

❑如果要求底面满足第三类边界条件，如何处理?

$$[\alpha u+\beta(\nabla u)\cdot\boldsymbol{n}]_{\vartheta=\pi/2}=0$$

底面法向和梯度算子为

$$\boldsymbol{n}=\boldsymbol{e}_\vartheta|_{\vartheta=\pi/2};\quad \nabla=\frac{\partial}{\partial r}\boldsymbol{e}_r+\frac{1}{r}\frac{\partial}{\partial\vartheta}\boldsymbol{e}_\vartheta$$

$$\left(\alpha u+\frac{\beta}{r}\frac{\partial u}{\partial\vartheta}\right)_{\vartheta=\pi/2}=0$$

与径向r有关

设分离变量解为

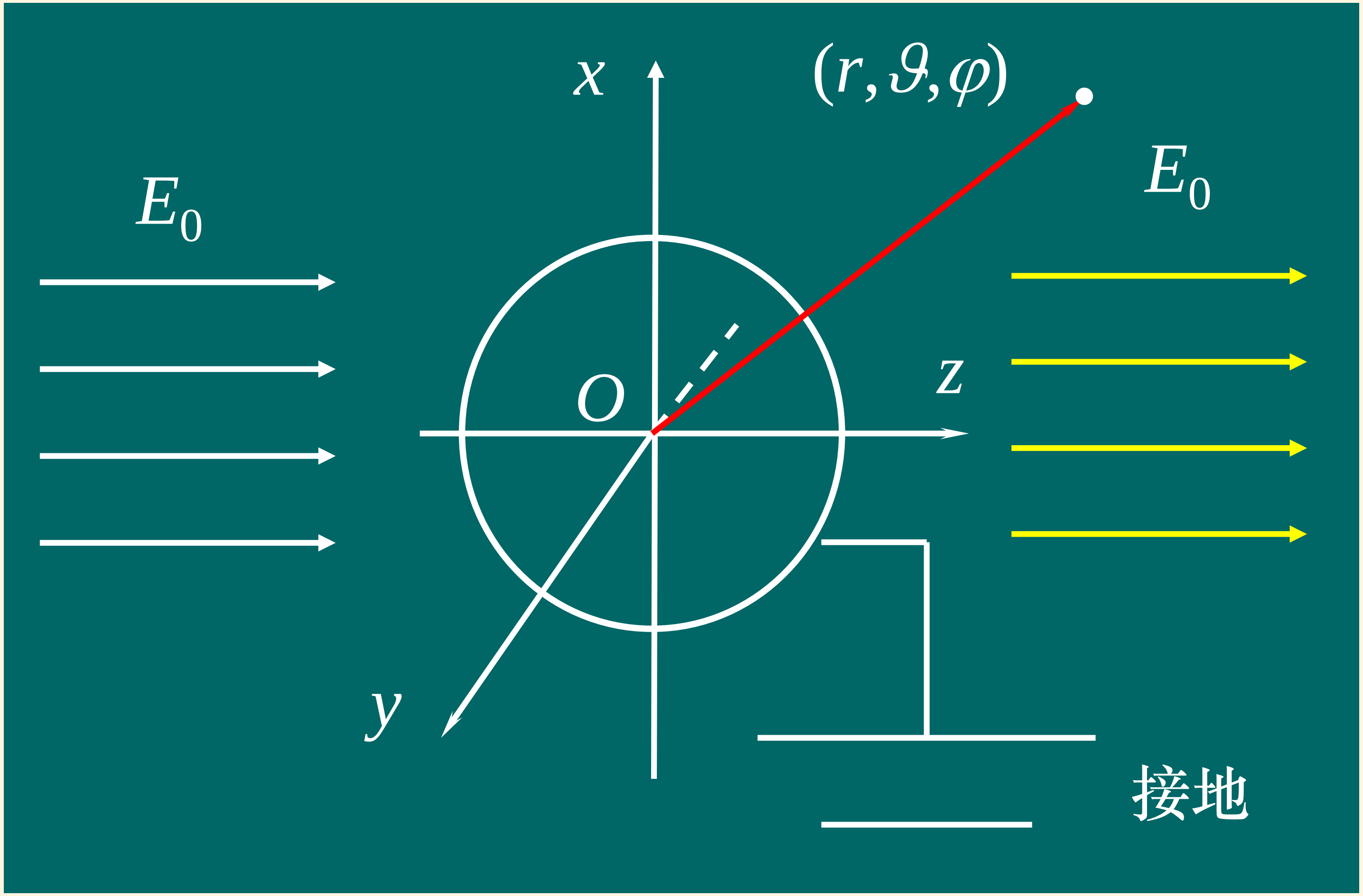
$$u(r,\vartheta)=R(r)\Theta(\vartheta)$$

$$R(r)\left[\alpha\Theta(\vartheta)+\frac{\beta}{r}\frac{\partial\Theta(\vartheta)}{\partial\vartheta}\right]_{\vartheta=\pi/2}=0$$

可见：无法得分离变量解，分离变量法失败, 即使系数α和β与极角无关(有关时,更无法分离变量)

■对$0<\vartheta<\pi/2$的圆锥形区域，见11.4节讨论，半球可看作为$\vartheta\to\pi/2$的特殊情况.

例3 在均匀电场 E_0 中放一接地导体球，球半径为 a，求球外电场的分布。



分析

■ 泛定方程

接地导体由于外电场作用而在球表面产生感应电荷，该分布影响原来的均匀电场。在球外，无自由电荷，电势满足 Laplace 方程

$$\nabla^2 u(r,\vartheta,\varphi)=0$$

■ 边界条件

①球接地 $$u(r,\vartheta,\varphi)\big|_{r=a}=0$$

②无限远处：均匀电场

$$-\left.\frac{\partial u(r,\vartheta,\varphi)}{\partial z}\right|_{r\to\infty}\to E_0$$

$$u(r,\vartheta,\varphi)\big|_{r\to\infty}\to E_0 z + C = -E_0 r\cos\vartheta + C$$

——如果假定在放置导体球前，原点的电势为u_0，则$C=u_0$

- **对称性：u 与方位角 φ 无关** $u(r,\vartheta,\varphi)=u(r,\vartheta)$

(注意：坐标选择的重要性)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial u(r,\vartheta)}{\partial\vartheta}\right]=0$$

$$(r>a,\ \ 0\le\vartheta\le\pi)$$

解：分离变量

$$u(r,\vartheta)=R(r)\Theta(\vartheta)$$

■ 径向方程和解

Euler方程

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}-l(l+1)R=0,\ (r>a)$$

$$R_l(r)=A_l r^l+B_l r^{-(l+1)}$$

■ 极角方向方程和解 $(x=\cos\vartheta)$

Legendre方程

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]+l(l+1)\Theta=0\\ \text{自然边界条件}:\Theta(x)\big|_{x=\pm1}=\text{有限}.\end{cases}$$

$$\Theta_l(\vartheta)=P_l(\cos\vartheta)\ (l=0,1,2,\ldots)$$

■ 通解

$$u(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]P_l(\cos\vartheta)$$

■ 系数 A_l 和 B_l 由边界条件决定

$$u(r,\vartheta)|_{r=a}=\sum_{l=0}^{\infty}[A_l a^l + B_l a^{-(l+1)}]P_l(\cos\vartheta)=0$$

$$u(r,\vartheta)|_{r\to\infty}\to\sum_{l=0}^{\infty}A_l r^l P_l(\cos\vartheta)=u_0-E_0 r\cos\vartheta$$

$$\Downarrow$$

$$A_0=u_0,\quad A_1=-E_0,\quad A_2=A_3=\ldots.=0$$

$$B_0=-au_0,\quad B_1=E_0a^3,\quad B_l=0\quad(l\geq 2)$$

■ **因此**

$$u(r,\vartheta)=u_0-E_0r\cos\vartheta-\frac{u_0a}{r}+E_0a^3\frac{\cos\vartheta}{r^2}\ (r\geq a)$$

$$=u_0\left(1-\frac{a}{r}\right)-E_0\left(r-\frac{a^3}{r^2}\right)\cos\vartheta\ (r\geq a)$$

■ **物理意义**

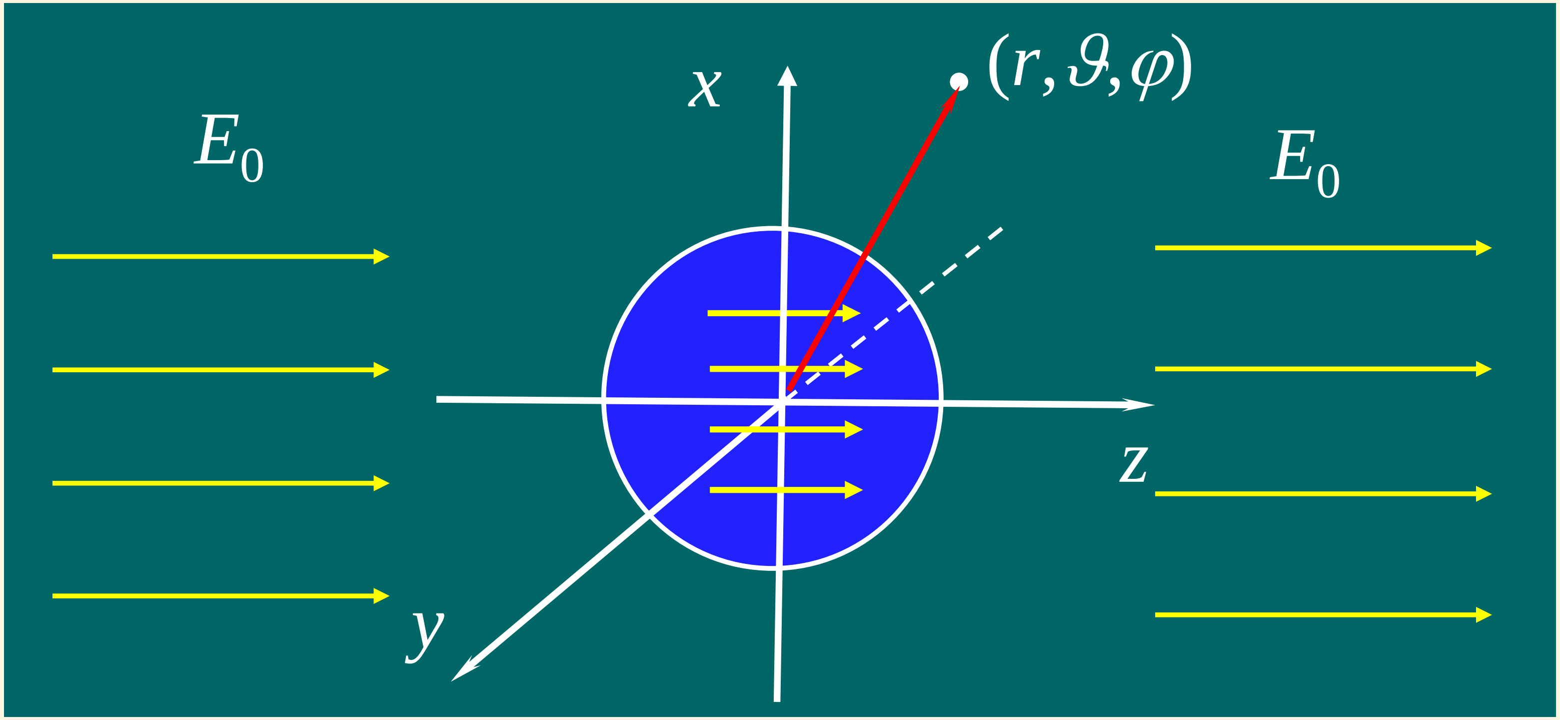
(1）前二项，原来的均匀电场

(2）第三项，由于导体球接地而带电荷 $-u_0a$ 产生的电场

(3）第四项，导体球受均匀电场感应，成为电偶极子产生的场，偶极矩的大小为 E_0a^3

例4 在均匀电场 E_0 中放一介电常数为 ε 的介质球，球半径为 a,求球内外电场的分布。

分析：由于受均匀电场极化，介质球表面出现束缚电荷. 在球的内外电势满足Laplace方程。球的表面有连接条件。



■ 泛定方程

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u_e(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial u_e(r,\vartheta)}{\partial\vartheta}\right]=0\quad(r>a)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u_i(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial u_i(r,\vartheta)}{\partial\vartheta}\right]=0\quad(r<a)$$

■ 边界条件

①原点有限 $u_i(r,\vartheta)\big|_{r=0}<\infty$

②无限远处：均匀电场 $u(r,\vartheta)\big|_{r\to\infty}\to u_0-E_0 r\cos\vartheta$

③球表面连接条件

□ 电场矢量$E=-\nabla u$的切向连续，当与φ无关时

$$\frac{1}{r}\frac{\partial u_e(r,\vartheta)}{\partial\vartheta}\bigg|_{r=a}=\frac{1}{r}\frac{\partial u_i(r,\vartheta)}{\partial\vartheta}\bigg|_{r=a}$$

- 电位移矢量$D=\varepsilon E=-\varepsilon\nabla u$的法向连续(假定球表面无自由电荷)

$$\varepsilon\frac{\partial u_i}{\partial r}\bigg|_{r=a}=\varepsilon_0\frac{\partial u_e}{\partial r}\bigg|_{r=a}$$

解：由于问题的对称性：u 与角度φ无关，因此，一般解为

$$u_i(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}]P_l(\cos\vartheta)\quad(r<a)$$

$$u_e(r,\vartheta)=\sum_{l=0}^{\infty}[C_l r^l+D_l r^{-(l+1)}]P_l(\cos\vartheta)\quad(r>a)$$

①球内：当r=0时，u 应该有限，因此B_l=0

$$u_i(r,\vartheta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\vartheta) \quad (r<a)$$

②球外无限远处

$$u_e(r,\vartheta)|_{r\to\infty}\approx\sum_{l=0}^{\infty}C_l r^l P_l(\cos\vartheta)=u_0-E_0 rP_1(\cos\vartheta)$$

$$C_0=u_0;C_1=-E_0,C_l=0 \quad (l\neq 0,1)$$

$$u_e(r,\vartheta)=u_0-E_0 rP_1(\cos\vartheta)+\sum_{l=0}^{\infty}D_l r^{-(l+1)}P_l(\cos\vartheta) \ (r>a)$$

③球表面连接条件

$$\sum_{l=1}^{\infty} A_l a^l \frac{\partial P_l(\cos\vartheta)}{\partial\vartheta} = -E_0 a \frac{\partial P_1(\cos\vartheta)}{\partial\vartheta} + \sum_{l=0}^{\infty} D_l a^{-(l+1)} \frac{\partial P_l(\cos\vartheta)}{\partial\vartheta}$$

$$\varepsilon \sum_{l=1}^{\infty} l A_l a^{l-1} P_l(\cos\vartheta) = \varepsilon_0 \left[-E_0 P_1(\cos\vartheta) - \sum_{l=0}^{\infty} (l+1) D_l a^{-(l+2)} P_l(\cos\vartheta) \right]$$

$$\Downarrow$$

$$D_0 = 0; \varepsilon A_1 = \varepsilon_0(-E_0 - 2D_1 a^{-3}); A_1 a = -E_0 a + D_1 a^{-2}$$

$$A_l a^l = D_l a^{-(l+1)}; \varepsilon l A_l a^{l-1} = -\varepsilon_0 (l+1) D_l a^{-(l+2)} (l \geq 2)$$

$$\Downarrow$$

$$D_0 = 0; D_1 = \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 a^3; D_l = 0 \ (l \geq 2)$$

$$A_1 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0; A_l = 0 \ (l \geq 2)$$

$$u_i(r,\vartheta) = A_0 - \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r\cos\vartheta \quad (r < a)$$

$$u_e(r,\vartheta) = u_0 - E_0 r\cos\vartheta + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 a^3 \frac{\cos\vartheta}{r^2} \quad (r > a)$$

物理分析

- **球内仍然是均匀电场**

$$\boldsymbol{E}_i = -\boldsymbol{e}_z \frac{\partial u_i}{\partial z} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \boldsymbol{e}_z$$

- **介质球表面的束缚电荷为一电偶极子，偶极矩的大小为**

$$\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 E_0$$

- **如果规定原点的电势为零，则A_0=0和u_0=0.**

11.2 母函数和递推公式

Legendre 多项式的母函数

$$\frac{1}{\sqrt{1-2t\cos\vartheta+t^2}}=\sum_{l=0}^{\infty}t^l P_l(\cos\vartheta)$$

幂级数展开

$$\frac{1}{\sqrt{1-2t\cos\vartheta+t^2}}=\sum_{l=0}^{\infty}a_l t^l \Rightarrow a_l=\frac{1}{l!}\left(\frac{1}{\sqrt{1-2t\cos\vartheta+t^2}}\right)^{(l)}\Bigg|_{t=0}$$

母函数：函数序列 $Q_n(x)$

$$f(t,x)=\sum_{n=-\infty}^{\infty}Q_n(x)t^n$$

研究序列的基本性质

■ Legendre展开

$$\frac{1}{\sqrt{1-2t\cos\vartheta+t^2}}=\sum_{l=0}^{\infty}f_lP_l(\cos\vartheta) \Rightarrow f_l=\frac{2l+1}{2}\int_{-1}^{1}\frac{P_l(x)}{\sqrt{1-2tx+t^2}}\mathrm{d}x$$

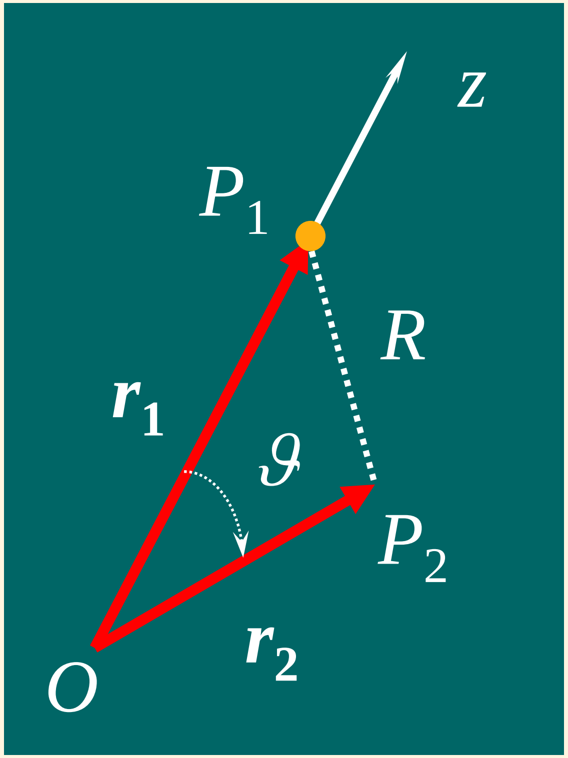
■ 技巧方法

P_1: 点电荷, P_2点产生的电势

$$u=\frac{1}{R}=\frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\vartheta}}$$

P_2点电势满足Laplace方程．当点电荷在z轴上,场仅与极角有关

$$u(r,\vartheta)=\sum_{l=0}^{\infty}[A_lr_2^l+B_lr_2^{-(l+1)}]P_l(\cos\vartheta)$$



两者相等, 应该有

$$\sum_{l=0}^{\infty}[A_l r_2^l + B_l r_2^{-(l+1)}]P_l(\cos\vartheta) = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2\cos\vartheta}}$$

(1) $r_2<r_1$：原点自然边界条件要求 $B_l=0$, 故

$$\sum_{l=0}^{\infty} A_l r_2^l P_l(\cos\vartheta) = \frac{1}{r_1\sqrt{1-2t\cos\vartheta+t^2}}$$

式中 $t=r_2/r_1<1$. 两边令 $\vartheta=0$

$$\sum_{l=0}^{\infty} r_1^l A_l t^l P_l(1) = \frac{1}{r_1\sqrt{1-2t+t^2}} = \frac{1}{r_1}\cdot\frac{1}{1-t} = \frac{1}{r_1}\sum_{l=0}^{\infty} t^l$$

比较级数二边 $A_l = r_1^{-(l+1)}[P_l(1)]^{-1}$ ⇐ $P_l(1) = 1$

最后，得到

$$\frac{1}{\sqrt{1-2t\cos\vartheta+t^2}}=\sum_{l=0}^{\infty}t^l P_l(\cos\vartheta)$$

(2)$r_2>r_1$：无限远处自然边界条件得 $A_l=0$，并令$t=r_1/r_2<1$，同样可得到上式.

母函数展开公式的应用

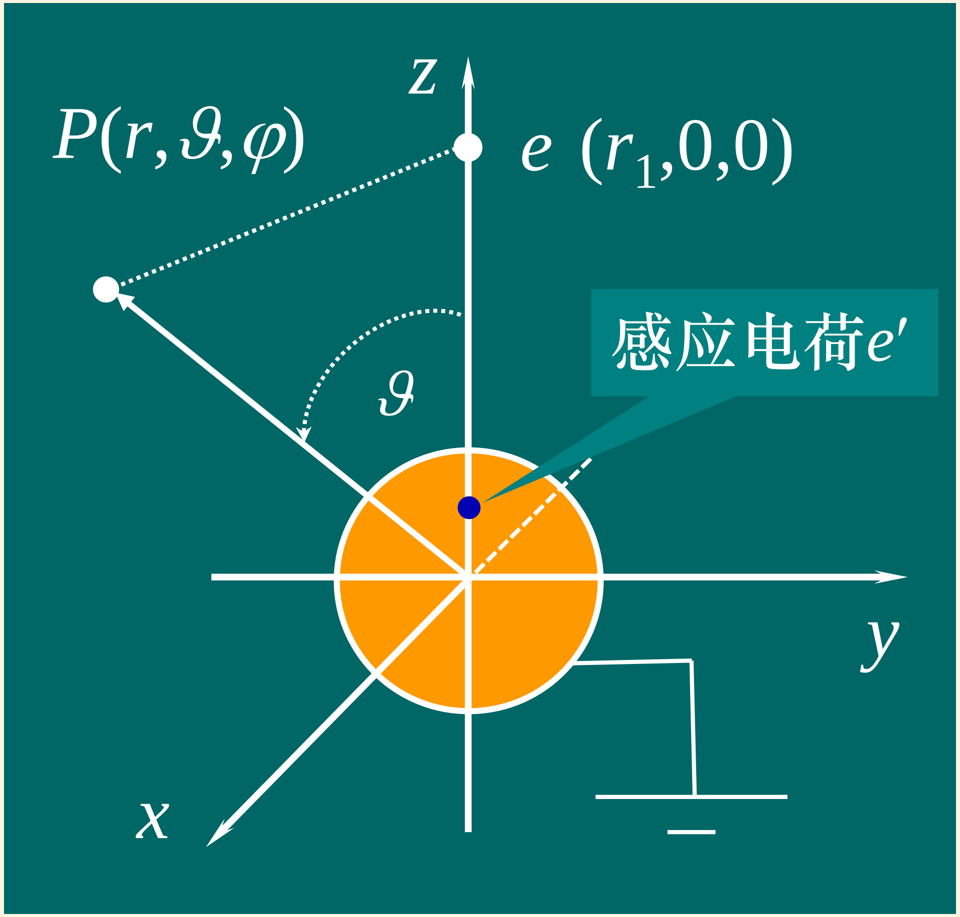
例1 点电荷电场中放置接地导体球，求电场的分布。满足Poisson方程和边界条件

$$\nabla^2 u=-\rho/\varepsilon_0 (r>a)$$

$$u|_{r=a}=0;\ \lim_{r\to\infty}u\to 0$$

$$\rho(r,\vartheta,\varphi)=\frac{1}{2\pi r^2\sin\vartheta}\delta(r-r_1)\delta(\vartheta)\quad(r_1>0)$$

P点场由二部分组成



■点电荷产生的场

$$u_e=\frac{e}{4\pi\sqrt{r_1^2-2r_1r\cos\vartheta+r^2}}$$

■感应电荷产生的场

$$v(r,\vartheta)\ \Rightarrow\ \nabla^2 v(r,\vartheta)=0$$

■P点总场

$$u(r,\vartheta)=u_e+v(r,\vartheta)$$

□ 感应电荷产生的场满足Laplace方程和边界条件

$$\frac{1}{r^2}\frac{\partial}{\partial}\left[r^2\frac{\partial v(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial v(r,\vartheta)}{\partial\vartheta}\right]=0$$

$$(r>a,\quad 0\le\vartheta\le\pi)$$

$$v\big|_{r=a}=-u_e\big|_{r=a}=-\frac{e}{4\pi\sqrt{r_1^2-2r_1a\cos\vartheta+a^2}},\qquad \lim_{r\to\infty}v\to 0$$

$$\Downarrow$$

$$v(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}]P_l(\cos\vartheta)$$

$$\lim_{r\to\infty}v\to 0\Rightarrow A_l\equiv 0 \quad\Rightarrow\quad v(r,\vartheta)=\sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\vartheta)$$

$$\sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\vartheta) = -\frac{e}{4\pi r_1\sqrt{1-2t\cos\vartheta+t^2}} \quad (t = a/r_1)$$

$$\frac{1}{\sqrt{1-2t\cos\vartheta+t^2}} = \sum_{l=0}^{\infty} t^l P_l(\cos\vartheta) \Rightarrow B_l = -\frac{ea^{2l+1}}{4\pi} r_1^{-(l+1)}$$

$$v(r,\vartheta) = -\frac{e}{4\pi}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{r_1^{l+1}}\cdot\frac{P_l(\cos\vartheta)}{r^{l+1}}$$

$$u(r,\vartheta) = \frac{e}{4\pi\sqrt{r_1^2-2r_1 r\cos\vartheta+r^2}} + \frac{(-e)(a/r_1)}{4\pi r}\sum_{l=0}^{\infty}\left(\frac{a^2}{r_1 r}\right)^l P_l(\cos\vartheta)$$

令 $\bar{t} = a^2/(r_1 r)$

$$u(r,\vartheta) = \frac{e}{4\pi\sqrt{r_1^2 - 2r_1 r\cos\vartheta + r^2}} + \frac{(-e)(a/r_1)}{4\pi r}\frac{1}{\sqrt{1-2\bar{t}\cos\vartheta + \bar{t}^2}}$$

$$= \frac{e}{4\pi\sqrt{r_1^2 - 2r_1 r\cos\vartheta + r^2}} + \frac{(-e)(a/r_1)}{4\pi\sqrt{\left(\frac{a^2}{r_1}\right)^2 - 2\left(\frac{a^2}{r_1}\right) r\cos\vartheta + r^2}}$$

$$u(r,\vartheta) = \frac{e}{4\pi\sqrt{r_1^2 - 2r_1 r\cos\vartheta + r^2}} + \frac{-e(a/r_1)}{4\pi\sqrt{\left(\frac{a^2}{r_1}\right)^2 - 2\left(\frac{a^2}{r_1}\right) r\cos\vartheta + r^2}}$$

——可见感应电荷等效于电荷 $e' = -e(a/r_1)$, 且位于 $r_0 = a^2/r_1$ ($r_0 < a$，故在球内)——电像

递推公式

$$\frac{1}{(1-2tx+t^2)^{1/2}}=\sum_{l=0}^{\infty}t^l P_l(x)$$

对t求导

$$\frac{x-t}{(1-2tx+t^2)^{3/2}}=\sum_{l=0}^{\infty}lt^{l-1}P_l(x)$$

两边乘$(1-2xt+t^2)$, 并再利用母函数

$$(x-t)\sum_{l=0}^{\infty}t^l P_l(x)=(1-2xt+t^2)\sum_{l=0}^{\infty}lt^{l-1}P_l(x)$$

比较两边得到递推公式

$$(l+1)P_{l+1}(x)-(2l+1)xP_l(x)+lP_{l-1}(x)=0,\ (l\ge 1)$$

□ 对 x 求导

$$\frac{t}{(1-2tx+t^2)^{3/2}}=\sum_{l=0}^{\infty}t^l P_l'(x)$$

两边乘$(1-2xt+t^2)$, 并再利用母函数

$$t\sum_{l=0}^{\infty}t^l P_l(x)=(1-2xt+t^2)\sum_{l=0}^{\infty}t^l P_l'(x)$$

比较两边得到

$$P_l(x)=P_{l+1}'(x)-2xP_l'(x)+P_{l-1}'(x),\ (l\geq 1)$$

对第一个递推公式求导，可得到导数递推公式

$$P'_{l+1}(x) = xP'_l(x) + (l+1)P_l(x)$$

□ 其它递推公式

$$xP'_l(x) - P'_{l-1}(x) = lP_l(x)$$

$$P'_{l+1}(x) - P'_{l-1}(x) = (2l+1)P_l(x)$$

——递推公式一般用于积分运算

例1 求积分

$$I = \int_{-1}^{1} xP_m(x)P_n(x)\mathrm{d}x$$

利用递推公式

$$(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0,\ (l \ge 1)$$

因此

$$I=\frac{m+1}{2m+1}\int_{-1}^{1}P_{m+1}(x)P_n(x)\mathrm{d}x+\frac{m}{2m+1}\int_{-1}^{1}P_{m-1}(x)P_n(x)\mathrm{d}x$$

利用正交关系

$$\int_{-1}^{1}P_k(x)P_l(x)\mathrm{d}x=\frac{2}{2l+1}\delta_{kl}$$

$$I=\int_{-1}^{1}xP_m(x)P_n(x)\mathrm{d}x=\begin{cases}\dfrac{2n}{4n^2-1}, & m-n=-1\\ \dfrac{2(n+1)}{(2n+3)(2n+1)}, & m-n=1\\ 0, & m-n\neq\pm1\end{cases}$$

例2 已知

$$P_{2k-1}(0)=0;\ P_{2k}(0)=\frac{(-1)^k(2k-1)!!}{(2k)!}$$

求零点的导数值。

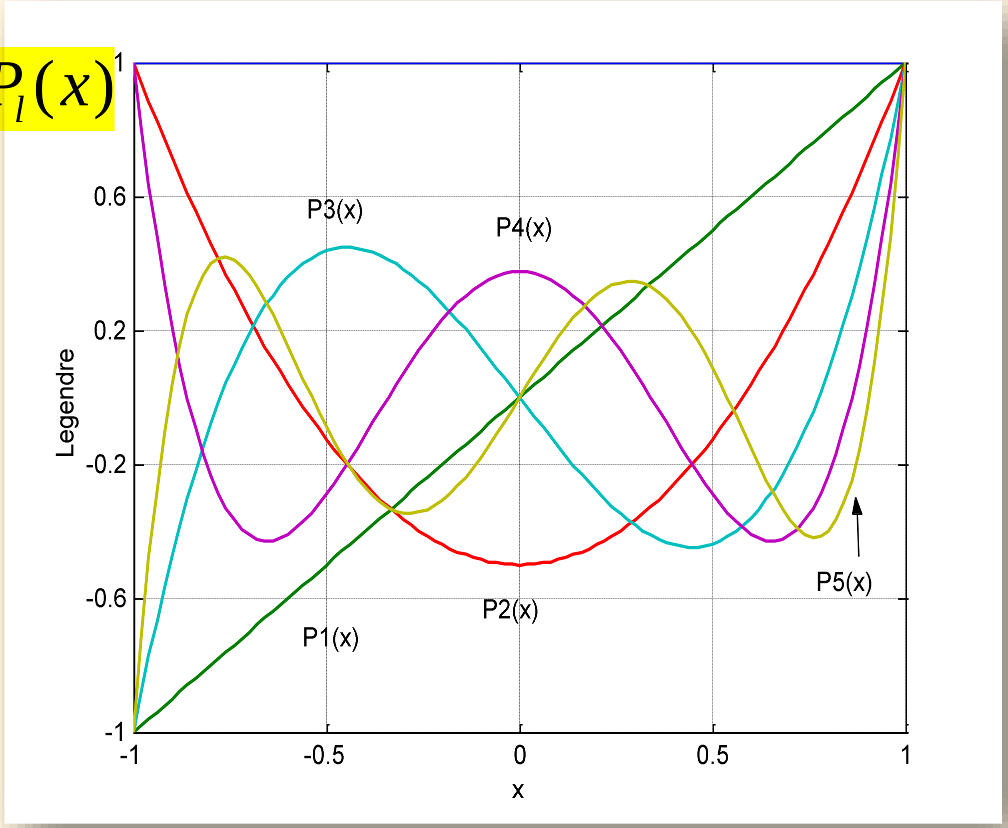
$$P'_{l+1}(x)=xP'_l(x)+(l+1)P_l(x)$$

$$P'_{l+1}(0)=(l+1)P_l(0)$$

$$P'_{2k+1}(0)=(2k+1)P_{2k}(0)$$

$$=\frac{(-1)^k(2k+1)!!}{(2k)!}$$

$$P'_{2k}(0)=2kP_{2k-1}(0)=0$$



11.4 连带 Legendre函数和球谐函数

❑连带 Legendre函数

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]+\left[\nu(\nu+1)-\dfrac{m^2}{1-x^2}\right]\Theta=0 \\ \Theta(x)\big|_{x=\pm1}<\infty \end{cases}$$

$$\Downarrow$$

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[\nu(\nu+1)-\frac{m^2}{1-x^2}\right]y=0$$

作变换

$$y(x)=(1-x^2)^{|m|/2}v(x)$$

为什么？在$\xi=1-x\sim1$附近

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[\nu(\nu+1)-\frac{m^2}{1-x^2}\right]y=0$$

$$\Downarrow$$

$$2(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{m^2}{2(1-x)}y\approx0$$

$$\Downarrow$$

$$\xi^2\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}+\xi\frac{\mathrm{d}y}{\mathrm{d}\xi}-\frac{m^2}{4}y=0 \Leftarrow$$ **Euler方程**

$$\Downarrow$$

$$y\approx a(1-x)^{|m|/2}+b(1-x)^{-|m|/2}\Rightarrow y\approx(1-x)^{|m|/2}$$

在x~−1附近
$$y \approx (1+x)^{|m|/2}$$

为了去掉x=±1二点的部分奇性，令

$$y(x) = (1-x)^{|m|/2}(1+x)^{|m|/2}v(x)$$
$$= (1-x^2)^{|m|/2}v(x)$$

$$(1-x^2)v'' - 2(|m|+1)xv' + \left[\nu(\nu+1) - |m|(|m|+1)\right]v = 0$$

- **归纳法**

上式可有 Legendre方程求m次导数得到

①Legendre方程两边求 1 次导数

$$\left[(1-x^2)y''\right]' - \left(2xy'\right)' + \nu(\nu+1)y' = 0$$

$$(1-x^2)(y^{[1]})'' - 2(1+1)x(y^{[1]})'$$
$$+[\nu(\nu+1)-1\cdot(1+1)]y^{[1]}=0$$

m=1时成立！

②设求m=k次导数, 下列方程成立

$$(1-x^2)(y^{[k]})'' - 2(k+1)x(y^{[k]})'$$
$$+[\nu(\nu+1)-k(k+1)]y^{[k]}=0$$

设m=k时成立

③上式两边再求 1 次导数

$$(1-x^2)(y^{[k+1]})'' - 2[(k+1)+1]x(y^{[k+1]})'$$
$$+[\nu(\nu+1)-(k+1)(k+1+1)]y^{[k+1]}=0$$

m=k+1时也成立！

因此方程的解为

$$v_1(x) = y_1^{[|m|]}(x) = \frac{\mathrm{d}^{|m}P_\nu(x)}{\mathrm{d}x^{|m|}}; v_2(x) = y_2^{[|m|]}(x) = \frac{\mathrm{d}^{|m}Q_\nu(x)}{\mathrm{d}x^{|m|}}$$

- 连带 Legendre 方程的解为

$$y_1(x) \equiv P_\nu^{|m|}(x) = (1-x^2)^{|m|/2}\frac{\mathrm{d}^{|m|}P_\nu(x)}{\mathrm{d}x^{|m|}}$$

$$y_2(x) \equiv Q_\nu^{|m|}(x) = (1-x^2)^{|m|/2}\frac{\mathrm{d}^{|m|}Q_\nu(x)}{\mathrm{d}x^{|m|}}$$

第一和二类连带 Legendre 函数

——绝对值是因为连带 Legendre 方程只出现 m^2, 对$(+m)$ 和 $(-m)$ 应该得到同样的结果.

特别注意：以上过程已经假定$|m|$是整数，否则必须严格求解方程, 但没有要求ν必须是整数.

4个第一类连带 Legendre 函数

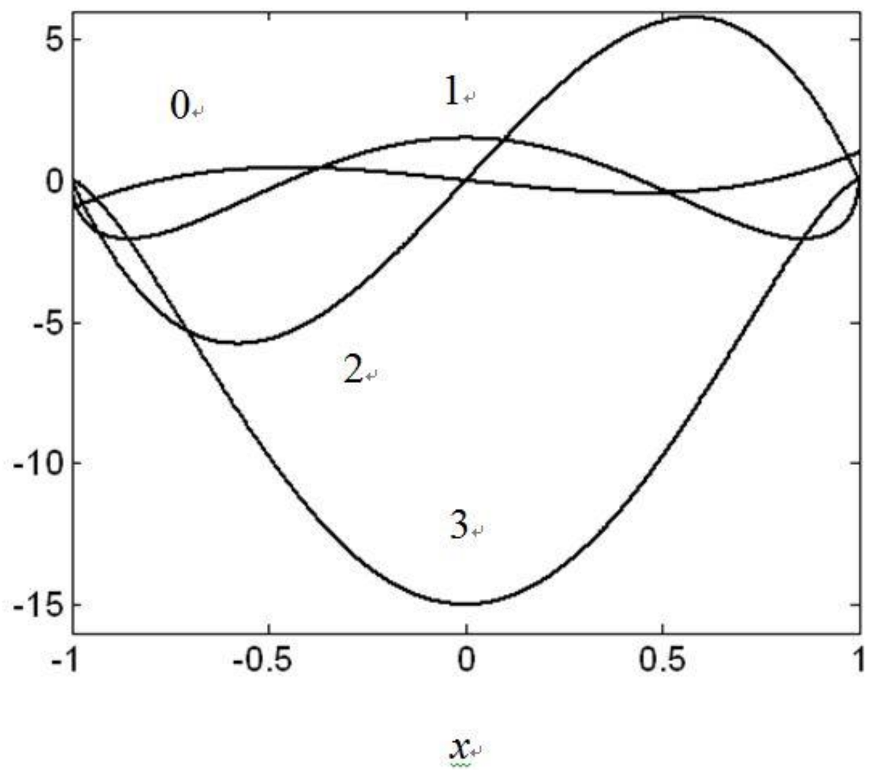


图 2.3.2 $P_3^0(x)=P_3(x)$(曲线 0)； $P_3^1(x)$(曲线 1)；

$P_3^2(x)$(曲线 2)； 和 $P_3^3(x)$(曲线 3)

□ 连带 Legendre算子的本征值问题

■ **奇异的S-L本征值问题，存在自然边界条件**

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]+\dfrac{m^2}{1-x^2}\Theta=\nu(\nu+1)\Theta \\ \Theta(x)\big|_{x=\pm1}<\infty \end{cases}$$

本征值 $\lambda_l\equiv\nu(\nu+1)=l(l+1)(l=0,1,2,3,\ldots)$

本征函数

$$\Theta_l(x)\equiv P_\nu^{|m|}(x)=(1-x^2)^{|m|/2}\frac{\mathrm{d}^{|m|}P_l(x)}{\mathrm{d}x^{|m|}},\quad (l=|m|,|m|+1,\ldots.)$$

注意：此时l是整数，$P_l(x)$是l阶多项式，故$l\geq|m|$

① 给定m, l: $m, m+1, m+2, \ldots$(无限个);

② 给定l, m: $-l, -l+1, \ldots, 0, l-1, l-2, \ldots l$, ($2l+1$个);

③ 本征值$\lambda_l = l(l+1)$ 与m无直接关系，说明简并度为($2l+1$).

■ 正交性关系

$$\int_{-1}^{1} P_l^{|m|}(x) P_k^{|m|}(x) \mathrm{d}x = \frac{(l+|m|)!}{(l-|m|)!} \frac{2}{2l+1} \delta_{lk}$$

式中 $|m|$ 应相同

■ 广义 Fourier 展开

定义在[–1,+1]上的平方可积函数 $f(x)$可展成广义 Fourier 级数

$$
\begin{cases}
f(x) = \sum_{l=|m|}^{\infty} f_l P_l^{|m|}(x) \\
f_l = \dfrac{2l+1}{2} \dfrac{(l-|m|)!}{(l+|m|)!} \int_{-1}^{1} f(x) P_l^{|m|}(x) \mathrm{d}x
\end{cases}
$$

$$\Downarrow$$

$$
\begin{cases}
f(\vartheta) = \sum_{l=|m|}^{\infty} f_l P_l^{|m|}(\cos\vartheta) \\
f_l = \dfrac{2l+1}{2} \dfrac{(l-|m|)!}{(l+|m|)!} \int_{0}^{\pi} f(\vartheta) P_l^{|m|}(\cos\vartheta) \sin\vartheta \mathrm{d}\vartheta
\end{cases}
$$

——注意：(1)对不同的 m 可得到许多完备系; (2)展式从 $l>|m|$的项开始.

❑球谐函数

■ 单位球面上的本征值问题

$$-\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]Y(\vartheta,\varphi)=\nu(\nu+1)Y(\vartheta,\varphi)$$

$$Y(\vartheta,\varphi)|_{\vartheta=0,\pi}<\infty;Y(\vartheta,\varphi)=Y(\vartheta,\varphi+2\pi)$$

本征值 $\lambda_l\equiv\nu(\nu+1)=l(l+1)(l=0,1,2,3,\ldots)$

本征函数—球谐函数

$$Y_l^m(\vartheta,\varphi)=P_l^{|m|}(\cos\vartheta)e^{\mathrm{i}m\varphi}$$

$$\left(\begin{array}{c} l=0,1,2,\ldots \\ |m|\le l:-l,-l+1,\ldots,0,1,2,\ldots,l \end{array}\right)$$

对每一个本征值λ_l, 有 $(2l+1)$ 个独立的球谐函数—简并度为$(2l+1)$

■归一化球谐函数

$$Y_{lm}(\vartheta,\varphi)=\left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2}Y_l^m(\vartheta,\varphi)$$

■正交关系

$$\iint_{球面}Y_l^m(\vartheta,\varphi)[Y_k^n(\vartheta,\varphi)]^*\mathrm{d}\Omega=\int_0^{\pi}\int_0^{2\pi}Y_l^m(\vartheta,\varphi)[Y_k^n(\vartheta,\varphi)]^*\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi$$

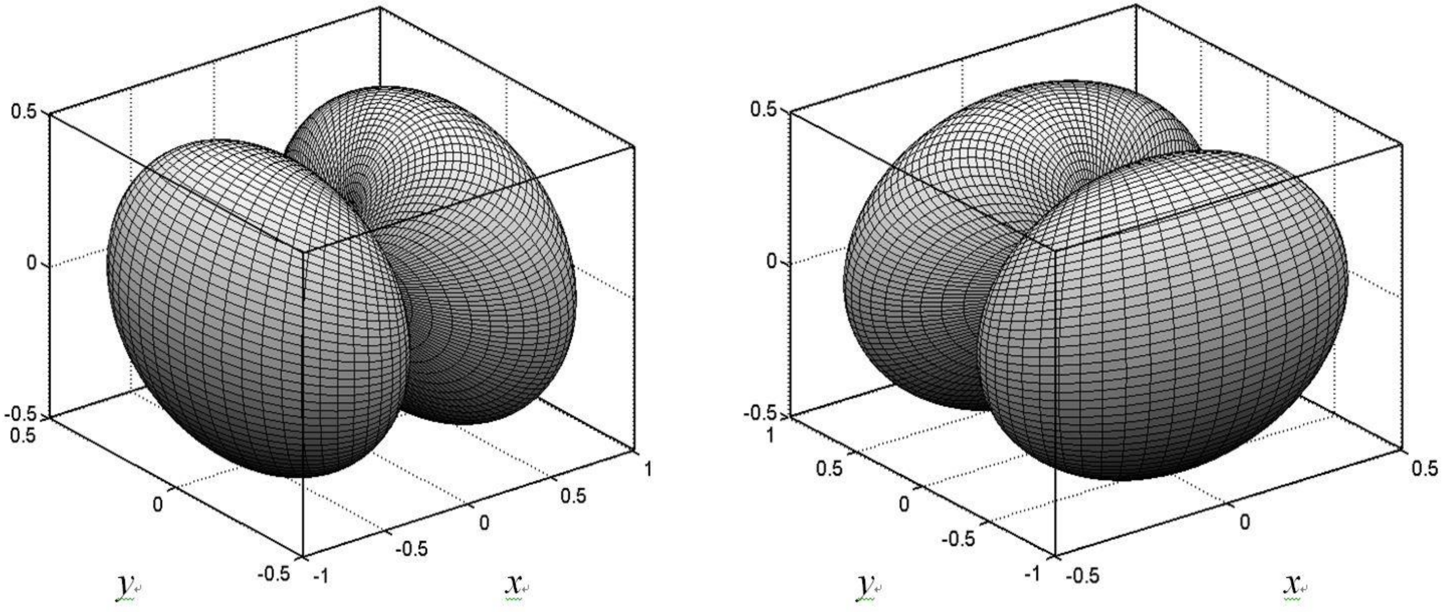
$$=\int_0^{\pi}P_l^{|m|}(\cos\vartheta)P_k^{|n|}(\cos\vartheta)\sin\vartheta\mathrm{d}\vartheta\cdot\int_0^{2\pi}e^{\mathrm{i}(m-n)\varphi}\mathrm{d}\varphi$$

$$=(N_l^m)^2\delta_{mn}\delta_{lk}$$

$$\iint_{球面}Y_l^m(\vartheta,\varphi)[Y_k^n(\vartheta,\varphi)]^*\mathrm{d}\Omega=(N_l^m)^2\delta_{mn}\delta_{lk}$$

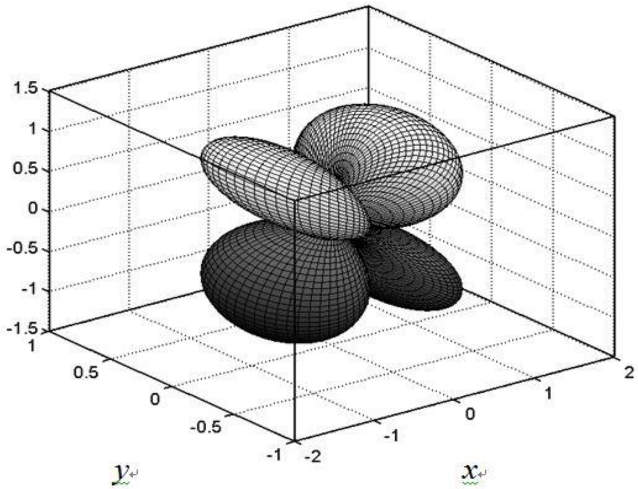
■球谐函数的模

$$(N_l^m)^2=\iint_{球面}|Y_l^m(\vartheta,\varphi)|^2\,\mathrm{d}\Omega==\frac{4\pi}{2l+1}\cdot\frac{(l+|m|)!}{(l-|m|)!}$$

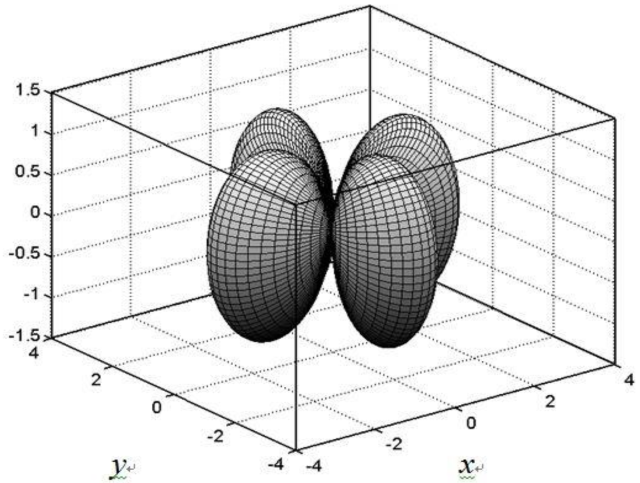


(a) (b)

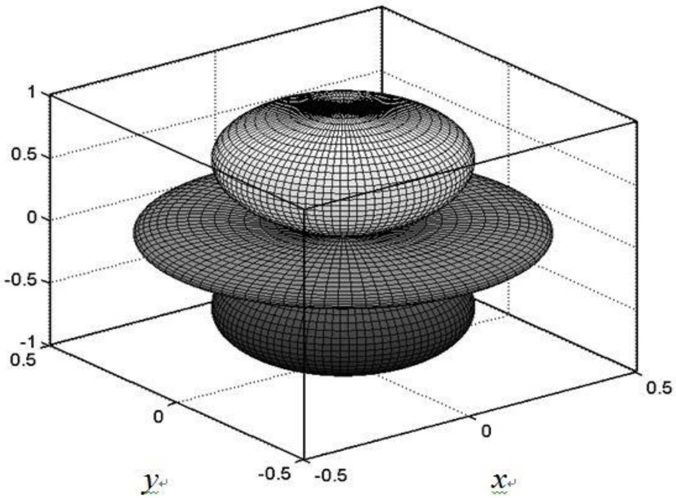
(a) $\mathrm{Re}\left[Y_{11}(\vartheta,\varphi)\right]$;(b) $\mathrm{Im}\left[Y_{11}(\vartheta,\varphi)\right]$



(a)



(b)



(c)

(a) $\mathrm{Re}\left[Y_{21}(\vartheta,\varphi)\right]$;

(b) $\mathrm{Re}\left[Y_{22}(\vartheta,\varphi)\right]$; (c) $Y_{20}(\vartheta,\varphi)$

■球面上的广义Fourier 展开

定义在球面 *S*: $(0\le\vartheta\le\pi,\ 0\le\varphi\le2\pi)$上的平方可积函数 $f(\vartheta,\varphi)$

$$\int_0^{\pi}\int_0^{2\pi} f^2(\vartheta,\varphi)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi<\infty$$

可展成广义 Fourier 级数

$$f(\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}C_l^m Y_l^m(\vartheta,\varphi)$$

$$C_l^m=\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\int_0^{\pi}\int_0^{2\pi} f(\vartheta,\varphi)[Y_l^m(\vartheta,\varphi)]^*\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

例1 把球面上的Dirac Delta函数展开成球函数。

$$\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_l^m Y_l^m(\vartheta,\varphi)$$

$$A_l^m=\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\int_0^{\pi}\int_0^{2\pi}\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')[Y_l^m(\vartheta,\varphi)]^*\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

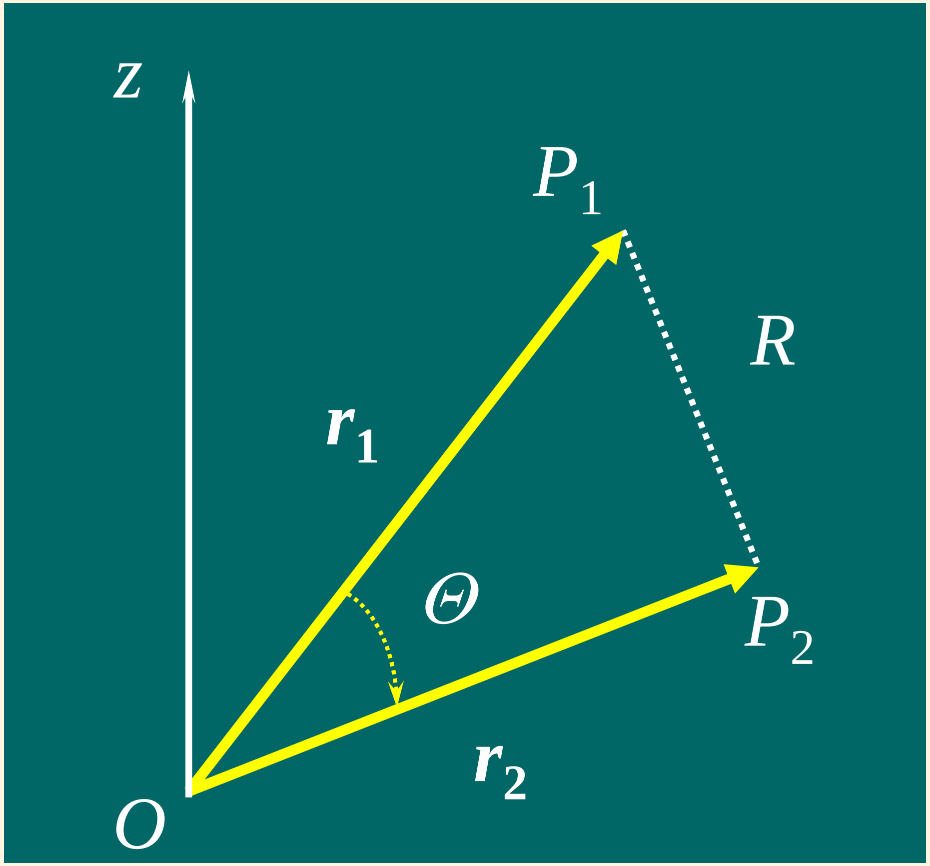
$$=\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}[Y_l^m(\vartheta',\varphi')]^*$$

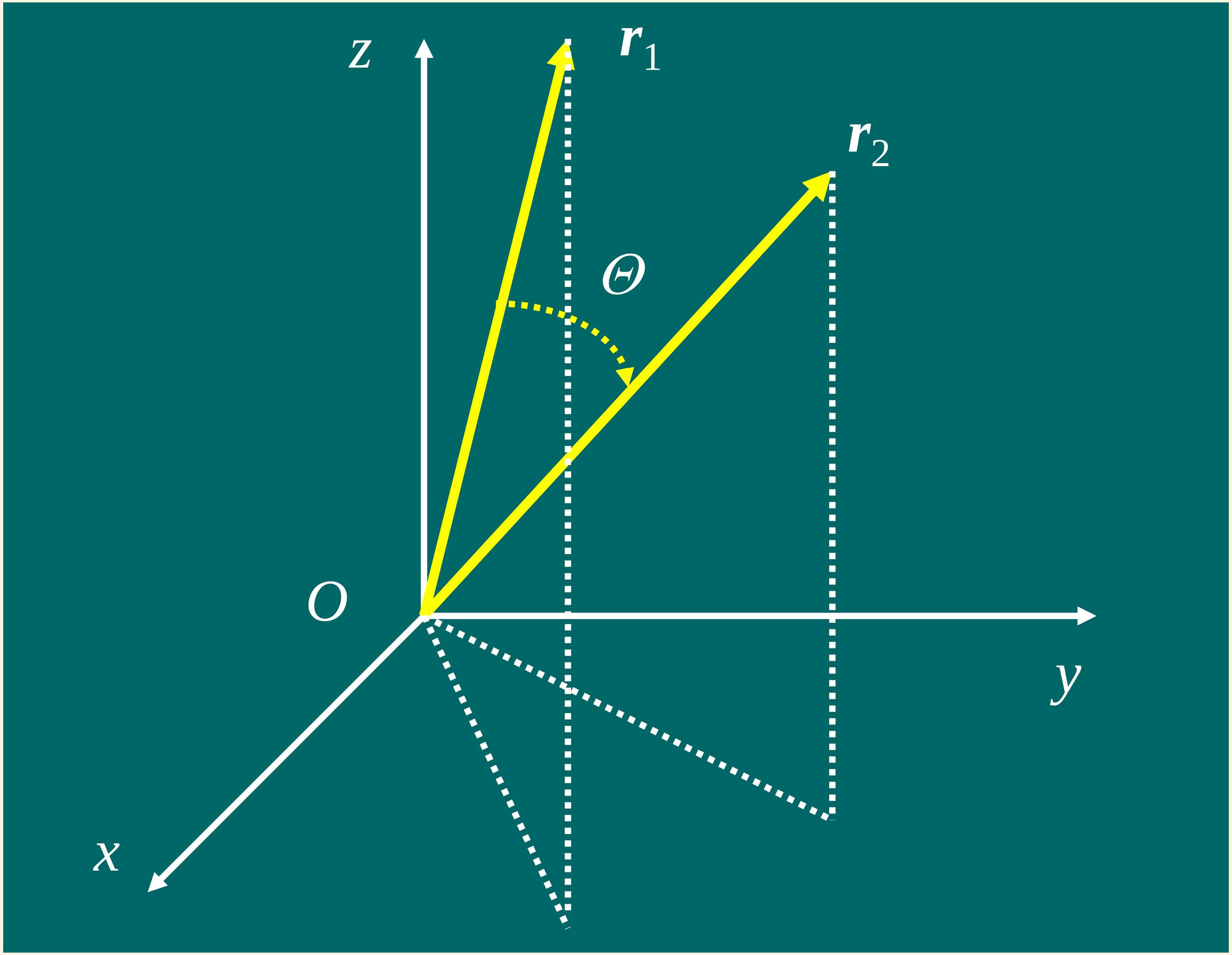
$$Y_{lm}(\vartheta,\phi)=\left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2}Y_l^m(\vartheta,\varphi)$$

$$\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^*(\vartheta',\varphi')Y_{lm}(\vartheta,\varphi)$$

例2 证明加法公式

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^{*}(\vartheta_1,\varphi_1) Y_{lm}(\vartheta_2,\varphi_2)$$




z
r1
r2
Θ
O
y
x

■P_1点电荷在P_2点产生的电势

$$u(r_2,\vartheta_2,\varphi_2)=\frac{1}{4\pi\sqrt{r_1^2+r_2^2-2r_1r_2\cos\Theta}}$$

$$\cos\Theta=\boldsymbol{r}_1\cdot\boldsymbol{r}_2/|\boldsymbol{r}_1||\boldsymbol{r}_2|=\boldsymbol{e}_{r_1}\cdot\boldsymbol{e}_{r_2}$$

$$=\cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos(\varphi_1-\varphi_2)$$

$$\boldsymbol{e}_{r_1}=\sin\vartheta_1\cos\varphi_1\boldsymbol{e}_x+\sin\vartheta_1\sin\varphi_1\boldsymbol{e}_y+\cos\vartheta_1\boldsymbol{e}_z$$

$$\boldsymbol{e}_{r_2}=\sin\vartheta_2\cos\varphi_2\boldsymbol{e}_x+\sin\vartheta_2\sin\varphi_2\boldsymbol{e}_y+\cos\vartheta_2\boldsymbol{e}_z$$

$$u(r_2,\vartheta_2,\varphi_2)=\frac{1}{4\pi r_1\sqrt{1-2t\cos\Theta+t^2}}=\frac{1}{4\pi r_1}\sum_{l=0}^{\infty}t^l P_l(\cos\Theta)$$

■ **P_2点的电势满足Poisson方程**

$$\nabla^2 u=-\frac{\delta(r-r_1)\delta(\vartheta-\vartheta_1)\delta(\varphi-\varphi_1)}{r^2\sin\vartheta}$$

$$u\big|_{r=0}<\infty;\quad \lim_{r\to\infty}u\to 0$$

$$\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta_1)\delta(\varphi-\varphi_1)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta,\varphi)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}$$

$$=-\frac{\delta(r-r_1)\delta(\vartheta-\vartheta_1)\delta(\varphi-\varphi_1)}{r^2\sin\vartheta}$$

令

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_{lm}(r)Y_{lm}(\vartheta,\varphi)$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g_{lm}}{\partial r}\right)-l(l+1)\frac{g_{lm}}{r^2}\right]Y_{lm}(\vartheta,\varphi)$$

$$=-\frac{1}{r^2}\delta(r,r_1)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta,\varphi)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g_{lm}}{\partial r}\right)-l(l+1)\frac{g_{lm}}{r^2}=-\frac{1}{r^2}\delta(r-r_1)Y_{lm}^*(\vartheta_1,\varphi_1)$$

(1)$r<r_1$——原点包含在区域内

$$r^2\frac{\partial^2 g_{lm}}{\partial r^2}+2r\frac{\partial g_{lm}}{\partial r}-l(l+1)g_{lm}=0$$

Euler 方程

$$g_{lm}(r)=A_l r^l+B_l r^{-(l+1)}=A_l r^l$$

(2)$r>r_1$——无限远包含在区域内

$$g_{lm}(r)=C_l r^l+D_l r^{-(l+1)}=D_l r^{-(l+1)}$$

(3)$r=r_1$连接条件

(A)函数必须连续，否则方程出现Dirac Delta函数的导数

$$g_{lm}(r)\big|_{r=r_1+\varepsilon}=g_{lm}(r)\big|_{r=r_1-\varepsilon}$$

(B)一阶导数必须间断，二阶导数后方程才能出现Dirac Delta函数

$$\int_{r_1-\varepsilon}^{r_1+\varepsilon}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g_{lm}}{\partial r}\right)-l(l+1)\frac{g_{lm}}{r^2}\right]\mathrm{d}r=-\int_{r_1-\varepsilon}^{r_1+\varepsilon}\left[\frac{1}{r^2}\delta(r-r_1)Y_{lm}^*(\vartheta_1,\varphi_1)\right]\mathrm{d}r$$

$$\Downarrow$$

$$\left.\frac{\partial g_{lm}}{\partial r}\right|_{r_1+\varepsilon}-\left.\frac{\partial g_{lm}}{\partial r}\right|_{r_1-\varepsilon}=-\frac{1}{r_1^2}Y_{lm}^*(\vartheta_1,\varphi_1)$$

$$D_l r_1^{-(l+1)} = A_l r_1^l$$

$$-(l+1)D_l r_1^{-(l+2)} - lA_l r_1^{l-1} = -\frac{1}{r_1^2} Y_{lm}^*(\vartheta_1, \varphi_1)$$

$$\Downarrow$$

$$A_l = \frac{r_1^{-(l+1)}}{2l+1} Y_{lm}^*(\vartheta_1, \varphi_1); D_l = \frac{r_1^l}{2l+1} Y_{lm}^*(\vartheta_1, \varphi_1)$$

$$\Downarrow$$

$$g_{lm}(r) = \frac{1}{2l+1} Y_{lm}^*(\vartheta_1, \varphi_1) \begin{cases} \dfrac{1}{r_1}\left(\dfrac{r}{r_1}\right)^l, (r < r_1) \\ \dfrac{1}{r}\left(\dfrac{r_1}{r}\right)^l, (r > r_1) \end{cases}$$

■P_2点的电势

$$u(r_2,\vartheta_2,\varphi_2)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tilde{g}_{lm}(r_2)Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta_2,\varphi_2)$$

$$\tilde{g}_{lm}(r_2)=\frac{1}{2l+1}\begin{cases}\dfrac{1}{r_1}\left(\dfrac{r_2}{r_1}\right)^l,(r_2<r_1)\\[2ex]\dfrac{1}{r_2}\left(\dfrac{r_1}{r_2}\right)^l,(r_2>r_1)\end{cases}$$

■加法公式(取 $r_2<r_1$)

$$u(r_2,\vartheta_2,\varphi_2)=\sum_{l=0}^{\infty}\frac{1}{(2l+1)r_1}t^l\sum_{m=-l}^{l}Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta_2,\varphi_2)$$

$$P_l(\cos\Theta)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}^{*}(\vartheta_1,\varphi_1)Y_{lm}(\vartheta_2,\varphi_2)$$

例3 证明正交性关系

$$\iint_{球面} \nabla Y_l^m(\vartheta,\varphi)\cdot[\nabla Y_k^n(\vartheta,\varphi)]^* \mathrm{d}\Omega = l(l+1)(N_l^m)^2\delta_{mn}\delta_{lk}$$

矢量恒等式

$$r\nabla\cdot[rY_k^{n*}\nabla Y_l^m] = \nabla[rY_k^{n*}]\cdot[r\nabla Y_l^m] + r^2 Y_k^{n*}\nabla^2 Y_l^m$$

$$\nabla[rY_k^{n*}] = Y_k^{n*}\nabla r + r\nabla Y_k^{n*}$$

注意到∇Y_l^m仅有e_ϑ和e_φ量，而∇r仅有e_r，故

$$\nabla[rY_k^{n*}]\cdot\nabla Y_l^m = \left(Y_k^{n*}\nabla r + r\nabla Y_k^{n*}\right)\cdot\nabla Y_l^m = r\nabla Y_k^{n*}\cdot\nabla Y_l^m$$

$$\Downarrow$$

$$r\nabla\cdot[rY_k^{n*}\nabla Y_l^m] = [r\nabla Y_k^{n*}]\cdot[r\nabla Y_l^m] + r^2 Y_k^{n*}\nabla^2 Y_l^m$$

$$\Downarrow$$

$$\iint_{球面} r\nabla\cdot[rY_k^{n*}\nabla Y_l^m]\mathrm{d}\Omega = \iint_{球面}[r\nabla Y_k^{n*}]\cdot[r\nabla Y_l^m]\mathrm{d}\Omega + \iint_{球面} r^2 Y_k^{n*}\nabla^2 Y_l^m \mathrm{d}\Omega$$

取单位球面取值r=1

$$\iint_{球面} \nabla\cdot[rY_k^{n*}\nabla Y_l^m]\mathrm{d}\Omega = \iint_{球面} \nabla Y_k^{n*}\cdot\nabla Y_l^m \mathrm{d}\Omega + \iint_{球面} Y_k^{n*}\nabla^2 Y_l^m \mathrm{d}\Omega$$

(A)右边第二项

$$\nabla^2 Y_l^m = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y_l^m}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_l^m}{\partial\varphi^2} = -l(l+1)Y_l^m$$

(B)左边第一项：必须微分运算后才能取r=1。注意到

$$Y_k^{*n}\nabla Y_l^m \equiv f(\vartheta,\varphi)\boldsymbol{e}_\vartheta + g(\vartheta,\varphi)\boldsymbol{e}_\varphi$$

$$\Downarrow$$

$$\nabla\cdot[rY_k^{*n}\nabla Y_l^m] = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta f) + \frac{1}{\sin\vartheta}\frac{\partial g}{\partial\varphi}$$

直接计算

$$\iint_{球面} \nabla\cdot[rY_k^{*n}\nabla Y_l^m]\mathrm{d}\Omega=\iint_{球面}\left[\frac{\partial(\sin\vartheta f)}{\partial\vartheta}+\frac{\partial g}{\partial\varphi}\right]\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$=\int_0^{2\pi}\int_0^{\pi}\frac{\partial(\sin\vartheta f)}{\partial\vartheta}\mathrm{d}\vartheta\mathrm{d}\varphi+\int_0^{2\pi}\int_0^{\pi}\frac{\partial g}{\partial\varphi}\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$=\int_0^{2\pi}\sin\vartheta f(\vartheta,\varphi)\Big|_0^{\pi}\mathrm{d}\varphi+\int_0^{\pi}g\Big|_0^{2\pi}\mathrm{d}\vartheta=0$$

(C)最后得到

$$\iint_{球面}\nabla Y_l^m\cdot\nabla Y_k^{n*}\mathrm{d}\Omega=l(l+1)\iint_{球面}Y_l^mY_k^{n*}\mathrm{d}\Omega$$

$$=l(l+1)(N_l^m)^2\delta_{mn}\delta_{lk}$$

$$\Downarrow$$

$$\iint_{球面}\nabla Y_{lm}\cdot\nabla Y_{kn}^*\mathrm{d}\Omega=l(l+1)\delta_{mn}\delta_{lk}$$

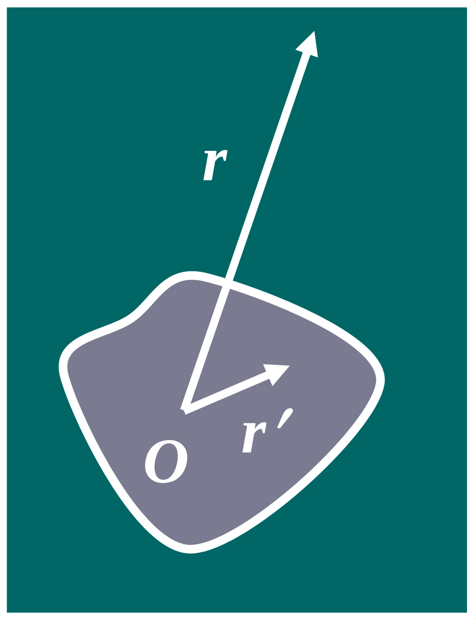
例4 球坐标中的多极展开关系

Laplace方程 无限空间

$$-\nabla^2 u(\boldsymbol{r}) = \rho(\boldsymbol{r}) \Rightarrow u(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'$$

$$\Uparrow$$

$$f(\boldsymbol{r}-\boldsymbol{r}') = f(\boldsymbol{r}) - \sum_{i=1}^{3} x_i' \frac{\partial f(\boldsymbol{r})}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{3} x_i' x_j' \frac{\partial^2 f(\boldsymbol{r})}{\partial x_i \partial x_j} + \cdots$$



- **小区域源，远场展开**

$$u(\boldsymbol{r}) = \frac{1}{4\pi|\boldsymbol{r}|}\int \rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}' - \frac{1}{4\pi}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{1}{|\boldsymbol{r}|}\right)\int x_i'\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'$$

$$+\frac{1}{8\pi}\sum_{i,j=1}^{3}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{|\boldsymbol{r}|}\right)\int x_i' x_j' \rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}' + \ldots.$$

球坐标展开

$$\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{1}{r}\left(\frac{r'}{r}\right)^{l}Y_{lm}^{*}(\vartheta',\varphi')Y_{lm}(\vartheta,\varphi),(r>r')$$

区域外场的分布

$$u(\boldsymbol{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Q_{lm}\frac{Y_{lm}(\vartheta,\varphi)}{r^{l+1}}$$

$$Q_{lm}\equiv\frac{1}{2l+1}\int r'^{l}\rho(r',\vartheta',\varphi')Y_{lm}^{*}(\vartheta',\varphi')r'^{2}\sin\vartheta'\mathrm{d}r'\mathrm{d}\vartheta'\mathrm{d}\varphi'$$

$$\Downarrow$$

$$u(\boldsymbol{r})=\underbrace{\frac{1}{r}Q_{00}Y_{00}(\vartheta,\varphi)}_{\text{平均电荷}}+\underbrace{\frac{1}{r^{2}}\sum_{m=-1}^{+1}Q_{1m}Y_{1m}(\vartheta,\varphi)}_{\text{偶极矩}}+\underbrace{\frac{1}{r^{3}}\sum_{m=-2}^{+2}Q_{2m}Y_{2m}(\vartheta,\varphi)}_{\text{四极矩}}+\ldots.$$

$$u_d(\boldsymbol{r})=\frac{1}{r^2}[Q_{10}Y_{10}(\vartheta,\varphi)+Q_{1+1}Y_{1+1}(\vartheta,\varphi)+Q_{1-1}Y_{1-1}(\vartheta,\varphi)]$$

$$Y_{10}(\vartheta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\vartheta;Y_{1\pm1}(\vartheta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\vartheta\exp(\pm\mathrm{i}\varphi)$$

第一项：*z*轴上的偶极矩产生的场

$$\frac{1}{r^2}Q_{10}Y_{10}(\vartheta,\varphi)=\sqrt{\frac{3}{4\pi}}\frac{1}{r^2}Q_{10}\cos\vartheta$$

第二和三项：*x*和*y*轴上的偶极矩

$$\frac{1}{r^2}[Q_{1+1}Y_{1+1}(\vartheta,\varphi)+Q_{1-1}Y_{1-1}(\vartheta,\varphi)]$$

$$=-\sqrt{\frac{3}{8\pi}}\frac{1}{r^2}\left[(Q_{1+1}+Q_{1-1})\frac{x}{r}+\mathrm{i}(Q_{1+1}-Q_{1-1})\frac{y}{r}\right]$$

■ 球对称分布 $\rho(r',\vartheta',\varphi')=\rho(r')$

$$Q_{lm}=\frac{\sqrt{4\pi}}{2l+1}\int_0^a r'^l\rho(r')r'^2\mathrm{d}r'\int Y_{00}(\vartheta',\varphi')Y_{lm}^*(\vartheta',\varphi')\sin\vartheta'\mathrm{d}\vartheta'\mathrm{d}\varphi'$$

因此

$$Q_{00}=\sqrt{4\pi}\int_0^a\rho(r')r'^2\mathrm{d}r';\quad Q_{lm}=0,\ (l>0,m\neq 0)$$

——不存在偶极矩以及以上的场

①位于原点的电荷

$$\rho(\boldsymbol{r})=e\delta(\boldsymbol{r})=e\delta(x)\delta(y)\delta(z)\ =\frac{e}{4\pi r^2}\delta(r)$$

$$Q_{lm}\equiv\frac{1}{2l+1}\int r'^l\rho(r',\vartheta',\varphi')Y_{lm}^*(\vartheta',\varphi')r'^2\sin\vartheta'\mathrm{d}r'\mathrm{d}\vartheta'\mathrm{d}\varphi'$$

$$=\frac{e}{4\pi}\frac{1}{2l+1}\int r'^l\delta(r')Y_{lm}^*(\vartheta',\varphi')\sin\vartheta'\mathrm{d}r'\mathrm{d}\vartheta'\mathrm{d}\varphi'$$

$$Q_{00}=\frac{e}{4\pi}\int Y_{00}^{*}(\vartheta',\varphi')\sin\vartheta'\mathrm{d}\vartheta'\mathrm{d}\varphi'=\frac{e}{\sqrt{4\pi}};Q_{lm}=0,(l\neq 0,m\neq 0))$$

$$u(\boldsymbol{r})=Q_{00}\frac{Y_{00}(\vartheta,\varphi)}{r}=\frac{e}{4\pi r}$$

②位于z的二个正负点电荷(z_0>0)

$$\rho(\boldsymbol{r})=\frac{e}{2\pi r^2\sin\vartheta}\delta(r-z_0)\delta(\vartheta)-\frac{e}{2\pi r^2\sin\vartheta}\delta(r-z_0)\delta(\vartheta-\pi)$$

$$Q_{lm}=\frac{e}{2\pi}\frac{1}{2l+1}\int r'^l\delta(r'-z_0)\delta(\vartheta')Y_{lm}^{*}(\vartheta',\varphi')\mathrm{d}r'\mathrm{d}\vartheta'\mathrm{d}\varphi'$$

$$-\frac{e}{2\pi}\frac{1}{2l+1}\int r'^l\delta(r-z_0)\delta(\vartheta'-\pi)Y_{lm}^{*}(\vartheta',\varphi')\mathrm{d}r'\mathrm{d}\vartheta'\mathrm{d}\varphi'$$

$$=\frac{e}{2\pi}\frac{z_0^l}{2l+1}\left[\int_0^{2\pi}Y_{lm}^{*}(0,\varphi')\mathrm{d}\varphi'-\int_0^{2\pi}Y_{lm}^{*}(\pi,\varphi')\mathrm{d}\varphi'\right]$$

$$\int_0^{2\pi} Y_{lm}^*(0,\varphi')\mathrm{d}\varphi' = 2\pi\sqrt{\frac{2l+1}{4\pi}}P_l^0(1)\delta_{m0}$$

$$\int_0^{2\pi} Y_{lm}^*(\pi,\varphi')\mathrm{d}\varphi' = 2\pi\sqrt{\frac{2l+1}{4\pi}}P_l^0(-1)\delta_{m0}$$

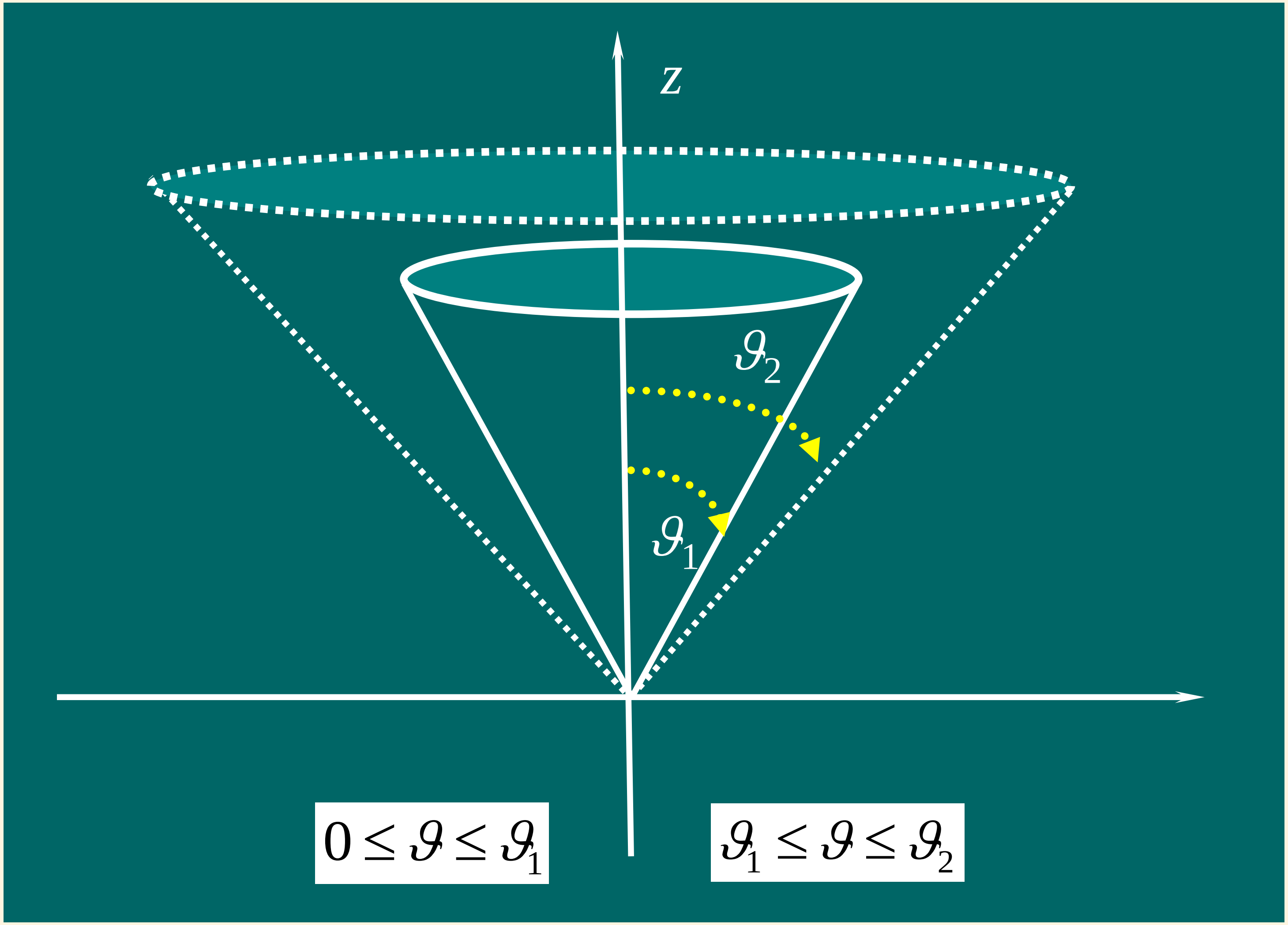
$$Q_{lm} = \frac{e}{\sqrt{4\pi}}\frac{z_0^l}{\sqrt{2l+1}}[1-(-1)^l]\delta_{m0}$$

$$u(\boldsymbol{r}) = \frac{e}{4\pi}\sum_{l=0}^{\infty} z_0^l[1-(-1)^l]\frac{P_l(\cos\vartheta)}{r^{l+1}}$$

$$= \frac{p_0}{4\pi r^2}\cos\vartheta + \frac{p_0 z_0^2}{4\pi}\frac{P_3(\cos\vartheta)}{r^4} + \ldots\ldots,(p_0 \equiv 2ez_0)$$

——平均电荷为零，只有偶极矩以上的场

11.4 Legendre函数：圆锥形区



问题：圆锥形区域的Laplace方程

$$\nabla^2 u(r,\vartheta)=0,(0<\vartheta<\vartheta_1,a<r<b)$$

$$u(r,\vartheta)|_{r=a}=\Theta_1(\vartheta);\quad u(r,\vartheta)|_{r=b}=\Theta_1(\vartheta)$$

$$u(r,\vartheta)|_{\vartheta=\vartheta_1}=0$$

$$u(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}][C_l P_l(\cos\vartheta)+D_l Q_l(\cos\vartheta)]$$

$$u(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}]P_l(\cos\vartheta)$$

■ **由极角方向边界条件**

任意r成立

$$u(r,\vartheta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}]P_l(\cos\vartheta_1)=0 \Rightarrow \{A_l,B_l\}=0$$

■ 由径向边界条件

$$u(a,\vartheta)=\sum_{l=0}^{\infty}[A_l a^l + B_l a^{-(l+1)}]P_l(\cos\vartheta)=\Theta_1(\vartheta)$$

$$u(b,\vartheta)=\sum_{l=0}^{\infty}[A_l b^l + B_l b^{-(l+1)}]P_l(\cos\vartheta)=\Theta_2(\vartheta)$$

$$\Downarrow$$

$$\{A_l, B_l\}$$

但不可能满足锥面边界条件

$$u(r,\vartheta_1)=\sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]P_l(\cos\vartheta_1)=0$$

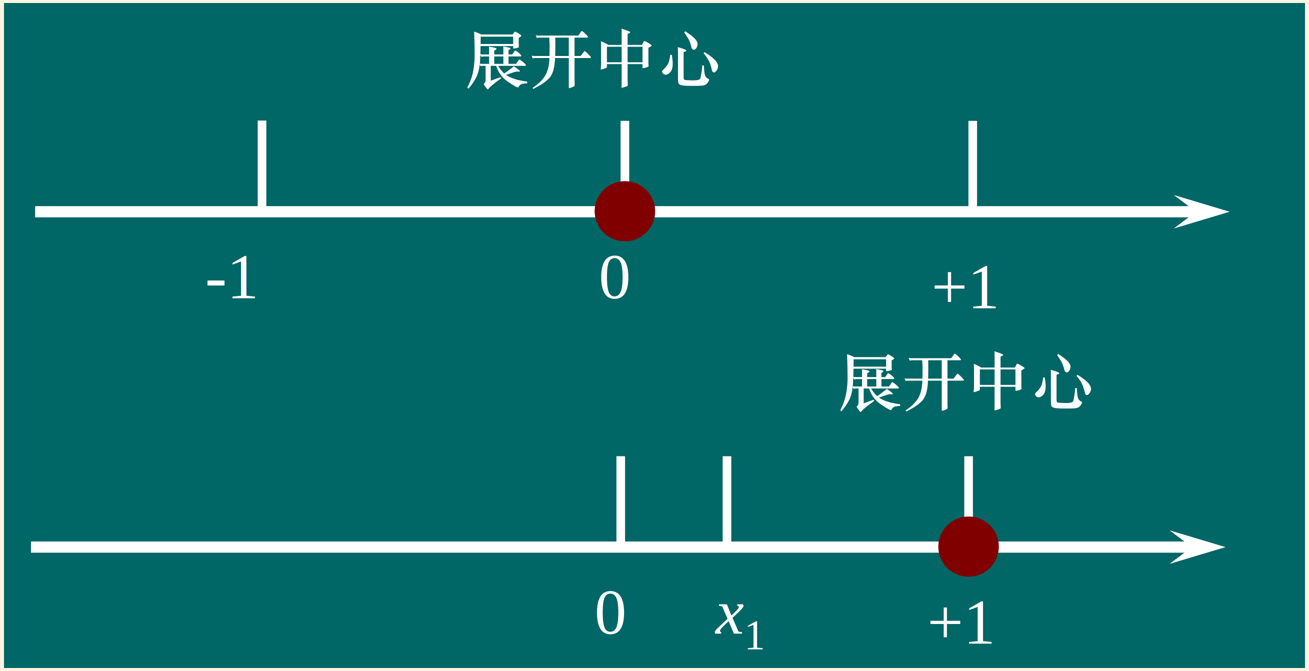
——问题出在哪里？解的形式错误！所取的解尽管满足Laplace方程，但是不可能同时满足3个边界条件！必须寻求解新的形式解。

$$\vartheta \in [0,\pi] \Rightarrow x \in [-1,+1]$$

$$\vartheta \in [0,\vartheta_1](\vartheta_1 < \pi/2)$$

$$\Downarrow$$

$$x \in [x_1,+1], x_1 = \cos\vartheta_1 > 0$$



假定问题关于z轴对称

$$(1-x^2)\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y(x)}{\mathrm{d}x}+\lambda(\lambda+1)y(x)=0$$

$$p(x)=-\frac{2x}{1-x^2};q(x)=\frac{\lambda(\lambda+1)}{1-x^2}$$

***x*=1正则奇点：一阶极点，存在一个正则解**

- **以正则奇点*x*=1为展开中心的解为**

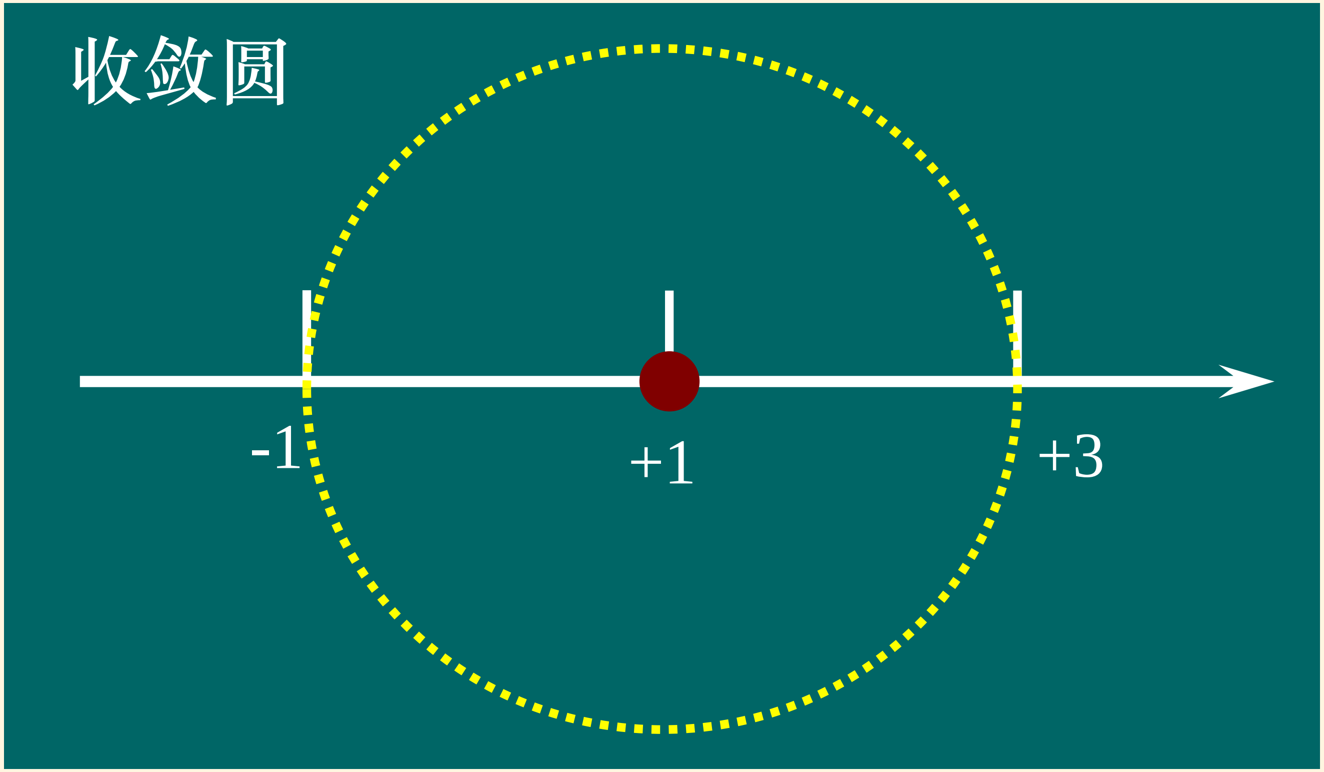
$$y(x)=(x-1)^{\rho}\sum_{n=0}^{\infty}c_n(x-1)^n$$

$$\rho(\rho-1)+\rho=0 \Rightarrow \rho=0$$

指标方程给出一个根

$$y_1(x)=P_\lambda(x)=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda-n+1)}\left(\frac{x-1}{2}\right)^n$$

——寻到一个在x=1点收敛的无限级数解(第一类Legendre函数)，但$\lambda \neq l$ (正整数)。



■ 另外一个解：第二类Legendre 函数

$$y_2(x) = Q_\lambda(x) = P_\lambda(x)\int \frac{1}{(1-x^2)[P_\lambda(x)]^2}\mathrm{d}x$$

注意：

- **$P_\lambda(x)$在x=–1处仍然发散; 如果要求$P_\lambda(x)$在x=–1处也有限,只有当λ=l (正整数) (与在x=0展开得到同样结果);**
- **$Q_\lambda(x)$在x=±1处始终发散!**

■ Laplace方程的解

$$u(r,\vartheta) = \sum_\lambda [A_\lambda r^\lambda + B_\lambda r^{-(\lambda+1)}]P_\lambda(\cos\vartheta)$$

■本征值λ的决定：具体问题有关

■$\vartheta=\vartheta_1$面上齐次边界条件(假定第一类边界条件)

$$u(r,\vartheta_1)=\sum_{\lambda}[A_\lambda r^{\lambda}+B_\lambda r^{-(\lambda+1)}]P_\lambda(\cos\vartheta_1)\equiv 0$$

$$P_\lambda(\cos\vartheta_1)=0 \Rightarrow \{\lambda_1,\lambda_2,\ldots,\}=\{\lambda_n\}$$

具有S-L本征值问题的基本性质

(1)正交性：

$$\int_{x_1}^{+1}P_{\lambda_i}(x)P_{\lambda_j}(x)\mathrm{d}x=\|P_{\lambda_i}(x)\|^2\,\delta_{ij}$$

(2)完备性：

$$f(x)=\sum_{n=1}^{\infty}a_nP_{\lambda_n}(x)=\sum_{n=1}^{\infty}\left[\frac{1}{\|P_{\lambda_n}\|^2}\int_{x_1}^{+1}f(x')P_{\lambda_n}(x')\mathrm{d}x'\right]P_{\lambda_n}(x)$$

例1 圆锥形区域的Laplace方程

$$\nabla^2 u(r,\vartheta)=0,(0<\vartheta<\vartheta_1,a<r<b)$$

$$u(r,\vartheta)|_{r=a}=\Theta_1(\vartheta);\quad u(r,\vartheta)|_{r=b}=\Theta_1(\vartheta)$$

$$u(r,\vartheta)|_{\vartheta=\vartheta_1}=0$$

$$u(r,\vartheta)=\sum_{\lambda}A_\lambda R_\lambda(r)P_\lambda(\cos\vartheta) \Rightarrow P_\lambda(\cos\vartheta_1)=0$$

存在无限个正根: $\{\lambda_1,\lambda_2,\ldots,\}=\{\lambda_n\}$

- **径向方程和解**

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R_n}{\mathrm{d}r}\right)-\lambda_n(\lambda_n+1)R_n=0,\ (a<r<b)$$

$$R_n(r)=A_n r^{\lambda_n}+B_n r^{-(\lambda_n+1)}$$

$$u(r,\vartheta)=\sum_{n=1}^{\infty}[A_n r^{\lambda_n}+B_n r^{-(\lambda_n+1)}]P_{\lambda_n}(\cos\vartheta)$$

$$u(r,\vartheta)|_{r=a}=\sum_{n=0}^{\infty}[A_n a^{\lambda_n}+B_n a^{-(\lambda_n+1)}]P_{\lambda_n}(\cos\vartheta)=\Theta_1(\vartheta)$$

$$u(r,\vartheta)|_{r=b}=\sum_{n=0}^{\infty}[A_n b^{\lambda_n}+B_n b^{-(\lambda_n+1)}]P_{\lambda_n}(\cos\vartheta)=\Theta_2(\vartheta)$$

$$A_n a^{\lambda_n}+B_n a^{-(\lambda_n+1)}=\frac{1}{\|P_{\lambda_n}\|^2}\int_0^{\vartheta_1}\Theta_1(\vartheta)P_{\lambda_n}(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta\equiv f_n$$

$$A_n b^{\lambda_n}+B_n b^{-(\lambda_n+1)}=\frac{1}{\|P_{\lambda_n}\|^2}\int_0^{\vartheta_1}\Theta_2(\vartheta)P_{\lambda_n}(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta\equiv g_n$$

$$A_n=\frac{g_n a^{-(\lambda_n+1)}-f_n b^{-(\lambda_n+1)}}{a^{-(\lambda_n+1)}b^{\lambda_n}-a^{\lambda_n}b^{-(\lambda_n+1)}};B_n=\frac{f_n b^{\lambda_n}-a^{\lambda_n}g_n}{a^{-(\lambda_n+1)}b^{\lambda_n}-a^{\lambda_n}b^{-(\lambda_n+1)}}$$

□ 本征函数正交性证明

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\varTheta}{\mathrm{d}x}\right]=\lambda(\lambda+1)\varTheta, x\in(\cos\vartheta_1,1) \\ \varTheta|_{x=1}<\infty;\ \varTheta|_{x=\cos\vartheta_1}=0 \end{cases}$$

$$\varTheta_{\lambda_n}(x)=P_{\lambda_n}(x),(n=1,2,\ldots)$$

$$P_\lambda(\cos\vartheta_1)=0\Rightarrow\lambda=\lambda_n,(n=1,2,\ldots)$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_{\lambda_i}}{\mathrm{d}x}\right]=\lambda_i(\lambda_i+1)P_{\lambda_i}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_{\lambda_j}}{\mathrm{d}x}\right]=\lambda_j(\lambda_j+1)P_{\lambda_j}$$

$$-P_{\lambda_j}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_{\lambda_i}}{\mathrm{d}x}\right]=\lambda_i(\lambda_i+1)P_{\lambda_i}P_{\lambda_j}$$

$$-P_{\lambda_i}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_{\lambda_j}}{\mathrm{d}x}\right]=\lambda_j(\lambda_j+1)P_{\lambda_i}P_{\lambda_j}$$

$$[\lambda_i(\lambda_i+1)-\lambda_j(\lambda_j+1)]\int_{\cos\vartheta_1}^{1} P_{\lambda_i}P_{\lambda_j}\,\mathrm{d}x$$

$$=\int_{\cos\vartheta_1}^{1}\left\{P_{\lambda_i}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_{\lambda_j}}{\mathrm{d}x}\right]-P_{\lambda_j}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_{\lambda_i}}{\mathrm{d}x}\right]\right\}\mathrm{d}x$$

$$=\int_{\cos\vartheta_1}^{1}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)P_{\lambda_i}\frac{\mathrm{d}P_{\lambda_j}}{\mathrm{d}x}\right]-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)P_{\lambda_j}\frac{\mathrm{d}P_{\lambda_i}}{\mathrm{d}x}\right]\right\}\mathrm{d}x$$

$$=(1-x^2)\left[P_{\lambda_i}\frac{\mathrm{d}P_{\lambda_j}}{\mathrm{d}x}-P_{\lambda_j}\frac{\mathrm{d}P_{\lambda_i}}{\mathrm{d}x}\right]_{\cos\vartheta_1}^{1}=0$$

$$\Downarrow$$

$$\int_{\cos\vartheta_1}^{1} P_{\lambda_i}(x)P_{\lambda_j}(x)\mathrm{d}x=0,\ (\lambda_i\neq\lambda_j)$$

注意：当x=1时，1–x^2=0, 当x=cosϑ_1时，满足边界条件——第二类边界条件也成立。

■径向齐次边界条件(假定第一类边界条件)

例2 圆锥形区域的Laplace方程

$$\nabla^2 u(r,\vartheta)=0,(0<\vartheta<\vartheta_1,a<r<b)$$

$$u(r,\vartheta)|_{r=a}=u(r,\vartheta)|_{r=b}=0$$

$$u(r,\vartheta)|_{\vartheta=\vartheta_1}=f(r)$$

①径向本征值问题

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\lambda(\lambda+1)R,\ (a<r<b)$$

$$R(r)=Ar^{\lambda}+Br^{-(\lambda+1)}$$

②边界条件

$$R(a)=Aa^{\lambda}+Ba^{-(\lambda+1)}=0$$

$$R(b)=Ab^{\lambda}+Bb^{-(\lambda+1)}=0$$

存在非零解条件

$$\begin{vmatrix} a^{\lambda} & a^{-(\lambda+1)} \\ b^{\lambda} & b^{-(\lambda+1)} \end{vmatrix} = a^{\lambda} b^{-(\lambda+1)} - b^{\lambda} a^{-(\lambda+1)}$$

$$= \frac{1}{b}\left(\frac{a}{b}\right)^{\lambda} - \frac{1}{a}\left(\frac{b}{a}\right)^{\lambda} = 0$$

$$\mu_n \equiv -\lambda_n(\lambda_n + 1) = \frac{1}{4} + \left[\frac{n\pi}{\ln(a/b)}\right]^2 > 0$$

$$\left(\frac{a}{b}\right)^{2\lambda+1} = 1 = e^{2in\pi}, (n = 0,1,2,\ldots) \Rightarrow \lambda_n = -\frac{1}{2} + \mathrm{i}\frac{n\pi}{\ln(a/b)}$$

$$R_n(r) = \left[r^{\lambda_n} - a^{\lambda_n}\left(\frac{a}{r}\right)^{\lambda_n+1}\right]; \lambda_n = -\frac{1}{2} + \mathrm{i}\frac{n\pi}{\ln(a/b)}$$

$$x^{\mathrm{i}\mu} = e^{\mathrm{i}\mu \ln x}$$

$$R_n(r) = 2\mathrm{i}r^{-\frac{1}{2}} e^{\mathrm{i}\frac{n\pi}{\ln(a/b)}\ln a} \sin\left[\frac{n\pi}{\ln(a/b)}\ln\left(\frac{r}{a}\right)\right]$$

$$R_n(r) = \frac{1}{\sqrt{r}}\sin\left[\frac{n\pi}{\ln(a/b)}\ln\left(\frac{r}{a}\right)\right]$$

——取为实值函数, 满足S-L本征值问题性质

③Laplace方程的解

$$u(r,\vartheta) = \sum_{n=0}^{\infty} A_n R_n(r) P_{\lambda_n}(\cos\vartheta); \quad \sum_{n=0}^{\infty} A_n R_n(r) P_{\lambda_n}(\cos\vartheta_1) = f(r)$$

$$u(r,\vartheta) = \sum_{n=0}^{\infty}\left[\frac{1}{\| R_n \|^2}\int_a^b f(r')R_n(r')\mathrm{d}r'\right] R_n(r)\frac{P_{\lambda_n}(\cos\vartheta)}{P_{\lambda_n}(\cos\vartheta_1)}$$

注意：径向本征值问题(S-L问题)

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\mu R,\ (a<r<b)$$

$$R(a)=R(b)=0,\mu\equiv-\lambda(\lambda+1)$$

$$\Rightarrow$$

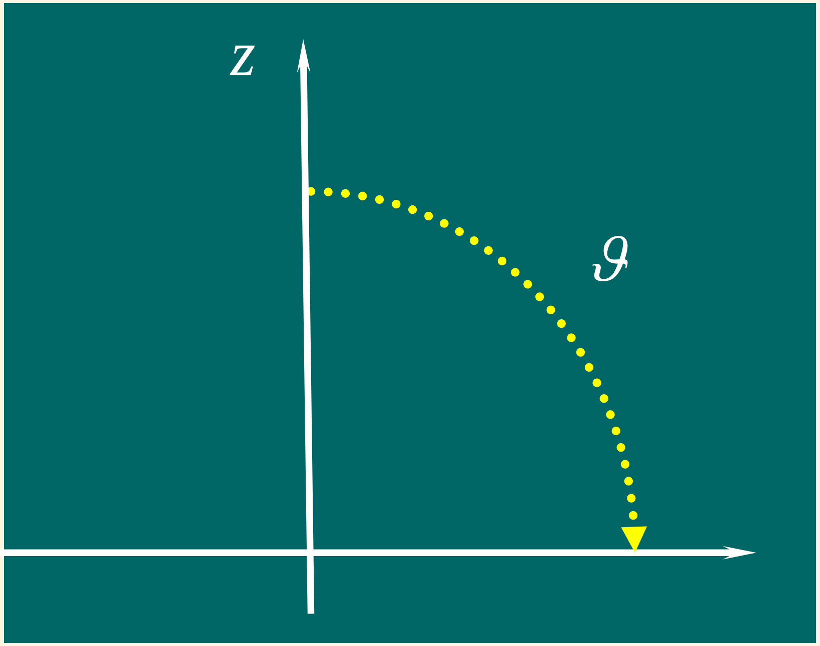
$$\mu_n=-\lambda_n(\lambda_n+1)=\frac{1}{4}+\left[\frac{n\pi}{\ln(a/b)}\right]^2>0$$

■ **半球问题**

圆锥区的特例: $\vartheta_1\to\pi/2$

$$P_\lambda(0)=\frac{1}{\sqrt{\pi}}\frac{\Gamma[(1+\lambda)/2]}{\Gamma(1+\lambda/2)}\cos\frac{\lambda\pi}{2}$$

$$P_\lambda'(0)=\frac{2}{\sqrt{\pi}}\frac{\Gamma(1+\lambda/2)}{\Gamma[(1+\lambda)/2]}\sin\frac{\lambda\pi}{2}$$



$$0\le\vartheta\le\pi/2\Rightarrow x=\cos\vartheta\in[0,1]$$

球底面($\vartheta=\pi/2,\ x=0$)边界条件

- 第一类 边界条件

$$\cos\frac{\lambda\pi}{2}=0\Rightarrow\frac{\lambda\pi}{2}=(2k+1)\frac{\pi}{2}\Rightarrow\lambda=2k+1,(k=0,1,2,\ldots)$$

本征值问题

$$\begin{cases}-\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]=\lambda(\lambda+1)\Theta,x\in(0,1)\\ \Theta\,|_{x=1}<\infty;\ \Theta\,|_{x=0}=0\end{cases}$$

$$\Downarrow$$

$$\Theta_{2k+1}(x)=P_{2k+1}(x),(k=0,1,2,\ldots)$$

$$\lambda_{2k+1}=(2k+1)(2k+2),(k=0,1,2,\ldots)$$

第二类 边界条件

$$\sin\frac{\lambda\pi}{2}=0\Rightarrow\frac{\lambda\pi}{2}=2k\frac{\pi}{2}\Rightarrow\lambda=2k,(k=0,1,2,\ldots)$$

本征值问题

$$\begin{cases}-\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right]=\lambda(\lambda+1)\Theta, x\in(0,1)\\ \Theta\big|_{x=1}<\infty;\ \left.\dfrac{\mathrm{d}\Theta}{\mathrm{d}x}\right|_{x=0}=0\end{cases}$$

$$\Theta_{2k}(x)=P_{2k}(x),(k=0,1,2,\ldots)$$

$$\lambda_{2k}=2k(2k+1),(k=0,1,2,\ldots)$$

本章小结

■**Legendre多项式**

微分形式，积分形式，递推公式，母函数公式

正交完备性

■**Legendre方程**

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right]=\mu(\mu+1)y,\ (-1\le x\le 1)$$

$$\Downarrow$$

$$y(x)=AP_\mu(x)+BQ_\mu(x)$$

自然边界条件 $y(x)=AP_l(x)\ (l=0,1,2,\ldots)$

■连带 Legendre函数

微分形式，递推公式，　正交完备性

■连带 Legendre 方程的通解

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right]+\frac{m^2}{1-x^2}y=\mu(\mu+1)y$$

$$y(x)=C_1P_\mu^{|m|}(x)+C_2Q_\mu^{|m|}(x)$$

自然边界条件

$$y(x)=AP_l^{|m|}(x)$$

$$(l=0,1,2,\ldots;|m|\leq l)$$

■球谐函数:正交完备性

$$\begin{cases} f(\vartheta,\varphi)=\displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} C_l^m Y_l^m(\vartheta,\varphi) \\ C_l^m=\dfrac{2l+1}{4\pi}\dfrac{(l-|m|)!}{(l+|m|)!}\displaystyle\int_0^{\pi}\int_0^{2\pi} f(\vartheta,\phi)[Y_l^m(\vartheta,\varphi)]^* \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi \end{cases}$$

■Legendre函数：圆锥形区

$$P_\lambda(x)=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda-n+1)}\left(\frac{x-1}{2}\right)^n$$

$$P_\lambda(0)=\frac{1}{\sqrt{\pi}}\frac{\Gamma[(1+\lambda)/2]}{\Gamma(1+\lambda/2)}\cos\frac{\lambda\pi}{2}$$

$$P_\lambda'(0)=\frac{2}{\sqrt{\pi}}\frac{\Gamma(1+\lambda/2)}{\Gamma[(1+\lambda)/2]}\sin\frac{\lambda\pi}{2}$$