第10章 定解问题的分离变量法

10.1 分离变量法及其应用

分量变量法，非齐次问题，本征函数展开法

10.2 分离变量法—连续谱问题

无限空间，半无限空间，连续谱+离散谱

10.3 一般模式展开解

Laplace方程，波动方程，零本征值讨论

10.4 柱坐标中的分离变量

Laplace方程，Helmholtz方程，分离变量解

10.5 球坐标中的分离变量

Laplace方程，Helmholtz方程，分离变量解

定解问题常用解法

分离变量法、Green函数、积分变换、复变函数法、数值方法(差分法,有限元法,边界元法, 无限元法)

- **分离变量法的基本思想**

偏微分方程的定解问题 → 分离变量 → 常微分方程初值问题边值问题

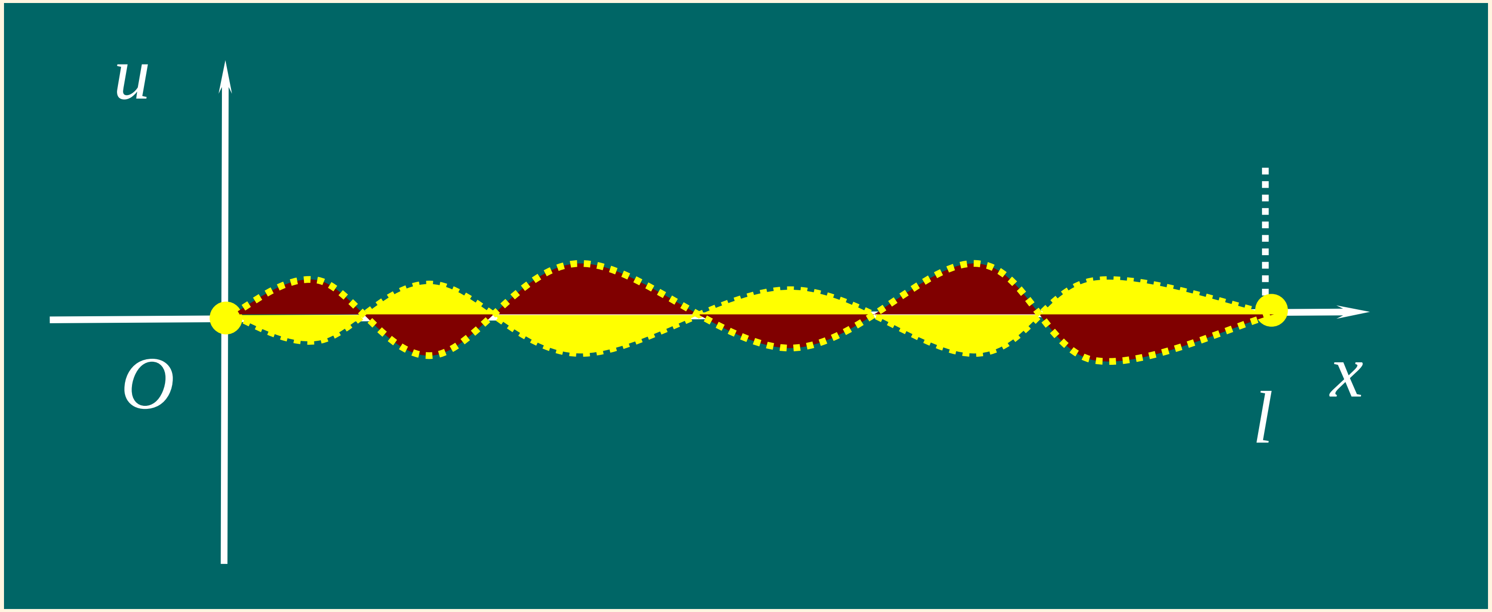
常微分方程：求出通解，然后由初始条件或边界条件求待定系数

偏微分方程：一般不可能求通解，而且通解中含有待定函数，因此直接求满足初始条件或边界条件的特解

10.1 分离变量法及其应用

基本概念: 驻波、波节、波腹，基频、本征频率、波的叠加，等等。

❑两端固定弦的自由振动



(1)泛定方程

$$u_{tt}-a^2u_{xx}=0\ (t>0,\ 0<x<l)$$

(2)初始条件

$$u(x,t)|_{t=0}=\varphi(x);\ u_t(x,t)|_{t=0}=\psi(x)\ (0<x<l)$$

(3)边界条件

$$u(x,t)|_{x=0}=u(x,t)|_{x=l}=0\ (t\geq 0)$$

第1步：泛定方程的分离变量

考虑如下形式的特解

$$u(x,t)=X(x)T(t)$$

为什么?

代入方程泛定方程

$$X(x)T''(t) - a^2 X''(x)T(t) = 0$$

$$\Downarrow$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} \equiv -\lambda$$

分析　左边：x 的函数；右边 t 的函数， 而 x 和 t 是独立变量，故只有两边为同一常数 ($-\lambda$)。由此得到二个常微分方程

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + \lambda a^2 T(t) = 0$$

第2步：边界条件的分离变量

$$u(x,t)=X(x)T(t) \Rightarrow u(x,t)|_{x=0}=u(x,t)|_{x=l}=0$$

$$\Downarrow$$

$$X(0)T(t)=X(l)T(t)=0 \quad (t\geq 0)$$

因此，只能 $X(0)=X(l)=0$

问题：能否对初始条件也进行分离变量呢？把分离变量解代入初始条件得到

$$X(x)T(0)=\varphi(x); \quad X(x)T'(0)=\psi(x)$$

而 $\varphi(x)$ 和 $\psi(x)$ 是任意函数，一般不可能满足.

第3步：解本征值问题

$$X''(x)+\lambda X(x)=0$$

$$X(0)=X(l)=0$$

本征振动模式

$$\boxed{X_n(x)=\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right);\quad \lambda_n=\left(\frac{n\pi}{l}\right)^2,\ (n=1,2,\ldots)}$$

第4步：解时间部分

$$T''(t)+\lambda a^2T(t)=0$$

$$T_n(t)=E_n\cos\left(\frac{n\pi at}{l}\right)+F_n\sin\left(\frac{n\pi at}{l}\right)$$

因此，泛定方程且满足边界条件的特解为

$$u_n(x,t)=\left[E_n\cos\left(\frac{n\pi at}{l}\right)+F_n\sin\left(\frac{n\pi at}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

问题：上述特解能否满足初始条件?

$$u_n(x,t)\big|_{t=0}=E_n\sin\left(\frac{n\pi x}{l}\right);\quad \left.\frac{\partial u_n(x,t)}{\partial t}\right|_{t=0}=F_n\frac{n\pi a}{l}\sin\left(\frac{n\pi x}{l}\right)$$

——显然，$u_n(x,t)$ 是不可能满足初始条件的, 因为$\varphi(x)$和$\psi(x)$ 是任意函数。

第5步：叠加原理

因为泛定方程和边界条件是线性齐次的，故

$$u(x,t)=\sum_{n=1}^{\infty}\left[E_n\cos\left(\frac{n\pi at}{l}\right)+F_n\sin\left(\frac{n\pi at}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

也是泛定方程且满足边界条件的解。其中系数 E_n, F_n 试由初始条件决定.

第6步：正交展开

$$u(x,0)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{l}\right)=\varphi(x)$$

$$u_t(x,0)=\sum_{n=1}^{n}F_n\frac{n\pi a}{l}\sin\left(\frac{n\pi x}{l}\right)=\psi(x)$$

——能否求出二组系数$\{E_n\}$和$\{F_n\}$?

两边乘$\sin\left(\frac{m\pi x}{l}\right)$并积分，假定无限求和与积分可交换

$$\sum_{n=1}^{\infty} E_n \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x = \int_0^l \varphi(x)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x$$

$$\sum_{n=1}^{n} F_n \frac{n\pi a}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x = \int_0^l \psi(x)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x$$

利用正交性关系

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x = \frac{l}{2}\delta_{nm}$$

$$E_m = \frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x; \quad F_m = \frac{2}{m\pi a}\int_0^l \psi(x)\sin\left(\frac{m\pi x}{l}\right)\mathrm{d}x$$

第7步：级数形式的解和积分解

$$u(x,t)=\sum_{n=1}^{\infty}\left[E_n\cos\left(\frac{n\pi at}{l}\right)+F_n\sin\left(\frac{n\pi at}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

$$E_m=\frac{2}{l}\int_0^l\varphi(\xi)\sin\left(\frac{m\pi\xi}{l}\right)\mathrm{d}\xi;\quad F_n=\frac{2}{n\pi a}\int_0^l\psi(\xi)\sin\left(\frac{m\pi\xi}{l}\right)\mathrm{d}\xi$$

这里写成Green函数形式，Green函数作用可以看作把微分方程转化为积分方程

$$u(x,t)=\frac{\partial}{\partial t}\int_0^l G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi+\int_0^l G(x,\xi,\tau)\psi(\xi)\mathrm{d}\xi$$

$$G(x,\xi,t)\equiv\frac{2}{l}\sum_{n=1}^{\infty}\frac{l}{n\pi a}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

❑解的物理意义

可把 $u_n(x,t)$ 改写作

$$u_n(x,t)=\left[E_n\cos\left(\frac{n\pi at}{l}\right)+F_n\sin\left(\frac{n\pi at}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

$$=A_n\cos(\omega_n t-\delta_n)\sin(k_n x)\quad (n=1,2,\ldots)$$

其中

$$A_n=\sqrt{E_n^2+F_n^2};\ \delta_n=\arctan\left(\frac{F_n}{E_n}\right);\ \omega_n=k_n a=\frac{n\pi a}{l}$$

——可见 $u_n(x,t)$ 代表n阶驻波

■**波节：振动中始终不动的点称为**

$$\sin(k_n x)=0\Rightarrow k_n x_{\text{node}}=m\pi\Rightarrow x_{\text{node}}=\frac{m}{n}l$$

波腹： **$|u_n(x,t)|$ 极大点**

$$\sin(k_n x) = 1 \Rightarrow k_n x_{\text{antinode}} = (m + 1/2)\pi$$

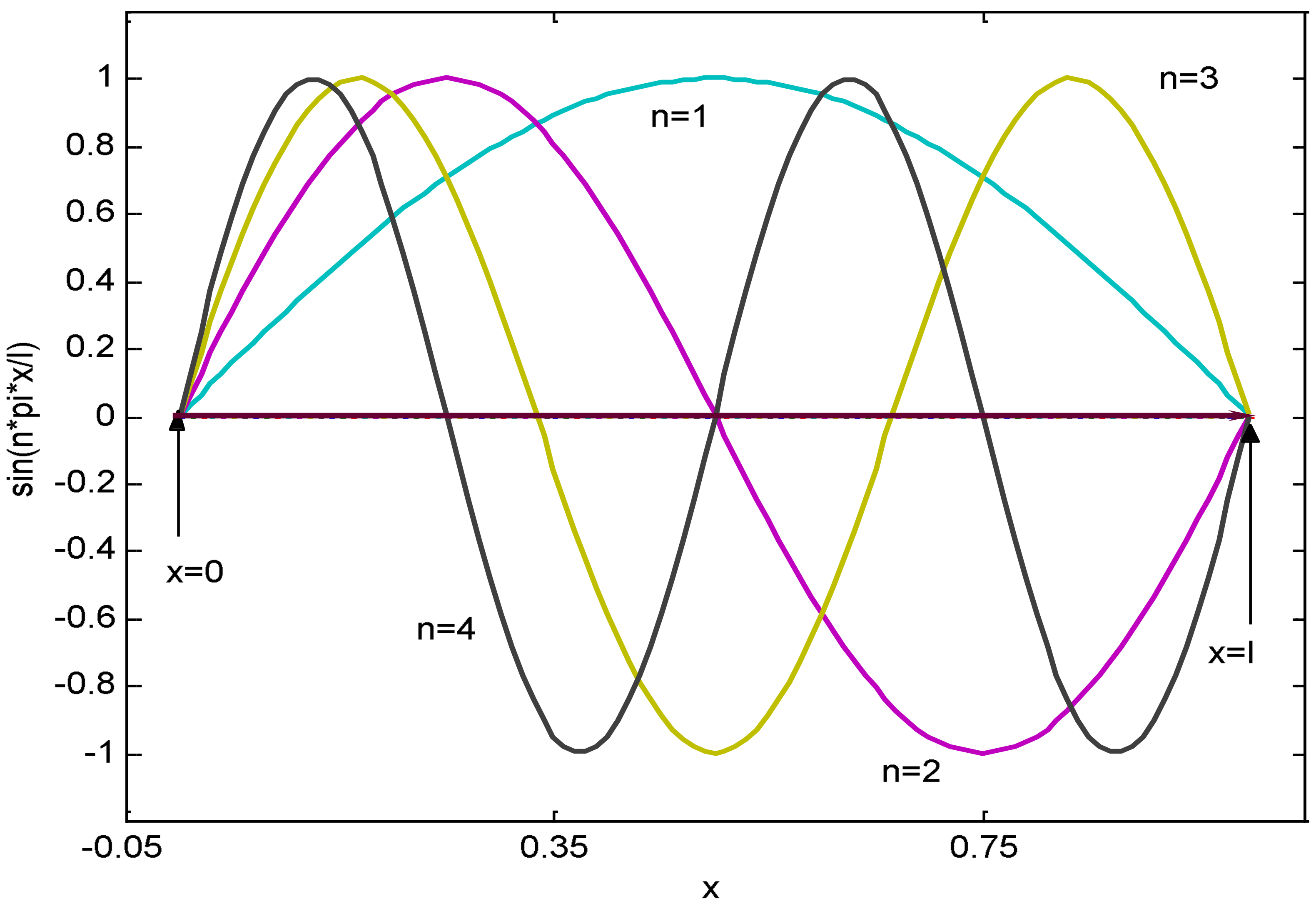
$$x_{\text{antinode}} = \frac{(m+1/2)}{n} l$$

本征频率： **系统振动的固有频率**

$$\omega_n = k_n a = \frac{n\pi a}{l}$$

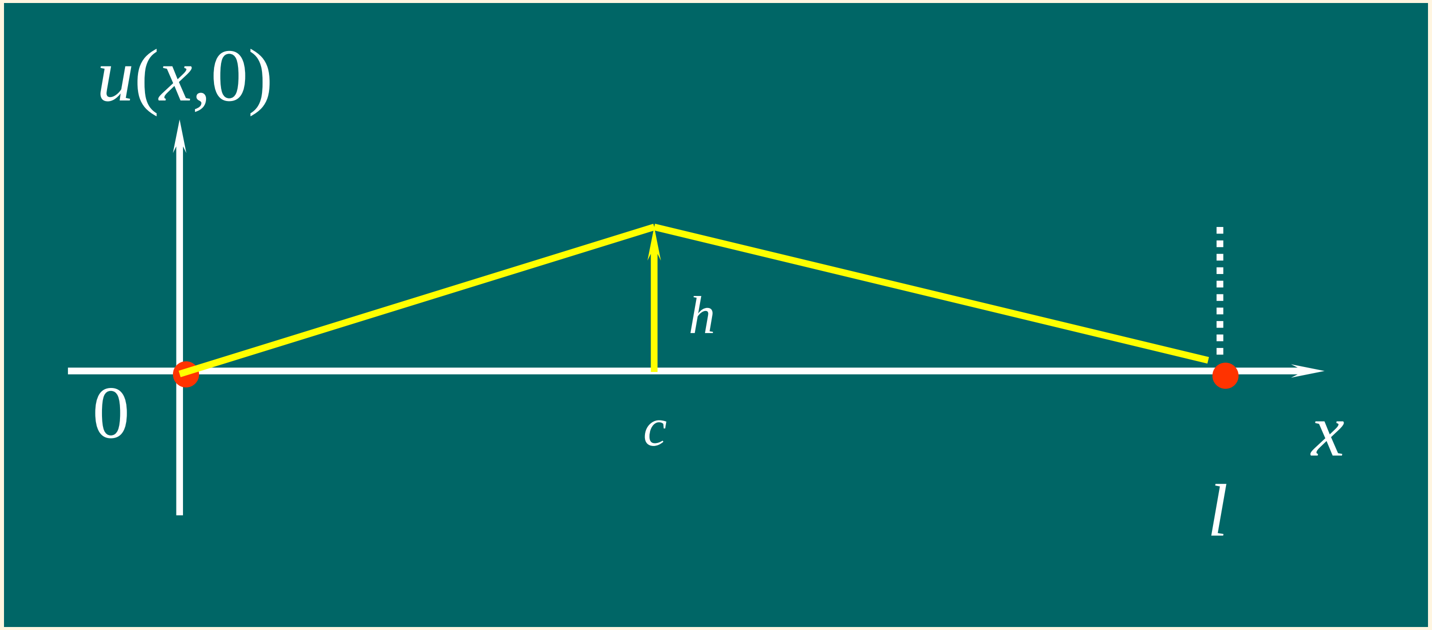
基频： **最小的本征振动频率**

$$\omega_0 = \min \omega_n = \frac{\pi a}{l}$$



n=3
n=1
n=4
n=2
x=0
x=l
sin(n*pi*x/l)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.05
0.35
0.75
x

■ 广义解：强解



初始条件

$$u(x,0)=\begin{cases}\dfrac{h}{c}x, & 0\le x\le c\\[2ex] \dfrac{h}{l-c}(l-x), & c\le x\le l\end{cases}\;;\quad u_t(x,0)=0$$

级数形式的解

$$u(x,t)=\frac{2hl^2}{\pi^2 c(l-c)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

- **满足边界条件：**

$$u(x,t)\,|_{x=0,l}=0$$

- **满足初始条件：**

$$u(x,0)=\frac{2hl^2}{\pi^2 c(l-c)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)=\begin{cases}\dfrac{h}{c}x, & 0\le x\le c\\ \dfrac{h}{l-c}(l-x), & c\le x\le l\end{cases}$$

$$u_t(x,t)\,|_{t=0}=\frac{2hla}{\pi c(l-c)}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)\Bigg|_{t=0}=0$$

但二阶偏导数

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -\frac{2h}{c(l-c)}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -\frac{2ha^2}{c(l-c)}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

——不收敛，求和与微分不能交换,无法验证满足波动过程。

■另一方面，令序列

$$u_n(x,0) = \frac{2hl^2}{\pi^2 c(l-c)}\sum_{k=1}^{n}\frac{1}{k^2}\sin\left(\frac{k\pi c}{l}\right)\sin\left(\frac{k\pi x}{l}\right)$$

$$u_n(x,t) = \frac{2hl^2}{\pi^2 c(l-c)}\sum_{k=1}^{n}\frac{1}{k^2}\sin\left(\frac{k\pi c}{l}\right)\sin\left(\frac{k\pi x}{l}\right)\cos\left(\frac{k\pi at}{l}\right)$$

显然

$$\frac{\partial^2 u_n(x,t)}{\partial t^2}-a^2\frac{\partial^2 u_n(x,t)}{\partial x^2}=0$$

而且

$$\lim_{n\to\infty}u_n(x,0)=\frac{2hl^2}{\pi^2 c(l-c)}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\left(\frac{k\pi c}{l}\right)\sin\left(\frac{k\pi x}{l}\right)=u(x,0)$$

因此

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$

$$=\frac{2hl^2}{\pi^2 c(l-c)}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\left(\frac{k\pi c}{l}\right)\sin\left(\frac{k\pi x}{l}\right)\cos\left(\frac{k\pi at}{l}\right)$$

——这样的级数解可以看作一类广义解-强解！

□两端固定弦的强迫振动

外力作用下弦振动：定解问题

$$u_{tt}(x,t)-a^2u_{xx}(x,t)=f(x,t),\ (t>0,\quad 0<x<l)$$

$$u(0,t)=u(l,t)=0$$

$$u(x,0)=\varphi(x),u_t(x,0)=\psi(x)$$

如果用上面的方法设 $u(x,t)=X(x)T(t)$，而直接分离变量，无法分离成二个常微分方程。

$$u(x,t)=X(x)T(t)$$

$$X(x)T''(t)-a^2X''(x)T(t)=f(x,t)$$

- **物理分析：外力$f(x,t)$激发弦振动，是各个振动模式的叠加**

$$u(x,t)=\sum_{n=1}^{\infty}a_n(t)X_n(x)$$

$$X_n(x)=\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right);\quad \lambda_n=\left(\frac{n\pi}{l}\right)^2,\ (n=1,2,\ldots).$$

- **数学分析：本征模式在[0,l]是完备的基函数，任意平方可积函数都可以展开成Fourier级数**

$$u(x,t)=\sum_{n=1}^{\infty}a_n(t)X_n(x)$$

第1步：满足非齐次方程

$$u(x,t)=\sum_{n=1}^{\infty}a_n(t)X_n(x)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi x}{l}\right)$$

——满足边界条件

$$\sum_{n=1}^{\infty}\left[T_n''(t)+\left(\frac{n\pi a}{l}\right)^2T_n(t)\right]\sin\left(\frac{n\pi x}{l}\right)=f(x,t)$$

$$T_n''(t)+\left(\frac{n\pi a}{l}\right)^2T_n(t)=\frac{2}{l}\int_0^l f(x,t)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x\equiv f_n(t)$$

第2步：满足初始条件

$$u(x,0)=\sum_{n=1}^{\infty}T_n(0)\sin\left(\frac{n\pi x}{l}\right)=\varphi(x)$$

$$u'(x,0)=\sum_{n=1}^{\infty}T_n'(0)\sin\left(\frac{n\pi x}{l}\right)=\psi(x)$$

$$T_n(0)=\frac{2}{l}\int_0^l\varphi(x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x\equiv\varphi_n$$

$$T_n'(0)=\frac{2}{l}\int_0^l\psi(x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x\equiv\psi_n$$

第3步：解非齐次常微分方程的初值问题

$$T_n''(t)+\left(\frac{n\pi a}{l}\right)^2 T_n(t)=f_n(t)$$

$$T_n(0)=\varphi_n;\ T_n'(0)=\psi_n$$

常数变易法
见P.29

$$T_n(t)=\frac{l}{n\pi a}\psi_n\sin\left(\frac{n\pi at}{l}\right)+\varphi_n\cos\left(\frac{n\pi at}{l}\right)$$

$$+\frac{l}{n\pi a}\int_0^t f_n(\tau)\sin\left[\frac{n\pi a(t-\tau)}{l}\right]\mathrm{d}\tau$$

第4步：级数形式的解

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi x}{l}\right)$$

$$T_n(t)=\varphi_n\cos\left(\frac{n\pi at}{l}\right)+\frac{l}{n\pi a}\psi_n\sin\left(\frac{n\pi at}{l}\right)$$

$$+\frac{l}{n\pi a}\int_0^t f_n(\tau)\sin\left[\frac{n\pi a(t-\tau)}{l}\right]\mathrm{d}\tau$$

$$\varphi_n=\frac{2}{l}\int_0^l\varphi(x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x;\quad \psi_n=\frac{2}{l}\int_0^l\psi(x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$f_n(t)=\frac{2}{l}\int_0^l f(x,t)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x$$

——每个激发模式的“大小”很明显：①初值分布中含有的“分量”；②激发外力分布中含有某一个的“分量”。

例1 初值为零分布，外力为作用在一点的冲击力

$$f(x,t)=f_0\delta(x-x_0)\delta(t)$$

$$f_n(t)=\frac{2f_0}{l}\int_0^l \delta(x-x_0)\delta(t)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$=\frac{2f_0}{l}\delta(t)\sin\left(\frac{n\pi x_0}{l}\right)$$

$$T_n(t)=\frac{l}{n\pi a}\frac{2f_0}{l}\sin\left(\frac{n\pi x_0}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

$$u(x,t)=\frac{2f_0}{l}\sum_{n=1}^{\infty}\frac{l}{n\pi a}\sin\left(\frac{n\pi x_0}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

①时间脉冲：含有丰富频率成分，每一个模式都能够激发，但1/*n*衰减;

②空间脉冲：激发模式与作用点密切相关，如果刚好作用在某一个模式的波节上，则该模式不能激发;

例2 初值为零分布，外力为作用在一点的简谐力(零时刻开始作用-或者初值可不考虑)

$$f(x,t)=f_0\delta(x-x_0)\sin(\omega t)$$

$$f_n(t)=\frac{2f_0}{l}\sin(\omega t)\sin\left(\frac{n\pi x_0}{l}\right)$$

$$T_n(t)=\frac{l}{n\pi a}\frac{2f_0}{l}\sin\left(\frac{n\pi x_0}{l}\right)\int_0^t \sin(\omega\tau)\sin\left[\frac{n\pi a(t-\tau)}{l}\right]\mathrm{d}\tau$$

$$=-\frac{2l}{n\pi a}f_0\sin\left(\frac{n\pi x_0}{l}\right)\frac{1}{\omega^2-\left(\frac{n\pi a}{l}\right)^2}\left[\frac{n\pi a}{l}\sin(\omega t)-\omega\sin\left(\frac{n\pi at}{l}\right)\right]$$

当激发频率刚好等于第N个模式的本征频率

$$\omega\to N\pi a/l$$

$$T_N(t)\to -f_0\sin\left(\frac{N\pi x_0}{l}\right)\left[\frac{N\pi a}{l}t\cos\left(\frac{N\pi at}{l}\right)-\sin\left(\frac{N\pi at}{l}\right)\right]$$

——第N个模式线性增长——而这是非物理的——系统必须考虑阻尼或者非线性！

第5步：积分形式解(微分方程转化成积分方程)

$$u(x,t)=\frac{\partial}{\partial t}\int_0^l G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi+\int_0^l G(x,\xi,\tau)\psi(\xi)\mathrm{d}\xi$$

$$+\int_0^t\int_0^l G(x,\xi,t-\tau)f(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi$$

$$G(x,\xi,t)\equiv\frac{2}{l}\sum_{n=1}^{\infty}\frac{l}{n\pi a}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

——含时Green函数

——前二项是初值问题的解，后一项为非齐次项的解——叠加原理.

■ 常数变易法

$$y''(t)+py'(t)+qy(t)=f(t)$$

■ **非齐次方程的解：齐次方程的通解+特解**

$$y(t)=Ay_1(t)+By_2(t)+w(t)$$

$$y_1''(t)+py_1'(t)+qy_1(t)=0$$

$$y_2''(t)+py_2'(t)+qy_2(t)=0$$

■ **特解—常数变易法**

$$w(t)=c_1(t)y_1(t)+c_2(t)y_2(t)$$

代入原方程

$$[c_1'(t)y_1(t)+c_2'(t)y_2(t)]'+p[c_1'(t)y_1(t)+c_2'(t)y_2(t)]$$
$$+[c_1'(t)y_1'(t)+c_2'(t)y_2'(t)]=f(t)$$

由上式决定二个函数是欠定的，表明变系数有一定的任意性，取系数满足（充分条件）

$$c_1'(t)y_1(t)+c_2'(t)y_2(t)=0$$
$$c_1'(t)y_1'(t)+c_2'(t)y_2'(t)=f(t)$$

$$c_2(t)=\int_{t_0}^{t}\frac{y_1(\tau)}{W(y_1,y_2)}f(\tau)\mathrm{d}\tau;\quad c_1(t)=-\int_{t_0}^{t}\frac{y_2(\tau)}{W(y_1,y_2)}f(\tau)\mathrm{d}\tau$$

$$W(y_1,y_2)\equiv y_1(\tau)y_2'(\tau)-y_1'(\tau)y_2(\tau)$$

$$w(t)=\int_{t_0}^{t}\frac{[y_1(\tau)y_2(t)-y_2(\tau)y_1(t)]}{W(y_1,y_2)}f(\tau)\mathrm{d}\tau$$

❑两端运动弦的强迫振动—非齐次边界条件

如果弦的端点不固定，而是按一定的规律作横向运动。定解问题为

$$u_{tt}(x,t)-a^2u_{xx}(x,t)=f(x,t)$$

$$\boxed{u(0,t)=\mu(t);u(l,t)=\nu(t)}$$

$$u(x,0)=\varphi(x);u_t(x,0)=\psi(x)$$

问题：不仅仅方程，边界条件也不能分离变量

⇓

$$u(x,t)=X(x)T(t)$$

$$X(0)T(t)=\mu(t);X(l)T(t)=\nu(t)$$

与上节区别：非齐次边界条件——能否齐次化?

$$u(x,t) = v(x,t) + P(x,t)$$

其中*P*(*x*,*t*) 满足

$$P(0,t) = \mu(t); P(l,t) = \nu(t)$$

$$v_{tt}(x,t) - a^2 v_{xx}(x,t) = f(x,t) - P_{tt}(x,t) + a^2 P_{xx}(x,t)$$

$$\boxed{v(0,t) = 0; v(l,t) = 0}$$

$$v(x,0) = \varphi(x) - P(x,0); v_t(x,0) = \psi(x) - P_t(x,0)$$

齐次边界条件问题

$P(x,t)$ 的选择有任意性，最简单的是 x 的线性函数

$$P(x,t)=A(t)x+B(t)$$

由边界条件： $B(t)=\mu(t)$; $A(t)l+B(t)=\nu(t)$

$$P(x,t)=\mu(t)+\frac{x}{l}[\nu(t)-\mu(t)]$$

问题：能否用本征函数展开法直接求解呢?

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x),\quad T_n(t)=\int_0^l u(x,t)X_n(x)\,\mathrm{d}\,x$$

$$X_n(x)=\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right)$$

—数学上，任意平方可积函数都可以展成Fourier级数

不满足边界条件 → 不能直接代入方程 → 不能得到展开系数

$$\frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty}T_n(t)X_n(x)\xrightarrow{?}\sum_{n=1}^{\infty}\frac{\mathrm{d}^2T_n(t)}{\mathrm{d}t^2}X_n(x)$$

$$\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty}T_n(t)X_n(x)\xrightarrow{?}\sum_{n=1}^{\infty}T_n(t)\frac{\mathrm{d}^2X_n(t)}{\mathrm{d}x^2}$$

——**端点不满足边界条件**，**Fourier级数不收敛到真值，没有一致收敛性, 求导与求和不能交换。**

- **解决方法：利用Lagrange恒等式**

$$\int_0^l \left(u\frac{\partial^2 X_n}{\partial x^2} - X_n\frac{\partial^2 u}{\partial x^2} \right) \mathrm{d}x = \int_0^l \frac{\mathrm{d}}{\mathrm{d}x}\left(u\frac{\partial X_n}{\partial x} - X_n\frac{\partial u}{\partial x} \right) \mathrm{d}x$$

$$= \left(u\frac{\partial X_n}{\partial x} - X_n\frac{\partial u}{\partial x} \right)_0^l$$

$$T_n(t) = \int_0^l u(x,t)X_n(x)\mathrm{d}x; \quad X_n''(x) + \lambda_n X_n(x) = 0; \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$\int_0^l u\frac{\partial^2 X_n}{\partial x^2}\mathrm{d}x = -\int_0^l \lambda_n u(x,t)X_n(x)\mathrm{d}x = -\lambda_n T_n(t)$$

$$\int_0^l X_n\frac{\partial^2 u}{\partial x^2}\mathrm{d}x = \frac{1}{a^2}\int_0^l X_n\left[u_{tt}(x,t) - f(x,t)\right]\mathrm{d}x$$

$$= \frac{1}{a^2}\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2} - \frac{1}{a^2}\int_0^l f(x,t)X_n(x)\mathrm{d}x$$

仅仅需要微分与积分交换

$$-\lambda_n T_n(t) - \frac{1}{a^2}\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2}dx = \left(u\frac{\partial X_n}{\partial x} - X_n\frac{\partial u}{\partial x}\right)_0^l$$

$$-\frac{1}{a^2}\int_0^l f(x,t)X_n(x)\mathrm{d}x$$

$$\boxed{\begin{aligned}&\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2} + a^2\lambda_n T_n(t) = \int_0^l X_n(x)f(x,t)dx \\ &\quad - a^2\left[v(t)\frac{\mathrm{d}X_n(l)}{\mathrm{d}x} - \mu(t)\frac{\mathrm{d}X_n(0)}{\mathrm{d}x}\right] \equiv f_n(t) + b_n(t)\end{aligned}}$$

$$T_n(0) = \int_0^l \varphi(x)X_n(x)\mathrm{d}x \equiv \varphi_n; T_n'(0) = \int_0^l \psi(x)X_n(x)\mathrm{d}x \equiv \psi_n$$

——非齐次方程的初值问题

$$T_n(t)=\frac{l}{n\pi a}\psi_n\sin\left(\frac{n\pi at}{l}\right)+\varphi_n\cos\left(\frac{n\pi at}{l}\right)$$
$$+\frac{l}{n\pi a}\int_0^t[f_n(\tau)+b_n(\tau)]\sin\left[\frac{n\pi a(t-\tau)}{l}\right]\mathrm{d}\tau$$

$$u(x,t)=\frac{\partial}{\partial t}\int_0^l G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi+\int_0^l G(x,\xi,\tau)\psi(\xi)\mathrm{d}\xi$$
$$+\int_0^t\int_0^l G(x,\xi,t-\tau)f(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi+u_b(x,t)$$

$$u_b(x,t)=a^2\int_0^t\left[\mu(\tau)\frac{\partial G(x,\xi,t-\tau)}{\partial\xi}\bigg|_{\xi=0}-\nu(\tau)\frac{\partial G(x,\xi,t-\tau)}{\partial\xi}\bigg|_{\xi=l}\right]\mathrm{d}\tau$$
$$G(x,\xi,t)\equiv\sum_{n=1}^{\infty}\frac{l}{n\pi a}X_n(x)X_n(\xi)\sin\left(\frac{n\pi at}{l}\right)$$

❑矩形区域上的波动方程

四边固定的膜的横向振动，*t*=0 时受外力作用。定解问题

$$u_{tt} - c^2(u_{xx} + u_{yy}) = f(x,y,t)\ (t>0)$$

$$u(x,y,t)\,|_{x=0,a} = u(x,y,t)\big|_{y=0,b} = 0$$

$$u(x,y,t)\,|_{t=0} = u_t(x,y,t)\big|_{t=0} = 0$$

■首先考虑齐次问题

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0$$

$$u\,|_{x=0,a} = u|_{y=0,b} = 0$$

$$\Uparrow$$

$$u(x, y, t) = T(t)U(x, y)$$

$$\Downarrow$$

$$-[U_{xx}(x, y) + U_{yy}(x, y)] = \lambda U(x, y)$$

$$U(x, y)\,|_{x=0,a} = 0;\quad U(x, y)\,|_{y=0,b} = 0$$

——二维Laplace算子的本征值问题

■ 继续分离变量

$$U(x,y)=X(x)Y(y)$$

$$X''(x)+\lambda_x X(x)=0$$
$$X(x)\,|_{x=0,a}=0$$

$$X(x)=\sin\left(\frac{n\pi x}{a}\right);\lambda_x=\left(\frac{n\pi}{a}\right)^2$$

$$Y''(y)+\lambda_y X(y)=0$$
$$Y(y)\,|_{y=0,b}=0$$

$$Y(y)=\sin\left(\frac{m\pi y}{b}\right);\lambda_y=\left(\frac{m\pi}{b}\right)^2$$

$$U_{nm}(x,y)=\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right);\ \lambda_{nm}=\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2$$

■其次考虑非齐次问题

■ 作二维 Fourier 变换

$$u(x,y,t)=\sum_{n,m=1}^{\infty}T_{nm}(t)U_{nm}(x,y)$$

——注意：解自动满足边界条件

$$\sum_{n,m=1}^{\infty}\left[T''_{nm}(t)+c^2\lambda_{nm}T_{nm}(t)\right]U_{nm}(x,y)=f(x,y,t)$$

■ 把 $f(x,y,t)$作二维 Fourier 变换

$$f(x,y,t)=\sum_{n,m=1}^{\infty}f_{nm}(t)U_{nm}(x,y)$$

$$f_{nm}(t)=\frac{1}{\|U_{nm}\|^2}\int_0^a\int_0^b f(\xi,\eta,t)U_{nm}(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta$$

$$\|U_{nm}\|^2\equiv\int_0^a\int_0^b|U_{nm}(\xi,\eta)|^2\mathrm{d}\xi\mathrm{d}\eta=\frac{ab}{4}$$

■ **因此，时间部分满足非齐次方程**

$$T''_{nm}(t)+c^2\lambda_{nm}T_{nm}(t)=f_{nm}(t)$$

零初始条件

$$u(x,y,0)=\sum_{n,m=1}^{\infty}T_{nm}(0)U_{nm}(x,y)=0$$

$$u_t(x,y,0)=\sum_{n,m=1}^{\infty}T'_{nm}(0)U_{nm}(x,y)=0$$

$$\Downarrow$$

$$T_{nm}(0)=T'_{nm}(0)=0$$

■ 于是，时间部分的解为

$$T_{nm}(t)=\frac{1}{\omega_{nm}}\int_0^t f_{nm}(\tau)\sin[\omega_{nm}(t-\tau)]\mathrm{d}\tau$$

其中ω_{nm}为本征频率

$$\omega_{nm}=c\sqrt{\lambda_{nm}}=c\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$$

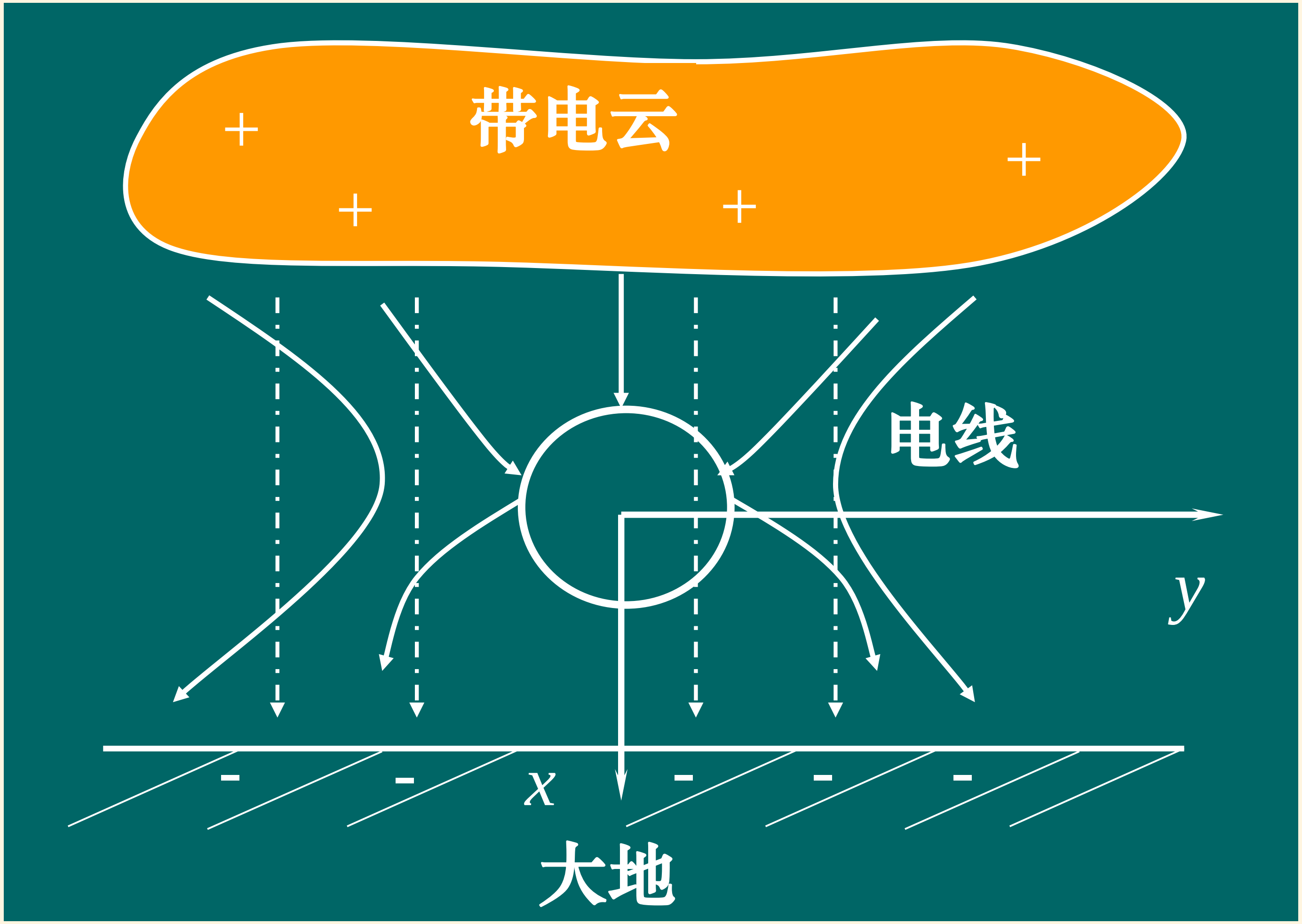
■积分形式的解

$$u(x,y,t)=\int_0^t\int_0^a\int_0^b f(\xi,\eta,\tau)G(x,y;\xi,\eta;t-\tau)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau$$

$$G(x,y;\xi,\eta;t-\tau)\equiv\sum_{n,m=1}^{\infty}\frac{4}{ab\omega_{nm}}U_{nm}(x,y)U_{nm}(\xi,\eta)\sin[\omega_{nm}(t-\tau)]$$

❑极坐标中的二维 Laplace 方程

- **物理问题**



■ 定解问题

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}=0 \quad (\rho > a)$$

$$u(\rho,\varphi)\big|_{\rho=a}=0$$

$$\lim_{\rho\to\infty} u(\rho,\varphi)=-E_0\rho\cos\varphi$$

分析:(1)导体表面等电位，故可假定在导体表面处电位为零；(2)在远离导体处，导体对电场分布的影响很小，故当 $\rho\to\infty$ 时，E_y=0, E_x=E_0, 即

$$-\lim_{\rho\to\infty}\frac{\partial u}{\partial x}=E_0 \Rightarrow \lim_{\rho\to\infty} u=-E_0 x=-E_0\rho\cos\varphi$$

解：分离变量解

$$u(\rho,\varphi)=R(\rho)\Phi(\varphi)$$

■ **方位角部分**

$$\Phi''(\varphi)+\lambda\Phi(\varphi)=0$$

$$\Phi(\varphi)=\Phi(2\pi+\varphi)$$

$$\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi}, m=0,\pm1,\pm2,\ldots$$

$$\Phi_m(\varphi)=A_m\sin(m\varphi)+B_m\cos(m\varphi), m=0,1,2,\ldots$$

■ **径向部分**

$$\rho^2\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}-m^2R=0$$ **—Euler方程**

$$R(\rho)=\begin{cases} C_m\rho^{|m|}+D_m\rho^{-|m|}, & m\neq 0 \\ E+F\ln\rho, & m=0 \end{cases}$$

- **分离变量通解为**

$$u(\rho,\varphi)=E+F\ln\rho+\sum_{m=-\infty}^{\infty}(C_m\rho^{|m|}+D_m\rho^{-|m|})e^{\mathrm{i}m\varphi}$$

或者

$$u(\rho,\varphi)=E+F\ln\rho+\sum_{m=1}^{\infty}(C_m\rho^{m}+D_m\rho^{-m})\begin{Bmatrix}\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}$$

- **问题关于方位角对称**

$$u(\rho,\varphi)=E+F\ln\rho+\sum_{m=1}^{\infty}(C_m\rho^{m}+D_m\rho^{-m})\cos(m\varphi)$$

■ 代入边界条件

$$u(\rho,\varphi)|_{\rho=a}=C_0+D_0\ln a+\sum_{m=1}^{\infty}(C_m a^m+D_m a^{-m})\cos(m\varphi)=0$$

$$u(\rho,\varphi)|_{\rho\to\infty}=\sum_{m=1}^{\infty}C_m\rho^m\cos(m\varphi)=-E_0\rho\cos\varphi$$

$$C_0+D_0\ln a=0;\quad C_m a^m+D_m a^{-m}=0$$

$$C_0=0;C_1=-E_0;C_m=0,(m\neq 1)$$

$$C_0=-D_0\ln a;\ C_1=-E_0,D_1=a^2E_0$$

$$C_m=0,D_m=0,(m\neq 1)$$

■ 最后，得到电位分布

$$u(\rho,\varphi)=D_0\ln\frac{\rho}{a}-E_0\left(\rho-\frac{a^2}{\rho}\right)\cos\varphi$$

物理分析:

第1项：无限长圆柱导体产生的场(带电时)

第2项：均匀电场

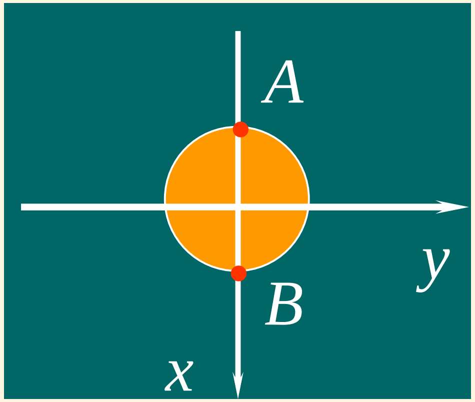
第3项：由于导线的存在，对均匀电场的影响

——感应电荷相当于电偶极子

■ A和B 二点的电场

$$(E_\rho)_{A,B}=-\frac{\partial u}{\partial\rho}\bigg|_{\rho=a;\varphi=\pi,0}=\mp 2E_0$$

——原来电场的二倍



10.2 分离变量法——连续谱问题

没有边界条件的限制——本征值构成连续谱

级数求和 → 积 分运算

❑一维波动方程的初值问题

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0,\ (t>0,-\infty<x<\infty)$$

$$u(x,t)\big|_{t=0}=\varphi(x);\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\psi(x)$$

■ 分离变量解

$$u(x,t)=X(x)T(t)$$

代入波动方程

$$X''(x)T(t)=X(x)T''(t)\Rightarrow\frac{X''(x)}{X(x)}=\frac{T''(t)}{T(t)}\equiv-\lambda^2$$

$$\Downarrow$$

$$T''(t)+\lambda^2T(t)=0$$

$$\Downarrow$$

$$T(t)=A\cos(\lambda t)+B\sin(\lambda t),(\lambda^2>0)$$

$$T(t)=A\cosh(\mu t)+B\sinh(\mu t),(\lambda^2<0,\lambda^2=-\mu^2)$$

$$X''(x)+\lambda^2 X(x)=0$$

——没有边界条件，对本征值λ没有限制

$$X(x)=Ae^{\mathrm{i}\lambda x}$$ ——λ可正可负

■ 方程的通解

$$u(x,t)=\sum_{\lambda} X_{\lambda}(x)T_{\lambda}(t)$$

$$=\sum_{\lambda}[A_{\lambda}\sin(\lambda t)+B_{\lambda}\cos(\lambda t)]e^{\mathrm{i}\lambda x}$$

$$u(x,t)=\int_{-\infty}^{\infty}[A(\lambda)\sin(\lambda t)+B(\lambda)\cos(\lambda t)e^{\mathrm{i}\lambda x}\mathrm{d}\lambda$$

■ 满足初始条件的特解

$$u(x,0)=\int_{-\infty}^{\infty}B(\lambda)e^{\mathrm{i}\lambda x}\mathrm{d}\lambda=\varphi(x)$$

$$u_t(x,0)=\int_{-\infty}^{\infty}\lambda A(\lambda)e^{\mathrm{i}\lambda x}\mathrm{d}\lambda=\psi(x)$$

到这里其实已经完成了求解的步骤，但是我们希望研究这个解和d'Alembert解是等价的

$$B(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x)e^{-\mathrm{i}\lambda x}\mathrm{d}x$$

$$A(\lambda)=\frac{1}{2\pi\lambda}\int_{-\infty}^{\infty}\psi(x)e^{-\mathrm{i}\lambda x}\mathrm{d}x$$

■ d'Alembert解

$$u(x,t)=\int_{-\infty}^{\infty}\left[A(\lambda)\sin(\lambda t)+B(\lambda)\cos(\lambda t)\right]e^{\mathrm{i}\lambda x}\mathrm{d}\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x')\left[\int_{-\infty}^{\infty}e^{-\mathrm{i}\lambda(x'-x)}\cos(\lambda t)\mathrm{d}\lambda\right]\mathrm{d}x'$$

$$+\frac{1}{2\pi}\int_{0}^{t}\int_{-\infty}^{\infty}\psi(x')\left[\int_{-\infty}^{\infty}e^{-\mathrm{i}\lambda(x'-x)}\cos(\lambda t')\mathrm{d}\lambda\right]\mathrm{d}x'\mathrm{d}t'$$

$$\int_{-\infty}^{\infty}e^{-\mathrm{i}\lambda(x'-x)}\cos(\lambda t)\mathrm{d}\lambda=\frac{1}{2}\int_{-\infty}^{\infty}\left\{e^{\mathrm{i}\lambda[-(x'-x)+t]}+e^{-\mathrm{i}\lambda[(x'-x)+t]}\right\}\mathrm{d}\lambda$$

$$=\pi\delta[(x'-x)-t]+\pi\delta[(x'-x)+t]$$

$$u(x,t)=\frac{1}{2}\int_{-\infty}^{\infty}\varphi(x')\left\{\delta[(x'-x)-t]+\delta[(x'-x)+t]\right\}\mathrm{d}x'$$

$$+\frac{1}{2}\int_{0}^{t}\int_{-\infty}^{\infty}\psi(x')\left\{\delta[(x'-x)-t']+\delta[(x'-x)+t']\right\}\mathrm{d}x'\mathrm{d}t'$$

$$=\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\frac{1}{2}\int_{0}^{t}\left[\psi(x+t)+\psi(x-t)\right]\mathrm{d}t'$$

$$=\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(s)\mathrm{d}s$$

$$\Downarrow$$

$$u(x,t)=\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds$$

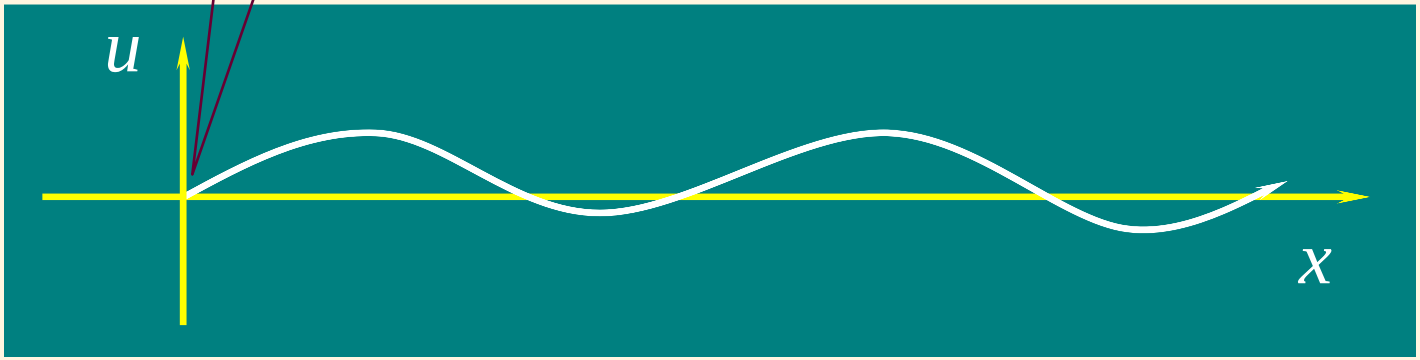
❑一维半空间的波动方程

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0,(t>0;0<x<\infty)$$

$$u(x,t)\big|_{t=0}=\varphi(x);\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\psi(x)$$

固定

$$u(x,t)\big|_{x=0}=0$$



■ 分离变量解

$$u(x,t)=X(x)T(t)$$

代入波动方程

$$X''(x)T(t)=X(x)T''(t)\Rightarrow\frac{X''(x)}{X(x)}=\frac{T''(t)}{T(t)}\equiv-\lambda^2$$

$$\Downarrow$$

$$T''(t)+\lambda^2T(t)=0$$

$$\Downarrow$$

$$T(t)=A\cos(\lambda t)+B\sin(\lambda t),(\lambda^2>0)$$

$$T(t)=A\cosh(\mu t)+B\sinh(\mu t),(\lambda^2<0,\lambda^2=-\mu^2)$$

$$X''(x)+\lambda^2 X(x)=0$$

$$X(x)=A\cos(\lambda x)+B\sin(\lambda x)$$

注意，这里λ>0是因为λ<0也没啥意义，只相差负号，归一化就消去了。而之前则是λ的正负有相位差的结果

——$\lambda>0$

■ 一端边界条件

$$u(x,t)\big|_{x=0}=0\Rightarrow X(0)=0\Rightarrow A\equiv 0$$

$$X(x)=B\sin(\lambda x)$$

■ 方程的通解

$$u(x,t)=\sum_{\lambda}[A_\lambda\sin(\lambda t)+B_\lambda\cos(\lambda t)]\sin(\lambda x)$$

$$u(x,t)=\int_0^\infty\left[A(\lambda)\sin(\lambda t)+B(\lambda)\cos(\lambda t)\right]\sin(\lambda x)\mathrm{d}\lambda$$

■ 满足初始条件的特解

$$u(x,0)=\int_0^{\infty}B(\lambda)\sin(\lambda x)\mathrm{d}\lambda=\varphi(x)$$

$$u_t(x,0)=\int_0^{\infty}\lambda A(\lambda)\sin(\lambda x)\mathrm{d}\lambda=\psi(x)$$

$$\Downarrow$$

$$B(\lambda)=\frac{2}{\pi}\int_0^{\infty}\varphi(x)\sin(\lambda x)\mathrm{d}x$$

$$A(\lambda)=\frac{2}{\pi\lambda}\int_0^{\infty}\psi(x)\sin(\lambda x)\mathrm{d}x$$

■ 积分解

$$u(x,t)=\int_0^{\infty}\left[A(\lambda)\sin(\lambda t)+B(\lambda)\cos(\lambda t)\right]\sin(\lambda x)\mathrm{d}\lambda$$

$$u(x,t)=\int_0^{\infty}\left[A(\lambda)\sin(\lambda t)+B(\lambda)\cos(\lambda t)\right]\sin(\lambda x)\mathrm{d}\lambda$$

$$=\frac{2}{\pi}\int_0^{\infty}\varphi(x')\left[\int_0^{\infty}\sin(\lambda x')\sin(\lambda x)\cos(\lambda t)\mathrm{d}\lambda\right]\mathrm{d}x'$$

$$+\frac{2}{\pi}\int_0^{t}\int_0^{\infty}\psi(x')\left[\int_0^{\infty}\sin(\lambda x')\sin(\lambda x)\cos(\lambda t')\ \ \mathrm{d}\lambda\right]\mathrm{d}x'\mathrm{d}t'$$

这里步骤不全，可以自行填补

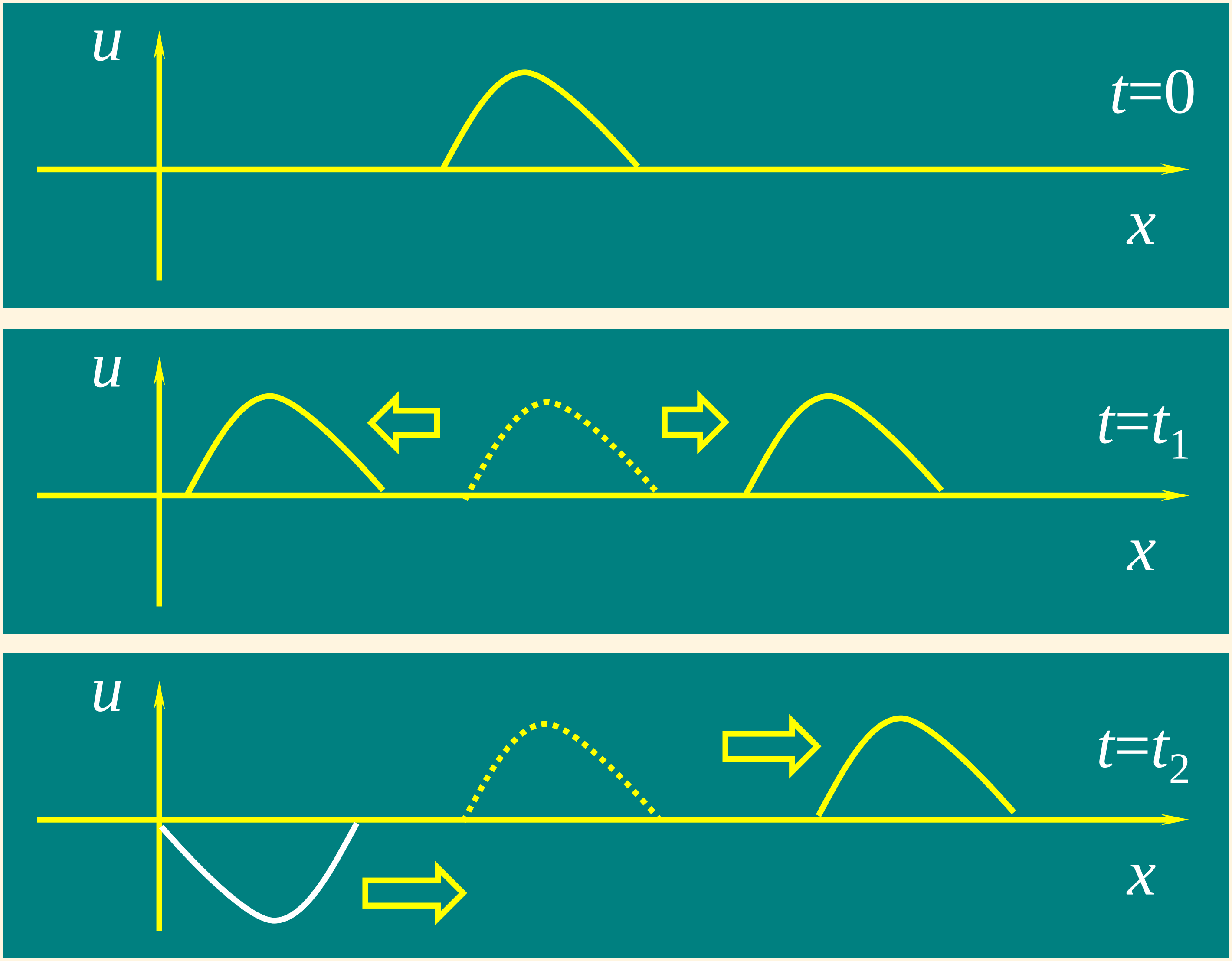
$$\int_0^{\infty}\sin(\lambda x')\sin(\lambda x)\cos(\lambda t')\mathrm{d}\lambda$$

$$=-\frac{\pi}{4}\left[\delta(x+x'+t')+\delta(x+x'-t')-\delta(x-x'+t')-\delta(x-x'-t')\right]$$

$$=-\frac{\pi}{4}\left[\delta(x+x'+t')+\delta(x+x'-t')-\delta(x-x'+t')-\delta(x-x'-t')\right]$$

$$=-\frac{\pi}{4}\left[\delta(x+x'-t')-\delta(x-x'+t')-\delta(x-x'-t')\right]$$

■ 边界反射解



$$u(x,t)=-\frac{1}{2}\int_0^{\infty}\varphi(x')\left[\delta(x+x'-t)-\delta(x-x'+t)-\delta(x-x'-t)\right]\mathrm{d}x'$$

$$-\frac{1}{2}\int_0^{t}\int_0^{\infty}\psi(x')\left[\delta(x+x'-t')-\delta(x-x'+t')-\delta(x-x'-t')\right]\mathrm{d}x'\mathrm{d}t'$$

$$x+x'-t=0\Rightarrow x'=t-x \qquad x+x'-t'=0\Rightarrow x'=t'-x$$

$$x-x'+t=0\Rightarrow x'=t+x \qquad x-x'+t'=0\Rightarrow x'=t'+x$$

$$x-x'-t=0\Rightarrow x'=x-t \qquad x-x'-t'=0\Rightarrow x'=x-t'$$

- **如果** $x>t$ **注意：** $t>t'$

$$u_I(x,t)=\frac{1}{2}\int_0^{\infty}\varphi(x')\left[\delta(x-x'+t)+\delta(x-x'-t)\right]\mathrm{d}x'$$

$$=\frac{1}{2}[\varphi(x+t)+\varphi(x-t)]$$

$$u_{II}(x,t)=\frac{1}{2}\int_0^t\int_0^\infty \psi(x')\left[\delta(x-x'+t')+\delta(x-x'-t')\right]\mathrm{d}x'\mathrm{d}t'$$

$$=\frac{1}{2}\int_{x-t}^{x+t}\psi(\eta)\mathrm{d}\eta$$

$$\Downarrow$$

$$u(x,t)=\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(\eta)\mathrm{d}\eta$$

—与d‘Alembert解相同——在边界反射前

- **如果** $x<t$

$$u_I(x,t)=-\frac{1}{2}\int_0^\infty \varphi(x')\left[\delta(x+x'-t)-\delta(x-x'+t)\right]\mathrm{d}x'$$

$$=\frac{1}{2}\left[\varphi(x+t)-\varphi(t-x)\right]$$

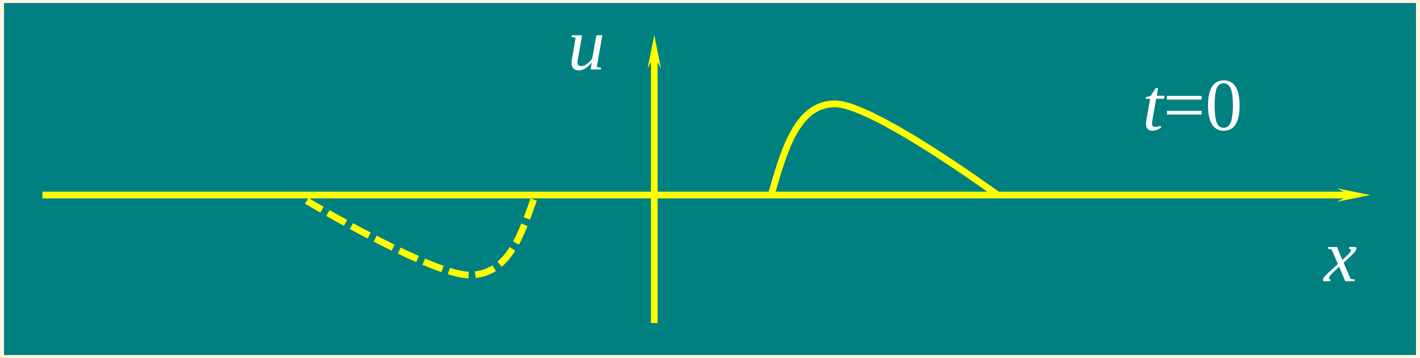
$$u_{II}(x,t)=-\frac{1}{2}\int_0^x\int_0^\infty \psi(x')\left[-\delta(x-x'+t')-\delta(x-x'-t')\right]\mathrm{d}x'\mathrm{d}t'$$

$$-\frac{1}{2}\int_x^t\int_0^\infty \psi(x')\left[\delta(x+x'-t')-\delta(x-x'+t')\right]\mathrm{d}x'\mathrm{d}t'$$

$$=\frac{1}{2}\int_{t-x}^{x+t}\psi(\eta)\mathrm{d}\eta$$

$$u(x,t)=\frac{1}{2}[\varphi(x+t)-\varphi(t-x)]+\frac{1}{2}\int_{t-x}^{x+t}\psi(\eta)\mathrm{d}\eta$$

- **边界奇延拓解**



$$u(x,t)|_{t=0}=\tilde{\varphi}(x)=\begin{cases}\varphi(x), & (0<x<\infty)\\ -\varphi(-x), & (-\infty<x<0)\end{cases}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\tilde{\psi}(x)=\begin{cases}\psi(x), & (0<x<\infty)\\ -\psi(-x), & (-\infty<x<0)\end{cases}$$

$$\Downarrow$$

$$u(x,t)=\frac{1}{2}[\tilde{\varphi}(x+t)+\tilde{\varphi}(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}\tilde{\psi}(\eta)\mathrm{d}\eta$$

在$x>0$和$t>0$区域：$x+t>0$恒成立.

□ 如果 $x>t$ $\Rightarrow$ **$x+t>0$和$x-t>0$**

$$u(x,t)=\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(\eta)\mathrm{d}\eta$$

□ 如果 $x<t$ $\Rightarrow$ $x+t>0$和$x-t<0$

$$\tilde{\varphi}(x-t)=-\varphi(t-x);\quad \tilde{\psi}(\eta)=-\psi(-\eta)$$

$$\Downarrow$$

$$u(x,t)=\frac{1}{2}[\varphi(x+t)-\varphi(t-x)]+\frac{1}{2}\left[\int_{-(t-x)}^{0}\tilde{\psi}(\eta)\mathrm{d}\eta+\int_{0}^{x+t}\tilde{\psi}(\eta)\mathrm{d}\eta\right]$$

$$=\frac{1}{2}[\varphi(x+t)-\varphi(t-x)]+\frac{1}{2}\left[\int_{t-x}^{0}\psi(\eta)\mathrm{d}\eta+\int_{0}^{x+t}\psi(\eta)\mathrm{d}\eta\right]$$

$$=\frac{1}{2}[\varphi(x+t)-\varphi(t-x)]+\frac{1}{2}\int_{t-x}^{x+t}\psi(\eta)\mathrm{d}\eta$$

$$\Downarrow$$

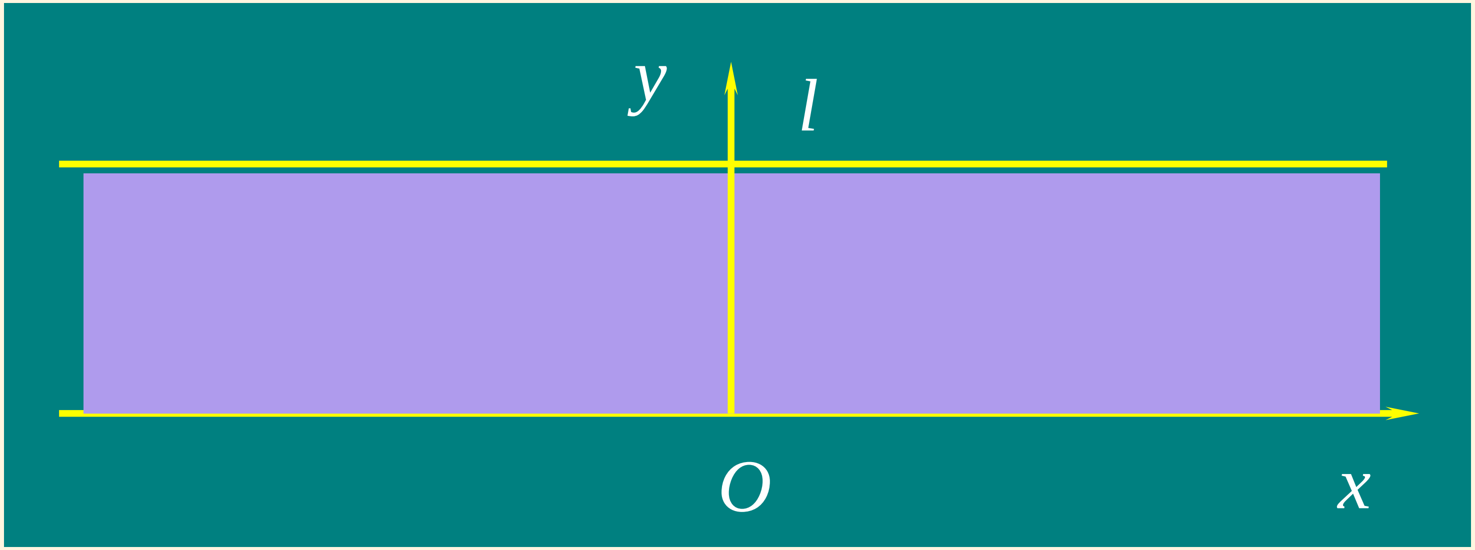
$$u(x,t)=\frac{1}{2}\begin{cases}[\varphi(x+t)+\varphi(x-t)]+\int_{x-t}^{x+t}\psi(\eta)\mathrm{d}\eta,(x>t)\\ [\varphi(x+t)-\varphi(t-x)]+\int_{t-x}^{x+t}\psi(\eta)\mathrm{d}\eta,(t>x)\end{cases}$$

❑二维波动方程

$$\frac{\partial^2 u}{\partial t^2}-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0$$

$$u(x,y,t)\big|_{t=0}=\varphi(x,y);u_t(x,y,t)\big|_{t=0}=\psi(x,y)$$

$$u(x,0,t)=u(x,l,t)=0$$



- **时间-空间分离变量**

$$u(x,y,t)=U(x,y)T(t)$$

$$\Downarrow$$

$$T''(t)+k^2T(t)=0\Rightarrow T(t)=A\sin(kt)+B\cos(kt)$$

$$-\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)=k^2U;\ U(x,0)=U(x,l)=0$$

——二维Laplace算子的本征值问题

- **空间进一步分离变量**

$$U(x,y)=X(x)Y(y)$$

$$X''(x)+\lambda^2 X(x)=0 \Rightarrow X(x)=Ae^{\mathrm{i}\lambda x}$$

——λ可正可负,x方向连续谱

$$\begin{aligned}&Y''(y)+\mu^2 Y(y)=0\\&Y(0)=Y(l)=0\end{aligned} \Rightarrow \begin{aligned}&Y_n(y)=\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi y}{l}\right)\\&\mu_n=\frac{n\pi}{l},(n=1,2,3,\ldots)\end{aligned}$$

——y方向分立谱，S-L本征值问题

□ 二维本征值

$$[k_n(\lambda)]^2=\lambda^2+\mu_n^2=\lambda^2+\left(\frac{n\pi}{l}\right)^2$$

■ 方程的通解

$$u(x,y,t)=\sum_k U_k(x,y)T_k(t)=\sum_{\lambda,\mu} X_\lambda(x)Y_\mu(y)T_{\lambda,\mu}(t)$$

$$=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\left\{A_n(\lambda)\sin[k_n(\lambda)t]+B_n(\lambda)\cos[k_n(\lambda)t]\right\}Y_n(y)e^{\mathrm{i}\lambda x}\mathrm{d}\lambda$$

——*y*方向：分立谱；*x*方向：连续谱

■ 满足初始条件的特解

$$u(x,y,0)=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}B_n(\lambda)Y_n(y)e^{\mathrm{i}\lambda x}\mathrm{d}\lambda=\varphi(x,y)$$

$$u_t(x,y,0)=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}k_n(\lambda)A_n(\lambda)Y_n(y)e^{\mathrm{i}\lambda x}\mathrm{d}\lambda=\psi(x,y)$$

$$B_n(\lambda)=\frac{1}{2\pi}\int_0^l\int_{-\infty}^{\infty}\varphi(x,y)Y_n(y)\mathrm{e}^{-\mathrm{i}\lambda x}\mathrm{d}x\mathrm{d}y$$

$$A_n(\lambda)=\frac{1}{2\pi k_n(\lambda)}\int_0^l\int_{-\infty}^{\infty}\psi(x,y)Y_n(y)\mathrm{e}^{-\mathrm{i}\lambda x}\mathrm{d}x\mathrm{d}y$$

■ 积分形式的解

$$u(x,y,t)=\frac{\partial}{\partial t}\int_0^l\int_{-\infty}^{\infty}G(x-\xi;y,\eta;t)\varphi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta$$

$$+\int_0^l\int_{-\infty}^{\infty}G(x-\xi;y,\eta;t)\psi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta$$

$$G(x-\xi;y,\eta;t)\equiv\frac{1}{2\pi}\sum_{n=1}^{\infty}Y_n(y)Y_n(\eta)\int_{-\infty}^{\infty}\frac{\sin[k_n(\lambda)t]}{k_n(\lambda)}\mathrm{e}^{\mathrm{i}\lambda(x-\xi)}\mathrm{d}\lambda$$

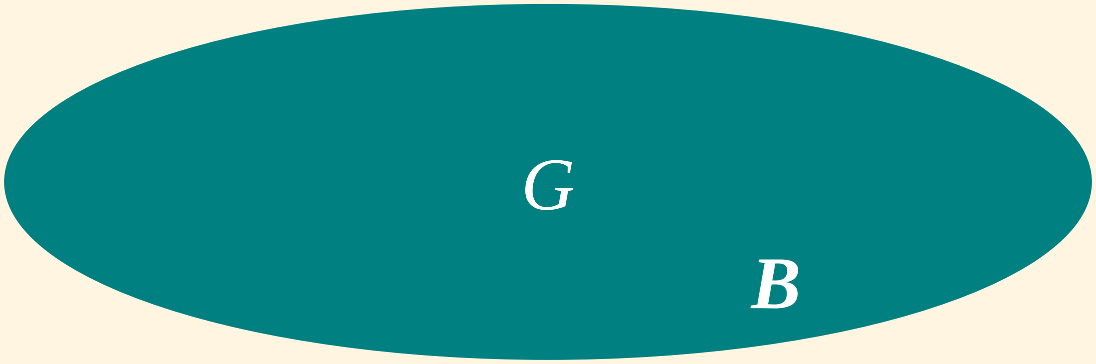
10.3 一般模式展开解

■Laplace 方程

$$-\nabla^2 u = f(\mathbf{r}), \mathbf{r} \in G$$

$$\left.\left(\alpha u + \beta \frac{\partial u}{\partial n}\right)\right|_B = b(\mathbf{r}), \mathbf{r} \in B$$

基本思想在右边f和b相当于源，激发内部不同本征振动模态，所以我们有理由把解按照本征模态展开，但有时这样的解不能直接带入边界条件求解系数（如果边界条件是非齐次的，而本征模式对应的是齐次问题）



□ Hermite对称算子，存在完备的函数系

$$-\nabla^2\psi_m(\boldsymbol{r})=\lambda_m\psi_m(\boldsymbol{r}),\boldsymbol{r}\in G$$

$$\left.\left(\alpha\psi_m+\beta\frac{\partial\psi_m}{\partial n}\right)\right|_B=0,\boldsymbol{r}\in B$$

m **是指标集合**

□ 模式展开解

$$u(\boldsymbol{r})=\sum_{m=0}^{\infty}a_m\psi_m(\boldsymbol{r}),\ \boldsymbol{r}\in G$$

$$a_m=\int_G u(\boldsymbol{r})\psi_m^*(\boldsymbol{r})\mathrm{d}\tau$$

边界条件不同，不能直接代入方程，求和与微分不能交换次序)

□ Green公式

$$\int_G (\varphi_1^* \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1^*) \mathrm{d}\tau = \iint_B \left(\varphi_1^* \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1^*}{\partial n} \right) \mathrm{d}S$$

⇧ ——为什么加复共轭?

$$\varphi_1^*(\boldsymbol{r}) = \psi_m^*(\boldsymbol{r}); \varphi_2(\boldsymbol{r}) = u(\boldsymbol{r})$$

⇩

$$\int_G (\psi_m^* \nabla^2 u - u \nabla^2 \psi_m^*) \mathrm{d}\tau = \iint_B \left(\psi_m^* \frac{\partial u}{\partial n} - u \frac{\partial \psi_m^*}{\partial n} \right) \mathrm{d}S$$

⇧

$$-\nabla^2 u = f(\boldsymbol{r}); -\nabla^2 \psi_m(\boldsymbol{r}) = \lambda_m \psi_m(\boldsymbol{r})$$

$$a_m = \int_G u(\boldsymbol{r}) \psi_m^*(\boldsymbol{r}) \mathrm{d}\tau$$

$$\lambda_m a_m = \int_G \psi_m^*(\boldsymbol{r}) f(\boldsymbol{r}) \mathrm{d}\tau + \iint_B \left(\psi_m^* \frac{\partial u}{\partial n} - u \frac{\partial \psi_m^*}{\partial n} \right) \mathrm{d}S$$

- **第一类边界条件：**

$$\beta(\boldsymbol{r}) = 0 \Rightarrow \lambda_m \neq 0; \quad \psi_m^* |_B = 0; \quad u |_B = \frac{b(\boldsymbol{r})}{\alpha(\boldsymbol{r})}$$

$$a_m = \frac{1}{\lambda_m} \int_G \psi_m^*(\boldsymbol{r}) f(\boldsymbol{r}) \mathrm{d}\tau - \frac{1}{\lambda_m} \iint_B \frac{b(\boldsymbol{r})}{\alpha(\boldsymbol{r})} \frac{\partial \psi_m^*}{\partial n} \mathrm{d}S$$

$$u(\boldsymbol{r}) = \int_G G(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}') \mathrm{d}\tau' - \iint_B \frac{b(\boldsymbol{r}')}{\alpha(\boldsymbol{r}')} \frac{\partial G(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'} \mathrm{d}S', \ \boldsymbol{r} \in G$$

$$G(\boldsymbol{r}, \boldsymbol{r}') \equiv \sum_{m=0}^{\infty} \frac{1}{\lambda_m} \psi_m(\boldsymbol{r}) \psi_m^*(\boldsymbol{r}')$$

□ 第二类边界条件：

$$\alpha(\mathbf{r})=0 \Rightarrow \lambda_0=0; \quad \left.\frac{\partial \psi_m^*}{\partial n}\right|_B=0; \quad \left.\frac{\partial u}{\partial n}\right|_B=\frac{b(\mathbf{r})}{\beta(\mathbf{r})}$$

$$\lambda_m a_m=\int_G \psi_m^*(\mathbf{r}) f(\mathbf{r}) \mathrm{d} \tau+\iint_B \frac{b(\mathbf{r})}{\beta(\mathbf{r})} \psi_m^* \mathrm{d} S$$

①如果*m*≠0

$$a_m=\frac{1}{\lambda_m} \int_G \psi_m^*(\mathbf{r}) f(\mathbf{r}) \mathrm{d} \tau+\frac{1}{\lambda_m} \iint_B \frac{b(\mathbf{r})}{\beta(\mathbf{r})} \psi_m^* \mathrm{d} S$$

②如果*m*=0

$$\lambda_0=0, \quad \psi_0(\mathbf{r})=\frac{1}{\sqrt{V}}, a_0=\text{任意}$$

$$0=\int_G f(\boldsymbol{r})\mathrm{d}\tau+\iint_B \frac{b(\boldsymbol{r})}{\beta(\boldsymbol{r})}\mathrm{d}S$$

——相容性条件——解存在的必要条件

$$u(\boldsymbol{r})=a_0\psi_0(\boldsymbol{r})+\sum_{m=1}^{\infty}a_m\psi_m(\boldsymbol{r}),\ \boldsymbol{r}\in G$$

$$a_m=\frac{1}{\lambda_m}\int_G \psi_m^*(\boldsymbol{r})f(\boldsymbol{r})\mathrm{d}\tau+\frac{1}{\lambda_m}\iint_{\partial G}\frac{b(\boldsymbol{r})}{\beta(\boldsymbol{r})}\psi_m^*(\boldsymbol{r})\mathrm{d}S\ \ (m>0)$$

$$u(\boldsymbol{r})=a_0\psi_0(\boldsymbol{r})+\int_G \tilde{G}(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\mathrm{d}\tau'+\iint_{\partial G}\frac{b(\boldsymbol{r}')}{\beta(\boldsymbol{r}')}\tilde{G}(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S'\ ,\ \boldsymbol{r}\in G$$

$$\tilde{G}(\boldsymbol{r},\boldsymbol{r}')\equiv\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\psi_m(\boldsymbol{r})\psi_m^*(\boldsymbol{r}')$$

注意：二者的区别

$$G(\mathbf{r},\mathbf{r}') \equiv \sum_{m=0}^{\infty} \frac{1}{\lambda_m} \psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')$$

所 有 的本征函数

$$\tilde{G}(\mathbf{r},\mathbf{r}') \equiv \sum_{m=1}^{\infty} \frac{1}{\lambda_m} \psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')$$

不包括0本征值的本征函数

$$-\nabla^2 G(\mathbf{r},\mathbf{r}') = \sum_{m=0}^{\infty} \psi_m(\mathbf{r})\psi_m^*(\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}')$$

本征函数的完备性

$$-\nabla^2 \tilde{G}(\mathbf{r},\mathbf{r}') = \sum_{m=0}^{\infty} \psi_m(\mathbf{r})\psi_m^*(\mathbf{r}') - \psi_0(\mathbf{r})\psi_0^*(\mathbf{r}')$$

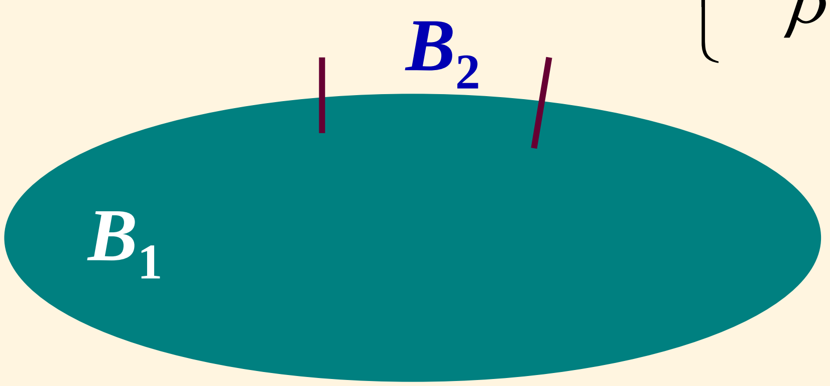
$$= \delta(\mathbf{r},\mathbf{r}') - \psi_0(\mathbf{r})\psi_0^*(\mathbf{r}')$$

——称为广义Green函数，零本征值的重要性:
经典波动(声学)中的意义；量子力学中的意义.

□ 第三类边界条件

$$\left.\left(\alpha\psi_m^* + \beta\frac{\partial\psi_m^*}{\partial n}\right)\right|_B = 0;\quad \left.\left(\alpha u + \beta\frac{\partial u}{\partial n}\right)\right|_B = b(\mathbf{r})$$

$$\left.\left(\psi_m^*\frac{\partial u}{\partial n} - u\frac{\partial\psi_m^*}{\partial n}\right)\right|_B = \begin{cases} -\dfrac{b(\mathbf{r})}{\alpha(\mathbf{r})}\left.\dfrac{\partial\psi_m^*}{\partial n}\right|_{B_1}, & \alpha(\mathbf{r})\neq 0 \\ +\dfrac{b(\mathbf{r})}{\beta(\mathbf{r})}\left.\psi_m^*\right|_{B_2}, & \beta(\mathbf{r})\neq 0 \end{cases}$$



可能：B_1上第一类边界；B_2上第二类边界

$$a_m = \frac{1}{\lambda_m}\int_G \psi_m^*(\mathbf{r}) f(\mathbf{r})\mathrm{d}\tau$$

$$+\frac{1}{\lambda_m}\left[\iint_{B_2}\frac{b(\mathbf{r})}{\beta(\mathbf{r})}\psi_m^*\mathrm{d}S - \iint_{B_1}\frac{b(\mathbf{r})}{\alpha(\mathbf{r})}\frac{\partial\psi_m^*}{\partial n}\mathrm{d}S\right]$$

□ 积分形式的解

$$u(\mathbf{r}) = \int_G G(\mathbf{r},\mathbf{r}') f(\mathbf{r}')\mathrm{d}\tau' + B(\mathbf{r}); G(\mathbf{r},\mathbf{r}') \equiv \sum_{m=0}^{\infty}\frac{1}{\lambda_m}\psi_m(\mathbf{r})\psi_m^*(\mathbf{r}')$$

$$\Uparrow$$

$$B(\mathbf{r}) \equiv \iint_{B_2}\frac{b(\mathbf{r}')}{\beta(\mathbf{r}')}G(\mathbf{r},\mathbf{r}')\mathrm{d}S' - \iint_{B_1}\frac{b(\mathbf{r}')}{\alpha(\mathbf{r}')}\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\mathrm{d}S'$$

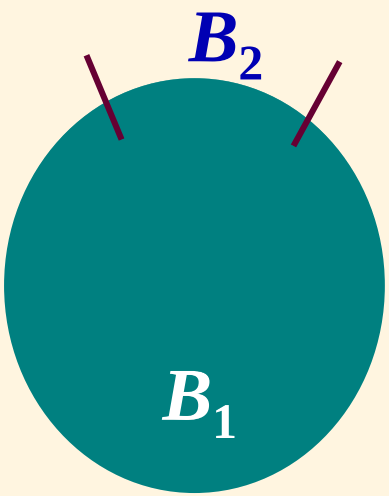
——注意：尽管解的形式雷同，但本征函数满足的边界条件不同.

■波动方程

$$\frac{\partial^2 u}{\partial t^2}-\nabla^2 u=f(\mathbf{r},t),\mathbf{r}\in G,t>0$$

$$\left(\alpha u+\beta\frac{\partial u}{\partial n}\right)\bigg|_B=b(\mathbf{r},t),\mathbf{r}\in B,t\geq 0$$

$$u(\mathbf{r},t)\,|_{t=0}=\psi_1(\mathbf{r});\frac{\partial u}{\partial t}\bigg|_{t=0}=\psi_2(\mathbf{r})$$



$$-\nabla^2\psi_m(\mathbf{r})=\lambda_m\psi_m(\mathbf{r}),\mathbf{r}\in G$$

$$\left(\alpha\psi_m+\beta\frac{\partial\psi_m}{\partial n}\right)\bigg|_B=0,\mathbf{r}\in B$$

$$\Rightarrow\{\psi_m(\mathbf{r}),\lambda_m\}$$

□ 模式展开解

$$u(\mathbf{r},t)=\sum_{m=0}^{\infty}a_m(t)\psi_m(\mathbf{r}),\ \mathbf{r}\in G$$

$$a_m(t)=\int_G u(\mathbf{r},t)\psi_m^*(\mathbf{r})\mathrm{d}\tau$$

$$\int_G(\psi_m^*\nabla^2 u-u\nabla^2\psi_m^*)\mathrm{d}\tau=\iint_B\left(\psi_m^*\frac{\partial u}{\partial n}-u\frac{\partial\psi_m^*}{\partial n}\right)\mathrm{d}S$$

$$\frac{\mathrm{d}^2 a_m(t)}{\mathrm{d}t^2}+\lambda_m a_m(t)=f_m(t)+b_m(t)$$

$$f_m(t)\equiv\int_G f(\mathbf{r},t)\psi_m^*(\mathbf{r})\mathrm{d}\tau;b_m(t)\equiv\iint_B\left(\psi_m^*\frac{\partial u}{\partial n}-u\frac{\partial\psi_m^*}{\partial n}\right)\mathrm{d}S$$

$$\left.\left(\psi_m^* \frac{\partial u}{\partial n} - u \frac{\partial \psi_m^*}{\partial n}\right)\right|_B = \begin{cases} -\dfrac{b(\mathbf{r},t)}{\alpha(\mathbf{r})} \dfrac{\partial \psi_m^*}{\partial n}, \alpha(\mathbf{r}) \neq 0 \\ +\dfrac{b(\mathbf{r},t)}{\beta(\mathbf{r})} \psi_m^*, \qquad \beta(\mathbf{r}) \neq 0 \end{cases}$$

$$b_m(t) \equiv \iint_{B_1} \frac{b(\mathbf{r},t)}{\beta(\mathbf{r})} \psi_m^* \mathrm{d}S - \iint_{B_2} \frac{b(\mathbf{r},t)}{\alpha(\mathbf{r})} \frac{\partial \psi_m^*}{\partial n} \mathrm{d}S$$

□ 非齐次常微分方程的初值问题

$$\frac{\mathrm{d}^2 a_m(t)}{\mathrm{d}t^2} + \lambda_m a_m(t) = f_m(t) + b_m(t)$$

$$a_m(0) = \int_G \psi_1(\mathbf{r}) \psi_m^*(\mathbf{r}) \mathrm{d}\tau ; a_m'(0) = \int_G \psi_2(\mathbf{r}) \psi_m^*(\mathbf{r}) \mathrm{d}\tau$$

$$u(\boldsymbol{r},t)=\sum_{m=0}^{\infty}a_m(t)\psi_m(\boldsymbol{r}),\ \boldsymbol{r}\in G$$

$$a_m(t)=\frac{1}{\sqrt{\lambda_m}}a_m'(0)\sin\left(\sqrt{\lambda_m}t\right)+a_m(0)\cos\left(\sqrt{\lambda_m}t\right)$$

$$+\frac{1}{\sqrt{\lambda_m}}\int_0^t[f_m(\tau)+b_m(\tau)]\sin\left[\sqrt{\lambda_m}(t-\tau)\right]\mathrm{d}\tau$$

□ 积分形式的解

$$u(\boldsymbol{r},t)=\int_G\psi_1(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r},\boldsymbol{r}',t)}{\partial t}\mathrm{d}\tau'+\int_G\psi_2(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}',t)\mathrm{d}\tau'$$

$$+\int_0^t\int_G f(\boldsymbol{r}',\tau)G(\boldsymbol{r},\boldsymbol{r}',t-\tau)\mathrm{d}\tau'\mathrm{d}\tau+u_B(\boldsymbol{r},t)$$

$$u_B \equiv \int_0^t \left[\begin{array}{l} +\iint_{B_1} \dfrac{b(\mathbf{r}',\tau)}{\beta(\mathbf{r}')} G(\mathbf{r},\mathbf{r}',t-\tau)\mathrm{d}S' \\ -\iint_{B_2} \dfrac{b(\mathbf{r}',\tau)}{\alpha(\mathbf{r}')} \dfrac{\partial G(\mathbf{r},\mathbf{r}',t-\tau)}{\partial n'} \mathrm{d}S' \end{array} \right] \mathrm{d}\tau$$

$$G(\mathbf{r},\mathbf{r}',t) \equiv \sum_{m=0}^{\infty} \frac{1}{\sqrt{\lambda_m}} \psi_m(\mathbf{r}) \psi_m^*(\mathbf{r}') \sin\left(\sqrt{\lambda_m} t\right)$$

例1 三维无限大空间的Cauchy问题

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(\mathbf{r},t), t > 0$$

$$u(\mathbf{r},t)\big|_{t=0} = h(\mathbf{r}); \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(\mathbf{r})$$

- 无限大空间：面积分项为零

$$u(\boldsymbol{r},t)=\int_G g(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}',t)\mathrm{d}\tau'+\frac{\partial}{\partial t}\int_G h(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}',t)\mathrm{d}\tau'$$

$$+\int_0^t\int_G f(\boldsymbol{r}',\tau)G(\boldsymbol{r},\boldsymbol{r}',t-\tau)\mathrm{d}\tau'\mathrm{d}\tau$$

- 关键是求Green函数

$$G(\boldsymbol{r},\boldsymbol{r}',t)\equiv\sum_{m=0}^{\infty}\frac{1}{\sqrt{\lambda_m}}\psi_m(\boldsymbol{r})\psi_m^*(\boldsymbol{r}')\sin\left(\sqrt{\lambda_m}t\right)$$

- 三维无限大空间Laplace算子的本征值问题解

$$\psi_{\boldsymbol{k}}(\boldsymbol{r})=\frac{1}{(2\pi)^{3/2}}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$$

$$m=(k_1,k_2,k_3),\lambda_m=k^2=k_1^2+k_2^2+k_3^2$$ ——连续谱

$$G(\boldsymbol{r},\boldsymbol{r}',t)\equiv\frac{1}{(2\pi)^3}\sum_{\boldsymbol{k}}\frac{1}{k}\exp[\mathrm{i}(\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\sin(kt)$$

$$\Downarrow$$

$$G(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(kt)}{k}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\,\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

三维k-空间的球坐标积分：积分过程中$(\boldsymbol{r}-\boldsymbol{r}')$是常矢量，取为$k$-空间的$k_z$方向，于是

$$G(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{(2\pi)^3}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(kt)}{k}\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta_k}\,k^2\sin\vartheta_k\mathrm{d}k\mathrm{d}\vartheta_k\mathrm{d}\varphi_k$$

$$=\frac{1}{(2\pi)^2}\int_0^{\infty}\frac{\sin(kt)}{k}\left[\int_0^{\pi}\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|\cos\vartheta_k}\sin\vartheta_k\mathrm{d}\vartheta_k\right]k^2\mathrm{d}k$$

$$G(\boldsymbol{r},\boldsymbol{r}',t)=\frac{1}{4\pi^2}\cdot\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\int_{-\infty}^{\infty}\sin(kt)\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}k$$

$$=\frac{1}{4\pi}\cdot\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}[\delta(t-|\boldsymbol{r}-\boldsymbol{r}'|)-\delta(t+|\boldsymbol{r}-\boldsymbol{r}'|)]$$

$$=\frac{1}{4\pi}\cdot\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta(t-|\boldsymbol{r}-\boldsymbol{r}'|)$$

如果不考虑初值的影响

$$u(\boldsymbol{r},t)=\frac{1}{4\pi}\int_0^t\int_G\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta[(t-\tau)-|\boldsymbol{r}-\boldsymbol{r}'|]f(\boldsymbol{r}',\tau)\mathrm{d}\tau'\mathrm{d}\tau$$

$$=\frac{1}{4\pi}\int_G\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}f(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|)\mathrm{d}\tau'$$

10.4 柱坐标中的分离变量

□柱坐标:(1)径向对称问题; (2)曲面在柱坐标很容易表达——物理问题的零级近似.

$$x=\rho\cos\varphi;\ y=\rho\sin\varphi;\ z=z$$

$$0<\rho<\infty,\ 0\le\varphi\le 2\pi,\ -\infty<z<\infty$$

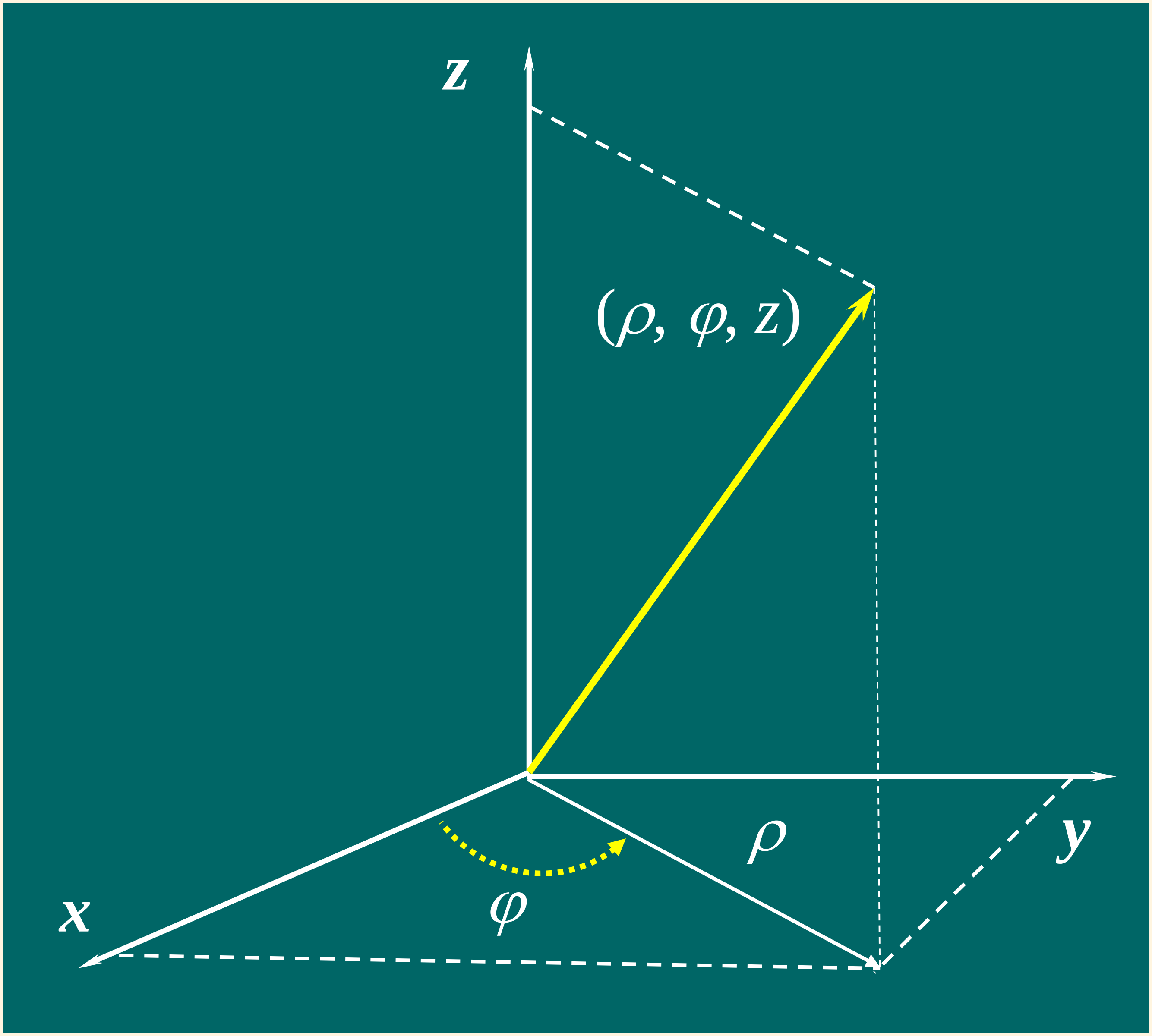
单位矢量的变换

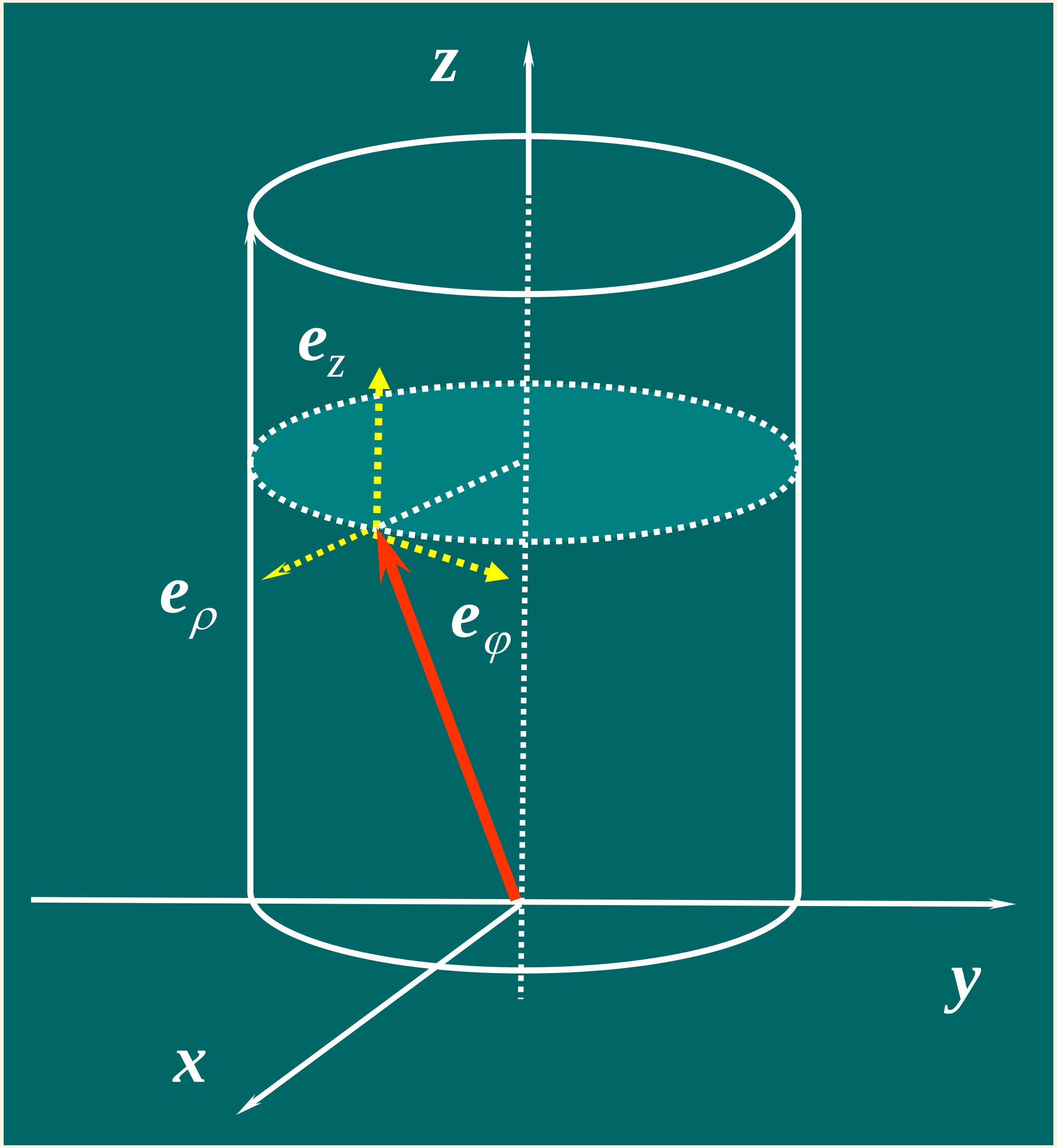
$$\boldsymbol{e}_\rho=\cos\varphi\boldsymbol{e}_x+\sin\varphi\boldsymbol{e}_y$$

$$\boldsymbol{e}_\varphi=-\sin\varphi\boldsymbol{e}_x+\cos\varphi\boldsymbol{e}_y$$

$$\boldsymbol{e}_z=\boldsymbol{e}_z$$

$$\Rightarrow\quad \begin{bmatrix}\boldsymbol{e}_\rho\\ \boldsymbol{e}_\varphi\\ \boldsymbol{e}_z\end{bmatrix}=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix}$$


z
(ρ, φ, z)
ρ
y
x
φ



z
e_z
e_ρ
e_φ
y
x

❑Laplace 方程在柱坐标中的分离变量

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=0$$

- **轴向分离变量**

$$u(\rho,\varphi,z)=\Xi(\rho,\varphi)\mathrm{Z}(z)$$

$$\frac{1}{\Xi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Xi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Xi}{\partial\varphi^2}\right]+\frac{1}{\mathrm{Z}}\frac{\mathrm{d}^2\mathrm{Z}}{\mathrm{d}z^2}=0$$

$$\frac{\mathrm{d}^2\mathrm{Z}(z)}{\mathrm{d}z^2}=\mu\mathrm{Z}(z);\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Xi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Xi}{\partial\varphi^2}+\mu\Xi=0$$

■ 径向与方位角方向分离变量

$$\varXi(\rho,\varphi)=R(\rho)\varPhi(\varphi)$$

$$\frac{\rho}{R(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right]+\mu\rho^2+\frac{1}{\varPhi(\varphi)}\frac{\mathrm{d}^2\varPhi(\varphi)}{\mathrm{d}\varphi^2}=0$$

$$\frac{\mathrm{d}^2\varPhi(\varphi)}{\mathrm{d}\varphi^2}+\lambda\varPhi(\varphi)=0$$

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right]+\left(\mu-\frac{\lambda}{\rho^2}\right)R(\rho)=0$$

$$\frac{\mathrm{d}^2Z(z)}{\mathrm{d}z^2}-\mu Z(z)=0$$

1个偏微分方程分离变量成3个常微分方程

■ 方位角存在本征值问题

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}+\lambda\Phi(\varphi)=0$$

$$\Phi(\varphi)=\Phi(2\pi+\varphi)$$

本征值问题的解 $\lambda_m=m^2$

$$\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi},(m=0,\pm1,\pm2,\ldots)$$

$$\Phi_m(\varphi)=A_m\cos(m\varphi)+B_m\sin(m\varphi),(m=0,1,2,\ldots)$$

注意：当*m*=0时

$$\Phi_0(\varphi)=A_0+C_0\varphi$$

——第2项不满足周期边界条件，但有时有意义，如不可压缩流体绕圆柱旋转

■ 轴向解

$$\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2} - \mu Z(z) = 0$$

(A)μ=0

$$Z_0(z) = C_0 + D_0 z$$

(B)μ >0

$$Z_\mu(z) = Ce^{-\sqrt{\mu}z} + De^{\sqrt{\mu}z}$$

$$Z_\mu(z) = C\sinh\left(\sqrt{\mu}z\right) + D\cosh\left(\sqrt{\mu}z\right)$$

(C)μ<0

$$Z_{|\mu|}(z) = Ce^{\mathrm{i}\sqrt{|\mu|}z} + De^{-\mathrm{i}\sqrt{|\mu|}z}$$

$$Z_{|\mu|}(z) = C\cos\left(\sqrt{|\mu|}z\right) + D\sin\left(\sqrt{|\mu|}z\right)$$

■ 径向解

$$\frac{\mathrm{d}^2 R(\rho)}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}+\left(\mu-\frac{m^2}{\rho^2}\right)R(\rho)=0$$

(A)μ=0

$$\frac{\mathrm{d}^2 R(\rho)}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}-\frac{m^2}{\rho^2}R(\rho)=0$$

——$m\neq 0$, Euler方程； m=0，可积方程

$$R_m(\rho)=\begin{cases}V_m\rho^{|m|}+U_m D\rho^{-|m|}, & m\neq 0\\ E_0+F_0\ln\rho, & m=0\end{cases}$$

(B)$\mu>0$，令 $x=\sqrt{\mu}\rho$

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x}+\left(1-\frac{m^2}{x^2}\right)R(x)=0$$

——m阶Bessel方程

通解

$$R_m(\sqrt{\mu}\rho)=L_mJ_m\left(\sqrt{\mu}\rho\right)+M_mN_m\left(\sqrt{\mu}\rho\right)$$

(C)$\mu<0$，令 $x=\sqrt{|\mu|}\rho$

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x}-\left(1+\frac{m^2}{x^2}\right)R(x)=0$$

——m 阶 虚宗量 Bessel 方程

通解

$$R_m\left(\sqrt{|\mu|}\rho\right)=O_m I_m\left(\sqrt{|\mu|}\rho\right)+P_m K_m\left(\sqrt{|\mu|}\rho\right)$$

- **Laplace方程的分离变量解的一般形式**

$$u(\rho,\varphi,z)=\boxed{\begin{aligned}&(C_0+D_0 z)(E_0+F_0\ln\rho) \quad \text{二维问题}\\&+\sum_m (C_m+D_m z)(V_m\rho^m+U_m\rho^{-m})\Phi_m(\varphi)\end{aligned}}$$

$$+\sum_m\sum_{\mu>0} Z_\mu(z)R_m\left(\sqrt{\mu}\rho\right)\Phi_m(\varphi)$$

$$+\sum_m\sum_{\mu<0} Z_{|\mu|}(z)R_m\left(\sqrt{|\mu|}\rho\right)\Phi_m(\varphi)$$

根据边界条件的要求，取舍不同的系数——Bessel函数和虚宗量Bessel函数的性质

事实上，以后将看到

(A) μ=0：对应于与 z 轴无关的二维情况($D_m \equiv 0$)；

(B) μ >0：对应于 z 轴方向无限长情况，此时ρ 方向的边界条件与 $R(\rho)$的方程构成本征值问题，而决定μ 的数值；

(C) μ <0：对应于 z 轴方向有限长，此时 z 轴方向的边界条件与 $Z(z)$ 的方程构成本征值问题，而决定μ的数值。

❑Helmholtz方程在柱坐标中的分离变量

$$\nabla^2 v + k^2 v = 0$$

三维波动方程和扩散方程，经时间与空间分离变量后空间部分满足的是 Helmholtz方程。

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

- **对φ方向, 同样有本征值问题**

$$\frac{\mathrm{d}^2 \Phi(\varphi)}{\mathrm{d}\varphi^2} + m^2 \Phi(\varphi) = 0; \quad \Phi(\varphi) = \Phi(2\pi + \varphi)$$

$$\Downarrow$$

$$\Phi_m(\varphi) = A_m e^{\mathrm{i}m\varphi}, (m = 0, \pm 1, \pm 2, \ldots)$$

$$\Phi_m(\varphi) = A_m \cos(m\varphi) + B_m \sin(m\varphi), (m = 0, 1, 2, \ldots)$$

■ 对z方向

$$\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2} - \mu Z(z) = 0$$

(A)μ=0

$$Z_0(z) = C_0 + D_0 z$$

z方向的倏逝波

(B)μ >0

$$Z_\mu(z) = Ce^{-\sqrt{\mu}z} + De^{\sqrt{\mu}z}$$

$$Z_\mu(z) = C\sinh\left(\sqrt{\mu}z\right) + D\cosh\left(\sqrt{\mu}z\right)$$

(C)μ<0

$$Z_{|\mu|}(z) = Ce^{\mathrm{i}\sqrt{|\mu|}z} + De^{-\mathrm{i}\sqrt{|\mu|}z}$$

$$Z_{|\mu|}(z) = C\cos\left(\sqrt{|\mu|}z\right) + D\sin\left(\sqrt{|\mu|}z\right)$$

■ 对 ρ 方向

$$\frac{\mathrm{d}^2R(\rho)}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}+\left(k_\rho^2-\frac{m^2}{\rho^2}\right)R(\rho)=0$$

$$k^2+\mu\equiv k_\rho^2$$

(A) $k_\rho^2>0\Rightarrow x=k_\rho\rho$

$$x^2\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+x\frac{\mathrm{d}R(x)}{\mathrm{d}x}+(x^2-m^2)R(x)=0$$

$$R_m(k_\rho\rho)=L_mJ_m(k_\rho\rho)+M_mN_m(k_\rho\rho)$$

□ **辐射形式的通解(——为什么？)**

$$R_m(k_\rho\rho)=L_mH_m^{(1)}(k_\rho\rho)+M_mH_m^{(2)}(k_\rho\rho)$$

其中，第一、二类Hankel函数定义为

$$H_m^{(1)}(k_\rho \rho) = J_m(k_\rho \rho) + \mathrm{i} N_m(k_\rho \rho)$$

$$H_m^{(2)}(k_\rho \rho) = J_m(k_\rho \rho) - \mathrm{i} N_m(k_\rho \rho)$$

- 一维类比

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + y(x) = 0$$

驻波形式解

$$y(x) = A\sin x + B\cos x$$

行波形式解——方便满足远场辐射条件

$$e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x; \ e^{-\mathrm{i}x} = \cos x - \mathrm{i}\sin x$$

$$y(x) = Ae^{\mathrm{i}x} + Be^{-\mathrm{i}x}$$

(B) $k_\rho^2 < 0 \Rightarrow k_\rho = \mathrm{i}\kappa_\rho \Rightarrow x = \kappa_\rho \rho$

$$x^2 \frac{\mathrm{d}^2 R(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}R(x)}{\mathrm{d}x} - (x^2 + m^2)R(x) = 0$$

$$R_m(\kappa_\rho \rho) = L_m I_m(\kappa_\rho \rho) + M_m K_m(\kappa_\rho \rho)$$

——不存在远场辐射条件问题，表示近场倏逝波

一维类比

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} - y(x) = 0 \Rightarrow \begin{aligned} & y(x) = A\sinh x + B\cosh x \\ & y(x) = Ae^{-x} + Be^{x} \end{aligned}$$

——波动中的非均匀波或者近场倏逝波

■ Helmholtz方程的分离变量解的一般形式

$$u(\rho,\varphi,z)=(C+Dz)\boxed{\sum_m R_m(k\rho)\Phi_m(\varphi)}$$

二维问题

$$+\sum_m\sum_{k_\rho>0,\mu>0} Z_\mu(z)R_m(k_\rho\rho)\Phi_m(\varphi)$$

$$+\sum_m\sum_{k_\rho>0,\mu<0} Z_{|\mu|}(z)R_m(k_\rho\rho)\Phi_m(\varphi)$$

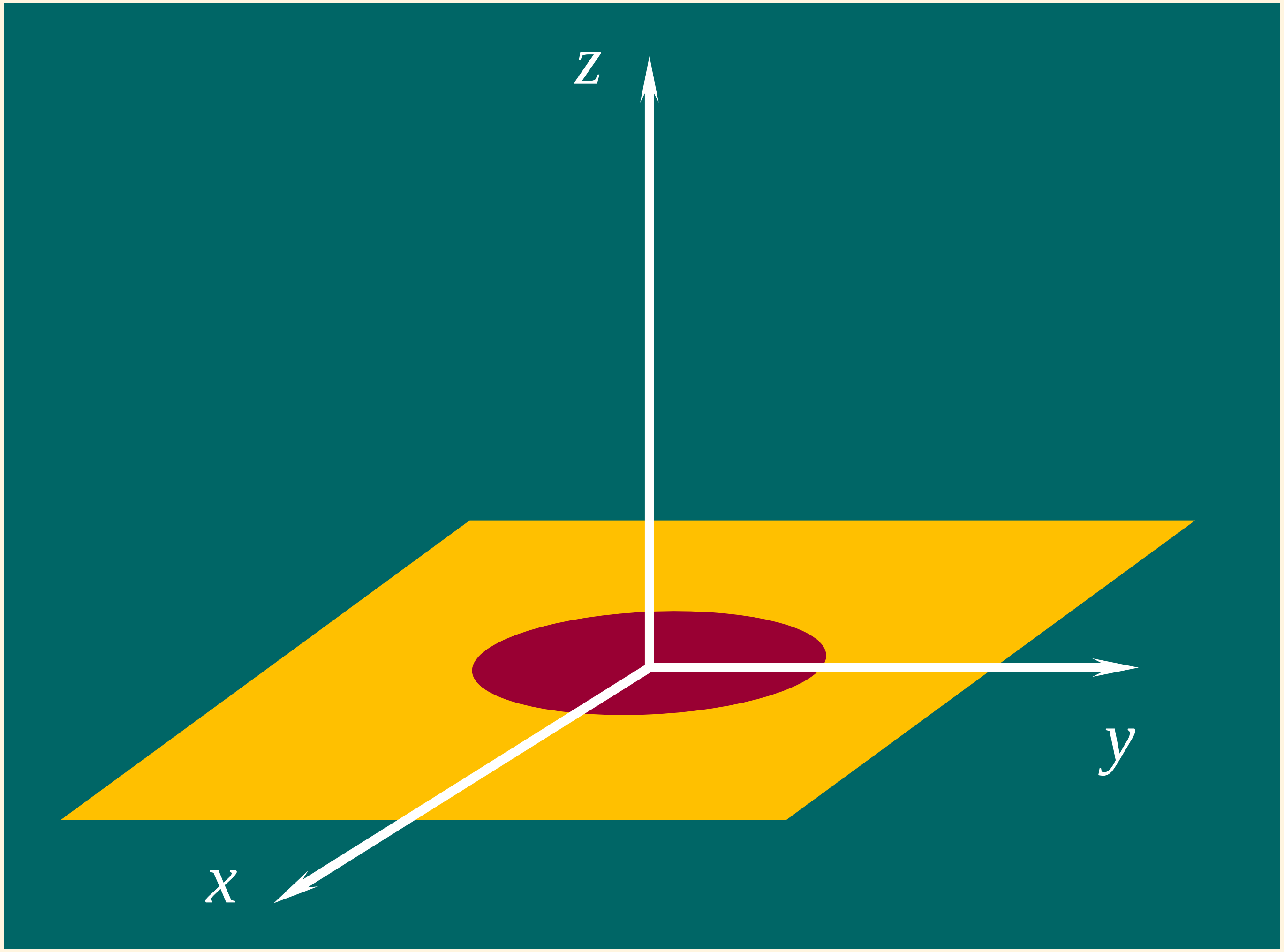
$$+\sum_m\sum_{k_\rho<0,\mu<0} Z_{|\mu|}(z)R_m(\kappa_\rho\rho)\Phi_m(\varphi)$$

根据不同的问题，取舍不同的系数——Bessel函数的性质

□ 柱坐标中常见问题

- 无限长柱内部问题(OK)
- 无限长柱外部问题(OK)
- 有限长柱内部问题 (OK)
- 有限长柱外部问题 (??)
- 有限高偏平区域问题(OK)

柱对称源的辐射



10.5 球坐标中的分离变量

❑球坐标：(1)球对称问题；(2)曲面在球坐标很容易表达——物理问题的零级近似.

$$x = r\sin\vartheta\cos\varphi; y = r\sin\vartheta\sin\varphi; z = r\cos\vartheta$$

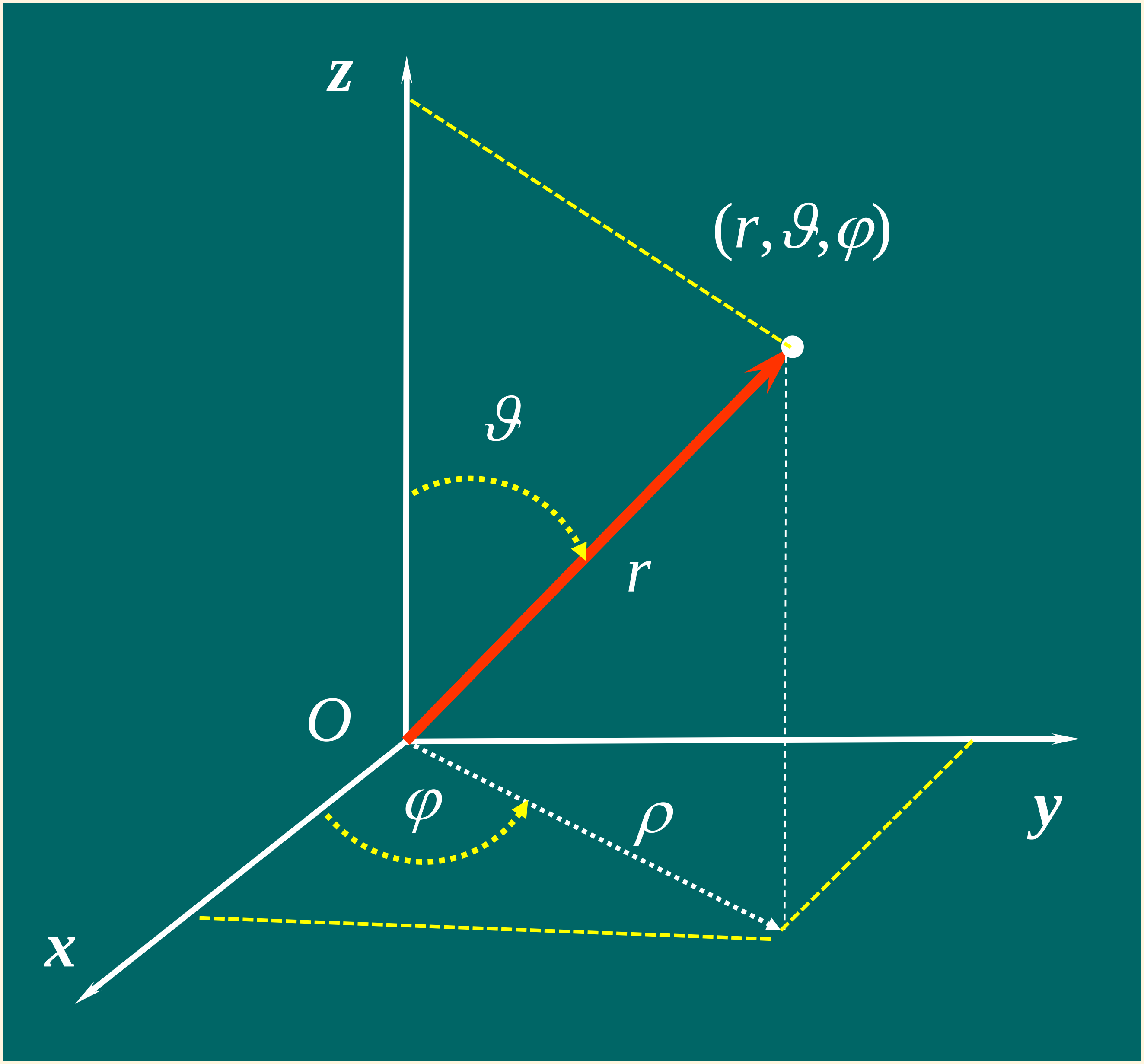
$$0 < \rho < \infty,\ 0 \le \vartheta \le \pi,\ 0 \le \varphi \le 2\pi$$

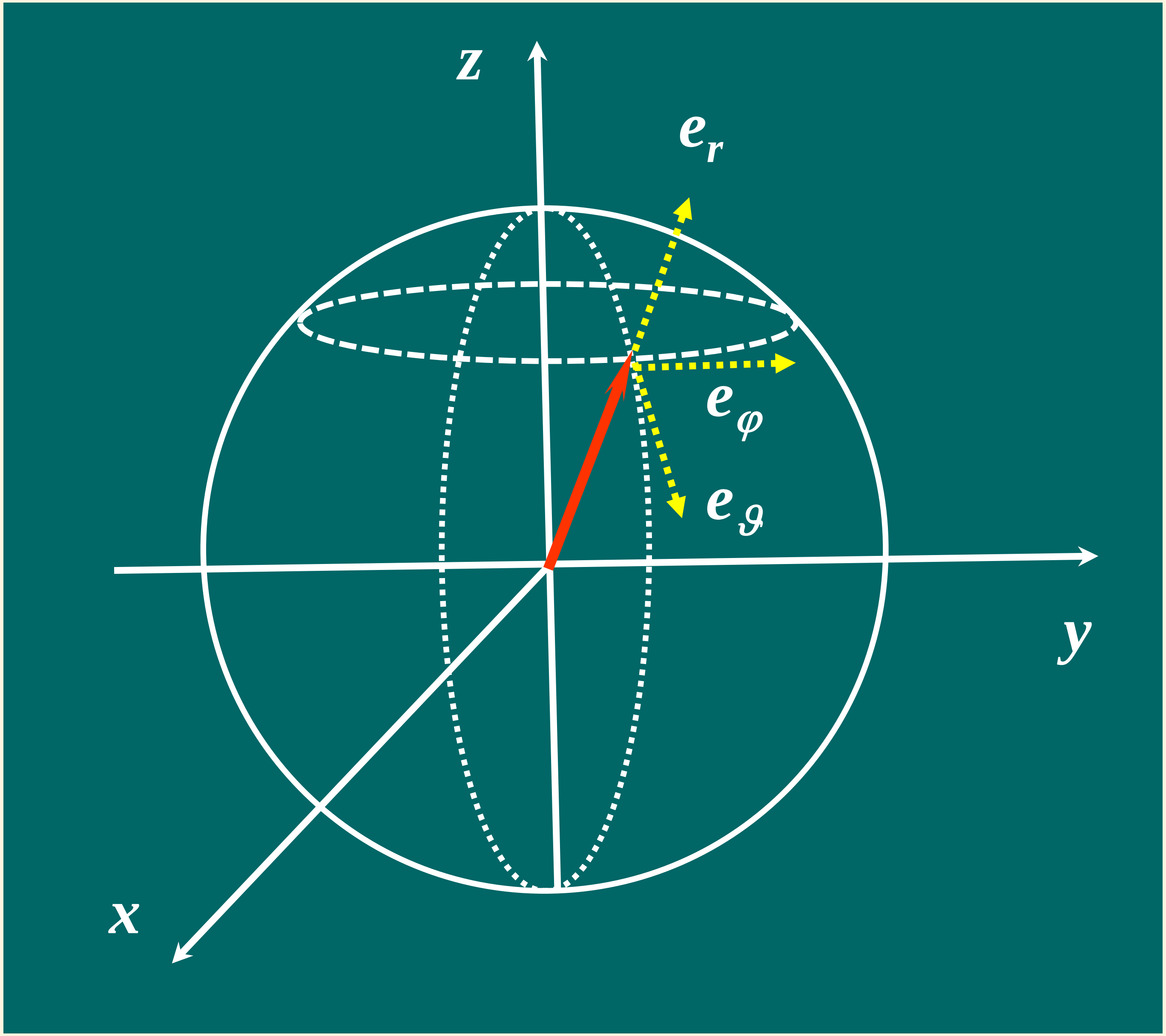
单位矢量的变换

$$\boldsymbol{e}_r = \sin\vartheta\cos\varphi\boldsymbol{e}_x + \sin\vartheta\sin\varphi\boldsymbol{e}_y + \cos\vartheta\boldsymbol{e}_z$$

$$\boldsymbol{e}_\vartheta = \cos\vartheta\cos\varphi\boldsymbol{e}_x + \cos\vartheta\sin\varphi\boldsymbol{e}_y - \sin\vartheta\boldsymbol{e}_z$$

$$\boldsymbol{e}_\varphi = -\sin\varphi\boldsymbol{e}_x + \cos\varphi\boldsymbol{e}_y$$


z
(r, ϑ, φ)
ϑ
r
O
φ
ρ
y
x



z
e_r
e_φ
e_ϑ
y
x

❑Laplace方程在球坐标中的分离变量

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}=0$$

- **分离变量解**

$$u(r,\vartheta,\varphi)=R(r)Y(\vartheta,\varphi)$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{1}{Y}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\nu(\nu+1)R=0$$

$$-\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\right]=\nu(\nu+1)Y$$

■ 径向方程

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}-\nu(\nu+1)R=0$$

$$\Downarrow$$

$$R(r)=Cr^{\nu}+Dr^{-(\nu+1)}$$

■ 单位球面上方程

$$-\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\right]=\nu(\nu+1)Y$$

可以进一步分离变量

$$Y(\vartheta,\varphi)=\Theta(\vartheta)\Phi(\varphi)$$

$$\frac{\sin^2\vartheta}{\Theta(\vartheta)\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\sin\vartheta\frac{\mathrm{d}\Theta(\vartheta)}{\mathrm{d}\vartheta}\right]+\nu(\nu+1)\sin^2\vartheta$$

$$=-\frac{1}{\Phi(\varphi)}\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}\equiv\lambda$$

$$\Downarrow$$

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}+\lambda\Phi(\varphi)=0$$

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\sin\vartheta\frac{\mathrm{d}\Theta(\vartheta)}{\mathrm{d}\vartheta}\right]+\left[\nu(\nu+1)-\frac{\lambda}{\sin^2\vartheta}\right]\Theta(\vartheta)=0$$

- **方位角方向的本征值问题**

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}+\lambda\Phi(\varphi)=0;\ \ \Phi(\varphi)=\Phi(\varphi+2\pi)$$

由周期性边界条件，可得本征值问题的解

$$\lambda_m = m^2; \quad \Phi_m(\varphi) = A_m e^{\mathrm{i}m\varphi}, (m = 0, \pm 1, \pm 2, \ldots)$$

$$\Phi_m(\varphi) = A_m \cos(m\varphi) + B_m \sin(m\varphi), (m = 0, 1, 2, \ldots)$$

■ 极角方向

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\sin\vartheta\frac{\mathrm{d}\Theta(\vartheta)}{\mathrm{d}\vartheta}\right] + \left[\nu(\nu+1) - \frac{m^2}{\sin^2\vartheta}\right]\Theta(\vartheta) = 0$$

令 $x=\cos\vartheta$ **$\vartheta \in [0, \pi]$, $x \in [-1, 1]$**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right] + \left[\nu(\nu+1) - \frac{m^2}{1-x^2}\right]\Theta(x) = 0$$

——连带 Legendre 方程

■ Legendre方程的本征值问题(当 m=0 时)

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right]=\nu(\nu+1)\Theta(x);\quad \Theta(\pm 1)<\infty$$

$$\Theta_l^0(x)=P_l(x)\Rightarrow\Theta(\cos\vartheta)=P_l(\cos\vartheta)$$

$$\lambda_l=\nu(\nu+1)=l(l+1),\ (l=0,1,2,\ldots)$$

■ 连带 Legendre 方程的本征值问题

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right]+\frac{m^2}{1-x^2}\Theta(x)=\nu(\nu+1)\Theta(x)$$

$$\Theta(\pm 1)<\infty$$

$$\Theta_l^m(x) = P_l^{|m|}(x) = (1-x^2)^{|m|/2}\frac{\mathrm{d}^{|m|}P_l(x)}{\mathrm{d}x^{|m|}}$$

$$\lambda_l = \nu(\nu+1) = l(l+1),\ (l = |m|, |m|+1, |m|+2, \ldots)$$

■ 单位球面方程的本征值问题

$$-\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = \nu(\nu+1)Y$$

$$Y(\vartheta,\varphi)|_{\vartheta=0,\pi} < \infty;\ Y(\vartheta,\varphi) = Y(\vartheta,\varphi+2\pi)$$

$$\Downarrow$$

$$Y_l^m(\vartheta,\varphi) = P_l^{|m|}(\cos\vartheta)\exp(\mathrm{i}m\varphi)$$

$$\lambda_l = \nu(\nu+1) = l(l+1)$$

$$(l = |m|, |m|+1, |m|+2, \ldots)$$

■ Laplace 方程分离变量的通解

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}r^{l}+B_{lm}r^{-(l+1)}]Y_l^m(\vartheta,\varphi)$$

❑ Helmholtz方程在球坐标中的分离变量

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}+k^2u=0$$

■ 分离变量解

$$u(r,\vartheta,\varphi)=R(r)Y(\vartheta,\varphi)$$

■ 单位球面方程—与Laplace方程结果相同

$$-\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\right]=\nu(\nu+1)Y$$

■ 径向方程

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right]+l(l+1)R(r)=k^2r^2R(r)$$

$$r^2\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R(r)}{\mathrm{d}r}+[k^2r^2-l(l+1)]R(r)=0$$

——球Bessel方程

令

$$x=kr;y(x)=\sqrt{\frac{2kr}{\pi}}R(r)$$

球Bessel方程变化成(l+1/2)阶Bessel方程—半奇数阶Bessel方程

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[x^2-\left(l+\frac{1}{2}\right)^2\right]y=0$$

$$y(x)=C_1 J_{l+1/2}(x)+C_2 N_{l+1/2}(x)$$

$$R_l(r)=\sqrt{\frac{\pi}{2kr}}C_1 J_{l+1/2}(kr)+C_2 N_{l+1/2}(kr)$$

$$R_l(r)=C_1 j_l(kr)+C_2 n_l(kr)$$ 球内驻波解

其中，球Bessel函数和球Neumann函数为

$$j_l(kr)\equiv\sqrt{\frac{\pi}{2kr}}J_{l+1/2}(kr);n_{l+1/2}(kr)\equiv\sqrt{\frac{\pi}{2kr}}N_{(l+1/2)}(kr)$$

■ **球外行波解**

$$R_l(r)=C_1 h_l^{(1)}(kr)+C_2 h_l^{(2)}(kr)$$

其中，第一、二类球Hankel函数定义为

$$h_l^{(1)}(kr) = j_l(kr) + \mathrm{i}n_l(kr)$$

$$h_l^{(2)}(kr) = j_l(kr) - \mathrm{i}n_l(kr)$$

- **Helmholtz方程球内驻波解**

$$u(r,\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}j_l(kr) + B_{lm}n_l(kr)]Y_{lm}(\vartheta,\varphi)$$

- **Helmholtz方程球外行波解**

$$u(r,\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}h_l^{(1)}(kr) + B_{lm}h_l^{(2)}(kr)]Y_{lm}(\vartheta,\varphi)$$

Laplace方程和Helmholtz方程分离变量

方程	球坐标	柱坐标
Laplace 方程	$R=\{r^{l},r^{-(l+1)}\}$ $\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi}$ $\Theta(\vartheta)$:连带 Legendre 方程	$R(\rho)$:m阶Bessel或虚宗量Bessel方程 $\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi}$ $Z''-\mu Z=0$
Helmholtz 方程	$R(r)$: l阶球Bessel方程 $\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi}$ $\Theta(\vartheta)$:连带 Legendre 方程	$R(\rho)$:m阶Bessel或虚宗量Bessel方程 $\Phi_m(\varphi)=A_m e^{\mathrm{i}m\varphi}$ $Z''-\mu Z=0$

■ 可分离变量的一般原则

- 方程可分离变量: 线性方程(特殊的变系数方程，特殊的非线性方程也可以);齐次方程;
- 边界条件可分离变量: 线性边界条件;齐次边界条件;规则的边界;
- 关键: 物理问题的解可以表示为模式展开的形式(——激发本征模式);
- 不是所有物理问题的解都可以表示成模式展开的形式.
- 核心：基函数或者模式展开(叠加原理)

例1 考虑无初始值问题

$$u_t - a^2 u_{xx} = 0, (t > 0, x > 0)$$
$$u|_{x=0} = A\cos(\omega t)$$

非齐次边界条件

解：为了方便，首先求解下列方程，然后取实部

$$u_t - a^2 u_{xx} = 0, (t > 0, x > 0)$$
$$u|_{x=0} = Ae^{\mathrm{i}\omega t}$$

设解为

$$u(x,t) = X(x)e^{\mathrm{i}\omega t}$$

$$a^2 X''(x) - \mathrm{i}\omega X(x) = 0$$
$$X(0) = A$$

$$X(x) = Ce^{\sqrt{\mathrm{i}\omega/a^2}\,x} + De^{-\sqrt{\mathrm{i}\omega/a^2}\,x}$$

$$X(x)=Ce^{\sqrt{\mathrm{i}\omega/a^2}x}+De^{-\sqrt{\mathrm{i}\omega/a^2}x}$$

$$=Ce^{(1+\mathrm{i})\sqrt{\omega/2a^2}x}+De^{-(1+\mathrm{i})\sqrt{\omega/2a^2}x}$$

$$C=0,\ D=A$$

$$u(x,t)=Ae^{-(1+\mathrm{i})\sqrt{\omega/2a^2}x}e^{\mathrm{i}\omega t}$$

取实部

$$u(x,t)=A\mathrm{Re}\left[e^{-(1+\mathrm{i})\sqrt{\omega/2a^2}x}e^{\mathrm{i}\omega t}\right]$$

$$=Ae^{-\sqrt{\omega/2a^2}x}\,\mathrm{Re}\left[e^{\mathrm{i}(\omega t-\sqrt{\omega/2a^2}x)}\right]$$

$$=Ae^{-\sqrt{\omega/2a^2}x}\cos\left(\omega t-\sqrt{\frac{\omega}{2a^2}}x\right)$$

不是简单的模式展开的形式

例2 求下列问题的稳态解

$$u_{tt} - a^2 u_{xx} = 0, (0 < x < l)$$

$$u|_{x=0} = 0; u|_{x=l} = A\cos(\omega t)$$

非齐次边界条件
无法分离变量

解：设解为

$$u(x,t) = X(x)\cos(\omega t)$$

$$X''(x) + (\omega / a)^2 X(x) = 0$$

$$X(0) = 0; X(l) = A$$

$$X(x) = C\sin\left(\frac{\omega}{a}x\right) + D\cos\left(\frac{\omega}{a}x\right) \Rightarrow D = 0; C = A\sin^{-1}\left(\frac{\omega}{a}l\right)$$

$$u(x,t) = A\sin^{-1}\left(\frac{\omega}{a}l\right)\sin\left(\frac{\omega}{a}x\right)\cos(\omega t)$$