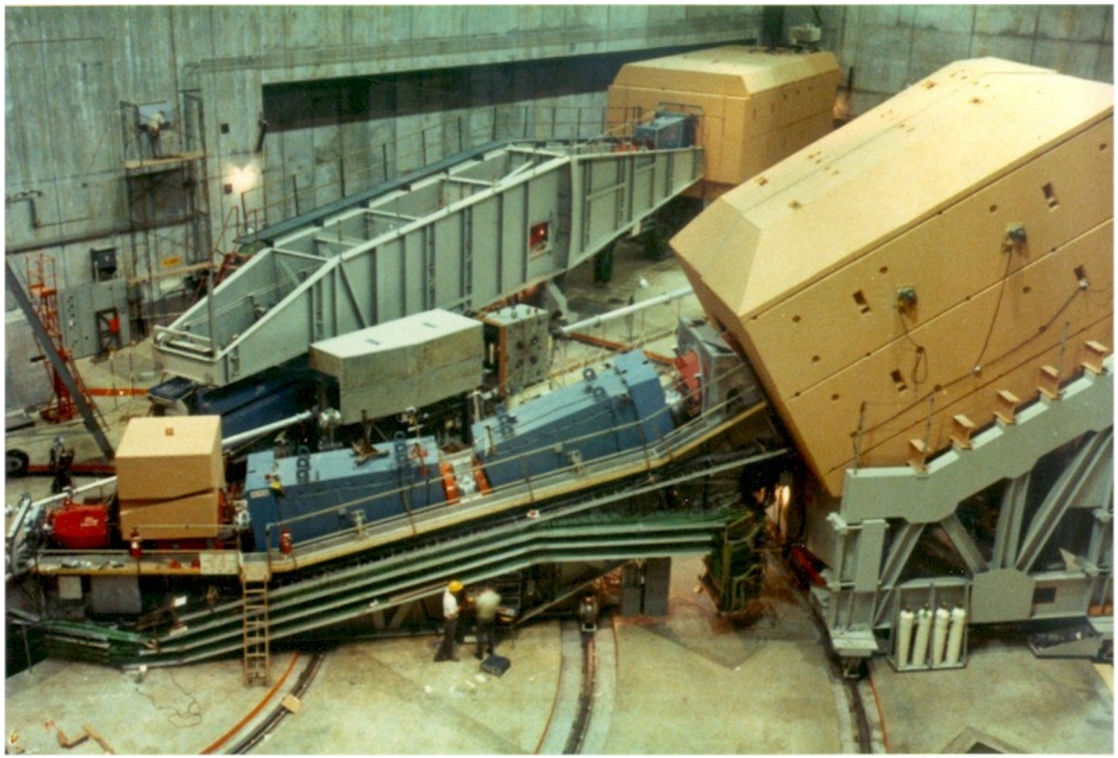
粒子物理学

第 5 章： 电子-质子弹性散射



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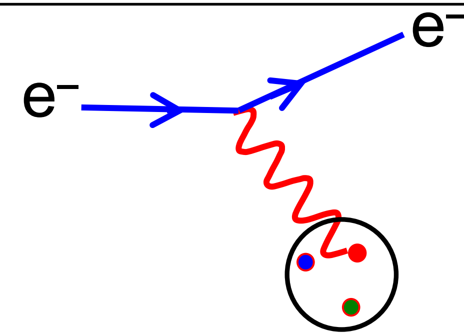
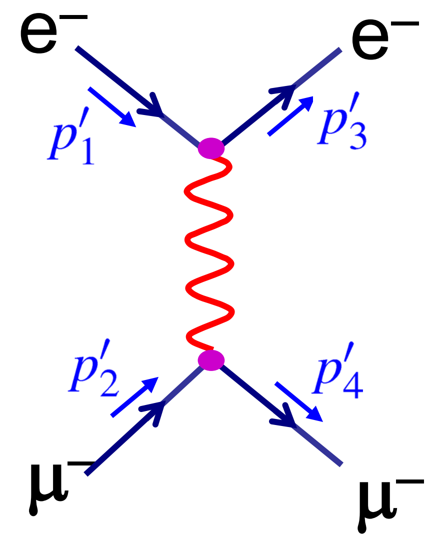
Based on M. Thomson’s notes

Electron-Proton Scattering



- 电子–质子 散射作为探测质子结构的探针
- 两大主题：
  - $e^-p \to e^-p$ 弹性散射（本节课）
  - $e\text{-}p \to e^-X$ 深度非弹性散射（下一节）



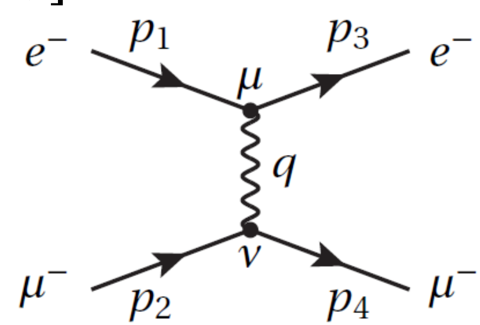
- 首先考虑点状粒子的散射，如 $e^-\mu^- \to e^-\mu^-$
  - 即 $(e^-q \to e^-q)$ 的QED部分
- 两种处理方法：
  - 从头开始QED计算

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3)\right] \qquad (1)$$

Calculate matrix element with trace techniques

$$-iM_{fi} = [\bar{u}(p_3 s_3)(ie\gamma^\mu)u(p_1 s_1)]\frac{-ig_{\mu\nu}}{q^2 + i\epsilon}[\bar{u}(p_4 s_4)(ie\gamma^\nu)u(p_2 s_2)]$$

$$M_{fi} = -e^2[\bar{u}(p_3 s_3)\gamma^\mu u(p_1 s_1)]\frac{1}{q^2 + i\epsilon}[\bar{u}(p_4 s_4)\gamma_\mu u(p_2 s_2)]$$



- 对上式做模的平方，并且需要对自旋态求和：

$$\overline{|M_{fi}|^2} = \frac{1}{4}\sum_{s_1 s_2}\sum_{s_3 s_4}|M_{fi}|^2$$

$$= \frac{e^4}{q^4}\frac{1}{4}\sum_{s_1 s_2}\sum_{s_3 s_4}\{[\bar{u}(p_3 s_3)\gamma^\mu u(p_1 s_1)][\bar{u}(p_4 s_4)\gamma_\mu u(p_2 s_2)]\}^\dagger$$
$$\cdot\{[\bar{u}(p_3 s_3)\gamma^\nu u(p_1 s_1)][\bar{u}(p_4 s_4)\gamma_\nu u(p_2 s_2)]\}$$

$$= \frac{e^4}{q^4}\frac{1}{2}\sum_{s_1 s_3}[\bar{u}(p_3 s_3)\gamma^\mu u(p_1 s_1)]^\dagger[\bar{u}(p_3 s_3)\gamma^\nu u(p_1 s_1)]$$
$$\cdot\frac{1}{2}\sum_{s_2 s_4}[\bar{u}(p_4 s_4)\gamma_\mu u(p_2 s_2)]^\dagger[\bar{u}(p_4 s_4)\gamma_\nu u(p_2 s_2)]$$

- 定义两个张量：

$$L_{\mu\nu}=\frac{1}{2}\sum_{s_1s_3}[\bar{u}(p_3s_3)\gamma^{\mu}u(p_1s_1)]^{\dagger}[\bar{u}(p_3s_3)\gamma^{\nu}u(p_1s_1)]$$

$$W_{\mu\nu}=\frac{1}{2}\sum_{s_2s_4}[\bar{u}(p_4s_4)\gamma_{\mu}u(p_2s_2)]^{\dagger}[\bar{u}(p_4s_4)\gamma_{\nu}u(p_2s_2)]$$

- 接下来分别计算两个张量。在此之前，先计算：

$$[\bar{u}(p_3s_3)\gamma^{\mu}u(p_1s_1)]^{\dagger}$$

$$=\left[u^{\dagger}(p_3s_3)\gamma^{0}\gamma^{\mu}u(p_1s_1)\right]^{\dagger}$$

$$=u^{\dagger}(p_1s_1)(\gamma^{\mu})^{\dagger}\gamma^{0}u(p_3s_3)$$

$$=\bar{u}(p_1s_1)\gamma^{\mu}u(p_3s_3)$$

- 因此：$L_{\mu\nu}=\frac{1}{2}\sum_{s_1s_3}[\bar{u}(p_1s_1)\gamma^{\mu}u(p_3s_3)][\bar{u}(p_3s_3)\gamma^{\nu}u(p_1s_1)]$

(optional) ME calculation with trace techniques

- 需要补充介绍一个化简：（自旋求和）

$$\sum_s u(ps)\,\bar{u}(ps) = u_1(p)\bar{u}_1(p) + u_2(p)\bar{u}_2(p)$$

$$u(ps) = \sqrt{E+m}\begin{pmatrix} \phi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi_s \end{pmatrix}, \text{with } \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{u}(ps) = u^\dagger(ps)\gamma^0 = \sqrt{E+m}\begin{pmatrix} \phi_s^T & \phi_s^T \dfrac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^\dagger}{E+m} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} \phi_s^T & -\phi_s^T \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \end{pmatrix}$$

$$\sum_s u(ps)\,\bar{u}(ps) = (E+m)\sum_s \begin{pmatrix} \phi_s\phi_s^T & -\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi_s\phi_s^T \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi_s\phi_s^T & -\dfrac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{(E+m)^2}\phi_s\phi_s^T \end{pmatrix}$$

- 利用 $\sum_s \phi_s\phi_s^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \boldsymbol{p}^2 = (E+m)(E-m)$:

$$\sum_s u(ps)\,\bar{u}(ps) = \begin{pmatrix} (E+m)I & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & (-E+m)I \end{pmatrix} = (\gamma^\mu p_\mu + mI) = \not{p} + m$$

(optional) ME calculation with trace techniques

- 开始利用矩阵的性质：

$$L_{\mu\nu}=\frac{1}{2}\sum_{s_1s_3}[\bar{u}(p_1s_1)]_j[\gamma^\mu]_{ji}[u(p_3s_3)]_i[\bar{u}(p_3s_3)]_n[\gamma^\nu]_{nm}[u(p_1s_1)]_m$$

此处全是数字

$$=\frac{1}{2}\sum_{s_1}[u(p_1s_1)]_m[\bar{u}(p_1s_1)]_j\sum_{s_3}[u(p_3s_3)]_i[\bar{u}(p_3s_3)]_n\,[\gamma^\mu]_{ji}[\gamma^\nu]_{nm}$$

$$=\frac{1}{2}[\not{p}_1+m]_{mj}[\not{p}_3+m]_{in}[\gamma^\mu]_{ji}[\gamma^\nu]_{nm}$$

$$L_{\mu\nu}=\frac{1}{2}\sum_{s_1s_3}[\bar{u}(p_1s_1)\gamma^\mu u(p_3s_3)][\bar{u}(p_3s_3)\gamma^\nu u(p_1s_1)]$$

$$\sum_s u(ps)\,\bar{u}(ps)=\not{p}+m$$

$$=\frac{1}{2}[\not{p}_1+m]_{mj}[\gamma^\mu]_{ji}[\not{p}_3+m]_{in}[\gamma^\nu]_{nm}$$

$$=\frac{1}{2}[(\not{p}_1+m)\gamma^\mu(\not{p}_3+m)\gamma^\nu]_{mm}$$

$$=\frac{1}{2}\mathrm{Tr}[(\not{p}_1+m)\gamma^\mu(\not{p}_3+m)\gamma^\nu]$$

- 类似的：

$$W_{\mu\nu}=\frac{1}{2}\mathrm{Tr}\big[(\not{p}_2+M)\gamma_\mu(\not{p}_4+M)\gamma_\nu\big]$$

(optional) ME calculation with trace techniques

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\text{odd number of } \gamma\text{s}) = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$$

- 开始计算迹：

$$L_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\not p_1 + m)\gamma^\mu(\not p_3 + m)\gamma^\nu]$$

$$= \frac{1}{2}[\mathrm{Tr}(\not p_1\gamma^\mu\not p_3\gamma^\nu) + \mathrm{Tr}(m\gamma^\mu\not p_3\gamma^\nu) + \mathrm{Tr}(\not p_1\gamma^\mu m\gamma^\nu) + \mathrm{Tr}(m^2\gamma^\mu\gamma^\nu)]$$

$$= \frac{1}{2}\left[4p_{1\rho}p_{3\sigma}(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) + 4m^2g^{\mu\nu}\right]$$

$$= 2\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}(m^2 - p_1p_3)\right]$$

- 用同样的方法计算$W_{\mu\nu}$：

$$W_{\mu\nu} = 2\left[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} + g_{\mu\nu}(M^2 - p_2p_4)\right]$$

(optional) ME calculation with trace techniques

$$g_{\mu\nu}g^{\mu\nu} = 4$$

$$p_1^\mu p_3^\nu g_{\mu\nu} = p_1 p_3$$

$$p_1^\mu p_3^\nu p_{2\mu} p_{4\nu} = (p_1 p_2)(p_3 p_4)$$

- 回到矩阵元:

$$\overline{|M_{fi}|^2} = \frac{4e^4}{q^4}\left[{p_1}^\mu {p_3}^\nu + {p_1}^\nu {p_3}^\mu + g^{\mu\nu}(m^2 - p_1 p_3)\right]$$

$$\left[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} + g_{\mu\nu}(M^2 - p_2 p_4)\right]$$

$$\overline{|M_{fi}|^2} = \frac{8e^4}{q^4}\left[(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) + M^2 p_1 p_3 + m^2 p_2 p_4 + 2m^2M^2\right]$$

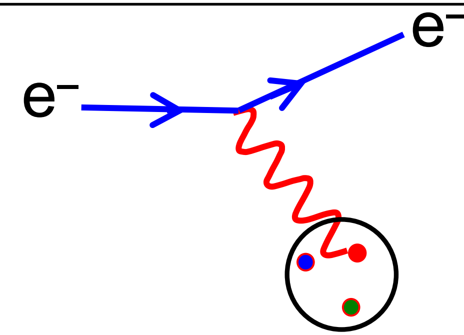
- 取极端相对论条件 $m \approx M \approx 0$:

$$\overline{|M_{fi}|^2} = \frac{8e^4}{q^4}\left[(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3)\right]$$

$$q^2 = t = (p_1 - p_3)^2 \approx -2p_1p_3, \qquad s \approx 2p_1p_2 = 2p_3p_4, \qquad u \approx 2p_1p_4 = 2p_2p_3$$

$$\overline{|M_{fi}|^2} = 2e^4\left(\frac{s^2 + u^2}{t^4}\right)$$

Electron-Proton Scattering



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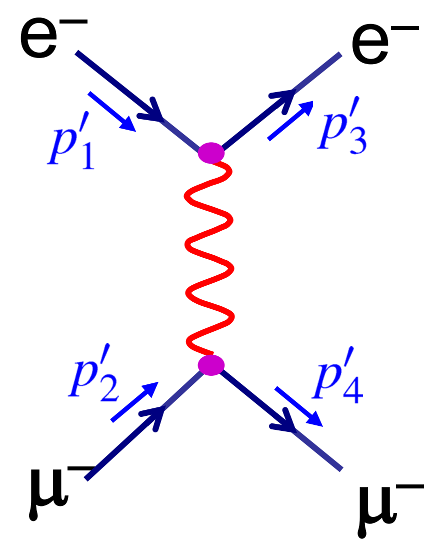

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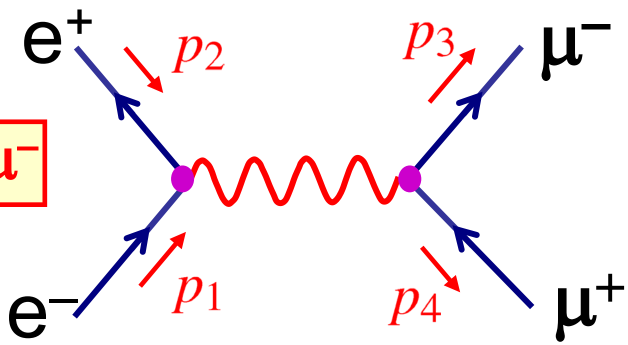
  - 从 $e^-e^+ \to \mu^-\mu^+$ 出发，利用“交叉对称性”得到 $e^-\mu^- \to e^-\mu^-$ 的矩阵元（附录I）

Appendix I : Crossing Symmetry

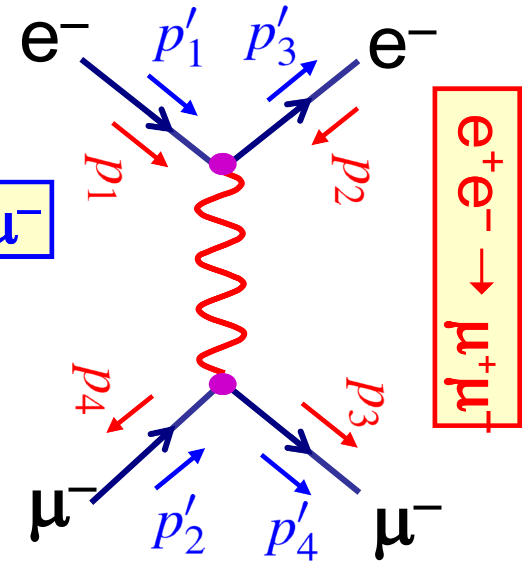
➢ 基于已经得到的$e^-e^+ \to \mu^-\mu^+$洛伦兹不变矩阵元，

- “旋转”得到$e^-\mu^- \to e^-\mu^-$相应的图，采用交叉对称性原理写出对应的矩阵元

$e^+e^- \to \mu^+\mu^-$



$e^-\mu^- \to e^-\mu^-$



★ **Transformation:**

$$p_1 \to p'_1;\ p_2 \to -p'_3;\ p_3 \to p'_4;\ p_4 \to -p'_2$$

Changes spin averaged ME for $\underset{p_1\ p_2\quad\ p_3\ p_4}{e^-e^+ \to \mu^-\mu^+}$ ➡ $\underset{p'_1\ p'_2\quad\ p'_3\ p'_4}{e^-\mu^- \to e^-\mu^-}$

•**Take ME for $e^+e^- \to \mu^+\mu^-$ (附录) and apply crossing symmetry:**

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2} \quad ➡ \quad \boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p'_1.p'_4)^2+(p'_1.p'_2)^2}{(p'_1.p'_3)^2}} \qquad \textbf{(App:1)}$$

Electron-Proton Scattering

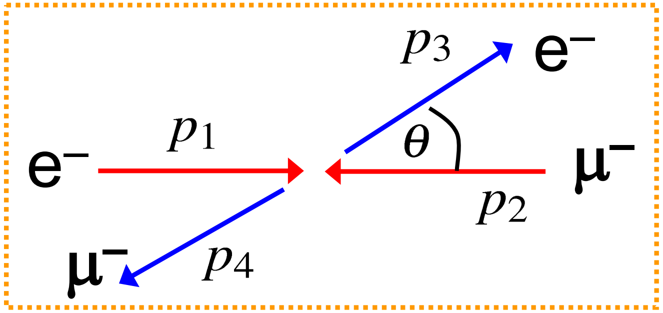
$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_4)^2+(p_1.p_2)^2}{(p_1.p_3)^2} \quad \textbf{(2)}$$

$$\equiv 2e^4\left(\frac{s^2+u^2}{t^2}\right)$$

- **质心系** $p_1=(E,0,0,E)$ $p_2=(E,0,0,-E)$

$$p_3=(E,E\sin\theta,0,E\cos\theta)$$

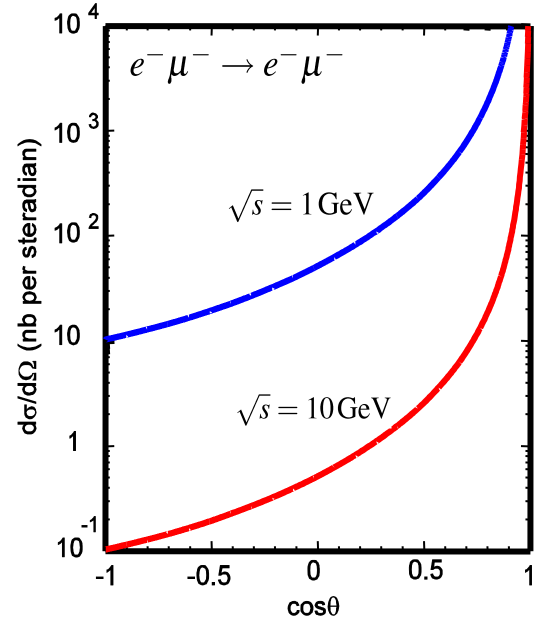
$$p_4=(E,-E\sin\theta,0,-E\cos\theta)$$



得到 $p_1.p_2=2E^2$; $p_1.p_3=E^2(1-\cos\theta)$; $p_1.p_4=E^2(1+\cos\theta)$

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{E^4(1+\cos\theta)^2+4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2 s}\langle|M_{fi}|^2\rangle=\frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$



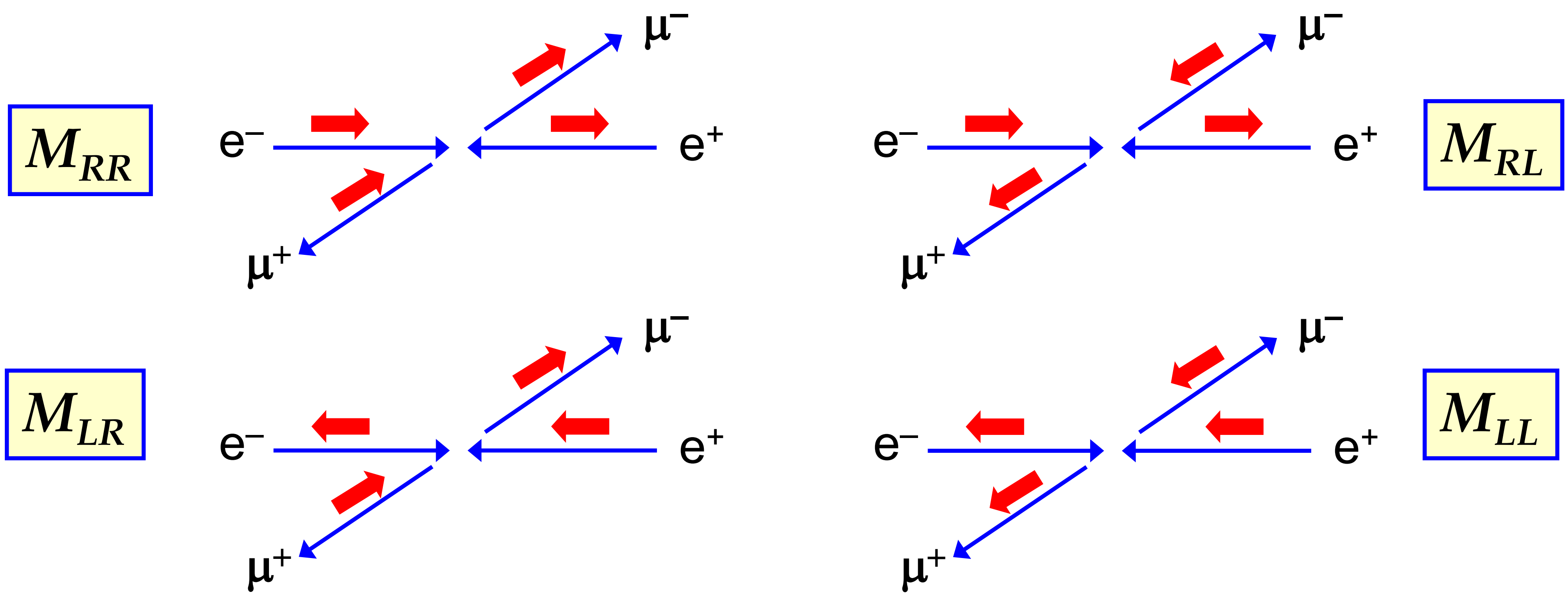
- **传播子得到的分母** $-ig_{\mu\nu}/q^2$

这里 $q^2=(p_1-p_3)^2=E^2(1-\cos\theta)$

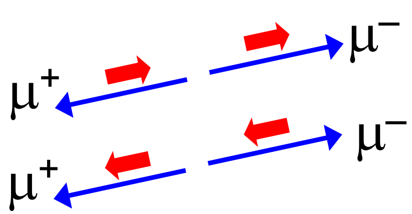
$q^2\to 0$ **时，截面趋于无穷大**

回顾上节课：Electron Positron Annihilation cont.

- 对于$ee^- \rightarrow \mu^+\mu^-$ 现在只需要考虑4个矩阵元（R,L代表e- mu-的helicity）



- 之前，推导出了允许螺旋度组合的缪子流：



$$\begin{array}{lcccc} \mu_R^- \mu_L^+ & : & \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) & = & 2E(0, -\cos\theta, i, \sin\theta) \\ \mu_L^- \mu_R^+ & : & \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) & = & 2E(0, -\cos\theta, -i, \sin\theta) \end{array}$$

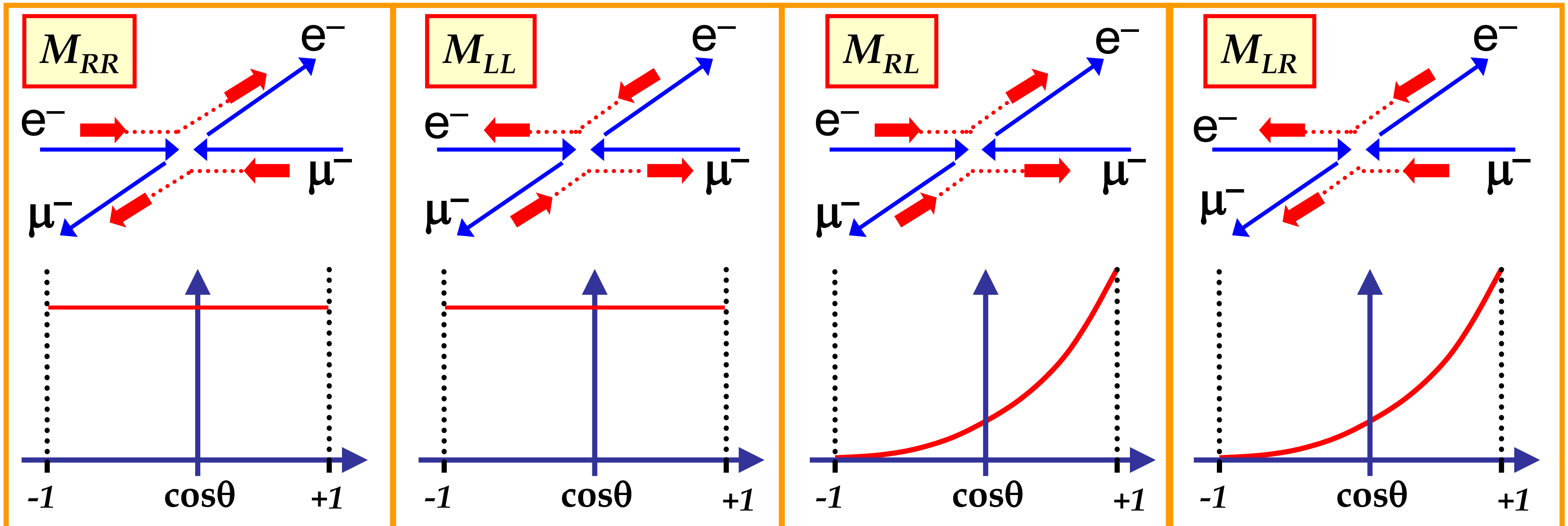
- 现在需要考虑电子流

Electron-Proton Scattering

- 分子上的角度依赖： $\dfrac{d\sigma}{d\Omega} = \dfrac{e^4}{8\pi^2 s}\dfrac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$

因子 $1+\frac{1}{4}(1+\cos\theta)^2$ 反映QED的螺旋度结构（本质是手征）

- 16种可能的螺旋度组合中只有4种非零：



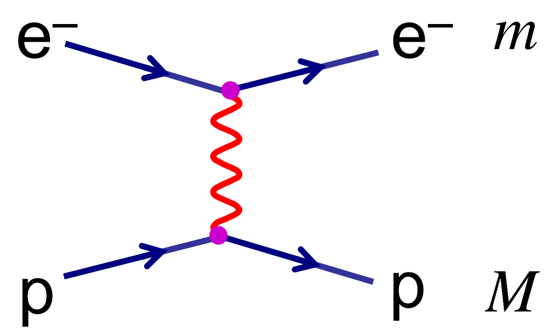
$S_z = 0$ $S_z = +1$ $S_z = -1$

$\dfrac{d\sigma}{d\Omega} \propto 1$ i.e. no preferred polar angle

$\dfrac{d\sigma}{d\Omega} \propto \dfrac{1}{4}(1+\cos\theta)^2$ spin 1rotation again

Electron-Proton Scattering

- **上述计算的截面适用于，在（忽略电子和缪子质量的）相对论极限下，两个自旋1/2狄拉克粒子（点状粒子）的散射**
- **矩阵元公式为：** $\langle|M_{fi}|^2\rangle = 2e^4 \frac{(p_1.p_4)^2+(p_1.p_2)^2}{(p_1.p_3)^2}$
  - “深度非弹散射”也将使用该公式描述电子与质子中夸克的散射（下节课）
- **首先考虑电子与质子(复合粒子)的散射：**
  - 如何知道质子不是基本“点”粒子？
- **先求矩阵元表达式（详见附录）**

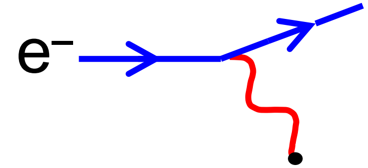


$$\boxed{\langle|M_{fi}|^2\rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1.p_2)(p_3.p_4)+(p_1.p_4)(p_2.p_3)-(p_1.p_3)M^2-(p_1.p_4)m^2+2m^2M^2\right]} \quad (3)$$

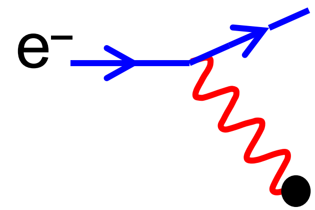
Probing the Structure of Proton

➢ 在 $e^-p \to e^-p$ 散射中，虚光子与质子的相互作用本质强烈依赖其波长

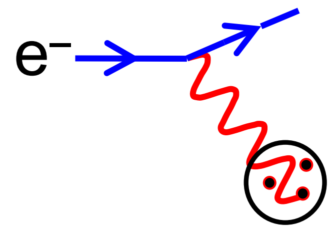
- 在**极低的**电子能量时 $\lambda \gg r_p$ ：
  等效于对一个无自旋“点粒子”散射



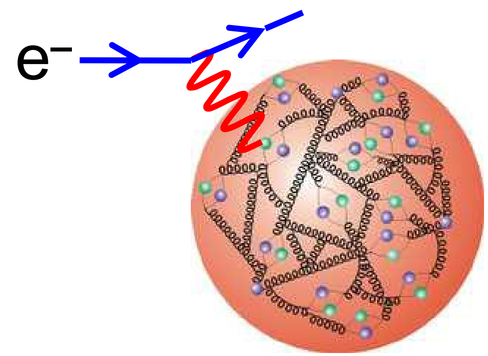
- 在**较低的**电子能量时 $\lambda \sim r_p$ ：
  等效于对一个扩散的带电体散射



- 在**较高的**电子能量时 $\lambda < r_p$ ：
  波长短到足够分辨子结构，与组分夸克散射



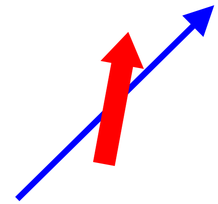
- 在**极高的**电子能量时 $\lambda \ll r_p$ ：
  质子表现为夸克和胶子的海洋



（回顾）Helicity

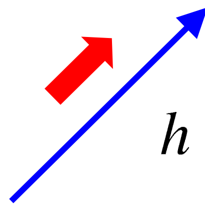
- **粒子自旋延飞行方向的分量是好量子数：** $[\hat{H},\hat{S}.\hat{p}]=0$

- **定义“粒子自旋延其飞行方向的分量”为 螺旋度：** $h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$



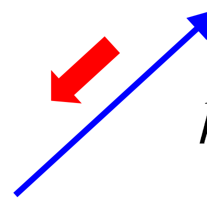
- **自旋延任何轴的测量分量只有两个值 $\pm$1/2**
  - 因此自旋1/2粒子的螺旋度算符的本征值为：± 1

Often termed: **“right-handed”**



$h=+1$

“left-handed”

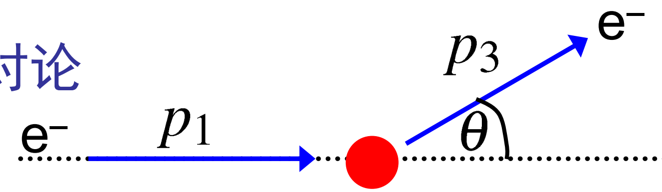


$h=-1$

- **注意：分“右手”手性 和“左手”手性 的螺旋度**
- **后续课程将讨论“右手”和“左手”手征(CHIRAL) 本征态**
  - 只在光速极限$(v \approx c)$时， 螺旋度本征态与手征本征态相同

Rutherford Scattering Revisited

- **卢瑟福散射：**
  - 质子反冲可以忽略的低能极限, 且电子为非相对论



(neglect proton recoil)

- **从右手和左手螺旋度粒子旋量出发**

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad \begin{aligned} N &= \sqrt{E+m}; \\ s &= \sin(\theta/2); \quad c = \cos(\theta/2) \end{aligned}$$

- **写成形式为：** $\alpha = \dfrac{|\vec{p}|}{E+m_e}$

非相对论极限： $\alpha \to 0$

极端相对论极限： $\alpha \to 1$

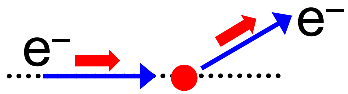
$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

可能得初态和末态电子旋量：

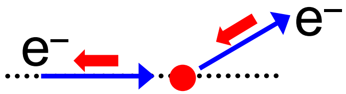
$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

Rutherford Scattering Revisited

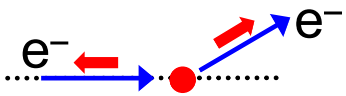
- 考虑所有四种可能电子流，即，螺旋度 R→R, L→L, L→R, R→L



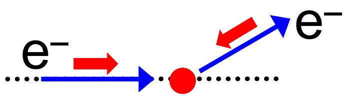
$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \qquad (4)$$



$$\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \qquad (5)$$



$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \qquad (6)$$



$$\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \qquad (7)$$

- 相对论极限(α=1)，即 E≫m，(6) 和 (7) 都是0，只有 R→R 和 L→L 组合非零

- 在非相对论极限($|\vec{p}| \ll E$)下，α=0

$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = \bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (2m_e)\left[c, 0, 0, 0\right]$$

$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = -\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (2m_e)\left[s, 0, 0, 0\right]$$

All four electron helicity combinations have non-zero Matrix Element

i.e. Helicity eigenstates ≠ Chirality eigenstates

Rutherford Scattering Revisited

- 初态和末态质子旋量：（假设无反冲） $u_{\uparrow}(0)=\sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $u_{\downarrow}(0)=\sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

粒子静止时的狄拉克方程解

给出质子流： $j_{p\uparrow\uparrow}=j_{p\downarrow\downarrow}=2M_p(1,0,0,0)$ $\quad j_{p\uparrow\downarrow}=j_{p\downarrow\uparrow}=0$

$$\mathcal{M}_{fi}=\frac{e^2}{q^2}j_e\cdot j_p$$

$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1)=\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1)=(2m_e)[c,0,0,0]$$

$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1)=-\bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1)=(2m_e)[s,0,0,0]$$

- 自旋平均的矩阵元对八种允许的螺旋度态求和 $\langle|M_{fi}^2|\rangle=\frac{1}{4}\frac{e^4}{q^4}(16M_p^2m_e^2)(4c^2+4s^2)=\frac{16M_p^2m_e^2e^4}{q^4}$

Rutherford Scattering Revisited

- 初态和末态质子旋量：（假设无反冲）

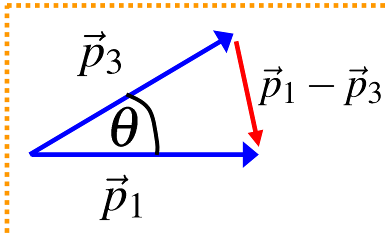
$$u_\uparrow(0)=\sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad u_\downarrow(0)=\sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

粒子静止时的狄拉克方程解

给出质子流： $j_{p\uparrow\uparrow}=j_{p\downarrow\downarrow}=2M_p(1,0,0,0)\qquad j_{p\uparrow\downarrow}=j_{p\downarrow\uparrow}=0$

- 自旋平均的矩阵元对八种允许的螺旋度态求和

$$\langle|M_{fi}^2|\rangle=\frac{1}{4}\frac{e^4}{q^4}(16M_p^2m_e^2)\underline{(4c^2+4s^2)}=\frac{16M_p^2m_e^2e^4}{q^4}$$



其中 $q^2=(p_1-p_3)^2=(0,\vec{p}_1-\vec{p}_3)^2=-4|\vec{p}|^2\sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle=\frac{M_p^2m_e^2e^4}{|\vec{p}|^4\sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- 实验室系的微分散射截面（第二节课导出）

$$\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2|M_{fi}|^2 \qquad (8)$$

Rutherford Scattering Revisited

- 因为电子为非相对论，所以 $E \approx me \ll Mp$，可以忽略公式(8) 分母中的 E_1

$$\Rightarrow \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

  - 写出 $e^2=4\pi\alpha$ 且电子的动能为 $E_K = p^2/2m_e$

$$\Rightarrow \quad \boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4\theta/2}} \qquad (9)$$

- 这是常规的卢瑟福散射截面表达式。可通过考虑非相对论粒子在质子的静态库伦势能$V(\vec{r})$中的散射推导出，不需要考虑电子或质子内秉磁矩导致的相互作用
- 由此可以得出，在非相对论极限下，只需要考虑粒子的电荷相互作用

Mott Scattering Cross Section

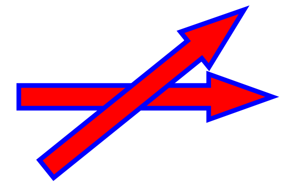
- 卢瑟福散射：靶的反弹可忽略且被散射粒子为非相对论（$E_K \ll m$）
- 莫特散射：靶的反弹可忽略且被散射粒子为相对论（即，忽略电子质量）
  - 在此条件下，电子流方程(4)和(6)变成：

$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = 2E\,[c, s, -is, c]$$
$$\bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = E\,[0, 0, 0, 0]$$

相对论效应 ➡ 电子“螺旋度守恒”

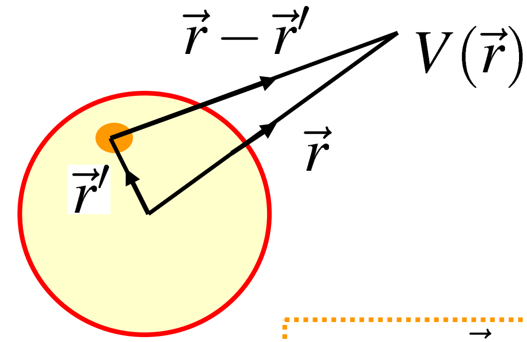
  - 结果为：

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \underbrace{\frac{\alpha^2}{4E^2\sin^4\theta/2}}_{\text{卢瑟福公式，其中 } E_K = E \; (E \gg m_e)} \underbrace{\cos^2\frac{\theta}{2}}_{\text{初末态电子波函数的重叠 Just QM of spin ½}} \qquad (10)$$



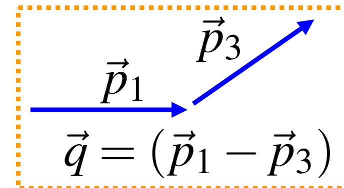
- 注意：该表达式也可通过电子与固定点静态势$V(\vec{r})$的散射推导出
  - 相互作用的本质是电性而非磁性的(spin-spin)

Form Factors



- 描述电子在扩展分布电荷导致的静态势下的散射
- 距中心为 $\vec{r}$ 处的势为：

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{其中} \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1$$



- 一阶微扰论的矩阵元为：

$$\begin{aligned} M_{fi} &= \langle\psi_f|V(\vec{r})|\psi_i\rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} \mathrm{d}^3\vec{r} \\ &= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' \mathrm{d}^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' \mathrm{d}^3\vec{r} \end{aligned}$$

- 固定 $\vec{r}$，并做 $\vec{R} = \vec{r} - \vec{r}'$ 替换后对 $\mathrm{d}^3\vec{r}$ 积分

$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} \mathrm{d}^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

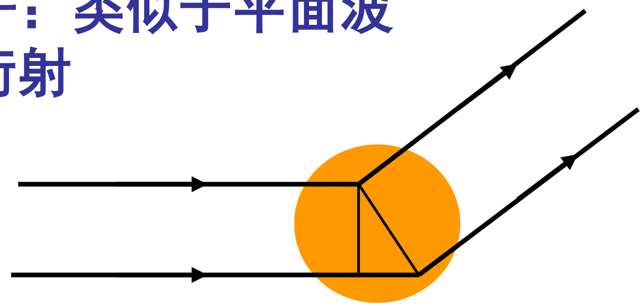
- 等价于**点源**散射矩阵元乘以**形状因子**

$$\boxed{F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} \mathrm{d}^3\vec{r}}$$

Form Factors

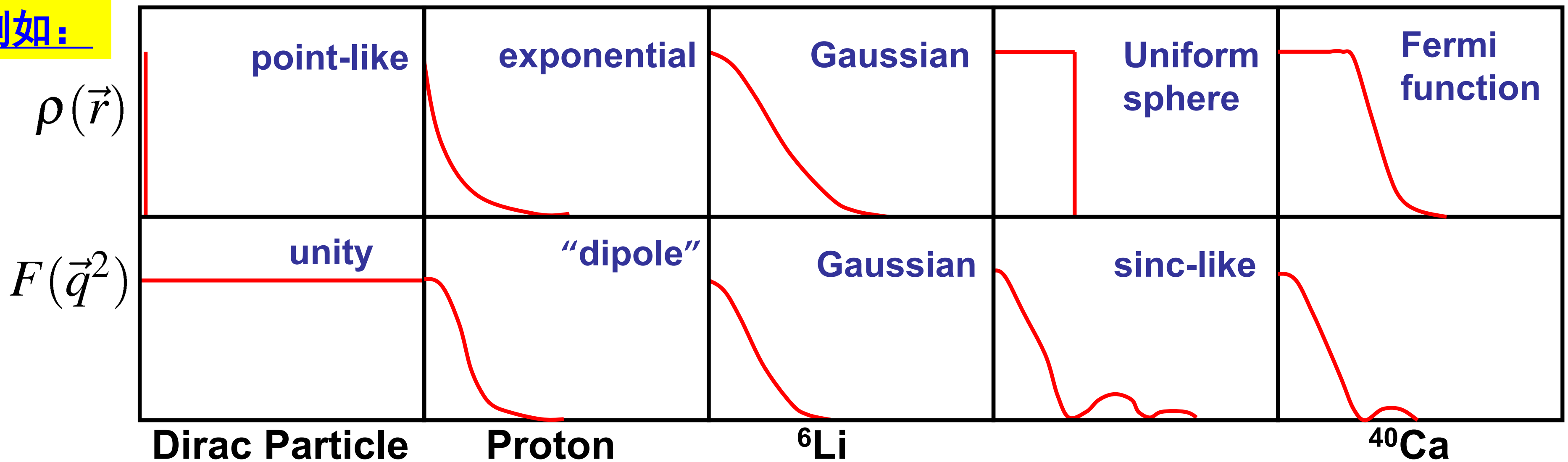
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} \rightarrow \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}\left|F(\vec{q}^2)\right|^2$$

- 形状因子：类似于平面波的光学衍射



- 有限尺寸的散射中心导致“空间不同点散射的平面波”的相位差
- 如果波长与被散射体尺寸相当，所有波同相位且 $F(\vec{\mathbf{q}}^2)=1$

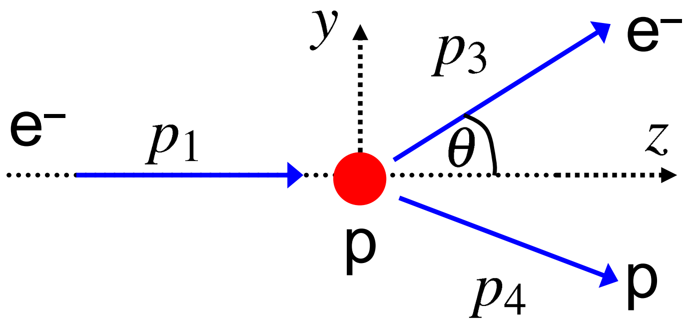
例如：



注意：对于点电荷形状因子是单位为一

Point-like Electron-Proton Elastic Scattering

- 考虑质子有反冲的一般情况，即$E_1 \gg m_e$



$$
\begin{aligned}
p_1 &= (E_1, 0, 0, E_1) \\
p_2 &= (M, 0, 0, 0) \\
p_3 &= \boxed{(E_3, 0, E_3 \sin\theta, E_3 \cos\theta)} \\
p_4 &= (E_4, \vec{p}_4)
\end{aligned}
$$

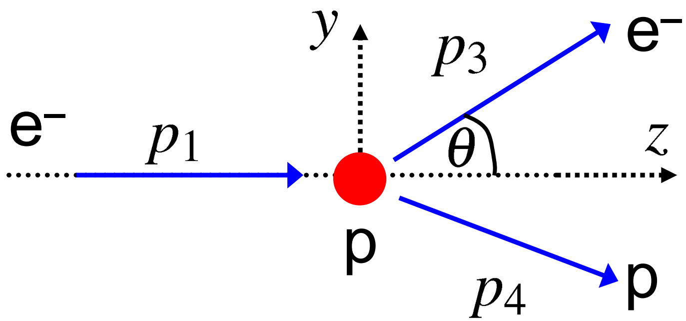
- $m=m_e=0$的情况下，由方程**(3)**可以得到该过程的矩阵元：

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \quad \textbf{(11)}$$

$$\boxed{\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2 \right]}$$

Point-like Electron-Proton Elastic Scattering

- 考虑质子有反冲的一般情况，即$E_1 \gg m_e$



$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) \\ p_2 &= (M, 0, 0, 0) \\ p_3 &= \boxed{(E_3, 0, E_3 \sin\theta, E_3\cos\theta)} \\ p_4 &= (E_4, \vec{p}_4) \end{aligned}$$

- $m=m_e=0$的情况下，由方程(3)可以得到该过程的矩阵元：

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1.p_2)(p_3.p_4)+(p_1.p_4)(p_2.p_3)-(p_1.p_3)M^2\right] \quad \textbf{(11)}$$

- 实验上不观测质子, 利用四动量守恒消去 $p_4=p_1+p_2-p_3$

- 不含p_4的标量积：$p_1.p_2 = E_1 M \quad p_2.p_3 = E_3 M \quad p_1.p_3 = E_1E_3(1-\cos\theta)$

包含p_4的表示为：

$$\begin{aligned} p_3.p_4 &= p_3.p_1 + p_3.p_2 - \cancel{p_3.p_3} = E_1E_3(1-\cos\theta) + E_3 M \\ p_1.p_4 &= \boxed{\cancel{p_1.p_1}} + p_1.p_2 - p_1.p_3 = E_1 M - E_1E_3(1-\cos\theta) \end{aligned}$$

$$\boxed{p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0}$$ **i.e. neglect** m_e

Point-like Electron-Proton Elastic Scattering

- 显式替换公式(11)的标量积后，得到

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{8e^4}{(p_1-p_3)^4} M E_1 E_3 \left[(E_1-E_3)(1-\cos\theta)+M(1+\cos\theta)\right] \\ &= \frac{8e^4}{(p_1-p_3)^4} 2M E_1 E_3 \left[(E_1-E_3)\sin^2(\theta/2)+M\cos^2(\theta/2)\right] \qquad \textbf{(12)}\end{aligned}$$

得到表达式 $q^4=(p_1-p_3)^4$ 和 (E_1-E_3)

$$q^2=(p_1-p_3)^2=\cancel{p_1^2}+\cancel{p_3^2}-2p_1.p_3=-2E_1E_3(1-\cos\theta) \qquad \textbf{(13)}$$

注意: $q^2<0$ $\boxed{\text{Space-like}}$

$$=-4E_1E_3\sin^2\theta/2 \qquad \textbf{(14)}$$

- 对于 (E_1-E_3)，利用 $q.p_2=(p_1-p_3).p_2=M(E_1-E_3)$

再利用

$$q=(p_1-p_3)=(p_4-p_2)$$

$$\begin{aligned}(q+p_2)^2 &= p_4^2 \\ q^2+p_2^2+2q.p_2 &= p_4^2 \\ q^2+M^2+2q.p_2 &= M^2\end{aligned}$$

$$\Rightarrow \quad q.p_2 = -q^2/2$$

$$\boxed{E_1-E_3=-\frac{q^2}{2M}}$$

Point-like Electron-Proton Elastic Scattering

- 因此转移到质子的能量为： $$\boxed{E_1 - E_3 = -\frac{q^2}{2M}}$$ (15)

  因为 q^2 总为负 E_1-E_3>0，所有散射后的电子能量总是低于入射电子

- 联合公式(11)，(12), (14), (15)：

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2\right]$$

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} 2ME_1E_3\left[(E_1 - E_3)\sin^2(\theta/2) + M\cos^2(\theta/2)\right]$$

$$q^2 = (p_1 - p_3)^2 = -4E_1E_3\sin^2\theta/2$$

Point-like Electron-Proton Elastic Scattering

- 因此转移到质子的能量为： $$\boxed{E_1 - E_3 = -\frac{q^2}{2M}} \qquad (15)$$

因为 q^2 总为负 $E_1-E_3>0$，所有散射后的电子能量总是低于入射电子

- 联合公式(11)，(12), (14), (15)：

$$\langle|M_{fi}|^2\rangle = \frac{8e^4}{16E_1^2E_3^2\sin^4\theta/2}2ME_1E_3\left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2\right]$$

$$E \gg m_e \quad = \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]$$

- 对于 $E \gg m_e$，有 (见第二课)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)} \qquad (16)$$

Interpretation

- 假设狄拉克自旋1/2点粒子，推导得 $e^-p \to e^-p$ 弹性散射的微分散射截面

如何理解该方程？

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

对比

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$$

要点提醒：Mott截面等效于自旋1/2电子对于固定**静电势**的散射。

此处 E_3/E_1 是由于质子反冲

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\boxed{\frac{E_3}{E_1}}\left(\cos^2\theta/2 \underbrace{- \frac{q^2}{2M^2}\sin^2\theta/2}_{\substack{\text{磁相互作用：}\\ \text{由自旋-自旋相互作用导致}}}\right)$$

- 新的项：$\propto \sin^2\frac{\theta}{2}$ ⟺ 磁相互作用：由自旋-自旋相互作用导致

Interpretation

- **上述微分截面只依赖单一参数**
  - 对于散射角位θ的电子，q^2 和能量 E_3 都被运动学固定了

提醒：

$$q^2 = -2E_1E_3(1-\cos\theta) \quad \textbf{(13)}$$

$$E_1 - E_3 = -\frac{q^2}{2M} \quad \textbf{(15)}$$

•**Equating (13) and (15)**

$$-2M(E_1-E_3) = -2E_1E_3(1-\cos\theta)$$

$$\Rightarrow \quad \frac{E_3}{E_1} = \frac{M}{M+E_1(1-\cos\theta)}$$

•**Substituting back into (13):**

$$\Rightarrow \quad q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M+E_1(1-\cos\theta)}$$

Interpretation

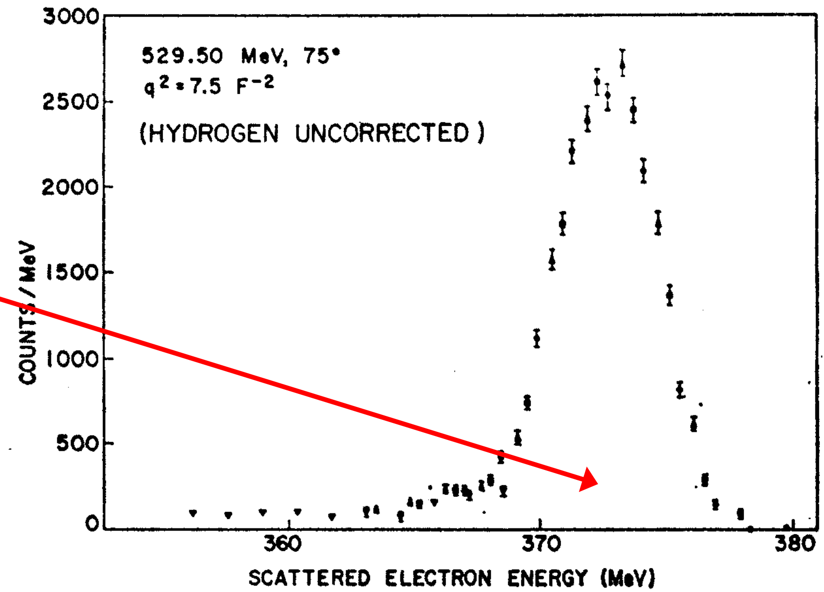
- 例如，束流能量为E_{beam}= 529.5 MeV的$e^-p \to e^-p$。测量 θ = 75° 处的被散射电子

对于**弹性**散射预期：

E.B.Hughes et al., Phys. Rev. 139 (1965) B458

$$E_3 = \frac{ME_1}{M + E_1(1-\cos\theta)}$$

$$E_3 = \frac{938\times 529}{938+529(1-\cos 75^\circ)} = 373\ \text{MeV}$$



通过能量确定该散射为弹性的，
也得到四动量转移的平方

$$|q^2| = \frac{2\times 938\times 529^2(1-\cos 75^\circ)}{938+529(1-\cos 75^\circ)} = 294\,\text{MeV}^2$$

Elastic Scattering from a Finite Size Proton

- **一般地，质子的有限尺寸可以通过引入2个结构函数来描述**
  - $G_E(q^2)$：与质子的电荷分布有关；$G_M(q^2)$：与质子的磁矩分布有关
  - 将公式**(16)**扩展为 **ROSENBLUTH FORMULA**

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

其中洛伦兹不变量：

$$\tau=-\frac{q^2}{4M^2}>0$$

- 与之前不同，这里的形状因子是q^2而不是$|\vec{q}|^2$的函数，不能被简单认为是电荷和磁矩分布的傅里叶变换
  - **但是利用** $q^2=(E_1-E_3)^2-\vec{q}^2$ **和公式(15)，得到**

$$-\vec{q}^2=q^2\left[1-\left(\frac{q}{2M}\right)^2\right]$$

$$E_1-E_3=-\frac{q^2}{2M}$$

$$\boxed{\text{因此，对于 } \frac{q^2}{4M^2}\ll 1 \text{ ： } q^2\approx-\vec{q}^2 \text{ 和 } G(q^2)\approx G(\vec{q}^2)}$$

Elastic Scattering from a Finite Size Proton

- 因此在 $q^2/4M^2 \ll 1$ 极限下，结构函数可以被诠释为电荷和磁矩分布的傅里叶变换

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \rho(\vec{r}) \mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \mu(\vec{r}) \mathrm{d}^3\vec{r}$$

- 注意在推导Rosenbluth公式时，
  假设质子为自旋1/2的狄拉克粒子，即 $\vec{\mu} = \frac{e}{M}\vec{S}$

- 但是，质子磁矩的实验测量值大于
  （按照点状狄拉克粒子的）理论预期： $\vec{\mu} = 2.79\frac{e}{M}\vec{S}$

因此对于质子 $G_E(0) = \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1 \quad G_M(0) = \int \mu(\vec{r}) \mathrm{d}^3\vec{r} = \mu_p = +2.79$

- 反常磁矩表明质子并非点状粒子！

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Rosenbluth公式可以表达为 $$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_0\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}+2\tau G_M^2\tan^2\frac{\theta}{2}\right)$$

其中 $$\left(\frac{d\sigma}{d\Omega}\right)_0=\frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

即，**Mott** 截面包含质子反冲。
它对应来自零自旋为质子的散射

- q^2 极低时： $\tau=-q^2/4M^2\approx 0$

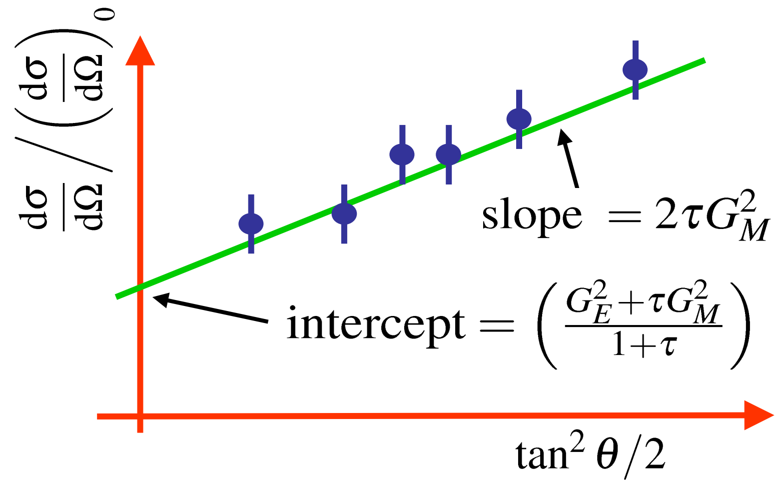
$$\frac{d\sigma}{d\Omega}\Big/\left(\frac{d\sigma}{d\Omega}\right)_0\approx G_E^2(q^2)$$

- 高 q^2 时： $\tau\gg 1$

$$\frac{d\sigma}{d\Omega}\Big/\left(\frac{d\sigma}{d\Omega}\right)_0\approx\left(1+2\tau\tan^2\frac{\theta}{2}\right)G_M^2(q^2)$$

➢ **一般情况则同时依赖两个结果函数！**

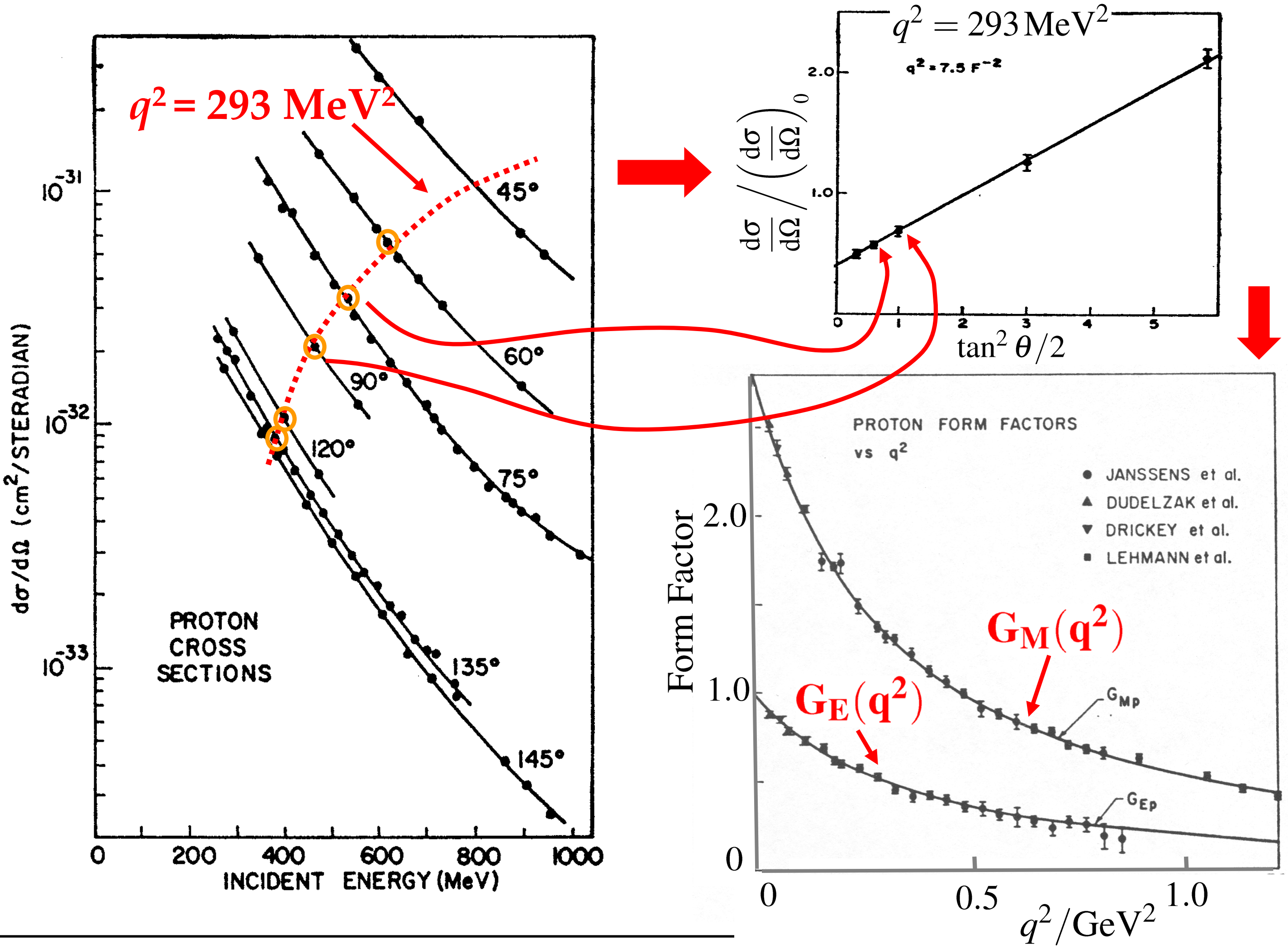
- q²固定时散射截面随角度依赖的依赖可以区分二者



Measuring $G_E(q^2)$ and $G_M(q^2)$

- **举例：** E_{beam} = 529.5 MeV 的 $e^- p \to e^- p$
  - 调整电子束流能量以获得具体的 q^2

E.B.Hughes et al., Phys. Rev. 139 (1965) B458

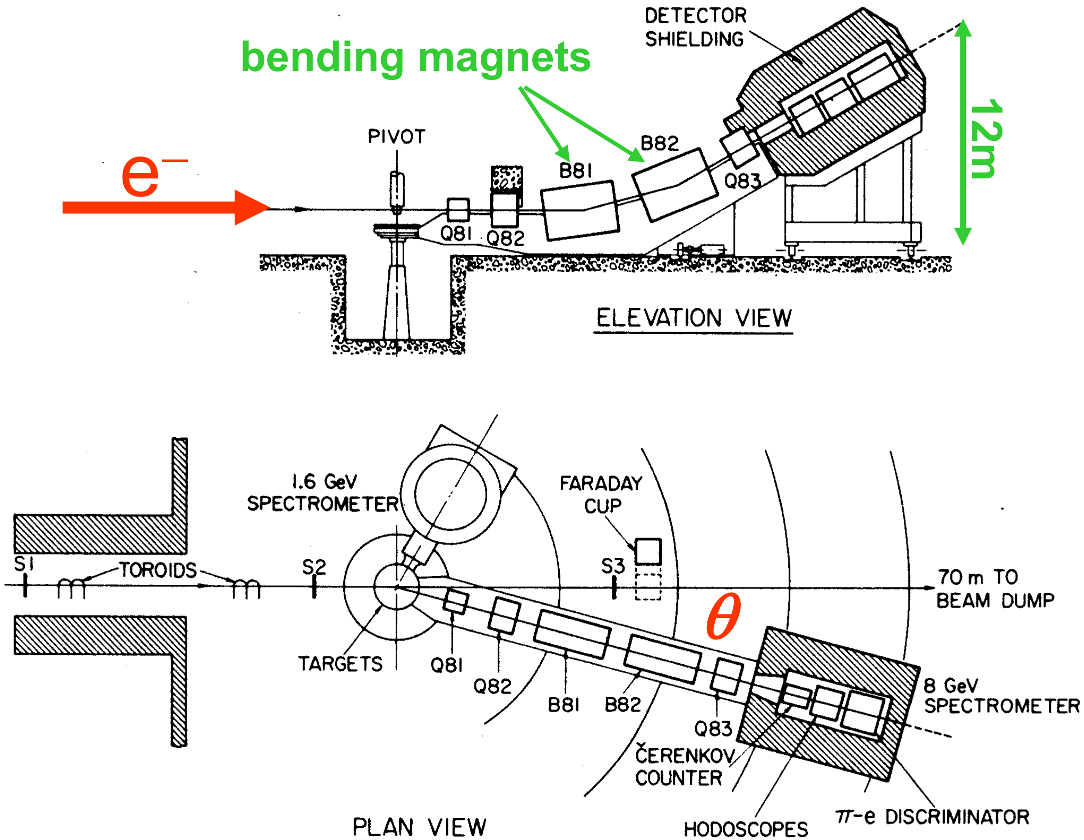


注意：

- 实验发现$G_M(q^2)$ = $2.79G_E(q^2)$，即，电和磁的形状因子具有相同分布

Higher Energy Electron-Proton Scattering

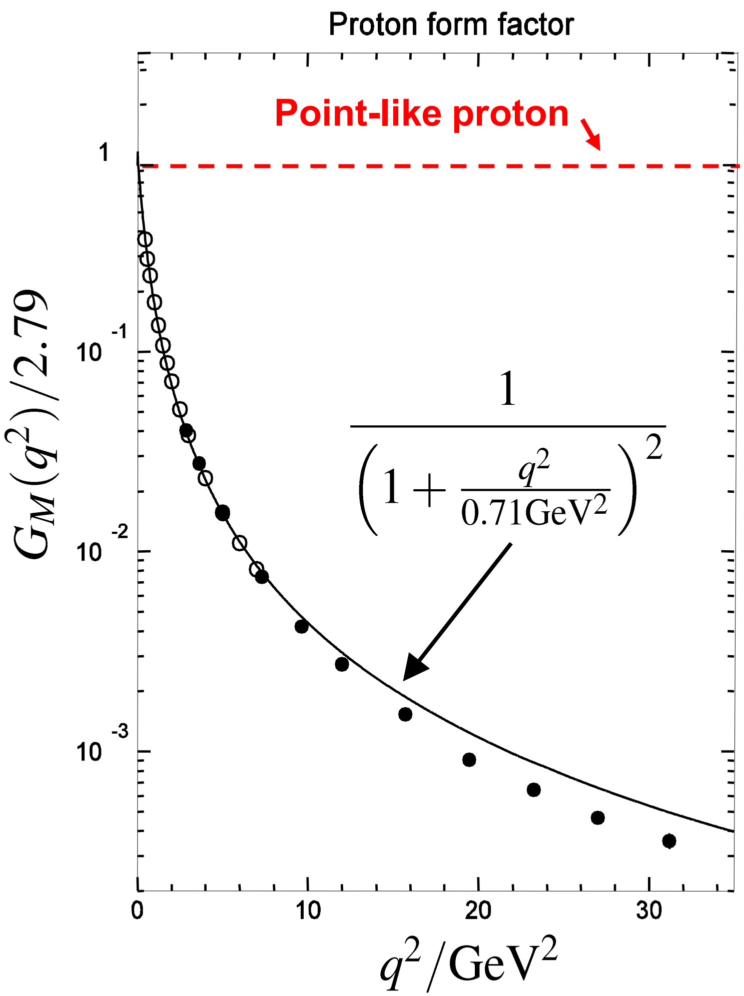
- SLAC的LINAC实验的电子束流：$5 < E_{beam} < 20$ GeV
  - 探测器（谱仪），测量散射后的电子



High q^2 → Measure $G_M(q^2)$

P.N.Kirk et al., Phys Rev D8 (1973) 63

High q^2 Results



R.C.Walker et al., PRD 49 (1994) 5671
A.F.Sill et al., PRD 48 (1993) 29

★形状因子随q²迅速下降=>质子并非点状

•数据很好地符合 "偶极形式":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

★傅里叶变换后，发现电荷和磁矩分布满足

$$\boxed{\rho(r) \approx \rho_0 e^{-r/a}} \quad \text{with} \quad a \approx 0.24\ \mathrm{fm}$$

对应电荷的分布均方根差RMS为电荷半径 $r_{rms} \approx 0.8\ \mathrm{fm}$

★虽然有迹象，但是不能说明质子是复合粒子

★注意：

- ★ 目前只考虑弹性散射;
- ★ 下节课讲深度非弹性散射

Summary: Elastic Scattering

- 相对论电子在点状狄拉克型质子的弹性散射：

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2 \sin^4\theta/2}}_{\text{Rutherford}} \underbrace{\frac{E_3}{E_1}}_{\text{Proton recoil}} \left(\underbrace{\cos^2\frac{\theta}{2}}_{\text{Electric/ Magnetic scattering}} - \underbrace{\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}}_{\text{Magnetic term due to spin}} \right)$$

- 相对论电子在扩展型质子的弹性散射：

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)}$$

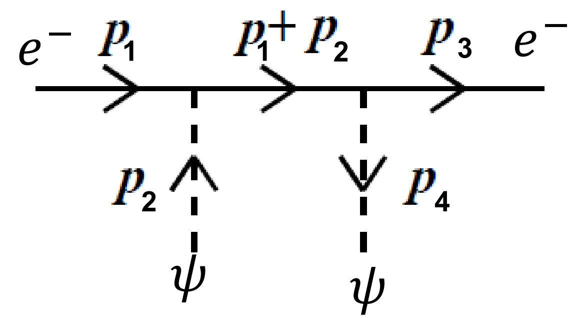
Rosenbluth Formula

- 相对论电子在质子的弹性散射显式质子是扩展型的，且电荷半径为 ~0.8 fm

App 1: ME with trace techniques

Scattering between e and scalar particle

- 考虑$e^- \psi \to e^- \psi$的散射过程，根据费曼规则写出矩阵元



for each incoming or outgoing scalar, 1;

for each incoming electron, $u_{s_i}(\mathbf{p}_i)$;

for each outgoing electron, $\overline{u}_{s_i'}(\mathbf{p}_i')$;

for each incoming positron, $\overline{v}_{s_i}(\mathbf{p}_i)$;

for each outgoing positron, $v_{s_i'}(\mathbf{p}_i')$;

for each vertex, ig;

for each internal scalar, $-i/(k^2 + M^2 - i\epsilon)$;

for each internal fermion, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$.

$$-iM_{fi} = [(ig)u(p_1 s_1)]\frac{-i(\not{q} + m)}{q^2 - m^2}[\bar{u}(p_3 s_3)(ig)]$$

- 做一个简单的化简，$(\not{p} - m)u = 0$：$(\not{q} + m) \to (\not{p}_2 + 2m)$

$$\overline{|M_{fi}|^2} = \frac{g^4}{(q^2 - m^2)^2}\frac{1}{2}\sum_{s_1}[\bar{u}(p_1 s_1)(\not{p}_2 + 2m)u(p_3 s_3)][\bar{u}(p_3 s_3)(\not{p}_2 + 2m)u(p_1 s_1)]$$

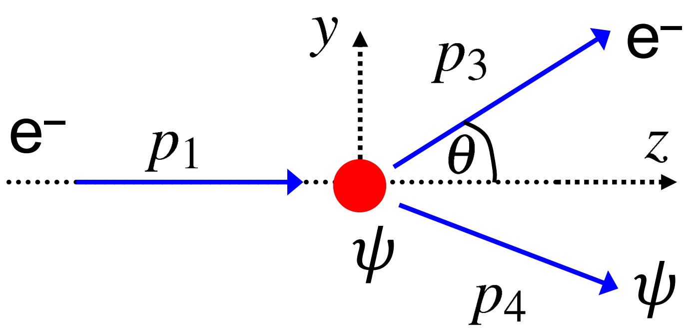
$$\overline{|M_{fi}|^2} = \frac{g^4}{2(q^2 - m^2)^2}\mathrm{Tr}[(\not{p}_1 + m)(\not{p}_2 + 2m)(\not{p}_3 + m)(\not{p}_2 + 2m)]$$

Scattering between e and scalar particle

$$\overline{|M_{fi}|^2} = \frac{g^4}{2(q^2-m^2)^2}\mathrm{Tr}[(p_1+m)(p_2+2m)(p_3+m)(p_2+2m)]$$

- 取相对论极限，忽略电子质量：

$$\overline{|M_{fi}|^2} = \frac{g^4}{2q^4}4[2(p_1p_2)(p_2p_3)-(p_1p_3)M^2]$$



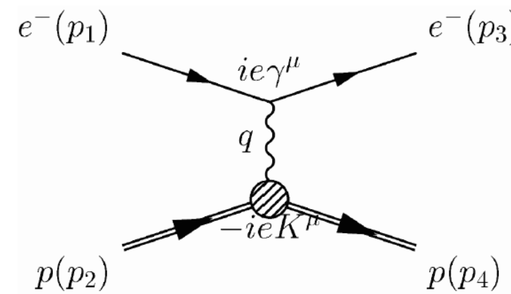
$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) \\ p_2 &= (M, 0, 0, 0) \\ p_3 &= (E_3, 0, E_3\sin\theta, E_3\cos\theta) \\ p_4 &= (E_4, \vec{p}_4) \end{aligned}$$

$$\overline{|M_{fi}|^2} = \frac{2g^4}{s^2}[2(E_1M)(E_3M)-(E_1E_3(1-\cos\theta))M^2]$$

$$= \frac{2g^4}{s^2}[E_1E_3M^2(1+\cos\theta)]$$

$$= \frac{g^4}{s^2}E_1E_3M^2\cos^2\frac{\theta}{2}$$

Rosenbluth Formula 的推导



- 回顾假设质子是点状粒子：

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- 现在需要考虑质子的结构，写出如图费曼图的矩阵元：

$$M_{fi} = -e^2[\bar{u}(p_3s_3)\gamma^\mu u(p_1s_1)]\frac{1}{q^2+i\epsilon}[\bar{u}(p_4s_4)K_\mu u(p_2s_2)]$$

- 只有原来的γ_μ被替换为了K_μ，质子的结构肯定比γ_μ复杂，需要试图用$\gamma^\mu, p_2^\mu, p_4^\mu$组合出$K_\mu$：

$$K^\mu = \gamma^\mu\cdot A + \left(p_2^\mu + p_4^\mu\right)\cdot B + \left(p_2^\mu - p_4^\mu\right)\cdot C$$

- 考虑流守恒：$\delta^\mu J_\mu = \delta^\mu\left[\bar{u}(p_4s_4)K_\mu u(p_2s_2)\right] = 0$可以得到$C = 0$

- 回顾作业题：$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')(p+p')^\mu u(p) + \frac{i}{2m}\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)$

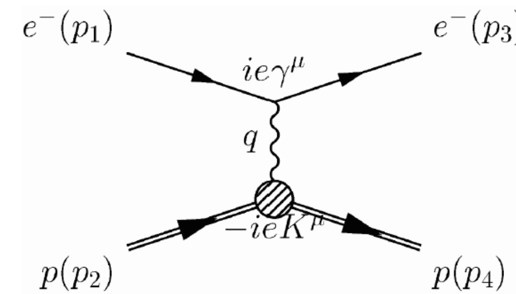
  可以将B项替换，最后得到K^μ的形式：

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$K^\mu = \gamma^\mu F_1(q^2) + \frac{i\kappa}{2M}\sigma^{\mu\nu}q_\nu F_2(q^2)$$

Rosenbluth Formula 的推导

$$K^{\mu}=\gamma^{\mu}F_1(q^2)+\frac{i\kappa}{2M}\sigma^{\mu\nu}q_{\nu}F_2(q^2)$$



- 其中$F_1(q^2), F_2(q^2)$为形状因子，κ为质子的反常磁矩：

$$\mu_p=(1+\kappa)\frac{e}{2M}$$

- 现在可以计算$e^-p\rightarrow e^-p$的矩阵元：

$$W_{\mu\nu}=\frac{1}{2}\sum_{s_2s_4}[\bar{u}(p_4s_4)K_{\mu}u(p_2s_2)]^{\dagger}[\bar{u}(p_4s_4)K_{\nu}u(p_2s_2)]$$

$$=\frac{1}{2}\mathrm{Tr}\big[(\not{p}_2+m)K_{\mu}(\not{p}_4+m)K_{\nu}\big]$$

$$=\frac{1}{2}\mathrm{Tr}[\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}]$$

$$\begin{aligned}\mathrm{tr}(\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) &= g^{\kappa\lambda}\times\mathrm{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma})-g^{\kappa\mu}\times\mathrm{tr}(\gamma^{\lambda}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma})+g^{\kappa\nu}\times\mathrm{tr}(\gamma^{\lambda}\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma})\\ &\quad -g^{\kappa\rho}\times\mathrm{tr}(\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma})+g^{\kappa\sigma}\times\mathrm{tr}(\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho})\\ &= 4g^{\kappa\lambda}\times\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right)\\ &\quad -4g^{\kappa\mu}\times\left(g^{\lambda\nu}g^{\rho\sigma}-g^{\lambda\rho}g^{\nu\sigma}+g^{\lambda\sigma}g^{\nu\rho}\right)\\ &\quad +4g^{\kappa\nu}\times\left(g^{\lambda\mu}g^{\rho\sigma}-g^{\lambda\rho}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\rho}\right)\\ &\quad -4g^{\kappa\rho}\times\left(g^{\lambda\mu}g^{\nu\sigma}-g^{\lambda\nu}g^{\mu\sigma}+g^{\lambda\sigma}g^{\mu\nu}\right)\\ &\quad +4g^{\kappa\sigma}\times\left(g^{\lambda\mu}g^{\nu\rho}-g^{\lambda\nu}g^{\mu\rho}+g^{\lambda\rho}g^{\mu\nu}\right).\end{aligned}$$

$$\mathrm{Tr}[\mathrm{I}]=F_1^2(q^2)\mathrm{Tr}\big[(\not{p}_2+M)\gamma_{\mu}(\not{p}_4+M)\gamma_{\nu}\big]$$

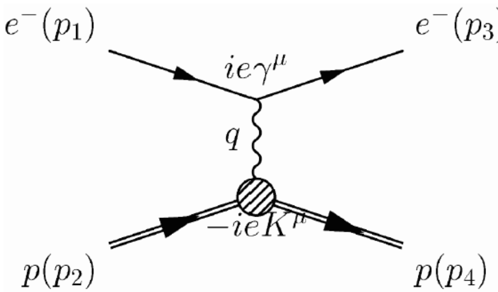
$$\mathrm{Tr}[\mathrm{II}]=\frac{i\kappa}{2M}F_2(q^2)\mathrm{Tr}\big[(\not{p}_2+M)\gamma_{\mu}(\not{p}_4+M)\sigma_{\nu\sigma}q^{\sigma}\big]$$

$$\mathrm{Tr}[\mathrm{IV}]=\left(\frac{i\kappa}{2M}\right)^2F_2^2(q^2)\mathrm{Tr}\big[(\not{p}_2+M)\sigma_{\mu a}q^{a}(\not{p}_4+M)\sigma_{\nu b}q^{b}\big]$$

Rosenbluth Formula 的推导

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E_1^2\sin^2\theta/2}\frac{E_3}{E_1}\left[\left(F_1^2-\frac{\kappa^2q^2}{4M}\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+\kappa F_2)^2\sin^2\frac{\theta}{2}\right]$$

- 更普遍使用的是$F_1(q^2), F_2(q^2)$的线性组合：



$$G_E(q^2)\equiv F_1+\frac{\kappa q^2}{4M^2}F_2$$

$$G_M(q^2)\equiv F_1+\kappa F_2$$

- 由此得到熟悉的**Rosenbluth Formula**

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

其中洛伦兹不变量：

$$\tau=-\frac{q^2}{4M^2}>0$$