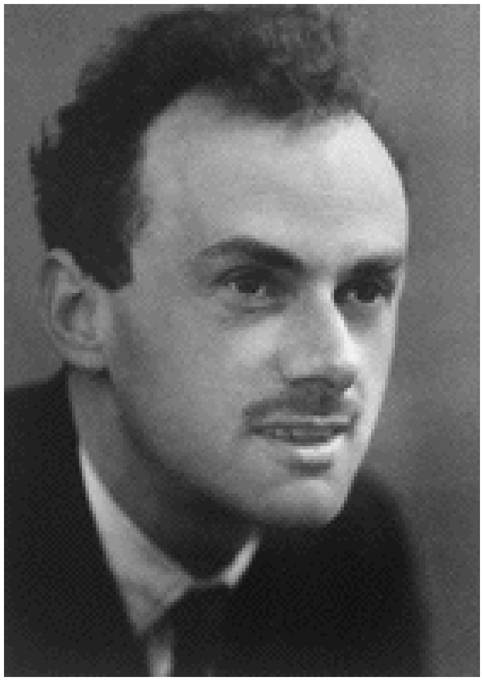
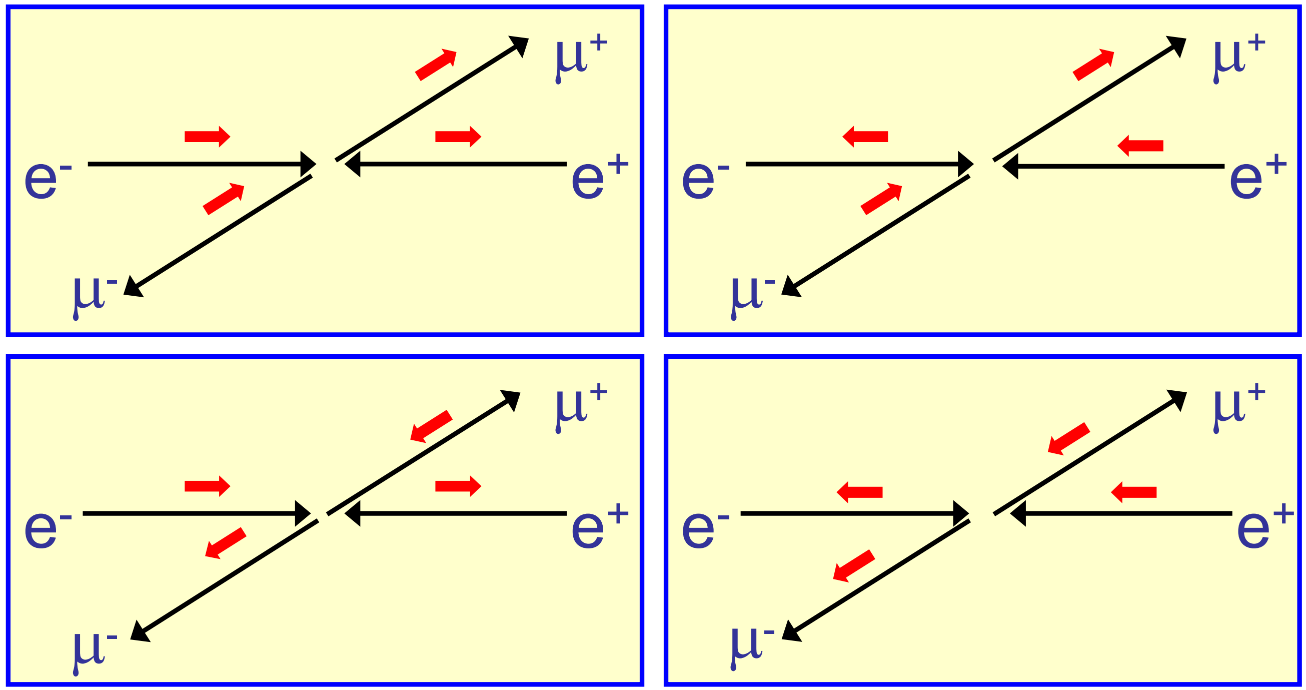
粒子物理学

第 2 章: 狄拉克方程





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Based on M. Thomson's notes

Non-Relativistic QM (Revision)

- 粒子物理需要相对论形式的量子力学，但先回顾一下非相对论量子力学
  - 非相对论的能量： $E = T + V = \frac{\vec{p}^2}{2m} + V$
  - 量子力学中的动量和能量算符 $\vec{p} \to -i\vec{\nabla}, \quad E \to i\frac{\partial}{\partial t}$

    给出含时薛定谔方程
    (take V=0 for simplicity)

    $$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t} \quad \textbf{(S1)}$$

    平面波解： $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$ **where** $\begin{cases} -i\nabla\psi = \vec{p}\psi \\ i\frac{\partial\psi}{\partial t} = E\psi \end{cases}$

  - 薛定谔方程：一阶时间导数和二阶空间导数。显然不满足洛伦兹不变

NR QM probability density/current

- 后续广泛用到**概率密度/流**。非相对论情况的推导如下：

(S1)* ➡ $$-\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial\psi^*}{\partial t}$$ (S2)

$$\psi^*\times(\mathbf{S1})-\psi\times(\mathbf{S2}):\quad \begin{aligned}-\frac{1}{2m}\left(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right) &= i\left(\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}\right)\\ -\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*\right) &= i\frac{\partial}{\partial t}(\psi^*\psi)\end{aligned}$$

- 对比连续方程 $\boxed{\vec{\nabla}.\vec{j}+\frac{\partial\rho}{\partial t}=0}$ 得到概率密度和概率流表达式：

$$\rho=\psi^*\psi=|\psi|^2 \text{ 和 } \vec{j}=\frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*\right)$$

- 对于平面波: $\psi=Ne^{i(\vec{p}.\vec{r}-Et)}$ $\qquad \rho=|N|^2$ 和 $\vec{j}=|N|^2\frac{\vec{p}}{m}=|N|^2\vec{v}$

- 单位体积内的粒子数 $|N|^2$ 以速度 $\vec{v}$ 运动，则单位时间穿过单位面积的粒子数（粒子通量Flux）为 $|N|^2\vec{v}$。因此，$\vec{j}$ 是粒子通量的矢量

Klein-Gordon Equation

- 将 $\vec{p} \to -i\vec{\nabla}, \quad E \to i\partial/\partial t$ 带入相对论能量方程 $E^2 = |\vec{p}|^2 + m^2$ **(KG1)**

  得到 Klein-Gordon 方程 $$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi$$ **(KG2)**

- 使用符号 $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Rightarrow \partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

  KG 方程表示为 $$\boxed{(\partial^\mu \partial_\mu + m^2)\psi = 0}$$ **(KG3)**

- 对于平面波解 $\psi = Ne^{i(\vec{p}.\vec{r} - Et)}$ KG方程得到：

$$-E^2 \psi = -|\vec{p}|^2 \psi - m^2 \psi \Rightarrow E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

➢ KG方程有“负能量解”

- 除了负能解，KG方程的另一个问题是“概率密度”

Klein-Gordon Equation

- **类似地，计算概率密度和概率流:** **(KG2)***
$$\frac{\partial^2 \psi^*}{\partial t^2} = \vec{\nabla}^2 \psi^* - m^2 \psi^* \qquad \textbf{(KG4)}$$

$\boxed{\psi^* \times (\mathbf{KG2}) - \psi \times (\mathbf{KG4}) :}$
$$\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} = \psi^* (\nabla^2 \psi - m^2 \psi) - \psi (\nabla^2 \psi^* - m^2 \psi^*)$$
$$\frac{\partial}{\partial t} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) = \vec{\nabla} . (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

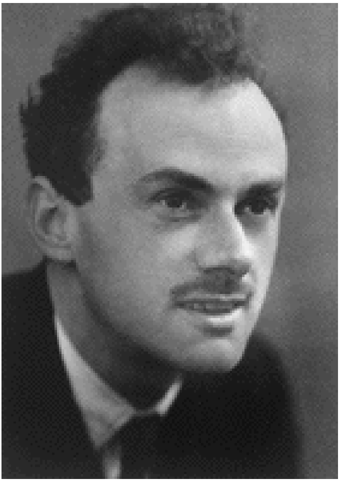
- **对比连续性方程，得：** $\rho = i \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right)$ and $\vec{j} = i (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$

- **对于平面波：** $\psi = N e^{i(\vec{p}.\vec{r} - Et)}$ ➡ $\rho = 2E|N|^2$ **and** $\vec{j} = |N|^2 \vec{p}$
  - 粒子密度正比与 E，符合洛伦兹不变相空间的预期
  - 但存在非物理的负几率密度

- **注：在量子场论中，这两个问题被克服，KG 方程被用于描述自旋-0 的粒子**

(inherently single particle description ➞ multi-particle quantum excitations of a scalar field)

Dirac Equation



- 这些问题促使 Dirac (1928) 去寻找不同形式的相对论量子力学，以使粒子密度都为正
  - 该波动方程不但解决了这些问题，还完整地描述电子的内秉自旋和磁矩
- 薛定谔方程 $-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t}$ 1st order in $\partial/\partial t$ 2nd order in $\partial/\partial x, \partial/\partial y, \partial/\partial z$
- KG方程 $(\partial^\mu\partial_\mu + m^2)\psi = 0$ 2nd order throughout

- 狄拉克寻找一种完全的一阶形式:

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \qquad \text{(D1)}$$

其中$\hat{H}$ 是哈密顿算符，而 $\vec{p} = -i\vec{\nabla}$

•Writing (D1) in full:

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

Dirac Equation：

- “平方”该方程 $$\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\psi=-\frac{\partial^2\psi}{\partial t^2}$$

- 展开后， $$\begin{aligned}-\frac{\partial^2\psi}{\partial t^2} \;=\; & -\alpha_x^2\frac{\partial^2\psi}{\partial x^2}-\alpha_y^2\frac{\partial^2\psi}{\partial y^2}-\alpha_z^2\frac{\partial^2\psi}{\partial z^2}+\beta^2m^2\psi \\ & -(\alpha_x\alpha_y+\alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y}-(\alpha_y\alpha_z+\alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z}-(\alpha_z\alpha_x+\alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\ & -(\alpha_x\beta+\beta\alpha_x)m\frac{\partial\psi}{\partial x}-(\alpha_y\beta+\beta\alpha_y)m\frac{\partial\psi}{\partial y}-(\alpha_z\beta+\beta\alpha_z)m\frac{\partial\psi}{\partial z}\end{aligned}$$

- 平方后需要满足相对论质能关系 $E^2=\vec{p}^2+m^2$，

  即满足KG方程 $$-\frac{\partial^2\psi}{\partial t^2}=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}+m^2\psi$$

  得到如下关系式 (D2, D3, D4)：

  $$\alpha_x^2=\alpha_y^2=\alpha_z^2=\beta^2=1 \qquad \alpha_j\beta+\beta\alpha_j=0 \qquad \alpha_j\alpha_k+\alpha_k\alpha_j=0 \qquad (j\neq k)$$

- 可以看出α_j和 β 不是数字，是4个相互反对易的矩阵，至少是 4x4 矩阵（附录1）

Appendix I : Dimensions of the Dirac Matrices

non-examinable

回顾 $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$

1. 对于所有 $\vec{p}$ 哈密顿量 $\hat{H}$ 是厄米的，则要求 $\alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger$

2. 为了符合KG 方程：

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$

$$\alpha_j\beta + \beta\alpha_j = 0 \qquad \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad (j \neq k)$$

- **考虑：**

$$\begin{aligned} Tr(B^\dagger AB) &= B^\dagger_{ij}A_{jk}B_{ki} \\ &= B_{ki}B^\dagger_{ij}A_{jk} \\ &= \delta_{jk}A_{jk} \\ &= Tr(A) \end{aligned}$$

利用 $B^\dagger B = 1$

使用交换律

$$\begin{aligned} Tr(\alpha) &= Tr(\alpha_j^\dagger \alpha_i \alpha_j) \\ &= -Tr(\alpha_j^\dagger \alpha_j \alpha_i) \\ &= -Tr(\alpha_i) \end{aligned}$$

$$\Rightarrow \quad Tr(\alpha_i) = 0$$

- **类似可得** $$Tr(\beta) = 0$$

3. 考虑关系式 $Tr(\alpha) = \sum_i \lambda_i$

Appendix I : Dimensions of the Dirac Matrices

- 右边本征值方程可以说明这些矩阵具有偶数维度 $\alpha\vec{x} = \lambda\vec{x}$

$$\vec{x}^{\dagger}\vec{x} = \vec{x}\alpha^{\dagger}\alpha\vec{x} = \lambda^{*}\lambda\vec{x}^{\dagger}\vec{x}$$

- 厄米矩阵的本征值为实数，因此 $\lambda^2 = 1 \quad \rightarrow \quad \lambda = \pm 1$

- 由于α_j和 β 是本征值为±1的无迹、厄米矩阵，其维度一定是偶数

- N=2 的情况，即3 个泡利自旋矩阵，满足 $\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad (j \neq i)$

4. 但是，要求4个反对易矩阵。因此矩阵 α_j和 β 的维度必须是 4, 6, 8,…
   - 最简单的选择是架设 α_j和 β 的维度为4

Dirac Equation：

- 相应地，波函数也必须是“四分量”

  Dirac Spinor（狄拉克旋量） $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

  方程为一阶时空导数的结果是波函数有新的自由度

- 哈密顿量满足厄米性条件 $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\partial\psi/\partial t$

  得 $\boxed{\alpha_x = \alpha_x^\dagger; \quad \alpha_y = \alpha_y^\dagger; \quad \alpha_z = \alpha_z^\dagger; \quad \beta = \beta^\dagger;}$

  四个反对易 4x4厄米矩阵 **(D5)**

- 求 α_j和 β 的具体形式 注意：物理结果不依赖特定形式 – 任何满足对易关系的都可以

  方便的选择是基于泡利矩阵 $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$ 厄米且相互对易

  with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Dirac Equation: Probability Density and Current

•**Start with Dirac equation**

$$-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+m\beta\psi=i\frac{\partial\psi}{\partial t}\qquad\text{(D6)}$$

厄米共轭

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger+m\psi^\dagger\beta^\dagger=-i\frac{\partial\psi^\dagger}{\partial t}\qquad\text{(D7)}$$

1. 如下操作 $\psi^\dagger\times(\mathbf{D6})-(\mathbf{D7})\times\psi$ 　　**提醒：α_j和 β是厄米的** ➡

$$\psi^\dagger\left(-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+\beta m\psi\right)-\left(i\frac{\partial\psi^\dagger}{\partial x}\alpha_x+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z+m\psi^\dagger\beta\right)\psi=i\psi^\dagger\frac{\partial\psi}{\partial t}+i\frac{\partial\psi^\dagger}{\partial t}\psi$$

$$\Rightarrow\quad \underbrace{\psi^\dagger\left(\alpha_x\frac{\partial\psi}{\partial x}\right.}+\left.\alpha_y\frac{\partial\psi}{\partial y}+\alpha_z\frac{\partial\psi}{\partial z}\right)+\underbrace{\left(\frac{\partial\psi^\dagger}{\partial x}\alpha_x\right.}+\left.\frac{\partial\psi^\dagger}{\partial y}\alpha_y+\frac{\partial\psi^\dagger}{\partial z}\alpha_z\right)\psi+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0$$

2. 利用等式：

$$\psi^\dagger\alpha_x\frac{\partial\psi}{\partial x}+\frac{\partial\psi^\dagger}{\partial x}\alpha_x\psi\equiv\frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x}$$

Dirac Equation: Probability Density and Current

- 得到连续方程：

$$\boxed{\vec{\nabla}.(\psi^\dagger\vec{\alpha}\psi)+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0}$$

(D8)

其中 $\psi^\dagger=(\psi_1^*,\psi_2^*,\psi_3^*,\psi_4^*)$

- 概率密度和流为： $\boxed{\rho=\psi^\dagger\psi}$ and $\boxed{\vec{j}=\psi^\dagger\vec{\alpha}\psi}$

其中 $\rho=\psi^\dagger\psi=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2>0$

- 不同于KG 方程，狄拉克方程的概率密度永远为正

- 此外，狄拉克方程的解为四分量的狄拉克旋量，可以自然地引入内秉自旋
  - 狄拉克方程可以表示半整数自旋粒子 (appendix II)
  - 内秉的磁矩为： $\boxed{\vec{\mu}=\frac{q}{m}\vec{S}}$ (appendix III)

Covariant Notation: Dirac γ Matrices

- 定义：四个狄拉克矩阵 $\gamma^0 \equiv \beta;\ \gamma^1 \equiv \beta\alpha_x;\ \gamma^2 \equiv \beta\alpha_y;\ \gamma^3 \equiv \beta\alpha_z$

在狄拉克方程(D6)前乘以 β $$i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} - \beta^2 m\psi = -i\beta\frac{\partial\psi}{\partial t}$$

$$\Rightarrow\quad i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

利用 $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ ，得：

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0} \qquad \text{(D9)}$$

★ 注意：Dirac gamma 矩阵 <u>不是</u> 四矢量

- 是不变的常数矩阵。但狄拉克方程本身是洛伦兹不变的 (Appendix IV)

Properties of the γ matrices

- 从 α_j和 β 矩阵的性质 (D2)-(D4)，可得：

$$(\gamma^0)^2 = \beta^2 = 1 \quad \text{和} \quad (\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -1$$

完整关系式

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \quad (j \neq k) \end{aligned}$$

表达为

$$\Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

(defines the algebra)

- γ 矩阵的厄米性：
  - β是厄米，因此γ^0 是厄米
  - α 是厄米，得 $\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1$

♦ **Hence γ^1, γ^2, γ^3** 是反厄米的

$$\boxed{\gamma^{0\dagger} = \gamma^0, \quad \gamma^{1\dagger} = -\gamma^1, \quad \gamma^{2\dagger} = -\gamma^2, \quad \gamma^{3\dagger} = -\gamma^3}$$

Pauli-Dirac Representation

- 本课程将使用γ矩阵的泡利-狄拉克表示 $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$

  展开后为：

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \; \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \; \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \; \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- 利用 γ矩阵 $\rho = \psi^\dagger \psi$ 和 $\vec{j} = \psi^\dagger \vec{\alpha} \psi$ 可被写为：

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

- 连续性方程 $\partial_\mu j^\mu = 0$ 　　其中j^μ是四矢量流 (Proof in Appendix V.)

Adjoint Spinor

- 定义伴随旋量：$\boxed{\overline{\psi} = \psi^{\dagger}\gamma^{0}}$ 将简化四矢量流 $j^{\mu} = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi$

即 $\overline{\psi} = \psi^{\dagger}\gamma^{0} = (\psi^{*})^{T}\gamma^{0} = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$\boxed{\overline{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)}$$

- 四矢量流表达为：$\boxed{j^{\mu} = \overline{\psi}\gamma^{\mu}\psi}$

★该表达式将被用于推导费曼规则，以计算基本相互作用的洛伦兹不变矩阵元

> 首先求解狄拉克方程的自由粒子解

Dirac Equation: Free Particle at Rest

- **设狄拉克方程自由粒子解的形式为：** $\boxed{\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}}$ $\quad u(\vec{p},E)$ 常数四分量旋量

狄拉克方程 $\quad (i\gamma^\mu \partial_\mu - m)\psi = 0$

- **自由粒子解的推导** $\quad \partial_0\psi = \dfrac{\partial\psi}{\partial t} = -iE\psi; \quad \partial_1\psi = \dfrac{\partial\psi}{\partial x} = ip_x\psi, \quad \ldots$

代换入狄拉克方程 $\quad (\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$

可写为 $\quad \boxed{(\gamma^\mu p_\mu - m)u = 0}$ $\quad$ **(D10)** $\quad$ “动量空间”狄拉克方程 --- 注意：其不含导数

- **对于静止粒子** $\vec{p}$ =0 **且** $\psi = u(E,0)e^{-iEt}$

$$\Rightarrow \quad E\gamma^0 u - mu = 0 \quad \Rightarrow \quad E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \quad \textbf{(D11)}$$

eq. (D10)

Dirac Equation: Free Particle at Rest

- 方程有四个正交的解：

$$\underbrace{u_1(m,0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2(m,0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{}\ ;\ \underbrace{u_3(m,0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ u_4(m,0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{}$$

(D11) ➡ $\boxed{E = m}$　　(D11) ➡ $\boxed{E = -m}$ • 仍然有 负能解

- 包括时间依赖性 $\psi = u(E,0)e^{-iEt}$ 得到

$$\underbrace{\psi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt};\quad \psi_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}}_{\text{Two spin states with } E>0};\quad \underbrace{\psi_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt};\ \text{and}\ \ \psi_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}}_{\text{Two spin states with } E<0}$$

★根据量子力学，我们需要一个态的完备集 -- 即“四个解”

★不能因为非物理性而丢弃E<0的解，

Dirac Equation: Plane Wave Solutions

- 推导一般的平面波解

$$\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$$

1. 考虑狄拉克方程 (D10): $(\gamma^\mu p_\mu - m)u = 0$

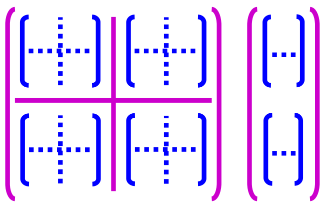
并利用： $\gamma^\mu p_\mu - m = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m$

Note: 4x4 矩阵写为4个 2x2 子矩阵

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} .\vec{p} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}$$

Note $\vec{\sigma}.\vec{p} = p_x\sigma_x + p_y\sigma_y + p_z\sigma_z$



$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

写为四分量旋量

$$(\gamma^\mu p_\mu - m)u = 0 \Rightarrow \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & (-E-m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Giving two coupled simultaneous equations

$$\left.\begin{aligned} (\vec{\sigma}.\vec{p})u_B &= (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B \end{aligned}\right\} \quad \textbf{(D12)}$$

Dirac Equation: Plane Wave Solutions

展开，得到 $\vec{\sigma}.\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$

$$\begin{aligned} (\vec{\sigma}.\vec{p})u_B &= (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B \end{aligned} \Rightarrow u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}u_A = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

2. 任意但最简单的选择 u_A :

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ 或 } u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{得到}} u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \text{ 和 } u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

注：$\vec{p}$ =0 的解对应 静止的 E>0 粒子

N: 波函数归一化

★ u_A 的选择是“随意的”，但是可以通过线性组合出任意的形式

★ 类似于，自旋基的选择（S_x, S_y or S_z本征函数）

3. 对 $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 和 $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 重复上述，得到 u_3 和 u_4

Dirac Equation: Plane Wave Solutions

- 四个解：$\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix};\quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix};\quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix};\quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

代回狄拉克方程，得到： $E^2 = \vec{p}^2 + m^2$

•疑问：四个解是否全都是正能解？

- 答案：否！
  - 如果全是相同能量，即E=+|E|，只有2个独立解
  - 只有包含2个E<0解后，才能得到4个独立的解

$$\boxed{u_1 = \frac{p_z}{E+m}u_3 + \frac{p_x+ip_y}{E+m}u_4}$$

- 参考静止粒子的解 (eq. D11)，可以确定哪些是E<0解
  - 对于 $\vec{p}=0$：u_1 和 u_2 对应 E>0 静止粒子解，u_3 和 u_4 则对应 E<0静止粒子解

- u_1 和 u_2是 正能解， u_3 和 u_4 是 负 能解

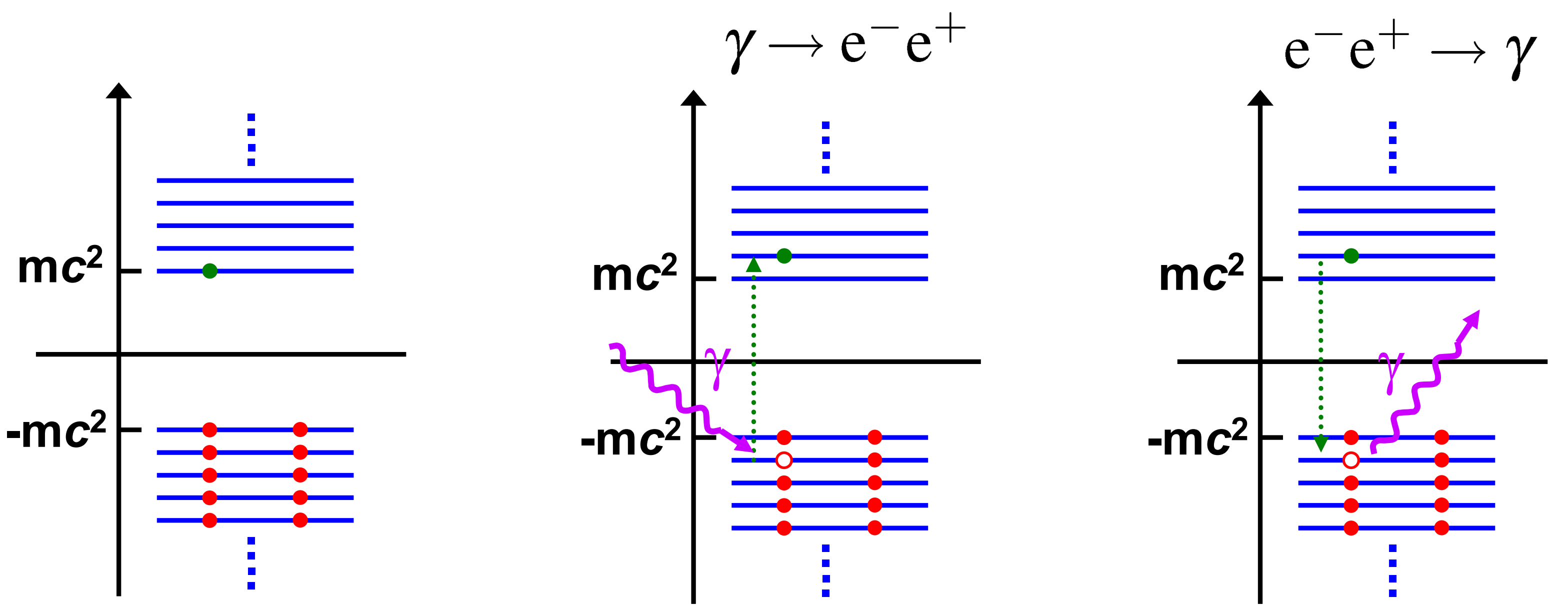
Interpretation of –ve Energy Solutions

- **狄拉克方程的概率密度为正，不像 KG方程。但如何理解负能解呢？**
  - 为什么不是所有的正能电子都落进更低的负能态呢？

➢ **<u>狄拉克的诠释</u>：**

  ➢ 真空对应负能解全部被占据，由于泡利不相容原理阻止电子进一步落入入负能态。负能态的“空穴”对应携带相反电荷的正能反粒子。

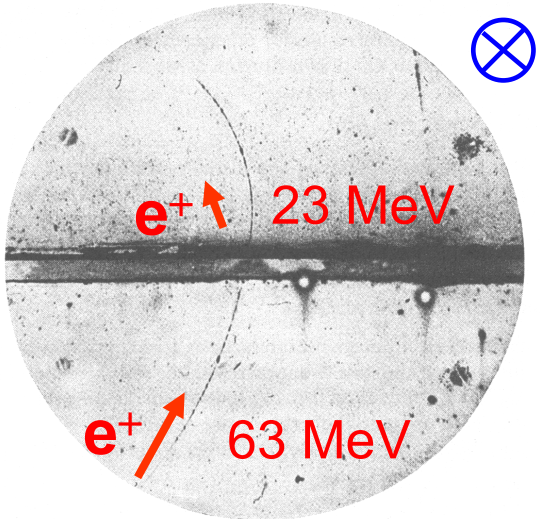
  ➢ 提供了对产生 (pair-production) 和湮灭 (annihilation) 的图像



Discovery of the Positron

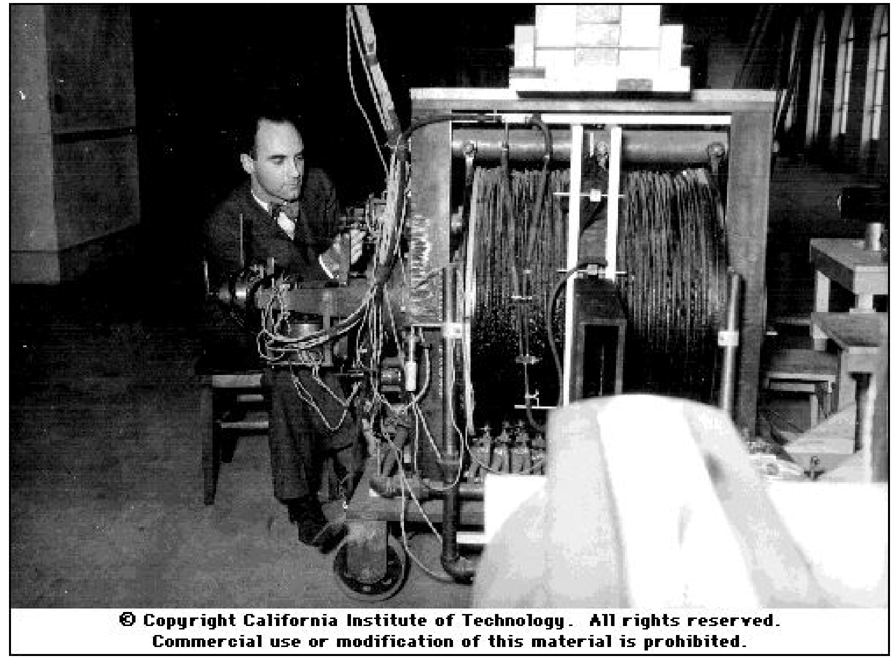
- “云室”观测到的宇宙线

6 mm
Lead
Plate



$\otimes \vec{B}$

C.D.Anderson, Phys Rev 43 (1933) 491



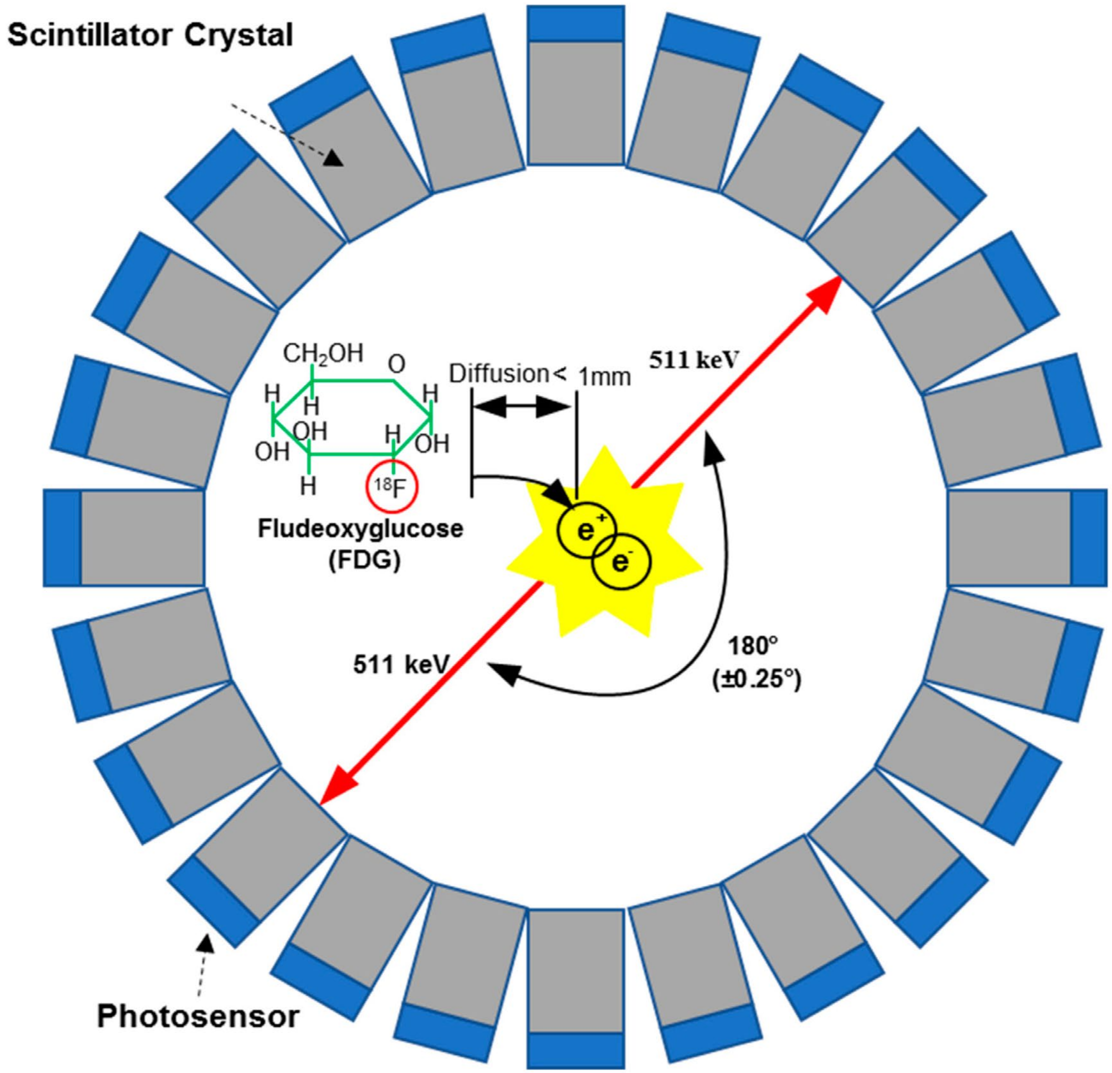
- e+ 从底部入射，在铅板中被减速
  - 从而确定入射方向
- 在磁场中弯曲方向，证明其携带正电荷
  - 质子不能被铅板减速，排除！

➡ 检验了狄拉克方程的预言

Anti-particle solutions exist !

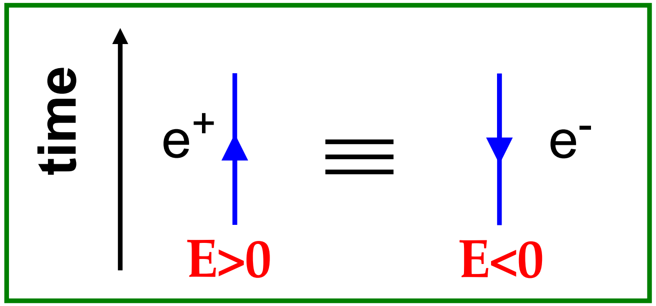
➢ 狄拉克海图像有缺陷，如不能解释反玻色子（不受泡利不相容原理约束）

Positron emission tomography (*PET*)

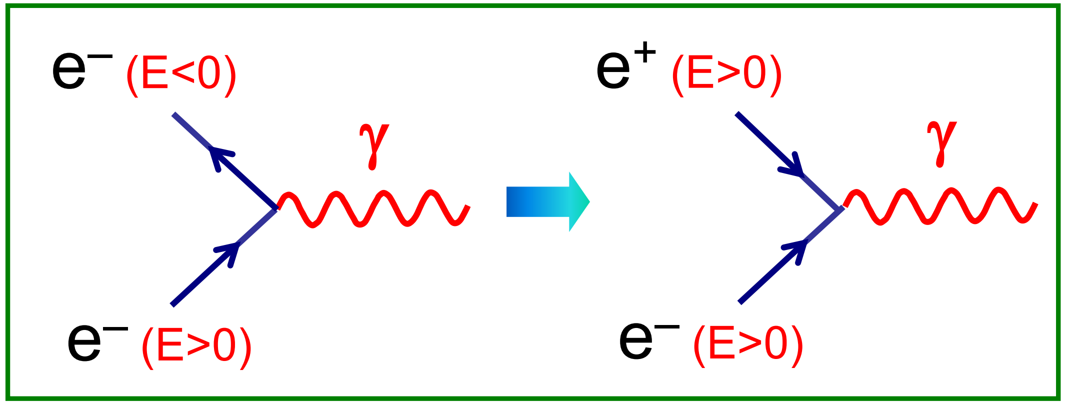


Feynman-Stückelberg Interpretation

- 将负能量解理解为延时间反向传播的负能量粒子
  - 或者等价地，延时间正向传播的正能量反粒子



$$e^{-i(-E)(-t)} \rightarrow e^{-iEt}$$



注意：费曼图上，反粒子箭头仍然与时间反向，以标记其是反粒子解（如左图）

➢ 这种诠释提供了更方便的正能量反粒子波函数

$$E = \left|\sqrt{|\vec{p}|^2 + m^2}\right|$$

Anti-Particle Spinors

- 重新定义负能量解，以使得： $E = |\sqrt{|\vec{p}|^2 + m^2}|$ 即物理性反粒子的能量

<u>We can look at this in two ways:</u>

❶ 从负能解

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Where E is negative

- “定义”反粒子波函数：
  - 根据Feynman-Stückelburg诠释，反转能量 E 和动量 $\vec{p}$ 的符号

$$v_1(E, \vec{p})e^{-i(\vec{p}.\vec{r} - Et)} = u_4(-E, -\vec{p})e^{i(\vec{p}.\vec{r} - Et)}$$

$$v_2(E, \vec{p})e^{-i(\vec{p}.\vec{r} - Et)} = u_3(-E, -\vec{p})e^{i(\vec{p}.\vec{r} - Et)}$$

其中 E 为正： $E = |\sqrt{|\vec{p}|^2 + m^2}|$

Anti-Particle Spinors

❷ 寻找如下形式的狄拉克方程的负能量平面波解：

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} \quad 其中 \quad E = \left|\sqrt{|\vec{p}|^2+m^2}\right|$$

注意：即使E>0，上述仍然是负能解： $\hat{H}v_1 = i\frac{\partial}{\partial t}v_1 = -Ev_1$

- 解狄拉克方程 $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)v = 0 \quad \Rightarrow \quad \boxed{(\gamma^\mu p_\mu + m)v = 0} \qquad \text{(D13)}$$

反粒子的动量空间狄拉克方程 (对比 D10，$- \rightarrow +$)

- 如前操作：

$$\left.\begin{aligned}(\vec{\sigma}.\vec{p})v_A &= (E-m)v_B \\ (\vec{\sigma}.\vec{p})v_B &= (E+m)v_A\end{aligned}\right\} \text{etc., ...} \Rightarrow v_1 = N_1'\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\\ 0\\ 1\end{pmatrix};\; v_2 = N_2'\begin{pmatrix}\frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\\ 1\\ 0\end{pmatrix}$$

➢ 与上一页的波动方程一样

Particle and anti-particle Spinors

- **如右形式的4个解** $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix};\ u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\ \underbrace{u_3 = N\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix};\ u_4 = N\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

- **如右形式的4个解** $\psi_i = v_i(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{v_1 = N\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix};\ v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\ \underbrace{v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \end{pmatrix};\ v_4 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

- **根据四分量旋量性质，只有4个线性独立的**
  - **可以任意选择** $\{u_1,u_2,u_3,u_4\}$ **或** $\{v_1,v_2,v_3,v_4\}$ **或 …**
  - **自然地，选择全部正能解** $\{u_1,u_2,v_1,v_2\}$

Wave-Function Normalisation

- 考虑 $\psi = u_1 e^{+i(\vec{p}.\vec{r}-Et)}$ 概率密度 $\rho = \psi^\dagger\psi = (\psi^*)^T\psi = u_1^\dagger u_1$

$$\boxed{u_1 = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}}$$

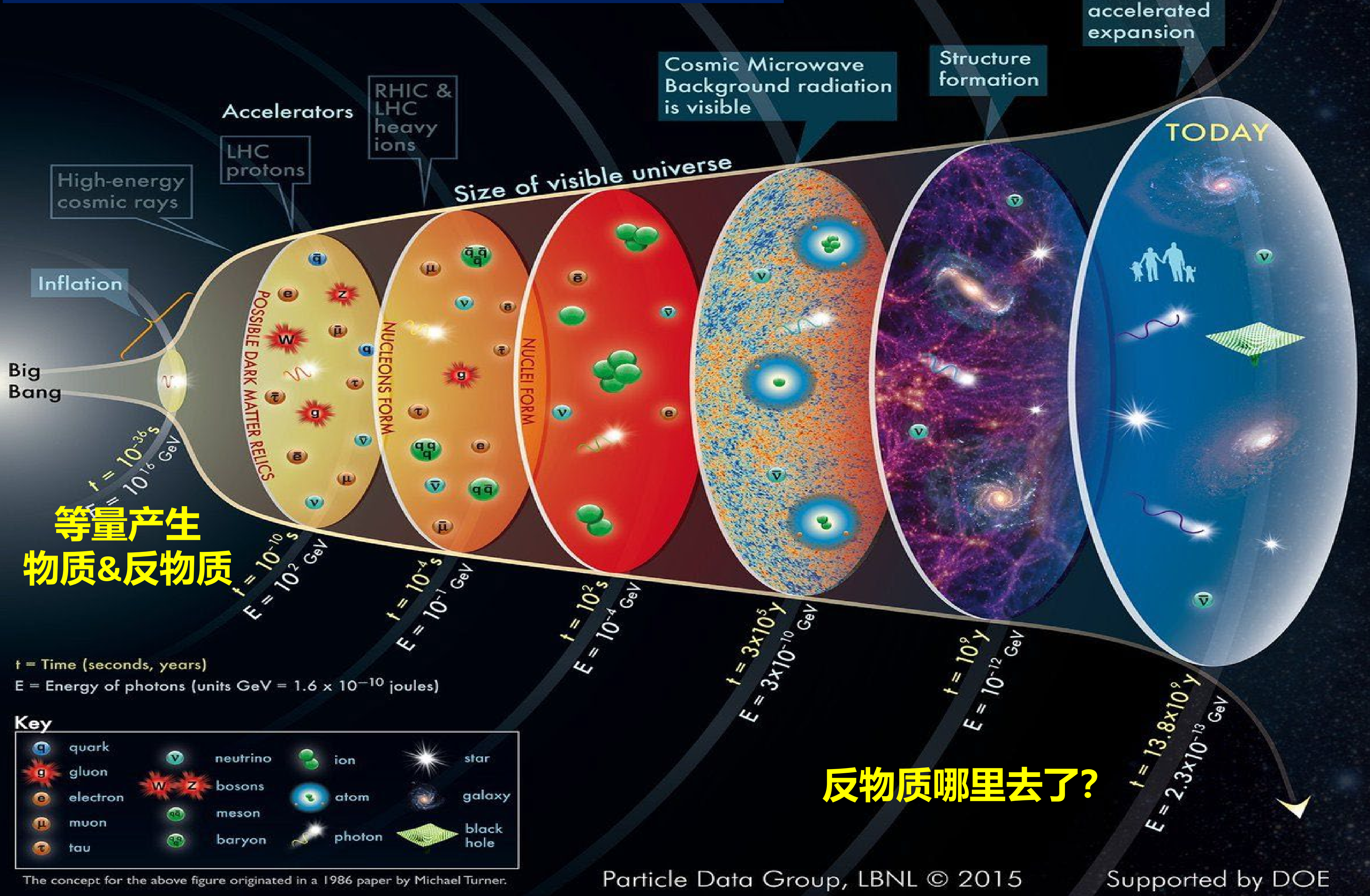
$$\begin{aligned}u_1^\dagger u_1 &= |N|^2\left(1+\frac{p_z^2}{(E+m)^2}+\frac{p_x^2+p_y^2}{(E+m)^2}\right)\\&= |N|^2\left(\frac{(E+m)^2+|\vec{p}|^2}{(E+m)^2}\right) = |N|^2\left(\frac{(E+m)^2+E^2-m^2}{(E+m)^2}\right)\\&= |N|^2\frac{2E^2+2Em}{(E+m)^2} = |N|^2\frac{2E}{E+m}\end{aligned}$$

（上节课）波函数归一化到单位体积内 $2E$ 个粒子

单位体积内2E个粒子要求 $\boxed{N = \sqrt{E+m}}$

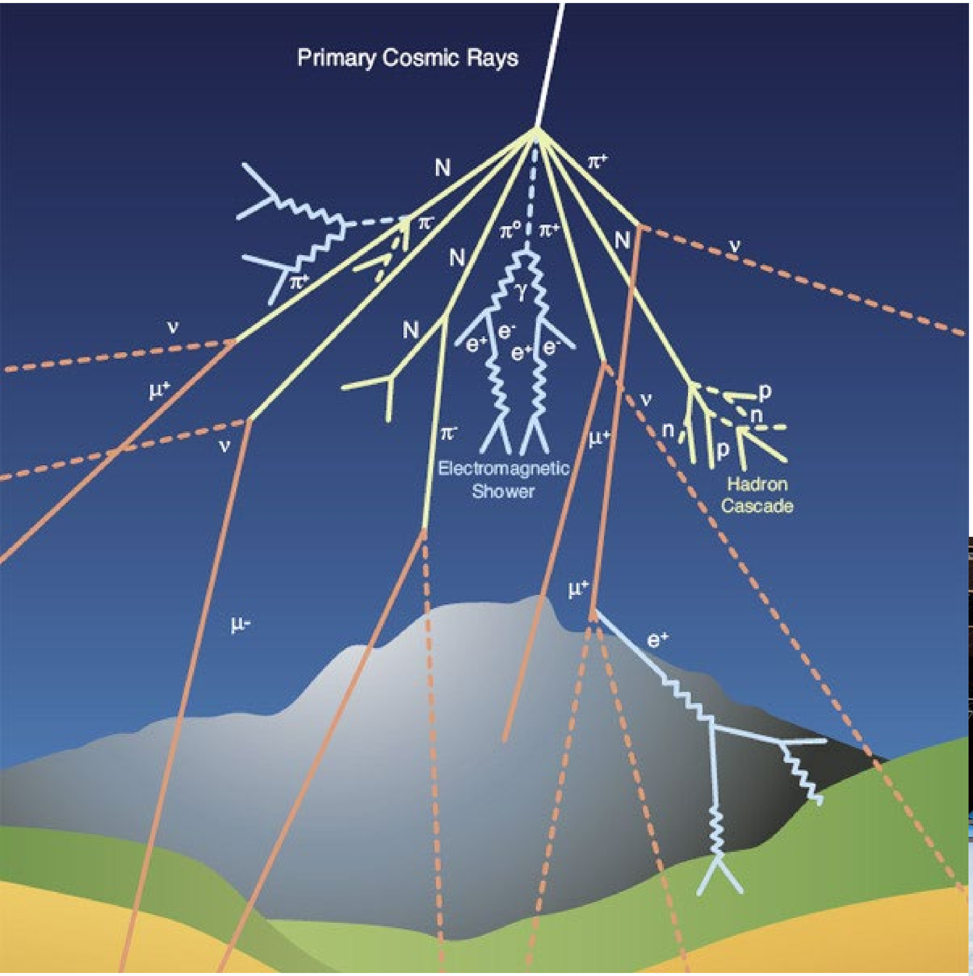
- 其他解类似 u_1，u_2，v_1，v_2

宇宙简史



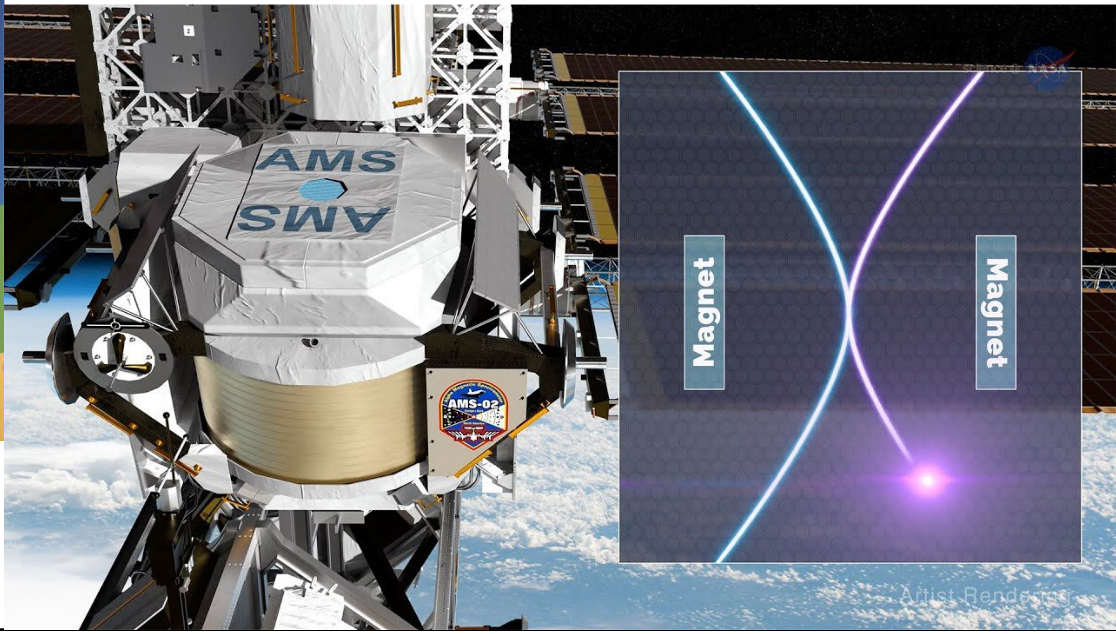
Dark energy accelerated expansion
Structure formation
Cosmic Microwave Background radiation is visible
RHIC & LHC heavy ions
Accelerators
LHC protons
High-energy cosmic rays
Size of visible universe
TODAY
Inflation
Big Bang
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
$t = 10^{-36}$ s
$E = 10^{16}$ GeV
$t = 10^{-10}$ s
$E = 10^{2}$ GeV
$t = 10^{-4}$ s
$E = 10^{-1}$ GeV
$t = 10^{2}$ s
$E = 10^{-4}$ GeV
$t = 3\times10^{5}$ y
$E = 3\times10^{-10}$ GeV
$t = 10^{9}$ y
$E = 10^{-12}$ GeV
$t = 13.8\times10^{9}$ y
$E = 2.3\times10^{-13}$ GeV
等量产生
物质&反物质
反物质哪里去了?
t = Time (seconds, years)
E = Energy of photons (units GeV = 1.6×10^{-10} joules)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
The concept for the above figure originated in a 1986 paper by Michael Turner.
Particle Data Group, LBNL © 2015
Supported by DOE

Anti-Matter Searches: AMS



大爆炸：

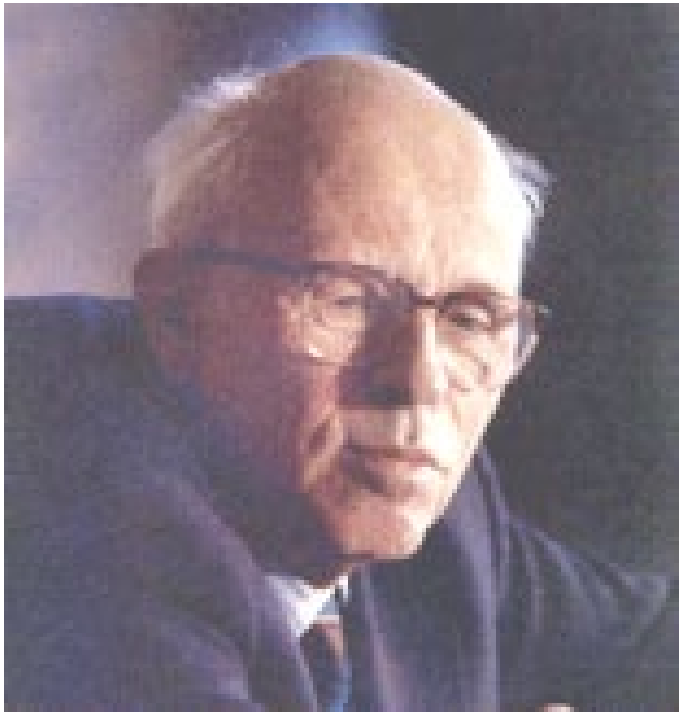
- 产生等量正反物质
- 某种未知原因使得演化过程更倾向于保留物质
- 结果：一点点物质和大量的光子
- $N_{baryons}/N_{photons} \cong 6\times10^{-10}$



Sakharov's conditions on Big Bang

In 1967, Sakharov formulated three necessary conditions to generate universe with a baryon asymmetry:

1. **a process that violates baryon number**
2. **C and CP violation, i.e. breaking of the C and CP symmetries**
3. **1 & 2 should occur during a phase which is NOT in thermal equilibrium**



Andrei Sakharov
"Father" of Soviet hydrogen bomb
& Nobel Peace Prize Winner

Charge Conjugation

- 在相对论和电磁学中，带电粒子与电磁场$A^\mu = (\phi, \vec{A})$ 的相互作用，可以通过**最小替换 minimal substitution**

$$\vec{p} \to \vec{p} - e\vec{A}; \quad E \to E - e\phi \quad \text{其中} \quad \vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

记为 $\boxed{\partial_\mu \to \partial_\mu + ieA_\mu}$ 狄拉克方程为 $\boxed{\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0}$

- 取“复共轭”并乘以$-i\gamma^2$ $\Rightarrow$ $-i\gamma^2\gamma^{\mu *}(\partial_\mu - ieA_\mu)\psi^* - m\gamma^2\psi^* = 0$

But $\gamma^{0*} = \gamma^0;\ \gamma^{1*} = \gamma^1;\ \gamma^{2*} = -\gamma^2;\ \gamma^{3*} = \gamma^3$ and $\boxed{\gamma^2\gamma^{\mu *} = -\gamma^\mu\gamma^2}$

$$\Rightarrow \quad \gamma^\mu(\partial_\mu - ieA_\mu)\underbrace{i\gamma^2\psi^*} + im\underbrace{i\gamma^2\psi^*} = 0 \qquad \text{(D14)}$$

定义“电荷共轭”算符 $\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*}$

D14变成：$\boxed{\gamma^\mu(\partial_\mu - ieA_\mu)\psi' + im\psi' = 0}$ 对比原方程 $\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$

➢ 旋量Ψ' 描述具有相同质量但是相反“荷”的粒子，即 反粒子！

Charge Conjugation

$\hat{C}$ ➡ $\boxed{\text{particle spinor} \leftrightarrow \text{anti-particle spinor}}$

- 将 $\hat{C}$ 作用到自由粒子波函数： $\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)}$ ⇨ $\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{-i(\vec{p}.\vec{r}-Et)}}$

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1$$

因此 $\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\vec{p}.\vec{r}-Et)}$

类似 $\psi = u_2 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\vec{p}.\vec{r}-Et)}$

- 电荷共轭操作下，粒子旋量 u_1 and u_2 变换为反粒子旋量 v_1 and v_2

Using anti-particle solutions

- **反粒子解** $\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$ **需要注意的一个要点：**

  施加常规的QM能量、动量算符 $\hat{p} = -i\vec{\nabla}, \quad \hat{H} = i\partial/\partial t$

  得到：$\hat{H}v_1 = i\partial v_1/\partial t = -Ev_1$ 和 $\hat{p}v_1 = -i\vec{\nabla}v_1 = -\vec{p}v_1$

- **因此，反粒子的物理的能量、动量算符为：** $\boxed{\hat{H}^{(v)} = -i\partial/\partial t \text{ 和 } \hat{p}^{(v)} = i\vec{\nabla}}$

  根据此变换 $(E,\vec{p}) \rightarrow (-E,-\vec{p})$ 得到：$\vec{L} = \vec{r} \wedge \vec{p} \rightarrow -\vec{L}$

  总角动量守恒 $[H,\vec{L}+\vec{S}] = 0$ ➡ $\boxed{\hat{S}^{(v)} \rightarrow -\hat{S}}$

  ➢ **反粒子解的物理自旋**

Summary of Solutions to Dirac Equation

- **Normalised free PARTICLE solutions to the Dirac equation:**

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)}$$ 满足 $$\boxed{(\gamma^\mu p_\mu - m)u = 0}$$

其中

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- **ANTI-PARTICLE solutions in terms of physical energy and momentum:**

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$$ 满足 $$\boxed{(\gamma^\mu p_\mu + m)v = 0}$$

其中

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For these states the spin is given by $\hat{S}^{(v)} = -\hat{S}$

- **For both particle and anti-particle solutions:** $E = \sqrt{|\vec{p}|^2 + m^2}$

Spin States

- 一般而言，旋量 u_1, u_2, v_1, v_2 不是 $\hat{S}_z$ 的本征态

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(Appendix II)

- 设(反)粒子延 Z 方向飞行 $p_z = \pm|\vec{p}|$

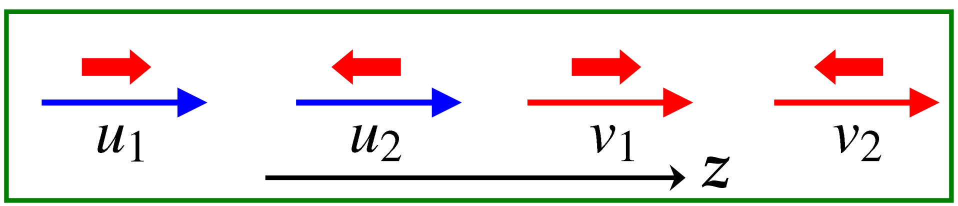
$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix}; \quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

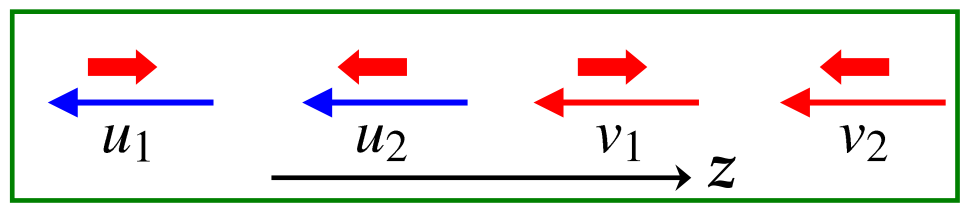
是 $\hat{S}_z$ 的本征态

$$\hat{S}_z u_1 = +\frac{1}{2}u_1 \qquad \hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\frac{1}{2}v_1$$

$$\hat{S}_z u_2 = -\frac{1}{2}u_2 \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\frac{1}{2}v_2$$

注意：对于反粒子旋量 $\hat{S}_z$ 符号改变





➢ 只在 $p_z = \pm|\vec{p}|$ 时为旋量 u_1, u_2, v_1, v_2 才是 $\hat{S}_z$ 的本征态

Pause for Breath…

- 得到了为$\hat{S}_Z$本质态的狄拉克方程解，但只对于延 Z 轴传播的粒子适用
- 希望能找到更一般可以标记态的“好量子数”，即 一组对易的观测量

    - 不能用自旋的z分量：$[\hat{H}, \hat{S}_z] \neq 0$ (Appendix II)

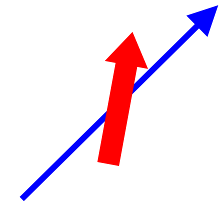
- 引入重要的概念 “螺旋度 (HELICITY) ”

Helicity plays an important role in much that follows

Helicity

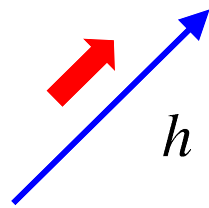
- 粒子自旋延飞行方向的分量是好量子数： $[\hat{H}, \hat{S}.\hat{p}] = 0$

- 定义“粒子自旋延其飞行方向的分量”为 **螺旋度：** $h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$



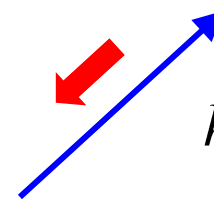
- 自旋延任何轴的测量分量只有两个值 ±1/2
  - 因此自旋1/2粒子的螺旋度算符的本征值为：±1

Often termed: **“right-handed”**



$h = +1$

“left-handed”



$h = -1$

- 注意：分“右手”手性 和“左手”手性 的螺旋度
- 后续课程将讨论“右手”和“左手”手征(CHIRAL) 本征态
  - 只在光速极限 $(v \approx c)$ 时， 螺旋度本征态与手征本征态相同

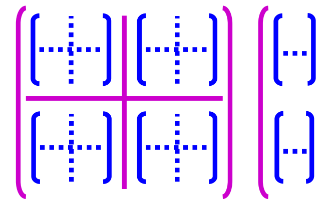
Helicity Eigenstates

- **寻找满足狄拉克方程的螺旋度本征态：** $(\vec{\Sigma}.\hat{p})u_\uparrow = +u_\uparrow \quad (\vec{\Sigma}.\hat{p})u_\downarrow = -u_\downarrow$

  $u_\uparrow$ 和 $u_\downarrow$ 是右手、左手 螺旋度态，$\hat{p}$ 是单位矢量

1. 本征方程：

$$\begin{pmatrix} \vec{\sigma}.\hat{p} & 0 \\ 0 & \vec{\sigma}.\hat{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

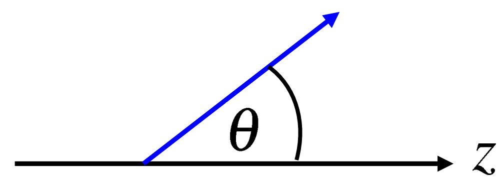


导出耦合方程：

$$\left.\begin{array}{rcl} (\vec{\sigma}.\hat{p})u_A & = & \pm u_A \\ (\vec{\sigma}.\hat{p})u_B & = & \pm u_B \end{array}\right\} \qquad \textbf{(D15)}$$

2. 设粒子延 (θ, ϕ) 方向传播

$$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



$$\vec{\sigma}.\hat{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Helicity Eigenstates

3. 解的形式，取 $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ **或** $u_B = \begin{pmatrix} a \\ b \end{pmatrix}$

利用 (D15) 得到关系式 $\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$ **(For helicity ± 1)**

因此u_A和u_B的分量都为 $\dfrac{b}{a} = \dfrac{\boxed{\pm 1} - \cos\theta}{\sin\theta} e^{i\phi}$

4. 对于右手螺旋度态，即 helicity +1:

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta} e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad u_{A\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_{B\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

5. 加入比例常数后，得到：

$$u_\uparrow = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \kappa_1 \cos\left(\frac{\theta}{2}\right) \\ \kappa_1 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \kappa_2 \cos\left(\frac{\theta}{2}\right) \\ \kappa_2 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Helicity Eigenstates

6. 由狄拉克方程 (D12)，得 $(\vec{\sigma}.\vec{p})u_A = (E+m)u_B$

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}u_A = \frac{|\vec{p}|}{E+m}\underbrace{(\vec{\sigma}.\hat{p})}_{\boxed{\text{Helicity}}}u_A = \pm\frac{|\vec{p}|}{E+m}u_A \qquad \textbf{(D16)}$$

- 确定 u_A和 u_B的相对归一化，如 正螺旋度

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

负螺旋度粒子态 用同样方法得到

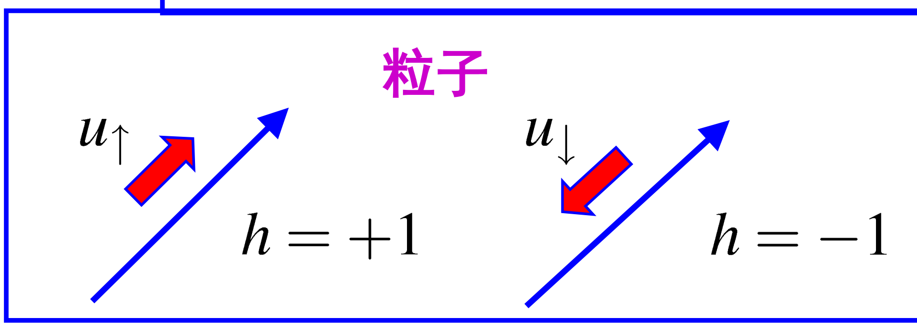
7. 反粒子态的方法相同，需要记住：

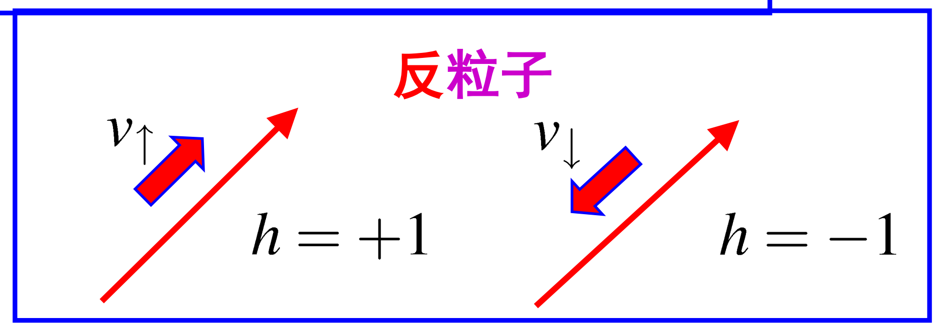
$$\hat{S}^{(v)} = -\hat{S} \qquad \text{即} \quad \hat{h}^{(v)} = -(\vec{\Sigma}.\hat{p}) \Rightarrow (\vec{\Sigma}.\hat{p})v_\uparrow = -v_\uparrow$$

粒子和反粒子的螺旋度本征态：

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$





- 全部四个态归一化到单位体积 $2E$ 个粒子，得到 $\boxed{N=\sqrt{E+m}}$

★ 螺旋度本征态将在后续计算中被广泛运用

Intrinsic Parity of Dirac Particles

non-examinable

- “宇称”操作：空间延原点反演 $x' \equiv -x; \quad y' \equiv -y; \quad z' \equiv -z; \quad t' \equiv t$

  - 设旋量 $\psi(x,y,z,t)$ 满足狄拉克方程

$$i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

宇称变换

$$\psi'(x',y',z',t') = \hat{P}\psi(x,y,z,t)$$

尝试 $\hat{P} = \gamma^0 \qquad \psi'(x',y',z',t') = \gamma^0\psi(x,y,z,t)$ **(D17)**

$(\gamma^0)^2 = 1$ 因此 $\psi(x,y,z,t) = \gamma^0\psi'(x',y',z',t')$

(D17) ➡

$$i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x} + i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y} + i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z} - m\gamma^0\psi' = -i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t}$$

- 新坐标系下表示导数

$$-i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x'} - i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y'} - i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z'} - m\gamma^0\psi' = -i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t'}$$

由于 γ^0 与 $\gamma^1, \gamma^2, \gamma^3$ 得

$$+i\gamma^0\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^0\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^0\gamma^3\frac{\partial\psi'}{\partial z'} - m\gamma^0\psi' = -i\frac{\partial\psi'}{\partial t'}$$

- 前乘 γ^0 ⇨

$$i\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^3\frac{\partial\psi'}{\partial z'} - m\psi' = -i\gamma^0\frac{\partial\psi'}{\partial t'}$$

新坐标系的狄拉克方程

Intrinsic Parity of Dirac Particles

- **如狄拉克旋量，满足如右宇称变换形式：** $\boxed{\psi \to \hat{P}\psi = \pm\gamma^0\psi}$
  - 则狄拉克方程形式不变

(注：上述推算不依赖 $\hat{p} = \pm\gamma^0$ 的选择)

- **静止粒子/反粒子的狄拉克方程解：**

$$\psi = u_1 e^{-imt};\ \psi = u_2 e^{-imt};\ \ \psi = v_1 e^{+imt}; \psi = v_2 e^{+imt}$$

其中 $u_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ v_1 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\ v_2 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix};$

$$\hat{P}u_1 = \pm\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \pm u_1$$ **etc.** ➡ $$\boxed{\begin{array}{ll}\hat{P}u_1 = \pm u_1 & \hat{P}v_1 = \mp v_1\\ \hat{P}u_2 = \pm u_2 & \hat{P}v_2 = \mp v_2\end{array}}$$

➢ **静止的反粒子与粒子的内禀宇称相反。约定：粒子宇称为正，对应于 $\hat{p} = +\gamma^0$**

Summary

- 从线性的狄拉克方程开始构建相对论量子力学的架构 $$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

新自由度：描述自旋 ½ 粒子

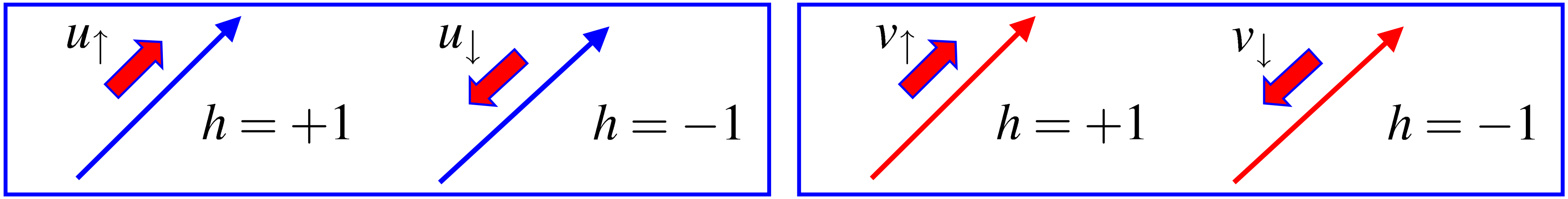
- 狄拉克方程写作 4x4 的γ矩阵 $$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

- 引入四矢量流和伴随旋量 $$j^{\mu} = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi = \overline{\psi}\gamma^{\mu}\psi$$

- 狄拉克方程: 要求含有两个正能量解 和 两个负能量解

- Feynman-Stückelberg 诠释： u_1, u_2, v_1, v_2
  - 延时间反向传播负能粒子解对应延时间正向传播的正能量反粒子解

- 非常有用的基：粒子 和 反粒子的螺旋度本征态



- 对于四分量旋量，电荷共轭 和 宇称操作：

$$\boxed{\psi \rightarrow \hat{C}\psi = i\gamma^2\psi^\dagger} \qquad \boxed{\psi \rightarrow \hat{P}\psi = \gamma^0\psi}$$

★得到了粒子所需的相对论描述…　后面讨论粒子相互作用和QED

Appendix I : Dimensions of the Dirac Matrices

non-examinable

Starting from $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$

For $\hat{H}$ **to be Hermitian for all** $\vec{p}$ **requires** $\alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger$

To recover the KG equation: $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

$$\beta\alpha_j + \alpha_j\beta = 0$$

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \qquad (j \neq k)$$

Consider

with $B^\dagger B = 1$

$$\begin{aligned} Tr(B^\dagger AB) &= B^\dagger_{ij}A_{jk}B_{ki} \\ &= B_{ki}B^\dagger_{ij}A_{jk} \\ &= \delta_{jk}A_{jk} \\ &= Tr(A) \end{aligned}$$

Therefore

$$\begin{aligned} Tr(\alpha) &= Tr(\alpha_j^\dagger \alpha_i \alpha_j) \\ &= -Tr(\alpha_j^\dagger \alpha_j \alpha_i) \qquad \text{(using commutation relation)} \\ &= -Tr(\alpha_i) \\ \Rightarrow \quad Tr(\alpha_i) &= 0 \end{aligned}$$

similarly $Tr(\beta) = 0$

We can now show that the matrices are of even dimension by considering the eigenvalue equation, e.g. $\alpha\vec{x} = \lambda\vec{x}$

$$\vec{x}^{\dagger}\vec{x} = \vec{x}\alpha^{\dagger}\alpha\vec{x} = \lambda^{*}\lambda\vec{x}^{\dagger}\vec{x}$$

Eigenvalues of a Hermitian matrix are real so $\lambda^2 = 1 \quad \rightarrow \quad \lambda = \pm 1$

but $$Tr(\alpha) = \sum_i \lambda_i$$

Since the α_i, β are trace zero Hermitian matrices with eigenvalues of ± 1 they must be of even dimension

For **N=2** the 3 Pauli spin matrices satisfy

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad (j \neq i)$$

But we require 4 anti-commuting matrices. Consequently the α_i, β of the Dirac equation must be of dimension **4**, **6**, **8**,….. The simplest choice for is to assume that the α_i, β are of dimension **4**.

Appendix II : Spin

non-examinable

- **For a Dirac spinor is orbital angular momentum a good quantum number?**
  i.e. does $L = \vec{r} \wedge \vec{p}$ commute with the Hamiltonian?

$$
\begin{aligned}
[H, \vec{L}] &= [\vec{\alpha}.\vec{p} + \beta m, \vec{r} \wedge \vec{p}] \\
&= [\vec{\alpha}.\vec{p}, \vec{r} \wedge \vec{p}]
\end{aligned}
$$

Consider the x component of L:

$$
\begin{aligned}
[H, L_x] &= [\vec{\alpha}.\vec{p}, (\vec{r} \wedge \vec{p})_x] \\
&= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]
\end{aligned}
$$

The only non-zero contributions come from: $[x, p_x] = [y, p_y] = [z, p_z] = i$

$$
\begin{aligned}
[H, L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\
&= -i(\alpha_y p_z - \alpha_z p_y) \\
&= -i(\vec{\alpha} \wedge \vec{p})_x
\end{aligned}
$$

Therefore $$\boxed{[H, \vec{L}] = -i\vec{\alpha} \wedge \vec{p}} \qquad \textbf{(A.1)}$$

★ **Hence the angular momentum does not commute with the Hamiltonian and is not a constant of motion**

Introduce a new 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ **are the Pauli spin matrices: i.e.**

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H,\vec{\Sigma}] = [\vec{\alpha}.\vec{p} + \beta m, \vec{\Sigma}]$$

here $$[\beta,\vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence $$[H,\vec{\Sigma}] = [\vec{\alpha}.\vec{p},\vec{\Sigma}]$$

Consider the x comp: $$\begin{aligned} [H,\Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x[\alpha_x,\Sigma_x] + p_y[\alpha_y,\Sigma_x] + p_z[\alpha_z,\Sigma_x] \end{aligned}$$

Taking each of the commutators in turn:

$$
\begin{aligned}
[\alpha_x, \Sigma_x] &= \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = 0 \\
[\alpha_y, \Sigma_x] &= \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_y\sigma_y - \sigma_y\sigma_x \\ \sigma_y\sigma_x - \sigma_x\sigma_y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} \\
&= -2i\alpha_z \\
[\alpha_z, \Sigma_x] &= 2i\alpha_y
\end{aligned}
$$

Hence

$$
\begin{aligned}
[H, \Sigma_x] &= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x] \\
&= -2ip_y\alpha_x + 2ip_z\alpha_y \\
&= 2i(\vec{\alpha} \wedge \vec{p})_x
\end{aligned}
$$

$$
[H, \vec{\Sigma}] = 2i\vec{\alpha} \wedge \vec{p}
$$

• Hence the observable corresponding to the operator $\vec{\Sigma}$ is also **not** a constant of motion. However, referring back to (A.1)

$$[H,\vec{S}] = \frac{1}{2}[H,\vec{\Sigma}] = i\vec{\alpha}\wedge\vec{p} = -[H,\vec{L}]$$

Therefore: $$\boxed{[H,\vec{L}+\vec{S}] = 0}$$

• Because $$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

the commutation relationships for $\vec{S}$ are the same as for the $\vec{\sigma}$, e.g. $[S_x, S_y] = iS_z$. Furthermore both S^2 and S_z are diagonal

$$S^2 = \tfrac{1}{4}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \tfrac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad S_z = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

• Consequently $S^2\psi = S(S+1)\psi = \frac{3}{4}$ and for a particle travelling along the z direction $S_z\psi = \pm\frac{1}{2}\psi$

★ S has all the properties of spin in quantum mechanics and therefore the Dirac equation provides a natural account of the intrinsic angular momentum of fermions

Appendix III：Magnetic Moment

$$\begin{aligned}(\vec{\sigma}.\vec{p})u_B &= (E-m)u_A\\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B\end{aligned}$$

- 相对论和电磁学曾讲过电磁场中的带电粒子的运动可以通过最小替换得到 $\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$

- 应用到方程 (D12)，得到

$$(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B = (E - m - q\phi)u_A \quad \text{(A.2.1)}$$

$$(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A = (E + m - q\phi)u_B \quad \text{(A.2.2)}$$

对 (A.2.1) 乘以 $(E + m - q\phi)$，再运用 (A.2.2)

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(E + m - q\phi)u_B &= (E - m - q\phi)(E + m - q\phi)u_A\\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (T - q\phi)(T + 2m - q\phi)u_A \quad \text{(A.3)}\end{aligned}$$

其中动能：$T = E - m$

- 在非相对论极限 $T \ll m$ (A.3) 变成

$$(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A \approx 2m(T - q\phi)u_A$$

$$\left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right]u_A \approx 2m(T - q\phi)u_A \quad \text{(A.4)}$$

Appendix III : Magnetic Moment

- **Now** $\vec{\sigma}.\vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$; $\vec{\sigma}.\vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$;

  which leads to $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \overrightarrow{(A \cdot B)} I + i\vec{\sigma} \cdot (\vec{A} \times \vec{B})$

  and $(\vec{\sigma}.\vec{A})^2 = |\vec{A}|^2$

$$\boxed{\left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right] u_A \quad \approx \quad 2m(T - q\phi)u_A}$$

$$= \left[\vec{p}^2 - q\left[\vec{A} \cdot \vec{p} + i\vec{\sigma} \cdot \left(\vec{A} \times \vec{p}\right) + \vec{p} \cdot \vec{A} + i\vec{\sigma} \cdot \left(\vec{p} \times \vec{A}\right)\right] + q^2\vec{A}^2\right]u_A$$

$$= \left[(\vec{p} - q\vec{A})^2 - iq\vec{\sigma} \cdot \left[\vec{A} \times \vec{p} + \vec{p} \times \vec{A}\right]\right]u_A$$

Appendix III : Magnetic Moment

- **The operator on the LHS of (A.4):**

$$\left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right] u_A$$

$$= \left[\vec{p}^2 - q\left[\vec{A}\cdot\vec{p} + i\vec{\sigma}\cdot\left(\vec{A}\times\vec{p}\right) + \vec{p}\cdot\vec{A} + i\vec{\sigma}\cdot\left(\vec{p}\times\vec{A}\right)\right] + q^2\vec{A}^2\right]u_A$$

$$= \left[(\vec{p} - q\vec{A})^2 - iq\vec{\sigma}\cdot\left[\vec{A}\times\vec{p} + \vec{p}\times\vec{A}\right]\right]u_A \qquad \vec{p} = -i\vec{\nabla}$$

$$= \left[(\vec{p} - q\vec{A})^2 - q\vec{\sigma}\cdot\left[\vec{A}\times\vec{\nabla} + \vec{\nabla}\times\vec{A}\right]\right]u_A \qquad \vec{\nabla}\times\left(\vec{A}\psi\right) = \left(\vec{\nabla}\times\vec{A}\right)\psi + \left(\vec{\nabla}\psi\right)\times\vec{A}$$

$$= (\vec{p} - q\vec{A})^2 u_A - q\vec{\sigma}\cdot\left[\left(\vec{\nabla}\times\vec{A}\right)u_A + \left(\vec{\nabla}u_A\right)\times\vec{A} + \vec{A}\times\left(\vec{\nabla}u_A\right)\right]$$

$$= (\vec{p} - q\vec{A})^2 u_A - q\vec{\sigma}\cdot\left(\vec{\nabla}\times\vec{A}\right)u_A \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \left[\left(\vec{p} - q\vec{A}\right)^2 - q\vec{\sigma}\cdot\vec{B}\right]u_A$$

- **替换回 (A.4) 得到薛定谔-泡利方程 描述电磁场中的非相对论自旋 ½ 粒子**

$$\boxed{\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right]u_A = Tu_A}$$

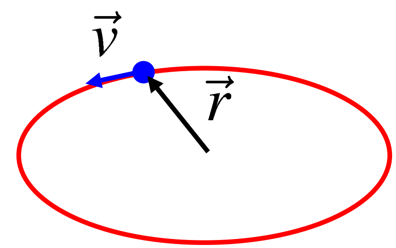
Appendix III : Magnetic Moment

$$\boxed{\left[\frac{1}{2m}(\vec{p}-q\vec{A})^2-\frac{q}{2m}\vec{\sigma}.\vec{B}+q\phi\right]u_A=Tu_A}$$

- 在磁场 $\vec{B}$ 中磁矩的能量为 $\vec{\mu}\cdot\vec{B}$,
  因此得到自旋 ½ 粒子的内秉磁矩为：
  $$\vec{\mu}=\frac{q}{2m}\vec{\sigma}$$

  根据自旋 $\vec{S}=\frac{1}{2}\vec{\sigma}$ 得到：
  $$\boxed{\vec{\mu}=\frac{q}{m}\vec{S}}$$

- 经典地，带电粒子的电流回路 $\vec{\mu}=\pi r^2\frac{qv}{2\pi r}\hat{z}=\frac{q}{2m}\vec{L}$



- 自旋 ½ 狄拉克粒子的内秉磁矩是经典物理预言的**两倍**,
  常用 **磁旋比** g=2表达
  $$\boxed{\vec{\mu}=g\frac{q}{2m}\vec{S}}$$

Appendix III : Magnetic Moment

- **Electron g-factors**

Electron spin g-factor

$$\overrightarrow{\mu_s} = -g_s \frac{e}{2m} \vec{S}$$

Classical, non-relativistic $g_s = 1$

Dirac equation $g_s = 2$

QED $g_s = 2.002\ 319\ 304\ \ldots$

Electron orbital g-factor

$$\overrightarrow{\mu_L} = -g_L \frac{e}{2m} \vec{L}$$

$g_L = 1$

Total angular momentum (Landé) g-factor

$$\left|\overrightarrow{\mu_J}\right| = g_J \frac{e}{2m} \left|\vec{J}\right|$$

$$g_J = g_L \frac{J(J+1) - S(S+1) + L(L+1)}{2J(J+1)} + g_S \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

- **Muon g-factor**

$$\vec{\mu} = g \frac{e}{2m_\mu} \vec{S}$$

learn more g-2 from Fermilab

Appendix IV：Covariance of Dirac Equation

- 展示狄拉克方程在洛伦兹变换下的协变性（协同变换）

$$i\gamma^\mu \partial_\mu \psi = m\psi \quad \text{(A.5)} \qquad \text{变换到} \qquad i\gamma^\mu \partial'_\mu \psi' = m\psi' \quad \text{(A.6)}$$

其中 $\partial'_\mu \equiv \dfrac{\partial}{\partial x'^\mu} = \left(\dfrac{\partial}{\partial t'}, \dfrac{\partial}{\partial x'}, \dfrac{\partial}{\partial y'}, \dfrac{\partial}{\partial z'}\right)$ 且 $\psi'(x') = S\psi(x)$ 是变换后的旋量

- 如果存在 4x4 矩阵 S，狄拉克方程有协变性

- Consider a Lorentz transformation with the primed frame moving with velocity v along the x axis

$$\partial'_\mu = \Lambda^\nu_\mu \partial_\nu \quad \text{其中} \quad \Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With this transformation equation (A.6)

$$i\gamma^\nu \partial'_\nu \psi' \;=\; m\psi' \;\Rightarrow\; i\gamma^\nu \Lambda^\mu_\nu \partial_\mu S\psi \;=\; mS\psi$$

which should be compared to the matrix S multiplying (A.5)

$$iS\gamma^\mu \partial_\mu \psi = mS\psi$$

★ **Therefore the covariance of the Dirac equation will be demonstrated if we can find a matrix S such that**

$$\begin{aligned} i\gamma^\nu \Lambda^\mu_\nu \partial_\mu S\psi &= iS\gamma^\mu \partial_\mu \psi \\ \Rightarrow \quad \gamma^\nu \Lambda^\mu_\nu S\partial_\mu \psi &= S\gamma^\mu \partial_\mu \psi \\ \Rightarrow \quad \boxed{S\gamma^\mu = \gamma^\nu S \Lambda^\mu_\nu} \end{aligned} \qquad \textbf{(A.7)}$$

• **Considering each value of** $\mu = 0,\ 1,\ 2,\ 3$

$$\left.\begin{aligned} S\gamma^0 &= \gamma\gamma^0 S - \beta\gamma\gamma^1 S \\ S\gamma^1 &= -\beta\gamma\gamma^0 S + \gamma\gamma^1 S \\ S\gamma^2 &= \gamma^2 S \\ S\gamma^3 &= \gamma^3 S. \end{aligned}\right\}$$

where $\gamma = (1-\beta^2)^{-1/2}$

and $\beta = v/c$

Appendix IV : Covariance of Dirac Equation

- It is easy (although tedious) to demonstrate that the matrix:

$$\boxed{S = aI + b\gamma^0\gamma^1} \quad \text{with} \quad \boxed{a = \sqrt{\tfrac{1}{2}(\gamma+1)}, \quad b = \sqrt{\tfrac{1}{2}(\gamma-1)}}$$

satisfies the above simultaneous equations

NOTE: For a transformation along in the $-x$ direction $b = -\sqrt{\tfrac{1}{2}(\gamma-1)}$

Now consider the effect of this transformation on the spinor for a particle at rest

$$u_1(p) = u_1(m,0) = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

In the Dirac–Pauli representation, the matrix $S = aI + b\gamma^0\gamma^1$ is

$$S = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}$$

Appendix IV : Covariance of Dirac Equation

Therefore, the transformed spinor of the particle is

$$u_1'(p') = Su_1(p) = \sqrt{2m}\begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} = \sqrt{E'+m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ b/a \end{pmatrix}$$

$$2ma^2 = m(\gamma+1) = E'+m$$

$$\frac{b}{a} = \sqrt{\frac{\gamma-1}{\gamma+1}} + \sqrt{\frac{\gamma^2-1}{(\gamma+1)^2}} = \frac{\beta\gamma}{(\gamma+1)}$$

As the primed frame moving with velocity v along the x axis, the velocity of the particle is $v' = -v\hat{x}$ and therefore $p_x' = m\beta\gamma$ and $E = m\gamma$

$$u_1'(p') = \sqrt{E'+m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dfrac{p_x'}{E'+m} \end{pmatrix}$$

which is, as expected, the corresponding general solution to the Dirac equation for a particle with momentum in the x-direction, as given by P.32

★ **To summarise, under a Lorentz transformation a spinor $\psi(x)$ transforms to $\psi'(x') = S\psi(x)$. This transformation preserves the mathematical form of the Dirac equation**

Appendix V : Transformation of Dirac Current

non-examinable

★ The Dirac current $j^\mu = \overline{\psi}\gamma^\mu\psi$ plays an important rôle in the description of particle interactions. Here we consider its transformation properties.

• Under a Lorentz transformation we have $\psi' = S\psi$

and for the adjoint spinor: $\overline{\psi}' = \psi'^\dagger\gamma^0 = S\psi^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$

• First consider the transformation properties of $\overline{\psi'}\psi'$

$$\overline{\psi'}\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi$$

where $$S^\dagger = aI + b\gamma^{1\dagger}\gamma^{0\dagger} = aI - b\gamma^1\gamma^0$$

giving
$$\begin{aligned} S^\dagger\gamma^0 S &= (aI - b\gamma^1\gamma^0)\gamma^0(aI + b\gamma^0\gamma^1) \\ &= a^2\gamma^0 - b^2\gamma^1\gamma^0\gamma^0\gamma^0\gamma^1 + ab\gamma^0\gamma^0\gamma^1 - b\gamma^1\gamma^0\gamma^0 \\ &= a^2\gamma^0 + b^2\gamma^0(\gamma^0)^2(\gamma^1)^2 + ab\gamma^1 - ab\gamma^1 \\ &= (a^2 - b^2)\gamma^0 \\ &= \gamma^0 \end{aligned}$$

hence $$\overline{\psi'}\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0\psi = \overline{\psi}\psi$$

★ The product $\overline{\psi}\psi$ is therefore a Lorentz invariant. More generally, the product $\overline{\psi_1}\psi_2$ is Lorentz covariant

Appendix V : Transformation of Dirac Current

★ Now consider
$$j'^{\mu} = \overline{\psi'}\gamma^{\mu}\psi'$$
$$= (\psi^{\dagger}S^{\dagger}\gamma^{0})\gamma^{\mu}S\psi$$

• To evaluate this wish to express $\gamma^{\mu}S$ in terms of $S\gamma^{\mu}$

(A.7)
$$S\gamma^{\mu} = \gamma^{\nu}S\Lambda^{\mu}_{\nu}$$

➡
$$S\gamma^{\mu}\Lambda^{\rho}_{\mu} = \gamma^{\nu}S\Lambda^{\mu}_{\nu}\Lambda^{\rho}_{\mu} = \gamma^{\nu}S\delta^{\rho}_{\nu} = \gamma^{\rho}S$$

where we used $\Lambda^{\mu}_{\nu}\Lambda^{\rho}_{\mu} = \delta^{\rho}_{\nu}$

• Rearranging the labels and reordering gives:

$$\boxed{\gamma^{\mu}S = \Lambda^{\mu}_{\nu}S\gamma^{\nu}}$$

$$j'^{\mu} = (\psi^{\dagger}S^{\dagger}\gamma^{0})\gamma^{\mu}S\psi = \psi^{\dagger}S^{\dagger}\gamma^{0}(\Lambda^{\mu}_{\nu}S\gamma^{\nu})\psi$$
$$= \Lambda^{\mu}_{\nu}\psi^{\dagger}(S^{\dagger}\gamma^{0}S)\gamma^{\nu}\psi = \Lambda^{\mu}_{\nu}\psi^{\dagger}\gamma^{0}\gamma^{\nu}\psi$$
$$= \Lambda^{\mu}_{\nu}\overline{\psi}\gamma^{\nu}\psi = \Lambda^{\mu}_{\nu}j^{\nu}$$

➡
$$\boxed{\overline{\psi'}\gamma^{\mu}\psi = \Lambda^{\mu}_{\nu}\overline{\psi}\gamma^{\nu}\psi}$$

★ Hence the Dirac current, $\overline{\psi}\gamma^{\mu}\psi$, transforms as a four-vector

Helicity

For particles at rest, the spinors $u_1(E,0)$ and $u_2(E,0)$ of (4.42) are clearly eigenstates of

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for particles/antiparticles travelling in the $\pm z$-direction ($\mathbf{p} = \pm\vec{p}_z$), the u and v spinors are

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm \mathrm{p}}{E+m} \\ 0 \end{pmatrix},\ u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp \mathrm{p}}{E+m} \end{pmatrix},\ v_1 = N\begin{pmatrix} 0 \\ \frac{\mp \mathrm{p}}{E+m} \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = N\begin{pmatrix} \frac{\pm \mathrm{p}}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{S}_z\, u_1(E,0,0,\pm\mathrm{p}) = +\tfrac{1}{2}u_1(E,0,0,\pm\mathrm{p})$$
$$\hat{S}_z\, u_2(E,0,0,\pm\mathrm{p}) = -\tfrac{1}{2}u_2(E,0,0,\pm\mathrm{p})$$

For antiparticle spinors, the *physical* spin is given by the operator $\hat{S}_z^{(v)} = -\hat{S}_z$ and therefore

$$\hat{S}_z^{(v)} v_1(E,0,0,\pm\mathrm{p}) \equiv -\hat{S}_z v_1(E,0,0,\pm\mathrm{p}) = +\tfrac{1}{2}v_1(E,0,0,\pm\mathrm{p})$$
$$\hat{S}_z^{(v)} v_2(E,0,0,\pm\mathrm{p}) \equiv -\hat{S}_z v_2(E,0,0,\pm\mathrm{p}) = -\tfrac{1}{2}v_2(E,0,0,\pm\mathrm{p})$$