■ 第5章小结

■ Fourier 级数

周期函数: Fourier级数(周期内平方可积)

复指数形式(系数的共轭对称性)，三角形式

收敛性(充分条件)；Gibbs现象

有限区域的Fourier级数

几个典型周期函数的Fourier级数及其性质

功率型信号

$$\frac{1}{T}\int_{-T/2}^{T/2}|f(t)|^2\,\mathrm{d}t=\lim_{N\to\infty}\sum_{m=-N}^{N}|c_m|^2$$

■ Fourier积分

非周期函数：Fourier积分，二个典型信号

时域信号；空间域信号；时-空信号

收敛性(充分条件)

能量型信号

Parseval等式： $$\int_{-\infty}^{\infty}[f(t)]^2\mathrm{d}t=\int_{-\infty}^{\infty}|F(\omega)|^2\,\mathrm{d}\omega$$

Fourier积分的性质：微分性质，积分性质

分数导数和分数积分

Fourier积分算子；逆算子；酉算子；本征值问题

分数Fourier积分

■时频分析

Fourier积分的时频分析能力?

短时Fourier分析（Gabor变换）

频域-时域不确定关系（量子力学比较）

短时Fourier分析高、低频率的分辨能力?

小波变换

■函数变换的本质

不同基函数展开——Fourier分析，分数Fourier分析，短时Fourier分析，短时分数Fourier分析，小波变换

第6章：广义函数

6.1 广义函数的定义

经典函数的困难, 广义函数的定义

6.2 广义函数的运算法则

Dirac Delta函数，广义函数的导数

6.3 广义函数的Fourier变换

速降函数空间，广函FT的基本性质

6.4 弱收敛和Dirac Delta 函数

典型序列，多维Dirac Delta 函数

□经典函数的困难之一 一类物理意义明确但非平方可积函数如何求Fourier变换

$$f(t)=\sin(\omega_0 t),\ (-\infty<t<\infty) \Rightarrow \int_{-\infty}^{\infty}|f(t)|^2\,\mathrm{d}t\to\infty$$

$$\Downarrow$$

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(\omega_0 t)e^{-\mathrm{i}\omega t}\mathrm{d}t$$

$$=\frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\int_{-T}^{T}\sin(\omega_0 t)e^{-\mathrm{i}\omega t}\mathrm{d}t$$

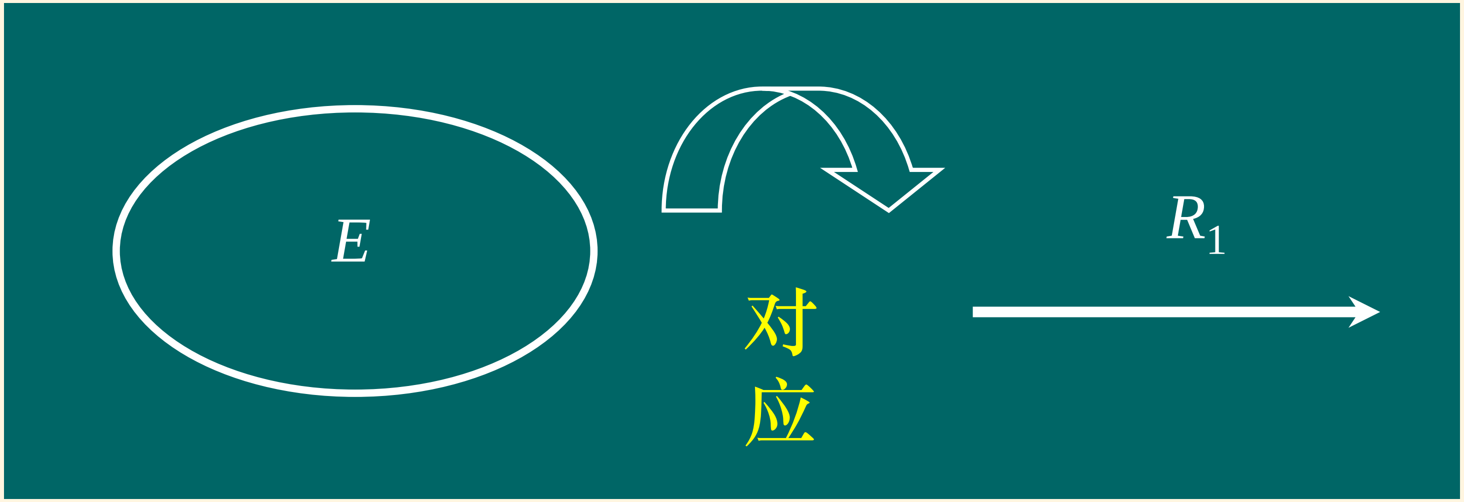
$$=\frac{1}{\mathrm{i}\sqrt{2\pi}}\lim_{T\to\infty}\left(\frac{\sin[(\omega_0-\omega)T]}{(\omega_0-\omega)}-\frac{\sin[(\omega_0+\omega)T]}{(\omega_0+\omega)}\right)$$

——经典函数意义下，极限不存在

6.1 广义函数的定义

■ 经典函数

对每一个$x \in E$， 有唯一确定的数 $f(x) \in R_1$ 与之对应，则称 f 是定义在 E 上的一个函数

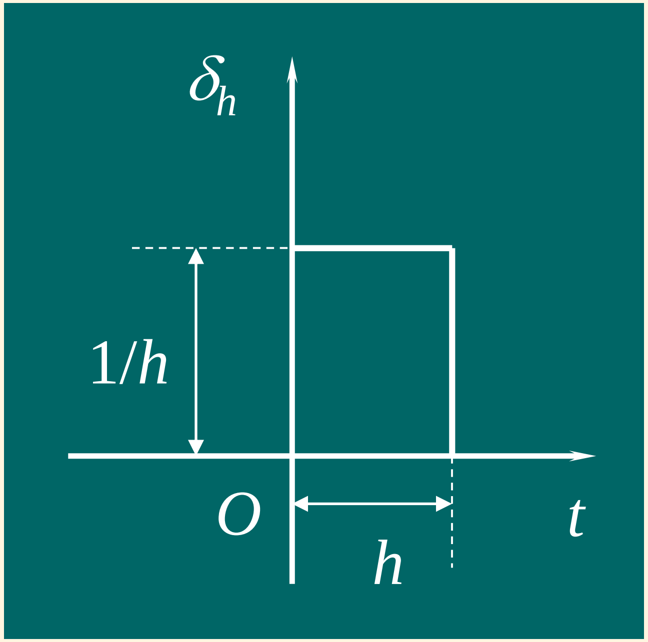


经典函数——数与数的对应关系

□经典函数的困难(1): 描述一些物理现象的困难—物理上的“点源”、“点电荷”、“质点”，以及“脉冲”，经典函数无法描述

例

$$\delta_h = \begin{cases} 0, & \text{if } t < 0 \\ 1/h, & \text{if } 0 < t < h \\ 0, & \text{if } t > h \end{cases}$$



显然函数的积分为“1”并且与 h 无关

$$\int_{-\infty}^{\infty} \delta_h(t)\mathrm{d}t = \int_0^h \frac{1}{h}\mathrm{d}t = 1$$

但当 $h\to 0$, 函数本身的变化

$$\lim_{h\to 0}\delta_h=\begin{cases}0, & \text{if } t<0\\ \infty, & \text{if } t=0\\ 0, & \text{if } t>h\end{cases} \Leftarrow \boxed{=\delta(x)}$$

显然，这样的极限无意义！但是，函数的积分与 h 无关，而有意义！

物理上，可以认为 $h\to 0$ 的过程为：信号宽度变窄，但能量保持不变。

❑经典函数的困难(2): 求导与无限求和的交换困难

- **锯齿波的Fourier展开**

$$f(t)=\frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(\frac{n\pi}{l}t\right) \Rightarrow \frac{\mathrm{d}f(t)}{\mathrm{d}t}=2\sum_{n=1}^{\infty}(-1)^{n-1}\sin\left(\frac{n\pi}{l}t\right)$$

—不收敛!

- **方程的解**

$$u(x,t)=\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{l}c\right)\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}at\right)$$

$$\Downarrow$$

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\pi^2}{l^2}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{l}c\right)\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}at\right)$$

—不收敛!

❑**经典函数的困难(3): 一类有物理意义但非平方可积函数如何求积分变换（如Fourier变换）**

- **正弦信号**

$$f(t)=\sin(\omega_0 t),\ (-\infty<t<\infty) \Rightarrow \int_{-\infty}^{\infty}|f(t)|^2\,\mathrm{d}t\to\infty$$

因此，必须推广函数的定义，新的定义：

① 反映通常的数量关系，能包含经典函数在内，且又能反映物理上“点源”分布问题;

② 可求任意阶导数，对经典函数，新定义应与之一致;

③ 推广的函数对求导、求积和求极限可任意交换运算;

④ 推广的函数能作积分变换.

■ 基本函数（试验函数）

空间D为所有在R^n中无穷可微且在不同有界域外恒等于零的函数组成的空间

$$D(R^n) \equiv C_0^\infty(R^n)$$

D中函数序列$\{\varphi_n\}$收敛于零定义为

(1)所有 φ_n 在某同一有界域K外恒为零

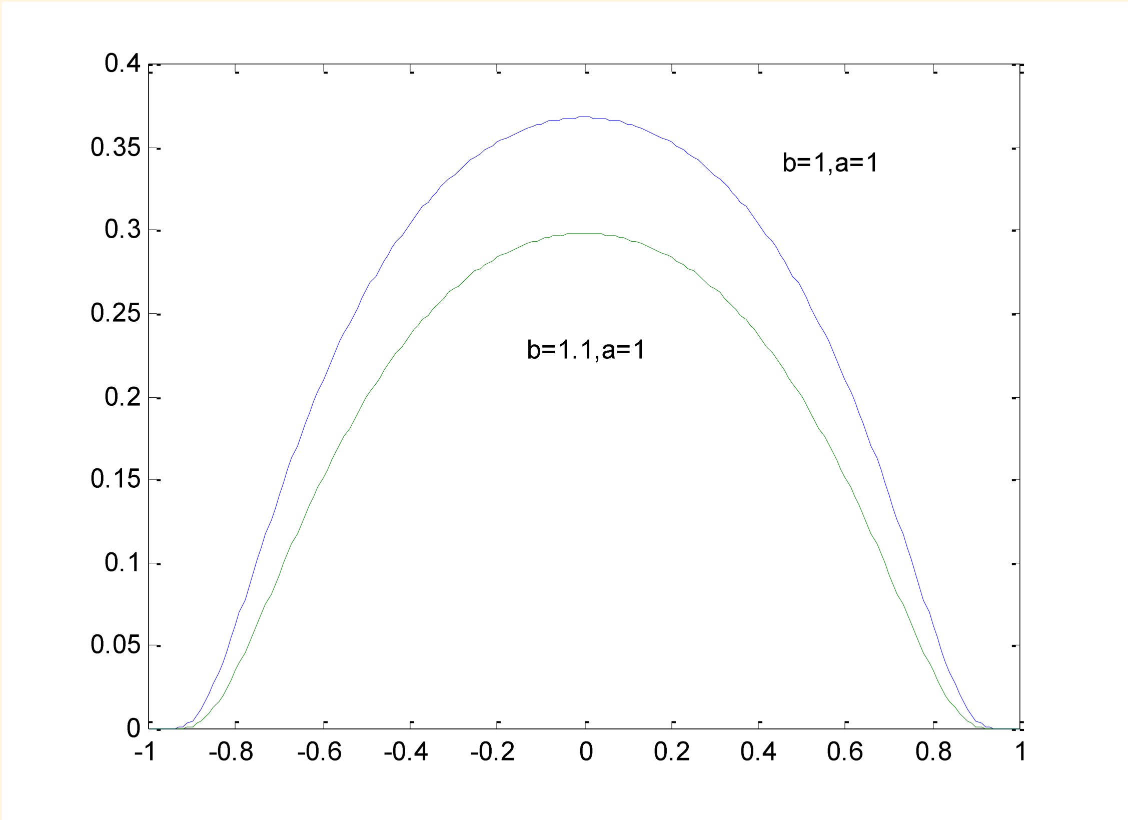
(2) $\{\varphi_n\}$及各阶导数在K上一致收敛于零, 记作

$$\varphi_n \to 0(D)$$

*D*中函数例子

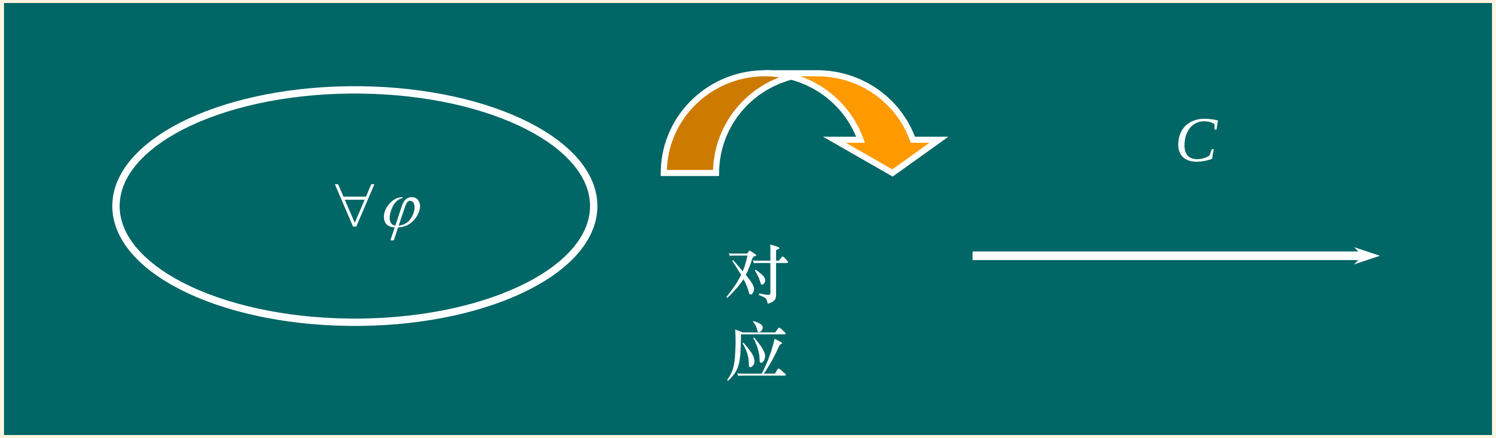
$$\phi(x,a,b)=\begin{cases}\exp\left(-\dfrac{b^2}{a^2-x^2}\right), & |x|<a\\ 0, & |x|\geq a\end{cases}$$

高维：*x* 用矢量长度代替即可



■ 广义函数

You can take it as an operator!



经典函数：数——数的关系

广义函数：检验函数——数的关系

严格地说, 广函 f 不是x的函数, 即对每一个x并不对应一个值, 而是对每一个检验函数φ对应一个值

❑广义函数

广义函数*f* 定义为*D*上的连续线性泛函

$$f(\varphi)\equiv(f,\varphi)=c(f,\varphi),\quad \forall\varphi\in D$$

对*D*中每个元素*φ*, 有确定的实或复数 *c*(*f*, *φ*)与之一一相应——广义函数关系

连续线性泛函

(1)线性, 对任意二个实或复数

$$f(\alpha\phi+\beta\psi)=\alpha f(\phi)+\beta f(\psi),\ \forall\ \phi\text{ 和 }\psi\in D$$

(2)连续性, 即当 $\varphi_n\to 0$ **时, 有** $f(\varphi_n)\to 0$

■ 正则广义函数

一般的可积函数，可定义泛函为线性积分

$$(f,\varphi)=\int f(x)\varphi(x)\mathrm{d}x,\quad \forall\varphi\in D$$

■奇异广义函数

不能用可积函数来表示的广义函数

例：下列泛函定义一个广义函数

$f(\varphi)=(f,\varphi)=\varphi(0)$——**Dirac Delta $\delta(x)$**

——泛函关系不能简单表示为线性积分关系

6.2 广义函数的运算法则

- **加法**

$$(f+g,\varphi)=(f,\varphi)+(g,\varphi),\quad \forall\varphi\in D$$

- **乘法**

$$(\alpha f,\varphi)=\alpha(f,\varphi)=(f,\alpha\varphi),\forall\varphi\in D$$

- **坐标扩展**　首先考虑可积函数

$$\left[f(cx),\varphi(x)\right]=\int_{-\infty}^{\infty}f(cx)\varphi(x)\mathrm{d}x$$

$$\Downarrow$$

$$\left[f(cx),\varphi(x)\right]=\frac{1}{|c|}\left[f(x),\varphi\left(\frac{x}{c}\right)\right]$$

对不是一般可积函数时，直接定义是下列泛函

$$\left[f(cx),\varphi(x)\right]=\frac{1}{|c|}\left[f(x),\varphi\left(\frac{x}{c}\right)\right]$$

例：$\delta(x)$函数

$$\left[\delta(cx),\varphi(x)\right]=\frac{1}{|c|}\left[\delta(x),\varphi\left(\frac{x}{c}\right)\right]$$

$$=\frac{1}{|c|}\varphi(0)=\left(\frac{1}{|c|}\delta(x),\varphi(x)\right)$$

$$\delta(cx)=\frac{\delta(x)}{|c|} \Rightarrow \boxed{\delta(-x)=\delta(x)}$$

所以说： Dirac Delta函数是偶函数

函数相乘　考虑可积情形

$$\left[g(x)f(x),\varphi(x)\right]=\int f(x)\left[g(x)\varphi(x)\right]\mathrm{d}x=\left[f(x),g(x)\varphi(x)\right]$$

当$f(x)$不是正则的广函时, 定义函数相乘

$$(gf,\varphi)=(f,g\varphi)$$

例 对$\delta(x)$函数

$$\left[g(x)\delta(x-y),\varphi(x)\right]=\left[\delta(x-y),g(x)\varphi(x)\right]$$

$$=g(y)\varphi(y)=\left[g(y)\delta(x-y),\varphi(x)\right]$$

$$g(x)\delta(x-y)=g(y)\delta(x-y)$$

$$\Downarrow$$

$$\boxed{x\delta(x)=0\cdot\delta(x)=0}$$

与传统的 $0\cdot\infty$ 不定式不同

广义函数的相等 对所有的基本函数φ，恒有

$$f(\varphi)=g(\varphi),\ \forall\varphi\in D$$

则我们说广义函数f和g相等，写成 $f=g.$

——与经典函数相等的区别：经典函数相等强调逐点相等，而广义函数相等强调的是对基本函数的整体作用。

广义函数的卷积 首先考虑可积函数

$$f*g=\int_{R^n}f(y)g(x-y)\mathrm{d}y=\int_{R^n}g(y)f(x-y)\mathrm{d}y$$

对基本函数 $\forall \varphi \in D(R^n)$

$$[f*g,\varphi(x)] = \int_{R^n}\int_{R^n} f(y)g(x-y)\mathrm{d}y\varphi(x)\mathrm{d}x$$

$$= \int_{R^n} f(y)\left[\int_{R^n} g(x-y)\varphi(x)\mathrm{d}x\right]\mathrm{d}y$$

$$= \int_{R^n} f(y)\left[\int_{R^n} g(x)\varphi(x+y)\mathrm{d}x\right]\mathrm{d}y$$

$$= \{f(x),[g(y),\varphi(x+y)]\}$$

对一般非可积函数，直接定义广义函数的卷积

$$[f*g,\varphi(x)] = \{f(x),[g(y),\varphi(x+y)]\}$$

例1 求连续函数$f(x)$与$\delta(x)$的卷积

解 取$g(x)=\delta(x)$函数，由卷积定义

$$[f*\delta,\varphi(x)]=\{f(x),[\delta(y),\varphi(x+y)]\}$$
$$=[f(x),\varphi(x)]$$

$$\boxed{f*\delta=f(x)}$$

该式也可以作为$\delta(x)$函数的简单定义

$$\int_{-\infty}^{\infty}f(y)\delta(x-y)\mathrm{d}y=f(x)$$

$$\int_{-\infty}^{\infty}f(x-y)\delta(y)\mathrm{d}y=f(x)$$

f 不是检验函数，而是任意连续函数

例2 求$f=\delta(x-a)$与$g=\delta(x)$的卷积

由卷积定义

$$[f*g,\varphi(x)]=\{f(x),[g(y),\varphi(x+y)]\}$$

$$[f*g,\varphi(x)]=\{\delta(x-a),[\delta(y),\varphi(x+y)]\}$$

$$=[\delta(x-a),\varphi(x)]=\varphi(a)$$

形式上可以写成

$$f*g=\delta(a)$$

$$\int_{-\infty}^{\infty}\delta(\eta-a)\delta(x-\eta)\mathrm{d}\eta=\delta(a)$$

注意：定义二个 Dirac delta函数乘积是困难的

广函的合复函数：设$g(x)$在x_0为零, 即$g(x_0)=0$

$$\delta[g(x)]=\frac{1}{|g'(x_0)|}\delta(x-x_0) \Rightarrow \delta[g(x)]=\sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n)$$

证明 由定义

$$(\delta[g(x)],\varphi)=\int_{-\infty}^{\infty}\delta[g(x)]\varphi(x)\mathrm{d}x$$

设$g(x)$有N个零点x_n ($n=1,2,\ldots,N$)，把x轴分成N个区间l_n，每个区间包含一个零点

$$(\delta[g(x)],\varphi)=\sum_{n=1}^{N}\int_{l_n}\delta[g(x)]\varphi(x)\mathrm{d}x$$

作积分变换 $y=g(x)\Rightarrow x=x(y);\ \mathrm{d}y=g'(x)\mathrm{d}x$

$$(\delta[g(x)],\varphi)=\sum_{n=1}^{N}\begin{cases}\displaystyle\int_{l_n}\frac{1}{g'[x(y)]}\delta(y)\varphi[x(y)]\mathrm{d}y; & g'[x(y)]>0\\ \displaystyle-\int_{l_n}\frac{1}{g'[x(y)]}\delta(y)\varphi[x(y)]\mathrm{d}y; & g'[x(y)]<0\end{cases}$$

$$=\sum_{n=1}^{N}\int_{l_n}\frac{1}{|g'[x(y)]|}\delta(y)\varphi[x(y)]\mathrm{d}y$$

在零点附近是减函数，积分方向变化

注意到： $y=0$就是$g(x)=0$, $x_n=x_n(y)\big|_{y=0}$

$$(\delta[g(x)],\varphi)=\sum_{n=1}^{N}\frac{\varphi[x_n(y)]}{|g'[x_n(y)]|}\Bigg|_{y=0}=\sum_{n=1}^{N}\frac{\varphi(x_n)}{|g'(x_n)|}$$

$$\delta[g(x)]=\sum_{n=0}^{N}\frac{1}{|g'(x_n)|}\delta(x-x_n)$$

广义函数的导数：先考虑经典的连续可微函数

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x},\varphi\right)=\int_{-\infty}^{\infty}\frac{\mathrm{d}f}{\mathrm{d}x}\varphi(x)\mathrm{d}x=\left[f(x)\varphi(x)\right]\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}f(x)\varphi''(x)\mathrm{d}x$$

$$=-\int_{-\infty}^{\infty}f(x)\varphi'(x)\mathrm{d}x=(f,-\varphi')$$

推广到任意广义函数　　　**高维偏导数**

$$(f',\varphi)=-(f,\varphi');\ (D^{\alpha}f,\varphi)=(-1)^{|\alpha|}(f,D^{\alpha}\varphi)$$

例1 在广函意义下，求Heaviside函数的导数

$$H(x)=\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$$

$$\left[H'(x),\varphi(x)\right]=-\int_0^{\infty}\varphi'(x)\mathrm{d}x=\varphi(0)=\left[\delta(x),\varphi\right]$$

$$H'(x)=\delta(x)$$

例2 计算广函 $\delta(x-a)$ 的导数

$$\left[\delta'(x-a),\varphi(x)\right]=-\left[\delta(x-a),\varphi'(x)\right]=-\varphi'(a)$$

$$\left[\delta^{(k)}(x-a),\varphi(x)\right]=(-1)^k\varphi^{(k)}(a)$$

——可见δ函数的导数只能用泛函来表示, 而$H(x)$的导数可用δ函数写成显式. 形式上, δ函数的导数可表示成微分算子

$$\boxed{\frac{\mathrm{d}\delta(x-a)}{\mathrm{d}x} = -\delta(x-a)\frac{\mathrm{d}}{\mathrm{d}x}}$$

例3 $g(x)=\delta'(x)$函数, 卷积为

$$[f * g, \varphi(x)] = \{f(x), [\delta'(y), \varphi(x+y)]\}$$

$$= [f(x), -\varphi'(x)] = [f'(x), \varphi(x)]$$

$$f * \delta' = \delta' * f = f'(x) \Rightarrow \delta^{(n)} * f = f^{(n)}(x)$$

例4 对存在第一类间断点的函数$f(x)$, 证明卷积为

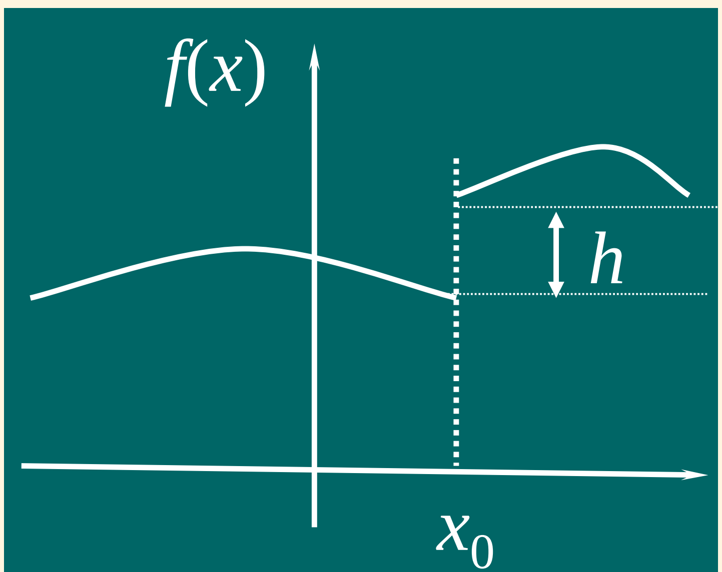
$$\int_{-\infty}^{\infty} f(y)\delta(y-x)\mathrm{d}y = \frac{1}{2}[f(x+0) + f(x-0)]$$

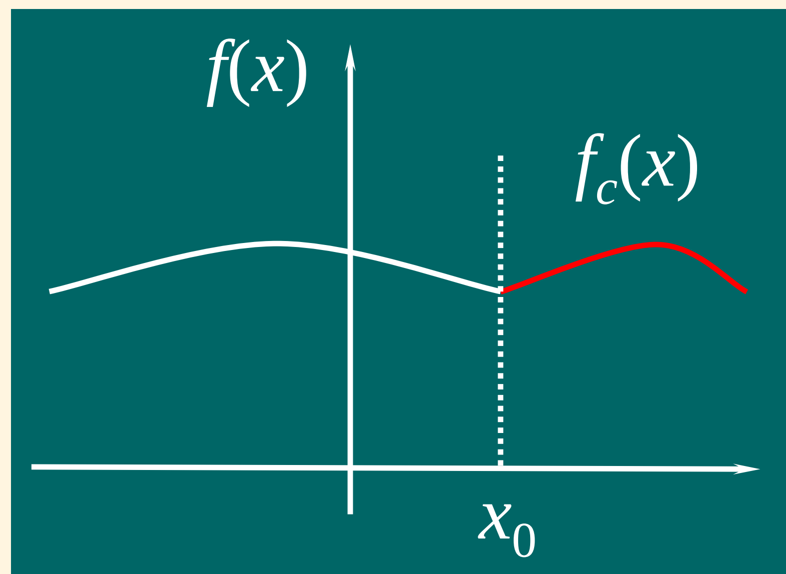
证明：设$f(x)$在x_0点存在第一类间断点

令：$f(x)=f_c(x)+hH(x-x_0)$ ——$f_c(x)$是连续函数

$$\int_{-\infty}^{\infty} f(y)\delta(y-x_0)\mathrm{d}y=\int_{-\infty}^{\infty}\left[f_c(y)+hH(y-x_0)\right]\delta(y-x_0)\mathrm{d}y$$

$$=\int_{-\infty}^{\infty} f_c(y)\delta(y-x_0)\mathrm{d}y+h\int_{-\infty}^{\infty} H(y-x_0)\delta(y-x_0)\mathrm{d}y$$





首先看第二个积分

$$\int_{-\infty}^{\infty} H(y-x_0)\delta(y-x_0)\mathrm{d}y$$

$$=\int_{-\infty}^{\infty} H(y-x_0)\frac{\mathrm{d}H(y-x_0)}{\mathrm{d}y}\mathrm{d}y$$

$$=\int_{-\infty}^{\infty}\frac{1}{2}\mathrm{d}H^2(y-x_0)=\frac{1}{2}H^2(y-x_0)\Big|_{-\infty}^{\infty}=\frac{1}{2}$$

因此（注意到 $h=f(x_0+0)-f(x_0-0)$ **）**

$$\int_{-\infty}^{\infty} f(y)\delta(y-x_0)\mathrm{d}y$$

$$\boxed{f_c(x_0)=f(x_0-0)}$$

$$=f_c(x_0)+\frac{h}{2}=\frac{1}{2}[f(x_0+0)+f(x_0-0)]$$

对连续点$x\neq x_0$，上式显然成立

同样可得

$$\lim_{x\to x_0\pm 0}\frac{\mathrm{d}f(x)}{\mathrm{d}x}=\lim_{x\to x_0\pm 0}f_c'(x)+hH'(x-x_0)$$

$$=\lim_{x\to x_0\pm 0}f_c'(x)+[f(x_0+0)-f(x_0-0)]\delta(x-x_0)$$

因此存在第一类间断点的函数*f*(*x*)的导数为

$$\boxed{\frac{\mathrm{d}f(x)}{\mathrm{d}x}=[f(x_0+0)-f(x_0-0)]\delta(x-x_0)+\begin{cases}f_{左}'(x)\\ f_{右}'(x)\end{cases}}$$

例5 求下列函数的导数

$$\ln x=\begin{cases}\ln|x|, & x>0\\ \ln(-|x|), & x<0\end{cases}$$

因

$$\ln x=\begin{cases}\ln|x|, & x>0\\ \ln(e^{\pm i\pi}|x|), & x<0\end{cases}=\begin{cases}\ln|x|, & x>0\\ \pm i\pi+\ln|x|, & x<0\end{cases}$$

由

$$\frac{df(x)}{dx}=[f(x_0+0)-f(x_0-0)]\delta(x-x_0)+\begin{cases}f'_{左}(x)\\ f'_{右}(x)\end{cases}$$

得到

$$\frac{d\ln x}{dx}=\frac{1}{x}\mp i\pi\delta(x)$$

又

$$\frac{d\ln x}{dx}=\lim_{\varepsilon\to 0}\frac{d\ln(x\pm i\varepsilon)}{dx}=\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon}$$

因此

$$\boxed{\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon}=\frac{1}{x}\mp i\pi\delta(x)}$$

广义函数意义下

□ 广义函数意义下

$$(f,\varphi)=\int_{-\infty}^{\infty}\frac{1}{x}\varphi(x)\mathrm{d}x=\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon}\frac{1}{x}\varphi(x)\mathrm{d}x+\int_{\varepsilon}^{\infty}\frac{1}{x}\varphi(x)\mathrm{d}x+\int_{-\varepsilon}^{\varepsilon}\frac{1}{x}\varphi(x)\mathrm{d}x\right]$$

$$=P\int_{-\infty}^{\infty}\frac{1}{x}\varphi(x)\mathrm{d}x+\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{1}{x}\varphi(x)\mathrm{d}x$$

$$=P\int_{-\infty}^{\infty}\frac{1}{x}\varphi(x)\mathrm{d}x+\varphi(0)\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\frac{1}{x}\mathrm{d}x$$

$$=P\int_{-\infty}^{\infty}\frac{1}{x}\varphi(x)\mathrm{d}x+\varphi(0)[\ln\varepsilon-\ln(e^{\mp\mathrm{i}\pi}\varepsilon)]$$

$$=P\int_{-\infty}^{\infty}\frac{1}{x}\varphi(x)\mathrm{d}x\mp\mathrm{i}\pi\varphi(0)$$

$$\Downarrow$$

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm\mathrm{i}\varepsilon}=\frac{1}{x}\mp\mathrm{i}\pi\delta(x)$$

例6: 证明

$$-\nabla^2 \frac{1}{4\pi r} = -\delta(x)\delta(y)\delta(z) \equiv \delta(\mathbf{r})$$

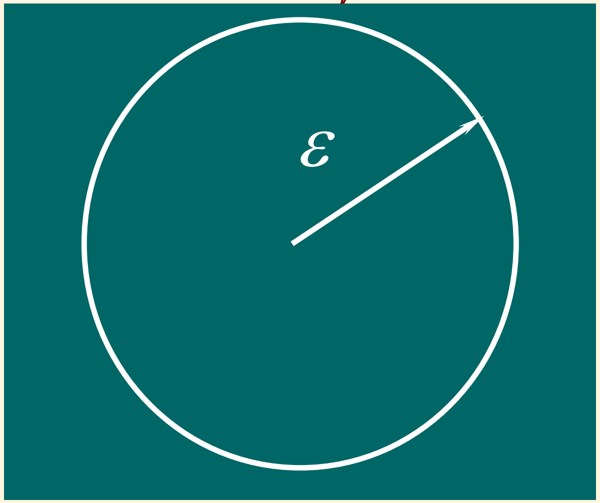
$$r = \sqrt{x^2 + y^2 + z^2}$$

挖去原点这个奇点

证明：在广函意义下

$$\left(\nabla^2 \frac{1}{r}, \varphi\right) = \left(\frac{1}{r}, \nabla^2 \varphi\right) = \iiint \frac{\nabla^2 \varphi}{r} \mathrm{d}\tau = \lim_{\varepsilon \to 0} \iiint_{r \geq \varepsilon} \frac{\nabla^2 \varphi}{r} \mathrm{d}\tau$$

由于φ是局部函数，存在a, 当 $r>a$, $\varphi=0$, 在半径为$r=a$和$r=\varepsilon$的球壳内应用Green公式



$$\iiint_{a-\varepsilon}\left(\frac{\nabla^2\varphi}{r}-\varphi\nabla^2\frac{1}{r}\right)\mathrm{d}\tau=\iint_{S_{a+\varepsilon}}\left[\varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\varphi}{\partial r}\right]\mathrm{d}S$$

①在$r>\varepsilon$区域，直接计算 $\nabla^2(1/r)=0$

②在$r=a$的球面上 $\varphi=\partial\varphi/\partial r=0$

因此只有$r=\varepsilon$球面上的贡献

$$\iiint_{a-\varepsilon}\frac{\nabla^2\varphi}{r}\mathrm{d}\tau=\iint_{S_\varepsilon}\left[\varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\varphi}{\partial r}\right]\mathrm{d}S$$

在$r=\varepsilon$球面上

注意这里检验函数φ的性质很好，不会发散

$$\iint_{r=\varepsilon}\frac{1}{r}\frac{\partial\varphi}{\partial r}\mathrm{d}S=\frac{1}{\varepsilon}\iint_{r=\varepsilon}\frac{\partial\varphi}{\partial r}\varepsilon^2\mathrm{d}\Omega=\varepsilon\iint_{r=\varepsilon}\frac{\partial\varphi}{\partial r}\mathrm{d}\Omega=O(\varepsilon)$$

$$\iint_{r=\varepsilon} \varphi \frac{\partial}{\partial r}\left(\frac{1}{r}\right) \mathrm{d}S = -\frac{1}{\varepsilon^2}\iint_{r=\varepsilon} \varphi\varepsilon^2 \mathrm{d}\Omega = -\iint_{r=\varepsilon} \varphi \mathrm{d}\Omega$$

因此

$$\left(\nabla^2 \frac{1}{r}, \phi\right) = \lim_{\varepsilon\to 0}\iiint_{r\geq\varepsilon} \frac{\nabla^2\phi}{r}\mathrm{d}\tau = -\lim_{\varepsilon\to 0}\iint_{r=\varepsilon} \varphi \mathrm{d}\Omega$$

$$= -4\pi\varphi(0,0,0) = -4\pi(\delta, \varphi)$$

$$-\nabla^2 \frac{1}{4\pi r} = \delta(x)\delta(y)\delta(z) \equiv \delta(\boldsymbol{r})$$

$$\boxed{-\nabla^2 \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} = \delta(\boldsymbol{r}-\boldsymbol{r}_0)}$$

6.3 广义函数的Fourier变换

首先考虑*n*维经典函数的FT

$$\boldsymbol{F}f=\frac{1}{(2\pi)^{n/2}}\int f(\boldsymbol{t})e^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{t}}\mathrm{d}^n\boldsymbol{t}$$

因为

$$(\boldsymbol{F}f,\varphi)=\int\left[\frac{1}{(2\pi)^{n/2}}\int f(\boldsymbol{t})e^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{t}}\mathrm{d}^n\boldsymbol{t}\right]\varphi(\boldsymbol{r})\mathrm{d}^n\boldsymbol{r}$$

$$=\int f(\boldsymbol{t})\left[\frac{1}{(2\pi)^{n/2}}\int\varphi(\boldsymbol{r})e^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{t}}\mathrm{d}^n\boldsymbol{r}\right]\mathrm{d}^n\boldsymbol{t}=(f,\boldsymbol{F}\varphi)$$

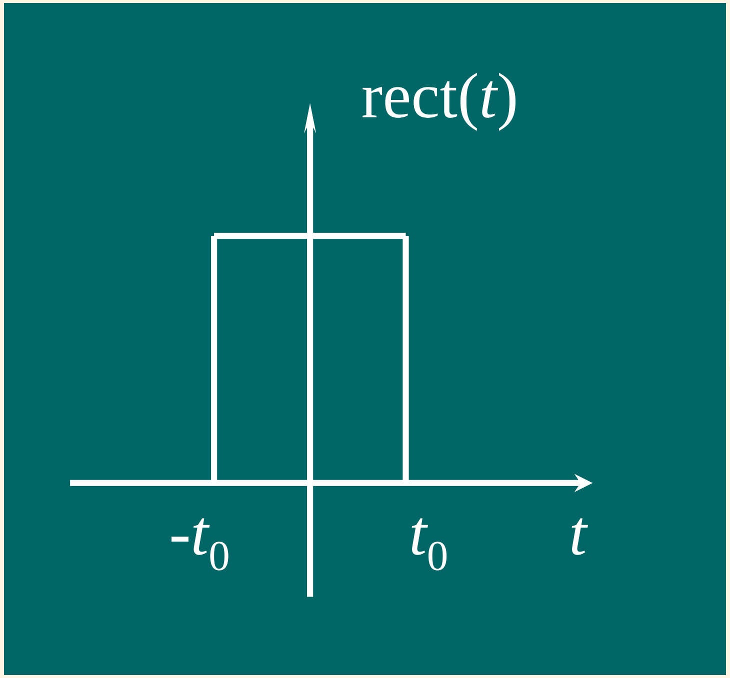
于是，对一般的广函 f, 可以定义其Fourier变换为广函

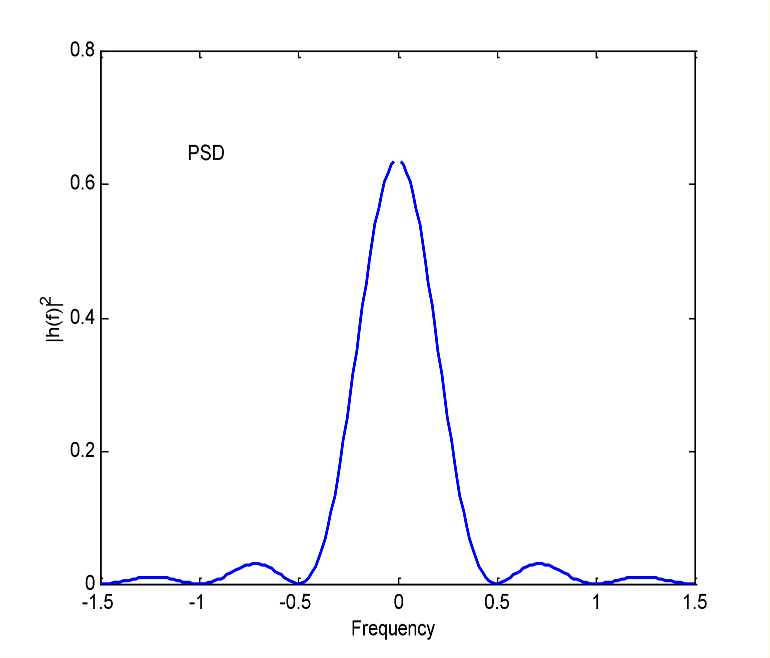
$$(\boldsymbol{F}f,\varphi)=(f,\boldsymbol{F}\varphi)$$

问题： $\boldsymbol{F}\varphi$ 不一定属于D, 因此 $\boldsymbol{F}\varphi$ 不一定都可作为 D 中的试验函数.

必须寻找新的函数空间，定义广义函数，其Fourier变换仍属这个空间, 这样就可以由上式定义广函的Fourier变换

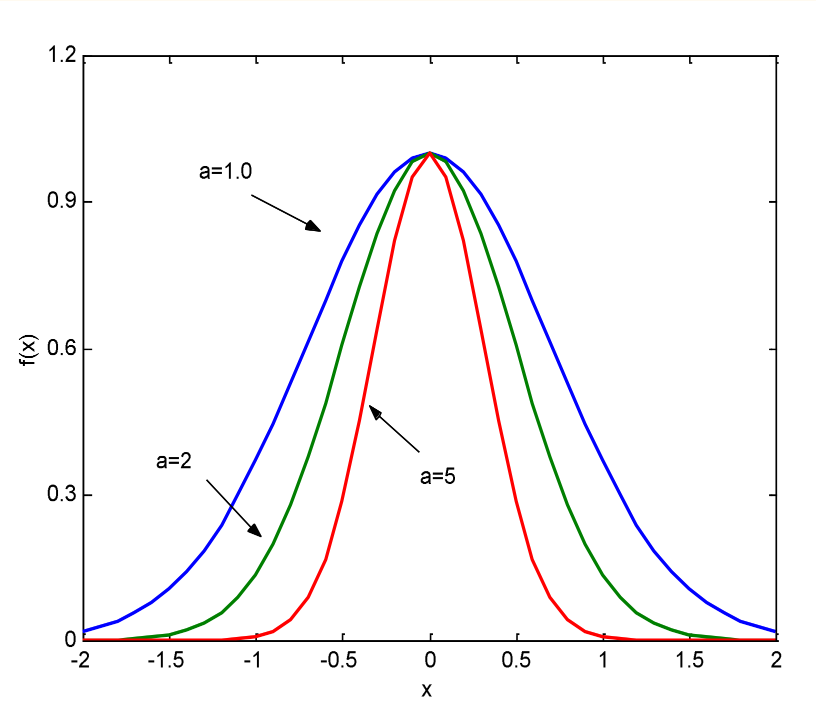
■ 空间局域函数 ⟷ 谱域扩散函数

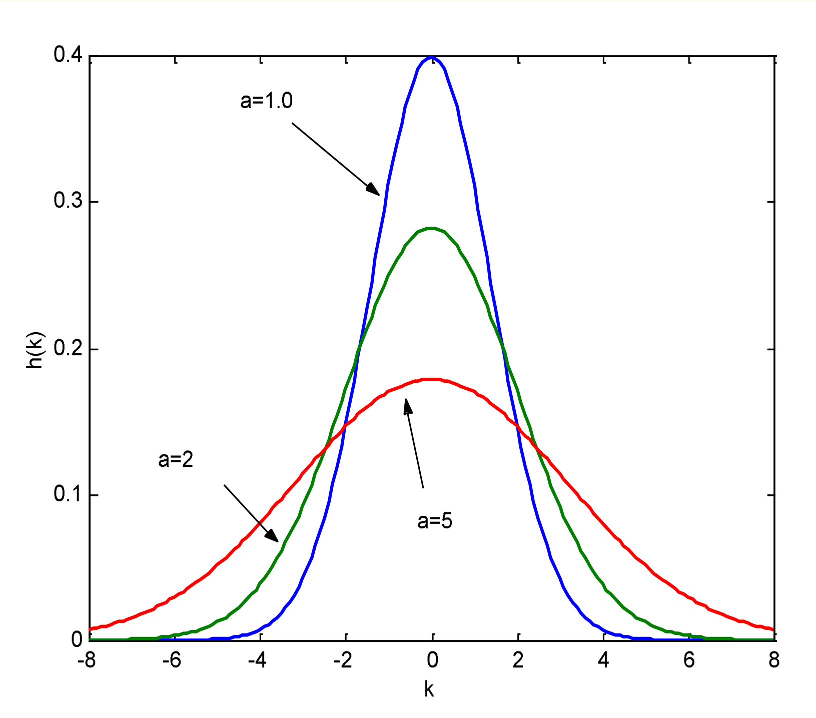




$$\mathrm{rect}(t)=\begin{cases}1, & |t|<t_0\\ 0, & |t|>t_0\end{cases} \Rightarrow F(\omega)=t_0\sqrt{\frac{2}{\pi}}\frac{\sin\omega t_0}{\omega t_0}$$

■ 空间速降函数 ⟺ 谱域速降函数





$$f(x)=e^{-ax^2} \Rightarrow F(k)=\frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}$$

- 由速降函数组成的空间$L(R^n)$中的函数具有这样好的性质. 显然$D(R^n)$是$L(R^n)$的一个子空间

$$L(R^n) \supset D(R^n)$$

- 因为D中的元素总可视为速降函数. 因此, 我们定义广函 f 的Fourier变换为广函

$$(\boldsymbol{F}f, \varphi) = (f, \boldsymbol{F}\varphi), \quad \forall \varphi \in L(R^n)$$

- 因速降函数的Fourier变换仍是速降函数, 故仍是试验函数. 上式右边确实能定义一个广函, 这个广函即是f的Fourier变换.

例1 求 $\delta(x-a)$ 的Fourier变换

$$(\boldsymbol{F}\delta,\varphi)=\left[\delta(x-a),\boldsymbol{F}\varphi\right]$$

$$=\left[\delta(x-a),\frac{1}{\sqrt{2\pi}}\int\varphi(\xi)e^{-\mathrm{i}x\xi}\mathrm{d}\xi\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int\varphi(\xi)e^{-\mathrm{i}a\xi}\mathrm{d}\xi=\frac{1}{\sqrt{2\pi}}(e^{-\mathrm{i}a\xi},\varphi)$$

$$\boldsymbol{F}[\delta(x-a)]=\frac{1}{\sqrt{2\pi}}e^{-\mathrm{i}a\xi} \Rightarrow \boldsymbol{F}[\delta(x)]=\frac{1}{\sqrt{2\pi}}$$

——Dirac delta 函数的谱为常数——脉冲含有丰富的频率成分

例2 求$f(x)=1$的Fourier变换. 根据经典的Fourier变换理论, $f(x)=1$的Fourier变换不存在, 但在广函意义下则存在.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1\cdot e^{ikx}\,\mathrm{d}k = \sqrt{2\pi}\,\delta(x)$$

证明: 由定义

$$[\boldsymbol{F}(1),\varphi] = (1,\boldsymbol{F}\varphi) \equiv (1,\phi)$$

$$= \int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \phi(x) e^{i0\cdot x}\,\mathrm{d}x$$

其中

$$\phi(k) \equiv \boldsymbol{F}\varphi \Rightarrow \varphi(x) = \boldsymbol{F}^{-1}(\phi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k) e^{ikx}\,\mathrm{d}k$$

即

$$\varphi(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{\mathrm{i}0\cdot k}\mathrm{d}k$$

因此

$$[\boldsymbol{F}(1),\varphi]=(1,\boldsymbol{F}\varphi)\equiv(1,\phi)$$
$$=\sqrt{2\pi}\varphi(0)=\sqrt{2\pi}(\delta,\varphi)$$

于是

$$\sqrt{2\pi}\delta(x)=\boldsymbol{F}(1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}1\cdot e^{\mathrm{i}kx}\mathrm{d}k$$

即

$$\boxed{\delta(x)=\frac{\boldsymbol{F}(1)}{\sqrt{2\pi}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathrm{i}kx}\mathrm{d}k}$$

$$\boxed{\delta(x)=\frac{1}{\pi}\int_{0}^{\infty}\cos(kx)\mathrm{d}k}$$

偶函数

■ 二维情况

$$\delta(\boldsymbol{r}) \equiv \delta(x)\delta(y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathrm{i}(k_x x + k_y y)}\mathrm{d}k_x\mathrm{d}k_y$$

■ 三维情况

$$\delta(\boldsymbol{r}) \equiv \delta(x)\delta(y)\delta(z) = \frac{1}{(2\pi)^3}\iiint_{\infty} e^{\mathrm{i}(k_x x + k_y y + k_z z)}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z$$

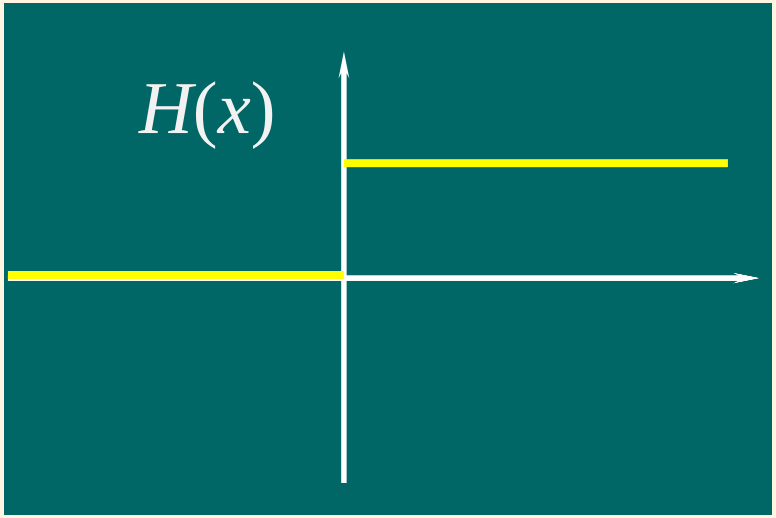
$$= \frac{1}{(2\pi)^3}\int e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k}$$

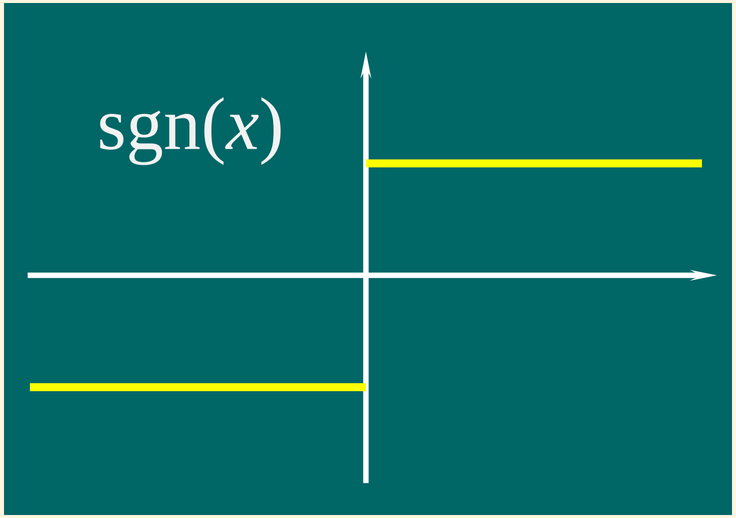
例3 求Heaviside函数的Fourier变换

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

解：注意到符号函数

$$\operatorname{sgn}(x)=\begin{cases}1, x>0\\0, x=0\\-1, x<0\end{cases} \Rightarrow H(x)=\frac{1}{2}[1+\operatorname{sgn}(x)]$$





$$\boldsymbol{F}[H(x)] = \frac{1}{2}\{[\boldsymbol{F}(1) + \boldsymbol{F}[\mathrm{sgn}(x)]\}$$

$$\boldsymbol{F}(1) = \sqrt{2\pi}\delta(x)$$

关键

$$\boldsymbol{F}[\mathrm{sgn}(x)] = ?$$

注意到积分关系

$$\mathrm{sgn}(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\sin\omega x}{\omega}\mathrm{d}\omega = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$\mathrm{sgn}(x) = \frac{\sqrt{2\pi}}{\mathrm{i}\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{\mathrm{i}\omega x}}{\omega}\mathrm{d}\omega\right) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

因此

$$\boldsymbol{F}[\mathrm{sgn}(x)] = \frac{\sqrt{2\pi}}{\mathrm{i}\pi\omega} = \sqrt{\frac{2}{\pi}}\frac{1}{\mathrm{i}\omega}$$

所以

$$\boldsymbol{F}[H(x)] = \frac{1}{2}\{[\boldsymbol{F}(1) + \boldsymbol{F}[\mathrm{sgn}(x)]\}$$

$$= \frac{1}{2}\left\{\sqrt{2\pi}\delta(\omega) + \frac{\sqrt{2\pi}}{\mathrm{i}\pi\omega}\right\} = \frac{\sqrt{2\pi}}{2\pi}\left\{\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right\}$$

$$\boxed{\boldsymbol{F}[H(x)] = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right]}$$

例4 求 $f(t)=\sin\omega_0 t$ 的 Fourier 变换

解

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin\omega_0 t e^{-\mathrm{i}\omega t}\mathrm{d}t$$

$$=\frac{1}{2\mathrm{i}\sqrt{2\pi}}\int_{-\infty}^{\infty}[e^{\mathrm{i}(\omega_0-\omega)t}-e^{-\mathrm{i}(\omega_0+\omega)t}]\mathrm{d}t$$

即

$$=\frac{\sqrt{2\pi}}{2\mathrm{i}}[\delta(\omega_0-\omega)-\delta(\omega_0+\omega)]$$

$$F(\omega)=\frac{1}{\mathrm{i}}\sqrt{\frac{\pi}{2}}[\delta(\omega_0-\omega)-\delta(\omega_0+\omega)]$$

例5 求$F(\omega)=\omega\sin\omega t_0$ 的逆 Fourier 变换

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\omega\sin\omega t_0 e^{\mathrm{i}\omega t}\mathrm{d}\omega=-\frac{\mathrm{i}}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\sin\omega t_0 e^{\mathrm{i}\omega t}\mathrm{d}\omega$$

$$=-\frac{\mathrm{i}}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\frac{1}{2\mathrm{i}}[e^{\mathrm{i}\omega(t_0+t)}-e^{-\mathrm{i}\omega(t_0-t)}]\mathrm{d}\omega$$

$$=\sqrt{\frac{\pi}{2}}\frac{\mathrm{d}}{\mathrm{d}t}[\delta(t-t_0)-\delta(t+t_0)]$$

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\omega\sin\omega t_0 e^{\mathrm{i}\omega t}\mathrm{d}\omega$$

$$==\sqrt{\frac{\pi}{2}}\frac{\mathrm{d}}{\mathrm{d}t}[\delta(t-t_0)-\delta(t+t_0)]$$

——微分算符的形式

■ 广函Fourier变换的基本性质

线性变换

$$\Im[c_1 f_1(t)+c_2 f_2(t)] = c_1\Im[f_1(t)]+c_2\Im[f_2(t)]$$

卷积定理

$$\Im\left[f(t)*g(t)\right] = \sqrt{2\pi}F(\omega)G(\omega)$$

$$\Im\left[f(t)g(t)\right] = \frac{1}{\sqrt{2\pi}}F(\omega)*G(\omega)$$

微分性质:

$$\Im[f'(t)] = \mathrm{i}\omega\Im[f(t)]$$

——与经典函数的Fourier变换性质类似

积分性质:——与经典函数的Fourier变换不同

$$\mathfrak{I}\left[\int_{-\infty}^{t} f(\xi)\mathrm{d}\xi\right] = \frac{1}{\mathrm{i}\omega}\mathfrak{I}[f(t)] + \pi\,\mathfrak{I}[f(t)]\Big|_{\omega=0}\delta(\omega)$$

证明

$$\int_{-\infty}^{t} f(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} f(\xi)H(t-\xi)\mathrm{d}\xi$$

$$\mathfrak{I}\left[\int_{-\infty}^{t} f(\xi)\mathrm{d}\xi\right] = \sqrt{2\pi}\,\mathfrak{I}[f(t)]\,\mathfrak{I}[H(t)]$$

卷积定理

$$\mathfrak{I}\left[\int_{-\infty}^{t} f(\xi)\mathrm{d}\xi\right] = \sqrt{2\pi}\,\mathfrak{I}[f(t)]\frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right\}$$

$$= \frac{1}{\mathrm{i}\omega}\mathfrak{I}[f(t)] + \pi\,\mathfrak{I}[f(t)]\delta(\omega)$$

$$\mathfrak{I}\left[\int_{-\infty}^{t} f(\xi)\mathrm{d}\xi\right] = \frac{1}{\mathrm{i}\omega}\mathfrak{I}[f(t)] + \pi\,\mathfrak{I}[f(t)]\big|_{\omega=0}\,\delta(\omega)$$

因为

$$\lim_{\omega\to 0}\mathfrak{I}[f(t)] = \frac{1}{\sqrt{2\pi}}\lim_{\omega\to 0}\int_{-\infty}^{\infty} f(\xi)e^{-\mathrm{i}\omega\xi}\mathrm{d}\xi = \int_{-\infty}^{\infty} f(\xi)\mathrm{d}\xi$$

如果平均值为零 $\mathfrak{I}[f]\big|_{\omega=0} = \lim_{\omega\to 0}\mathfrak{I}[f(t)] = 0$

积分性质与经典函数一样

例6 求下列函数的FT(频谱)

$$f(t) = \begin{cases} \sin(\omega_g t), & t < 0 \\ 0, & t > 0 \end{cases}$$

物理意义?

解 $$f(t)=\sin(\omega_g t)H(-t)$$

由卷积定理 $$F[f(t)]=F[\sin(\omega_g t)H(-t)]$$

$$=\frac{1}{\sqrt{2\pi}}F[\sin(\omega_g t)]*F[H(-t)]$$

注意到 $$F[H(-t)]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(\omega)-\frac{1}{\mathrm{i}\omega}\right]$$

$$F[\sin(\omega_g t)]=\frac{1}{\mathrm{i}}\sqrt{\frac{\pi}{2}}[\delta(\omega_g-\omega)-\delta(\omega_g+\omega)]$$

$$\Downarrow$$

$$F[f(t)]=\frac{1}{\sqrt{2\pi}}\frac{\omega_g}{\omega^2-\omega_g^2}+\frac{1}{2\mathrm{i}}\sqrt{\frac{\pi}{2}}\left[\delta(\omega_g-\omega)-\delta(\omega_g+\omega)\right]$$

6.4 弱收敛和Dirac Delta 函数

❑弱收敛 (一致收敛、逐点收敛，均方收敛)

给定D上的广函序列$\{f_k\}$, 当有

$$\lim_{k\to\infty}(f_k,\varphi)=(f,\varphi),\quad \forall\varphi\in D$$

称广函序列$\{f_k\}$弱收敛到 f

如果收敛到δ-函数

$$\lim_{k\to\infty}(f_k,\varphi)=(\delta,\varphi)=\varphi(0),\quad \forall\varphi\in D$$

称序列弱收敛到δ-函数

例 函数序列

$$f_k(x)=\begin{cases} k/2, & |x|<1/k \\ 0, & |x|>1/k \end{cases}$$

显然有

$$(f_k,\varphi)=\int_{-\infty}^{\infty} f_k(x)\varphi(x)\mathrm{d}x=\int_{-1/k}^{1/k} f_k(x)\varphi(x)\mathrm{d}x=\varphi(\bar{x})$$

其中 $\bar{x}\in(-1/k,1/k)$

$$k\to\infty,\ \varphi(\bar{x})=\varphi(0)\Rightarrow \lim_{k\to\infty}(f_k,\varphi)=\varphi(0)$$

$$\lim_{k\to\infty} f_k=\delta(x)$$

序列弱收敛到δ函数

常用的弱收敛到δ-函数的序列

$$\lim_{t\to 0}\frac{1}{2a\sqrt{\pi t}}\exp\left[-\frac{(x-\xi)^2}{4a^2t}\right]=\delta(x-\xi)$$

——应用于热传导方程

$$\lim_{r\to 1}\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2}=\delta(\vartheta-\varphi)$$

——应用于二维Laplace方程

$$\lim_{k\to\infty}\frac{1}{\pi}\frac{\sin kx}{x}=\delta(x);\lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}=\delta(x)$$

——应用于多个物理问题

■ **弱收敛序列的微分性质**　如果$\{f_k\}$弱收敛到 f，则微分和极限运算能交换次序

$$\lim_{k\to\infty}\frac{\partial f_k}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\lim_{k\to\infty}f_k\right)=\frac{\partial f}{\partial x_i}$$

$$\Uparrow$$

$$\lim_{k\to\infty}\left(\frac{\partial f_k}{\partial x_i},\varphi\right)=\lim_{k\to\infty}\left(f_k,-\frac{\partial \varphi}{\partial x_i}\right)$$

$$=\left(f,-\frac{\partial \varphi}{\partial x_i}\right)=\left(\frac{\partial f}{\partial x_i},\varphi\right),\quad \forall\varphi\in D$$

例 分析Fourier 级数

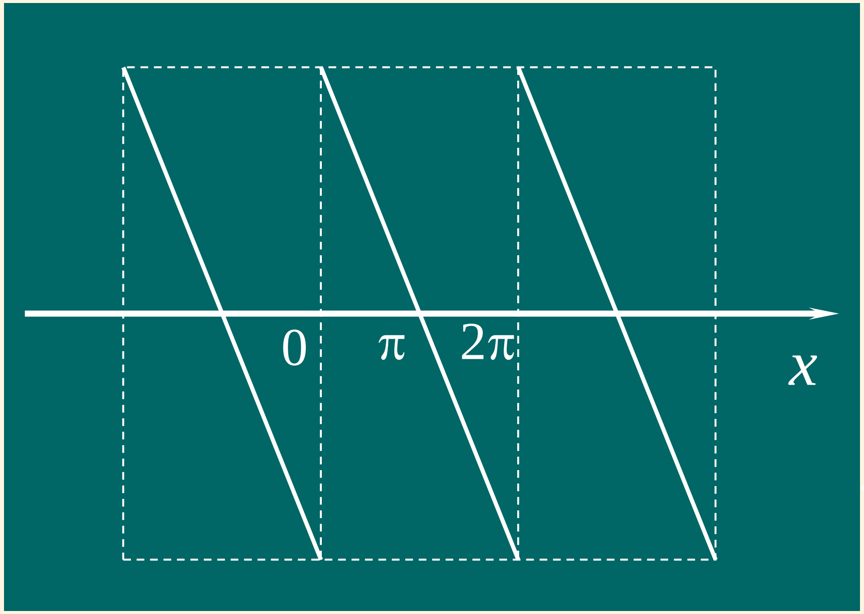
$$f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n}\;\Rightarrow\; f_k(x)=\sum_{n=1}^{k}\frac{\sin nx}{n}$$

因此

$$\frac{\partial f_k}{\partial x} = \sum_{n=1}^{k} \cos nx$$

另一方面，直接求导

$$\frac{\partial f}{\partial x} = -\frac{1}{2} + \pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n)$$



$$f(x) = \frac{1}{2}(\pi - x)$$

$$(0 < x < 2\pi)$$

2π周期函数

58

所以，在广义函数意义下

$$\sum_{n=1}^{\infty}\cos nx=-\frac{1}{2}+\pi\sum_{n=-\infty}^{\infty}\delta(x-2\pi n) \Leftarrow \lim_{k\to\infty}\frac{\partial f_k}{\partial x_i}=\frac{\partial f}{\partial x_i}$$

——在经典函数意义下，没有意义！

■证明序列弱收敛到Delta函数，只要证明

$$\lim_{K\to\infty}\int_{-\infty}^{\infty}\delta_K(t-t_0)\mathrm{d}t=1;\quad \lim_{K\to\infty}\int_{-\infty}^{\infty}f(t)\delta_K(t-t_0)\mathrm{d}t=f(t_0)$$

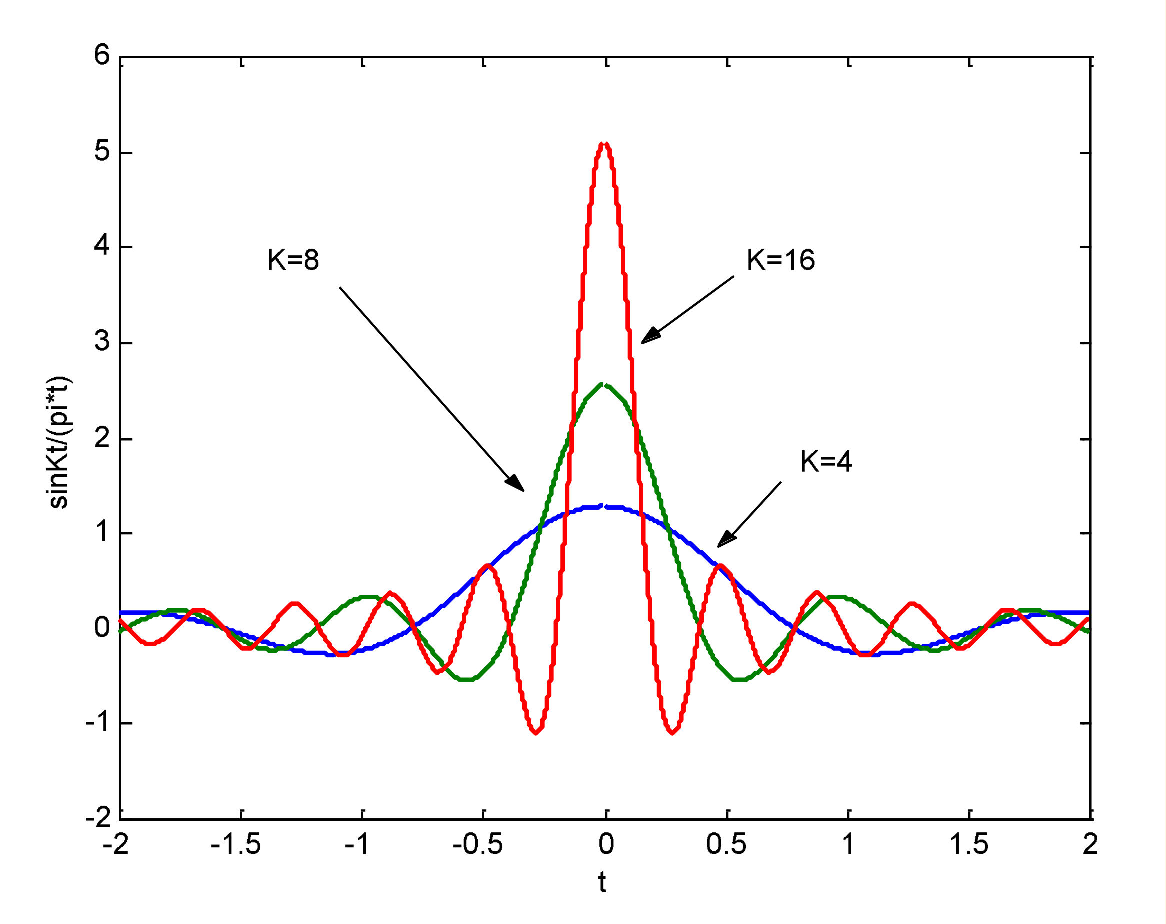
$$\Downarrow$$

$$\lim_{K\to\infty}\left|\int_{-\infty}^{\infty}f(t)\delta_K(t)\mathrm{d}t-f(0)\right|$$

$$=\lim_{K\to\infty}\left|\int_{-\infty}^{\infty}[f(t)-f(0)]\delta_K(t)\mathrm{d}t\right|=0$$

(1)sinc 函数序列

$$\delta(t)=\lim_{K\to\infty}\delta_K(t)=\lim_{K\to\infty}\frac{\sin Kt}{\pi t}$$



证明(不严格)

$$\int_{-\infty}^{\infty}\delta_K(t-t_0)\mathrm{d}t=\int_{-\infty}^{\infty}\frac{\sin K(t-t_0)}{\pi(t-t_0)}\mathrm{d}t=\int_{-\infty}^{\infty}\frac{\sin K(t-t_0)}{\pi K(t-t_0)t}\mathrm{d}[K(t-t_0)]$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin t'}{t'}dt'=1$$

$$\lim_{K\to\infty}\int_{-\infty}^{\infty}f(t)\delta_K(t-t_0)\mathrm{d}t=\lim_{K\to\infty}\int_{-\infty}^{\infty}f(t)\frac{\sin K(t-t_0)}{\pi K(t-t_0)}\mathrm{d}\left[K(t-t_0)\right]$$

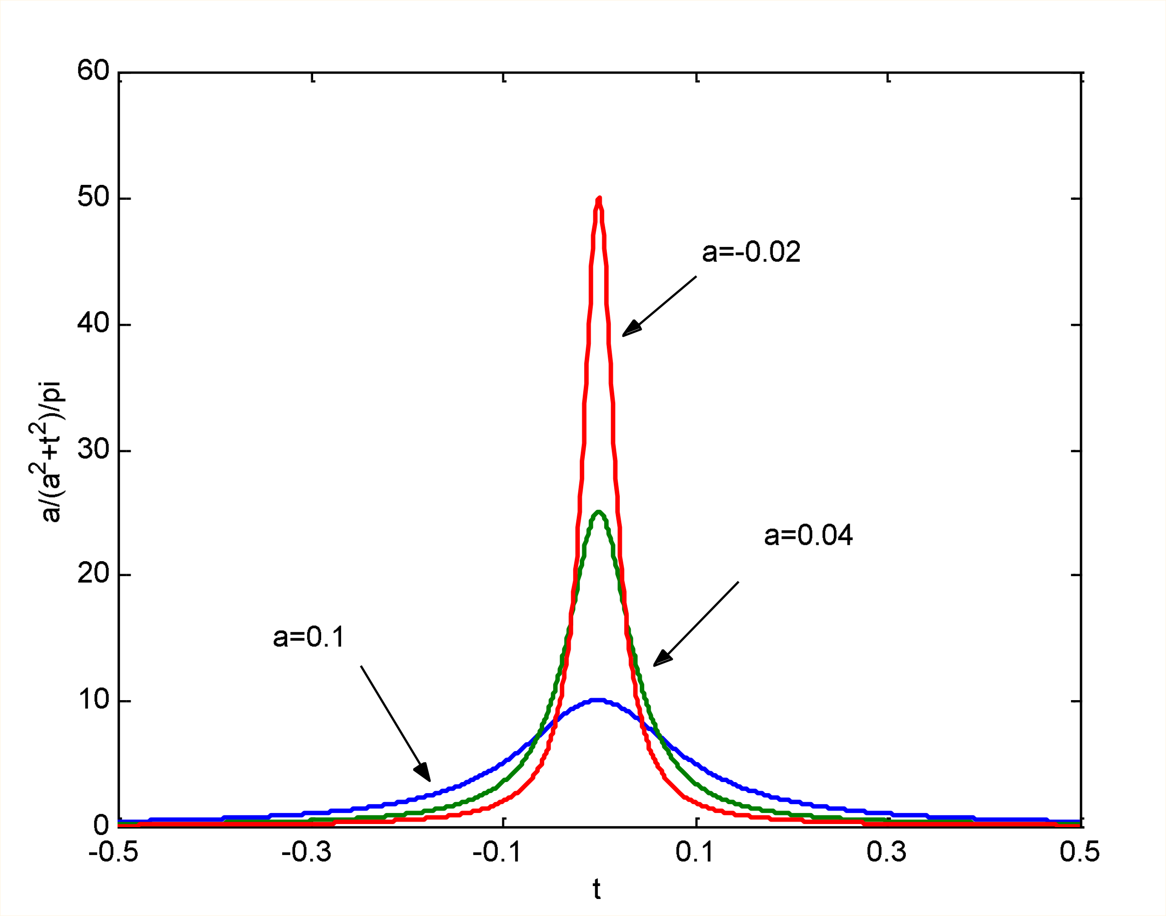
$$\overset{K(t-t_0)=t'}{=\!=\!=\!=}\lim_{K\to\infty}\int_{-\infty}^{\infty}f\left(t_0+\frac{t'}{K}\right)\frac{\sin t'}{\pi t'}\mathrm{d}t'=f(t_0)\int_{-\infty}^{\infty}\frac{\sin t'}{\pi t'}\mathrm{d}t'=f(t_0)$$

$$\lim_{K\to\infty}\int_{-\infty}^{\infty}\delta_K(t-t_0)\mathrm{d}t=1$$

$$\lim_{K\to\infty}\int_{-\infty}^{\infty}f(t)\delta_K(t-t_0)\mathrm{d}t=f(t_0)$$

(2)函数序列

$$\delta(t)=\lim_{a\to 0}\delta_a(t)=\frac{1}{\pi}\lim_{a\to 0}\frac{a}{a^2+t^2}$$



证明(不严格)

$$\int_{-\infty}^{\infty}\delta_a(t-t_0)\mathrm{d}t=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+(t-t_0)^2}\mathrm{d}t$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+t^2}\mathrm{d}t=\frac{2}{\pi}\int_{0}^{\infty}\frac{1}{1+(t/a)^2}\mathrm{d}(t/a)=\frac{2}{\pi}\arctan\left(\frac{t}{a}\right)\Bigg|_0^{\infty}=1$$

$$\lim_{a\to 0}\int_{-\infty}^{\infty}f(t)\delta_a(t-t_0)\mathrm{d}t=\lim_{a\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{a}{a^2+(t-t_0)^2}\mathrm{d}t$$

$$\overset{(t-t_0)/a=t'}{=\!=\!=\!=}\frac{1}{\pi}\lim_{a\to 0}\int_{-\infty}^{\infty}f(t_0+at')\frac{\mathrm{d}t'}{1+t'^2}=f(t_0)\frac{2}{\pi}\int_{0}^{\infty}\frac{\mathrm{d}t'}{1+t'^2}=f(t_0)$$

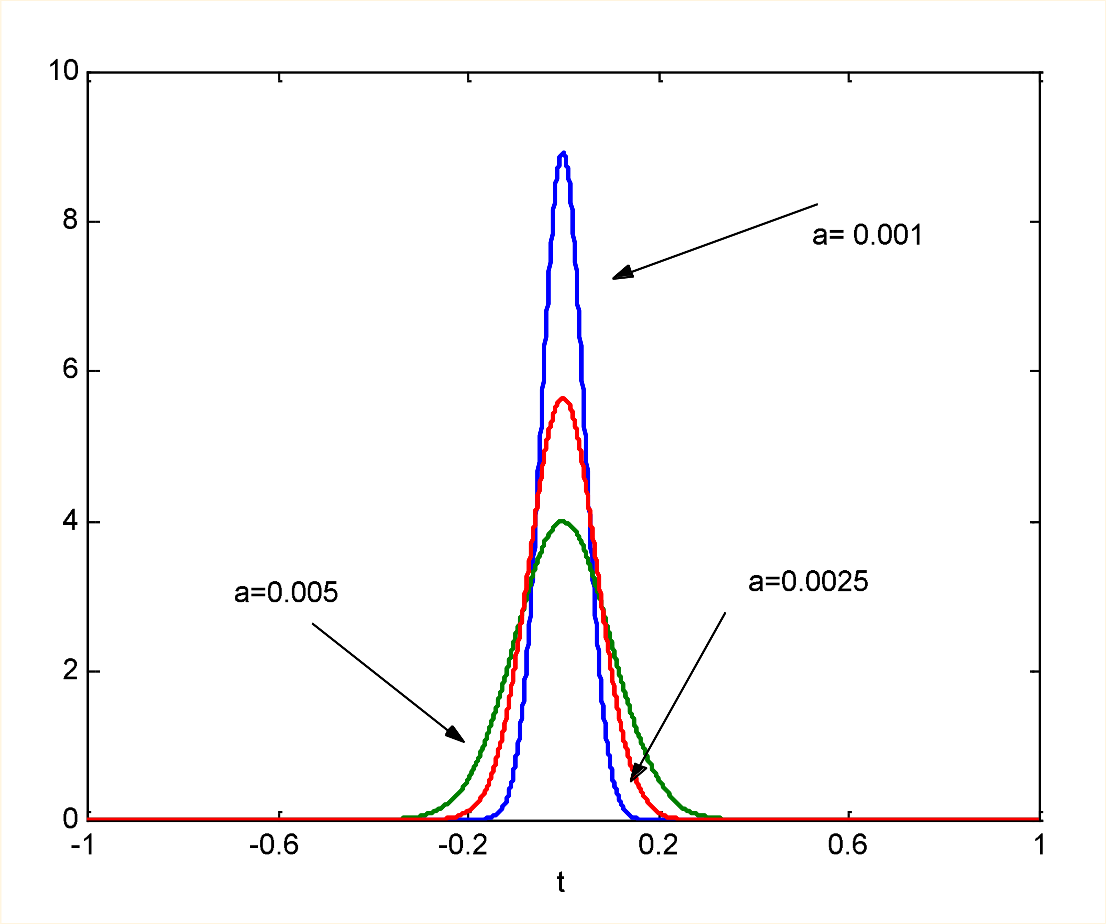
$$\Downarrow$$

$$\lim_{a\to 0}\int_{-\infty}^{\infty}\delta_a(t-t_0)\mathrm{d}t=1$$

$$\lim_{a\to 0}\int_{-\infty}^{\infty}f(t)\delta_a(t-t_0)\mathrm{d}t=f(t_0)$$

(3) 函数序列

$$\delta(t)=\lim_{a\to 0}\delta_a(t)=\lim_{a\to 0}\frac{1}{2\sqrt{\pi a}}\exp\left(-\frac{t^2}{4a}\right)$$



证明(不严格)

$$\int_{-\infty}^{\infty}\delta_a(t)\mathrm{d}t=\frac{1}{2\sqrt{\pi a}}\int_{-\infty}^{\infty}\exp\left(-\frac{t^2}{4a}\right)\mathrm{d}t$$

$$=\frac{1}{\sqrt{\pi a}}\int_{0}^{\infty}\exp\left[-\left(\frac{t}{\sqrt{4a}}\right)^2\right]\mathrm{d}t=\frac{2}{\sqrt{\pi}}\int_{0}^{\infty}\exp(-y^2)=1$$

$$\lim_{a\to 0}\int_{-\infty}^{\infty}f(t)\delta_a(t-t_0)\mathrm{d}t=\lim_{a\to 0}\frac{1}{2\sqrt{\pi a}}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(t-t_0)^2}{4a}\right)\mathrm{d}t$$

$$=\lim_{a\to 0}\frac{1}{2\sqrt{\pi a}}\int_{-\infty}^{\infty}f(t)\exp\left[-\left(\frac{t-t_0}{\sqrt{4a}}\right)^2\right]\mathrm{d}t$$

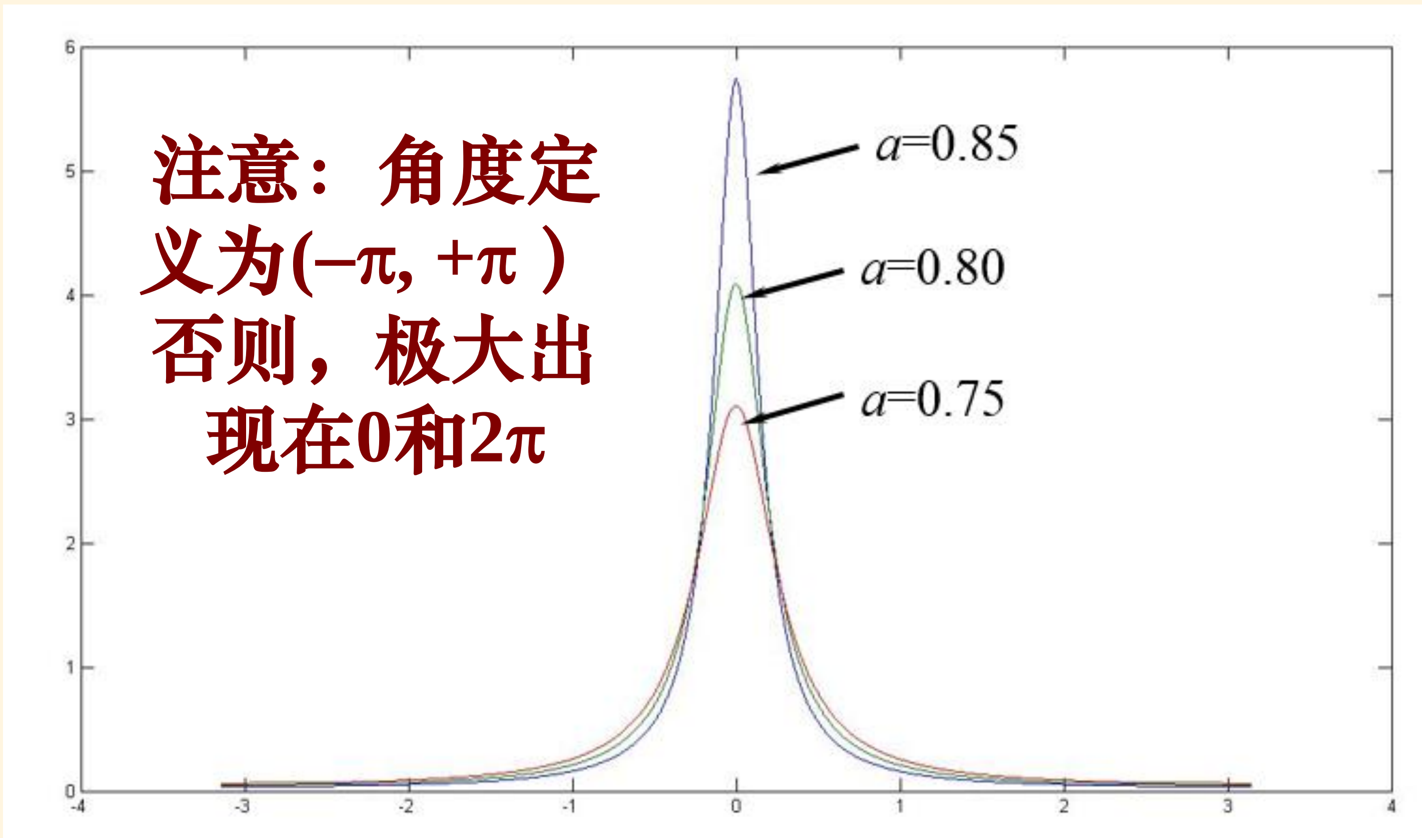
$$=\frac{1}{\sqrt{\pi}}\lim_{a\to 0}\int_{-\infty}^{\infty}f(t_0+\sqrt{4a}y)\exp(-y^2)\mathrm{d}y=f(t_0)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-y^2)\mathrm{d}y=f(t_0)$$

$$\lim_{a\to 0}\int_{-\infty}^{\infty}\delta_a(t-t_0)\mathrm{d}t=1$$

$$\lim_{a\to 0}\int_{-\infty}^{\infty}f(t)\delta_a(t-t_0)\mathrm{d}t=f(t_0)$$

(4)函数序列

$$\delta(\vartheta-\varphi)=\lim_{a\to1}\delta_a(t)=\lim_{a\to1}\frac{1}{2\pi}\frac{1-a^2}{1-2a\cos(\vartheta-\varphi)+a^2}$$



证明

$$\int_{-\pi}^{\pi}\delta_a(\vartheta)\mathrm{d}\vartheta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-a^2}{1-2a\cos(\vartheta-\varphi)+a^2}\mathrm{d}\vartheta$$

$$\frac{1-a^2}{1-2a\cos(\vartheta-\varphi)+a^2}=1+2\sum_{k=1}^{\infty}a^k\cos(k\vartheta)\ \ (a<1)$$

$$\int_{-\pi}^{\pi}\delta_a(\vartheta)\mathrm{d}\vartheta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1+2\sum_{k=1}^{\infty}a^k\cos(k\vartheta)\right]\mathrm{d}\vartheta$$

$$=1+\frac{1}{2\pi}\left[2\sum_{k=1}^{\infty}a^k\int_{-\pi}^{\pi}\cos(k\vartheta)\mathrm{d}\vartheta\right]=1$$

$$\int_{-\pi}^{\pi}\delta_a(\vartheta)\mathrm{d}\vartheta=1$$

因此

$$\left|\int_{-\pi}^{\pi} f(\vartheta)\delta_a(\vartheta)\mathrm{d}\vartheta - f(0)\right| = \left|\int_{-\pi}^{\pi}\left[f(\vartheta) - f(0)\right]\delta_a(\vartheta)\mathrm{d}\vartheta\right|$$

$$= \frac{1-a^2}{2\pi}\left|\int_{-\pi}^{\pi}\left[f(\vartheta) - f(0)\right]\frac{1}{1-2a\cos\vartheta + a^2}\mathrm{d}\vartheta\right|$$

由图看见：当*a*→1,积分主要是*ϑ*=0附近的贡献，其它部分由于*a*→1而为0

$$\left|\int_{-\pi}^{\pi} f(\vartheta)\delta_a(\vartheta)\mathrm{d}\vartheta - f(0)\right| = \frac{1-a^2}{2\pi}\left|\int_{-\delta}^{\delta}\frac{\left[f(\vartheta) - f(0)\right]\mathrm{d}\vartheta}{1-2a\cos\vartheta + a^2}\right|$$

$$\leq \frac{1-a^2}{2\pi}\int_{-\delta}^{\delta}\left|\frac{f(\vartheta) - f(0)}{\sin\vartheta}\right|\frac{|\sin\vartheta|}{1-2a\cos\vartheta + a^2}|\mathrm{d}\vartheta|$$

注意到：当$|\vartheta|<\pi/2$时， $|\sin\vartheta| > 2|\vartheta|/\pi$

$$\left|\int_{-\pi}^{\pi} f(\vartheta)\delta_a(\vartheta)\mathrm{d}\vartheta - f(0)\right| \le \frac{1-a^2}{4}\int_{-\delta}^{\delta}\left|\frac{f(\vartheta)-f(0)}{\vartheta}\right|\frac{|\sin\vartheta|}{|1-2a\cos\vartheta+a^2|}|\mathrm{d}\vartheta|$$

$$\le \frac{1-a^2}{2}|\max[f'(\vartheta)]|\int_0^{\pi}\frac{\sin\vartheta}{1-2a\cos\vartheta+a^2}\mathrm{d}\vartheta \sim (1-a)\ln(1-a)$$

积分区域再放大

积分直接求出，然后求极限

$$\lim_{a\to 1}\left|\int_{-\pi}^{\pi} f(\vartheta)\delta_a(\vartheta)\mathrm{d}\vartheta - f(0)\right| = 0$$

$$\lim_{a\to 1}\int_{-\pi}^{\pi} f(\vartheta)\delta_a(\vartheta)\mathrm{d}\vartheta = f(0)$$

■多维δ函数和其他形式的δ函数

■多维δ函数定义为

$$\delta(x_1 - x_1^0, x_2 - x_2^0, \ldots., x_n - x_n^0)$$
$$= \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\cdot\ldots\cdot\delta(x_n - x_n^0)$$

⇧

不能简单看作坐标分离

⇩

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(\mathbf{r}, \mathbf{r}_0) = \delta(x - x_0, y - y_0, z - z_0)$$
$$= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$
$$= \delta(x, x_0)\delta(y, y_0)\delta(z, z_0)$$

■曲线坐标中的δ函数

$$\int_V f(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}_0)\mathrm{d}^3\boldsymbol{r} = f(\boldsymbol{r}_0)$$

$$x = x(q_1, q_2, q_3); y = y(q_1, q_2, q_3); z = z(q_1, q_2, q_3)$$

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, q_3); \boldsymbol{r}_0 = \boldsymbol{r}_0(q_0^1, q_0^2, q_0^3)$$

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}x = |J|\,\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3$$

$$\int_V f(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\mathrm{d}V = f(x_0, y_0, z_0)$$

$$\int_V f(q_1,q_2,q_3)\delta(q_1-q_0^1)\delta(q_2-q_0^2)\delta(q_3-q_0^3)\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 = f(q_0^1, q_0^2, q_0^3)$$

改写成对体积元dV的 积分

$$\int_V f(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\mathrm{d}V = f(x_0,y_0,z_0)$$

$$\int_V f(q_1,q_2,q_3)\frac{\delta(q_1-q_0^1)\delta(q_2-q_0^2)\delta(q_3-q_0^3)}{|J|}\mathrm{d}V = f(q_0^1,q_0^2,q_0^3)$$

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{\delta(q_1-q_0^1)\delta(q_2-q_0^2)\delta(q_3-q_0^3)}{|J|}$$

$$J=\frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)}\neq 0$$

■ 注意Delta函数的量纲问题

$$\delta(x-x_0)\sim\frac{1}{x};\delta(y-y_0)\sim\frac{1}{y};\delta(z-z_0)\sim\frac{1}{z}$$

$$\delta(q_1-q_0^1)\sim\frac{1}{q_1};\delta(q_2-q_0^2)\sim\frac{1}{q_2};\delta(q_3-q_0^3)\sim\frac{1}{q_3}$$

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\sim\frac{1}{xyz}$$

$$\delta(q_1-q_0^1)\delta(q_2-q_0^2)\delta(q_3-q_0^3)\sim\frac{1}{q_1q_2q_3}$$

保持两边量纲一致

$$J\sim\frac{xyz}{q_1q_2q_3}\Rightarrow\frac{\delta(q_1-q_0^1)\delta(q_2-q_0^2)\delta(q_3-q_0^3)}{|J|}\sim\frac{1}{xyz}$$

■ 柱坐标

$$x=\rho\cos\varphi;\ y=\rho\sin\varphi;\ z=z \Rightarrow J=\frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)}=\rho$$

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{\delta(\rho-\rho_0)\delta(\varphi-\varphi_0)\delta(z-z_0)}{\rho}$$

① 当ρ_0>0时, (x_0,y_0,z_0)与(ρ_0,φ_0, z_0)一一对应，故上式成立；

② 当ρ_0=0时，原点的φ_0没有定义.

$$\delta(\mathbf{r})=\delta(x)\delta(y)\delta(z)=\frac{1}{2\pi\rho}\delta(\rho)\delta(z)$$

■ 二维极坐标

$$\delta(x-x_0)\delta(y-y_0)=\frac{\delta(\rho-\rho_0)\delta(\varphi-\varphi_0)}{\rho}$$

$$\delta(x)\delta(y)=\frac{1}{2\pi\rho}\delta(\rho)$$

■ 球坐标

$$x=r\sin\vartheta\cos\varphi;\ y=r\sin\vartheta\sin\varphi;\ z=r\cos\vartheta$$

$$J=\frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)}=r^2\sin\vartheta$$

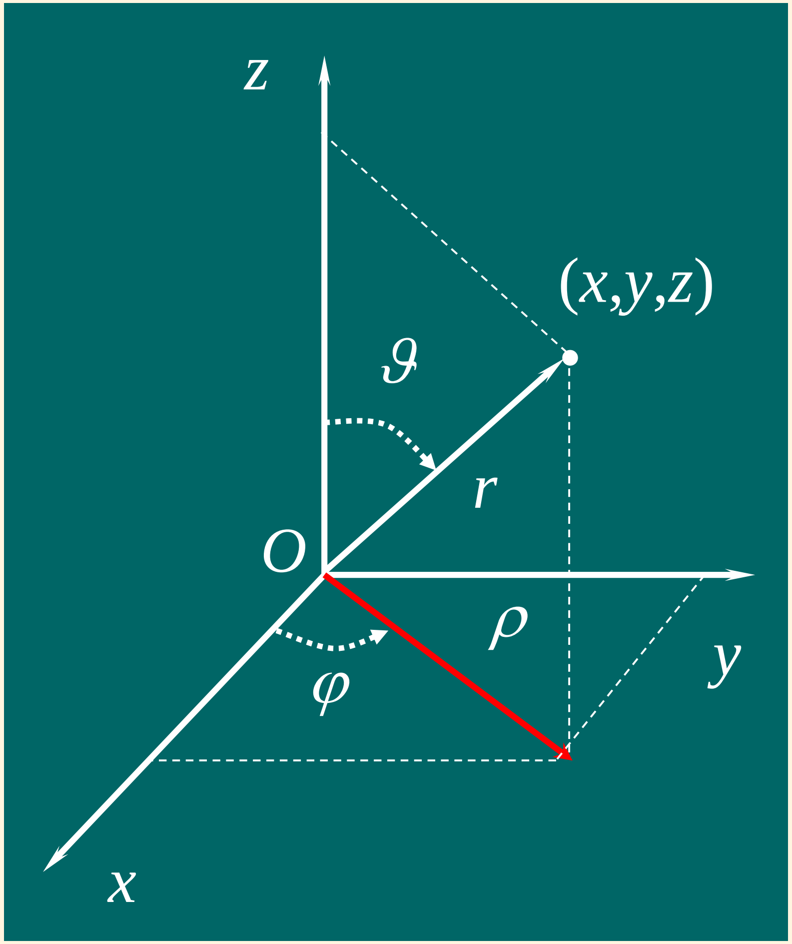
$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{1}{r^2\sin\vartheta}\delta(r-r_0)\delta(\vartheta-\vartheta_0)\delta(\varphi-\varphi_0)$$

① 当$r_0>0$时，(x_0,y_0,z_0)与$(r_0,\vartheta_0,\varphi_0)$一一对应，故上式成立；

② 当$r_0=0$时，原点的(ϑ_0,φ_0)没有定义

$$\delta(\mathbf{r})=\delta(x)\delta(y)\delta(z)$$

$$=\frac{1}{4\pi r^2}\delta(r)$$



——一般先假定$r_0>0$，然后把最后结果取$r_0=0$

□原点的Dirac delta函数

$$\int_V \delta(x)\delta(y)\delta(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 1 \Rightarrow \mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2 \sin\vartheta \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$\delta(x)\delta(y)\delta(z) \sim \frac{A}{r^2}\delta(r+0)$$

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{A}{r^2}\delta(r+0) r^2 \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi\mathrm{d}r = 1$$

$$A\int_0^\pi \int_0^{2\pi} \sin\vartheta \mathrm{d}\vartheta\mathrm{d}\varphi = 1 \Rightarrow A = \frac{1}{4\pi}$$

$$\delta(x)\delta(y)\delta(z) = \frac{1}{4\pi r^2}\delta(r)$$

□极轴上的Dirac delta 函数

$$\int_V \delta(x)\delta(y)\delta(z-z_0)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 1 \Rightarrow \mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2 \sin\vartheta \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$\delta(x)\delta(y)\delta(z-z_0) \sim \frac{B}{r^2\sin\vartheta}\delta(r-r_0)\delta(\vartheta)$$

$$\int_0^\infty \int_{0-}^{\pi} \int_0^{2\pi} \frac{B}{r^2\sin\vartheta}\delta(r-r_0)\delta(\vartheta) r^2\sin\vartheta \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi = 1$$

$$B\int_0^{2\pi}\mathrm{d}\varphi = 1 \Rightarrow A = 1/2\pi$$

$$\delta(x)\delta(y)\delta(z-z_0) = \frac{1}{2\pi r^2\sin\vartheta}\delta(r-r_0)\delta(\vartheta) \quad (z_0 > 0)$$

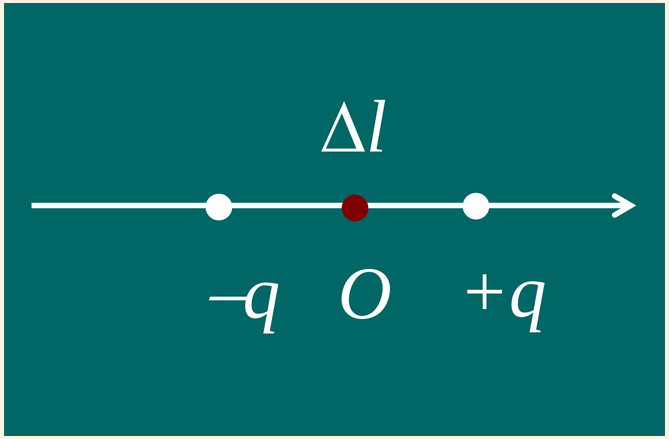
$$\delta(x)\delta(y)\delta(z-z_0) = \frac{1}{2\pi r^2\sin\vartheta}\delta(r-r_0)\delta(\vartheta-\pi) \quad (z_0 < 0)$$

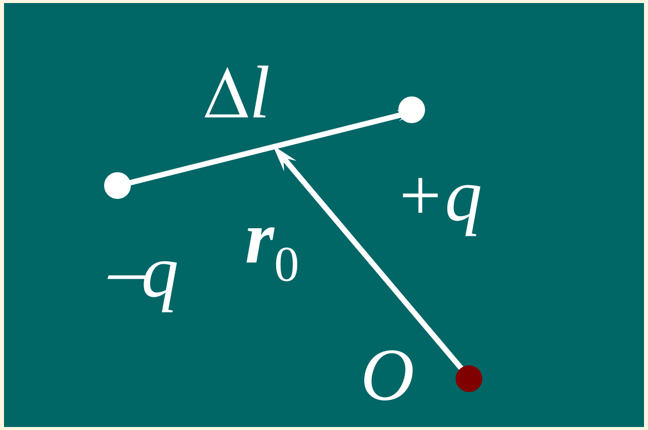
例1 求电偶极矩的表达式

$$\rho(x) = -q\delta\left(x+\frac{\Delta l}{2}\right) + q\delta\left(x-\frac{\Delta l}{2}\right)$$

$$= -\lim_{\Delta l \to 0}(q\Delta l)\cdot\lim_{\Delta l \to 0}\left[\frac{\delta(x+\Delta l/2)-\delta(x-\Delta l/2)}{\Delta l}\right]$$

$$= -p\delta'(x)$$





■三维情况

$$\rho(\boldsymbol{r}) = -q\delta\left(\boldsymbol{r}+\frac{\Delta\boldsymbol{l}}{2}\right)+q\delta\left(\boldsymbol{r}-\frac{\Delta\boldsymbol{l}}{2}\right)$$

$$\approx -q\left[\delta(\boldsymbol{r})+\frac{1}{2}\Delta\boldsymbol{l}\cdot\nabla\delta(\boldsymbol{r})\right]+q\left[\delta(\boldsymbol{r})-\frac{1}{2}\Delta\boldsymbol{l}\cdot\nabla\delta(\boldsymbol{r})\right]$$

$$= -q\left[\Delta\boldsymbol{l}\cdot\nabla\delta(\boldsymbol{r})\right] = -\boldsymbol{p}\cdot\nabla\delta(\boldsymbol{r})$$

空间任意$\boldsymbol{r}_0$点 $\rho(\boldsymbol{r}) = -\boldsymbol{p}\cdot\nabla\delta(\boldsymbol{r}-\boldsymbol{r}_0)$

例2 求四偶极矩的表达式

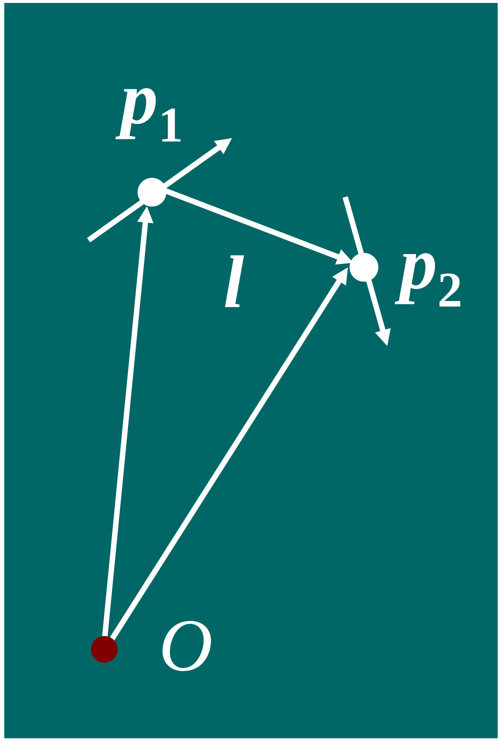
二个偶极子相距 $\boldsymbol{l}=\Delta\boldsymbol{r}$ **，设偶极子为**

$\boldsymbol{p}=p(t)\boldsymbol{d}$**——$\boldsymbol{d}$为偶极子方向的单位矢量**

空间电荷密度

$$\rho(\boldsymbol{r}) = -\boldsymbol{p}\cdot\nabla\delta\left(\boldsymbol{r}+\frac{\Delta\boldsymbol{r}}{2}\right)+\boldsymbol{p}\cdot\nabla\delta\left(\boldsymbol{r}-\frac{\Delta\boldsymbol{r}}{2}\right)$$

$$\begin{aligned}\rho(\boldsymbol{r}) &\approx -\Delta\boldsymbol{r}\cdot\nabla[\boldsymbol{p}\cdot\nabla\delta(\boldsymbol{r})]\\ &= -p(t)(\boldsymbol{l}\cdot\nabla)(\boldsymbol{d}\cdot\nabla)\delta(\boldsymbol{r})\\ &= -p(t)\sum_{i=1,j=1}^{3} d_i l_j \frac{\partial^2\delta(\boldsymbol{r})}{\partial x_i \partial x_j}\end{aligned}$$



空间任意$\boldsymbol{r}_0$点

$$\rho(\boldsymbol{r}) = -p(t)\sum_{i=1,j=1}^{3} d_i l_j \frac{\partial^2\delta(\boldsymbol{r}-\boldsymbol{r}_0)}{\partial x_i \partial x_j}$$

纵向四极子，横向四极子

■ 小结

■经典函数存在的问题?

■广义函数：检验函数——数的对应关系

■奇异广义函数(解决了点源表示问题)

$$f(\varphi)=(f,\varphi)=\varphi(0)\Rightarrow\delta(x)$$

注意f不是检验函数

■重要关系

■卷积关系 $\int_{-\infty}^{\infty} f(y)\delta(x-y)\mathrm{d}y = f(x)$

■合复函数

$$\delta[g(x)]=\sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n)$$

■导数关系——(解决了求任意阶导数问题)

$$(D^\alpha f,\varphi)=(-1)^{|\alpha|}(f,D^\alpha\varphi)$$

■广义函数的Fourier变换 (解决了一般函数的Fourier积分问题)

■速降函数空间

$$(\boldsymbol{F}f,\varphi)=(f,\boldsymbol{F}\varphi),\quad \forall\varphi\in L(R^n)$$

■重要关系

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathrm{i}kx}\mathrm{d}k$$

■ 弱收敛和Dirac Delta 函数 (解决了微分与求和交换问题)

- **弱收敛(一致收敛、逐点收敛，均方收敛)**

$$\lim_{k\to\infty}(f_k,\varphi)=(f,\varphi),\quad \forall\varphi\in D$$

$$f=\lim_{k\to\infty}f_k$$

$$\frac{\partial f}{\partial x_i}=\frac{\partial}{\partial x_i}\lim_{k\to\infty}f_k=\lim_{k\to\infty}\frac{\partial f_k}{\partial x_i}$$

- **几个典型的弱收敛系列**

■曲线坐标中的Dirac delta δ-函数

柱坐标

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{\delta(\rho-\rho_0)\delta(\varphi-\varphi_0)\delta(z-z_0)}{\rho}$$

$$\Downarrow$$

$$\delta(x)\delta(y)\delta(z-z_0)=\frac{1}{2\pi\rho}\delta(\rho)\delta(z-z_0)$$

球坐标

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{\delta(r-r_0)\delta(\vartheta-\vartheta_0)\delta(\varphi-\varphi_0)}{r^2\sin\vartheta}$$

$$\Downarrow$$

$$\delta(\mathbf{r})=\delta(x)\delta(y)\delta(z)\ =\frac{1}{4\pi r^2}\delta(r)$$

极轴上?