第12章 柱函数及其应用

12.1 Bessel函数和Neumann 函数

渐进展开，母函数，Fourier-Bessel展开

12.2 Hankel 函数及其应用

辐射解，倏逝波，Sommerfeld辐射条件

12.3 虚宗量 Bessel 函数

渐进展开,与Bessel和Hankel函数的关系

12.4 球Bessel 函数和球Hankel函数

球Bessel方程的本征值问题，辐射解

12.5 小结

12.1 Bessel函数和Neumann 函数

- 柱坐标中Laplace方程和Helmholtz方程分离变量后，径向部分满足

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2}+x\frac{\mathrm{d}R}{\mathrm{d}x}+(x^2-m^2)R=0,\ (x=\sqrt{\mu}\ \rho)$$

——Bessel 方程

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2}+x\frac{\mathrm{d}R}{\mathrm{d}x}-(x^2+m^2)R=0,\quad (x=\sqrt{|\mu|}\rho)$$

——虚宗量 Bessel 方程

- 球坐标中Helmholtz方程分离变量后，径向部分满足

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+[k^2r^2-l(l+1)]R=0$$

——球Bessel 方程

$$x=kr;\ y(x)=\sqrt{\frac{2kr}{\pi}}R(r)$$

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[x^2-\left(l+\frac{1}{2}\right)^2\right]y=0$$

——半奇数阶Bessel 方程

■一般形式的Bessel 方程

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\nu^2)y=0$$

通解

$$y(x)=C_1J_\nu(x)+C_2N_\nu(x)$$

$$N_\nu(x)=\frac{J_\nu(x)\cos(\nu\pi)-J_{-\nu}(x)}{\sin(\nu\pi)}$$

为什么这样定义?

$$J_\nu(x)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(k+1+\nu)}\left(\frac{x}{2}\right)^{\nu+2k}$$

■Bessel 和 Neumann 函数的重要性质

■ $x\to 0$特性(见第7章讨论)

$$J_0(x)\approx 1-\frac{1}{4}x^2;\quad J_\nu(x)\approx\frac{1}{2^\nu\Gamma(\nu+1)}x^\nu\ (\nu\neq 0)$$

$$N_0(x)\approx\frac{2}{\pi}\ln\frac{x}{2}\ ;\ N_\nu(x)\approx-\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu\ (\nu\neq 0)$$

因此：在研究圆柱内部问题时 (包含 ρ=0),存在自然边界条件，只能取零阶和正整数阶 Bessel 函数。

■ $x\to\infty$ 特性

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

可见

①Bessel 函数: 振荡特性, 讨论封闭空间的驻波问题,类似于一维的cos(kx);

②Neumann 函数: 在x=0有奇异性, 讨论不包括原点的问题, 振荡特性: 类似于一维的sin(kx).

远场特性进一步讨论

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-v^2)y=0$$

$$g(x)=\sqrt{x}\,y(x)\Rightarrow y(x)=\frac{1}{\sqrt{x}}g(x)$$

$$\frac{\mathrm{d}^2g(x)}{\mathrm{d}x^2}+\left(1+\frac{1/4-v^2}{x^2}\right)g(x)=0$$

远场条件

$$\frac{\left|1/4-v^2\right|}{x^2}\ll1\Rightarrow x^2\gg\left|v^2-1/4\right|\Rightarrow \boxed{x\gg v\sqrt{\left|1-\frac{1}{4v^2}\right|}}$$

$$\frac{\mathrm{d}^2 g(x)}{\mathrm{d}x^2} + g(x) \approx 0 \Rightarrow \begin{aligned} g_1(x) &\approx a\cos(x+b) \\ g_2(x) &\approx a\sin(x+b) \end{aligned}$$

$$\Downarrow$$

$$y_1(x) \approx \frac{a}{\sqrt{x}}\cos(x+b); \quad y_2(x) \approx \frac{a}{\sqrt{x}}\sin(x+b)$$

$$\Downarrow$$

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right); J_{-\nu}(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$\Downarrow$$

$$\cos\left(x + \nu\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos(\nu\pi)\cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) - \sin(\nu\pi)\sin\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)$$

①如果ν=m (整数): $\sin(\nu\pi) = 0$

$$\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)=(-1)^{\nu}\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)$$

——$J_\nu(x)$与$J_{-\nu}(x)$线性相关

②如果$\nu\neq m$ (整数): $\sin(\nu\pi)\neq 0$

$$\left\{\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right);\sin\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\right\} \Rightarrow$$ **线性独立**

——$J_\nu(x)$与$J_{-\nu}(x)$线性独立

□ 为什么定义Neumann函数?

$$\sin\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\cos(\nu\pi)\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)-\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{\sin(\nu\pi)}$$

考虑下列极限

$$\lim_{\nu\to m} f(\nu) \equiv \lim_{\nu\to m} \frac{\cos(\nu\pi)\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)-\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{\sin(\nu\pi)}$$

如果$\nu\to m$ (整数)，上式是0/0型，于是

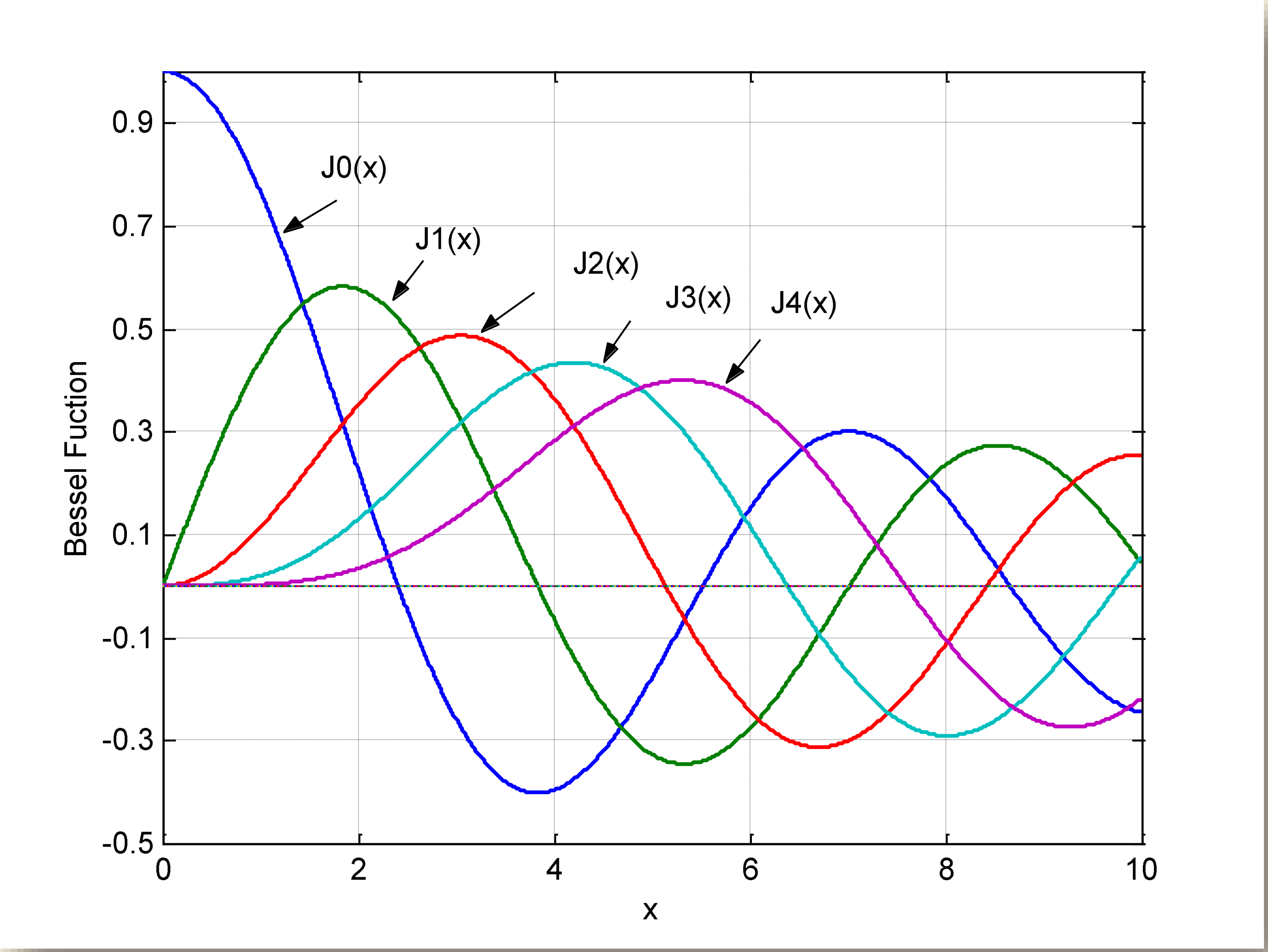
$$\lim_{\nu\to m} f(\nu) = \lim_{\nu\to m} \frac{\frac{\partial}{\partial \nu}\left[\cos(\nu\pi)\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)-\cos\left(x+\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\right]}{\frac{\partial \sin(\nu\pi)}{\partial \nu}}$$

$$=\sin\left(x-m\frac{\pi}{2}-\frac{\pi}{4}\right)$$

近场极限同样存在

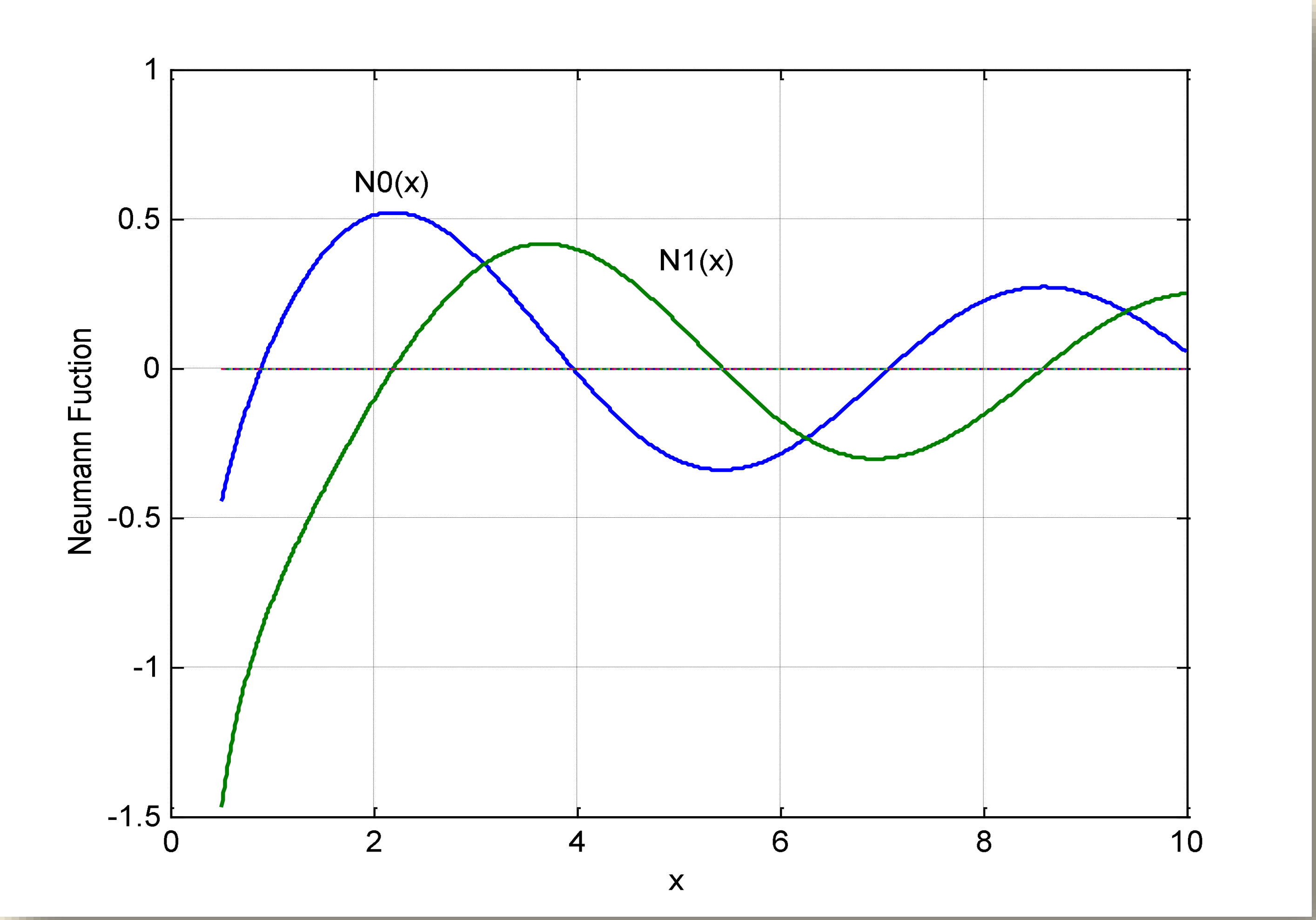
$$N_\nu(x)=\frac{\cos(\nu\pi)J_\nu(x)-J_{-\nu}(x)}{\sin(\nu\pi)}$$

□ 5个Bessel函数



J0(x)
J1(x)
J2(x)
J3(x)
J4(x)
Bessel Fuction
x
0.9
0.7
0.5
0.3
0.1
-0.1
-0.3
-0.5
0
2
4
6
8
10

☐ 2个Neumann函数



■ 大参数、大阶数(x~ν)Bessel函数

远场条件 $x \gg \nu\sqrt{|1-1/(4\nu^2)|} \sim \nu$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-\frac{\nu^2}{x^2}\right)y=0$$

⇑

$$x=\nu+x';\quad x'/\nu \ll 1$$

⇓

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x'^2}+\frac{2}{\nu}x'y\approx 0$$

——**Airy方程**

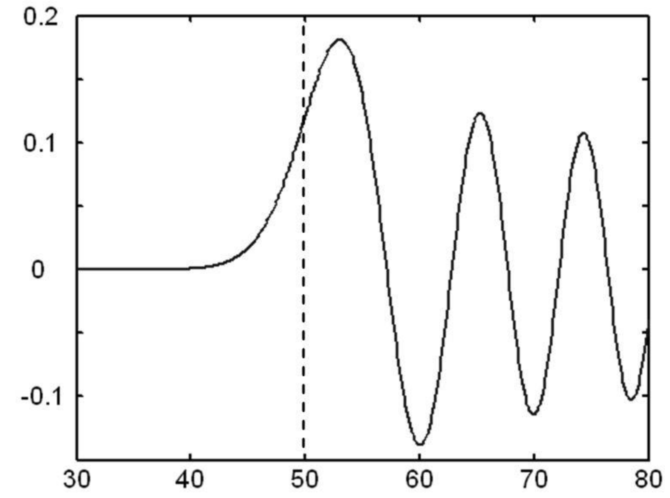


图 2.2.3 $J_{50}(x)$ 在 $x=50$ 附近的图像，只有当 $x>60$ 时才呈现震荡特性

① 当$x'<0$: 类似指数函数衰减或发散;

② 当$x'>0$: 类似正弦或余弦函数振荡，且衰减.

——**具体关系较复杂**

■ Airy 方程

$$\boxed{\frac{\mathrm{d}^2 Z(\eta)}{\mathrm{d}\eta^2}+\eta Z(\eta)=0}$$

① $\eta<0$, 函数变换 $Z(\eta)=\sqrt{-\eta}\tilde{Z}(\eta)$

$$\frac{\mathrm{d}^2\tilde{Z}(\eta)}{\mathrm{d}\eta^2}+\frac{1}{\eta}\frac{\mathrm{d}\tilde{Z}(\eta)}{\mathrm{d}\eta}-\left(\eta+\frac{1}{4\eta^2}\right)\tilde{Z}(\eta)=0 \Leftarrow \xi=\frac{2}{3}(-\eta)^{3/2}$$

$$\Downarrow$$

$$\frac{\mathrm{d}^2\tilde{Z}}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}\xi}-\left[1+\frac{(1/3)^2}{\xi^2}\right]\tilde{Z}=0 \Leftarrow$$

虚宗量 Bessel方程

$$\Downarrow$$

$$\tilde{Z}(\xi)=AI_{1/3}(\xi)+BI_{-1/3}(\xi)$$

$$\Downarrow$$

$$Z(\eta)=A\sqrt{-\eta}I_{1/3}\left[\frac{2}{3}(-\eta)^{3/2}\right]+B\sqrt{-\eta}I_{-1/3}\left[\frac{2}{3}(-\eta)^{3/2}\right],\ (\eta<0)$$

②η>0, 函数变换 $Z(\eta)=\sqrt{\eta}\tilde{Z}(\eta)$

$$\frac{\mathrm{d}^2\tilde{Z}(\eta)}{\mathrm{d}\eta^2}+\frac{1}{\eta}\frac{\mathrm{d}\tilde{Z}(\eta)}{\mathrm{d}\eta}+\left(\eta-\frac{1}{4\eta^2}\right)\tilde{Z}(\eta)=0 \Leftarrow \xi=\frac{2}{3}\eta^{3/2}$$

$$\Downarrow$$

$$\frac{\mathrm{d}^2\tilde{Z}}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}\xi}+\left[1-\frac{(1/3)^2}{\xi^2}\right]\tilde{Z}=0 \Leftarrow$$ **Bessel方程**

$$\Downarrow$$

$$\tilde{Z}(\xi)=AJ_{1/3}(\xi)+BJ_{-1/3}(\xi)$$

$$\Downarrow$$

$$Z(\eta)=A\sqrt{\eta}J_{1/3}\left(\frac{2}{3}\eta^{3/2}\right)+B\sqrt{\eta}J_{-1/3}\left(\frac{2}{3}\eta^{3/2}\right),\ (\eta>0)$$

■ 定义第一和第二类Airy函数

$$\mathrm{Ai}(\eta)=\begin{cases}\dfrac{\sqrt{\eta}}{3}\left[J_{-1/3}\left(\dfrac{2}{3}\eta^{3/2}\right)+J_{1/3}\left(\dfrac{2}{3}\eta^{3/2}\right)\right], & (\eta>0)\\ \dfrac{\sqrt{|\eta|}}{3}\left[I_{-1/3}\left(\dfrac{2}{3}|\eta|^{3/2}\right)-I_{1/3}\left(\dfrac{2}{3}|\eta|^{3/2}\right)\right], & (\eta<0)\end{cases}$$

$$\mathrm{Bi}(\eta)=\begin{cases}\sqrt{\dfrac{\eta}{3}}\left[J_{-1/3}\left(\dfrac{2}{3}\eta^{3/2}\right)-J_{1/3}\left(\dfrac{2}{3}\eta^{3/2}\right)\right], & (\eta>0)\\ \sqrt{\dfrac{|\eta|}{3}}\left[I_{-1/3}\left(\dfrac{2}{3}|\eta|^{3/2}\right)+I_{1/3}\left(\dfrac{2}{3}|\eta|^{3/2}\right)\right], & (\eta<0)\end{cases}$$

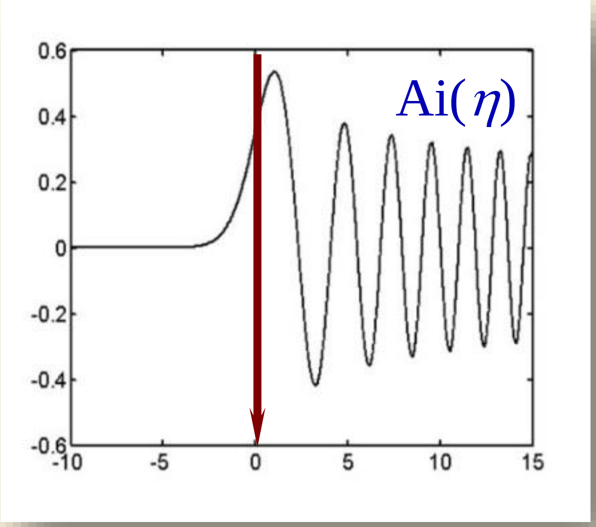
■ Airy方程的解

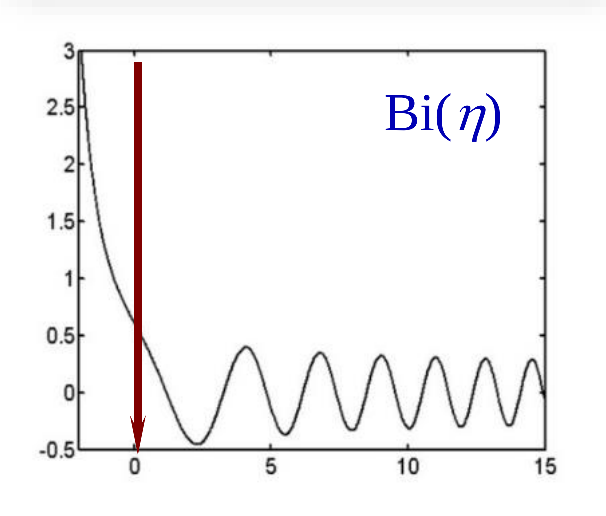
$$Z(\eta)=C_1\mathrm{Ai}(\eta)+C_2\mathrm{Bi}(\eta)$$

■ Airy函数的渐近表达式

$$\mathrm{Ai}(\eta)\approx\begin{cases}\dfrac{1}{\sqrt{\pi}\eta^{1/4}}\sin\left(\dfrac{2}{3}\eta^{3/2}+\dfrac{\pi}{4}\right), & (\eta\to\infty)\\ \dfrac{1}{2\sqrt{\pi}|\eta|^{1/4}}\exp\left(-\dfrac{2}{3}|\eta|^{3/2}\right), & (\eta\to-\infty)\end{cases}$$

$$\mathrm{Bi}(\eta)\approx\begin{cases}\dfrac{1}{\sqrt{\pi}\eta^{1/4}}\cos\left(\dfrac{2}{3}\eta^{3/2}+\dfrac{\pi}{4}\right), & (\eta\to\infty)\\ \dfrac{1}{\sqrt{\pi}|\eta|^{1/4}}\exp\left(\dfrac{2}{3}|\eta|^{3/2}\right), & (\eta\to-\infty)\end{cases}$$





注：

$$\frac{\mathrm{d}^2Z(\eta)}{\mathrm{d}\eta^2}+\eta^m Z(\eta)=0$$

$$Z=\sqrt{\eta}\,y(\xi);\ \ \xi=\frac{2}{m+2}\eta^{(m+2)/2}$$

$$y''(\xi)+\frac{1}{\xi}y'(\xi)+\left[1-\frac{1/(m+2)^2}{\xi^2}\right]y(\xi)=0$$

❑柱函数的递推公式(直接从表达式推导)

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\nu}Z_{\nu}(x)] = x^{\nu}Z_{\nu-1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu}Z_{\nu}(x)] = -x^{-\nu}Z_{\nu+1}(x)$$

$$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x}Z_{\nu}(x)$$

$$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z_{\nu}'(x)$$

式中 Z_ν 为 $J_\nu(x)$ 或 $N_\nu(x)$. 最常用的有 $J_0'(x) = -J_1(x)$

定义：满足上述递推关系的函数称为柱函数

注意：柱函数一定满足Bessel方程，但Bessel方程的解不一定是柱函数, 如$J_\nu(x)+\nu N_\nu(x)$满足Bessel方程，但不是柱函数.

❑Bessel 函数的母函数(直接作Laurent展开)

$$\exp\left[\frac{1}{2}x(z-z^{-1})\right]=\sum_{m=-\infty}^{\infty}J_m(x)z^m,\ (0<|z|<\infty)$$

取 $z=e^{\mathrm{i}(\varphi+\pi/2)}$ 得到(极坐标中x方向传播的平面波)

$$\exp(\mathrm{i}x\cos\varphi)=\sum_{m=-\infty}^{\infty}\mathrm{i}^m J_m(x)e^{\mathrm{i}m\varphi}$$

取 $z=e^{\mathrm{i}\varphi}$ 得到 (极坐标中y方向传播的平面波)

$$\exp(\mathrm{i}x\sin\varphi)=\sum_{m=-\infty}^{\infty}J_m(x)e^{\mathrm{i}m\varphi}$$

φ的周期函数
周期为2π

Fourier级数
形式

❑Bessel函数的积分形式

$$J_m(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{\mathrm{i}(x\sin\varphi-m\varphi)}\mathrm{d}\varphi=\frac{1}{\pi}\int_{0}^{\pi}\cos(x\sin\varphi-m\varphi)\mathrm{d}\varphi$$

$$J_0(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{\mathrm{i}(0\sin\varphi-0\varphi)}\mathrm{d}\varphi=1$$

$$J_m(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-\mathrm{i}m\varphi}\mathrm{d}\varphi=0,(m>0)$$

Anger函数

注意：*m*不是整数，而为任意复数*ν*时

$$\frac{1}{\pi}\int_{0}^{\pi}\cos(x\sin\varphi-\nu\varphi)\mathrm{d}\varphi\equiv\tilde{J}_\nu(x)\neq J_\nu(x)$$

$$x^2\frac{\mathrm{d}^2\tilde{J}_\nu}{\mathrm{d}x^2}+x\frac{\mathrm{d}\tilde{J}_\nu}{\mathrm{d}x}+(x^2-\nu^2)\tilde{J}_\nu=\frac{x-\nu}{\pi}\sin(\nu\pi)$$

$$E_\nu(x) \equiv \frac{1}{\pi}\int_0^\pi \sin(x\sin\varphi - \nu\varphi)\mathrm{d}\varphi$$

Weber 函数

$$x^2\frac{\mathrm{d}^2\tilde{J}_\nu}{\mathrm{d}x^2} + x\frac{\mathrm{d}\tilde{J}_\nu}{\mathrm{d}x} + (x^2-\nu^2)\tilde{J}_\nu = -\frac{x+\nu}{\pi} - \frac{x-\nu}{\pi}\cos(\nu\pi)$$

❑Bessel函数的加法公式

$$\exp\left[\frac{1}{2}(x+y)(z-z^{-1})\right] = \sum_{m=-\infty}^{\infty} J_m(x+y)z^m,\ (0<|z|<\infty)$$

$$J_m(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{m-k}(y)$$

$$R = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\varphi}$$

$$J_0(R) = J_0(r_1)J_0(r_2) + \sum_{m=1}^{\infty} J_m(r_1)J_m(r_2)\cos(m\varphi)$$

□ Bessel方程的本征值问题

本征函数和本征值

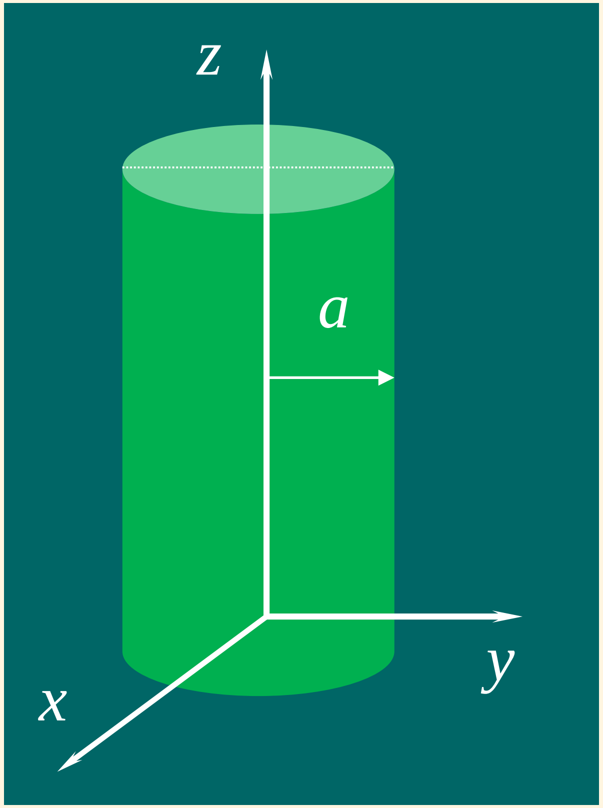
$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right]+\frac{m^2}{\rho}R(\rho)=\mu\rho R(\rho)$$

$$R(0)<\infty;\ \left[\alpha R(\rho)+\beta R'(\rho)\right]\Big|_{\rho=a}=0$$

$$\begin{cases}\dfrac{\mathrm{d}^2R(\rho)}{\mathrm{d}\rho^2}+\dfrac{1}{\rho}\dfrac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}+\left(\mu-\dfrac{m^2}{\rho^2}\right)R(\rho)=0\\ R(\rho)<\infty;\ \ \alpha R(\rho)+\beta R'(\rho)\Big|_{\rho=a}=0\end{cases}$$

$$R_m(\rho)=A_mJ_m\left(\sqrt{\mu}\rho\right)+B_mN_m\left(\sqrt{\mu}\rho\right)$$

$$R(0)<\infty \Rightarrow B_m\equiv 0 \Rightarrow \boxed{R_m(\rho)=A_mJ_m\left(\sqrt{\mu}\rho\right)}$$



本征方程：本征值是下列方程的正根

$$\alpha J_m\left(\sqrt{\mu}a\right)+\beta\left.\frac{\mathrm{d}J_m\left(\sqrt{\mu}\rho\right)}{\mathrm{d}\rho}\right|_{\rho=a}=0 \Rightarrow \left[\alpha J_m(x)+\frac{\beta}{a}x\frac{\mathrm{d}J_m(x)}{\mathrm{d}x}\right]\Bigg|_{x=\sqrt{\mu}a}=0$$

第一类边界条件

$$J_m\left(\sqrt{\mu}a\right)=0$$

设x_{mn}是$J_m(x)$的第n个零点，则本征值为

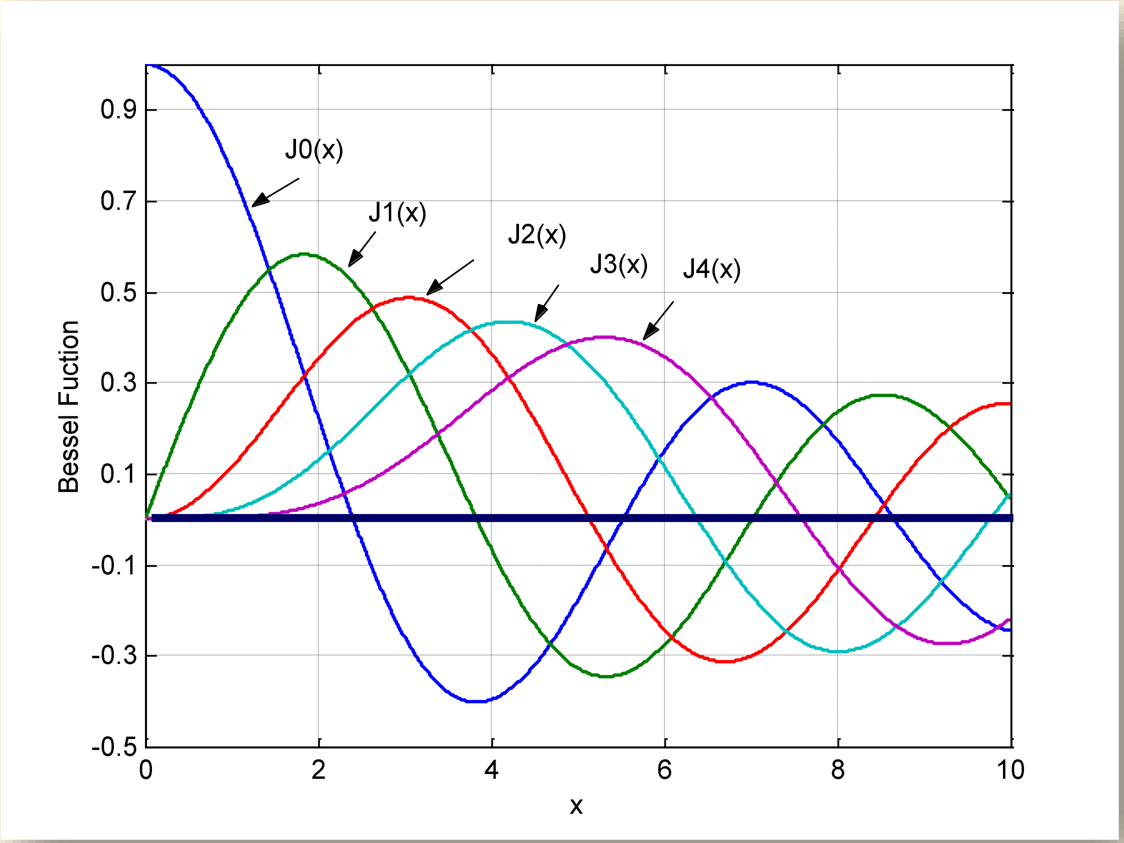
$$\mu_{mn}=\left(\frac{x_{mn}}{a}\right)^2,\ (n=1,2,\ldots)$$

——可见：Bessel 函数零点的分布是非常重要的，从Bessel函数的曲线上可清楚地看出 Bessel 的零点分布特性：

① $J_m(x)$ 有无限多个零点；

② $J_m(x)$ 的两个零点之间必有$J_{m+1}(x)$ 的零点；

③ $x=0$是 $J_m(x)(m\geq 1)$的零点，但不是$J_0(x)$的零点。



第二类边界条件

$$J'_m(x)|_{x=\sqrt{\mu}a}=0$$

设x_{mn}是$J'_m(x)$的第n个零点，则本征值为

$$\mu_{mn}=\left(\frac{x_{mn}}{a}\right)^2,(n=0,1,2,\ldots)$$

特别注意：对第二类边界条件, 当$m=0$时(与方位角无关)，$J'_0(x)=-J_1(x)$, 本征值由$J_1(x)$的零点决定, 而$J_1(0)=0$, 所以第一个本征值是零本征值.

第三类边界条件

$$\left[\alpha J_m(x)+\frac{\beta}{a}x\frac{\mathrm{d}J_m(x)}{\mathrm{d}x}\right]\Bigg|_{x=\sqrt{\mu}a}=0$$

❑Bessel 函数的正交关系

$$\int_0^a J_m\left(\sqrt{\mu_n}\rho\right)J_m\left(\sqrt{\mu_l}\rho\right)\rho\mathrm{d}\rho = N_{mn}^2\delta_{nl}$$

其中N_{mn}为 Bessel 函数的模。证明:

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}J_m\left(\sqrt{\mu_n}\rho\right)}{\mathrm{d}\rho}\right]+\frac{m^2}{\rho}J_m\left(\sqrt{\mu_n}\rho\right)=\mu_{mn}\rho J_m\left(\sqrt{\mu_n}\rho\right)$$

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}J_m\left(\sqrt{\mu_l}\rho\right)}{\mathrm{d}\rho}\right]+\frac{m^2}{\rho}J_m\left(\sqrt{\mu_l}\rho\right)=\mu_{ml}\rho J_m\left(\sqrt{\mu_l}\rho\right)$$

第一式$\times J_m\left(\sqrt{\mu_l}\rho\right)-$**第二式** $\times J_m\left(\sqrt{\mu_n}\rho\right)$,**且积分**

$$\int_0^a \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho J_m\left(\sqrt{\mu_n}\rho\right)\frac{\mathrm{d}J_m\left(\sqrt{\mu_l}\rho\right)}{\mathrm{d}\rho}-\rho J_m\left(\sqrt{\mu_l}\rho\right)\frac{\mathrm{d}J_m\left(\sqrt{\mu_n}\rho\right)}{\mathrm{d}\rho}\right]\mathrm{d}\rho$$

$$=(\mu_{mn}-\mu_{ml})\int_0^a \rho J_m\left(\sqrt{\mu_l}\rho\right)J_m\left(\sqrt{\mu_n}\rho\right)\mathrm{d}\rho$$

$$(\mu_{mn}-\mu_{ml})\int_0^a \rho J_m\left(\sqrt{\mu_l}\rho\right)J_m\left(\sqrt{\mu_n}\rho\right)\mathrm{d}\rho$$

$$=\rho\left[J_m\left(\sqrt{\mu_n}\rho\right)\frac{\mathrm{d}J_m\left(\sqrt{\mu_l}\rho\right)}{\mathrm{d}\rho}-J_m\left(\sqrt{\mu_l}\rho\right)\frac{\mathrm{d}J_m\left(\sqrt{\mu_n}\rho\right)}{\mathrm{d}\rho}\right]_0^a$$

——积分上限代入后利用边界条件，结果为零；下限代入时恰好为零——奇异S-L本征值问题.

$$\int_0^a \rho J_m\left(\sqrt{\mu_l}\rho\right)J_m\left(\sqrt{\mu_n}\rho\right)\mathrm{d}\rho=0,\ (n\neq l)$$

❑Bessel 函数的模

$$N_{mn}^2=\int_0^a\left[J_m\left(\sqrt{\mu_n}\rho\right)\right]^2\rho\mathrm{d}\rho=\frac{1}{2}\left(a^2-\frac{m^2}{\mu_{mn}}\right)\left[J_m\left(\sqrt{\mu_{mn}}a\right)\right]^2$$

$$+\frac{1}{2}a^2\left[\left.\frac{\mathrm{d}J_m(x)}{\mathrm{d}x}\right|_{x=\sqrt{\mu_{mn}}a}\right]^2$$

第一类边界条件

$$N_{mn}^{2}=\frac{1}{2}a^{2}\left[J'_{m}\left(\sqrt{\mu_{mn}}a\right)\right]^{2}=\frac{1}{2}a^{2}\left[J_{m+1}\left(\sqrt{\mu_{mn}}a\right)\right]^{2}$$

第二类边界条件

$$N_{mn}^{2}=\frac{1}{2}\left(a^{2}-\frac{m^{2}}{\mu_{mn}}\right)\left[J_{m}\left(\sqrt{\mu_{mn}}a\right)\right]^{2}$$

第三类边界条件

$$\left.\frac{\mathrm{d}J_{m}(x)}{\mathrm{d}x}\right|_{x=\sqrt{\mu_{mn}}a}=-\frac{\alpha}{\beta}\frac{J_{m}\left(\sqrt{\mu_{mn}}a\right)}{\sqrt{\mu_{mn}}}$$

$$\Downarrow$$

$$N_{mn}^{2}=\frac{1}{2}\left[a^{2}-\frac{m^{2}}{\mu_{mn}}+\frac{a^{2}}{\mu_{mn}}\left(\frac{\alpha}{\beta}\right)^{2}\right]\left[J_{m}\left(\sqrt{\mu_{mn}}a\right)\right]^{2}$$

❑ Fourier-Bessel 展开

函数系

$$\left\{J_m\left(\sqrt{\mu_{mn}}\rho\right),\ n=1,2,3,\ldots\right\}$$

是完备系。对 [0, *a*]上带权*ρ*的平方可积函数 *f*(*ρ*)

$$\int_0^a |f(\rho)|^2\rho\mathrm{d}\rho<\infty$$

存在Fourier-Bessel 展开

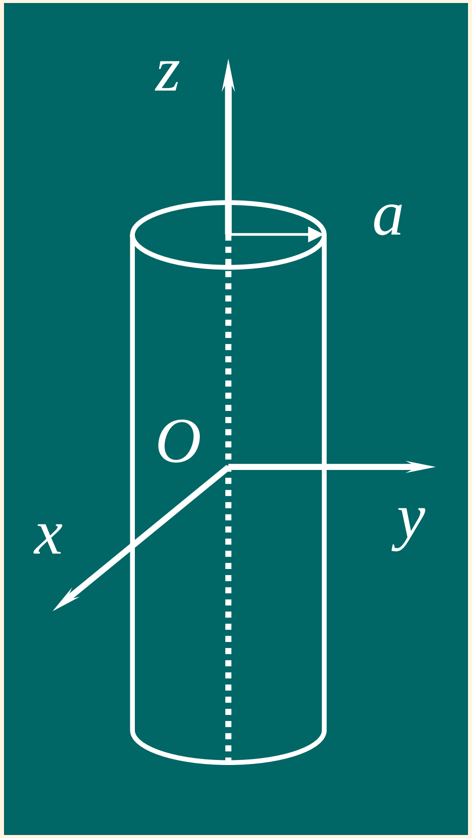
$$\begin{cases} f(\rho)\cong\sum_{n=1}^{\infty} f_n J_m\left(\sqrt{\mu_{nm}}\rho\right)=\frac{1}{2}[f(\rho-0)+f(\rho+0)] \\ f_n=\frac{1}{N_{mn}^2}\int_0^a f(\rho)J_m\left(\sqrt{\mu_{nm}}\rho\right)\rho\mathrm{d}\rho \end{cases}$$

❑Bessel 和Neumann 函数的应用

例1 圆柱体的冷却：无限长圆柱体半径为 a，初始温度为 u_0, 表面温度维持为零，求圆柱体内温度的变化。

解：定解问题

$$\rho c_V \frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0; u|_{t=0} = u_0, u|_{\rho=a} = 0$$



第1步：时间-空间分离变量

$$u = v(\rho, \varphi, z) T(t)$$

$$\frac{\nabla^2 v}{v} = \frac{T'}{\chi^2 T} \equiv -k^2, \left(\chi^2 = \frac{\kappa}{\rho c_V} \right)$$

因此有

$$\nabla^2 v + k^2 v = 0;\ T' + k^2\chi^2 T = 0$$

$$\Downarrow$$

$$T(t) = C\exp(-k^2\chi^2 t)$$

第2步：边界条件分离变量

$$u|_{\rho=a} = vT(t)|_{\rho=a} = 0 \Rightarrow v|_{\rho=a} = 0$$

因此，变成 Helmholtz方程第一类边值问题

$$-\nabla^2 v = k^2 v;\ v|_{\rho=a} = 0$$

第3步：空间变量的分离变量

$$v(\rho,\varphi,z) = R(\rho)\Phi(\varphi)Z(z)$$

引进二个常数 m^2 和 λ

$$\Phi'' + m^2\Phi = 0;\quad Z'' + \lambda Z = 0$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \left(k^2 - \lambda - \frac{m^2}{\rho^2}\right)R = 0$$

第4步：分析引进常数

(1)由周期条件 $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$$\Downarrow$$

$$\Phi(\varphi) = A\mathrm{e}^{\mathrm{i}m\varphi}, (m = 0, \pm1, \pm2, \ldots)$$

——本题中，因为u_0=常数，与φ无关，问题与φ无关，因此必须选择 m=0.

(2)分三种情况：$\lambda<0$, $\lambda=0$, $\lambda>0$讨论

$$Z(z)=Ae^{-\sqrt{|\lambda|}z}+Be^{\sqrt{|\lambda|}z},\ (\lambda<0)$$

$$Z(z)=A\cos\left(\sqrt{\lambda}z\right)+B\sin\left(\sqrt{\lambda}z\right),\ (\lambda>0)$$

$$Z(z)=Az+B,\ \ (\lambda=0)$$

——本题中，因为 u_0=常数，与 z 无关，问题与 z 无关，因此必须选择 $\lambda=0$, 并且 $Z(z)$=常数

第5步：径向方程

$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}+k^2R=0,\ (0\le\rho<a)$$

——$k^2>0$,否则时间部分$T(t)\Rightarrow$ 无限

一般解

$$R(\rho)=AJ_0(k\rho)+BN_0(k\rho)$$

由 ρ=0 自然边界条件 B=0, 所以径向解为

$$R(\rho)=AJ_0(k\rho)$$

第6步：边界条件

$$R(a)=AJ_0(ka)=0$$

设 μ_j是 $J_0(x)$的零点(j=1,2,3,…),本征值k

$$k_j=\frac{\mu_j}{a},\ (j=1,2,\ldots;0<\mu_1<\mu_2<\ldots)$$

问题的通解

$$u(\rho,t)=\sum_{j=0}^{\infty}A_j e^{-\frac{\chi^2\mu_j^2}{a}t}J_0\left(\frac{\mu_j}{a}\rho\right)$$

第7步：初始条件决定 A_j

$$u(\rho,t)|_{t=0}=\sum_{j=0}^{\infty}A_j J_0\left(\frac{\mu_j}{a}\rho\right)=u_0$$

由Bessel函数的正交关系

$$\begin{aligned}A_j&=2u_0\left[aJ_0'(\mu_j)\right]^{-2}\int_0^a J_0(k_j\rho)\rho\,\mathrm{d}\rho\\&=2u_0\left[k_j aJ_0'(\mu_j)\right]^{-2}\int_0^{k_j a}J_0(x)x\,\mathrm{d}x\end{aligned}$$

利用递推关系

$$\int_0^{k_j a} J_0(x)x\mathrm{d}x = \int_0^{k_j a} \frac{\mathrm{d}}{\mathrm{d}x}[xJ_1(x)]\mathrm{d}x \Rightarrow A_j = \frac{2u_0}{\mu_j J_1(\mu_j)}$$

$$= xJ_1(x)\big|_0^{k_j a} = \mu_j J_1(\mu_j)$$

因此, 问题的解为

$$u(\rho,t) = 2u_0 \sum_{j=0}^{\infty} \frac{J_0\left(\mu_j \rho / a\right)}{\mu_j J_1(\mu_1)} e^{-\frac{\chi^2 \mu_j^2}{a} t}$$

问题:

(1)如果u_0与 ρ有关,结果如何?——容易修改

(2)如果u_0与φ有关,结果如何?——包括$m \neq 0$的项

(3)如果u_0与ρ和φ都有关，结果又如何?

当问题与z有关，如何处理?

$$\begin{cases} \rho c_V \dfrac{\partial u}{\partial t} - \kappa \nabla^2 u = 0 \\ u|_{t=0} = u_0(\rho,\varphi,z), u|_{\rho=a} = 0 \end{cases}$$

角度部分

$$\Phi_m(\varphi) = A_m e^{\mathrm{i}m\varphi}, (m = 0, \pm 1, \pm 2, \ldots)$$

z方向部分

$$\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2} + \delta^2 Z(z) = 0 \Leftarrow \lambda = \delta^2$$

不存在边界：连续谱 $(-\infty < \delta < \infty)$

本征函数: $Z(z) = Ae^{\mathrm{i}\delta z}$

■ 径向部分

$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}+\left(k'^2-\frac{m^2}{\rho^2}\right)R=0,(k'^2\equiv k^2-\delta^2)$$

$$R(\rho)=AJ_m(k'\rho) \Rightarrow J_m(k'a)=0$$

设 μ_{mj}是 $J_m(x)$ 的第j个零点(j=1,2,3,…),本征值 k'

$$k'=\mu_{mj}/a \Rightarrow k^2=k'^2+\delta^2=(\mu_{mj}/a)^2+\delta^2$$

■ 模式解

$$u_{mj\delta}(\rho,\varphi,z,t)=A_{mj}(\delta)e^{-\chi^2\left[(\mu_{mj}/a)^2+\delta^2\right]t}J_m\left(\frac{\mu_{mj}}{a}\rho\right)e^{\mathrm{i}(m\varphi+\delta z)}$$

■ 通解 叠加原理

$$u(\rho,\varphi,z,t)=\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{mj}(\delta)e^{-\chi^2\left[(\mu_{mj}/a)^2+\delta^2\right]t}J_m\left(\frac{\mu_{mj}}{a}\rho\right)e^{\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\delta$$

①径向:离散谱(广义Fourier级数); ②周向:离散谱(Fourier级数);③轴向:连续谱(Fourier积分)

$$\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{mj}(\delta)J_m\left(\frac{\mu_{mj}}{a}\rho\right)e^{\mathrm{i}(m\varphi+\delta z)}=u_0(\rho,\varphi,z)$$

$$A_{mj}(\delta)=\frac{1}{(2\pi)^2(N_{mj})^2}\int_0^a\int_0^{2\pi}\int_{-\infty}^{\infty}u_0(\rho,\varphi,z)J_m\left(\frac{\mu_{mj}}{a}\rho\right)$$

$$\times e^{-\mathrm{i}(m\varphi+\delta z)}\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z$$

■ 积分形式的解

$$u(\rho,\varphi,z,t)=\int_0^a\int_0^{2\pi}\int_{-\infty}^{\infty}G(\rho,\varphi,z;\rho',\varphi',z',t)u_0(\rho',\varphi',z')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$$

$$G(\rho,\varphi,z;\rho',\varphi',z',t)\equiv\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{-\chi^2(\mu_{mj}/a)^2t}e^{\mathrm{i}m(\varphi-\varphi')}}{2\pi(N_{mj})^2\sqrt{4\pi\chi^2t}}\exp\left[-\frac{(z-z')^2}{4\chi^2t}\right]$$

$$\times J_m\left(\frac{\mu_{mj}}{a}\rho\right)J_m\left(\frac{\mu_{mj}}{a}\rho'\right)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\chi^2\delta^2t+\mathrm{i}\delta(z-z')}\mathrm{d}\delta=\frac{1}{\sqrt{4\pi\chi^2t}}\exp\left[-\frac{(z-z')^2}{4\chi^2t}\right]$$

验证： t=0

$$\lim_{t\to0}\frac{1}{\sqrt{4\pi\chi^2t}}\exp\left[-\frac{(z-z')^2}{4\chi^2t}\right]=\delta(z-z')$$

$$G(\rho,\varphi,z;\rho',\varphi',z',0)\equiv\sum_{m=-\infty}^{\infty}\frac{e^{\mathrm{im}(\varphi-\varphi')}}{2\pi}\left[\sum_{j=1}^{\infty}\frac{1}{(N_{mj})^2}J_m\left(\frac{\mu_{mj}}{a}\rho\right)J_m\left(\frac{\mu_{mj}}{a}\rho'\right)\right]\delta(z-z')$$

$$=\frac{1}{\rho}\delta(\rho,\rho')\delta(\varphi,\varphi')\delta(z-z')$$

$$\sum_{j=1}^{\infty}\frac{1}{(N_{mj})^2}J_m\left(\frac{\mu_{mj}}{a}\rho\right)J_m\left(\frac{\mu_{mj}}{a}\rho'\right)=\frac{1}{\rho'}\delta(\rho,\rho')$$

$$\sum_{m=-\infty}^{\infty}\frac{e^{\mathrm{im}\varphi}}{\sqrt{2\pi}}\cdot\frac{e^{-\mathrm{im}\varphi'}}{\sqrt{2\pi}}=\sum_{m=-\infty}^{\infty}\Phi_m(\varphi)\Phi_m^*(\varphi')=\delta(\varphi,\varphi')$$

本征函数的完备性关系

$$u(\rho,\varphi,z,t)\big|_{t=0}=\int_0^a\int_0^{2\pi}\int_{-\infty}^{\infty}G(\rho,\varphi,z;\rho',\varphi',z',0)u_0(\rho',\varphi',z')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$$

$$=\int_0^a\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{1}{\rho'}\delta(\rho,\rho')\delta(\varphi,\varphi')\delta(z-z')u_0(\rho',\varphi',z')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$$

$$=u_0(\rho,\varphi,z)$$

有限长圆柱体(上下底面也为0)，如何处理?

$$\begin{cases} \rho c_V \dfrac{\partial u}{\partial t} - \kappa \nabla^2 u = 0 \\ u|_{t=0} = u_0(\rho,\varphi,z); u|_{\rho=a} = 0; u|_{z=0} = u|_{z=l} = 0 \end{cases}$$

- **轴向** $Z_n(z) = B_n \sin\left(\dfrac{n\pi z}{l}\right); \delta_n = \dfrac{n\pi}{l}, (n = 1, 2, \ldots)$
- **通解**

$$u(\rho,\varphi,z,t) = \sum_{j=1}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mjn} e^{-\chi^2\left[(\mu_{mj}/a)^2 + (n\pi/l)^2\right]t}$$

$$\times J_m\left(\frac{\mu_{mj}}{a}\rho\right) \sin\left(\frac{n\pi z}{l}\right) e^{im\varphi}$$

■ 初始条件

$$u(\rho,\varphi,z,0)=\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}A_{mjn}J_m\left(\frac{\mu_{jm}}{a}\rho\right)\sin\left(\frac{n\pi z}{l}\right)e^{im\varphi}=u_0(\rho,\varphi,z)$$

①径向:离散谱(广义Fourier级数); ②周向:离散谱(Fourier级数);③轴向:离散谱(Fourier级数)

上式两边同乘 $J_M\left(\frac{\mu_{MJ}}{a}\rho\right)e^{-\mathrm{i}M\varphi}\sin\left(\frac{N\pi z}{l}\right)$,且在柱内积分

$$\int_0^a J_M\left(\frac{\mu_{Mj}}{a}\rho\right)J_M\left(\frac{\mu_{MJ}}{a}\rho\right)\rho\mathrm{d}\rho=N_{MJ}^2\delta_{jJ}$$

$$\int_0^l \sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{N\pi z}{l}\right)\mathrm{d}z=\frac{l}{2}\delta_{nN}$$

$$\int_0^{2\pi} e^{\mathrm{i}(m-M)\varphi}\mathrm{d}\varphi=2\pi\delta_{mM}$$

$$A_{MJN}=\frac{1}{\pi l N_{MJ}^2}\int_0^a\int_0^l\int_0^{2\pi}u_0(\rho,\varphi,z)J_M\left(\frac{\mu_{MJ}}{a}\rho\right)\sin\left(\frac{N\pi z}{l}\right)e^{-\mathrm{i}M\varphi}\rho\mathrm{d}\rho\mathrm{d}z\mathrm{d}\varphi$$

$$\Downarrow$$

$$A_{mjn}=\frac{1}{\pi l N_{mj}^2}\int_0^a\int_0^l\int_0^{2\pi}u_0(\rho,\varphi,z)J_m\left(\frac{\mu_{mj}}{a}\rho\right)\sin\left(\frac{n\pi z}{l}\right)e^{-\mathrm{i}m\varphi}\rho\mathrm{d}\rho\mathrm{d}z\mathrm{d}\varphi$$

■ 积分形式解

$$u(\rho,\varphi,z,t)=\int_0^a\int_0^l\int_0^{2\pi}G(\rho,\varphi,z;\rho',\varphi',z',t)u_0(\rho',\varphi',z')\rho'\mathrm{d}\rho'\mathrm{d}z'\mathrm{d}\varphi'$$

$$\Uparrow$$

$$G(\rho,\varphi,z;\rho',\varphi',z',t)\equiv\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\pi l N_{mj}^2}J_m\left(\frac{\mu_{mj}}{a}\rho\right)J_m\left(\frac{\mu_{mj}}{a}\rho'\right)$$

$$\times\sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi z'}{l}\right)e^{-\chi^2\left[(\mu_{mj}/a)^2+(n\pi/l)^2\right]t}e^{\mathrm{i}m(\varphi-\varphi')}$$

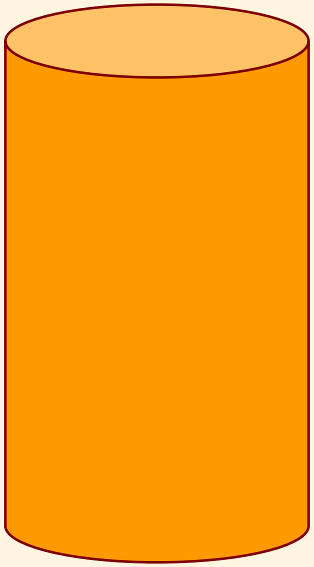
■有限长圆柱体——非齐次边界条件，非齐次方程，如何处理?

$$
\begin{cases}
\rho c_V \dfrac{\partial u}{\partial t} - \kappa \nabla^2 u = f(\rho,\varphi,z,t) \\
u|_{t=0} = u_0(\rho,\varphi,z); u|_{\rho=a} = u_1(\varphi,z,t) \\
u|_{z=0} = u_2(\rho,\varphi,t); u|_{z=l} = u_3(\rho,\varphi,t)
\end{cases}
$$

■本征函数展开方法

Laplace算子在柱内的本征值问题

$$
\begin{cases}
-\nabla^2 U(\rho,\varphi,z) = k^2 U(\rho,\varphi,z) \\
U(\rho,\varphi,z)|_{\rho=a} = 0 \\
U(\rho,\varphi,z)|_{z=0} = U(\rho,\varphi,z)|_{z=l} = 0
\end{cases}
$$



□ 柱内本征函数和本征值

$$U_{mjn}(\rho,\varphi,z)=\frac{1}{N_{mjn}}J_m\left(\mu_{mj}\frac{\rho}{a}\right)\sin\left(\frac{n\pi z}{l}\right)e^{im\varphi}$$

$$k_{mjn}^2=\left(\frac{\mu_{mj}}{a}\right)^2+\left(\frac{n\pi}{l}\right)^2$$

作为习题

$$(j=1,2,3,\ldots.;n=1,2,\ldots.;m=0,\pm1,\pm2,\ldots.)$$

□ 本征函数展开解

$$u(\rho,\varphi,z,t)=\sum_{mjn}a_{mjn}(t)U_{mjn}(\rho,\varphi,z)$$

$$a_{mjn}(t)=\int_G u(\rho,\varphi,z,t)U_{mjn}^*(\rho,\varphi,z)\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z$$

$$\int_G (\varphi_1^* \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1^*) \mathrm{d}\tau = \iint_B \left(\varphi_1^* \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1^*}{\partial n} \right) \mathrm{d}S$$

$$\varphi_1^*(\boldsymbol{r}) = U_{mjn}^*(\boldsymbol{r}); \varphi_2(\boldsymbol{r}) = u(\boldsymbol{r}, t)$$

$$\int_G (U_{mjn}^* \nabla^2 u - u \nabla^2 U_{mjn}^*) \mathrm{d}\tau = \iint_{\partial G} \left(U_{mjn}^* \frac{\partial u}{\partial n} - u \frac{\partial U_{mjn}^*}{\partial n} \right) \mathrm{d}S$$

$$\begin{cases} -\nabla^2 U_{mjn}^*(\rho, \varphi, z) = k_{mjn}^2 U_{mjn}^*(\rho, \varphi, z) \\ U_{mjn}^*(\rho, \varphi, z)\big|_{\rho=a} = 0 \\ U_{mjn}^*(\rho, \varphi, z)\big|_{z=0} = U_{mjn}^*(\rho, \varphi, z)\big|_{z=l} = 0 \end{cases}$$

$$\frac{\mathrm{d}a_{mjn}(t)}{\mathrm{d}t}+\chi^2 k_{mjn}^2 a_{mjn}(t)=\chi^2 F_{mjn}(t)$$

$$a_{mjn}(t)\big|_{t=0}=\int_G u_0(\rho,\varphi,z)U_{mjn}^*(\rho,\varphi,z)\mathrm{d}\tau\equiv a_{mjn}(0)$$

$$\Uparrow$$

$$F_{mjn}(t)\equiv B_{mjn}(t)+\frac{1}{\kappa}\int_G f(\rho,\varphi,z,t)U_{mjn}^*(\rho,\varphi,z)\mathrm{d}\tau$$

$$B_{mjn}(t)\equiv-\iint_{\partial G}u\frac{\partial U_{mjn}^*}{\partial n}\mathrm{d}S=-\int_0^l\int_0^{2\pi}u_1(\varphi,z,t)\frac{\partial U_{mjn}^*}{\partial\rho}\bigg|_{\rho=a}a\mathrm{d}z\mathrm{d}\varphi$$

$$+\int_0^a\int_0^{2\pi}u_2(\rho,\varphi,t)\frac{\partial U_{mjn}^*}{\partial z}\bigg|_{z=0}\rho\mathrm{d}\rho\mathrm{d}\varphi-\int_0^a\int_0^{2\pi}u_3(\rho,\varphi,t)\frac{\partial U_{mjn}^*}{\partial z}\bigg|_{z=l}\rho\mathrm{d}\rho\mathrm{d}\varphi$$

$$\Downarrow$$

$$a_{mjn}(t)=a_{mjn}(0)e^{-\chi^2k_{mjn}^2t}+\chi^2\int_0^t F_{mjn}(t')e^{-\chi^2k_{mjn}^2(t-t')}\mathrm{d}t'$$

□ 积分形式解

$$u(\rho,\varphi,z,t)=\int_G u_0(\rho',\varphi',z')G(\rho,\varphi,z;\rho',\varphi',z';t)\mathrm{d}\tau'$$

$$+\frac{\chi^2}{\kappa}\int_0^t\int_G f(\rho',\varphi',z',t')G(\rho,\varphi,z;\rho',\varphi',z';t-t')\mathrm{d}\tau'\mathrm{d}t'$$

$$+\chi^2\int_0^t\left[\begin{array}{l}-\int_0^l\int_0^{2\pi}u_1(\varphi,z,t)\left.\dfrac{\partial G(\rho,\varphi,z;\rho',\varphi',z';t-t')}{\partial\rho'}\right|_{\rho'=a}a\mathrm{d}z'\mathrm{d}\varphi'\\ +\int_0^a\int_0^{2\pi}u_2(\rho',\varphi',t)\left.\dfrac{\partial G(\rho,\varphi,z;\rho',\varphi',z';t-t')}{\partial z'}\right|_{z'=0}\rho\mathrm{d}\rho\mathrm{d}\varphi\\ -\int_0^a\int_0^{2\pi}u_3(\rho,\varphi,t)\left.\dfrac{\partial G(\rho,\varphi,z;\rho',\varphi',z';t-t')}{\partial z'}\right|_{z'=l}\rho\mathrm{d}\rho\mathrm{d}\varphi\end{array}\right]\mathrm{d}t'$$

$$\Uparrow$$

$$G(\rho,\varphi,z;\rho',\varphi',z';t)\equiv\sum_{mjn}U_{mjn}(\rho,\varphi,z)U^*_{mjn}(\rho',\varphi',z')e^{-\chi^2k^2_{mjn}t}$$

■ 无限长圆柱体——非齐次边界条件，非齐次方程，如何处理?

$$\begin{cases} \rho c_V \dfrac{\partial u}{\partial t} - \kappa \nabla^2 u = f(\rho, \varphi, z, t) \\ u|_{t=0} = u_0(\rho, \varphi, z); u|_{\rho=a} = u_1(\varphi, z, t) \end{cases}$$

■z方向Fourier积分

$$u(\rho, \varphi, z, t) = \int_{-\infty}^{\infty} U(\rho, \varphi, \delta, t) \exp(\mathrm{i}\delta z) \mathrm{d}\delta$$

$$\Downarrow$$

$$\begin{cases} \int_{-\infty}^{\infty} \left\{ \rho c_V U_t(\rho, \varphi, \delta, t) - \kappa \left[\nabla_T^2 U(\rho, \varphi, \delta, t) - \delta^2 U(\rho, \varphi, \delta, t) \right] \right\} \\ \qquad \cdot \exp(\mathrm{i}\delta z) \mathrm{d}\delta = f(\rho, \varphi, z, t) \\ \int_{-\infty}^{\infty} U(\rho, \varphi, \delta, t)|_{t=0} \exp(\mathrm{i}\delta z) \mathrm{d}\delta = u_0(\rho, \varphi, z) \\ \int_{-\infty}^{\infty} U(\rho, \varphi, \delta, t)|_{\rho=a} \exp(\mathrm{i}\delta z) \mathrm{d}\delta = u_1(\varphi, z, t) \end{cases}$$

$$
\begin{cases}
\rho c_V U_t(\rho,\varphi,\delta,t)-\kappa\left[\nabla_T^2 U(\rho,\varphi,\delta,t)-\delta^2 U(\rho,\varphi,\delta,t)\right] \\
\qquad = F(\rho,\varphi,\delta,t);\ \nabla_T^2 \equiv \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\varphi^2} \\
U(\rho,\varphi,\delta,t)\big|_{t=0}=U_0(\rho,\varphi,\delta) \\
U(\rho,\varphi,\delta,t)\big|_{\rho=a}=U_1(\varphi,\delta,t)
\end{cases}
$$

$$
F(\rho,\varphi,\delta,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\rho,\varphi,z,t)\exp(-\mathrm{i}\delta z)\mathrm{d}z
$$

$$
U_0(\rho,\varphi,\delta)=\frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(\rho,\varphi,z)\exp(-\mathrm{i}\delta z)\mathrm{d}z
$$

$$
U_1(\varphi,\delta,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} u_1(\varphi,z,t)\exp(-\mathrm{i}\delta z)\mathrm{d}z
$$

■ **二维本征函数**

$$\begin{cases} -\nabla_T^2 \psi(\rho,\varphi) = k^2 \psi(\rho,\varphi) \\ \psi(\rho,\varphi)\big|_{\rho=a} = 0 \end{cases}$$

$$\psi_{mj}(\rho,\varphi) = \frac{1}{N_{mj}} J_m\left(\mu_{mj}\frac{\rho}{a}\right)\exp(\mathrm{i}m\varphi)$$

$$k_{mj} = \mu_{mj}/a; N_{mj} = \sqrt{2\pi\int_0^a \left[J_m\left(\mu_{mj}\frac{\rho}{a}\right)\right]^2 \rho \mathrm{d}\rho}$$

■二维本征函数展开解

$$U(\rho,\varphi,\delta,t) = \sum_{mj} a_{mj}(t,\delta)\psi_{mj}(\rho,\varphi)$$

$$a_{mj}(t,\delta) = \int_S U(\rho,\varphi,\delta,t)\psi_{mj}^*(\rho,\varphi)\rho \mathrm{d}\rho \mathrm{d}\varphi$$

$$\int_S (\varphi_1^* \nabla_T^2 \varphi_2 - \varphi_2 \nabla_T^2 \varphi_1^*) \mathrm{d}S = \int_L \left(\varphi_1^* \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1^*}{\partial n} \right) \mathrm{d}L$$

$$\varphi_1^* = \psi_{mj}^*; \varphi_2 = U(\rho, \varphi, \delta, t)$$

$$\int_S (\psi_{mj}^* \nabla_T^2 U - U \nabla_T^2 \psi_{mj}^*) \mathrm{d}S = \int_L \int_0^{2\pi} \left(\psi_{mj}^* \frac{\partial U}{\partial \rho} - U \frac{\partial \psi_{mj}^*}{\partial \rho} \right) \Bigg|_{\rho=a} a \mathrm{d}\varphi$$

$$\frac{\mathrm{d}a_{mj}(t,\delta)}{\mathrm{d}t} + \chi^2 (k_{mj}^2 + \delta^2) a_{mj}(t,\delta) = B_{mj}(t,\delta)$$

$$a_{mj}(t,\delta)\big|_{t=0} = \int_S U_0(\rho,\varphi,\delta) \psi_{mj}^*(\rho,\varphi) \rho \mathrm{d}\rho \mathrm{d}\varphi = a_{mj}(0,\delta)$$

$$B_{mj}(t,\delta) \equiv \frac{\chi^2}{\kappa} \int_0^a \int_0^{2\pi} \psi_{mj}^*(\rho,\varphi) F(\rho,\varphi,\delta,t) \rho \mathrm{d}\rho \mathrm{d}\varphi$$

$$- \int_0^{2\pi} \left[U_1(\varphi,\delta,t) \frac{\partial \psi_{mj}^*}{\partial \rho} \right] \Bigg|_{\rho=a} a \mathrm{d}\varphi$$

$$a_{mj}(t,\delta)=a_{mj}(0,\delta)e^{-\chi^2(k_m^2+\delta^2)t}+\int_0^t B_{mj}(t')e^{-\chi^2(k_{mj}^2+\delta^2)(t-t')}\mathrm{d}t'$$

□ 积分形式解

$$u(\rho,\varphi,z,t)=\int_{-\infty}^{\infty}\int_0^a\int_0^{2\pi}\int_{-\infty}^{\infty}u_0(\rho',\varphi',z')G(\rho,\varphi,z;\rho',\varphi',z',t)\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$$

$$+\frac{\chi^2}{\kappa}\int_0^t\int_{-\infty}^{\infty}\int_0^a\int_0^{2\pi}f(\rho',\varphi',z',t')G(\rho,\varphi,z;\rho',\varphi',z',t-t')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'\mathrm{d}t'$$

$$-\int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}u_1(\varphi',z',t')\left.\frac{\partial G(\rho,\varphi,z;\rho',\varphi',z',t-t')}{\partial\rho'}\right|_{\rho'=a}a\mathrm{d}\varphi'\mathrm{d}z'\mathrm{d}t'$$

$$G\equiv\frac{1}{\sqrt{4\pi\chi^2t}}\exp\left[-\frac{(z-z')^2}{4\chi^2t}\right]\sum_{mj}e^{-\chi^2k_m^2t}\psi_{mj}(\rho,\varphi)\psi_{mj}^*(\rho',\varphi')$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\chi^2\delta^2t+\mathrm{i}\delta(z-z')}\mathrm{d}\delta=\frac{1}{\sqrt{4\pi\chi^2t}}\exp\left[-\frac{(z-z')^2}{4\chi^2t}\right]$$

例2 有限厚度的无限大圆盘

$$\begin{cases} \rho c_V \dfrac{\partial u}{\partial t} - \kappa \nabla^2 u = 0, (t > 0, 0 < \rho < \infty, 0 < z < l) \\ u|_{t=0} = u_0(\rho, \varphi, z); u|_{z=0} = 0; u|_{z=l} = 0 \end{cases}$$

- **角度部分**

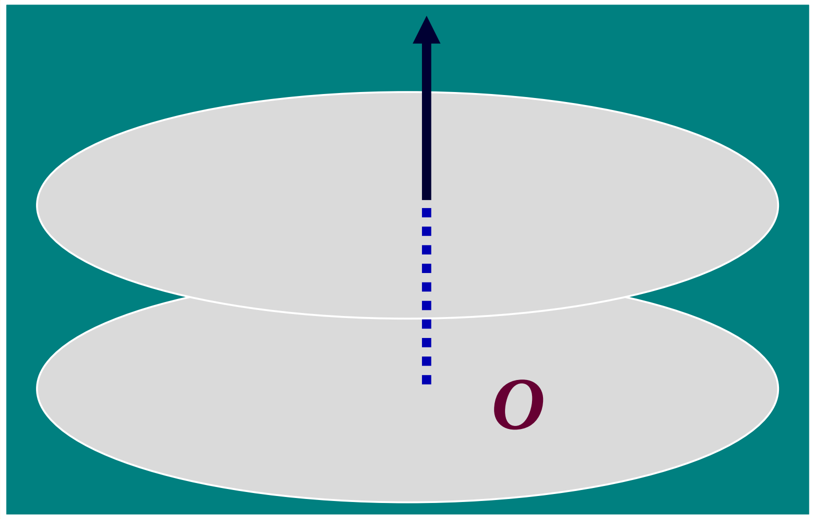
$$\Phi_m(\varphi) = A_m e^{\mathrm{i}m\varphi}, (m = 0, \pm 1, \pm 2, \ldots)$$

- **轴向部分**

$$Z_n(z) = A_n \sin\left(\frac{n\pi z}{l}\right); \ \delta_n = \frac{n\pi}{l}$$

- **径向部分**

$$k_\rho^2 \equiv k^2 - \delta_n^2$$



$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}+\left(k_\rho^2-\frac{m^2}{\rho^2}\right)R=0$$

$$R_m(\rho)=A_mJ_m(k_\rho\rho)+B_mN_m(k_\rho\rho)\Rightarrow B_m=0$$

——不存在ρ边界条件：k_ρ为连续谱$0<k_\rho<\infty$

$$u(\rho,\varphi,z,t)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^\infty A_{mn}(k_\rho)e^{-\chi^2\left[k_\rho^2+(n\pi/l)^2\right]t}J_m(k_\rho\rho)k_\rho\mathrm{d}k_\rho$$

$$\cdot\sin\left(\frac{n\pi z}{l}\right)e^{\mathrm{i}m\varphi}$$

初始条件

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^\infty A_{mn}(k_\rho)J_m\left(k_\rho\rho\right)k_\rho\mathrm{d}k_\rho\sin\left(\frac{n\pi z}{l}\right)e^{\mathrm{i}m\varphi}=u_0(\rho,\varphi,z)$$

$$\bar{u}_{mn}(\rho)=\frac{1}{\pi l}\int_0^{2\pi}\int_0^{l}u_0(\rho,\varphi,z)\sin\left(\frac{n\pi z}{l}\right)e^{-\mathrm{i}m\varphi}\mathrm{d}\varphi\mathrm{d}z$$

$$\int_0^{\infty}A_{mn}(k_\rho)J_m\left(k_\rho\rho\right)k_\rho\mathrm{d}k_\rho=\bar{u}_{mn}(\rho)$$ **能否求出系数?**

两边× $\rho J_m\left(k'_\rho\rho\right)$ **并且积分**

$$\int_0^{\infty}A_{mn}(k_\rho)\left[\int_0^{\infty}J_m\left(k_\rho\rho\right)J_m\left(k'_\rho\rho\right)\rho\mathrm{d}\rho\right]k_\rho\mathrm{d}k_\rho$$

$$=\int_0^{\infty}\bar{u}_{mn}(\rho)J_m\left(k'_\rho\rho\right)\rho\mathrm{d}\rho$$

$$\int_0^{\infty}J_m\left(k_\rho\rho\right)J_m\left(k'_\rho\rho\right)\rho\mathrm{d}\rho=\frac{\delta(k_\rho-k'_\rho)}{k_\rho}$$

$$\int_0^{\infty}A_{mn}(k_\rho)\left[\frac{\delta(k_\rho-k'_\rho)}{k_\rho}\right]k_\rho\mathrm{d}k_\rho=A_{mn}(k'_\rho)$$

$$A_{mn}(k_\rho)=\int_0^\infty \bar{u}_{mn}(\rho)J_m\left(k_\rho\rho\right)\rho\mathrm{d}\rho$$

$$\bar{u}_{mn}(\rho)=\int_0^\infty A_{mn}(k_\rho)J_m\left(k_\rho\rho\right)k_\rho\mathrm{d}k_\rho$$

$$f_m(\rho)=\int_0^\infty g_m(k_\rho)J_m(k_\rho\rho)k_\rho\mathrm{d}k_\rho$$

$$g_m(k_\rho)=\int_0^\infty f_m(\rho)J_m(k_\rho\rho)\rho\mathrm{d}\rho$$

——m阶 Hankel 变换对

积分解形式

$$u(\rho,\varphi,z,t)=\int_0^\infty\int_0^{2\pi}\int_0^l u_0(\rho',\varphi',z')G(\rho,\varphi,z;\rho',\varphi',z',t)\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$$

$$G(\rho,\varphi,z;\rho',\varphi',z',t)\equiv\frac{1}{\pi l}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^\infty e^{-\chi^2[k_\rho^2+(n\pi/l)^2]t}J_m(k_\rho\rho)J_m(k_\rho\rho')k_\rho\mathrm{d}k_\rho$$

$$\times\sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi z'}{l}\right)e^{\mathrm{i}m(\varphi-\varphi')}$$

利用积分关系

$$\int_0^\infty e^{-\chi^2 k_\rho^2 t} J_m(k_\rho \rho) J_m(k_\rho \rho') k_\rho \mathrm{d}k_\rho = \frac{1}{2\chi^2 t} \exp\left(-\frac{\rho^2+\rho'^2}{4\chi^2 t}\right) I_m\left(\frac{\rho\rho'}{2\chi^2 t}\right)$$

$$G(\rho,\varphi,z;\rho',\varphi',z',t) \equiv \frac{1}{\pi l}\frac{1}{2\chi^2 t}\exp\left(-\frac{\rho^2+\rho'^2}{4\chi^2 t}\right)$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{\rho\rho'}{2\chi^2 t}\right) e^{-\chi^2 (n\pi/l)^2 t} \sin\left(\frac{n\pi z}{l}\right)\sin\left(\frac{n\pi z'}{l}\right) e^{\mathrm{i}m(\varphi-\varphi')}$$

■ t=0的特性

$$G(\rho,\varphi,z;\rho',\varphi',z',t=0) = \delta(z-z')\delta(\varphi-\varphi')\lim_{t\to 0}\frac{1}{2\sqrt{\pi\rho\rho'\chi^2 t}}\exp\left[-\frac{(\rho-\rho')^2}{4\chi^2 t}\right]$$

$$= \frac{1}{\rho}\delta(\rho-\rho')\delta(z-z')\delta(\varphi-\varphi')$$

$$I_m\left(\frac{\rho\rho'}{2\chi^2 t}\right) \to \frac{\sqrt{\chi^2 t}}{\sqrt{\pi\rho\rho'}}\exp\left(\frac{\rho\rho'}{2\chi^2 t}\right); \delta(x) = \lim_{t\to 0}\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right)$$

12.2 Hankel 函数及其应用

❑ 定义：Bessel 函数 与 Neumann 函数的线性组合也是 Bessel 方程的解

$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + \mathrm{i}N_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - \mathrm{i}N_\nu(x) \end{cases}$$

Bessel方程的通解也可表示为

$$y(x) = C_1 H_\nu^{(1)}(x) + C_2 H_\nu^{(2)}(x)$$

■ 柱函数：Bessel函数$J_\nu(x)$、Neumann函数$N_\nu(x)$、Hankel函数 $H_\nu^{(1)}(x), H_\nu^{(2)}(x)$

——满足柱函数的递推公式

柱坐标

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-\frac{v^2}{x^2}\right)y=0$$

■ **驻波解**

$$y(x)=C_1 J_v(x)+C_2 N_v(x)$$

$$\begin{cases} H_v^{(1)}(x)=J_v(x)+\mathrm{i}N_v(x) \\ H_v^{(2)}(x)=J_v(x)-\mathrm{i}N_v(x) \end{cases}$$

■ **行波解**

$$y(x)=C_1 H_v^{(1)}(x)+C_2 H_v^{(2)}(x)$$

直角坐标

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+y=0$$

■ **驻波解**

$$y(x)=C_1\cos x+C_2\sin x$$

$$\begin{cases} e^{\mathrm{i}x}=\cos x+\mathrm{i}\sin x \\ e^{-\mathrm{i}x}=\cos x-\mathrm{i}\sin x \end{cases}$$

■ **行波解**

$$y(x)=C_1 e^{\mathrm{i}x}+C_2 e^{-\mathrm{i}x}$$

❑无限远的特性

$$H_\nu^{(1)}(x)\sim\sqrt{\frac{2}{\pi x}}e^{\mathrm{i}(x-\nu\pi/2-\pi/4)};H_\nu^{(2)}(x)\sim\sqrt{\frac{2}{\pi x}}e^{-\mathrm{i}(x-\nu\pi/2-\pi/4)}$$

不同用途:

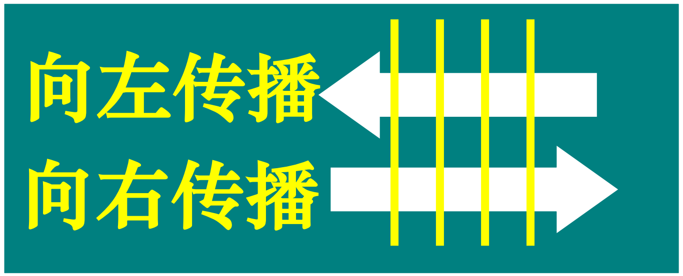
（1）Bessel 函数：振荡特性，讨论封闭空间的驻波问题，类似于 cos*x*

（2）Neumann 函数：在 ρ=0 有奇异性，讨论不包括原点的问题, 振荡特性: 类似于 sin*x*

（3）Hankel 函数：行波特性，讨论开空间波的传播和散射问题，类似于 e^{ix} 和 e^{-ix}

■一维平面，振幅不变化

$$y(x)=C_1e^{\mathrm{i}x}+C_2e^{-\mathrm{i}x}$$



■二维柱面波，波阵面扩散

$$H_v^{(1)}(x)\sim\sqrt{\frac{2}{\pi x}}e^{\mathrm{i}(x-v\pi/2-\pi/4)};H_v^{(2)}(x)\sim\sqrt{\frac{2}{\pi x}}e^{-\mathrm{i}(x-v\pi/2-\pi/4)}$$

远场 $\sim 1/\sqrt{x}$ 衰减，近场变化很复杂

例1 半径为 a 的无限长圆柱面，其径向速度为

$$v = v_0 \cos(\omega t); v = v_0 e^{-\mathrm{i}\omega t}$$

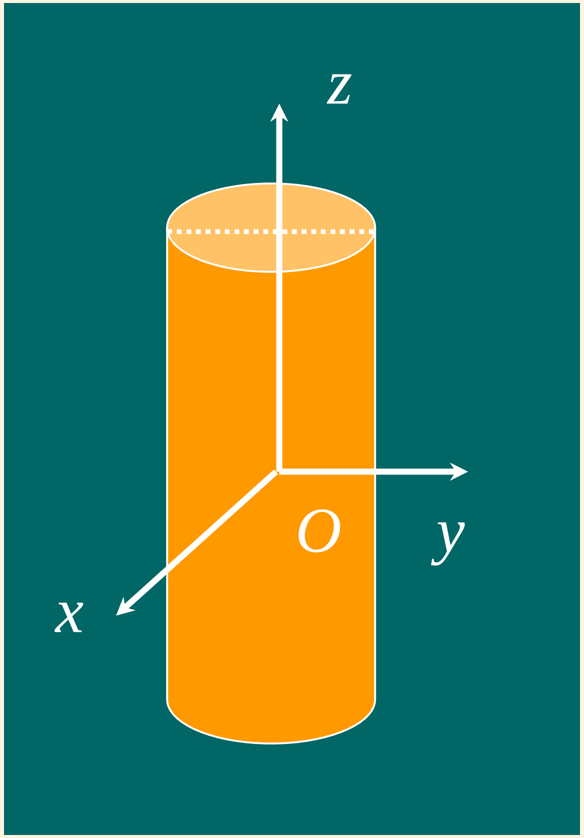
求向外辐射的声场。

解：问题与 z 和 φ 无关

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial u}{\partial \rho}\right) = 0 \\ \rho_0 \dfrac{\partial v}{\partial t} = -\dfrac{\partial u}{\partial \rho} \Rightarrow \left.\dfrac{\partial u}{\partial \rho}\right|_{\rho=a} = \mathrm{i}\rho_0 \omega v_0 e^{-\mathrm{i}\omega t} \end{cases}$$

求稳态解
无须分离
变量

$$u(\rho, t) = R(\rho) e^{-\mathrm{i}\omega t}$$



$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right)+k^2R=0,\left(k=\frac{\omega}{c},\quad \rho>a\right);\quad \left.\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right|_{\rho=a}=\mathrm{i}\rho_0\omega v_0$$

$$R(\rho)=C_1H_0^{(1)}(k\rho)+C_2H_0^{(2)}(k\rho)$$

——Hankel 函数的取舍：决定于时间部分的形式。在无限远处

$$\exp[\mathrm{i}(k\rho-\omega t)]$$

$$k\rho-\omega t=C\Rightarrow\frac{\mathrm{d}\rho}{\mathrm{d}t}=\frac{\omega}{k}=+c$$

—原点向外辐射柱面波

$$\exp[-\mathrm{i}(k\rho+\omega t)]$$

$$k\rho+\omega t=C\Rightarrow\frac{\mathrm{d}\rho}{\mathrm{d}t}=-\frac{\omega}{k}=-c$$

—向原点会聚的柱面波

因此 （1）如果时间部分为 $e^{-\mathrm{i}\omega t}$

$H_v^{(1)}$ ——向外辐射的柱面波

$H_v^{(2)}$ ——向原点会聚的柱面波

（2）如果时间部分为 $e^{\mathrm{i}\omega t}$

$H_v^{(2)}$ ——向外辐射的柱面波

$H_v^{(1)}$ ——向原点会聚的柱面波

本问题取

$$R(\rho)=C_1H_0^{(1)}(k\rho);C_2\equiv 0$$

$$\Downarrow$$

$$C_1k\left[\frac{\mathrm{d}H_0^{(1)}(k\rho)}{\mathrm{d}(k\rho)}\right]\Bigg|_{\rho=a}=\mathrm{i}\rho_0\omega v_0\Rightarrow C_1=\frac{\mathrm{i}\rho_0\omega v_0}{kH_0'^{(1)}(ka)}$$

❑如果*ka*<<1（柱的半径远小于波长），利用

$$H_0^{(1)}(x)=J_0(x)+\mathrm{i}N_0(x)\approx 1+\mathrm{i}\frac{2}{\pi}\ln\frac{x}{2}$$

$$Ck\left[\frac{\mathrm{d}}{\mathrm{d}(k\rho)}\left(1+\mathrm{i}\frac{2}{\pi}\ln\frac{k\rho}{2}\right)\right]\Bigg|_{\rho=a}=\mathrm{i}\rho_0\omega v_0\Rightarrow \mathrm{i}C\frac{2}{\pi a}=\mathrm{i}\rho_0\omega v_0$$

于是，声场的分布为

$$u(\rho,t)=\frac{\pi a}{2}\rho_0\omega v_0 H_0^{(1)}\left(\frac{\omega}{c}\rho\right)e^{-\mathrm{i}\omega t}$$

远场近似 $\omega\rho/c>>1$

$$u(\rho,t)\sim\rho_0\omega v_0 a e^{\mathrm{i}\pi/4}\sqrt{\frac{\pi c}{2\omega\rho}}\exp\left[\mathrm{i}\left(\frac{\omega}{c}\rho-\omega t\right)\right]$$

$$\mathrm{Re}[u(\rho,t)] = \mathrm{Re}\left[\frac{\pi\rho_0\omega v_0 a}{2} H_0^{(1)}\left(\frac{\omega}{c}\rho\right)e^{-\mathrm{i}\omega t}\right]$$

□ 如果不利用复数进行，会怎么样?

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) = 0;\ \left.\frac{\partial u}{\partial\rho}\right|_{\rho=a} = -\rho_0\left.\frac{\partial v}{\partial t}\right|_{\rho=a} = \rho_0\omega v_0\sin(\omega t)$$

$$u(\rho,t) = R_1(\rho)\sin(\omega t) + R_2(\rho)\cos(\omega t)$$

为了满足 Sommerfeld 辐射条件

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R_j}{\mathrm{d}\rho}\right) + k^2 R_j = 0, (j = 1,2)$$

$$R_j(\rho) = C_{1j}H_0^{(1)}(k\rho) + C_{2j}H_0^{(2)}(k\rho)$$

$$R_j(\rho) = C_{1j}J_0(k\rho) + C_{2j}N_0(k\rho)$$

如何决定4个系数?

边界条件：$\rho=a$

$$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=a} = \rho_0 \omega v_0 \sin(\omega t) = R_1'(\rho)\sin(\omega t) + R_2'(\rho)\cos(\omega t)$$

$$\left.\frac{\mathrm{d}R_1(\rho)}{\mathrm{d}\rho}\right|_{\rho=a} = \rho_0 \omega v_0; \left.\frac{\mathrm{d}R_2(\rho)}{\mathrm{d}\rho}\right|_{\rho=a} = 0$$

边界条件：$\rho\to\infty$，Sommerfeld辐射条件

$$\lim_{\rho\to\infty}\sqrt{\rho}\left(\frac{\partial u}{\partial \rho}+\frac{1}{c}\frac{\partial u}{\partial t}\right)=0$$

$$\lim_{\rho\to\infty}\sqrt{\rho}\left[\frac{\mathrm{d}R_1(\rho)}{\mathrm{d}\rho}-kR_2(\rho)\right]=0$$

$$\lim_{\rho\to\infty}\sqrt{\rho}\left[\frac{\mathrm{d}R_2(\rho)}{\mathrm{d}\rho}+kR_1(\rho)\right]=0$$

4个方程决定4个系数！——可见复数运算的必要性

■问题1：与角度有关？如何处理

$$v(\varphi,t)=v_0(\varphi)e^{-\mathrm{i}\omega t}$$

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-c^2\left[\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\varphi^2}\right]=0\\ \left.\dfrac{\partial u}{\partial\rho}\right|_{\rho=a}=\mathrm{i}\rho_0\omega v_0(\varphi)e^{-\mathrm{i}\omega t};\lim\limits_{\rho\to\infty}\sqrt{\rho}\left(\dfrac{\partial u}{\partial\rho}+\dfrac{1}{c}\dfrac{\partial u}{\partial t}\right)=0\end{cases}$$

$$u(\rho,\varphi,t)=\sum_{m=-\infty}^{\infty}C_m H_m^{(1)}(k\rho)e^{\mathrm{i}m\varphi}\exp(-\mathrm{i}\omega t)$$

对称性

$$v_0(\varphi)=+v_0(-\varphi)$$
$$v_0(\varphi)=-v_0(-\varphi)$$

$$u(\rho,\varphi,t)=\sum_{m=0}^{\infty}C_m H_m^{(1)}(k\rho)\cos(m\varphi)\exp(-\mathrm{i}\omega t)$$

$$u(\rho,\varphi,t)=\sum_{m=1}^{\infty}C_m H_m^{(1)}(k\rho)\sin(m\varphi)\exp(-\mathrm{i}\omega t)$$

■问题2：与z有关？如何处理

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\left[\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\varphi^2} + \dfrac{\partial^2 u}{\partial z^2}\right] = 0 \\ \left.\dfrac{\partial u}{\partial\rho}\right|_{\rho=a} = \mathrm{i}\rho_0\omega v_0(z,\varphi)e^{-\mathrm{i}\omega t}; \lim\limits_{\rho\to\infty}\sqrt{\rho}\left(\dfrac{\partial u}{\partial\rho} + \dfrac{1}{c}\dfrac{\partial u}{\partial t}\right) = 0 \end{cases}$$

$$\Downarrow$$

$$u(\rho,\varphi,t) = \int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_m(\delta)H_m^{(1)}(k_\rho\rho)e^{\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\delta\exp(-\mathrm{i}\omega t)$$

$$\Downarrow$$

$$\boxed{k_\rho \equiv \sqrt{(\omega/c)^2 - \delta^2}}$$

$$\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} k_\rho C_m(\delta)\frac{\mathrm{d}H_m^{(1)}(k_\rho a)}{\mathrm{d}(k_\rho a)}e^{\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\delta = \mathrm{i}\rho_0\omega v_0(z,\varphi)$$

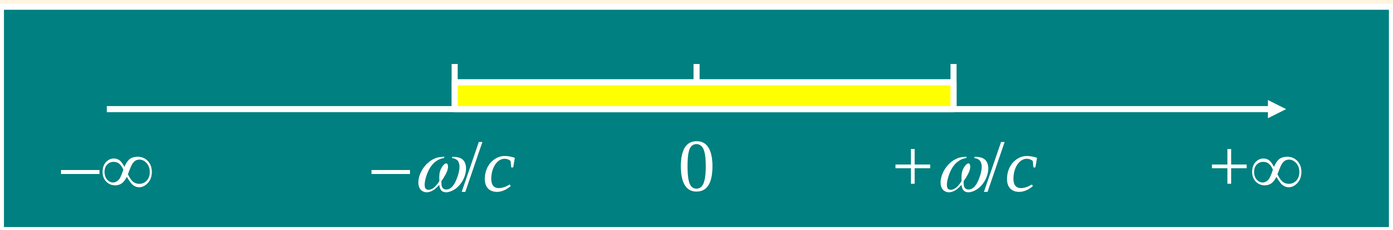
$$\Downarrow$$

$$C_m(\delta) = \frac{\mathrm{i}\rho_0\omega}{(2\pi)^2 k_\rho}\left[\frac{\mathrm{d}H_m^{(1)}(k_\rho a)}{\mathrm{d}(k_\rho a)}\right]^{-1}\int_0^{2\pi}\int_{-\infty}^{\infty} v_0(z,\varphi)e^{-\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\varphi\mathrm{d}z$$

$$u(\rho,\varphi,t)=\int_{-\omega/c}^{\omega/c}\sum_{m=-\infty}^{\infty}C_m(\delta)H_m^{(1)}\left(\sqrt{\frac{\omega^2}{c^2}-\delta^2}\,\rho\right)e^{\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\delta\exp(-\mathrm{i}\omega t)$$

$$+\int_{-\infty}^{-\omega/c}\sum_{m=-\infty}^{\infty}C_m(\delta)H_m^{(1)}\left(\mathrm{i}\sqrt{\delta^2-\frac{\omega^2}{c^2}}\,\rho\right)e^{\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\delta\exp(-\mathrm{i}\omega t)$$

$$+\int_{\omega/c}^{\infty}\sum_{m=-\infty}^{\infty}C_m(\delta)H_m^{(1)}\left(\mathrm{i}\sqrt{\delta^2-\frac{\omega^2}{c^2}}\,\rho\right)e^{\mathrm{i}(m\varphi+\delta z)}\mathrm{d}\delta\exp(-\mathrm{i}\omega t)$$



倏逝波 **传播波** **倏逝波**

——高频分布(指空间频率)的振动速度不产生向外传播的声辐射，仅仅存在于近场

$$v_0(z,\varphi)=v_{01}(\varphi)e^{\mathrm{i}k_1z}+v_{02}(\varphi)e^{\mathrm{i}k_2z}$$

$$u(\rho,\varphi,t)=\sum_{m=-\infty}^{\infty}\sum_{j=1,2}\frac{\mathrm{i}\rho_0\omega}{2\pi k_\rho^j}\left[\frac{\mathrm{d}H_m^{(1)}(k_\rho^j a)}{\mathrm{d}(k_\rho^j a)}\right]^{-1}H_m^{(1)}(k_\rho^j\rho)e^{\mathrm{i}k_j z}\exp(-\mathrm{i}\omega t)$$

$$\times\left[\int_0^{2\pi}v_{0j}(\varphi')e^{\mathrm{i}m(\varphi-\varphi')}\mathrm{d}\varphi'\right];\left(k_\rho^j\equiv\sqrt{(\omega/c)^2-k_j^2}\right)$$

如果 $$k_1<\frac{\omega}{c};\quad k_2>\frac{\omega}{c}$$

$$k_\rho^1=\sqrt{(\omega/c)^2-k_1^2};k_\rho^2=\sqrt{(\omega/c)^2-k_2^2}=\mathrm{i}\sqrt{k_2^2-(\omega/c)^2}\equiv\mathrm{i}\kappa_2$$

$$H_m^{(1)}(k_\rho^2\rho)=H_m^{(1)}(\mathrm{i}\kappa_2\rho)\Leftarrow K_\nu(x)=\frac{\pi}{2}\mathrm{i}^{\nu+1}H_\nu^{(1)}(\mathrm{i}x)$$

——k_2模式(高频)产生的辐射是倏逝波——只有近场辐射——不是所有的振动都产生有效的声辐射

■ 如果刚好$k_2=\omega/c$，结果如何?

$$k_\rho^2=\sqrt{(\omega/c)^2-k_2^2}\to 0\Rightarrow H_m^{(1)}(k_\rho^2\rho)\to\infty$$

——此时，必须考虑振动柱体与激发声场的耦合问题，修改物理模型

■问题3：与时间一般关系？如何处理

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-c^2\left[\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\varphi^2}+\dfrac{\partial^2 u}{\partial z^2}\right]=0\\ \left.\dfrac{\partial u}{\partial\rho}\right|_{\rho=a}=-\rho_0\dfrac{\partial v}{\partial t}\end{cases}$$

$$\boxed{k_\rho=\sqrt{\left(\frac{\omega}{c}\right)^2-\delta^2}}$$

$$u(\rho,\varphi,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}C_m(\delta,\omega)H_m^{(1)}(k_\rho\rho)e^{\mathrm{i}(m\varphi+\delta z-\omega t)}\mathrm{d}\delta\mathrm{d}\omega$$

例2 刚性圆柱体对平面声波的散射

解：声波方程

$$\frac{\partial^2 p}{\partial t^2} - c^2\nabla^2 p = 0$$

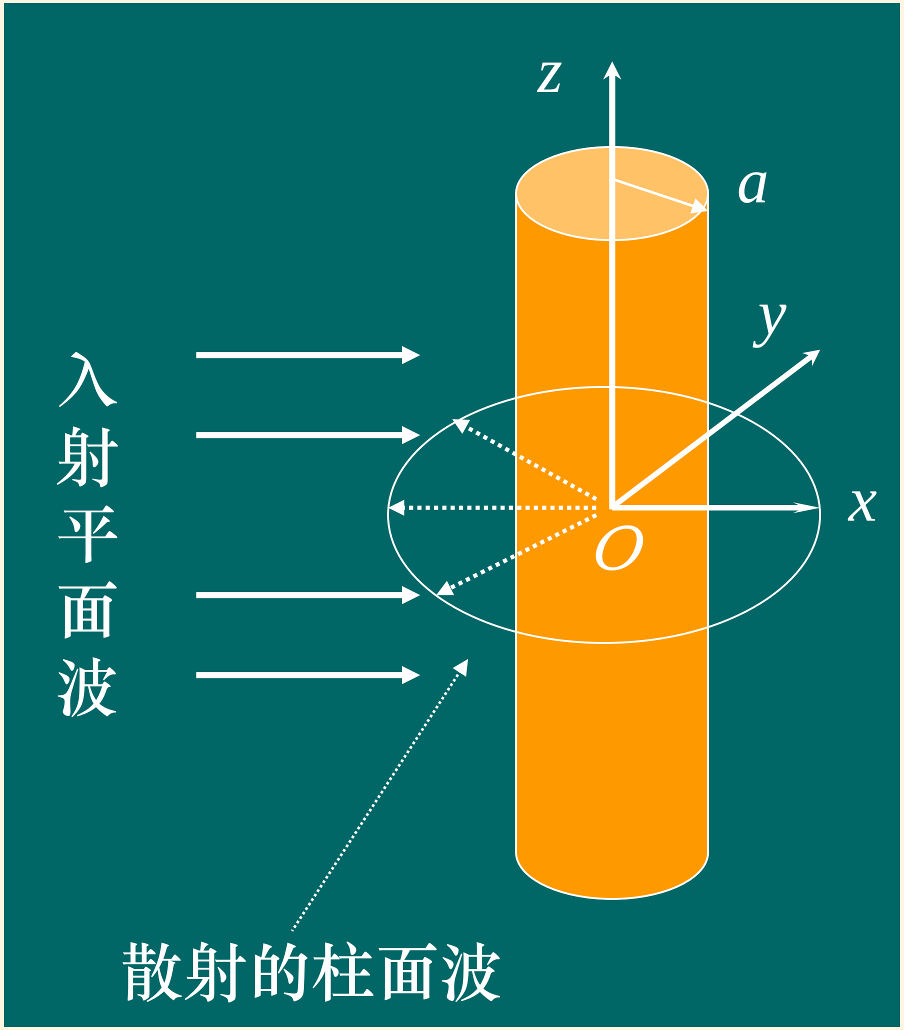
考虑单频平面波的入射

$$p_i = p_0 \exp\left[\mathrm{i}\left(\frac{\omega}{c}x - \omega t\right)\right]$$

$$= p_0 \exp\left[\mathrm{i}\left(\frac{\omega}{c}\rho\cos\varphi - \omega t\right)\right]$$

整个声场由入射场和散射场组成

$$p = p_i + p_s e^{-\mathrm{i}\omega t}$$



代入波动方程

$$\nabla^2 p_s + k^2 p_s = 0, \left(k = \frac{\omega}{c} \right)$$

由对称性，问题与 z 无关

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial p_s}{\partial \rho} \right) + \frac{1}{\rho^2}\frac{\partial^2 p_s}{\partial \varphi^2} + k^2 p_s = 0$$

分离变量 $p_s = R(\rho)\Phi(\varphi)$

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}R}{\mathrm{d}\rho} \right) + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0; \quad \frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + m^2\Phi = 0$$

考虑到散射波是向外辐射的波，因此通解取为

$$p_s(\rho,\varphi) = \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(k\rho)e^{\mathrm{i}m\varphi}$$

系数$\{A_m\}$由圆柱面上的条件决定：因为圆柱体是刚性的，总声压的法向导数为零

$$\frac{\partial}{\partial\rho}\left(p_0 e^{\mathrm{i}k\rho\cos\varphi}+p_s\right)\Big|_{\rho=a}=0$$

利用

$$\exp(\mathrm{i}k\rho\cos\varphi)=\sum_{m=-\infty}^{\infty}\mathrm{i}^m J_m(k\rho)e^{\mathrm{i}m\varphi}$$

因此

$$\sum_{m=-\infty}^{\infty}A_m\frac{\mathrm{d}H_m^{(1)}(k\rho)}{\mathrm{d}(k\rho)}\Bigg|_{\rho=a}e^{\mathrm{i}m\varphi}=-p_0\sum_{m=-\infty}^{\infty}\mathrm{i}^m\frac{\mathrm{d}J_m(k\rho)}{\mathrm{d}(k\rho)}\Bigg|_{\rho=a}e^{\mathrm{i}m\varphi}$$

$$\Downarrow$$

$$A_m=-p_0\mathrm{i}^m\left[\frac{\mathrm{d}J_m(k\rho)}{\mathrm{d}(k\rho)}\right]\left[\frac{\mathrm{d}H_m^{(1)}(ka)}{\mathrm{d}(ka)}\right]^{-1}$$

散射场的计算问题

$$p_s(\rho,\varphi) = -p_0 \sum_{m=-\infty}^{\infty} \mathrm{i}^m \frac{\mathrm{d}J_m(ka)}{\mathrm{d}(ka)} \left[\frac{\mathrm{d}H_m^{(1)}(ka)}{\mathrm{d}(ka)} \right]^{-1} H_m^{(1)}(k\rho) e^{\mathrm{i}m\varphi}$$

远场近似

$$p_s(\rho,\varphi) \approx -p_{0i}(\omega)\sqrt{\frac{2}{\pi k\rho}} \exp\left[\mathrm{i}\left(k\rho - \frac{\pi}{4}\right)\right]\psi_s(\varphi,\omega)$$

$$\psi_s(\varphi,\omega) \equiv \frac{J_1(ka)}{H_1^{(1)}(ka)} + 2\sum_{m=1}^{\infty} \frac{[J_m(ka)]'}{[H_m^{(1)}(ka)]'} \cos(m\varphi)$$

- **低频散射** $(ka << 1)$ **小参数展开**

$$J_0(x) \approx 1 - \frac{x^2}{2}; \quad J_\nu(x) \approx \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}, (\nu \neq -1, -2, -3, \ldots)$$

$$H_0^{(1)}(x)\approx \mathrm{i}N_0(x)\approx \frac{2\mathrm{i}}{\pi}\ln\frac{x}{2};\ H_\nu^{(1)}(x)\approx \mathrm{i}N_\nu(x)\approx -\frac{\mathrm{i}\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu,(\nu\neq 0)$$

$$\psi_s(\varphi,\omega)\approx \mathrm{i}\frac{\pi}{4}(ka)^2(1-2\cos\varphi)$$

- **高频散射** $(ka>>1)$ **能否用大参数展开?**

$$J_\nu(x)\approx\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right);\ H_\nu^{(1)}(x)\approx\sqrt{\frac{2}{\pi x}}\exp\left[\mathrm{i}\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right]$$

$$\psi_s(\varphi,\omega)=\frac{J_1(ka)}{H_1^{(1)}(ka)}+2\sum_{m=1}^{\infty}\frac{[J_m(ka)]'}{[H_m^{(1)}(ka)]'}\cos(m\varphi)$$

$$\approx\frac{\cos\left(ka-\frac{3\pi}{4}\right)}{\exp\left[\mathrm{i}\left(ka-\frac{3\pi}{4}\right)\right]}+2\mathrm{i}\sum_{m=1}^{\infty}\frac{\sin\left(ka-\frac{m\pi}{2}-\frac{\pi}{4}\right)}{\exp\left[\mathrm{i}\left(ka-\frac{m\pi}{2}-\frac{\pi}{4}\right)\right]}\cos(m\varphi)$$

收敛?

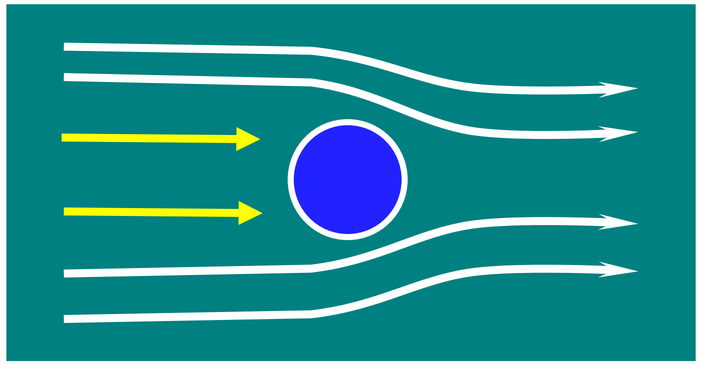
——当求和项足够多，可能$ka \sim m$—Bessel函数的大参数、大阶数展开！

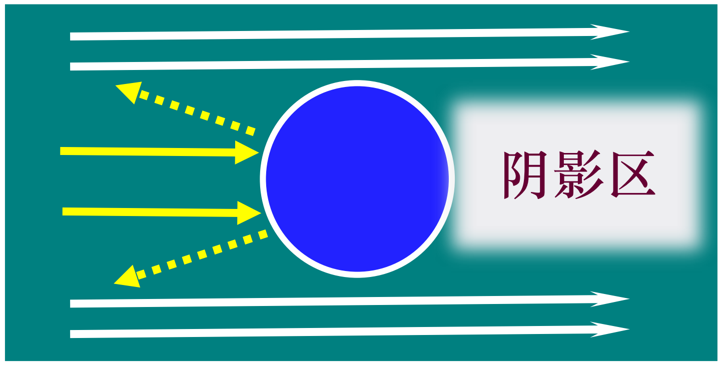
——数值计算中出现∞/∞, 解决方法：

(1)理论推导$[J_m(ka)]'/[H_m^{(1)}(ka)]'$

(2)衍射几何理论。

物理本质：低频——衍射；高频——反射场





阴影区散射场与入射场抵消，求和项必须足够多

12.3 虚宗量Bessel函数

❑**虚宗量 Bessel 函数：Laplace方程分离变量时出现，当μ<0 时：径向方程为虚宗量 Bessel 方程**

令ξ=ix

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2}+x\frac{\mathrm{d}R}{\mathrm{d}x}-(x^2+\nu^2)R=0,\quad\left(x=\sqrt{|\mu|}\rho\right)$$

$$\xi^2\frac{\mathrm{d}^2R}{\mathrm{d}\xi^2}+\xi\frac{\mathrm{d}R}{\mathrm{d}\xi}+(\xi^2-\nu^2)R=0$$

$$\left\{J_\nu(\mathrm{i}x),J_{-\nu}(\mathrm{i}x),N_\nu(\mathrm{i}x),H_\nu^{(1)}(\mathrm{i}x),H_\nu^{(2)}(\mathrm{i}x)\right\}$$

——是否线性独立?

■第一个:ν阶Bessel函数

$$R_1(x)=J_\nu(\mathrm{i}x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{\mathrm{i}x}{2}\right)^{\nu+2k}$$

$$=\mathrm{i}^\nu\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k}$$

定义虚宗量 Bessel函数

$$I_\nu(x)\equiv \mathrm{i}^{-\nu}J_\nu(\mathrm{i}x)=\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k}$$

■第二个: $-\nu$阶Bessel函数

$$R_2(x)=J_{-\nu}(\mathrm{i}x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)}\left(\frac{\mathrm{i}x}{2}\right)^{-\nu+2k}$$

$$=\mathrm{i}^{-\nu}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{-\nu+2k}$$

$$I_{-\nu}(x) = \mathrm{i}^{\nu} J_{-\nu}(\mathrm{i}x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{-\nu+2k}$$

但是当ν=m=0或整数

$$I_{-m}(x) = \mathrm{i}^m J_{-m}(\mathrm{i}x) = \mathrm{i}^m (-1)^m J_m(\mathrm{i}x)$$

$$= \mathrm{i}^m (-1)^m \mathrm{i}^m I_m(x) = I_m(x)$$

■第三个: Neumann函数

$$N_m(\mathrm{i}x) = \lim_{\nu \to m} \frac{J_\nu(\mathrm{i}x)\cos(\nu\pi) - J_{-\nu}(\mathrm{i}x)}{\sin \nu\pi}$$

$$= \lim_{\nu \to m} \frac{\mathrm{i}^{\nu} I_\nu(x)\cos(\nu\pi) - \mathrm{i}^{-\nu} I_{-\nu}(x)}{\sin(\nu\pi)}$$

——可见 当ν=2k+1, 极限不存在. 因此, 这时Neumann函数不能作为第二个解。

■**第四个:第一类Hankel函数**

$$H_\nu^{(1)}(\mathrm{i}x)=J_\nu(\mathrm{i}x)+\mathrm{i}N_\nu(\mathrm{i}x)=-\mathrm{i}e^{-\mathrm{i}\nu\pi/2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu\pi)}$$

$$K_\nu(x)\equiv\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu\pi)} \Rightarrow K_\nu(x)=\frac{\pi}{2}\mathrm{i}^{\nu+1}H_\nu^{(1)}(\mathrm{i}x)$$

■**第五个:第二类Hankel函数，与$K_\nu(x)$类似**

$$H_\nu^{(2)}(-\mathrm{i}x)=J_\nu(-\mathrm{i}x)-\mathrm{i}N_\nu(-\mathrm{i}x)=\mathrm{i}e^{\mathrm{i}\nu\pi/2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu\pi)}$$

$$=\frac{2\mathrm{i}}{\pi}e^{\mathrm{i}\nu\pi/2}K_\nu(x)\Rightarrow K_\nu(x)=-\frac{\pi\mathrm{i}}{2}e^{-\mathrm{i}\nu\pi/2}H_\nu^{(2)}(-\mathrm{i}x)$$

因此，虚宗量Bessel方程的一般解为

$$R(x)=A_\nu I_\nu(x)+B_\nu K_\nu(x)$$ ——**无论ν是何值**

■ 虚宗量 Bessel和Hankel函数的特性

□ 当 $x \to 0$ 时

$$I_0(0)=1;\quad I_m(0)=0\quad (m>0)$$

$$K_0(x)\sim -\ln\frac{x}{2};\quad K_m(x)\sim\frac{(m-1)!}{2}\left(\frac{x}{2}\right)^{-m}$$

——可见：K_m 在原点发散，当研究的区域包括原点时，只能取 $I_m(x)$.

□ 当 $x \to \infty$ 时

$$I_m(x)\approx\frac{1}{2\sqrt{x}}e^{x};\quad K_m(x)\approx\frac{1}{2\sqrt{x}}e^{-x}$$

——可见：I_m 在无限远发散，当研究的区域是开区域时，只能取 $K_m(x)$.

柱坐标

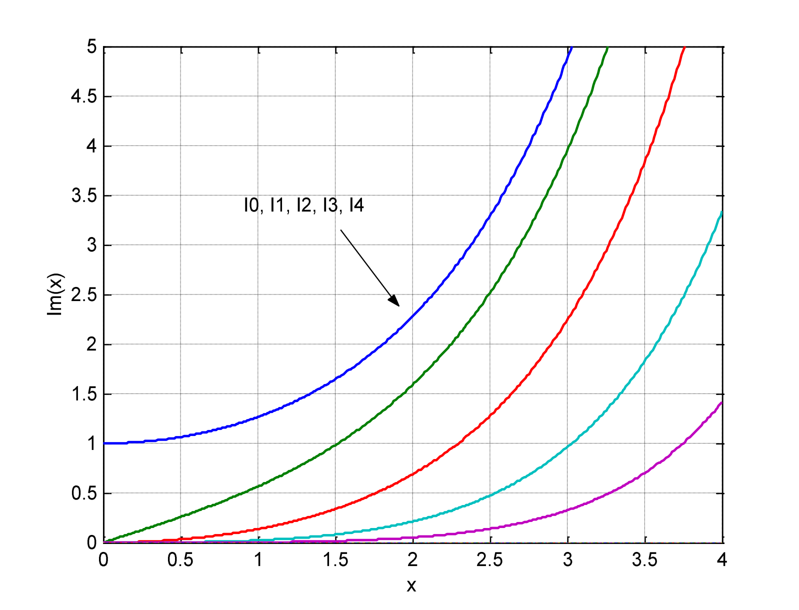
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\left(1+\frac{\nu^2}{x^2}\right)y=0$$

$$y(x)=C_1 I_\nu(x)+C_2 K_\nu(x)$$

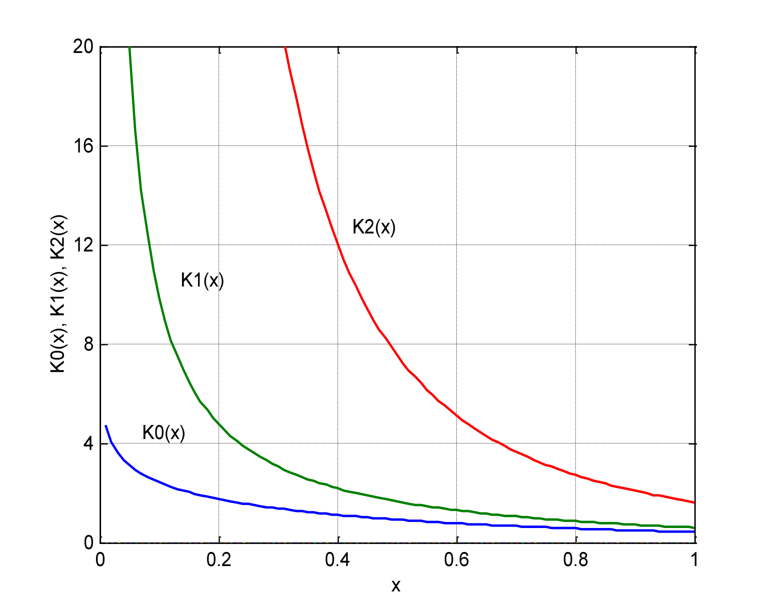
一维

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-y=0$$

$$y(x)=C_1 e^{x}+C_2 e^{-x}$$



虚宗量Bessel函数曲线



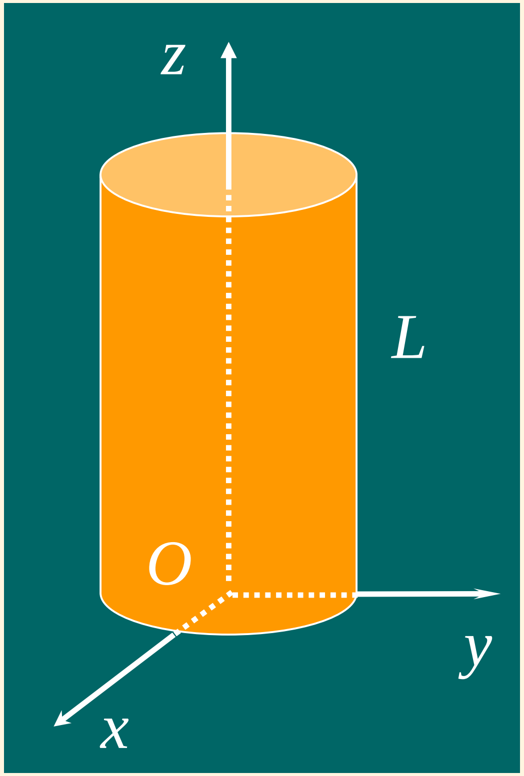
虚宗量Hankel函数曲线

❑虚宗量 Bessel 的应用

圆柱体：半径为 a, 高为 L， 柱侧面法向有均匀分布的恒定热流 q_0, 圆柱上下面温度分布保持为 $f_1(\rho)$ 和 $f_2(\rho)$. 求圆柱体中温度场的分布

解：定解问题

$$\begin{cases} \nabla^2 u = 0, \quad (0 < z < L, \rho < a) \\ -\kappa \dfrac{\partial u}{\partial \rho}\Big|_{\rho=a} = -q_0; \quad u|_{\rho=0} < \infty \\ u|_{z=0} = f_1(\rho); \; u|_{z=L} = f_2(\rho) \end{cases}$$



——边界条件全是非齐次的

一个复杂的定解问题化为二个简单的定解问题

令：$u=v+w$，其中

v：上下面是齐次的

w：径向是齐次的

$$
(\mathrm{I})\qquad \begin{cases} \nabla^2 v = 0,\ (0 < z < L, \rho < a) \\ -\kappa \left.\dfrac{\partial v}{\partial \rho}\right|_{\rho=a} = -q_0;\ v|_{\rho=0} < \infty \\ v|_{z=0} = 0;\ v|_{z=L} = 0 \end{cases}
$$

——上、下底面齐次

$$
\text{(II)} \qquad \begin{cases} \nabla^2 w = 0, \quad (0 < z < L, \rho < a) \\ -\kappa \left.\dfrac{\partial w}{\partial \rho}\right|_{\rho=a} = 0; \quad u\,|_{\rho=0} < \infty \\ w\,|_{z=0} = f_1(\rho); w\,|_{z=L} = f_2(\rho) \end{cases}
$$

——柱面齐次

❑ Laplace方程的一般解

$$
u(\rho,\varphi,z) = (C_0 + D_0 z)(E_0 + F_0 \ln\rho) + \sum_{m=-\infty}^{\infty} (C_m + D_m z)\rho^m e^{im\varphi}
$$

$$
+ \sum_{m=-\infty}^{\infty} \sum_{\mu>0} (Ae^{-\sqrt{\mu}z} + Be^{\sqrt{\mu}z}) \left[LJ_m\left(\sqrt{\mu}\rho\right) + MN_m\left(\sqrt{\mu}\rho\right) \right] e^{im\varphi}
$$

$$
+ \sum_{m=-\infty}^{\infty} \sum_{\mu<0} \left[H\sin\left(\sqrt{|\mu|}z\right) + G\cos\left(\sqrt{|\mu|}z\right) \right] \left[OI_m\left(\sqrt{|\mu|}\rho\right) + PK_m\left(\sqrt{|\mu|}\rho\right) \right] e^{im\varphi}
$$

根据具体的物理问题，选择不同的函数。本题

(1)要求原点有限，因此

$$F_0=0,\ C_{-|m|}=D_{-|m|}=0,\ M=0,\ P=0$$

(2)关于极角对称：$m=0$。

因此

$$u(\rho,z)=(C_0+D_0 z)+\sum_{\mu>0}\left(Ae^{-\sqrt{\mu}z}+Be^{\sqrt{\mu}z}\right)J_0\left(\sqrt{\mu}\rho\right)$$

$$+\sum_{\mu<0}\left[H\sin\left(\sqrt{|\mu|}z\right)+G\cos\left(\sqrt{|\mu|}z\right)\right]I_0\left(\sqrt{|\mu|}\rho\right)$$

(一)v 的解：要求上下面边界齐次，因此取

$$v(\rho,z)=(C_0+D_0 z)+\sum_{\mu<0}\left[H\sin\left(\sqrt{|\mu|}z\right)+G\cos\left(\sqrt{|\mu|}z\right)\right]I_0\left(\sqrt{|\mu|}\rho\right)$$

由

$$v\big|_{z=0}=C_0+\sum_{\mu<0}GI_0\left(\sqrt{|\mu|}\rho\right)=0$$

$$v\big|_{z=L}=(C_0+D_0L)+\sum_{\mu<0}\left[H\sin\left(\sqrt{|\mu|}L\right)+G\cos\left(\sqrt{|\mu|}L\right)\right]I_0\left(\sqrt{|\mu|}\rho\right)=0$$

$$\Downarrow$$

$$C_0=0; G=0; D=0; H\sin\left(\sqrt{|\mu|}L\right)=0$$

$$\Downarrow$$

$$\sin\left(\sqrt{|\mu|}L\right)=0 \Rightarrow \sqrt{|\mu|}=\frac{n\pi}{L},\quad (n=1,2,\ldots)$$

因此

$$v(\rho,z)=\sum_{n=1}^{\infty}H_n\sin\left(\frac{n\pi z}{L}\right)I_0\left(\frac{n\pi}{L}\rho\right)$$

——*z*方向构成本征值问题

由径向边界条件

$$\left.\frac{\partial v(\rho,z)}{\partial \rho}\right|_{\rho=a}=\sum_{n=1}^{\infty}H_n\frac{n\pi}{L}\sin\left(\frac{n\pi z}{L}\right)\left.\frac{\mathrm{d}I_0(x)}{\mathrm{d}x}\right|_{x=n\pi a/L}=\frac{q_0}{\kappa}$$

于是

$$H_n=\frac{L}{n\pi}\frac{1}{I_0'(n\pi a/L)}\frac{2}{L}\int_0^L\frac{q_0}{\kappa}\sin\left(\frac{n\pi z}{L}\right)\mathrm{d}z$$

$$=\frac{2Lq_0}{n^2\pi^2\kappa}\frac{1}{I_0'(n\pi a/L)}[1-(-1)^n]$$

最后

$$v(\rho,z)=\frac{4Lq_0}{\kappa\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\frac{I_0[(2l+1)\pi\rho/L]}{I_0'[(2l+1)\pi a/L]}$$

$$\times\sin\left[\frac{(2l+1)\pi z}{L}\right]$$

(二)w 的解：要求侧面边界齐次，因此取

$$w(\rho,z)=(C_0+D_0 z)+\sum_{\mu}\left(Ae^{-\sqrt{\mu}z}+Be^{\sqrt{\mu}z}\right)J_0\left(\sqrt{\mu}\rho\right)$$

由边界条件得到

$$\left.\frac{\partial w(\rho,z)}{\partial\rho}\right|_{\rho=a}=\sum_{\mu>0}\left(Ae^{-\sqrt{\mu}z}+Be^{\sqrt{\mu}z}\right)\sqrt{\mu}J_0'\left(\sqrt{\mu}a\right)=0$$

设 x_n是 $J_0'(x)=0$ 的第n个根，则 $\sqrt{\mu}=x_n/a$

$$w(\rho,z)=(C_0+D_0 z)+\sum_{n=1}^{\infty}(A_n e^{-x_n z/a}+B_n e^{x_n z/a})J_0\left(x_n\frac{\rho}{a}\right)$$

上式系数由上下面的边界条件决定

$$w(\rho,z)|_{z=0}=C_0+\sum_{n=1}^{\infty}(A_n+B_n)J_0\left(x_n\frac{\rho}{a}\right)=f_1(\rho)$$

$$w(\rho,z)|_{z=L}=(C_0+D_0L)$$

$$+\sum_{n=1}^{\infty}\left(A_ne^{-x_nL/a}+B_ne^{x_nL/a}\right)J_0\left(x_n\frac{\rho}{a}\right)=f_2(\rho)$$

注意：当问题是第二类边界条件时，(C_0+D_0z) **项一般要考虑——对应μ=0**

$J_0'(x)=-J_1(x)=0$ ➡ **第一个根为零** $x_0=0$

$$f(\rho)=f_0+\sum_{n=1}^{\infty}f_nJ_0\left(x_n\frac{\rho}{a}\right);\ f_0=\frac{2}{a^2}\int_0^a f(\rho)\rho\mathrm{d}\rho$$

$$f_n=\frac{1}{[N_n^{(0)}]^2}\int_0^a f(\rho)J_0\left(x_n\frac{\rho}{a}\right)\rho\mathrm{d}\rho$$

■ “直流”项

$$C_0 = \frac{2}{a^2}\int_0^a f_1(\rho)\rho\mathrm{d}\rho \equiv \overline{f}_{10}$$

$$(C_0 + D_0 L) = \frac{2}{a^2}\int_0^a f_2(\rho)\rho\mathrm{d}\rho \equiv \overline{f}_{02}$$

■ “交流”项

$$A_n + B_n = \frac{1}{[N_n^{(0)}]^2}\int_0^a f_1(\rho)J_0\left(x_n\frac{\rho}{a}\right)\rho\mathrm{d}\rho \equiv f_{1n}$$

$$A_n e^{-x_n L/a} + B_n e^{x_n L/a} = \frac{1}{[N_n^{(0)}]^2}\int_0^a f_1(\rho)J_0\left(x_n\frac{\rho}{a}\right)\rho\mathrm{d}\rho \equiv f_{2n}$$

$$\Uparrow$$

$$[N_n^{(0)}]^2 = \frac{a^2}{2}\left[J_0(x_n)\right]^2$$

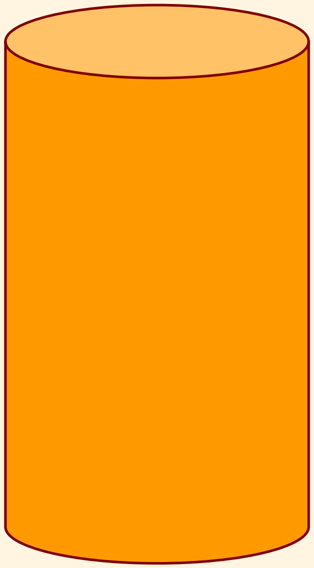
问题1：如果不分成二部分，如何解?

$$
\begin{cases}
\nabla^2 u = 0, \quad (0 < z < L, \rho < a) \\
-\kappa \left.\dfrac{\partial u}{\partial \rho}\right|_{\rho=a} = -q(z); \quad u|_{\rho=0} < \infty \\
u|_{z=0} = f_1(\rho); \; u|_{z=L} = f_2(\rho)
\end{cases}
$$

■本征函数展开方法

Laplace算子在柱内的本征值问题

$$
\begin{cases}
-\nabla^2 U(\rho,\varphi,z) = k^2 U(\rho,\varphi,z) \\
\left.\dfrac{\partial U(\rho,\varphi,z)}{\partial \rho}\right|_{\rho=a} = 0 \\
U(\rho,\varphi,z)|_{z=0} = U(\rho,\varphi,z)|_{z=L} = 0
\end{cases}
$$



$$U(\rho,\varphi,z)=Z_{|\mu|}(z)R_m(k_\rho\rho)\Phi_m(\varphi),(k_\rho^2=k^2+\mu)$$

$$Z_{|\mu|}(z)=C\cos\left(\sqrt{|\mu|}z\right)+D\sin\left(\sqrt{|\mu|}z\right)$$

$$R_m(k_\rho\rho)=L_mJ_m(k_\rho\rho)+M_mN_m(k_\rho\rho)$$

$$\Phi_m(\varphi)=A_me^{\mathrm{i}m\varphi},(m=0,\pm1,\pm2,\ldots)$$

(1)问题与方位角无关，求*m*=0的本征函数即可
(2)问题包含原点，取Bessel函数即可
(3)上下是第一类边界条件，取*C*=0即可

$$U(\rho,z)=J_0(k_\rho\rho)\sin\left(\sqrt{|\mu|}z\right),(k_\rho^2=k^2+\mu)$$

由上下边界条件

$$\sin\left(\sqrt{|\mu|}L\right)=0\Rightarrow|\mu|=\left(\frac{n\pi}{L}\right)^2,(n=1,2,\ldots)$$

由径向边界条件

$$\left.\frac{\mathrm{d}J_0(x)}{\mathrm{d}x}\right|_{x=k_\rho a}=0 \Rightarrow J_0'(x)=0 \Rightarrow x_j,(j=0,1,2,\ldots)$$

$$J_0'(x)=-J_1(x)=0 \Rightarrow$$ 第一个根为零 $x_0=0$

径向本征值

$$k_\rho^j=x_j/a,(j=0,1,2,\ldots)$$

本征值

$$(k_{jn})^2=\left(\frac{x_j}{a}\right)^2+\left(\frac{n\pi}{L}\right)^2,(j=0,1,2,\ldots;n=1,2,\ldots)$$

本征函数

$$U_{jn}(\rho,z)=\frac{1}{\|U_{jn}\|}J_0\left(x_j\frac{\rho}{a}\right)\sin\left(\frac{n\pi z}{L}\right),(j=0,1,2,\ldots;n=1,2,\ldots)$$

$$\|U_{jn}\|^2=\frac{L}{2}\cdot\frac{a^2}{2}\left[J_0(x_n)\right]^2$$

□ 本征函数展开解

$$u(\rho, z) = \sum_{jn} a_{jn} U_{jn}(\rho, z)$$

$$a_{jn} = \int_G u(\rho, z) U_{jn}^*(\rho, z) \rho \mathrm{d}\rho \mathrm{d}z$$

□ Green公式

$$\int_G (\varphi_1^* \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1^*) \mathrm{d}\tau = \iint_B \left(\varphi_1^* \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1^*}{\partial n} \right) \mathrm{d}S$$

$$\Uparrow$$

$$\varphi_1^*(\boldsymbol{r}) = U_{jn}^*(\rho, z); \varphi_2(\boldsymbol{r}) = u(\rho, z)$$

$$\Downarrow$$

$$\int_G (U_{jn}^* \nabla^2 u - u \nabla^2 U_{jn}^*) \rho \mathrm{d}\rho \mathrm{d}z \mathrm{d}\varphi = \iint_{\partial G} \left(U_{jn}^* \frac{\partial u}{\partial n} - u \frac{\partial U_{jn}^*}{\partial n} \right) \mathrm{d}S$$

$$\begin{cases} -\nabla^2 U_{jn}^*(\rho,z) = k_{jn}^2 U_{jn}^*(\rho,z) \\ \left.\dfrac{\partial U_{jn}^*(\rho,z)}{\partial\rho}\right|_{\rho=a} = 0 \\ U_{jn}^*(\rho,z)\,|_{z=0} = U_{jn}^*(\rho,z)\,|_{z=l} = 0 \end{cases}$$

$$\begin{cases} \nabla^2 u = 0, \quad (0 < z < L, \rho < a) \\ -\kappa \left.\dfrac{\partial u}{\partial\rho}\right|_{\rho=a} = -q(z); \quad u\,|_{\rho=0} < \infty \\ u\,|_{z=0} = f_1(\rho);\ u\,|_{z=L} = f_2(\rho,) \end{cases}$$

$$a_{jn} = \frac{1}{k_{jn}^2}\left\{ \frac{a}{\kappa}\int_0^L q(z) U_{jn}^*(a,z)\mathrm{d}z + \int_0^a \left[f_2(\rho)\frac{\partial U_{jn}^*}{\partial z}\right]_{z=0} \rho \mathrm{d}\rho \right. \\ \left. - \int_0^a \left[f_2(\rho)\frac{\partial U_{jn}^*}{\partial z}\right]_{z=L} \rho \mathrm{d}\rho \right\}$$

$$u(\rho,z) = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} a_{jn} U_{jn}(\rho,z)$$

□ 积分解

$$u(\rho,z)=\frac{a}{\kappa}\int_0^L q(z')G(\rho,z;a,z')\mathrm{d}z'$$

$$+\int_0^a\left[f_2(\rho')\frac{\partial G(\rho,z;\rho',z')}{\partial z'}\right]_{z'=0}\rho'\mathrm{d}\rho'$$

$$-\int_0^a\left[f_2(\rho')\frac{\partial G(\rho,z;\rho',z')}{\partial z'}\right]_{z'=L}\rho'\mathrm{d}\rho'$$

$$G(\rho,z;\rho',z')\equiv\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k_{jn}^2}U_{jn}(\rho,z)U_{jn}^*(a,z')$$

——注意：本题中，尽管$x_0=0$, 但是$k_{01}\neq0$, 零不是本征值，因为上、下满足的是第一类边界条件.

问题2：对下列问题，结果如何？

$$\begin{cases} \nabla^2 u = 0, \quad (0 < z < L, \rho < a) \\ -\kappa \left.\dfrac{\partial u}{\partial \rho}\right|_{\rho=a} = -q(z); \quad u\,|_{\rho=0} < \infty \\ \kappa \left.\dfrac{\partial u}{\partial z}\right|_{z=0} = f_1(\rho); \ -\kappa \left.\dfrac{\partial u}{\partial z}\right|_{z=L} = f_2(\rho) \end{cases}$$

对称情况下，Laplace算子在柱内的本征值问题

$$\begin{cases} -\nabla^2 U(\rho, z) = k^2 U(\rho, z) \\ \left.\dfrac{\partial U(\rho, z)}{\partial \rho}\right|_{\rho=a} = 0 \\ \kappa \left.\dfrac{\partial U(\rho, z)}{\partial z}\right|_{z=0} = 0; \ -\kappa \left.\dfrac{\partial U(\rho, z)}{\partial z}\right|_{z=L} = 0 \end{cases}$$

本征值

$$(k_{jn})^2=\left(\frac{x_j}{a}\right)^2+\left(\frac{n\pi}{L}\right)^2,(j=0,1,2,\ldots;n=1,2,\ldots)$$

$$J_0'(x_j)=-J_1(x_j)=0,(j=0,1,2,\ldots)$$

本征函数

$$U_{jn}(\rho,z)=\frac{1}{\|U_{jn}\|}J_0\left(x_j\frac{\rho}{a}\right)\cos\left(\frac{n\pi z}{L}\right),(j=0,1,2,\ldots;n=0,1,2,\ldots)$$

$$\|U_{jn}\|^2=\frac{L\varepsilon_n}{2}\cdot\frac{a^2}{2}\left[J_0(x_j)\right]^2,\varepsilon_0=2;\varepsilon_n=1,(n>0)$$

□ 本征函数展开解

$$u(\rho,z)=\sum_{jn}a_{jn}U_{jn}(\rho,z)$$

$$a_{jn}=\int_G u(\rho,z)U_{jn}^*(\rho,z)\rho\,\mathrm{d}\rho\,\mathrm{d}z$$

□ Green公式

$$\int_G (U_{jn}^* \nabla^2 u - u \nabla^2 U_{jn}^*) \rho \mathrm{d}\rho \mathrm{d}z \mathrm{d}\varphi = \iint_{\partial G} \left(U_{jn}^* \frac{\partial u}{\partial n} - u \frac{\partial U_{jn}^*}{\partial n} \right) \mathrm{d}S$$

$$\begin{cases} -\nabla^2 U_{jn}^*(\rho,z) = k_{jn}^2 U_{jn}^*(\rho,z) \\ \left.\dfrac{\partial U_{jn}^*(\rho,z)}{\partial \rho}\right|_{\rho=a} = 0 \\ \kappa \left.\dfrac{\partial U_{jn}^*(\rho,z)}{\partial z}\right|_{z=0} = 0; \quad -\kappa \left.\dfrac{\partial U_{jn}^*(\rho,z)}{\partial z}\right|_{z=L} = 0 \end{cases}$$

$$\begin{cases} \nabla^2 u = 0, \quad (0 < z < L, \rho < a) \\ -\kappa \left.\dfrac{\partial u}{\partial \rho}\right|_{\rho=a} = -q(z); \quad u|_{\rho=0} < \infty \\ \kappa \left.\dfrac{\partial u}{\partial z}\right|_{z=0} = f_1(\rho); \quad -\kappa \left.\dfrac{\partial u}{\partial z}\right|_{z=L} = f_2(\rho) \end{cases}$$

$$\kappa k_{jn}^2 a_{jn} = \frac{1}{2\pi} \iint_{\partial G} \left(U_{jn}^* \frac{\partial u}{\partial n} - u \frac{\partial U_{jn}^*}{\partial n} \right) \mathrm{d}S$$

$$= a \int_0^L q(z) U_{jn}^*(a,z) \mathrm{d}z - \int_0^a U_{jn}^*(\rho,0) f_1(\rho) \rho \mathrm{d}\rho$$

$$+ \int_0^a U_{jn}^*(\rho,L) f_2(\rho) \rho \mathrm{d}\rho$$

(1) j和n不同时为0

$$a_{jn}=\frac{1}{\kappa k_{jn}^2}\left[\begin{array}{l} a\int_0^L q(z)U_{jn}^*(a,z)\mathrm{d}z-\int_0^a U_{jn}^*(\rho,0)f_1(\rho)\rho\mathrm{d}\rho \\ +\int_0^a U_{jn}^*(\rho,L)f_2(\rho)\rho\mathrm{d}\rho \end{array}\right]$$

(2) j和n同时为0 (j=n=0)

$$a\int_0^L q(z)U_{00}^*(a,z)\mathrm{d}z-\int_0^a U_{00}^*(\rho,0)f_1(\rho)\rho\mathrm{d}\rho$$

$$+\int_0^a U_{00}^*(\rho,L)f_2(\rho)\rho\mathrm{d}\rho=0$$ ⟸ **相容性条件**

⇓

$$u(\rho,z)=a_{00}U_{00}(\rho,z)+\sum_{jn}{}^{*}a_{jn}U_{jn}(\rho,z)$$

——其中求和中*表示j和n不同时为零

□ 积分解

$$u(\rho,z)=a_{00}U_{00}(\rho,z)+\frac{a}{\kappa}\int_0^L q(z')\tilde{G}(\rho,z;a,z')\mathrm{d}z$$

$$-\frac{1}{\kappa}\int_0^a \tilde{G}(\rho,z;\rho',0)\,f_1(\rho')\rho'\mathrm{d}\rho'+\frac{1}{\kappa}\int_0^a \tilde{G}(\rho,z;\rho',L)\,f_2(\rho')\rho'\mathrm{d}\rho'$$

$$\tilde{G}(\rho,z;\rho',z')\equiv\sum_{jn}{}^{*}\frac{1}{k_{jn}^2}U_{jn}(\rho,z)U_{jn}^{*}(\rho',z')$$

——广义Green函数，满足方程

$$-\nabla^2\tilde{G}(\rho,z;\rho',z')=\sum_{jn}{}^{*}U_{jn}(\rho,z)U_{jn}^{*}(\rho',z')$$

$$=\sum_{jn}U_{jn}(\rho,z)U_{jn}^{*}(\rho',z')-U_{00}(\rho,z)U_{00}^{*}(\rho',z')$$

$$=\frac{1}{\rho}\delta(\rho,\rho')\delta(z,z')-U_{00}(\rho,z)U_{00}^{*}(\rho',z')$$

12.4 球Bessel函数和球Hankel函数

❑基本形式：球坐标中 Helmholtz 方程分离变量

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+l(l+1)R=k^2r^2R$$

S-L形式

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}+[k^2r^2-l(l+1)]R=0$$

$$y(x)=\sqrt{\frac{2kr}{\pi}}R(r),\ x=kr\Rightarrow R(r)=\sqrt{\frac{\pi}{2kr}}y(kr)$$

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[x^2-\left(l+\frac{1}{2}\right)^2\right]y=0,$$

■如果 $k=0$

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}+l(L+1)R=0$$ —Euler 方程

$$R(r)=Ar^l+Br^{-(l+1)}$$

■$k\neq0$情况：四种形式的解——两组成独立解

$$\{J_{l+1/2}(x),\ N_{l+1/2}(x)\};\ \{\ H^{(1)}_{l+1/2}(x),\ H^{(2)}_{l+1/2}(x)\}$$

因此，球 Bessel方程的二组独立解为

(1)球Bessel函数和球Neumann函数

$$j_l(x)\equiv\sqrt{\frac{\pi}{2x}}J_{l+1/2}(x);n_l(x)\equiv\sqrt{\frac{\pi}{2x}}N_{l+1/2}(x)$$

(2)球Hankel函数

$$h_l^{(1)}(x)=\sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(1)}(x);\ h_l^{(2)}(x)=\sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(2)}(x)$$

二种形式的解：前者有驻波形式，在封闭空间使用；后者有行波形式，波在无限空间中的传播和散射使用。

球Bessel方程的通解

❑驻波形式解

$$R_l(r)=A_l j_l(kr)+B_l n_l(kr)$$

$$=\sqrt{\frac{\pi}{2kr}}\left[A_l J_{l+1/2}(kr)+B_l N_{l+1/2}(kr)\right]$$

$$j_l(x) \sim \frac{1}{x}\cos\left[x-\left(l+\frac{1}{2}\right)\frac{\pi}{2}-\frac{\pi}{4}\right]$$

$$n_l(x) \sim \frac{1}{x}\sin\left[x-\left(l+\frac{1}{2}\right)\frac{\pi}{2}-\frac{\pi}{4}\right]$$

□行波形式解

$$R_l(r) = A_l h_l^{(1)}(kr) + B_l h_l^{(2)}(kr)$$

$$= \sqrt{\frac{\pi}{2kr}}\left[A_l H_{l+1/2}^{(1)}(kr) + B_l H_{l+1/2}^{(2)}(kr)\right]$$

$$h_l^{(1)}(x) \sim \frac{1}{x}\exp\left\{\mathrm{i}\left[x-\left(l+\frac{1}{2}\right)\frac{\pi}{2}-\frac{\pi}{4}\right]\right\}$$

$$h_l^{(2)}(x) \sim \frac{1}{x}\exp\left\{-\mathrm{i}\left[x-\left(l+\frac{1}{2}\right)\frac{\pi}{2}-\frac{\pi}{4}\right]\right\}$$

球坐标

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[k^2-\frac{l(l+1)}{r^2}\right]R=0$$

$$R_l(r)=A_l j_l(kr)+B_l n_l(kr)$$

$$h_l^{(1)}(kr)=j_l(kr)+\mathrm{i}j_l(kr)$$

$$h_l^{(2)}(kr)=j_l(kr)-\mathrm{i}j_l(kr)$$

$$R_l(kr)=A_l h_l^{(1)}(x)+B_l h_l^{(2)}(kr)$$

一维

$$\frac{d^2y}{dx^2}+k^2y=0$$

$$y(x)=A\cos(kx)+B\sin(kx)$$

$$e^{\mathrm{i}kx}=\cos(kx)+\mathrm{i}\sin(kx)$$

$$e^{-\mathrm{i}kx}=\cos(kx)-\mathrm{i}\sin(kx)$$

$$y(x)=Ae^{\mathrm{i}kx}+Be^{-\mathrm{i}kx}$$

——驻波解，行波解

■平面波，振幅不变化

$$y(x)=Ae^{\mathrm{i}kx}+Be^{-\mathrm{i}kx};\ y(\mathbf{r})=Ae^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}+Be^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$$

振幅不变

■柱面波，波阵面 $1/\sqrt{\rho}$ 扩散

$$H_n^{(1)}(k_\rho\rho)\approx\sqrt{\frac{2}{\pi k_\rho\rho}}\exp\left[\mathrm{i}\left(k_\rho\rho-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right]$$

$$H_\nu^{(2)}(k_\rho\rho)\approx\sqrt{\frac{2}{\pi k_\rho\rho}}\exp\left[-\mathrm{i}\left(k_\rho\rho-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right]$$

近场变化很复杂

■球面波，波阵面 $1/r$ 扩散

$$h_l^{(1)}(kr)\sim\frac{1}{kr}\exp\left\{\mathrm{i}\left[kr-\left(l+\frac{1}{2}\right)\frac{\pi}{2}-\frac{\pi}{4}\right]\right\}$$

$$h_l^{(2)}(kr)\sim\frac{1}{kr}\exp\left\{-\mathrm{i}\left[kr-\left(l+\frac{1}{2}\right)\frac{\pi}{2}-\frac{\pi}{4}\right]\right\}$$

近场变化很复杂

递推公式：球函数统一写作

$$z_l(x)=\sqrt{\frac{\pi}{2x}}Z_{l+1/2}(x)$$

从柱函数的递推公式，可得到

$$z_{l-1}+z_{l+1}=\frac{2l+1}{x}z_l$$

$$lz_{l-1}-(l+1)z_{l+1}=(2l+1)z_l'$$

初等函数形式：当 *l* 是整数，可用初等函数来表达球函数

$$j_0(x)=\frac{\sin x}{x};\ n_0(x)=-\frac{\cos x}{x}$$

$$j_1(x)=\frac{\sin x-x\cos x}{x^2};\quad n_1(x)=-\frac{\cos x+x\sin x}{x^2}$$

球 Hankel函数的初等函数形式为

$$h_0^{(1)}(x) = -\frac{\mathrm{i}}{x}e^{\mathrm{i}x};\ h_0^{(2)}(x) = \frac{\mathrm{i}}{x}e^{-\mathrm{i}x}$$

$$h_1^{(1)}(x) = -\left(\frac{\mathrm{i}}{x^2}+\frac{1}{x}\right)e^{\mathrm{i}x};\quad h_1^{(2)}(x) = \left(\frac{\mathrm{i}}{x^2}-\frac{1}{x}\right)e^{-\mathrm{i}x}$$

渐近形式

(1)$x\to 0$

$$j_0(0)=1;\ j_l(0)=0,\ (l>0)$$

$$\lim_{x\to 0} n_l(x)\to\infty$$

因此，在原点存在自然边界条件

(2)$x\to\infty$

$$j_l(x)\sim\frac{1}{x}\cos\left(x-\frac{l+1}{2}\pi\right);\quad n_l(x)\sim\frac{1}{x}\sin\left(x-\frac{l+1}{2}\pi\right)$$

$$h_l^{(1)}(x) \sim \frac{1}{x}(-\mathrm{i})^{l+1}e^{\mathrm{i}x}; \quad h_l^{(2)}(x) \sim \frac{1}{x}(\mathrm{i})^{l+1}e^{-\mathrm{i}x}$$

平面波展为球面波(z方向传播)

$$\exp(\mathrm{i}kr\cos\vartheta) = \sqrt{\frac{\pi}{2kr}}\sum_{l=0}^{\infty}(2l+1)\mathrm{i}^l J_{l+1/2}(kr)P_l(\cos\vartheta)$$

$$= \sum_{l=0}^{\infty}(2l+1)\mathrm{i}^l j_l(kr)P_l(\cos\vartheta)$$

简单证明见下一章

$$\exp(\mathrm{i}k\rho\cos\varphi) = \sum_{m=-\infty}^{\infty}\mathrm{i}^m J_m(k\rho)e^{\mathrm{i}m\varphi}$$

x方向传播的平面波

球Bessel方程的本征值问题

注意：本征值—k^2；权函数—r^2

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r^2\dfrac{\mathrm{d}R}{\mathrm{d}r}\right)+l(l+1)R=k^2r^2R \\ \left.\left(\alpha R+\beta\dfrac{\mathrm{d}R}{\mathrm{d}r}\right)\right|_{r=a}=0,\ R|_{r=0}<\infty \end{cases}$$

(1)一般解

$$R_l(r)=A_l j_l(kr)+B_l n_l(kr)$$

(2)由原点自然边界条件：*B_l*=0

$$R_l(r)=A_l j_l(kr)$$

(3)由*r*=*a*处边界条件 ——决定本征值*k^2*的方程

$$\left.\left[\alpha j_l(ka)+k\beta\frac{\mathrm{d}j_l(kr)}{\mathrm{d}(kr)}\right]\right|_{r=a}=0 \quad \Rightarrow \quad k_n^{(l)},(n=1,2,\ldots,\infty)$$

作为Sturm-Liouville本征值问题，应有

正交性

$$\int_0^a j_l[k_m^{(l)}r]j_l[k_n^{(l)}r]r^2\mathrm{d}r=[N_{nn}^{(l)}]^2\delta_{mn}$$

——注意：带权 r^2正交

其中：模的平方为

$$[N_{nn}^{(l)}]^2=\int_0^a[j_l(k_n^{(l)}r)]^2r^2\mathrm{d}r$$

证明: (忽略上标(l))

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\left[r^2\dfrac{\mathrm{d}j_l(k_\nu r)}{\mathrm{d}r}\right]+l(l+1)j_l(k_\nu r)=k_\nu^2r^2j_l(k_\nu r) \\ \alpha j_l(k_\nu a)+k_\nu\beta\dfrac{\mathrm{d}j_l(k_\nu a)}{\mathrm{d}(k_\nu a)}=0 \end{cases}\qquad(\nu=m,n)$$

取 $\nu=m$ 的方程 $\times j_l(k_n r)$ – 取 $\nu=n$ 的方程 $\times j_l(k_m r)$, 并且积分得到

$$(k_m^2-k_n^2)\int_0^a j_l(k_m r)\,j_l(k_n r)r^2\mathrm{d}r$$

$$=\int_0^a \frac{\mathrm{d}}{\mathrm{d}r}\left\{r^2\left[j_l(k_m r)\frac{\mathrm{d}j_l(k_n r)}{\mathrm{d}r}-j_l(k_n r)\frac{\mathrm{d}j_l(k_m r)}{\mathrm{d}r}\right]\right\}\mathrm{d}r$$

$$=r^2\left[j_l(k_m r)\frac{\mathrm{d}j_l(k_n r)}{\mathrm{d}r}-j_l(k_n r)\frac{\mathrm{d}j_l(k_m r)}{\mathrm{d}r}\right]_0^a$$

$$=a^2\left[k_n j_l(k_m a)\frac{\mathrm{d}j_l(k_n a)}{\mathrm{d}(k_n a)}-k_m j_l(k_n a)\frac{\mathrm{d}j_l(k_m a)}{\mathrm{d}(k_m a)}\right]=0$$

$$\Downarrow$$

$$\int_0^a j_l(k_m r)\,j_l(k_n r)r^2\mathrm{d}r=0,(m\neq n)$$

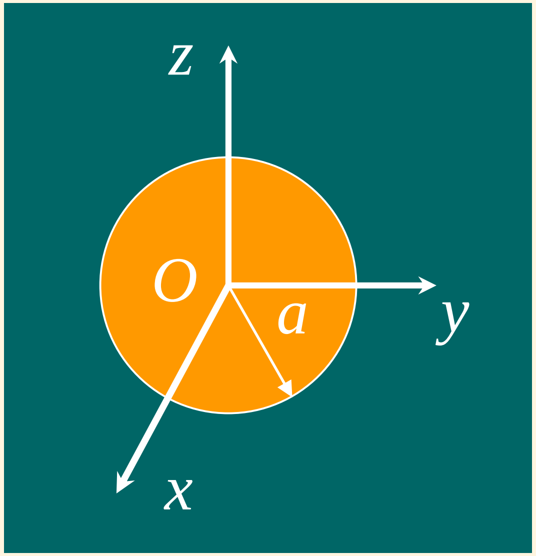
——注意：尽管是奇异的S-L问题，正交性仍然成立

完备性：对[0, *a*]上的平方可积函数 *f*(*r*)存在广义 Fourier 展开

$$f(r)\approx\sum_{n=1}^{\infty} f_n j_l(k_n r);\quad f_n=\frac{1}{[N_n]^2}\int_0^a f(r)\, j_l(k_n r) r^2 \mathrm{d}r$$

❑球Bessel和球Hankel函数的应用

例1 半径为 a 的球，初始温度为 u_0. 放入温度为 U_0 的烘箱，求球内温度分布



解：定解问题

$$\begin{cases} u_t-\alpha^2\nabla^2 u=0,\ (\alpha^2=\kappa/(\rho c_V)) \\ u|_{r=a}=U_0, u|_{r=0}<\infty \\ u|_{t=0}=u_0 \end{cases}$$

(1)化成齐次边界：$u=U_0+w$

$$\begin{cases} w_t - \alpha^2\nabla^2 w = 0 \\ w|_{r=a} = 0,\ w|_{r=0} < \infty, w|_{t=0} = u_0 - U_0 \end{cases}$$

(2)显然问题仅与 径向有关：$m=0$, $l=0$, 因此特解为

$$w(r,t) = Aj_0(kr)e^{-k^2\alpha^2 t} = A\frac{\sin(kr)}{kr}e^{-k^2\alpha^2 t}$$

(3)由径向边界条件

$$w(r,t)|_{r=a} = A\frac{\sin(ka)}{ka}e^{-k^2\alpha^2 t} = 0$$

$$\Downarrow$$

$$k = k_n = \frac{n\pi}{a},\ (n = 1,2,3,\ldots)$$

(4)一般解为

$$w(r,t)=\sum_{n=1}^{\infty}A_n\frac{\sin(k_nr)}{k_nr}e^{-k_n^2\alpha^2t}$$

(5)由初始条件

$$w(r,t)|_{t=0}=\sum_{n=1}^{\infty}A_n\frac{\sin(k_nr)}{k_nr}=u_0-U_0$$

$$\Downarrow$$

$$A_n=\frac{1}{N_n^2}\int_0^a(u_0-U_0)\frac{\sin(k_nr)}{k_nr}r^2\mathrm{d}r=(-1)^n2(U_0-u_0)$$

$$N_n^2=\int_0^a\left[\frac{\sin(k_nr)}{k_nr}\right]^2r^2\mathrm{d}r=\frac{1}{k_n^2}\int_0^a\sin^2(k_nr)\mathrm{d}r=\frac{a^3}{2(n\pi)^2}$$

(6)最后得到

$$u(r,t)=U_0+\frac{2(U_0-u_0)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1}{r}\sin\left(\frac{n\pi}{a}r\right)e^{-k_n^2\alpha^2t}$$

问题1：非齐次方程，并且与角度有关

$$\begin{cases} u_t - \alpha^2\nabla^2 u = f(\mathbf{r},t),\ \alpha^2 = \kappa/(\rho c_V) \\ u|_{r=0} < \infty;\ \ u|_{r=a} = U_0(\vartheta,\varphi,t) \\ u|_{t=0} = u_0(r,\vartheta,\varphi) \end{cases}$$

■**Green函数方法**

■**本征函数展开方法：Laplace算子在球内的本征值问题**

$$\begin{cases} -\nabla^2 U(r,\vartheta,\varphi) = \lambda^2 U(r,\vartheta,\varphi) \\ U(r,\vartheta,\varphi)|_{r=a} = 0 \end{cases}$$

$$U(r,\vartheta,\varphi)=j_l(\lambda r)Y_{lm}(\vartheta,\varphi)$$

本征方程 $$j_l(\lambda a)=0\Rightarrow\lambda_n^l=x_n^l/a,(n=1,2,\ldots)$$

——径向本征值与*m*无关

本征函数

$$U_{nlm}(\boldsymbol{r})=U_{nlm}(r,\vartheta,\varphi)=\frac{1}{N_{nl}}j_l\left(x_n^l\frac{r}{a}\right)Y_{lm}(\vartheta,\varphi)$$

$$N_{nl}=\sqrt{\int_0^a\left[j_l\left(x_n^l\frac{r}{a}\right)\right]^2r^2\mathrm{d}r}$$

本征函数展开解

$$u(r,\vartheta,\varphi,t)=\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{nlm}(t)U_{nlm}(r,\vartheta,\varphi)$$

$$a_{nlm}(t)=\int_G u(r,\vartheta,\varphi,t)U_{nlm}^{*}(r,\vartheta,\varphi)\mathrm{d}\tau$$

$$\mathrm{d}\tau=r^2\mathrm{d}r\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$\int_G(\varphi_1^{*}\nabla^2\varphi_2-\varphi_2\nabla^2\varphi_1^{*})\mathrm{d}\tau=\iint_{\partial G}\left(\varphi_1^{*}\frac{\partial\varphi_2}{\partial n}-\varphi_2\frac{\partial\varphi_1^{*}}{\partial n}\right)\mathrm{d}S$$

$$\Uparrow$$

$$\varphi_1^{*}=U_{nlm}^{*};\varphi_2=u$$

$$\Downarrow$$

$$\int_G(U_{nlm}^{*}\nabla^2u-u\nabla^2U_{nlm}^{*})\mathrm{d}\tau=\iint_{\partial G}\left(U_{nlm}^{*}\frac{\partial u}{\partial n}-u\frac{\partial U_{nlm}^{*}}{\partial n}\right)\mathrm{d}S$$

$$\begin{cases} -\nabla^2 U^*_{nlm}(r,\vartheta,\varphi) = \left(\lambda_n^l\right)^2 U^*_{nlm}(r,\vartheta,\varphi) \\ U^*_{nlm}(r,\vartheta,\varphi)\,|_{\rho=a} = 0 \end{cases}; \begin{cases} u_t - \alpha^2\nabla^2 u = f(\mathbf{r},t) \\ u\,|_{r=a} = U_0(\vartheta,\varphi), u\,|_{r=0} < \infty \\ u\,|_{t=0} = u_0(\vartheta,\varphi) \end{cases}$$

$$\frac{\mathrm{d}a_{nlm}(t)}{\mathrm{d}t} + \left(\alpha\lambda_n^l\right)^2 a_{nlm}(t) = \int_G f(\mathbf{r},t)U^*_{nlm}\mathrm{d}\tau$$

$$-\alpha^2\iint_{\partial G}\left[U_0(\vartheta,\varphi,t)\frac{\partial U^*_{nlm}}{\partial r}\right]_{r=a} a^2\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi \equiv \Im(t)$$

$$a_{nlm}(t)\,|_{t=0} = \iiint_{r<a} u_0(r,\vartheta,\varphi)U^*_{nlm}(r,\vartheta,\varphi)\mathrm{d}\tau$$

$$a_{nlm}(t) = a_{nlm}(t)\,|_{t=0}\,\exp(-\alpha\lambda_n^l t) + \int_0^t \Im(t')\exp[-\alpha\lambda_n^l(t-t')]\mathrm{d}t'$$

积分形式解

$$u(r,\vartheta,\varphi,t)=\int_G u_0(\mathbf{r}')G(\mathbf{r},\mathbf{r}',t)\mathrm{d}\tau'$$

$$+\int_0^t\int_G f(\mathbf{r}',t')G(\mathbf{r},\mathbf{r}',t-t')\mathrm{d}\tau'\mathrm{d}t'$$

$$-\alpha^2\int_0^t\iint_{\partial G}\left[U_0(\vartheta',\varphi',t')\frac{\partial G^*(\mathbf{r},\mathbf{r}',t)}{\partial r'}\right]_{r'=a}\mathrm{d}S'\mathrm{d}t'$$

$$G(\mathbf{r},\mathbf{r}',t)\equiv\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}U_{nlm}(\mathbf{r})U_{nlm}^*(\mathbf{r}')\exp(-\alpha\lambda_n^l t)$$

问题2：不可分离变量的例子——非均匀球面

$$\left[\alpha(\vartheta,\varphi)U+\beta(\vartheta,\varphi)\frac{\partial U}{\partial r}\right]_{r=a}=0$$

$$-\nabla^2 U(r,\vartheta,\varphi)=\lambda^2 U(r,\vartheta,\varphi)$$

$$U\,|_{r=a}=0,\left(0<\vartheta<\pi/2\right)$$

$$\left.\frac{\partial U}{\partial r}\right|_{r=a}=0,\left(\pi/2<\vartheta<\pi\right)$$

- **上半球：第一类边界条件**
- **下半球：第二类边界条件**

例2 在半球内部 $r<a$，　$0<\vartheta<\pi/2$ 求解 Laplace方程使满足边界条件：(1) 半球面 $f(\vartheta)$; (2)底面存在热流。

解:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u(r,\vartheta)}{\partial r}\right]+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\frac{\partial u(r,\vartheta)}{\partial\vartheta}\right]=0$$

$$(r<a,\quad 0\le\vartheta\le\pi/2)$$

$$u\big|_{r=a} = f(\vartheta)\ (0<\vartheta<\pi/2);\quad -\kappa\frac{1}{r}\frac{\partial u}{\partial \vartheta}\bigg|_{\vartheta=\pi/2} = g(r)\ (r<a)$$

$$-\kappa(\nabla u)\cdot\boldsymbol{n} = -\kappa\left(\frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \vartheta}\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial u}{\partial \varphi}\boldsymbol{e}_\varphi\right)\cdot\boldsymbol{e}_\vartheta\bigg|_{\vartheta=\pi/2}$$

$$= -\kappa\frac{1}{r}\frac{\partial u}{\partial \vartheta}\bigg|_{\vartheta=\pi/2}$$

底面满足非齐次边界条件，无法用上章的延拓方法

■ **本征函数展开法——先求本征函数**

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left[\sin\vartheta\frac{\partial U}{\partial \vartheta}\right] + \lambda^2 U = 0$$

$$(r<a,\quad 0\le\vartheta\le\pi/2)$$

$$U\big|_{r=a} = 0\ (0<\vartheta<\pi/2);\quad \frac{\partial U}{\partial \vartheta}\bigg|_{\vartheta=\pi/2} = 0\ (r<a)$$

与方位角无关,取m=0的解

$$U(r,\vartheta)=Aj_l(\lambda r)P_l(\cos\vartheta)$$

上式一定满足本征方程，问题是：能否满足边界条件?

- **半球面上** $U\big|_{r=a}=0\ (0<\vartheta<\pi/2)$

$$\Downarrow$$

$$j_l(\lambda a)=0\Rightarrow\lambda_n^l=x_n^l/a,(n=1,2,\ldots)$$

- **半球底面**

$$\left.\frac{\partial U}{\partial\vartheta}\right|_{\vartheta=\pi/2}=j_l(x_n^l r/a)\left.\frac{\mathrm{d}P_l(\cos\vartheta)}{\mathrm{d}\vartheta}\right|_{\vartheta=\pi/2}$$

$$=j_l(x_n^l r/a)\frac{\mathrm{d}\cos\vartheta}{\mathrm{d}\vartheta}\left.\frac{\mathrm{d}P_l(\cos\vartheta)}{\mathrm{d}\cos\vartheta}\right|_{\vartheta=\pi/2}=-j_l(x_n^l r/a)P_l'(0)=0$$

$$\left.\frac{\mathrm{d}P_{2k}(x)}{\mathrm{d}x}\right|_{x=0}=\frac{2k}{\pi}\mathrm{i}^{2k-1}\int_0^{\pi}\cos^{2k-1}\psi\,\mathrm{d}\psi=0$$

$$\left.\frac{\mathrm{d}P_{2k+1}(x)}{\mathrm{d}x}\right|_{x=0}=\frac{2k+1}{\pi}\mathrm{i}^{2k}\int_0^{\pi}\cos^{2k}\psi\,\mathrm{d}\psi\neq 0$$

$$\Rightarrow l=2k,(k=0,1,2,\ldots)$$

本征函数

$$U_{nk}(r,\vartheta)=\frac{1}{N_{nk}}j_{2k}\left(x_n^{2k}\frac{r}{a}\right)P_{2k}(\cos\vartheta)$$

$$(n=1,2,\ldots;k=0,1,2,3,\ldots)$$

$$N_{nk}=\sqrt{\int_G\left[j_{2k}\left(x_n^{2k}\frac{r}{a}\right)P_{2k}(\cos\vartheta)\right]^2\mathrm{d}\tau}$$

■ 原问题的解

$$u(r,\vartheta)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}a_{nk}U_{nk}(r,\vartheta);a_{nk}\equiv\int_G u(r,\vartheta)U_{nk}(r,\vartheta)\mathrm{d}\tau$$

$$\int_G (\varphi_1^* \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1^*) \mathrm{d}\tau = \iint_{\partial G} \left(\varphi_1^* \frac{\partial \varphi_2}{\partial n} - \varphi_2 \frac{\partial \varphi_1^*}{\partial n} \right) \mathrm{d}S$$

$$\Uparrow$$

$$\varphi_1^* = U_{nk}; \varphi_2 = u$$

$$\Downarrow$$

$$\int_G (U_{nk} \nabla^2 u - u \nabla^2 U_{nk}) \mathrm{d}\tau = \iint_{\partial G} \left(U_{nk} \frac{\partial u}{\partial n} - u \frac{\partial U_{nk}}{\partial n} \right) \mathrm{d}S$$

$$\Uparrow$$

$$\nabla^2 u = 0; \nabla^2 U_{nk} = -\left(x_n^{2k} / a \right)^2 U_{nk}$$

$$\Downarrow$$

$$a_{nk} = \frac{1}{\left(x_n^{2k} / a \right)^2} \iint_{\partial G} \left(U_{nk} \frac{\partial u}{\partial n} - u \frac{\partial U_{nk}}{\partial n} \right) \mathrm{d}S$$

$$= -\frac{1}{\left(x_n^{2k} / a \right)^2} \iint_{r=a} f(\vartheta) \frac{\partial U_{nk}}{\partial r} \mathrm{d}S + \frac{1}{\left(x_n^{2k} / a \right)^2} \iint_{\vartheta=\pi/2} \left(U_{nk} \frac{\partial u}{\partial n} \right) \mathrm{d}S$$

$$\frac{\partial u}{\partial n} = (\nabla u)\cdot \boldsymbol{n} = \frac{1}{r}\frac{\partial u}{\partial \vartheta}\bigg|_{\vartheta=\pi/2} = -\frac{g(r)}{\kappa}$$

$$a_{nk} = -\frac{2\pi}{\left(x_n^{2k}/a\right)^2}\int_0^{\pi/2} f(\vartheta)\frac{\partial U_{nk}(r,\vartheta)}{\partial r}\bigg|_{r=a} a^2\sin\vartheta \mathrm{d}\vartheta$$

$$-\frac{2\pi}{\left(x_n^{2k}/a\right)^2}\int_0^{a} U_{nk}(r,\pi/2)\frac{g(r)}{\kappa} r\mathrm{d}r$$

■ 积分解

$$u(r,\vartheta) = -\int_0^{\pi/2} f(\vartheta')\frac{\partial g(r,\vartheta;r',\vartheta')}{\partial r'}\bigg|_{r'=a} a^2\sin\vartheta'\mathrm{d}\vartheta'$$

$$-\int_0^{a} g(r,\vartheta;r',\vartheta')\big|_{\vartheta'=\pi/2}\frac{g(r')}{\kappa} r'\mathrm{d}r'$$

$$g(r,\vartheta;r',\vartheta') \equiv \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{2\pi}{\left(x_n^{2k}/a\right)^2}U_{nk}(r,\vartheta)U_{nk}(r',\vartheta')$$

例3 半径为 a 的球，球面径向振动速度分布为

$$v = v_0 e^{-\mathrm{i}\omega t}$$

求辐射的声场分布。

解：显然问题与 ϑ 和 φ 无关：$m=0, l=0$，声压场

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r}\right) = 0; \rho_0 \left.\frac{\partial v}{\partial t}\right|_{r=a} = -\left.\frac{\partial p}{\partial r}\right|_{r=a}$$

- **求稳态解** $p(r,t) = R(r)e^{-\mathrm{i}\omega t}$

$$\downarrow$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + k^2 R = 0, \quad \left(k = \frac{\omega}{c}, \quad r > a\right)$$

$$\left.\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right|_{r=a} = \mathrm{i}\rho_0 \omega v_0$$

零阶球Bessel 方程的解为

$$R(r)=C_1h_0^{(1)}(kr)+C_2h_0^{(2)}(kr)$$

- 球Hankel 函数的取舍：决定于时间部分的形式。在无限远处

$$h_l^{(1)}(x)\sim\frac{1}{x}(-\mathrm{i})^{l+1}e^{\mathrm{i}x};\quad h_l^{(2)}(x)\sim\frac{1}{x}(\mathrm{i})^{l+1}e^{-\mathrm{i}x}$$

$e^{+\mathrm{i}(kr-\omega t)}$ ——向外辐射的球面波

$e^{-\mathrm{i}(kr+\omega t)}$ ——向原点会聚的球面波

因此：(1)如果时间部分为 $e^{-\mathrm{i}\omega t}$

$h_l^{(1)}$ ——向外辐射的球面波

$h_l^{(2)}$ ——向原点会聚的球面波

(2)如果时间部分为 $e^{+\mathrm{i}\omega t}$

$h_l^{(2)}$ ——向外辐射的球面波

$h_l^{(1)}$ ——向原点会聚的球面波

- 本问题取 $R(r)=Ch_0^{(1)}(kr)$
- 利用边界条件

$$Ck\left[\frac{\mathrm{d}h_0^{(1)}(kr)}{\mathrm{d}(kr)}\right]\Bigg|_{r=a}=\mathrm{i}\rho_0\omega v_0 \Rightarrow Ck\left(\frac{\mathrm{i}}{ka}+1\right)\frac{1}{ka}e^{\mathrm{i}ka}=\mathrm{i}\rho_0\omega v_0$$

- 因此辐射的声场为

$$p(r,t)=\frac{\rho_0\omega 4\pi a^2 v_0}{\mathrm{i}+ka}\frac{1}{4\pi r}e^{\mathrm{i}(kr-ka-\omega t)}$$

例4 球面辐射

$$\nabla^2 p(r,\vartheta,\varphi,\omega)+k_0^2 p(r,\vartheta,\varphi,\omega)=0,\ (r>a)$$

$$\frac{1}{\mathrm{i}\rho_0\omega}\frac{\partial p(r,\vartheta,\varphi,\omega)}{\partial r}\bigg|_{r=a}=U_0(\vartheta,\varphi,\omega)$$

$$\Downarrow$$

$$p(r,\vartheta,\varphi,\omega)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}h_l^{(1)}(k_0 r)Y_{lm}(\vartheta,\varphi)$$

$$\Downarrow$$

$$\frac{1}{\mathrm{i}\rho_0 c_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}\left[\frac{\mathrm{d}h_l^{(1)}(k_0 r)}{\mathrm{d}(k_0 r)}\right]_{r=a}Y_{lm}(\vartheta,\varphi)=U_0(\vartheta,\varphi,\omega)$$

$$\Downarrow$$

$$A_{lm}=\mathrm{i}\sqrt{4\pi}\rho_0 c_0 U_{lm}\left[\frac{\mathrm{d}h_l^{(1)}(k_0 r)}{\mathrm{d}(k_0 r)}\right]_{r=a}^{-1}$$

$$U_{lm} \equiv \frac{1}{\sqrt{4\pi}} \int_0^{2\pi} \int_0^{\pi} U_0(\vartheta, \varphi) Y_{lm}^*(\vartheta, \varphi) \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$$

$$p = \mathrm{i}\sqrt{4\pi}\rho_0 c_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_{lm} \left[\frac{\mathrm{d}h_l^{(1)}(k_0 r)}{\mathrm{d}(k_0 r)} \right]_{r=a}^{-1} h_l^{(1)}(k_0 r) Y_{lm}(\vartheta, \varphi)$$

- **低频 ($k_0 a << 1$)**

$$p(r, \vartheta, \varphi, \omega) \approx A\rho_0 c_0 (k_0 a)^2 \bar{U}_0 h_0^{(1)}(k_0 r)$$

$$\bar{U}_0 \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} U_0(\vartheta, \varphi) \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$$

——正比于球面平均速度

■ 低频、远场 ($k_0a<<1$， $k_0r>>1$)

$$p(r,\vartheta,\varphi,\omega)\approx -4\pi \mathrm{i}\rho_0 c_0 k_0 a^2 \bar{U}_0 \frac{1}{4\pi r}\exp(\mathrm{i}k_0 r)$$

■ 一般频率、远场 ($k_0r>>1$)

$$p(r,\vartheta,\varphi,\omega)\approx -\frac{\mathrm{i}}{k_0 r}\exp(\mathrm{i}k_0 r)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mathrm{e}^{-\mathrm{i}l\pi/2}A_{lm}Y_{lm}(\vartheta,\varphi)$$

$$p(r,\vartheta,\varphi,\omega)\approx \boxed{-\mathrm{i}\frac{\exp(\mathrm{i}k_0 r)}{k_0 r}}\boxed{[F_\mathrm{s}(\vartheta,\varphi)+F_\mathrm{d}(\vartheta,\varphi)+F_\mathrm{q}(\vartheta,\varphi)+\cdots]}$$

远场球面波　　　　远场方向性因子

$$F_{\mathrm{s}}(\vartheta,\varphi) \equiv \sqrt{\frac{1}{4\pi}} A_{00}$$

$$F_{\mathrm{d}}(\vartheta,\varphi) \equiv -\mathrm{i}[A_{10}Y_{10}(\vartheta,\varphi) + A_{11}Y_{11}(\vartheta,\varphi) + A_{1-1}Y_{1-1}(\vartheta,\varphi)]$$

$$F_{\mathrm{q}}(\vartheta,\varphi) \equiv A_{22}Y_{22}(\vartheta,\varphi) + A_{2-2}Y_{2-2}(\vartheta,\varphi)$$

$$+A_{21}Y_{21}(\vartheta,\varphi) + A_{2-1}Y_{2-1}(\vartheta,\varphi) + A_{20}Y_{20}(\vartheta,\varphi)$$

$$\cdots\cdots$$

■单极辐射

$$U_0(\vartheta,\varphi,\omega) = U_0(\omega)$$

辐射的功率

$$\overline{P}_{\mathrm{s}} \approx 2\pi\rho_0 c_0 a^2 \overline{U}_0^2 (k_0 a)^2 \sim \omega^2$$

——正比于频率的2次方

■偶极辐射

$$U_0(\vartheta,\varphi,\omega)=\begin{cases}+U_0 & \left(-\dfrac{\pi}{2}<\varphi<\dfrac{\pi}{2}\right)\\[2ex] -U_0 & \left(\dfrac{\pi}{2}<\varphi<\dfrac{3\pi}{2}\right)\end{cases}$$

$$A_{10}=0;\ A_{1+1}=A_{1-1}=-\rho_0c_0U_0\sqrt{\frac{3\pi}{8}}(k_0a)^3$$

辐射的功率

$$\bar{P}_{\mathrm{d}}=\frac{c_0}{2\omega^2\rho_0}\left(|A_{1-1}|^2+|A_{1+1}|^2\right)=\frac{3}{8}\rho_0c_0\pi a^2U_0^2(k_0a)^4\sim\omega^4$$

——正比于频率的4次方

■四极辐射

$$U_0(\vartheta,\varphi,\omega)=\begin{cases} +U_0 & \left(0<\varphi<\dfrac{\pi}{2}\right) \\ -U_0 & \left(\dfrac{\pi}{2}<\varphi<\pi\right) \\ +U_0 & \left(\pi<\varphi<\dfrac{3\pi}{2}\right) \\ -U_0 & \left(\dfrac{3\pi}{2}<\varphi<2\pi\right) \end{cases}$$

$$A_{2\pm2}=\pm\sqrt{\frac{15}{8\pi}}\rho_0 c_0\left[\frac{\mathrm{d}h_2^{(1)}(k_0 r)}{\mathrm{d}(k_0 r)}\right]_{r=a}^{-1}\approx\pm\frac{\mathrm{i}}{9}\sqrt{\frac{15}{8\pi}}\rho_0 c_0 U_0 (k_0 a)^4$$

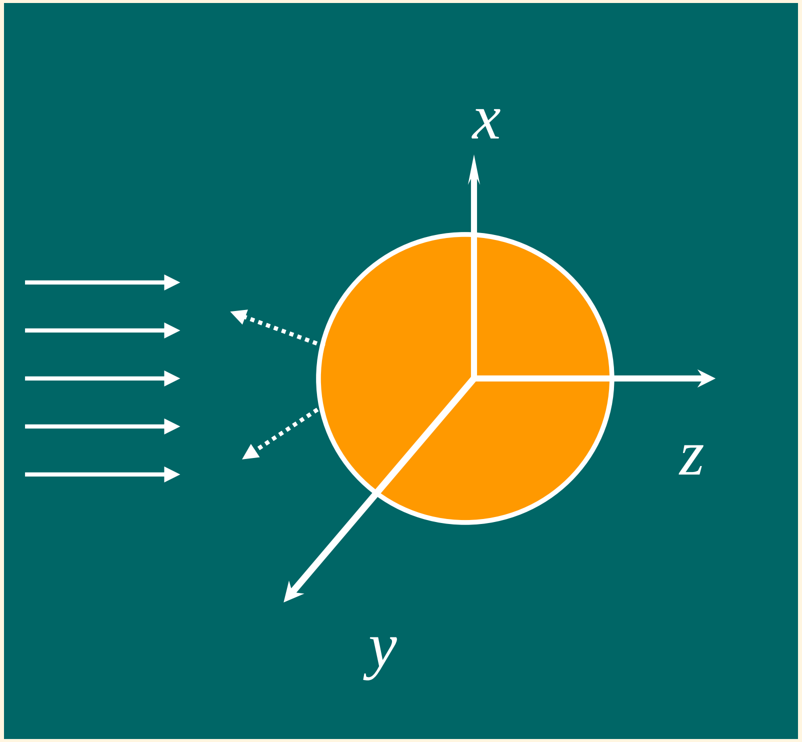
辐射的功率 ——正比于频率的6次方

$$\bar{P}_{\mathrm{q}}=\frac{c_0}{2\omega^2\rho_0}\left(|A_{2-2}|^2+|A_{2+2}|^2\right)=\frac{15}{72\pi}\rho_0 c_0 a^2 U_0^2 (k_0 a)^6\sim\omega^6$$

例5 半径为 a 的刚性球，对平面波的散射。

解：空间声场满足方程

$$\frac{\partial^2 p}{\partial t^2} - c_0^2 \nabla^2 p = 0$$



考虑单频平面波的入射

$$p_i = p_0 \exp[\mathrm{i}(k_0 z - \omega t)]$$
$$= p_0 \exp[\mathrm{i}(k_0 r \cos\vartheta - \omega t)]$$

$$\frac{\partial^2 p_i}{\partial t^2} - c_0^2 \nabla^2 p_i = 0$$

整个声场由入射场和散射场组成

$$p = p_i + p_s e^{-\mathrm{i}\omega t}$$

- **散射场满足的波动方程**

$$\nabla^2 p_s + k_0^2 p_s = 0, \quad (k_0 = \omega / c)$$

- **散射场满足的波动方程** **因为球是刚性的，球面总声压的法向导数为零**

$$\frac{\partial}{\partial r}\left(p_0 e^{\mathrm{i}k_0 r\cos\vartheta} + p_s\right)\Big|_{r=a} = 0$$

- **散射场的通解**

$$p_s(r,\vartheta) = \sum_{l=0}^{\infty} A_l h_l^{(1)}(k_0 r) P_l(\cos\vartheta)$$

——散射场满足满足Sommerfeld辐射条件.

注意：总声场不满足

■ 代入边界条件

$$k_0 \sum_{l=0}^{\infty} A_l \frac{\mathrm{d}h_l^{(1)}(k_0 a)}{\mathrm{d}(k_0 a)} P_l(\cos\vartheta) = -\frac{\partial}{\partial r}\left(p_0 e^{\mathrm{i}k_0 r\cos\vartheta} \right)\Big|_{r=a}$$

■ 利用平面波展开公式

$$e^{\mathrm{i}k_0 r\cos\vartheta} = \sum_{l=0}^{\infty} (2l+1)\mathrm{i}^l j_l(k_0 r) P_l(\cos\vartheta)$$

$$\Downarrow$$

$$k_0 \sum_{l=0}^{\infty} A_l \frac{\mathrm{d}h_l^{(1)}(k_0 a)}{\mathrm{d}(k_0 a)} P_l(\cos\vartheta) = p_0 k_0 \sum_{l=0}^{\infty} (2l+1)\mathrm{i}^l \frac{\mathrm{d}j_l(k_0 a)}{\mathrm{d}(k_0 a)} P_l(\cos\vartheta)$$

$$\Downarrow$$

$$A_l = -p_0 (2l+1)\mathrm{i}^l \left[\frac{\mathrm{d}h_l^{(1)}(k_0 a)}{\mathrm{d}(k_0 a)} \right]^{-1} \frac{\mathrm{d}j_l(k_0 a)}{\mathrm{d}(k_0 a)}$$

■ 散射声场

$$p_s(r,\vartheta)=-p_0\sum_{l=0}^{\infty}(2l+1)\mathrm{i}^l\frac{j_l'(k_0a)}{h_l'^{(1)}(k_0a)}h_l^{(1)}(k_0r)P_l(\cos\vartheta)$$

$$h_l'^{(1)}(k_0a)\equiv\frac{\mathrm{d}h_l^{(1)}(k_0a)}{\mathrm{d}(k_0a)};\ j_l'(k_0a)\equiv\frac{\mathrm{d}j_l(k_0a)}{\mathrm{d}(k_0a)}$$

■ 远场散射声场 $k_0r>>1$

$$p_s(r,\vartheta,\omega)\approx\mathrm{i}p_{0i}(\omega)\boxed{\frac{\exp(\mathrm{i}k_0r)}{k_0r}}\psi_s(\vartheta,\omega)$$

方向因子

远场球面波

$$\psi_s(\vartheta,\omega)\equiv\sum_{l=0}^{\infty}(2l+1)\frac{j_l'(k_0a)}{h_l'^{(1)}(k_0a)}P_l(\cos\vartheta)$$

■ 低频散射 $k_0 a << 1$

小参数展开

$$j_l(x) \approx \frac{x^l}{(2l+1)!!}; \quad n_l(x) \approx -\frac{(2l-1)!!}{x^{l+1}}; \quad (x \to 0)$$

$$\psi_s(\vartheta,\omega) \approx \frac{j_0'(k_0 a)}{h_0'^{(1)}(k_0 a)} + 3\frac{j_1'(k_0 a)}{h_1'^{(1)}(k_0 a)}\cos\vartheta \approx -\frac{(k_0 a)^3}{3\mathrm{i}} + \frac{(k_0 a)^3}{2\mathrm{i}}\cos\vartheta$$

$$= -\frac{(k_0 a)^3}{3\mathrm{i}}\left(1 - \frac{3}{2}\cos\vartheta\right)$$

远场散射声强的分布

$$I_s(r,\vartheta,\omega) = \frac{I_{0i}}{(k_0 r)^2}\,|\psi_s(\vartheta,\omega)|^2 \approx \frac{I_{0i}}{r^2}\frac{\omega^4 a^6}{9c_0^4}\left(1 - \frac{3}{2}\cos\vartheta\right)^2$$

——球的散射功率与频率的4次方成正比，这是低频散射的基本特征，称为Rayleigh散射。

- 中频散射 $k_0 a \sim 1$ ——Mie散射
- 高频散射 $k_0 a >> 1$ ——讨论与柱体散射类似

问题1 非刚性球：球内驻波解；球外行波解

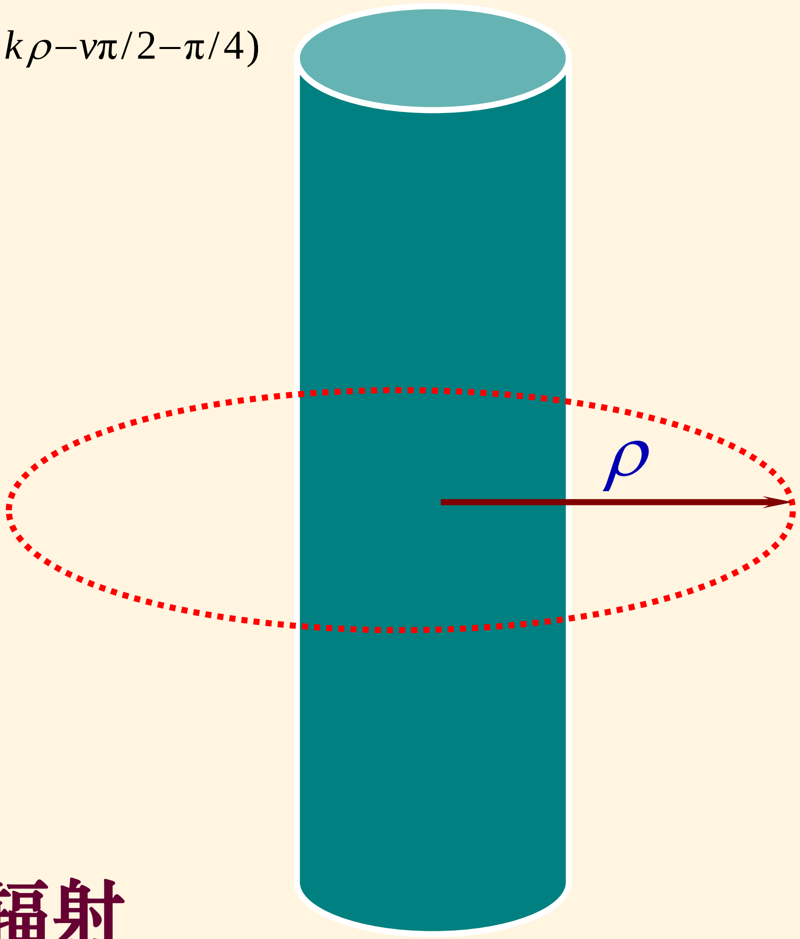
$$p_s(r,\vartheta)=\sum_{l=0}^{\infty} A_l h_l^{(1)}(k_0 r) P_l(\cos\vartheta);(r>a)$$

$$p_s(r,\vartheta)=\sum_{l=0}^{\infty} B_l j_l(k_0 r) P_l(\cos\vartheta);(r<a)$$

问题2 任意散射体，如何求散射场？——积分方程

■ 柱面波远场特征

$$p(\rho) \sim H_{\nu}^{(1)}(k_0\rho) \sim \sqrt{\frac{2}{\pi k_0 \rho}} e^{\mathrm{i}(k\rho - \nu\pi/2 - \pi/4)}$$



通过单位长柱面的能量

$$E = \iint_S |p|^2 \mathrm{d}S$$

$$= \int_0^{2\pi} p(\rho) p^*(\rho) \rho \mathrm{d}\varphi$$

$$= \text{常数}$$

——与柱面半径无关，辐射场——注意：近场比较复杂

- **距离增加一倍，幅度下降多少dB?**

$$20\log\frac{|p(\rho_1)|}{|p(\rho_2)|}=20\log\sqrt{\frac{\rho_2}{\rho_1}}=-10\log 2\sim -3\text{dB}$$

- 柱面波传得更远——声柱
- 柱面波形式的噪声传播更远——降噪困难——高速公路噪声

- **球面波的远场特征**

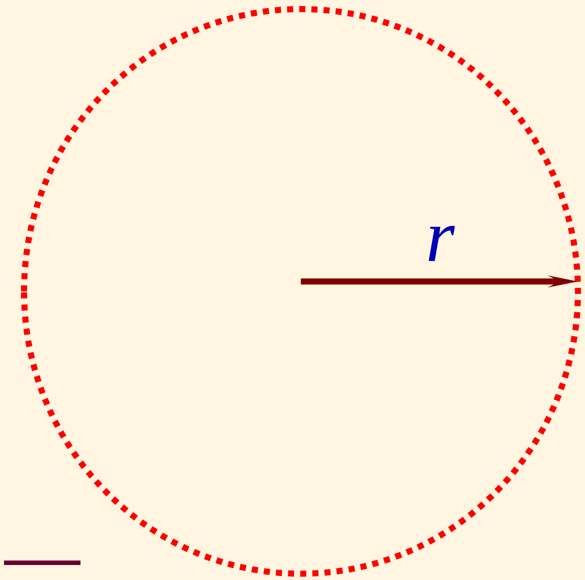
$$p(r)\sim h_l^{(1)}(k_0 r)\sim\frac{1}{k_0 r}e^{\mathrm{i}[kr-(l+1/2)\pi/2-\pi/4]}$$

通过球面的能量

$$E = \iint_S |p|^2 \mathrm{d}S$$

$$= \int_0^{\pi}\int_0^{2\pi} p(r)\,p^*(r)\,r^2 \sin\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\varphi$$

$$= \text{常数}$$



——与球面半径无关，辐射场——注意：近场比较复杂

- **距离增加一倍，幅度下降多少dB?**

$$20\log\frac{|p(r_1)|}{|p(r_2)|} = 20\log\frac{r_1}{r_2} = -20\log 2 \sim -6\text{dB}$$

第12章　小 结

- **Laplace方程在球坐标的分离变量解**

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}[A_{lm}r^{l}+B_{lm}r^{-(l+1)}]Y_{lm}(\vartheta,\varphi)$$

- **球内部**

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}r^{l}Y_{lm}(\vartheta,\varphi)$$

- **球外部**

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}B_{lm}r^{-(l+1)}Y_{lm}(\vartheta,\varphi)$$

Laplace方程在柱坐标中的分离变量解

$$u(\rho,\varphi,z)=(C_0+D_0z)(E_0+F_0\ln\rho)+\sum_m (C_m+D_mz)\rho^m\Phi_m(\varphi)$$

$$+\sum_m\sum_{\mu>0}\begin{pmatrix}A_m e^{-\sqrt{\mu}z}\\ B_m e^{\sqrt{\mu}z}\end{pmatrix}\begin{bmatrix}L_m J_m(\sqrt{\mu}\rho)\\ M_m N_m(\sqrt{\mu}\rho)\end{bmatrix}\Phi_m(\varphi)$$

$$+\sum_m\sum_{\mu<0}\begin{pmatrix}H_m \sin\sqrt{|\mu|}z\\ G_m \cos\sqrt{|\mu|}z\end{pmatrix}\begin{bmatrix}O_m I_m(\sqrt{|\mu|}\rho)\\ P_m K_m(\sqrt{|\mu|}\rho)\end{bmatrix}\Phi_m(\varphi)$$

极角方向

$$\Phi_m(\varphi)=e^{im\varphi},(m=-\infty,\ldots,\infty)$$

$$\Phi_m(\varphi)=A_m\cos m\varphi+B_m\sin m\varphi,(m=0,\ldots,\infty)$$

■ Helmholtz方程球坐标在中的驻波解

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left[A_{lm}j_l(k_0r)+B_{lm}n_l(k_0r)\right]Y_{lm}(\vartheta,\varphi)$$

■ Helmholtz方程球坐标在中的行波解

$$u(r,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left[A_{lm}h_l^{(1)}(k_0r)+B_{lm}h_l^{(2)}(k_0r)\right]Y_{lm}(\vartheta,\varphi)$$

■ Helmholtz方程在柱坐标中的解

$$u(\rho,\varphi,z)=\sum_{m}\sum_{k_z,k_\rho}Z_m(z)R_m(\rho)\Phi_m(\varphi)$$

$$k^2=k_z^2+k_\rho^2$$

4种不同组合

- **轴向驻波和径向驻波(有限长柱体腔内声场)**

$$Z_m(z) = H_m \sin k_z z + G_m \cos k_z z$$

$$R_m(\rho) = O_m J_m(k_\rho \rho) + P_m N_m(k_\rho \rho)$$

- **轴向行波和径向驻波(无限长场柱体内)**

$$Z_m(z) = H_m \exp(\mathrm{i}k_z z) + G_m(-\mathrm{i}k_z z)$$

$$R_m(\rho) = O_m J_m(k_\rho \rho) + P_m N_m(k_\rho \rho)$$

- **轴向行波和径向行波(无限长场柱体外)**

$$Z_m(z) = H_m \exp(\mathrm{i}k_z z) + G_m(-\mathrm{i}k_z z)$$

$$R_m(\rho) = O_m H_m^{(1)}(k_\rho \rho) + P_m H_m^{(2)}(k_\rho \rho)$$

- **轴向驻波和径向行波(平面波导)**

$$Z_m(z) = H_m \sin k_z z + G_m \cos k_z z$$

$$R_m(\rho) = O_m H_m^{(1)}(k_\rho \rho) + P_m H_m^{(2)}(k_\rho \rho)$$

例1 无限长圆柱体的辐射

$$\nabla^2 u + k_0^2 u = 0, \rho > a$$

$$u|_{\rho=a} = f(\varphi, z)$$

解

$$u(\rho,\varphi,z) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A_m \exp(\mathrm{i}k_z z) H_m^{(1)}\left(\sqrt{k_0^2 - k_z^2}\,\rho\right) \mathrm{d}k_z e^{\mathrm{i}m\varphi}$$

$$\Downarrow$$

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A_m \exp(\mathrm{i}k_z z) H_m^{(1)}\left(\sqrt{k_0^2 - k_z^2}\,a\right) \mathrm{d}k_z e^{\mathrm{i}m\varphi} = f(\varphi, z)$$

$$\Downarrow$$

$$A_m = \frac{1}{(2\pi)^2 H_m^{(1)}\left(\sqrt{k_0^2 - k_z^2}\,a\right)} \int_{-\infty}^{\infty} \int_0^{2\pi} f(\varphi, z) e^{-\mathrm{i}(k_z z + m\varphi)} \mathrm{d}\varphi \mathrm{d}z$$

$$u(\rho,\varphi,z)=u_1(\rho,\varphi,z)+u_2(\rho,\varphi,z)$$

$$u_1(\rho,\varphi,z)=\sum_{m=-\infty}^{\infty}\int_{-k}^{k}A_m H_m^{(1)}(\sqrt{k_0^2-k_z^2}\,\rho)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

$$u_2(\rho,\varphi,z)=\sum_{m=-\infty}^{\infty}\int_{-\infty}^{-k}A_m H_m^{(1)}(\sqrt{k_0^2-k_z^2}\,\rho)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

$$+\sum_{m=-\infty}^{\infty}\int_{k}^{\infty}A_m H_m^{(1)}(\sqrt{k_0^2-k_z^2}\,\rho)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

倏逝波模式

$$u_2(\rho,\varphi,z)=\sum_{m=-\infty}^{\infty}\int_{-\infty}^{-k_0}A_m H_m^{(1)}(\mathrm{i}\sqrt{k_z^2-k_0^2}\,\rho)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

$$+\sum_{m=-\infty}^{\infty}\int_{k_0}^{\infty}A_m H_m^{(1)}(\mathrm{i}\sqrt{k_z^2-k_0^2}\,\rho)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

一个简单的逆问题：测量的场数据反演 $f(\varphi, z)$

- **测量远场数据**

$$u_1(\rho,\varphi,z) \approx \sum_{m=-\infty}^{\infty}\int_{-k}^{k}\tilde{A}_m H_m^{(1)}(\sqrt{k^2-k_z^2}\rho)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

$$\Downarrow$$

$$f(\varphi,z) \approx \sum_{m=-\infty}^{\infty}\int_{-k}^{k}\tilde{A}_m H_m^{(1)}(\sqrt{k^2-k_z^2}a)e^{\mathrm{i}(k_z z+m\varphi)}\mathrm{d}k_z$$

- **测量近场数据**

$$u(\rho,\varphi,z) = u_1(\rho,\varphi,z) + u_2(\rho,\varphi,z)$$

$$f(\varphi,z) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_m H_m^{(1)}(\sqrt{k^2-k_z^2}a)\mathrm{d}k_z e^{\mathrm{i}(k_z z+m\varphi)}$$

例2 平面波导

$$\nabla^2 u + k_0^2 u = 0, \rho > a, 0 < z < L$$

$$u|_{\rho=a} = f(\varphi, z); u|_{z=0} = u|_{z=L} = 0$$

$$u(\rho,\varphi,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_m \sin\left(\frac{n\pi z}{L}\right) H_m^{(1)}\left(\sqrt{k_0^2 - \left(\frac{n\pi}{L}\right)^2}\rho\right) e^{im\varphi}$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_m \sin\left(\frac{n\pi z}{L}\right) \cdot H_m^{(1)}\left(\sqrt{k_0^2 - \left(\frac{n\pi}{L}\right)^2}a\right) e^{im\varphi} = f(\varphi, z)$$

$$A_m=\frac{1}{\pi L}\left[H_m^{(1)}\left(\sqrt{k_0^2-\left(\frac{n\pi}{L}\right)^2}a\right)\right]^{-1}\int_0^L\int_0^{2\pi} f(\varphi,z)\sin\left(\frac{n\pi z}{L}\right)e^{-\mathrm{i}m\varphi}\mathrm{d}\varphi\mathrm{d}z$$

$$u(\rho,\varphi,z)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{n_c}A_m\sin\left(\frac{n\pi z}{L}\right)H_m^{(1)}\left(\sqrt{k_0^2-\left(\frac{n\pi}{L}\right)^2}\rho\right)e^{\mathrm{i}m\varphi}$$

$$+\sum_{m=-\infty}^{\infty}\sum_{n=n_c+1}^{\infty}A_m\sin\left(\frac{n\pi z}{L}\right)H_m^{(1)}\left(\mathrm{i}\sqrt{\left(\frac{n\pi}{L}\right)^2-k_0^2}\right)e^{\mathrm{i}m\varphi}$$

——倏逝波模式。当 $k_0^2<(\pi/L)^2$ 时，所有模式都不能向外辐射——截止频率

例3 平面辐射

$$\nabla^2 u + k_0^2 u = 0, 0 < \rho < \infty; z > 0$$

$$u\big|_{z=0} = f(\rho,\varphi)$$

$$\Downarrow$$

$$u(\rho,\varphi,z) = \sum_{m=-\infty}^{\infty}\sum_{k_\rho} Z_m(k_\rho) e^{\mathrm{i}\sqrt{k^2-k_\rho^2}z} J_m(k_\rho\rho) e^{\mathrm{i}m\varphi}$$

$$\Downarrow$$

$$u(\rho,\varphi,z) = \sum_{m=-\infty}^{\infty}\int_0^{\infty} Z_m(k_\rho) \ e^{\mathrm{i}\sqrt{k^2-k_\rho^2}z} J_m(k_\rho\rho) k_\rho \mathrm{d}k_\rho e^{\mathrm{i}m\varphi}$$

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty} Z_m(k_\rho)\ J_m(k_\rho\rho)k_\rho \mathrm{d}k_\rho e^{\mathrm{i}m\varphi} = f(\rho,\varphi)$$

$$Z_m(k_\rho) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\mathrm{i}m\varphi} f(\rho,\varphi) J_m(k_\rho\rho)\rho\mathrm{d}\rho\mathrm{d}\varphi$$

$$u(\rho,\varphi,z) = \sum_{m=-\infty}^{\infty}\int_0^{k_0} Z_m(k_\rho)\exp\left(\mathrm{i}\sqrt{k_0^2-k_\rho^2}\,z\right) J_m(k_\rho\rho)k_\rho\mathrm{d}k_\rho e^{\mathrm{i}m\varphi}$$

$$+\sum_{m=-\infty}^{\infty}\int_{k_0}^{\infty} Z_m(k_\rho)\exp\left(-\sqrt{k_\rho^2-k_0^2}\,z\right) J_m(k_\rho\rho)k_\rho\mathrm{d}k_\rho e^{\mathrm{i}m\varphi}$$

——倏逝波模式