Lorentz Group

2024年9月14日 10:46

1. Lorentz 群的定义和结构
2. Lorentz 变换的回顾(Four-vector 版本)
3. Lorentz 群的李代数和生成元
4. Lorentz 群的表示理论

$$\Lambda^{\rho}{}_{\mu}\eta_{\rho\sigma}\Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}$$

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right)$$

一、Definition of Lorentz group.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$= \eta_{\mu\nu}dx^\mu dx^\nu \qquad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Minkowski spacetime $\mathbb{R}^{(1,3)}$

△ motivation：寻找保内积的变换 Λ，

$\forall x^\mu \in \mathbb{R}^{(1,3)}$ 有 $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

满足 $x^\mu x_\mu = x'^\mu x'_\mu$

$$\Leftrightarrow \quad \eta_{\mu\nu}x^\mu x^\nu = \Lambda^\sigma{}_\mu x^\mu \cdot \eta_{\sigma\rho}x'^\rho = \Lambda^\sigma{}_\mu x^\mu \eta_{\sigma\rho}\Lambda^\rho{}_\nu x^\nu$$

$$\Rightarrow \boxed{\eta_{\mu\nu} = \Lambda^\sigma{}_\mu \eta_{\sigma\rho}\Lambda^\rho{}_\nu \quad (*)} \xrightarrow{\text{matrix form}} \boxed{\Lambda^T\eta\Lambda = \eta}$$

All the Λ satisfying $(*)$ compose the Lorentz group $O(1,3)$

给出 Lorentz group 定义

◎ 变换的逆 $\Lambda^{-1} = \Lambda^T$

Pf: $\Lambda^\sigma{}_\mu \eta_{\sigma\rho}\Lambda^\rho{}_\nu = \eta_{\mu\nu}$ 两边同乘 $\eta^{\nu\gamma}$

$$\Lambda^\sigma{}_\mu \Lambda_{\sigma\nu}$$

$$\Downarrow$$

$$\Lambda^\sigma{}_\mu \Lambda_\sigma{}^\gamma = \delta_\mu{}^\gamma$$

矩阵

$$\Lambda^T\Lambda = I$$

$$\Rightarrow \Lambda^{-1} = \Lambda^T \qquad \square$$

△ Structure of Lorentz group

$\eta_{\mu\nu} = \Lambda^{\sigma}{}_{\mu}\, \eta_{\sigma\rho}\, \Lambda^{\rho}{}_{\nu}$

① 取 det，$\det\eta = -1$，$RHS = \det(\Lambda)^2 \det(\eta)$

$\Rightarrow \det(\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$

$x' = \Lambda x$ 若时间分量 trivial $\begin{pmatrix} 1 & 0\ 0\ 0 \\ 0 & \\ 0 & (\quad)_{3\times3} \\ 0 & \end{pmatrix}$

类比 $O(3)$ $\det\Lambda = +1$ 右手系未翻转

$\det\Lambda = -1$ 右手系 $\to$ 左手系

② 取 $(0,0)$ 位置

$\eta_{00} = 1 \qquad RHS = \Lambda^{\sigma}{}_{0}\, \eta_{\sigma\rho}\, \Lambda^{\rho}{}_{0} = \Lambda^{0}{}_{0}\, \eta_{00}\, \Lambda^{0}{}_{0} + \Lambda^{i}{}_{0}\, \eta_{ij}\, \Lambda^{j}{}_{0}$

$= (\Lambda^{0}{}_{0})^2 - \sum_i (\Lambda^{i}{}_{0})^2$

$\Rightarrow (\Lambda^{0}{}_{0})^2 = 1 + \sum_i (\Lambda^{i}{}_{0})^2 \Rightarrow \Lambda^{0}{}_{0} = \pm\sqrt{1 + \sum_i (\Lambda^{i}{}_{0})^2}$

$x^{0\prime} = \Lambda^{0}{}_{0} x^0 + \Lambda^{0}{}_{i} x^i$

if $\Lambda^{0}{}_{0} \geq 1$ t', t 符号一致

$\Lambda^{0}{}_{0} \leq -1$ t', t 变号

由 ①. ② Lorentz group $O(1,3)$ 含 4 个分支：

(类似于 $O(3)$ 分为 2 个分支)

(1) proper orthochronous ($L_{+}^{\uparrow}$), 对应 $\det L = 1, \Lambda^{0}{}_{0} \geq 1$;

(2) proper non-orthochronous ($L_{+}^{\downarrow}$), 对应 $\det L = 1, \Lambda^{0}{}_{0} \leq -1$;

(3) improper orthochronous ($L_{-}^{\uparrow}$), 对应 $\det L = -1, \Lambda^{0}{}_{0} \geq 1$;

(4) improper non-orthochronous ($L_{-}^{\downarrow}$), 对应 $\det L = -1, \Lambda^{0}{}_{0} \leq -1$.

$\Lambda = I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

之间由空间反演 Λ_P，时间反演 Λ_T 变换沟通

$$\begin{tikzcd}
I \in SO(1,3)^{\uparrow} \arrow[r, dash, "\Lambda_P"] \arrow[d, dash, "\Lambda_T"'] & \Lambda_P SO(1,3)^{\uparrow} \arrow[d, dash, "\Lambda_T"] \\
\Lambda_T SO(1,3)^{\uparrow} \arrow[r, dash, "\Lambda_P"'] & \Lambda_T \Lambda_P SO(1,3)^{\uparrow}
\end{tikzcd}$$

$\Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\Lambda_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\Rightarrow \quad O(1,3) = \{ SO(1,3)^{\uparrow},\ \Lambda_P SO(1,3)^{\uparrow},\ \Lambda_T SO(1,3)^{\uparrow},\ \Lambda_T \Lambda_P SO(1,3)^{\uparrow} \}$

每个分支可由 $SO(1,3)^{\uparrow}$ 元素得到

接下来关注 $SO(1,3)^{\uparrow}$ 为主！

二. Lorentz 变换实例

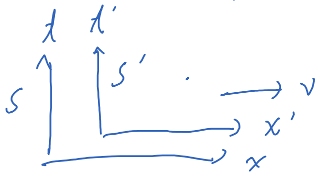
① Rotation $\Lambda_R = \begin{pmatrix} 1 & \\ & \hat{O} \end{pmatrix}$ $\quad \hat{O} \in SO(3) \longrightarrow \begin{cases} \hat{O}^T \hat{O} = I \\ \det \hat{O} = 1 \end{cases}$

$\Lambda^T \eta \Lambda = \eta \xrightarrow[\text{part}]{\text{空间}} \Lambda_R^T \Lambda_R = I.$ $\quad \Downarrow$

$\det \Lambda_R = 1 \cdot \det(\hat{O}) = 1 \qquad \Lambda_R \in SO(1,3)^{\uparrow}$

$SO(3) \qquad J_x = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \quad —\cdots$

② Boost



$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c} \overset{c=1}{=\!=} v \qquad \gamma^2 - (\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$

令 $\gamma = \cosh w$, $\gamma\beta = \sinh w$ $\Rightarrow w \equiv \operatorname{arctanh} \beta$ $\downarrow$ rapidity

$\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu} \eta_{\rho\sigma} \Lambda^{\sigma}{}_{\nu} \quad (*)$

$\Lambda_x = \begin{pmatrix} \cosh w & -\sinh w & & \\ -\sinh w & \cosh w & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \xrightarrow{\text{显然}}$ 满足 $\eta = \Lambda^T \eta \Lambda$

$\Lambda_b \in SO(1,3)^{\uparrow} \qquad \cosh w \geq 1 \ ✓$

$\det \Lambda_x = 1 \ ✓$

三. Generators of Lorentz group

Λ DoF 4×4 Real $\longrightarrow 16$ 分量 $\backslash$ D.F = 6

△ DoF 4×4 Real $\longrightarrow$ 16 分量

$\eta = \underline{\Lambda^T \eta \Lambda} \longrightarrow$ 10 约束 $\Big\}$ DoF = 6 $\longrightarrow$ 6 线性无关生成元!

$\quad\quad\ \ \overset{\shortparallel}{(\Lambda^T\eta\Lambda)^T}$

① $J_i = \begin{pmatrix} 0 & \\ & J_i^{SO(3)} \end{pmatrix}$ $i=1,2,3$ $\quad \Lambda_R = Exp(-i\vec{\theta}\cdot\vec{J})$

② $\Lambda_x = Exp(-iw_x K_x) \longrightarrow K_x = i\left.\dfrac{\partial\Lambda_x}{\partial w_x}\right|_{w_x\to 0}$

$= i\dfrac{\partial}{\partial w_x}\left.\begin{pmatrix} \cosh w_x & -\sinh w_x & & \\ -\sinh w_x & \cosh w_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right|_{w_x\to 0}$

$= i\begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

同理

$K_y = -i\begin{pmatrix} 0 & 0 & 1 & \\ 0 & 0 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix}$

$K_z = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & & \\ 0 & & 0 & \\ 1 & & & 0 \end{pmatrix}$

生成元

$[J_i, J_j] = i\varepsilon_{ijk}J_k$

$[K_i, J_j] = i\varepsilon_{ijk}K_k \Longleftrightarrow [J_i, K_j] = i\varepsilon_{ijk}K_k$

Pf: $[J_j, K_i] = -i\varepsilon_{ijk}K_k$

$\downarrow$

$[J_i, K_j] = -i\varepsilon_{jik}K_k = i\varepsilon_{ijk}K_k$ □

$[K_i, K_j] = -i\varepsilon_{ijk}J_k$

$\Rightarrow \forall \Lambda \in SO(1,3)^\uparrow$ 可写为: $\Lambda = Exp(-i\vec{\theta}\cdot\vec{J} - i\vec{w}\cdot\vec{K})$

K_i, J_i $i=1,2,3$ 张成 $SO(1,3)^\uparrow$ Lie algebra!

△. $N_i^{\pm} = \frac{1}{2}[J_i \pm iK_i]$

① $[N_i^+, N_j^+] = \frac{1}{4}\left([J_i + iK_i, J_j + iK_j]\right)$

$= \frac{1}{4}\left([J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] - [K_i, K_j]\right)$

$= \frac{1}{2}\left(i\varepsilon_{ijk} J_k - \varepsilon_{ijk} K_k\right) = \frac{i}{2}\varepsilon_{ijk}[J_k + iK_k]$

$= i\varepsilon_{ijk} N_k^+$

$[N_i^+, N_j^-] = \frac{1}{4}[J_i + iK_i, J_j - iK_j]$

$= \frac{1}{4}\left([J_i, J_j] - i[J_i, K_j] + i[K_i, J_j] + [K_i, K_j]\right)$

$= 0$

$[N_i^-, N_j^-] = i\varepsilon_{ijk} N_k^-$

$SO(1,3)^{\uparrow}$ 李代数 $\to$ $2\times SU(2)$ 李代数

△ 无穷小形式 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \underline{\omega^\mu{}_\nu} = e^{-i\vec{\theta}\cdot\vec{J} - i\vec{w}\cdot\vec{K}}$ $(\vec{\theta}, \vec{w} \to 0)$

$\eta^{\mu\nu} = \Lambda^\mu{}_\sigma \eta^{\sigma\rho} \Lambda^\nu{}_\rho$

$= (\delta^\mu{}_\sigma + \omega^\mu{}_\sigma)\eta^{\sigma\rho}(\delta^\nu{}_\rho + \omega^\nu{}_\rho)$

$= \eta^{\mu\nu} + (\omega^{\mu\nu} + \omega^{\nu\mu})$ $\Longrightarrow$ $\omega^{\mu\nu} + \omega^{\nu\mu} = 0$

$\vec{\theta}, \vec{w} \longrightarrow \omega^{\mu\nu}$

$\underline{\vec{J}, \vec{K}}$ $M^{\rho\sigma}$ 类比 Electrodynamics

$\vec{B}, \vec{E} \longrightarrow F^{\mu\nu}$

$F^{\mu\nu} = -F^{\nu\mu}$

$M_{ij} = \varepsilon_{ijk} J_k$

$M_{0i} = K_i$

$$M = \begin{pmatrix} 0 & k_x & k_y & k_z \\ k_x & 0 & j_z & -j_y \\ -k_y & -j_z & 0 & j_x \\ -k_z & j_y & -j_x & 0 \end{pmatrix}$$

$\omega^\mu{}_\nu = -\frac{i}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$

$\Omega_{\rho\sigma}$ 反对称 $\longrightarrow$ DoF=6

eg. ① $\Omega = \begin{bmatrix} 0 & w & & \\ -w & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$

$$\omega^\mu{}_\nu = -\frac{i}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$$
$$= -\frac{i}{2}\left[\Omega_{01}(M^{01})^\mu{}_\nu + \Omega_{10}(M^{10})^\mu{}_\nu\right]$$
$$= -i\, w K_x$$

$$Exp(-iwK_x) = I - iwK_x \quad (w\to 0)$$
$$\| $$
$$\delta^\mu{}_\nu - iw(K_x)^\mu{}_\nu$$

② $\Omega = \begin{bmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{bmatrix}$

$$-\frac{i}{2}(\Omega_{\rho\sigma})(M^{\rho\sigma})^\mu{}_\nu = -i\theta(J_z)^\mu{}_\nu$$

绕z轴无穷小 rotation $\quad \delta^\mu{}_\nu - i\theta(J_z)^\mu{}_\nu$

$$\Rightarrow \Lambda = Exp(-i\vec\theta\cdot\vec J - i\vec w\cdot\vec k) = Exp\left(-\frac{i}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

\# $[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = i\left(\eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}\right)$

Pf: $\rho\sigma \to 0i \quad \tau\nu \to 0j \quad i\neq j$

$LHS = [K_i, K_j]$

$RHS = i(\cancel{\eta^{i0}M^{0j}} - \eta^{00}M^{ij} + \cancel{\eta^{0j}M^{i0}} - \cancel{\eta^{ij}M^{00}})$

$= -iM^{ij} = -i\varepsilon_{ijk}J_k$

$\Rightarrow [K_i, K_j] = -i\varepsilon_{ijk}J_k \qquad \square$

其余同理可证

四 群表示

回顾 $SU(2)$, $SO(3)$ $\qquad [J_i, J_j] = i\varepsilon_{ijk}J_k$

$SO(1,3)^\uparrow$ 的李代数 $\longrightarrow$ g 不100%是 $SO(1,3)^\uparrow$

g 一定 $SO(1,3)^\uparrow$ 覆盖群元素

类比 $\mathfrak{su}(2) \longrightarrow SU(2)$ 的表示

$[N_i^{\pm}, N_j^{\pm}] = i\varepsilon_{ijk} N_k^{\pm}$

$[N_i^{+}, N_j^{-}] = 0$

$s \longrightarrow 2s+1 = \dim$

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

1° $(0,0)$ $\quad s_1 = s_2 = 0$

trivial ! $\quad N_i^+ = N_i^- = 0 \quad \forall i = 1,2,3$

$$e^{N_i^+} = e^{N_i^-} = I$$

$(0,0)$ 表示 $\longrightarrow$ Lorentz scalar

2° $(\frac{1}{2}, 0)$ $\quad N_i^+ \longrightarrow SU(2)$ 2维表示 $\quad N_i^+ = \frac{\sigma_i}{2} = \frac{1}{2}[J_i + iK_i]$

left-handed spinor $\quad N_i^- \longrightarrow 0$ 维 $\quad N_i^- = 0 = \frac{1}{2}[J_i - iK_i]$

$$\Longrightarrow J_i = iK_i = \frac{\sigma_i}{2}$$

$$\Lambda_{(\frac{1}{2},0)} = \mathrm{Exp}\left[-i(\vec{\theta}\cdot\vec{J} + \vec{w}\cdot\vec{k})\right] = \mathrm{Exp}\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} \underline{- \frac{1}{2}\vec{w}\cdot\vec{\sigma}}\right)$$

3° $(0, \frac{1}{2})$ 同理 $\qquad$ boost !

right-handed spinor $\quad J_i = -iK_i = \frac{\sigma_i}{2} \quad \Longleftarrow \begin{cases} \frac{1}{2}[J_i + iK_i] = 0 \\ \frac{1}{2}[J_i - iK_i] = \frac{\sigma_i}{2} \end{cases}$

$$\Lambda_{(0,\frac{1}{2})} = \mathrm{Exp}\left[-i(\vec{\theta}\cdot\vec{J} + \vec{w}\cdot\vec{k})\right] = \mathrm{Exp}\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} \underline{+ \frac{1}{2}\vec{w}\cdot\vec{\sigma}}\right)$$

4° $(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$

$\quad \downarrow$

见代码 $\qquad$ 4-vector 表示 □

$$\Lambda_x = \begin{pmatrix} \mathrm{Exp}(-\frac{1}{2}\vec{w}\cdot\vec{\sigma}) & \\ & \mathrm{Exp}(\frac{1}{2}\vec{w}\cdot\vec{\sigma}) \end{pmatrix}$$

5°. Dirac spinor

$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$$\Lambda_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})} = \begin{pmatrix} \mathrm{Exp}(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{w}\cdot\vec{\sigma}) & 0 \\ 0 & \mathrm{Exp}(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{w}\cdot\vec{\sigma}) \end{pmatrix}$$

$\vec{j} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{pmatrix} \qquad \Lambda_R = \mathrm{Exp}(-i\vec{\theta}\cdot\vec{j})$

$\vec{k} = \frac{i}{2}\begin{pmatrix} -\vec{\sigma} & \\ & \end{pmatrix} \qquad \Lambda_B = \mathrm{Exp}(-i\vec{\omega}\cdot\vec{k})$

$$\vec{k} = \frac{i}{2}\begin{pmatrix} -\vec{\sigma} & 0 \\ & \vec{\sigma} \end{pmatrix} \qquad \Lambda_B = Exp(-i\vec{\theta}\cdot\vec{k}) \downarrow \quad \downarrow$$

更紧结

△ 定义 Dirac γ_s $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

习表示 $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Chiral 表示

$$\frac{i}{2}\gamma^0\gamma^i = \frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} = k^i$$

$$\frac{i}{2}\gamma^i\gamma^j = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} \sigma^i\sigma^j & \\ & \sigma^i\sigma^j \end{pmatrix}$$

$i \neq j$

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \qquad i \neq j$$
$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \qquad \Rightarrow \sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k$$

$$\Rightarrow \frac{i}{2}\gamma^i\gamma^j = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix} = \epsilon_{ijk} j^k$$

$$S^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & \rho = \sigma \\ \frac{i}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{cases}$$

$$\left[\rho \neq \sigma \quad \begin{matrix} \gamma^\rho\gamma^\sigma + \gamma^\sigma\gamma^\rho = 0 \\ \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho = 2\gamma^\rho\gamma^\sigma \end{matrix}\right]$$

$S^{\rho\sigma}$ 反对称

$$S^{0i} = \frac{i}{2}\gamma^0\gamma^i = k^i$$

$$S^{ij} = \frac{i}{2}\gamma^i\gamma^j = \epsilon_{ijk} j^k.$$

$$S^{\rho\sigma} = \begin{pmatrix} 0 & k_x & k_y & k_z \\ k_x & 0 & j_z & -j_y \\ -k_y & -j_z & 0 & j_x \\ -k_z & j_y & -j_x & 0 \end{pmatrix}$$

$$\Lambda_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})} = Exp(-\frac{i}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$$

$\Uparrow$

Dirac Spinor

Lorentz transform

$\Updownarrow$ Lorentz transform

$$\Lambda_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})} = \begin{pmatrix} \mathrm{Exp}\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{w}\cdot\vec{\sigma}\right) & 0 \\ 0 & \mathrm{Exp}\left(-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{w}\cdot\vec{\sigma}\right) \end{pmatrix}$$