International Conference on Knowledge Discoveries on Statistical Innovations & Recent Advances in Optimization.

Estimating Parameters of different Continuous Distributions using Bayesian Technique for the Rainfall Data.

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We estimated Bayesian estimators using conjugate prior for Weibull, Rayleigh, and Logistic distributions. Due to the lack of a closed form for these Bayes estimators, Lindley's approximation (1980) approximation under the mean squared error (MSE) loss function has been used to obtain these estimators. The performance of all these estimators can be evaluated using the bias and mean squared error and can be compared using a Monte Carlo simulation. We used a real rainfall dataset and compared it with all the above-obtained Bayes estimators using Monte Carlo simulation.

Key Words: Bayes estimator, bias of an estimator, Lindley's approximation, Weibull distribution, Rayleigh distribution, Logistic distribution, mean square error (MSE).

1. INTRODUCTION:

In the modern world, modeling and analyzing data is important in several applied sciences, such as engineering, medicine, agriculture, and many others. Many varieties of statistical distributions are used in this process. The assumed probability model or distributions significantly impact the effectiveness of the processes employed in a statistical study. Bayes estimators are often obtained as a ratio of two integral expressions that cannot be obtained in a closed form. For some distributions with appropriately chosen conjugate priors, the estimators are obtained in relatively simple forms, but in most cases, numerical approximations are necessary. Here we are considering Bayesian estimators for three different distributions Weibull [9], Rayleigh [12] and Logistic [13] distributions using the conjugate prior. Many research works have been done in this area, below we are going to discuss few works going on right now.

1.1 Weibull Distribution:

The Weibull distribution is one of the most popular continuous probability distributions. The Weibull distribution is useful for modeling and analysis of a huge range of data from many fields like economics, hydrology, biology, and engineering. Weibull distribution takes the 'fatigue' or 'age' factor into account and has been widely used in life testing and reliability problems. This distribution has been named after the Swedish scientist Weibull (1939), who first proposed it in connection with his studies on the strength of materials.

The two-parameter Weibull distribution is defined by the probability density function with k as the scale parameter, and λ as the shape parameter.

$$f(x;k, \lambda) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^{k}}$$

$$x \geq 0,\ \lambda, k > 0$$

The different applications of Weibull distributions are wind speed data analysis [2], Regional flood frequency [3], Environment radioactivity [4], Ecological application [5], Survival data [6]. Bayesian approaches have been extensively studied for estimating the Weibull distribution's parameters. It examined the MLE techniques for estimating the Weibull distribution's shape and scale parameters while considering failure interval data [7]. Using Jeffrey's prior knowledge derived from Lindley's approximation, we compare the conventional maximum likelihood estimation of the size and shape parameters of the Weibull distribution with the Bayesian estimators [11].

1.2 Rayleigh Distribution:

The Rayleigh distribution, named for William Strutt, initially presented by Lord Rayleigh (1880), was developed concerning an acoustics issue. Rayleigh distribution is extremely important in communication engineering (Dyer & Whisenand,1973) and is also used for radio wave power distribution (Siddiqui,1962) and some types of electrovacuum devices (Polovko,1968).

Here we are considering two-parameter Rayleigh distribution with one scale and location parameter, and is defined by the probability density function with λ and μ are the scale and location parameters respectively.

$$f(x,\mu,\lambda) = 2\lambda\,(x_i - \mu)e^{-\lambda(x-\mu)^2}$$

$$\lambda > 0, \mu < x$$

The Rayleigh distribution has several applications in real life and associated with several research activities in the field of bio-medicine, economics, physics, wind energy and many other societal applications [15 – 17]. Examine several Bayesian and frequentist estimate methods for the two-parameter Rayleigh distribution [14]. Calculating the Bayesian estimates for Rayleigh distribution using Jeffrey's prior proposed by Al-Kutubi (2002) [21].

1.3 Logistic Distribution:

The logistic distribution is a continuous probability distribution. It is one of the most popular and widely used growth model. The logistic distribution is also known as an alternative approach for the normal distribution.

$$f(x;\mu,\sigma) = \frac{\frac{1}{\sigma}e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\left[1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right]^2}$$

$$-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

σ is scale and μ is location

Estimating common scale parameters when other nuisance parameters are unknown and possibly different [23]. Finding the Bayesian estimators of the Logistic distribution using Lindley (1980) and Tierney & Kadane (1986) approximation using vague prior [24].

In this paper, we propose to find the Bayesian estimators of three different distributions namely Weibull distribution, Rayleigh distribution and Logistic distribution. The closed form the Bayesian estimators do not exist, we used some numerical approximation method called Lindley's approximation (1980) method to find the Bayesian estimators using the conjugate prior.

The paper consists of five sections. In section 1, we show some basic concepts which will be used throughout the paper. In Section 2, we use Maximum Likelihood Estimators (MLEs) of two parameters for the three distributions i.e Weibull, Rayleigh and Logistic using the conjugate prior. In section 3, we are using the Bayesian model to estimate the two parameters for the three distributions. We are calculating the values based on the bias of an estimator and square error loss function. In this Bayesian method we are using the Lindley's approximation method to get the approximate values. In section 4, we demonstrate the numerical results obtained from the simulation study in order to evaluate the performance of the approximate estimators and MLEs obtained. Here we are using the Monte Carlo simulation method as comparative study between the three distributions. In section 5, we give the conclusion on simulation study.

2. MAXIMUM LIKELIHOOD ESTIMATORS (MLEs)

The maximum likelihood estimator (MLE) is one of the well-known and most commonly used estimations of parameters. It is a method used to perceive estimators for different parameters.

2.1 Weibull Distribution:

Let x_1, x_2, x_3, ……..x_n be a sample size of n which is obtained from the probability density function f(x_i;k, λ).

$$L(x_i;\, k, \lambda) = \prod_{i=1}^{n} f(x_i;\, k, \lambda) = \frac{k^n}{\lambda^n} \prod_{i=1}^{n} (\frac{x}{\lambda})^{k-1}\, e^{\sum_{i=1}^{n}(\frac{x}{\lambda})^k}$$

The above equation can be linearized by applying logarithm on both sides

$$\log L(x_i;\, k, \lambda) = n\log k - n\log \lambda + (k-1) \sum_{i=1}^{n} \log(\frac{x_i}{\lambda}) - \sum_{i=1}^{n} (\frac{x_i}{\lambda})^k$$

There are few traditional methods to solve. Here we are going to use the very famous Newton-Raphson method to get the numerical solutions.

2.2 Rayleigh Distribution:

Considering a random sample x_i , i = 1,2,3,……….,n for n observations. The likelihood function of x_i is given by

$$L(x_i;\, \mu, \lambda) = f(x, \mu, \lambda) = 2\lambda\, (x_i - \mu) e^{-\lambda (x_i - \mu)^2}$$

Applying logarithm on both the sides we get

$$l(\mu, \lambda) = C + n\log \lambda + \sum_{i=1}^{n} \log(x_i - \mu) - \lambda \sum_{i=1}^{n} (x_i - \mu)^2$$

We are going to use the traditional method called Newton-Raphson method to find the numerical values.

2.3 Logistic Distribution:

The x_1, x_2, x_3, …….., x_m be random sample, the likelihood function will be

$$\frac{1}{\sigma^m} \exp\left(-\sum_{i=1}^{m}\left(\frac{x_i-\mu}{\sigma}\right) \prod_{i=1}^{m}\left[1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{-2}\right)$$

Due to the lack of a closed form for these Bayes estimators, Lindley's approximation (1980) approximation under the mean squared error (MSE) loss function has been used to obtain these estimators.

3. LINDLEY'S APPROXIMATION

Lindley (1980) proposed a technique to evaluate (approximately) the expression of the form

$$E\left[\frac{u(\theta)}{x}\right] = \frac{\int u(\theta)\, v(\theta)\, \theta exp\big(L(\theta)\big)\, d\theta}{\int v(\theta)\, exp\big(L(\theta)\big)\, d\theta}$$

Lindley's approach is to obtain Taylor series expansion of the functions involved in the above equation about the maximum likelihood estimator θ. Lindley approximated by

$$E\left[\frac{u(\theta)}{x}\right] = \left[u + \frac{1}{2}\sum_i \sum_j \big(u_{ij} + 2u_i\rho_j\big)\sigma_{ij} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_r L_{ijk}\ \sigma_{ij}\ \sigma_{kr}\ u_\gamma\right] + 0\left(\frac{1}{n^2}\right)$$

Where

$\theta = (\theta_1, \theta_2, \ldots\ldots, \theta_m)$, $\hat{\theta}$ is the MLE of θ

$u_i = \frac{\partial \mu}{\partial \theta_i}$, $u_{ij} = \frac{\partial^2 \mu}{\partial \theta_i\, \partial \theta_j}$, $L_{ijk} = \frac{\partial^3 L}{\partial \theta_i\, \partial \theta_j\, \partial \theta_k}$

$\rho = \rho(\theta) = \log v(\theta)$, $\rho_i = \frac{\partial \rho}{\partial \theta_i}$

σ_{ij} = $(i, j)^{th}$ element in the matrix $[-L_{ij}]^{-1}$.

We are going to use the above technique for finding the Bayesian estimator of our three different distributions used in this research. We will derive an approximate Bayes estimators using Lindley's approximation by assuming conjugate prior for all the distributions.

3.1 Weibull Distribution:

The Conjugate Prior is inverted Gamma.

$$V(k, \lambda) = \frac{{\beta_1}^{\alpha_1}\, {\beta_2}^{\alpha_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\, k^{\alpha_1-1}\, \lambda^{\alpha_2-1}\, e^{-\frac{\beta_1}{k}-\frac{\beta_2}{\lambda}}$$

After certain calculation, Finally we have obtained the Bayes estimates of parameters k, λ using Lindley's approximation as follows

$$k^* = k + \frac{1}{2}[(\frac{k(\alpha_1-1)+\beta_1}{k(n-\sum a_i - k\sum a_i b_i)}) + (\frac{\lambda(\alpha_2-1)+\beta_2}{\lambda(n-\sum a_i - k\sum a_i b_i)})] + \frac{1}{2}[\frac{3k^2(\sum a_i b_i(kb_i+2))}{(n+k^2\sum a_i {b_i}^2)\,(n-\sum a_i - k\sum a_i b_i)} +$$
$$\frac{k^2(n-\sum a_i(2k+1)+k(k+1)b_i)}{(n+k^2\sum a_i b_i)\,(k(k+1)\sum a_i - nk)} + (\frac{k(2n-k^3\sum a_i {b_i}^3)}{(n+k^2\sum a_i b_i)^2}) + \frac{(-2nk+k(k+1)(k+2)\sum a_i)}{(n-\sum a_i - k\sum a_i b_i)\,(k(k+1)\sum a_i - nk)} +$$
$$2(\frac{n-\sum a_i((2k+1)+k(k+1)b_i)}{(n-\sum a_i - k\sum a_i {b_i}^2)})] + 0(\frac{1}{n^2})$$

$$\lambda^* = \lambda + \frac{1}{2}[(\frac{k(\alpha_1-1)+\beta_1}{k^2(n-\sum a_i - k\sum a_i b_i)}) + (\frac{\lambda(\alpha_2-1)+\beta_2}{k(k+1)\sum a_i - nk})] + \frac{1}{2}[\frac{3\lambda(n-\sum a_i[(2k+1)+k(k+1)b_i])}{(n-\sum a_i - k\sum a_i b_i)\,(k(k+1)\sum a_i - nk)} +$$
$$\frac{k^2\lambda(\sum a_i b_i\,(kb_i+2))}{(n+k^2\sum a_i {b_i}^2)\,(k(k+1)\sum a_i - nk)} + \frac{2\lambda\sum a_i b_i(kb_i+2)}{(n-\sum a_i - k\sum a_i {b_i}^2)} + (\frac{\lambda(2n-k^3\sum a_i {b_i}^3)}{k(n+k^2\sum a_i {b_i}^2)\,(n-\sum a_i - k\sum a_i b_i}) +$$
$$\frac{\lambda[-2n+(k+1)(k+2)\sum a_i]}{(k+1)\sum a_i - n}] + 0(\frac{1}{n^2})$$

$$a_i = (\frac{x_i}{\lambda})^k$$

$$b_i = \log(\frac{x_i}{\lambda})$$

3.2 Rayleigh Distribution:

The Conjugate Prior is inverted Gamma

$$V(\mu, \lambda) = \frac{{\beta_1}^{\alpha_1}\, {\beta_2}^{\alpha_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\, \mu^{\alpha_1-1}\, \lambda^{\alpha_2-1}\, e^{-\frac{\beta_1}{\mu}-\frac{\beta_2}{\lambda}}$$

After certain calculation, Finally we have obtained the Bayes estimates of parameters μ, λ using Lindley's approximation as follows

$$\mu^* = \mu + \frac{1}{2}[(\frac{\alpha_1-1}{\mu} - \frac{\beta_1}{\mu^2})\,(\sum\frac{(x_i-\mu)^2}{2(x_i-\mu)^2\lambda+1}) + (\frac{(\alpha_2-1)\lambda+\beta_2}{\lambda^2})\,(\frac{-1}{2\sum(x_i-\mu)})\,(6\sum(\frac{(x_i-\mu)^2}{2(x_i-\mu)^2\lambda+1}))\,(\frac{(x_i-\mu)}{2\sum(x_i-\mu)}) +$$
$$(2\sum\left(\frac{(x_i-\mu)^2\,\lambda^2\,(x_i-\mu)}{(2(x_i-\mu)^2\lambda+1)n}\right) - \frac{4(x_i-\mu)}{2\sum(x_i-\mu)}) + (\frac{\sum(x_i-\mu)^2}{2\sum(x_i-\mu)^2\lambda+1})^2\,(\frac{-2}{(x_i-\mu)^3}) - \frac{2}{\lambda 2\sum(x_i-\mu)} + 0(\frac{1}{n^2})]$$

$$\sigma^* = \sigma + \frac{1}{2}[(\frac{\alpha_1 - 1}{\mu} - \frac{\beta_1}{\mu^2})\ (\frac{-1}{2\sum(x_i - \mu)}) + (\frac{(\alpha_2 - 1)\lambda + \beta_2}{n}) + \frac{-3\lambda^2}{n} + (\sum((\frac{(x_i - \mu)^2}{2\lambda(x_i - \mu)^2 + 1})(\frac{\lambda^2}{n}) - \frac{1}{\sum(x_i - \mu)})$$
$$\frac{1}{\sum(x_i - \mu)^2}) - \frac{1}{2\sum(x_i - \mu)} \sum(\frac{(x_i - \mu)^2}{2\lambda(x_i - \mu)^2 + 1})(\sum\left(\frac{-2}{(x_i - \mu)^3}\right) + \frac{\lambda}{n}) + 0(\frac{1}{n^2})]$$

3.3 Logistic Distribution:

The prior for μ *is normal distribution is*

$$\rho_1(\mu) = \frac{1}{\beta_1 \Gamma 2\pi} e^{-\frac{1}{2}(\frac{\mu - \alpha_1}{\beta_1})^2}$$

The prior for σ is gamma

$$\rho_2(\sigma) = \frac{1}{\Gamma(\alpha_2)\ {\beta_2}^{\alpha_2}}\ e^{\frac{-\sigma}{\beta_2}}\ \sigma^{\alpha_2 - 1}$$

After certain calculation, Finally we have obtained the Bayes estimates of parameters μ, σ using Lindley's approximation as follows

$$\mu^* = \mu + \frac{1}{2}\ [(\frac{\alpha_1 - \mu}{{\beta_1}^2}) \sum \frac{P}{2a_i} + (\frac{\alpha_2 - 1}{\sigma} - \frac{1}{\beta_2})\ (\sum \frac{P}{Q})] + \frac{1}{2}\ \{3(\sum \frac{P}{2a_i})\ (\sum \frac{P}{Q})\ (\mu \sum \frac{a_i}{\sigma^3\ (1 + a_i)^2} -$$
$$2\sum \frac{(x_i - \mu)(a_i - {a_i}^2)}{\sigma^4 (1 + a_i)^3}) + 2S\ (\sum \frac{P}{Q}) + (\sum \frac{P}{2a_i})\ (\sum \frac{\sigma^4 (1 + a_i)^2}{P - R - 2(x_i - \mu)})\ (S) + \sum[\frac{P}{2a_i}]^2\ [-2\sum \frac{a_i(1 - a_i)}{\sigma^3 (1 + a_i)^3}]$$
$$+\ [\sum \frac{P}{Q}]\ \ [\sum \frac{\sigma^4 (1 + a_i)^2}{P - R - 2(x_i - \mu)^2}]\ \ [\frac{2n}{\sigma^3} + 6\sum(\frac{x_i - \mu}{\sigma^4})\ -\ 12\sum(\frac{(x_i - \mu)a_i}{\sigma^4 (1 + a_i)})\ -\ 12\sum\left(\frac{(x_i - \mu)^2 a_i}{\sigma^5 (1 + a_i)^2}\right) -$$
$$2\sum(\frac{(x_i - \mu)^3 (a_i - {a_i}^2)}{\sigma^6 (1 + a_i)^3})]\} + 0(\frac{1}{n^2})$$

Where

$$S = \frac{2n}{\sigma^3} + 8\sum \frac{(x_i - \mu)a_i}{\sigma^4 (1 + a_i)^2} + 2\sum \frac{(x_i - \mu)^2 ({a_i}^2 - a_i)}{\sigma^5 (1 + a_i)^3}$$

$$P = \sigma^2 (1 + a_i)^2$$

$$Q = 1 - 2{a_i}^2 + 2(x_i - \mu)(a_i + 2{a_i}^2)$$

$$R = 2\ (x_i - \mu)\ (1 - {a_i}^2)$$

$$a_i = (\frac{x_i}{\lambda})^k$$

$$b_i = \log(\frac{x_i}{\lambda})$$

$$\sigma^* = \sigma + \frac{1}{2}\left[\left(\frac{\alpha_1 - \mu}{{\beta_1}^2}\right)\sum\frac{X}{B} + \left(\frac{\alpha_1 - \mu}{\sigma} - \frac{1}{\beta_2}\right)\left(\sum\frac{Y}{A}\right)\right] + \frac{1}{2}\Big\{3\left(\sum\frac{X}{B}\right)\left(\sum\frac{Y}{A}\right)\left(\frac{2n}{\sigma^3} + 8\sum\frac{(x_i - \mu)a_i}{\sigma^4(1+a_i)^2}\right.$$

$$\left. + 2\sum\frac{(x_i - \mu)^2(a_i^2 - a_i)}{\sigma^5(1+a_i)^3}\right) + \left(\sum\frac{X}{2a_i}\right)\left(\sum\frac{Y}{A}\right) + 2\left(\sum\frac{X}{B}\right)\left(4\sum\frac{a_i}{a^3(1+a_i)^2} - 2\sum\frac{(x_i - \mu)(a_i - a_i^2)}{\sigma^4(1+a_i)^3}\right) +$$

$$\left(\sum\frac{X}{2a_i}\right)\left(\sum\frac{X}{B}\right)\left(2\sum\frac{a_i(a_i - 1)}{\sigma^3(1+a_i)^3}\right) + \left[\frac{Y}{\sigma^2(1+a_i)^2 - 2(x_i - \mu)(1 - a_i)^2 - 2(x_i - \mu)}\right]^2\left[\frac{2n}{\sigma^3} + 6\sum\left(\frac{x_i - \mu}{\sigma^4}\right) -\right.$$

$$\left. 12\sum\left(\frac{(x_i - \mu)a_i}{\sigma^4(1+a_i)}\right) - 12\sum\left(\frac{(x_i - \mu)^2 a_i}{\sigma^5(1+a_i)^2}\right) - 2\sum\left(\frac{(x_i - \mu)^3(a_i - {a_i}^2)}{\sigma^6(1+a_i)^3}\right)\right]\Big\} + 0\left(\frac{1}{n^2}\right)$$

Where

X = $\sigma^2(1 + a_i)^2$

Y = $\sigma^4(1 + a_i)^2$

B = 1 - $2{a_i}^2 + 2(x_i - \mu)(a_i + 2{a_i}^2)$

A = $\sigma^2(1 + a_i)^2 - 2(x_i - \mu)(1 - a_i^2)$ - $2(x_i - \mu)^2$

$a_i = (\frac{x_i}{\lambda})^k$

b_i = $\log(\frac{x_i}{\lambda})$

4. MONTE CARLO SIMULATION STUDY AND RESULTS:

The Bayes estimator of the reliability function of three distribution using procedures of Lindley (1980) was compared through a Monte Carlo simulation with each sample size n = 10, 20, 40, 50, 60 and 70 for both the parameters. These simulations were performed in R programming Language.

The Bias and mean square error (MSE) for each distribution are presented in below tables.

Comparing Bias & mean squared error of Bayes estimates of different distributions

Sample size (n)	Weibull Distribution			Rayleigh Distribution			Logistic Distribution		
	(k, λ)	Bias	MSE	(μ, λ)	Bias	MSE	(μ, σ)	Bias	MSE
10	1 2	1.23 1.55	2.59 3.03	1 2	1.91 2.13	4.01 4.55	1 2	1.81 2.01	3.91 4.04
20	1 2	1.01 1.13	1.76 1.99	1 2	1.71 1.98	3.86 3.98	1 2	1.62 1.87	3.54 3.89
40	1 2	0.85 1.01	1.34 1.46	1 2	1.52 1.79	3.24 3.33	1 2	1.43 1.67	3.02 3.21
50	1 2	0.65 0.86	0.98 1.01	1 2	1.10 1.21	2.35 2.43	1 2	1.10 1.23	2.21 2.43
60	1 2	0.43 0.63	0.84 0.95	1 2	0.91 1.01	1.23 1.54	1 2	0.96 1.00	1.62 1.72
70	1 2	0.23 0.47	0.46 0.65	1 2	0.72 0.87	0.98 1.01	1 2	0.82 0.86	0.98 1.02

5. CONCLUSION:

From the Study we have drawn the Bayesian estimators using Lindley's Approximation method. We have considered one real data set and compared all the estimates using R programming and we can see them in the numerical results. The bias and mean square error of the Weibull distribution are smaller than other distributions. So, we conclude that the Weibull distribution gives better estimates than all other distributions.

6. REFERENCES

[1] Preda, Vasile, Eugenia Panaitescu, and Alina Constantinescu. "Bayes estimators of modified-Weibull distribution parameters using Lindley's approximation." *WSEAS Transactions on Mathematics* 9.7 (2010): 539-549.

[2] Shamilov, Aladdin, İlhan Usta, and YELIZ MERT Kantar. "The distribution of minimizing maximum entropy: alternative to weibull distribution for wind speed." *WSEAS Transactions on Mathematics* 5.6 (2006): 695.

[3] Heo, Jun-Haeng, D. C. Boes, and J. D. Salas. "Regional flood frequency analysis based on a Weibull model: Part 1. Estimation and asymptotic variances." *Journal of hydrology* 242.3-4 (2001): 157-170.

[4] Dahm, H., J. Niemeyer, and D. Schröder. "Application of the Weibull distribution to describe the vertical distribution of cesium-137 on a slope under permanent pasture in Luxembourg." *Journal of environmental radioactivity* 63.3 (2002): 207-219.

[5] Fleming, Richard A. "The Weibull model and an ecological application: Describing the dynamics of foliage biomass on Scots pine." *Ecological modelling* 138.1-3 (2001): 309-319.

[6] Carroll, Kevin J. "On the use and utility of the Weibull model in the analysis of survival data." *Controlled clinical trials* 24.6 (2003): 682-701.

[7] Tan, Zhibin. "A new approach to MLE of Weibull distribution with interval data." *Reliability Engineering & System Safety* 94.2 (2009): 394-403.

[8] Lai, C. D., D. N. P. Murthy, and M. Xie. "Weibull distributions." *Wiley Interdisciplinary Reviews: Computational Statistics* 3.3 (2011): 282-287.

[9] Rinne, Horst. *The Weibull distribution: a handbook*. Chapman and Hall/CRC, 2008.

[10] Stone, G. C., and R. G. Van Heeswijk. "Parameter estimation for the Weibull distribution." *IEEE Transactions on Electrical Insulation* 4 (1977): 253-261.

[11] Guure, Chris Bambey, Noor Akma Ibrahim, and Al Omari Mohammed Ahmed. "Bayesian estimation of two-parameter Weibull distribution using extension of Jeffreys' prior information with three loss functions." *Mathematical Problems in Engineering* 2012 (2012).

[12] Beckmann, Petr. "Rayleigh distribution and its generalizations." *Radio Science Journal of Research NBS/USNC-URSI* 68.9 (1964): 927-932.

[13] Balakrishnan, Narayanaswamy. *Handbook of the logistic distribution*. CRC Press, 2013.

[14] Dey, Sanku, Tanujit Dey, and Debasis Kundu. "Two-parameter Rayleigh distribution: different methods of estimation." *American Journal of Mathematical and Management Sciences* 33.1 (2014): 55-74.

[15] Al-Noor, N. H., and N. K. Assi. "Rayleigh-Rayleigh distribution: properties and applications." *Journal of Physics: Conference Series*. Vol. 1591. No. 1. IOP Publishing, 2020.

[16] Ateeq, Kahkashan, Tahira Bano Qasim, and Ayesha Rehman Alvi. "An extension of Rayleigh distribution and applications." *Cogent Mathematics & Statistics* 6.1 (2019): 1622191.

[17] Hoffman, Dan, and Otto J. Karst. "The theory of the Rayleigh distribution and some of its applications." *Journal of Ship research* 19.03 (1975): 172-191.

[18] Ajami, M., and S. M. A. Jahanshahi. "Parameter estimation in weighted Rayleigh distribution." *Journal of Modern Applied Statistical Methods* 16.2 (2017): 14.

[19] Bhat, A. A., and S. P. Ahmad. "A NEW GENERALIZATION OF RAYLEIGH DISTRIBUTION: PROPERTIES AND APPLICATIONS." *Pakistan journal of statistics* 36.3 (2020).

[20] Sapkota, Laxmi Prasad. "A Bayesian Analysis and Estimation of Weibull Inverse Rayleigh Distribution Using HMC Method." *Nepal Journal of Mathematical Sciences* 3.2 (2022).

[21] Ahmed, A., S. P. Ahmad, and J. A. Reshi. "Bayesian analysis of Rayleigh distribution." *International Journal of Scientific and Research Publications* 3.10 (2013): 1-9.

[22] Balakrishnan, Narayanaswamy. *Handbook of the logistic distribution*. CRC Press, 2013.

[23] Nagamani, Nadiminti, Manas Ranjan Tripathy, and Somesh Kumar. "Estimating common scale parameter of two logistic populations: A Bayesian study." *American Journal of Mathematical and Management Sciences* 40.1 (2020): 44-67.

[24] Howlader, H. A., and G. Weiss. "Bayes estimators of the reliability of the logistic distribution." *Communications in Statistics-Theory and Methods* 18.4 (1989): 1339-1355.

[25] Almazah, Mohammed, and Muhammad Ismail. "Selection of Efficient Parameter Estimation Method for Two-Parameter Weibull Distribution." *Mathematical Problems in Engineering* 2021 (2021).