

High frequency trading, noise herding and market quality

Yi Xue^a, Ramazan Gençay^{b,c,*}, Chenyu Wang^a

^a *Department of Finance, School of International Trade and Economics, University of International Business and Economics, #10 Huixin Dongjie, Chaoyang District, Beijing, China.*

^b *Department of Economics, Simon Fraser University, 8888 University Drive, Burnaby, British Columbia, V5A 1S6, Canada.*

First Version: September, 2013

This version: April 2014

Abstract

We develop a discrete-time infinite-horizon model to investigate the role of a high frequency trader's speed advantage when he interacts with a low frequency trader and with noise traders who herd on informed trading. We find that speed in trading rather than an informational advantage can make the high frequency trader a temporary monopolist in the market, and thus lead to a higher trading volume and trading profits for the high frequency trader. Furthermore, noise traders herding behavior combined with high frequency trading leads to slower price discovery, i.e., the market is not semi-strong as in Chau and Vayanos (2008). In addition, when the high frequency trader can adopt an order splitting strategy, he can generate more profit, and the market quality, for instance, in terms of price discovery and market depth is further improved.

JEL classification: G11; G12; G14

Keywords: High frequency trading; Low latency trading; Herding; Market quality.

Yi Xue gratefully acknowledge financial support from the National Natural Science Foundation of China (NSFC), project No.71101031. Ramazan Gençay gratefully acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada and the Social Sciences and Humanities Research Council of Canada. All errors are ours.

* Corresponding author: Tel.: +1-778-782-3934.

Email addresses: yxue@uibe.edu.cn (Yi Xue), rgencay@sfu.ca (Ramazan Gençay), chenyuwan-guibe@gmail.com (Chenyu Wang).

1. Introduction

Our paper primarily addresses the following question — what unique benefits can high frequency trading (HFTN) bring to traders, and what influence does HFTN have on market quality, particularly, in terms of price discovery and market liquidity? With the rapid development of automation technology and low latency infrastructure, the volume of HFTN has been increasing tremendously over the past decade. Therefore, it is very important to investigate how this technological progress influences institutional investors' trading strategies and how the changes in strategies, in turn, affect price evolution. This question has generated a number of research. Kirilenko *et al.* (2010) defines high frequency traders as those whose number of transactions rank in the top 7% among intermediaries. In addition, Duhigg (2010) finds that trading volume in the NYSE increased by approximately 164% between 2005 and 2009, and this increase might be attributable to HFTN. Other studies have suggested that high frequency firms accounted for 50% of all US equity trading volume in 2012.

To understand the impact of HFTN, the difference between HFTs and ordinary traders, who we call low frequency traders (LFTs), must be analyzed. Theoretically, speed is generally considered as a means to help traders to exploit their informational advantage. For example, Martinez and Roşu (2013) builds a model in which both HFTs and LFTs trade on news, but in each period, LFTs receive a signal with one-period delay relative to HFTs. They show that even if LFTs only obtain news one instant later, they earn considerably lower profits. Thus, the authors argue that faster trading speed generates higher profits. However, little empirical evidence has been found yet on asymmetric information advantage for HFTs; therefore, it might be more relevant to instead examine the pure speed advantage of HFTs— separate from the informational advantage. According to SEC (2010), HFTN has two distinctive characteristics: first, the use of extraordinary, high speed and sophisticated computer programs for generating, routing, and executing orders¹, and, second, the employment of co-location services and individual data feeds offered by exchanges and other data

¹Appearing at least since 1999, after the SEC authorized electronic exchanges in 1998, HFTN had an initial execution time of several seconds. Whereas by 2010, this had decreased to milli- and even micro-seconds.

providers to minimize network and other types of trading latencies. For instance, electronic data of company news which can be extracted from sources such as Bloomberg, and Twitter feeds, is available nowadays and automated systems can be used to identify and analyze these electronic data; this provides HFTs a relative advantage to human traders in terms of speed. We conjecture that speed itself can be beneficial for HFTs, in that given the same signal, they can trade earlier than the rest of the market and thus act as a temporary monopolist in the spirit of Kyle (1985). As argued in Hasbrouck and Saar (2010), high speed may bring two possible benefits to trading: first, the ability to rebalance positions faster on the basis of inherent fundamental volatility of financial securities could result in higher utility; second, by being faster than others in responding to news or signals, traders could generate more profits.² Therefore, competing on speed might lead to an “arms race”, i.e., traders may make substantial investments to implement advanced technology and specific algorithms that ordinary traders cannot implement in order to decrease the latency of orders submission and cancellation. As a result, currently, high frequency traders had an execution time of milliseconds, Hasbrouck and Saar (2010).

We build a discrete-time infinite-horizon model to investigate the role of HFTs’ speed advantage under symmetric information among informed traders. Specifically, in our model, there are two risk-neutral informed traders who are continuously trading on an asset and a competitive market maker who sets the price to clear the market. The only difference between these two informed traders is the trading speed. The one who always trades first in any given trading period is called the HFT, while the other informed trader who always trades second is called the LFT. In the model, the insiders receive a stream of news, while the dealer cannot observe such information. The HFT trades before the LFT in each period. The market maker sets the market clearing price every time the informed traders trade. Different from the standard literature, we allow herding behavior for noise traders, i.e., the variance of their trading volume is a function of the informed traders’ trading.

Our model can be solved analytically and has a unique linear equilibrium. Both traders have a linear strategy, where their trading volume is determined by their trading intensity and

²By observing a flow of quotes, HFTs are capable of extracting information that has not yet crossed the news screen.

the difference between the fundamental value of the asset and the current market price. To characterize the role of the HFT's speed advantage relative to the LFT, we start by examining the HFT's and the LFT's order submission strategies. Intuitively, three effects arise when informed traders deliver a large order. First, if they trade intensively, they can exploit their private information, which can generate more profit. Second, a large order will reveal more information to the market maker, which facilitates price adjustments and diminishes pricing errors. Third, according to our assumption, aggressive trading might generate more noise trading, which hinders the market maker's ability to distinguish informed trading from noise trading. Apparently, the second effect is negative, and the other two effects are positive. We show that in equilibrium, the second effect is dominated by the other two effects. Therefore, the HFT trades much more aggressively than the LFT in every trading period. Note that the market maker can improve price discovery by learning from each trade. As a result, the pricing error at the beginning of each period achieves its maximum. Hence, the HFT has more incentive to trade on his signal and to submit a large order if the price substantially deviates from the true fundamental value. The LFT trades second in each trading period and thus faces a much smaller pricing error owing to price discovery. Therefore, the LFT is less inclined to trade intensively. Therefore, speed advantage helps HFT trade with a larger pricing errors before sufficient price discovery and earn more profit. As a result, the HFT trades more aggressively than the LFT because he can exploit the signal earlier and the market maker's uncertainty regarding the asset's value will decrease after the HFT's order is observed. Our result contrasts with that of Chau and Vayanos (2008) because the HFT's profit does not depend on an infinite trading volume. In addition, the equilibrium in our model is not semi-strong efficient because the price discovery in our model is not instantaneous, i.e., the difference between the value of asset can never equal the price even after infinite trading periods. Therefore, HFT makes a per-trade profit. Why does the market maker not know the asset's value even after a very long trading period? This lack of knowledge results from our assumption regarding noise herding. Intuitively, the market maker can only learn the fundamental value from the aggregate order flow because he cannot observe any individual signals. As in the literature, the aggregate order flow consists of two components: the informed traders' orders which reveal information, and the noise traders'

orders which provide camouflage and thus obscure information. If the noise traders' orders are affected by insiders' strategies, i.e., noise traders herd in the market, it will be more difficult for the market maker to differentiate informed trading from noise trading. Even if an informed trader submits a large order that should reveal substantial information regarding the fundamental value of the security, given the herding behavior of noise trading, the larger order size delivered by the noise traders prevents information diffusion. Thus, the uncertainty of the market maker toward the value of the underlying asset is always positive.

If we incorporate an order-splitting strategy by allowing the HFT to trade twice before the LFT, we find that the HFT will trade more effectively by lowering the price impact, hence, increasing market liquidity. To understand this result, we should first note that the market maker can never learn the true value of the asset. Thus, the pricing errors always exist. The HFT, who observes these pricing errors, will always want to submit an order to exploit the market maker's pricing mistakes. If he is only allowed to trade once in a period, the HFT needs to deliver a large order to generate more revenue. With respect to the three effects of a large order discussed above, the second effect can erode his profit. However, if the HFT can trade twice, he can mitigate the second effect by splitting the order and delivering smaller-sized orders in each trading event. The market maker, observing the smaller aggregate order, will learn less and will not change the price drastically. Therefore, HFTN helps traders to move down the market maker's demand curve and lower the price impact.

Our model offers a set of interesting empirical implications. First, we show that the market price can never equal the fundamental value.³ De-Long *et al.* (1990) establish a model to show that noise trading can substantially deviate the price from fundamental value. Empirically, Malkiel (1977) and Herzfeld (1980) discover that closed-end funds sell and have sold at large and substantially fluctuating discounts (De-Long *et al.* (1990)). This finding is inconsistent with efficient markets hypothesis, which states that the price of an asset should equal its fundamental value. Second, HFTs are able to earn more profit than others. Baron *et al.* (2012) use transaction level data with user identification to show that HFTN is highly

³If we allow at least one of the informed traders to trade an infinite number of times in a given period, the market maker can learn the true value of the asset despite noise herding.

profitable. They generate an unusually high average Sharpe ratio of 9.2. Additionally, a 2010 report from Barron’s estimates that Renaissance Technology’s Medallion hedge fund—a quantitative HFTN fund—achieved a 62.8% annual compound return in the three years prior to the report. Third, more frequent trading from HFTs can increase liquidity. Several papers confirm this effect. For instance, Hasbrouck and Saar (2010) and Hendershott *et al.* (2011) find that HFTs have a positive impact on market liquidity.

There is a growing theoretical and empirical literature on HFTN. Foucault *et al.* (2012) analyze HFTs’ effect on market quality by allowing them to observe signals more quickly than the market maker. They argue that liquidity will decrease if HFTs receive news earlier because their trading on news reveals more information to the market maker. Martinez and Roşu (2013) shows that high speed traders can own more revenue than low speed traders if they observe the latest information, which is inaccessible to other traders. Hendershott *et al.* (2011) shows that HFTN can improve liquidity and reduce adverse selection by using algorithmic trading as an exogenous instrument to measure its causal effect on market quality. Different from them, we build our model based on Kyle (1985), Chau and Vayanos (2008) and Back and Pedersen (1998) to examine how price evolves given the HFT’s order. We adopt a multi-period dynamic auction environment to characterize the HFT’s strategies to exploit his speed advantage. Our model contributes to the literature by showing the differing impacts of high speed and high frequency on traders’ strategies and profits. The benchmark model shows that a pure speed advantage enables the HFT to trade aggressively and earn more profit. The extension model shows that the HFT can deliver orders more effectively and, thus, avoid a large price impact, by trading twice in a period.

The rest of this paper is organized as follows. Section 2 presents the benchmark model. Section 3 describes the resulting trading strategies and market quality in the equilibrium. Section 4 presents and analyzes the equilibrium in the extension model. Section 5 concludes the paper.

2. Benchmark Model

The trading environment is an infinite-horizon dynamic model, where trading occurs in each time interval. The length of every period is $3h$, where h is an arbitrary positive constant. The risk-free rate is set to zero. There is only one risky financial asset with the fundamental value ν_i in period i , which is only observable to insiders. We assume that ν_i follows from the process given by

$$\nu_i = \nu_{i-1} + \Delta\nu_i = \nu_0 + \sum_{t=0}^i \Delta\nu_t, \quad \text{with} \quad \Delta\nu_i \sim N(0, h\sigma_\nu^2) \quad (1)$$

where ν_0 is normally distributed with mean zero and variance Σ_0^0 .

Four types of traders exist: a HFT, a LFT, noise traders and a market maker. Here we refer to the HFT as someone who has more advanced technology and more efficient automation. In other words, he has two advantages over other traders: (i) he trades faster, and (ii) he is able to trade more frequently in a given period. In the benchmark model, we only take the first advantage into account. Both the HFT and the LFT are risk-neutral. Both informed traders observe ν_0 at time 0 and, receive the same signal on $\Delta\nu_i$ at the beginning of each period i . Only the technology differs between them.⁴

In each trading period, there are two rounds of trading. Specifically, trade occurs at h , $2h$. The HFT delivers his order in the first round (at h), while LFT does so in the second (at $2h$). Each informed trader submits order x_i^s ($s = H, L$) to maximize the expected utility given by

$$U_i^s = E \left(\sum_{\tau=i}^{\infty} (\nu_\tau - p_\tau^1) x_\tau^s | \Omega_\tau^I \right) \quad (2)$$

where $\Omega_\tau^I = \{\nu_0, \Delta\nu_1, \dots, \Delta\nu_{\tau-1}, \Delta\nu_\tau\}$ is the information set available to insiders at time τ ; and p_τ^1 and p_τ^2 are the prices at which orders x_τ^H , and, x_τ^L are executed, respectively.

What distinguishes our model from previous literature as in Kyle (1985), Foster and Viswanathan (1996) is the assumption concerning noise trading. We no longer assume that

⁴HFTs may use automatic trading technology which can lower latency, but LFTs cannot access such technology. Therefore, HFTs can respond to signals more quickly and have low latency. The LFT is an ordinary trader without sophisticated trading algorithms.

the order of liquidity traders denoted by u_i is independent from other factors; rather, it is defined as follows:

Definition 1 *The order delivered by noise traders is a normal distribution with mean zero and variance $f(\sigma_u, z)$:*

$$u_i \sim N(0, f(\sigma_u, z)) \quad (3)$$

where z refers to the influence of informed traders' trading on noise trading, and $f(\cdot)$ is any real-valued function: $R^2 \rightarrow R^+$.

We can interpret σ_u as the innate trading inclination of noise traders. That is, σ_u cannot be altered by exogenous forces. Additionally, we account for the possible impact of other factors on the behavior of noise traders. In particular, we assume that insiders' trading behavior may influence noise trading. When a number of orders are delivered to buy, noise traders may choose to jump on the bandwagon. However, if a large amount of sell orders are submitted, noise traders may also choose to ride the wave. In fact, this assumption is not singular. Shleifer and Summers (1990) concludes that biased noise traders' investment strategies can be greatly influenced through sentiment. They further argue that some common signals such as expert advice and analyst forecasts, can play the role of distorting sentiment. Shiller (1984), De-Long and Shleifer (1991) and Birkchandani *et al.* (1992) posit that fad and fashion may be the key to changing that sentiment and, hence may affect investment strategy. In our model, we allow noise traders to herd on the informed trading because the informed trading might also be a signal for noise traders to extract and to trade on. Previous literature has analyzed this approach. For instance, Palomino (1996) builds an overlapping generation model to show that when the expected return of the informed traders is higher than that of noise traders, new entrants may imitate the strategies of informed traders even if they should be noise traders. Banerjee (1992) argues that traders' actions can be influenced by their predecessors' strategies, which leads to herd behavior among investors.

The market maker does not observe the fundamental value ν , but see the aggregate order flow submitted by insiders and noise traders $X_i^1 = x_i^H + u_i^1$ and $X_i^2 = x_i^L + u_i^2$, and sets a price equal to his/her expected value of ν_i conditional on the information available at each

auction. Hence, the market maker earns zero expected profit. This process can be expressed in the following formulas:

$$p_i^j = E(\nu_i | \Psi_i^j), \quad j = 1, 2$$

where $\Psi_i^1 = \{X_0^1, X_0^2, X_1^1, X_1^2, \dots, X_{i-1}^1, X_{i-1}^2, X_i^1\}$, and $\Psi_i^2 = \Psi_i^1 \cup \{X_i^2\}$. Ψ_i^1 is the market maker's information at the beginning of period i . He observes aggregate order flow prior to period i . Ψ_i^2 is the information set that is available to the market maker after the HFT's trade, which involves the aggregate order in the first trading round in this period. We also denote

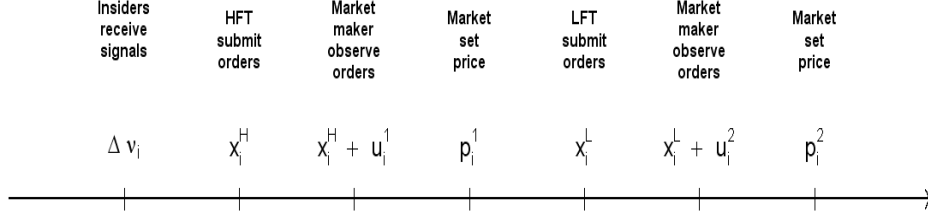
$$\Sigma_i^0 = Var\{\nu_i | \Psi_{i-1}^2\}, \quad \Sigma_i^1 = Var\{\nu_i | \Psi_i^1\}, \quad \Sigma_i^2 = Var\{\nu_i | \Psi_i^2\}$$

to be the uncertainty of the market maker toward the fundamental value ν_i under different information sets. Σ_i^0 represents the market maker's uncertainty at the beginning of period i . At this time, because he cannot observe any signals, his information set is the same as that at the end of period $i - 1$. Σ_i^1 represents the market maker's uncertainty after the first trading round in period i and Σ_i^2 represents the market maker's uncertainty at the end of period i .

The timing of events in period i is demonstrated in Figure 1. First, both insiders receive the signal $\Delta\nu_i$. Then, the HFT submits a market order x_i^H , and the noise traders deliver their order u_i^1 . Next, the market maker sets a price p_i^1 based on his information set. Second, the LFT and some informed traders submit market orders x_L^2 , and u_i^2 , respectively. Then, the market maker updates his information and sets a price p_i^2 .

Thus, we follow most of the settings as in Kyle (1985) and Chau and Vayanos (2008). Informed traders have superior information relative to the market maker, and they trade on pricing errors. The market maker updates his information and corrects the price through aggregate order flow. However, our model differs from previous models in the assumption concerning the type of insiders. We discriminate two types of insiders based on speed nuances. In other words, we allow the HFT to trade on signals first; thus, we can examine the benefit of the HFT's speed advantage.

Figure 1: **Timing of Events in Benchmark Model.** This figure shows the order of events in a period in the benchmark model. At the beginning, both insiders receive a new signal $\Delta\nu_i$ ($i = 1, 2, \dots, \infty$). Then HFT first submit his order x_i^H due to advanced technology. Market maker observes aggregate order of size $x_i^H + u_i^1$, and sets a price p_i^1 . Next, LFT delivers his order x_i^L . Market maker sees the aggregate order $x_i^L + u_i^2$, and sets a price p_i^2 .



3. Equilibrium

3.1. Equilibrium Concept

In this model, we search for an equilibrium in which the market maker uses a pricing rule that is linear in order flow:

$$p_i^1 = p_{i-1}^2 + \lambda_i^1 X_i^1, \quad p_i^2 = p_i^1 + \lambda_i^2 X_i^2$$

Additionally, we search for the linear strategies of informed traders in the following form:

$$x_i^H = \beta_i^H (\nu_i - p_{i-1}^2)h, \quad x_i^L = \beta_i^L (\nu_i - p_i^1)h \quad (4)$$

We solve for β_i^H and β_i^L to maximize the expected utility. $\nu_i - p_{i-1}^2$ and $\nu_i - p_i^1$ are deviations in the market maker's pricing in the spirit of Kyle (1985), Back *et al.* (2000), and Chau and Vayanos (2008). Foucault *et al.* (2012) calls this type of trading strategy *level trading*.

Definition 2 *If the market maker's pricing rule is linear and informed traders adopt linear strategies in an equilibrium, then this equilibrium is called a linear equilibrium.*

In the next theorem, we derive the value function of both informed traders and prove, given the linear pricing rule defined above, that insiders' strategies are also linear. We conjecture that the variance of noise trading is of the form: $Var(u_i) = \beta_i^2 \sigma_u^2 h$.

Theorem 1 Any linear equilibrium must be of the form

$$\begin{aligned} x_i^H &= \beta_i^H(\nu_i - p_{i-1}^2)h, & x_i^L &= \beta_i^L(\nu_i - p_i^1)h \\ p_i^1 &= p_{i-1}^2 + \lambda_i^1(x_i^H + u_i^1), & p_i^2 &= p_i^1 + \lambda_i^2(x_i^L + u_i^2) \end{aligned}$$

for $i = 0, \dots, \infty$, where $\beta_i^H, \beta_i^L, \lambda_i^1, \lambda_i^2, \Sigma_i^0, \Sigma_i^1, \Sigma_i^2$ are constants that satisfy

$$\lambda_i^1 = \frac{\Sigma_i^0}{\beta_i^H(\Sigma_i^0 h + \sigma_u^2)}, \quad \lambda_i^2 = \frac{\Sigma_i^1}{\beta_i^L(\Sigma_i^1 h + \sigma_u^2)} \quad (5)$$

$$\Sigma_i^0 = \Sigma_{i-1}^2 + \sigma_\nu^2 h, \quad \Sigma_i^1 = \Sigma_i^0 - \frac{(\Sigma_i^0)^2 h}{\Sigma_i^0 h + \sigma_u^2}, \quad \Sigma_i^2 = \Sigma_i^1 - \frac{(\Sigma_i^1)^2 h}{\Sigma_i^1 h + \sigma_u^2} \quad (6)$$

The Bellman value functions of informed traders are quadratic:

$$\pi_i^S = B_i^S(\nu_{i-1} - p_{i-1}^1)^2 + C_i^S(\Delta\nu_i)^2 + D_i^S, \quad S = H, L \quad (7)$$

The coefficients in the optimal trading strategies and value equations are determined by the following difference equation system:

$$\beta_i^S = \frac{-1 + 2B_{i+1}^S \lambda_i^1}{2h\lambda_i^1(-1 + B_{i+1}^S \lambda_i^1)}, \quad S = H, L \quad (8)$$

$$\begin{aligned} B_i^H &= \beta_i^H(1 - h\lambda_{i-1}^2\beta_{i-1}^L)^2(1 - h\lambda_i^1\beta_i^H)h + B_{i+1}^H(1 - h\lambda_{i-1}^2\beta_{i-1}^L)^2(1 - h\lambda_i^1\beta_i^H)^2 \\ C_i^H &= \beta_i^H(1 - h\lambda_i^1\beta_i^H)h + B_{i+1}^H(1 - h\lambda_i^1\beta_i^H)^2 \\ B_i^L &= \beta_i^L(1 - h\lambda_i^1\beta_i^H)^2(1 - h\lambda_i^2\beta_i^L)h + B_{i+1}^L(1 - h\lambda_i^1\beta_i^H)^2(1 - h\lambda_i^2\beta_i^L)^2 \\ C_i^L &= \beta_i^L(1 - h\lambda_i^1\beta_i^H)^2(1 - h\lambda_i^2\beta_i^L)h + B_{i+1}^L(1 - h\lambda_i^1\beta_i^H)^2(1 - h\lambda_i^2\beta_i^L)^2 \end{aligned} \quad (9)$$

$$\begin{aligned} D_i^H &= \beta_i^H(\beta_{i-1}^L \lambda_{i-1}^2)^2(1 - \lambda_i^1\beta_i^H)\sigma_u^2 h^2 + B_{i+1}^H(\lambda_{i-1}^2\beta_{i-1}^L)^2(1 - \lambda_i^1\beta_i^H)\sigma_u^2 h^2 \\ &\quad + B_{i+1}^H(\lambda_i^1\beta_i^H)^2\sigma_u^2 h + 3C_{i+1}^H\sigma_\nu^2 h + D_{i+1}^H \\ D_i^L &= \beta_i^L(\beta_i^H \lambda_i^1)^2(1 - \lambda_i^2\beta_i^L)\sigma_u^2 h^2 + B_{i+1}^L(\lambda_i^1\beta_i^H)^2(1 - \lambda_i^2\beta_i^L)\sigma_u^2 h^2 \\ &\quad + B_{i+1}^L(\lambda_i^2\beta_i^L)^2\sigma_u^2 h + 3C_{i+1}^L\sigma_\nu^2 h + D_{i+1}^L \end{aligned}$$

The terminal conditions are:

$\lim_{i \rightarrow \infty} B_i, \lim_{i \rightarrow \infty} C_i, \lim_{i \rightarrow \infty} D_i$ are constants for both insiders.

The second order condition is

$$\lambda_i^1(1 - B_{i+1}^H \lambda_i^1) > 0, \quad \lambda_i^2(1 - B_{i+1}^L \lambda_i^2) > 0 \quad (10)$$

Proof: See Appendix.

In the following analysis, for simplicity, if ignoring the superscript of $\beta_i, \pi_i, x_i, B_i, C_i, D_i$, we indicate that both informed traders should adopt the same form of strategy. The HFT's and LFT's trading strategies are of the same form because, in their trading round, they act as a monopolistic informed trader. Additionally, both informed traders have superior information relative to the market maker, exploiting pricing loopholes $\nu_i - p_i^1$ and $\nu_i - p_i^2$ is profitable. Our result is consistent with Kyle (1985) and Chau and Vayanos (2008) in that given the market maker's belief, both informed traders take full advantage of pricing deviations. Moreover, note that trading intensity is negatively correlated with the price impact. Intuitively, given a large λ , a large order can induce an adverse price impact and reveal more information, and both insiders will be inclined to trade slowly but more frequently— “going down” the market maker's demand curve (Chau and Vayanos (2008)). However, because of the difference in speed, the LFT's level trading is different from that of the HFT because the market maker will update his information after the HFT's order. Thus, the LFT is less able to take advantage of pricing errors if the HFT's trading reveals a great amount of information in the trade. Another remarkable characteristic in this theorem is the evolution of the market maker's uncertainty, which has no direct connection with trading intensity β . The trading intensity has two effects on market maker's uncertainty. Firstly, the information learned by market maker from the first order in each period is $Cov(\nu_i - p_{i-1}^2, X_i^1) = \beta_i^H \Sigma_i^0 h$. Therefore, larger trading intensity increases the informativeness of the order. Secondly, the noise incorporated in the first order is $Var X_i^1 = (\beta_i^H)^2 (\Sigma_i^0 h + \sigma_u^2)$. So, a larger trading intensity increases the noise in each order. If $\beta_i^H = 1$, the noise in each order will equal to that in Chau and Vayanos (2008). If $\beta_i^H > 1$, the noise will be greater. These two effects offset each other. Therefore, when informed traders increase their trading intensity, noise traders herd on the informed traders' trading. The price discovery from aggressive insiders' trading is

offset by the camouflage effect from noise trading. In contrast, in Chau and Vayanos (2008), if h does not converge to zero, trading intensity β directly influences the market maker's uncertainty.⁵

3.2. Existence and Uniqueness

From Theorem 1, the following lemma can easily be obtained:

Lemma 1 *There exists a unique trading strategy for each insider; their trading intensity is determined by the endogenous parameters B_{i+1}^H , B_{i+1}^L and the market maker's uncertainty over the fundamental value, respectively,*

$$\beta_i^H = \frac{2B_{i+1}^H \sigma_u^2 \Sigma_i^0}{\sigma_u^4 - h^2 (\Sigma_i^0)^2}, \quad \beta_i^L = \frac{2B_{i+1}^L \sigma_u^2 \Sigma_i^1}{\sigma_u^4 - h^2 (\Sigma_i^1)^2} \quad (11)$$

Proof: See Appendix.

Intuitively, each informed trader maximizes his present value of aggregate future earnings given the current information. The HFT does not need to concern himself with the LFT because in the next period, they will both trade on the new signal. In other words, his current trading decision only influences the market maker's price impact and thus his profit in the current period, rather than his profit in the next period. Regarding the LFT, given the speed disadvantage, he can do no better than make full use of his private information, which has not been revealed.

Corollary 1 *Given initial value of the market maker's uncertainty over the fundamental value, the difference equation system for estimating variance can be solved analytically:*

$$\Sigma_0^0 = R \quad (12)$$

$$\Sigma_i^0 = \frac{f(i, h, R, \sigma_u, \sigma_\nu)}{g(i, h, R, \sigma_u, \sigma_\nu)} \quad (13)$$

$$\Sigma_i^1 = \frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^2 + h \Sigma_i^0}, \quad \Sigma_i^2 = \frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^2 + 2h \Sigma_i^0}$$

⁵In their paper,

$$(\Sigma_g^2 - \sigma_g^2 h) [\Sigma_g^2 (\beta^2 \sigma_D^2 + \nu^2 \sigma_u^2 h^2) + \sigma_D^2 \sigma_u^2 h] - (1 - kh)^2 \Sigma_g^2 \sigma_D^2 \sigma_u^2 h = 0$$

where $f(\cdot)$, $g(\cdot)$ are two functions of i , h , R , σ_u , σ_ν .

$$f(\cdot) = h^3 R \sigma_\nu^2 k_1^i + h^2 \sigma_u^2 \sigma_\nu^2 k_1^i - h^3 R \sigma_\nu^2 k_2^i - h^2 \sigma_u^2 \sigma_\nu^2 k_2^i + R k_3 k_1^i + R k_3 k_2^i \quad (14)$$

$$g(\cdot) = 2h^2 R k_1^i - h^3 \sigma_\nu^2 k_1^i - 2h^2 R k_2^i + h^3 \sigma_\nu^2 k_2^i + k_3 k_1^i + k_3 k_2^i \quad (15)$$

$$k_1 = \frac{-h\sigma_u^2 - h^3\sigma_\nu^2 - \sqrt{h^4\sigma_\nu^2(2\sigma_u^2 + h^2\sigma_\nu^2)}}{\sigma_u^4} \quad (16)$$

$$k_2 = \frac{-h\sigma_u^2 - h^3\sigma_\nu^2 + \sqrt{h^4\sigma_\nu^2(2\sigma_u^2 + h^2\sigma_\nu^2)}}{\sigma_u^4} \quad (17)$$

$$k_3 = \sqrt{h^4\sigma_\nu^2(2\sigma_u^2 + h^2\sigma_\nu^2)} \quad (18)$$

Proof: See Appendix.

Lemma 2 *There exists a uniformly asymptotically stable equilibrium for Σ_i^0 as well as Σ_i^1 and Σ_i^2 .*

Proof: See Appendix.

Based on Theorem 1, Corollary 1, Lemma 1, and Lemma 2, the following theorem can be derived directly.

Theorem 2 *The linear equilibrium is uniquely determined by the difference equation system defined in Theorem 1.*

Proof: The proof of this theorem is quite straightforward. We can find that, in fact, the product of β_i^H and λ_i^1 , β_i^L , and λ_i^2 is only correlated with forecast uncertainty.⁶ Given the result in Theorem 1 and the terminal conditions, substitute the expression Σ_i^0 in Corollary 1 into the difference equations for the six value function coefficients. Note that the difference equations for B_i , C_i , and D_i are simply first-order linear.⁷ Thus, after calculating the value of uncertainty, we can solve their explicit expression through backward induction. Then, combining these results into the trading intensity and price impact equations will complete the proof.

Q.E.D

⁶ $\beta_i^H \lambda_i^1 = \frac{\Sigma_i^0}{h\Sigma_i^0 + \sigma_u^2}$, $\beta_i^L \lambda_i^2 = \frac{\Sigma_i^1}{h\Sigma_i^1 + \sigma_u^2}$.

⁷ We can easily derive that $B_i^H = \frac{B_{i+1}^H(\sigma_u^2 + h(h\sigma_\nu^2 - \Sigma_i^0))^2}{\sigma_u^4 - (h\Sigma_i^0)^2}$ and $B_i^L = \frac{B_{i+1}^L \sigma_u^2}{\sigma_u^2 + 2h\Sigma_i^0} = C_i^L$. Substituting these expressions into the difference equations for D_i^H and D_i^L , equations (13) and (14) will be transformed into first-order difference equations.

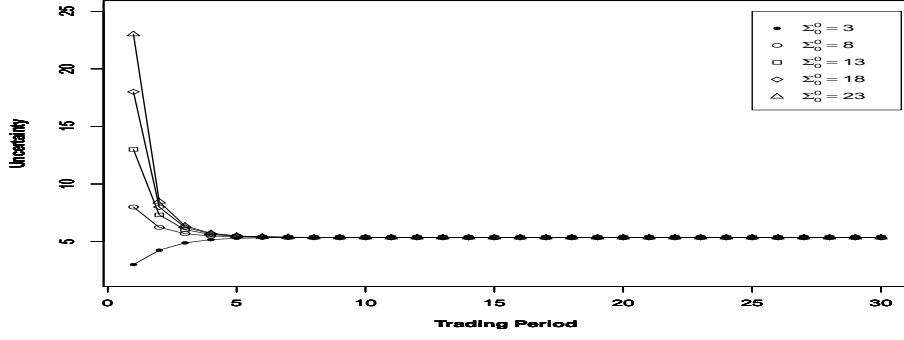


Figure 2: **Evolution of Market Maker's Uncertainty Towards ν_i .** The figure plots market maker's uncertainty (Σ_i^0) towards ν_i under information set Ψ_{i-1}^2 against horizon $i \in [1, 30]$ in the benchmark model. Five different dotted lines show the evolution of Σ_i^0 when given distinct initial uncertainty Σ_0^0 . The parameters used are $\sigma_u = 3, \sigma_\nu = 2, h = 0.5$.

4. Properties of Equilibrium

In this section, we investigate the properties of the equilibrium, by starting with the insiders' trading behavior. Then, we address the role of the HFT's speed advantage. Next, we analyze the impact of the HFTN on the market qualities. Finally, we investigate the influence of noise trading on the micro-market structure and how informed traders respond to the behavior of noise traders.

4.1. Market Maker's Uncertainty

Proposition 1 *As the trading period goes to infinity, the market maker's uncertainty towards the fundamental value converges to a positive constant.*

$$\lim_{i \rightarrow \infty} \Sigma_i^0 = \frac{h\sigma_\nu^2 + \sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4}}{2} \quad (19)$$

$$\lim_{i \rightarrow \infty} \Sigma_i^1 = \frac{\sigma_u^2(h\sigma_\nu^2 + \sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4})}{2\sigma_u^2 + h^2\sigma_\nu^2 + h\sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4}} \quad (20)$$

$$\lim_{i \rightarrow \infty} \Sigma_i^2 = \frac{\sigma_u^2(h\sigma_\nu^2 + \sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4})}{2\sigma_u^2 + h^2\sigma_\nu^2 + 2h\sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4}} \quad (21)$$

Proof: See Appendix.

From the formula for insiders' trading strategies, we can see that the HFT and LFT can use their information in two ways. First, they can trade intensively, and thus make full use

of their information. However, this strategy will rapidly reveal their information, and the market maker may set a high price, preventing such an adverse selection problem. Second, insiders can choose a relatively small trading intensity to avoid information leakage and a larger price impact. This strategy may not generate as large a profit in the early stage as the former strategy but it can generate a larger profit in the future owing to the slow speed of information revelation.

As demonstrated in the above formulas and in Figure 2, in our model, the market maker's uncertainty regarding the fundamental value of the asset, ν_i , decreases dramatically during the first several trading periods if it is initially very large. In Figure 2, after nearly ten trading periods, Σ_i^0 almost converges to a constant level, which is not zero. However, regardless of how the market maker initially learns about the value of the asset, he can never know its fundamental value. The first trade in each period reduces uncertainty by $\frac{(\Sigma_i^0)^2 h}{\Sigma_i^0 h + \sigma_u^2}$. If initial uncertainty is small, the informativeness of the first trade is small because $\frac{\partial \frac{(\Sigma_i^0)^2 h}{\Sigma_i^0 h + \sigma_u^2}}{\partial \Sigma_i^0} = \frac{h \Sigma_i^0 (h \Sigma_i^0 + 2 \sigma_u^2 h)}{(h \Sigma_i^0 + \sigma_u^2)^2} > 0$. It can be shown that if initial uncertainty is smaller than $\frac{h \sigma_\nu^2 + \sqrt{2 \sigma_u^2 \sigma_\nu^2 + h^2 \sigma_\nu^4}}{2}$, the decline in uncertainty is smaller than $\nu_i^2 h$, leading to a larger Σ_{i+1}^0 . Because the market maker cannot distinguish informed trading from noise trading, the noise traders' orders can camouflage the informed traders' strategies. However, in the long run, as long as the noise trading is independent, the market maker can eventually learn the fundamental value. In our model, the camouflage effect of noise trading is intensified by noise herding. Because noise trading herds on the informed traders' trading, the market maker has no way to solve this identification problem. He has only one instrument, i.e., the aggregate order flow. Therefore, the market maker has no way to "separate" or "extract" informed trading component from the aggregate order flow.

This proposition is in sharp contrast to Chau and Vayanos (2008), Li (2012), and Holden and Subrahmanyam (1992). In Chau and Vayanos (2008), information is revealed instantaneously. Owing to their assumption regarding the constancy of the price impact, insiders decide to trade an infinite volume because doing so will not affect pricing rules but can generate larger profit. The non-cooperative competition in Holden and Subrahmanyam (1992) forces all insiders to trade aggressively on their signals to squeeze revenue. This type of Cournot

game leads to the immediate revelation of private information. Thus, our assumption on noise herding is crucial to prevent the market from being semi-strong efficient. Additionally, the convergent value positively correlates to $\sigma_{nu_i}^2$ and σ_u^2 , which contrasts that in Chau and Vayanos (2008). This can be seen from the uncertainty reduction in each trade. If h goes to zero, the amount of uncertainty decline is $\frac{1}{\sigma_u^2}$. Therefore, intensive noise trading prevents market maker's learning. However, if we do not allow noise herding, and let h goes to zero, according to Chau and Vayanos (2008), $\beta_i h$ will go to infinity. So noise term σ_u^2 can be ignored.

Proposition 2 *Market maker's uncertainty increases in σ_ν , σ_u and Σ_0^0 .*

Proof: See Appendix.

This proposition is quite straightforward. An increase in σ_ν represents a signal containing more information. Then, more time is required for the HFT and LFT to reveal their private information. Therefore, it is more difficult for the market maker to incorporate most of the private information into the price. Second, when σ_u increases, the market maker will be more uncertain regarding the fundamental value because he cannot distinguish noise traders from insiders. Additionally, by introducing more noise into the market, insiders' strategies will be camouflaged. As a result, the price revelation process is encumbered, and information asymmetry among informed traders and the market maker is enhanced. If Σ_0^0 is high, then a severe disparity exists between the market maker's information set and that of insiders at the beginning period. Although the HFT and LFT may choose to trade intensively, and thus reveal information at a faster pace, they still leave a considerable amount of information unrevealed to the market maker. In summary, the effect of an increase in uncertainty dominates that of information leakage. Figure 2 clearly depicts the evolution path of Σ_i^0 under the circumstance of different Σ_0^0 . Note that although the speed with which information is revealed increases as Σ_0^0 increases, uncertainty is still much greater with a larger Σ_0^0 than with a smaller Σ_0^0 .

Definition 3 *We define the speed with which uncertainty declines in each trading round as*

the uncertainty decreasing ratio in each round

$$velocity_i^1 = \frac{\Sigma_i^0 - \Sigma_i^1}{\Sigma_i^0} \quad (22)$$

$$velocity_i^2 = \frac{\Sigma_i^1 - \Sigma_i^2}{\Sigma_i^1} \quad (23)$$

Proposition 3 *velocity_i¹ > velocity_i², i.e., the market maker's uncertainty level diminishes much faster when the HFT trades. A larger Σ_i^0 and smaller σ_u can increase velocity.*

Proof: See Appendix.

The proposition indicates that the HFT reveals more information through his trading. To generate more revenue, the HFT must exploit his information advantage. Therefore, he may trade aggressively to take the edge off of pricing errors. However, his large order will reveal more of his private information to the market maker. Thus, the market maker can learn this private information very quickly. However, when the LFT trades, the uncertainty level has reduced greatly; thus the LFT has less incentive than the HFT to deliver a large order. Although the noise trading also becomes milder in this situation, the aggregate effect decreases $velocity_i^2$. Similar to this scenario, if the market maker's initial uncertainty Σ_i^0 increases, then both the HFT and the LFT will be inclined to trade more, and therefore, pricing errors will be corrected. In such a case, even though noise trading can be more severe, but the dominant effect is the former one, velocity will increase drastically. If σ_u decreases, the market maker can learn more from the aggregate order, and the informed traders' order will reveal more private information. Therefore, velocity will increase.

4.2. Profitability

Proposition 4 *The HFT trades more aggressively than the LFT in each period if informed traders cannot trade in the opposite direction with signals ($\beta_i^H > 0, \beta_i^L > 0$). In other words, $\beta_i^H > \beta_i^L$.*

Proof: See Appendix.

The proposition indicates that the HFT is more inclined to submit a large order. The intuition is that, at the first trading round of each period, the market maker's uncertainty and

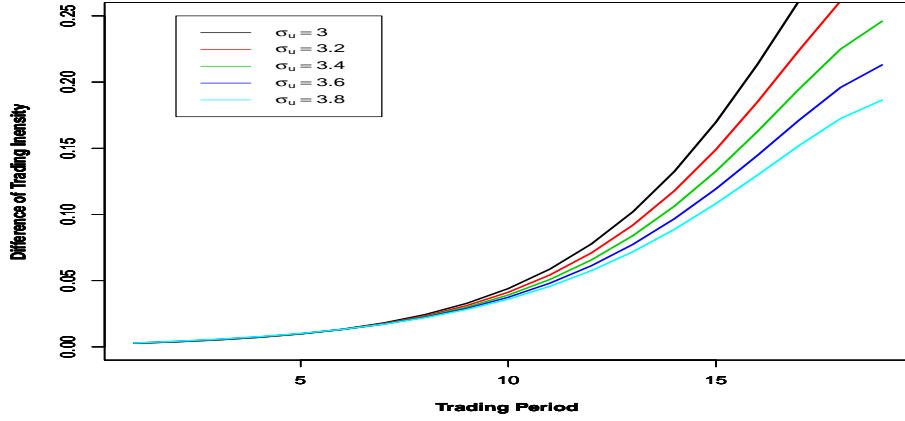


Figure 3: **Difference of Trading Intensity.** This figure depicts the difference of trading intensity between two informed traders ($\beta_i^H - \beta_i^L$) from $i = 1$ to $i = 19$ under the condition of distinct σ_u . The parameters used are $\Sigma_0^0 = 5, \sigma_\nu = 2, h = 0.5, B_{20} = 1$.

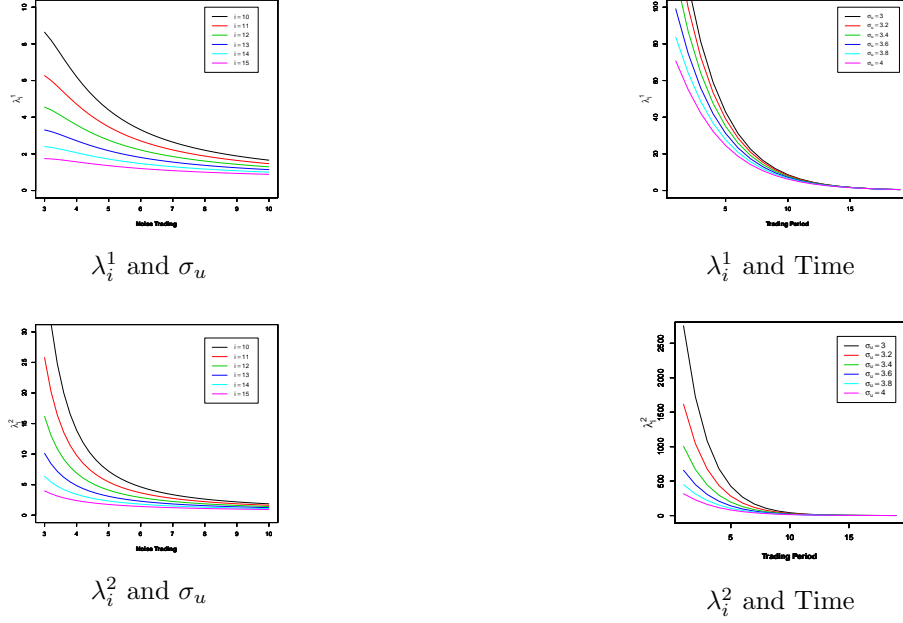
price deviation are greater. So both B_i^H and Σ_i^0 can be greater. The HFT chooses a larger β_i^H and an order of larger size. This results in a huge decline in market maker's uncertainty level. Then, when the LFT delivers his order, the price will be closer to the fundamental value. If the LFT trades as aggressively as the HFT, the market maker may increase the price impact to prevent adverse selection. Therefore, the LFT prefers to trade less aggressively. Note that Figure 3 shows that when σ_u increases, the HFT's speed advantage gradually vanishes. This result can be explained as follows: σ_u is the factor that hinders the market maker's learning ν_i . Therefore, a rising σ_u presages a low pace of information leakage. The LFT, who was struggling previously because of the speed with which information was revealed, is now more comfortable. The market maker learns little from the first round of trading, and the price deviation may still be huge. Hence, the LFT will choose a more aggressive strategy, and thus, the difference between the two insiders will diminish.

Proposition 5 *The aggregate expected profit of the HFT dominates that of the LFT.*

Proof See Appendix.

The profit of insiders comes from three parts. The first is from market maker's pricing errors in previous period, represented by B_i . The second is from the new signal $\delta\nu_i$. The third source is from term D_i which can stand for other exogenous factors. Clearly, the ability to own profit is determined by the three coefficients: B_i, C_i, D_i . Their magnitude is determined

Figure 4: Evolution of Price Impact. This figure plots two inverse measure of market depth, λ_i^1 and λ_i^2 against time and noise trading intensity σ_u . Two graphs above show the paths of λ_i^1 along σ_u and time respectively, while graphs below tracks the evolution of λ_i^2 against σ_u and trading period respectively. The parameters used are $\Sigma_0^0 = 5$, $\sigma_\nu = 2$, $h = 0.5$, $B_{20} = 1$.



by market maker's uncertainty. Larger uncertainty level increases the magnitude of all of them. Therefore, the HFT who always faces a market maker with greater uncertainty level can generate more profit.

4.3. Liquidity

Proposition 6 *Price impact decreases in σ_u .*

Proof: See Appendix.

Figure 4, clearly shows that, both λ_i^1 and λ_i^2 decline along with σ_u . The intuition for this result is that an increasing σ_u implies more noise, so the market maker learns less from the aggregate order. On the condition that the order reveals little information about the underlying asset, the market maker should not change the price substantially, so the price impact decreases, but the market liquidity increases dramatically. Next, from the graphs on the right, we can note that the price impact diminishes with time. This result is quite straightforward because increasingly more information is revealed as the HFT and LFT

continue their trading. The gap between the insiders and the market maker toward the fundamental value is narrowed, and most information has been incorporated into price. As a result, the price is close to ν_i , and there is no need to dramatically change the price. Furthermore, what is interesting in Figure 4 is that λ_i^1 is almost always smaller than λ_i^2 . This result is due to noise trading. Because the variance of noise trading is proportional to the insiders' trading intensity β_i , aggressive trading can result in more serious noise trading despite its role as a catalyst for information revelation. Notice that the slope in the above two graphs are much flatter, indicating that intensive trading by the HFT can greatly decrease liquidity. These two effects together generate a small price impact in the first trading round and a larger price impact in the second round.

5. Extension Model

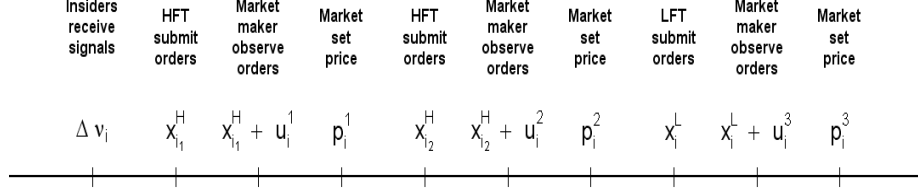
In this section, we extend the benchmark model by allowing the HFT the option to trade twice before the LFT. The transaction cost is zero. We want to analyze the benefit brought to the HFT by the second advantage of automated trading. In this extension model, the set up is the same as that for the benchmark model, except that the HFT is allowed to trade two times before the LFT trades. The sequence of events is as follows. At $i = 0$, informed traders observe ν_0 . At each trading period, insiders observe $\Delta\nu_i = \nu_i - \nu_{i-1}$. The HFT first submits the order x_{i1}^H and the noise traders submit u_i^1 . The market maker only observes the aggregate order flow, $X_i^1 = x_{i1}^H + u_i^1$, and sets the price p_i^1 equal to his expectation for the fundamental value ν_i . Then, the HFT delivers another order x_{i2}^H and the noise traders submit in aggregate u_i^2 . The market maker observes the aggregate order flow and sets a price p_i^2 . Finally, the LFT submits his order x_i^L , and the market maker sets a price p_i^3 at which trading takes place. Figure 5 displays a detailed order of events.

5.1. Equilibrium

Because the market maker is competitive, he generates zero profit, implying that

$$p_i^1 = E(\nu_i | \Psi_i^1), \quad \Psi_i^1 = \{X_0^1, X_0^2, X_0^3, X_1^1, \dots, X_{i-1}^1, X_{i-1}^2, X_{i-1}^3, X_i^1\}$$

Figure 5: Timing of Events in Extension Model. This figure shows the order of events in a period in the extension model. At the beginning, both insiders receive a new signal $\Delta\nu_i$ ($i = 0, 1, 2, \dots, \infty$). Then HFT first submits his order x_{i1}^H due to advanced technology. Market maker observes aggregate order of size $X_i^1 = x_{i1}^H + u_i^1$, and sets a price p_i^1 . Next, HFT submits his another order x_{i2}^H . Market maker observes aggregate order of size $X_i^2 = x_{i2}^H + u_i^2$, and sets a price p_i^2 . Later, LFT delivers his order x_i^L . Market maker observes the aggregate order $x_i^L + u_i^3$, and sets a price p_i^3 .



$$\begin{aligned}
 p_i^2 &= E(\nu_i | \Psi_i^2), & \Psi_i^1 &= \{X_0^1, X_0^2, X_0^3, X_1^1, \dots, X_{i-1}^1, X_{i-1}^2, X_{i-1}^3, X_i^1, X_i^2\} \\
 p_i^3 &= E(\nu_i | \Psi_i^3), & \Psi_i^2 &= \{X_0^1, X_0^2, X_0^3, X_1^1, \dots, X_{i-1}^1, X_{i-1}^2, X_{i-1}^3, X_i^1, X_i^2, X_i^3\}
 \end{aligned}$$

Denote the market maker's uncertainty toward ν_i by

$$\Sigma_i^0 = \text{Var}(\nu_i | \Psi_{i-1}^3), \quad \Sigma_i^j = \text{Var}(\nu_i | \Psi_i^j), \quad j = 1, 2, 3$$

The next theorem shows that if the pricing rule is linear, the informed trader's strategy is also linear.

Theorem 3 *Any linear equilibrium must be of the form*

$$\begin{aligned}
 x_{i1}^H &= \beta_{i1}^H(\nu_i - p_{i-1}^3), & x_{i2}^H &= \beta_{i2}^H(\nu_i - p_i^1), & x_i^L &= \beta_i^L(\nu_i - p_i^2) \\
 p_i^1 &= p_{i-1}^3 + \lambda_i^1(x_{i1}^H + u_i^1), & p_i^2 &= p_i^1 + \lambda_i^2(x_{i2}^H + u_i^2), & p_i^3 &= p_i^2 + \lambda_i^3(x_i^L + u_i^3)
 \end{aligned}$$

for $i = 0, 1, 2, \dots, \infty$, and $\beta_{i1}^H, \beta_{i2}^H, \beta_i^L, \lambda_i^1, \lambda_i^2, \lambda_i^3, \Sigma_i^0, \Sigma_i^1, \Sigma_i^2, \Sigma_i^3$ satisfy

$$\begin{aligned}
 \lambda_i^1 &= \frac{\Sigma_i^0}{\beta_{i1}^H(\sigma_u^2 + h\Sigma_i^0)}, & \lambda_i^2 &= \frac{\Sigma_i^1}{\beta_{i2}^H(\sigma_u^2 + h\Sigma_i^1)}, & \lambda_i^3 &= \frac{\Sigma_i^2}{\beta_i^L(\sigma_u^2 + h\Sigma_i^2)} \\
 \Sigma_i^0 &= \Sigma_{i-1}^3 + h\sigma_\nu^2, & \Sigma_i^1 &= \Sigma_i^0 - \frac{(\Sigma_i^0)^2 h}{\Sigma_i^0 h + \sigma_u^2}
 \end{aligned}$$

$$\Sigma_i^2 = \Sigma_i^1 - \frac{(\Sigma_i^1)^2 h}{\Sigma_i^1 h + \sigma_u^2}, \quad \Sigma_i^3 = \Sigma_i^2 - \frac{(\Sigma_i^2)^2 h}{\Sigma_i^2 h + \sigma_u^2}$$

The value function of insiders are quadratic for all i :

$$\pi_i^H = B_i^H(\nu_{i-1} - p_{i-1}^2)^2 + C_i^H(\Delta\nu_i)^2 + D_i^H$$

$$\pi_i^L = B_i^L(\nu_{i-1} - p_{i-1}^3)^2 + C_i^L(\Delta\nu_i)^2 + D_i^L$$

The coefficients of the optimal trading strategy and value function satisfy

$$\beta_{i_1}^H = \frac{\lambda_i^1 + 2\lambda_i^2(-1 + B_{i+1}^H\lambda_i^2)}{h\lambda_i^1(\lambda_i^1 + 4\lambda_i^2(-1 + B_{i+1}^H\lambda_i^2))} \quad (24)$$

$$\beta_{i_2}^H = \frac{-1 + 2B_{i+1}^H\lambda_i^2}{2h\lambda_i^2(-1 + B_{i+1}^H\lambda_i^2)} \quad (25)$$

$$\beta_i^L = \frac{-1 + 2B_{i+1}^L\lambda_i^3}{2h\lambda_i^3(-1 + B_{i+1}^L\lambda_i^3)} \quad (26)$$

$$B_i^H = h\beta_{i_1}^H k_0^2 k_1 + h\beta_{i_2}^H k_0^2 k_1^2 k_2 + B_{i+1}^H k_0^2 k_1^2 k_2^2$$

$$C_i^H = h\beta_{i_1}^H k_1 + h\beta_{i_2}^H k_1^2 k_2^2 + B_{i+1}^H k_1^2 k_2^2$$

$$D_i^H = D_{i+1}^H + f(\cdot)$$

$$B_i^L = k_1^2 k_2^2 k_3 (h\beta_i^L + B_{i+1}^L k_3)$$

$$C_i^L = k_1^2 k_2^2 k_3 (h\beta_i^L + B_{i+1}^L k_3)$$

$$D_i^L = D_{i+1}^L + g(\cdot)$$

where

$$k_0 = 1 - h\beta_{i-1}^L \lambda_{i-1}^3, \quad k_1 = 1 - h\beta_{i_1}^H \lambda_i^1$$

$$k_2 = 1 - h\beta_{i_2}^H \lambda_i^2, \quad k_3 = 1 - h\beta_i^L \lambda_i^3$$

$f(\cdot)$ and $g(\cdot)$ are functions of the endogenous variables.

Proof: See Appendix.

Similar to Theorem 1, this theorem indicates that the insiders trade on the market maker's pricing errors. Note that the expressions for $\beta_{i_2}^H$ and β_i^L in this theorem are similar to those

in Theorem 1. These two coefficients imply that, both the HFT's second trade and the LFT's trade only need to fully take advantage of their own private information. In other words, they care little about others' trading strategies. However, from the formula for β_{i1}^H , we can easily find that λ_i^2 influences the choice of the size for β_{i1}^H . To exploit his extra information, the HFT must allocate his order optimally in two trade opportunities. His first order size affects the second order trade and price. Therefore, to avoid a huge price change in the second trading round, the HFT needs to take λ_i^2 (price impact) into account. This theorem essentially shows that the HFT's trading direction is consistent with the price change, which is in line with empirical evidence. Brogaard *et al.* (2013) finds that, overall, HFTs facilitate price efficiency by trading in the direction of permanent price changes and in the opposite direction of transitory pricing errors, both on average and on the highest volatility days.

From Theorem 3, the following lemma can be easily obtained

Lemma 3 *There exists a unique trading strategy for each insider; their trading intensity β_i is determined by B_i and market maker's uncertainty regarding ν_i*

$$\beta_{i1}^H = \frac{2B_{i+1}^H \Sigma_i^0 (\sigma_u^2 + h \Sigma_i^0)}{(\sigma_u^2 - h \Sigma_i^0)(\sigma_u^2 + 2h \Sigma_i^0)} \quad (27)$$

$$\beta_{i2}^H = \frac{2B_{i+1}^H \sigma_u^2 \Sigma_i^1}{\sigma_u^4 - h^2 (\Sigma_i^1)^2} \quad (28)$$

$$\beta_i^L = \frac{2B_{i+1}^L \sigma_u^2 \Sigma_i^2}{\sigma_u^4 - h^2 (\Sigma_i^2)^2} \quad (29)$$

Proof: See Appendix.

Similarly, this lemma indicates that the HFT's trading intensity in the second trading round β_{i2}^H is of the same form as the LFT's trading intensity β_i^L . However, because the HFT needs to account for the first trade's impact on the second trade, the form of β_{i1}^H differs from that of β_{i2}^H and β_i^L .

Solving the difference equation for Σ_i^0 in Theorem 3, we obtain

Corollary 2 *Given the initial value of the market maker's uncertainty regarding fundamental value, the difference equation system for estimating variance can be solved analytically:*

$$\Sigma_0^0 = R$$

$$\Sigma_i^0 = \frac{f(i, h, R, \sigma_u, \sigma_\nu)}{g(i, h, R, \sigma_u, \sigma_\nu)} \quad (30)$$

where $f(\cdot)$ and $g(\cdot)$ are two functions of $i, h, R, \sigma_u, \sigma_\nu$.

Lemma 4 *There exists an asymptotically stable equilibrium for Σ_i^0 , Σ_i^1 , Σ_i^2 and Σ_i^3 .*

The proof for this lemma is similar to that for Lemma 2. Similarly, we can again derive the following theorem to ensure the uniqueness of the equilibrium.

Theorem 4 *The linear equilibrium is uniquely determined by the difference equation system defined in Theorem 3.*

The key factor leading to this equilibrium is that the informed traders' trading intensity β_i does not directly affect the market maker's uncertainty level. This lack of a direct effect can be seen from the evolution functions of Σ_i^0 . Intuitively, aggressive trading can lead to more noise trading owing to herd behavior. Therefore, the increasing noise camouflages the extra information revealed by the large order. In this case, regardless of how many times an HFT chooses to trade and what order size he decides to deliver, the market maker's uncertainty toward ν_i evolves in the same pattern.

5.2. Properties of Equilibrium

In this subsection, we discuss the impact of one more trade by the HFT on the market maker's uncertainty, the informed traders' strategies, and the market quality. Additionally, we investigate whether the HFT can generate more revenue through more frequent trading.

5.2.1. Market Maker's Uncertainty

Proposition 7 *As trading period goes to ∞ , the market maker's uncertainty regarding the fundamental value converges to a positive constant.*

$$\lim_{i \rightarrow \infty} \Sigma_i^0 = \frac{3h\sigma_\nu^2 + \sqrt{3}\sqrt{4\sigma_u^2\sigma_\nu^2 + 3h^2\sigma_\nu^4}}{6} \quad (31)$$

$$\lim_{i \rightarrow \infty} \Sigma_i^1 = \frac{\sigma_u^2(3h\sigma_\nu^2 + \sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4})}{6\sigma_u^2 + 3h^2\sigma_\nu^2 + h\sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4}} \quad (32)$$

$$\lim_{i \rightarrow \infty} \Sigma_i^2 = \frac{\sigma_u^2(3h\sigma_\nu^2 + \sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4})}{6\sigma_u^2 + 6h^2\sigma_\nu^2 + 2h\sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4}} \quad (33)$$

$$\lim_{i \rightarrow \infty} \Sigma_i^3 = \frac{\sigma_u^2(3h\sigma_\nu^2 + \sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4})}{6\sigma_u^2 + 9h^2\sigma_\nu^2 + 3h\sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4}} \quad (34)$$

This proposition implies a result similar to that of Proposition 1. Comparing $\lim_{i \rightarrow \infty} \Sigma_i^0$ in this extension model with that in the benchmark model, we note that the market maker's uncertainty converges to a smaller value in this proposition. Explicitly,

$$\begin{aligned} & \frac{3h\sigma_\nu^2 + \sqrt{3}\sqrt{4\sigma_u^2\sigma_\nu^2 + 3h^2\sigma_\nu^4}}{6} - \frac{h\sigma_\nu^2 + \sqrt{\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4}}{2} \\ &= \frac{-3\sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4} + \sqrt{12\sigma_u^2\sigma_\nu^2 + 9h^2\sigma_\nu^4}}{6} < 0 \end{aligned}$$

Each trade made by the informed trader will leak some information to the market maker. Therefore, when the HFT is allowed to trade more times in a given period, the uncertainty level of the market maker will decline. Note that, if we let the HFT trade infinitely, then the market maker's initial uncertainty Σ_i^0 becomes $h\sigma_\nu^2$, and the market maker will eventually know ν_i in each period because $\lim_{k \rightarrow \infty} \Sigma_i^k = \lim_{k \rightarrow \infty} \frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^2 + kh \Sigma_i^0} = 0$. Hence, even if noise trading shows herd behavior, when there are infinite auctions in a given period, our result is similar to previous literature such as Kyle (1985) and Holden and Subrahmanyam (1992), where uncertainty converges to zero.

Proposition 8 *The market maker's uncertainty increases in σ_u, σ_ν and Σ_0^0 . And the influence of σ_u, σ_ν on $\lim_{i \rightarrow \infty} \Sigma_i$ in the extension model is smaller than that in the benchmark one.*

Proof: See Appendix.

This proposition indicates that when the HFT trades more frequently, the impact of noise on the market maker's estimating variance declines. The result occurs because one more order delivered by informed traders in a period will surely endow the market maker with more information on the asset. In that case, even though an increase in σ_u may hinder information revelation, its effect is diminished by the HFT's trading. Similarly, as news containing more information arrives in each period, the HFT's frequent trading will

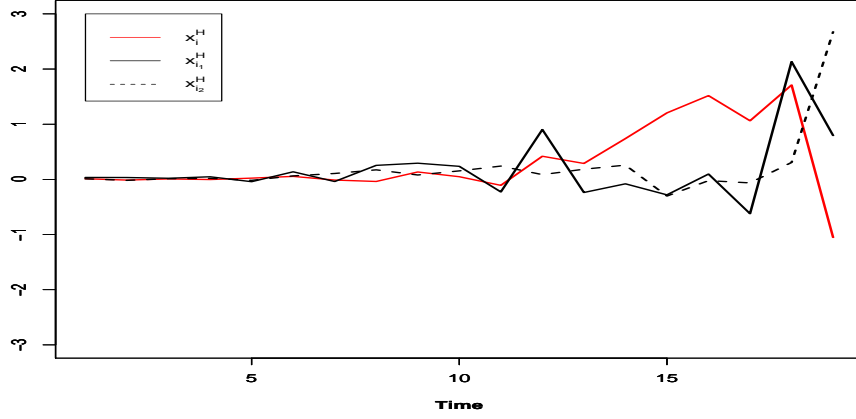


Figure 6: **Trading Volume of HFT in Two Models.** This figure plots the trading volume traded by HFT from period 1 to 19 in two models. The parameters used are $\Sigma_0^0 = 5, \sigma_\nu = 2, h = 0.5, \sigma_u = 3$.

incorporate more his private information into his order. Therefore, the market maker's uncertainty in the extension model is less influenced by σ_u and σ_ν .

5.2.2. Profitability

Proposition 9 *The HFT trades more aggressively in his first trade than in his second one in each period. In other words, $\beta_{i_1}^H > \beta_{i_2}^H$.*

Proof. According to Theorem 3 and Lemma 3, it is easy to obtain the ratio of $\beta_{i_1}^H$ to $\beta_{i_2}^H$:

$$\frac{\beta_{i_1}^H}{\beta_{i_2}^H} = \frac{\sigma_u^2}{\sigma_u^2 - h\Sigma_i^0} > 1 \quad (35)$$

■

The intuition underlying this expression is that at the beginning of each period, the market maker's uncertainty toward ν_i is at its maximum, which implies a larger pricing errors. In the HFT's first trading round, he is inclined to deliver a large order to take advantage of the pricing deviation. In his second trading round, the market maker will learn more information concerning the asset through the order previously submitted by the HFT, so the deviation $\nu_i - p_i$ can be small. Although a large trading intensity β_i^H may create a greater price impact, it also causes more noise trading. In general, these three effects

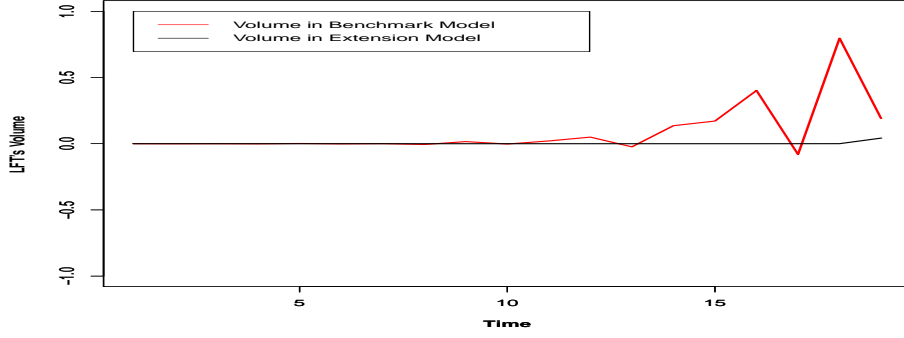


Figure 7: **Trading Volume of LFT in Two Models.** This figure plots the trading volume traded by LFT from period 1 to 19 in two models. The parameters used are $\Sigma_0^0 = 5, \sigma_\nu = 2, h = 0.5, \sigma_u = 3$.

together determine the HFT's more aggressive trading in the first round. Note that if h approaches zero, $\beta_{i_1}^H$ will be close to $\beta_{i_2}^H$ because the insiders' order size is proportional to the trading length. If h is small, the order size will also be small. Therefore, the market maker learns little in the first trading round, and the HFT will prefer to submit an order in the second round of similar size to that in the first order. However, if Σ_i^0 is large, $\beta_{i_2}^H$ can be trivial relative to $\beta_{i_2}^H$; thus, the HFT would prefer to exploit all his private information in the first trading round. From Figure 6, we note that x_i^H in the benchmark model is larger than $x_{i_1}^H$ and $x_{i_2}^H$. This result can be explained as follows: because the HFT has two chances to exploit the market maker's pricing errors, he may split his order to avoid a large price impact resulting from a large order. In other words, splitting the order can help the HFT to move down the demand curve of the market maker.

Proposition 10 *The LFT makes smaller trades in the extension model compared to the benchmark model.*

Proof: See Appendix.

As in Figure 7, compared to the order delivered in the benchmark model by the LFT, the order size x_i^L in the extension model is extremely small. The black line in the graph depicting the evolution of x_i^L essentially remains at approximately zero, because the HFT's two previous trades greatly reduce information asymmetry. Therefore, the LFT has little incentive to submit a large order given small pricing errors.

Proposition 11 *The HFT generates much more profit in the extension model.*

Figure 8: **Price Impact in two models.** This figure plots the difference in price impact if HFT is allowed to trade twice. The parameters used are $\Sigma_0^0 = 5$, $\sigma_\nu = 2$, $h = 0.5$, $B_{20} = 1$.



Proof: See Appendix.

Under Proposition 7, the market maker can never know the fundamental value of the asset, so pricing errors always exist. If the HFT is allowed to trade twice, his second trade will exploit the market maker's pricing loopholes that increases his aggregate revenue. Intuitively, even if he finds that the second trade is not profitable, he can choose not to trade and the profit will be the same as in the benchmark model. Moreover, his two trading chances can help him avoid large price impact. Because, in our model, there is no waiting cost, a smaller price impact implies larger market maker's uncertainty level. Large uncertainty level increases B_i , C_i , and D_i which measure profitability. Therefore, the HFT in the extension model is able to make more profit.

5.2.3. Liquidity

Proposition 12 *The price impacts λ_i^1 and λ_i^2 in the extension model are much smaller than those in the benchmark model.*

Proof: See Appendix.

From Figure 8, we note that market liquidity which is the inverse measure of price impact, is greater in the extension model, because the HFT's second trading chance helps him to better avoid the market maker's large price impact. Furthermore, each of the HFT's orders will be accompanied by the corresponding noise trading, and thus, distinguishing informed trading from liquidity trading and incorporating information into price will be more difficult. Therefore, the price impact decreases sharply, leading to high liquidity. Previous empirical

work provides different conclusions for the effect of HFTN on liquidity. Hasbrouck and Saar (2010) and Hendershott *et al.* (2011) find a positive effect as we do, while others, for example, Hendershott and Moulton (2011) find the opposite effect.

6. Conclusion

In this article, we consider two discrete-time, infinite-horizon models with two different types of insiders. The insiders privately observe the signals for asset values in each period and trade with competitive market makers in the presence of noise herding. We find that the market maker can never learn the true value of the asset because of the camouflage effect of noise herding if the traders are allowed a finite number of trades in a period.

To characterize the influence of the speed advantage and high frequency, we allow the HFT to trade earlier than the other insider in both models and allow the HFT to trade twice per period in the model extension. The benchmark model shows that the HFT trades much more intensively than the LFT. As a result of this speed advantage, the HFT can generate more profit than the LFT. The extension model also shows that the HFT can make full use of his two trading opportunities to avoid a large price impact.

As a consequence, market liquidity is higher in the extension model than that in the benchmark model. Furthermore, the market maker's uncertainty regarding the asset's fundamental value is smaller in the extension model than in the benchmark model. As a result of the low price impact in the extension, the HFT can earn more revenue in the extension model than in the benchmark model.

References

- Back, K. and Pedersen, H. (1998). Long-lived information and intraday patterns. *Journal of Financial Markets*, **1**, 385–402.
- Back, K., Cao, H., and Willard, G. A. (2000). Imperfect competition among informed traders. *Journal of Finance*, **55**, 2117–2155.
- Banerjee, A. V. (1992). A simple model of herd behavior. *The Quarterly Journal of Economics*, **107**, 797–817.
- Baron, M., Brogaard, J., and Kirilenko, A. (2012). The trading profits of high frequency traders. Working Paper.
- Birkchandani, S., Hirshleifer, D., and Welch, I. (1992). A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, **100**, 992–1026.
- Brogaard, J., Hendershott, T., and Riordan, R. (2013). High frequency trading and price discovery. Working Paper.
- Chau, M. and Vayanos, D. (2008). Strong-form efficiency with monopolistic insiders. *Review of Financial Studies*, **21**, 2275–2306.
- De-Long, J., Shleifer, A., Summers, L. H., and Waldmann, R. J. (1990). Noise trader risk in financial markets. *Journal of Political Economy*, **98**, 703–738.
- De-Long, J. B. and Shleifer, A. (1991). The stock market bubble of 1929: Evidence from closed-end funds. *Journal of Economic History*, **52**, 675–700.
- Duhigg, C. (2010). Stock traders find speed pays, in milliseconds. *New York Times*.
- Foster, F. D. and Viswanathan, S. (1996). Strategic trading when agents forecast the forecasts of others. *Journal of Finance*, **LI**, 1437–1478.
- Foucault, T., Hombert, J., and Roşu, I. (2012). News trading and speed. Working Paper.
- Hasbrouck, J. and Saar, G. (2010). Low latency trading. Working Paper.

- Hendershott, T. and Moulton, P. (2011). Automation, speed, and stock market quality: The NYSE’s hybrid. *Journal of Financial Markets*, **14**, 568–604.
- Hendershott, T., Jones, C., and Menkveld, A. (2011). Does algorithm trading improve liquidity. *Journal of Finance*, **66**, 1–33.
- Herzfeld, T. J. (1980). *The Investor’s Guide to Closed-End Funds: The Herzfeld Edge*. McGraw-Hill, New York.
- Holden, C. W. and Subrahmanyam, A. (1992). Long-lived private information and imperfect competition. *Journal of Finance*, **47**, 247–270.
- Kirilenko, A., Kyle, A. S., Samadi, M., and Tuzun, T. (2010). The flash crash: The impact of high frequency trading on an electronic market. Working Paper.
- Kyle, A. S. (1985). Continuous auctions and insider trading. *Econometrica*, **53**, 1315–1335.
- Li, S. (2012). Speculative dynamics I: Imperfect competition, and the implications for high frequency trading. Working Paper.
- Malkiel, B. G. (1977). The valuation of closed-end investment companies shares. *Journal of Finance*, **32**, 847–859.
- Martinez, V. H. and Roşu, I. (2013). High-frequency traders, news and volatility. Working Paper.
- Palomino, F. (1996). Noise trading in small markets. *Journal of Finance*, **51**, 1537–1550.
- SEC (2010). Concept release on equity market structure. Release No. 34-61358; File No. S7-02-10.
- Shiller, R. J. (1984). Stock prices and social dynamics. *Brookings Papers on Economic Activity*, **2**, 457–510.
- Shleifer, A. and Summers, L. H. (1990). The noise trader approach to finance. *Journal of Economic Perspectives*, **4**, 19–33.

Appendix

Proof of Theorem 1:

Proof. The proof contains two steps which is similar to Foucault *et al.* (2012). First, we show that Equations (5)-(6) can be derived from zero profit condition of market maker. Secondly, we show that Equations (8)-(14) are equivalent to insiders' optimal trading strategies.

Step 1. At the beginning of period i , market maker's information set is Ψ_{i-1}^2 . Condition on that, ν_i has a normal distribution, $\nu_i|\Psi_{i-1}^2 \sim N(p_{i-1}^2, \Sigma_i^0)$. The aggregate order flow at the first trading round is of the form $X_i^1 = \beta_i^H(\nu_i - p_{i-1}^2)h + u_i^1$.

Denote by

$$\Phi_i^1 = Cov(\nu_i - p_{i-1}^2, X_i^1) = \beta_i^H \Sigma_i^0 h \quad (36)$$

Then, condition on $\Psi_i^1 = \Psi_{i-1}^2 \cup \{X_i^1\}$, $\nu_i|\Psi_i^1 \sim N(p_i^1, \Sigma_i^1)$, with

$$p_i^1 = p_{i-1}^2 + \lambda_i^1 X_i^1 \quad (37)$$

$$\lambda_i^1 = \Phi_i^1 Var(X_i^1)^{-1} = \frac{\Sigma_i^0}{\beta_i^H(\Sigma_i^0 h + \sigma_u^2)} \quad (38)$$

$$\begin{aligned} \Sigma_i^1 &= Var(\nu_i - p_{i-1}^2) - \Phi_i^1 Var(X_i^1)^{-1} \Phi_i^1 \\ &= \Sigma_i^0 - \frac{(\Sigma_i^0)^2}{\Sigma_i^0 h + \sigma_u^2} h \\ &= \frac{\Sigma_i^0 \sigma_u^2}{\Sigma_i^0 h + \sigma_u^2} \end{aligned} \quad (39)$$

which proves Equations (5) and (6). Next, after the second trading round, the market maker observes the order flow of the form $X_i^2 = \beta_i^L(\nu_i - p_i^1)h + u_i^2$. Denote by

$$\Phi_i^2 = Cov(\nu_i - p_i^1, X_i^2) = \beta_i^L \Sigma_i^1 h \quad (40)$$

Then, condition on $\Psi_i^2 = \Psi_i^1 \cup \{X_i^2\}$, $\nu_i|\Psi_i^2 \sim N(p_i^2, \Sigma_i^2)$, with

$$p_i^2 = p_i^1 + \lambda_i^2 X_i^2 \quad (41)$$

$$\lambda_i^2 = \Phi_i^2 \text{Var}(X_i^2)^{-1} = \frac{\Sigma_i^1}{\beta_i^L(\Sigma_i^1 h + \sigma_u^2)} \quad (42)$$

$$\begin{aligned} \Sigma_i^2 &= \text{Var}(\nu_i - p_i^1) - \Phi_i^2 \text{Var}(X_i^2)^{-1} \Phi_i^2 \\ &= \Sigma_i^1 - \frac{(\Sigma_i^1)^2}{\Sigma_i^1 h + \sigma_u^2} h \\ &= \frac{\Sigma_i^1 \sigma_u^2}{\Sigma_i^1 h + \sigma_u^2} \end{aligned} \quad (43)$$

Step 2. At each period i , HFT and LFT maximize their expected profit:

$$\pi_i^H = \max E \left(\sum_{\tau=i}^{\infty} (\nu_{\tau} - p_{\tau}^1) x_{\tau}^H | \Omega_{\tau}^I \right) \quad (44)$$

$$\pi_i^L = \max E \left(\sum_{\tau=i}^{\infty} (\nu_{\tau} - p_{\tau}^2) x_{\tau}^L | \Omega_{\tau}^I \right) \quad (45)$$

We prove that the Bellman value function is in the form given in Theorem 1 by backward induction. In the induction step, if $\tau=1,2,\dots,i-1$, we assume that $\pi_{\tau+1}$ is of the desired form. The Bellman principle of intertemporal optimization implies

$$\pi_i^H = \max_{x_i^H} E \left((\nu_i - p_i^1) x_i^H + \pi_{i+1}^H | \Omega_{\tau}^I \right) \quad (46)$$

$$\pi_i^L = \max_{x_i^L} E \left((\nu_i - p_i^2) x_i^L + \pi_{i+1}^L | \Omega_{\tau}^I \right) \quad (47)$$

Pricing strategies of market maker show that the insiders' trading strategies affects execution price. Substituting them into Bellman equation, we get

$$\pi_i^H = \max_{x_i^H} E \{ (\nu_i - p_{i-1}^2 - \lambda_i^1 x_i^H - \lambda_i^1 u_i^1) x_i^H \quad (48)$$

$$\begin{aligned} &+ B_{i+1}^H (\nu_i - p_{i-1}^2 - \lambda_i^1 x_i^H - \lambda_i^1 u_i^1)^2 + C_{i+1}^H (\Delta \nu_i)^2 + D_{i+1}^H \} \\ &= \max_{x_i^H} x_i^H (\nu_i - p_{i-1}^2 - \lambda_i^1 x_i^H) \\ &+ B_{i+1}^H (\nu_i - p_{i-1}^2 - \lambda_i^1 x_i^H)^2 + B_{i+1}^H (\lambda_i^1 \sigma_u)^2 h \\ &+ C_{i+1}^H (\Delta \nu_i)^2 + D_{i+1}^H \\ \pi_i^L &= \max_{x_i^L} E \{ (\nu_i - p_i^1 - \lambda_i^2 x_i^L - \lambda_i^2 u_i^2) x_i^L \\ &+ B_{i+1}^L (\nu_i - p_i^1 - \lambda_i^2 x_i^L - \lambda_i^2 u_i^2)^2 \\ &+ C_{i+1}^L (\Delta \nu_i)^2 + D_{i+1}^L \} \end{aligned} \quad (49)$$

$$\begin{aligned}
&= \max_{x_i^L} x_i^L (\nu_i - p_i^1 - \lambda_i^2 x_i^L) \\
&+ B_{i+1}^L (\nu_i - p_i^1 - \lambda_i^2 x_i^L)^2 \\
&+ B_{i+1}^L (\lambda_i^2 \sigma_u)^2 h + C_{i+1}^L (\Delta \nu_i)^2 + D_{i+1}^L
\end{aligned}$$

The first order condition with respect to x_i^H , x_i^L is

$$x_i^H = \frac{-1 + 2B_{i+1}^H \lambda_i^1}{2\lambda_i^1(-1 + B_{i+1}^H \lambda_i^1)} (\nu_i - p_{i-1}^2) \quad (50)$$

$$x_i^L = \frac{-1 + 2B_{i+1}^L \lambda_i^2}{2\lambda_i^2(-1 + B_{i+1}^L \lambda_i^2)} (\nu_i - p_i^1) \quad (51)$$

The second order conditions for maximum are

$$\lambda_i^1(1 - B_{i+1}^H \lambda_i^1) > 0 \quad (52)$$

$$\lambda_i^2(1 - B_{i+1}^L \lambda_i^2) > 0 \quad (53)$$

which are Equation (15). Thus the optimal x_i^H and x_i^L are indeed of the form $x_i^H = \beta_i^H (\nu_i - p_{i-1}^2)h$, $x_i^L = \beta_i^L (\nu_i - p_i^1)h$, where β_i^H and β_i^L satisfy Equations (8) respectively. We substitute x_i into the formula for π_i to obtain

$$\begin{aligned}
\pi_i^H &= (\beta_i^H h(1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) (\nu_i - p_{i-1}^2)^2 \\
&+ B_{i+1}^H (\lambda_i^1 \beta_i^H \sigma_u)^2 h + C_{i+1}^H (\Delta \nu_{i+1})^2 + D_{i+1}^H \\
&= (\beta_i^H h(1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) \\
&\cdot (\nu_i - p_{i-1}^1 - \lambda_{i-1}^2 x_{i-1}^L - \lambda_{i-1}^2 u_{i-1}^2)^2 \\
&+ B_{i+1}^H (\lambda_i^1 \beta_i^H \sigma_u)^2 h + C_{i+1}^H (\Delta \nu_{i+1})^2 + D_{i+1}^H \\
&= (\beta_i^H h(1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) \\
&\cdot (\nu_i - p_{i-1}^1 - \lambda_{i-1}^2 \beta_{i-1}^L h(\nu_{i-1} - p_{i-1}^1) - \lambda_{i-1}^2 u_{i-1}^2)^2 \\
&+ B_{i+1}^H (\lambda_i^1 \beta_i^H \sigma_u)^2 h + C_{i+1}^H (\Delta \nu_i)^2 + D_{i+1}^H \\
&= (\beta_i^H h(1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) \\
&\cdot (\nu_{i-1} - p_{i-1}^1 + \Delta \nu_i - \lambda_{i-1}^2 \beta_{i-1}^L h(\nu_{i-1} - p_{i-1}^1))^2 \\
&+ (\beta_i^H h(1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) (\lambda_{i-1}^2 \beta_{i-1}^H)^2 \sigma_u^2 h
\end{aligned} \quad (54)$$

$$\begin{aligned}
& + B_{i+1}^H (\lambda_i^1 \beta_i^H \sigma_u)^2 h + C_{i+1}^H (\Delta \nu_{i+1})^2 + D_{i+1}^H \\
& = (\beta_i^H h (1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) \\
& \quad \cdot (1 - \lambda_{i-1}^2 \beta_{i-1}^L h)^2 (\nu_{i-1} - p_{i-1}^1)^2 \\
& + (\beta_i^H h (1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) (\Delta \nu_i)^2 \\
& + (\beta_i^H h (1 - \lambda_i^1 \beta_i^H h) + B_{i+1}^H (1 - \lambda_i^1 \beta_i^H h)^2) (\lambda_{i-1}^2 \beta_{i-1}^H)^2 \sigma_u^2 h \\
& + B_{i+1}^H (\lambda_i^1 \beta_i^H \sigma_u)^2 h + C_{i+1}^H \sigma_v^2 h + D_{i+1}^H
\end{aligned} \tag{55}$$

$$\pi_i^L = (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) (\nu_i - p_i^1)^2 \tag{56}$$

$$\begin{aligned}
& + B_{i+1}^L (\lambda_i^2 \beta_i^L \sigma_u)^2 h + C_{i+1}^L (\Delta \nu_{i+1})^2 + D_{i+1}^L \\
& = (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) \\
& \quad \cdot (\nu_{i-1} - p_{i-1}^2 + \Delta \nu_i - \lambda_i^1 x_i^H - \lambda_i^1 u_i^1)^2 \\
& + B_{i+1}^L (\lambda_i^2 \beta_i^L \sigma_u)^2 h + C_{i+1}^L (\Delta \nu_{i+1})^2 + D_{i+1}^L \\
& = (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) \\
& \quad \cdot (\nu_{i-1} - p_{i-1}^2 + \Delta \nu_i - \lambda_i^1 \beta_i^1 (\nu_{i-1} + \Delta \nu_i - p_{i-1}^2))^2 \\
& + (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) (\lambda_i^1 \beta_i^H)^2 \sigma_u^2 h \\
& + B_{i+1}^L (\lambda_i^2 \beta_i^L \sigma_u)^2 h + C_{i+1}^L (\Delta \nu_{i+1})^2 + D_{i+1}^L \\
& = (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) \\
& \quad \cdot (1 - \lambda_i^1 \beta_i^1)^2 (\nu_{i-1} - p_{i-1}^2)^2 \\
& + (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) (1 - \lambda_i^1 \beta_i^1)^2 (\Delta \nu_i)^2 \\
& + (\beta_i^L h (1 - \lambda_i^2 \beta_i^L h) + B_{i+1}^L (1 - \lambda_i^2 \beta_i^L h)^2) (\lambda_i^1 \beta_i^H)^2 \sigma_u^2 h \\
& + B_{i+1}^L (\lambda_i^2 \beta_i^L \sigma_u)^2 h + C_{i+1}^L \sigma_v^2 h + D_{i+1}^L
\end{aligned} \tag{57}$$

This proves that indeed π_i^H , and π_i^L are of the form shown in Theorem 1. Comparing with the coefficients in the value function, Equations (9)-(14) can be obtained. ■

Proof of Lemma 1:

Proof. From Theorem 1, we have the equations for β_i^H, λ_i^1 . Substituting Equation (5) into the Equation determining β_i^H , we have

$$\beta_i^H = \frac{\beta_i^H(\sigma_u^2 + h\Sigma_i^0)(\sigma_u^2\beta_i^H - 2B_{i+1}^H\Sigma_i^0 + h\beta_i^H\Sigma_i^0)}{2\Sigma_i^0(\sigma_u^2\beta_i^H - B_{i+1}^H\Sigma_i^0 + h\beta_i^H\Sigma_i^0)} \quad (58)$$

It is a quadratic formula, and there exists two solutions

$$\beta_i^H = 0 \quad (59)$$

or

$$\beta_i^H = \frac{2B_{i+1}^H\sigma_u^2\Sigma_i^0}{\sigma_u^4 - h^2(\Sigma_i^0)^2} \quad (60)$$

Therefore, if HFT chooses to participate in the auction, his optimal trading intensity in each period is unique. The formula for β_i^L can be derived in the same way and we do not discuss it in detail. ■

Proof of Corollary 1:

Proof. The difference equation for Σ_i^0 is as follows (in the following proof, we use x_n to stand for Σ_i^0 for simplicity):

$$x_n - \sigma_\nu^2 h = \frac{\sigma_u^2 x_{n-1}}{\sigma_u^2 + 2hx_{n-1}} \quad (61)$$

The proof consists of three steps. First of all, we transform Equation (61) into a linear difference equation. Second, we solve the transformed equation. Finally, substituting the solution derived in step two into Equation (61), find the answer we need.

Step 1. We rewrite Equation (61):

$$x_n = \frac{\sigma_u^2 \sigma_\nu^2 h + (2\sigma_\nu^2 h^2 + \sigma_u^2)x_{n-1}}{\sigma_u^2 + 2hx_{n-1}} \quad (62)$$

Next, we define $\frac{y_n}{y_{n-1}}$ as the denominator of RHS of Equation (62):

$$\frac{y_n}{y_{n-1}} = 2hx_{n-1} + \sigma_u^2 \quad (63)$$

Then, solve x_{n-1} from Equation (63):

$$x_{n-1} = \frac{y_n}{2hy_{n-1}} - \frac{\sigma_u^2}{2h} \quad (64)$$

Substitute Equation (64) into difference Equation (61):

$$\frac{y_{n+1}}{y_n} - \frac{\sigma_u^2}{2h} = \frac{(\sigma_\nu^2 h^2 + \sigma_u^2)(\frac{y_n}{y_{n-1}} - \frac{\sigma_u^2}{2h}) + \sigma_u^2 \sigma_\nu^2 h}{\frac{y_n}{y_{n-1}}} \quad (65)$$

Arrange the above equation, we get:

$$y_{n+1} - (2\sigma_u^2 + 2h\sigma_\nu^2)y_n + \sigma_u^2 y_{n-1} = 0 \quad (66)$$

Step 2. Notice that, difference Equation (66) is second-order, so there exists two characteristic roots (λ_1, λ_2) and the solution for y_n has the form:

$$y_n = C_1(\lambda_1)^n + C_2(\lambda_2)^n \quad (67)$$

$$\lambda_1 = \frac{2\sigma_u^2 + 2\sigma_\nu^2 h + \sqrt{4\sigma_\nu^2 h^2(2\sigma_u^2 + h^2\sigma_\nu^2)}}{2} \quad (68)$$

$$\lambda_2 = \frac{2\sigma_u^2 + 2\sigma_\nu^2 h - \sqrt{4\sigma_\nu^2 h^2(2\sigma_u^2 + h^2\sigma_\nu^2)}}{2} \quad (69)$$

where C_1 and C_2 are two constants, which can be determined by initial condition.

Step 3. Thus, substituting the expression of y_n into Equation (64), we have:

$$\begin{aligned} x_n &= \frac{y_{n+1}}{2hy_n} - \frac{\sigma_u^2}{2h} \\ &= \frac{C_1(\lambda_1)^{n+1} + C_2(\lambda_2)^{n+1}}{2hC_1(\lambda_1)^n + 2hC_2(\lambda_2)^n} - \frac{\sigma_u^2}{2h} \\ &= \frac{C_1(\lambda_1 - \sigma_u^2)(\lambda_1)^n + C_2(\lambda_2 - \sigma_u^2)\lambda_2^n}{2hC_1(\lambda_1)^n + 2hC_2(\lambda_2)^n} \end{aligned} \quad (70)$$

Virtually, we only need one constant to be determined by initial condition. Dividing the above equation by C_1 , there exists one constant $\frac{C_2}{C_1}$. Given $x_0 = R$, we have

$$R = \frac{(\lambda_1 - \sigma_u^2) + C(\lambda_2 - \sigma_u^2)}{2h + 2hC} \quad (71)$$

where $C = \frac{C_1}{C_2}$. It is easy to solve the following equation for C

$$C = \frac{-\sigma_u^2 + \lambda_1 - 2hR}{\sigma_u^2 - \lambda_2 + 2hR} \quad (72)$$

Substituting Equation (72) into the equation for σ_i^0 will finish the proof. ■

Proof of Lemma 2:

Proof. First, we calculate all the equilibrium points determined by Equation (61)

$$\Sigma^* - \sigma_\nu^2 h = \frac{\sigma_u^2 \Sigma^*}{\sigma_u^2 + 2h\Sigma^*} \quad (73)$$

where Σ^* is the fixed point for Σ_i^0 . There are two solutions to the above equation

$$\Sigma^* = \frac{h\sigma_\nu^2 - \sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4}}{2} \quad (74)$$

and

$$\Sigma^* = \frac{h\sigma_\nu^2 + \sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4}}{2} \quad (75)$$

Note that the first equilibrium point is negative since

$$\sqrt{2\sigma_u^2\sigma_\nu^2 + h^2\sigma_\nu^4} > \sqrt{h^2\sigma_\nu^4} > h\sigma_\nu^2.$$

But, market maker's uncertainty toward ν_i cannot become negative. Therefore, only the second fixed point is appropriate. Then we show that this equilibrium is asymptotically

stable. The derivative with respect of x_{n-1} of RHS of Equation (61) is

$$-\frac{2h\sigma_u^2 x_{n-1}}{(\sigma_u^2 + 2h)^2} + \frac{\sigma_u^2}{\sigma_u^2 + 2hx_{n-1}} \quad (76)$$

Substituting the solution gotten in Equation (75), we have

$$\left(\frac{\sigma_u^2 x_{n-1}}{\sigma_u^2 + 2hx_{n-1}} \right)' \bigg|_{x_{n-1}=\Sigma^*} = \frac{\sigma_u^2 (h\sigma_\nu^2 + \sqrt{2\sigma_u^2 \sigma_\nu^2 + h^2 \sigma_\nu^4})}{\sigma_u^2 + h^2 \sigma_\nu^2 + h\sqrt{2\sigma_u^2 \sigma_\nu^2 + h^2 \sigma_\nu^4}} \quad (77)$$

Notice that the derivative in Equation (77) is smaller than 1. Hence, according to the notion of Liapunov stability, this equilibrium is asymptotically stable (AS). Finally, we prove that this equilibrium is also uniformly asymptotically stable (UAS). Denote $x(n, n_0, x_0)$ and $y(n, m_0, x_0)$ are two solutions to Equation (61), where $m_0 = n_0 + r_0$, $r_0 > 0$, x_0 satisfies $\|x_0 - \Sigma^*\| < \delta$. $x(n, n_0, x_0)$ and $y(n, m_0, x_0)$ satisfy $\|x(n, n_0, x_0) - \Sigma^*\| < \epsilon$ and $\|y(n, m_0, x_0) - \Sigma^*\| < \epsilon$. Note that $x(n - r_0, n_0, x_0)$ intersects with $y(n, m_0, x_0)$ at $n = m_0$. By the uniqueness of solutions, it follows that $y(n, m_0, x_0) = x(n - r_0, n_0, x_0)$. This implies that the δ in the definition of uniform stability is independent of initial time n_0 which finishes our proof. Based on the stability of Σ_i^0 , the stability of equilibrium of Σ_i^1 and Σ_i^2 is quite easy to derive. ■

Proof of Proposition 1:

In order to derive this proposition, we need to adopt *Poincarè-Perron Theorem*. We introduce it as the following lemma.

Lemma 5 (*Poincarè-Perron*) *Considering Poincarè type difference equation with nonconstant coefficients*

$$x(n+k) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0 \quad (78)$$

such that there are real numbers p_i , $1 \leq i \leq k$, with

$$\lim_{n \rightarrow \infty} p_i(n) = p_i, \quad 1 \leq i \leq k. \quad (79)$$

Suppose that the characteristic roots $\lambda_1, \lambda_2, \dots, \lambda_k$ of Equation (78) have distinct moduli and $p_k(n) \neq 0$ for all $n \in \mathbb{Z}^+$. Then Poincarè type difference equation has a fundamental set of solutions $\{x_1(n), x_2(n), \dots, x_k(n)\}$ with the property

$$\lim_{n \rightarrow \infty} \frac{x_i(n+1)}{x_i(n)} = \lambda_i, \quad 1 \leq i \leq k. \quad (80)$$

Applying Lemma 5, it is easy to have the Proposition 1.

Proof. We first consider the transformed difference equation in Equation (66). It is one of the simplest Poincarè type difference equation, since the coefficients are constant. Moreover, it has two distinct characteristic roots. Therefore, this linear equation satisfies all the condition in Lemma 5, and there are two fundamental sets of solutions $\{x_1(n), x_2(n)\}$ with the property

$$\lim_{n \rightarrow \infty} \frac{x_1(n+1)}{x_1(n)} = \lambda_1 = \frac{2\sigma_u^2 + 2\sigma_\nu^2 h + \sqrt{4\sigma_\nu^2 h^2 (2\sigma_u^2 + h^2 \sigma_\nu^2)}}{2} \quad (81)$$

$$\lim_{n \rightarrow \infty} \frac{x_2(n+1)}{x_2(n)} = \lambda_2 = \frac{2\sigma_u^2 + 2\sigma_\nu^2 h - \sqrt{4\sigma_\nu^2 h^2 (2\sigma_u^2 + h^2 \sigma_\nu^2)}}{2} \quad (82)$$

Substituting Equation (81) and Equation (82) into Equation (64), we will have the limit value of Σ_i^0

$$\lim_{i \rightarrow \infty} \Sigma_i^0 = \lim_{n \rightarrow \infty} \frac{x_1(n+1)}{2hx_1(n)} - \frac{\sigma_u^2}{2h} = \frac{h\sigma_\nu^2 + \sqrt{2\sigma_u^2 \sigma_\nu^2 + h^2 \sigma_\nu^4}}{2} \quad (83)$$

$$\lim_{i \rightarrow \infty} \Sigma_i^0 = \lim_{n \rightarrow \infty} \frac{x_2(n+1)}{2hx_2(n)} - \frac{\sigma_u^2}{2h} = \frac{h\sigma_\nu^2 - \sqrt{2\sigma_u^2 \sigma_\nu^2 + h^2 \sigma_\nu^4}}{2} \quad (84)$$

From Lemma 2, we know that the second convergent value is impossible because market maker's uncertainty cannot be negative. Thus, Σ_i^0 converges to an equilibrium point which

is global stable. With Equations (6), the convergent value of Σ_i^1 and Σ_i^2 can also be found. ■

Proof of Proposition 2:

Proof. Given the complicated form of solution to the difference equation of Σ_i^0 , we do not prove this proposition directly, but in an indirect recursive way. Denote by

$$f(\sigma_u, \sigma_\nu, \Sigma_i^0, h) = \sigma_\nu^2 h + \frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^2 + 2h \Sigma_i^0} \quad (85)$$

First we show that an increase in Σ_0^0 can raise Σ_i^0 for all $i < \infty$

$$\frac{\partial f(\cdot)}{\partial \Sigma_i^0} = -\frac{2h\sigma_u^2 \Sigma_i^0}{(\sigma_u^2 + 2h \Sigma_i^0)^2} + \frac{\sigma_u^2}{\sigma_u^2 + 2h \Sigma_i^0} = \frac{\sigma_u^4}{(\sigma_u^2 + 2h \Sigma_i^0)^2} > 0 \quad (86)$$

From Equation (86), it is easy to conclude that a larger uncertainty in this period will lead to a higher level of uncertainty in the next period too. By adopting mathematical induction, we finish proving that Σ_i^0 increases in Σ_0^0 .

Then, start from the above proof, it will be natural to see that given fixed Σ_0^0 , signals containing more information can lift Σ_i^0 . Note that, when fixing Σ_0^0 , Σ_1^0 will become bigger as σ_ν rises. Now, from period $i (i > 1)$, there exists two factors resulting in a larger Σ_{i+1}^0 . One is due to a larger σ_ν and the other is owing to a higher previous level of uncertainty (see Equation (86)). Again, by employing mathematical induction, we can reach our conclusion.

Finally, we prove that a large liquidity trading intensity can also aggravate information asymmetry. Taking partial derivative of Equation (85), it is easy to have

$$\begin{aligned} \frac{\partial f(\cdot)}{\partial \sigma_u} &= \frac{(2\sigma_u \Sigma_i^0 + \sigma_u^2 \frac{\partial \Sigma_i^0}{\partial \sigma_u})(\sigma_u^2 + 2h \Sigma_i^0) - \sigma_u^2 \Sigma_i^0 (2\sigma_u + 2h \frac{\partial \Sigma_i^0}{\partial \sigma_u})}{(\sigma_u^2 + 2h \Sigma_i^0)^2} \\ &= \frac{4h\sigma_u(\Sigma_i^0)^2 + \sigma_u^3 \frac{\partial \Sigma_i^0}{\partial \sigma_u}}{(\sigma_u^2 + 2h \Sigma_i^0)^2} > 0 \end{aligned}$$

Similar to the proof concerning increase of σ_ν , given other factors fixed, more noise trading hinder private information revelation. Applying mathematical induction again will finish the whole proof. ■

Proof of Proposition 3:

Proof. According to Theorem 1, it is easy to obtain

$$\begin{aligned} velocity_i^1 &= \frac{\Sigma_i^0 - \Sigma_i^1}{\Sigma_i^0} = \frac{h\Sigma_i^0}{\sigma_u^2 + h\Sigma_i^0} \\ velocity_i^2 &= \frac{\Sigma_i^1 - \Sigma_i^2}{\Sigma_i^1} = \frac{h\Sigma_i^1}{\sigma_u^2 + h\Sigma_i^1} \end{aligned}$$

Substituting Σ_i^1 with Σ_i^0 , we get

$$velocity_i^2 = \frac{\Sigma_i^1 - \Sigma_i^2}{\Sigma_i^1} = \frac{h\Sigma_i^0}{\sigma_u^2 + 2h\Sigma_i^0} < \frac{h\Sigma_i^0}{\sigma_u^2 + h\Sigma_i^0} = velocity_i^1 \quad (87)$$

From two equations for $velocity_i^1$, and $velocity_i^2$, we can see clearly the σ_u only appears at the denominator.⁸ In the proof of Proposition 7, we prove that $\frac{\sigma_u^2}{\Sigma_i^0}$ increases against σ_u . Therefore, $velocity_i$ decreases against σ_u . The influence of Σ_i^0 on velocity can be obtained through differentiating the expressions for $velocity_i$ with respect to Σ_i^0

$$\frac{\partial velocity_i^1}{\partial \Sigma_i^0} = \frac{h\sigma_u^2}{(\sigma_u^2 + h\Sigma_i^0)^2} > 0 \quad (88)$$

$$\frac{\partial velocity_i^2}{\partial \Sigma_i^0} = \frac{h\sigma_u^2}{(\sigma_u^2 + 2h\Sigma_i^0)^2} > 0 \quad (89)$$

■

Proof of Proposition 4:

Proof. We finish this proof in two steps. First, we show that if β_i^H and $\beta_i^L > 0$, then $B_i^H > B_i^L$. Secondly, we prove $\beta_i^H > \beta_i^L$.

Step 1. Substituting Equations (5), and (16) into Equations (9) and (11), we have

$$B_i^H = \frac{B_{i+1}^H(\sigma_u^2 + h(h\sigma_u^2 - \Sigma_i^0))^2}{\sigma_u^4 - (h\Sigma_i^0)^2} \quad (90)$$

⁸Actually, we can rewrite these two equations as $velocity_i^1 = \frac{h}{\frac{\sigma_u^2}{\Sigma_i^0} + h}$ and $velocity_i^2 = \frac{h}{\frac{\sigma_u^2}{\Sigma_i^0} + 2h}$, respectively.

$$B_i^L = \frac{B_{i+1}^L \sigma_u^2}{\sigma_u^2 + 2h\Sigma_i^0} \quad (91)$$

To ensure the positiveness of trading intensity, σ_u^2 must be larger than $h\Sigma_i^0$ from Lemma 1. Comparing the coefficients before B_{i+1}^H and B_{i+1}^L , we can obtain

$$\frac{\frac{(\sigma_u^2 + h^2\sigma_\nu^2 - h\Sigma_i^0)^2}{\sigma_u^4 - (h\Sigma_i^0)^2}}{\frac{\sigma_u^2}{\sigma_u^2 + 2h\Sigma_i^0}} = \frac{(\sigma_u^2 + h(h\sigma_\nu^2 - \Sigma_i^0))^2(\sigma_u^2 + 2h\Sigma_i^0)}{\sigma_u^6 - (h\sigma_u\Sigma_i^0)^2} \quad (92)$$

According to Proposition 1, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{\frac{\sigma_u^2}{\sigma_u^2 + 2h\Sigma_i^0}}{\frac{(\sigma_u^2 + h^2\sigma_\nu^2 - h\Sigma_i^0)^2}{\sigma_u^4 - (h\Sigma_i^0)^2}} & \quad (93) \\ = \frac{2\sigma_u^4 - (h\sigma_u\sigma_\nu)^2 - h^3\sigma_\nu^2(h\sigma_\nu^2 + \sqrt{2(\sigma_u\sigma_\nu)^2 + (h\sigma_u^2)^2})}{2\sigma_u^4 + (h\sigma_u\sigma_\nu)^2} < 1 \end{aligned}$$

If we restrict h to be very small, then the above expression is positive. From difference equation system, it is easy to calculate B_i^H, B_i^L

$$B_i^H = \prod_{t=i}^{\infty} \frac{B_\infty^H(\sigma_u^2 + h^2\sigma_\nu^2 - h\Sigma_t^0)}{\sigma_u^4 - (h\Sigma_t^0)^2} \quad (94)$$

$$B_i^L = \prod_{t=i}^{\infty} \frac{B_\infty^L \sigma_u^2}{\sigma_u^2 + 2h\Sigma_t^0} \quad (95)$$

Therefore

$$\frac{B_i^L}{B_i^H} = \frac{B_\infty^L}{B_\infty^H} \prod_{t=i}^{\infty} \frac{\sigma_u^6 - (h\sigma_u\Sigma_t^0)^2}{(\sigma_u^2 + h(h\sigma_\nu^2 - \Sigma_t^0))^2(\sigma_u^2 + 2h\Sigma_t^0)} \quad (96)$$

From Equation (93), we can derive that the infinite product diverges. We now prove that it diverges to zero. Since Σ_i^0 is continuous with respect to i and function $\frac{\sigma_u^6 - (h\sigma_u\Sigma_i^0)^2}{(\sigma_u^2 + h(h\sigma_\nu^2 - \Sigma_i^0))^2(\sigma_u^2 + 2h\Sigma_i^0)}$ is continuous with respect to Σ_i^0 , so from Equation (93), it can be shown that when Σ_i^0 lies in interval $(\Sigma_0^* - \epsilon, \Sigma_0^* + \epsilon)$, $\frac{\sigma_u^6 - (h\sigma_u\Sigma_i^0)^2}{(\sigma_u^2 + h(h\sigma_\nu^2 - \Sigma_i^0))^2(\sigma_u^2 + 2h\Sigma_i^0)} < 1$. Moreover, since $\lim_{i \rightarrow \infty} \Sigma_i^0 = \Sigma_0^*$, given any ϵ , there exists an integer N and for any $i > N$, $|\Sigma_i^0 - \Sigma_0^*| < \epsilon$. Hence for any $i > N$,

$\frac{\sigma_u^6 - (h\sigma_u \Sigma_i^0)^2}{(\sigma_u^2 + h(h\sigma_u^2 - \Sigma_i^0))^2(\sigma_u^2 + 2h\Sigma_i^0)} < 1$. Clearly

$$\prod_{i>N} \frac{\sigma_u^6 - (h\sigma_u \Sigma_i^0)^2}{(\sigma_u^2 + h(h\sigma_u^2 - \Sigma_i^0))^2(\sigma_u^2 + 2h\Sigma_i^0)} = 0 \quad (97)$$

This proves that this infinite product diverges to zero. In other words, $B_i^H > B_i^L$.

Step 2. From Lemma 1, we can see that if $\frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^4 - (h\Sigma_i^0)^2} > \frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^4 - (h\Sigma_i^0)^2}$, then β_i^H is larger than β_i^L for any i . Derive $\frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^4 - (h\Sigma_i^0)^2}$ with respect to Σ_i^0 , we have

$$\frac{\partial \frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^4 - (h\Sigma_i^0)^2}}{\partial \Sigma_i^0} = 2 \frac{\sigma_u^2(\sigma_u^4 + (h\Sigma_i^0)^2)}{(\sigma_u^4 - (h\Sigma_i^0)^2)^2} > 0 \quad (98)$$

Therefore, $\frac{\sigma_u^2 \Sigma_i^0}{\sigma_u^4 - (h\Sigma_i^0)^2}$ is increasing in Σ_i^0 . Notice that $\Sigma_i^0 > \Sigma_i^1$, which finishes the whole proof. ■

Proof of Proposition 5:

Proof. The profit for HFT is of the form:

$$\begin{aligned} E(\pi_0^H) &= \Sigma_{i=0}^\infty E(x_i^H(\nu_i - p_i^1)) = \Sigma_{i=0}^\infty E(\beta_i^H(\nu_i - p_{i-1}^2)(\nu_i - p_i^1)) \\ &= \Sigma_{i=0}^\infty \beta_i^H \Sigma_i^0 (1 - \lambda_i^1 \beta_i^H) \end{aligned} \quad (99)$$

Similar, LFT's profit takes the form:

$$\begin{aligned} E(\pi_0^L) &= \Sigma_{i=0}^\infty E(x_i^L(\nu_i - p_i^2)) = \Sigma_{i=0}^\infty E(\beta_i^L(\nu_i - p_i^1)(\nu_i - p_i^2)) \\ &= \Sigma_{i=0}^\infty \beta_i^L \Sigma_i^1 (1 - \lambda_i^2 \beta_i^L) \end{aligned} \quad (100)$$

Notice that $\frac{\Sigma_i^1 \beta_i^L (1 - \lambda_i^2 \beta_i^L)}{\Sigma_i^0 \beta_i^H (1 - \lambda_i^1 \beta_i^H)} = 0$ since $B_i^H \gg B_i^L$. Therefore $E(\pi_0^H) \gg E(\pi_0^L)$ ■

Proof of Proposition 6:

Proof. We first prove that $\frac{\Sigma_i^0}{\sigma_u^2}$ decreases against σ_u . From the difference equation for Σ_i^0 , we

have

$$\frac{\Sigma_1^0}{\sigma_u^2} = \frac{\frac{\Sigma_0^0}{\sigma_u^2}}{1 + \frac{\Sigma_0^0}{\sigma_u^2}} + \frac{\sigma_\nu^2 h}{\sigma_u^2}.$$

When σ_u goes up, clearly, $\frac{\Sigma_1^0}{\sigma_u^2}$ goes down. Since

$$\frac{\Sigma_2^0}{\sigma_u^2} = \frac{\frac{\Sigma_1^0}{\sigma_u^2}}{1 + \frac{\Sigma_1^0}{\sigma_u^2}} + \frac{\sigma_\nu^2 h}{\sigma_u^2}.$$

when σ_u rises $\frac{\Sigma_1^0}{\sigma_u^2}$ and $\frac{\Sigma_2^0}{\sigma_u^2}$ drops. Therefore, we can show that $\frac{\Sigma_i^0}{\sigma_u^2}$ decreases in σ_u in the same way.

Next, we prove that B_i^H and B_i^L increases along σ_u . From the proof of Proposition 4, we have

$$\begin{aligned} B_i^H &= \frac{B_{i+1}^H(\sigma_u^2 + h^2\sigma_\nu^2 - h\Sigma_i^0)^2}{\sigma_u^4 - (h\Sigma_i^0)^2} \\ B_i^L &= \frac{B_{i+1}^L\sigma_u^2}{\sigma_u^2 + 2h\Sigma_i^0} \end{aligned}$$

Given h can be arbitrarily small, we approximate $\frac{(\sigma_u^2 + h^2\sigma_\nu^2 - h\Sigma_i^0)^2}{\sigma_u^4 - (h\Sigma_i^0)^2}$ by $1 - \frac{2h\Sigma_i^0}{\sigma_u^2}$, where we ignore terms having h with order larger than two in its Taylor series. Since $\frac{\Sigma_i^0}{\sigma_u^2}$ decreases in σ_u , $1 - \frac{2h\Sigma_i^0}{\sigma_u^2}$ increases in σ_u and B_i^H increases in σ_u . Finally, from the difference equation for B_i^L , it can be derived easily that a larger σ_u raises B_i^L .

Then, given $\lambda_i^1 = \frac{\Sigma_i^0}{\beta_i^H(\sigma_u^2 + h\Sigma_i^0)}$ and $\beta_i^H = \frac{2B_{i+1}^H\sigma_u^2\Sigma_i^0}{\sigma_u^4 - (h\Sigma_i^0)^2}$, we can obtain

$$\lambda_i^1 = \frac{\sigma_u^2 - h\Sigma_i^0}{2B_{i+1}^H\sigma_u^2}$$

Differentiating it with respect to σ_u , we have

$$\frac{\partial \lambda_i^1}{\partial \sigma_u} = \frac{2B_{i+1}^H(2\sigma_u - h\frac{\partial \Sigma_i^0}{\partial \sigma_u})\sigma_u^2 - (\sigma_u^2 - h\Sigma_i^0)(4\sigma_u B_{i+1}^H + 2\frac{\partial B_{i+1}^H}{\partial \sigma_u}\sigma_u^2)}{4(\sigma_u^2 B_{i+1}^H)^2}$$

Since h can be arbitrarily small, we can ensure RHS of the above equation is negative given $\frac{\partial B_{i+1}^H}{\partial \sigma_u} > 0$ and $\frac{\partial \Sigma_i^0}{\partial \sigma_u} > 0$. With similar approach, $\frac{\partial \lambda_i^2}{\partial \sigma_u} < 0$ can be proved. ■

Proof of Proposition 8:

Proof. If we allow the HFT trades k times before the LFT, then it is easy to derive that

$$\lim_{i \rightarrow \infty} \Sigma_i^0 = \frac{hk\sigma_\nu^2 + \sqrt{k}\sigma_\nu\sqrt{4\sigma_u^2 + h^2k\sigma_\nu^2}}{2k} \quad (101)$$

Differentiating Equation (101) with respect to σ , we have

$$\frac{\partial \frac{hk\sigma_\nu^2 + \sqrt{k}\sigma_\nu\sqrt{4\sigma_u^2 + h^2k\sigma_\nu^2}}{2k}}{\partial \sigma_u} = \frac{2\sigma_u\sigma_\nu}{\sqrt{4k^2\sigma_u^2 + h^2k^3\sigma_\nu^2}} \quad (102)$$

Obviously, RHS of Equation (101) decreases in k , so the influence of σ_u on Σ_i^0 is larger in the benchmark model. Differentiating Equation (101) with respect to σ , we have

$$\frac{\partial \frac{hk\sigma_\nu^2 + \sqrt{k}\sigma_\nu\sqrt{4\sigma_u^2 + h^2k\sigma_\nu^2}}{2k}}{\partial \sigma_\nu} = h\sigma_\nu + \frac{2\sigma_u^2}{\sqrt{4k\sigma_u^2 + h^2k\sigma_\nu^2}} + \frac{h^2\sigma_\nu^2\sqrt{k}}{\sqrt{4\sigma_u^2 + h^2k\sigma_\nu^2}} \quad (103)$$

Taking partial derivatives of RHS of the Equation (103) with respect to k , we obtain

$$\frac{\partial h\sigma_\nu + \frac{2\sigma_u^2}{\sqrt{4k\sigma_u^2 + h^2k\sigma_\nu^2}} + \frac{h^2\sigma_\nu^2\sqrt{k}}{\sqrt{4\sigma_u^2 + h^2k\sigma_\nu^2}}}{\partial k} = -\frac{4\sigma_u^4}{k^{3/2}(4\sigma_u^2 + h^2k\sigma_\nu^2)^{3/2}} < 0 \quad (104)$$

Therefore, the influence of σ_ν on Σ_i^0 is larger in the benchmark model. The influence of σ_ν and σ_u on Σ_i^1, Σ_i^2 can be derived similarly.

■

Proof of Proposition 10:

Proof. With Lemma 3, we rewrite the difference equation for B_i^L in the extension

$$B_i^L = \frac{B_{i+1}^L\sigma_u^2}{(1 + \frac{h\Sigma_i^0}{\sigma_u^2})(\sigma_u^2 + 3h\Sigma_i^0)} \quad (105)$$

Since $\frac{\lim_{i \rightarrow \infty} (\Sigma_i^0)_{Benchmark}}{\lim_{i \rightarrow \infty} (\Sigma_i^0)_{Extension}} < \frac{3}{2}$, we can obtain that $\frac{(B_i^L)_{Extension}}{(B_i^L)_{Benchmark}} \approx 0$. Notice that in the expressions for β_i^L in the two models, other terms are bounded. So $(\beta_i^L)_{Benchmark} \gg (\beta_i^L)_{Extension}$.

■

Proof of Proposition 11:

Proof. Similar to the proof of Proposition 10, it can be shown that $\frac{(B_i^H)_{Extension}}{(B_i^L)_{Benchmark}} \approx 0$. Substituting this into the value functions for HFT, given other terms are bounded, $\frac{(\pi_i^H)_{Benchmark}}{(\pi_i^H)_{Extension}} \approx 0$.

■

Proof of Proposition 12:

Proof. Note that $\lambda_i^1 = \frac{\Sigma_i^0}{\sigma_u^2 + h \Sigma_i^0}$ and $\lambda_i^2 = \frac{\Sigma_i^1}{\sigma_u^2 + h \Sigma_i^1}$. Given $\frac{(B_i^H)_{Benchmark}}{(B_i^L)_{Extension}} \approx 0$, and $\frac{\Sigma_i^0}{\sigma_u^2 + h \Sigma_i^0}$ and $\frac{\Sigma_i^1}{\sigma_u^2 + h \Sigma_i^1}$ are bounded, clearly price impact in the extension should be smaller than that in the benchmark model ■