1 Geometric Brownian Motion (BS)

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy.* 81(3), 637-654.

A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t, \qquad W_t \sim \mathcal{N}(0,\, t) \tag{1}$$

Applying Itô's lemma

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right)\, dt + dW_t \tag{2}$$

Discretization

$$\ln S_{t+\Delta t} = \ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\, \Delta t + \sigma\sqrt{\Delta t}Z, \qquad Z \sim \mathcal{N}(0,\, 1) \tag{3}$$

Or

$$S_{t+\Delta t} = S_t \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\, \Delta t + \sigma\sqrt{\Delta t}Z\right] \tag{4}$$

In risk-neutral world

$$\mathrm{E}[S_t] = S_0 e^{(r-q)t} \quad \Rightarrow \quad \mathrm{E}[e^{R_t}] = e^{(r-q)t} \tag{5}$$

Hence,

$$\mu = r - q \tag{6}$$

2 Jump Diffusion Model (Merton)

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics.* 3(1-2), 125-144.

In jump-diffusion model, S_t is assumed to follow the stochastic differential equation (SDE)

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t + S_t\, dJ_t \tag{7}$$

where

$$J_t = \sum_{j=1}^{N_t}(Y_j - 1), \qquad N_t \sim \mathrm{Poisson}(\lambda) \tag{8}$$

The SDE can be solved to obtain

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + \sum_{j=1}^{N_t} \ln Y_j\right] \tag{9}$$

where

$$\ln Y \sim \mathcal{N}(\gamma,\, \delta^2) \tag{10}$$

Discretization

$$S_{t+\Delta t} = S_t \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\, \Delta t + \sigma\sqrt{\Delta t}Z_D + X\right] \tag{11}$$

where

$$X \sim \mathcal{N}(n\gamma,\, n\delta^2) = n\gamma + \sqrt{n}\delta Z_J, \qquad n \sim \mathrm{Poisson}(\lambda\Delta t) \tag{12}$$

In risk-neutral world

$$\mathrm{E}[e^{R_t}] = e^{(r-q)t} \quad \Rightarrow \quad e^{\left(\mu - \frac{\sigma^2}{2}\right)t}\mathrm{E}[e^{\sigma W_t}]\mathrm{E}[e^{X_t}] = e^{(r-q)t} \tag{13}$$

Hence,

$$\mu = (r - q) - \frac{\ln\left(\mathrm{E}[e^{X_t}]\right)}{t} = r - q - \lambda k \tag{14}$$

where

$$k = \mathrm{E}\left[e^{\ln Y}\right] - 1 = e^{\gamma + \frac{1}{2}\delta^2} - 1 \tag{15}$$

3 Stochastic Volatility Model (Heston)

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies.* 6(2), 327-343.

SDEs of the Heston model are given by

$$dS_t = \mu S_t\,dt + \sqrt{V_t} S_t\,dW_{1,t} \tag{16}$$

$$d\nu_t = \kappa(\theta - \nu_t)\,dt + \sigma\sqrt{\nu_t}\,dW_{2,t} \tag{17}$$

where $\mathrm{E}[dW_{1,t}\,dW_{2,t}] = \rho\,dt$

The parameters of the model are

$\kappa > 0$	the mean reversion speed for the variance
$\theta > 0$	the mean reversion level for the variance
$\sigma > 0$	the volatility of the variance
$\nu_0 > 0$	the initial level of the variance
$\rho \in [-1, 1]$	the correlation between the two Brownian motions W_1 and W_2

Mean Reversion

$$d\nu_t = \kappa(\theta - \nu_t)\,dt \;\Rightarrow\; \frac{d\nu_t}{dt} = \kappa(\theta - \nu_t) \tag{18}$$

If ν_t is greater than θ, then $(\theta - \nu_t)$ becomes negative, and $\frac{d\nu_t}{dt}$ becomes negative as well. This means that the rate of change of ν_t will be negative, leading ν_t to decrease over time. Conversely, if ν_t is less than θ, $(\theta - \nu_t)$ becomes positive, and $\frac{d\nu_t}{dt}$ becomes positive too. In this case, the rate of change of ν_t will be positive, causing ν_t to increase.

Discretization

Generate two independent random variables Z_1 and Z_2, and define $Z_\mathrm{S} = Z_1$ and $Z_\mathrm{V} = \rho Z_\mathrm{S} + \sqrt{1-\rho^2} Z_2$.

for the variance

Euler Scheme

$$\nu_{t+\Delta t} = \nu_t + \kappa(\theta - \nu_t)\Delta t + \sigma\sqrt{\nu_t \Delta t} Z_\mathrm{V} \tag{19}$$

Milstein Scheme

$$\nu_{t+\Delta t} = \left(\sqrt{\nu_t} + \frac{1}{2}\sigma\sqrt{\Delta t} Z_\mathrm{V}\right)^2 + \kappa(\theta - \nu_t)\Delta t - \frac{1}{4}\sigma^2\Delta t \tag{20}$$

for the log-stock price

$$S_{t+\Delta t} = S_t \exp\left[\left(\mu - \frac{1}{2}\nu_t\right)\Delta t + \sqrt{\nu_t \Delta t} Z_\mathrm{S}\right] \tag{21}$$

In risk-neutral world

$$\mu = r - q \tag{22}$$