2019 Linear Algebra Handouts

教科書：　Linear Algebra With Applications 9th edition

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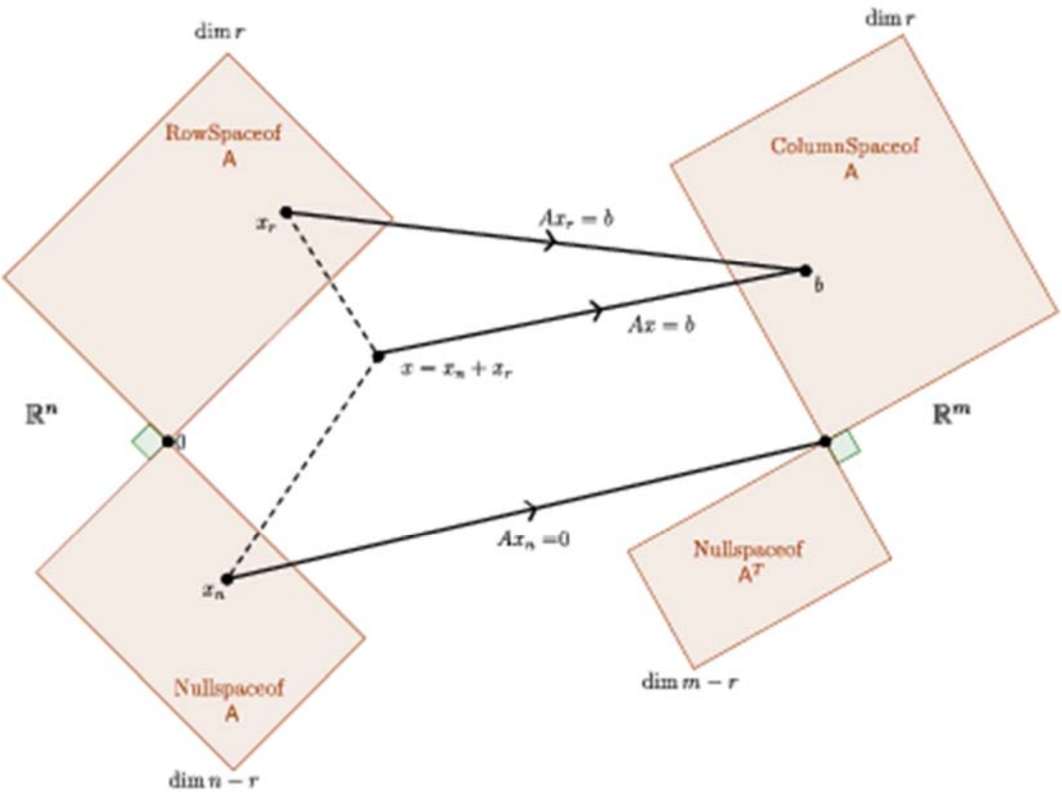


http://mathworld.wolfram.com/FundamentalTheoremofLinearAlgebra.html

We know the central role of linear algebra. It is much more than a random math course. It's applications touch many more students than calculus. We are in a digital world now.

Gilbert Strang in "Too Much Calculus"

Chapter 1 Matrixes and Systems of Equations

這一章在介紹矩陣 Matrix 的運算，像是加法、減法、乘法、除法等。而除法的部分，和解多元一次聯立方程式有直接的相關，每一個多元一次聯立方程式都可以寫成矩陣的形式，而一個多元一次聯立方程式有唯一解等同於相對應的矩陣可以做〝除法〞。

1.1 Systems of Linear Equations

複習中學所學的多元一次聯立方程式求解。要熟悉（1）何時有（唯一）解，（2）如何有系統的解一個多元一次聯立方程式，（3）一個多元一次聯立方程式如何寫成矩陣形式（4）了解名詞意義：Consistent, strict triangular form, ELEMENTARY ROW OPERATIONS 等。

- A linear equation in n unknowns is of the form

  $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables.
- A linear system of m equations in n unknowns is of the form

  $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\;\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

  where the a_{ij}'s and the b_i's are all real numbers.

Examples:

(a) $$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$$
is a 2×2 system

(b) $$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$
is a 2×3 system

(c) $$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$
is a 3×2 system

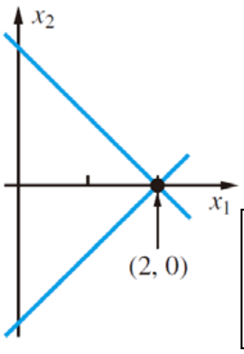
Try to find the solutions of (a) (b) (c).

Note:

- If a linear system has <u>no solution</u>, the system is **inconsistent**; if the system has <u>at least one solution</u>, it is **consistent.** The set of all solutions to a linear system is called the **solution set** of the system.
- In the above examples, the system ____________ is inconsistent, and the systems ________________ are consistent.

Example: A 2x2 linear system and the line equations.

$$\boxed{\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}}$$



$$\boxed{\begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}}$$ __________

$$\boxed{\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned}}$$ __________

- Two systems of equations involving the same variables are said to be **equivalent** if they have <u>the same solution set</u>.

Example: Check if those two linear systems are equivalent

$$\text{(a)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

- A system is said to be in **strict triangular form** if in the *k*th Equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero ($k = 1, 2, \ldots, n$).

Example: How to solve it? (back substitution)

$$\boxed{\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}}$$

- From linear systems to matrices

Example:

$$\boxed{\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}}$$

The *coefficient matrix* of the system

The *augmented matrix* of the system

- **ELEMENTARY ROW OPERATIONS**
  - **I. Interchange two rows.**
  - **II. Multiply a row by a nonzero real number.**
  - **III. Replace a row by it's sum with a multiple of another row.**

Example: Solve the following system

$$\begin{array}{rcrcrcrcr} & - & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 6 \\ 2x_1 & + & 4x_2 & + & x_3 & - & 2x_4 & = & -1 \\ 3x_1 & + & x_2 & - & 2x_3 & + & 2x_4 & = & 3 \end{array}$$

1.2 Row Echelon Form

如果可以把係數矩陣變成三角矩陣或是 Row Echelon Form，線性系統就很容易解了，而這個轉變過程就是著名的高斯消去法(Gaussian Elimination)。電腦中解線性系統目前主要的演算法還是高斯消去法，資工系的不可不會。

Example: Consider the system represented by the following augmented matrix:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right]$$

Using the "elementary row operations", we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is equivalent to the system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 1 \\ & & & & x_3 & + & x_4 & + & 2x_5 & = & 0 \\ & & & & & & & & x_5 & = & 3 \end{array}$$

- x_1, x_3, x_5 are the **lead variable** (the variables corresponding to the first nonzero elements in each row of the augmented matrix)
- x_2, x_4 are the **free variables** (the remaining variables corresponding to the columns skipped in the reduction process)

- If we transfer the free variables over to the right-hand side, we get

$$\begin{array}{ccccccccccc} x_1 & + & x_3 & + & x_5 & = & 1 & - & x_2 & - & x_4 \\ & & x_3 & + & 2x_5 & = & -x_4 & & & & \\ & & & & x_5 & = & 3 & & & & \end{array}$$

- Set free variables $x_2 = a$, $x_4 = b$, the solution set is $(-a+4, a, -b-6, b, 3)$

- A matrix is said to be in **row echelon form** if
  I. The first nonzero entry in each row is 1.
  II. If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
  III. If there are rows whose entries are all zero, they are below the rows having nonzero entries.

- The process of using elementary row operations I, II, and III to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination**.

Example: which matrices are in the row echelon form?

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- If the row echelon form of the augmented matrix contains a row of the form

  [0 0 … 0 | 1]

  the system is____________________. Otherwise the system will be________________.
  If the system is consistent and the nonzero rows of the row echelon form of the matrix form a strictly triangular system, the system will have ___________________________.

- A linear system of m equations and n unknowns is said to be ***overdetermined*** if there are more equations than unknowns ($m > n$). A linear system is said to be ***underdetermined*** if there are fewer equations than unknowns ($m < n$).
- A consistent *underdetermined* system will have__________________________.

- A matrix is said to be in **reduced row echelon form** if:
  (1) the matrix is in row echelon form
  (2) the first nonzero entry is the only nonzero entry in its column.

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The process of using elementary row operations to transform a matrix into reduced row echelon form is called **Gauss-Jordan reduction**.

- A system of linear equations is said to be **homogeneous** if the constants on the right–hand side are all zero.
- Homogeneous systems are always ______________since it must have the trivial solution _____.

- **Theorem 1.2.1** An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $n > m$ (underdetermined).

Proof:

1.3 Matrix Algebra

介紹矩陣的相等、加減法、轉置、線性組合、乘法等運算。

- An $m \times n$ matrix A can be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The entries of a matrix are called ***scalars*** (real or complex number). In general, a_{ij} will denote the entry of the matrix A that is in the ith row and the jth column. We will sometimes shorten this matrix to $A = (a_{ij})$.

- An n-tuple of real number is referred to as a **vector.**
  - **row vector**: a $1 \times n$ matrix, e.g., $[a_1, a_2, \ldots, a_n]$
  - **column vector**: an $n \times 1$ matrix , e.g., $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - The set of all $n \times 1$ matrices of real numbers is called **Euclidean *n*-space** and denoted by R^n.

Example:

$$A = \begin{bmatrix} 3 & 2 & 5 \\ -1 & 8 & 4 \end{bmatrix} \text{ then } \quad \mathbf{a}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\text{and} \quad \vec{\mathbf{a}}_1 = (3, \quad 2, \quad 5), \quad \vec{\mathbf{a}}_2 = (-1, \quad 8, \quad 4)$$

- **Equality:** Two $m\times n$ matrices A and B are said to be **equal** if $a_{ij} = b_{ij}$ for each i and j.
- **Scalar Multiplication:** If A is an m×n matrix and α is a scalar, then αA is the m×n matrix whose (i, j) entry is αa_{ij}.
- **Matrix Addition**: If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m\times n$ matrices, then the sum $A+B$ is the $m\times n$ matrix whose (i, j) entry is $a_{ij}+b_{ij}$ for each ordered pair (i, j).
- **Subtraction**: We can then define $A - B$ to be $A + (-1)B$
- If O represent a matrix, with the same dimension as A, whose entries are all 0, then the following properties must hold

  (1) O acts as the ***additive identity,*** i.e., $A + O = O + A = A$

  (2) each matrix A has an ***additive inverse***, $A + (-1)A = O = (-1)A + A$

  - It is commonly to denote the additive inverse by $-A$, thus $-A = (-1)A$.
- **Transpose**: The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix B defined by $b_{ji} = a_{ij}$ for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. The transpose of A is denoted by A^T.
- An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$.
- **Linear combination**: If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are vectors in R^m and $c_1, c_2, \ldots, c_n$ are scalars, then a sum of the form $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n$ is said to be a **linear combination** of the vectors of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.
- If A is an $m \times n$ matrix and $\mathbf{x} = [x_1, \quad x_2, \quad \ldots \quad x_n]^T$ is a vector in R^n, then $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$ is the linear combination of A's column vectors.
- If $A = (a_{ij})$ is a $m\times n$ matrix and $B = (b_{ij})$ is a $n\times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m\times r$ matrix whose entries are defined by

$$c_{ij} = \vec{\mathbf{a}}_i \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example:

$$\text{If} \quad A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3\cdot(-2)+(-2)\cdot 4 & 3\cdot 1+(-2)\cdot 1 & 3\cdot 3+(-2)\cdot 6 \\ 2\cdot(-2)+4\cdot 4 & 2\cdot 1+4\cdot 1 & 2\cdot 3+4\cdot 6 \\ 1\cdot(-2)+(-3)\cdot 4 & 1\cdot 1+(-3)\cdot 1 & 1\cdot 3+(-3)\cdot 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2\cdot 3+1\cdot 2+3\cdot 1 & -2\cdot(-2)+1\cdot 4+3\cdot(-3) \\ 4\cdot 3+1\cdot 2+6\cdot 1 & 4\cdot(-2)+1\cdot 4+6\cdot(-3) \end{bmatrix}$$

- Multiplication of matrices is NOT <u>commutative</u> (i.e., $AB \neq BA$).
- Linear system and matrix representation: Consider an $m\times n$ linear system

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

...

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$

It can be represented as a matrix equation: $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- **Theorem 1.3.1** A linear system Ax = b is <u>consistent</u> if and only if b can be written as a linear combination of the column vectors of A.

Proof:

1.4 Matrix Algebra

介紹矩陣的乘法性質、單位矩陣、與反矩陣（除法）。並不是每個矩陣都可以做除法，就像不可是每個實數都可以被除一樣，記住有反矩陣的條件，之後每一章都會學習一些新的方法，來了解矩陣是否可逆。

- **Theorm 1.4.1** Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A+B) = \alpha A + \alpha B$

Example: Rule 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}, \text{and } C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\rightarrow \quad A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix} \rightarrow \quad (AB)C = \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$$

Proof of Rule 3:

A be a $m \times n$ matrix, and B an $n \times r$ matrix and C an $r \times s$ matrix. Let $D = AB$ and $E = BC$.

$$d_{il} = \sum_{k=1}^{n} a_{ik}b_{kl} \quad \text{and} \quad e_{kj} = \sum_{l=1}^{r} b_{kl}c_{lj}$$

The ijth term of DC is $\sum_{l=1}^{r} d_{il}c_{lj} = \sum_{l=1}^{r}\left(\sum_{k=1}^{n} a_{ik}b_{kl}\right)c_{lj}$ and the ijth term of AE is $\sum_{k=1}^{n} a_{ik}e_{kj} = \sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{r} b_{kl}c_{lj}\right)$

Since

$$\sum_{l=1}^{r}\left(\sum_{k=1}^{n} a_{ik}b_{kl}\right)c_{lj} = \sum_{l=1}^{r}\left(\sum_{k=1}^{n} a_{ik}b_{kl}c_{lj}\right) = \sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{r} b_{kl}c_{lj}\right),$$

it follows that $(AB)C = DC = AE = A(BC)$.

- The **identity matrix** is the $n \times n$ matrix $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- The identity matrix I for matrix multiplication will serve as $IA = AI = A$

Example: 3x3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} \quad (IA = A) \quad \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} \quad (AI = A)$$

- In general, if B is any $m \times n$ matrix and C is any $n \times m$, then $BI = B$ and $IC = C$
- The column vectors of the $n \times n$ identity matrix I are the standard vectors used to define a coordinate system in Euclidean n-space and its standard notation for the jth column vector of I is $\mathbf{e}_j$, that is $I = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$

- **Matrix Inversion**: An $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there exists a matrix B such that $AB = BA = I$. The matrix B is said to be a **multiplicative inverse** of A. The **inverse** of A is denoted by A^{-1}.
- An $n \times n$ matrix is said to be **singular** if it does not have a multiplicative inverse.

Example:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The following matrix A has no inverse.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad A^{-1} = B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}. \quad BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- A matrix can have at most one multiplicative inverse. That is If B and C are both multiplicative inverse of A (i.e., $BA = AB = I$ and $CA = AC = I$), then

$$B = BI = B(AC) = (BA)C = IC = C$$

- **Theorem 1.4.2:** If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof:

- **Algebraic Rules for Transposes**

I. $(A^T)^T = A$ II. $(aA)^T = aA^T$ III. $(A+B)^T = A^T + B^T$ IV. $(AB)^T = B^T A^T$

Example:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 5 \\ 34 & 23 & 14 \\ 15 & 8 & 9 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & 8 \\ 5 & 14 & 9 \end{bmatrix} \quad B^T A^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & 8 \\ 5 & 14 & 9 \end{bmatrix}$$

1.5 Elementary Matrices

每個 elementary row operation 都可以用一個 elementary matrix 來表示。這裡可以看到矩陣的第二個用途：operator，（那第一個用途是甚麼？）這節的結果之後常被用到，例如定理 1.5.2，要背下來。

- Given an $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, if we multiply both sides of the equation by a <u>nonsingular</u> $m \times m$ matrix M: $A\mathbf{x} = \mathbf{b}$, $MA\mathbf{x} = M\mathbf{b}$. If x is a solution to $MA\mathbf{x} = M\mathbf{b}$, then x is also a solution to $A\mathbf{x} = \mathbf{b}$. So $A\mathbf{x} = \mathbf{b}$ and $MA\mathbf{x} = M\mathbf{b}$ are <u>equivalent.</u>
- A matrix obtained by performing <u>exactly one</u> elementary row operation on the identity matrix I is called an **elementary matrix**.
- **<u>Type I</u>**: Interchanging two rows of I

Example:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$AE_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$$

- **<u>Type II</u>**: Multiplying a row of *I* by a nonzero constant

Example:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & a_{32} & 3a_{33} \end{bmatrix}$$

$$AE_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{bmatrix}$$

- **<u>Type III</u>**: Adding a multiple of one row of *I* to another row

Example:

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Row operation:} \quad E_3A = \begin{bmatrix} a_{11}+3a_{31} & a_{12}+3a_{32} & a_{13}+3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Column operation:} \quad AE_3 = \begin{bmatrix} a_{11} & a_{12} & 3a_{11}+a_{13} \\ a_{21} & a_{22} & 3a_{21}+a_{23} \\ a_{31} & a_{32} & 3a_{31}+a_{33} \end{bmatrix}$$

- Suppose that *E* is an $n \times n$ elementary matrix.
  If *A* is an $n \times r$ matrix, <u>premultiplying</u> *A* by *E* has the effect of performing that same <u>row operation</u> on *A*. If *B* is an $m \times n$ matrix, <u>postmultiplying</u> *B* by *E* is equivalent to performing that same <u>column operation</u> on *B*.
- **Theorem 1.5.1:** If *E* is an elementary matrix, then *E* is nonsingular and E^{-1} is an elementary matrix of the same type.

Proof: **Type I**: interchange of two rows $EE = I \Rightarrow E^{-1} = E$

<u>Type II</u>: multiplying the *i*th row of *I* by a nonzero scalar α

$$E = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \quad i\text{th row} \qquad E^{-1} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & 1 & & & & \\ & & & \boxed{} & & & \\ & & & & 1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \quad i\text{th row}$$

- **<u>Type III</u>**: adding *m* times the *i*th row to the *j*th row

$$E = \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & \mathbf{0} & & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix} \qquad E^{-1} = \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & \mathbf{0} & & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & \boxed{\quad} & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix}$$

- A matrix B is row equivalent to A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that $B = E_kE_{k-1} \ldots E_1A$.

- Two augmented matrices $(A \mid \mathbf{b})$ and $(B \mid \mathbf{c})$ are row equivalent iff $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ are equivalent systems.

  I. If A is row equivalent to B, B is row equivalent to A.

  II. If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

- **Theorem 1.5.2** Let A be an $n\times n$ matrix. The following are equivalent:

  (a) A is nonsingular.

  (b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.

  (c) A is row equivalent to I.

Proof: (a) $\Rightarrow$ (b) If A is nonsingular (i.e., A^{-1} exists), then for $A\mathrm{x} = 0$

____

(b) $\Rightarrow$ (c) If $A\mathrm{x} = 0$ has only the trivial solution 0

$A\mathrm{x} = 0 \Rightarrow U\mathrm{x} = 0$ where U is in row echelon form and $U = E_k \ldots E_1A$. From Theorem 1.2.1.

In U, ____

Thus, U must be a strictly triangular matrix with diagonal elements all equal to 1.

$\therefore$ I will be the reduced row echelon form of A, so A is row equivalent to I.

(c) $\Rightarrow$ (a) If A is row equivalent to I, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $A = E_kE_{k-1}\ldots E_1I = E_kE_{k-1}\ldots E_1$. A is nonsingular (invertible) because $A^{-1} =$ __________________

- **Corollary 1.5.3**: The system of n linear equations in n unknown $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is nonsingular.

Proof: ($\Leftarrow$) If A is nonsingular, and x_0 is any solution of $A\mathbf{x} = \mathbf{b}$, then

____

($\Rightarrow$) Suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution x_0 and <u>A is singular</u>

$\therefore$ $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{z}\neq 0$ (i.e., $A\mathbf{z} = \mathbf{0}$). Let $\mathrm{y} = \mathrm{x}0 + \mathrm{z}$.

____

- If A is nonsingular, A is row equivalent to I, so there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k E_{k-1} \dots E_1 A = I$. So A^{-1} = ____.

- The same series of elementary row operations that **transform a nonsingular matrix** A **into** I **will transform** I **into** A^{-1}. That is, the reduced row echelon form of the augmented matrix $(A \mid I)$ will be $(I \mid A^{-1})$.

Example:

$$A=\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix},\quad \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[I \,\middle|\, A^{-1}\right],\quad A^{-1}=\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

- An $n \times n$ matrix A is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$.
- An $n \times n$ matrix A is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$.
- A matrix is said to be **triangular** if it is either upper triangular or lower triangular
- An $n \times n$ matrix $\boldsymbol{A}$ is said to be **diagonal** if $a_{ij} = 0$ whenever $i \neq j$.
- If an $n\times n$ matrix $\boldsymbol{A}$ can be reduced to <u>strict upper triangular form</u> using **only row operation Ⅲ**, then it is possible to represent the reduction process in terms of a **matrix factorization**.

Example: LU factorization

$$A=\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[l_{31}=2]{l_{21}=1/2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{l_{32}=-3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix},\quad LU=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}=\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}=A$$

1.6 Partitioned Matrices

矩陣切塊之後，加減乘除還是有規律可循，這部分對於之後證明的寫法很有幫助，可以大幅簡化證明寫法，之後在演算法中的快速矩陣乘法，Strassen algorithm 也會用到，對於高效能運算也很重要（想去參加 SC 比賽的同學要特別留意）。

- A matrix can be partitioned into small submatrices (called **blocks**)

Example:

$$C=\begin{bmatrix} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{bmatrix}=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}=\left[\begin{array}{ccc|cc} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{array}\right]$$

$$B=\begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}=\left[\mathbf{b}_1, \quad \mathbf{b}_2, \quad \mathbf{b}_3\right]=\left[\begin{array}{c|c|c} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{array}\right]$$

- If A is an $m \times n$ matrix and B is $n \times r$ which has been partitioned into columns $[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r]$, then the block multiplication of A times B is given by $AB = A\,[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r] = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_r]$

- Let A be an $m \times n$ matrix and B an $n \times r$ matrix, consider the following 4 cases

CASE 1. $B = [B_1, B_2]$, where B_1 is an $n \times t$ matrix and B_2 is an $n \times (r-t)$ matrix, then

$AB = A[\mathbf{b}_1, \ldots, \mathbf{b}_t, \mathbf{b}_{t+1}, \ldots, \mathbf{b}_r] = [A\mathbf{b}_1, \ldots, A\mathbf{b}_t, A\mathbf{b}_{t+1}, \ldots, A\mathbf{b}_r] = [A[\mathbf{b}_1, \ldots, \mathbf{b}_t], A[\mathbf{b}_{t+1}, \ldots, \mathbf{b}_r]] = [AB_1, AB_2]$

Thus, $A[B_1, B_2] = [AB_1, AB_2]$

CASE 2.

$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_1 is a $k \times n$ matrix and A_2 is a $(m-k) \times n$ matrix.

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \left[\begin{array}{c} \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_k \\ \hline \vec{\mathbf{a}}_{k+1} \\ \vdots \\ \vec{\mathbf{a}}_m \end{array}\right] B = \left[\begin{array}{c} \vec{\mathbf{a}}_1 B \\ \vdots \\ \vec{\mathbf{a}}_k B \\ \hline \vec{\mathbf{a}}_{k+1} B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{array}\right] = \begin{bmatrix} \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_k \end{bmatrix} B \\ \begin{bmatrix} \vec{\mathbf{a}}_{k+1} \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B \end{bmatrix} = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

CASE 3.

$A = [A_1, A_2]$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, where A_1 is an $m \times s$ matrix, A_2 is an $m \times (n-s)$ matrix,

B_1 is an $s \times r$ matrix, and B_2 is an $(n-s) \times r$ matrix, if $C = AB$, then

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} = \sum_{l=1}^{s} a_{il} b_{lj} + \sum_{l=s+1}^{n} a_{il} b_{lj}$$

Thus c_{ij} is the sum of the (i, j) entry of A_1B_1 and the (i, j) entry of A_2B_2. Therefore,

$$C = AB = [A_1 \quad A_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1 B_1 + A_2 B_2$$

CASE 4. Let A and B be partitioned as follows:

$$A = \begin{array}{c} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} k \\ m-k \end{array} \\ \begin{array}{cc} s & n-s \end{array} \end{array}, \quad B = \begin{array}{c} \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \begin{array}{l} s \\ n-s \end{array} \\ \begin{array}{cc} t & r-t \end{array} \end{array}$$

$$A_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}, B_1 = [B_{11} \quad B_{12}], \quad B_2 = [B_{21} \quad B_{22}] \quad AB = [A_1 \quad A_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1 B_1 + A_2 B_2$$

$$A_1 B_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} B_1 = \begin{bmatrix} A_{11} B_1 \\ A_{21} B_1 \end{bmatrix} = \begin{bmatrix} A_{11}[B_{11} \quad B_{12}] \\ A_{21}[B_{11} \quad B_{12}] \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix}$$

$$A_2 B_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} B_2 = \begin{bmatrix} A_{12} B_2 \\ A_{22} B_2 \end{bmatrix} = \begin{bmatrix} A_{12}[B_{21} \quad B_{22}] \\ A_{22}[B_{21} \quad B_{22}] \end{bmatrix} = \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

- In summary, if the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & & \\ A_{s1} & \cdots & A_{st} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & & \\ B_{t1} & \cdots & B_{tr} \end{bmatrix},$$

$$\text{then } C = AB = \begin{bmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & & \\ C_{s1} & \cdots & C_{sr} \end{bmatrix}, \text{ where } C_{ij} = \sum_{k=1}^{t} A_{ik}B_{kj}$$

- Given two vectors **x** and **y** in R^n, the scalar product or the inner product is defined as the matrix product $\mathbf{x}^T\mathbf{y}$, which is the product of a row vector (a $1 \times n$ matrix) times a column vector (an $n \times 1$ matrix) and results in a 1×1 matrix or simply a scalar

$$\mathbf{x}^T\mathbf{y} = [x_1, x_2, \cdots, x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

- The **outer product** is defined as the matrix product $\mathbf{x}\mathbf{y}^T$, which is the product of an $n \times 1$ matrix times an $1 \times n$ matrix and results in an $n \times n$ matrix:

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}[y_1, y_2, \cdots, y_n] = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & & & \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix}$$

  - Each of the rows is a multiple of $\mathbf{y}^T$ and each of the column vectors is a multiple of **x**

Chapter 2 Determinant

一個 nxn 的矩陣是否可逆，定理 1.5.2 說的很複雜，有沒有簡單的方法可以判定，如果可逆，他的反矩陣有沒有公式解，這些問題，行列式提供了解答。行列式是一個遞迴的的定義，所以證明多用歸納法。

2.1 The Determinant of a Matrix

定義行列式，和證明一些基本性質。

- The determinant of an $n \times n$ matrix A, det(A), will tell us <u>whether the matrix is nonsingular</u> (its multiplicative inverse exists or not).

<u>Case 1. 1×1 Matrces</u>

- If $A = [a]$ is a 1×1 matrix then A will have a multiplicative inverse iff $a \neq 0$ (i.e., $\det(A) \neq 0$)
- Define $\det(A) = a$
- A is nonsingular iff $\det(A) \neq 0$

<u>Case 2. 2×2 Matrices</u>

- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. By Theorem 1.5.2, A will be nonsingular iff it is row equivalent to I.
- (1) If $a_{11} \neq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

  A is row equivalent to I iff $a_{11}a_{22} - a_{21}a_{12} \neq 0$

- (2) If $a_{11} = 0$, switching the two rows of A, $A' = \begin{bmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{bmatrix}$

  A' is row equivalent to I iff $a_{21} \neq 0$ and $a_{12} \neq 0$. Also can be written as $a_{11}a_{22} - a_{21}a_{12} \neq 0$

<u>Case 3. 3×3 Matrices</u>

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{If } a_{11} \neq 0, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \\ 0 & \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \frac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{bmatrix}$$

- The matrix will be row equivalent to I if and only if

$$a_{11} \begin{vmatrix} \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \\ \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \frac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0$$

- Define $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix},\ M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix},\ M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

- Let $A = (a_{ij})$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the **minor** of a_{ij}. We define the **cofactor** A_{ij} of a_{ij} by $A_{ij} = (-1)^{i+j}\det(M_{ij})$

- The determinant of an $n \times n$ matrix A, denoted $\det(A)$, is a <u>scalar</u> associated with the matrix A that is defined inductively as follows:

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

  where

$$A_{1j} = (-1)^{1+j}\det(M_{1j}), \qquad j = 1, 2, \ldots, n$$

  are the cofactors associated with the entries in the first row of A.

Example:

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$=$ ____

$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$=$ ____

- **Theorem 2.1.1**: If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using <u>any row or any column</u> of A

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$$

  for $i, j = 1, 2, \ldots, n$

- **Theorem 2.1.2**: If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Proof: (Hint: using Theorem 2.1.1 and induction)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\ A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- **Theorem 2.1.3:** If A is an $n \times n$ <u>triangular matrix</u>, then $\det(A)$ = the product of the diagonal elements of A.

Proof: (Hint: induction)

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- **Theorem 2.1.4:** Let A be an $n \times n$ matrix.

  (i) If A has a row or column consisting entirely of zeros, then det(A) = 0

  (ii) If A has two identical rows or two identical columns, then det(A) = 0

Proof: (Hint: prove by induction)

$$A=\begin{bmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & & 0 & & a_{2n} \\ a_{31} & & 0 & & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$A'=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & \vdots & & & \vdots \\ a_{11} & a_{12} & a_{13} & & a_{1n} \\ & \vdots & & & \vdots \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$

2.2 Properties of Determinants

證明更多行列式的基本性質。重要結果，例如矩陣 A 可逆等價於 det(A)≠ 0，det(AB) = det(A) det(B)，要背下來。

- **Lemma 2.2.1** Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for k = 1, 2, ..., n, then

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=\begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Proof:

(i) If $i = j$, it is just a cofactor expansion along the i-th row of A.

(ii) If $i \neq j$, Let A^* be the matrix obtained by replacing the j-th row of A by the i-th row of A:

$$\boldsymbol{A}^*=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{array}{l} \\ \\ j-\text{th row : replaced by } i\text{ - th row} \\ \\ \\ \\ \\ \end{array}$$

- Since A^* has two rows are of the same, det(A^*) = 0. If we expand A^* along the j-th row:

$0 = \det(A^*) =$ ____

- Let E denote the elementary matrix of a row operation, then $\det(EA) = \det(E)\det(A)$ where

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

- Remarks:

– Interchanging two rows of a matrix changes the sign of the determinant.

– Multiplying a single row of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.

– Adding a multiple of one row to another does not change the value of the determinant.

- **Row Operation II**: *A row of A is multiplied by a nonzero constant.*
  - Let E denote the elementary matrix formed from I by multiplying the i-th row by the nonzero constant α. $\det(E) = \det(EI) = \alpha \det(I) = \alpha$
  - If $\det(EA)$ is expanded by cofactors along the i-th row:

    $\det(A_2) = \det(EA) = \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \ldots + \alpha a_{in}A_{in}$

    $= \alpha (a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}) = \alpha \det(A)$
  - $\det(EA) = \alpha \det(A) = \det(E)\det(A)$

$$\mathbf{A}_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & \vdots & & & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \alpha a_{i3} & & \alpha a_{in} \\ & \vdots & & & \vdots \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$

- **Row Operation III**: *A multiple of one row is added to another row.*
  - Let E is formed from I by adding c times the i-th row to the j-th row, $\det(E) = 1$.
  - If $\det(EA)$ is expanded by cofactors along the j-th row:

    $\det(EA) = (a_{j1} + c\, a_{i1})A_{j1} + (a_{j2} + c\, a_{i2})A_{j2} + \ldots + (a_{jn} + c\, a_{in})A_{jn}$

    $= (a_{j1}A_{j1} + a_{j2}A_{j2} + \ldots + a_{jn}A_{jn}) + c\,(a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}) = \det(A)$
  - $\det(EA) = \det(A) = \det(E)\det(A)$

- **Row Operation I**: *Two rows of A are interchanged.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[\text{from 3rd row}]{\text{2nd row is subtracted}} A^{(1)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\xrightarrow{\text{add 2nd row to 3rd row}} A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\xrightarrow{\text{2nd row is subtracted from 3rd row}} A^{(3)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -a_{31} & -a_{32} & -a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\xrightarrow{\text{2nd row is multiplied by (-1)}} A^{(4)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- $\det(A) = \det(A^{(1)}) = \det(A^{(2)}) = \det(A^{(3)})$, $\det(A^{(4)}) = (-1)\det(A^{(3)}) = -\det(A)$
- Let E_{ij} is formed from I by interchanging the i-th row and j-th row of I, then $\det(E_{ij}) = \det(E_{ij}I) = -\det(I) = -1$.
- $\det(E_{ij}A) = -\det(A) = \det(E)\det(A)$

- **Theorem 2.2.2**: An $n\times n$ matrix A is singular iff $\det(A) = 0$

Proof: Let $U = E_kE_{k-1} \ldots E_1A$ is in reduced echelon form, then

$$\det(U) = \det(E_k)\det(E_{k-1}) \ldots \det(E_1)\det(A)$$

Since $\det(E_i) \neq 0$, then $\det(U) = 0$ iff $\det(A) = 0$

If A is singular,

If A is nonsingular,

- **Theorem 2.2.3**: If A and B are $n\times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Proof: If B is nonsingular, B can be written as a product of elementary matrices.

2.3 Additional Topics and Applications

給出 A 反矩陣的公式解，和線性系統的公式解。數學上很漂亮，但是計算上很慢（還記的目前電腦用來解線性系統的方法是甚麼吧！）。

- Let A be an $n \times n$ matrix, define the *adjoint* of A by

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

- Theorem 2.3.1 $\boxed{A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj} A \quad \text{when } \det(A) \neq 0}$

Proof: By Lemma 2.2.1

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + \cdots + a_{1n}A_{1n} & 0 & \cdots & 0 \\ 0 & a_{21}A_{21} + \cdots + a_{2n}A_{2n} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{n1}A_{n1} + \cdots + a_{nn}A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \det\ (A) & 0 & \cdots & 0 \\ 0 & \det\ (A) & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \det\ (A) \end{bmatrix} = \det\ (A) \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \det\ (A)\, I$$

Example: For a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

- **Theorem 2.3.1 (Cramer's Rule)**: Let A be an $n \times n$ nonsingular matrix and let $\mathbf{b} \in R^n$, and let A_i be the matrix obtained by replacing the i-th column of A by $\mathbf{b}$. If $\mathbf{x}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$,

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad \text{for } i = 1, 2, \cdots, n$$

Proof: Since $\mathbf{x} = A^{-1}\mathbf{b} = \dfrac{1}{\det(A)} (\text{adj } A)\, \mathbf{b}$

$$= \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A_i = \begin{bmatrix} a_{11} & \cdots & \boxed{b_1} & \cdots & a_{1n} \\ a_{21} & & b_2 & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & & a_{2i} & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

Example: Use Cramer's rule to solve

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$