Honor Calculus, Math 1450 – Midterm I – Solution.

(1) Given $f(x) = 1 + (x-2)^{\frac{1}{3}}$ on $[0,3]$.

$f'(x) = \frac{1}{3}\frac{1}{(x-2)^{\frac{2}{3}}}$ $\Leftrightarrow$ Critical number: $f'(x)$ DNE $\Rightarrow$ $x=2$.

f': $+$ on $(0,2)$, $+$ on $(2,3)$ $\Rightarrow$ f is always increasing ($f'>0$ except $x=2$) $\Rightarrow$ No local extreme.

For abs. extreme, check the endpoint we have

$f(0) = 1-\sqrt[3]{2}$ is abs. min, $f(3) = 2$ is abs. max.

(2) Assume $f(x)$ is twice differentiable on $\mathbb{R}$ and "a, b, c" are three roots of f. W.L.O.G. we assume $a<b<c$, that is, $f(a)=0$, $f(b)=0$, $f(c)=0$. Then

By MVT, there exist $d \in (a,b)$, $e \in (b,c)$ such that

$$f'(d) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0, \quad f'(e) = \frac{f(c)-f(b)}{c-b} = \frac{0-0}{c-b} = 0$$

which means "d" and "e" are two roots of f'.

Thus f' has at least two roots.

Since f' is differentiable and $f'(d)=0$, $f'(e)=0$, there exists $h \in (d,e)$ such that

$f''(h) = \frac{f'(e)-f'(d)}{e-d} = 0$ which means h is a root of f''.

So f'' has at least one root.

(3) Given $x^2+\frac{y^2}{4}=1$ and an point $(0,\frac{1}{2})$. ($y\in[-2,2]$).

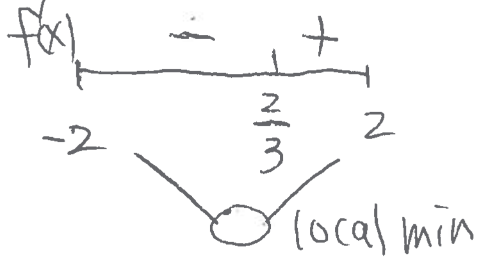
let (x,y) be a point on $x^2+\frac{y^2}{4}=1$, we have $x^2=1-\frac{y^2}{4}$

Distance from (x,y) to $(0,\frac{1}{2})$ $\Rightarrow d=\sqrt{x^2+(y-\frac{1}{2})^2}$

$$\Rightarrow d^2 = 1-\frac{y^2}{4}+(y-\tfrac{1}{2})^2 = 1-\frac{y^2}{4}+y^2-y+\frac{1}{4} = -\frac{3}{4}y^2-y+\frac{5}{4}.$$

Find the closest point $\Leftrightarrow$ Find the smallest d (or d^2)

let $f(y)=-\frac{3}{4}y^2-y+\frac{5}{4}$. $f'(y)=-\frac{3}{2}y-1=0 \Rightarrow y=\frac{2}{3}$.



So check endpoints

$f(-2)=\left(\frac{5}{2}\right)^2 \Rightarrow$ distance $=\frac{5}{2}$

$f(2)=\left(\frac{3}{2}\right)^2 \Rightarrow$ distance $=\frac{3}{2}$

$f(\frac{2}{3})=$

(4) Given $PV^{\frac{3}{2}}=K$ where K is a constant.

(a) At time t_1 we have $P|_{t_1}=5$, $\frac{dP}{dt}\Big|_{t_1}=+0.5$, $V|_{t_1}=4$.

Then, do "$\frac{d}{dt}$", we obtain

$$V^{\frac{3}{2}}\frac{dP}{dt}+P\cdot\frac{3}{2}V^{\frac{1}{2}}\frac{dV}{dt}=0 \xrightarrow{\text{at time } t_1}$$

$$\underbrace{4^{\frac{3}{2}}}_{8}\cdot 0.5+5\cdot\frac{3}{2}\cdot\underbrace{4^{\frac{1}{2}}}_{2}\frac{dV}{dt}=0 \Rightarrow \frac{dV}{dt}=\frac{-4}{15}$$

(4)

(b)

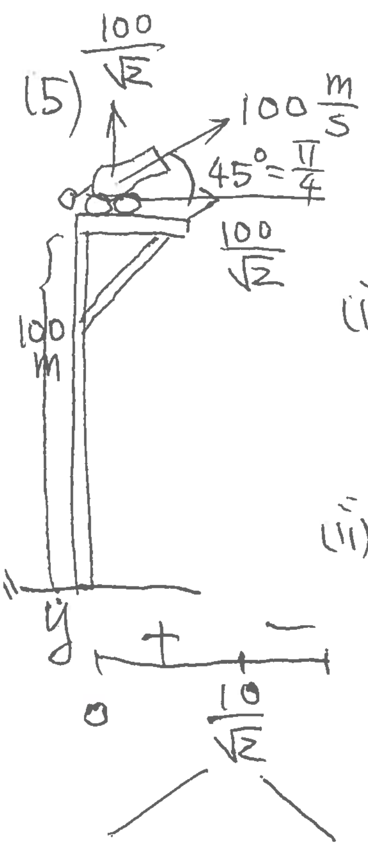
(i) $\lim_{x\to 0} x\ln(x^2) \overset{0\cdot(-\infty)}{=} \lim_{x\to 0}\frac{2\ln(x)}{\frac{1}{x}} \underset{(L')}{\overset{(\frac{-\infty}{\infty})}{=}} \lim_{x\to 0}\frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -\frac{2x^2}{x} = 0$

(ii) $\lim_{x\to\infty}\frac{2x+1}{x+3\sqrt{x}+1} \underset{\text{leading coefficient}}{=} \frac{2}{1} = 2$

(iii) $\lim_{x\to 0} e^{x}\sin(x) = 1\cdot 0 = 0$

(iv) $\lim_{x\to\infty} e^{-\sqrt{3}x}(x^{10}+12x^2) \overset{0\cdot\infty}{=} \lim_{x\to\infty}\frac{x^{10}+12x^2}{e^{\sqrt{3}x}} \underset{L'}{\overset{(\frac{\infty}{\infty})}{=}} \lim_{x\to\infty}\frac{10x^9+24x}{\sqrt{3}\,e^{\sqrt{3}x}} = 0$

(5)



Assume

$\ddot{y} = -10\frac{m}{s^2}$, $\ddot{x} = 0$, $y(0) = 100$, $x(0) = 0$.

↓ anti-derivative $\qquad \dot{y}(0) = \frac{100}{\sqrt{2}}$, $\dot{x}(0) = \frac{100}{\sqrt{2}}$

(i) $\dot{y}(t) = -10t + \dot{y}(0) = -10t + \frac{100}{\sqrt{2}}$

$y(t) = -5t^2 + \frac{100}{\sqrt{2}}t + y(0) = -5t^2 + \frac{100}{\sqrt{2}}t + 100$.

(ii) Find the greatest height → max y.

$\dot{y} = -10t + \frac{100}{\sqrt{2}} = 0 \Rightarrow t = \frac{10}{\sqrt{2}}$

local max, as $t = \frac{10}{\sqrt{2}}$ $\quad y(\frac{10}{\sqrt{2}}) = -5\cdot\frac{\overset{50}{\cancel{100}}}{\cancel{2}} + \frac{\overset{500}{\cancel{1000}}}{\cancel{2}} + 100$

$= 350$ m

is the greatest height.

(iii) horizontal distance before it hits the ground $\Rightarrow y(t) = 0$.

$\Rightarrow -5t^2 + \frac{100}{\sqrt{2}}t + 100 = 0$, $t = \frac{-\frac{100}{\sqrt{2}} \pm \sqrt{7000}}{-10} = \frac{10}{\sqrt{2}} \pm \sqrt{70}$

$\overset{t>0}{\Rightarrow} t = \frac{10}{\sqrt{2}} + \sqrt{70}$.

Since $\dot{x}(t) = \frac{100}{\sqrt{2}}$, $x(0) = 0$ $\Rightarrow$ $x(t) = \frac{100}{\sqrt{2}} t$

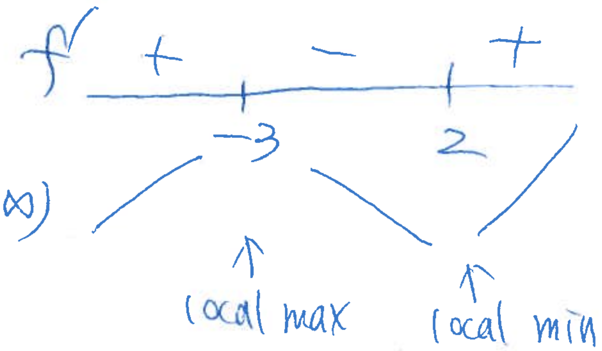
as $t = \frac{10}{\sqrt{2}} + \sqrt{70}$, we have.

$$x\left(\frac{10}{\sqrt{2}} + \sqrt{70}\right) = \frac{100}{\sqrt{2}}\left(\frac{10}{\sqrt{2}} + \sqrt{70}\right) = 500 + 100\sqrt{35} \text{ (m)}$$

(6) Given $f(x) = 2x^3 + 3x^2 - 36x$.

$f'(x) = 6x^2 + 6x - 36 = 0 \Rightarrow x^2 + x - 6 = 0$, $(x+3)(x-2) = 0$

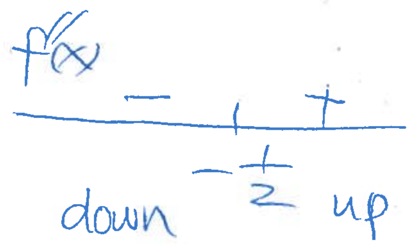
$x = -3$ or 2



(a) Increasing intervals: $(-\infty, -3) \cup (2, \infty)$

(b) decreasing interval $(-3, 2)$.

$f''(x) = 12x + 6 = 0 \Rightarrow x = -\frac{1}{2}$



(c) Concave up interval $(-\frac{1}{2}, \infty)$

(d) concave down interval $(-\infty, -\frac{1}{2})$

$f(0) = 0$