MAT1375, Classwork17, Fall2025

Ch16. Compound interest and half-life

1. The function to describe the population growths or decline:

$$\boxed{f(t) = c \cdot b^t,}$$

$r = 3\% = 0.03$

$b = e^{0.03}$

where $f(0) =$ __c__ which is initial condition at $t =$ __0__, and $b > 0, b \neq 1$.

If the **rate of growth** r is given, then the base $b =$ __e^r__ and $f(t) = c \cdot ($__e^r__$)^t$.

2. The population size of a country was 12.7 million in the year 2010, and 14.3 million in the year 2020.

(a) Assuming an exponential growth for the population size, find the formula for the population depending on the year t.

① Exponential growth $P(t) = c \cdot b^t$

② 12.7 m in year 2010

③ 14.3 m in year 2020

$t=0$, $P(0) = 12.7$, $\boxed{12.7} = P(0) = c \cdot b^0 = c \cdot 1 = \boxed{c}$

$P(t) = (12.7) \cdot b^t$

$t = 2020 - 2010 = 10$, $P(10) = 14.3$, $14.3 = P(10) = (12.7) b^{10}$

$14.3 = (12.7) b^{10} \Rightarrow \sqrt[10]{b^{10}} = \sqrt[10]{\frac{14.3}{12.7}} \Rightarrow b = \sqrt[10]{\frac{14.3}{12.7}} = \left(\frac{14.3}{12.7}\right)^{\frac{1}{10}}$ By ①②③. $P(t) = (12.7) \cdot \left(\frac{14.3}{12.7}\right)^{\frac{t}{10}}$

(b) What will the population size be in the year 2025, assuming the formula holds until then?

in year 2025, it is $t = 2025 - 2010 = 15$

$P(15) = (12.7) \cdot \left(\frac{14.3}{12.7}\right)^{\frac{15}{10}}$

(c) When will the population reach 18 million?

$18 = P(t) = (12.7) \cdot \left(\frac{14.3}{12.7}\right)^{\frac{t}{10}} \Rightarrow \frac{18}{12.7} = \left(\frac{14.3}{12.7}\right)^{\frac{t}{10}}$

Take "ln" on the both sides

$\ln\left(\frac{18}{12.7}\right) = \ln\left(\left(\frac{14.3}{12.7}\right)^{\frac{t}{10}}\right)$

$\boxed{\ln\left(\frac{18}{12.7}\right)} = \boxed{\frac{t}{10} \cdot \ln\left(\frac{14.3}{12.7}\right)}$

$t = 10 \cdot \frac{\ln\left(\frac{18}{12.7}\right)}{\ln\left(\frac{14.3}{12.7}\right)} = 29.4 \Rightarrow$ year $2010 + 29 = 2039$

3. The population of a country grows exponentially at a rate of 2.6% per year. If the population was 35.7 million in the year 2000, then what is the population size of this country in the year 2027?

① grow exponentially at a rate 2.6% : $r = 0.026 \Rightarrow b = e^{0.026}$

② 35.7 m in year 2000 $\Rightarrow t=0$, $P(0) = 35.7 \Rightarrow c = 35.7$

$P(t) = (35.7) \cdot e^{0.026t}$

$P(t)$ in year 2027 $\Rightarrow t = 2027 - 2000 = 27$

$P(27) = (35.7) \cdot e^{0.026(27)}$ million

$(e = 2.718 \cdots\cdots)$

4. If a principal (i.e. initial amount) $\boldsymbol{P}$ is invested for $\boldsymbol{t}$ years at a rate $\boldsymbol{r}$ and compounded $\boldsymbol{n}$ times per year, then the final amount $\boldsymbol{A}$ is given by

① ② ③

$$A(t) = P \cdot \left(\boxed{1+\frac{r}{n}}\right)^{n\cdot t}.$$

If a **continuous compounding** ($n \to$ ∞) is considered, then the final amount $\boldsymbol{A}$ with a continuous compounding is given by $A(t) = P \cdot (e^r)^t$.

5. How much do you get if you invest \$500 today at 3% compounded quarterly in 3 years?

① $P = 500$

② $r = 0.03$

③ $n = 4$ (how many quarters in a year)

④ $t = 3$

$A(t) = 500 \cdot \left(1+\frac{0.03}{4}\right)^{4t}$

$\longrightarrow A(3) = 500 \cdot \left(1+\frac{0.03}{4}\right)^{4\cdot 3} = 500\left(1+\frac{0.03}{4}\right)^{12}$

6. The exponential function $f(t)$ has a half-life of $\boldsymbol{h}$ if the base is given by

$$b = \left(\frac{1}{2}\right)^{\frac{1}{h}},$$

and $f(t) = c \cdot \left(\left(\frac{1}{2}\right)^{\frac{1}{h}}\right)^t = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$.

7. The half-life of carbon-14 is $h=$ 5730 years. If a dead tree trunk has 86% (0.86) of its original ($t=0$) carbon-14. how many years ago did the tree die?

$f(t) = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$c \Rightarrow t=0,\ f(0) = c$ original

$t = ?$ (current) $f(t) = 0.86 \cdot c$

$\boxed{0.86 \cdot c = f(t) = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}}}$

$\frac{0.86c}{c} = \frac{c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}}}{c} \Rightarrow 0.86 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

Take "ln": $\ln(0.86) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right)$

$\ln(0.86) = \frac{t}{5730} \cdot \ln\left(\frac{1}{2}\right)$

$5730 \cdot \frac{\ln(0.86)}{\ln\left(\frac{1}{2}\right)} = \frac{t}{\cancel{5730}} \cdot \cancel{5730} \Rightarrow t = 5730 \cdot \frac{\ln(0.86)}{\ln(0.5)}$

$= 1247$ years