Mat 1372 HW10

4.11 Is it Bernoulli? Determine if each trial can be considered an independent Bernoulli trial for the following situations.

(a) Cards dealt in a hand of poker.

(b) Outcome of each roll of a die.

Sol (a) No. The cards are not independent. Once a card has been picked, it will not be able to pick again.

(b) NO. There are six events from rolling a die and there can only be 2 events for Bernoulli trial.

4.12 With and without replacement. In the following situations assume that half of the specified population is male and the other half is female.

(a) Suppose you're sampling from a room with 10 people. What is the probability of sampling two females in a row when sampling with replacement? What is the probability when sampling without replacement?

(b) Now suppose you're sampling from a stadium with 10,000 people. What is the probability of sampling two females in a row when sampling with replacement? What is the probability when sampling without replacement?

(c) We often treat individuals who are sampled from a large population as independent. Using your findings from parts (a) and (b), explain whether or not this assumption is reasonable.

Sol: (a) With replacement: $P(2 \text{ females in a row}) = \frac{5}{10} \cdot \frac{5}{10} = \frac{1}{4}$

without replacement: $P(2 \text{ females in a row}) = \frac{5}{10} \cdot \frac{4}{9} = \frac{20}{90} = 0.222$

(b) with replacement: $P(2 \text{ females in a row}) = \frac{5000}{10000} \cdot \frac{5000}{10000} = \frac{1}{4} = 0.25$

without replacement: $P(2 \text{ females in a row}) = \frac{5000}{10000} \cdot \frac{4999}{9999} = 0.249975$

(c) When the population size is small, the results of sampling with and without replacement are quite different. But if the population size is large, the results are almost identical

4.15 Bernoulli, the mean. Use the probability rules from Section 3.4 to derive the mean of a Bernoulli random variable, i.e. a random variable X that takes value 1 with probability p and value 0 with probability $1-p$. That is, compute the expected value of a generic Bernoulli random variable.

Sol: Assume X be a Bernoulli Random Variable.

We have $P(X=1)=p$

$P(X=0)=1-p$

Then the expected value of X is

$$E(X) = 1\cdot P(X=1) + 0\cdot P(X=0)$$
$$= 1\cdot p + 0\cdot (1-p) = p.$$

4.16 Bernoulli, the standard deviation. Use the probability rules from Section 3.4 to derive the standard deviation of a Bernoulli random variable, i.e. a random variable X that takes value 1 with probability p and value 0 with probability $1-p$. That is, compute the square root of the variance of a generic Bernoulli random variable.

Sol: Assume X be a Bernoulli Random Variable.

We have $P(X=1)=p$

$P(X=0)=1-p$

Then we have the standard deviation of X is

$$SD(X) = (1-E(X))^2 P(X=1) + (0-E(X))^2 P(X=0)$$
$$= (1-p)^2\cdot p + (0-p)^2\cdot (1-p) = (1-p)^2 p + p^2(1-p)$$
$$= (1-p)\cdot p\,[(1-p)+p] = p(1-p)$$

4.17 Underage drinking, Part I. Data collected by the Substance Abuse and Mental Health Services Administration (SAMSHA) suggests that 69.7% of 18-20 year olds consumed alcoholic beverages in any given year.[31]

(a) Suppose a random sample of ten 18-20 year olds is taken. Is the use of the binomial distribution appropriate for calculating the probability that exactly six consumed alcoholic beverages? Explain.

(b) Calculate the probability that exactly 6 out of 10 randomly sampled 18- 20 year olds consumed an alcoholic drink.

(c) What is the probability that exactly four out of ten 18-20 year olds have *not* consumed an alcoholic beverage?

(d) What is the probability that at most 2 out of 5 randomly sampled 18-20 year olds have consumed alcoholic beverages?

(e) What is the probability that at least 1 out of 5 randomly sampled 18-20 year olds have consumed alcoholic beverages?

Sol: (a) Yes if the following conditions are met:

① Independent trails: In a random sample, whether or not one sample has consumed alcohol doesn't depend on whether or not another one has

② Fixed number of trails: $n=10$

③ Only two outcomes at each trail: consumed or not consumed alcohol.

④ Probability of "consumed" (a sucess) is the same for each trail: $p=0.697$.

(b) $P(\text{exactly 6 consumed out of } 10) = \frac{10!}{6!4!}(0.697)^6(1-0.697)^{10-6}$

$$= \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}(0.697)^6(0.303)^4 = 210\cdot(0.697)^6(0.303)^4 = 0.203$$

(c) $P(\text{exactly 4 "not consumed" out of } 10) = \frac{10!}{4!6!}(0.303)^4\cdot(0.697)^{10-4}$

$$= 210\cdot(0.303)^4\cdot(0.697)^6 = 0.203$$

(d) $P(\text{at most 2 "consumed" out of } 5)$

$= P(\text{exactly 0 "consumed" out of 5}) + P(\text{exactly 1 "consumed" out of 5}) + P(\text{exactly 2 "consumed" out of 5})$

$$= \frac{5!}{0!5!}(0.697)^0\cdot(0.303)^{5-0} + \frac{5!}{1!4!}(0.697)^1(0.303)^{5-1} + \frac{5!}{2!3!}(0.697)^2(0.303)^{5-2}$$

$$= 0.0026 + 0.0293 + 0.1351 = 0.167$$

(e) P(at least 1 "consumed" out of 5)

$= 1 - P(\text{exactly } 0 \text{ "consumed" out of } 5)$

$= 1 - \frac{5!}{0!5!}(0.697)^0(0.303)^{5-0} = 1 - 0.0026 = 0.9974$

4.19 Underage drinking, Part II. We learned in Exercise 4.17 that about 70% of 18-20 year olds consumed alcoholic beverages in any given year. We now consider a random sample of fifty 18-20 year olds.

50

(a) How many people would you expect to have consumed alcoholic beverages? And with what standard deviation?

(b) Would you be surprised if there were 45 or more people who have consumed alcoholic beverages?

(c) What is the probability that 45 or more people in this sample have consumed alcoholic beverages? How does this probability relate to your answer to part (b)?

Sol: $n=50$. $p=0.7$. Assume X be the number of sample who consumed.

(a) Expected value of the sample who consumed:

$E(X) = np = 50 \cdot 0.7 = 35$

$SD(X) = \sqrt{np(1-p)} = \sqrt{50 \cdot 0.7 \cdot 0.3} = 3.24$

If $p = 0.697$
$E(X) = 34.85$
$SD(X) = 3.25$

(b) $Z = \frac{45-35}{3.24} = \frac{10}{3.24} = 3.09$ which means 45 is 3 SD above the mean and it is an unusual observation. So Yes!

(c) $P(X \geq 45) = P(X=45) + P(X=46) + P(X=47) + \cdots + P(X=50)$

We can either find out all the terms above or we can check if we can use normal distribution to approximate it:

$np = 50 \cdot 0.7 = 35 > 10$

$n(1-p) = 50(1-0.7) = 15 > 10$

$\Rightarrow$ Yes, we can approximate it by

$N(\mu = 35, \sigma = 3.24)$

Then $P(X \geq 45) = P(Z \geq 3.09)$

$= 1 - P(Z < 3.09)$

$= 1 - 0.9990 = 0.0009.$

This is why we said in (b) that we would be surprized if there are 45 people concumed.

4.21 Game of dreidel. A dreidel is a four-sided spinning top with the Hebrew letters *nun*, *gimel*, *hei*, and *shin*, one on each side. Each side is equally likely to come up in a single spin of the dreidel. Suppose you spin a dreidel three times. Calculate the probability of getting

(a) at least one *nun*?
(b) exactly 2 *nuns*?
(c) exactly 1 *hei*?
(d) at most 2 *gimels*?

Photo by Staccabees, cropped
(http://flic.kr/p/7gLZTf)

Sol: (a) $p(\text{nun}) = \frac{1}{4}$. $p(\text{not nun}) = 1 - \frac{1}{4} = \frac{3}{4}$.

Assume X be the number of nun. Then X is a Binomial R.V.

$p(\text{at least one num out of } 3) = 1 - P(X=0)$

$= 1 - \frac{3!}{0!3!}\left(\frac{1}{4}\right)^0 \cdot \left(\frac{3}{4}\right)^{3-0} = 1 - (0.75)^3 = 0.5781$

(b) $P(\text{eatly 2 nuns out of } 3) = \frac{3!}{2!1!}\left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{3-2}$

$= 3 \cdot \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) = \frac{9}{64} = 0.1406$

(c) $p(\text{hei}) = \frac{1}{4}$, $p(\text{not hei}) = 1 - \frac{1}{4} = \frac{3}{4}$

$p(\text{eactly 1 hei out of } 3) = \frac{3!}{1!2!}\left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{3-1}$

$= 3 \cdot \left(\frac{1}{4}\right)^1 \cdot \left(\frac{3}{4}\right)^2 = \frac{27}{64} = 0.4219$

(d) $p(\text{gimel}) = \frac{1}{4}$, $p(\text{not gimel}) = \frac{3}{4}$

$p(\text{at most 2 gimels out of } 3)$

$= 1 - P(\text{3 gimels out of } 3) = 1 - \frac{3!}{3!0!}\left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^{3-3}$

$= 1 - \frac{1}{64} = \frac{63}{64} = 0.9844$

4.24 Sickle cell anemia. Sickle cell anemia is a genetic blood disorder where red blood cells lose their flexibility and assume an abnormal, rigid, "sickle" shape, which results in a risk of various complications. If both parents are carriers of the disease, then a child has a 25% chance of having the disease, 50% chance of being a carrier, and 25% chance of neither having the disease nor being a carrier. If two parents who are carriers of the disease have 3 children, what is the probability that

(a) two will have the disease?

(b) none will have the disease?

(c) at least one will neither have the disease nor be a carrier?

(d) the first child with the disease will the be 3^{rd} child?

Sol: Both parents are carrier and they have the child that

25% having the disease

50% being a carrier } 75% not having the disease

25% neither

Assume X be the number of children who have disease, then

X is a binomial R.V.

(a) $P(X=2) = \frac{3!}{2!1!}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{3-2} = 3\cdot\frac{1}{16}\cdot\frac{3}{4} = \frac{9}{64} = 0.1406$

(eactly 2 out of 3)

(b) $P(X=0) = \frac{3!}{0!3!}\left(\frac{1}{4}\right)^0\cdot\left(\frac{3}{4}\right)^{3-0} = 1\cdot1\cdot\frac{27}{64} = 0.4219$

(c) 25% having the disease } 75% either disease or carrier

50% being a carrier

25% neither

$P(\text{neither}) = \frac{1}{4}$; $P(\text{either or}) = \frac{3}{4}$

$P(\text{at least one be neither out of } 3)$

$= 1 - P(\text{no one is neither}) = 1 - \frac{3!}{0!3!}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{3-0} = 1 - \frac{27}{64} = \frac{37}{64}$

$= 0.5781$

(d) $P(\text{the first one has disease is the third one})$

$= \underset{\uparrow\,1^{st}\text{ healthy}}{(0.75)}\cdot\underset{\uparrow\,2^{nd}\text{ healthy}}{(0.75)}\cdot\underset{\uparrow\,3^{rd}\text{ disease}}{(0.25)} = 0.1406.$

4.26 Male children. While it is often assumed that the probabilities of having a boy or a girl are the same, the actual probability of having a boy is slightly higher at 0.51. Suppose a couple plans to have 3 kids.

(a) Use the binomial model to calculate the probability that two of them will be boys.

(b) Write out all possible orderings of 3 children, 2 of whom are boys. Use these scenarios to calculate the same probability from part (a) but using the addition rule for disjoint outcomes. Confirm that your answers from parts (a) and (b) match.

(c) If we wanted to calculate the probability that a couple who plans to have 8 kids will have 3 boys, briefly describe why the approach from part (b) would be more tedious than the approach from part (a).

Sol: $P(\text{baby boy}) = 0.51$, $P(\text{baby girl}) = 1-0.51 = 0.49$

Assume X be the number of boys out of 3 kids. Then X is a Binomial R.V.

(a) $\underbrace{P(X=2)}_{\text{2 boys out of 3}} = \frac{3!}{2!\,1!}(0.51)^2(0.49)^{3-1} = 3\cdot(0.51)^2(0.49)$
$= 0.3823$

(b) $P(B.B.G) = (0.51)(0.51)(0.49) = 0.12744$

$P(B,G,B) = (0.51)(0.49)(0.51) = 0.12744$

$P(G,B,B) = (0.49)(0.51)(0.51) = 0.12744$

$\Rightarrow P(\text{2 boys out of 3}) = P(B,B,G) + P(B,G,B) + P(G,B,B)$
$= 3\cdot 0.12744 = 0.3823$

(c) If we only want to know the probabiltiey of how many boys and girls they have, Using part (a) will be more efficient than part (b).