1 Riemann Sum

Given a continuous function f and a partition $P = \{x_0 = a, x_1, x_2, \cdots, x_{n-1}, x_n = b\}$ on $[a, b]$. Then we can estimate $\int_a^b f(x)\,dx$ by Riemann Sum:

$$\sum [(\text{ length of the subinterval}) \times (\text{ value of } f \text{ on this subinterval})]$$

(1) Upper sum (U_f)

$$U_f = \sum [(\text{ length of the subinterval}) \times (\text{ maximum value of } f \text{ on this subinterval})]$$

(2) Lower sum (L_f)

$$L_f = \sum [(\text{ length of the subinterval}) \times (\text{ minimum value of } f \text{ on this subinterval})]$$

(3) $U_f \geq L_f$

(2) Specific points:(left endpoint, right endpoint, midpoint)

$$\text{Sum } = \sum [(\text{ length of the subinterval}) \times (\text{ specific point value of } f \text{ on this subinterval})]$$

2 Basic Integration Properties

Assume f, g are continuous on $[a, b]$ and α, β are constants.

(1) If $a < c < b$, then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

(2) The integration value will change of sign if we integrate in the different directions:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

(3) The integral from any number to itself is defined to be zero:

$$\int_c^c f(x)\,dx = 0.$$

(4) Linearity of integration:

$$\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

3 Fundamental Theorem of Calculus

3.1 First Fundamental Theorem of Calculus

Theorem 3.1 *Let f be continuous on $[a, b]$. The function F defined on $[a, b]$ by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$, differentiable on (a, b), and has derivative $F'(x) = f(x)$ for all x in (a, b).

Assume a function f is defined as above in the theorem and function $u(x), v(x)$ are differentiable, we have

(1) If $F(x) = \int_a^x f(t)\,dt$, then

$$F'(x) = f(x).$$

(2) If $F(x) = \int_x^a f(t)\,dt$, so $F(x) = -\int_a^x f(t)\,dt$, then

$$F'(x) = -f(x).$$

(3) If $F(x) = \int_a^{u(x)} f(t)\,dt$, then

$$F'(x) = u'(x) \cdot f(u(x)).$$

(3) If $F(x) = \int_{v(x)}^{u(x)} f(t)\,dt$, there is a constant c such that

$$F(x) = \int_c^{u(x)} f(t)\,dt - \int_c^{v(x)} f(t)\,dt$$

then we have

$$F'(x) = u'(x) \cdot f(u(x)) - v'(x) \cdot f(v(x)).$$

3.2 Second Fundamental Theorem of Calculus

Definition 3.2 Let f be continuous on $[a, b]$. A function is called an antiderivative for f on $[a, b]$ if

$$F \text{ is continuous on } [a, b] \text{ and } F'(x) = f(x) \text{ for all } x \in (a, b).$$

Theorem 3.3 *Let f be continuous on $[a, b]$. If F is any antiderivative for f on $[a, b]$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

4 Differential

Given $f(a + h)$. we can estimate $f(a + h) - f(a)$ by differential df:

$$f(a + h) - f(a) \approx df = f'(a) \cdot h,$$

then

$$f(a + h) \approx f(a) + f'(a) \cdot h.$$