MAT2440, Classwork46, Spring2025

ID:______________________________ Name:______________________________

II. Examples of Mathematical Induction. → $P(n)$

1. Prove that $\boxed{n^2 + n \text{ is divisible by } 2}$ for all $n \in \mathbb{Z}^+$.

Proof ① show $P(1)$ is true:

$(1)^2+1$ is divisible by 2 True (2 is divisible by 2)

② Assume $P(k)$ is true.

$P(k)$: $\boxed{k^2+k \text{ is divisible by } 2} \Leftrightarrow \frac{k^2+k}{2} = m \Leftrightarrow k^2+k=2m$

③ Prove $P(k) \to P(k+1)$

P(k+1) looks like: $(k+1)^2+(k+1)$ is divisible by 2

$$L.H.S = (k+1)^2+(k+1) = k^2+2k+1+k+1$$
$$= k^2+k+2k+1+1$$
$$= 2m + 2k + 2 \quad \text{the assumption in ②}$$
$$= 2(m+k+1)$$
$$\Rightarrow (k+1)^2+(k+1) \text{ is divisible by } 2$$

The statement is true for all $n \in \mathbb{Z}^+$ by induction

2. Prove that $\boxed{n^3 + 2n \text{ is divisible by } 3}$ $P(n)$ for every integer $n \geq 3$.

Proof: ① show $P(3)$ is true

$3^3+2\cdot 3 = 27+6 = 33$ is divisible by 3 True.

② Assume $P(k)$ is true

k^3+2k is divisible by 3 $\Leftrightarrow k^3+2k = 3m$, $m \in \mathbb{Z}$.

③ Prove $P(k) \to P(k+1)$

P(k+1) looks like $(k+1)^3+2(k+1)$ is divisible by 3

$$L.H.S = (k+1)^3+2(k+1)$$
$$= k^3+3k^2+3k+1+2k+2$$
$$= k^3+2k+3k^2+3k+1+2$$
$$= 3m+3k^2+3k+3 \quad \text{based on the assumption in ②}$$
$$= 3(m+k^2+k+1) \Rightarrow \text{This is divisible by } 3$$

By induction, based on ①,②,③, this statement is true for $n \geq 3$

3. Use mathematical induction to prove the inequality for all positive integer n

$\boxed{n < 2^n.}$ $P(n)$ $n \in \mathbb{Z}^+$

Proof: ① show $P(1)$ is true.

$1 < 2^1$ True

② Assume $P(k)$ is true. : $k < 2^k$ is true. $\rightarrow P(k)$

③ Prove $P(k) \rightarrow P(k+1)$

P(k+1) looks like

$\boxed{k+1 < 2^{k+1}}$

(Left Hand side)

$L.H.S = k+1$

based on the assumption in ②

$< 2^k + 1$

$1 < 2^k$, for $k \in \mathbb{Z}^+$

$< 2^k + 2^k$

$= 2 \cdot 2^k = 2^{k+1} = R.H.S$

By induction, based ①, ②, ③, this statement is true for all $n \in \mathbb{Z}^+$

4. Use mathematical induction to prove the inequality for every positive integer $n \geq 4$

Proof:

$\boxed{2^n < n!} \rightarrow P(n)$

① when $n = 4$. $2^4 = 16$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$16 < 24 \Rightarrow P(4)$ is true

② Assume $P(k)$ is true $(k \geq 4)$ $(2^k < k!)$

③ show $P(k) \rightarrow P(k+1)$

$L.H.S$ of $P(k+1) = 2^{k+1} = 2 \cdot 2^k$

P(k+1) looks like

$\boxed{2^{k+1} \overset{?}{<} (k+1)!}$

$(k+1) \cdot k!$

the assumption in ②

$< 2 \cdot k!$

$k+1 > 2$

$< (k+1)\, k! = (k+1)! = R.H.S.$ of $P(k+1)$

By induction, based on ①, ②, ③, the statement is true for $n \geq 4$.