Mat 2540 HW2

1. Find $f(1), f(2), f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, \ldots$

a) $f(n+1) = f(n) + 2$.
b) $f(n+1) = 3f(n)$.
c) $f(n+1) = 2^{f(n)}$.
d) $f(n+1) = f(n)^2 + f(n) + 1$.

Sol: a) $n=0$, $f(1) = f(0+1) = f(0) + 2 = 1 + 2 = 3$
$n=1$, $f(2) = f(1+1) = f(1) + 2 = 3 + 2 = 5$
$n=2$, $f(3) = f(2+1) = f(2) + 2 = 5 + 2 = 7$
$n=3$, $f(4) = f(3+1) = f(3) + 2 = 7 + 2 = 9$

b) $n=0$, $f(1) = f(0+1) = 3 \cdot f(0) = 3 \cdot 1 = 3$
$n=1$, $f(2) = f(1+1) = 3 \cdot f(1) = 3 \cdot 3 = 9$
$n=2$, $f(3) = f(2+1) = 3 \cdot f(2) = 3 \cdot 9 = 27$
$n=3$, $f(4) = f(3+1) = 3 \cdot f(3) = 3 \cdot 27 = 81$

c) $n=0$, $f(1) = f(0+1) = 2^{f(0)} = 2^1 = 2$
$n=1$, $f(2) = f(1+1) = 2^{f(1)} = 2^2 = 4$
$n=2$, $f(3) = f(2+1) = 2^{f(2)} = 2^4 = 16$
$n=3$, $f(4) = f(3+1) = 2^{f(3)} = 2^{16}$

d) $n=0$, $f(1) = f(0+1) = f(0)^2 + f(0) + 1 = 1^2 + 1 + 1 = 3$
$n=1$, $f(2) = f(1+1) = f(1)^2 + f(1) + 1 = 3^2 + 3 + 1 = 13$
$n=2$, $f(3) = f(2+1) = f(2)^2 + f(2) + 1 = 13^2 + 13 + 1 = 183$
$n=3$, $f(4) = f(3+1) = f(3)^2 + f(3) + 1 = 183^2 + 183 + 1$

3. Find $f(2), f(3), f(4)$, and $f(5)$ if f is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, \ldots$

a) $f(n+1) = f(n) + 3f(n-1)$.
b) $f(n+1) = f(n)^2 f(n-1)$.
c) $f(n+1) = 3f(n)^2 - 4f(n-1)^2$.
d) $f(n+1) = f(n-1)/f(n)$.

a) $n=1$ $f(2) = f(1+1) = f(1) + 3f(0) = 2 + 3 \cdot (-1) = -1$
$n=2$ $f(3) = f(2+1) = f(2) + 3f(1) = (-1) + 3 \cdot 2 = 5$
$n=3$ $f(4) = f(3+1) = f(3) + 3f(2) = 5 + 3 \cdot (-1) = 2$
$n=4$, $f(5) = f(4+1) = f(4) + 3f(4) = 2 + 3 \cdot 5 = 17$

b) $n=1,\ f(2)=f(1+1)=f(1)^2\cdot f(0) = 2^2\cdot(-1) = -4$

$n=2,\ f(3)=f(2+1)=f(2)^2\cdot f(1) = (-4)^2\cdot 2 = 32$

$n=3,\ f(4)=f(3+1)=f(3)^2\cdot f(2) = (32)^2\cdot(-4) = -4096$

$n=4\ \ f(5)=f(4+1)=f(4)^2\cdot f(3) = (-4096)^2\cdot 32 = 2^{29}$

c) $n=1,\ f(2)=f(1+1)=3f(1)^2-4f(0)^2 = 3(2)^2-4(-1)^2 = 8$

$n=2,\ f(3)=f(2+1)=3f(2)^2-4f(1)^2 = 3(8)^2-4(2)^2 = 176$

$n=3,\ f(4)=f(3+1)=3f(3)^2-4f(2)^2 = 3(176)^2-4(8)^2 = 92672$

$n=4,\ f(5)=f(4+1)=3f(4)^2-4f(3)^2 = 3(92672)^2-4(92672)^2$

d) $n=1,\ f(2)=f(1+1)=\dfrac{f(0)}{f(1)} = \dfrac{-1}{2} = -\dfrac{1}{2}$

$n=2,\ f(3)=f(2+1)=\dfrac{f(1)}{f(2)} = \dfrac{2}{-\frac{1}{2}} = -4$

$n=3,\ f(4)=f(3+1)=\dfrac{f(2)}{f(3)} = \dfrac{-\frac{1}{2}}{-4} = \dfrac{1}{8}$

$n=4,\ f(5)=f(4+1)=\dfrac{f(3)}{f(4)} = \dfrac{-4}{\frac{1}{8}} = -32$

5. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.

a) $f(0) = 0, f(n) = 2f(n-2)$ for $n \geq 1$

b) $f(0) = 1, f(n) = f(n-1) - 1$ for $n \geq 1$

c) $f(0) = 2, f(1) = 3, f(n) = f(n-1) - 1$ for $n \geq 2$

d) $f(0) = 1, f(1) = 2, f(n) = 2f(n-2)$ for $n \geq 2$

e) $f(0) = 1, f(n) = 3f(n-1)$ if n is odd and $n \geq 1$ and $f(n) = 9f(n-2)$ if n is even and $n \geq 2$

a) When $n=1$, we have $f(1)=2f(1-2)=2f(-1)$.
However, we don't know $f(-1)$. Thus, this is NOT a valid one.

b) $f(1)=f(1-1)-1=f(0)-1=1-1=0$

$f(2)=f(2-1)-1=f(1)-1=0-1=-1$

$f(3)=f(3-1)-1=f(2)-1=-1-1=-2$

$f(4)=f(4-1)-1=f(3)-1=-2-1=-3$

$\vdots$

$-1,\ -1,\ -1 \Rightarrow$ Assume $f(n)=1+n\cdot(-1)\quad n\geq 1$

Proof: $f(n)=f(n-1)-1 \Rightarrow f(n)-f(n-1)=-1$ and it means that there is a common difference $d=-1$ from previous term to the current term, and $f(n)$ is an arithmetic sequence with initial term $f(0)=1$

Thus, the exact formula of $f(n)$ is

$$f(n)=f(0)+n\cdot d = 1+n\cdot(-1) \quad \text{for } n\geq 1$$

c) $f(0)=2$, $f(1)=3$, $f(n)=f(n-1)-1$ for $n\geq 2$

Since $f(n)=f(n-1)-1$, we have $f(n)-f(n-1)=-1$, which means $f(n)$ is an arithmetic sequence with common difference $d=-1$ with $f(0)=2$, $f(1)=3$.

Therefore, we have $f(n)=3+(n-1)(-1)=3+1-n=4-n$ for $n\geq 1$ and $f(0)=2$

Prove $f(n)=4-n$ for $n\geq 1$ by induction, we have

- Basis Step: $f(1)=4-1=3$ and given $f(0)=2$.
- Inductive Step: Assume $f(k)=4-k$, $k\geq 1$. To prove $f(k+1)=4-(k+1)$,

we have $f(k+1)=f(k)-1=4-k-1=4-(k+1)$

which shows $f(k+1)=4-(k+1)$ is true and

$f(n)=4-n$, $n\geq 1$ and $f(0)=2$.

d) Given $f(0)=1$, $f(1)=2$, $f(n)=2f(n-2)$, $n\geq 2$. We have

$f(0)=1$, $f(1)=2$

$f(2)=2\cdot f(2-2)=2\cdot f(0)=2\cdot 1=2=2^1$

$f(3)=2\cdot f(3-2)=2\cdot f(1)=2\cdot 2=4=2^2$ $\Rightarrow$ Assume $f(n)=2^{\lfloor\frac{n+1}{2}\rfloor}$, $n\geq 0$ (floor function)

$f(4)=2\cdot f(4-2)=2\cdot f(2)=2\cdot 2=4=2^2$

$f(5)=2\cdot f(5-2)=2\cdot f(3)=2\cdot 4=8=2^3$

$\vdots$

To prove $f(n)=2^{\lfloor\frac{n+1}{2}\rfloor}$ for $n\geq 0$ by strong induction, we have

Basis Step: $f(0)=2^{\lfloor\frac{0+1}{2}\rfloor}=2^0=1$ (checked)

$f(1)=2^{\lfloor\frac{1+1}{2}\rfloor}=2^1=2$ (checked)

Inductive step: Assume $f(j)=2^{\lfloor\frac{j+1}{2}\rfloor}$ for $0\leq j\leq k$ with $k\geq 1$.

To prove $f(k+1) = 2^{\lfloor \frac{k+1+1}{2} \rfloor}$, we have

$k \geq 1 \Rightarrow k-1 \geq 0$ and $f(k-1) = 2^{\lfloor \frac{(k-1)+1}{2} \rfloor} = 2^{\lfloor \frac{k}{2} \rfloor}$.

Then, by the original formula,

$$f(k+1) = 2 \cdot f(k-1) = 2 \cdot 2^{\lfloor \frac{k}{2} \rfloor} = 2^{\lfloor \frac{k}{2} \rfloor + 1} = 2^{\lfloor \frac{k}{2} + 1 \rfloor}$$

$$= 2^{\lfloor \frac{k+2}{2} \rfloor} = 2^{\lfloor \frac{(k+1)+1}{2} \rfloor} \text{ (checked)}.$$

By strong induction, $f(n) = 2^{\lfloor \frac{n+1}{2} \rfloor}$, $n \geq 0$.

e) Given $f(0) = 1$, $f(n) = \begin{cases} 3 \cdot f(n-1) & , n \text{ is odd and } n \geq 1 \\ 9 \cdot f(n-2) & , n \text{ is even and } n \geq 2 \end{cases}$

$f(1) = 3 \cdot f(0) = 3 \cdot 1 = 3 = 3^1$

$f(2) = 9 \cdot f(0) = 9 \cdot 1 = 9 = 3^2$

$f(3) = 3 \cdot f(2) = 3 \cdot 9 = 27 = 3^3$ $\Rightarrow$ Assume $f(n) = 3^n$, $n \geq 0$.

$f(4) = 9 \cdot f(2) = 9 \cdot 9 = 81 = 3^4$

$f(5) = 3 \cdot f(4) = 3 \cdot 81 = 3^5$

To prove $f(n) = 3^n$, $n \geq 0$ by strong induction, we have

Basis step: $f(0) = 3^0 = 1$ (checked)

$f(1) = 3 \cdot f(0) = 3 = 3^1$ (checked)

$f(2) = 9 \cdot f(0) = 9 = 3^2$ (checked)

Inductive step: Assume $f(j) = 3^j$ for $0 \leq j \leq k$, $k \geq 2$.

To prove $f(k+1) = 3^{k+1}$, $k+1 \geq 3$ we have two cases:

① If $k+1$ is odd, then

$f(k+1) = 3 \cdot f(k) = 3 \cdot 3^k = 3^{k+1}$

② If $k+1$ is even, then $k-1 \geq 1$

$f(k+1) = 9 \cdot f(k-1) = 9 \cdot 3^{k-1} = 3^{k+1}$

By strong induction, $f(n) = 3^n$ for $n \geq 0$.

7. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n$. **b)** $a_n = 2n + 1$.
c) $a_n = 10^n$. **d)** $a_n = 5$.

a) Given $a_n = 6n$. $a_1 = 6$, $a_2 = 6\cdot 2 = 12$, $a_3 = 6\cdot 3 = 18$, $a_4 = 6\cdot 4 = 24$, $\ldots$, $a_n = 6\cdot n$, $a_{n+1} = 6(n+1)$

$a_2 - a_1 = 6$ $\quad a_3 - a_2 = 6$ $\quad a_4 - a_3 = 6$ $\ldots$ $\quad a_{n+1} - a_n = 6$

$\Rightarrow a_{n+1} = a_n + 6$ with $a_1 = 6$, $n \geq 1$

b) $a_n = 2n+1$, $a_1 = 2\cdot 1+1 = 3$, $a_2 = 2\cdot 2+1 = 5$, $a_3 = 2\cdot 3+1 = 7$, $a_4 = 2\cdot 4+1 = 9$, $\ldots$, $a_n = 2\cdot n+1$, $a_{n+1} = 2(n+1)+1 = 2n+3$

$a_2 - a_1 = 2$ $\quad a_3 - a_2 = 2$ $\quad a_4 - a_3 = 2$ $\quad a_{n+1} - a_n = 2$

$\Rightarrow a_{n+1} = a_n + 2$ $\quad n \geq 1$ with $a_1 = 3$

c) $a_n = 10^n$, $a_1 = 10^1 = 10$, $a_2 = 10^2 = 100$, $a_3 = 10^3 = 1000$, $a_4 = 10^4$, $\ldots$, $a_n = 10^n$, $a_{n+1} = 10^{n+1}$

$\frac{a_2}{a_1} = 10$ $\quad \frac{a_3}{a_2} = 10$ $\quad \frac{a_4}{a_3} = 10$ $\quad \frac{a_{n+1}}{a_n} = 10$

$\Rightarrow a_{n+1} = 10 \cdot a_n$ $\quad n \geq 1$ with $a_1 = 10$

d) $a_n = 5$, $a_1 = 5$, $a_2 = 5$, $a_3 = 5$ $\cdots$ $a_n = 5$, $a_{n+1} = 5$ $\Rightarrow a_{n+1} = a_n$ $\quad n \geq 1$ with $a_1 = 5$.

$a_2 = a_1$ $\quad a_3 = a_2$ $\quad a_{n+1} = a_n$

9. Let F be the function such that $F(n)$ is the sum of the first n positive integers. Give a recursive definition of $F(n)$.

Given $F(n) = \sum_{k=1}^{n} k$, we have

$F(1) = 1$, $F(2) = 1+2 = 3$, $F(3) = 1+2+3 = 6$, $\cdots$, $F(n) = 1+\cdots+n$, $F(n+1) = 1+\cdots+n+(n+1)$

$\Rightarrow F(n+1) = F(n) + (n+1)$. with $F(1) = 1$ and $n \geq 1$

***20.** Give a recursive definition of the functions max and min so that $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, \ldots, a_n$, respectively.

① If there is only 1 element, the max or min value is itself.

② If there are 2 elements, denoted by a_1, a_2, then

$$\max(a_1, a_2) = \begin{cases} a_1 , & a_1 \geq a_2 \\ a_2 , & a_2 > a_1 \end{cases}$$

③ If there are $n+1$ elements, we have

$$\max(a_1, a_2, a_3, \cdots, a_n, a_{n+1}) = \max(\max(a_1, a_2, \cdots, a_n), a_{n+1})$$

(Similarly, $\min(a_1, a_2, a_3, \cdots, a_n, a_{n+1}) = \min(\min(a_1, a_2, \cdots, a_n), a_{n+1})$)