MAT2540, Classwork1, Spring2026

5.2 Strong Induction and Well-Ordering

In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

1. **Mathematical Induction**. To prove that $P(n)$ is true for all positive integers n, we complete 2 steps.

   BASIS STEP: We verify that $P(1)$ is true.

   INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

2. An Example of Proof using Mathematical Induction: *Prove that $n^2 + n$ is divisible by 2 for all $n \in \mathbb{Z}^+$.* belongs to

   Recognize $P(n)$: n^2+n is divisible by 2 (for all $n \in \mathbb{Z}^+$)

   Basis Step. Show $P(1)$ is true: $n=1$ Is $1^2+1=2$ divisible by 2? Yes, it is true.

   Inductive Step. Assume $P(k)$ is true for $k>0$, then prove $P(k+1)$ is true

   k^2+k is divisible by 2 $\Rightarrow \frac{k^2+k}{2} = m$ where m is an integer $\Rightarrow k^2+k=2m$

   Then we have $(k+1)^2+(k+1) = k^2+2k+1+k+1 = k^2+3k+2$

   $= \boxed{k^2+k}+2k+2 = 2m+2k+2 = 2(m+k+1) \Rightarrow (k+1)^2+(k+1)$ is divisible by 2

   Conclusion: $P(n)$ is true for all $n \in \mathbb{Z}^+$

3. **Strong Induction**. To prove that $P(n)$ is true for all positive integers n, we complete 2 steps:

   BASIS STEP: We verify that $P(1)$ is true.

   INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

4. An Example of Proof using Strong Induction.

   Show that if n is an integer greater than 1, then n can be written as the product of primes.

Recognize $P(n)$: if n is an integer greater than 1, then n can be written as the product of primes

Basis Step. Show $P(2)$ is true: Because $2 = 1 \times 2$, this is true.

Inductive Step: Assume $P(j)$ is true for $2 \le j \le k$, it means any j can be a product of primes. To show that $P(k+1)$ is true, that is, $k+1$ is the product of primes. There are two cases:

(1) If $k+1$ is prime. we immediately see that $P(k+1)$ is true.

(2) If $k+1$ is composite, it means $(k+1) = a \cdot b$, $2 \le a \le b \le k$. We can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k+1 = a \cdot b$, then $k+1$ is a product of primes. It means that $P(k+1)$ is true.

5. How to decide which induction to use?

6. **Modified Strong Induction**

Let b be a fixed integer. To prove that $P(n)$ is true for all positive integers n with $n \geq b$, we complete 2 steps:

BASIS STEP: We verify that the propositions p(b), p(b+1), ..., b(j) are true where j is a fixed positive integer and j > b.

INDUCTIVE STEP: We show that the conditional statement [p(b) ∧ p(b+1) ∧ .... ∧ p(k)] → p(k+1)

is true for every positive integer k > j.

7. Prove that *every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps*.

(a) Prove the statement by **Mathematical Induction**.

Basis Step. Show __________ is true: ____.

Inductive Step. Assume __________________________. Then to prove $P(k+1)$ is true, we have 2 cases:

(1) k-cent is formed using ___________ 4-cent stamp: ____

____.

(2) k-cent is formed __________ 4-cent stamps: ____

____

____

____.

Conclusion: ____.

(b) Prove the statement by **Strong Induction**. ________________________________

Basis Step. Show ________________________ are true: Since we have ____________________________

____.

Inductive Step. Assume ____.

To prove $P(k+1)$ is true, we do the following:

____

____

____

____

____

____.

8. The Well-Ordering Property: ____.

Example: $A = \{5, 8, 3, 11\}$. The least element is _____.

9. The Equivalent Principles.