Mat 2540 HW1

3. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 8$.

a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$.

b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$?

c) What do you need to prove in the inductive step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$?

d) Complete the inductive step for $k \geq 10$.

e) Explain why these steps show that $P(n)$ is true whenever $n \geq 8$.

Sol:

(a) Since $8 = 3+5$, then $P(8)$ is true.
since $9 = 3+3+3$, then $P(9)$ is true.
since $10 = 5+5$, then $P(10)$ is true.

(b) Inductive hypothesis:
Assume $P(j)$ is true for $8 \leq j \leq k$ with $k \geq 10$

(c) $[P(8) \wedge P(9) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ for $k \geq 10$.

(d) Since $k \geq 10$, $k-2 \geq 8$, then $P(k-2)$ is true which means $k-2$ can be formed by 3-cent and 5-cent stamps. Then $k+1 = k-2+3$ can also be formed by 3-cent & 5-cent stamps and it means $P(k+1)$ is true

(e) By strong induction, once the basis step and the inductive step are proven true, $P(n)$ is true for $n \geq 8$.

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.

a) Show that the statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$.

b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$?

c) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all integers $n \geq 18$?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that $P(n)$ is true for all integers $n \geq 18$.

Sol: (a) Since $18 = 7+7+4$, $19 = 7+4+4+4$, $20 = 4+4+4+4+4$, $21 = 7+7+7$, then $P(18)$, $P(19)$, $P(20)$, $P(21)$ are true

(b) Inductive hypothesis: Assume $P(j)$ is true for $18 \leq j \leq k$ with $k \geq 21$.

(c) $[P(18) \wedge P(19) \wedge P(20) \wedge P(21) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$, $k \geq 21$

(d) Since $k \geq 21$, then $k-3 \geq 18$ and $P(k-3)$ is true which means $k-3$ is formed by 4-cent & 7-cent stamps by Inductive hypothesis. Therefore, $k+1 = k-3+4$ is formed by 4-cent & 7-cent stamps and it means $P(k+1)$ is true

(e) By strong induction, once the basis step and the inductive step are proven true, $P(n)$ is true for $n \geq 18$.

5. a) Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

Sol: (a) To find a number c such that n is formed by 4-cent and 11-cent stamps for $n \geq c$, we have

① $\gcd(4, 11) = 1 \Rightarrow 3 \times 4 - 1 \times 11 = 1$ by Euclidean Algorithm.

② By ①, we know that it can "gain 1 cent" when replace a 11-cent by 3 4-cent stamps

③ If we want to make strong induction work, we need to find at least four consecutive integers which are formed by 4-cent and 11-cent stamps:

22	23	24	25	26	27	28	29
11+11	11+4+4+4	4+4+4+4+4+4	X	11+11+4	11+3x4+4	3x4+3x4+4	X

30	31	32	33
11+11+4+4	11+3x4+4+4	3x4+3x4+4+4	3x11

$\Rightarrow C = 30$

Let $P(n)$ be "n-cent is formed by 4-cent and 11-cent stamps for $n \geq 30$".

(b) Prove $P(n)$ is true for $n \geq 30$ by math induction:

Basis step: To show $P(30)$ is true, we have $30 = 11 + 11 + 4 + 4$ which implies $P(30)$ is true.

inductive hypothesis

inductive step: $\boxed{\text{Assume } P(K) \text{ is true for } K \geq 30}$,

To prove $P(K+1)$ is true, we consider 2 cases:

① If K is formed by at least 1 11-cent stamp:

$K = i + 11$ where i is formed by 4-cent and 11-cent stamps.

Then we have $K+1 = i + 4 + 4 + 4$ is also formed by 4-cent & 11-cent stamps. which implies $P(K+1)$ is true.

② If K is formed without any 11-cent stamps:

Since $K \geq 30$ and K is only formed by 4-cent stamps, then

$K = i + 8 \times 4$ (K has to be formed by more than 8 4-cent stamps)

Then $K+1 = i + 11 + 11 + 11$ is also formed by 4-cent & 11-cent stamps, which means $P(K+1)$ is true.

By basis, inductive step, $P(n)$ is true for $n \geq 30$.

(c) Prove $P(n)$ is true for $n \geq 30$ by strong induction.

Basis step. To show $P(30), P(31), P(32), P(33)$ are true:

$30 = 2 \times 11 + 2 \times 4$, $31 = 11 + 3 \times 4 + 2 \times 4$, $32 = 3 \times 4 + 3 \times 4 + 2 \times 4$, and

$33 = 3 \times 11$

Thus, $P(30), P(31), P(32), P(33)$ are true.

Inductive hypothesis

Inductive step: $\boxed{\text{Assume } P(j) \text{ is true for } 30 \leq j \leq K \text{ with } K \geq 33}$

To prove $P(K+1)$ is true, we have

① By inductive hypothesis, $K \geq 33$ implies $K-3 \geq 30$ and $P(K-3)$ is true: $K-3$ is formed by 4-cent and 11-cent stamps.

② Since $K+1 = K-3+4$, then $K+1$ is formed by 4-cent and 11-cent stamps which implies $P(K+1)$ is true.

By basis, inductive steps, $P(n)$ is true for $n \geq 30$.

6. a) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

(a) To find a number c such that n is formed by 3-cent and 10-cent stamps for $n \geq c$, we have

① $\gcd(3, 10) = 1 \Rightarrow 1 \times 10 - 3 \times 3 = 1$ by Euclidean Algorithm.

② By ①, we know that it can "gain 1 cent" when replace 3 3-cent by 1 10-cent stamp

③ If we want to make strong induction work, we need to find at least <u>three</u> consecutive integers which are formed by

3-cent and 10-cent stamps:

12	13	14	15	16	17
$3+3\times3$	$3+10$	$\times$	$3\times2+3\times3$	$3\times2+10$	$\times$

$$\boxed{\begin{array}{ccc} 18 & 19 & 20 \\ 3\times3+3\times3 & 3\times3+10 & 10+10 \end{array}} \Rightarrow C=18.$$

Let $P(n)$ be " n-cent is formed by 3-cent and 10-cent stamps for $n\geq18$"

(b) To prove $P(n)$ is true for $n\geq18$ by math induction:

Basis step: To show $P(18)$ is true: $18=6\times3$ and it means $P(18)$ is true.

Inductive hypothesis

Inductive step: $\boxed{\text{Assume } P(K) \text{ is true for } K\geq18.}$

To prove $P(K+1)$ is true, we consider 2 cases:

① K is formed by at least three 3-cent stamps:

If $K=C+3+3+3$ where C is formed by 3-cent and 10-cent stamps

Then $K+1=C+10$ and it means $K+1$ is formed by 3-cent and 10-cent stamps

② K is formed by 10-cent and less than three 3-cent stamps:

$K\geq18$ ⓐ there are two 3-cent: $K-6\geq12$ is formed by at least two 10-cent stamps

If $K-6=C+10+10 \Rightarrow K=C+6+10+10$,

then $K+1=C+6+3\times7$

ⓑ there are one 3-cent: $K-3\geq16$ is formed by at least two 10-cent stamps

If $K-3=C+10+10 \Rightarrow K=C+3+10+10$

then $K+1=C+3+3\times7$

ⓒ there are no 3-cent: $K\geq18$ is formed by at least two 10-cent stamps

If $K=C+10+10$ then $K+1=C+3\times7$

By basis, and inductive step. $P(n)$ is true for $n\geq18$

(c) To prove $P(n)$ is true for $n\geq18$ by strong induction.

Basis Step: To prove $P(18)$, $P(19)$, $P(20)$ are true:

$18=3\times6$, $19=3\times3+10$, $20=10+10$. which imply

$P(18)$, $P(19)$, $P(20)$ are true.

Inductive hypothesis

Inductive step: $\boxed{\text{Assume } P(j) \text{ is true for } 18 \le j \le k \text{ with } k \ge 20.}$

To prove $P(k+1)$ is true, we have

Based on inductive hypothesis, $k \ge 20 \Rightarrow k-2 \ge 18$ and $P(k-2)$ is true. Thus

$k+1 = k-2+3$ is formed by 3-cent and 10-cent stamps.

which implies $P(k+1)$ is true.

By strong induction, $P(n)$ is true for $n \ge 18$.

7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

① Since $\gcd(2,5) = 1 \Rightarrow -2\times2 + 1\times5 = 1$

② By ①, we can gain "one dollar" when replacing two 2-dollar bills by one 5-dollar bill.

③ If we want to use strong induction, we need two consecutive integers which are formed by 2-dollar and 5-dollar bills:

4	5	6
2×2	1×5	2×3

which means $P(n)$: n is formed by 2-dollar and 5-dollar bills if $n \ge 4$

④ To prove $P(n)$ is true for $n \ge 4$ by strong induction:

Basis step: To show $P(4)$, $P(5)$ are true, we have

$4 = 2\times2$, $5 = 1\times5$ which imply $P(4)$, $P(5)$ are true.

Inductive step: Assume $P(j)$ is true for $4 \le j \le k$ with $k \ge 5$.

To prove $P(k+1)$ is true, we have:

By inductive hypothesis, $k \ge 5$, $k-1 \ge 4$ and $P(k-1)$ is true.

Then $k+1 = k-1 + 2$ and it means $k+1$ is formed by 2-dollar and 5-dollar bills. Thus $P(k+1)$ is true

By strong induction, $P(n)$ is true for $n \geqslant 4$