Mat 1375 HW 16

Exercise 16.2

Determine the final amount in a savings account under the given conditions.

a) $700, compounded quarterly, at 3%, for 7 years
b) $1400, compounded annually, at 2.25%, for 5 years
c) $1400, compounded continuously, at 2.25%, for 5 years

a) Initial = \$700 compounded quarterly $\to \frac{1}{4}$ rate $= 0{,}03$. $n = 7$ years

$$\text{Final amount} = 700 \cdot \left(1 + \frac{0{,}03}{4}\right)^{4 \cdot 7}$$

b) Initial = \$1400 compounded annually $\to 1$ rate $= 0{,}0225$ $n = 5$ years

$$\text{Final amount} = 1400 \left(1 + \frac{0{,}0225}{1}\right)^{1 \cdot 5}$$

c) Initial = \$1400 compounded continuously rate $= 0{,}0225$ $n = 5$ years

$$\text{Final amount} = 1400 \cdot e^{0{,}0225 \cdot 5}$$

Exercise 16.3

a) Find the amount P that needs to be invested at a rate of 5% compounded quarterly for 6 years to give a final amount of \$2000.

$\frac{1}{4}$ $n = 6$ $r = 0{,}05$

$$P \cdot \left(1 + \frac{0{,}05}{4}\right)^{4 \cdot 6} = 2000$$

$$\Rightarrow P = \frac{2000}{\left(1 + \frac{0{,}05}{4}\right)^{24}} = 1484{,}39\ldots$$

b) Find the present value P of a future amount of $A = \$3500$ invested at 6% compounded annually for 3 years.

$0{,}06$ $\qquad 1 \qquad n=3$

$$P \cdot \left(1+\frac{0{,}06}{1}\right)^{1\cdot 3} = 3500$$

$$\Rightarrow P = \frac{3500}{(1.06)^3}$$

c) Find the present value P of a future amount of $\$1000$ invested at a rate of 4.9% compounded continuously for 7 years.

$r = 0{,}049 \qquad e^r$

$$P \cdot e^{0{,}049\cdot 7} = 1000 \Rightarrow P = \frac{1000}{e^{0{,}049\cdot 7}}$$

$r = ?$

d) At what rate do we have to invest $\$1900$ for 4 years compounded monthly to obtain a final amount of $\$2250$?

$\frac{1}{12}$

$$1900 \cdot \left(1+\frac{r}{12}\right)^{12\cdot 4} = 2250$$

$$\Rightarrow \left(1+\frac{r}{12}\right)^{48} = \frac{2250}{1900} \Rightarrow 1+\frac{r}{12} = \sqrt[48]{\frac{2250}{1900}}$$

$-1 \qquad -1$

$$\Rightarrow \frac{r}{12} = \sqrt[48]{\frac{2250}{1900}} - 1 \Rightarrow r = 12\left(\sqrt[48]{\frac{2250}{1900}} - 1\right)$$

$$= 0{,}042343\ldots \doteqdot 4{,}23\%$$

e) At what rate do we have to invest $\$1300$ for 10 years compounded continuously to obtain a final amount of $\$2000$?

$r = ?$ e

$$1300 \cdot e^{r \cdot 10} = 2000 \Rightarrow e^{10r} = \frac{2000}{1300}$$

$$\Rightarrow \ln\left(e^{10r}\right) = \ln\left(\frac{2000}{1300}\right) \Rightarrow 10r \cdot \underbrace{\ln(e)}_{=1} = \ln\left(\frac{2000}{1300}\right)$$

$$\Rightarrow r = \frac{1}{10} \cdot \ln\left(\frac{2000}{1300}\right) = 0{,}043078\cdots \fallingdotseq 4{,}30\%$$

Exercise 16.4

An unstable element decays at a rate of 5.9% per minute. If 40mg of this element has been produced, how long will it take until 2mg of the element are left? Round your answer to the nearest thousandth.

$r = 0{,}059 / \text{min}$ initial $= 40\,\text{mg}$ $t =$ how long until 2mg left?
(n's unit is minute)

$$f(t) = 40 \cdot (1 - 0{,}059)^t$$

If $f(t) = 2$, what is t?

$$\Rightarrow 40 \cdot (1 - 0{,}059)^t = 2 \Rightarrow (1 - 0{,}059)^t = \frac{2}{40}$$

Take "ln" on the both sides: $\ln (1 - 0{,}059)^t = \ln\left(\frac{2}{40}\right)$

$$\Rightarrow t \cdot \ln(1 - 0{,}059) = \ln\left(\frac{2}{40}\right)$$

$$\Rightarrow t = \frac{\ln\left(\frac{2}{40}\right)}{\ln(1 - 0{,}059)} = 49{,}262096\cdots \fallingdotseq \boxed{49{,}262} \text{ (mins)}$$

Exercise 16.5

A substance decays radioactively with a half-life of 232.5 days. How much of 6.8 grams of this substance is left after 1 year?

$b = \left(\frac{1}{2}\right)^{\frac{1}{232.5}}$ $t = 1$ year $= 365$ days.

$$6.8 \cdot \left(\frac{1}{2}\right)^{\frac{1}{232.5} \cdot 365} = 6.8 \cdot \left(\frac{1}{2}\right)^{\frac{365}{232.5}} = 2.290467\cdots \fallingdotseq \boxed{2.290 \text{ grams}}$$

Exercise 16.6

Fermium-252 decays in 10 minutes to 76.1% of its original mass. Find the half-life of fermium-252.

$1 \xrightarrow{\text{after 10 mins}} 0.761$, half time $h = ?$ (h's unit is minute) $b = \left(\frac{1}{2}\right)^{\frac{1}{h}}$

$$1 \cdot \left(\frac{1}{2}\right)^{\frac{1}{h} \cdot 10} = 0.761 \Rightarrow \left(\frac{1}{2}\right)^{\frac{10}{h}} = 0.761$$

Take "ln" on the both sides: $\ln\left(\frac{1}{2}\right)^{\frac{10}{h}} = \ln(0.761)$

$$\frac{10}{h} \cdot \ln\left(\frac{1}{2}\right) = \ln(0.761) \Rightarrow \frac{10}{h} = \frac{\ln(0.761)}{\ln(0.5)}$$

$$h = \frac{10 \cdot \ln(0.5)}{\ln(0.761)} = 25.3786\cdots \fallingdotseq \boxed{25.379 \text{ (mins)}}$$

Exercise 16.7

How long do you have to wait until 15mg of beryllium-7 have decayed to 4mg if the half-life of beryllium-7 is 53.12 days?

$t = ?$ initial

$t = ?$ (t's unit is day) $b = \left(\frac{1}{2}\right)^{\frac{1}{53.12}}$

$$15 \cdot \left(\frac{1}{2}\right)^{\frac{1}{53.12} \cdot t} = 4 \Rightarrow \left(\frac{1}{2}\right)^{\frac{t}{53.12}} = \frac{4}{15}$$

Take "ln" on the both sides:

$$\ln\left(\frac{1}{2}\right)^{\frac{t}{53.12}} = \ln\left(\frac{4}{15}\right)$$

$$\frac{t}{53.12} \cdot \ln(0.5) = \ln\left(\frac{4}{15}\right) \Rightarrow \frac{t}{53.12} = \frac{\ln\left(\frac{4}{15}\right)}{\ln(0.5)}$$

$\Rightarrow t = 53.12 \cdot \dfrac{\ln(\frac{4}{15})}{\ln(0.5)} = 101.294028\cdots \fallingdotseq \boxed{101.294 \text{ (days)}}$

Exercise 16.8

If Pharaoh Ramses II died in the year 1213 BC, then what percent of the carbon-14 was left in the mummy of Ramses II in the year 2000?

Half-life of Carbon-14 is 5730 years (based on Google ☺)

$1 \longrightarrow ?$ $\Rightarrow 1 \cdot \left(\frac{1}{2}\right)^{\frac{1}{5730} \cdot 3213} = ?$

1213 BC 2000 AD

t = total passing $2000 + 1213 = 3213$ years

$\Rightarrow \left(\frac{1}{2}\right)^{\frac{3213}{5730}} = ? \Rightarrow ? = 0.677957\cdots \fallingdotseq \boxed{67.796\%}$

Exercise 16.9

In order to determine the age of a piece of wood, the amount of carbon-14 was measured. It was determined that the wood had lost 33.1% of its carbon-14. $\boxed{\text{How old}}$ is this piece of wood?

$t = ?$ (t's unit is year)

Half-life of Carbon-14 is 5730 years (based on Google ☺)

lost 33.1% → remain $1 - 0.331 = 0.669$

$1 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}} = 0.669.$

Take "ln" on the both sides: $\ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right) = \ln(0.669)$

$\Rightarrow \dfrac{t}{5730} \cdot \ln(\tfrac{1}{2}) = \ln(0.669)$

$\Rightarrow t = 5730 \cdot \dfrac{\ln(0.669)}{\ln(0.5)} = 3322.95239\cdots \fallingdotseq \boxed{3322.952 \text{ years}}$

Exercise 16.10

Archaeologists uncovered a bone at an ancient resting ground. It was determined that 62% of the carbon-14 was left in the bone. How old is the bone?

$t = ?$

Half-life of Carbon-14 is 5730 years

$$1 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}} = 0.62$$

Take "ln" on the both sides: $\ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right) = \ln(0.62)$

$$\Rightarrow \frac{t}{5730} \cdot \ln\left(\frac{1}{2}\right) = \ln(0.62)$$

$$\Rightarrow t = 5730 \cdot \frac{\ln(0.62)}{\ln(0.5)} = 3951.75110\cdots \fallingdotseq \boxed{3951.751 \text{ years}}$$