Mat 1375 HW15

Exercise 15.1

Solve for x without using a calculator.

a) $6^{x-2}=36$ b) $2^{3x-8}=16$
c) $10^{5-x}=0.0001$ d) $5^{5x+7}=\frac{1}{125}$
e) $2^{x}=64^{x+1}$ f) $4^{x+3}=32^{x}$

Sol:

a) $6^{x-2}=36$

$\Rightarrow 6^{x-2}=6^{2}$

By one-to-one property,

$x-2=2$

$\Rightarrow \boxed{x=4}$

b) $2^{3x-8}=16 \Rightarrow 2^{3x-8}=2^{4}$

By one-to-one property, $3x-8=4 \Rightarrow 3x=12 \Rightarrow \boxed{x=4}$

c) $10^{5-x}=0.0001 \Rightarrow 10^{5-x}=10^{-4}$

By one-to-one property, $5-x=-4 \Rightarrow -x=-9 \Rightarrow \boxed{x=9}$

e) $2^{x}=64^{x+1} \Rightarrow 2^{x}=(2^{6})^{(x+1)} \Rightarrow 2^{x}=2^{6(x+1)}$

By one-to-one property, $x=6(x+1) \Rightarrow x=6x+6$

$\Rightarrow -5x=6 \Rightarrow \boxed{x=-\frac{6}{5}}$

Exercise 15.2

Solve for x. First find the exact answer as an expression involving logarithms. Then approximate the answer to the nearest hundredth using a calculator.

a) $4^{x}=57$ b) $9^{x-2}=7$ c) $2^{x+1}=31$
d) $3.8^{2x+7}=63$ e) $5^{x+5}=8^{x}$ f) $3^{x+2}=0.4^{x}$

Sol: a) $4^{x}=57$ (has different bases on the both sides)

Take "ln" on the both sides: $\ln(4^{x})=\ln(57)$

power rule $\Rightarrow x\cdot\ln(4)=\ln(57) \Rightarrow \boxed{x=\frac{\ln(57)}{\ln(4)}}$

b) $9^{x-2}=7$ (has different bases on the both sides)

Take "ln" on the both sides: $\ln(9^{x-2})=\ln(7)$

power rule $\Rightarrow (x-2)\cdot\ln(9)=\ln(7)$

$\Rightarrow x\cdot\ln(9)-2\ln(9)=\ln(7)$

$\rightarrow x\cdot\boxed{\ln(9)}=\ln(7)+2\ln(9)$

$\Rightarrow \boxed{x=\frac{\ln(7)+2\ln(9)}{\ln(9)}}$

c) $2^{x+1} = 31$ (has different bases on the both sides)

Take "ln" on the both sides $\ln(2^{x+1}) = \ln(31)$

power rule $\Rightarrow$ $(x+2)\ln(2) = \ln(31)$

$\Rightarrow x\ln(2) + 2\ln(2) = \ln(31) \Rightarrow x\ln(2) = \ln(31) - 2\ln(2)$

$$\Rightarrow \boxed{x = \frac{\ln(31) - 2\ln(2)}{\ln(2)}}$$

d) $3.8^{2x+7} = 63$ (has different bases on the both sides)

Take "ln" on the both sides $\ln(3.8^{2x+7}) = \ln(63)$

power rule $\Rightarrow$ $(2x+7)\ln(3.8) = \ln(63)$

$\Rightarrow 2x\ln(3.8) + 7\ln(3.8) = \ln(63)$

$\Rightarrow x\ln(3.8^2) + 7\ln(3.8) = \ln(63)$

$\Rightarrow x\ln(3.8^2) = \ln(63) - 7\ln(3.8)$

$$\Rightarrow \boxed{x = \frac{\ln(63) - 7\ln(3.8)}{\ln(3.8^2)}}$$

e) $5^{x+5} = 8^x$ (has different bases on the both sides)

Take "ln" on the both sides: $\ln(5^{x+5}) = \ln(8^x)$

power rule $\Rightarrow$ $(x+5)\ln(5) = x\ln(8)$

$\Rightarrow x\cdot\ln(5) + 5\ln(5) = x\ln(8)$

$\Rightarrow x\cdot\ln(5) - x\ln(8) = -5\ln(5)$

$\Rightarrow x(\ln(5) - \ln(8)) = -5\ln(5)$

$$\Rightarrow \boxed{x = \frac{-5\ln(5)}{\ln(5) - \ln(8)}}$$

Exercise 15.3

Assuming that $f(x) = c \cdot b^x$ is an exponential function, find the constants c and b from the given conditions.

a) $f(0) = 4, \quad f(1) = 12$ b) $f(0) = 5, \quad f(3) = 40$

a) $f(x) = c \cdot b^x$

- $f(0) = 4 \Rightarrow c \cdot b^0 = 4 \Rightarrow c \cdot 1 = 4 \Rightarrow c = 4$
- $f(1) = 12 \Rightarrow 4 \cdot b^1 = 12 \Rightarrow b = \frac{12}{4} = 3$

$\rightarrow f(x) = 4 \cdot 3^x$

b) $f(x) = c \cdot b^x$

- $f(0) = 5 \Rightarrow c \cdot b^0 = 5 \Rightarrow c \cdot 1 = 5 \Rightarrow c = 5$
- $f(3) = 40 \Rightarrow 5 \cdot b^3 = 40 \Rightarrow b^3 = \frac{40}{5} = 8 \Rightarrow b = 2$

$\rightarrow f(x) = 5 \cdot 2^x$

Exercise 15.5

initial $(t=0)$

The population size of a city was $\boxed{79,000}$ in the year 1998 and 136,000 in the $\boxed{\text{year } 2013}$. Assume that the population size follows an exponential function.

if year 1998 is "t=0", then year 2013 is "$t = 2013 - 1998 = 15$"

a) Find the formula for the population size.

b) What is the population size in the year 2030?

c) What is the population size in the year 2035?

d) When will the city reach its expected maximum capacity of one million residents?

Sol: a) Let the growth rate be r

$$P(t) = 79000 \cdot e^{rt} \Rightarrow 79000 \cdot e^{r \cdot 15} = 136000$$

$$\Rightarrow e^{15r} = \frac{136000}{79000}$$

Take "ln" on the both sides

$$\Rightarrow \ln(e^{15r}) = \ln\left(\frac{136000}{79000}\right) \Rightarrow 15r \cdot \underbrace{\ln(e)}_{1} = \ln\left(\frac{136000}{79000}\right)$$

$$\Rightarrow r = \frac{1}{15} \cdot \ln\left(\frac{136000}{79000}\right) \Rightarrow P(t) = 79000\, e^{0.036213\,t}$$

$$= 0.036213 \cdots \doteqdot \boxed{3.62\%}$$

b) Year 2030 is $t = 2030 - 1998 = 32$

$$P(32) = 79000 \cdot e^{0.036213 \cdot 32} = 251712.99 \fallingdotseq \boxed{251713.}$$

c) Year 2035 is $t = 2035 - 1998 = 37$

$$P(37) = 79000 \cdot e^{0.036213 \cdot 37} = 301677.49 \cdots \fallingdotseq \boxed{301677}$$

d) If $P =$ one million $= 1{,}000{,}000$, find $t = ?$

$$P(t) = 79000 \cdot e^{0.036213t} = 1\,000\,000.$$

$$\Rightarrow e^{0.036213t} = \frac{1000000}{79000}$$

$$\Rightarrow 0.036213t \cdot \underbrace{\ln(e)}_{=1} = \ln\left(\frac{1000000}{79000}\right)$$

$$\Rightarrow t = \frac{1}{0.036213} \cdot \ln\left(\frac{1000000}{79000}\right) = 70.093 \cdots \fallingdotseq 70$$

$\Rightarrow$ In year $(1998 + 70 =)$ $\boxed{2068}$, the city will reach the maximum capacity.

✓Exercise 15.6

The population of a city decreases at a rate of 2.3% per year. After how many years will the population be at 90% of its current size? Round your answer to the nearest tenth.

$r = -0.023$

$$P(t) = 1 \cdot e^{-0.023t} = 0.9$$

$$\Rightarrow \ln(e^{-0.023t}) = \ln(0.9)$$

$$\Rightarrow -0.023t \cdot \underbrace{\ln(e)}_{=1} = \ln(0.9)$$

$$\Rightarrow t = \frac{\ln(0.9)}{-0.023} = 4.5808 \cdots \fallingdotseq \boxed{4.6 \text{ years}}$$

Exercise 15.7

A big company plans to expand its franchise and, with this, its number of employees. For tax reasons, it is most beneficial to expand the number of employees at a rate of 5% per year. If the company currently has 4730 employees, how many years will it take until the company has 6000 employees? Round your answer to the nearest hundredth.

$$P(t) = 4730 \cdot e^{0.05t} = 6000$$

$$\Rightarrow e^{0.05t} = \frac{6000}{4730} \Rightarrow \ln\left(e^{0.05t}\right) = \frac{6000}{4730}$$

$$\Rightarrow 0.05t \cdot \underbrace{\ln(e)}_{=1} = \ln\left(\frac{6000}{4730}\right) \Rightarrow t = \frac{1}{0.05} \cdot \ln\left(\frac{6000}{4730}\right) = 4.7566\ldots$$

$$\fallingdotseq \boxed{4.76 \text{ years}}$$

Exercise 15.8

An ant colony has a population size of 4000 ants and is increasing at a rate of 3% per week. How long will it take until the ant population has doubled? Round your answer to the nearest tenth.

doubled of 4000 = 8000

$$P(t) = 4000 \cdot e^{0.03t} = 8000$$

$$\Rightarrow e^{0.03t} = \frac{8000}{4000} \Rightarrow \ln\left(e^{0.03t}\right) = \ln(2)$$

$$\Rightarrow 0.03t \cdot \underbrace{\ln(e)}_{=1} = \ln(2) \Rightarrow t = \frac{\ln(2)}{0.03} = 23.1049\ldots$$

$$\fallingdotseq \boxed{23.1 \text{ week}}$$

Exercise 15.9

The size of a beehive is decreasing at a rate of 15% per month. How long will it take for the beehive to be at half of its current size? Round your answer to the nearest hundredth.

$r = -0.15$

$$P(t) = 1 \cdot e^{-0.15t} = \frac{1}{2}$$

$$\Rightarrow \ln\left(e^{-0.15t}\right) = \ln(0.5)$$

$$\Rightarrow -0.15t \cdot \underbrace{\ln(e)}_{=1} = \ln(0.5) \Rightarrow t = \frac{\ln(0.5)}{-0.15}$$

$$= 4.62098 \doteqdot \boxed{4.62 \text{ months}}$$

Exercise 15.10

If the population size of the world is increasing at a rate of 0.5% per year, how long does it take until the world population doubles in size? Round your answer to the nearest tenth.

$r = 0.005$

$$P(t) = 1 \cdot e^{0.005t} = 2$$

$$\Rightarrow \ln\left(e^{0.005t}\right) = \ln(2)$$

$$\Rightarrow 0.005t \cdot \underbrace{\ln(e)}_{=1} = \ln(2) \Rightarrow t = \frac{\ln(2)}{0.005} = 138.6294\ldots$$

$$\doteqdot \boxed{138.6 \text{ years}}$$