Homework 3

Math 1451 Accelerated Calculus Spring 2016

Problem 1. Showing Kepler's 2nd Law
Solution 1.
Given $e_r = (cos\theta, sin\theta)$ and $e_\theta = (-sin\theta, cos\theta)$ we have:

$$r = re_r \text{ and } \dot{e_r} = \dot{\theta}(-sin\theta, cos\theta) = \dot{\theta}e_\theta.$$

Our goal is to find acceleration.

$$v = \tfrac{dr}{dt} = \dot{r}e_r + r\dot{e_r} = \dot{r}e_r + r\dot{\theta}e_\theta$$
$$a = \tfrac{dv}{dt} = \ddot{r}e_r + \dot{r}\dot{e_r} + (\dot{r}\dot{\theta} + r\ddot{\theta})e_\theta + r\dot{\theta}\dot{e_\theta}$$

Now,

$$\dot{e_\theta} = \dot{\theta}(-cos\theta, -sin\theta) = -\dot{\theta}e_r$$

Rewriting acceleration gives:

$$a = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta = fe_r$$

We are looking good now! Since e_r and e_θ are orthogonal we can break acceleration into 2 parts:

$$\ddot{r} - r\dot{\theta}^2 = f \text{ and}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \text{ since all force is in the radial direction.}$$

We conclude

$$\tfrac{d(r^2\dot{\theta})}{dt} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Solution 2.
Given that angular momentum is conserved we have:

$$C = r \times \dot{r} = (rcos\theta i + rsin\theta j) \times (\dot{r}cos\theta - rsin\theta\dot{\theta})i + (\dot{r}sin\theta + rcos\theta)j$$

i	j	k
$rcos\theta$	$rsin\theta$	0
$\dot{r}cos\theta - rsin\theta$	$\dot{r}sin\theta + rcos\theta\dot{\theta}$	0

$$= (r\dot{r}cos\theta sin\theta + r^2cos^2\theta\dot{\theta} - r\dot{r}cos\theta sin\theta + r^2sin^2\theta)k$$
$$= r\dot{r}(e_r \times e_r) + r(r\dot{\theta})k = r(r\dot{\theta})k$$