MAT1375, Classwork25, Fall2025

Ch23. Complex Numbers

1. The Imaginary Unit and the Complex Number:

We define the **Imaginary Unit** or **complex unit** to be

$$i = \underline{\sqrt{-1}} \quad (\text{since } i^2 = \underline{-1}).$$

A complex number is a number with the form

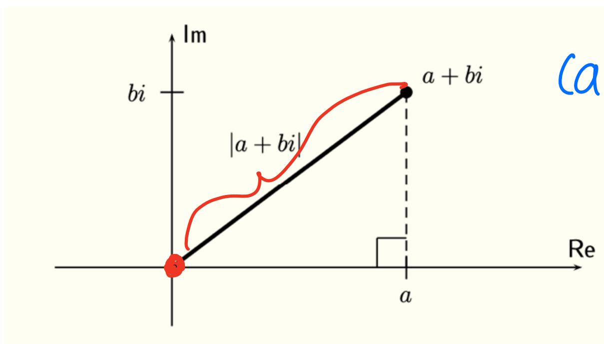
$$a + bi$$

where a and b are any <u>real</u> numbers, i is the <u>imaginary</u> unit. The number a is called the <u>real</u> part of $a + bi$, and b is called the <u>imaginary</u> part of $a + bi$.

real number : $\mathbb{R}$

The set of all complex numbers is denoted by <u>$\mathbb{C}$</u>.

2. Complex Plane:



(a, b)

A complex number $z = a + bi$ can be represented as a point <u>(a, b)</u> in a **Coordinate Plane** with the horizontal axis which is called <u>real</u> axis and the vertical axis which is called <u>imaginary</u> axis.

The absolute value or <u>the length</u> of $z = a + bi$ is the <u>distance</u> between z (a,b) in the complex plane and the origin (0,0) and it denoted by

$$|z| = |a + bi| = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}.$$

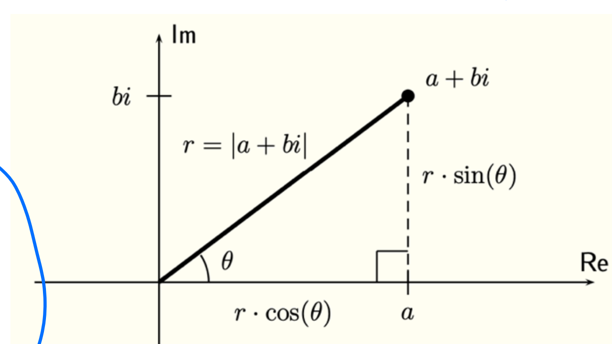
3. Polar Form of a Complex number:

(a.b)

The complex number $z = a + bi$ is written in **Polar Form** as

$$z = a + bi = \underline{r\cos(\theta)} + i(\underline{r\sin(\theta)})$$

where $\tan(\theta) = \frac{b}{a}$ and $r = |z|$.



$\cos(\theta) = \frac{a}{r}$, $\sin(\theta) = \frac{b}{r}$

$a = r\cos(\theta)$ $\quad$ $b = r\sin(\theta)$

4. Product and Quotient in polar form:

$r_1\cos(\theta_1) + i\, r_1 \sin(\theta_1)$ $\qquad$ $r_2\cos(\theta_2) + i\, r_2\sin(\theta_2)$

Let $r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $r_2(\cos(\theta_2) + i\sin(\theta_2))$ be two complex numbers in polar form. We have

(De Moivre's Theorem)

$$r_1(\cos(\theta_1) + i\sin(\theta_1)) \cdot r_2(\cos(\theta_2) + i\sin(\theta_2)) = \underline{r_1 r_2} \cdot (\underline{\cos(\theta_1+\theta_2)} + i\,\underline{\sin(\theta_1+\theta_2)}).$$

$$\frac{r_1(\cos(\theta_1)+i\sin(\theta_1))}{r_2(\cos(\theta_2)+i\sin(\theta_2))} = \frac{r_1}{r_2} \cdot (\underline{\cos(\theta_1-\theta_2)} + i\,\underline{\sin(\theta_1-\theta_2)}).$$

standard complex form

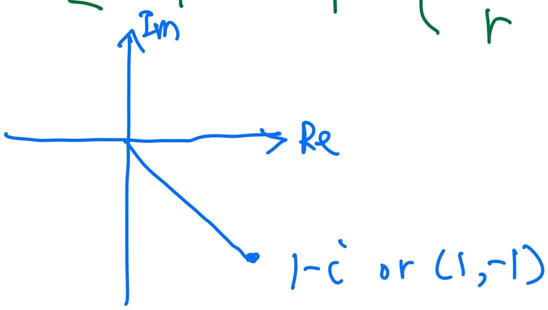
5. Let $z = 1 - i$. Find the polar form of z. $\quad a=1,\ b=-1$

The length of z: $r = |z| = \sqrt{a^2+b^2} = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$$z = 1 - i = r\left(\frac{1}{r} + i\frac{-1}{r}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\left(\frac{-1}{\sqrt{2}}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$$

polar form



$$\cos\left(\frac{7\pi}{4}\right) = \frac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

6. Let $z_1 = 2(\cos(210°) + i\sin(210°))$ and $z_2 = 4(\cos(90°) + i\sin(90°))$. $= 4i$

$\frac{7\pi}{6}$ $\quad$ $\frac{\pi}{2}$ $\quad$ $-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}$ $\quad$ 0 $\quad$ 1

Find a) $z_1 \cdot z_2$ in standard complex form, and b) $\frac{z_1}{z_2}$ in standard complex form.

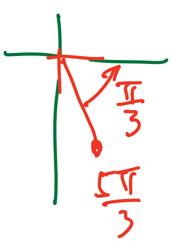
a) $z_1 \cdot z_2 = 2(\cos(210°) + i\sin(210°)) \cdot 4(\cos(90°) + i\sin(90°))$

$= 2\cdot 4\,(\cos(210°+90°) + i\sin(210°+90°))$

$= 8\,(\cos(300°) + i\sin(300°))$

$= 8\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

$= 8\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = \frac{8}{2} - i\,\frac{8\sqrt{3}}{2} = 4 - 4\sqrt{3}\,i$



b) $\frac{z_1}{z_2} = \frac{2}{4}\left(\cos(210°-90°) + i\sin(210°-90°)\right)$

$= \frac{1}{2}\left(\cos(120°) + i\sin(120°)\right) = \frac{1}{2}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$= \frac{1}{2}\left(\left(-\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)\right) = -\frac{1}{4} + \frac{\sqrt{3}}{4}\,i$