Section 2.4

3. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $2^n + 1$? **b)** $(n+1)^{n+1}$?
c) $\lfloor n/2 \rfloor$? **d)** $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?

Sol: a) $a_n = 2^n + 1$

$a_0 = 2^0 + 1 = 1+1 = 2$

$a_1 = 2^1 + 1 = 2+1 = 3$

$a_2 = 2^2 + 1 = 4+1 = 5$

$a_3 = 2^3 + 1 = 8+1 = 9$

b) $a_n = (n+1)^{n+1}$

$a_0 = (0+1)^{0+1} = 1^1 = 1$

$a_1 = (1+1)^{1+1} = 2^2 = 4$

$a_2 = (2+1)^{2+1} = 3^3 = 27$

$a_3 = (3+1)^{3+1} = 4^4 = 256$

c) $a_n = \lfloor \frac{n}{2} \rfloor$

$a_0 = \lfloor \frac{0}{2} \rfloor = 0$

$a_1 = \lfloor \frac{1}{2} \rfloor = 0$

$a_2 = \lfloor \frac{2}{2} \rfloor = 1$

$a_3 = \lfloor \frac{3}{2} \rfloor = 1$

d) $a_n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$

$a_0 = \lfloor \frac{0}{2} \rfloor + \lceil \frac{0}{2} \rceil = 0+0 = 0$

$a_1 = \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil = 0+1 = 1$

$a_2 = \lfloor \frac{2}{2} \rfloor + \lceil \frac{2}{2} \rceil = 1+1 = 2$

$a_3 = \lfloor \frac{3}{2} \rfloor + \lceil \frac{3}{2} \rceil = 1+2 = 3$

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$

Sol: $a_0 = 2$

$a_1 = 6a_{1-1} = 6a_0 = 6 \cdot 2 = 12$

$a_2 = 6a_{2-1} = 6a_1 = 6 \cdot 12 = 72$

$a_3 = 6a_{3-1} = 6a_2 = 6 \cdot 72 = 432$

$a_4 = 6a_{4-1} = 6 \cdot a_3 = 6 \cdot 432 = 2592$

b) $a_n = a_{n-1}^2, a_1 = 2$
c) $a_n = a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = 2$
d) $a_n = na_{n-1} + n^2 a_{n-2}, a_0 = 1, a_1 = 1$
e) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$

Sol b) $a_1 = 2$

$a_2 = a_{2-1}^2 = a_1^2 = 2^2 = 4$

$a_3 = a_{3-1}^2 = a_2^2 = 4^2 = 16$

$a_4 = a_{4-1}^2 = a_3^2 = 16^2 = 256$

$a_5 = a_{5-1}^2 = a_4^2 = 256^2$

c) $a_0 = 1$

$a_1 = 2$

$a_2 = a_{2-1} + 3a_{2-2} = a_1 + 3a_0 = 2 + 3 \cdot 1 = 5$

$a_3 = a_{3-1} + 3a_{3-2} = a_2 + 3a_1 = 5 + 3 \cdot 2 = 11$

$a_4 = a_{4-1} + 3a_{4-2} = a_3 + 3a_2$
$= 11 + 3 \cdot 5 = 26.$

d) $a_n = na_{n-1} + n^2 a_{n-2}$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2 \cdot a_{2-1} + 2^2 a_{2-2}$
$= 2 \cdot a_1 + 4a_0 = 2 \cdot 1 + 4 \cdot 1 = 6$

$a_3 = 3 \cdot a_{3-1} + 3^2 a_{3-2}$
$= 3 \cdot a_2 + 9a_1 = 3 \cdot 6 + 9 \cdot 1 = 27$

$a_4 = 4 \cdot a_{4-1} + 4^2 a_{4-2}$
$= 4a_3 + 16a_2 = 4 \cdot 27 + 16 \cdot 6$
$= 204$

e) $a_n = a_{n-1} + a_{n-3}$

$a_0 = 1$

$a_1 = 2$

$a_2 = 0$

$a_3 = a_{3-1} + a_{3-3} = a_2 + a_0$
$= 0 + 1 = 1$

$a_4 = a_{4-1} + a_{4-3} = a_3 + a_1$
$= 1 + 1 = 2$

25. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

Sol: The rule of this list: one 1 and one zero, then two 1s and two 0s, then three 1s and three 0s, ...

The next three terms are 1, 1, 1, ...

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

Sol: The rule of this list: it has an order as 1, 2, 3, 4, 5, ⋯ but when it hits even numbers (2, 4, 6, 8, ⋯), that number will be repeated and shows twice.

The next three terms are 9, 10, 10.

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

Sol: Formula: $a_n = 2^{\lfloor \frac{n}{2} \rfloor} \cdot \frac{1-(-1)^n}{2}$

$$\left(\begin{array}{l} a_1 = 2^{\lfloor \frac{1}{2} \rfloor} \cdot \frac{1-(-1)^1}{2} = 2^0 \cdot \frac{2}{2} = 1 \cdot 1 = 1 \\ a_2 = 2^{\lfloor \frac{2}{2} \rfloor} \cdot \frac{1-(-1)^2}{2} = 2^0 \cdot \frac{0}{2} = 1 \cdot 0 = 0 \\ \vdots \end{array} \right.$$

The next three terms are $a_{11} = 2^{\lfloor \frac{11}{2} \rfloor} \cdot \frac{1-(-1)^{11}}{2} = 2^5 \cdot 1 = 32$

$a_{12} = 0$

$a_{13} = 2^{\lfloor \frac{13}{2} \rfloor} \cdot \frac{1-(-1)^{13}}{2} = 2^6 \cdot 1 = 64.$

d) 3, 6, 12, 24, 48, 96, 192, …

Sol: $a_1 = 3$, $a_2 = 6$, $a_3 = 12$, ⋯

The formula is $a_n = 3 \cdot 2^{n-1}$.

The next three terms are $a_8 = 3 \cdot 2^{8-1} = 3 \cdot 2^7$

$a_9 = 3 \cdot 2^{9-1} = 3 \cdot 2^8$

$a_{10} = 3 \cdot 2^{10-1} = 3 \cdot 2^9$

e) 15, 8, 1, −6, −13, −20, −27, …

Sol: $a_1 = 15$, $a_2 = 8$, $a_3 = 1$, $a_4 = -6$, ⋯

The formula is $a_n = 15 - 7(n-1)$ or $22 - 7n$

The next three terms are $a_8 = 22 - 7 \cdot 8 = -34$

$a_9 = 22 - 7 \cdot 9 = -41$

$a_{10} = 22 - 7 \cdot 10 = -48$

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …

The formula is

Sol: $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \longrightarrow a_n = 2 + \frac{n\cdot(n+1)}{2}$

$a_1 = 3,\ a_2 = 5,\ a_3 = 8,\ a_4 = 12,\ a_5 = 17,\ a_6 = 23,\ a_7 = 30,\ a_8 = 38,\ a_9 = 47$

2, 3, 5, 8, 12, 17, 23, 30, 38, 47

difference: $+1,\ +2,\ +3,\ +4,\ +5,\ +6,\ +7,\ +8,\ +9 \rightarrow \sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$

the difference's difference: $+1,\ +1,\ +1,\ +1,\ +1,\ +1,\ +1,\ +1 \longrightarrow +1$

Check: if $a_n = 2 + \frac{n(n+1)}{2}$, $\therefore$ $a_1 = 2 + \frac{1\cdot(1+1)}{2} = 2+1 = 3$

$a_2 = 2 + \frac{2\cdot(2+1)}{2} = 2+3 = 5$

$a_3 = 2 + \frac{3\cdot(3+1)}{2} = 2+6 = 8$

$\vdots$

The next three terms are $a_{10} = 2 + \frac{10\cdot(10+1)}{2} = 2+55 = 57$

$a_{11} = 2 + \frac{11\cdot(11+1)}{2} = 2+66 = 68$

$a_{12} = 2 + \frac{12\cdot(12+1)}{2} = 2+78 = 80$

g) 2, 16, 54, 128, 250, 432, 686, …

The formula is

Sol: $a_1, a_2, a_3, a_4, a_5, a_6, a_7$

$a_1 = 2 = 2\times 1 = 2\cdot 1^2 \times 1$

$a_2 = 16 = 4\times 4 = 2\cdot 2\times 2^2$

$a_3 = 54 = 6\times 9 = 2\cdot 3\times 3^2$

$a_4 = 128 = 8\times 16 = 2\cdot 4\times 4^2$

$a_5 = 250 = 10\times 25 = 2\cdot 5\times 5^2$

$a_6 = 432 = 12\times 36 = 2\cdot 6\times 6^2$

$a_7 = 686 = 14\times 49 = 2\cdot 7\times 7^2$

$\Rightarrow a_n = 2\cdot n\cdot n^2 = 2n^3$

The next three terms are $a_8 = 2\cdot 8^3$

$a_9 = 2\cdot 9^3$

$a_{10} = 2\cdot 10^3$

h) 2, 3, 7, 25, 121, 721, 5041, 40321, …

$a_1 = 2$
$a_2 = 2 \cdot a_1 - 1 = 4 - 1 = 3$
$a_3 = 3 \cdot a_2 - 2 = 3 \cdot 3 - 2 = 7$
$a_4 = 4 \cdot a_3 - 3 = 4 \cdot 7 - 3 = 25$
$a_5 = 5 \cdot a_4 - 4 = 5 \cdot 25 - 4 = 121$
$a_6 = 6 \cdot a_5 - 5 = 6 \cdot 121 - 5 = 721$
$a_7 = 7 \cdot a_6 - 6 = 7 \cdot 721 - 6 = 5041$
$a_8 = 8 \cdot a_7 - 7 = 8 \cdot 5041 - 7 = 40321$

$\Rightarrow$ The formula is

$a_n = n \cdot a_{n-1} - (n-1)$

and the next three terms are

$a_9 = 9a_8 - 8$
$a_{10} = 10a_9 - 9$
$a_{11} = 11a_{10} - 10$

29. What are the values of these sums?

a) $\sum_{k=1}^{5} (k+1)$ **b)** $\sum_{j=0}^{4} (-2)^j$

c) $\sum_{i=1}^{10} 3$ **d)** $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

Sol:

a) $\sum_{k=1}^{5} (k+1) = 2+3+4+5+6 = 20$

b) $\sum_{j=0}^{4} (-2)^j = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4$
$= 1 - 2 + 4 - 8 + 16 = 11$

c) $\sum_{i=1}^{10} 3 = \underset{i=1}{3} + \underset{i=2}{3} + \underset{i=3}{3} + \underset{i=4}{3} + \underset{i=5}{3} + \underset{i=6}{3} + \underset{i=7}{3} + \overset{i=8}{3} + \underset{i=9}{3} + \overset{i=10}{3} = 30$

d) $\sum_{j=0}^{8} (2^{j+1} - 2^j) = (2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + (2^4 - 2^3) + (2^5 - 2^4)$
$+ (2^6 - 2^5) + (2^7 - 2^6) + (2^8 - 2^7) + (2^9 - 2^8)$
$= 2^9 - 2^0 = 512 - 1 = 511$

30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

a) $\sum_{j \in S} j$ **b)** $\sum_{j \in S} j^2$

c) $\sum_{j \in S} (1/j)$ **d)** $\sum_{j \in S} 1$

Sol. a) $\sum_{j \in S} j = 1+3+5+7 = 16$

b) $\sum_{j \in S} j^2 = 1^2+3^2+5^2+7^2 = 1+9+25+49 = 84$

c) $\sum_{j \in S} \frac{1}{j} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{105+35+21+15}{105} = \frac{176}{105}$

d) $\sum_{j \in S} 1 = 1+1+1+1 = 4$

31. What is the value of each of these sums of terms of a geometric progression?

a) $\sum_{j=0}^{8} 3 \cdot 2^j$ **b)** $\sum_{j=1}^{8} 2^j$

c) $\sum_{j=2}^{8} (-3)^j$ **d)** $\sum_{j=0}^{8} 2 \cdot (-3)^j$

Sol: a) $\sum_{j=0}^{8} 3 \cdot 2^j = 3(2^0+2^1+2^2+2^3+2^4+2^5+2^6+2^7+2^8)$

$= 3 \cdot \frac{2^9-1}{2-1} = 3 \cdot 511 = 1533$

b) $\sum_{j=1}^{8} 2^j = 2^1+2^2+2^3+\cdots+2^8 = 2 \cdot \frac{2^8-1}{2-1} = 2 \cdot 255 = 510$

c) $\sum_{j=2}^{8} (-3)^j = (-3)^2+(-3)^3+(-3)^4+\cdots+(-3)^8 = (-3)^2\left(1+(-3)+(-3)^2+\cdots+(-3)^6\right)$

$= (-3)^2 \cdot \frac{1-(-3)^7}{1-(-3)} = 9 \cdot \frac{1+2187}{4} = 4932$

d) $\sum_{j=0}^{8} 2\cdot(-3)^j = 2\cdot\left((-3)^0+(-3)^1+(-3)^2+\cdots+(-3)^8\right)$

$= 2\cdot\frac{1-(-3)^9}{1-(-3)} = 2\cdot\frac{1+19683}{4} = 9842$

33. Compute each of these double sums.

a) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$ **b)** $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} i$ **d)** $\sum_{i=0}^{2}\sum_{j=1}^{3} ij$

Sol:

a) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) = \sum_{i=1}^{2}\left((i+1)+(i+2)+(i+3)\right)$

$= \sum_{i=1}^{2} 3i+6 = (3\cdot1+6)+(3\cdot2+6)$

$= 9+12=21$

b) $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j) = \sum_{i=0}^{2}\left((2i+3\cdot0)+(2i+3\cdot1)+(2i+3\cdot2)+(2i+3\cdot3)\right)$

$= \sum_{i=0}^{2}(8i+18) = (8\cdot0+18)+(8\cdot1+18)+(8\cdot2+18)$

$= 24+54 = 78$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} i = \sum_{i=1}^{3} \underset{j=0}{i} + \underset{j=1}{i} + \underset{j=2}{i} = \sum_{i=1}^{3} 3i = 3\cdot1+3\cdot2+3\cdot3$

$= 3+6+9 = 18$

d) $\sum_{i=0}^{2}\sum_{j=1}^{3} ij = \sum_{i=0}^{2}\left(1\cdot i+2\cdot i+3\cdot i\right) = \sum_{i=0}^{2} 6i = 6\cdot0+6\cdot1+6\cdot2$

$= 0+6+12 = 18$

35. Show that $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called **telescoping**.

Sol:
$$\sum_{j=1}^{n}(a_j - a_{j-1}) = \underset{j=1}{(a_1 - a_0)} + \underset{j=2}{(a_2 - a_1)} + \underset{j=3}{(a_3 - a_2)} + \cdots + \underset{j=n}{(a_n - a_{n-1})}$$

$$= -a_0 + a_n = a_n - a_0$$

There is also a special notation for products. The product of $a_m, a_{m+1}, \ldots, a_n$ is represented by $\prod_{j=m}^{n} a_j$, read as the product from $j = m$ to $j = n$ of a_j.

45. What are the values of the following products?

a) $\prod_{i=0}^{10} i$ **b)** $\prod_{i=5}^{8} i$

c) $\prod_{i=1}^{100}(-1)^i$ **d)** $\prod_{i=1}^{10} 2$

Sol

a) $\prod_{i=0}^{10} i = 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot 10 = 0$

b) $\prod_{i=5}^{8} i = 5 \cdot 6 \cdot 7 \cdot 8 = 1680$

c) $\prod_{i=1}^{100}(-1)^i = (-1)^1 \cdot (-1)^2 \cdot (-1)^3 \cdot (-1)^4 \cdots (-1)^{100} = (-1)^{1+2+3+\cdots+100} = (-1)^{5050} = 1$

d) $\prod_{i=1}^{10} 2 = \underset{i=1}{2} \cdot \underset{i=2}{2} \cdot \underset{i=3}{2} \cdot \underset{i=4}{2} \cdot \underset{i=5}{2} \cdot \underset{i=6}{2} \cdot \underset{i=7}{2} \cdot \underset{i=8}{2} \cdot \underset{i=9}{2} \cdot \underset{i=10}{2} = 2^{10} = 1024.$