MAT1375, Classwork10, Fall2025

Ch10. Rational Functions I

1. Definition of the **Rational function**:

A rational function is a fraction of two polynomials $f(x) = \frac{p(x)}{g(x)}$, where $p(x)$ and $g(x)$ are both polynomials, and $g(x) \neq 0$.

The **domain of a rational function** f is all real numbers for which the denominator $g(x)$ is not zero:

$$D_f = \{ x \in \mathbb{R} \mid D_p \cap D_g \text{ and } g(x) \neq 0 \}$$

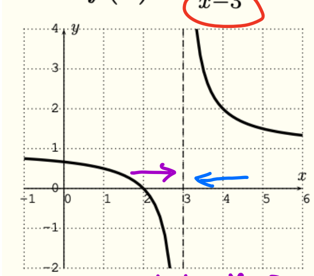
2. **Arrow Notation**: Given a constant a and we have

$x \to a^+$:	x approaches a from the **right** (x is very closed to a but $x \neq a$ and x $>$ a)
$x \to a^-$:	x approaches a from the **left** (x is very closed to a but $x \neq a$ and x $<$ a)
$x \to \infty$:	x approaches infinity (x increases without bound)
$x \to -\infty$:	x approaches negative infinity (x decreases without bound)

3. The definition of a **Vertical Asymptote**:

The line $\boldsymbol{x = a}$ is a vertical asymptote of the graph of a function f if $f(x)$ **increases or decreases without bound** as $\boldsymbol{x}$ **approaches** $\boldsymbol{a}$. Here are three examples:

$f(x) = \frac{x-2}{x-3}$



V.A.: X=3

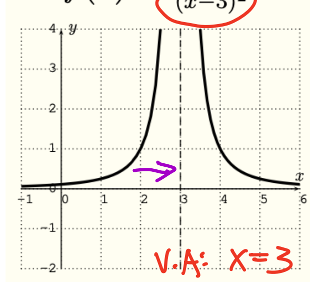
closed to 3 from the left

As $x \to$ 3^-, $f(x) \to$ $-\infty$;

As $x \to$ 3^+, $f(x) \to$ ∞.

closed to 3 from the right

$f(x) = \frac{1}{(x-3)^2}$

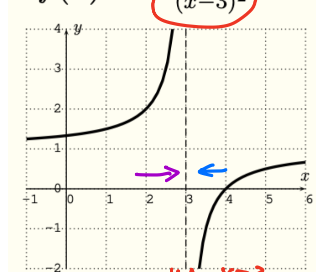


V.A: X=3

As $x \to 3^-$, $f(x) \to \infty$

As $x \to 3^+$, $f(x) \to \infty$

$f(x) = \frac{(x-3)(x-4)}{(x-3)^2}$



V.A.: X=3

As $x \to 3^-$, $f(x) \to \infty$

As $x \to 3^+$, $f(x) \to -\infty$

4. How to locate Vertical Asymptotes: Let $f(x) = \frac{p(x)}{g(x)}$ be a rational function.

(1) If $p(x)$ and $g(x)$ have no common factor(s), and $\boldsymbol{a}$ is a **zero** of $g(x)$ which makes $f(x)$ undefined (∞ or $-\infty$ or unbounded), then $x = a$ is a vertical asymptote of the graph of $f(x)$.

(2) If $\boldsymbol{a}$ is a **zero** of both $p(x)$ and $g(x)$ ($p(a) =$ 0, $g(a) =$ 0.) which means $x - a$ is the common factor of $p(x)$ and $g(x)$, then there is a jump/ removable discontinuity / singularity at $x = a$ and there is no vertical asymptote at $x = a$.

5. Find the vertical asymptotes of the graph of each rational function:

a) $f(x) = \frac{x}{x^2-1}$ b) $g(x) = \frac{x-1}{x^2-1}$ c) $h(x) = \frac{x-1}{x^2+1}$

a) $f(x) = \frac{x}{x^2-1}$, $p(x) = x$, $q(x) = x^2-1$, $f(x) = \frac{p(x)}{q(x)}$

To find vertical asymptote(s), let $q(x) = x^2 - \underbrace{1}_{\text{factor of } 1 \text{ is "1" or "-1"}} = 0$

$q(x) = (x+1)(x-1) = 0$

$x+1=0$ and $x-1=0 \Rightarrow$

$x=-1$, $x=1$ are zeros of $q(x)$ which makes $f(x)$ undefined

(Since there is no common factor of $p(x)$ and $q(x)$), the line $x=-1$ and $x=1$ are vertical asymptote(s) of $f(x)$

b) $g(x) = \frac{x-1}{x^2-1}$, $p(x) = x-1$, $q(x) = x^2-1$, $g(x) = \frac{p(x)}{q(x)}$

To find vertical asymptote(s), let $q(x) = x^2-1 = 0$

$q(x) = (x-1)(x+1) = 0 \Rightarrow x-1=0$ or $x+1=0$

$\Rightarrow x=-1$ and $x=1$ are zeros of $q(x)$ which makes $f(x)$ undefined

Since $(x-1)$ is a common factor of $p(x)$, $q(x)$,

$g(x) = \frac{p(x)}{q(x)} = \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{1}{x+1}$ but $x-1 \neq 0$,

then $x=1$ is a removable discontinuity and

$x=-1$ is a vertical asymptote of $g(x)$

c) $h(x) = \frac{x-1}{x^2+1}$, $p(x) = x-1$, $q(x) = x^2+1$, $h(x) = \frac{p(x)}{q(x)}$

To find vertical asymptote(s), let $q(x) = x^2+1 = 0$

But $q(x) = x^2+1 \geq 1$ which could never be zero

Thus, there is root of $q(x)$ which makes $h(x)$ undefined.

There is NO vertical asymptote of $h(x)$.