MAT2540, Classwork2, Spring2026

5.3 Recursive Definitions and Structural Induction Part 1 (p. 365-370)

1. Review of 2.4: Define a Sequence by **Recursive Relations**:

   Another popular method to define a sequence is to provide one or more initial terms together with a recursive rule for determining subsequent terms from those that precede them.

2. Let $\{a_n\}$ be a sequence that satisfies the **initial term** $a_0 = 2$ and the **recurrence relation**

$$a_n = a_{n-1} + 3 \text{ for } n = 1, 2, 3, \cdots.$$

$a_0 = 2$

What are a_1, a_2, and a_3?

$a_1 = a_0 + 3 = 2+3 = 5$ $+3$

$a_2 = a_1 + 3 = 5+3 = 8$ $+3$

$a_3 = a_2 + 3 = 8+3 = 11$ $+3$

$\Rightarrow$ Arithmetic Sequence with a common difference $d = 3$

Explicit formula of a_n:

$a_n = a_0 + n \cdot d = 2 + 3n$

3. Let $\{a_n\}$ be a sequence that satisfies the **initial term** $a_0 = 3$ and the **recurrence relation**

$$a_n = \frac{1}{3} a_{n-1} \text{ for } n = 1, 2, 3, \cdots.$$

What are a_1, a_2, and a_3?

$a_1 = \frac{1}{3} a_0 = \frac{1}{3} \cdot 3 = 1$ $\times \frac{1}{3}$

$a_2 = \frac{1}{3} a_1 = \frac{1}{3} \cdot 1 = \frac{1}{3}$ $\times \frac{1}{3}$

$a_3 = \frac{1}{3} a_2 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$\Rightarrow$ Geometric Sequence with a common ratio $r = \frac{1}{3}$

Explicit formula of a_n:

$\Rightarrow a_n = a_0 \cdot r^n = 3 \cdot \left(\frac{1}{3}\right)^n$ or $\left(\frac{1}{3}\right)^{n-1}$

4. (*Fibonacci sequence*) Let $\{f_n\}$ be a sequence that satisfies the **initial term** $f_0 = 0$, $f_1 = 1$, and **recurrence relation**

$$f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, \cdots.$$

What are the first five terms?

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

Explicit formula (also called a closed formula) of *Fibonacci sequence*:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$n = 1, 2, 3, 4, \cdots$

5. Recursively Defined Function.

   We use two steps to define a function with the set of nonnegative integers as its domain:

   BASIS STEP: Specify the value of the function at zero.

   RECURSIVE STEP: Give a rule for finding its value at an integer from its value at smallest integers

   Such a definition is called a recursive or inductive definition.

6. The Recursive Sequences and the Induction.

When we define a sequence recursively by specifying how terms of the sequence are found from previous terms, we can use induction to prove results about the sequence.

7. Let $\{f_n\}$ be the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \cdots$.

Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, *where* $\alpha = (1+\sqrt{5})/2$. (Hint: $\alpha^2 - \alpha - 1 = 0$)

Recognize $P(n)$: $f_n > \alpha^{n-2}$ where $n \geq 3$, $\alpha = \frac{1+\sqrt{5}}{2}$

Basis Step. Show P(3) and P(4) are true

Inductive Step:

8. **The Euclidean Algorithm:** Let $a = qb + r$, where a, b, q, r are integers. Then gcd(,) =gcd(,).

9. Find GCD by using the Euclidean Algorithm. Let $d = \gcd(24, 36)$. We have $d|24$ and $d|36$.

$\mathbf{36} = \_\_ \times \mathbf{24} + \_\_\_$, then $\_\_\_ = \_\_\_$ **mod** $\_\_\_$ and $d|\_\_\_$. It implies $d = \gcd(24, \_\_\_\_)$.

$\mathbf{24} = \_\_ \times \mathbf{12} + \_\_\_$, then $\_\_\_ = \_\_\_$ **mod** $\_\_\_$ and $d|\_\_\_$. It implies $d = \gcd(12, \_\_\_)$. Thus, $d = \_\_\_\_$.

10. (*LAMÉ's Theorem*) Let a and b be positive integers with $a \geq b$. Then the number of divisions used by the Euclidean algorithm to find $\gcd(a, b)$ is less than or equal to five times the number of decimal digits in b.