Honors Calculus, Final Exam 2015. — Solution

(1)

(a) Let $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ and R be the radius of convergence for this power series.

So if $x \in (a-R, a+R)$, say x_0, we have

$\sum_{n=0}^{\infty} a_n (x_0-a)^n$ is convergent and

$$f(x_0) = \sum_{n=0}^{\infty} a_n (x_0-a)^n.$$

For differentiability, we have

$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) = \sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1}$ and

$\sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1}$ is convergent as $x \in (a-R, a+R)$.

This means, if $x_0 \in (a-R, a+R)$, we have

$$f'(x_0) = \sum_{n=1}^{\infty} n\, a_n (x_0-a)^{n-1}.$$

(b) Since, as $x \in (a-R, a+R)$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

By (a), since $a \in (a-R, a+R)$, we have $f(a) = a_0$.

Furthermore, for $f'(x)$, by (a), we have, as $x \in (a-R, a+R)$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots \Rightarrow f'(a) = a_1$$

Doing this process continuously,

We can get

$$f^{(n)}(a) = n!\, a_n \iff a_n = \frac{f^{(n)}(a)}{n!}$$

So a_n is uniquely determined by $f^{(n)}(a)$.

(c) (i) Let $f(x) = \cos(x)$, $\Rightarrow f(0) = 1$

$f'(x) = -\sin(x) \Rightarrow f'(0) = 0$

$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$.

The third Taylor expansion T_2 of $f(x)$ is

$$1 + \frac{0}{1!}x - \frac{1}{2!}x^2$$

Which implies $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2!} = -\frac{1}{2}$

(ii) By Remainder Theorem, we have

$$|R_n(x)| = |f(x) - T_n(x)| \leq \frac{M \cdot |x-a|^{n+1}}{(n+1)!}$$ where $|f^{(n+1)}(x)| \leq M$ as $|x-a| \leq d$ for some d.

By (i); we have $a = 0$, $n = 2$.

and $|\cos^{(n+1)}(x)| \leq 1 = M$

As $|x| < \frac{1}{5}$, we obtain

$$|R_2(x)| \leq \frac{1 \cdot |x|^3}{3!} < \frac{1}{5 \cdot 3!} = \frac{1}{30}$$

(2)

(a) Let $a_n = \frac{(-1)^n}{n+1}$ Then

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ is convergent by Alternating Series Test.

(① $\frac{1}{n+1} \to 0$ as $n \to \infty$ ② $\frac{1}{n+1} > \frac{1}{n+2} \geq 0 \ \forall n \in \mathbb{N}$)

But $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n+1}$ is divergent by Limit Comparison Test which compares with $\sum \frac{1}{n}$.

(b) let $f(x) = \frac{1}{x(\ln(x))^2}$ and given $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

We have $\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x(\ln(x))^2}$

$\boxed{\text{Let } u = \ln(x),\ du = \frac{dx}{x}}$

$$= \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln(x))^2} \overset{\downarrow}{=} \lim_{b\to\infty} -\frac{1}{\ln(x)}\Big|_2^b = \frac{1}{\ln(2)} - \lim_{b\to\infty} \frac{1}{\ln(b)} = \frac{1}{\ln(2)}$$

Then, by the Integral Test, since $\int_2^{\infty} \frac{dx}{x(\ln(x))^2}$ is finite,

so $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges.

(c) (i) It is ture, By Basic Comparison Test,

if $0 < a_n \leq b_n$ and $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Example:

let $a_n = \frac{1}{n}$ and $b_n = \frac{2}{n}$.

(2) (c) (ii) This statement is false.

Counterexample:

let $a_n = \frac{1}{n+1}$, we have

$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ converges by p-series Test

but $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

(3) (a) Given $\sum_{n=1}^{\infty} \frac{n^3+2n+1}{n^4+n+2}$

Let $a_n = \frac{n^3+2n+1}{n^4+n+2}$ and $b_n = \frac{1}{n}$. we have

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{n^3+2n+1}{n^4+n+2} \cdot \frac{n}{1}\right| = \left|\frac{n^4+2n^2+n}{n^4+n+2}\right| \to 1 \text{ as } n\to\infty$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series, then,

by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n^3+2n+1}{n^4+n+2}$ diverges.

(b) Given $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$

Let $b_n = \frac{1}{\ln(n)}$. Since ① $b_n > b_{n+1} > 0$ ② $b_n \to 0$ as $n\to\infty$.

Then, by Alternating Series Test, $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges.

(3)

(c) Given $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$.

Let $a_n = \frac{n^2}{e^n}$, we have $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}\right| = \left|\frac{1}{e} \cdot \frac{(n+1)^2}{n^2}\right| \to \frac{1}{e} < 1$ as $n \to \infty$.

Then, by Ratio Test, $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ Converges.

(d) Given $\sum_{n=1}^{\infty} \frac{2^{\frac{1}{n}}}{(1.2)^n} = \sum_{n=1}^{\infty} \frac{\sqrt[n]{2}}{(1.2)^n}$

let $a_n = \frac{\sqrt[n]{2}}{(1.2)^n}$, we have $\sqrt[n]{|a_n|} = \left|\frac{\sqrt[n^2]{2}}{1.2}\right| \to \frac{1}{1.2} < 1$ as $n \to \infty$.

Then, by Root Test, $\sum_{n=1}^{\infty} \frac{\sqrt[n]{2}}{(1.2)^n}$ converges.

(4)

(i) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$

Then $e^{2x} = 1 + (2x) + \frac{4x^2}{2} + \frac{8x^3}{6} + \frac{16x^4}{24} + \frac{32x^5}{120} + \cdots$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Thus.

T_5 of $x^2 e^{2x}$ is $x^2\left(1 + 2x + 2x^2 + \frac{4}{3}x^3\right)$

$$= x^2 + 2x^3 + 2x^4 + \frac{4}{3}x^5.$$

(ii) By Taylor expansion, $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$

This means $\frac{f^{(5)}(0)}{5!} = \frac{4}{3} \Rightarrow f^{(5)}(0) = \frac{4}{3} \cdot 120 = 160$.

(5)

(i) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$. for $|x|<1$.

(ii) Since $\frac{1}{1+x^2} = \frac{d}{dx}[\arctan(x)]$, we have

$$\arctan(x) = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n (x^{2n})\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

As $x=0$, we have $0 = \arctan(0) = C + 0 \Rightarrow C = 0$

Thus, $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $|x|<1$.