Honors Calculus, Midterm 2. — Solution.

(1) (a) $\displaystyle\int_0^1 x^2\sqrt{1-x^3}\,dx = -\frac{1}{3}\int_1^0 \sqrt{u}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_0^1 = \frac{2}{9}.$

$\boxed{\text{Let } u=1-x^3,\ du=-3x^2dx}$

(b) $\displaystyle\int \frac{1}{\sqrt{1+x}}\,dx = 2(1+x)^{\frac{1}{2}}+C$

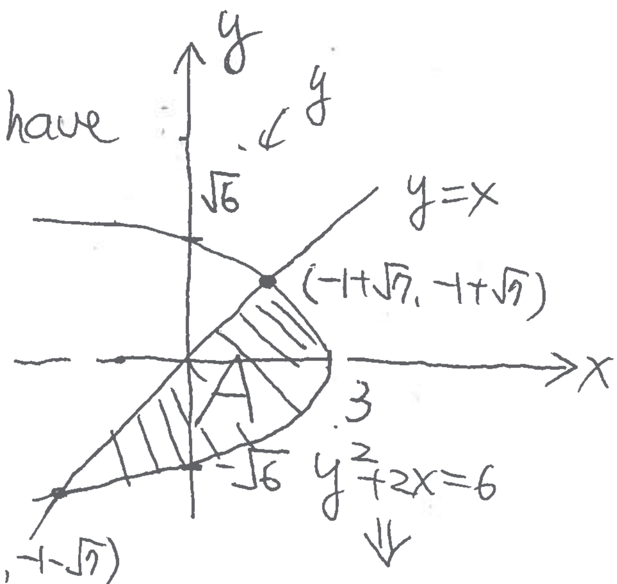
(c) $\displaystyle\int (\ln x)^2dx = x(\ln x)^2 - 2\int(\ln x)\,dx = x(\ln x)^2 - 2x\ln x + 2\int dx$

$\boxed{\begin{matrix}\text{Let } u=(\ln x)^2, & dv=dx\\ du=2(\ln x)\cdot\frac{dx}{x}, & v=x\end{matrix}}$ $\boxed{\begin{matrix}\text{Let } u=\ln x, & dv=dx\\ du=\frac{dx}{x} & v=x\end{matrix}}$

$= x(\ln x)^2 - 2x\ln x + 2x + C.$

(d) $\displaystyle\int \frac{1}{x^2+9}\,dx = \int\frac{1}{x^2+3^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C.$

(2) Given $y=x$ and $y^2+2x=6$, we have



$\Downarrow$

$x=\dfrac{6-y^2}{2}$

Intersection points: $y^2+2y-6=0$

$y=\dfrac{-2\pm\sqrt{28}}{2} = -1\pm\sqrt{7}.$

$$A=\int_{-1-\sqrt{7}}^{-1+\sqrt{7}} \frac{6-y^2}{2}-y\,dy$$

$$= 3y-\frac{y^3}{6}-\frac{y^2}{2}\Big|_{-1-\sqrt{7}}^{-1+\sqrt{7}} = 6\sqrt{7}-\frac{1}{6}\left[(-1+\sqrt{7})^3-(-1-\sqrt{7})^3\right]-\frac{1}{2}\left[(-1+\sqrt{7})^2-(-1-\sqrt{7})^2\right]$$

$$=\frac{14}{3}\sqrt{7}.$$

(3)

① Yes, since

$$\int_0^1 \ln x\,dx = \lim_{a\to 0}\int_a^1 \ln x\,dx = \lim_{a\to 0}\left[x\ln x - x\right]_a^1$$

$$= (1\cdot\ln 1 - 1) - \lim_{a\to 0}\left[a\ln a - a\right] = -1 - \lim_{a\to 0}\left[\frac{\ln a}{\frac{1}{a}}\right] + 0$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{L'}{=}} -1 - \lim_{a\to 0}\left[\frac{\frac{1}{a}}{-\frac{1}{a^2}}\right] = -1 - \lim_{a\to 0}[a] = -1.$$

② Yes, since

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \lim_{b\to\infty}\lim_{a\to -\infty}\int_a^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\lim_{a\to-\infty} \arctan(x)\Big|_a^b$$

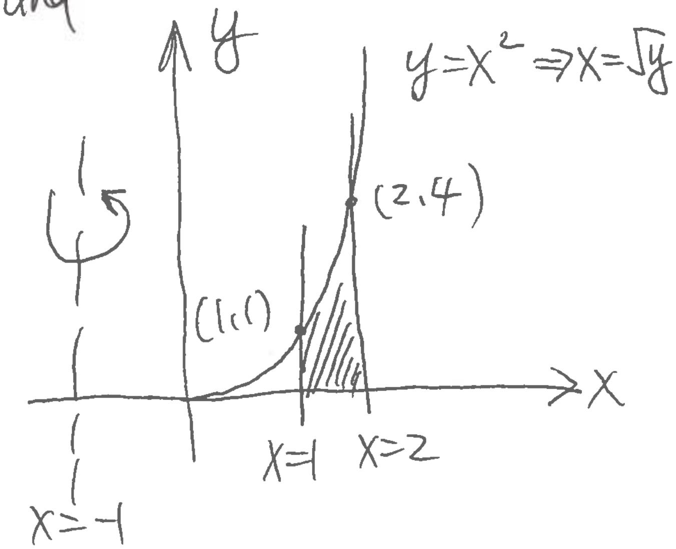
$$= \lim_{b\to\infty}\arctan(b) - \lim_{a\to-\infty}\arctan(a) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

(4) Given $y=x^2$, $x=1$, $x=2$ and rotated axis $x=-1$

We have

(a) Method of cylindrical shells,

$h(x) = x^2$, $r(x) = x-(-1) = x+1$



$$V = 2\pi\int h(x)\,r(x)\,dx = 2\pi\int_1^2 x^2(x+1)\,dx$$

$$= 2\pi\cdot\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_1^2 = 2\pi\left[\frac{15}{4}+\frac{7}{3}\right] = \frac{73}{6}\pi$$

(4)

(b) Method of cross-section.

$R(y) = 2-(-1) = 3,\quad r(y) = \begin{cases} \sqrt{y}+1 & 1 \le y \le 4 \\ 2 & \end{cases}$

$$V = \pi \int_1^4 3^2 - (\sqrt{y}+1)^2 \, dy + \pi \int_0^1 3^2 - 2^2 \, dy$$

$$= \pi \left[\int_1^4 9 - y - 2\sqrt{y} - 1 \, dy + 5y \Big|_0^1 \right]$$

$$= \pi \left[-\frac{y^2}{2} - \frac{4}{3} y^{\frac{3}{2}} + 8y \Big|_1^4 + 5 \right] = \pi \left[-\frac{15}{2} - \frac{28}{3} + 24 + 5 \right] = \frac{73}{6}\pi$$

(5)

(a) $$\int \frac{x}{(x-1)(x+1)(x+2)} dx = \int \frac{\frac{1}{6}}{(x-1)} + \frac{\frac{1}{2}}{(x+1)} + \frac{\frac{-2}{3}}{(x+2)} dx$$

$$= \frac{\ln|x-1|}{6} + \frac{\ln|x+1|}{2} - \frac{2}{3} \ln|x+2| + C$$

(b) since $\sqrt{x^4+1} \geq 1$,

Then $\int_0^1 \sqrt{x^4+1} \geq \int_0^1 1 \, dx = 1 \geq \frac{1}{2}$.

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or using $x^4 - 2x^2 + 1 \geq 0$

$\Rightarrow x^4 + 1 \geq 2x^2 \geq x^2 \Rightarrow \sqrt{x^4+1} \geq \sqrt{x^2} = x$ as $x \in [0,1]$,

$\Rightarrow \int_0^1 \sqrt{x^4+1} \, dx \geq \int_0^1 x \, dx = \frac{1}{2}$