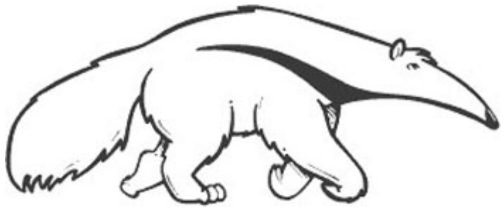
Machine Learning and Data Mining

VC Dimension

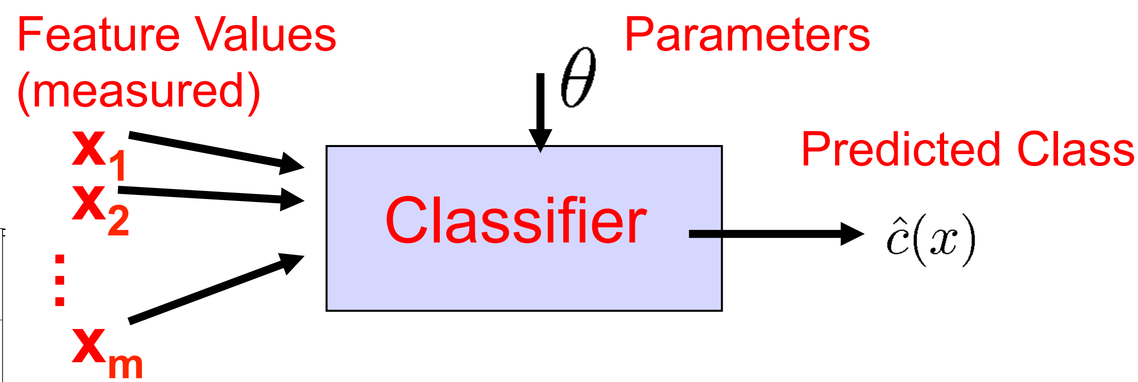
Prof. Alexander Ihler
Fall 2012

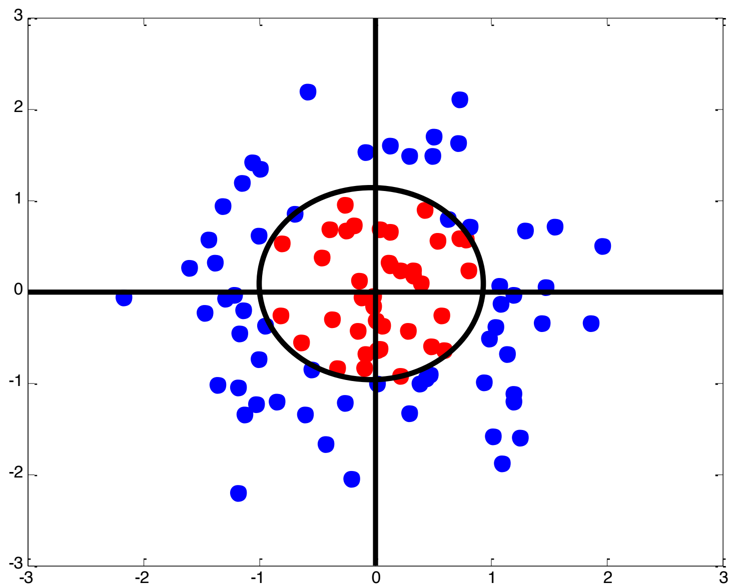


Slides based on Andrew Moore's

Learners and Complexity

- We've seen many versions of underfit/overfit trade-off
  – Complexity of the learner
  – "Representational Power"
- Different learners have different power





Example:

$$\hat{c}(x) = \text{sign}(x^T x - \theta_0)$$

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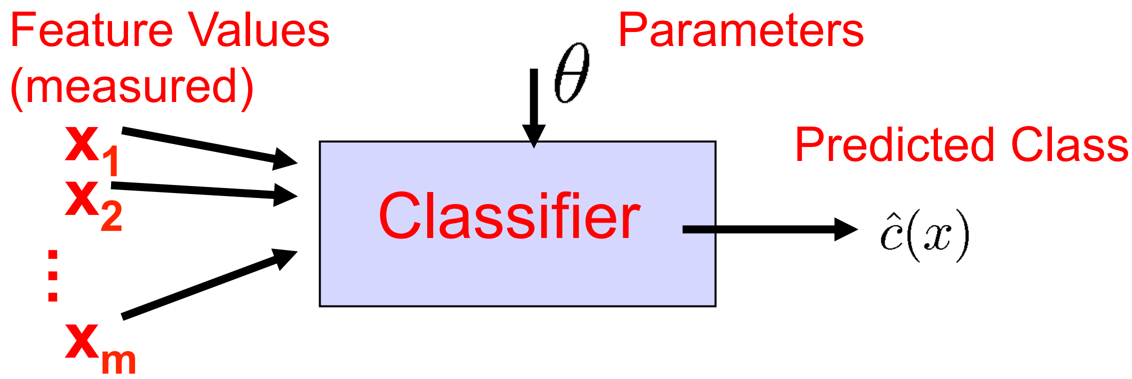

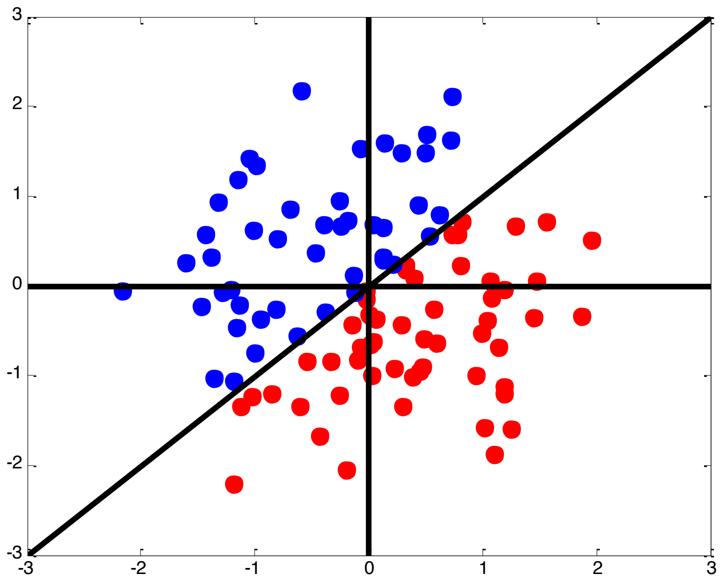

Example:

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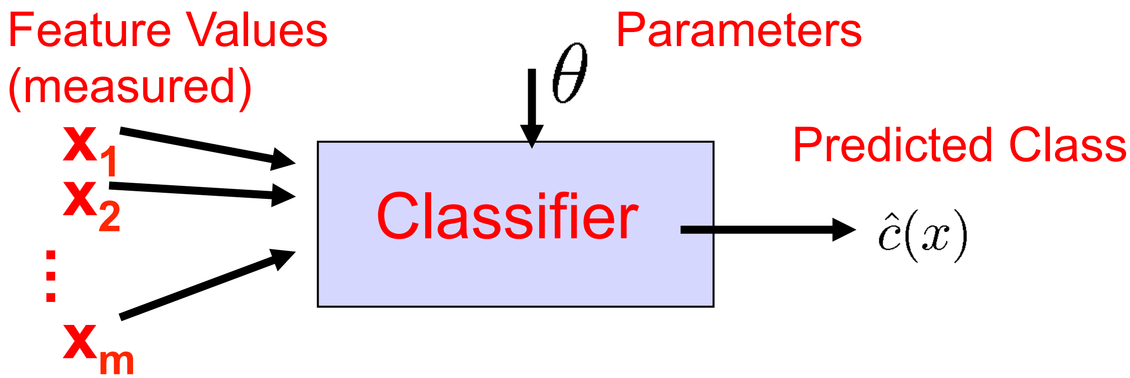

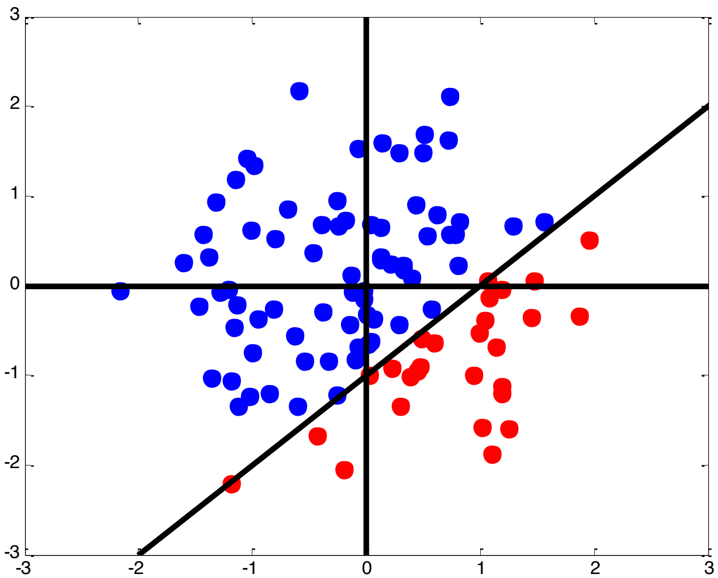

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Learners and Complexity

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- Usual trade-off:
  - More power = represent more complex systems, might overfit
  - Less power = won’t overfit, but may not find “best” learner

- How can we quantify representational power?
  - Not easily…
  - One solution is VC (Vapnik-Chervonenkis) dimension

Some notation

- Let's assume our training data are iid from some distribution p(x)
- Define "risk" and "empirical risk"
  - These are just "long term" test and observed training error

$$R(\theta) = \mathrm{TestError} = \mathbb{E}[\delta(c \neq \hat{c}(x\,;\,\theta))]$$

$$R^{\mathrm{emp}}(\theta) = \mathrm{TrainError} = \frac{1}{N}\sum_i \delta(c^{(i)} \neq \hat{c}(x^{(i)}\,;\,\theta))$$

- How are these related? Depends on overfitting…
  - Underfitting domain: pretty similar…
  - Overfitting domain: test error might be lots worse!

VC Dimension and Risk

- Given some classifier, let H be its VC dimension
  - Represents “representational power” of classifier

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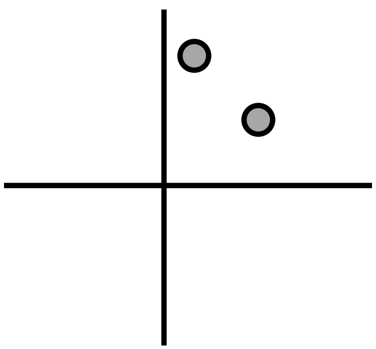
- With “high probability” (1-η), Vapnik showed

$$\text{TestError} \leq \text{TrainError} + \sqrt{\frac{H\log(2N/H) + H - \log(\eta/4)}{N}}$$

Shattering

- We say a classifier f(x) can shatter points x1…xN iff
  For *all* y1…yN, f(x) can achieve zero error on
  training data (x1,y1), (x2,y2), … (xN,yN)
  (i.e., there exists some θ that gets zero error)

- Can $f(x;\theta) = \text{sign}(\theta\, x^T)$ shatter these points?



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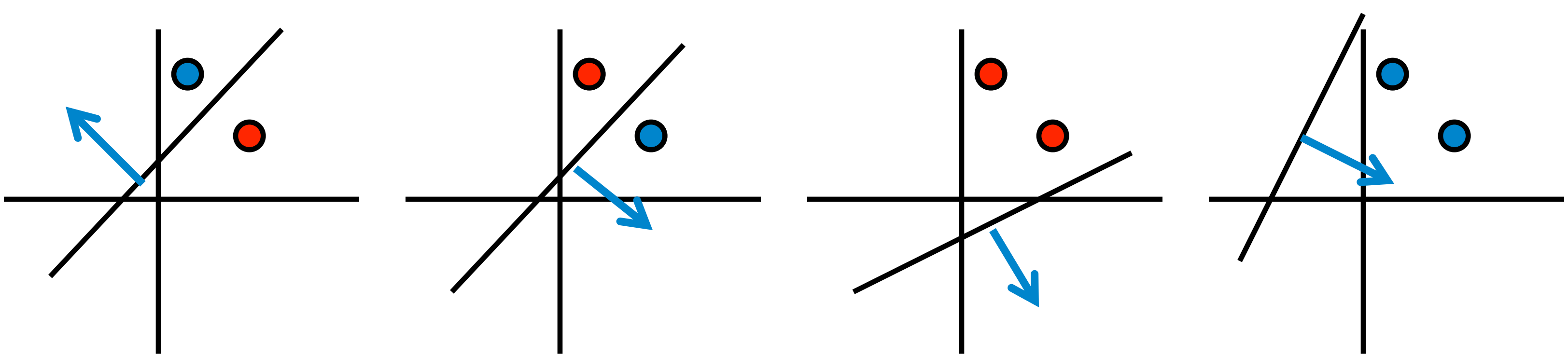
- Can $f(x;\theta) = \text{sign}(\theta_0 + \theta_1 x_1 + \theta_2 x_2)$ shatter these points?
- Yes: there are 4 possible training sets…



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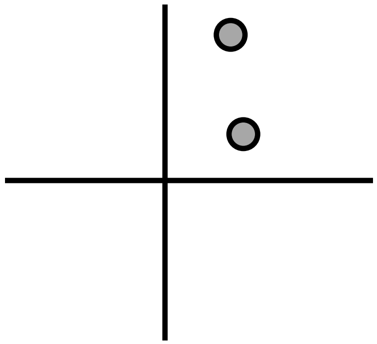

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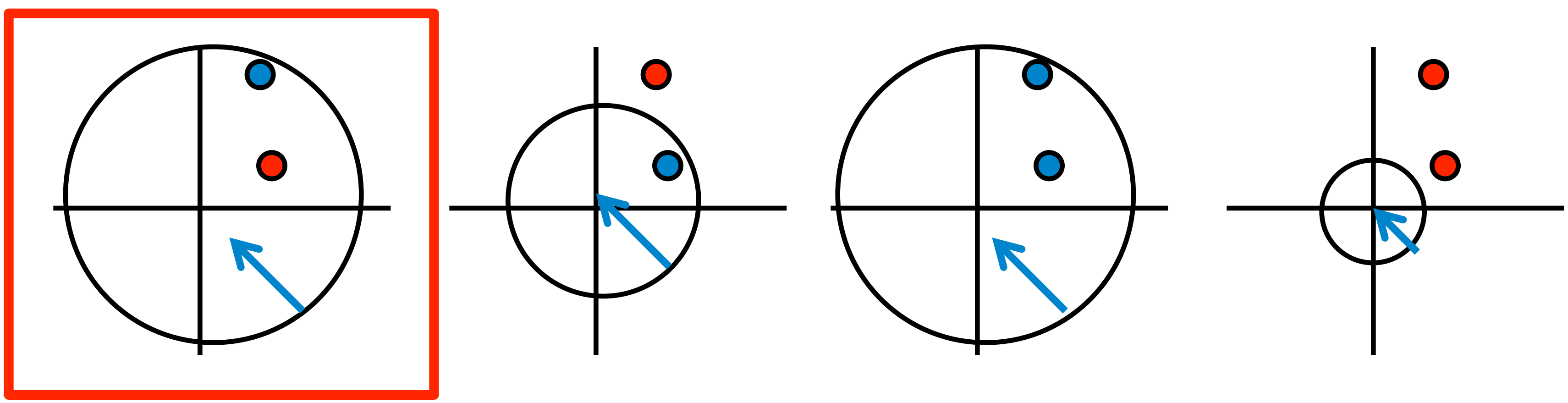
- Nope!



VC Dimension

- The VC dimension is defined as

  The maximum number of points that *can be arranged* so that f(x) can shatter them

- Example: what's the VC dimension of the (zero-centered) circle, $f(x;\theta) = \text{sign}(x^T x + \theta)$?

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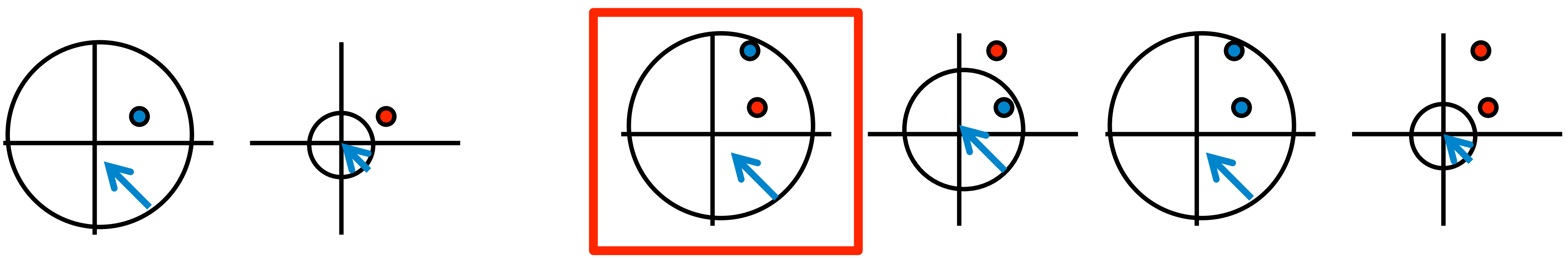
- VCdim = 1 : can arrange one point, cannot arrange two (previous example was general)



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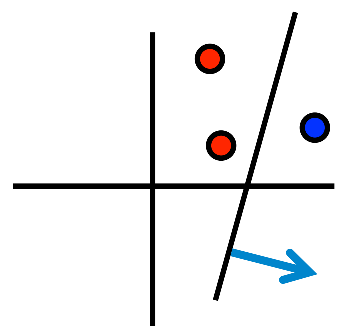

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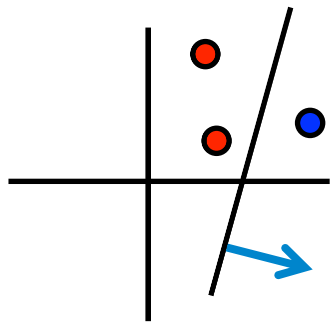
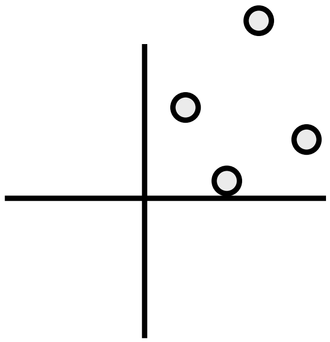


- VC dim >= 4?



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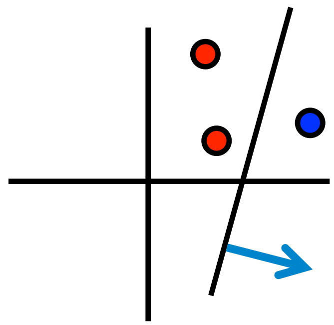
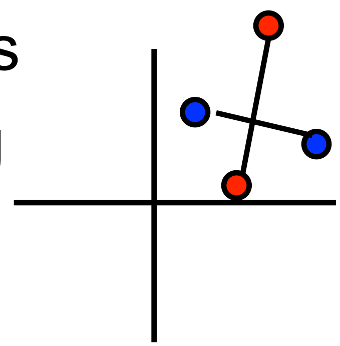


- VC dim >= 4? No…

  Any line through these points must split one pair (by crossing one of the lines)



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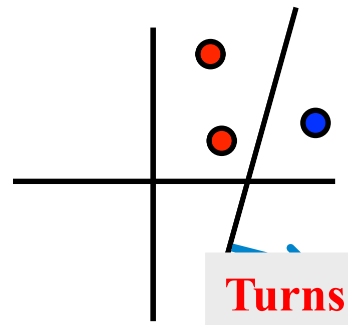

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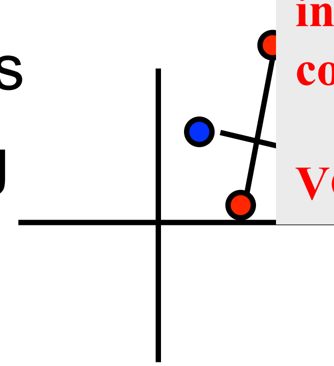

Turns out:
For a general , linear classifier (perceptron) in d dimensions with a constant term:

VC dim = d+1

VC dimension

- VC dimension measures the “power” of the learner
- Does *not* necessarily equal the # of parameters!

- Number of parameters does not necessarily equal complexity
  - Can define a classifier with a lot of parameters but not much power (how?)
  - Can define a classifier with one parameter but lots of power (how?)

- Lots of work to determine what the VC dimension of various learners is…

Using VC dimension

- Recall how we used validation data, or cross-validation error rates to select a complexity

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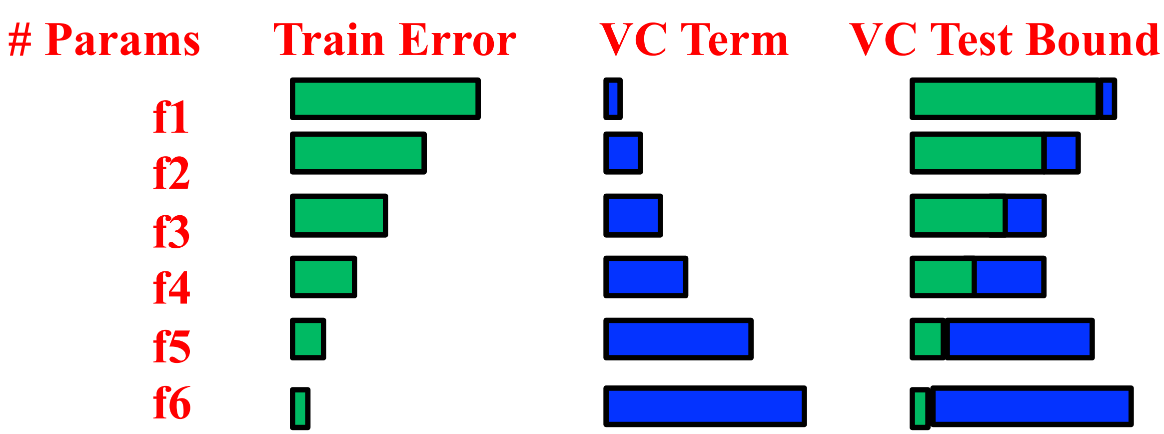
- Use VC dimension based bound on test error similarly
- “Structural Risk Minimization” (SRM)



Using VC dimension

- Recall how we used validation data, or cross-validation error rates to select a complexity
- Use VC dimension based bound on test error similarly
- Other Alternatives
  - Probabilistic models: likelihood under model (rather than classification error)
  - AIC (Aikike Information Criterion)
    - Log-likelihood of training data - # of parameters
  - BIC (Bayesian Information Criterion)
    - Log-likelihood of training data - (# of parameters)*log(N)
- Similar to VC dimension: performance + penalty
- BIC conservative; SRM very conservative
- Also, “true Bayesian” methods (take prob. learning…)