L1-Norm Minimization

Xiaoyi Liu
Email:xiaoyil3@uci.edu

Theorem 1: Given two vectors $\boldsymbol{x},\boldsymbol{y} \in R^n$, the optimal scalar β that minimizes $||\beta\boldsymbol{x} - \boldsymbol{y}||_1$ is

$$\beta^\star = \frac{y_{j_{t^\star}}}{x_{j_{t^\star}}},$$

where:

1. x_i or y_i is the i-th element of $\boldsymbol{x}$ or $\boldsymbol{y}$, respectively;
2. $\{j_1,\dots,j_m\}$ is a permutation of the indices of non-zero elements in $\boldsymbol{x}$ ($\{i_1,\dots,i_m\}$, where $m \le n$) satisfying $\frac{y_{j_1}}{x_{j_1}} \le \dots \le \frac{y_{j_m}}{x_{j_m}}$;
3. $t^\star$ satisfies $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| < 0$ for $t < t^\star$ and $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| \ge 0$ for $t^\star \le t \le m$.

Proof: $f(\beta) = ||\beta\boldsymbol{x} - \boldsymbol{y}||_1$ is equivalent to

$$\begin{aligned} f(\beta) &= \sum_{i=1}^{n} |\beta x_i - y_i| = \sum_{k=1}^{m} |\beta x_{i_k} - y_{i_k}| \\ &= \sum_{k=1}^{m} |x_{i_k}| \times \left|\beta - \frac{y_{i_k}}{x_{i_k}}\right| = \sum_{k=1}^{m} \left|x_{j_k}\right| \times \left|\beta - \frac{y_{j_k}}{x_{j_k}}\right|. \end{aligned} \tag{1}$$

First, we show the optimal value of β is one of $\frac{y_{j_1}}{x_{j_1}},\dots,\frac{y_{j_m}}{x_{j_m}}$.
When $\beta \le \frac{y_{j_1}}{x_{j_1}}$, $f(\beta)$ in (1) is equal to

$$f(\beta) = \sum_{k=1}^{m} \left|x_{j_k}\right| \times \left(\frac{y_{j_k}}{x_{j_k}} - \beta\right) = \sum_{k=1}^{m} \left|x_{j_k}\right| \times \frac{y_{j_k}}{x_{j_k}} - \left(\sum_{k=1}^{m} \left|x_{j_k}\right|\right) \times \beta. \tag{2}$$

The minimum of (2) is attained when $\beta = \frac{y_{j_1}}{x_{j_1}}$.

Similarly, when $\beta \geq \frac{y_{j_m}}{x_{j_m}}$, we obtain

$$f(\beta) = \sum_{k=1}^{m} \left|x_{j_k}\right| \times \left(\beta - \frac{y_{j_k}}{x_{j_k}}\right) = \left(\sum_{k=1}^{m} \left|x_{j_k}\right|\right) \times \beta - \sum_{k=1}^{m} \left|x_{j_k}\right| \times \frac{y_{j_k}}{x_{j_k}}. \tag{3}$$

The minimum of (3) is achieved when $\beta = \frac{y_{j_m}}{x_{j_m}}$.

When $\frac{y_{j_t}}{x_{j_t}} \leq \beta \leq \frac{y_{j_{t+1}}}{x_{j_{t+1}}}$ for $1 \leq t \leq m-1$, we have

$$\begin{aligned} f(\beta) &= \sum_{k=1}^{t} \left|x_{j_k}\right| \times \left(\beta - \frac{y_{j_k}}{x_{j_k}}\right) + \sum_{k=t+1}^{m} \left|x_{j_k}\right| \times \left(\frac{y_{j_k}}{x_{j_k}} - \beta\right) \\ &= \left(\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right|\right) \times \beta + \sum_{k=t+1}^{m} \left|x_{j_k}\right| \times \frac{y_{j_k}}{x_{j_k}} - \sum_{k=1}^{t} \left|x_{j_k}\right| \times \frac{y_{j_k}}{x_{j_k}}. \end{aligned} \tag{4}$$

The minimum of (4) is obtained when $\beta = \frac{y_{j_t}}{x_{j_t}}$ if $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| \geq 0$ and $\beta = \frac{y_{j_{t+1}}}{x_{j_{t+1}}}$ if $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| < 0$. Therefore, we have shown the optimal β should be one of $\frac{y_{j_1}}{x_{j_1}}, \ldots, \frac{y_{j_m}}{x_{j_m}}$.

Second, we show the optimal β is $\beta^\star = \frac{y_{j_{t^\star}}}{x_{j_{t^\star}}}$ such that $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| < 0$ for $t < t^\star$ and $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| \geq 0$ for $t^\star \leq t \leq m$. By subtracting $f\left(\frac{y_{j_t}}{x_{j_t}}\right)$ from $f\left(\frac{y_{j_{t+1}}}{x_{j_{t+1}}}\right)$, we get

$$\begin{aligned} f\left(\frac{y_{j_{t+1}}}{x_{j_{t+1}}}\right) - f\left(\frac{y_{j_t}}{x_{j_t}}\right) &= \sum_{k=1}^{t} \left|x_{j_k}\right| \times \left(\frac{y_{j_{t+1}}}{x_{j_{t+1}}} - \frac{y_{j_k}}{x_{j_k}}\right) + \sum_{k=t+2}^{m} \left|x_{j_k}\right| \times \left(\frac{y_{j_k}}{x_{j_k}} - \frac{y_{j_{t+1}}}{x_{j_{t+1}}}\right) \\ &\quad - \sum_{k=1}^{t-1} \left|x_{j_k}\right| \times \left(\frac{y_{j_t}}{x_{j_t}} - \frac{y_{j_k}}{x_{j_k}}\right) - \sum_{k=t+1}^{m} \left|x_{j_k}\right| \times \left(\frac{y_{j_k}}{x_{j_k}} - \frac{y_{j_t}}{x_{j_t}}\right) \\ &= \left(\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right|\right) \times \left(\frac{y_{j_{t+1}}}{x_{j_{t+1}}} - \frac{y_{j_t}}{x_{j_t}}\right). \end{aligned}$$

Note that $\frac{y_{j_{t+1}}}{x_{j_{t+1}}} - \frac{y_{j_t}}{x_{j_t}} \geq 0$. For any $t < t^\star$, $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| < 0$, thus, $f\left(\frac{y_{j_t}}{x_{j_t}}\right) \geq f\left(\frac{y_{j_{t+1}}}{x_{j_{t+1}}}\right) \geq \cdots \geq f(\beta^\star)$. For any $t > t^\star$, $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right| \geq 0$, thus, $f\left(\frac{y_{j_t}}{x_{j_t}}\right) \geq f\left(\frac{y_{j_{t-1}}}{x_{j_{t-1}}}\right) \geq \cdots \geq f(\beta^\star)$. Furthermore, since $\sum_{k=1}^{t} \left|x_{j_k}\right| - \sum_{k=t+1}^{m} \left|x_{j_k}\right|$ is strictly increasing on t, there exists only one such $t^\star$. The proof is complete. ■