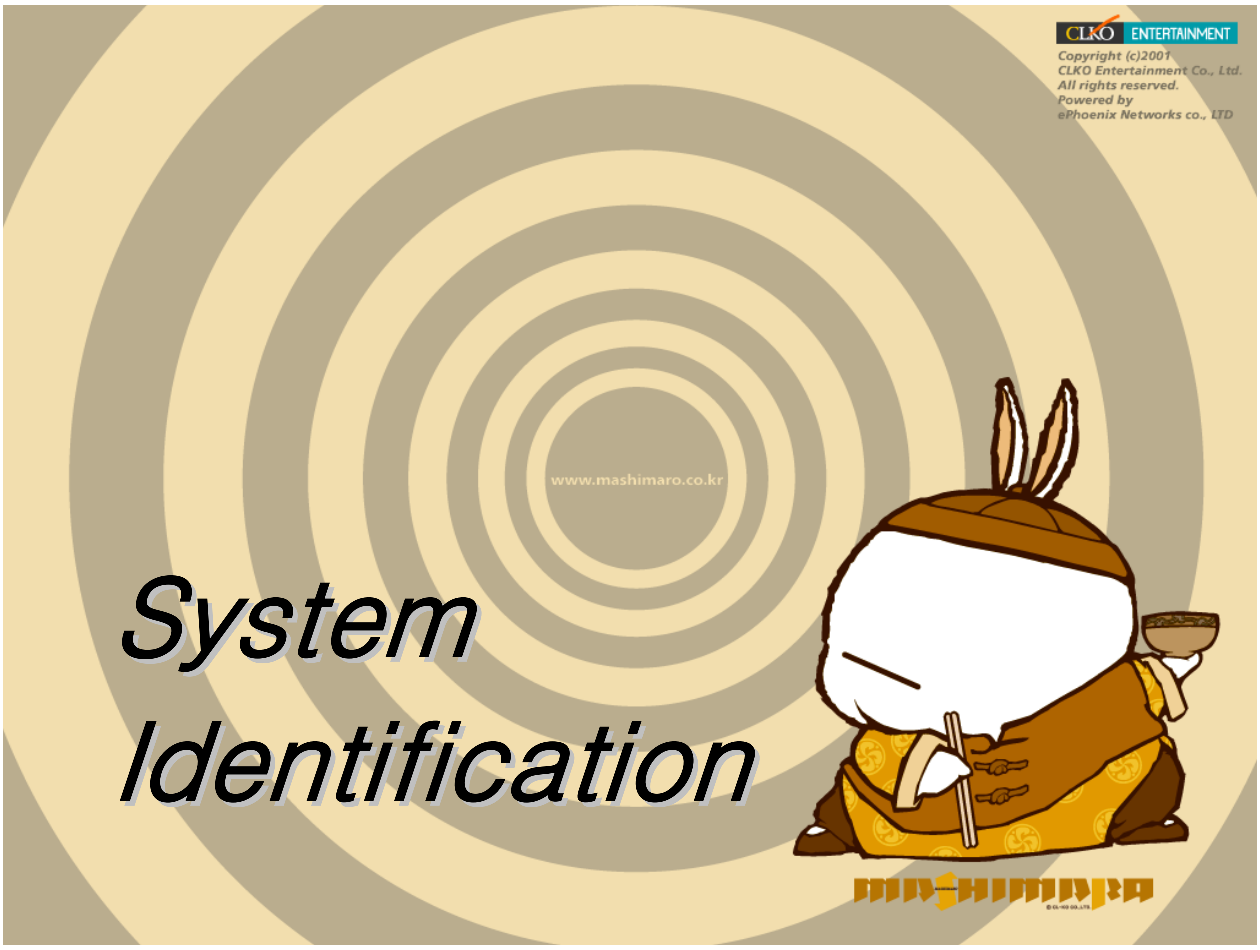

江士標老師 SBJ 講義 系統建模及識別 丁崇武電腦繪圖文字編輯 20100315

1-INT introduction s8 p1-1, 2-TDM time domain model s13 p1-6,

3-FDM frequency domain model s17 p1-14,

4-FRF frequency response function s30 p1-26,

5-RT recursive technology s24 p2-1 (SBJ proprietary),

6-SRT system realization theory s22 p2-22,

7-OKID observer / Kalman filter identification s15 p3-1,

8-OCID observer / controller identification s5 p3-13,

9-FDSS frequency domain state space identification s10 p3-18;

Ref: 莊哲男老師 Jer-Nan Juang, Applied System Identification, 1st ed, Prentice Hall, 1994. *阿標老師手寫講義 system_ID_sbj20060916.pdf , * 丁崇武學長 電腦排版 阿標老師的講義 CH?SysIDsbj 丁崇武*.doc , *p.doc 為 友善列印版: (待勘誤與補充 修訂時會再上傳更新版) ; *系統建模與識別 OM6023

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***note: equations are edited by mathtype5 add-on to msOffice2003, figures were plotted by visio2003 and matlab (source files are provided by 丁崇武).**

1 Introduction

1.1 Real Application:

The procedures used to Active control of Aerospace Structure

1. System modeling
- FEM; (Finite Elements method) for rough model construction.
2. System Identification testing
- Select control and measurement points to perform dynamic test.
- Construct mathematical model.
3. Controller design
- Design controller based on mathematical model.
4. Verification
- Verify closed loop performance by conducting closed loop dynamic testing.

1.2 Identification:

The process used "experimental" data develops or improves a mathematical representation of a physical system.

Three types of ID used in Aerospace structure

- Modal parameters identification.
- Structure-Modal parameters identification; Structure engineer.
- Control-Modal parameters identification; Control engineer.

Numerical model (10% error)

- 1st ID iteration refine → 5% Error.
- 2nd ID iteration refine → 1~2%Error. It's good enough for analysis of flutter and dynamic load but not enough for control.

For control → directly use the ID plant model.

1.3 ID process:

Purpose of ID:

- For control:

ID the model adequately describing the output and input mapping; sensor and actuator.

- Number and location requisition is not very high.
- Phase information is important; avoid non-minimum phase system.

- For modal testing:

Analyze the dynamic properties; stiffness, damping and frequency.
Sensor and actuator;

- Need enough number and good-location for measure and excite the modal.
- Phase information is not important.

Six key steps for system ID task:

1. Develop analytical model (FEM. Etc.).
2. Simulation. Excite analytical model by anticipated sources to establish the levels of structure dynamic response.
3. Chose instrument according to the information given in 2.
4. Perform experiment.
5. Use ID techniques to identify characteristic of the structure and noise.
6. Refine the analytical model if necessary.

1.4 ID methods:

- Modal parameter ID (modal testing)
  - Processing the measuring signal produced by a structure and identifying modal parameters (damping, frequency mode shapes).
  - Well developed discipline with strong experimental foundation.
- System ID in control
  - Processing the measuring signal produced by a system and building a model to represent the system for control design.
  - Well developed with solid theoretical and methodological foundation.
- Correlation between these two fields
  - If linear: Eigen solution of model → modal parameters.
  - Application: Active control of a flexible structure requires the combined efforts.

1.5 Modal testing:

- Excitation:
  - Multiple input random excitation; 45%.
  - Single input random approach; 30%.
  - Multi-input sine dwell; <10%.
  - Natural ambient excitation 5%.
- Analysis:
  - FRF; Frequency Response Function; 80%, Reason: tradition.
  - Direct analysis of time history of response.
- Modal identification:
  - Multi-Degree-of-Freedom (MDOF), time domain 40%.
  - Multi-Degree-of-Freedom (MDOF), frequency domain 30%.
  - Single-Degree-of-Freedom (SDOF), 20%.
- System ID starts to be important discipline in control since 196? because:
  - Modern control need accurate model.
  - Availability of computer.
  - Time domain $\leftrightarrow$ Frequency domain.
- Frequency domain:

It dominated before 1960; sinusoidal transfer function analyzer.

- Advantage:

It can reject low frequency drift and harmonic distortion due to nonline-arity.

- Disadvantage:

1. Testing time is very long under sweep sine wave; FFT (Fast Fourier transformation), from analog to digital.
2. Non-parametric ID.

- Time domain:
  - Provide modal parameters.
  - Least square.
  - For linear system.
  - Most time domain approaches at the EE field (low order) but no other field.

- Two techniques which have been extended and applied to modal parameter ID of structures.

1. Minimum realization.
2. Lattice filtering.

- Minimum realization:
  - *Gilbert*, *Kalman*. Controllability and Observability.
  - *Ho*, *Kalman*. Minimum realization = Represent model in *Markov* parameters (plus response samples).
  - *Tether*, *Silverman*, *Rosen Lepidus*. They study different I/O minimum partial realization.

2 Time domain model

2.1 Introduction:

- Purpose of ID:
  - Developing mathematical model of a physical system.
  - Using experimental data.
  - Describe relationship between input, output and noise.
- Steps of ID:

1. Chose a family of candidate models.
2. Determine the particular member in this family base on observed data and criterion.
3. Transformed the format of model to desired form for control and modal analysis.

- Mathematical format of dynamic system:
  - Continuous time:

Differential equation → Time domain.

Algebraic equation → Frequency domain.

- Discrete time; There are two forms of weighting sequence and state space.

Difference equation → Time domain.

Algebraic equation → Frequency domain.

2.2 Continuous-time state-space model:

- Spring-Mass System;

$M\ddot{\omega}+\zeta\dot{\omega}+K\omega=f(\omega,t)=B_2u(t)$. We use displacement ω and velocity $\dot{\omega}$ as states; $x=\begin{bmatrix}\omega\\ \dot{\omega}\end{bmatrix}\Rightarrow \dot{x}=\begin{bmatrix}0 & I\\ -M^{-1}K & -M^{-1}\zeta\end{bmatrix}x+\begin{bmatrix}0\\ M^{-1}B_2\end{bmatrix}u$.

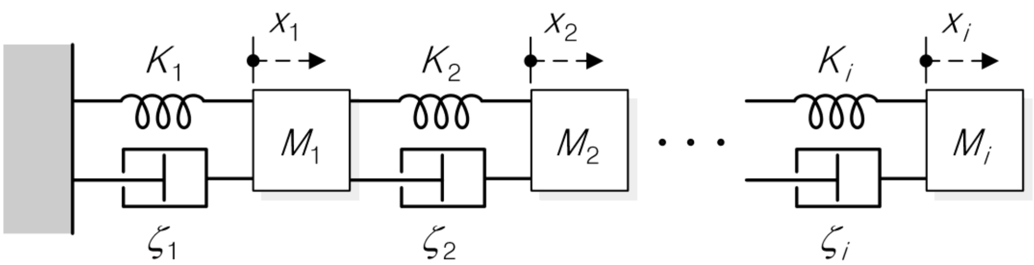
And use output $y(t)$ as combination of accelerometer, tachometer and strain gages; $y=C_a\ddot{\omega}+C_v\dot{\omega}+C_d\omega$, where C_a, C_v, C_d are influence matrices (contain conversion factors). It's implies that y is a function of states only.

$$y=C_aM^{-1}\left[B_2u-\zeta\dot{\omega}-K\omega\right]+C_v\dot{\omega}+C_d\omega$$

$$=\left[C_d-C_aM^{-1}K \quad C_v-C_aM^{-1}\zeta\right]\begin{bmatrix}\omega\\ \dot{\omega}\end{bmatrix}+C_aM^{-1}B_2u=Cx+Du.$$

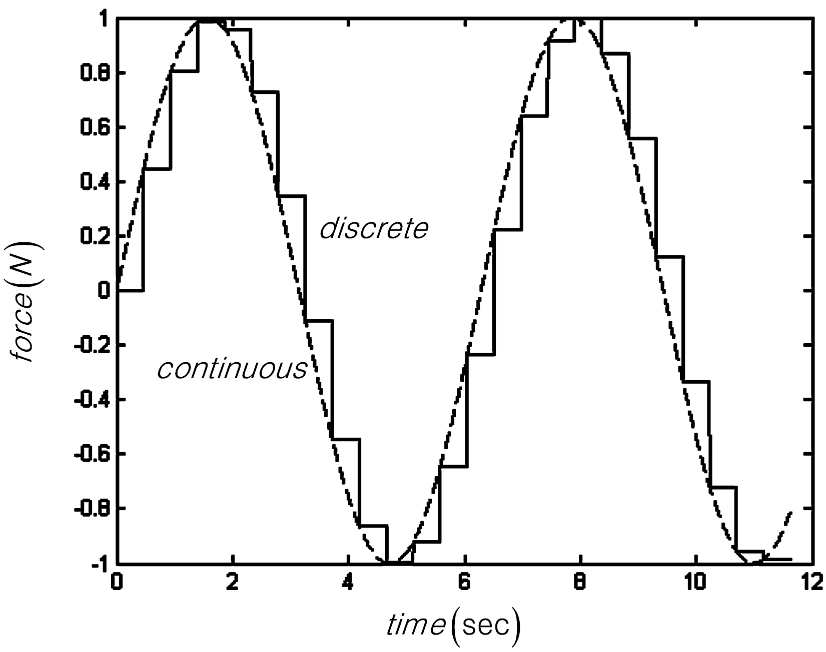
- Homework 1:

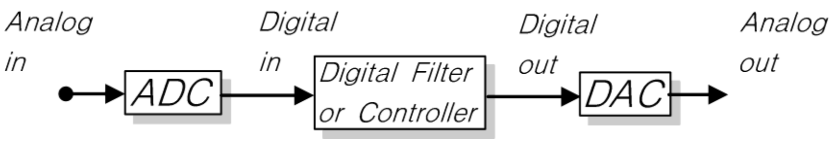
Please write a program under MatLab which can automatically transform the spring-mass system to the state-space matrix form.

Given: 

Object: $\dot{x}=A_cx+B_cu,\quad y=C_cx+D_cu$.

2.3 Discrete-time state-space model:





- Given C.T. linear system:

$\dot{x}(t) = A_c x(t) + B_c u(t)$, $y(t) = Cx(t) + Du(t)$, B_c and D are diagonal matrices. Discretized D.T. linear system with zero-order holder for $x(k) = x(k\Delta t)$, where Δt : sampling period, we obtain

$$\begin{matrix} \dot{x}(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{matrix} \cdots (0), \text{ where } \begin{matrix} A = e^{A_c \Delta t} \\ B = \int_0^{\Delta t} e^{A_c \Delta \tau} d\tau B_c \end{matrix},$$

$$\begin{matrix} x(k+1) = x\left[(k+1)\Delta t\right] \\ u(k) = u(k\Delta t) \end{matrix}.$$

- Homework 2:

From the toolbox of MatLab, please find the subroutine which can transform a continuous time state space to a discrete time state space and recover it to the ordinary algorithm. In addition, please derive and prove the formula according to the algorithm.

2.4 Weighting sequence description:

- Response of a relaxed discrete linear system

$$x(0) = 0 \qquad x(1) = Bu(0) \qquad x(2) = ABu(0) + Bu(1)$$

$$y(0) = Du(0) \quad y(1) = CBu(0) + Du(1) \quad y(2) = CABu(0) + CBu(1) + Du(2)$$

$$\cdots \quad x(k) = \sum_{i=1}^{k} A^{i-1} Bu(k-i) \quad y(k) = \sum_{i=1}^{k} CA^{i-1} Bu(k-i) + Du(k).$$

If $u(k) = \left[u_1(k) \ \ u_2(k) \ \ \cdots \ \ u_r(k)\right]_{1\times r}^T$, $\forall i = 1,2,\cdots,r$ $u_i(0) = 1$ and $u_i(k) = 0$ when $k > 0$, then $y(0) = Y_0 = D$, $y(1) = Y_1 = CB$, $Y_2 = CAB$, $\cdots$, $Y_k = CA^{k-1}B \Rightarrow$ Pulse response Y_0 , Y_1 , Y_2 , $\cdots$, Y_k . They're also called System *Markov* parameter or *Markov* parameters and unique for given system; $Y_k = \hat{C}\hat{A}^{k-1}\hat{B} = CT^{-1}\left(TAT^{-1}\right)^{k-1} TB = CA^{k-1}B$.

- Represent the system response with *Markov* parameters;

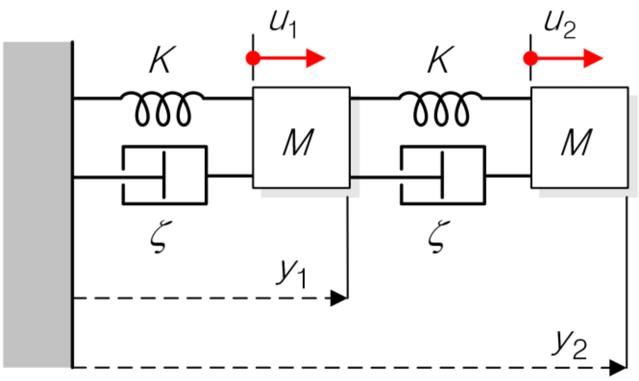
$y(k) = \sum_{i=0}^{\infty} Y_i u(k-i)$, where $Y_0 = D$. It seems the weighting summation of the input. We also call *Markov* parameters as weighting sequence. Theoretically, weighting sequence is infinite long. However, it can be approximate with finite terms as follow if the system is asymptotically stable. $y(k)=\sum_{i=0}^{\hat{k}} Y_i u(k-i)\cdots(*)$, where $\hat{k}$ is small under high damping. In the other hand, $\hat{k}$ is large under low damping.

- The equation $(*)$ is so-call FIR model because we use it to represent the given system. So, *Markov* parameters are also called FIR parameters.
- The limitation of FIR model:

For steady state only, no initial condition influence.

- Homework 3:

Please write a program using MatLab in order to obtain the output of the system which's A, B, C, D and $u(k)$ are known and $y(k)$ is desired. In addition, please use the programs of the last homework to compute the *Markov* parameters of the following system to 100 terms.



In which, $M=1$ $K=1$ $\zeta=0.1$ $y(k)=\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}$ $u(k)=\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$ and $u_1(k)$, $u_2(k)$ are force.

2.5 State space observer model:

- System: $\begin{matrix} x(k+1)=Ax(k)+Bu(k)\cdots(1) \\ y(k)=Cx(k)+Du(k)\cdots(2) \end{matrix}$, only $\begin{matrix} y(k) \\ u(k) \end{matrix}$ are available,

$x(k+1)$ can not be measured directly. We estimate $x(k)$ with $\hat{x}(k)$ using $y(k)$ and $u(k)$

- Observer:

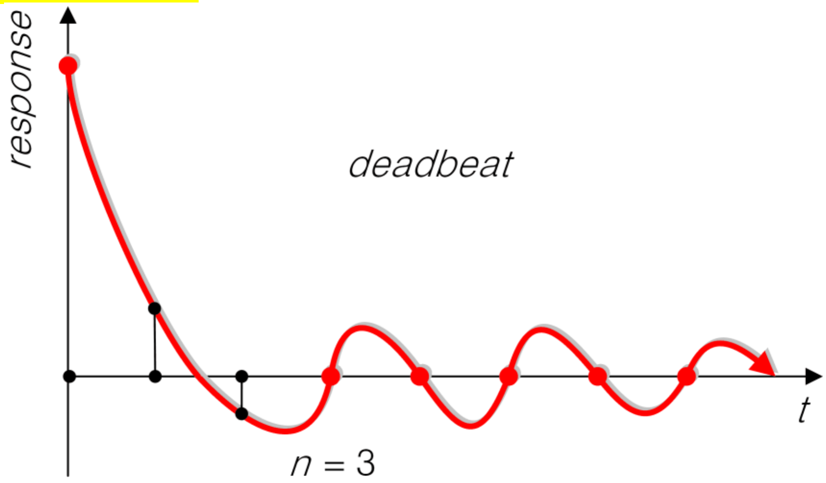
The $\hat{x}(k)$ has the same dynamic of $x(k)$ when we choose an observer such that $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) - G\left[y(k) - \hat{y}(k)\right] \cdots (3)$ and modify the estimation by the error between $y(k)$ and the estimated output $\hat{y}(k) = C\hat{x}(k) + Du(k) \cdots (4)$. In (3), G is the observer feedback gain. Rewrite (3) and (4) taking $y(k)$, $u(k)$ as input and $\hat{x}(k)$ as state:

$\hat{x}(k+1) = (A+GC)\hat{x}(k) + (B+GD)u(k) - Gy(k) = \bar{A}\hat{x}(k) + \bar{B}v(k) \cdots (5)$,

where $\bar{A} = A + GC$, $\bar{B} = \left[B+GD \quad -G\right]$, $v(k) = \left[u^T(k) \quad y^T(k)\right]^T$.

- Check the state estimation error:

$e(k+1) \triangleq x(k+1) - \hat{x}(k+1) = Ax(k) + Bu(k) - \left[\bar{A}\hat{x}(k) + \bar{B}v(k)\right]$
$= Ax(k) + GCx(k) + Bu(k) + GDu(k) - A\hat{x}(k) - GC\hat{x}(k) - Bu(k)$
$-GDu(k) = (A+GC)\left[x(k) - \hat{x}(k)\right] = \bar{A}e(k)$. The result suggests that the converging speed of $e(k)$ depends on $\bar{A}$. We choose G such that all the eigenvalues of $\bar{A}$ are zero. Now, the result implies that $\bar{A}^k = \mathit{0}$, $\forall k \ge n$ if $\bar{A}$ is a $n \times n$ matrix. From $e(1) = \bar{A}e(0)$, $e(2) = \bar{A}^2 e(0)$, $\cdots$, $e(n-1) = \bar{A}^{n-1}e(0)$, $e(n) = \bar{A}^n e(0) = \mathit{0}$, $\cdots$, we can see that $e(k)$ will converge to $\mathit{0}$ for $k \ge n$. The response, the resulting observer and the gain G are called as deadbeat response, deadbeat observer and deadbeat observer gain in this condition, respectively.



- The *Markov* parameters of an observer:

Since equations (5) and (2) are similar to equation (0), we will define the observer *Markov* parameters as $\bar{Y}_0 = D$, $\bar{Y}_1 = C\bar{B}$, $\bar{Y}_2 = C\bar{A}\bar{B}$, $\cdots$, $\bar{Y}_k = C\bar{A}^{k-1}\bar{B}$. They can be represented in A, B, C and D:

$\bar{Y}_0 = D,\ \ \bar{Y}_1 = C\left[B+GD \quad -G\right],$ $\begin{aligned}\bar{Y}_2 &= C(A+GC)\left[B+GD \quad -G\right] \\ &= \left[C(A+GC)(B+GD) \quad -C(A+GC)G\right],\end{aligned}$

$\cdots,\ \ \bar{Y}_k = \left[C(A+GC)^{k-1}(B+GD) \quad -C(A+GC)^{k-1}G\right] \square \left[\bar{Y}_k^{(1)} \quad -\bar{Y}_k^{(2)}\right],$

where $\bar{Y}_k,\ \forall k>0$ and $\bar{Y}_0$ are $m\times(m+r)$ and $m\times r$ matrixes, re-spe-ctively. They can be used as the basis for computing system *Markov* parameter.

- Homework 4:

Please design an observer using the discrete version of A, B, C and D in homework 3 such that its G makes all the eigenvalues of $\bar{A}$ are zero and use the program of homework 3 to compute the observer *Markov* parameters.

2.6 Linear difference model:

- Since observer *Markov* parameters are $\bar{Y}_0 = D$ and

$\bar{Y}_k = \left[C\bar{A}^{k-1}\bar{B} \quad -C\bar{A}^{k-1}G\right] \square \left[\bar{Y}_k^{(1)} \quad -\bar{Y}_k^{(2)}\right]$ and *Markov* parameters are the weighting sequence used by $v(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$, we can derive

$$y(k) = \sum_{i=0}^{k} \bar{Y}_i v(k-i) = \sum_{i=1}^{k} \left[\bar{Y}_i^{(1)} \quad -\bar{Y}_i^{(2)}\right] \begin{bmatrix} u(k-i) \\ y(k-i) \end{bmatrix} + \bar{Y}_0 v(k)$$

$$= -\sum_{i=1}^{k} \bar{Y}_i^{(2)} y(k-i) + \sum_{i=1}^{k} \bar{Y}_i^{(1)} u(k-i) + Du(k).$$

If the order of system is n, we get $\lambda^n = 0 \Rightarrow \bar{A}^n = (A+GC)^n = 0$ $\Rightarrow \bar{Y}_k = 0,\ \forall k>n$ for deadbeat observer. This result will be such that $\ni\ y(k) + \sum_{i=1}^{n} \bar{Y}_i^{(2)} y(k-i) = \sum_{i=1}^{n} \bar{Y}_i^{(1)} u(k-i) + Du(k) \cdots (6)$. Equation (6) is a linear difference model and called as ARX model (Auto-Regressing eXogenous model).

- Homework 5:

Please write a program which can transform the representation of a system from state space model to the linear difference model.

3 Frequency domain model

3.1 Introduction:

This topic is about the frequency response function, s-Transform and z-Transform. *Laplace* Transform (one side; causality) is defined as:

$F(s) \triangleq \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$ and then differential function's *Laplace* transform under relaxation is $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\mathcal{L}\{f(t)\} - f(0) = s\mathcal{L}\{f(t)\}$. So, we can replace differential equations in time domain with algebraic equation in *s* domain.

3.2 s-Transform of second-order dynamic equation:

- s-domain dynamic equation $\left(Ms^2 + \zeta s + K\right)\boldsymbol{\omega}(s) = F(s)$, where M, ζ and K are $n_2 \times n_2$ matrices, $\left(Ms^2 + \zeta s + K\right)$ is lambda matrix:

The analytic solution of the dynamic equ： $\boldsymbol{\omega}(s) = \left(Ms^2 + \zeta s + K\right)^{-1} F(s)$,

The generally experimental solution： $\boldsymbol{\omega}(s) = \left(\sum_{i=0}^{2n_2-2} N_i \frac{s^{2n_2-2-i}}{d(s)}\right) F(s)$,

where N_i is also a $n_2 \times n_2$ matrix. The relationship between them is

$$\left(Ms^2 + \zeta s + K\right)^{-1} = \sum_{i=0}^{2n_2-2} N_i \frac{s^{2n_2-2-i}}{d(s)} \triangleq \frac{N(s)}{d(s)}.$$

$$\Rightarrow \begin{array}{l} N(s) = N_0 s^{2n_2-2} + N_1 s^{2n_2-2-1} + \cdots + N_{2n_2-2} \\ d(s) = s^{2n_2} + d_1 s^{2n_2-1} + d_2 s^{2n_2-2} + \cdots + d_{2n_2} \end{array},$$

$$\Rightarrow M = N_0^{-1}, \quad \zeta = \left[d_1 I - MN_1\right]M, \quad K = \left[d_2 I - MN_2 - \zeta N_1\right]M.$$

The knowledge of N_0, N_1, N_2, d_1 and d_2 is sufficient to completely determine the mass, damping, and stiffness matrices.

[Remark]:

If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, then $A^{-1} = \frac{\mathrm{adj}A}{\det A}$, where $\mathrm{adj}A = \begin{bmatrix} & \vdots & \\ \cdots & \hat{a}_{ij} & \cdots \\ & \vdots & \end{bmatrix}^T$;

adjugate of A, $\hat{a}_{ij} = (-1)^{i+j} \det M_{ij}$; cofactor of a_{ij}, $M_{ij} = \begin{bmatrix} & / & \\ / & \cancel{a_{ij}} & / \\ & / & \end{bmatrix}$; minor of A.

- Limitation: fully measurable of $\boldsymbol{\omega}(s)$; including displacements, velocity, and acceleration: $y(t) = C_a \ddot{\boldsymbol{\omega}}(t) + C_v \dot{\boldsymbol{\omega}}(t) + C_d \boldsymbol{\omega}(t)$

$y(s) = (C_a s^2 + C_v s + C_d)\boldsymbol{\omega}(s)$

$= (C_a s^2 + C_v s + C_d)(Ms^2 + \zeta s + K)^{-1} B_2 u(s)$. There will not be a simple mapping between M, ζ, K and N_0, N_1, N_2, d_1, d_2 here.

3.3 s-Transform of continuous state space model:

- s.s.m: $\begin{matrix} sx(s) = A_c x(s) + B_c u(s) \\ y(s) = Cx(s) + Du(s) \end{matrix}$, with zero initial condition;

Solution: $x(s) = [sI - A_c]^{-1} B_c u(s)$; transfer function for $u \to x$ and $y(s) = \left(C[sI - A_c]^{-1} B_c + D\right) u(s)$; transfer function for $u \to y$.

- Computation of $[sI - A_c]^{-1}$:

1. *Laplace* transform of $e^{A_c t}$:

$\because \mathcal{L}\{e^{A_c t}\} = \int_0^\infty e^{A_c t} e^{-st} dt = \int_0^\infty e^{-(sI - A_c)t} dt = -[sI - A_c]^{-1} e^{-(sI - A_c)t}\Big|_0^\infty$

$= [sI - A_c]^{-1}$.

2. Sequence:

$$[sI - A_c]^{-1} = \mathcal{L}\{e^{A_c t}\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} \frac{1}{k!} t^k A_c^k\right\} = \sum_{k=0}^{\infty} \mathcal{L}\left\{\frac{1}{k!} t^k A_c^k\right\} = \sum_{k=0}^{\infty} s^{-(k+1)} A_c^k.$$

3. An algorithm in linear system?

- Computation of $e^{A_c t}$:

1. Sequence: $e^{A_c t} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A_c^k = I + A_c t + \frac{1}{2} A_c^2 t^2 + \cdots$.

2. Eigen decomposition:

$$e^{A_c t} = e^{\psi \Lambda \psi^{-1} t} = \psi e^{\Lambda t} \psi^{-1} = \psi \,\text{diag}\left[e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}\right] \psi^{-1}.$$

3. Using *Cayley Halmilton* theorem:

If $\dim\left[A_c\right] = n \times n$ and $f\left(A_c\right) = 0 \Rightarrow f\left(\lambda_j\right) = 0$, then $\exists$ a polynomial of order up to *n*, $\ni f\left(A_c\right) = \sum_{i=0}^{n} p_i A_c^i$. $\Rightarrow e^{A_c} = \sum_{i=0}^{n} p_i \left(A_c\right)^i \Rightarrow e^{\lambda_j} = \sum_{i=0}^{n} p_i \left(\lambda_j\right)^i$, $\forall j = 1 \sim n$ to solve p_i. Then we get $e^{A_c t} = p_0 I t + p_1 A_c t + \cdots + p_n A_c^n t$.

- Transfer function:

Analytic; $y(s) = \left(C\left[sI - A_c\right]^{-1} B_c + D\right) u(s)$.

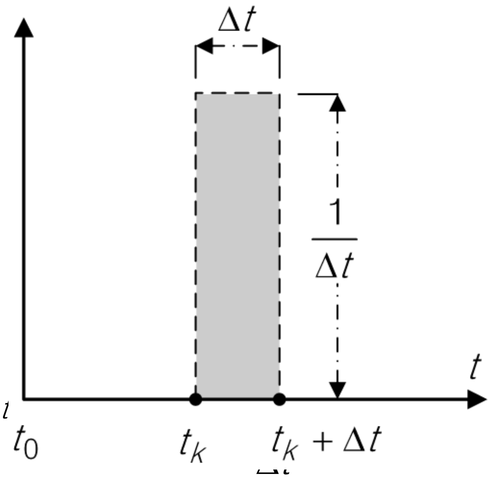
Experimental; $y(s) = \left(Y_0 + Y_1 s^{-1} + \cdots + Y_n s^{-n} + \cdots\right) u(s)$.

- Four kinds of responses:

Impulse response, Pulse response, Sampled Impulse response, Sampled Pulse response.

➢ Pulse:

$$\delta_\Delta\left(t - t_k\right) = \begin{cases} 0 & \forall t < t_k \\ \dfrac{1}{\Delta t} & \forall t_k \le t \le t_k + \Delta t \\ 0 & \forall t > t_k + \Delta t \end{cases},$$



$$\Rightarrow \int_0^\infty \delta_\Delta\left(t - t_k\right) dt = \int_{t_k}^{t_k + \Delta t} \frac{1}{\Delta t} dt = \frac{\Delta t}{\Delta t} = 1,$$

$$\int_0^\infty f(t)\,\delta_\Delta\left(t - t_k\right) dt = \int_{t_k}^{t_k + \Delta t} \frac{f(t)}{\Delta t} dt = \frac{1}{\Delta t} \int_{t_k}^{t_k + \Delta t}$$

, where $t_k < t_i < t_k + \Delta t$.

➢ Impulse: $\delta\left(t - t_k\right) = \lim_{\Delta t \to 0} \delta_\Delta\left(t - t_k\right)$

$$\Rightarrow \int_0^\infty \delta\left(t - t_k\right) dt = 1 \text{ and } \int_0^\infty f(t)\,\delta\left(t - t_k\right) dt = \lim_{\Delta t \to 0} f\left(t_i\right) = f\left(t_k\right).$$

3.4 Impulse response s-Transform:

$$\delta(s) = \mathcal{L}\left\{\delta\left(t - t_k\right)\right\} = \int_0^\infty e^{-st} \delta\left(t - t_k\right) dt = e^{-s t_k}.$$

Impulse response: applied Impulse at time $t_k = 0$ for every single input

$$u(s) = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \Rightarrow \text{ Impulse response; } Y_\delta(s) = \left[C\left(sI - A_c\right)^{-1} B_c + D\right] \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}.$$

3.5 Pulse response s-Transform:

$$\delta_\Delta(s)=\int_0^\infty e^{-st}\delta_\Delta(t-t_k)\,dt=\int_{t_k}^{t_k+\Delta t}e^{-st}\frac{1}{\Delta t}dt=\frac{-1}{s\Delta t}e^{-st}\Big|_{t_k}^{t_k+\Delta t}$$

$$=\frac{e^{-st_k}-e^{-s(t_k+\Delta t)}}{s\Delta t}=\frac{e^{-st_k}}{s}\frac{1-e^{-s\Delta t}}{\Delta t}.$$

Pulse response: applied Pulse at time $t_k=0$ for every single input

$$u(s)=\Delta t\begin{bmatrix}u_1\\ \vdots\\ u_r\end{bmatrix}\bullet\delta_\Delta(s)=\begin{bmatrix}u_1\\ \vdots\\ u_r\end{bmatrix}\frac{1-e^{-s\Delta t}}{s}\Rightarrow \text{ Pulse response;}$$

$$Y_{\delta_\Delta}(s)=\frac{C(sI-A_c)^{-1}B_c+D}{s}\begin{bmatrix}u_1\\ \vdots\\ u_r\end{bmatrix}-e^{-s\Delta t}\frac{C(sI-A_c)^{-1}B_c+D}{s}\begin{bmatrix}u_1\\ \vdots\\ u_r\end{bmatrix}, \text{ where}$$

$$\frac{C(sI-A_c)^{-1}B_c+D}{s}\begin{bmatrix}u_1\\ \vdots\\ u_r\end{bmatrix}: \qquad e^{-s\Delta t}\frac{C(sI-A_c)^{-1}B_c+D}{s}\begin{bmatrix}u_1\\ \vdots\\ u_r\end{bmatrix}:$$

Step response at $t_k=0$ Step response with delay Δt.

3.6 Sampled Impulse response s-Transform:

- Sampled representation of $y(t)$ with period Δt.

$y^*(t)=\sum_{k=0}^{\infty}y(t)\delta(t-k\Delta t)$, it means that $y^*(t)=0,\ \forall t\neq k\Delta t$,

$y^*(t)=y(t)\lim_{\Delta t\to 0}\delta(t)=\infty$, if $t=k\Delta t$. *Laplace* transform of $y^*(t)$ is

$$y^*(s)=\int_0^\infty y^*(t)e^{-st}dt=\int_0^\infty e^{-st}\sum_{k=0}^{\infty}y(t)\delta(t-k\Delta t)dt$$

$$=\sum_{k=0}^{\infty}\int_0^\infty e^{-st}y(t)\delta(t-k\Delta t)dt=\sum_{k=0}^{\infty}y(k\Delta t)e^{-sk\Delta t}=\sum_{k=0}^{\infty}y(k\Delta t)\left(e^{s\Delta t}\right)^{-k}.$$

- Impulse and Step:

➢ $\delta(t-t_k)$ is not realistic in practical implement; $\delta(t-t_k)=0,\ \forall t\neq t_k$,

$\delta(t-t_k)=\infty,\ t=t_k$. For a step, $S(t-t_0)=\begin{cases}1 & \forall t\geq t_0\\ 0 & \forall t<t_0\end{cases}$.

$$\int_0^t\delta(\tau-0)d\tau=\begin{cases}1 & \forall t\geq 0\\ 0 & \forall t<0\end{cases}=S(t-0)\Rightarrow\delta(t-0)=\frac{d}{dt}S(t-0).$$

➢ Step and Impulse response:

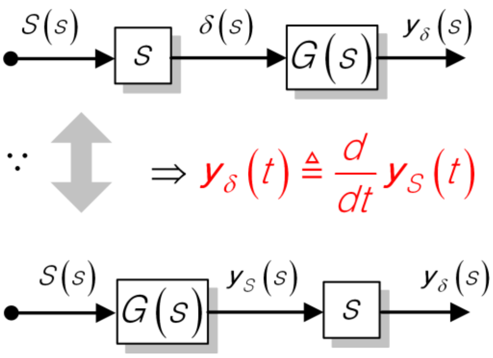
Step response:

$$y_S(t)=C\int_0^t e^{A_c(t-\tau)}B_c u(\tau)d\tau+Du(t)$$

$$=C\int_0^t e^{A_c(t-\tau)}B_c I_{r\times1}d\tau+DI_{r\times1}S(t-0)$$

$$=C(-A_c)^{-1}\bullet e^{A_c(t-\tau)}\Big|_0^t\bullet B_c+DI_{r\times1}S(t-0)$$

$$=C(-A_c)^{-1}\left(e^{A_ct}-\cancel{e^0}_I\right)B_c+DI_{r\times1}S(t-0).$$



$\because$ $\Rightarrow y_\delta(t)\triangleq\dfrac{d}{dt}y_S(t)$

So, Impulse response: $y_\delta(t)=Ce^{A_ct}B_c+DI_{r\times1}\delta(t-0)$, $\forall t\ge0$. In practical measurement is after Impulse $y_\delta(t)=Ce^{A_ct}B_c$, $\forall t>0$. Sampling of Impulse response; $y^*(t)=\sum_{k=0}^{\infty}y_\delta(t)\delta(t-k\Delta t)$.

- *Laplace* Transform of Sampled Impulse response:

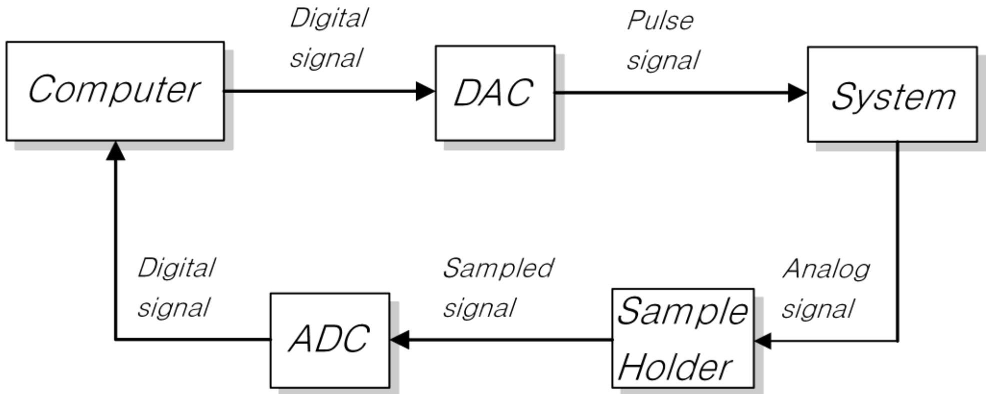
$$y_\delta^*(s)=\int_{0^+}^{\infty}y_\delta^*(t)e^{-st}dt=\int_{0^+}^{\infty}\sum_{k=0}^{\infty}y_\delta(t)\delta(t-k\Delta t)e^{-st}dt$$

$$=\sum_{k=0}^{\infty}y_\delta(k\Delta t)\left(e^{s\Delta t}\right)^{-k}=CB_c+\sum_{k=1}^{\infty}Ce^{A_ck\Delta t}B_c\left(e^{s\Delta t}\right)^{-k}$$

□ $CB_c+\sum_{k=1}^{\infty}CA^{k-1}\bar{B}_c\left(e^{s\Delta t}\right)^{-k}=C\Big(\underbrace{e^{s\Delta t}}_{=z}I-A\Big)^{-1}\bar{B}_c+CB_c$, where $A=e^{A_c\Delta t}$; Discrete-Time State Matrix, $\bar{B}_c=AB_c$.

3.7 Sampled Pulse response s-Transform:

Typical testing setup as follow, testing result is Sampled Pulse response.



Input: $u(t)=\delta_\Delta(t-0)\begin{bmatrix}1\\\vdots\\1\end{bmatrix}_{r\times1}=\begin{cases}\begin{bmatrix}1&\cdots&1\end{bmatrix}_{1\times r}^T & 0\le t\le\Delta t\\ 0 & t>\Delta t\end{cases}.$

Response: $y(t)=C\int_0^t e^{A_c(t-\tau)}B_c u(\tau)\,d\tau+Du(t)\Rightarrow y_{\delta_\Delta}(0)=0+D\bullet\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}_{r\times 1}$

$=D$, $\forall t\ge \Delta t$, $y_{\delta_\Delta}(t)=C\int_0^{\Delta t} e^{A_c(t-\tau)}B_c\,d\tau=Ce^{A_c(t-\Delta t)}\int_0^{\Delta t} e^{A_c(\Delta t-\tau)}B_c\,d\tau$

$=Ce^{A_c(t-\Delta t)}\int_0^{\Delta t} e^{A_c(\Delta t-\tau)}B_c\,d\tau=Ce^{A_c(t-\Delta t)}\int_0^{\Delta t} e^{A_c\hat{\tau}}B_c\,d\hat{\tau},\ \left(\hat{\tau}=\Delta t-\tau\right)$

$=Ce^{A_c(t-\Delta t)}B$, where $B\triangleq\int_0^{\Delta t} e^{A_c\hat{\tau}}B_c\,d\hat{\tau}$.

Sampled response: $y^*_{\delta_\Delta}=\sum_{k=0}^{\infty} y_{\delta_\Delta}(t)\,\delta(t-k\Delta t)$.

Laplace Transform of Sampled Pulse response:

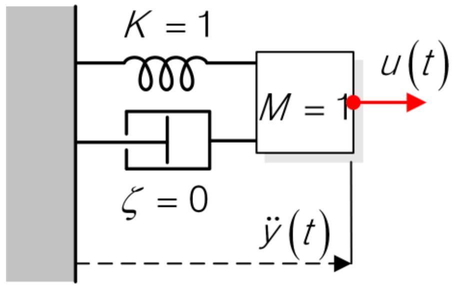
$$y^*_{\delta_\Delta}(s)=\int_0^{\infty} y^*_{\delta_\Delta}(t)e^{-st}dt$$

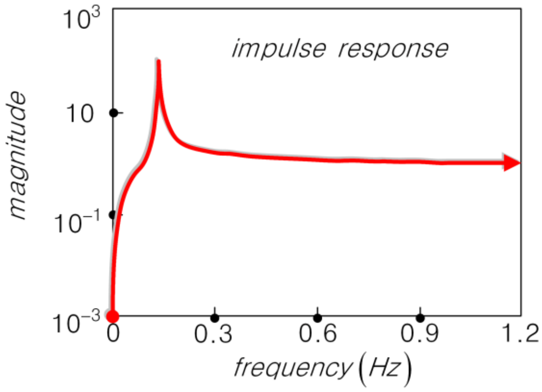
$$=\sum_{k=0}^{\infty}\int_0^{\infty} y_{\delta_\Delta}(t)e^{-st}\delta(t-k\Delta t)\,dt$$

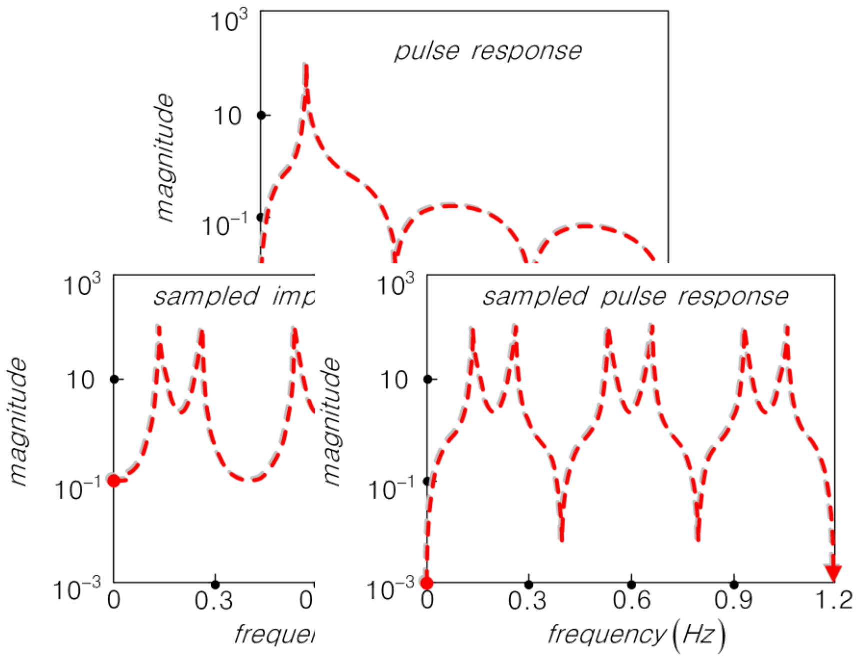
$$=\sum_{k=0}^{\infty} y_{\delta_\Delta}(k\Delta t)\left(e^{s\Delta t}\right)^{-k}$$

$$=D+\sum_{K=1}^{\infty} CA^{K-1}B\left(e^{s\Delta t}\right)^{-k},\ \left(A=e^{A_c\Delta t}\right)$$

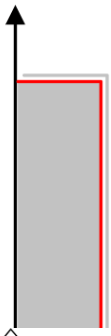
$$=C\Big[\underbrace{e^{s\Delta t}}_{=z}I-A\Big]^{-1}B+D.$$



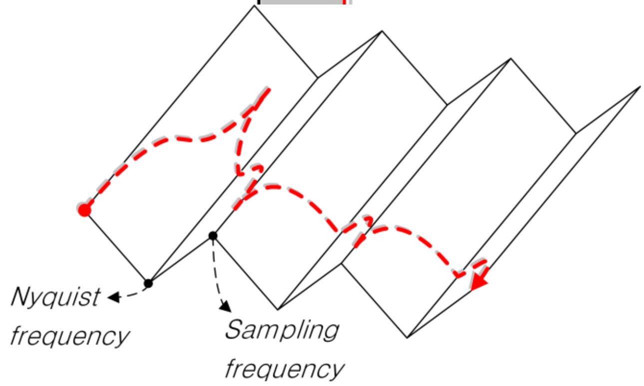




- Impulse → Pulse: introduces zeros at $1-e^{j\omega\Delta t}=0 \Rightarrow \omega\Delta t = 2\pi f\Delta t$ $=0,\ 2\pi,\ 4\pi,\ \cdots \Rightarrow f=\dfrac{n}{\Delta t}=0,\ 0.4,\ 0.8,\ \cdots$
- Impulse: excites every mode for all frequencies.
- Pulse: excites most modes except at frequency which is the multiply of the sampling frequency because it doesn't repeat at sampling frequency and multiply of it as shown in the following.



- Sampling: the action will introduce Alias on frequency domain (folder) at $i\left[f_{Nyq}\pm\left(f_{Nyq}-f_n\right)\right]$.



3.8 Summary:

State-Space Model (Continuous Time)	$\dot{x}=A_c x + B_c u,\ y=Cx+Du$
Impulse Response	$y_\delta(t)=Ce^{A_c t}B_c,\ \ t\ge 0^+$
Pulse Response	$y_{\delta_\Delta}(0)=D,$ $y_{\delta_\Delta}(t)=Ce^{A_c(t-\Delta t)}B,\ \ t\ge\Delta t$
Impulse Response s-Transform	$y_\delta(s)=C\left(sI-A_c\right)^{-1}B_c+D$
Pulse Response s-transform	$y_{\delta_\Delta}(s)=y_\delta(s)\dfrac{1-e^{-s\Delta t}}{s}$
Sampled Impulse Response s-Transform	$y_\delta^*(s)=CB_c+\sum_{k=1}^{\infty}CA^{k-1}\bar{B}_c\left(e^{s\Delta t}\right)^{-k}$ $=C\left(e^{s\Delta t}I-A\right)^{-1}\bar{B}_c+CB_c$
Sampled Pulse Response s-Transform	$y_{\delta_\Delta}^*(s)=D+\sum_{K=1}^{\infty}CA^{K-1}B\left(e^{s\Delta t}\right)^{-k}$ $=C\left(e^{s\Delta t}I-A\right)^{-1}B+D$
Definition of some notation shown above	

$$A = e^{A_c \Delta t}, \quad \bar{B}_c = AB_c, \quad B = \int_0^{\Delta t} e^{A_c \hat{\tau}} B_c d\hat{\tau}$$

3.9 z-Transforms:

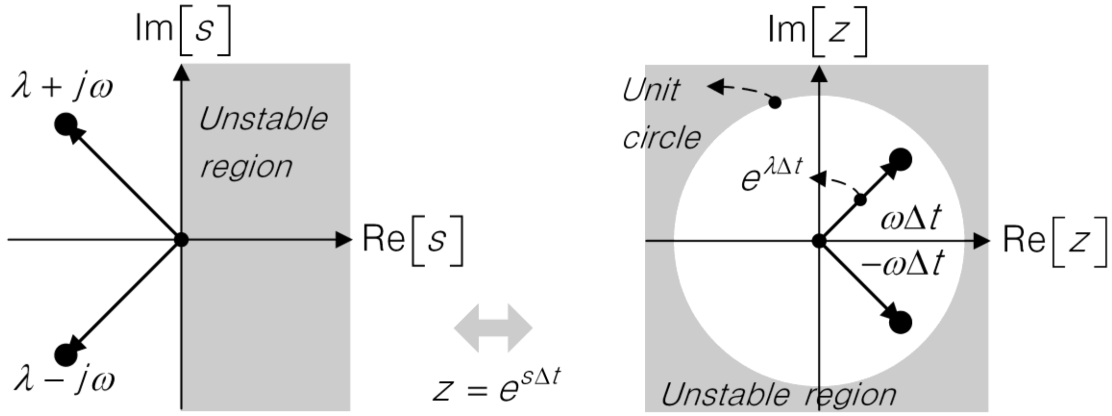
- The definition of z-transform: $y(z) \triangleq \mathcal{Z}\{y(k\Delta t)\} \triangleq \sum_{k=0}^{\infty} y(k\Delta t) z^{-k}$

➢ z-Transform vs. s-Transform of sampled signal;

$$\{y(k\Delta t) : k = 0, 1, 2, \cdots\} \cong \left\{ \sum_{k=0}^{\infty} y(t)\delta(t - k\Delta t) \right\}$$

$$\Rightarrow y^*(s) = \sum_{k=0}^{\infty} y(k\Delta t)\left(e^{s\Delta t}\right)^{-k} \triangleq y(z) \text{ with } z = e^{s\Delta t}.$$

➢ $z = e^{s\Delta t}$ map s domain to z domain;



- The properties of z-Transform:

$$\mathcal{Z}\{y(k-n)\} = \sum_{k=0}^{\infty} y(k-n) z^{-k} = y(-n) + \cdots + y(-2) z^{-n+2} + y(-1) z^{-n+1}$$

$+z^{-n}\mathcal{Z}\{y(k)\} \Rightarrow \mathcal{Z}\{y(k-1)\} = y(-1)z^0 + z^{-1}\{y(k)\}$. If $k \to k+1$, then

$\Rightarrow \mathcal{Z}\{y(k+1-1)\} = y(-1+1)z^0 + z^{-1}\{y(k+1)\} \Rightarrow \mathcal{Z}\{y(k)\} = y(0)z^0$

$+z^{-1}\{y(k+1)\} \Rightarrow \mathcal{Z}\{y(k+1)\} = z\left[\mathcal{Z}\{y(k)\} - y(0)\right]$.

- z-Transform of State-Space:

$\mathcal{Z}\{x(k+1) = Ax(k) + Bu(k)\} \Rightarrow z\left[x(z) - x(t=0)\right] = Ax(z) + Bu(z)$,

where $x(z) = \mathcal{Z}\{x(k)\}$. If $x(t=0) = 0 \Rightarrow (zI - A)x(z) = Bu(z)$

$\Rightarrow x(z) = (zI - A)^{-1} Bu(z) \Rightarrow y(z) = \left[C(zI - A)^{-1} B + D\right] u(z)$.

- z-Transform of weighting sequence:

$\mathcal{Z}\left\{ y(k) = \sum_{i=0}^{\infty} Y_i u(k-i) \right\} \Rightarrow y(z) = \sum_{i=0}^{\infty} Y_i \mathcal{Z}\{u(k-i)\}$. If $u(-i) = \cdots$

$= u(-1) = 0$, then $\mathcal{Z}\{u(k-i)\} = z^{-i}u(z) \Rightarrow y(z) = \left(\sum_{i=0}^{\infty} Y_i z^{-i}\right)u(z)$.

- z–transform of State–Space observer model:

$$\begin{aligned} x(k+1) &= \bar{A}x(k) + \bar{B}v(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}, \Rightarrow \begin{aligned} \bar{A} &= A + GC \\ \bar{B} &= \begin{bmatrix} B+GD & -G \end{bmatrix} \end{aligned}, \quad v = \begin{bmatrix} u \\ y \end{bmatrix}$$

. Let us take z–transfer function between $u(t)$ and $y(t)$.

$$\Rightarrow x(z) = \left(zI - \bar{A}\right)^{-1}\bar{B}v(z),\ y(z) = Cx(z) + Du(z)$$

$$\Rightarrow y(z) = C\left(zI - \bar{A}\right)^{-1}\begin{bmatrix} B+GD & -G \end{bmatrix}\begin{bmatrix} u(z) \\ y(z) \end{bmatrix} + Du(z)$$

$$= -C\left(zI - \bar{A}\right)^{-1}Gy(z) + \left[C\left(zI - \bar{A}\right)^{-1}(B+GD) + D\right]u(z)$$

$$\Rightarrow y(z) = \left[I + C\left(zI - \bar{A}\right)^{-1}G\right]^{-1}\left[C\left(zI - \bar{A}\right)^{-1}(B+GD) + D\right]u(z).$$

Whether this one equal to the transfer function derived from state space;

$y(z) = \left[C\left(zI - A\right)^{-1}B + D\right]u(z)$?

[Ans.]:

$$\left[I + C\left(zI - \bar{A}\right)^{-1}G\right]^{-1}\left[C\left(zI - \bar{A}\right)^{-1}(B+GD) + D\right]$$

$$= \left[I - C\left(zI - A\right)^{-1}G\right]\left[C\left(zI - \bar{A}\right)^{-1}(B+GD) + D\right]$$

$$= C\left[I - \left(zI - A\right)^{-1}GC\right]\left(zI - \bar{A}\right)^{-1}(B+GD) + \left[D - C\left(zI - A\right)^{-1}GD\right]$$

$$= C\left(zI - A\right)^{-1}\left[zI - A - GC\right]\left(zI - \bar{A}\right)^{-1}(B+GD) + D - C\left(zI - A\right)^{-1}GD$$

$$= C\left(zI - A\right)^{-1}B + C\left(zI - A\right)^{-1}GD + D - C\left(zI - A\right)^{-1}GD$$

$= C\left(zI - A\right)^{-1}B + D$. They are the same.

- z–Transform of linear difference model:

$$y(k) = \sum_{i=1}^{k}\bar{Y}_i v(k-i) + Du(k) = \sum_{i=1}^{k}\begin{bmatrix} \bar{Y}_i^{(1)} & -\bar{Y}_i^{(2)} \end{bmatrix}\begin{bmatrix} u(k-i) \\ y(k-i) \end{bmatrix} + Du(k)$$

$$\Rightarrow y(z) = \left(-\sum_{i=1}^{k}\bar{Y}_i^{(2)}z^{-i}\right)y(z) + \left(\sum_{i=1}^{k}\bar{Y}_i^{(1)}z^{-i}\right)u(z) + Du(z)$$

$$\Rightarrow \left(I + \sum_{i=1}^{k} \bar{Y}_i^{(2)} z^{-i} \right) y(z) = \left(\sum_{i=1}^{k} \bar{Y}_i^{(1)} z^{-i} + D \right) u(z)$$. If $\bar{Y}_i$ converge such that $\bar{A}^i \approx 0,\ \forall i > p$, $\Rightarrow \left(I + \sum_{i=1}^{p} \bar{Y}_i^{(2)} z^{-i} \right) y(z) = \left(\sum_{i=1}^{p} \bar{Y}_i^{(1)} z^{-i} + D \right) u(z)$.

4 Frequency Response Functions; FRF

4.1 Introduction:

- FRF: *Fourier* Transform of impulse or pulse response (*Markov* parameters).
- FRF vs. Transfer Function:

FRF; *Fourier* transform s is replaced by $s = j\omega$.

Transfer function; *Laplace* transform s approximates to $s \square \sigma + j\omega$.

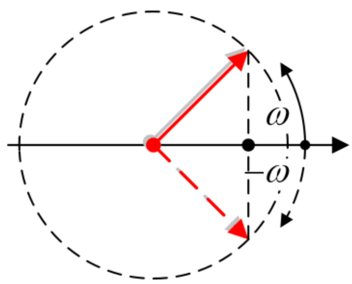
- FRF provide simple to see information for modal testing personnel.

4.2 Basic Formula:

- Response to Single Sinusoid Wace:

➢ Weighting sequence model of I/O relationship;

$$y(k) = \sum_{i=0}^{\infty} Y_i u(k-i).$$



Input is sine wave; $u(k) = \sin(\omega k \Delta t)$

$$= \frac{1}{2j}\left(e^{j\omega k \Delta t} - e^{-j\omega k \Delta t} \right)$$. Response;

$$y(k) = \sum_{i=0}^{\infty} Y_i u(k-i) = \sum_{i=0}^{\infty} \frac{Y_i}{2j}\left[e^{j\omega(k-i)\Delta t} - e^{-j\omega(k-i)\Delta t} \right]$$

$$= \frac{e^{j\omega k \Delta t}}{2j} \sum_{i=0}^{\infty} Y_i e^{-j\omega i \Delta t} - \frac{e^{-j\omega k \Delta t}}{2j} \sum_{i=0}^{\infty} Y_i e^{j\omega i \Delta t}$$; if $z \square e^{j\omega \Delta t}$

$$= \frac{e^{j\omega k \Delta t}}{2j} G(z) - \frac{e^{-j\omega k \Delta t}}{2j} \hat{G}(z)$$; $\hat{G}$ denotes complex conjugate

$$= \frac{\cos(\omega k \Delta t)}{2j} \mathrm{Re}\{G(z)\} + \frac{\sin(\omega k \Delta t)}{2} \mathrm{Re}\{G(z)\} + \frac{\cos(\omega k \Delta t)}{2j} j \,\mathrm{Im}\{G(z)\}$$

$$- \frac{\sin(\omega k \Delta t)}{2j} \mathrm{Im}\{G(z)\} - \frac{\cos(\omega k \Delta t)}{2j} \mathrm{Re}\{G(z)\} + \frac{\sin(\omega k \Delta t)}{2} \mathrm{Re}\{G(z)\}$$

$$+\frac{\cos(\omega k\Delta t)}{2j}j\,\mathrm{Im}\{G(z)\}+\frac{\sin(\omega k\Delta t)}{2j}\mathrm{Im}\{G(z)\}=\sin(\omega k\Delta t)\mathrm{Re}\{G(z)\}$$

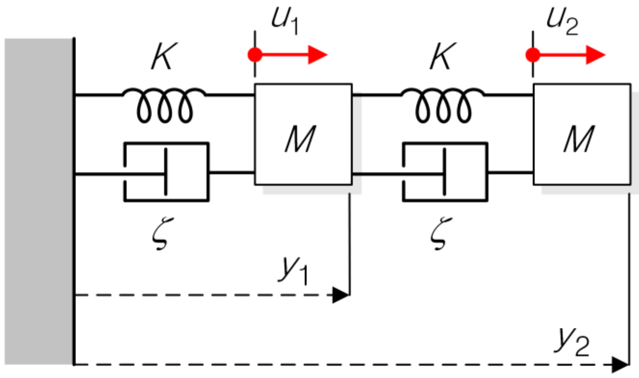
$+\cos(\omega k\Delta t)\mathrm{Im}\{G(z)\}=|G(z)|\sin(\omega k\Delta t+\psi)$, where $\psi=\angle G(z)$. So, the output is sine wave with gain $|G(z)|$ and phase shift $\psi=\angle G(z)$ if input is sine wave. Then, we can say that $G(z)=G(e^{j\omega\Delta t})$ completely discribes the frequency response of system. It's so-called FRF.

- Bode plot: $\log(\omega)$ vs. $\log|G(z)|$ and $\log(\omega)$ vs. $\angle G(z)$.
- Nyquist plot: $G(z)$ in complex plane.
- Frequency Analysis with sweep sine method:

Changing ω of input sine wave; $\sin(\omega k\Delta t)$ and recording $|G(z)|$ and $\angle G(z)$. Its advantages are simple and robust. Its disadvantage is time consuming.

- Homework 6:

Use the resulting file; .m of the above homework to the sweep sine simulation of the figure shown as right-hand side.



$M=1 \quad K=1 \quad \zeta=0.01$

Steps:

1. Decide the sampling rate according to the analog system.
2. Digitize the system with the above sampling rate to obtain its digital model.
3. Change the frequency of the input sine wave from low to high for generating output sine wave.
4. Compare the gain and phase shift between input and output, draw a Bode plot.
5. Compare the resulting Bode plots of analog and digital models.

- Use combination of different sine waves as input:

Every periodic wave can be represented in *Fourier* series expansion;

$u(t)=\sum_{i=-\infty}^{\infty}\tilde{U}(i)e^{j(\omega t)i}$ and $\tilde{U}(i)=\frac{\omega}{2\pi}\int_{d}^{d+\frac{2\pi}{\omega}}u(t)e^{-j(\omega t)i}dt$. Then, we make the following two assumptions:

1. $u(t)$ is a signal with period $T=\dfrac{2\pi}{\omega}$.

2. We sample $u(t)$ with sampling period $\Delta t=\dfrac{T}{\boldsymbol{\ell}}$ for $\boldsymbol{\ell}\left(\dfrac{samples}{period}\right)$.

$$u(\Delta t,k)=u(k)=\sum_{i=-\infty}^{\infty}\tilde{U}(i)e^{j(\omega k\Delta t)i}=\sum_{i=-\infty}^{\infty}\tilde{U}(i)e^{j\frac{2\pi}{\boldsymbol{\ell}}ki}.$$

- $e^{j\frac{2\pi}{\boldsymbol{\ell}}ki}$ is periodic, $\because\ \forall\ n=0,1,\cdots,\boldsymbol{\ell}-1$ and $\boldsymbol{\upsilon}=-\infty,\cdots,-1,0,1,\cdots,\infty$

$\Rightarrow e^{j\frac{2\pi}{\boldsymbol{\ell}}kn}=e^{j\frac{2\pi}{\boldsymbol{\ell}}k(n+\boldsymbol{\ell\upsilon})}$.

- Let $i=n+\boldsymbol{\ell\upsilon}\Rightarrow u(k)=\displaystyle\sum_{i=-\infty}^{\infty}\tilde{U}(i)e^{j\frac{2\pi}{\boldsymbol{\ell}}ki}=\sum_{n+\boldsymbol{\ell\upsilon}=-\infty}^{\infty}\tilde{U}(n+\boldsymbol{\ell\upsilon})e^{j\frac{2\pi}{\boldsymbol{\ell}}k(n+\boldsymbol{\ell\upsilon})}$

$$=\sum_{n=0}^{\boldsymbol{\ell}-1}\sum_{\boldsymbol{\upsilon}=-\infty}^{\infty}\tilde{U}(n+\boldsymbol{\ell\upsilon})e^{j\frac{2\pi}{\boldsymbol{\ell}}k(n+\boldsymbol{\ell\upsilon})}=\sum_{n=0}^{\boldsymbol{\ell}-1}\sum_{\boldsymbol{\upsilon}=-\infty}^{\infty}\tilde{U}(n+\boldsymbol{\ell\upsilon})e^{j\frac{2\pi}{\boldsymbol{\ell}}kn}$$

$=\displaystyle\sum_{n=0}^{\boldsymbol{\ell}-1}\left[\sum_{\boldsymbol{\upsilon}=-\infty}^{\infty}\tilde{U}(n+\boldsymbol{\ell\upsilon})\right]e^{j\frac{2\pi}{\boldsymbol{\ell}}kn}\ \square\ \sum_{n=0}^{\boldsymbol{\ell}-1}U(n)e^{j\frac{2\pi}{\boldsymbol{\ell}}kn}$. It is so-called discrete *Fourier* series or Inverse Discrete Time *Fourier* Transform; IDTFT. Where $U(n)\ \square\ \displaystyle\sum_{\boldsymbol{\upsilon}=-\infty}^{\infty}\tilde{U}(n+\boldsymbol{\ell\upsilon})$.

- $\hat{U}(n)$ can also be represented as linear combination of $u(k)$;

$U(n)=\dfrac{1}{\boldsymbol{\ell}}\displaystyle\sum_{k=0}^{\boldsymbol{\ell}-1}u(k)e^{-j\frac{2\pi}{\boldsymbol{\ell}}kn}$. It is so-called Discrete Time Fourier Transform; DTFT. We verify it as follow; $u(k)=\displaystyle\sum_{n=0}^{\boldsymbol{\ell}-1}U(n)e^{j\frac{2\pi}{\boldsymbol{\ell}}kn}$

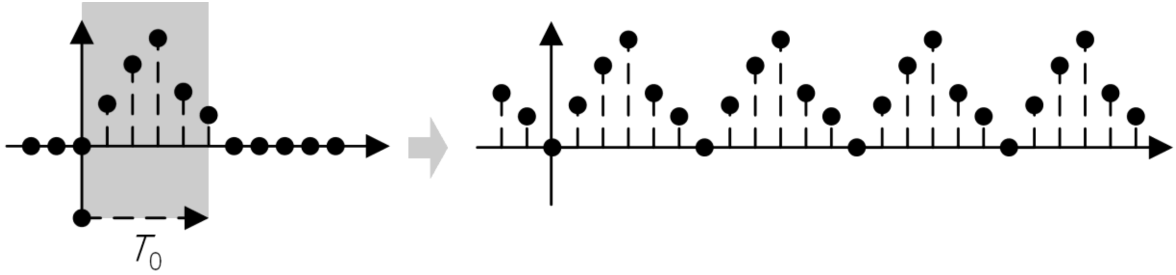
$$=\sum_{n=0}^{\boldsymbol{\ell}-1}\frac{1}{\boldsymbol{\ell}}\sum_{\hat{k}=0}^{\boldsymbol{\ell}-1}u(\hat{k})e^{-j\frac{2\pi}{\boldsymbol{\ell}}\hat{k}n}e^{j\frac{2\pi}{\boldsymbol{\ell}}kn}=\frac{1}{\boldsymbol{\ell}}\sum_{n=0}^{\boldsymbol{\ell}-1}\sum_{\hat{k}=0}^{\boldsymbol{\ell}-1}u(\hat{k})e^{j\frac{2\pi}{\boldsymbol{\ell}}n(\hat{k}-k)}$$

$$=\frac{1}{\boldsymbol{\ell}}\sum_{\hat{k}=0}^{\boldsymbol{\ell}-1}u(\hat{k})\sum_{n=0}^{\boldsymbol{\ell}-1}e^{j\frac{2\pi}{\boldsymbol{\ell}}n(\hat{k}-k)}=\frac{1}{\boldsymbol{\ell}}\sum_{\hat{k}=0}^{\boldsymbol{\ell}-1}u(\hat{k})\frac{1-e^{j\frac{2\pi}{\boldsymbol{\ell}}(\hat{k}-k)\boldsymbol{\ell}}}{1-e^{j\frac{2\pi}{\boldsymbol{\ell}}(\hat{k}-k)}}=\frac{1}{\boldsymbol{\ell}}\boldsymbol{\ell}u(k)=u(k)$$

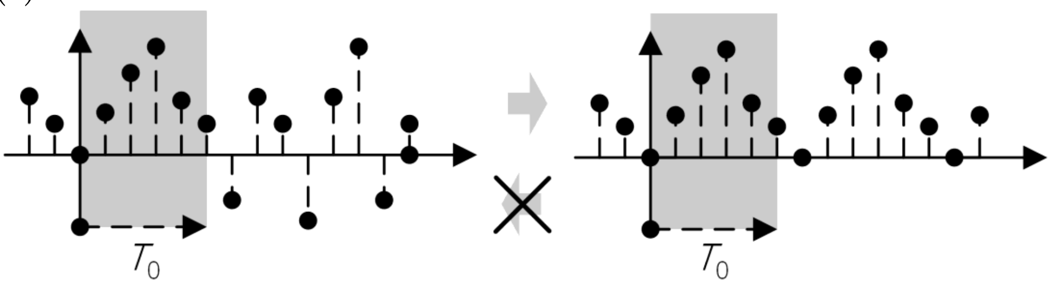
$$\because\ \frac{1-e^{j2\pi(\hat{k}-k)}}{1-e^{j\frac{2\pi}{\boldsymbol{\ell}}(\hat{k}-k)}}=\begin{cases}\boldsymbol{\ell} & \hat{k}=k\\ 0 & \hat{k}\neq k\end{cases}.$$

- From the assumption that $u(t)$ is periodic, we can obtain that the DTFT; $\hat{U}(n)$ of $u(t)$ exists and $\hat{U}(n)$ is periodic. However, there are two cases shown below if the periodicity of $u(t)$ doesn't exist.

1. We can assume $u(t)$ is periodic with period $T \ge T_0$ when $u(k)$ is finite; i.e. $u(t),\ 0 \le t \le T_0$, elsewhere $u(t) = 0$. Now, DTFT still exist.



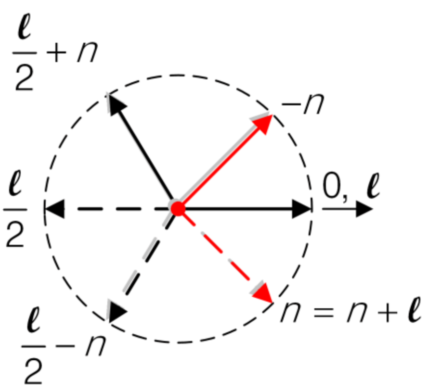
2. If the duration of $u(t)$ is infinite and it isn't periodic, the DTFT of $u(t)$ doesn't exist. Otherwise, the inverse will detune unless the range of $0 \le t \le T_0$ when we choose a section of $u(t)$; i.e. take $u(t),\ 0 \le t \le T_0$ to DTFT.



- Properties of DTFT:

➢ $U(-n) = \frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1} u(\hat{k}) e^{-j\frac{2\pi}{\ell}\hat{k}(-n)} = \frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1} u(\hat{k}) e^{j\frac{2\pi}{\ell}\hat{k}(n)} = \bar{U}(n)$.

➢ $U(n+\ell) = \frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1} u(\hat{k}) e^{-j\left(\frac{2\pi}{\ell}n+2\pi\right)\hat{k}} = U(n)$.



➢ $U\left(\ell/2 + n\right) = \frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1} u(\hat{k}) e^{-j\frac{2\pi}{\ell}\left[\ell-\left(\frac{\ell}{2}-n\right)\right]\hat{k}}$

$= \frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1} u(\hat{k}) e^{-j\left[2\pi-\frac{2\pi}{\ell}\left(\frac{\ell}{2}-n\right)\right]\hat{k}} = \frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1} u(\hat{k}) e^{j\frac{2\pi}{\ell}\left(\frac{\ell}{2}-n\right)\hat{k}}$

$= \bar{U}\left(\ell/2 - n\right)$.

➢ $U(n)$ are complex conjugate pairs except that $U(0)$, $U\left(\ell/2\right)$, $U(\ell)$ are real.

➢ Only half of circle are independent; $\left[0\ \ \frac{\ell}{2}\right]$, $\left[0\ \ \pi\right]$, or $\left[0\ \ \frac{f_s}{2} = f_{Nyq}\right]$.

- Sampling frequency $\uparrow$ $\Rightarrow$Digital $\rightarrow$ analog mode. $\Rightarrow$More modes in digital mode. Therefore, harder for ID algorithm to converge. Trade off sampling frequency f_s is 2~3 times f_c; cut off frequency.

4.3 FRF from Input/Output DTFT:

We have known that for a linear system:

1. Sine wave input $\rightarrow$ sine wave output.
2. Periodic signal $\xrightarrow{DTFT}$ combination of sine waves.

- DTFT of periodic output:

$$Y(n)=\frac{1}{\ell}\sum_{k=0}^{\ell-1}y(k)e^{-j\frac{2\pi}{\ell}kn}=\frac{1}{\ell}\sum_{k=0}^{\ell-1}\sum_{i=0}^{\infty}Y_i u(k-i)e^{-j\frac{2\pi}{\ell}kn}$$

$$=\frac{1}{\ell}\sum_{k=0}^{\ell-1}\sum_{i=0}^{\infty}Y_i e^{-j\frac{2\pi}{\ell}in}u(k-i)e^{-j\frac{2\pi}{\ell}(k-i)n}=\frac{1}{\ell}\sum_{i=0}^{\infty}Y_i e^{-j\frac{2\pi}{\ell}in}\sum_{k=0}^{\ell-1}u(k-i)e^{-j\frac{2\pi}{\ell}(k-i)n}$$

$$=\frac{1}{\ell}\sum_{i=0}^{\infty}Y_i e^{-j\frac{2\pi}{\ell}in}\sum_{\hat{k}=-i}^{\ell-1-i}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}.$$

- If $u(\hat{k})$ is periodic in $\ell \Rightarrow u(\hat{k})=u(\hat{k}+\ell)$,

$$\Rightarrow \frac{1}{\ell}\sum_{\hat{k}=-i}^{\ell-1-i}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}=\frac{1}{\ell}\sum_{\hat{k}=-i}^{-1}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}+\frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1-i}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}$$

$$=\frac{1}{\ell}\sum_{\hat{k}=\ell-i}^{\ell-1}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}+\frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1-i}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}=\frac{1}{\ell}\sum_{\hat{k}=0}^{\ell-1}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}=U(n).$$

$\therefore Y(n)=\left(\sum_{i=0}^{\infty}Y_i z^{-in}\right)U(n)=G(z^n)U(n)$. Where $Y(n)=DTFT\{y(t)\}$,

$z_n \triangleq e^{j\frac{2\pi}{\ell}n}=z^n$, $G(z^n)\triangleq \mathcal{Z}\{Y_i\}\big|_{z=z^n}$; z-Transform of system transfer function with $z=z^n$ and $U(n)=DTFT\{u(t)\}$.

- If $u(\hat{k})$ is non-periodic or period is different from ℓ,

$$\Rightarrow DTFT_\ell\{y(t)\}=Y(n)=\frac{1}{\ell}\sum_{i=0}^{\infty}Y_i e^{-j\frac{2\pi}{\ell}in}\sum_{\hat{k}=-i}^{\ell-1-i}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}$$

$$=\sum_{i=0}^{\infty}Y_i e^{-j\frac{2\pi}{\ell}in}\frac{1}{\ell}\left[\sum_{\hat{k}=0}^{\ell-1}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}+\sum_{\hat{k}=-i}^{\ell-1-i}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}-\sum_{\hat{k}=0}^{\ell-1}u(\hat{k})e^{-j\frac{2\pi}{\ell}\hat{k}n}\right]$$

$=G(z^n)U(n)-\varepsilon_u(n)$. Where

$$\varepsilon_u(n) \triangleq \sum_{i=0}^{\infty} Y_i e^{-j\frac{2\pi}{\ell}in} \frac{1}{\ell} \sum_{\hat{k}=-i}^{-1} \left[u(\hat{k}+\ell) - u(\hat{k})\right] e^{-j\frac{2\pi}{\ell}\hat{k}n} \triangleq \sum_{i=0}^{\infty} Y_i e^{-j\frac{2\pi}{\ell}in} e_i(n)$$

and $\left|e_i(n)\right| \le \frac{1}{\ell}\left|\sum_{\hat{k}=\ell-i}^{\ell-1} u(\hat{k}) e^{-j\frac{2\pi}{\ell}\hat{k}n}\right| + \frac{1}{\ell}\left|\sum_{\hat{k}=-i}^{-1} u(\hat{k}) e^{-j\frac{2\pi}{\ell}\hat{k}n}\right| \le \frac{2}{\ell} C_u i$, where

$\forall \hat{k} \in [0, \ell]$, $\left|u(\hat{k})\right| \le C_u$. $\Rightarrow \left|\varepsilon_u(n)\right| \le \left|\sum_{i=0}^{\infty} Y_i e^{-j\frac{2\pi}{\ell}in} e_i(n)\right|$

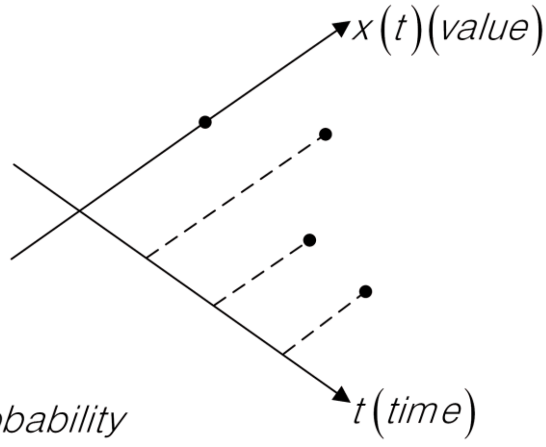
$\le \sum_{i=0}^{\infty} \left|Y_i e^{-j\frac{2\pi}{\ell}in}\right| \left|e_i(n)\right| \le \frac{2}{\ell} C_u C_G$, where $C_G \triangleq \sum_{i=0}^{\infty} i\left|Y_i\right|$. Thus, error is bounded by power bound of $u(\hat{k})$; i.e. C_u and bound of $\sum_{i=0}^{\infty} i\left|Y_i\right|$; i.e. C_G. In addition, $\ell \uparrow \;\Rightarrow\; \left|\varepsilon_u(n)\right| \downarrow$.

- Properties of I/O DTFT:

➢ If SISO system and periodic input then $G(z^n) = \frac{Y(n)}{U(n)}$; $U(n)$ should be non-zero. $\rightarrow$ Random signal.

➢ For MIMO system and non-periodic input that has r inputs and m outputs, we obtain $Y^{(\tau)}_{m\times 1}(n) = G(z^n)_{m\times r} U^{(\tau)}_{r\times 1}(n) - \varepsilon^{(\tau)}_u(n)_{m\times 1}$; $\tau = 1, \cdots, N$ if we have N experiments or N pairs of I/O sets. It can be written as

$\left[Y^{(1)}_{m\times 1}(n) \;\; \cdots \;\; Y^{(N)}_{m\times 1}(n)\right] = G(z^n)\left[U\right.$

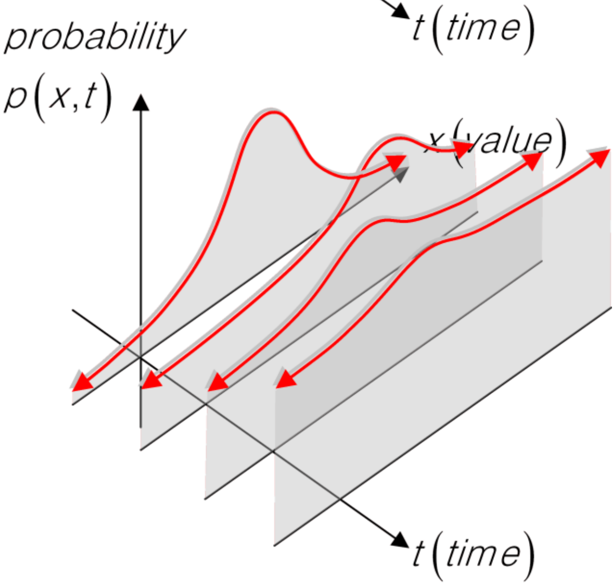


$-\left[\varepsilon^{(1)}_u(n)_{m\times 1} \;\; \cdots \;\; \varepsilon^{(N)}_u(n)_{m\times 1}\right] \Rightarrow \Psi$

.

$\Rightarrow (\Psi\Phi^*)_{m\times r} = G(z^n)_{m\times r} (\Phi\Phi^*)_{r\times r}$

, where $\Phi^* = \hat{\Phi}^T$ is the conju-gate and transpose of Φ. In the other hand, $\Theta_{m\times N} = 0$ if inputs are periodic then equa-



tion (1) can be written as $G\left(z^n\right)_{m\times r} = \left(\boldsymbol{\Psi\Phi}^*\right)_{m\times r}\left(\boldsymbol{\Phi\Phi}^*\right)_{r\times r}^{-1}$,

where $\boldsymbol{\Psi\Phi}^* = \sum_{i=1}^{N} Y^{(i)}(k)U^{(i)*}(k)$

and $\boldsymbol{\Phi\Phi}^* = \sum_{i=1}^{N} U^{(i)}(k)U^{(i)*}(k)$.

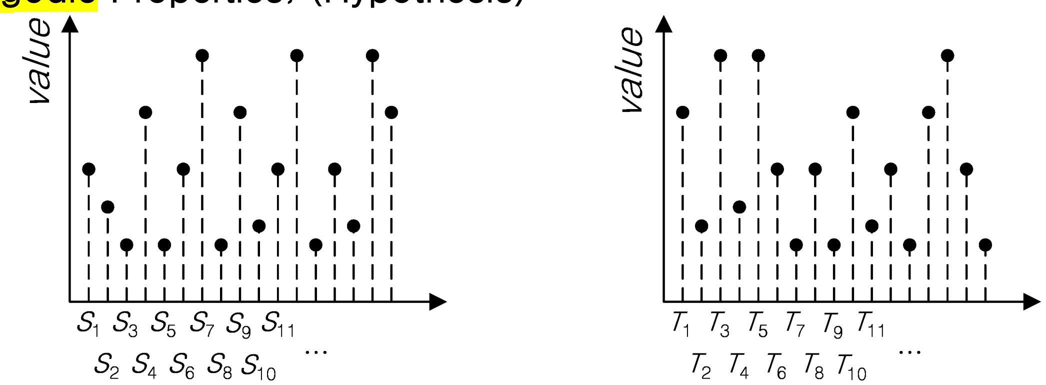
4.4 Input/Output Correlation:

- Review Random Process and Correlation:
- Determinist number vs. Random number. 1, 2, 3, 4, ⋯ "give a number as value"; $p(1)$, $p(2)$, ⋯ "give a probability as function of values".
- Determinist sequence vs. Random sequence; (Random process; Stochastic process). D.S.: $x(t)$; $\forall t \in [-\infty,\infty]$, R.S.: $p(x,t)$; $\forall t \in [-\infty,\infty] \cap x \in \mathbb{R}$.
- Stationary Random Sequence; $p(x,t_1) = p(x,t_2)$; $\forall x, t_1 \neq t_2$.
- Correlation; only stationary random sequence:

  Auto correlation; $R_{uu}(\tau) \triangleq E\left\{\mathbf{u}(k+\tau)\mathbf{u}^T(k)\right\}$,

  Cross correlation; $R_{yu}(\tau) \triangleq E\left\{\mathbf{y}(k+\tau)\mathbf{u}^T(k)\right\}$.

- Ergodic Properties; (Hypothesis):



$S \to \infty$; *ensamble axis* $\qquad$ $T \to \infty$; *times axis*

(a) $\qquad$ (b)

When we focus on the result of one experiment, we consider the following two cases: (a) Making infinite clones of the experiment first, we execute all of the clones in the same period and record their results, (b) We repeat the only one experiment infinite times and record its result every times.

- For stationary random process, the Ergodic hypothesis assumes the probability of different tests can be represented by several times of the same test.

- For stationary, cross correlation: $R_{yu}(\tau,k) \triangleq E\left\{y(k)u^T(k-\tau)\right\}$ $= \lim\limits_{S\to\infty}\frac{1}{S}\sum\limits_{i=1}^{S} y_i(k)u_i^T(k-\tau) = \lim\limits_{T\to\infty}\frac{1}{T}\sum\limits_{i=1}^{T} y(i)u^T(i-\tau) = R_{yu}(\tau)$.
- Correlation of sine wave:

  Auto correlation: a sinusoid wave still with same frequency.

  Cross correlation of two sinusoid waves:

  1. Same frequency: sinusoid wave with same frequency.
  2. Nearby frequency: sinusoid wave with smaller amplitude.
  3. Different frequency: near zero sequence.
- Correlation of random noise:

  Auto correlation: near pulse signal with first $R_{nn}(0) \approx 1$, $R_{nn}(\tau) \approx 0$; $\forall \tau \neq 0$.

  Cross correlation of random noise and sinusoid wave: a sinusoid wave with same frequency.

  Cross correlation of two different random noise: zero sequence.

● Weighting Sequence model $y(k) = \sum\limits_{i=0}^{\infty} Y_i u(k-i)$:

For FIR, $y(k) = \sum\limits_{i=0}^{\ell-1} Y_i u(k-i)$. $(y_T)_{m\times T} \triangleq \begin{bmatrix} y_{m\times 1}(1) & \cdots & y_{m\times 1}(T) \end{bmatrix}$

$Y_{m\times r\ell} \triangleq \begin{bmatrix} (Y_0)_{m\times r} & \cdots & (Y_{\ell-1})_{m\times r} \end{bmatrix} = \begin{bmatrix} D & CB & CAB & \cdots & CA^{\ell-1}B \end{bmatrix}$ and

$$(u_T)_{r\ell\times T} \triangleq \begin{bmatrix} u_{r\times 1}(1) & u_{r\times 1}(2) & \cdots & u_{r\times 1}(T) \\ u_{r\times 1}(0) & u_{r\times 1}(1) & \cdots & u_{r\times 1}(T-1) \\ \vdots & \vdots & & \vdots \\ u_{r\times 1}(2-\ell) & u_{r\times 1}(3-\ell) & \cdots & u_{r\times 1}(1+T-\ell) \end{bmatrix} \Rightarrow \begin{matrix} (y_T)_{m\times T} = \\ Y_{m\times r\ell}\cdot(u_T)_{r\ell\times T} \end{matrix}.$$

- Correlation matrix:

$$\because \frac{1}{T} y_T u_T^T = \frac{1}{T}\begin{bmatrix} y(1)u^T(1) & y(1)u^T(0) & & y(1)u^T(2-\ell) \\ + & + & & + \\ y(2)u^T(2) & y(2)u^T(1) & & y(2)u^T(3-\ell) \\ \vdots & \vdots & \cdots & \vdots \\ + & + & & + \\ y(T)u^T(T) & y(T)u^T(T-1) & & y(T)u^T(1+T-\ell) \end{bmatrix}_{m\times r\ell}$$

$$= \frac{1}{T}\begin{bmatrix} \sum_{i=1}^{T} y(i)u^T(i) & \sum_{i=1}^{T} y(i)u^T(i-1) & \cdots & \sum_{i=1}^{T} y(i)u^T(i-(\ell-1)) \end{bmatrix}$$

$$\therefore \lim_{T\to\infty}\frac{1}{T}(y_T)_{m\times T}\left(u_T^T\right)_{T\times r\ell} = \begin{bmatrix} R_{yu}(0) & R_{yu}(1) & \cdots & R_{yu}(\ell-1) \end{bmatrix} \triangleq R_{y_T u_T}.$$

$$\because \frac{1}{T} u_T u_T^T$$

$$= \frac{1}{T}\begin{bmatrix} \sum_{i=1}^{T} u(i)u^T(i) & \sum_{i=1}^{T} u(i)u^T(i-1) & \cdots & \sum_{i=1}^{T} u(i)u^T(i-(\ell-1)) \\ \sum_{i=0}^{T-1} u(i)u^T(i+1) & \sum_{i=0}^{T-1} u(i)u^T(i) & \cdots & \sum_{i=0}^{T-1} u(i)u^T(i-(\ell-2)) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1-(\ell-1)}^{T-(\ell-1)} u(i)u^T(i+(\ell-1)) & \sum_{i=1-(\ell-1)}^{T-(\ell-1)} u(i)u^T(i+(\ell-2)) & \cdots & \sum_{i=1-(\ell-1)}^{T-(\ell-1)} u(i)u^T(i) \end{bmatrix}_{r\ell\times r\ell}$$

Let ${}^{k}\overleftarrow{R}_{uu}(\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\sum_{i=-k}^{T-k-1} u(i)u^T(i-\tau) = \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T} u(i)u^T(i-\tau) = R_{uu}(\tau)$

and $R_{uu}(-\tau) = R_{uu}^T(\tau)$ then

$$\therefore \lim_{T\to\infty}\frac{1}{T}(u_T)_{r\ell\times T}\left(u_T^T\right)_{T\times r\ell} = \begin{bmatrix} R_{uu}(0) & R_{uu}(1) & \cdots & R_{uu}(\ell-1) \\ R_{uu}(-1) & {}^{1}\overleftarrow{R}_{uu}(0) & \cdots & R_{uu}(\ell-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{uu}(-(\ell-1)) & R_{uu}(-(\ell-2)) & \cdots & {}^{\ell-1}\overleftarrow{R}_{uu}(0) \end{bmatrix}$$

$$
=\begin{bmatrix} R_{uu}(0) & R_{uu}(1) & \cdots & R_{uu}(\ell-1) \\ R_{uu}^{T}(1) & R_{uu}(0) & \cdots & R_{uu}(\ell-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{uu}^{T}(\ell-1) & R_{uu}^{T}(\ell-2) & \cdots & R_{uu}(0) \end{bmatrix} \square\, R_{u_T u_T}. \quad \begin{array}{l} \because\ y_T u_T^T = Y_T u_T u_T^T \\ \Rightarrow R_{y_T u_T} = Y R_{u_T u_T} \end{array}
$$

$$
\therefore \begin{bmatrix} R_{yu}(0) & R_{yu}(1) & \cdots & R_{yu}(\ell-1) \end{bmatrix} = \begin{bmatrix} (Y_0)_{m\times r} & (Y_1)_{m\times r} & \cdots & (Y_{\ell-1})_{m\times r} \end{bmatrix}
$$

$$
\times \begin{bmatrix} R_{uu}(0)_{r\times r} & R_{uu}(1)_{r\times r} & \cdots & R_{uu}(\ell-1)_{r\times r} \\ R_{uu}(-1)_{r\times r} & R_{uu}(0)_{r\times r} & \cdots & R_{uu}(\ell-2)_{r\times r} \\ \vdots & \vdots & \ddots & \vdots \\ R_{uu}(-(\ell-1))_{r\times r} & R_{uu}(-(\ell-2))_{r\times r} & \cdots & R_{uu}(0)_{r\times r} \end{bmatrix}
$$

$$
= \begin{bmatrix} \sum_{\tau=0}^{\ell-1} Y_\tau R_{uu}(0-\tau) & \sum_{\tau=0}^{\ell-1} Y_\tau R_{uu}(1-\tau) & \cdots & \sum_{\tau=0}^{\ell-1} Y_\tau R_{uu}(\ell-1-\tau) \end{bmatrix}
$$

$\Rightarrow R_{yu}(i) = \sum_{\tau=0}^{\ell-1} Y_\tau R_{uu}(i-\tau)$; Correlation description of a weighting sequence model.

- Since correlation of random noise is pulse $\Rightarrow R_{u_T u_T} = I$ $\Rightarrow R_{y_T u_T} = Y \bullet I = Y$; it is impossible because we require infinite length of $u(k)$ and $y(k)$ as $T \to \infty$.
- If $u(k)$ isn't a random noise $\Rightarrow Y = R_{y_T u_T} R_{u_T u_T}^{-1}$; $R_{u_T u_T}$ need to be invertible; (= full rank).

  For a finite sample length; $T < \infty$, let $R_{y_T u_T} = \frac{1}{T} y_T u_T^T$, $R_{u_T u_T} = \frac{1}{T} u_T u_T^T$ and $\dim\{y_T\} = m \times T$, $\dim\{u_T\} = r\ell \times T$, where m: output channel number, r: input channel number, and ℓ: maximum non-zero *Markov* matrix number.
- Rank of $R_{u_T u_T} \le \min\{r\ell, T\}$. For a full rank of $R_{u_T u_T}$ $\Rightarrow T \ge r\ell$, we can get an invertible $R_{u_T u_T}$ and unique $Y = R_{y_T u_T}\left(R_{u_T u_T}\right)^{-1}$.

  For multi-experimental case; N experiments,

  $$
  y_{TN} \,\square \begin{bmatrix} y^{(1)} & y^{(2)} & \cdots & y^{(N)} \end{bmatrix}_{m\times TN} = Y \begin{bmatrix} u^{(1)} & u^{(2)} & \cdots & u^{(N)} \end{bmatrix}_{r\ell\times TN} \square\, Y u_{TN}
  $$

  $\Rightarrow \mathrm{Rank}\{R_{u_{TN} u_{TN}}\} = \frac{1}{TN} u_{TN} u_{TN}^T \le \min\{r\ell, TN\}$. For a full rank of $R_{u_{TN} u_{TN}}$

$\Rightarrow TN \geq r\ell$.

4.5 DTFT of Correlation Function:

- Given periodic signal $u_c(k)$ and $y_c(k)$ with period ℓ;

Def: Circular Correlation of $u_c(k)$ and $y_c(k)$;

$$R_{yu}^{c}(\tau) \triangleq \frac{1}{\ell}\sum_{i=0}^{\ell-1} y_c(i)u_c^T(i-\tau) \Rightarrow \lim_{\ell\to\infty} R_{yu}^{c}(\tau) = R_{yu}(\tau).$$

➢ $R_{yu}^{c}(\tau)$ is also periodic with period ℓ.

➢ If $u(k)$ and $y(k)$ are non-periodic then their circular correlation with period ℓ is defined as circular correlation of $u_c(k)$ and $y_c(k)$ where $u_c(k) = u(k)$ and $y_c(k) = y(k)$; $\forall k \in [0, \ell - 1]$.

- Given circular correlation function $R_{yu}^{c}(\tau)$;

Def: Spectral Density of $R_{yu}^{c}(\tau)$; $S_{yu}(n) \triangleq DTFT\left\{R_{yu}^{c}(\tau)\right\}$

$= \dfrac{1}{\ell}\displaystyle\sum_{\tau=0}^{\ell-1} R_{yu}^{c}(\tau)e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}$. Then, $S_{yu}(n) = Y(n)U^*(n)$ where $Y(n) = DTFT\left\{y_c(k)\right\}$ and $U(n) = DTFT\left\{u_c(k)\right\}$.

[pf]: $Y(n)U^*(n) = \dfrac{1}{\ell}\displaystyle\sum_{k=0}^{\ell-1} y_c(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n} \cdot \dfrac{1}{\ell}\displaystyle\sum_{\hat{k}=0}^{\ell-1} u_c^T(\hat{k})e^{j\left(\frac{2\pi}{\ell}\hat{k}\right)n}$

$$= \frac{1}{\ell^2}\sum_{k=0}^{\ell-1}\sum_{\hat{k}=0}^{\ell-1} y_c(k)u_c^T(\hat{k})e^{-j\left(\frac{2\pi}{\ell}n\right)(k-\hat{k})} \quad \text{let } k-\hat{k} \triangleq \tau \Rightarrow \hat{k} = k-\tau$$

$$= \frac{1}{\ell^2}\sum_{k=0}^{\ell-1}\sum_{\tau=k}^{k-\ell+1} y_c(k)u_c^T(k-\tau)e^{-j\left(\frac{2\pi}{\ell}n\right)\tau} \quad \because y_c(k),\ u_c^T(k-\tau) \text{ are periodic}$$

$$= \frac{1}{\ell^2}\sum_{k=0}^{\ell-1}\sum_{\tau=0}^{\ell-1} y_c(k)u_c^T(k-\tau)e^{-j\left(\frac{2\pi}{\ell}n\right)\tau} = \frac{1}{\ell^2}\sum_{\tau=0}^{\ell-1}\sum_{k=0}^{\ell-1} y_c(k)u_c^T(k-\tau)e^{-j\left(\frac{2\pi}{\ell}n\right)\tau}$$

$$= \frac{1}{\ell}\sum_{\tau=0}^{\ell-1}\left[\frac{1}{\ell}\sum_{k=0}^{\ell-1} y_c(k)u_c^T(k-\tau)\right]e^{-j\left(\frac{2\pi}{\ell}n\right)\tau} = \frac{1}{\ell}\sum_{\tau=0}^{\ell-1} R_{yu}^{c}(\tau)e^{-j\left(\frac{2\pi}{\ell}n\right)\tau} = S_{yu}(n)_{m\times r}.$$

➢ Since $u_c(k)$, $y_c(k)$ and $e^{-j\left(\frac{2\pi}{\ell}n\right)\tau}$ are all periodic;

$$\Rightarrow S_{yu}(n) = Y(n)U^*(n) = \frac{1}{\ell^2}\sum_{k=0}^{\ell-1} y_c(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n} \cdot \sum_{\hat{k}=0}^{\ell-1} u_c^T(\hat{k})e^{j\left(\frac{2\pi}{\ell}\hat{k}\right)n}$$

$$=\frac{1}{(p\ell)^2}\sum_{k=0}^{p\ell-1} y_c(k) e^{-j\left(\frac{2\pi}{\ell}k\right)n} \sum_{\hat{k}=0}^{p\ell-1} u_c^T(\hat{k}) e^{j\left(\frac{2\pi}{\ell}\hat{k}\right)n}$$, $p=1,2,\cdots,\infty$. It doesn't affect the "smoothness" of $S_{yu}(n)$ when we increase the period of data ℓ by $p=1,2,\cdots,\infty$.

- For "smooth" $S_{yu}(n)$, we chop $u(k)$ and $y(k)$ to generate different circular $u_c^{(1)}(k)$, $y_c^{(1)}(k)$, $u_c^{(2)}(k)$, $y_c^{(2)}(k)$, $\cdots$, $u_c^{(N)}(k)$, $y_c^{(N)}(k)$ and take them to DTFT; $U^{(1)}(n)$, $Y^{(1)}(n)$, $U^{(2)}(n)$, $Y^{(2)}(n)$, $\cdots$, $U^{(N)}(n)$, $Y^{(N)}(n)$. Then, we will get $\bar{S}_{yu}(n)=\frac{1}{N}\sum_{i=1}^{N} S_{yu}^{(i)}(n)$ $=\frac{1}{N}\sum_{i=1}^{N} Y^{(i)}(n) U^{*(i)}(n)$ when we average them.

- Homework 7:

1. Please write a program to take the circular correlation of two sequences; $u(k)$ and $y(k)$. [Note]: In the same way, you must denote the basic theory, usage, definitions, and variables in the file head and the function of every program section between program sections.
2. (a) Please use the FFT of Matlab function to compute the DTFT of digital impulse sequences and draw it including magnitude and phase. Please discuss the difference between DTFTs in different lengths and the variation caused by shifting the digital impulse sequence. (b) Please use the FFT of Matlab function to compute the DTFT of sine wave and inspect the difference 1.) sine waves are sin((1:90)*2pi./20) and sin((1:100)*2pi./20). 2.) before and after through shifting. 3.) different lengths e.g. sin((1:90)*2pi./20) and sin((1:290)*2pi./20).
3. Please use the above program to compute:

   (a) Auto correlation of a normal distributed random sequence and its DTFT.

   (b) Cross correlation and its DTFT of two different normal distributed random sequences.

   (c) Cross correlation and its DTFT of sine wave and a normal distributed random sequence.

(d) Auto correlation and its DTFT of sine wave.

(e) Cross correlation and its DTFT of two different sine waves in the same frequency but out of phase 90°.

(f) Cross correlation and its DTFT of two different sine waves in different frequencies. Please draw the variation under the difference from small to large.

4.6 FRF from Correlation DTFT:

- Generally, system output is the function of input and noise;

$y(k)=\sum_{i=0}^{\infty}Y_i u(k-i)+\varepsilon(k)\approx\sum_{i=0}^{p}Y_i u(k-i)+\varepsilon(k)$. We correlate this equation with $u(k)$ sequence;

$$\Rightarrow R_{yu}(i)=\sum_{\tau=0}^{\infty}Y_\tau R_{uu}(i-\tau)+R_{\varepsilon u}(i)\approx\sum_{\tau=0}^{p}Y_\tau R_{uu}(i-\tau)+R_{\varepsilon u}(i).$$

Benefit: If $\varepsilon(k)$ and $u(k)$ are un-correlated $\Rightarrow R_{\varepsilon u}(i)=0$. In general, $R_{\varepsilon u}(i)\ll R_{uu}(i-\tau)$.

Cost: Computation of $R_{yu}(i)$, $R_{uu}(i-\tau)$ and its convolution are time consuming.

- $DTFT_\ell$ the correlation equation of FIR; weighting sequence model:

$$S_{yu}(n)=\frac{1}{\ell}\sum_{i=0}^{\ell-1}R_{yu}(i)e^{-j\left(\frac{2\pi}{\ell}i\right)n}=\frac{1}{\ell}\sum_{i=0}^{\ell-1}\sum_{\tau=0}^{\infty}Y_\tau R_{uu}(i-\tau)e^{-j\left(\frac{2\pi}{\ell}i\right)n}$$

$$+\frac{1}{\ell}\sum_{i=0}^{\ell-1}R_{\varepsilon u}(i)e^{-j\left(\frac{2\pi}{\ell}i\right)n}=\frac{1}{\ell}\sum_{\tau=0}^{\infty}Y_\tau e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}\sum_{k=-\tau}^{\ell-1-\tau}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}+S_{\varepsilon u}(n)$$

$$=\frac{1}{\ell}\sum_{\tau=0}^{\infty}Y_\tau e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}\sum_{k=0}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}+\frac{1}{\ell}\sum_{\tau=0}^{\infty}Y_\tau e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}\sum_{k=-\tau}^{\ell-1-\tau}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}$$

$$-\frac{1}{\ell}\sum_{\tau=0}^{\infty}Y_\tau e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}\sum_{k=0}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}+S_{\varepsilon u}(n)=G\left(z^n\right)\cdot\frac{1}{\ell}\sum_{k=0}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}$$

$$-\sum_{\tau=0}^{\infty}Y_\tau e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}\frac{1}{\ell}\left[\sum_{k=0}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}-\sum_{k=-\tau}^{\ell-1-\tau}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}\right]+S_{\varepsilon u}(n)$$

$$\cdots(1)$$

$$\because\frac{1}{\ell}\sum_{k=0}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}-\frac{1}{\ell}\sum_{k=-\tau}^{\ell-1-\tau}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}=\frac{1}{\ell}\sum_{k=\ell-\tau}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}$$

$$-\frac{1}{\ell}\sum_{k=-\tau}^{-1} R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n} \text{; (let } k=i+\ell\text{)} =\frac{1}{\ell}\sum_{i=-\tau}^{-1} R_{uu}(i+\ell)e^{-j\left(\frac{2\pi}{\ell}(i+\ell)\right)n}$$

$$-\frac{1}{\ell}\sum_{k=-\tau}^{-1} R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n} =\frac{1}{\ell}\sum_{k=-\tau}^{-1} R_{uu}(k+\ell)e^{-j\left(\frac{2\pi}{\ell}k+2\pi\right)n}$$

$$-\frac{1}{\ell}\sum_{k=-\tau}^{-1} R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n} =\frac{1}{\ell}\sum_{k=-\tau}^{-1} R_{uu}(k+\ell)e^{-j\left(\frac{2\pi}{\ell}k\right)n}$$

$$-\frac{1}{\ell}\sum_{k=-\tau}^{-1} R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n} =\frac{1}{\ell}\sum_{k=-\tau}^{-1} \left[R_{uu}(k+\ell)-R_{uu}(k)\right]e^{-j\left(\frac{2\pi}{\ell}k\right)n} \triangleq e_{\tau}(n);$$

This error is caused by the time shift between $R_{uu}(k+\tau)$ and $R_{uu}(k)$ in equation (1). Then, equation (1) can be written as follow;

$$S_{yu}(n)=G(z^n)\cdot\frac{1}{\ell}\sum_{k=0}^{\ell-1} R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}-\sum_{\tau=0}^{\infty}Y_{\tau}e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}e_{\tau}(n)+S_{\varepsilon u}(n)$$

$=G(z^n)\cdot S_{uu}(n)-S_{eu}(n)+S_{\varepsilon u}(n)$, where $S_{eu}(n)\triangleq\sum_{\tau=0}^{\infty}Y_{\tau}e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}e_{\tau}(n)$.

- If $u(k)$ and $y(k)$ are periodic with period ℓ then $R_{yu}(i)$ and $R_{uu}(i)$ are also periodic with period $\ell \Rightarrow S_{eu}(n)=0$. $\therefore S_{yu}(n)$ $=G(z^n)\cdot S_{uu}(n)-\cancel{S_{eu}(n)}^{\approx 0}\cong G(z^n)\cdot S_{uu}(n)$. For SISO system, $\Rightarrow G(z^n)\cong\dfrac{S_{yu}(n)}{S_{uu}(n)}$.
- If $u(k)$ and $y(k)$ are non-periodic or period$\neq\ell$ then $\Rightarrow S_{eu}(n)\neq 0$ we need to find the bound of $S_{eu}(n)$ to verify the usage of $S_{yu}(n)\cong G(z^n)S_{uu}(n)$.

1. Bound of $e_{\tau}(n)$: If $\left\|R_{uu}(k)\right\|\leq C_{uu}$ then

$$\left\|e_{\tau}(n)\right\|\leq\frac{1}{\ell}\left\|\sum_{k=\ell-\tau}^{\ell-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}\right\|+\frac{1}{\ell}\left\|\sum_{k=-\tau}^{-1}R_{uu}(k)e^{-j\left(\frac{2\pi}{\ell}k\right)n}\right\|\leq\frac{2\tau}{\ell}C_{uu}.$$

2. Bound of $S_{eu}(n)$: $\left\|S_{eu}(n)\right\|\leq\left\|\sum_{\tau=0}^{\infty}Y_{\tau}e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}e_{\tau}(n)\right\|$

$$\le \sum_{\tau=0}^{\infty}\left\| Y_\tau e^{-j\left(\frac{2\pi}{\ell}\tau\right)n}\right\| \left\| e_\tau(n)\right\| \le \frac{2}{\ell} C_{uu}\sum_{\tau=0}^{\infty}\tau\left|Y_\tau\right| \le \frac{2}{\ell} C_{uu} C_G .$$

- Advantage of usage: $S_{yu}(n) = G(z^n) \cdot S_{uu}(n) - S_{eu}(n) + S_{\varepsilon u}(n)$;

It's simple to calculate the $R_{yu}(i) = \sum_{\tau=0}^{\infty} Y_\tau R_{uu}(i-\tau)$.

$$G(z^n) = S_{yu}(n) \cdot S_{uu}^{-1}(n) + S_{eu}(n) S_{uu}^{-1}(n) - S_{\varepsilon u}(n) S_{uu}^{-1}(n) \approx S_{yu}(n) \cdot S_{uu}^{-1}(n).$$

To improve the smoothness, using N experiments; $\bar{S}_{yu}(n) = \frac{1}{N}\sum_{i=1}^{N} S_{yu}^{(i)}(n)$,

$\bar{S}_{uu}(n) = \frac{1}{N}\sum_{i=1}^{N} S_{uu}^{(i)}(n)$, $\bar{S}_{eu}(n) = \frac{1}{N}\sum_{i=1}^{N} S_{eu}^{(i)}(n)$, and $\bar{S}_{\varepsilon u}(n) = \frac{1}{N}\sum_{i=1}^{N} S_{\varepsilon u}^{(i)}(n)$

$$G(z^n) = \bar{S}_{yu}(n) \cdot \bar{S}_{uu}^{-1}(n) + \bar{S}_{eu}(n) \bar{S}_{uu}^{-1}(n) - \bar{S}_{\varepsilon u}(n) \bar{S}_{uu}^{-1}(n) \approx \bar{S}_{yu}(n) \cdot \bar{S}_{uu}^{-1}(n).$$

Then, we get Y_τ by using $Y_\tau = \sum_{n=0}^{\infty} G(z^n) e^{j\left(\frac{2\pi}{\ell}n\right)\tau}$.

4.7 Coherence Function:

- Spectral Density: It decomposes the working energy on frequency domain.

IDTFT of signals; $u(k) = \sum_{n=0}^{\ell-1} U(n) e^{j\frac{2\pi}{\ell}nk}$ and $y(k) = \sum_{n=0}^{\ell-1} Y(n) e^{j\frac{2\pi}{\ell}nk}$.

Working energy (power): $\sum_{k=0}^{\ell-1} u(k) y^*(k)$ or $\sum_{k=0}^{\ell-1} u(k) u^*(k)$. So, we get

$$\sum_{k=0}^{\ell-1} u(k) y^*(k) = \sum_{k=0}^{\ell-1}\left(\sum_{n=0}^{\ell-1} U(n) e^{j\frac{2\pi}{\ell}nk} \cdot \sum_{\hat{n}=0}^{\ell-1} Y^*(\hat{n}) e^{-j\frac{2\pi}{\ell}\hat{n}k}\right)$$

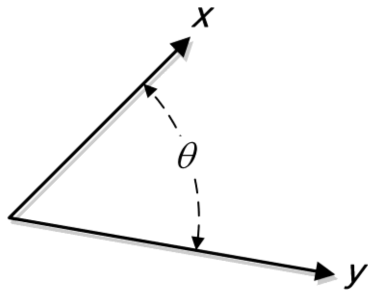
$$= \sum_{n=0}^{\ell-1}\sum_{\hat{n}=0}^{\ell-1}\sum_{k=0}^{\ell-1}\left[U(n) Y^*(\hat{n}) e^{j\frac{2\pi}{\ell}(n-\hat{n})k}\right] = \sum_{n=0}^{\ell-1}\sum_{\hat{n}=0}^{\ell-1}\left[U(n) Y^*(\hat{n}) \sum_{k=0}^{\ell-1} e^{j\frac{2\pi}{\ell}(n-\hat{n})k}\right]$$

$$= \sum_{n=0}^{\ell-1}\sum_{\hat{n}=0}^{\ell-1} U(n) Y^*(\hat{n}) \cdot \begin{cases} \ell & \forall n-\hat{n}=0 \\ 0 & \forall n-\hat{n}\ne 0 \end{cases} = \ell \sum_{n=0}^{\ell-1} U(n) Y^*(n).$$ Similarly,

$$\sum_{k=0}^{\ell-1} u(k) u^*(k) = \ell \sum_{n=0}^{\ell-1} U(n) U^*(n).$$

$$\Rightarrow E\left\{ \sum_{k=0}^{\ell-1} u(k) u^*(k)\right\} = \ell \sum_{n=0}^{\ell-1} E\left\{ U(n) U^*(n)\right\} = \ell \sum_{n=0}^{\ell-1} E\left\{ S_{uu}(n)\right\};$$ Separate

the $\frac{energy}{time} = power$ to be combination of energy from different fre-quency signals. So, Spectral Density is the energy density of different frequencies.



- Vectors and their Angle: Given vectors x and y, the cosine of their angle; $\cos\theta = \frac{\langle x, y\rangle}{\|x\|\|y\|}$.

$$\Rightarrow |\cos\theta|^2 = \frac{|\langle x,y\rangle|^2}{\langle x,x\rangle\langle y,y\rangle} = \frac{\langle x,y\rangle\langle x,y\rangle^*}{\langle x,x\rangle\langle y,y\rangle}$$. If we let $U_\upsilon(n)$ and $Y_\mu(n)$ that they are stochastic numbers expressed as infinite vectors in ensemble domain and define the inner product of them as $\langle U_\upsilon(n), Y_\mu(n)\rangle \triangleq E\{U_\upsilon(n)Y_\mu^*(n)\}$ then the Coherence between $u_\upsilon(k)$ and $y_\mu(k)$ is $\gamma_{y_\mu u_\upsilon}(k)$ satisfied $\gamma^2_{y_\mu u_\upsilon}(n)$

$$\triangleq \frac{\langle Y_\mu(n), U_\upsilon(n)\rangle\langle Y_\mu(n), U_\upsilon(n)\rangle^*}{\langle Y_\mu(n), Y_\mu(n)\rangle\langle U_\upsilon(n), U_\upsilon(n)\rangle} = \frac{E\{Y_\mu(n)U_\upsilon^*(n)\}E\{U_\upsilon(n)Y_\mu^*(n)\}}{E\{Y_\mu(n)Y_\mu^*(n)\}E\{U_\upsilon(n)U_\upsilon^*(n)\}}$$

$$= \frac{\bar{S}_{y_\mu u_\upsilon}(n)\bar{S}^*_{y_\mu u_\upsilon}(n)}{\bar{S}_{y_\mu y_\mu}(n)\bar{S}_{u_\upsilon u_\upsilon}(n)}$$; as $n = 0, \cdots, \ell - 1$. So, we have ℓ terms of them.

$\gamma^2_{y_\mu u_\upsilon}(n)$ satisfies in-equation; $0 \le \gamma^2_{y_\mu u_\upsilon}(n) = \frac{\bar{S}_{y_\mu u_\upsilon}(n)\bar{S}^*_{y_\mu u_\upsilon}(n)}{\bar{S}_{y_\mu y_\mu}(n)\bar{S}_{u_\upsilon u_\upsilon}(n)} \le 1$.

- Relationship between Input and Output Spectral: Given Input/Output relationship as $Y(n) = G(z^n)U(n) + \varepsilon(n)$, where $\varepsilon(n)$: error due to noise and non-periodic. $\Rightarrow Y(n)Y^*(n) = G(z^n)U(n)U^*(n)G^*(z^n)$ $+G(z^n)U(n)\varepsilon^*(n) + \varepsilon(n)U^*(n)G^*(z^n) + \varepsilon(n)\varepsilon^*(n)$, $\Rightarrow \bar{S}_{yy}(n)$

$$= E\{Y(n)Y^*(n)\} = G(z^n)E\{U(n)U^*(n)\}G^*(z^n) + E\{\varepsilon(n)\varepsilon^*(n)\}$$
$$+G(z^n)E\{U(n)\varepsilon^*(n)\} + E\{\varepsilon(n)U^*(n)\}G^*(z^n) = G(z^n)\bar{S}_{uu}(n)G^*(z^n)$$
$$+G(z^n)\bar{S}^*_{\varepsilon u}(n) + \bar{S}_{\varepsilon u}(n)G^*(z^n) + \bar{S}_{\varepsilon\varepsilon}(n)$$, where $\bar{S}_{yy}(n) \triangleq \frac{1}{N}\sum_{i=0}^{N} S_{yy}^{(i)}(n)$,

$\bar{S}_{uu}(n) \triangleq \frac{1}{N}\sum_{i=0}^{N} S_{uu}^{(i)}(n)$, $\bar{S}_{\varepsilon u}(n) \triangleq \frac{1}{N}\sum_{i=0}^{N} S_{\varepsilon u}^{(i)}(n)$ and $\bar{S}_{\varepsilon\varepsilon}(n) \triangleq \frac{1}{N}\sum_{i=0}^{N} S_{\varepsilon\varepsilon}^{(i)}(n)$.

- Coherence Function of SISO system:

If $y(k)$ and $u(k)$ are periodic then $\bar{S}_{yu}(n)=G(z^n)\bar{S}_{uu}(n)+\bar{S}_{\varepsilon u}(n)$,

$\bar{S}_{yy}(n)=G(z^n)\bar{S}_{uu}(n)G^*(z^n)+G(z^n)\bar{S}_{\varepsilon u}^*(n)+\bar{S}_{\varepsilon u}(n)G^*(z^n)+\bar{S}_{\varepsilon\varepsilon}(n)$.

- If $\varepsilon=0 \Rightarrow \bar{S}_{\varepsilon u}(n)=\bar{S}_{\varepsilon\varepsilon}(n)=0 \Rightarrow \bar{S}_{yu}(n)=G(z^n)\bar{S}_{uu}(n)$ and

  $\bar{S}_{yy}(n)=G(z^n)\bar{S}_{uu}(n)G^*(z^n)$;

  $$\Rightarrow \gamma_{yu}^2(n)=\frac{G(z^n)\bar{S}_{uu}(n)\bar{S}_{uu}^*(n)G^*(z^n)}{G(z^n)\bar{S}_{uu}(n)G^*(z^n)\bar{S}_{uu}(n)}=1. \quad \begin{array}{l}\because \bar{S}_{uu}\in \square\\ G(z^n)\in \square\end{array} \text{; are scalar.}$$

  So, Coherence equal to one when zero noise.

- If $\varepsilon\neq 0$ but $\varepsilon(k)$ and $u(k)$ are uncorrelated $\Rightarrow \bar{S}_{\varepsilon u}(n)=0$,

  $\Rightarrow \bar{S}_{yu}(n)=G(z^n)\bar{S}_{uu}(n)$, $\bar{S}_{yy}(n)=G(z^n)\bar{S}_{uu}(n)G^*(z^n)+\bar{S}_{\varepsilon\varepsilon}(n)$

  $$\Rightarrow \gamma_{yu}^2(n)=\frac{G(z^n)\bar{S}_{uu}(n)G^*(z^n)\bar{S}_{uu}(n)}{\bar{S}_{yy}(n)\bar{S}_{uu}(n)}=\frac{\bar{S}_{yy}(n)-\bar{S}_{\varepsilon\varepsilon}(n)}{\bar{S}_{yy}(n)}$$

  $=1-\dfrac{\bar{S}_{\varepsilon\varepsilon}(n)}{\bar{S}_{yy}(n)}$. If noise and input signal uncorrelated then $\gamma_{yu}^2(n)\rightarrow 1$

  when S–N ratio increasing.

- Coherence Function of MIMO system:

I/O relation; $\boldsymbol{Y}(n)_{m\times 1}=\boldsymbol{G}(z^n)_{m\times r}\boldsymbol{U}(n)_{r\times 1}+\boldsymbol{\varepsilon}(n)_{m\times 1}$

$$=\begin{bmatrix}\boldsymbol{G}_1(z^n)_{m\times 1} & \cdots & \boldsymbol{G}_r(z^n)_{m\times 1}\end{bmatrix}\begin{bmatrix}U_1(n)\\ \vdots \\ U_r(n)\end{bmatrix}+\boldsymbol{\varepsilon}(n)_{m\times 1}=\sum_{i=1}^{r}U_i(n)\boldsymbol{G}_i(z^n)+\boldsymbol{\varepsilon}(n).$$

$\boldsymbol{Y}(n)$ is a linear combination of $U_i(n)$ and $\boldsymbol{\varepsilon}(n)$. If all of them are uncorrelated then $\boldsymbol{Y}\boldsymbol{Y}^*=\left(\sum_i U_i\boldsymbol{G}_i+\boldsymbol{\varepsilon}\right)\left(\sum_{\hat{i}} U_{\hat{i}}\boldsymbol{G}_{\hat{i}}+\boldsymbol{\varepsilon}\right)^*=\left(\sum_i U_i\boldsymbol{G}_i+\boldsymbol{\varepsilon}\right)$

$$\square\left(\sum_{\hat{i}}\boldsymbol{G}_{\hat{i}}^*U_{\hat{i}}^*+\boldsymbol{\varepsilon}^*\right)=\sum_i U_i\boldsymbol{G}_i\sum_{\hat{i}}\boldsymbol{G}_{\hat{i}}^*U_{\hat{i}}^*+\left(\sum_i U_i\boldsymbol{G}_i\right)\boldsymbol{\varepsilon}^*+\boldsymbol{\varepsilon}\left(\sum_i \boldsymbol{G}_i^*U_i^*\right)+\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*$$

$$=\sum_i\sum_{\hat{i}}\boldsymbol{G}_iU_iU_{\hat{i}}^*\boldsymbol{G}_{\hat{i}}^*+\sum_i\boldsymbol{G}_iU_i\boldsymbol{\varepsilon}^*+\sum_i\boldsymbol{\varepsilon}U_i^*\boldsymbol{G}_i^*+\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*=\sum_i\sum_{\hat{i}}\boldsymbol{G}_iU_iU_{\hat{i}}^*\boldsymbol{G}_{\hat{i}}^*+\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*$$

$(\because \boldsymbol{S}_{U_i\boldsymbol{\varepsilon}}(n)=U_i(n)\boldsymbol{\varepsilon}^*(n)=0\Rightarrow\boldsymbol{\varepsilon}(n)U_i^*(n)=0)$ $=\sum_i\boldsymbol{G}_iU_iU_i^*\boldsymbol{G}_i^*+\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^*$

$(\because U_i(n)U_{\hat{i}}^*(n)=0,\ \forall i\neq\hat{i})\ \therefore\Rightarrow \langle Y,Y\rangle = E\{YY^*\} = E\left\{\sum_i G_iU_iU_i^*G_i^* + \varepsilon\varepsilon^*\right\}$

$=\sum_i G_iE\{U_iU_i^*\}G_i^* + E\{\varepsilon\varepsilon^*\} = \sum_{i=1}^{r} G_i\langle U_i,U_i\rangle G_i + \langle\varepsilon,\varepsilon\rangle$. And because

$\langle Y,U_i^*\rangle = E\{YU_i^*\} = E\left\{\left(\sum_{\hat{i}} U_{\hat{i}}G_{\hat{i}} + \varepsilon\right)U_i^*\right\} = E\left\{\sum_{\hat{i}} G_{\hat{i}}U_{\hat{i}}U_i^* + \varepsilon U_i^*\right\}$

$= E\{G_iU_iU_i^* + \varepsilon U_i^*\} = G_iE\{U_iU_i^*\} = G_i\langle U_i,U_i\rangle$. So, γ_i^2 □ $\cos\theta_i^2$

□ $\dfrac{\langle Y,U_i\rangle\langle Y,U_i\rangle^*}{\langle Y,Y\rangle\langle U_i,U_i\rangle} = \dfrac{G_i\langle U_i,U_i\rangle\langle U_i,U_i\rangle^* G_i^*}{\langle Y,Y\rangle\langle U_i,U_i\rangle} = \dfrac{G_i\langle U_i,U_i\rangle G_i^*}{\langle Y,Y\rangle}$,

$$\therefore\Rightarrow \sum_{i=1}^{r}\gamma_i^2 = \frac{1}{\langle Y,Y\rangle}\sum_{i=1}^{r} G_i\langle U_i,U_i\rangle G_i^* = \frac{\langle Y,Y\rangle - \langle\varepsilon,\varepsilon\rangle}{\langle Y,Y\rangle} = I - \frac{\langle\varepsilon,\varepsilon\rangle}{\langle Y,Y\rangle} = I - \frac{\bar{S}_{\varepsilon\varepsilon}}{\bar{S}_{YY}}.$$

- In addition, if $\varepsilon = 0$ then the sum of these coherence will be I.
- If coherence function between some one input and output less than I then it means that the follows will exists at least one;
  1. Extraneous noise is present in measurement.
  2. The system has significant uncertainties and nonlinearities.
  3. The output was generated by more than one output.

- Homework 8:

Please identify a system from frequency domain using the methods of DTFT and Spectral. For the MCK system provided above, please draw its FRF, compute its impulse response using IDTFT and compare it with real simulation.

[Hints]: take random noise as input, execute the DTFT and Spectral against one input and one output every times.

[Note]: