

Eigensystem Realization

CEE 629. System Identification

Duke University, Fall 2017

1 Markov Parameters

The response sequence of a system driven by a particular input sequence and assuming initial conditions of zero, can be found from $y(i) = Cx(i) + Du(i)$

$$\begin{aligned}
 y(1) &= Cx(1) + Du(1) = CBu(0) + Du(1) \\
 y(2) &= CAx(1) + CBu(1) + Du(2) = CABu(0) + CBu(1) + Du(2) \\
 y(3) &= CAx(2) + CBu(2) + Du(3) = CA^2Bu(0) + CABu(1) + CBu(2) + Du(3) \\
 &\vdots \\
 y(i) &= CA^{(i-1)}Bu(0) + \cdots + CA^2Bu(i-3) + CABu(i-2) + CBu(i-1) + Du(i)
 \end{aligned}$$

$$y(i) = \begin{bmatrix} D & CB & CAB & CA^2B & \cdots & CA^{(i-2)}B & CA^{(i-1)}B \end{bmatrix} \begin{bmatrix} u(i) \\ u(i-1) \\ u(i-2) \\ u(i-3) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \quad (1)$$

This is a matrix input-output relationship in which the matrix on the left of the product is called a matrix of *Markov parameters*, $Y(0) = D$, and $Y(i) = CA^{(i-1)}B$ for $i > 0$.

$$y(i) = \begin{bmatrix} Y(0) & Y(1) & Y(2) & Y(3) & \cdots & Y(i-1) & Y(i) \end{bmatrix} \begin{bmatrix} u(i) \\ u(i-1) \\ u(i-2) \\ u(i-3) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \quad (2)$$

$$= \sum_{k=0}^i Y(k) u(i-k) \quad (3)$$

This is a discrete-time convolution of an input sequence with a sequence of Markov parameters. So the sequence of Markov parameters can be interpreted as the unit impulse response of the discrete-time system. Given a sequence of Markov parameters and an input sequence (and assuming an initial state of zero), the output sequence can be found from a linear matrix operation.

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} D & 0 & 0 & 0 & 0 & \cdots & 0 \\ CB & D & 0 & 0 & 0 & \cdots & 0 \\ CAB & CB & D & 0 & 0 & \cdots & 0 \\ CA^2B & CAB & CB & D & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \\ CA^{N-1}B & CA^{N-2}B & CA^{N-3}B & \cdots & CAB & CB & D \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(N) \end{bmatrix} \quad (4)$$

This matrix equation can be analyzed via the singular value decomposition of its lower-triangular Toeplitz matrix.

$$[y_N] = [Y \Sigma U^T] [u_N] \quad (5)$$

Any input sequence equal to the j -th column of U results in an output sequence equal to $\Sigma_{j,j}$ times the j -th column of Y . The input sequence of the first column of U gives the largest magnification, in the sense that the ratio of norms ($\|y_n\|_2^2 / \|u_n\|_2^2$) is maximized and is $\Sigma_{1,1}$.

Given a long sequence of input and output data (long enough to strip-out the transient response from initial state $x(0)$), Markov parameters can be estimated in a least-squares sense by solving the Wiener-Hopf equations.

$$\begin{bmatrix} y(K) & y(K+1) & \cdots & y(N) \end{bmatrix} = \begin{bmatrix} Y(0) & Y(1) & \cdots & Y(K) \end{bmatrix} \begin{bmatrix} u(K) & u(K+1) & \cdots & u(N) \\ u(K-1) & u(K) & \cdots & u(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ u(0) & u(1) & \cdots & u(N-K) \end{bmatrix} \quad (6)$$

2 Block Hankel Matrices of Markov Parameters

Define a finite-dimensional Hankel matrix of the Markov parameters of an LTI system

$$H(0) = \begin{bmatrix} Y(1) & Y(2) & \cdots & Y(\beta) \\ Y(2) & Y(3) & \cdots & Y(\beta + 1) \\ \vdots & \vdots & \ddots & \vdots \\ Y(\alpha) & Y(\alpha + 1) & \cdots & Y(\alpha + \beta - 1) \end{bmatrix} \quad (7)$$

The Hankel matrix $H(0)$ is the product of the observability and controllability matrices,

$$H(0) = \mathcal{P}_\alpha \mathcal{Q}_\beta \quad (8)$$

The number of block-rows and block-columns of $H(0)$ (α and β) could be set to just under half the length of the sequence of Markov parameters, $\alpha = \beta = K/2 - 1$

If the LTI is observable and controllable, and the Markov parameters are noise-free, then the rank of $H(0)$ is the model order, n . If the Markov parameters contain measurement noise then $H(0)$ can be full-rank, but the spectrum of singular values of $H(0)$ can guide in the selection of the model order.

3 Product of Observability and Controllability and Eigensystem Realization

The product of the Observability and Controllability matrices is a Hankel matrix of Markov parameters, $Y(k) = CA^{(k-1)}B$.

$$\begin{aligned}
 H(0) &= \mathcal{P}_\alpha \mathcal{Q}_\beta \\
 &= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{(\alpha-1)} \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & A^{(\beta-2)}B & A^{(\beta-1)}B \end{bmatrix} \\
 &= \begin{bmatrix} CB & CAB & CA^2B & \dots & CA^{(\beta-2)}B & CA^{(\beta-1)}B \\ CAB & CA^2B & CA^3B & \dots & CA^{(\beta-1)}B & CA^{(\beta)}B \\ CA^2B & CA^3B & CA^4B & \dots & CA^{(\beta)}B & CA^{(\beta+1)}B \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ CA^{(\alpha-1)}B & CA^{(\alpha)}B & CA^{(\alpha+1)}B & \dots & CA^{(\alpha-1+\beta-2)}B & CA^{(\alpha-1+\beta-1)}B \end{bmatrix} \quad (9)
 \end{aligned}$$

$$\begin{aligned}
 &= \begin{bmatrix} Y(1) & Y(2) & Y(3) & \dots & Y(\beta-1) & Y(\beta) \\ Y(2) & Y(3) & Y(4) & \dots & Y(\beta) & Y(\beta+1) \\ Y(3) & Y(4) & Y(5) & \dots & Y(\beta+1) & Y(\beta+2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ Y(\alpha) & Y(\alpha+1) & Y(\alpha+2) & \dots & Y(\alpha+\beta-2) & Y(\alpha+\beta-1) \end{bmatrix} \quad (10)
 \end{aligned}$$

$$\begin{aligned}
 H(1) &= \mathcal{P}_\alpha A \mathcal{Q}_\beta \\
 &= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{(n-1)} \end{bmatrix} A \begin{bmatrix} B & AB & A^2B & \dots & A^{(n-2)}B & A^{(n-1)}B \end{bmatrix} \\
 &= \begin{bmatrix} CAB & CA^2B & CA^3B & \dots & CA^{(\beta-1)}B & CA^{(\beta)}B \\ CA^2B & CA^3B & CA^4B & \dots & CA^{(\beta)}B & CA^{(\beta+1)}B \\ CA^3B & CA^4B & CA^5B & \dots & CA^{(\beta+1)}B & CA^{(\beta+2)}B \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ CA^{(\alpha)}B & CA^{(\alpha+1)}B & CA^{(\alpha+2)}B & \dots & CA^{(\alpha+\beta-2)}B & CA^{(\alpha+\beta-1)}B \end{bmatrix} \quad (11)
 \end{aligned}$$

$$\begin{aligned}
 &= \begin{bmatrix} Y(2) & Y(3) & Y(4) & \dots & Y(n) & Y(n) \\ Y(3) & Y(4) & Y(5) & \dots & Y(n+1) & Y(n+2) \\ Y(4) & Y(5) & Y(6) & \dots & Y(n+2) & Y(n+3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ Y(\alpha+1) & Y(\alpha+2) & Y(\alpha+3) & \dots & Y(\alpha+\beta-1) & Y(\alpha+\beta) \end{bmatrix} \quad (12)
 \end{aligned}$$

- Neither $H(0)$ nor $H(1)$ is necessarily symmetric, since the Markov parameters $Y(k)$ are m by r matrices.
- The estimate of the feedthrough matrix $\hat{D} = Y(0)$ is not a part of these block Hankel matrices.

The observability and controllability matrices can be extracted from the SVD of $H(0)$ in many different ways. First, truncate the SVD expansion of $H(0)$ to the n most-significant singular values. A plot of the spectrum of singular values frequently reveals the order of the system represented by the data (i.e., the number of dynamic states in the system).

$$H(0) \approx U_n \Sigma_n V_n^T = (U_n \Sigma_n^{(1/2+p)}) T T^{-1} (\Sigma_n^{(1/2-p)} V_n^T) = \mathcal{P}_\alpha \mathcal{Q}_\beta \quad (13)$$

(for $-1/2 \leq p \leq 1/2$). The matrix T is an arbitrary unitary transformation matrix.

- for $p = -1/2$: $\mathcal{P}_\alpha = U_n T$ and $\mathcal{Q}_\beta = T^{-1} \Sigma_n V_n^T$ from which the gramians are $P_n = I_n$ and $Q_n = \Sigma_n^2$
- for $p = 0$: $\mathcal{P}_\alpha = U_n \Sigma_n^{1/2} T$ and $\mathcal{Q}_\beta = T^{-1} \Sigma_n^{1/2} V_n^T$, from which the gramians are diagonal and equal $P_n = \Sigma_n$ and $Q_n = \Sigma_n$. The resulting state-space realization is called *internally balanced* because it is as observable as it is controllable. (More on this later.)
- for $p = +1/2$: $\mathcal{P}_\alpha = U_n \Sigma_n T$ and $\mathcal{Q}_\beta = T^{-1} V_n^T$ from which the gramians are $P_n = \Sigma_n^2$ and $Q_n = I_n$.

(Recall that $U_n^T U_n = I_n$, $V_n^T V_n = I_n$, $\Sigma_n = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, $P_n = \mathcal{P}_\alpha^T \mathcal{P}_\alpha$, and $Q_n = \mathcal{Q}_\beta \mathcal{Q}_\beta^T$) Note that the singular values of the Hankel matrix of Markov parameters are invariant to way that the SVD is split, (i.e., invariant to the basis of the state space). These are called the *Hankel singular values* of the system.

The choice of p affects the gramians, but does not affect the state-space realization, to within an arbitrary unitary transformation and to within the effects of finite-precision computation. For the cases itemized above,

- for $p = -1/2$:

$$\begin{aligned} U \bar{T}^{-1} \bar{T} \Sigma V^T &= U \Sigma^{1/2} T^{-1} T \Sigma^{1/2} V^T \\ \mathcal{P} = U \bar{T}^{-1}, \quad \mathcal{Q} = \bar{T} \Sigma V^T &\quad \mathcal{P} = U \Sigma^{1/2} T^{-1}, \quad \mathcal{Q} = T \Sigma^{1/2} V^T \\ \mathcal{P} = U \bar{T}^{-1} = U \Sigma^{1/2} T^{-1} &\quad \mathcal{Q} = \bar{T} \Sigma V^T = T \Sigma^{1/2} V^T \\ \bar{T}^{-1} = \Sigma^{1/2} T^{-1} &\quad \bar{T} \Sigma = T \Sigma^{1/2} \\ T &= \Sigma^{1/2} \bar{T} \end{aligned}$$

- for $p = +1/2$:

$$\begin{aligned}
U\bar{\Sigma}T^{-1} \bar{T}V^T &= U\Sigma^{1/2}T^{-1} T\Sigma^{1/2}V^T \\
\mathcal{P} = U\Sigma\bar{T}^{-1}, \quad \mathcal{Q} = \bar{T}V^T &\quad \mathcal{P} = U\Sigma^{1/2}T^{-1}, \quad \mathcal{Q} = T\Sigma^{-1/2}V^T \\
\mathcal{P} = U\Sigma\bar{T}^{-1} = U\Sigma^{1/2}T^{-1} &\quad \mathcal{Q} = \bar{T}V^T = T\Sigma^{1/2}V^T \\
\Sigma\bar{T}^{-1} = \Sigma^{1/2}T^{-1} &\quad \bar{T} = T\Sigma^{1/2} \\
&\quad \bar{T} = T\Sigma^{1/2}
\end{aligned}$$

So, the choice made in splitting the SVD of $H(0)$ affects the resulting observability matrix and controllability matrix only up to an arbitrary unitary transformation.

Continuing with the case of balanced realizations, ($\mathcal{P}_\alpha = U_n\Sigma_n^{1/2}$ and $\mathcal{Q}_\beta = \Sigma_n^{1/2}V_n^T$), the dynamics matrix can be found from . . .

$$\begin{aligned}
H(1) &= \mathcal{P}_\alpha A \mathcal{Q}_\beta = U_n \Sigma_n^{1/2} A \Sigma_n^{1/2} V_n^T \\
U_n^T H(1) V_n &= U_n^T U_n \Sigma_n^{1/2} A \Sigma_n^{1/2} V_n^T V_n = \Sigma_n^{1/2} A \Sigma_n^{1/2} \\
\Sigma_n^{-1/2} U_n^T H(1) V_n \Sigma_n^{-1/2} &\equiv \hat{A}
\end{aligned} \tag{14}$$

The estimate of the input matrix \hat{B} is recovered as the first r columns of \mathcal{Q}_β and the estimate of the output matrix \hat{C} is recovered as the first m rows of \mathcal{P}_α . and the estimate of the feedthrough matrix \hat{D} is recovered from the first r columns of the sequence of Markov parameters, $Y(0)$ from (6). The resulting realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is minimal, meaning that no pole matches any zero of the transfer function matrix, and is therefore equivalent to any other realization via a change of basis of the state-space, e.g., $x = T\bar{x}$,

$$\bar{x}(k+1) = T^{-1}AT\bar{x}(k) + T^{-1}Bu(k)$$

$$y(k+1) = CT\bar{x}(k) + Du(k)$$

4 Eigensystem Realization with Data Correlation

In many cases, measurement noise will propagate to noise the sequence of estimated Markov parameters. In such cases, the application of the Eigensystem Realization Algorithm to correlation matrices of Markov parameters can serve to average-out the effect of noise on the estimated state-space realization.

Define the auto-correlation of Markov Parameters as

$$\mathcal{R}_{HH}(0) = H(0)H^T(0) \quad (15)$$

$$\begin{aligned} &= \begin{bmatrix} Y(1) & Y(2) & \cdots & Y(\beta) \\ Y(2) & Y(3) & \cdots & Y(\beta+1) \\ \vdots & \vdots & & \vdots \\ Y(\alpha) & Y(\alpha+1) & \cdots & Y(\alpha+\beta-1) \end{bmatrix} \begin{bmatrix} Y^T(1) & Y^T(2) & \cdots & Y^T(\alpha) \\ Y^T(2) & Y^T(3) & \cdots & Y^T(\alpha+1) \\ \vdots & \vdots & & \vdots \\ Y^T(\beta) & Y^T(\beta+1) & \cdots & Y^T(\alpha+\beta-1) \end{bmatrix} \\ &= \begin{bmatrix} \sum Y(i)Y^T(i) & \sum Y(i)Y^T(i+1) & \cdots & \sum Y(i)Y^T(i+n-1) \\ \sum Y(i+1)Y^T(i) & \sum Y(i+1)Y^T(i+1) & \cdots & \sum Y(i+1)Y^T(i+n-1) \\ \vdots & \vdots & & \vdots \\ \sum Y(i+n-1)Y^T(i) & \sum Y(i+n-1)Y^T(i+1) & \cdots & \sum Y(i+n-1)Y^T(i+n-1) \end{bmatrix} \\ &= \mathcal{P}_\alpha \mathcal{Q}_\beta \mathcal{Q}_\beta^T \mathcal{P}_\alpha^T = \mathcal{P}_\alpha \mathcal{Q}_\gamma \end{aligned} \quad (16)$$

and the cross correlation of Markov Parameters as

$$\mathcal{R}_{HH}(1) = H(1)H^T(0) = \mathcal{P}_\alpha A \mathcal{Q}_\beta \mathcal{Q}_\beta^T \mathcal{P}_\alpha^T = \mathcal{P}_\alpha A \mathcal{Q}_\gamma \quad (17)$$

where $\mathcal{Q}_\gamma \equiv \mathcal{Q}_\beta \mathcal{Q}_\beta^T \mathcal{P}_\alpha^T$.

Note here that larger values of β will result in more averaging in estimation of the auto-correlation values. For a fixed length of Markov parameters, K , increasing β will result in a smaller auto-correlation matrix, \mathcal{R}_{HH} , i.e., fewer time-lags (α) in the correlation matrices.

The Eigensystem Realization Algorithm can then proceed with a truncated SVD

$$\mathcal{R}_{HH}(0) = \mathcal{P}_\alpha \mathcal{Q}_\gamma \approx U_n \Sigma_n V_n^T.$$

From which, $\mathcal{P}_\alpha = U_n \Sigma_n^{1/2}$ and $\mathcal{Q}_\gamma = \Sigma_n^{1/2} V_n^T$. And since $\mathcal{R}_{HH}(1) = \mathcal{P}_\alpha A \mathcal{Q}_\gamma$,

$$\hat{A} = \Sigma_n^{-1/2} U_n^T \mathcal{R}_{HH}(1) V_n \Sigma_n^{-1/2}. \quad (18)$$

And, finally, since $\mathcal{P}_\alpha \mathcal{Q}_\beta = H(0)$,

$$\mathcal{Q}_\beta = \mathcal{P}_\alpha^+ H(0) \quad (19)$$

and \hat{B} is recovered from the first r columns of \mathcal{Q}_β and \hat{C} is recovered from the first m rows of \mathcal{P}_α .

Note that even though the SVD of $\mathcal{R}_{HH}(0)$ is split as for a balanced realization, the estimated realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is not internally balanced.

If the input sequence $u(k)$ is not measured, or, especially, if the system is known to be stochastically driven, or is in free-response, then the ERA-DC method can proceed from Markov parameters estimated as the auto-correlation of the measured output data

$$[Y(1), Y(2), \dots, Y(K)] \approx \mathcal{R}_{yy} \approx [y(K+1), \dots, y(N)] \cdot \begin{bmatrix} y(K) & y(K-1) & \dots & y(1) \\ y(K+1) & y(K) & \dots & y(2) \\ \vdots & \vdots & & \vdots \\ y(N-1) & y(N-2) & \dots & y(N-K) \end{bmatrix}$$

Since measurement noise can affect this estimate for the sequence of Markov parameters, It can be preferable to proceed with an ERA applied to $\mathcal{R}_{HH}(0)$ and $\mathcal{R}_{HH}(1)$. This is called Eigensystem Realization for Natural Excitation Technique (ERA-NExT).

5 Observers

Sometimes, it is helpful to identify a system that is a modification of the actual system from which measurements are taken, because the identification process can be more stable for the modified system.

For example, a discrete time linear system can be simply re-written as

$$x(k+1) = Ax(k) + Bu(k) + Gy(k) - Gy(k) \quad (20)$$

$$y(k) = Cx(k) + Du(k) \quad (21)$$

which can be re-written

$$x(k+1) = Ax(k) + Bu(k) + GCx(k) + GDu(k) - Gy(k) \quad (22)$$

$$= (A + GC)x(k) + (B + GD)u(k) - Gy(k) \quad (23)$$

$$= [A + GC]x(k) + [B + GD, -G] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \quad (24)$$

$$= \bar{A}x(k) + \bar{B}v(k) \quad (25)$$

$$y(k) = Cx(k) + Du(k) \quad (26)$$

where

$$\bar{A} = (A + GC), \quad (27)$$

$$\bar{B} = [B + GD, -G], \text{ and} \quad (28)$$

$$v(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \quad (29)$$

By designing the matrix G appropriately, the dynamics matrix of the modified system dynamics matrix $[A + GC]$ can be made much more heavily damped than the dynamics of the original system, A . For example in equation (25) if the matrix G is set equal to the Kalman filter gain, and if $y(k)$ is replaced by output estimation error, then the state $x(k)$ becomes an optimal estimate of the true state. This modification is intrinsic to the success of the Observer/Kalman Identification (OKID) method used in system identification.

6 Estimation of the Observer model Markov parameters

The inputs $(u(k), y(k))$ and outputs $(y(k))$ of the observer system (25) and (26) involve the same quantities as the input and output of the ERA system, but they are just organized in a different way.

A sequence of $k + 1$ Markov parameters for the observer model can be estimated from

$$\begin{bmatrix} y(k) & y(k+1) & \cdots & y(N) \end{bmatrix} = \begin{bmatrix} D & C\bar{B} & C\bar{A}\bar{B} & \cdots & C\bar{A}^{k-1}\bar{B} \end{bmatrix} \begin{bmatrix} u(k) & u(k+1) & \cdots & u(N) \\ v(k-1) & v(k) & \cdots & v(N-1) \\ v(k-2) & v(k-1) & \cdots & v(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ v(0) & v(1) & \cdots & v(N-k) \end{bmatrix} \quad (30)$$

$$y(k) = Du(k) + \sum_{i=1}^k \bar{Y}(i)v(k-i) \quad (31)$$

$$y(k) = Du(k) + \sum_{i=1}^k [\bar{Y}^{(1)}(i), \bar{Y}^{(2)}(i)] \begin{bmatrix} u(k-i) \\ y(k-i) \end{bmatrix} \quad (32)$$

$$y(k) = Du(k) + \sum_{i=1}^k \bar{Y}^{(1)}(i)u(k-i) + \sum_{i=1}^k \bar{Y}^{(2)}(i)y(k-i) \quad (33)$$

where the observer Markov parameters \bar{Y}_k are partitioned according to two submatrices. The first r columns fill $\bar{Y}^{(1)}(k)$ which multiplies the input $u(k)$ and the last m columns fill $\bar{Y}^{(2)}(k)$ which multiplies the output $y(k)$.

$$\bar{Y}(k) = \begin{bmatrix} \bar{Y}^{(1)}(k) & \bar{Y}^{(2)}(k) \end{bmatrix} \quad (34)$$

$$= \begin{bmatrix} C\bar{A}^{k-1}\bar{B} \end{bmatrix} \quad (35)$$

$$= \begin{bmatrix} C [A + GC]^{k-1} [B + GD] & -G \end{bmatrix} \quad (36)$$

$$= \begin{bmatrix} C [A + GC]^{k-1} [B + GD] & -C [A + GC]^{k-1} G \end{bmatrix} \quad (37)$$

Do the observer Markov parameters as estimated in equation (30) estimate the *one-step-ahead* predictor for $y(k)$? In other words, the model parameters are defined by a projection of the measured responses $y(k)$ onto a basis that involves the current and previous inputs $[u(k), u(k-1), \dots]$ and the previous *measured* responses $[y(k-1), y(k-2), \dots]$. (!?)

7 References

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8 .m-functions

```

1 function Q = ctrb(A,B,p)
2 % Q = ctrb(A,B,p)
3 % Form the controllability matrix Q = [B AB A^2B A^3B ... A^(p-1)B ]
4 % for the discrete time system, x(k+1) = A*x(k) + B*u(k)
5
6 [n,r] = size(B); % number of states and inputs
7 if nargin < 3, p = n; else p = max(p,n); end % number of blocks in Q
8 Q = zeros(n, r*p); % dimension of Q
9 Q(:,1:r) = B; % first block is input matrix
10 for k=1:p-1
11     AkB = Q(:,(k-1)*r+1:(k-1)*r+r);
12     Q(:,k*r+1:k*r+r) = A * AkB;
13 end
14 % ----- CTRB

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1 function P = obsv(A,C,p)
2 % P = obsv(A,C,p)
3 % Form the observability matrix P = [ C ; CA ; CA^2 ; CA^3 ; ... ; C A^(p-1) ]
4 % for the discrete time system, x(k+1) = A*x(k); y(k) = C*x(k)
5
6 [m,n] = size(C); % number of outputs and states
7 if nargin < 3, p = n; else p = max(p,n); end % number of blocks in P
8 P = zeros(m*p,n); % dimension of P
9 P(1:m,:) = C; % first block is output matrix
10 for k=1:p-1
11     CAk = P((k-1)*m+1:(k-1)*m+m,:);
12     P(k*m+1:k*m+m,:) = CAk * A;
13 end
14 % ----- OBSV

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1 >> A = [ 1 0.5 ; -0.5 0.7 ]; B = [ 1 ; -1 ]; C = [ 1 2 ]; D = 0; % discrete time SISO
2 >> n = 100;
3 >> dt = 0.05; % time step, s
4 >> t = [0:n]*dt;
5
6 >> P = obsv(A,C,n) % observability matrix
7 >> Q = ctrb(A,B,n) % controllability matrix
8
9 >> Ymp = [ D C*ctrb(A,B,n) ] % Markov parameters
10 Ymp = 0.00000 -1.00000 -1.90000 -2.28000 -2.07100 -1.35470 -0.33554 . . .
11
12 >> Yt = tril(toeplitz(Ymp)); % lower triangular Toeplitz matrix . . . works for SISO only
13 >> [Y,S,U] = svd(Yt); % singular value decomposition
14 >> j = 1;
15 >> plot(t,U(:,j),'--k', t,Yt*U(:,j),'-b', t,S(j,j)*Y(:,j),'--r')
16 >> legend('u_j', 'Ymp*U_j', '\Sigma_{jj}*Y_j')

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