CM 1015 Computational Mathematics

Kevin Juandi
email: kjuandi@gmail.com

April 30, 2023

Preface

I wrote this note after finishing the course, so the content might not reflect the current version of the course. It is mostly based on my handwritten personal notes. I personally feel this course should be called "Foundation Mathematics" instead of "Computational Mathematics" because of the lack of "Numerical Methods" and probably some other things people more familiar with the topic would say. If you spot any error please don't hesitate to contact me via slack or mail me. I will update the notes along with the pace of the course, every Saturday or Sunday before the start of the week. If you need help with the subject, don't hesitate to contact me.

Contents

Chapter 1

Number Bases, Conversion and Operations

Reading Materials:
Croft, A. and R. Davidson *Foundation maths.* (Harlow: Pearson, 2016) 6th edition. **Chapter 14 Number Bases**

1.1 Number Bases

Decimal System

The numbers that we commonly used are based on 10.
For example:

$$\begin{aligned} 253 &= 200 + 50 + 3 \\ &= 2(100) + 5(10) + 3 \\ &= 2(10^2) + 5(10^1) + 3(10^0) \end{aligned} \tag{1.1}$$

Binary System

A binary system uses base 2, it only consist of 2 digits, 0 and 1.
Numbers in base 2 are called binary digits or simply bits.
Consider the binary number 110101_2. As the base is 2, this means that power of 2 essentially replace powers of 10. Let us convert it to base 10.

$$\begin{aligned} 110101_2 &= 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0) \\ &= 1(32) + 1(16) + 0(8) + 1(4) + 0(2) + 1(1) \\ &= 32 + 16 + 4 + 1 \\ &= 53_{10} \end{aligned} \tag{1.2}$$

Octal System

Octal numbers use 8 as a base. The eight digits used in the octal system are 0, 1, 2, 3, 4, 5, 6 and 7. Octal numbers use powers of 8, just as decimal numbers use powers of 10 and binary numbers use powers of 2. Example:

$$\begin{aligned} 325_8 &= 3(8^2) + 2(8^1) + 5(8^0) \\ &= 3(64) + 2(8) + 5(1) \\ &= 192 + 16 + 5 \\ &= 213_{10} \end{aligned} \tag{1.3}$$

Hexadecimal System

Hexadecimal system use 16 as a base. The digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F. Example:

$$\begin{aligned} 93\mathrm{A}_{16} &= 9(16^2) + 3(16^1) + \mathrm{A}(16^0) \\ &= 9(256) + 3(16) + 10(1) \\ &= 2304 + 48 + 10 \\ &= 2362_{10} \end{aligned} \tag{1.4}$$

1.2 Number Conversion

1.2.1 Converting from Decimal to Other Number Base

The no-brainer way is to divide the number by the base, the remainder would be the last digit of the new number base. Keep dividing the quotient until it is smaller than the number base. Let us convert 253_{10} as example.

$$\begin{aligned} 2\overline{)\,253\,} &= 126 \text{ with remainder } 1 \\ 2\overline{)\,126\,} &= 63 \text{ with remainder } 0 \\ 2\overline{)\,63\,} &= 31 \text{ with remainder } 1 \\ 2\overline{)\,31\,} &= 15 \text{ with remainder } 1 \\ 2\overline{)\,15\,} &= 7 \text{ with remainder } 1 \\ 2\overline{)\,7\,} &= 3 \text{ with remainder } 1 \\ 2\overline{)\,3\,} &= 1 \text{ with remainder } 1 \\ 2\overline{)\,1\,} &= 0 \text{ with remainder } 1 \end{aligned} \tag{1.5}$$

Thus, 253_{10} is 11111101_2 in binary. We could do the same to other number bases.

Another method is by listing the powers of the base, compare and subtract. Using the same number as example.

$$\begin{array}{ccc} 2^0 = 1 & 2^1 = 2 & 2^2 = 4 \\ 2^3 = 8 & 2^4 = 16 & 2^5 = 32 \\ 2^6 = 64 & 2^7 = 128 & 2^8 = 256 \end{array}$$

From here we compare the number that we are going to convert with the list:

$$\begin{aligned} 253 - 1(128) &= 125 \\ 125 - 1(64) &= 61 \\ 61 - 1(32) &= 29 \\ 29 - 1(16) &= 13 \\ 13 - 1(8) &= 5 \\ 5 - 1(4) &= 1 \\ 1 - 0(2) &= 1 \\ 1 - 1(1) &= 0 \end{aligned} \tag{1.6}$$

Thus, 253_{10} is 11111101_2 in binary. Like the other method, we can also do this to convert to other number bases.

1.2.2 Conversion with Binary Number

Converting binary numbers to Octal or Hexadecimal and vice versa are very straightforward. It can be performed without converting to Decimal first. Let us use number 11100110_2 as example:

$$\begin{aligned} \underbrace{11}_{3}\underbrace{100}_{4}\underbrace{110}_{6} &= 346_8 \\ \underbrace{1110}_{\text{E}}\underbrace{0110}_{6} &= \text{E}6_{16} \end{aligned} \tag{1.7}$$

1.2.3 Non-integer Number Conversion

Converting non-integer number might look counterintuitive and intimidating. It is actually rather simple. Let us convert 17.375_{10} to binary.

$$\begin{aligned} 17.375 &= 10 + 7 + 0.3 + 0.07 + 0.005 \\ &= 1(10^1) + 7(10^0) + 3(10^{-1}) + 7(10^{-2}) + 5(10^{-3}) \end{aligned}$$

Converting to binary, $17_{10} = 10001_2$. But how about the decimal point? We multiply them by two until we are left with whole number

$$\begin{aligned} 0.375 \times 2 = 0.75 = 0 + 0.75 &\text{ we have 0 at power } -1 \\ 0.75 \times 2 = 1.5 = 1 + 0.5 &\text{ we have 1 at power } -2 \\ 0.5 \times 2 = 1.0 = 1 &\text{ we have 1 at power } -3 \end{aligned} \tag{1.8}$$

Thus, $17.375_{10} = 10001.011_2$
The reverse is much simpler.

$$\begin{aligned} 1101.101_2 &= 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) + 1(2^{-1}) + 0(2^{-2}) + 1(2^{-3}) \\ &= 1(8) + 1(4) + 0(2) + 1(1) + 1(0.5) + 0(0.25) + 1(0.125) \\ &= 8 + 4 + 1 + 0.5 + 0.125 \\ &= 13.625_{10} \end{aligned} \tag{1.9}$$

1.3 Operations with Binary Number

Addition

Addition is rather straightforward, just like with decimal numbers. Here we use 11001001_2 and 11111111_2 as example. In decimal they are 201 and 255 respectively.

$$\begin{array}{r} {\scriptstyle 1\,1111\,111} \\ 0\,1100\,1001 \\ +\,0\,1111\,1111 \\ \hline 1\,1100\,1000 \end{array} \tag{1.10}$$

$$\underbrace{11001001}_{201} + \underbrace{11111111}_{255} = \underbrace{111001000}_{456} \tag{1.11}$$

Subtraction

Likewise for subtraction, between 1110011_2 and 1010010_2

$$\begin{array}{r} 111\,0011 \\ -\,101\,0010 \\ \hline 010\,0001 \end{array} \tag{1.12}$$

$$\underbrace{1110011}_{115} - \underbrace{1010010}_{82} = \underbrace{100001}_{33} \tag{1.13}$$

Multiplication

Multiplication is essentially addition done multiple times. Let us have multiplication of 1100_2 and 1111_2. I would skip the carry so it wouldn't look too cramped.

$$\begin{array}{r} 1100 \\ \times \quad 1111 \\ \hline 1100 \\ 1100 \\ 1100 \\ 1100 \\ \hline 10110100 \end{array} \tag{1.14}$$

$$\underbrace{1100}_{12} \times \underbrace{1111}_{15} = \underbrace{10110100}_{180} \tag{1.15}$$

Division

Division is perhaps the one that feels the most unnatural and most likely to cause mistakes. Let us do this with 11100110_2 divided with 110_2 as example.

$$\begin{array}{rr}
 & 100110 \\ \cline{2-2}
110\big) & 11100110 \\
 & -\ 110 \\ \hline
 & 10 \\
 & -\ \ 0 \\ \hline
 & 100 \\
 & -\ \ 0 \\ \hline
 & 1001 \\
 & -\ 110 \\ \hline
 & 111 \\
 & -\ 110 \\ \hline
 & 10 \\
 & -\ \ 0 \\ \hline
 & 10
\end{array} \tag{1.16}$$

$$\underbrace{11100110}_{230} \div \underbrace{110}_{12} = \underbrace{100110}_{38} \text{ with remainder } \underbrace{10}_{2} \tag{1.17}$$

Chapter 2

Series and Sequence

Reading Materials:
Croft, A. and R. Davidson *Foundation maths.* (Harlow: Pearson, 2016) 6th edition. **Chapter 12 Sequences and series.**

2.1 Little Gauss

Skip this section if you have no interest in casual reading.

There is this story that is often told in mathematics classes. While the story itself is likely apocryphal, it likely have some pedagogical value. The story goes this way:

There was once a German school where a boy Carl Friedrich made mischief during mathematics lesson. Instead of corporal punishment that was common in that time, the teacher instead decided to give him mathematics assignment to keep him busy. He was asked to add up the numbers from one to a hundred. Most students would diligently start adding and be busy for a while. The young Carl Friedrich, on the other hand, answered after a few minutes. The teacher was surprised at the request to speak, since he had just kept the boy busy. He was all the more astonished when Carl Friedrich said that he had finished the task and was even able to say the correct result (5050).

How had he solved it?

How he did it so fast? Carl Friedrich discovered discovered the following - unfortunately I do not know what coincidence was behind it. He wrote the numbers down like this:

1	2	3	...	99	100
100	99	98	...	2	1

This still doesn't look interesting yet. He would then add up the numbers.

1	2	3	...	99	100
100	99	98	...	2	1
101	101	101	...	101	101

Each of them have the sum 101. This looks rather promising.

To sum it up, we write down the numbers from one to one hundred twice, once in increasing order and once in decreasing order, we would then sum them up and we can clearly see that we obtain the sum of $100 \cdot 101$. But we are not finished yet because we counted each numbers twice so we still have to divide the results by two. Then, we would have the sum of numbers from one to a hundred. And that's exactly how Carl Friedrich proceeded. Do we know Carl Friedrich? Hopefully that's the case, because Carl Friedrich was none other than Carl Friedrich Gauss. One of the most important German mathematicians (if not the most important German mathematician).

Let us talk about the formula

Mathematicians love formulas or should I say the general solution of a problem. The sum of the first n of natural numbers follows the formula:

$$\Sigma = \frac{n \cdot (n+1)}{2} \tag{2.1}$$

This is not as complicated as it looks. We could for example count the sum of 1 to 150, then we set n equals to 150.

$$\Sigma = \frac{150 \cdot (150+1)}{2} = \frac{150 \cdot (151)}{2} = \frac{22650}{2} = 11325 \tag{2.2}$$

This formula is today is still affectionately referred as "Der Kleine Gauss", German for "Little Gauss". Anyone studying higher mathematics would have to prove the validity of the formula. The formula itself is an arithmetic series. We would learn more about it later.

2.2 Sequence

A sequence is a set of numbers written down in a specific order. For example, 1,3,4,7,9. There need not be an obvious rule relating to the numbers in the

sequence. For example, 9, -11, $\frac{1}{2}$, 32.5 is a sequence.
A simple way of forming a sequence is to calculate each new term by adding a fixed amount to the previous term. For example, $1, 7, 13, 19, \ldots$.
Such sequence is called arithmetic progression or arithmetic sequence.

Geometric progression is a sequence formed by multiplying the previous term by fixed amount. For example, $2, 10, 50, 250, \ldots$.

Some sequences continue indefinitely, these are called infinite sequences/ It can happen that as we move along the sequence, the term get closer and closer to a fixed value. For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$.
This sequence can be rewritten in the form of:

$$x_k = \frac{1}{k} \text{ , for } k = 1, 2, 3, 4, 5, \ldots \tag{2.3}$$

As k get larger and larger, and approaches infinity, x_k get closer and closer to zero. We say that $\frac{1}{k}$ *tends to zero as* k *tends to infinity* or alternatively, *as* k *tends to infinity, the limit of the sequence is zero.* We write this down as:

$$\lim_{k \to \infty} \frac{1}{k} = 0 \tag{2.4}$$

When a sequence possess a limit, it is said to *converge*, when not, it is said to *diverge*.

2.3 Series

If the terms of a sequence are added the result is known as series. A series is a sum. If the series contains a finite number of terms, we are able to add them all up and obtain the sum.

A series is said to *converge* if it has finite sum.

The sum of first n terms of an arithmetic series with first term a and common difference d is denoted by $S - n$ and given by

$$S_n = \frac{n}{2}(2a + (n-1)d) \tag{2.5}$$

The sum of first n terms of an geometric series with first term a and common ratio r is denoted by $S - n$ and given by

$$S_n = \frac{a(1 - r^n}{1 - r} \text{ , for } r \neq 1 \tag{2.6}$$

for $n \to \infty$ and $-1 < r < 1$,

$$S_n = \frac{a}{1-r} \tag{2.7}$$

Beware that a converging sequence does not mean it's sum (series) converges. in our example of $x_k = \frac{1}{k}$, x converges to zero as k approaches infinity however, the sum of x does not. This is known as *Harmonic series*. I'm not going to prove it here. You can check yourself here.

Chapter 3

Modular Arithmetic

Reading Materials:
Yan, S.Y. *Number theory for computing.* (Berlin: Springer-Verlag, 2002) 2nd edition. **Section 1.2 Theory of divisibility pp.21-pp.24 , Section 1.6 Theory of congruences pp.111-119**

3.0.1 Congruence

Modular arithmetic is a system of arithmetic for integers where the number "wraps around" after reaching a certain value we call **modulus**.

Modular arithmetic is commonly used in number theory, algebra (group theory, ring theory, etc) and also in cryptography. An example in daily life is the clock. The numbers go from 1 to 12, but when you get to "13 o'clock", it actually becomes 1 o'clock again (think of how the 24 hour clock numbering works). So 13 becomes 1, 14 becomes 2, and so on. This can keep going, so when you get to "25 o'clock", you are actually back round to where 1 o'clock is on the clock face (and also where 13 o'clock was too), this is arithmetic modulo 12.

In our clock example, the number go from 1 to 12. In formal mathematics, we usually start from 0. In this case our clock would have 12 replaced with zero. Thus,

$$24 \equiv 0 \mod 12 \tag{3.1}$$

Two integers a and b are said to be **congruent** modulus k if when they are divided by k, they have the same remainder. More examples:

$$3 \equiv 5 \mod 2 \tag{3.2}$$

3 divided by 2 gives 1 with remainder 1, 5 divided by 2 gives 2 with remainder 1.

$$14 \equiv 2 \mod 12 \tag{3.3}$$

14 divided by 12 gives 1 with remainder 2, 2 divided by 12 gives 0 with remainder 2. How about negative numbers? $-17 \mod 12$ for example:

$$\begin{aligned} -17 \mod 12 &\equiv -5 \mod 12 \\ &\equiv 7 \mod 12 \end{aligned} \tag{3.4}$$

3.1 Operations in Modular Artithmetics

Chapter 4

Trigonometric Relations

Chapter 5

Functions

Chapter 6

Trigonometric Functions

Chapter 7

Exponential and Logarithmic Functions

Chapter 8

Limit and Differentiation

Chapter 9

Linear Algebra, Vector and Matrices

Chapter 10

Combinatorics and Probability

Bibliography