# CM 1015 Computational Mathematics

Kevin Juandi
email: kjuandi@gmail.com

April 21, 2023

# Preface

I wrote this note after finishing the course, so the content might not reflect the current version of the course. It is mostly based on my handwritten personal notes. I personally feel this course should be called "Foundation Mathematics" instead of "Computational Mathematics" because of the lack of "Numerical Methods" and probably some other things people more familiar with the topic would say. If you spot any error please don't hesitate to contact me via slack or mail me. I will update the notes along with the pace of the course, every Saturday or Sunday before the start of the week. If you need help with the subject, don't hesitate to contact me.

# Contents

# Chapter 1

# Number Bases, Conversion and Operations

Reading Materials:
Croft, A. and R. Davidson *Foundation maths.* (Harlow: Pearson, 2016) 6th edition. **Chapter 14 Number Bases**

## 1.1 Number Bases

### Decimal System

The numbers that we commonly used are based on 10.
For example:

$$\begin{aligned} 253 &= 200 + 50 + 3 \\ &= 2(100) + 5(10) + 3 \\ &= 2(10^2) + 5(10^1) + 3(10^0) \end{aligned} \tag{1.1}$$

### Binary System

A binary system uses base 2, it only consist of 2 digits, 0 and 1.
Numbers in base 2 are called binary digits or simply bits.
Consider the binary number $110101_2$. As the base is 2, this means that power of 2 essentially replace powers of 10. Let us convert it to base 10.

$$\begin{aligned} 110101_2 &= 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0) \\ &= 1(32) + 1(16) + 0(8) + 1(4) + 0(2) + 1(1) \\ &= 32 + 16 + 4 + 1 \\ &= 53_{10} \end{aligned} \tag{1.2}$$

### Octal System

Octal numbers use 8 as a base. The eight digits used in the octal system are 0, 1, 2, 3, 4, 5, 6 and 7. Octal numbers use powers of 8, just as decimal numbers use powers of 10 and binary numbers use powers of 2. Example:

$$\begin{aligned} 325_8 &= 3(8^2) + 2(8^1) + 5(8^0) \\ &= 3(64) + 2(8) + 5(1) \\ &= 192 + 16 + 5 \\ &= 213_{10} \end{aligned} \tag{1.3}$$

### Hexadecimal System

Hexadecimal system use 16 as a base. The digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F. Example:

$$\begin{aligned} 93\mathrm{A}_{16} &= 9(16^2) + 3(16^1) + \mathrm{A}(16^0) \\ &= 9(256) + 3(16) + 10(1) \\ &= 2304 + 48 + 10 \\ &= 2362_{10} \end{aligned} \tag{1.4}$$

## 1.2 Number Conversion

### 1.2.1 Converting from Decimal to Other Number Base

The no-brainer way is to divide the number by the base, the remainder would be the last digit of the new number base. Keep dividing the quotient until it is smaller than the number base. Let us convert $253_{10}$ as example.

$$\begin{aligned} 2\overline{)\,253} &= 126 \text{ with remainder } 1 \\ 2\overline{)\,126} &= 63 \text{ with remainder } 0 \\ 2\overline{)\,63} &= 31 \text{ with remainder } 1 \\ 2\overline{)\,31} &= 15 \text{ with remainder } 1 \\ 2\overline{)\,15} &= 7 \text{ with remainder } 1 \\ 2\overline{)\,7} &= 3 \text{ with remainder } 1 \\ 2\overline{)\,3} &= 1 \text{ with remainder } 1 \\ 2\overline{)\,1} &= 0 \text{ with remainder } 1 \end{aligned} \tag{1.5}$$

Thus, $253_{10}$ is $11111101_2$ in binary. We could do the same to other number bases.

Another method is by listing the powers of the base, compare and subtract. Using the same number as example.

$$
\begin{array}{ccc}
2^0 = 1 & 2^1 = 2 & 2^2 = 4 \\
2^3 = 8 & 2^4 = 16 & 2^5 = 32 \\
2^6 = 64 & 2^7 = 128 & 2^8 = 256
\end{array}
$$

From here we compare the number that we are going to convert with the list:

$$
\begin{gathered}
253 - 1(128) = 125 \\
125 - 1(64) = 61 \\
61 - 1(32) = 29 \\
29 - 1(16) = 13 \\
13 - 1(8) = 5 \\
5 - 1(4) = 1 \\
1 - 0(2) = 1 \\
1 - 1(1) = 0
\end{gathered} \tag{1.6}
$$

Thus, $253_{10}$ is $11111101_2$ in binary. Like the other method, we can also do this to convert to other number bases.

### 1.2.2 Conversion with Binary Number

Converting binary numbers to Octal or Hexadecimal and vice versa are very straightforward. It can be performed without converting to Decimal first. Let us use number $11100110_2$ as example:

$$
\begin{gathered}
\underbrace{11}_{3}\,\underbrace{100}_{4}\,\underbrace{110}_{6} = 346_8 \\
\underbrace{1110}_{\text{E}}\,\underbrace{0110}_{6} = \text{E}6_{16}
\end{gathered} \tag{1.7}
$$

### 1.2.3 Non-integer Number Conversion

Converting non-integer number might look counterintuitive and intimidating. It is actually rather simple. Let us convert $17.375_{10}$ to binary.

$$
\begin{aligned}
17.375 &= 10 + 7 + 0.3 + 0.07 + 0.005 \\
&= 1(10^1) + 7(10^0) + 3(10^{-1}) + 7(10^{-2}) + 5(10^{-3})
\end{aligned}
$$

Converting to binary, $17_{10} = 10001_2$. But how about the decimal point? We multiply them by two until we are left with whole number

$$
\begin{aligned}
0.375 \times 2 = 0.75 = 0 + 0.75 &\text{ we have 0 at power } -1 \\
0.75 \times 2 = 1.5 = 1 + 0.5 &\text{ we have 1 at power } -2 \\
0.5 \times 2 = 1.0 = 1 &\text{ we have 1 at power } -3
\end{aligned} \tag{1.8}
$$

Thus, $17.375_{10} = 10001.011_2$
The reverse is much simpler.

$$\begin{aligned}1101.101_2 &= 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) + 1(2^{-1}) + 0(2^{-2}) + 1(2^{-3}) \\ &= 1(8) + 1(4) + 0(2) + 1(1) + 1(0.5) + 0(0.25) + 1(0.125) \\ &= 8 + 4 + 1 + 0.5 + 0.125 \\ &= 13.625_{10}\end{aligned} \tag{1.9}$$

## 1.3 Operations with Binary Number

### Addition

Addition is rather straightforward, just like with decimal numbers. Here we use $11001001_2$ and $11111111_2$ as example. In decimal they are 201 and 255 respectively.

$$\begin{array}{r} {\scriptstyle 1\,1111\,111\phantom{1}} \\ 0\,1100\,1001 \\ +\,0\,1111\,1111 \\ \hline 1\,1100\,1000 \end{array} \tag{1.10}$$

Or simply

$$\underbrace{11001001}_{201} + \underbrace{11111111}_{255} = \underbrace{111001000}_{456} \tag{1.11}$$

### Subtraction

Likewise for subtraction, between $1110011_2$ and $1010010_2$

$$\begin{array}{r} 111\,0011 \\ -\,101\,0010 \\ \hline 010\,0001 \end{array} \tag{1.12}$$

Or simply

$$\underbrace{1110011}_{115} - \underbrace{1010010}_{82} = \underbrace{100001}_{33} \tag{1.13}$$

### Multiplication

Multiplication is essentially addition done multiple times. Let us have multiplication of $1100_2$ and $1111_2$. I would skip the carry so it wouldn't look too cramped.

$$\begin{array}{r} 1100 \\ \times\quad 1111 \\ \hline 1100 \\ 1100\phantom{0} \\ 1100\phantom{00} \\ 1100\phantom{000} \\ \hline 10110100 \end{array} \tag{1.14}$$

Or simply

$$\underbrace{1100}_{12} - \underbrace{1111}_{15} = \underbrace{10110100}_{180} \tag{1.15}$$

### Division

Division is perhaps the one that feels the most unnatural and most likely to cause mistakes. Let us do this with $11100110_2$ divided with $110_2$ as example.

## 1.4 Common Pitfalls

# Chapter 2

# Series and Sequence

Reading Materials:
Croft, A. and R. Davidson *Foundation maths.* (Harlow: Pearson, 2016) 6th edition. **Chapter 12 Sequences and series.**

## 2.1 Little Gauss

There is this story that is often told in mathematics classes. While the story itself is likely apocryphal, it likely have some pedagogical value. The story goes this way:

There was once a German school where a boy Carl Friedrich made mischief during mathematics lesson. Instead of corporal punishment that was common in that time, the teacher instead decided to give him mathematics assignment to keep him busy. He was asked to add up the numbers from one to a hundred. Most students would diligently start adding and be busy for a while. The young Carl Friedrich, on the other hand, answered after a few minutes. The teacher was surprised at the request to speak, since he had just kept the boy busy. He was all the more astonished when Carl Friedrich said that he had finished the task and was even able to say the correct result (5050).

### How had he solved it?

How he did it so fast? Carl Friedrich discovered discovered the following - unfortunately I do not know what coincidence was behind it. He wrote the numbers down like this:

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & 99 & 100 \\ 100 & 99 & 98 & \ldots & 2 & 1 \end{array}$$

This still doesn't look interesting yet. He would then add up the numbers.

$$\begin{array}{cccccc} 1 & 2 & 3 & \dots & 99 & 100 \\ 100 & 99 & 98 & \dots & 2 & 1 \\ 101 & 101 & 101 & \dots & 101 & 101 \end{array}$$

Each of them have the sum 101. This looks rather promising.

To sum it up, we write down the numbers from one to one hundred twice, once in increasing order and once in decreasing order, we would then sum them up and we can clearly see that we obtain the sum of $100 \cdot 101$. But we are not finished yet because we counted each numbers twice so we still have to divide the results by two. Then, we would have the sum of numbers from one to a hundred. And that's exactly how Carl Friedrich proceeded. Do we know Carl Friedrich? Hopefully that's the case, because Carl Friedrich was none other than Carl Friedrich Gauss. One of the most important German mathematicians (if not the most important German mathematician).

## Let us talk about the formula

Mathematicians love formulas or should I say the general solution of a problem. The sum of the first $n$ of natural numbers follows the formula:

$$\Sigma = \frac{n \cdot (n+1)}{2} \tag{2.1}$$

This is not as complicated as it looks. We could for example count the sum of 1 to 150, then we set $n$ equals to 150.

$$\Sigma = \frac{150 \cdot (150+1)}{2} = \frac{150 \cdot (151)}{2} = \frac{22650}{2} = 11325 \tag{2.2}$$

This formula is today is still affectionately referred as "Der Kleine Gauss", German for "Little Gauss". Anyone studying higher mathematics would have to prove the validity of the formula.

# Chapter 3

# Modular Mathematics

# Chapter 4

# Trigonometric Relations

# Chapter 5

# Functions

# Bibliography