Cheatsheet - Functions

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1. Intro

A function maps one element from a set to **exactly one** element of another (or the same) set.

For example, we have sets:

$$A = \{1, 2, 3\}$$
$$B = \{2, 4, 6, 7, 8\}$$

and the function:

$$f : A \to B$$

defined as:

$$f(x) = 2x$$

Now lets apply each element from A to f:

$$f(1) = 2$$
$$f(2) = 4$$
$$f(3) = 6$$

We call set A the **domain** and set B the **co-domain**. The set of all possible values when mapping elements from set A to set B, respectively set $R = \{2, 4, 6\}$, is called the **range**. Hence $R \subseteq B$.

Additionally, we say that 1 is the **pre-image** of 2, which in return is the **image** of 1. The element 2 is the pre-image of 4, which in return is the image of 2. And so on.

2. Injective, Surjective & Bijective Functions

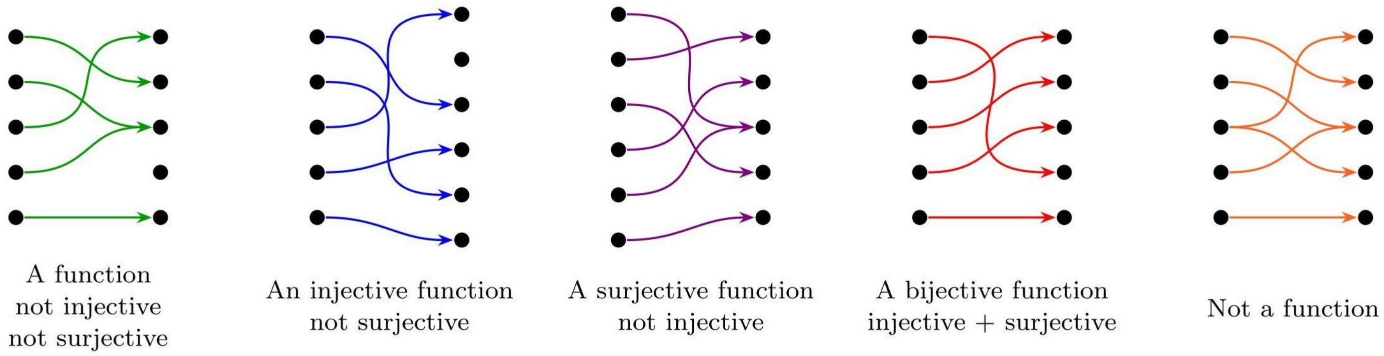

Figure 1. Source: https://twitter.com/JDHamkins/status/841318019397779456

- **General function**: A has at most one B (not injective, not surjective).
- **Injective**: A has exactly one B (not surjective).
- **Surjective**: *Each and every* B has one or many A (not injective).
- **Bijective**: *Each and every* B has exactly one A (injective and surjective).
- **NOT a function**: A has many B.

2.1. Proofs

We can prove whether a function is injective, surjective or bijective by solving an equation. Lets use the following function as an example:

$$f(x) = 2x + 3$$

Injective Proof

Let $a, b \in R$, show that if $a \neq b$ then $f(a) \neq f(b)$:

$$a \neq b \quad \times 2$$
$$2a \neq 2b \quad +3$$
$$2a + 3 \neq 2b + 3$$
$$\Rightarrow f(a) \neq f(b)$$

Or: Let $a, b \in R$, show that if $f(a) = f(b)$ then $a = b$:

$$f(a) = f(b) \Rightarrow$$
$$2a + 3 = 2b + 3 \quad -3$$
$$2a = 2b \quad \div 2$$
$$a = b$$

Hence, function f is injective.

Surjective Proof

Let $y \in R$, show that there exists $x \in R$ such that $f(x) = y$:

$$f(x) = y \Rightarrow$$
$$2x + 3 = y \quad -3$$
$$2x = y - 3 \quad \div 2$$
$$x = \frac{y-3}{2}$$

Hence, function f is surjective.

3. Composition

Function composition means we apply one function to the result of another.

For example:

$$(f \circ g)(x) = f(g(x))$$

which means that the result of $g()$ is passed on to $f()$. If we define $f(x) = 2x$ and $g(x) = x^2$, then:

$$(f \circ g)(5) = f(g(x)) = 2 \times \left(5^2\right) = 50$$

Do note that function composition is not commutative, meaning $f \circ g \neq g \circ f$:

$$(f \circ g)(5) = f(g(x)) = 2 \times \left(5^2\right) = 50$$
$$(g \circ f)(5) = g(f(x)) = (5 \times 2)^2 = 100$$

4. Inverse Function

If function f is bijective, then there exists an inverse function f^{-1}.

$$f: A \to B$$
$$f^{-1}: B \to A$$

For example, given $f(x) = 2x$, then $f^{-1}(x) = \frac{x}{2}$.

$$f(2) = 4$$
$$f^{-1}(4) = 2$$

Additionally:

$$\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$$

5. Exponential Functions

Properties of exponential the function:

$$y = f(x) = b^x \quad (b > 0 \quad \text{and} \quad b \neq 1)$$

- The domain is $(-\infty, \infty)$
- The range is $(0, \infty)$
- It passes through the point $(0, 1)$
- If $b > 1$ then it's increasing on $(-\infty, \infty)$ ("exponential growth")
- If $b < 1$ then it's decreasing on $(-\infty, \infty)$ ("exponential decay")

6. Logarithmic Functions

The logarithmic function with base b where $b > 0$ and $b \neq 1$ is defined as:

$$\log_b x = y \quad \text{if and only if} \quad x = b^y$$

Respectively:

$$x = b^y \Leftrightarrow \log_b(x) = y$$

For example:

$$81 = 3^4 \Leftrightarrow \log_3(81) = 4$$

6.1. Laws

$$\log_b(m \times n) = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b(m^n) = n \times \log_b(m)$$

$$\log_b(1) = 0$$

$$\log_b(b) = 1$$

Conventionally, we also define **natural logarithms** as:

$$\log = \log_{10}$$

$$\ln = \log_e$$

Where e is the "Euler number" (https://en.wikipedia.org/wiki/E_(mathematical_constant)) ($e = 2.71828$).

7. Floor and Ceiling Functions

We define the **floor** of the real number x as (round **down** to the previous integer **or equal**):

$$x = 3.6$$

$$\lfloor x \rfloor = 3$$

We define the **ceiling** of real number x as (round **up** to the next integer **or equal**):

$$x = 3.6$$

$$\lceil x \rceil = 4$$

Additionally (equal):

$$y = 5$$

$$\lfloor y \rfloor = 5$$

$$\lceil y \rceil = 5$$

and (negative numbers)

$$x = -3.5$$

$$\lfloor x \rfloor = -4$$

$$\lceil x \rceil = -3$$

Both the floor and the ceiling function convert a real number to an integer, respectively $\mathbb{R} \to \mathbb{Z}$.

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