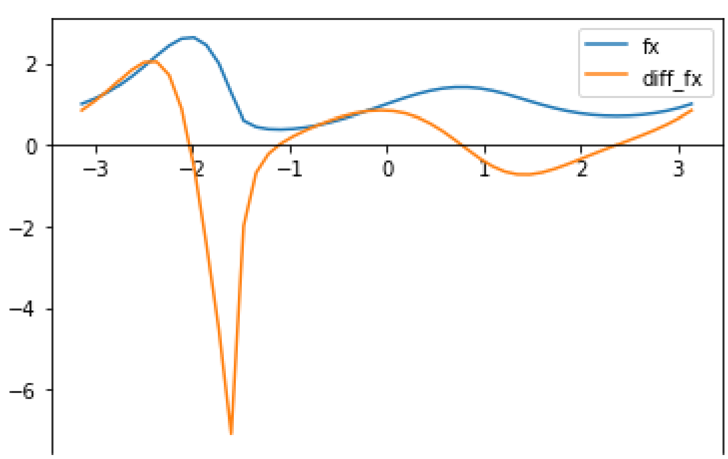
```
In [1]: import sympy as sy
        import numpy as np
        import matplotlib.pyplot as plt
        import pandas as pd
        import time
        from autograd import numpy as anp
        from autograd import grad
```

## Problem 1

```
In [2]: def get_func():

            # Define the function
            sym_x = sy.symbols('x')
            sym_fx = (sy.sin(sym_x)+1)**(sy.sin(sy.cos(sym_x)))

            # Lambdify the function
            fx = sy.lambdify(sym_x, sym_fx, 'numpy')

            # Lambdify the derivative function
            diff_fx = sy.lambdify(sym_x, sy.diff(sym_fx), 'numpy')

            # return two functions
            return fx, diff_fx
```

```
In [4]: # Plot f and f' on [-pi, pi]
        x = np.linspace(-np.pi, np.pi)
        fx, diff_fx = get_func()
        ax = plt.gca()
        ax.spines['bottom'].set_position('zero')
        plt.plot(x, fx(x), label='fx')
        plt.plot(x, diff_fx(x), label='diff_fx')
        plt.legend()
```

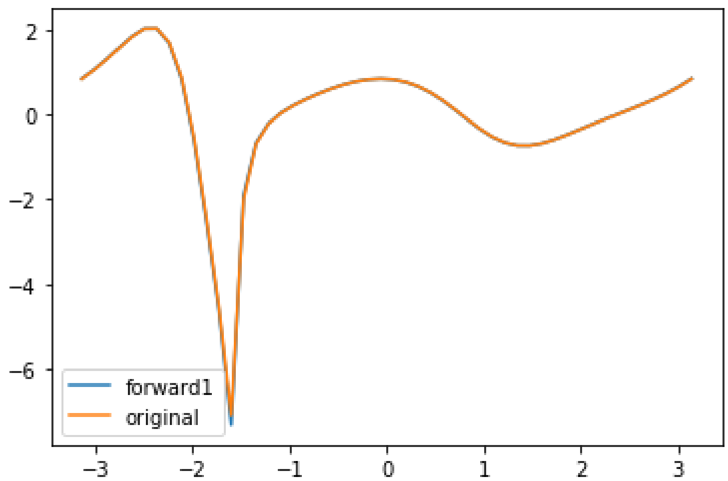
```
Out[4]: <matplotlib.legend.Legend at 0x11ec40a58>
```



## Problem 2

In [4]:
```
# Operate forward with order 1
def fwd1(f, x, h):
    return (f(x+h)-f(x))/h

# Plot forward approximation with order 1
x = np.linspace(-np.pi, np.pi)
f1=fwd1(fx,x,0.01)
plt.plot(x, f1, label='forward1')
plt.plot(x, diff_fx(x), label='original')
plt.legend()
```

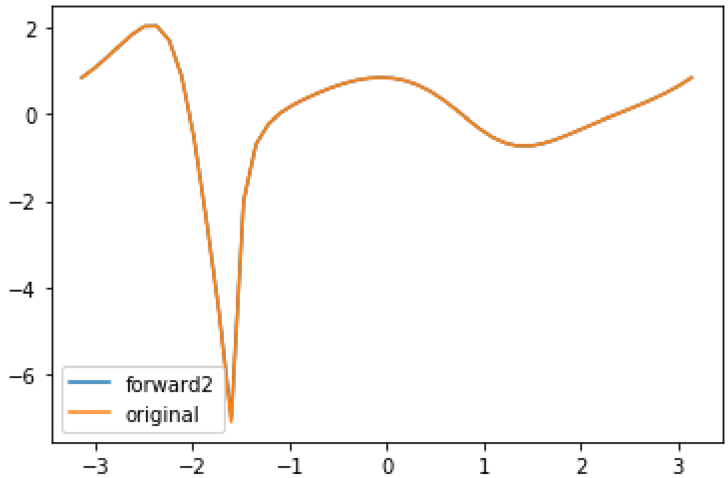
Out[4]:
```
<matplotlib.legend.Legend at 0x1dcdb884eb8>
```




The result of Foward 1 approximation is quite close to the exact result in this case.

In [5]:
```python
# Operate forward with order 2

def fwd2(f, x, h):
    return (-3*f(x)+4*f(x+h)-f(x+2*h))/2/h

# Plot forward approximation with order 2
x = np.linspace(-np.pi, np.pi)
f2=fwd2(fx,x,0.01)
plt.plot(x, f2, label='forward2')
plt.plot(x, diff_fx(x), label='original')
plt.legend()
```

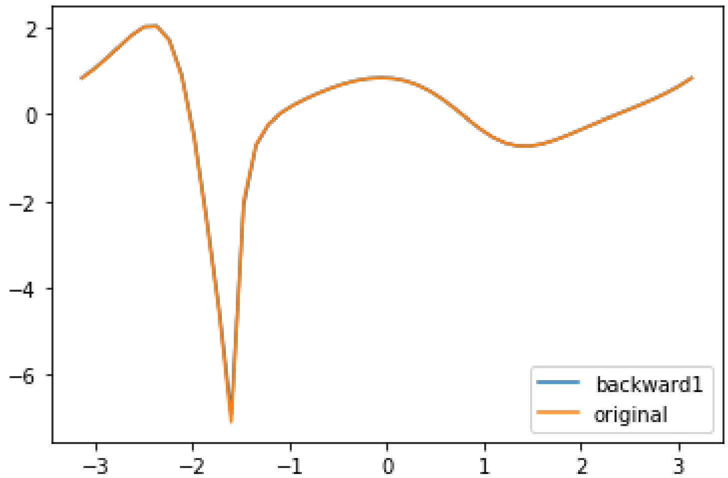
Out[5]: <matplotlib.legend.Legend at 0x1dcdb9160f0>




The result of Foward 2 approximation is quite close to the exact result in this case.

In [6]:
```
# Operate backward with order 1

def bwd1(f, x, h):
    return (f(x)-f(x-h))/h

# Plot backward approximation with order 1
x = np.linspace(-np.pi, np.pi)
b1=bwd1(fx,x,0.01)
plt.plot(x, b1, label='backward1')
plt.plot(x, diff_fx(x), label='original')
plt.legend()
```

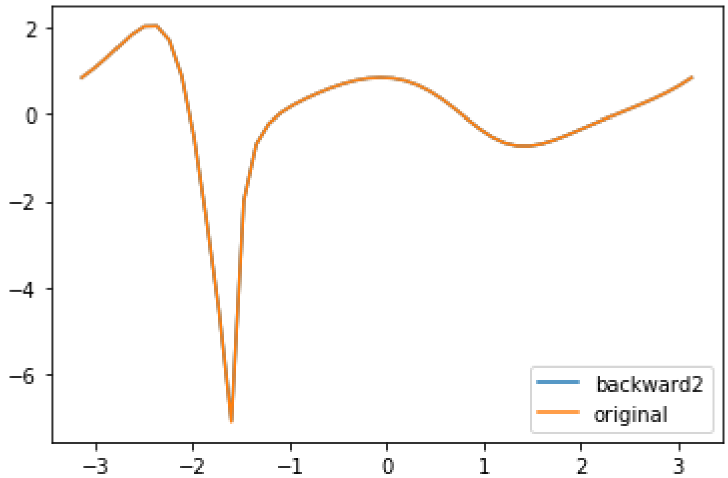
Out[6]: <matplotlib.legend.Legend at 0x1dcdb981828>




The result of Backward 1 approximation is quite close to the exact result in this case.

In [7]:
```
# Operate backward with order 2

def bwd2(f, x, h):
    return (3*f(x)-4*f(x-h)+f(x-2*h))/2/h

# Plot backward approximation with order 2
x = np.linspace(-np.pi, np.pi)
b2=bwd2(fx,x,0.01)
plt.plot(x, b2, label='backward2')
plt.plot(x, diff_fx(x), label='original')
plt.legend()
```

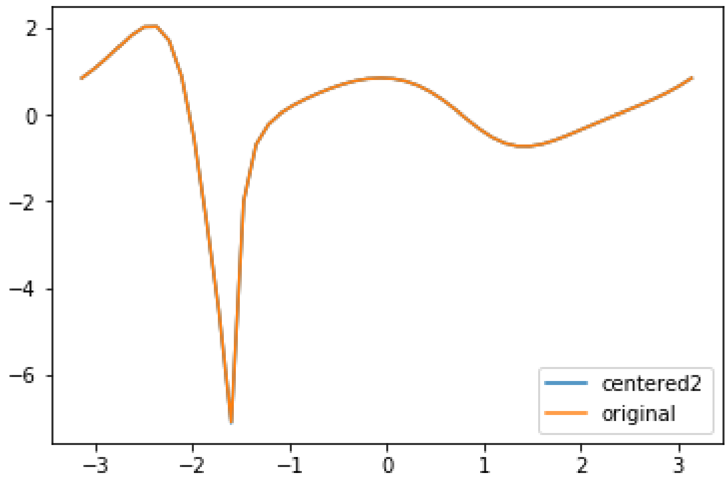
Out[7]: <matplotlib.legend.Legend at 0x1dcdb9c5f60>




The result of Backward 2 approximation is quite close to the exact result in this case.

In [5]:
```
# Operate centered with order 2

def ctr2(f, x, h):
    return (f(x+h)-f(x-h))/2/h

# Plot centered approximation with order 2
x = np.linspace(-np.pi, np.pi)
c2=ctr2(fx,x,0.01)
plt.plot(x, c2, label='centered2')
plt.plot(x, diff_fx(x), label='original')
plt.legend()
```

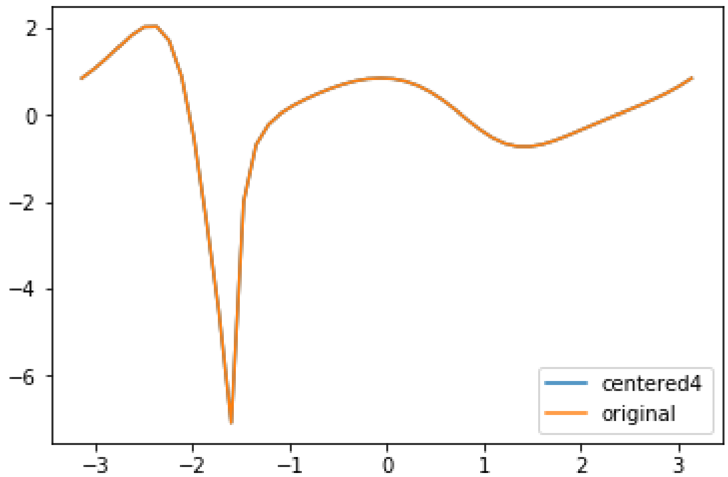
Out[5]:
```
<matplotlib.legend.Legend at 0x11ed72358>
```




The result of Center 2 approximation is quite close to the exact result in this case.

In [8]:
```
# Operate centered with order 4

def ctr4(f, x, h):
    return (f(x-2*h)-8*f(x-h)+8*f(x+h)-f(x+2*h))/12/h

# Plot centered approximation with order 4
x = np.linspace(-np.pi, np.pi)
c4=ctr4(fx,x,0.01)
plt.plot(x, c4, label='centered4')
plt.plot(x, diff_fx(x), label='original')
plt.legend()
```

Out[8]: <matplotlib.legend.Legend at 0x11ee71f98>



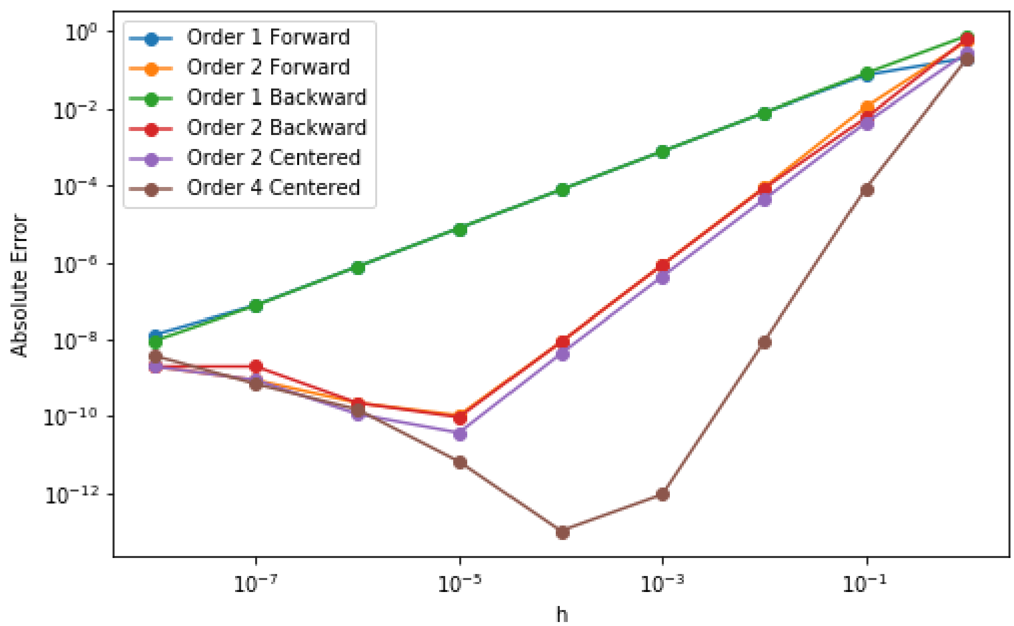

The result of Center 4 approximation is quite close to the exact result in this case.

We can see that all the six methods are good approximation in this case.

## Problem 3

In [10]:
```
def error(n):

    h = np.logspace(-8, 0, 9)
    plt.figure(figsize = (8,5))

    # plot six lines of absolute errors
    plt.loglog(h, np.abs(diff_fx(n)-fwd1(fx,n,h)), label='Order 1 Forwar
d', marker='o')
    plt.loglog(h, np.abs(diff_fx(n)-fwd2(fx,n,h)), label='Order 2 Forwar
d', marker='o')
    plt.loglog(h, np.abs(diff_fx(n)-bwd1(fx,n,h)), label='Order 1 Backwa
rd', marker='o')
    plt.loglog(h, np.abs(diff_fx(n)-bwd2(fx,n,h)), label='Order 2 Backwa
rd', marker='o')
    plt.loglog(h, np.abs(diff_fx(n)-ctr2(fx,n,h)), label='Order 2 Center
ed', marker='o')
    plt.loglog(h, np.abs(diff_fx(n)-ctr4(fx,n,h)), label='Order 4 Center
ed', marker='o')
    plt.xlabel('h')
    plt.ylabel('Absolute Error')
    plt.legend()
```

In [11]:
```
error(1)
```




## Problem 4

In [12]:
```
plane = np.load('plane.npy')
plane_df = pd.DataFrame(plane, columns=['time','alpha','beta'])
plane_df['alpha']=np.deg2rad(plane_df['alpha'])
plane_df['beta']=np.deg2rad(plane_df['beta'])
```

In [13]:
```
plane_df['x'] = (500*np.tan(plane_df['beta']))/(np.tan(plane_df['beta'])
-np.tan(plane_df['alpha']))
plane_df['y'] = (500*np.tan(plane_df['beta'])*np.tan(plane_df['alpha']))
/(np.tan(plane_df['beta'])-np.tan(plane_df['alpha']))
```

In [14]:
```
plane_df['x_prime'] = plane_df['x']
plane_df['y_prime'] = plane_df['y']

plane_df['x_prime'][0] = plane_df['x'][1]-plane_df['x'][0]
plane_df['y_prime'][0] = plane_df['y'][1]-plane_df['y'][0]

plane_df['x_prime'][7] = plane_df['x'][7]-plane_df['x'][6]
plane_df['y_prime'][7] = plane_df['y'][7]-plane_df['y'][6]

for i in range(1,7):
    plane_df['x_prime'][i] = (plane_df['x'][i+1]-plane_df['x'][i-1])/2
    plane_df['y_prime'][i] = (plane_df['y'][i+1]-plane_df['y'][i-1])/2

plane_df['speed'] = np.sqrt(plane_df['x_prime']**2+plane_df['y_prime']**
2)
```

In [15]:
```
plane_df[['time','speed']]
```

Out[15]:

| | time | speed |
|---|---|---|
| **0** | 7.0 | 46.424201 |
| **1** | 8.0 | 47.001039 |
| **2** | 9.0 | 48.998805 |
| **3** | 10.0 | 50.099442 |
| **4** | 11.0 | 48.290351 |
| **5** | 12.0 | 51.564559 |
| **6** | 13.0 | 53.923034 |
| **7** | 14.0 | 51.514801 |

## Problem 5

In [16]:
```
# Define the Jacobian function

def jb_mat(vec_f, vec_x, h):
    n = len(vec_x)
    m = len(vec_f)

    # define standard basis
    e = np.identity(n)
    # get a m*n matrix
    jacob = np.zeros((m,n))

    # calculate entry for the matrix
    for i in range(0,n):
        for j in range(0,m):
            jacob[j,i]=((vec_f[j](vec_x+h*e[i])-vec_f[j](vec_x-h*e[i])))
/2/h

    return jacob
```

In [17]:
```
# Test the Jacobian function
fset = [lambda x: x[0]**2,lambda x: x[0]**3-x[1]]
jb_mat(fset, [1,1], 0.0001)
```

Out[17]:
```
array([[ 2.        ,  0.        ],
       [ 3.00000001, -1.        ]])
```

In [18]:
```
np.full((3,2), 0, dtype = 'float')
```

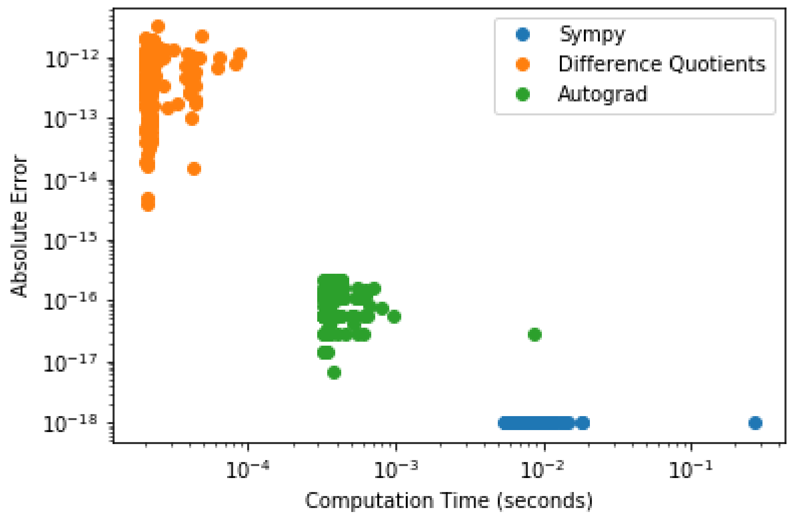
Out[18]:
```
array([[0., 0.],
       [0., 0.],
       [0., 0.]])
```

## Problem 7

In [6]:
```
def t_error(N):


    prob1_time = []
    prob1_error = [10**(-18)]*N
    prob3_time = []
    prob3_error = []
    prob7_time = []
    prob7_error = []

    # From problem 3, we can use h = e-18 for centred approximation with
order 4
    h = 10**(-4)

    for i in range(N):

        # choose a random value x0
        x0 = np.random.random()

        # time and absolute error for Sympy method
        start = time.time()
        fx, diff_fx = get_func()
        exact = diff_fx(x0)
        end = time.time()
        prob1_time.append(end-start)

        # time and absolute error for difference quotient method
        start = time.time()
        approx = ctr4(fx, x0, h)
        end = time.time()
        prob3_time.append(end-start)
        prob3_error.append(np.abs(exact-approx))

        # time and absolute error for autograd method
        start = time.time()
        g=lambda x0: (anp.sin(x0)+1)**(anp.sin(anp.cos(x0)))
        dg=grad(g)
        approx=dg(x0)
        end = time.time()
        prob7_time.append(end-start)
        prob7_error.append(np.abs(exact-approx))

    return prob1_time,prob1_error,prob3_time,prob3_error,prob7_time,prob
7_error
```

In [9]:
```
# Test the function when N = 200
prob1_time,prob1_error,prob3_time,prob3_error,prob7_time,prob7_error = t
_error(200)
plt.loglog(prob1_time,prob1_error,'o',label='Sympy')
plt.loglog(prob3_time,prob3_error,'o',label='Difference Quotients')
plt.loglog(prob7_time,prob7_error,'o',label='Autograd')
plt.xlabel('Computation Time (seconds)')
plt.ylabel('Absolute Error')
plt.legend()
```

Out[9]: <matplotlib.legend.Legend at 0x11ef94dd8>




We can see that the Difference Quotient has the higest absolute error but the fastest speed while the Sympy method has slowest speed, but the most accuracy.