Module 3 : Assignment mechanism

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- Science table provides observable outcomes :

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$$Y_i^{obs} = W_i^{obs} Y_i(W_i = 1) + (1 - W_i^{obs}) Y_i(W_i = 0)$$

Assignment mechanism

- The fundamental problem of causal inference is the presence of missing data :
  - Recall, for each unit, we can observe at most one of the potential outcomes.
- A key component in a causal analysis is the assignment mechanism :
  - The process that determines which units receive which treatments, hence which potential outcomes are observed, and which potential outcomes are missing.
- Formally, the assignment mechanism describes the probability of any vector of assignments as a function of all covariates and all potential outcomes.
- Taxonomy of assignment mechanisms (Rubin, 1974).

Assignment mechanism

- The assignment mechanism is the function that assigns probabilities to all 2^N possible values for the N-vector of assignments $\mathbf{W}$ (i.e., each unit can be assigned to treatment or control) given the N-vector of potential outcomes $\mathbf{Y(0)}$ and $\mathbf{Y(1)}$, the N*K matrix of covariates $\mathbf{X}$.

- Given a population of N units, the assignment mechanism is a row-exchangeable function $P(\mathbf{W}|\mathbf{X},\mathbf{Y(0)},\mathbf{Y(1)})$, taking values in [0,1], satisfying :

$$\sum_{\mathbf{W}\in\{0,1\}^N} P(\mathbf{W}|\mathbf{X},\mathbf{Y(0)},\mathbf{Y(1)}) = 1$$

  - for all $\mathbf{X}$,$\mathbf{Y(0)}$, and $\mathbf{Y(1)}$)
  - $\{0,1\}^N$ is the set of all N-vectors with all elements equal to 0 or 1.

Unit assignment probability

- The unit-level assignment probability for unit i is :

$$p_i(\mathbf{X}, \mathbf{Y(0)}, \mathbf{Y(1)}) = \sum_{\mathbf{W}:W_i=1} P(\mathbf{W}|\mathbf{X}, \mathbf{Y(0)}, \mathbf{Y(1)})$$

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Classical randomized experiments : the assignment mechanism satisfies all three restrictions below :

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  - For all units $i = 1, ..., N$:

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- Unconfounded assignment :
  - This disallows dependence of the assignment mechanisms on the potential outcomes.

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- Classical randomized experiments :
  - The assignment mechanism is individualistic, probabilistic, and unconfounded.
  - The researcher knows and controls the functional form of the assignment mechanism.

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- $\binom{N}{N_T} = \frac{N!}{N_T!(N-N_T)!} = \frac{N(N-1)...(N-N_T+1)}{N_T(N_T-1)...x1}$

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- There are "N choose N_T", i.e., $\binom{N}{N_T}$ number of ways to do this.
- Each participant has the same chance of receiving the active treatment, i.e., $\frac{N_T}{N}$.

Completely randomized experiment

- Assignment mechanism :

$$P(\mathbf{W}|\mathbf{X},\mathbf{Y(0)},\mathbf{Y(1)}) = \begin{cases} \binom{N}{N_t}^{-1} & \text{if } \sum_{i=1}^{N} W_i = N_t \\ 0 & \text{otherwise.} \end{cases}$$

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- Possible assignment vectors are $W = (W_1 = 0, ..., W_N = 0)$ and $W = (W_1 = 1, ..., W_N = 1)$.

Bernouilli trials

- Bernouilli experiment with a fair coin :

  For each unit,

$$P(W_i = 1 | X_i, Y_i(0), Y_i(1)) = 0.5$$

- Assignment mechanism :

$$P(\mathbf{W}|\mathbf{X},\mathbf{Y(0)},\mathbf{Y(1)}) = \prod_{i=1}^{N} P(W_i = 1|X_i)^{W_i}[1 - P(W_i = 1|X_i)]^{1-W_i}$$

Paired randomized experiment

- Each unit has probability $\frac{1}{2}$ of being assigned to the treatment group.
- Assignment mechanism :

$$P(\mathbf{W}|\mathbf{X},\mathbf{Y(0)},\mathbf{Y(1)}) = \begin{cases} 2^{-N/2} & \text{if } W \in W^+ \\ 0 & \text{otherwise.} \end{cases}$$