Understanding the Infection Dynamics of COVID-19 through a SuEIR Model and Deep Learning

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Presentation Structure

1. Project Objectives
2. COVID-19 SuEIR Model
3. Solving the Model: Numerical Approximation vs Deep Learning
4. Estimating Parameters
5. Discussion

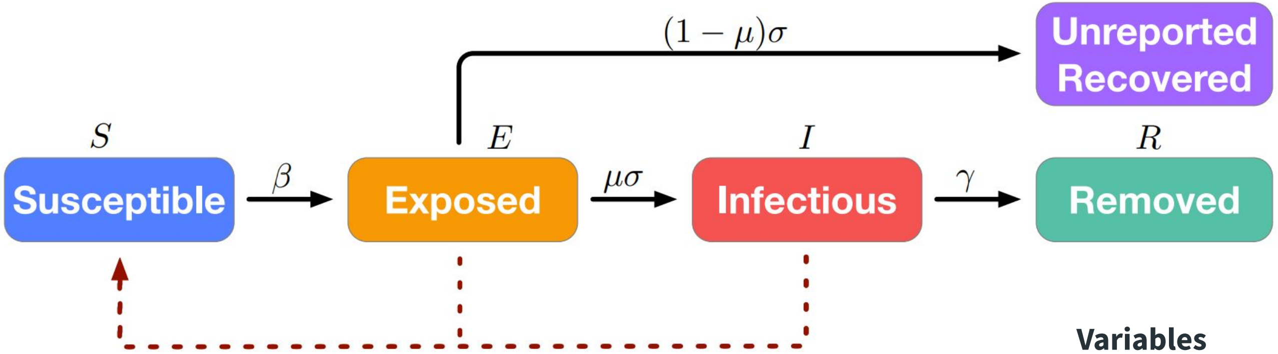
Project Objectives

1. Examine and explain the COVID-19 SuEIR Model presented in [Zou et al 2020].

2. Numerically solve the SuEIR Model via artificial neural network ODE solvers ([Chen et al 2020]) to:
   2.1. First, generate “ground truth data” via numerical simulation using given parameters θ = (β, σ, γ, μ) and initial conditions
   2.2. Next, compute the ground truth R_0
   2.3. Then, generate 10 simulations of 4 time series (S, E, I, R) by selectively perturbing the ground truth parameters

3. Reimplement and analyze the parameter learning method presented in Zou et al 2020 to do the following for each of the perturbed simulations:
   3.1. Find the parameters θ’ = (β’, σ’, γ’, μ’) to calculate R_0

COVID-19 SuEIR Model



$$\frac{dS_t}{dt} = -\frac{\beta(I_t + E_t)S_t}{N},$$
$$\frac{dE_t}{dt} = \frac{\beta(I_t + E_t)S_t}{N} - \sigma E_t,$$
$$\frac{dI_t}{dt} = \mu\sigma E_t - \gamma I_t,$$
$$\frac{dR_t}{dt} = \gamma I_t,$$

Variables

S : Susceptible population

E : Exposed population

I : Infectious population

R : Recovered population

N: Total population

β : Contact rate (E and I)

μ : Discovery rate

σ : Transition rate (E → I)

γ : Transition rate (I → R)

Zhou et al 2020

Simulation Parameters (ode45 vs NeuroDiffEq)

Simulation #	N	S_0	E_0	I_0	R_0	beta	mu	sigma	gamma	Explanation
0	1000	800	100	50	50	0.25	0.075	0.09	0.12	Ground truth
1	1000	800	100	50	50	0.5	0.05	0.09	0.12	Varying beta / low mu
2	1000	800	100	50	50	0.4	0.05	0.09	0.12	Varying beta / low mu
3	1000	800	100	50	50	0.3	0.05	0.09	0.12	Varying beta / low mu
4	1000	800	100	50	50	0.2	0.05	0.09	0.12	Varying beta / low mu
5	1000	800	100	50	50	0.1	0.05	0.09	0.12	Varying beta / low mu
6	1000	800	100	50	50	0.5	0.1	0.09	0.12	Varying beta / high mu
7	1000	800	100	50	50	0.4	0.1	0.09	0.12	Varying beta / high mu
8	1000	800	100	50	50	0.3	0.1	0.09	0.12	Varying beta / high mu
9	1000	800	100	50	50	0.2	0.1	0.09	0.12	Varying beta / high mu
10	1000	800	100	50	50	0.1	0.1	0.09	0.12	Varying beta / high mu

Solving Model via ode45 (Runge-Kutta (4,5))

Runge-Kutta Method of 4th Order:

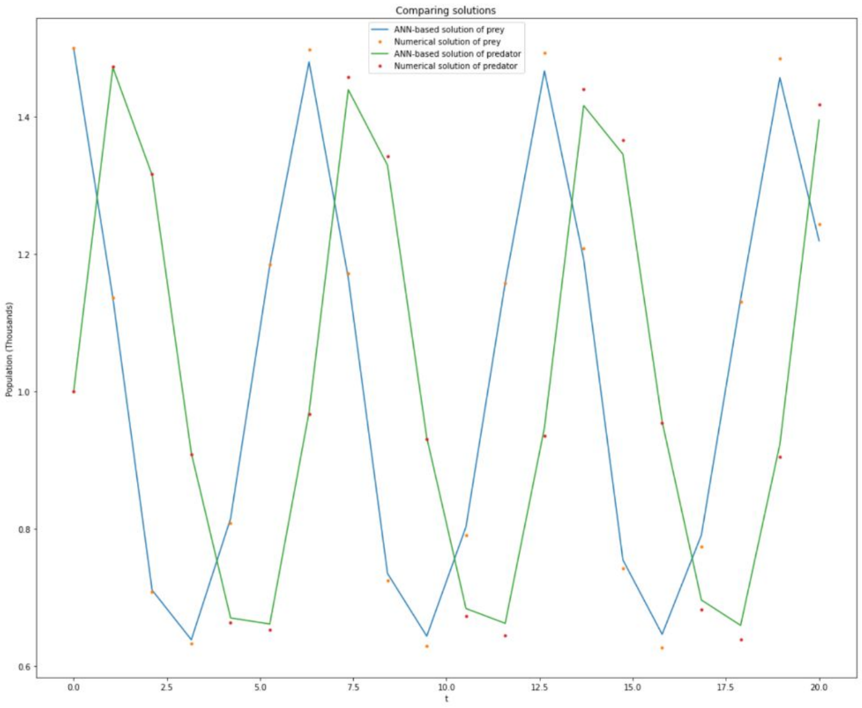
- Consider the differential Equation
  $$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$
  Calculate successively
- $k_1 = hf(x_0, y_0)$
- $k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$
- $k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
- $k_4 = hf(x_0 + h, y_0 + k_3)$

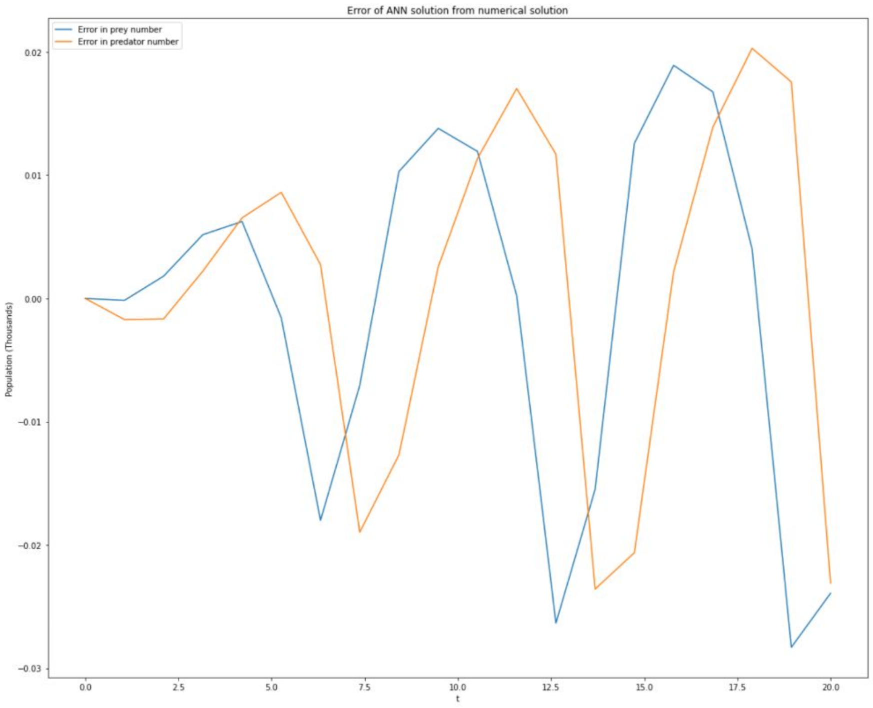
  Find $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore\ y_1 = y_0 + k$ and $x_1 = x_0 + h$

Shampine and Reichelt 1997

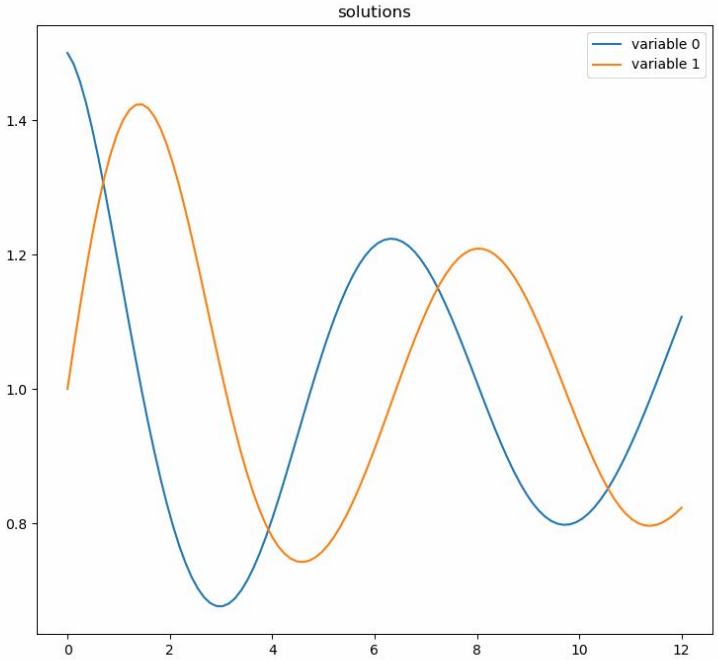
NeuroDiffEq: Solving DEs via Neural Networks

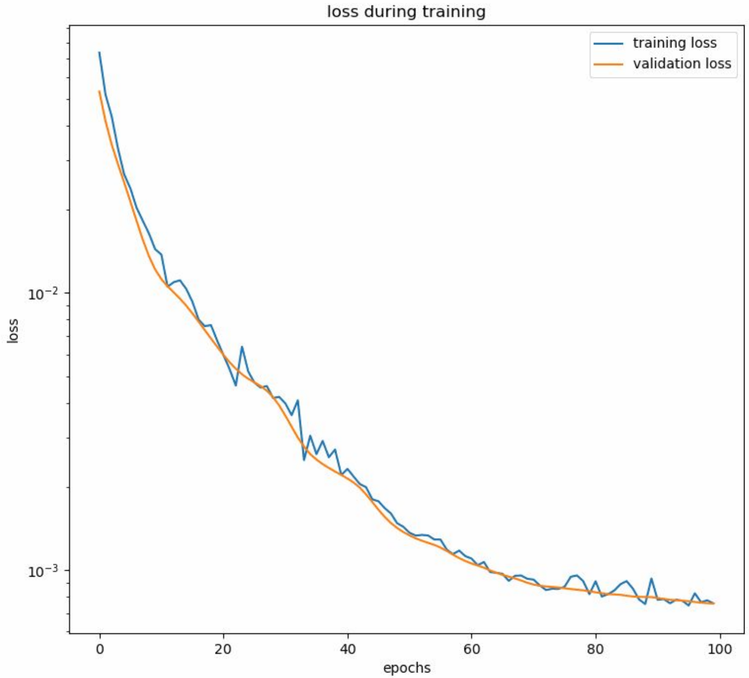




Chen et al 2020

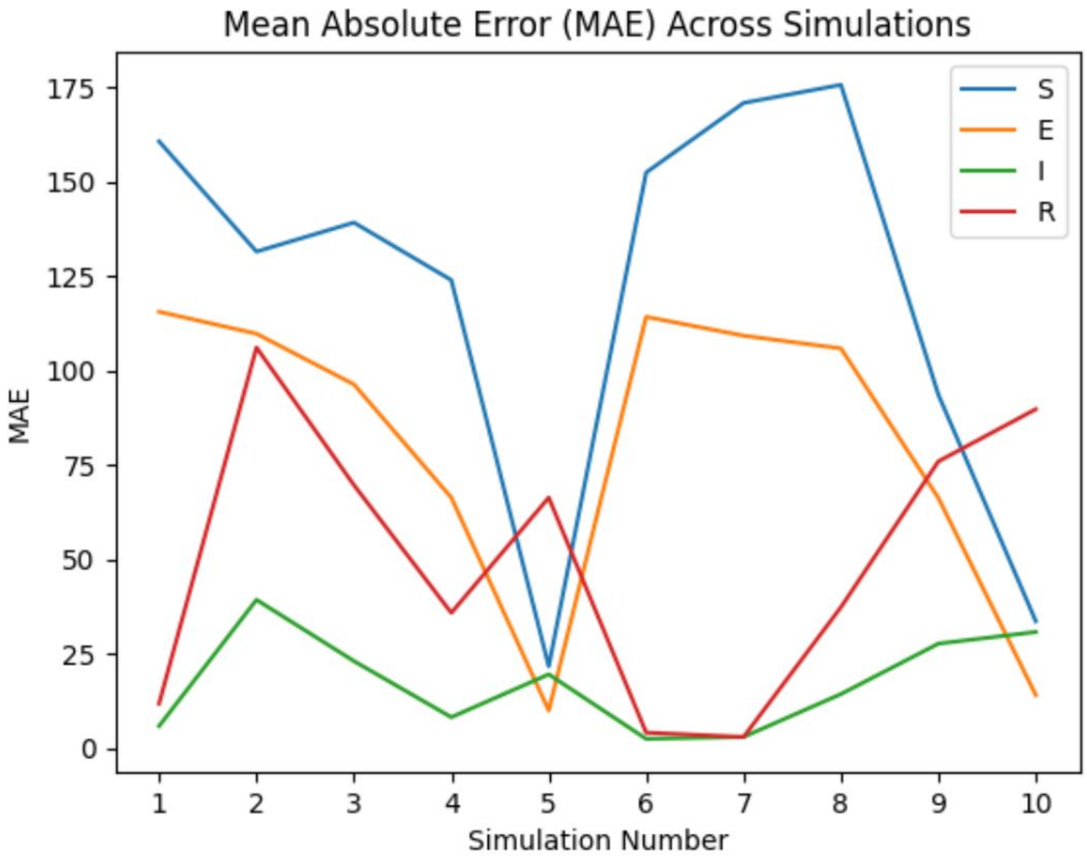
Solve Model via NeuroDiffEq



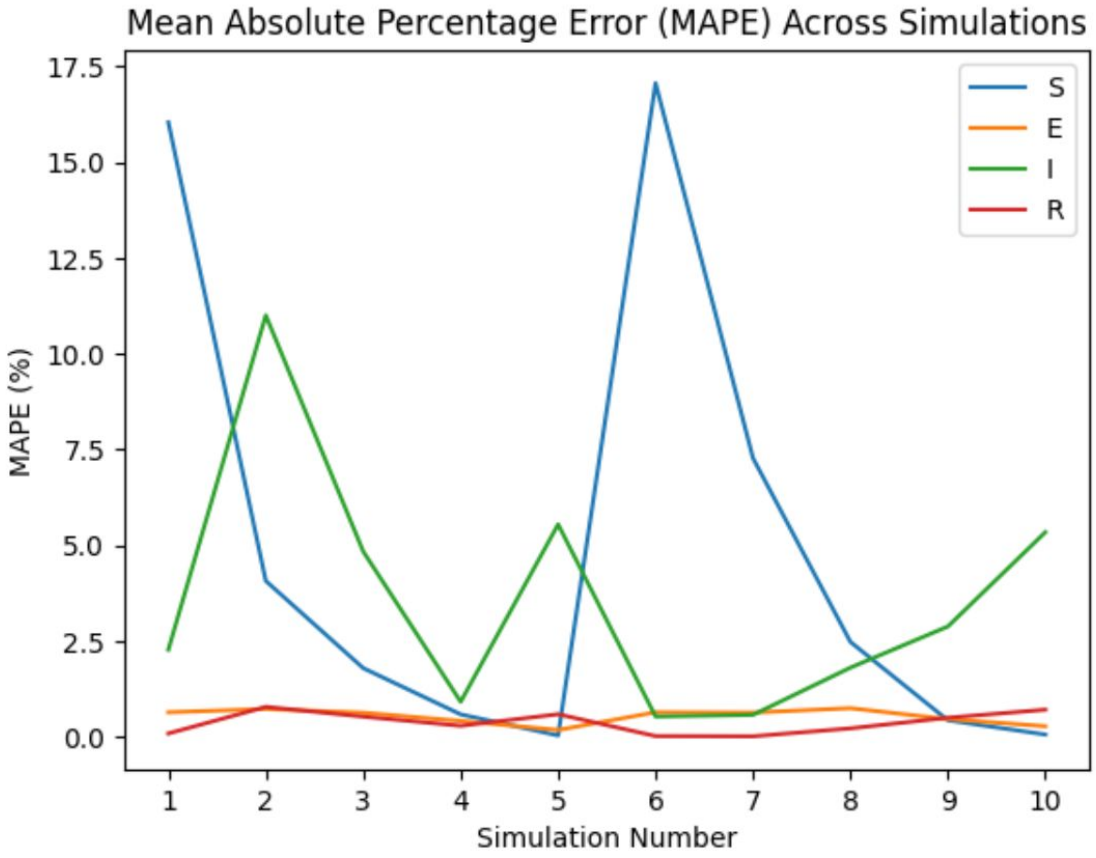


Chen et al 2020

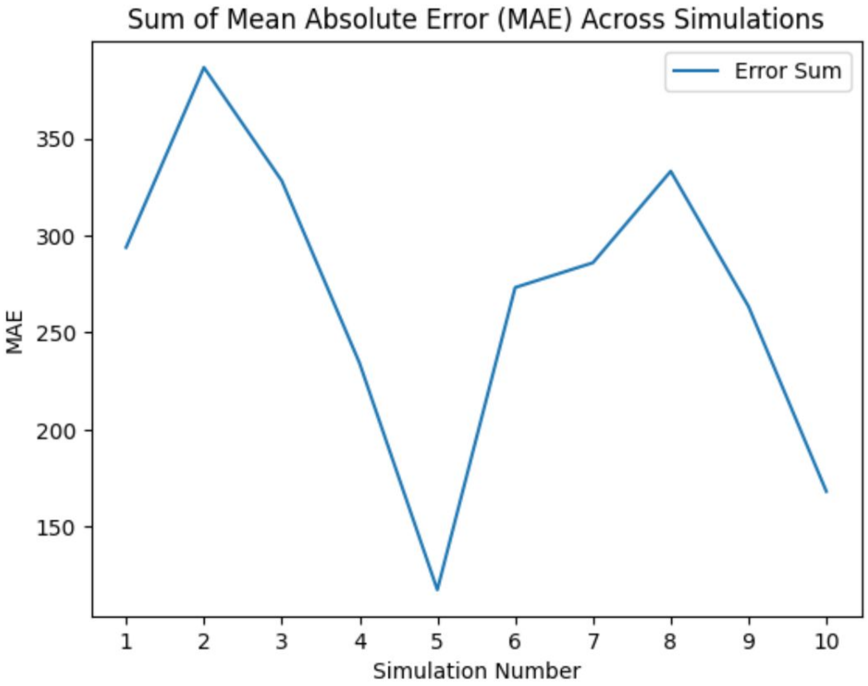
MAE: ode45 vs NeuroDiffEq

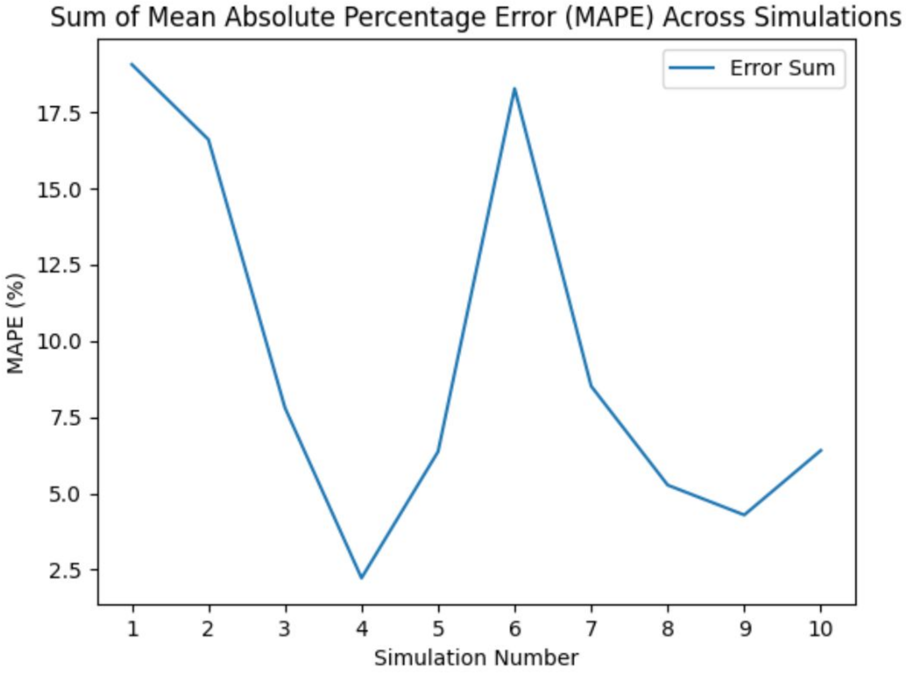


MAPE: ode45 vs NeuroDiffEq

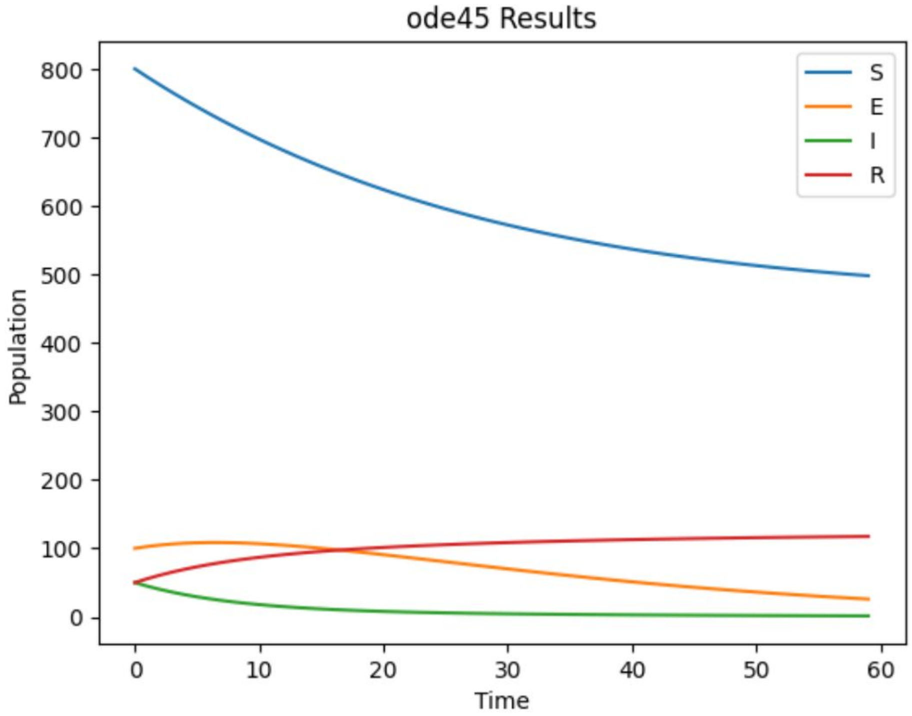


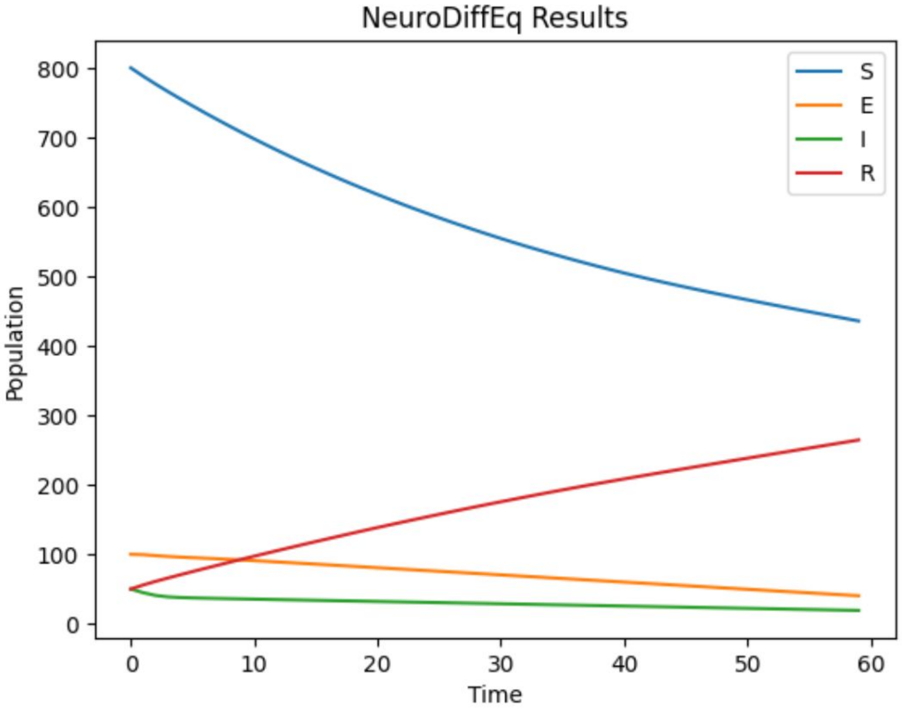
Summing Errors: ode45 vs NeuroDiffEq



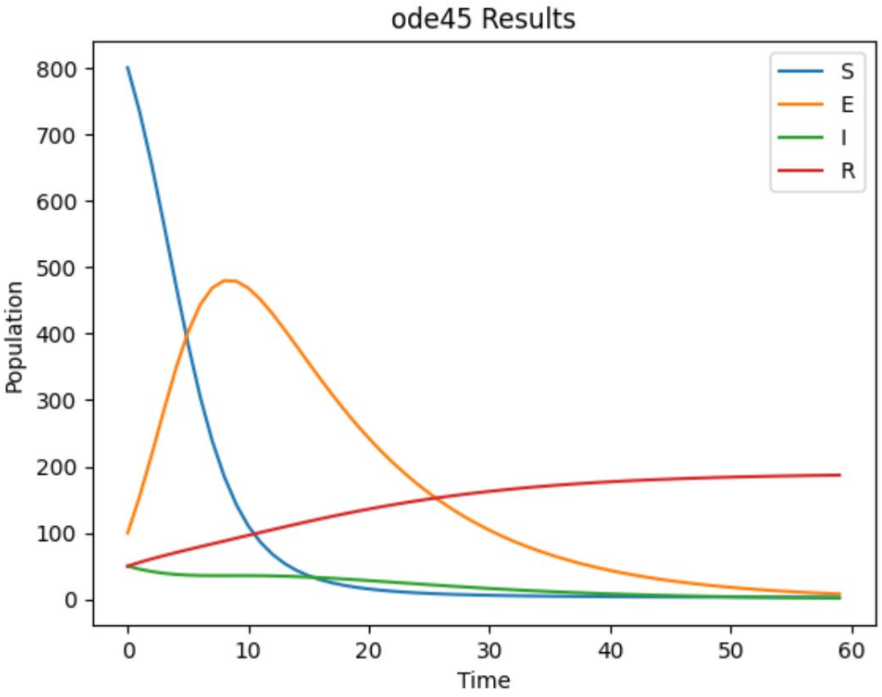


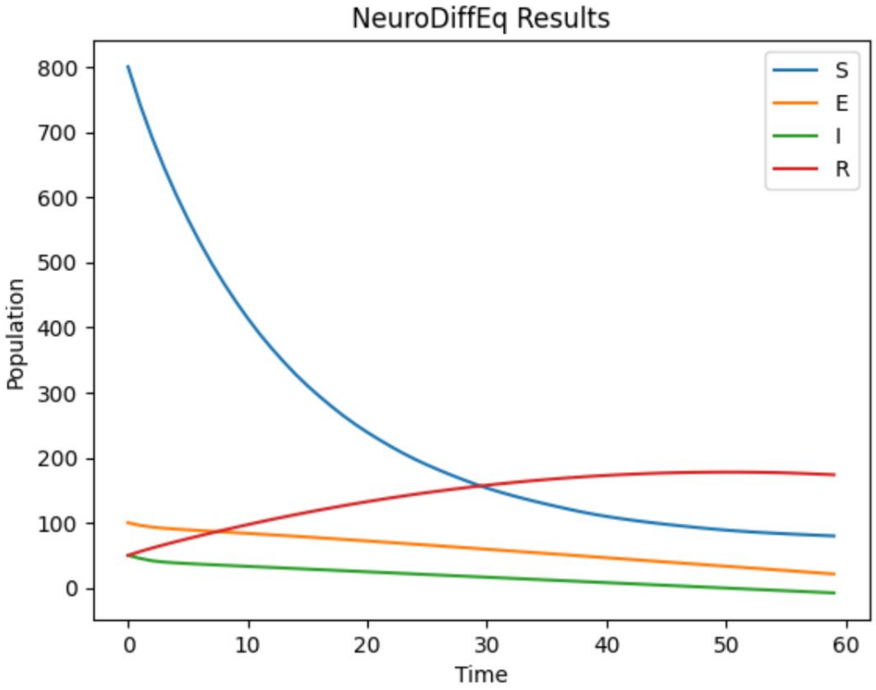
Good? Simulation 5 Comparison





Bad? Simulation 6 Comparison





Simulations with Higher Beta Perform Worse

Simulation #	N	S_0	E_0	I_0	R_0	beta	mu	sigma	gamma	Explanation
0	1000	800	100	50	50	0.25	0.075	0.09	0.12	Ground truth
1	1000	800	100	50	50	0.5	0.05	0.09	0.12	Varying beta / low mu
2	1000	800	100	50	50	0.4	0.05	0.09	0.12	Varying beta / low mu
3	1000	800	100	50	50	0.3	0.05	0.09	0.12	Varying beta / low mu
4	1000	800	100	50	50	0.2	0.05	0.09	0.12	Varying beta / low mu
5	1000	800	100	50	50	0.1	0.05	0.09	0.12	Varying beta / low mu
6	1000	800	100	50	50	0.5	0.1	0.09	0.12	Varying beta / high mu
7	1000	800	100	50	50	0.4	0.1	0.09	0.12	Varying beta / high mu
8	1000	800	100	50	50	0.3	0.1	0.09	0.12	Varying beta / high mu
9	1000	800	100	50	50	0.2	0.1	0.09	0.12	Varying beta / high mu
10	1000	800	100	50	50	0.1	0.1	0.09	0.12	Varying beta / high mu

Parameter Estimation Method

$$L(\boldsymbol{\theta}; \mathbf{I}, \mathbf{R}) = \frac{1}{T} \sum_{t=1}^{T} \left[\left(\log(\widehat{I}_t + p) - \log(I_t + p) \right)^2 + \left(\log(\widehat{R}_t + p) - \log(R_t + p) \right)^2 \right]$$

$$\widehat{\boldsymbol{\theta}} = (\widehat{\beta}, \widehat{\sigma}, \widehat{\gamma}, \widehat{\mu}) = \operatorname{argmin}_{\boldsymbol{\theta}} L(\boldsymbol{\theta}; \mathbf{I}, \mathbf{R})$$

- I_t and R_t denote the reported simulation numbers
- $\widehat{I}_t$ and $\widehat{R}_t$ are described as differentiable functions of the parameters. We only need to supply the initial conditions.
- Apply different optimizers onto the loss function under some constraints*
  - L-BFGS-B
  - Powell

Parameter Bounds

$$\widehat{\boldsymbol{\theta}} = (\widehat{\beta}, \widehat{\sigma}, \widehat{\gamma}, \widehat{\mu})$$

- β : contact rate between the susceptible and the E and I groups
  - (0 , 1)
- σ : ratio of cases in E that are either confirmed as infectious or dead/recovered without confirmation
  - (0.01, 1)
- γ : transition rate between I and R
  - **(0.05 , 0.3)**
  - assuming it takes 3 to 20 days to recover
- μ : discovery rate of the infected cases
  - (0, 1)

Parameter Estimation Results

	Neural ODE Solver			Traditional ODE Solver		
Simulation #	Estimated R_0	Diff of estimated and simulation specific R_0	Diff of estimated and ground truth R_0= 2.9	Estimated R_0	Diff of estimated and simulation specific R_0	Diff of estimated and ground truth R_0
1	3.24	-2.52	0.3	2.43	-3.334	-0.5
2	2.28	-2.33	-0.7	4.59	-0.021	1.7
3	1.59	-1.87	-1.3	3.46	0.002	0.5
4	2.12	-0.19	-0.8	2.31	0.004	-0.6
5	1.48	0.33	-1.5	1.15	-0.003	-1.8
6	2.85	-3.12	-0.1	2.51	-3.462	-0.4
7	2.89	-1.89	0.0	4.76	-0.018	1.8
8	1.60	-1.98	-1.3	3.58	-0.003	0.6
9	1.99	-0.40	-0.9	2.39	0.001	-0.5
10	1.91	0.72	-1.0	1.19	-0.004	-1.7

Table: estimated R_0 and bias. Using a **more informed initial parameter** (0.15, 0.075, 0.05, 0.1) for L-BFGS-B optimizer.

Parameter Estimation Results

	Neural ODE Solver			Traditional ODE Solver		
Simulation #	Estimated R_0	Diff of estimated and simulation specific R_0	Diff of estimated and ground truth R_0=2.9	Estimated R_0	Diff of estimated and simulation specific R_0	Diff of estimated and ground truth R_0
1	3.24	-2.52	0.3	2.43	-3.33	-0.5
2	2.28	-2.33	-0.7	4.59	-0.02	1.7
3	1.59	-1.87	-1.3	3.46	0.00	0.5
4	2.12	-0.19	-0.8	2.31	0.00	-0.6
5	0.77	-0.38	-2.2	1.16	0.01	-1.8
6	2.85	-3.12	-0.1	2.51	-3.46	-0.4
7	2.89	-1.89	0.0	4.76	-0.02	1.8
8	1.60	-1.98	-1.3	3.58	0.00	0.6
9	1.99	-0.40	-0.9	2.39	0.00	-0.5
10	1.91	0.72	-1.0	38.12	36.93	35.2

Table: estimated R_0 and bias. Using a **more uninformed initial parameter** (0.5, 0.5, 0.5, 0.05) for L-BFGS-B optimizer.

Discussion

- Parameter estimates using values from Neural ODE solvers weren’t as accurate as the traditional ODE solver, but were able to give R_0 ranges that are more realistic
- Uninformed initial parameter estimates might lead to unrealistic results
    - Nonconvex function so optimizer might’ve got stuck at local optimum

Appendix

Compute R_0 from Next Generation Matrix

Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^T$ denote the number of infected individuals in compartment S, E, I, R.

The ODE system can now be expressed as $d\boldsymbol{x}/dt = F(\boldsymbol{x}) - V(\boldsymbol{x})$ with $F(\mathbf{x}) = \begin{bmatrix} 0 \\ \frac{\beta(x_2+x_3)x_1}{N} \\ 0 \\ 0 \end{bmatrix}, \quad V(\mathbf{x}) = \begin{bmatrix} \frac{\beta(x_2+x_3)x_1}{N} \\ \sigma x_2 \\ \gamma x_3 - \mu\sigma x_2 \\ -\gamma x_3 \end{bmatrix}$

Let $\mathbf{F}$ and $\boldsymbol{V}$ be the partial Jacobian matrices of functions $F(\boldsymbol{x})$ and $V(\boldsymbol{x})$ wrt. x_i, i = 1, 2, 3, 4.

$$\mathbf{F} = \begin{bmatrix} \frac{\partial F_2(\mathbf{x}^*)}{\partial x_2} & \frac{\partial F_2(\mathbf{x}^*)}{\partial x_3} \\ \frac{\partial F_3(\mathbf{x}^*)}{\partial x_2} & \frac{\partial F_3(\mathbf{x}^*)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \beta & \beta \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \frac{\partial V_2(\mathbf{x}^*)}{\partial x_2} & \frac{\partial V_2(\mathbf{x}^*)}{\partial x_3} \\ \frac{\partial V_3(\mathbf{x}^*)}{\partial x_2} & \frac{\partial V_3(\mathbf{x}^*)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ -\mu\sigma & \gamma \end{bmatrix}$$

Then the next-generation matrix $\boldsymbol{G} = \mathbf{F}\boldsymbol{V}^{-1}$ can be computed as $\mathbf{G} = \mathbf{F}\mathbf{V}^{-1} = \begin{bmatrix} \frac{\beta}{\sigma} + \frac{\beta\mu}{\gamma} & \frac{\beta}{\gamma} \\ 0 & 0 \end{bmatrix}$

R_0 is given by the largest eigenvalue of next generation matrix $\boldsymbol{G}$:

$$\mathcal{R}_0 = \frac{\beta}{\sigma} + \frac{\beta\mu}{\gamma}$$

Heffernan et al. (2005)

Variation in beta leads to variation in simulated R_0

Simulation #	beta	mu	sigma	gamma	Explanation	simulation specific R0	diff from ground truth R0
0	0.25	0.075	0.09	0.12	Ground truth	2.9	
1	0.5	0.05	0.09	0.12	Varying beta / low mu	5.8	2.8
2	0.4	0.05	0.09	0.12	Varying beta / low mu	4.6	1.7
3	0.3	0.05	0.09	0.12	Varying beta / low mu	3.5	0.5
4	0.2	0.05	0.09	0.12	Varying beta / low mu	2.3	-0.6
5	0.1	0.05	0.09	0.12	Varying beta / low mu	1.2	-1.8
6	0.5	0.1	0.09	0.12	Varying beta / high mu	6.0	3.0
7	0.4	0.1	0.09	0.12	Varying beta / high mu	4.8	1.8
8	0.3	0.1	0.09	0.12	Varying beta / high mu	3.6	0.6
9	0.2	0.1	0.09	0.12	Varying beta / high mu	2.4	-0.5
10	0.1	0.1	0.09	0.12	Varying beta / high mu	1.2	-1.7

Powell seems to underestimate R_0 more

		Traditional ODE Solver		Neural ODE Solver	
Simulation #	simulation specific R0	Powell Estimated R_0	L-BFGS-B Estimated R_0	Powell Estimated R_0	L-BFGS-B Estimated R_0
1	5.8	1.02	3.24	3.25	2.43
2	4.6	5.27	2.28	2.29	4.59
3	3.5	3.49	1.59	1.59	3.46
4	2.3	2.31	2.12	0.36	2.31
5	1.2	1.04	0.77	1.52	1.16
6	6.0	0.28	2.85	0.002	2.51
7	4.8	0.34	2.89	0.003	4.76
8	3.6	0.45	1.60	1.93	3.58
9	2.4	1.01	1.99	2.00	2.39
10	1.2	0.99	1.91	1.91	38.12

Table: Using a **more uninformed initial parameter** (0.5, 0.5, 0.5, 0.05).

Similar performances but L-BFGS-B slight upper hand

		Traditional ODE Solver		Neural ODE Solver	
Simulation #	simulation specific R0	Powell Estimated R_0	L-BFGS-B Estimated R_0	Powell Estimated R_0	L-BFGS-B Estimated R_0
1	5.8	2.62	2.43	3.28	3.24
2	4.6	4.59	4.59	1.64	2.28
3	3.5	1.03	3.46	1.39	1.59
4	2.3	2.31	2.31	1.23	2.12
5	1.2	0.93	1.15	1.57	1.48
6	6.0	2.51	2.51	2.83	2.85
7	4.8	4.76	4.76	2.89	2.89
8	3.6	3.58	3.58	1.24	1.60
9	2.4	7.84	2.39	1.50	1.99
10	1.2	1.19	1.19	1.48	1.91

Table: Using a **more informed initial parameter** (0.15, 0.075, 0.05, 0.1).