VP150-RC3

Jiahe Huang, *sevenkishuang@sjtu.edu.cn*, June 28th 2023

Work, Power and Energy

Work

Definition

- **Differential Form:** $\delta W = \vec{F} \circ \mathrm{d}\vec{r}$
- **Integral Form:** $W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ \mathrm{d}\vec{r}$

> How to calculate?
>
> **We use line integral.**
>
> Let's consider a 3D environment:
>
> 1. Find the relationship between x, y, z and the trajectory.
> 2. Use one variable to represent the others (*e.g.* use x to represent y, z).
> 3. Take derivatives to obtain $\mathrm{d}y = g(x)\mathrm{d}x, \mathrm{d}z = h(x)\mathrm{d}x$.
> 4. $W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ \mathrm{d}\vec{r} = \int_{x_A}^{x_B} (F_x(x), F_y(x), F_z(x)) \circ (1, g(x), h(x))\mathrm{d}x$
>
> More information about line integral: https://en.wikipedia.org/wiki/Line_integral (Wikipedia)
>
> Also refer to VV255 & VV285!

Power

Definition

- **Average Power:** $P_{av} = \frac{W}{\Delta t}$
- **Instantaneous Power:** $P = \frac{\delta W}{\mathrm{d}t} = \vec{F} \circ \vec{v}$

Energy

Kinetic Energy

E_k

$$K = \frac{1}{2}mv^2$$

Potential Energy

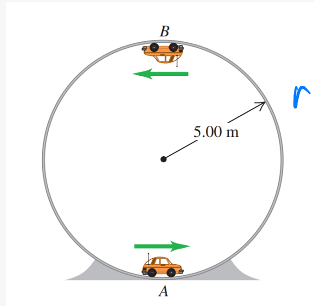
- **Gravitational potential energy:** (close to the surface of Earth) $U_{grav} = mgh$
- **Elastic potential energy:** (following the Hooke's Law) $U_{el} = \frac{1}{2}kx^2$

Work - Kinetic Energy Theorem

Definition

Work done by the net force on a particle is equal to the change in the particle's kinetic energy.

$$W = \Delta K$$

> Let the mass of the toy car to be $1kg$ and take the gravity acceleration as $g = 10m/s^2$. There is only gravity and normal force acting on the car. To prevent it from dropping down at point B, what's the minimum velocity of the car at point A?
>
> $\frac{v_B^2}{r} = g \Rightarrow v_B = \sqrt{gr}$
>
> $mg \cdot 2r = \frac{1}{2}m(v_A^2 - v_B^2)$
>
> $\Rightarrow v_A^2 = v_B^2 + 4gr$
>
> $v_A = \sqrt{5gr}$
>
> 
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>
> $r = 5m$

Conservation of Mechanical Energy

Definition

If only gravitational and elastic forces are acting, then

$$E = U_{grav} + U_{el} + K = const$$

If there are other forces acting on the particle, then

$$W_{other} = \Delta E$$

> m
>
> Δx
>
> A 2.00kg block is pushed against a spring with negligible mass and force constant $k = 400N/m$, compressing it $0.220m$. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope $37.0°$.
>
> 1. What is the speed of the block as it slides along the horizontal surface after having left the spring?
> 2. How far does the block travel up the incline before starting to slide back down?
>
> 1. $\frac{1}{2}k\Delta x^2 = \frac{1}{2}mv^2$
>
> 2. $\frac{1}{2}k\Delta x^2 = mgh$
>
> $= mg \cdot 0.6s$
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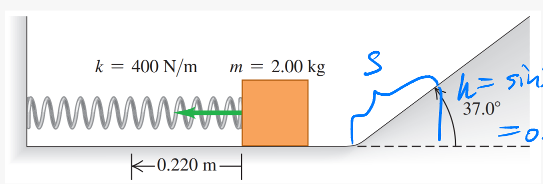
> 
>
>
> S
>
> $h = \sin 37° \cdot s$
>
> $= 0.6s$

Conservative Force

Conservative Force

Definition & Property

The work done by the conservative force is **independent of the path** of the body and depends on only the starting and ending points.

Properties

- $\oint_{\Gamma} \vec{F} \circ d\vec{r} = 0$
- $\int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} = W_{AB} = U_A - U_B$ ☆

A B

Non-conservative Force

e.g. Friction, Air-drag, Magnetic Force

Formulas

- **Conservative Force in 1D:** $F(x) = -\frac{\mathrm{d}U}{\mathrm{d}x}$
- **Conservative Force in 3D:** $\vec{F} = -\mathrm{grad}U = -\nabla U = (-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z})$

> For a hard, non-linear spring with $F = -3000(x + 160x^3)[N]$, what is the relationship between the Elastic potential energy U_{el} and the position x?
>
> $x=0 \to x$ $\quad U_{el}(x) - U_{el}(0) = -\int_0^x F\,dx = -\int_0^x -3000(\tau + 160\tau^3)\,d\tau = 1500x^2 + 120000x^4$

$U_{el}(x) = 1500x^2 + 120000x^4 + U_{el}(0)$

Conservation of Total Energy

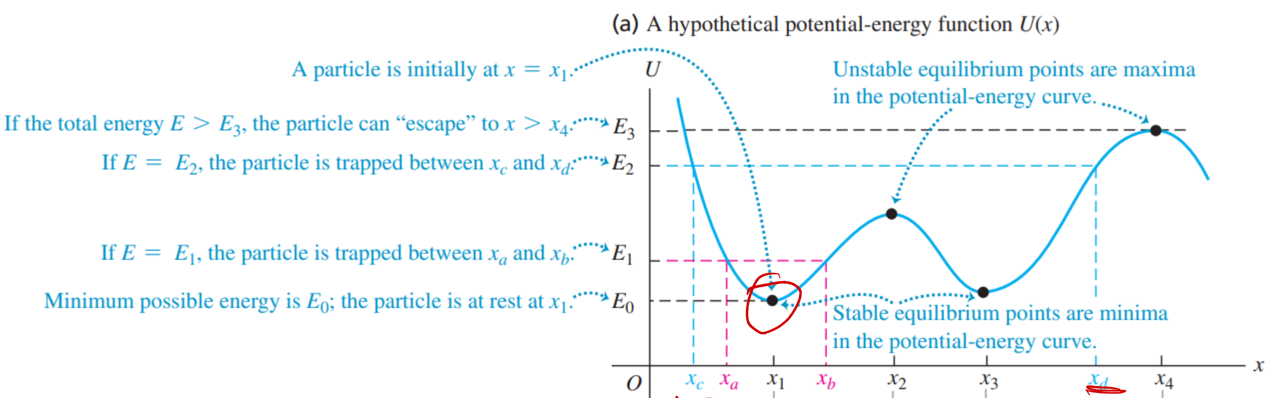
Definition

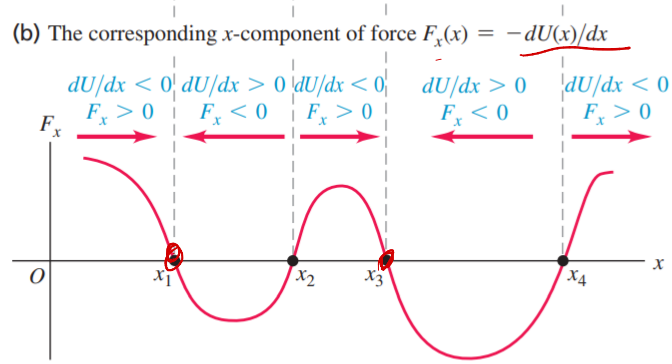
$$\Delta K + \Delta U + \Delta U_{int} = 0$$

where U_{int} is the internal energy.

Interpretation: Energy can be transformed between its different forms but the net change is always zero.

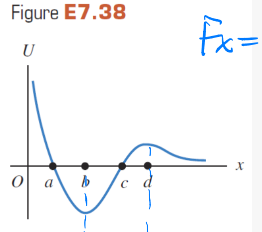
Energy Diagram





stable : x_1, x_3

unstable : x_2, x_4

> **7.38** • A marble moves along the x-axis. The potential-energy function is shown in Fig. E7.38. (a) At which of the labeled x-coordinates is the force on the marble zero? (b) Which of the labeled x-coordinates is a position of stable equilibrium? (c) Which of the labeled x-coordinates is a position of unstable equilibrium?
>
> Figure **E7.38**
>
> 
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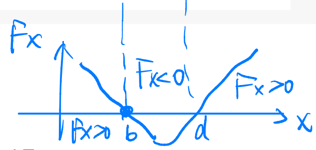
>
> $F_x = -\frac{du}{dx}$

b & d

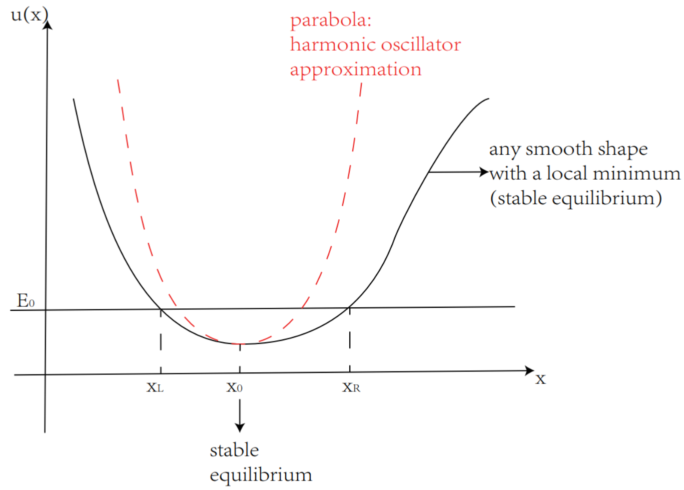
b

d



Harmonic Approximation

Energy Diagram



Approximation Equation

Take $U(x_0)$ as 0, and x_0 as 0:

$$\begin{cases} U(x) = \frac{1}{2}U''(0)x^2 \\ F(x) = -U''(0)x \end{cases}$$

- **Motion equation:** $x = A\cos(\omega_0 t + \varphi)$, where $\omega_0 = \sqrt{\frac{U''(0)}{m}}$

This method could be used iff the oscillation about the equilibrium point is small.

Momentum

Momentum

Expression

$\vec{p} = m\vec{v}$ (Unit: $kg \cdot m/s$)

Derived Formulas

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \text{ (Follows from Newton's II law)}$$
$$p_2 - p_1 = \int_{t_1}^{t_2} F\mathrm{d}t \text{ (Momentum-Impulse Theorem)}$$

Conservation of Momentum

Formula

$$\vec{p} = \Sigma\vec{p}_i = const$$
$$\textbf{Requirements:} \quad \Sigma\vec{F}_i = 0$$

Explanation

No external force on the **system** $\Rightarrow$ No change in the momentum of the **system**

- The momentum of part of the system might change.
- Remember that momentum is a vector, which can be decomposed.

Collision

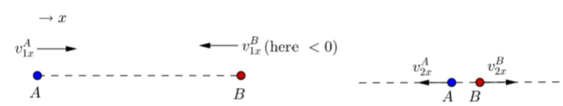
Classification

- Elastic collision: $K_{final} = K_{initial}$
- Inelastic collision: $K_{final} < K_{initial}$
  - Completely inelastic collision: stick to each other.
- Superelastic collision: $K_{final} > K_{initial}$

> How to solve problems?
>
> - Elastic collision: Conservation of Kinetic Energy and Momentum.
> - Completely inelastic collision: Conservation of Momentum.

Elastic Collision Case

General 1D case (head-on collision) $\to$ before/after the collision the velocities of both particles are colinear



1. Momentum conserved

$$m_A v_{1x}^A + m_B v_{1x}^B = m_A v_{2x}^A + m_B v_{2x}^B$$

2. Energy conserved

$$\frac{1}{2}m_A(v_{1x}^A)^2 + \frac{1}{2}m_B(v_{1x}^B)^2 = \frac{1}{2}m_A(v_{2x}^A)^2 + \frac{1}{2}m_B(v_{2x}^B)^2$$

solution
$$\boxed{\begin{aligned} v_{2x}^A &= \frac{m_A - m_B}{m_A + m_B}v_{1x}^A + \frac{2m_B}{m_A + m_B}v_{1x}^B \\ v_{2x}^B &= \frac{2m_A}{m_A + m_B}v_{1x}^A + \frac{m_B - m_A}{m_A + m_B}v_{1x}^B \end{aligned}}$$

Center of Mass

Formula

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

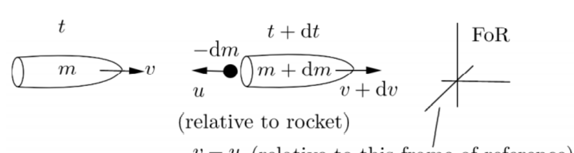
Properties

Σm_i

$$M\vec{v}_{cm} = \sum \vec{p}_i = \vec{p}$$

- The idea of center of mass is to introduce a **hypothetical particle** of mass M that has the same momentum as the whole system.
- No external force $\Rightarrow$ center of mass moves with a constant velocity

Rocket Propulsion



$mv = (m+dm)\cdot(v+dv)$
$\quad + (-dm)\cdot(v-u)$

$\Rightarrow \cancel{mv} = \cancel{mv} + \cancel{dm\cdot v} + m\cdot dv + dm\cdot dv - \cancel{dm\cdot v} + dm\cdot u$

$m\cdot dv = -u\cdot dm$

Results

$$\begin{cases} \vec{a} = -\frac{u}{m(t)}\frac{\mathrm{d}m}{\mathrm{d}t} \\ \vec{v} = v_0 + u\ln\frac{m_0}{m(t)} \end{cases}$$

Moment of Inertia

Rigid Body

Definition

The distance between any two points of the rigid body remains constant. (No compression, stretch, etc.)

Angular Quantities

- Angular displacement θ
- Angular velocity ω
- Angular acceleration ε β

Comments

- Note that θ, ω, and ε are vectors.
- When deriving the relationship between θ, ω, and ε, compare them with x, v, and a.
- When the axis of rotation is not fixed, $\vec{\varepsilon} \nparallel \vec{\omega}$.

Moment of Inertia

Definition

$$I = \int r_{\perp}^2 \mathrm{d}m$$

Relationship with Kinetic Energy

$$K = \frac{1}{2}I\omega^2$$

Comments

- I is the moment of inertia about the fixed axis of rotation A.
- Moment of inertia depends on the distribution (arrangement) of mass.

> How to find the moment of inertia?
>
> **General calculation:**
>
> An object rotate along z-axis with the density function $\rho(x, y, z)$
>
> $$I = \iiint_V \underbrace{(x^2 + y^2)}_{r_\perp^2} \cdot \rho(x, y, z)dxdydz$$
>
> **Practical calculation steps:**
>
> - Determine the axis. Find out how the body is symmetrical. Construct the equation based on the symmetry.
> - Find out the $\mathrm{d}m$. Do integration.
> - Substitute $\mathrm{d}m$ with m and other quantities.

Theories

Parallel Axis Theorem

A' // A

$$I_{A'} = I_A + mb^2$$

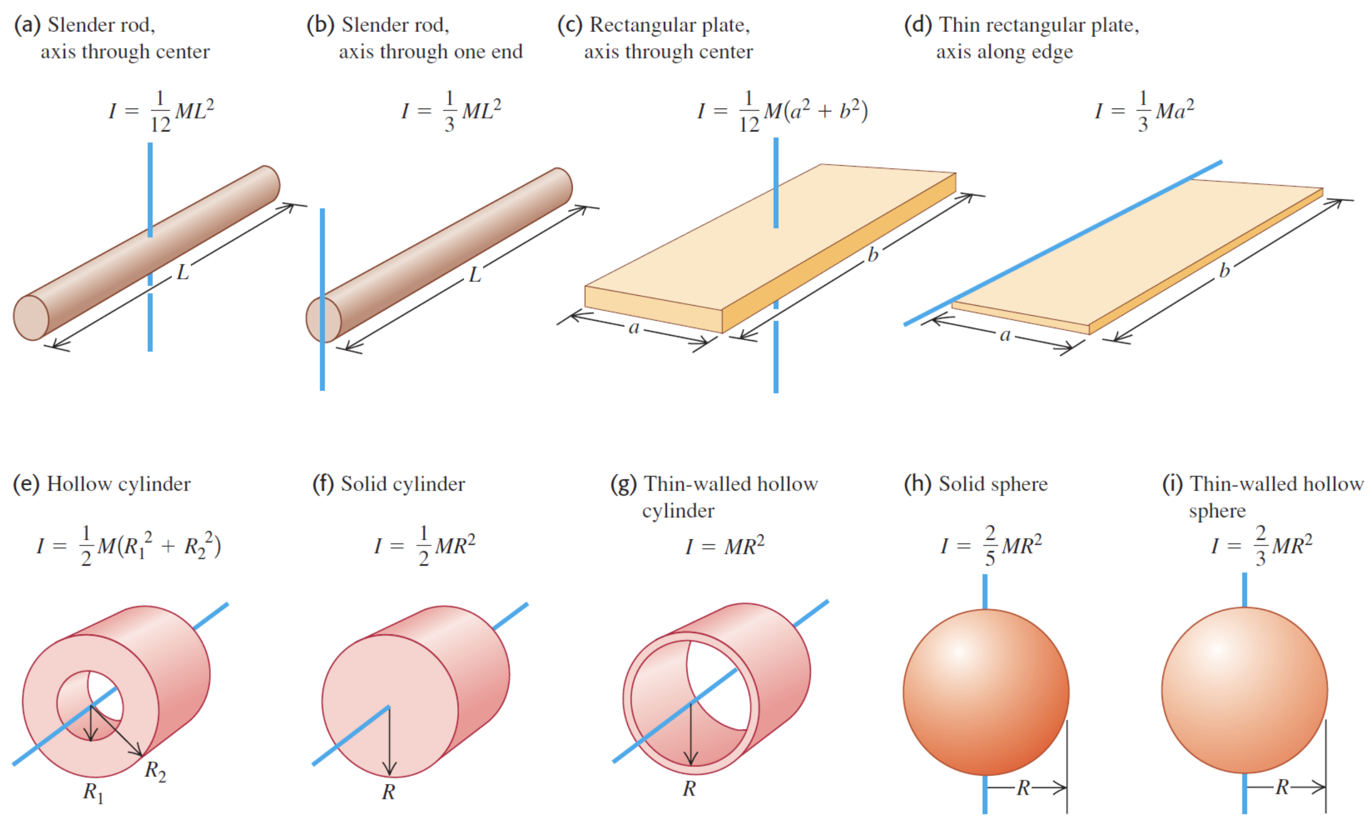
Caution: I_A is the MoI with the axis passing through the center of mass.

Perpendicular Axis Theorem

$$I_Z = I_X + I_Y$$

Caution: The rigid body is only on the plane of XoY.

MoI Table



Reference

1. He Yinghui, 2022SU VP150 RC.
2. Qu Zhemin, 2021SU VP150 RC.
3. Mateusz Krzyzosiak, 2023SU VP150 Slides.