VP150-RC6

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Moment of Inertia

Rigid Body

Definition

The distance between any two points of the rigid body remains constant. (No compression, stretch, etc.)

Angular Quantities

- Angular displacement θ
- Angular velocity ω $x - v - a$
- Angular acceleration ε β

Comments

- Note that θ, ω, and ε are vectors.
- When deriving the relationship between θ, ω, and ε, compare them with x, v, and a.
- When the axis of rotation is not fixed, $\vec{\varepsilon} \nparallel \vec{\omega}$.

Moment of Inertia

Definition

$$I = \int r_\perp^2 \mathrm{d}m \rightarrow \rho \cdot \underline{dxdydz}$$

dV

Relationship with Kinetic Energy

$$K = \frac{1}{2} I \omega^2$$

Comments

- I is the moment of inertia about the fixed axis of rotation A.
- Moment of inertia depends on the distribution (arrangement) of mass.

> How to find the moment of inertia?
>
> **General calculation:**
>
> An object rotate along z-axis with the density function $\rho(x, y, z)$
>
> $$I = \iiint_V (x^2 + y^2) \cdot \rho(x, y, z) dxdydz$$
>
> **Practical calculation steps:**
>
> - Determine the axis. Find out how the body is symmetrical. Construct the equation based on the symmetry.
> - Find out the $\mathrm{d}m$. Do integration.
> - Substitute $\mathrm{d}m$ with m and other quantities.

Theories

Parallel Axis Theorem

$$I_{A'} = I_A + mb^2$$

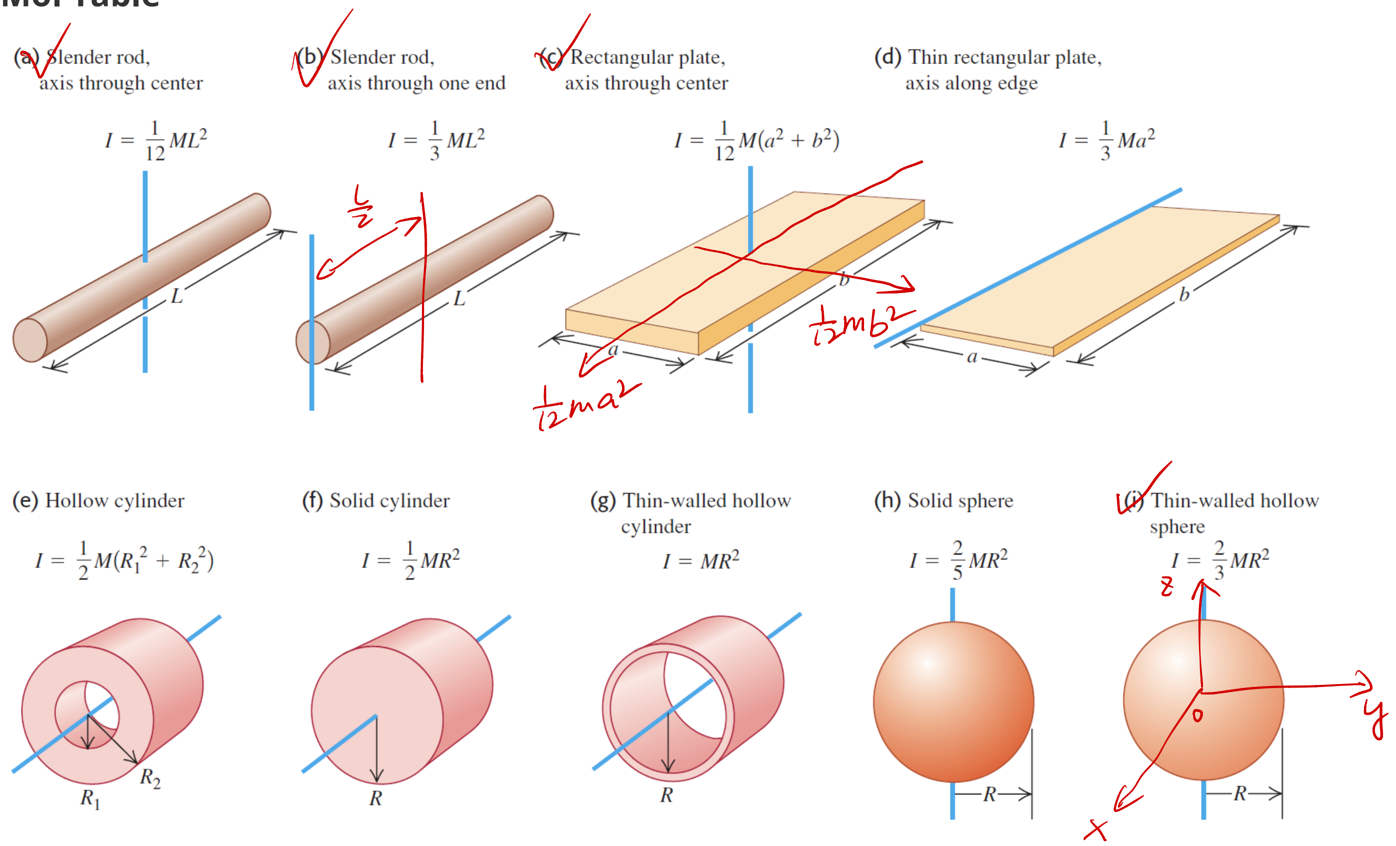
Caution: I_A is the MoI with the axis passing through the center of mass. $A' \parallel A$

Perpendicular Axis Theorem

$$I_Z = I_X + I_Y$$

Caution: The rigid body is only on the plane of XoY.

MoI Table



Try to calculate the MoI listed above (hint: you may use Parallel/Perpendicular Axis Theorem for some MoI):

(a)

$\lambda = \frac{m}{L}$ $\qquad \int_{-\frac{L}{2}}^{\frac{L}{2}} \lambda \cdot dx \cdot x^2 = \left[\frac{1}{3}\lambda x^3\right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{12}mL^2$

(b)

$\frac{1}{12}mL^2 + m\cdot\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$

(c)

(i)

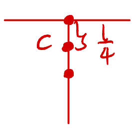
$I = I_z = \sum m_i \cdot (x_i^2 + y_i^2)$ $\quad I = I_x = \sum m_i \cdot (y_i^2 + z_i^2)$ $\quad I = I_y = \sum m_i (x_i^2 + z_i^2)$

$3I = 2\sum m_i \cdot \underbrace{(x_i^2 + y_i^2 + z_i^2)}_{R^2} = 2MR^2 \Rightarrow I = \frac{2}{3}MR^2$

Exercise

Ex.1

Two thin, uniform rods with mass m and length l are symmetrically connected to form a T-square ruler. Each part of the ruler is provided with a rotating shaft perpendicular to the ruler plane and the moment of inertia is I. Find I_{min} and I_{max}.

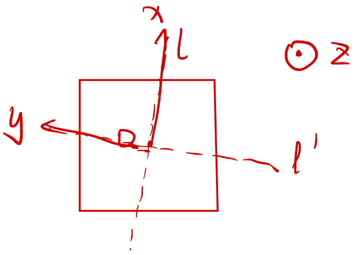


$I_{min} = \left[\frac{1}{12}ml^2 + m\cdot\left(\frac{l}{4}\right)^2\right] + \left[\frac{1}{12}ml^2 + m\cdot\left(\frac{l}{4}\right)^2\right] = \frac{7}{24}ml^2$

$I_{max} = \underbrace{I_c}_{I_{min}} + m\cdot\left(\frac{3}{4}l\right)^2 = \frac{17}{12}ml^2$

Ex.2

A uniform square thin plate has a mass of m and each side length of a. Take the rotating axis through center O on the plane of the plate, and find the moment of inertia of the plate with respect to the axis.



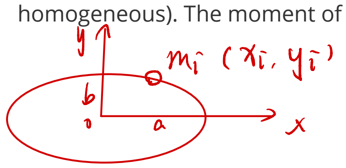
$$I_x = I_y = I$$

$$I_z = \frac{1}{12}m\cdot(a^2+a^2) = \frac{1}{6}ma^2 = I_x + I_y = 2I$$

$$\Rightarrow I = \frac{1}{12}ma^2$$

Ex.3

The semi-major axis length of an elliptical ring is a, the semi-minor axis length is b, and the mass is m (not necessarily homogeneous). The moment of inertia about the major axis is I_a, find the moment of inertia I_b about the minor axis.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \frac{x_i^2}{a^2}+\frac{y_i^2}{b^2}=1$$

$$I_x = I_a = \sum_i m_i\cdot y_i^2 \qquad I_y = I_b = \sum_i m_i\cdot x_i^2$$

$$\frac{I_b}{a^2}+\frac{I_a}{b^2} = \sum_i m_i\cdot\left(\frac{x_i^2}{a^2}+\frac{y_i^2}{b^2}\right) = \sum_i m_i = m$$

$$\Rightarrow I_b = ma^2 - \frac{a^2}{b^2}I_a$$

Reference

1. He Yinghui, 2022SU VP150 RC.
2. Qu Zhemin, 2021SU VP150 RC.
3. Mateusz Krzyzosiak, 2023SU VP150 Slides.