VP150-RC2

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Kinematics in 1D

Position, Velocity, and Acceleration

Average Velocity and Instantaneous Velocity

Since in a 1D system, the position can be determined by a function $x(t)$, the average velocity within Δt can be calculated:

$$v_{av}=\frac{x\left(t_0+\Delta t\right)-x\left(t_0\right)}{\Delta t}$$

When Δt goes to 0 , the instantaneous velocity at t_0 can be derived:

$$v\left(t_0\right)=\lim_{\Delta t\rightarrow 0}\frac{x\left(t_0+\Delta t\right)-x\left(t_0\right)}{\Delta t}=\dot{x}\left(t_0\right)$$

Generally, given a position function $x(t)$, the corresponding velocity function can be expressed as:

$$v_x(t)=\frac{\mathrm{d}x(t)}{\mathrm{d}t}=\dot{x}(t)$$

Average Acceleration and Instantaneous Acceleration

Since velocity represents the rate that the position changes, and acceleration represents the rate that velocity changes, the relationship between acceleration and velocity can be derived similarly.

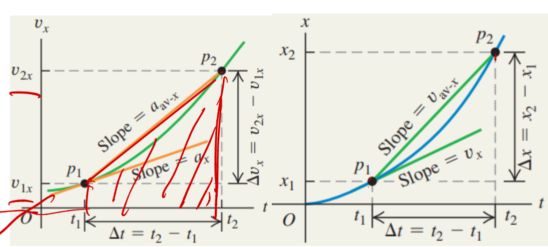
$$a_{av}=\frac{v\left(t_0+\Delta t\right)-v\left(t_0\right)}{\Delta t}$$
$$a_x(t)=\frac{\mathrm{d}v(t)}{\mathrm{d}t}=\dot{v}(t)=\ddot{x}(t)$$

$= \lim_{\Delta t\to 0} a_{av}$

$a-t$, $v-t$, and $x-t$ Figures

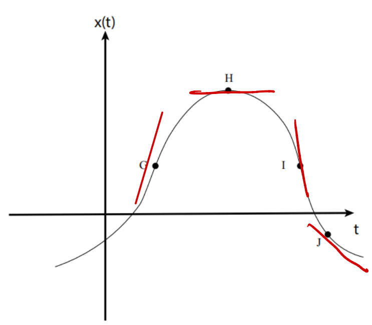
$v-t$**, and** $x-t$ **Figures**

$x=\int v\,dt$



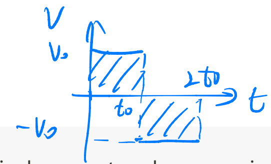
$\frac{x_2-x_1}{\Delta t}$

Analysis of an $a-t$ **figure**



1. $G: v_x > 0, a_x = 0$ **(inflection point)**
2. $H: v_x = 0, a_x < 0$
3. $I: v_x < 0, a_x = 0$ **(inflection point)**
4. $J: v_x < 0, a_x > 0$

$\vec{v}_{av} = \frac{\Delta \vec{x}}{\Delta t} = 0$



$x = \int v dt = 0$.

> 1. Can you have zero displacement and nonzero average velocity? Zero displacement and nonzero instantaneous velocity? Illustrate your answers on an $x - t$ graph.
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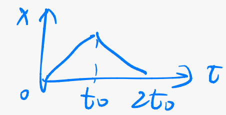
> 
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> 2.
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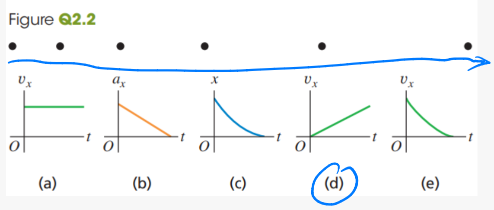
> 
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> **Q2.2** The black dots at the top of **Fig. Q2.2** represent a series of high-speed photographs of an insect flying in a straight line from left to right (in the positive x-direction). Which of the graphs in Fig. Q2.2 most plausibly depicts this insect's motion?
>
> Figure **Q2.2**
>
>

Distance and Speed

Distance

$$\text{distance} = \int_{t_0}^{t_1} |v_{(t)}| dt$$

Speed

The instantaneous speed is the magnitude of the instantaneous velocity.

$$\text{speed} = \frac{\int_{t_0}^{t_1} |v_{(t)}| dt}{t_1 - t_0} = \frac{\text{distance}}{\Delta t}$$

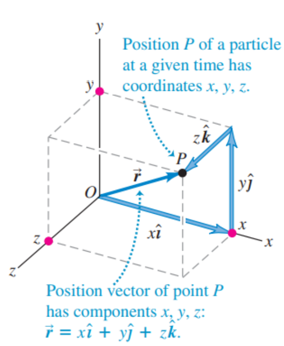
Compare with velocity:

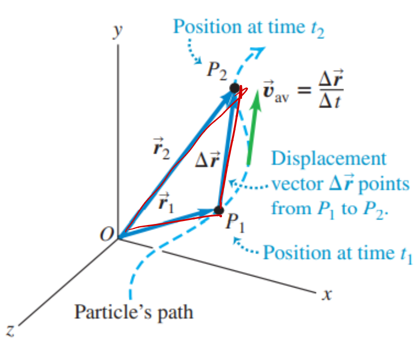
$$\text{average velocity} = \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}$$

Kinematics in 2D&3D

Trajectory

Position Vector and Trajectory

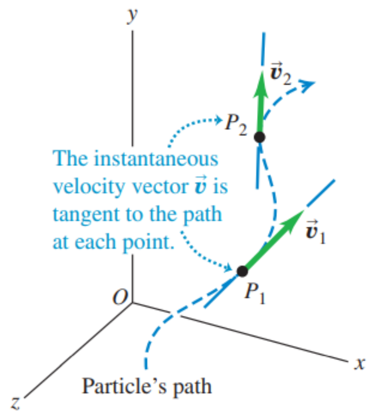




Displacement: $\Delta \vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Velocity

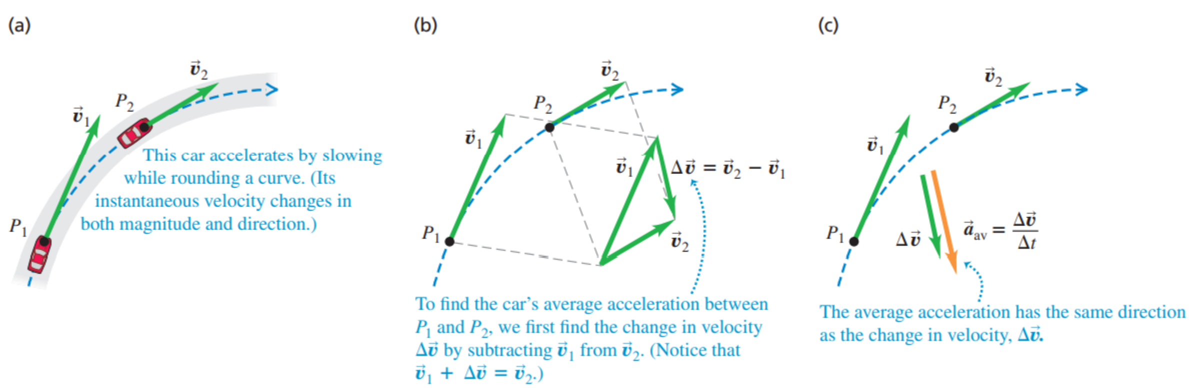
Velocity in 3D



Velocity: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$

Acceleration

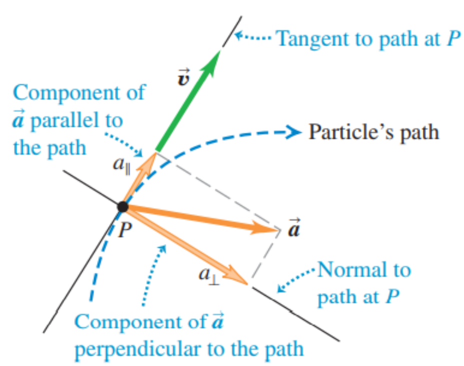
Acceleration in 3D



Acceleration: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = (\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt})$ $= \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k}$

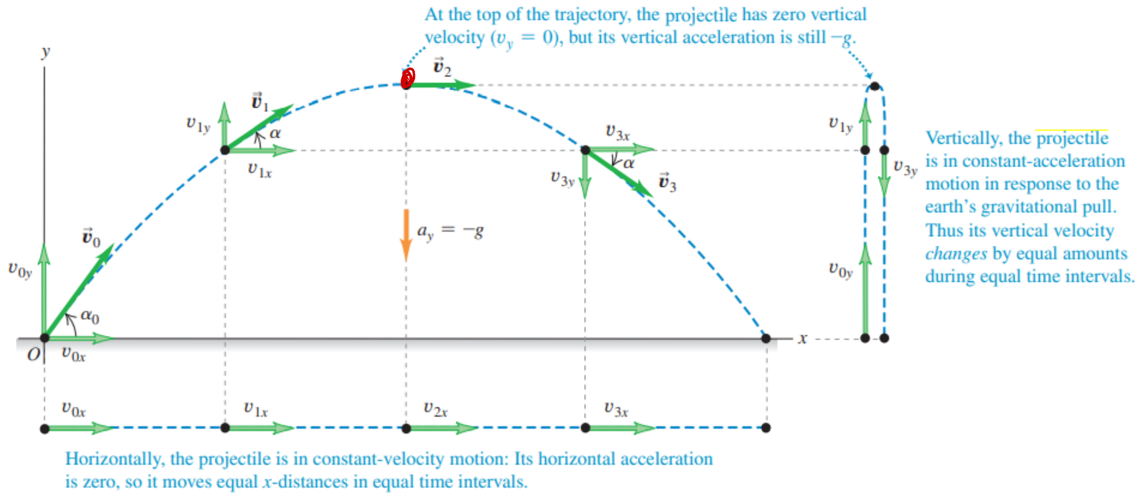
Components of Acceleration

Tangential and Normal Components



- Tangential Component: only changing the magnitude of the velocity.
- Normal Component: only changing the direction of the velocity.

Projectile Motion



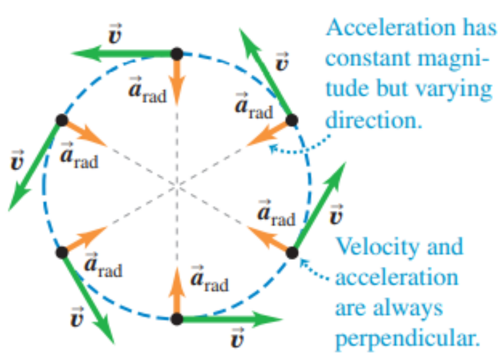
- Velocity:
  $v_x(t) = v_0 \cos\alpha, v_y(t) = v_0 \sin\alpha - gt$
- Displacement:
  $x(t) = v_0 t \cos\alpha, y(t) = v_0 t \sin\alpha - \frac{1}{2}gt^2$
- Maximum height:
  $t_h = \frac{v_0 \sin\alpha}{g}, y(t_h) = \frac{v_0^2 \sin^2\alpha}{2g} = h_{\max}$ $V_y(t)=0$
- Range:
  $x_R = \frac{v_0^2 \sin 2\alpha}{g}$ (**maximum** range for $\alpha = \pi/4$) $y=0$

10m/s   20m/s

> A $124kg$ balloon carrying a $22kg$ basket is descending with a constant downward velocity of $20.0m/s$ (V_0). A $1.0kg$ stone is thrown from the basket with an initial velocity of $15.0m/s$ perpendicular to the path of the descending balloon, as measured relative to a person at rest in the basket. That person sees the stone hit the ground $5.00s$ (t) after it was thrown. Assume that the balloon continues its downward descent with the same constant speed of $20.0m/s$. Take $g = 10\ m/s^2$.
>
> 1. How high is the balloon when the rock is thrown? $H_0 = v_0 t + \frac{1}{2}gt^2 = 225\ m$
> 2. How high is the balloon when the rock hits the ground? $H_1 = \frac{1}{2}gt^2 = 125\ m$
> 3. At the instant the rock hits the ground, how far is it from the basket? $x = 15 \cdot 5 = 75m$ $\sqrt{x^2 + H_1^2} = d$
> 4. Just before the rock hits the ground, find its horizontal and vertical velocity components as measured by an observer at rest on the ground.
>
> d H_1 x

Uniform Circular Motion

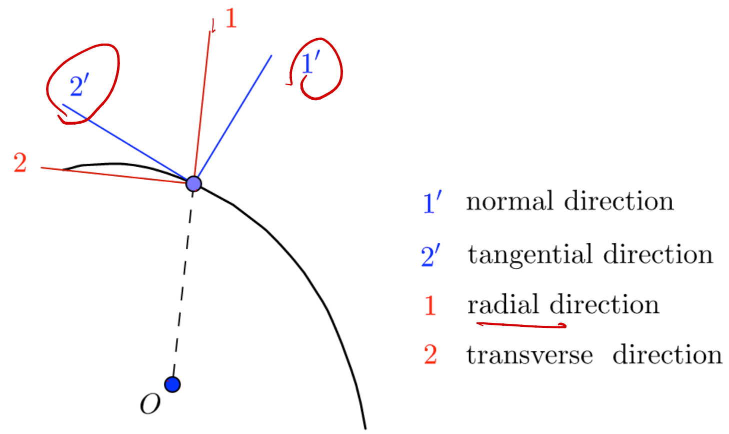
* directions of v & a vary



$$a_{rad} = \frac{v^2}{R} = \omega^2 R = \omega \cdot v$$

Radial & Transverse vs. Normal & Tangential

In general, radial $\neq$ normal, transverse $\neq$ tangential! (Though, it holds for uniform circular motion).



- Radial: tangent to the position vector.
  Transverse: normal to the position vector.
- Normal: normal to the trajectory.
  Tangential: tangent to the trajectory.

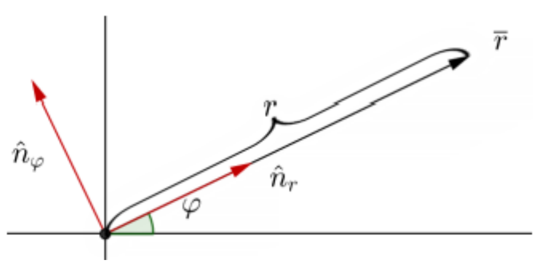
3 perspectives in 2D & 3D motion

1. x, y, z components $\longrightarrow$ Cartesian coordinates
2. radial & transverse components $\longrightarrow$ polar coordinates & cylinder coordinates
3. tangential & normal components $\longrightarrow$ natural coordinates

V_τ V_n

Polar Coordinates

Polar System



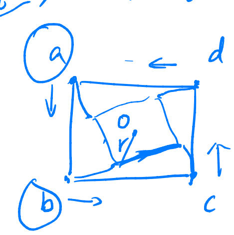
(r, φ)

(r, φ) in polar coordinate is equivalent as $(r\cos\varphi, r\sin\varphi)$ in rectangular coordinates.

- $\dot{\hat{n}}_r = \dot{\varphi}\hat{n}_\varphi$
- $\dot{\hat{n}}_\varphi = -\dot{\varphi}\hat{n}_r$
- $\vec{v} = \dot{r}\hat{n}_r + r\dot{\hat{n}}_r = \dot{r}\hat{n}_r + r\dot{\varphi}\hat{n}_\varphi$
- $\vec{a} = \dot{\vec{v}} = (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$

$\frac{d\varphi}{dt} = \dot{\varphi} = \frac{v}{L - Vt}$

$\varphi = \int_0^t \frac{v}{L - Vt}\, dt = \ln \frac{L}{L - Vt}$



$L'(t) = L - Vt$

$r = \frac{L'}{\sqrt{2}} = \frac{\sqrt{2}}{2}(L - Vt)$

$V^2 = \dot{r}^2 + (r\dot{\varphi})^2$

$\vec{v} = \dot{r}\hat{n}_r + r\dot{\varphi}\hat{n}_\varphi$

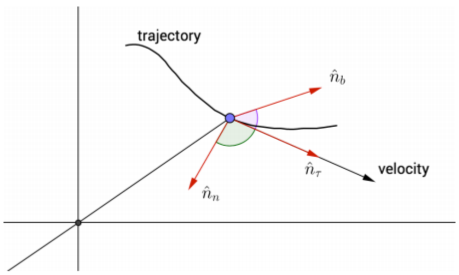
$\dot{r} = -\frac{\sqrt{2}}{2} v.$

$\forall t \quad \vec{v}_b \perp \vec{ab}$

> Four spiders are initially placed at the four corners of a square with side length L. The spiders crawl counter-clockwise at the same speed v and each spider crawls directly toward the next spider at all times. They approach the center of the square along spiral paths. Find
>
> 1. polar coordinates of a spider at any instant of time, assuming the origin is at the center of the square.
> 2. the time after which all spiders meet, $t=\frac{L}{v}$
> 3. the trajectory of a spider in polar coordinates.

Natural Coordinates

Unit Vectors



- $\hat{n}_\tau = \frac{\vec{v}}{v} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$
- $\hat{n}_n = \frac{\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}}{|\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}|}$
- $\hat{n}_b = \hat{n}_\tau \times \hat{n}_n$

Velocity and Acceleration

$$\begin{cases} \vec{v} = v\hat{n}_\tau \\ \vec{a} = \dot{v}\hat{n}_\tau + \frac{v^2}{R_c}\hat{n}_n \end{cases}$$

where $R_c = \frac{v}{|\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}|}$

Reference

1. He Yinghui, 2022SU VP150 RC.
2. Qu Zhemin, 2021SU VP150 RC.
3. Mateusz Krzyzosiak, 2023SU VP150 Slides.