VV156 RC

Gamma Funcion

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RC Overview

Gamma Function

Definition

Let a $>$ 0. The integral

$$\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} \mathrm{d}x$$

is called the gamma function.

Gamma Function

Basic properties

1. $\Gamma(1)=1$
2. $a\Gamma(a)=\Gamma(a+1)$

Try to prove:

$$\Gamma(n+1) = n! \qquad n \in \mathbb{N}$$

(Hint: apply mathematical induction)

Gamma Function

Recall the beta function: Let a, b>0. The integral

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$$

is called the beta function. And

$$B(a,1-a)=\frac{\pi}{\sin a\pi}$$

Optional

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Property

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

You **only** need to remember this conclusion in most cases.

Two methods to prove:

1. Consider the beta function.
2. Double integral (will be mentioned in VV255 & VV256).

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} u^{-\frac{1}{2}} e^{-u} \mathrm{d}u = 2\int_0^{+\infty} e^{-x^2} \mathrm{d}x \qquad (u = x^2)$$

$$= 2\sqrt{\int_0^{+\infty} e^{-x^2} \mathrm{d}x \int_0^{+\infty} e^{-y^2} \mathrm{d}y} = 2\sqrt{\int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r \mathrm{d}r \mathrm{d}\theta}$$

Q&A

References

[1] Cai, Runze. vv256-fall2022-hw05.pdf. 2022.