IIIT Vadodara
WINTER 2019-20
MA202 Numerical Techniques
LAB#3 Approximations and error analysis[1]

1. **Taylor Polynomial Approximations** It is desirable to know the size of error while making approximations in numerical computation. We would like to know the difference $R(x)$ between the original function $f(x)$ and our approximation $\hat{f}(x)$.

Taylor-Maclaurin Series Suppose $f(x)$ has a power series expansion at $x = a$ with radius of convergence $R > 0$, then the series expansion of $f(x)$ takes following form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + ... \tag{1}$$

, where $f^{(n)}(x)$ is the differentiation of $f(x)$ at n^{th} order. Here in equation (1), the coefficient in the expansion of $f(x)$ centered at $x = a$ and expansion is called Taylor series. If $a = 0$, then the expansion is called Maclaurin Series. Let us now consider one such classical Taylor series expansion.

For the following example we will assume that all of the functions involved can be expanded into power series. The function $f(x) = e^x$ satisfies $f^{(n)}(x) = e^x$ for any integer $n \geq 1$ and in particular $f^{(n)}(0) = 1$ for all n and then the Maclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{2}$$

Approximations and truncation errors:

Q. 1: (a) Write a MATLAB script that computes $e^{0.1}$ and compare with actual value by calculating error, i.e., absolute difference of true and approximated value
(b) Observe that the error reduces by retaining more number of terms. Plot the error between your approximation and the exact value for different number of terms, i.e., $n = 1$ to $n = 5$. Make sure to include labels and have different colors for different values of n.
(c) Plot the error using log-log scale for each of the step sizes. choose step sizes as $0.1, 0.05, 0.02, 0.01$.
(d) Explain the behaviour you see in both the graphs. Comment on error with number of terms as well as accuracy.
(e) What is the slope with respect to the accuracy?

Taylor's series approximation for forward difference method to approximate the derivatives of a function

Numerical differentiation for $f'(x)$ for the given function $f(x)$ is defined as,

$$f'(x) = \frac{f(x+h) - f(x)}{h}, \tag{3}$$

where h is the step size.

[1]submission deadline : 31^{st} January 11 PM

Q. 2: (a) Write a MATLAB script to calculate numerical derivative of $tan^{-1}(x)$
(b) Use stepsizes ranging from 10^{-1} to 10^{-16}. Plot the error using log-log scale for each of your step sizes.
(c) Comment on truncation error, machine precision, roundoff error with respect to h.

Home assignment:

Q. 3: Consider a linear system with

$$Ax = b \Leftrightarrow \begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \tag{4}$$

(a) Find solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of equation (5).
(b) Now, consider the system with a slightly perturbed measurement vector $\hat{b}$ such that

$$\hat{A}\hat{x} = \hat{b} \Leftrightarrow \begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix} \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix} = \begin{bmatrix} 2.02 \\ 1.98 \end{bmatrix}, \tag{5}$$

Find solution $\begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix}$ of equation (6).

(c) **Forward error analysis**: Find relative error in observation $\frac{\|\hat{b}-b\|}{\|b\|}$ and relative error in solution $\frac{\|\hat{x}-x\|}{\|x\|}$; and use eq. (4) to comment on ill-posedness of the problem. Let x denote the solution vector of a linear system Ax = b. If we choose a slightly perturbed observation vector $\hat{b}$ then we obtain a different solution vector $\hat{x}$ satisfying $A\hat{x} = \hat{b}$. We would like to know how the relative error in observation influences the relative error in solution. Show that condition number of the matrix A is,

$$cond(A) = \kappa_A = \|A\|\|A^{-1}\| = \frac{\|\hat{x} - x\|}{\|x\|} / \frac{\|\hat{b} - b\|}{\|b\|}, \tag{6}$$

where, / represents division operator. See that the value of κ_A determines (a) how much the relative error in observation can be amplified, i.e., affect the solution, and (b) how much A is close to a singular matrix.

Q. 4: In practice, referring to Q.3, x is not available (since need to be estimated!) and hence impossible to calculate $\|\hat{x} - x\|$ as done in Q.4. In order to check the accuracy of computed (estimated) solution $\hat{x}$, we measure backward error in terms of difference $\|\hat{b} - b\|$. This is possible by considering available (given) observation vector as the true (for reference) while a vector can be (re)constructed using the A and estimated x, i.e., $A\hat{x}$.
(a) To realize this, find the **backward error** $\|b - \hat{b}\|$ for Q.4 and comment.
(b) Show that, in general, $\hat{x}$ is the solution to $Ax = b$ exactly when the backward error $\|b - \hat{b}\|$ is zero.