Time Series
6. Volatility and GARCH models

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Volatility

Non-constant variance

Variance is not a function of the series level

Box-Cox transformations are not useful

Very common in financial data

Fiancial data: Prices or Indexes

Prices of assets:

Apple (AAPL), Coca Cola(KO), Nike (NKE),. . .

BBVA (BBVA.MC), Endesa (ELE.MC), Inditex (ITX.MC),. . .

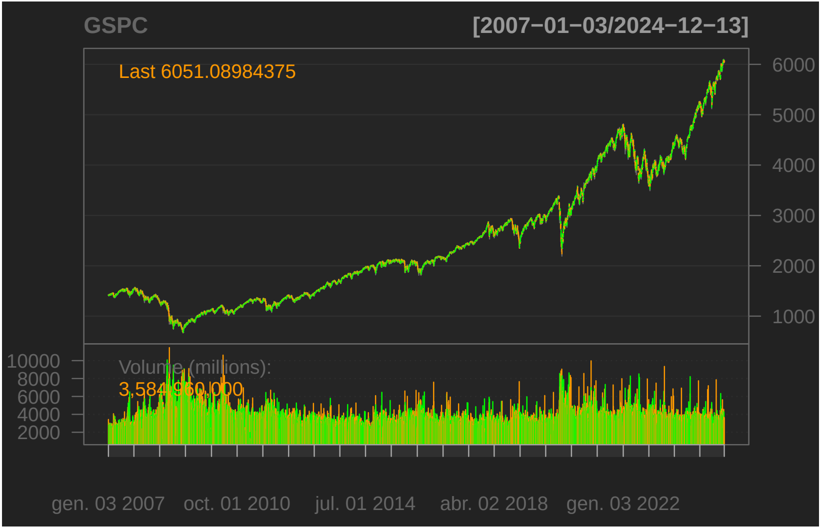
Indexes:

Dow Jones (ˆDJI), S&P500 (ˆGSPC), IBEX35 (ˆIBEX),. . .

Example: Standard & Poor's 500

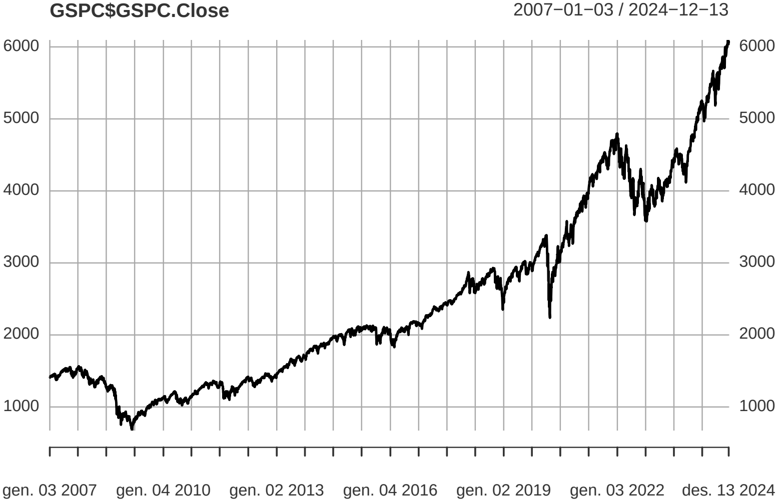
```
##            GSPC.Open GSPC.High GSPC.Low GSPC.Close GSPC.Volume GSPC.Adjusted
## 2007-01-03   1418.03   1429.42  1407.86    1416.60  3429160000       1416.60
## 2007-01-04   1416.60   1421.84  1408.43    1418.34  3004460000       1418.34
## 2007-01-05   1418.34   1418.34  1405.75    1409.71  2919400000       1409.71
## 2007-01-08   1409.26   1414.98  1403.97    1412.84  2763340000       1412.84
## 2007-01-09   1412.84   1415.61  1405.42    1412.11  3038380000       1412.11

##            GSPC.Open GSPC.High GSPC.Low GSPC.Close GSPC.Volume GSPC.Adjusted
## 2024-12-09   6083.01   6088.51  6048.63    6052.85  4556460000       6052.85
## 2024-12-10   6057.59   6065.40  6029.89    6034.91  4048410000       6034.91
## 2024-12-11   6060.15   6092.59  6060.15    6084.19  4269950000       6084.19
## 2024-12-12   6074.29   6079.68  6051.25    6051.25  3678010000       6051.25
## 2024-12-13   6068.17   6078.58  6035.77    6051.09  3584960000       6051.09
```

Example: Standard & Poor's 500



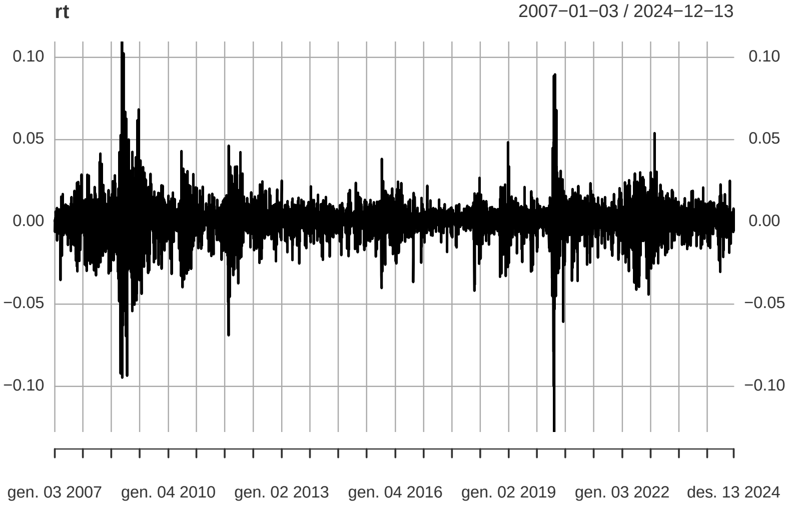
Example: Standard & Poor's 500

Closing Prices:

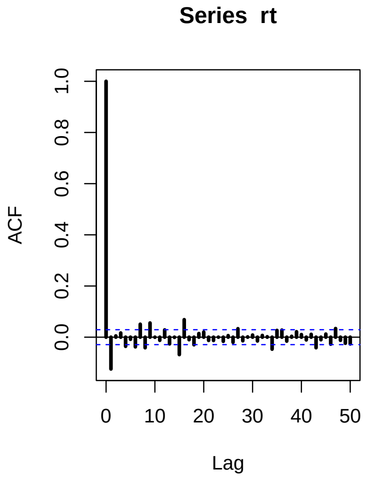


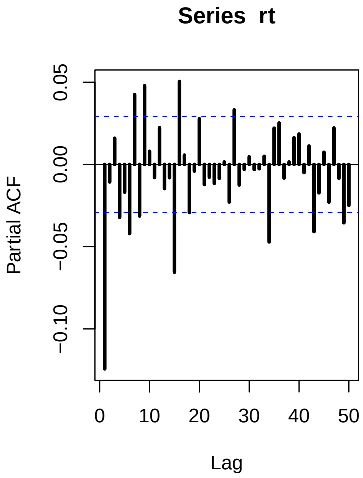
Returns

$$r_t = 100(1-B)\log(I_t) = 100\log(\frac{I_t}{I_{t-1}}) \sim \frac{I_t - I_{t-1}}{I_{t-1}}100$$



Usually, we work with AR(p) with large p

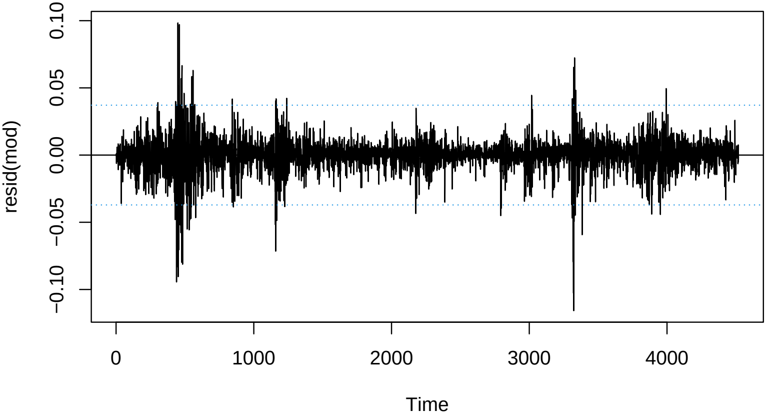




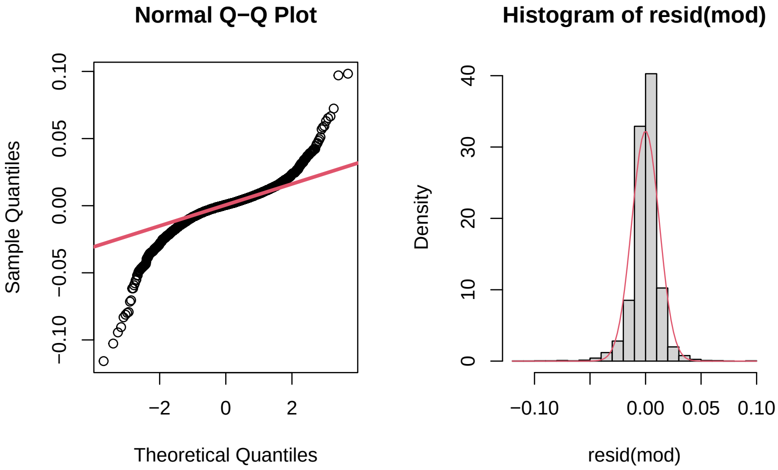
Model for returns

```
##
## Call:
## arima(x = rt, order = c(20, 0, 0))
##
## Coefficients:
##           ar1      ar2     ar3      ar4      ar5      ar6     ar7      ar8
##       -0.1188  -0.0092  0.0139  -0.0367  -0.0187  -0.0323  0.0352  -0.0220
## s.e.   0.0149   0.0150  0.0150   0.0150   0.0150   0.0149  0.0149   0.0149
##          ar9    ar10     ar11    ar12     ar13     ar14     ar15    ar16
##       0.0450  0.0067  -0.0066  0.0228  -0.0186  -0.0156  -0.0583  0.0517
## s.e.  0.0149  0.0150   0.0150  0.0149   0.0149   0.0149   0.0149  0.0149
##         ar17     ar18     ar19    ar20  intercept
##       0.0018  -0.0294  -0.0007  0.0276      3e-04
## s.e.  0.0149   0.0149   0.0150  0.0148      2e-04
##
## sigma^2 estimated as 0.000153:  log likelihood = 13437.13,  aic = -26830.25
```

1. Clusters of volatility



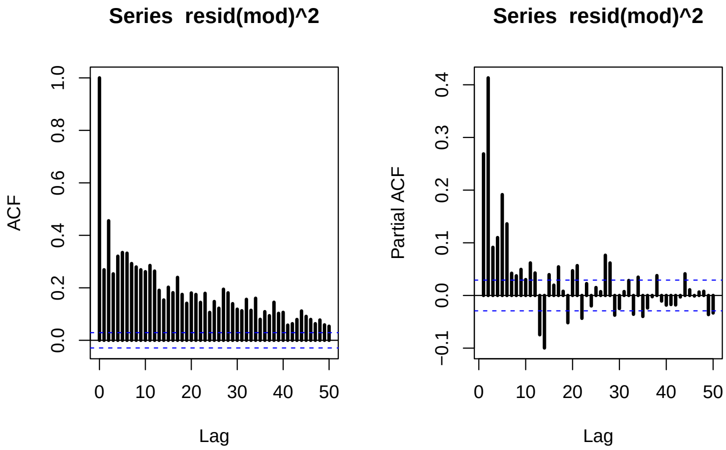
2) Distribution with heavy tails (Excess of Kurtosis)



Diagnosis of volatility

```
##                 resid.mod
## nobs          4519.000000
## NAs              0.000000
## Minimum         -0.115775
## Maximum          0.098307
## 1. Quartile     -0.004708
## 3. Quartile      0.005839
## Mean             0.000000
## Median           0.000603
## Sum              0.000953
## SE Mean          0.000184
## LCL Mean        -0.000361
## UCL Mean         0.000361
## Variance         0.000153
## Stdev            0.012372
## Skewness        -0.781829
## Kurtosis        10.428592
```

3) Autocorrelation in the square of residuals



Model for the variance

Residuals of the model (sometimes, the returns themselves)

$$Z_t = \sigma_t \epsilon_t \quad \epsilon_t \sim WN(0,1)$$

The process σ_t is stationary and reflects the changes in the variance

The two components (σ_t and ϵ_t) are independents

The variance at time t is determined by the previous values (and, possibly, other unobserved variables):

$$Var(Z_t|W_{t-1}) = f(W_{t-1}, \xi_t)$$

These models are called **Conditional Heterocedasticity Models**

Model for the variance

Properties of Conditional Heterocedasticity models:

- First and second moments

– $E(Z_t) = E(\sigma_t)E(\epsilon_t) = 0$

– $V(Z_t) = E(Z_t^2) = E(\sigma_t^2)E(\epsilon_t^2) = E(\sigma_t^2) = \sigma^2$

- Conditional moments

– $E(Z_t|W_{t-1}) = E(\sigma_t|W_{t-1})E(\epsilon_t) = 0$

– $V(Z_t|W_{t-1}) = E(Z_t^2|W_{t-1}) = E(\sigma_t^2|W_{t-1})E(\epsilon_t^2) = \sigma_t^2$

- Covariance

– $E(Z_t Z_{t-k}) = E(\sigma_t \sigma_{t-k})E(\epsilon_t)E(\epsilon_{t-k}) = 0$

Model ARCH(1)

Auto Regresive Conditional Heterocedasticity of order 1

$$E(z_t^2|W_{t-1}) = \sigma_t^2 = \alpha_0 + \alpha_1 Z_{t-1}^2 \quad \alpha_0 > 0 \quad \alpha_1 \geq 0$$

Marginal variance (σ^2):

$$\sigma^2 = E(E(Z_2|W_{t-1})) = \alpha_0 + \alpha_1 E(Z_{t-1}^2) = \alpha_0 + \alpha_1 \sigma^2$$

$$\sigma^2 = \frac{\alpha_0}{1-\alpha_1}$$

The ACF of squares of an ARCH(1) model follows an AR(1) pattern

The kurtosis of the Z_t process will be $K = 3\frac{1-\alpha_1^2}{1-3\alpha_1^2}$

Model ARCH(r)

Auto Regresive Conditional Heterocedasticity of order r

$$E(z_t^2|W_{t-1}) = \sigma_t^2 = \alpha_0 + \alpha_1 Z_{t-1}^2 + ... + \alpha_r Z_{t-r}^2$$

Marginal variance (σ^2):

$$\sigma^2 = E(E(Z_2|W_{t-1})) = \alpha_0 + \sum_{i=1}^{r} \alpha_i E(Z_{t-i}^2) = \alpha_0 + \alpha_1\sigma^2 + ... + \alpha_r\sigma^2$$

$$\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - ... - \alpha_r}$$

Restriction: $\sum_{i=1}^{r} \alpha_i < 1$

Model GARCH(1,1)

General Auto Regresive Conditional Heterocedasticity of orders 1 and 1

$$E(z_t^2|W_{t-1}) = \sigma_t^2 = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Marginal variance (σ^2):

$$\sigma^2 = E(E(Z_2|W_{t-1})) = \alpha_0 + \alpha_1 E(Z_{t-1}^2) + \beta_1 E(\sigma_{t-1}^2) = \alpha_0 + \alpha_1 \sigma^2 + \beta_1 \sigma^2$$

$$\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

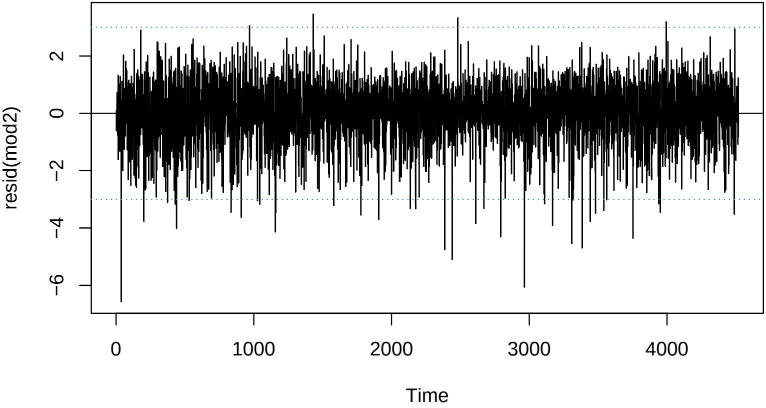
Restriction: $\alpha_1 + \beta_1 < 1$ (the parameter β_1 is called **persistence**)

AR(20)-GARCH(1,1) for S&P500

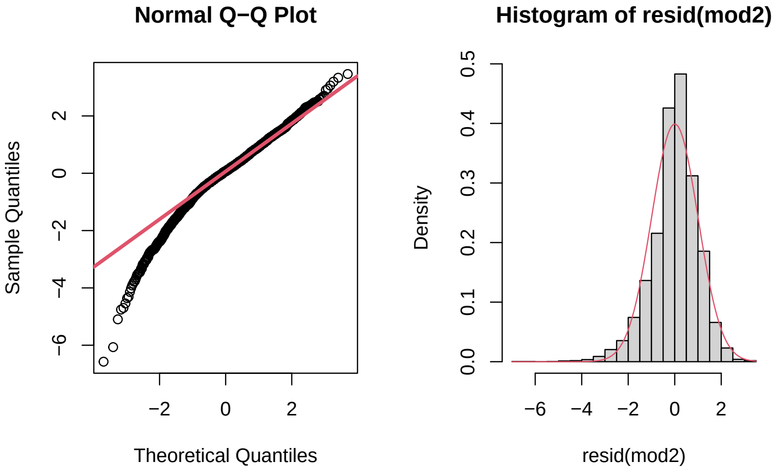
```
##
## Call:
## garch(x = resid(mod), order = c(1, 1))
##
## Model:
## GARCH(1,1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -6.57501 -0.49915  0.07268  0.62717  3.46296
##
## Coefficient(s):
##     Estimate  Std. Error  t value Pr(>|t|)
## a0 2.836e-06   2.504e-07    11.33   <2e-16 ***
## a1 1.337e-01   8.213e-03    16.28   <2e-16 ***
## b1 8.457e-01   8.790e-03    96.21   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Diagnostic Tests:
##  Jarque Bera Test
##
## data:  Residuals
## X-squared = 1051.5, df = 2, p-value < 2.2e-16
##
##
##  Box-Ljung test
##
## data:  Squared.Residuals
## X-squared = 0.78696, df = 1, p-value = 0.375
```

Validation of the standardized residuals

$$\epsilon_t = \frac{Z_t}{\sigma_t}$$



Validation of the standardized residuals



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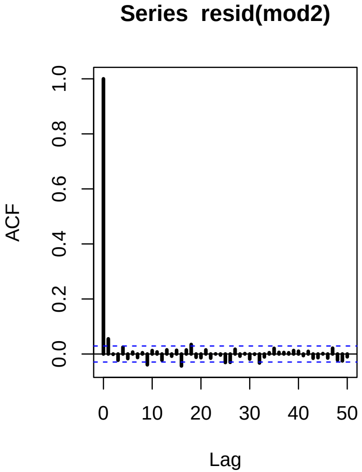

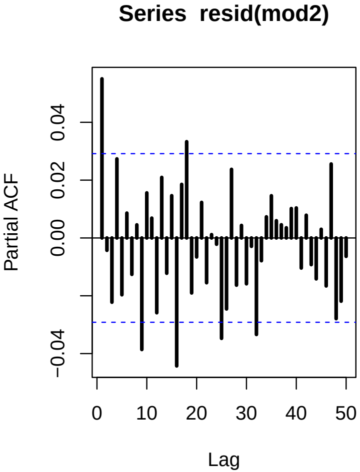

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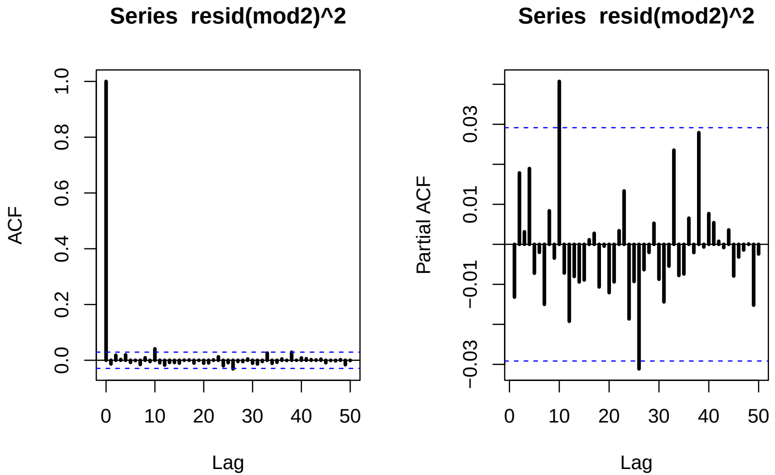

Validation of the standardized residuals