Time Series
3. Non-Stationary Processes

Josep A. Sanchez-Espigares

Department of Statistics and Operations Research
Universitat Politècnica de Catalunya
Barcelona, Spain

Outline

- Stationarity and Invertibility
- ARIMA models: $ARIMA(p, d, q)$
- Seasonal ARIMA models: $SARIMA(p, d, q)(P, D, Q)_s$

ARMA(p,q) models

Under certain conditions of stationarity, the ARMA(p, q) models can be expressed as an AR(∞) or MA(∞)model:

$$(1 - \phi_1 B - ... - \phi_p B^p)X_t = (1 + \theta_1 B + ... + \theta_q B^q)Z_t$$

Expression as an AR(∞):

$$\frac{1 - \phi_1 B - ... - \phi_p B^p}{1 + \theta_1 B + ... + \theta_q B^q} X_t = (1 - \pi_1 B - \pi_2 B^2 - ...)X_t = Z_t$$

Expression as an MA(∞):

$$X_t = \frac{1 + \theta_1 B + ... + \theta_q B^q}{1 - \phi_1 B - ... - \phi_p B^p} z_t = (1 + \psi_1 B + \psi_2 B^2 + ...)Z_t$$

$AR(\infty)$: π-weights

- The π weights come from the power expansion of the rational function in B: $\pi(B) = \frac{\phi_p(B)}{\theta_q(B)}$
- The expansion will converge $(\sum_{i=0}^{\infty} \pi_i^2 < \infty) \Leftrightarrow$ The module of all roots in $\theta_q(B)$ are greater than one
- This means that all complex roots of the characteristic polynomial of the MA part lie outside the unit circle.
- This condition implies that the model is **invertible**

$AR(\infty)$: π-weights

Example: $MA(1) \quad X_t = (1+\theta B)Z_t$

Expression as an $AR(\infty)$:

$$\frac{1}{1+\theta B}X_t = Z_t$$

$$(1 - \theta B + \theta^2 B^2 - ... + (-1)^k \theta^k B^k + ...)X_t = Z_t$$

In this case: $\pi_k = (-1)^k \theta^k$
The model is **invertible** $\Leftrightarrow |B| = |-\frac{1}{\theta}| > 1 \Leftrightarrow |\theta| < 1$

$MA(\infty)$: ψ-weights

- The ψ weights come from the power expansion of the rational function in B: $\psi(B) = \frac{\theta_q(B)}{\phi_p(B)}$
- The expansion will converge $(\sum_{i=0}^{\infty} \psi_i^2 < \infty) \Leftrightarrow$ The module of all roots in $\phi_p(B)$ are greater than one
- This means that all complex roots of the characteristic polynomial of the AR part lie outside the unit circle.
- This condition implies that the model is **causal** (stationary)

$MA(\infty)$: ψ-weights

Example: $AR(1) \quad (1-\phi B)X_t = Z_t$

Expression as an $MA(\infty)$:

$$X_t = \frac{1}{1-\phi B} Z_t$$

$$X_t = (1 + \phi B + \phi^2 B^2 - \ldots + \phi^k B^k + \ldots) Z_t$$

In this case: $\psi_k = \phi^k$

The model is **causal** $\Leftrightarrow |B| = |\frac{1}{\phi}| > 1 \Leftrightarrow |\phi| < 1$

Stationarity and Invertibility

- All $AR(p)$ models are **invertible**:

$$\pi_k = \phi_k \quad k = 1 ... p \Rightarrow \sum_{i=0}^{\infty} \pi_i^2 = \sum_{i=0}^{p} \pi_i^2 < \infty$$

- All $MA(q)$ models are **causal**:

$$\psi_k = \theta_k \quad k = 1 ... q \Rightarrow \sum_{i=0}^{\infty} \psi_i^2 = \sum_{i=0}^{q} \psi_i^2 < \infty$$

- An $ARMA(p, q)$ model will be...:
  - Invertible $\Leftrightarrow$ All roots of $\theta_q(B)$ have module greater than one
  - Causal $\Leftrightarrow$ All roots of $\phi_p(B)$ have module greater than one

Stationarity and Invertibility

For an **causal** and **invertible** $ARMA(p, q)$ model:

- Expression as an $AR(\infty)$ can be truncated when the π_k weight is very small.
  This is useful to calculate the point prediction by using past observations and the π-weights

- Expression as an $MA(\infty)$ can be truncated when the ψ_k weight is very small.
  This is useful to calculate the variance of the prediction by using the ψ-weights

Stationarity and Invertibility

Example: $ARMA(1,2) \quad X_t = 0.8X_{t-1} + Z_t - 3Z_{t-1} + 2Z_{t-2}$

$$(1 - 0.8B)X_t = (1 - 3B + 2B^2)Z_t$$

Roots of the AR-polynomial:
$1 - 0.8B = 0 \Rightarrow B = 1/0.8 = 1.25 > 1$

```
polyroot(c(1,-0.8))
```

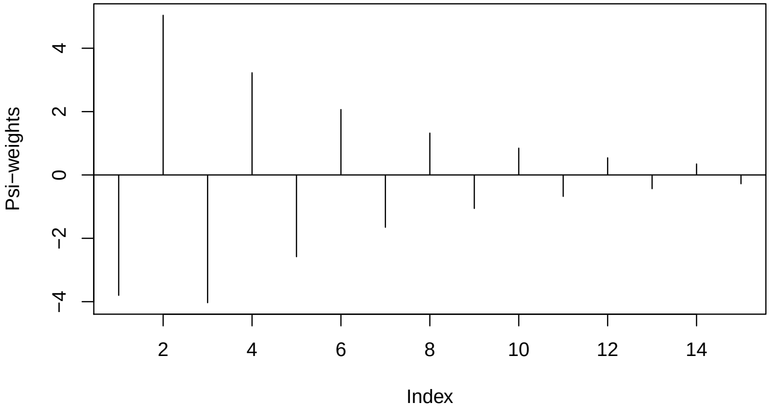
```
## [1] 1.25+0i
```

```
Mod(polyroot(c(1,-0.8)))
```

```
## [1] 1.25
```

Stationarity and Invertibility

The model $X_t = 0.8X_{t-1} + z_t - 3Z_{t-1} + 2Z_{t-2}$ is **causal**



Stationarity and Invertibility

Example: $ARMA(1,2)$ $\quad x_t = 0.8x_{t-1} + z_t - 3z_{t-1} + 2z_{t-2}$

$$(1 - 0.8B)x_t = (1 - 3B + 2B^2)z_t$$

Roots of the MA-polynomial: $1 - 3B + 2B^2 = 0 \Rightarrow B = \{0.5, 1\}$

```
polyroot(c(1,-3,2))
```

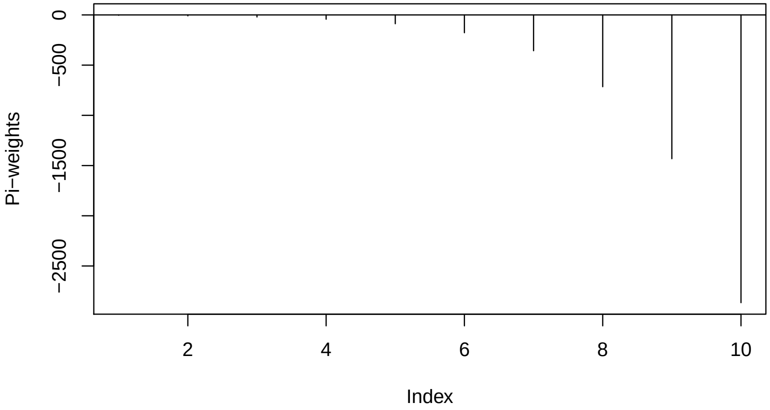
```
## [1] 0.5+0i 1.0-0i
```

```
Mod(polyroot(c(1,-3,2)))
```

```
## [1] 0.5 1.0
```

Stationarity and Invertibility

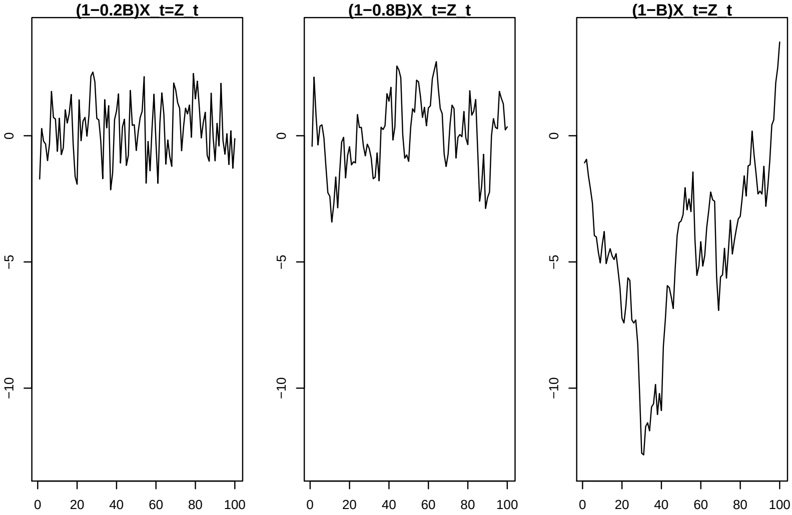
The model $X_t = 0.8X_{t-1} + z_t - 3Z_{t-1} + 2Z_{t-2}$ is not **invertible**



Non-Stationary models with unit roots

Let's consider several $AR(1)$ models

$$(1-0.2B)X_t = Z_t \quad (1-0.8B)X_t = Z_t \quad (1-B)X_t = Z_t$$



Non-Stationary models with unit roots:Random Walk

Random Walk:

$$X_t = X_{t-1} + Z_t \qquad (1-B)X_t = Z_t \qquad Z_t \sim WN(0, \sigma_Z^2)$$

$$X_t = Z_t + Z_{t-1} + ... + Z_1$$

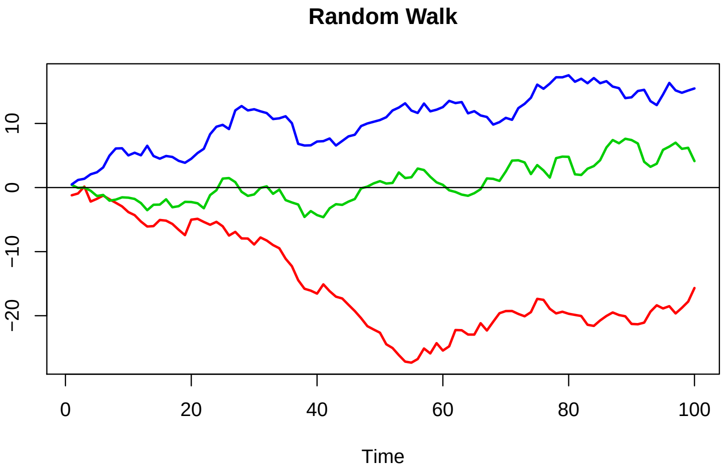
Non stationary process:

$$V(X_t) = t\sigma_Z^2$$

$$\gamma(X_t, X_{t+k}) = t\sigma_Z^2$$

Non-Stationary models with unit roots:Random Walk

Random Walk:



Non-Stationary models with unit roots

Non-stationary $ARMA(p, q)$ with 1 unit root:

$$\phi_p(B)X_t = \theta_q(B)Z_t$$

$$\phi_p(B) = 1 - \phi_1 B - ... - \phi_p B^p = (1 - \phi'_1 B - ... - \phi'_{p-1} B)(1 - B)$$

x_t is a process with 1 unit root: Integrated of order 1 ($x_t \sim I(1)$)

This means that $W_t = (1 - B)X_t$ is an stationary $ARMA(p - 1, q)$ process

x_t is an $ARIMA(p - 1, 1, q)$

Non-Stationary models with unit roots

Non-stationary $ARMA(p, q)$ with d unit roots:

$$\phi_p(B)X_t = \theta_q(B)Z_t$$

$$\phi_p(B) = 1 - \phi_1 B - ... - \phi_p B^p = (1 - \phi'_1 B - ... - \phi'_{p-d} B)(1 - B)^d$$

x_t is a process with d unit roots: Integrated of order d ($X_t \sim I(d)$)

This means that $W_t = (1 - B)^d X_t$ is an stationary $ARMA(p - d, q)$ process

X_t is an $ARIMA(p - d, d, q)$

Non-Stationary models

Non-stationary $ARIMA(p, d, q)$ with d unit roots:

$$\phi_p(B)(1-B)^d X_t = \theta_q(B) Z_t$$

$$(1-\phi_1 B - \ldots - \phi_p B^p)(1-B)^d X_t = (1+\theta_1 B + \ldots + \theta_q B^q) Z_t$$

In the Identification step:

- d $=$ number of regular differences to reach stationarity
- p,q $=$ analysis of ACF and PCF of the transformed series

Example: ARIMA(0,1,1)

$ARMA(1,1)$ with $\phi = 1 \Rightarrow$ Non-stationary and invertible (if $|\theta| < 1$)

$$(1 - B)X_t = (1 + \theta B)Z_t$$

The π-weights show an exponential decay:

$$X_t = (\theta + 1)X_{t-1} - \theta(\theta + 1)X_{t-2} + \theta^2(\theta + 1)X_{t-3} + ... + Z_t$$

This model is the basis for the **EWMA** filter (Exponential Weighted Moving Average)

Seasonal Models

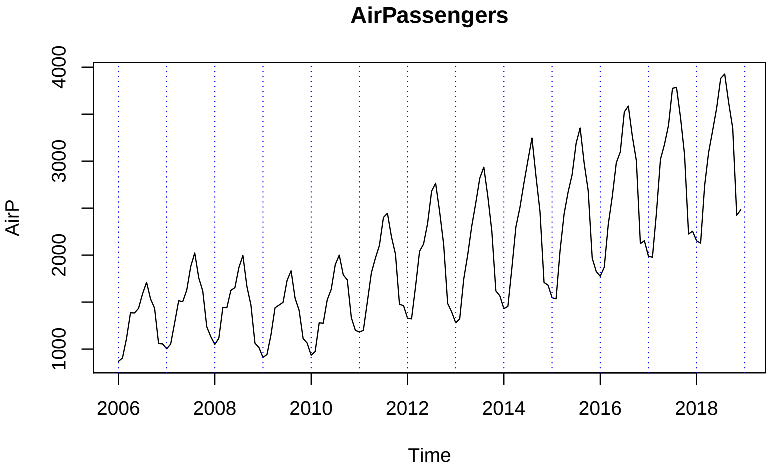
AirP: Number of monthly passengers (in thousands) of international air flights at El Prat between January 2006 and December 2018

Source: Ministry of Public Works of Spain (http://www.fomento.gob.es/BE/?nivel=2&orden=03000000)

```
##        Jan  Feb  Mar  Apr  May  Jun  Jul  Aug  Sep  Oct  Nov  Dec
## 2006   869  905 1110 1386 1384 1433 1587 1711 1533 1435 1058 1055
## 2007  1003 1052 1279 1514 1504 1625 1878 2023 1758 1617 1237 1132
## 2008  1050 1115 1441 1440 1625 1653 1867 1995 1665 1468 1063 1016
## 2009   910  943 1154 1439 1468 1497 1730 1833 1540 1413 1111 1064
## 2010   935  974 1279 1275 1524 1636 1898 2000 1789 1736 1336 1200
## 2011  1181 1199 1504 1812 1967 2103 2400 2445 2195 2010 1475 1463
## 2012  1329 1321 1665 2041 2118 2337 2680 2765 2462 2115 1485 1395
## 2013  1279 1322 1744 2005 2308 2554 2819 2936 2625 2255 1619 1566
## 2014  1430 1454 1865 2301 2506 2767 3011 3246 2838 2461 1709 1680
## 2015  1547 1534 2049 2437 2671 2856 3191 3353 2977 2686 1968 1826
## 2016  1775 1871 2327 2623 2982 3098 3523 3586 3263 3005 2122 2153
## 2017  1989 1977 2448 3019 3177 3381 3775 3784 3462 3068 2224 2254
## 2018  2152 2127 2737 3094 3325 3566 3880 3927 3619 3349 2424 2482
```

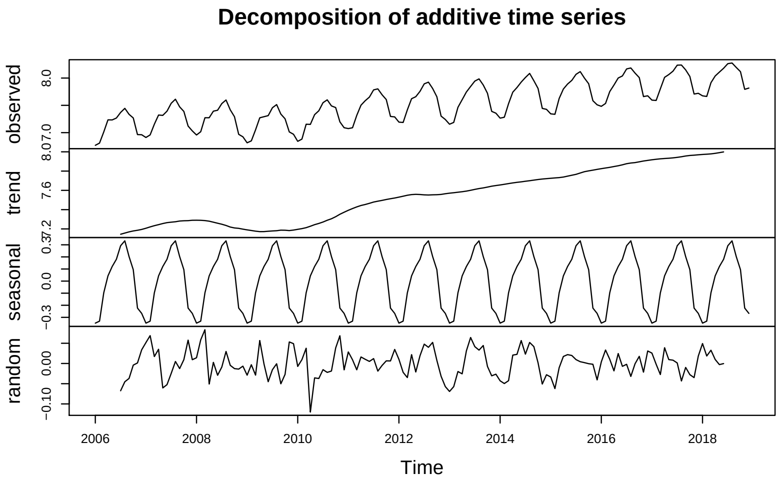
Seasonal Model: AirPassengers

X_t = Increasing Variance + Linear Trend + Seasonal component + stationary process



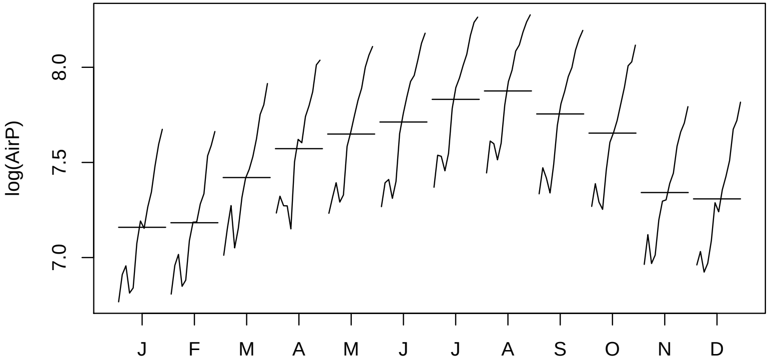
Seasonal Model: AirPassengers

$$log(X_t) = \text{Linear Trend} + \text{Seasonal component} + \text{stationary process}$$

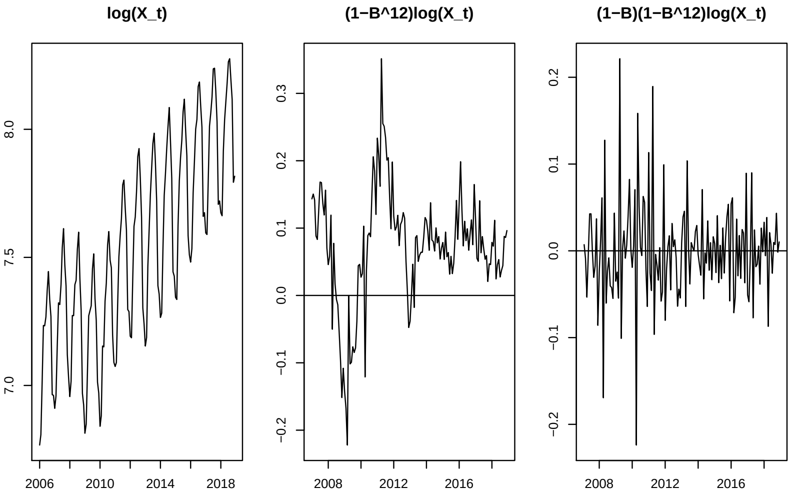


Seasonal Model: AirPassengers

$$log(X_t) = \text{Linear Trend} + \text{Seasonal component} + \text{stationary process}$$



Seasonal Model: AirPassengers



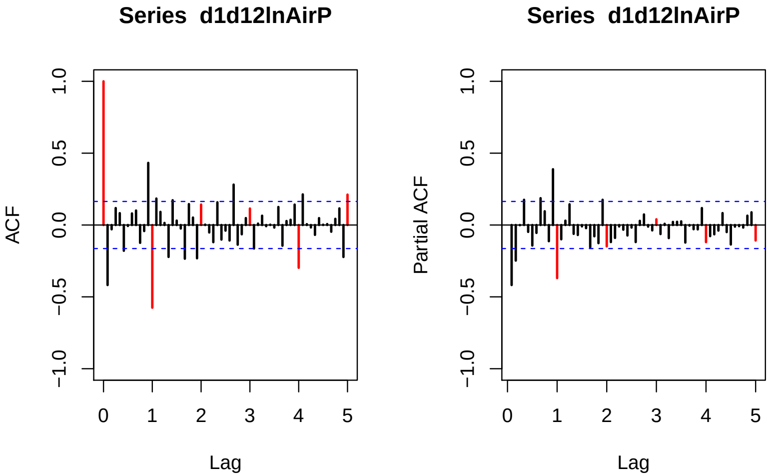
Seasonal Model: AirPassengers

Transformation into an stationary series:

$$W_t = (1 - B)(1 - B^{12}) log(X_t)$$

- Logarithm scale: to stabilize the variance
- Seasonal difference: remove the seasonal pattern (and perhaps a global linear trend: $(1 - B^{12}) = (1 - B)(1 + B + ... + B^{11})$)
- Regular difference: to reach a constant mean

Seasonal Model: AirPassengers



Seasonal ARIMA models

Non-stationary Seasonal $ARIMA(p,d,q)(P,D,Q)_s$ (or SARIMA):

$$\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D X_t = \theta_q(B)\Theta_Q(B^s)Z_t$$

$$(1-\phi_1 B-\ldots-\phi_p B^p)(1-\Phi_1 B^s-\ldots-\Phi_P B^s P)(1-B)^d(1-B^s)^D X_t$$
$$= (1+\theta_1 B+\ldots+\theta_q B^q)(1+\Theta_1 B^s+\ldots+\Theta_Q B^s Q)Z_t$$

In the Identification step:

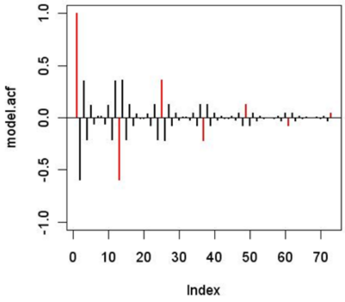
- d = number of regular differences to reach stationarity
- D = number of seasonal differences to reach stationarity (usually 0 or 1)
- p,q = analysis of ACF and PCF (only first lags)
- P,Q = analysis of ACF and PCF (only lags multiple of s)

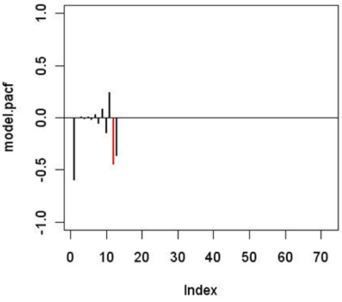
Seasonal ARIMA models

Example 1: The following Theoretical ACF and PACF belong to models $ARMA(p, q)(P, Q)_{12}$ of this form:

$$(1 - \phi B)(1 - \Phi B^{12})W_t = (1 + \theta B)(1 + \Theta B^{12})Z_t$$

and $\phi, \Phi, \theta, \Theta \in \{-0.6, 0, 0.6\}$



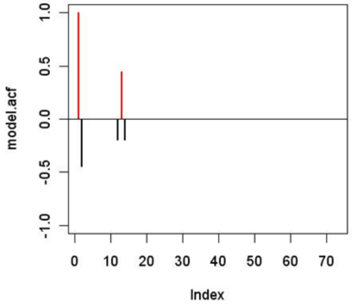


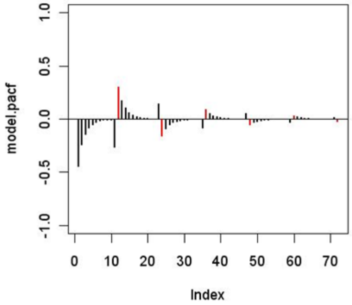
Seasonal ARIMA models

Example 2: The following Theoretical ACF and PACF belong to models $ARMA(p,q)(P,Q)_{12}$ of this form:

$$(1-\phi B)(1-\Phi B^{12})W_t = (1+\theta B)(1+\Theta B^{12})Z_t$$

and $\phi, \Phi, \theta, \Theta \in \{-0.6, 0, 0.6\}$



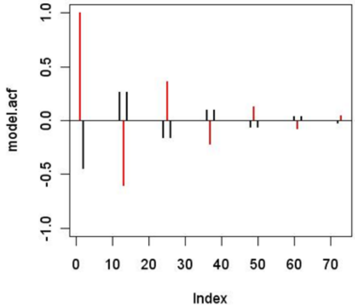


Seasonal ARIMA models

Example 3: The following Theoretical ACF and PACF belong to models $ARMA(p,q)(P,Q)_{12}$ of this form:

$$(1-\phi B)(1-\Phi B^{12})W_t = (1+\theta B)(1+\Theta B^{12})Z_t$$

and $\phi, \Phi, \theta, \Theta \in \{-0.6, 0, 0.6\}$



Seasonal ARIMA models

Example 1: $(1+0.6B)(1+0.6B^{12})W_t = Z_t$

Example 2: $W_t = (1-0.6B)(1+0.6B^{12})Z_t$

Example 3: $(1+0.6B^{12})W_t = (1-0.6B)Z_t$