Chapter 4
Hierarchical Models

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Chapter 4. Hierarchical Models

Bayesian Model (non-hierarchical)

$$p(\tilde{y} \mid \theta)$$
$$\pi(\theta)$$

Hierarchical Bayesian Model

$p(\tilde{y} \mid \theta)$ ← 1st Level

$\pi(\theta \mid \gamma)$ ← 2nd Level

$\pi(\gamma)$ ← 3rd Level

The hierarchical Bayesian Model treats the parameters of the prior distribution as random variables. This fact adds a new level to the model.

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If we take a random sample of size n, $y_1, \ldots, y_n$, then we can write the hierarchical bayesian model as:

$$\tilde{y}_i \mid \theta_i \sim p(\tilde{y}_i \mid \theta_i) \qquad \textit{for } i=1,..,n$$
$$\theta_i \mid \gamma \sim \pi(\theta_i \mid \gamma)$$
$$\gamma \sim \psi(\gamma)$$

$\tilde{y}_i$	observable variable	$p(\tilde{y}_i \mid \theta_i)$	probability distribution
θ_i	parameter	$\pi(\theta_i \mid \gamma)$	prior distribution
γ	hyperparameter	$\psi(\gamma)$	hyperprior distribution

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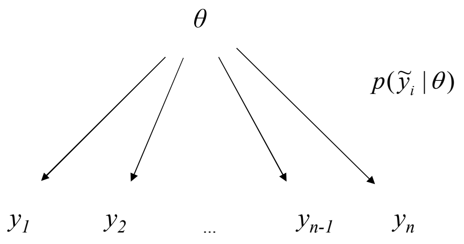
Bayesian Model (non-hierarchical)

$$\tilde{y}_i \mid \theta \sim p(\tilde{y}_i \mid \theta)$$
$$\theta \sim \pi(\theta)$$



The y 's are generated by the probability model $p(\tilde{y}_i \mid \theta)$

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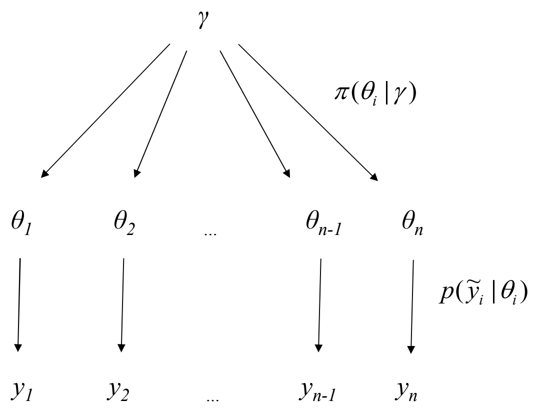
Hierarchical Bayesian Model

$$\tilde{y}_i \mid \theta_i \sim p(\tilde{y}_i \mid \theta_i)$$
$$\theta_i \mid \gamma \sim \pi(\theta_i \mid \gamma)$$
$$\gamma \sim \psi(\gamma)$$



The y 's are generated by the probability model $p(\tilde{y}_i \mid \theta_i)$, and the θ 's are generated by $\pi(\theta_i \mid \gamma)$

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Example: Hospitals

Exercise 6.2 Surgical: Institutional ranking. This exercise considers mortality rates in 12 hospitals performing cardiac surgery in babies. The data are shown below:

Hospital	No of ops	No of deaths
A	47	0
B	148	18
C	119	8
D	810	46
E	211	8
F	196	13
G	148	9
H	215	31
I	207	14
J	97	8
K	256	29
L	360	24

The objective of this study is to know the probability of death around all the hospitals in the country, not only in the hospitals that are in the sample.

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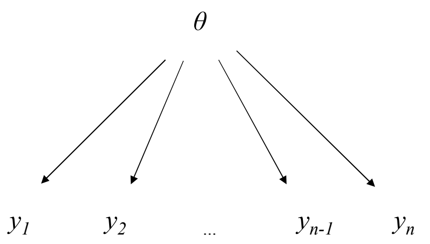
Example: Hospitals

y_i := number of deaths in the *i-th* hospital
n_i := number of surgeries in the *i-th* hospital

for i=1,..,12

Model A

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta)$$
$$\theta \sim Beta(a = 1, b = 1)$$



Where θ is the probability of dying, and it is the same for all hospitals

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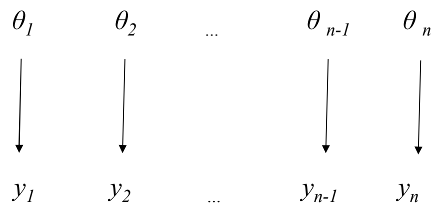
Example: Hospitals

Model B

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta_i)$$
$$\theta_i \sim Beta(1, 1)$$



Where θ_i is the probability of dying for the *i-th* hospital for i=1,...,12, but we don't know anything about the death probability from other hospitals.

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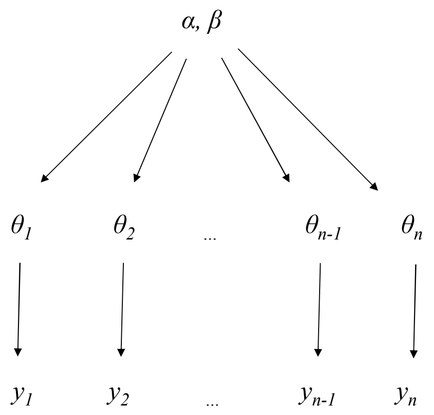
Example: Hospitals

Model C

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta_i)$$
$$\theta_i \mid \alpha, \beta \sim Beta(\alpha, \beta)$$
$$\alpha \sim Gamma(0.01, 0.001)$$
$$\beta \sim Gamma(0.01, 0.001)$$



We can make inference and prediction at different levels

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Example: Hospitals

Model C (hierarchical):

All we know about the probability of dying in the *i-th* hospital is in :

$$\pi(\theta_i \mid y)$$

And we can make inference of the probability of dying in a hospitat that is not in the sample using the posterior predictive distribution of the second level:

$$\pi(\tilde{\theta} \mid y) = \iint \pi(\tilde{\theta} \mid \alpha, \beta) \pi(\alpha, \beta \mid y) \partial\alpha\partial\beta$$

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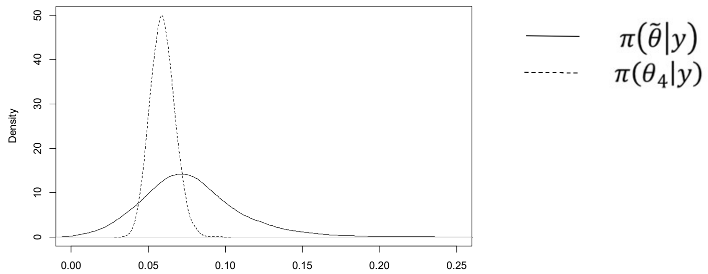
Example: Hospitals

Posterior probability distribution of the probability of dying for hospital nº 4

Posterior probability distribution of the probability of dying for a new hosptial



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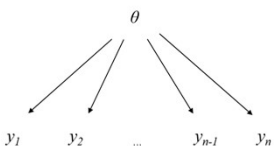
Example: Hospitals

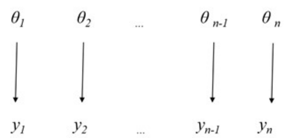
Model A

$$\tilde{y}_i \,|\, \theta_i \sim Binomial(n_i, \theta)$$
$$\theta \sim Beta(1, 1)$$



Model B

$$\tilde{y}_i \,|\, \theta_i \sim Binomial(n_i, \theta_i)$$
$$\theta_i \sim Beta(1, 1)$$



Model C

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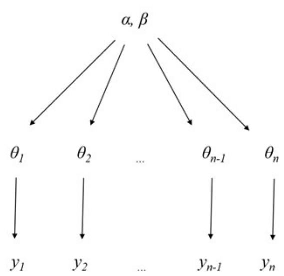

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Example: Hospitals

Model A

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta)$$
$$\theta \sim Beta(1,1)$$

parameters

$$\begin{array}{c|c|} & \theta \\ \text{simulations} & \begin{matrix} \theta^{(1)} \\ \theta^{(2)} \\ \vdots \\ \theta^{(M)} \end{matrix} \end{array}$$

Model B

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta_i)$$
$$\theta_i \sim Beta(1,1)$$

parameters

$$\begin{array}{c|cccc|} & \theta_1 & \theta_2 & \ldots & \theta_{12} \\ \hline & \theta_1^{(1)} & \theta_2^{(1)} & \ldots & \theta_{12}^{(1)} \\ \text{simulations} & \theta_1^{(2)} & \theta_2^{(2)} & \ldots & \theta_{12}^{(2)} \\ & \vdots & & & \\ & \theta_1^{(M)} & \theta_2^{(M)} & \ldots & \theta_{12}^{(M)} \\ \hline \end{array}$$

Model C

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta_i)$$
$$\theta_i \mid \alpha, \beta \sim Beta(\alpha, \beta)$$
$$\alpha \sim Gamma(0.01, 0.001)$$
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parameters

$$\begin{array}{c|cccccc|} & \theta_1 & \theta_2 & \ldots & \theta_{12} & \alpha & \beta \\ \hline & \theta_1^{(1)} & \theta_2^{(1)} & \ldots & \theta_{12}^{(1)} & \alpha^{(1)} & \beta^{(1)} \\ \text{simulations} & \theta_1^{(2)} & \theta_2^{(2)} & \ldots & \theta_{12}^{(2)} & \alpha^{(2)} & \beta^{(2)} \\ & \vdots & & & & & \\ & \theta_1^{(M)} & \theta_2^{(M)} & \ldots & \theta_{12}^{(M)} & \alpha^{(M)} & \beta^{(M)} \\ \hline \end{array}$$

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Example: Hospitals

$$\tilde{y}_i \mid \theta_i \sim Binomial(n_i, \theta_i)$$
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parameters

$$\begin{array}{c|cccccc|} & \theta_1 & \theta_2 & \ldots & \theta_{12} & \alpha & \beta \\ \hline & \theta_1^{(1)} & \theta_2^{(1)} & \ldots & \theta_{12}^{(1)} & \alpha^{(1)} & \beta^{(1)} \\ \text{simulations} & \theta_1^{(2)} & \theta_2^{(2)} & \ldots & \theta_{12}^{(2)} & \alpha^{(2)} & \beta^{(2)} \\ & \vdots & & & & & \\ & \theta_1^{(M)} & \theta_2^{(M)} & \ldots & \theta_{12}^{(M)} & \alpha^{(M)} & \beta^{(M)} \\ \hline \end{array}$$

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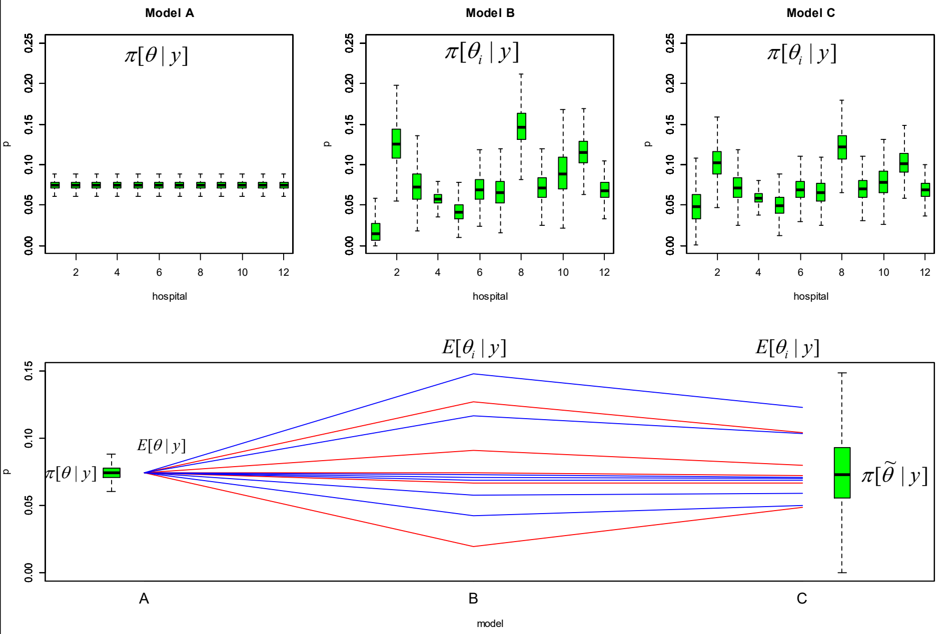

Chapter 4. Hierarchical Models

Example: Hospitals

Can we do a *ranking* for the sample hospitals?

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Example: Hospitals

parameter

$$\begin{array}{cccc} \theta_1 & \theta_2 & \dots & \theta_{12} \end{array}$$

simulations

$$\begin{array}{|cccc|} \theta_1^{(1)} & \theta_2^{(1)} & \dots & \theta_{12}^{(1)} \\ \theta_1^{(2)} & \theta_2^{(2)} & \dots & \theta_{12}^{(2)} \\ \vdots & & & \\ \theta_1^{(M)} & \theta_2^{(M)} & \dots & \theta_{12}^{(M)} \end{array}$$

Ranking

$$\begin{array}{cccc} R_1 & R_2 & \dots & R_{12} \end{array}$$

$$\begin{array}{|cccc|} R_1^{(1)} & R_2^{(1)} & \dots & R_{12}^{(1)} \\ R_1^{(2)} & R_2^{(2)} & \dots & R_{12}^{(2)} \\ \vdots & & & \\ R_1^{(M)} & R_2^{(M)} & \dots & R_{12}^{(M)} \end{array}$$

In each simulation we create a new *Ranking* variable for each hospital

For example, if in the first simulation after sorting the values of θ's we get:

$$\theta_2^{(1)} < \theta_3^{(1)} < \theta_5^{(1)} < \theta_{12}^{(1)} < \theta_4^{(1)} < \theta_{11}^{(1)} < \theta_9^{(1)} < \dots < \theta_1^{(1)}$$

Then the ranking variable will be:

$$R_1^{(1)} = 12,\ R_2^{(1)} = 1,\ R_3^{(1)} = 2,\ R_4^{(1)} = 5, \dots, R_{12}^{(1)} = 4$$

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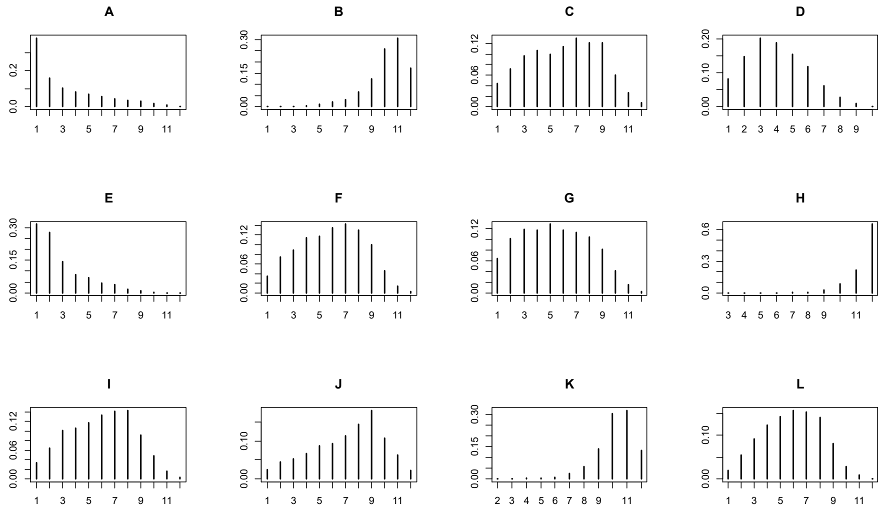
The Posterior probability Distribution for the *ranking* variable for each hospital



80% CI		A	B	C	D	E	F	G	H	I	J	K	L
	10%	1	8	2	2	1	2	2	10	3	3	9	3
	90%	7	12	9	7	6	9	9	12	9	10	12	9

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I encourage you to read chapter 5 (Hierarchical models) and chapter 15 (*Hierarchical Lineal Models*) from the book:

Gelman A, Carlin J, Stern H, Dunson D, Vehtari A, and Rubin D (2014). Bayesian Data Analysis (3rd ed). London: Chapman & Hall.