Uniwersytet
Wrocławski

A Simple and Efficient Implementation of Strong Call by Need by an Abstract Machine

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Why call by need?

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call by name

call by value

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don’t evaluate unneeded arguments

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$\times\ 2\ 2$

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$\times\ 2\ 2 \overset{\text{weak}}{\longrightarrow\!\!\!\!\longrightarrow} \lambda f.\ 2\ (2\ f) \overset{\text{weak}}{\not\longrightarrow}$

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strong reduction applied in type checking in proof assistants

In what sense <u>efficient</u>?

Time complexity as a function of the two parameters:

- size of the initial term
- number of β-reductions

[Accattoli, Barenbaum, Mazza, ICFP 2014]

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Implementation is *efficient* if it is linear in both parameters.

cf. [Accattoli, Guerrieri, FSEN 2017]

Abstract Machines
for Strong call by Need

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  implement
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both machines suffer from exponential overhead

Further Plan of the Presentation

- derivation of the efficient machine

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- derivation of the efficient machine (quasi-*efficient*)

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Functional Correspondence

Definitional Interpreter

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CPS translation

Definitional Interpreter

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CPS translation

defunctionalization

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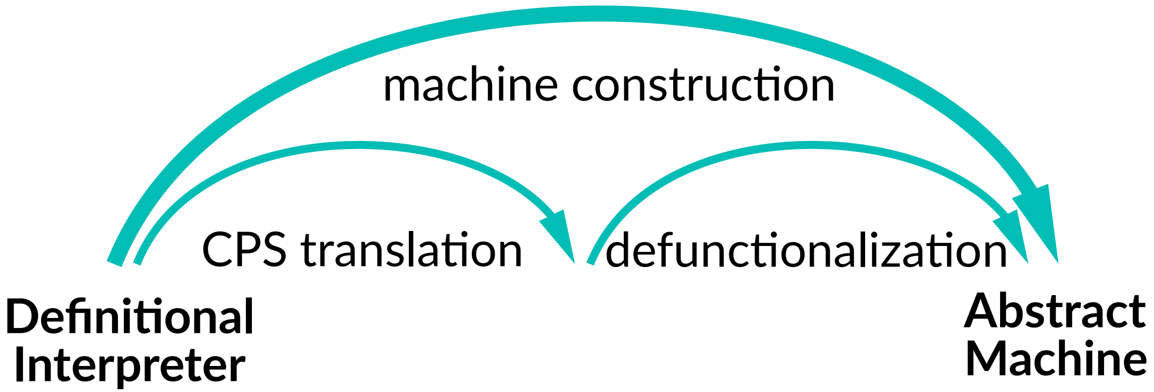
[John C. Reynolds, 1972]

Functional Correspondence



[John C. Reynolds, 1972]
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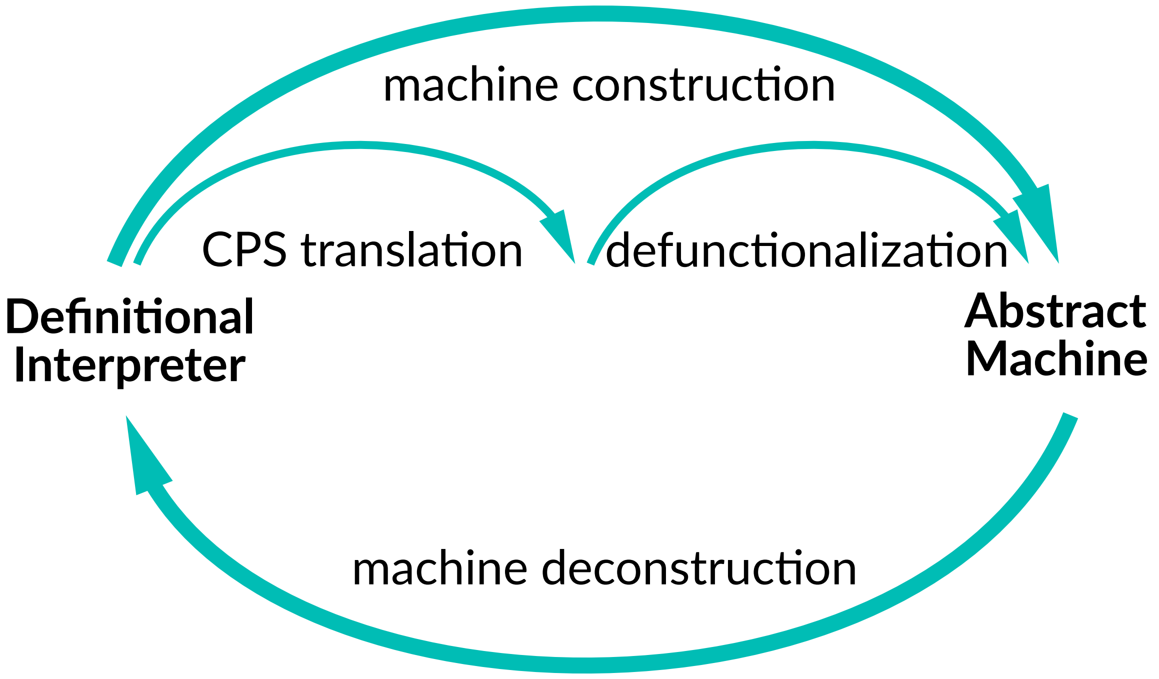

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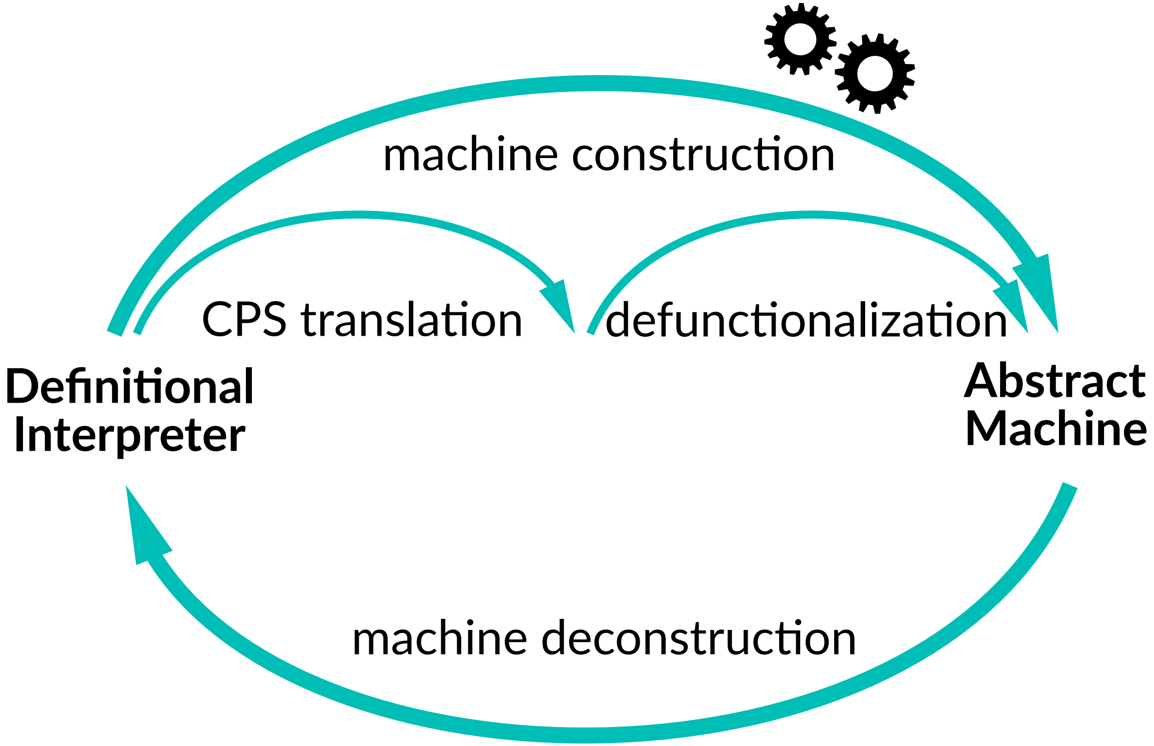

[John C. Reynolds, 1972]
[Ager, Danvy, Biernacki, Midtgaard, PPDP 2003]
[Buszka, Biernacki, PPDP 2021]

Derivation

[Crégut, 1990]

KN

abstract machine

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KN
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implements $\dashrightarrow$

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[Munk, 2008] [Crégut, 1990] [García-Pérez, Nogueira, JFP 2019]

<functional code> $\xleftarrow{\text{deconstruction}}$ KN $\xrightarrow{\text{implements}}$ normal order

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construction

RKNL

abstract machine

RKNL

$$\langle (t_1\ t_2, e), s, \sigma \rangle_\triangledown \longrightarrow \langle (t_1, e), \square\ (t_2, e) :: s, \sigma \rangle_\triangledown \tag{1}$$

$$\langle (\lambda x.t, e), s, \sigma \rangle_\triangledown \longrightarrow \langle \ell := (\lambda x.t, e), s, \sigma * [\ell \mapsto \bot_{\text{✗}}] \rangle_\vartriangle \tag{2}$$

$$\langle (x, e), s, \sigma \rangle_\triangledown \longrightarrow \langle (t, e_2), \ell := \square :: s, \sigma \rangle_\triangledown \text{ where } \ell = e(x),\ \sigma(\ell) = (t, e_2)_{\text{✗}} \tag{3}$$

$$\langle (x, e), s, \sigma \rangle_\triangledown \longrightarrow \langle v, s, \sigma \rangle_\vartriangle \qquad \text{where } \sigma(e(x)) = v_{\checkmark} \vee (v = x \notin e) \tag{4}$$

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$$\text{where } \sigma(\ell) = \bot_{\text{✗}} \tag{7}$$

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$$\langle t, \square\ (t_2, e_2) :: s, \sigma \rangle_\vartriangle \longrightarrow \langle (t_2, e_2), t\ \square :: s, \sigma \rangle_\triangledown \tag{9}$$

$$\langle t_2, t_1\ \square :: s, \sigma \rangle_\vartriangle \longrightarrow \langle t_1\ t_2, s, \sigma \rangle_\vartriangle \tag{10}$$

$$\langle t, \lambda x.\square :: s, \sigma \rangle_\vartriangle \longrightarrow \langle \lambda x.t, s, \sigma \rangle_\vartriangle \tag{11}$$

Elaborate Example Execution

$$0: \quad \langle((\lambda x.c\,x\,x)\,(A\,\Omega),[])\rangle_{\triangledown} \| [] \quad \xrightarrow{(1)}$$

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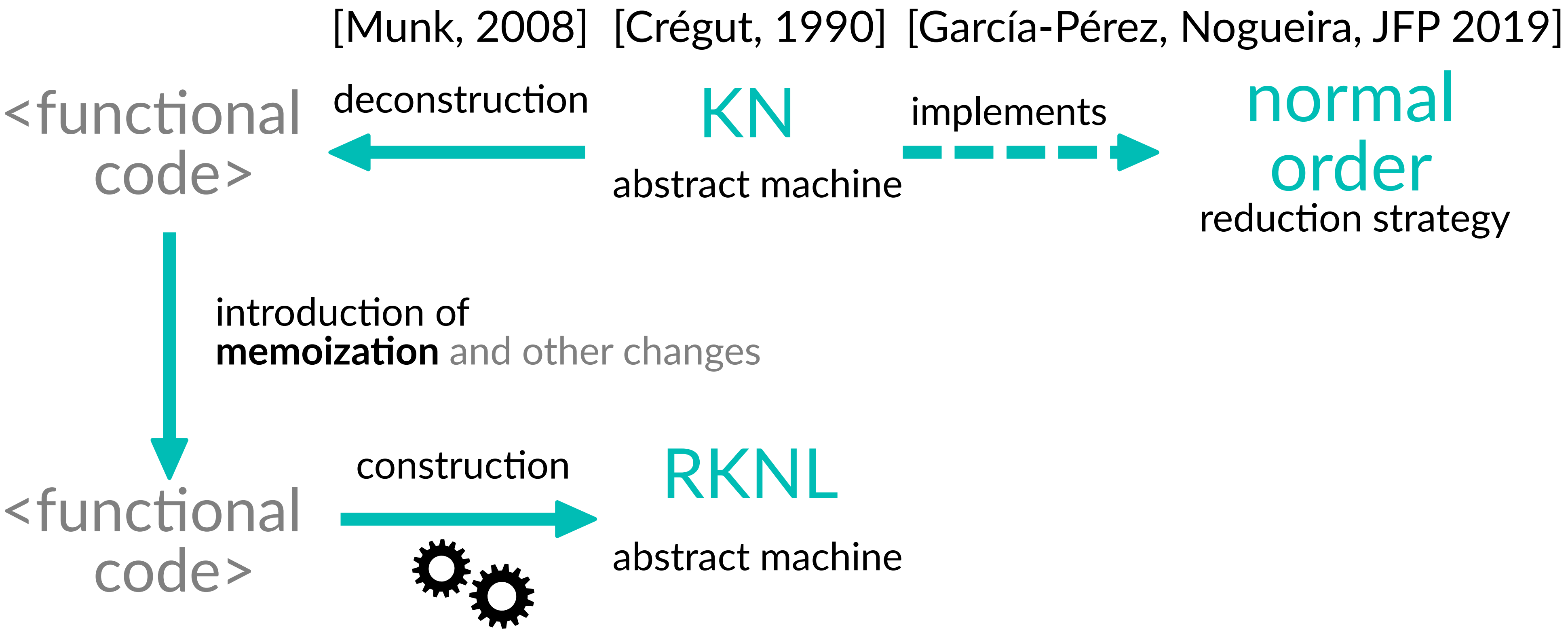
$$8: \quad c\ \left(\mathbb{x} := \langle(A\,\Omega,[])\rangle_{\triangledown}\right)\ x^{\mathbb{x}} \,|\, [\mathbb{x} \mapsto (A\,\Omega,[])\,✗] \quad \xrightarrow{(1)}$$

Correctness and Complexity



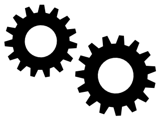
Correctness and Complexity

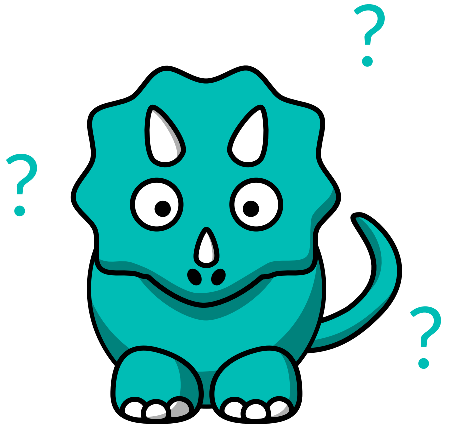
[Munk, 2008] [Crégut, 1990] [García-Pérez, Nogueira, JFP 2019]

<functional code> ← deconstruction — **KN** abstract machine — implements - - → **normal order** reduction strategy

↓ introduction of **memoization** and other changes

<functional code> — construction → **RKNL** abstract machine — implements - - →





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Ghost Abstract Machine

$$
\begin{aligned}
\mathit{Applicative\ Stacks} \ni\ & \alpha ::= \ell := \square :: \alpha \mid \square\, c :: \alpha \mid \lceil \square \rceil :: \nu \\
\mathit{Non\text{-}applicative\ Stacks} \ni\ & \nu ::= \ell' := \square :: \nu \mid [\,] \mid \lambda x.\square :: \nu \mid a\,\square :: \alpha \\
\mathit{Confs} \ni\ & k ::= \langle c, \alpha, \sigma, \sigma' \rangle_{\triangledown} \mid \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \mid \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \\
\mathit{Transitions} :\ &
\end{aligned}
$$

$$
\begin{aligned}
t &\mapsto \langle (t, [\,]), \lceil \square \rceil :: [\,], [\,] \rangle_{\triangledown} \\
\langle (t_1\, t_2, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle (t_1, e), \square\, (t_2, e) :: \alpha, \sigma, \sigma' \rangle_{\triangledown} && (1) \\
\langle (\lambda x.t, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle \ell' := (\lambda x.t, e), \alpha, \sigma, \sigma' * [\ell' \mapsto \bot_{\text{✗}}] \rangle_{\vartriangle} && (2) \\
\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle (t, e_2), \ell := \square :: \alpha, \sigma, \sigma' \rangle_{\triangledown} && (3) \\
&\qquad \text{where } \ell = e(x),\ \ \sigma(\ell) = (t, e_2)_{\text{✗}} \\
\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \qquad \text{where } \sigma(e(x)) = v_{\checkmark} \vee (v = x \notin e) && (4) \\
\langle v, \ell := \square :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle v, \alpha, \sigma[\ell := v_{\checkmark}], \sigma' \rangle_{\vartriangle} && (5) \\
\langle n, \ell' := \square :: \nu, \sigma, \sigma' \rangle_{\vartriangle'} &\to \langle n, \nu, \sigma, \sigma'[\ell' := n_{\checkmark}] \rangle_{\vartriangle'} && (5') \\
\langle \ell' := (\lambda x.t, e), \square\, (t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle (t, e[x := \ell_2]), \alpha, \sigma * [\ell_2 \mapsto (t_2, e_2)_{\text{✗}}], \sigma' \rangle_{\triangledown} && (6) \\
\langle \ell' := (\lambda x.t, e), \quad \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle (t, e[x := \ell_2]), \lceil \square \rceil :: \lambda \check{x}.\square :: \ell' := \square :: \nu, \\
&\qquad \sigma * [\ell_2 \mapsto \check{x}_{\checkmark}], \sigma' \rangle_{\triangledown} \qquad \text{where } \sigma'(\ell') = \bot_{\text{✗}} && (7) \\
\langle \ell' := (\lambda x.t, e), \quad \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \qquad \text{where } \sigma'(\ell') = n_{\checkmark} && (8) \\
\langle a, \square\, (t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle (t_2, e_2), \lceil \square \rceil :: a\,\square :: \alpha, \sigma, \sigma' \rangle_{\triangledown} && (9) \\
\langle a, \quad \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle \lceil a \rceil, \nu, \sigma, \sigma' \rangle_{\vartriangle'} && (9a)
\end{aligned}
$$

(10)

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\begin{aligned}
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t &\mapsto \langle (t, [\,]), \lceil \square \rceil :: [\,], [\,] \rangle_{\triangledown} \\
\langle (t_1\, t_2, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle (t_1, e), \square\, (t_2, e) :: \alpha, \sigma, \sigma' \rangle_{\triangledown} && (1) \\
\langle (\lambda x.t, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle \ell' := (\lambda x.t, e), \alpha, \sigma, \sigma' * [\ell' \mapsto \bot_{\text{✗}}] \rangle_{\vartriangle} && (2) \\
\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle (t, e_2), \ell := \square :: \alpha, \sigma, \sigma' \rangle_{\triangledown} && (3) \\
& \qquad \text{where } \ell = e(x),\ \sigma(\ell) = (t, e_2)_{\text{✗}} \\
\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} &\to \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \quad \text{where } \sigma(e(x)) = v_{\checkmark} \vee (v = x \notin e) && (4) \\
\langle v, \ell := \square :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle v, \alpha, \sigma[\ell := v_{\checkmark}], \sigma' \rangle_{\vartriangle} && (5) \\
\langle n, \ell' := \square :: \nu, \sigma, \sigma' \rangle_{\vartriangle'} &\to \langle n, \nu, \sigma, \sigma'[\ell' := n_{\checkmark}] \rangle_{\vartriangle'} && (5') \\
\langle \ell' := (\lambda x.t, e), \square\, (t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle (t, e[x := \ell_2]), \alpha, \sigma * [\ell_2 \mapsto (t_2, e_2)_{\text{✗}}], \sigma' \rangle_{\triangledown} && (6) \\
\langle \ell' := (\lambda x.t, e), \quad \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle (t, e[x := \ell_2]), \lceil \square \rceil :: \lambda \check{x}.\square :: \ell' := \square :: \nu, \\
& \qquad \sigma * [\ell_2 \mapsto \check{x}_{\checkmark}], \sigma' \rangle_{\triangledown} \quad \text{where } \sigma'(\ell') = \bot_{\text{✗}} && (7) \\
\langle \ell' := (\lambda x.t, e), \quad \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \quad \text{where } \sigma'(\ell') = n_{\checkmark} && (8) \\
\langle a, \square\, (t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle (t_2, e_2), \lceil \square \rceil :: a\, \square :: \alpha, \sigma, \sigma' \rangle_{\triangledown} && (9) \\
\langle a, \quad \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} &\to \langle \lceil a \rceil, \nu, \sigma, \sigma' \rangle_{\vartriangle'} && (9a)
\end{aligned}
$$

Ghost Abstract Machine

$$
\begin{array}{rrcl}
\textit{Applicative Stacks} \ni & \alpha & ::= & \ell := \Box :: \alpha \mid \Box\, c :: \alpha \mid \lceil \Box \rceil :: \nu \\
\textit{Non-applicative Stacks} \ni & \nu & ::= & \ell' := \Box :: \nu \mid [\,] \mid \lambda x.\Box :: \nu \mid a\,\Box :: \alpha \\
\textit{Confs} \ni & k & ::= & \langle c, \alpha, \sigma, \sigma' \rangle_{\triangledown} \mid \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \mid \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \\
\textit{Transitions :} & & &
\end{array}
$$

$$
\begin{array}{rcll}
t & \mapsto & \langle (t, [\,]), \lceil \Box \rceil :: [\,], [\,] \rangle_{\triangledown} & \\
\langle (t_1\, t_2, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} & \longrightarrow & \langle (t_1, e), \Box\,(t_2, e) :: \alpha, \sigma, \sigma' \rangle_{\triangledown} & (1) \\
\langle (\lambda x.t, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} & \longrightarrow & \langle \ell' := (\lambda x.t, e), \alpha, \sigma, \sigma' * [\ell' \mapsto \bot_{\times}] \rangle_{\vartriangle} & (2) \\
\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} & \longrightarrow & \langle (t, e_2), \ell := \Box :: \alpha, \sigma, \sigma' \rangle_{\triangledown} & (3) \\
 & & \quad \text{where } \ell = e(x),\ \ \sigma(\ell) = (t, e_2)_{\times} & \\
\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} & \longrightarrow & \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \quad \text{where } \sigma(e(x)) = v_{\checkmark} \vee (v = x \notin e) & (4) \\
\langle v, \ell := \Box :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} & \longrightarrow & \langle v, \alpha, \sigma[\ell := v_{\checkmark}], \sigma' \rangle_{\vartriangle} & (5) \\
\langle n, \ell' := \Box :: \nu, \sigma, \sigma' \rangle_{\vartriangle'} & \longrightarrow & \langle n, \nu, \sigma, \sigma'[\ell' := n_{\checkmark}] \rangle_{\vartriangle'} & (5') \\
\langle \ell' := (\lambda x.t, e), \Box\,(t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} & \longrightarrow & \langle (t, e[x := \ell_2]), \alpha, \sigma * [\ell_2 \mapsto (t_2, e_2)_{\times}], \sigma' \rangle_{\triangledown} & (6) \\
\langle \ell' := (\lambda x.t, e), \quad \lceil \Box \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} & \longrightarrow & \langle (t, e[x := \ell_2]), \lceil \Box \rceil :: \lambda \check{x}.\Box :: \ell' := \Box :: \nu, & \\
 & & \quad \sigma * [\ell_2 \mapsto \check{x}_{\checkmark}], \sigma' \rangle_{\triangledown} \quad \text{where } \sigma'(\ell') = \bot_{\times} & (7) \\
\langle \ell' := (\lambda x.t, e), \quad \lceil \Box \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} & \longrightarrow & \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \quad \text{where } \sigma'(\ell') = n_{\checkmark} & (8) \\
\langle a, \Box\,(t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} & \longrightarrow & \langle (t_2, e_2), \lceil \Box \rceil :: a\,\Box :: \alpha, \sigma, \sigma' \rangle_{\triangledown} & (9) \\
\langle a, \quad \lceil \Box \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} & \longrightarrow & \langle \lceil a \rceil, \nu, \sigma, \sigma' \rangle_{\vartriangle'} & (9a)
\end{array}
$$

Ghost Abstract Machine

$$
\begin{aligned}
\textit{Applicative Stacks} \ni \quad & \alpha ::= \ell := \square :: \alpha \;\mid\; \square\, c :: \alpha \;\mid\; \lceil \square \rceil :: \nu \\
\textit{Non-applicative Stacks} \ni \quad & \nu ::= \ell' := \square :: \nu \;\mid\; [\,] \;\mid\; \lambda x.\square :: \nu \;\mid\; a\,\square :: \alpha \\
\textit{Confs} \ni \quad & k ::= \langle c, \alpha, \sigma, \sigma' \rangle_{\triangledown} \;\mid\; \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \;\mid\; \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \\
\textit{Transitions} :
\end{aligned}
$$

$$t \mapsto \langle (t, [\,]), \lceil \square \rceil :: [\,], [\,] \rangle_{\triangledown}$$

$$\langle (t_1\, t_2, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} \to \langle (t_1, e), \square\, (t_2, e) :: \alpha, \sigma, \sigma' \rangle_{\triangledown} \quad (1)$$

$$\langle (\lambda x.t, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} \to \langle \ell' := (\lambda x.t, e), \alpha, \sigma, \sigma' * [\ell' \mapsto \bot_{\text{✗}}] \rangle_{\vartriangle} \quad (2)$$

$$\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} \to \langle (t, e_2), \ell := \square :: \alpha, \sigma, \sigma' \rangle_{\triangledown} \quad (3)$$

$$\text{where } \ell = e(x),\ \ \sigma(\ell) = (t, e_2)_{\text{✗}}$$

$$\langle (x, e), \alpha, \sigma, \sigma' \rangle_{\triangledown} \to \langle v, \alpha, \sigma, \sigma' \rangle_{\vartriangle} \quad \text{where } \sigma(e(x)) = v_{\checkmark} \vee (v = x \notin e) \quad (4)$$

$$\langle v, \ell := \square :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} \to \langle v, \alpha, \sigma[\ell := v_{\checkmark}], \sigma' \rangle_{\vartriangle} \quad (5)$$

$$\langle n, \ell' := \square :: \nu, \sigma, \sigma' \rangle_{\vartriangle'} \to \langle n, \nu, \sigma, \sigma'[\ell' := n_{\checkmark}] \rangle_{\vartriangle'} \quad (5')$$

$$\langle \ell' := (\lambda x.t, e), \square\, (t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} \to \langle (t, e[x := \ell_2]), \alpha, \sigma * [\ell_2 \mapsto (t_2, e_2)_{\text{✗}}], \sigma' \rangle_{\triangledown} \quad (6)$$

$$\langle \ell' := (\lambda x.t, e), \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} \to \langle (t, e[x := \ell_2]), \lceil \square \rceil :: \lambda \check{x}.\square :: \ell' := \square :: \nu,$$

$$\sigma * [\ell_2 \mapsto \check{x}_{\checkmark}], \sigma' \rangle_{\triangledown} \quad \text{where } \sigma'(\ell') = \bot_{\text{✗}} \quad (7)$$

$$\langle \ell' := (\lambda x.t, e), \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} \to \langle n, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \quad \text{where } \sigma'(\ell') = n_{\checkmark} \quad (8)$$

$$\langle a, \square\, (t_2, e_2) :: \alpha, \sigma, \sigma' \rangle_{\vartriangle} \to \langle (t_2, e_2), \lceil \square \rceil :: a\,\square :: \alpha, \sigma, \sigma' \rangle_{\triangledown} \quad (9)$$

$$\langle a, \lceil \square \rceil :: \nu, \sigma, \sigma' \rangle_{\vartriangle} \to \langle \lceil a \rceil, \nu, \sigma, \sigma' \rangle_{\vartriangle'} \quad (9a)$$

Potential Function

Table 6. The potential function for RKNL

$$\Phi_{\mathrm{t}}(t_1\ t_2) := 3 + \Phi_{\mathrm{t}}(t_1) + \Phi_{\mathrm{t}}(t_2)$$

$$\Phi_{\mathrm{t}}(\lambda x.t) := 4 + \Phi_{\mathrm{t}}(t)$$

$$\Phi_{\mathrm{t}}(x) := 2$$

$$\Phi_{\mathrm{v}}(t) := 0$$

$$\Phi_{\mathrm{v}}(\ell := (\lambda x.t, e)) := 1$$

$$\Phi_{\mathrm{s}}([\,]) := 0$$

$$\Phi_{\mathrm{s}}(\square\,(t, e) :: s) := 2 + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t)$$

$$\Phi_{\mathrm{s}}(t\,\square :: s) := 1 + \Phi_{\mathrm{s}}(s)$$

$$\Phi_{\mathrm{s}}(\lambda x.\square :: s) := 1 + \Phi_{\mathrm{s}}(s)$$

$$\Phi_{\mathrm{s}}(\ell := \square :: s) := 1 + \Phi_{\mathrm{s}}(s)$$

$$\Phi_{\sigma}(k) := \sum_{\ell \in k \wedge \sigma(\ell) = (t,e)_{\text{✗}} \wedge \ell := \square \notin s} \Phi_{\mathrm{t}}(t) + \sum_{\ell := (\lambda x.t, e) \in k \wedge \sigma(\ell) = \bot_{\text{✗}} \wedge \ell := \square \notin s} (2 + \Phi_{\mathrm{t}}(t))$$

$$\Phi_{\mathrm{k}}(\langle (t, e), s, \sigma \rangle_{\triangledown}) := \Phi_{\mathrm{t}}(t) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k)$$

$$\Phi_{\mathrm{k}}(\langle v, s, \sigma \rangle_{\vartriangle}) := \Phi_{\mathrm{v}}(v) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k)$$

Potential Function

Table 6. The potential function for RKNL

$$
\begin{aligned}
\Phi_{\mathrm{t}}(t_1\ t_2) &:= \boxed{3} + \Phi_{\mathrm{t}}(t_1) + \Phi_{\mathrm{t}}(t_2) \\
\Phi_{\mathrm{t}}(\lambda x.t) &:= 4 + \Phi_{\mathrm{t}}(t) \\
\Phi_{\mathrm{t}}(x) &:= 2 \\
\Phi_{\mathrm{v}}(t) &:= \boxed{0} \\
\Phi_{\mathrm{v}}(\ell := (\lambda x.t, e)) &:= 1
\end{aligned}
\qquad
\begin{aligned}
\Phi_{\mathrm{s}}([\,]) &:= 0 \\
\Phi_{\mathrm{s}}(\square\,(t, e) :: s) &:= \boxed{2} + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t) \\
\Phi_{\mathrm{s}}(t\,\square :: s) &:= \boxed{1} + \Phi_{\mathrm{s}}(s) \\
\Phi_{\mathrm{s}}(\lambda x.\square :: s) &:= 1 + \Phi_{\mathrm{s}}(s) \\
\Phi_{\mathrm{s}}(\ell := \square :: s) &:= 1 + \Phi_{\mathrm{s}}(s)
\end{aligned}
$$

$$
\Phi_\sigma(k) := \sum_{\ell \in k \wedge \sigma(\ell) = (t,e)_{\text{✗}} \wedge \ell := \square \notin s} \Phi_{\mathrm{t}}(t) + \sum_{\ell := (\lambda x.t, e) \in k \wedge \sigma(\ell) = \bot_{\text{✗}} \wedge \ell := \square \notin s} (2 + \Phi_{\mathrm{t}}(t))
$$

$$
\begin{aligned}
\Phi_{\mathrm{k}}(\langle (t, e), s, \sigma \rangle_{\triangledown}) &:= \Phi_{\mathrm{t}}(t) + \Phi_{\mathrm{s}}(s) + \Phi_\sigma(k) \\
\Phi_{\mathrm{k}}(\langle v, s, \sigma \rangle_{\vartriangle}) &:= \Phi_{\mathrm{v}}(v) + \Phi_{\mathrm{s}}(s) + \Phi_\sigma(k)
\end{aligned}
$$

RKNL

$$
\begin{aligned}
\langle (t_1\ t_2, e), s, \sigma\rangle_{\triangledown} &\longrightarrow \langle (t_1, e), \square\, (t_2, e) :: s, \sigma\rangle_{\triangledown} && (1)\\
\langle (\lambda x.t, e), s, \sigma\rangle_{\triangledown} &\longrightarrow \langle \ell := (\lambda x.t, e), s, \sigma * [\ell \mapsto \bot_{\text{✗}}]\rangle_{\triangle} && (2)\\
\langle (x, e), s, \sigma\rangle_{\triangledown} &\longrightarrow \langle (t, e_2), \ell := \square :: s, \sigma\rangle_{\triangledown} \text{ where } \ell = e(x),\ \sigma(\ell) = (t, e_2)_{\text{✗}} && (3)\\
\langle (x, e), s, \sigma\rangle_{\triangledown} &\longrightarrow \langle v, s, \sigma\rangle_{\triangle} \qquad \text{where } \sigma(e(x)) = v_{\checkmark} \lor (v = x \notin e) && (4)\\
\langle v, \ell := \square :: s, \sigma\rangle_{\triangle} &\longrightarrow \langle v, s, \sigma[\ell := v_{\checkmark}]\rangle_{\triangle} && (5)\\
\langle \ell := (\lambda x.t, e), \square\, (t_2, e_2) :: s, \sigma\rangle_{\triangle} &\longrightarrow \langle (t, e[x := \ell_2]), s, \sigma * [\ell_2 \mapsto (t_2, e_2)_{\text{✗}}]\rangle_{\triangledown} && (6)\\
\langle \ell := (\lambda x.t, e), \qquad s, \sigma\rangle_{\triangle} &\longrightarrow \langle (t, e[x := \ell_2]), \lambda\check{x}.\square :: \ell := \square :: s, \sigma * [\ell_2 \mapsto \check{x}_{\checkmark}]\rangle_{\triangledown} \\
&\qquad \text{where } \sigma(\ell) = \bot_{\text{✗}} && (7)\\
\langle \ell := (\lambda x.t, e), \qquad s, \sigma\rangle_{\triangle} &\longrightarrow \langle v, s, \sigma\rangle_{\triangle} \qquad \text{where } \sigma(\ell) = v_{\checkmark} && (8)\\
\langle t, \square\, (t_2, e_2) :: s, \sigma\rangle_{\triangle} &\longrightarrow \langle (t_2, e_2), t\, \square :: s, \sigma\rangle_{\triangledown} && (9)\\
\langle t_2, t_1\, \square :: s, \sigma\rangle_{\triangle} &\longrightarrow \langle t_1\ t_2, s, \sigma\rangle_{\triangle} && (10)\\
\langle t, \lambda x.\square :: s, \sigma\rangle_{\triangle} &\longrightarrow \langle \lambda x.t, s, \sigma\rangle_{\triangle} && (11)
\end{aligned}
$$

Potential Decrease

$$3 + \Phi_{\mathrm{t}}(t_1) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > \Phi_{\mathrm{t}}(t_1) + 2 + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\sigma}(k) \tag{1}$$

$$4 + \Phi_{\mathrm{t}}(t) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) + (2 + \Phi_{\mathrm{t}}(t)) \tag{2}$$

$$2 + \Phi_{\mathrm{s}}(s) + (\Phi_{\sigma}(k') + \Phi_{\mathrm{t}}(t)) > \Phi_{\mathrm{t}}(t) + (1 + \Phi_{\mathrm{s}}(s)) + \Phi_{\sigma}(k') \tag{3}$$

$$2 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > \Phi_{\mathrm{v}}(v) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{4}$$

$$\Phi_{\mathrm{v}}(v) + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > \Phi_{\mathrm{v}}(v) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{5}$$

$$1 + \Phi_{\mathrm{s}}(s) + (\Phi_{\sigma}(k') + 2 + \Phi_{\mathrm{t}}(t)) > \Phi_{\mathrm{t}}(t) + 1 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k') \tag{7}$$

$$1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 0 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{8}$$

$$0 + 2 + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\sigma}(k) > \Phi_{\mathrm{t}}(t_2) + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{9}$$

$$0 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 0 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{10}$$

$$0 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 0 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{11}$$

Potential Decrease and Increase

$$3 + \Phi_{\mathrm{t}}(t_1) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > \Phi_{\mathrm{t}}(t_1) + 2 + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\sigma}(k) \tag{1}$$

$$4 + \Phi_{\mathrm{t}}(t) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) + (2 + \Phi_{\mathrm{t}}(t)) \tag{2}$$

$$2 + \Phi_{\mathrm{s}}(s) + (\Phi_{\sigma}(k') + \Phi_{\mathrm{t}}(t)) > \Phi_{\mathrm{t}}(t) + (1 + \Phi_{\mathrm{s}}(s)) + \Phi_{\sigma}(k') \tag{3}$$

$$2 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > \Phi_{\mathrm{v}}(v) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{4}$$

$$\Phi_{\mathrm{v}}(v) + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > \Phi_{\mathrm{v}}(v) + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{5}$$

$$1 + 2 + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\sigma}(k) + \Phi_{\mathrm{t}}(t_0) > \Phi_{\mathrm{t}}(t) + 1 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) + \Phi_{\mathrm{t}}(t_2) \tag{6}$$

$$1 + \Phi_{\mathrm{s}}(s) + (\Phi_{\sigma}(k') + 2 + \Phi_{\mathrm{t}}(t)) > \Phi_{\mathrm{t}}(t) + 1 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k') \tag{7}$$

$$1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 0 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{8}$$

$$0 + 2 + \Phi_{\mathrm{s}}(s) + \Phi_{\mathrm{t}}(t_2) + \Phi_{\sigma}(k) > \Phi_{\mathrm{t}}(t_2) + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{9}$$

$$0 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 0 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{10}$$

$$0 + 1 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) > 0 + \Phi_{\mathrm{s}}(s) + \Phi_{\sigma}(k) \tag{11}$$

Efficient Implementation

Theorem 5.13.

Let ρ be a sequence of consecutive machine transitions starting from term t_0 to configuration k',

$|\rho|$ be the number of steps in ρ,

and $|\rho|_\beta$ be the number of normal-order β-reductions from t_0 to $\underline{k'}_{\mathrm{k}}$.

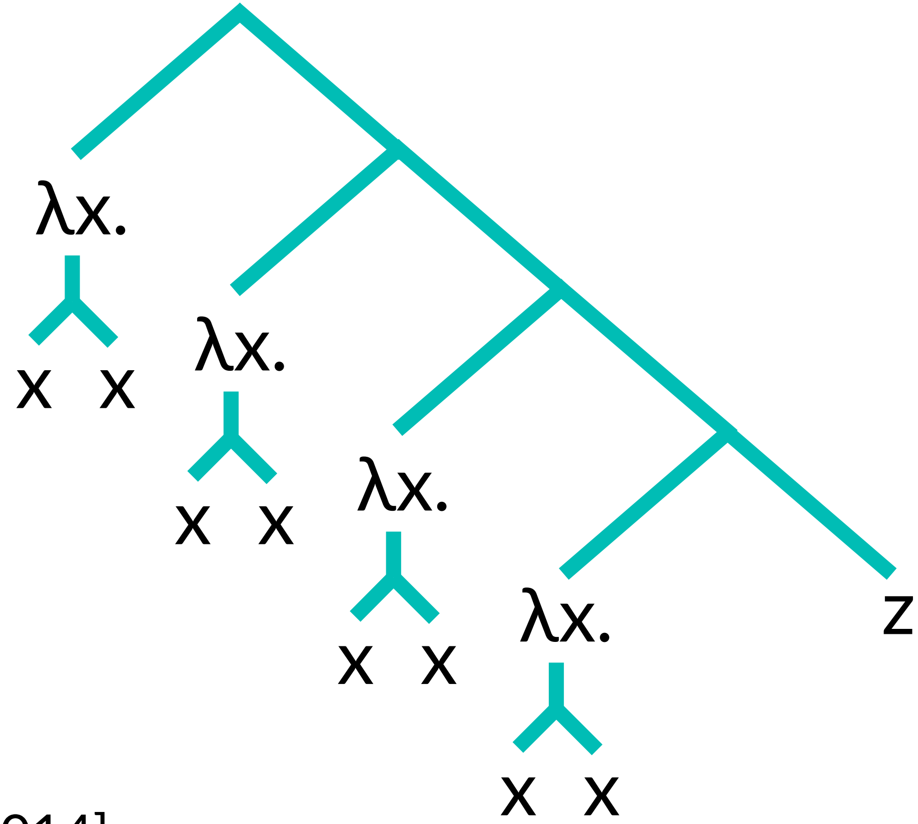
$$\text{Then } |\rho| \leq (|\rho|_\beta + 1) \cdot \Phi_t(t_0).$$

Empirical Method

Table 3. Empirical execution lengths for $1 \leq n \leq 9$

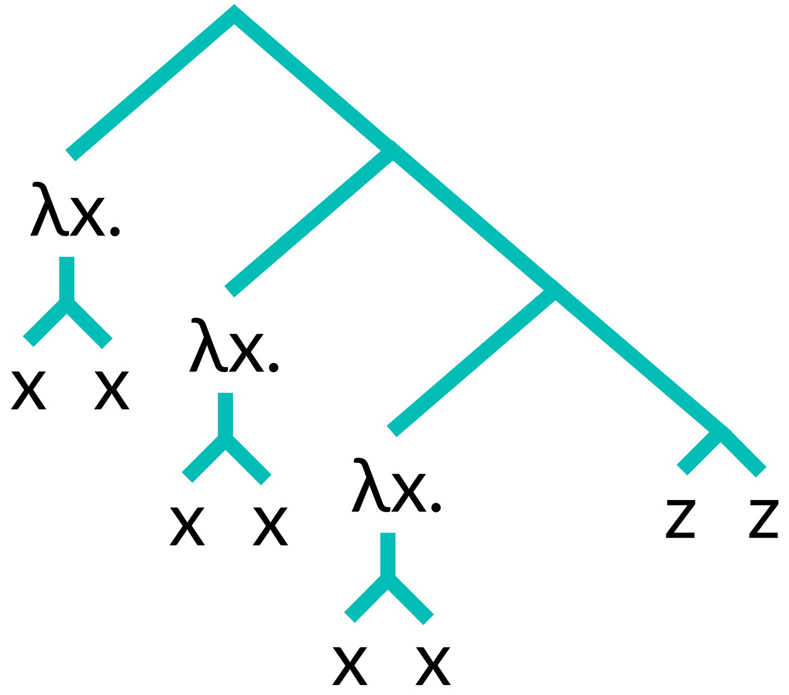
term family	$\xrightarrow{no}$	KN	RKNL
$c_n\, c_2\, I$	$3 \cdot 2^n - 1$	$15 \cdot 2^n - 6$	$10 \cdot 2^n + 5n + 5$
$pred\, c_n$	$6n + 8$	$26n + 25$	$30n + 41$
$\lambda x.c_n\, \omega\, x$	$2^n + 1$	$12 \cdot 2^n - 3$	$9n + 15$
$c_n\, dub\, I$	$2^n + 1$	$23 \cdot 2^n - 14$	$18n + 15$
$c_n\, dub\, (\lambda x.I\, x)$	$2 \cdot 2^n + 1$	$26 \cdot 2^n - 14$	$18n + 20$

Size Explosion Problem



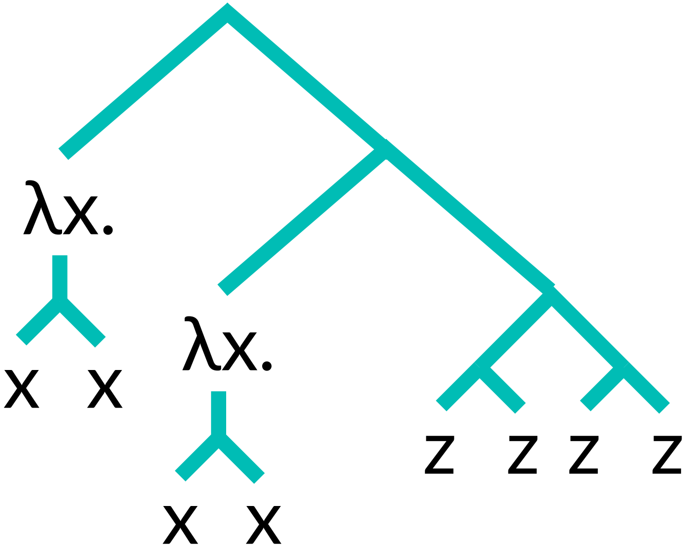
named in
[Accattoli, Dal Lago, CSL-LICS 2014]

Size Explosion Problem



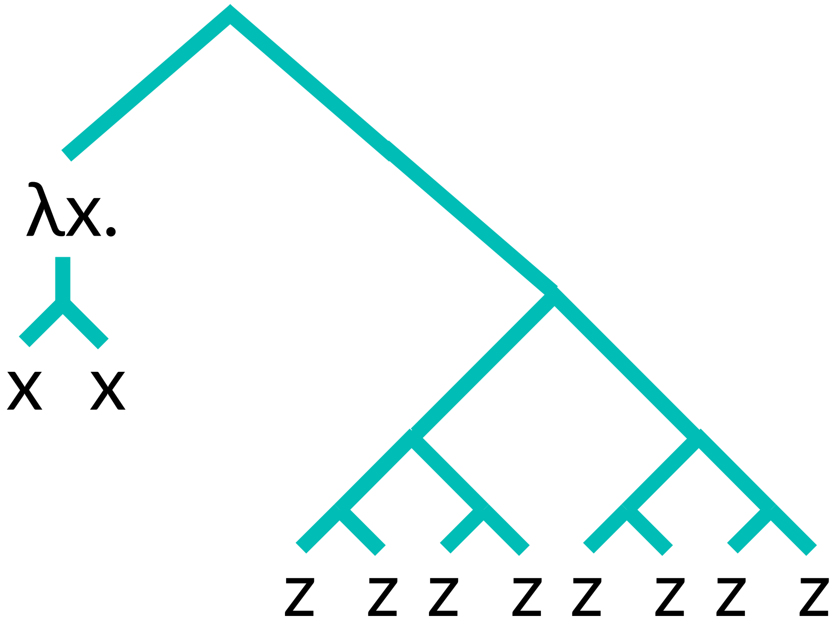
named in
[Accattoli, Dal Lago, CSL-LICS 2014]

Size Explosion Problem



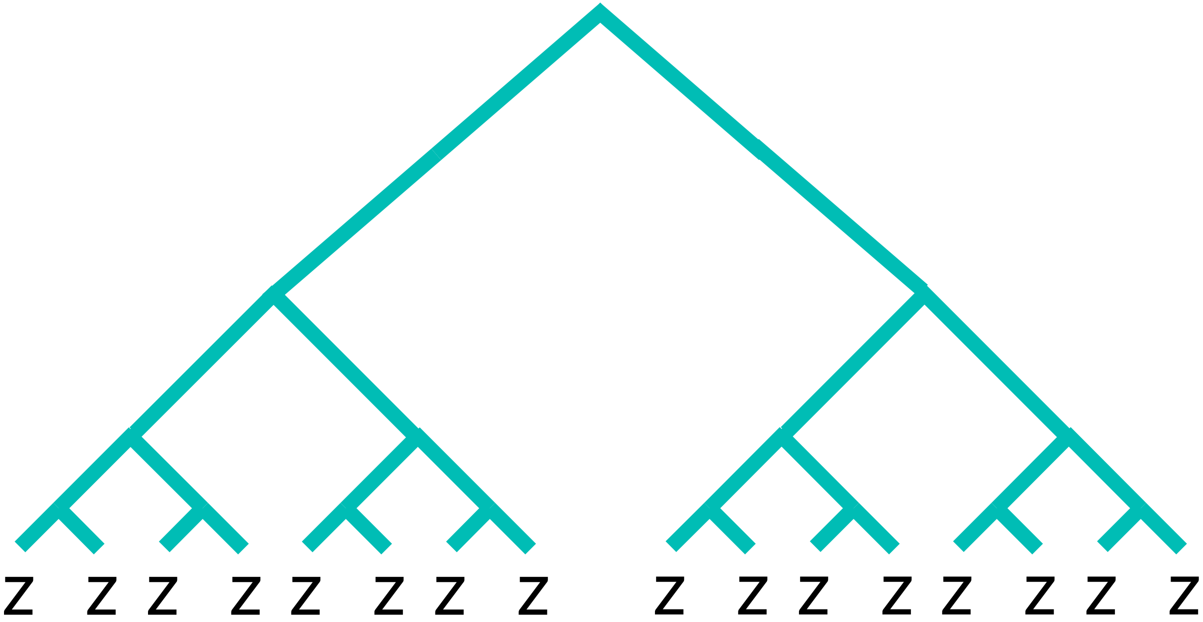
named in
[Accattoli, Dal Lago, CSL-LICS 2014]

Size Explosion Problem



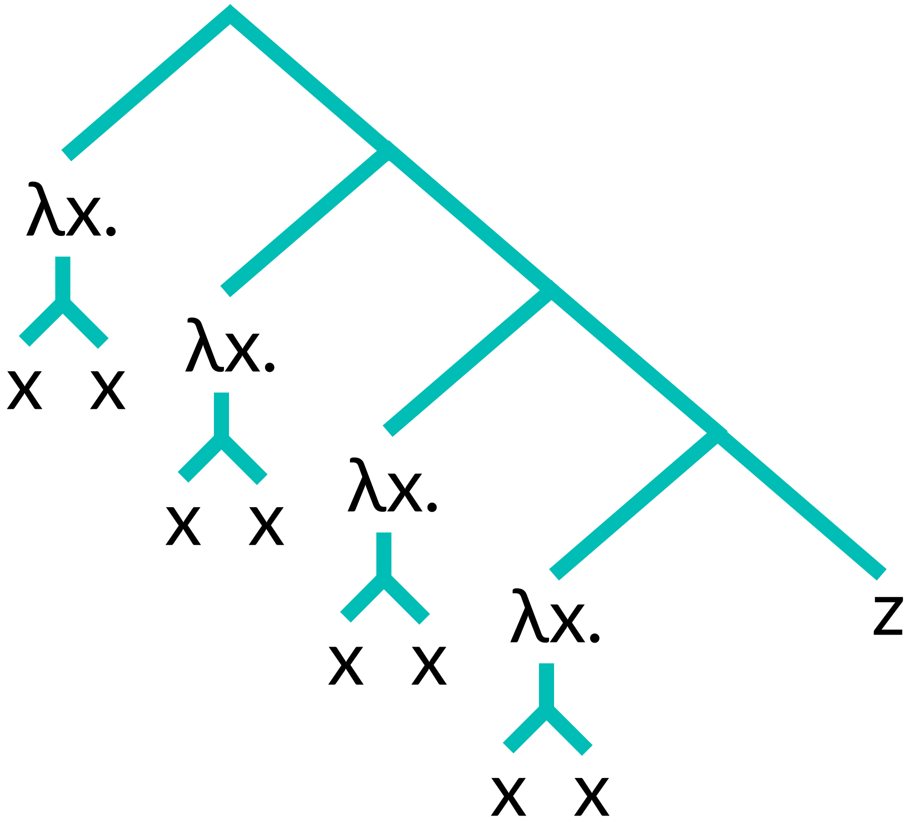
named in
[Accattoli, Dal Lago, CSL-LICS 2014]

Size Explosion Problem

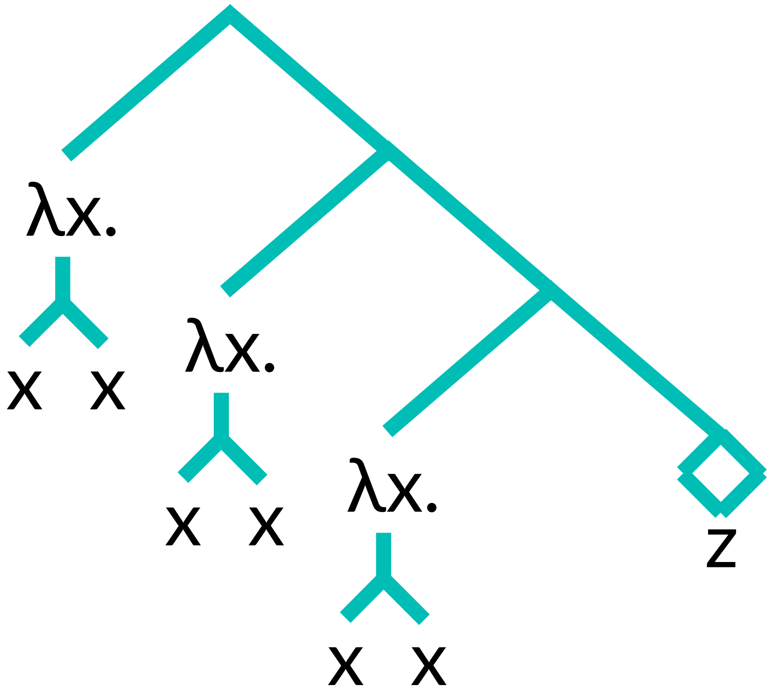


named in
[Accattoli, Dal Lago, CSL-LICS 2014]

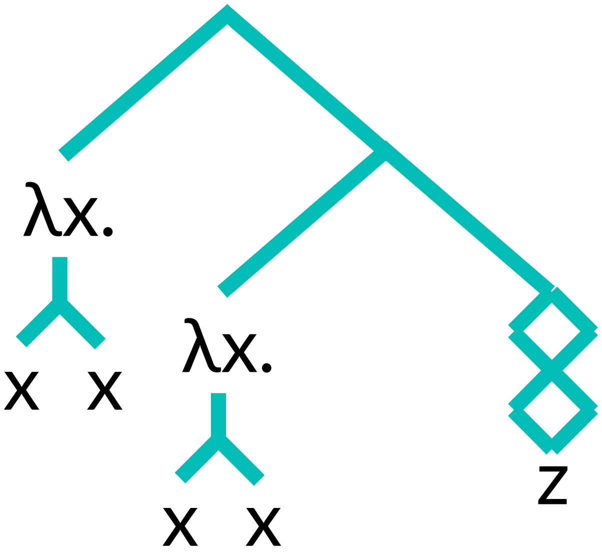
Implicit Sharing



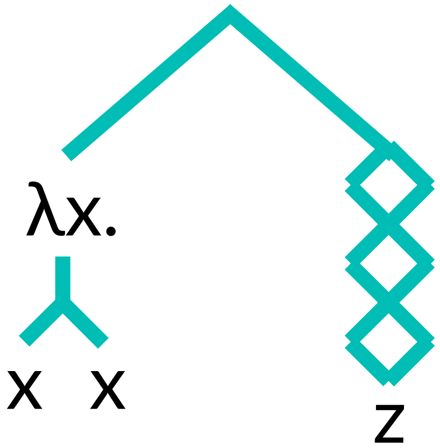
Implicit Sharing



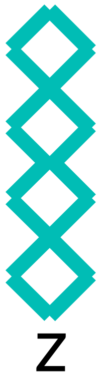
Implicit Sharing



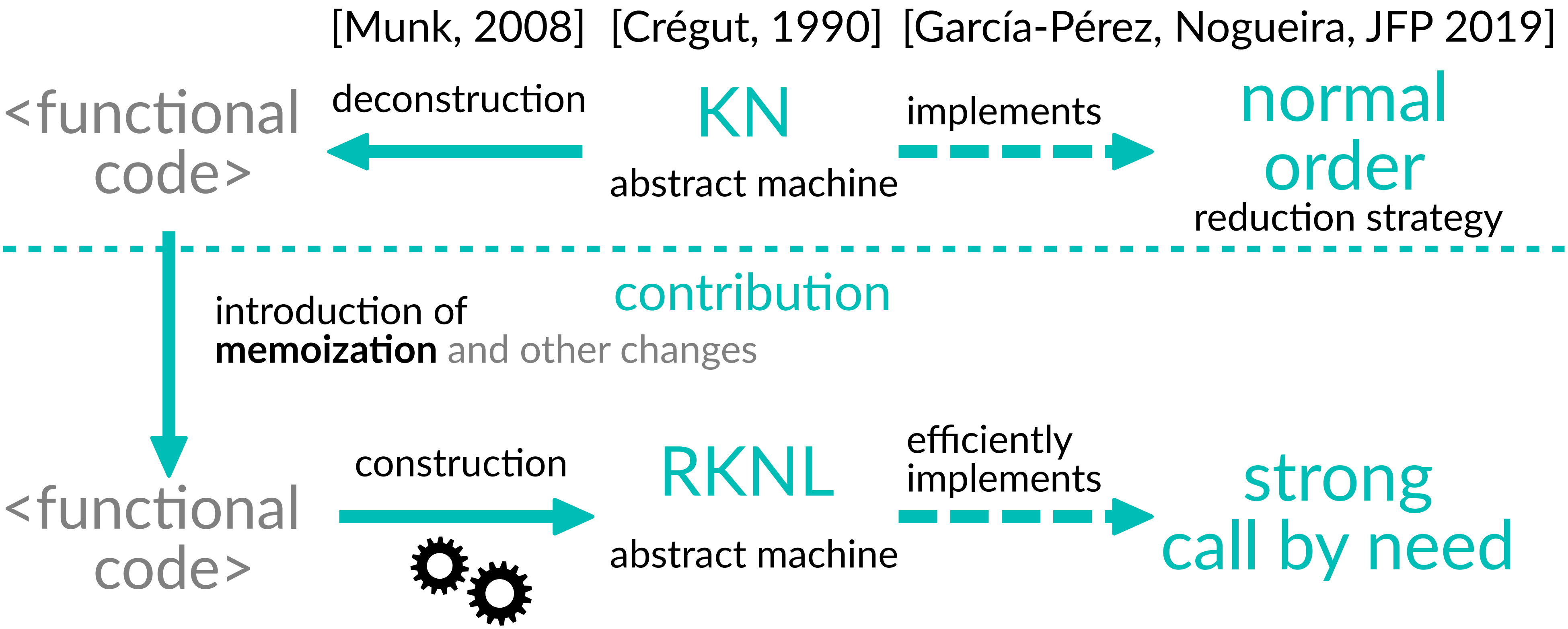
Implicit Sharing



Implicit Sharing



Summary


[Munk, 2008]
[Crégut, 1990]
[García-Pérez, Nogueira, JFP 2019]
<functional code>
deconstruction
KN
abstract machine
implements
normal order
reduction strategy
contribution
introduction of
memoization and other changes
<functional code>
construction
RKNL
abstract machine
efficiently implements
strong call by need

Bonus

Table 2. Elaborate example execution in refocusing notation

0 :	$\langle((\lambda x.c\,x\,x)\,(A\,\Omega),[])\rangle_{\triangledown}\,\|\,[]$	$\xrightarrow{(1)}$
1 :	$\langle(\lambda x.c\,x\,x,[])\rangle_{\triangledown}\,(A\,\Omega,[])\,\|\,[]$	$\xrightarrow{(2)}$
2 :	$\langle \mathbb{a} := (\lambda x.c\,x\,x,[])\rangle_{\vartriangle}\,(A\,\Omega,[])\,\|\,[\mathbb{a}\mapsto\bot\,\text{✗}]$	$\xrightarrow{(6)}$
3 :	$\langle(c\,x\,x,[x\mapsto\mathbb{x}])\rangle_{\triangledown}\,\|\,[\mathbb{x}\mapsto(A\,\Omega,[])\,\text{✗}]$	$\xrightarrow{(1)}$
4 :	$\langle(c\,x,[x\mapsto\mathbb{x}])\rangle_{\triangledown}\,x^{\mathbb{x}}\,\|\,[\mathbb{x}\mapsto(A\,\Omega,[])\,\text{✗}]$	$\xrightarrow{(1)}$
5 :	$\langle(c,[x\mapsto\mathbb{x}])\rangle_{\triangledown}\,x^{\mathbb{x}}\,x^{\mathbb{x}}\,\|\,[\mathbb{x}\mapsto(A\,\Omega,[])\,\text{✗}]$	$\xrightarrow{(4)}$
6 :	$\langle c\rangle_{\vartriangle}\,x^{\mathbb{x}}\,x^{\mathbb{x}}\,\|\,[\mathbb{x}\mapsto(A\,\Omega,[])\,\text{✗}]$	$\xrightarrow{(9)}$
7 :	$c\,\langle x^{\mathbb{x}}\rangle_{\triangledown}\,x^{\mathbb{x}}\,\|\,[\mathbb{x}\mapsto(A\,\Omega,[])\,\text{✗}]$	$\xrightarrow{(3)}$
8 :	$c\,\left(\mathbb{x} := \langle(A\,\Omega,[])\rangle_{\triangledown}\right)\,x^{\mathbb{x}}\,\|\,[\mathbb{x}\mapsto(A\,\Omega,[])\,\text{✗}]$	$\xrightarrow{(1)}$

$$9:\quad c\left(\mathbb{x}:=\langle(A,[])\rangle_{\triangledown}\ (\Omega,[])\right)\ x^{\mathbb{x}}\,|\,[\mathbb{x}\mapsto(A\,\Omega,[])_{\text{✗}}] \quad\xrightarrow{(2)}$$

$$10:\quad c\left(\mathbb{x}:=\langle\mathbb{b}:=(A,[])\rangle_{\vartriangle}\ (\Omega,[])\right)\ x^{\mathbb{x}}\,|\,[\mathbb{x}\mapsto(A\,\Omega,[])_{\text{✗}},\mathbb{b}\mapsto\bot_{\text{✗}}] \quad\xrightarrow{(6)}$$

$$11:\quad c\left(\mathbb{x}:=\langle(\lambda z.I\,z,[y\mapsto\mathbb{y}])\rangle_{\triangledown}\right)\ x^{\mathbb{x}}\,|\,[\mathbb{x}\mapsto(A\,\Omega,[])_{\text{✗}},\mathbb{y}\mapsto(\Omega,[])_{\text{✗}}] \quad\xrightarrow{(2)}$$

$$12:\quad c\left(\mathbb{x}:=\langle\mathbb{d}:=(\lambda z.I\,z,[y\mapsto\mathbb{y}])\rangle_{\vartriangle}\right)\ x^{\mathbb{x}}\,|\,[\mathbb{x}\mapsto(A\,\Omega,[])_{\text{✗}},\mathbb{d}\mapsto\bot_{\text{✗}},\mathbb{y}\mapsto(\Omega,[])_{\text{✗}}] \quad\xrightarrow{(5)}$$

$$13:\quad c\,\langle\mathbb{d}:=(\lambda z.I\,z,[y\mapsto\mathbb{y}])\rangle_{\vartriangle}\ x^{\mathbb{x}}\,|\,\sigma_1*[\mathbb{d}\mapsto\bot_{\text{✗}}] \quad\xrightarrow{(7)}$$

$$14:\quad c\left(\mathbb{d}:=\lambda z_0.\langle(I\,z,e_{yz})\rangle_{\triangledown}\right)\ x^{\mathbb{x}}\,|\,\sigma_1*[\mathbb{d}\mapsto\bot_{\text{✗}},\mathbb{z}\mapsto z_{0\checkmark}] \quad\xrightarrow{(1)}$$

$$15:\quad c\left(\mathbb{d}:=\lambda z_0.\langle(I,e_{yz})\rangle_{\triangledown}\ (z,e_{yz})\right)\ x^{\mathbb{x}}\,|\,\sigma_1*[\mathbb{d}\mapsto\bot_{\text{✗}},\mathbb{z}\mapsto z_{0\checkmark}] \quad\xrightarrow{(2)}$$

$$16:\quad c\left(\mathbb{d}:=\lambda z_0.\langle\mathbb{e}:=(I,e_{yz})\rangle_{\vartriangle}\ (z,e_{yz})\right)\ x^{\mathbb{x}}\,|\,\sigma_1*[\mathbb{d}\mapsto\bot_{\text{✗}},\mathbb{e}\mapsto\bot_{\text{✗}},\mathbb{z}\mapsto z_{0\checkmark}] \quad\xrightarrow{(6)}$$

$$17:\quad c\left(\mathbb{d}:=\lambda z_0.\langle(x,e_{yz}*[x\mapsto\mathbb{w}])\rangle_{\triangledown}\right)\ x^{\mathbb{x}}\,|\,\sigma_1*[\mathbb{d}\mapsto\bot_{\text{✗}},\mathbb{w}\mapsto(z,e_{yz})_{\text{✗}},\mathbb{z}\mapsto z_{0\checkmark}] \quad\xrightarrow{(3)}$$

$$18:\quad c\left(\mathbb{d}:=\lambda z_0.\mathbb{w}:=\langle(z,e_{yz})\rangle_{\triangledown}\right)\ x^{\mathbb{x}}\,|\,\sigma_1*[\mathbb{d}\mapsto\bot_{\text{✗}},\mathbb{w}\mapsto(z,e_{yz})_{\text{✗}},\mathbb{z}\mapsto z_{0\checkmark}] \quad\xrightarrow{(4)}$$

$$19:\quad c\left(\mathbb{d}:=\lambda z_0.\mathbb{w}:=\langle z_0\rangle_{\triangle}\right) x^{\mathbb{x}} \mid \sigma_1 * [\mathbb{d}\mapsto \bot_{\times}, \mathbb{w}\mapsto (z,e_{yz})_{\times}] \quad \xrightarrow{(5)}$$

$$20:\quad c\left(\mathbb{d}:=\lambda z_0.\langle z_0\rangle_{\triangle}\right) x^{\mathbb{x}} \mid \sigma_1 * [\mathbb{d}\mapsto \bot_{\times}] \quad \xrightarrow{(11)}$$

$$21:\quad c\left(\mathbb{d}:=\langle \lambda z_0.z_0\rangle_{\triangle}\right) x^{\mathbb{x}} \mid \sigma_1 * [\mathbb{d}\mapsto \bot_{\times}] \quad \xrightarrow{(5)}$$

$$22:\quad c\,\langle \lambda z_0.z_0\rangle_{\triangle}\, x^{\mathbb{x}} \mid \sigma_1 * [\mathbb{d}\mapsto \lambda z_0.z_{0\,\checkmark}] \quad \xrightarrow{(10)}$$

$$23:\quad \langle c\,\lambda z_0.z_0\rangle_{\triangle}\, x^{\mathbb{x}} \mid \sigma_1 * [\mathbb{d}\mapsto \lambda z_0.z_{0\,\checkmark}] \quad \xrightarrow{(9)}$$

$$24:\quad c\,(\lambda z_0.z_0)\,\langle x^{\mathbb{x}}\rangle_{\triangledown} \mid \sigma_1 * [\mathbb{d}\mapsto \lambda z_0.z_{0\,\checkmark}] \quad \xrightarrow{(4)}$$

$$25:\quad c\,(\lambda z_0.z_0)\,\langle \mathbb{d}:=(\lambda z.I\,z,[y\mapsto \mathbb{y}])\rangle_{\triangle} \mid [\mathbb{d}\mapsto \lambda z_0.z_{0\,\checkmark}, \mathbb{y}\mapsto (\Omega,[])_{\times}] \quad \xrightarrow{(8)}$$

$$26:\quad c\,(\lambda z_0.z_0)\,\langle \lambda z_0.z_0\rangle_{\triangle} \mid [] \quad \xrightarrow{(10)}$$

$$27:\quad \langle c\,(\lambda z_0.z_0)\,\lambda z_0.z_0\rangle_{\triangle} \mid [] \quad \not\rightarrow$$