Taller 10

1) Ganar con prob. p.
Perder con prob. $1-p$. $F_0 = x$.
$p > \frac{1}{2}$ se apuesta $(2p-1)x$

para $k=1$, $X_k = X_1$ y $X_0 = x$.

$$\mathbb{E}(X_1 | X_0 = x) = p\left(x + (2p-1)x\right) + (1-p)\left(x - (2p-1)x\right).$$

$$= px + px(2p-1) + (1-p)x - (1-p)(2p-1)x$$

$$= px + 2p^2x - px + x - px + (-2p + 1 + 2p^2 - p)x$$

$$= \cancel{px} + 2p^2x - \cancel{px} + x - px - 2px + x + 2p^2x - \cancel{px}$$

$$= -4px + 4p^2x + 2x$$

Ahora, $\mathbb{E}(X_k | X_{k-1}) = 4p^2 X_{k-1} - 4p X_{k-1} + 2X_{k-1}$

$$\mathbb{E}(X_k) = \mathbb{E}(\mathbb{E}(X_k | X_{k-1})) = 4p^2\,\mathbb{E}(X_{k-1}) - 4p\,\mathbb{E}(X_{k-1}) + 2\,\mathbb{E}(X_{k-1})$$

$$= \mathbb{E}(X_{k-1}) \cdot (4p^2 - 4p + 2)$$

Pero $\mathbb{E}(X_{k-1}) = \mathbb{E}(X_{k-2})(4p^2 - 4p + 2)$.

luego, $\mathbb{E}(X_k) = (4p^2 - 4p + 2)^2\, \mathbb{E}(X_{k-2})$

Generalizando.

$$\mathbb{E}(X_k) = (4p^2 - 4p + 2)^n X$$

2) N llega a tiempo.
P " una hora aleatoria unif. entre 8 y 10 pm.

X: tiempo entre las 8pm y la hora que llega P.
(X toma valores [0,2]

Si P llega antes de las 9 pm. la cita dura 3 horas
" " " desp. " " " " " unif. entre 0 y 3-X horas

N terminará si P llega más de 45 min tarde 2 veces.

a) ~~$E(X) = P(X\geq 1)E(X|X\geq 1) + P(X\leq 1)E(X|X\leq 1)$~~

Sea Y tiempo de espera.

$E(Y) = E(Y| X \leq 1)\, P(X \leq 1) + E(Y|X \geq 1)\, P(X \geq 1)$.

$= 0 + \frac{1}{2} \cdot E(Y / X \geq 1)$.

Si $X \geq 1$, $Y = X - 1$.

$\Rightarrow E(Y) = \frac{1}{2} \cdot E(X - 1 \mid X \geq 1)$.

$E(X/X\geq 1) = \int_1^2 x\, dx = \frac{x^2}{2}\Big|_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$.

$\Rightarrow E(Y) = \frac{1}{2} \cdot (E(X|X\geq 1) - 1) = \frac{1}{2}\left(\frac{3}{2} - 1\right) = \frac{1}{4}$.

Scanned by CamScanner

Si $X \geq 1$

$E(Y \mid X) = \frac{3-X}{2}$

b) Sea Y duración de cita.

$$E(Y) = E(Y \mid X \leq 1)\,P(X \leq 1) + E(Y \mid X \geq 1)\,P(X \geq 1)$$

$$= 3 \cdot \frac{1}{2} + E(Y \mid X \geq 1)\,\frac{1}{2}$$

$$= 3 \cdot \frac{1}{2} + E\left(\frac{3-X}{2} \,\Big|\, X \geq 1\right)\frac{1}{2}$$

$$= 3 \cdot \frac{1}{2} + \frac{1}{2}\int_1^2 \left(\frac{3-x}{2}\right) dx$$

$$= \frac{3}{2} + \left(\frac{1}{4}\left(3x - \frac{x^2}{2}\right)\Big|_1^2\right)$$

$$= \frac{3}{2} + \frac{1}{4}\left(3x - \frac{x^2}{2}\Big|_1^2\right)$$

$$= \frac{3}{2} + \frac{1}{4}\left(6 - 2 - 3 + \frac{1}{2}\right)$$

$$= \frac{3}{2} + \frac{6}{4} - \frac{2}{4} - \frac{3}{4} + \frac{1}{8}$$

$$= \frac{12 + 12 - 4 - 6 + 1}{8} = \frac{15}{8}.$$

c) Sea Y_1: # citas hasta que llega tarde 1ra vez.

Y_1 es geométrica con parametro $p = P(X > 1 + \frac{3}{4})$

$= P(X > \frac{7}{4}) = \frac{1}{8}$.

Luego $\mathbb{E}(Y_1) = 8$.

Ahora sea Y_2: # citas entre 1ra llegada tarde y 2da llegada tarde.

$\mathbb{E}(Y_2) = 8$.

Luego. $Y = Y_1 + Y_2$

$\mathbb{E}(Y) = 2 \cdot \mathbb{E}(Y_1) = 16$.

3) a) $\lambda(y) = 1/(5-y)$

Sea Y: tiempo después de las nueve. que llega prof.

T es exp. con param. $\lambda(y) = \frac{1}{5-y}$.

$\mathbb{E}(T \mid Y = y) = 5 - y \Rightarrow \mathbb{E}(T \mid Y) = 5 - Y$.

$\mathbb{E}(T) = \mathbb{E}(\mathbb{E}(T \mid Y)) = \mathbb{E}(5 - Y) = 5 - \mathbb{E}(Y) = 5 - 2 = 3$

b) Sea $Z = Y + T$

$\mathbb{E}(Z) = \mathbb{E}(Y + T) = 3 + \mathbb{E}(Y) = 3 + 2 = 5$. Luego la hora esperada es $9 + 5 = 14$. (2 pm).

4) X : v.a con f.m.p.

$$p_X(k) = \begin{cases} \frac{1}{2} & k=1 \\ \frac{1}{4} & k=2,3 \\ 0 & \text{d.l.c.} \end{cases}$$

Vamos a calcular la F.G.M de X.

$$M_X(s) = E(e^{sX}) = \frac{1}{2}e^{s} + \frac{1}{4}e^{2s} + \frac{1}{4}e^{3s}$$

Ahora $\frac{d}{ds}M_X(s) = \frac{1}{2}e^{s} + 2\cdot\frac{1}{4}e^{2s} + \frac{3}{4}e^{3s}$.

$$E(X) = \frac{d}{ds}M_X(s)\Big|_{s=0} = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = \frac{7}{4}$$

$$\frac{d^2}{ds^2}M_X(s) = \frac{1}{2}e^{s} + e^{2s} + \frac{9}{4}e^{3s}$$

$$E(X^2) = \frac{d^2}{ds^2}M_X(s)\Big|_{s=0} = \frac{1}{2} + 1 + \frac{9}{4} = \frac{15}{4}$$

$$\frac{d^3}{ds^3}M_X(s) = \frac{1}{2}e^{s} + 2e^{2s} + \frac{27}{4}e^{3s}$$

$$E(X^3) = \frac{d^3}{ds^3}M_X(s)\Big|_{s=0} = \frac{1}{2} + 2 + \frac{27}{4} = \frac{37}{4}.$$

5) Sabemos que la F.G.M de la normal estándar Z es ④

$M_Z(s) = e^{s^2/2}$.

Luego, derivando.

$$\frac{dM_Z(s)}{ds} = \frac{\cancel{2}s}{\cancel{2}} \cdot e^{s^2/2} = s \cdot e^{s^2/2}.$$

$$\frac{d^2M_Z(s)}{ds^2} = e^{s^2/2} + s\left(\frac{\cancel{2}s}{\cancel{2}}\right)e^{s^2/2} = e^{s^2/2} + s^2 e^{s^2/2}.$$

$$\frac{d^3M_Z(s)}{ds^3} = s e^{s^2/2} + \left(2s\, e^{s^2/2} + s^2 \cdot s\, e^{s^2/2}\right)$$

$$= s e^{s^2/2} + (2s) e^{s^2/2} + s^3 e^{s^2/2}$$

$$\frac{d^4M_Z(s)}{ds^4} = e^{s^2/2} + s^2 e^{s^2/2} + 2e^{s^2/2} + 2s\left(\frac{2s}{2} e^{s^2/2}\right) + 3s^2 e^{s^2/2} + s^3 \cdot s\, e^{s^2/2}$$

Luego

$$\mathbb{E}(Z^3) = \left.\frac{d^3M_Z(s)}{ds^3}\right|_{s=0} = 0.$$

$$\mathbb{E}(Z^4) = \frac{d^4M_Z(s)}{ds^4} = 1 + 2 = 3.$$

6) La F.G.M de X es. $M_X(r) = \dfrac{\lambda}{\lambda - s}$.

$$\frac{dM_X(s)}{ds} = \frac{\lambda}{(\lambda-s)^2}. \quad \frac{d^2M_X(s)}{ds^2} = 2\lambda \cdot (\lambda-s)^{-3}; \quad \frac{d^3M_X(s)}{ds^3} = 2\lambda \cdot 3 \cdot (\lambda-s)^{-4} = 6\lambda \cdot (\lambda-s)^{-4}.$$

$$\frac{d^4M_X(s)}{ds^4} = 24\lambda \cdot (\lambda-s)^{-5} \quad \frac{d^5M_X(s)}{ds^5} = 120\lambda \cdot (\lambda-s)^{-6}.$$

Luego $\mathbb{E}(X^3) = \dfrac{6\lambda}{\lambda^4} = \dfrac{6}{\lambda^3}$. $\mathbb{E}(X^4) = \dfrac{24\lambda}{\lambda^5} = \dfrac{24}{\lambda^4}$, $\mathbb{E}(X^5) = \dfrac{120\lambda}{\lambda^6} = \dfrac{120}{\lambda^5}$

7) a) $M_x^1(s) = e^{2(e^{e^s-1}-1)}$, $M_x^2(s) = e^{2(e^{e^s}-1)}$.

$M_x(s) = \mathbb{E}(e^{sx})$

$M_x(0) = \mathbb{E}(e^{s0}) = \mathbb{E}(1) = 1.$

$M_x^1(0) = e^{2(e^{1-1}-1)} = e^{2(0)} = 1.$

$M_x^2(0) = e^{2(e-1)} \neq 1.$

b) A demonstrar. $\lim_{s\to-\infty} M_x(s) = P(x=0)$

$M_x(s) = \mathbb{E}(e^{xs}) = \sum_{i=0}^{\infty} P(x=i)\, e^{is}$

$= \lim_{s\to-\infty} M_x(s) = \lim_{s\to-\infty} \sum_{i=0}^{\infty} P(x=i)\, e^{is}.$

$= \sum_{i=0}^{\infty} \lim_{s\to-\infty} P(x=i)\, e^{is}$

si $i \neq 0$. $\lim_{s\to-\infty} e^{is} = 0$. Luego

$\lim_{s\to-\infty} M_x(s) = \lim_{s\to-\infty} P(x=0)\, e^{0s} = P(x=0).$

c) $P(x=0) = \lim_{s\to-\infty} e^{2(e^{e^s-1}-1)} = e^{2(e^{-1}-1)}$

⑧

X tiene

$$M_X(s) = \frac{1}{3}\left(\frac{2}{2-s}\right) + \frac{2}{3}\left(\frac{3}{3-s}\right).$$

X con prob $\frac{1}{3}$ es exp con param. $\lambda = 2$

X con " $\frac{2}{3}$ " " " " $\lambda = 3$.

Luego $f_X(x) = \begin{cases} \frac{1}{3} \cdot 2e^{-2x} + \frac{2}{3} 3 e^{-3x} & x \geq 0 \\ 0 & d.l.c \end{cases}$

$$f_X(x) = \begin{cases} \frac{2}{3} e^{-2x} + 2e^{-3x} & x \geq 0 \\ 0 & d.l.c. \end{cases}$$

9)

a) $X = X_1 + X_2$. X_1 y X_2 son Bernoulli con parámetros p_1 y p_2.

$$p_X(k) = \sum_x p_{X_1}(x)\, p_{X_2}(k-x).$$

$$p_X(k) = \begin{cases} (1-p_1)(1-p_2) & k = 0 \\ (1-p_1)p_2 + p_1(1-p_2) & k = 1 \\ p_1 p_2 & k = 2 \end{cases}$$

b) $M_X(s) = M_{X_1}(s) M_{X_2}(s) = (1-p_1+p_1e^s)(1-p_2+p_2e^s)$

$$= \underbrace{(1-p_1)(1-p_2)}_{P(X=0)} e^{0s} + \underbrace{(1-p_2)p_1 e^s + (1-p_1)p_2 e^s}_{P(X=1)} + \underbrace{p_1p_2}_{P(X=2)} e^s.$$

10) X, Y, Z indep. X Bernoulli param $p = \frac{1}{3}$.
Y. exp. con parám 2. y Z Poisson con Param 3.

a) $U = XY + (1-X)Z$.

U toma valor Y con prob. $\frac{1}{3}$.
U " " Z con " $\frac{2}{3}$.

$$M_U(s) = \frac{1}{3} M_Y(s) + \frac{2}{3} M_Z(s).$$

$$= \frac{1}{3} \frac{2}{2-s} + \frac{2}{3} e^{3(e^{s-1})}$$

b) $W = 2Z + 3$.

$$M_W(s) = \mathbb{E}(e^{Ws}) = \mathbb{E}(e^{2Zs+3s}) =$$

$$= e^{3s} M_Z(2s)$$

$$= e^{3s} e^{3(e^{2s}-1)}$$

c) $W = Y + Z$.

$$M_W(s) = M_Y(s) M_Z(s) = \frac{2}{2-s} e^{3(e^s-1)}.$$

Scanned by CamScanner

(11) Sean $\{X_i\}_{i=1}^{n}$ $X_i = 1$ si al menos un cliente pide pizza i.

$X_i = 0$ si no.

$$X = X_1 + \dots + X_n.$$

$\mathbb{E}(X \mid k) \longrightarrow k = \#$ clientes. $M_k(s) = \mathbb{E}(e^{sk})$ conocida.

~~$\mathbb{E}(X\mid k)$~~

Sabemos $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid k)) = \mathbb{E}(\mathbb{E}(X_1 + \dots + X_n \mid k))$

* Cada X_i es Bernoulli con prob.

$$p = 1 - \left(\frac{n-1}{n}\right)^k.$$

Luego.

$$\mathbb{E}(X) = \mathbb{E}(n\,\mathbb{E}(X_1 \mid k)) = \mathbb{E}\left(n\left(1 - \left(\frac{n-1}{n}\right)^K\right)\right)$$

$$= n\,\mathbb{E}\left(1 - \left(\frac{n-1}{n}\right)^k\right) = n\left(1 - \mathbb{E}\left(\left(\frac{n-1}{n}\right)^k\right)\right)$$

$$= n\left(1 - \mathbb{E}\left(e^{\log\left(\frac{n-1}{n}\right)^k}\right)\right) = n\left(1 - \mathbb{E}\left(e^{k\log\left(\frac{n-1}{n}\right)}\right)\right)$$

$$= n\left(1 - M_k\left(\log\left(\frac{n-1}{n}\right)\right)\right)$$