AN EXPLORATION OF THE CAYLEY GRAPH AND GEODESICS OF THE BRAID GROUP ON FOUR STRANDS

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Abstract. Braid groups comprise an important class of groups in Group Theory with applications to Knot Theory and Geometry. Cayley Graphs are a vital tool used to understand group structure. In this light, we build the Cayley Graph Γ of a well-behaved subgroup of the Braid Group on four strands with its natural presentation. Further, we explicitly embed this Cayley Graph into $\mathbb{R}^3$. We use this result to explore the larger Cayley Graph of the Braid Group of four strands.

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1. Introduction

This paper is concerned with the Braid group on four strands given by the presentation

$$B_4 := \langle a, b, c \mid aba = bab, cbc = bcb, ac = ca \rangle.$$

We call this presentation of the Braid Group on four strands the *natural* presentation, since the generators and relations most naturally describe the braiding construction of the group.

The braiding construction that naturally generates this group involves braiding four strings together and fixing the endpoints. The generating elements a, b, and c are described in Figure 1 below:

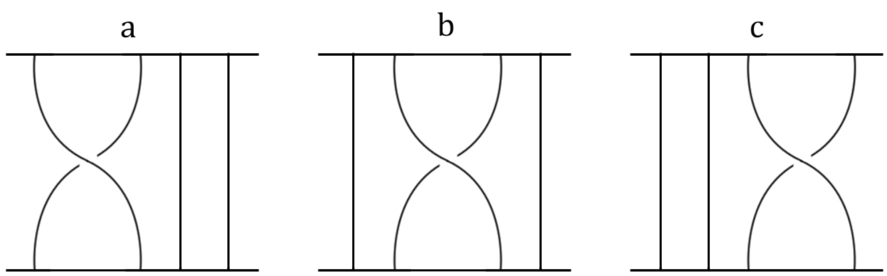

Figure 1. Generators of B_4: a, b, c.

The construction of an element of B_4 using the above generators follows the *sequential* rule. In particular, the braid is constructed using a, b, and c sequentially and in the direction of the braid. Figure 2 below is constructed from left to right, by placing the above generators in the sequence designated (in this case, ab^2c).

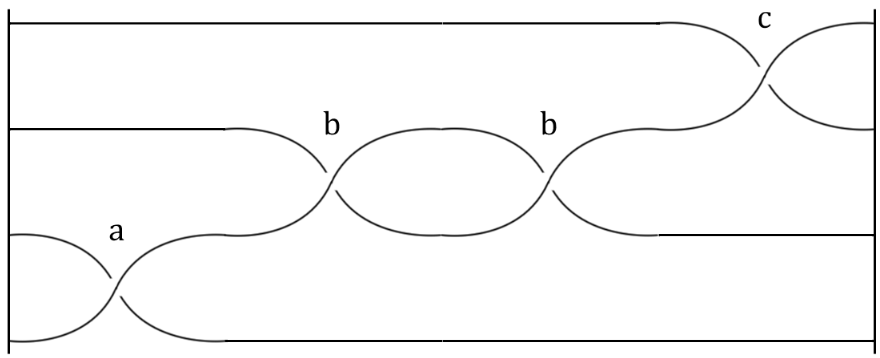

FIGURE 2. Construction of the element $ab^2c \in B_4$ using the natural generators.

A number of generating sets of the Braid group have been extensively studied, an example of which is discussed by Charney and Meier in [2]. We note here that the natural presentation of B_n has been less rigorously studied, and therefore provides us with an opportunity for discovery.

In [12, §3.2], Sabalka builds the Cayley Graph of the Braid Group on three strands via a well-behaved subgroup. Using the Cayley Graph of the subgroup, Sabalka forms a disjoint union of cosets of the subgroup which then connect to form the Cayley graph of B_3. In this paper, we explore the Braid Group on four strands in a similar way. We will examine a subgroup of B_4 and construct its Cayley graph.

Main Result: We explore the Braid Group on 4 strands via a subgroup $H = \langle a^2, b^2, c^2 \rangle \leq B_4$. We develop an algorithm φ which explicitly embeds the Cayley graph of H into $(-1,1)^3 \subseteq \mathbb{R}^3$.

The paper is organized as follows: Section 2 discusses fundamental definitions and concepts, including words and groups, Cayley graphs, geodesics, normal forms and rewriting systems. This background includes an exposition of the normal forms and geodesics of the group $\mathbb{Z} * \mathbb{Z}^2$, which plays a vital role in this paper. Then we proceed in Section 3 to embed the chosen subgroup H into the Euclidean cube $(-1,1)^3$ and discuss tools and strategies to construct the Cayley graph of B_4. In the last section, we discuss potential applications of the results developed in Section 3.

2. Background

2.1. **Words and Groups.** Let G be a group with presentation $\langle A \mid R\rangle$, where A represents a list of elements that generate G and R represents a list of relators that the generators satisfy from which all relators can be derived up to free reduction. Since the elements of A generate G, we can represent any element of G as a string of symbols from the alphabet $A^{\pm 1} := \{s^{\pm 1} \mid s \in A\}$. In general, we will denote the set of all strings of symbols from the alphabet A as A^*. Furthermore, for $x \in A$, we will often denote x^{-1} by X. So, each group element $g \in G$ can be represented by an element $\hat{g} \in (A^{\pm 1})^*$. This representation, however, is not unique. For instance, in the group $\langle a, b \mid ab = ba\rangle$, the group element a can be written as the strings $\hat{a_1} = a$ or $\hat{a_2} = aa^{-1}a$ or $\hat{a_3} = aaba^{-1}b^{-1}$. Notice in the last equation, the group element $aaba^{-1}b^{-1}$ is equal to the group element a since the relation $ab = ba$ implies that $(ab)a^{-1}b^{-1} = (ba)a^{-1}b^{-1} = 1$. This lack of unique representation of elements $g \in G$ in the set of strings $(A^{\pm 1})^*$ happens because $(A^{\pm 1})^*$ does not respect the group structure of G, namely that inverses should annihilate one another and relators should vanish.

Definition 1. *Let G be a group with presentation $\langle A \mid R\rangle$. Whenever a string $\hat{g} \in (A^{\pm 1})^*$ represents an element $g \in G$, we call $\hat{g}$ a* **word** *representing g. Whenever u and v are words in $(A^{\pm 1})^*$ that represent the same group element, we write* $\mathbf{u} \equiv \mathbf{v}$. *For any word $w \in (A^{\pm 1})^*$, we denote the number of symbols in w as $|w|$. Any subset $L \subseteq (A^{\pm 1})^*$ is called a* **language** *in G.*

2.2. **Cayley Graphs.** In this paper, we will use the word **graph** to mean a directed, edge-labeled graph. For a graph M, we call the set of its vertices V_M and the set of its edges E_M. An edge $e \in E_M$ from $u \in V_M$ to $v \in V_M$ with label s is denoted $e = [u, v, s]$.

Definition 2. *The* **Cayley Graph** $\Gamma = \Gamma(G, A)$ *for a finitely presented group $G := \langle A \mid R\rangle$ is a graph with vertices consisting of group elements and edges labeled by generators. More precisely, $V_\Gamma = G$ and $E_\Gamma = \{[g, gs, s] \mid g \in G, s \in A^{\pm 1}\}$. If a generating set is understood, we may simply write $\Gamma(G)$ to denote the Cayley Graph. We call $\alpha \in A$ a* **positive** *label in Γ and $\beta \in A^{-1}$ a negative label. Whenever drawing or defining Γ, we will neglect the negative labels.*

Notice that when we read the edge-labels of a path from the vertex $1 \in G$ to a vertex $g \in G$ in Γ, we have read a word from $(A^{\pm 1})^*$ that represents g. For this reason, we will not distinguish between the label of a path from the identity to a vertex g and the corresponding word that represents g. Furthermore, if we consider each edge in the Cayley Graph of B_4 to have a length of 1 unit, then the length of a path from the identity to a vertex g is exactly the word length of the corresponding word representing g. In particular, the set of geodesics, or shortest paths, in the Cayley Graph of a group is exactly the set of words that use the minimal number of letters to represent their corresponding group elements. This prompts the following definition.

Definition 3. *In general, for a finitely presented group $G = \langle A \mid R\rangle$, we call the set*

$$Geo(G,A) := \left\{w \in (A^{\pm 1})^* \mid \ |w| = \min\{\ |v| \ \mid v \equiv w\}\right\}$$

the set of **geodesics** *of G with respect to A.*

2.3. **Geodesics of $\mathbb{Z} * \mathbb{Z}^2$.** It will become important in this paper to understand the geodesics of $\mathbb{Z} * \mathbb{Z}^2$ presented by

$$\langle x, y, z \mid xz = zx\rangle.$$

The set of geodesics $Geo(\mathbb{Z} * \mathbb{Z}^2)$ is well-known (see, for example, [6] or [9]), and will be instrumental in creating a formula to embed the Cayley Graph of a particular subgroup of B_4 into $\mathbb{R}^3$.

In order to describe the geodesics of $\mathbb{Z} * \mathbb{Z}^2$ with respect to $A = \{x, y, z\}$, we must first define three functions:

$$\alpha : (A^{\pm 1})^* \to (\{x^{\pm 1}\} \cup \$)^* \text{ induced by } x^{\pm 1} \mapsto x^{\pm 1},\ y^{\pm 1} \mapsto \$,\ z^{\pm 1} \mapsto 1$$
$$\beta : (A^{\pm 1})^* \to (\{y^{\pm 1}\} \cup \$)^* \text{ induced by } x^{\pm 1} \mapsto \$,\ y^{\pm 1} \mapsto y^{\pm 1},\ z^{\pm 1} \mapsto \$$$
$$\gamma : (A^{\pm 1})^* \to (\{z^{\pm 1}\} \cup \$)^* \text{ induced by } x^{\pm 1} \mapsto 1,\ y^{\pm 1} \mapsto \$,\ z^{\pm 1} \mapsto z^{\pm 1},$$

where 1 is used to denote the empty word. We must also construct three sets:

$$U_x := \{u_0\$u_1\$\ldots\$u_n \mid n \in \mathbb{N}_0,\ u_i \in x^* \cup x^{-1^*}\}$$
$$U_y := \{v_0\$v_1\$\ldots\$v_n \mid n \in \mathbb{N}_0,\ v_i \in y^* \cup y^{-1^*}\}$$
$$U_z := \{w_0\$w_1\$\ldots\$w_n \mid n \in \mathbb{N}_0,\ w_i \in z^* \cup z^{-1^*}\}.$$

Then

$$Geo(\mathbb{Z} * \mathbb{Z}^2) = \alpha^{-1}(U_x) \cap \beta^{-1}(U_y) \cap \gamma^{-1}(U_z)$$

by [6] and [9].

In order to gain some intuition about why these are the geodesic words in $\mathbb{Z} * \mathbb{Z}^2$, it is helpful to think of $\mathbb{Z} * \mathbb{Z}^2$ as a combination of two "independent" objects: $\mathbb{Z}$ and $\mathbb{Z}^2$. That is to say, if one stands at an element of $\langle x, y, z \mid xz = zx\rangle$, there are two independent ways to move. One may move by way of a word from $(\{x, z\}^{\pm 1})^*$ to remain in the current xz-plane, or one may move by way of a word from $(\{y\}^{\pm 1})^*$ to leap to a new xz-plane. So, a geodesic adventure in $\mathbb{Z}*\mathbb{Z}^2$ is one that is geodesic within each xz-plane and does not backtrack upon leaping from one plane to the next. This idea manifests itself in the \$ dummy symbol above. The functions α and γ keep track of the $x's$ and $z's$ respectively and mark transitions between planes by \$. The β function keeps track of the transitions between planes, marking all movement by $x's$ and $z's$ within the plane as \$. So, the set $\alpha^{-1}(U_x)$ characterizes the paths in $\mathbb{Z} * \mathbb{Z}^2$ that keep a consistent x-direction within each xz-plane. These are the words that, within each xz-plane, contain only non-negative powers of x or non-positive powers of x but not both (i.e., there is no backtracking in the x direction for each xz-plane). Similarly, $\gamma^{-1}(U_z)$ characterizes the paths with no backtracking in the z direction for each xz-plane. The set $\beta^{-1}(U_y)$ characterizes the paths that do not contain a y adjacent to y^{-1}. This translates to paths that never jump to a new xz-plane followed by immediately turning around and jumping right back.

So we see that $Geo(\mathbb{Z}*\mathbb{Z}^2)$ characterizes the paths that do not backtrack in the x component of any xz-plane, do not backtrack in the z component of any xz-plane, and do not wastefully jump back and forth between any two xz-planes.

2.4. **Normal Forms and Rewriting Systems.** We discussed in §2.1 that for a group $G = \langle A \mid R\rangle$ and an element $g \in G$, there are many words in $(A^{\pm 1})^*$ that describe g. If we know the geodesics of G, then we can write g using the fewest possible symbols, but it still may be possible that there are many ways to write g using the fewest possible symbols. Think, for instance, of $\mathbb{Z}^2 = \langle a, b \mid ab = ba\rangle$. The element a^3b^4 can be written a^2bab^3 or ab^2abab, with both of these using the fewest letters from the alphabet $\{a, b\}^{\pm 1}$ possible. It is natural to ask the question: Is it possible to take two words and tell whether they represent the same group element? In braid groups, the answer to the question is "yes" by using a rewriting system and its normal forms.

Definition 4. *Let $G = \langle A \mid R\rangle$ be a finitely presented group. A* **set of normal forms** *N for G is a language in $(A^{\pm 1})^*$ such that every element of G is represented exactly once in N.*

Suppose we have a set of normal forms N for a group G and a way of inputting any word $w \in (A^{\pm 1})^*$ and outputting the normal form of the group element w represents. Then, we can tell if two words represent the same group element by looking at the normal form representative for each word and checking to see whether those normal forms are equal. If the normal forms are equal, then the two words represent exactly the same group element.

As an example of normal forms, we turn yet again to $\mathbb{Z}^2 = \langle a, b \mid ab = ba\rangle$. A set of normal forms for $\mathbb{Z}^2$ is $\{a^i b^j \mid i, j \in \mathbb{Z}\}$. These normal forms also happen to be geodesic, but this is not the case in general. In order to find these normal forms, we observe that the commutation relation $ab = ba$ suggests an algorithm for writing any element in a specific form. Since we can always commute any two letters in a word representing a given element without changing that element, we can choose to commute all powers of a to the left and all powers of b to the right. This idea extends to many groups with finite presentation using the idea of a complete rewriting system.

Additionally, the set of normal forms for $\mathbb{Z}^2$ can be used to find a set of normal forms for the group $\mathbb{Z} * \mathbb{Z}^2$ mentioned earlier. The set of geodesics described in §2.3 give representatives for every element of $\mathbb{Z} * \mathbb{Z}^2$ written with the fewest possible symbols from $\{x, y, z\}^{\pm 1}$, but there are many ways to represent elements with the fewest possible symbols. However, the description of normal forms for $\mathbb{Z}^2$ described in the last paragraph can be used to build a subset of $Geo(\mathbb{Z}*\mathbb{Z}^2)$ in which the elements of the group are represented only once. We accomplish this by insisting that our words are not only geodesic, but also that each sub-word over $\{x, z\}^{\pm 1}$ is written in normal form in $\mathbb{Z}^2$. In other words, we have a set of (geodesic) normal forms:

Definition 5. *Define the set of* **normal forms** *for $\mathbb{Z}*\mathbb{Z}^2$, $NF(\mathbb{Z}*\mathbb{Z}^2)$, as the set of elements of the form*

$$g_1 h_1 \dots g_n h_n,$$

with $g_i \in (\{y\})^* \cup (\{y^{-1}\})^*$ and $h_i = x^j z^k$ for some $j, k \in \mathbb{Z}$, where g_i and h_j are not the empty word whenever $i \neq 1$ and $j \neq n$. Furthermore, we will denote the normal form for $g \in \mathbb{Z} * \mathbb{Z}^2$ as $\tilde{g}$.

We will refer to these normal forms heavily in §3 when defining the vector algorithm.

Definition 6. *Let B be a finite set. Let $S \subseteq B^* \times B^*$, where we think of $(u,v) \in S$ as a "rewriting rule" $u \to v$, such that $xuy \to xvy$ whenever $u \to v$ for any words $x, y \in B^*$. Then we call S a* **rewriting system** *over B.*

Definition 7. *Let S be a rewriting system over B that satisfies the following:*

(1) *For every sequence of rewritings $u_1 \to \ldots$, there is some $N \in \mathbb{N}$ such that u_N cannot be rewritten. (In other words, every sequence of rewritings* **terminates***. Also, we call u_N an* **irreducible word***. Termination ensures that we will always rewrite to a word that can't be rewritten any further).*
(2) *Whenever $(t_1 s, r_1), (st_2, r_2) \in S$, there is some $z \in B^*$ with rewritings $t_1 s t_2 \to r_1 t_2 \xrightarrow{*} z$ and $t_1 s t_2 \to t_1 r_2 \xrightarrow{*} z$, where $x \xrightarrow{*} y$ means either $x = y$ or there is some finite sequence of rewritings $x \to \ldots \to y$.*
(3) *Whenever $(rst, u), (s, v) \in S$, then there is some $z \in B^*$ with rewritings $rst \to u \xrightarrow{*} z$ and $rst \to rvt \xrightarrow{*} z$. (Properties (2) and (3) together are called* **confluence***. Confluence ensures that a word doesn't rewrite to two different words).*

Then we call S a **complete rewriting system** *over B.*

Theorem 1. *Let S be a complete rewriting system over $(A^{\pm 1})^*$ including the rules $aa^{-1} \to 1$ and $a^{-1}a \to 1$ for every $a \in A$ and let G be the group presented by $\langle A \mid u = v \ \ \forall (u,v) \in S \rangle$. Then the set of irreducible words with respect to S is a set of normal forms for G. (See* [8],[10]*).*

This means that if one can find a complete rewriting system over the generators of a finitely presented group G, then taking any word representing $g \in G$ and applying the rewriting rules repeatedly will eventually yield a normal form for g. As mentioned earlier in this section, these normal forms allow us to compare different words to see if they represent the same group element. This is a valuable tool for constructing a Cayley Graph of a group G from the Cayley Graph of a subgroup.

2.4.1. *Shortlex Ordering.* From the previous section, we have learned that complete rewriting systems help to produce normal forms. One useful tool in the quest to make a rewriting system for a group $G = \langle A|R \rangle$ is to impose an order relation on the set $A^{\pm 1}$ and make rewriting rules from the relations so that words rewrite to smaller words with respect to that order relation. We will give an example of a commonly used order relation, as well as exhibit its usefulness with an application.

Definition 8. *Let $v = v_1v_2\dots v_n$ and $w = w_1w_2\dots w_m$ be distinct words in B^* where each $v_i, w_j \in B$. The* **shortlex ordering** $\prec$ *is defined in the following way: Since B is finite, we write $B = \{b_1, b_2, \dots, b_p\}$ and impose $1 \prec b_1 \prec b_2 \prec \dots \prec b_p$, where 1 is the empty word. Then, we say that $v \prec w$ whenever $|v| < |w|$ or $|v| = |w|$ and $v_k \prec w_k$, where k is the smallest index i for which $v_i \neq w_i$.*

So shortlex is an ordering on words according to length, with ties being broken by looking at the first letter for which the two words differ and consulting the imposed ordering on those letters. For instance, if $A = \{a, b, c\}$ with $a \prec b \prec c$, then $abab \prec abac$ since $b \prec c$, but $ccc \prec aaaa$ since $|ccc| = 3 < 4 = |aaaa|$. It would be as if the dictionary were organized by word length with words of a given length being ordered in the ordinary fashion.

In [1, Chapter 9], one encounters a set $D \subset B_4$ that is a list of all of the nontrivial elements of B_4 that correspond to braids where any two strands cross at most once. It is well-known that the set D is a set of canonical representatives for S_4 in B_4. Below is a sketch of a proof of this fact that relies on shortlex ordering.

Lemma 1. *The canonical homomorphism from B_4 to S_4 restricts to a bijection from $\mathcal{D} \cup \{1\}$ to S_4.*

Proof. (Sketch): Impose the ordering $a \prec b \prec c$ for $\{a, b, c\}$. Next, write down all words of length 1 in shortlex order from smallest to largest (namely, a b c). Now, cross out all words that can be reduced to words that have appeared earlier in the list according to the relations of B_4 or which have strands that cross twice or more (in this case, there are no words to cross out). Now, write all length 2 words in shortlex order by taking each length 1 word that hasn't been crossed out and appending a generator to the end to get three length 2 words for each length 1 word (i.e. we have that a spawns aa, ab, ac). Now, cross out all words that can be reduced to a word that has already appeared in the list according to the relations of B_4 or which have strands that cross twice or more (in this case, aa gets crossed out since the first and second braids cross twice).

We proceed in this way so that when we have a shortlex ordered list of all length n words over $\{a, b, c\}$ that represent elements of $\mathcal{D}$ with no repetitions, call it L_n, we construct another minimal, exhaustive, shortlex ordered list of all length $n + 1$ words that represent elements of $\mathcal{D}$, call it L_{n+1} in the following way: First, append a, b, and c to each word in L_n to spawn three new words in shortlext order of length $n+1$. Next, cross out words that can be reduced to words that have already appeared in the list by the relations of B_4 or whose braids have strands that cross twice or more.

Now, we show that L_{n+1} is an exhaustive list of all length $n+1$ words representing elements of $\mathcal{D}$. Let w be a word over $\{a, b, c\}$ with $|w| = n+1$. Note that $w \in \{va, vb, vc\}$ where v is a positive word of length n. Observe that the statement "w represents a braid in $\mathcal{D}$" implies the statement "v represents a braid in $\mathcal{D}$." To see this, assume by contrapositive that v does not represent a braid in $\mathcal{D}$. That means v represents a braid for which there are two strands that cross twice or more. So, when we make another crossing by appending $a, b,$, or c to v, the situation can only get worse. That is to say, the two strands that crossed twice or more either still cross twice or more, or cross three times or more after appending v with a, b, or c. So $w \in \{va, vb, vc\}$ also represents a braid for which there are two strands that cross twice or more. Next, let u be a word over $\{a, b, c\}$ of length $n+1$ that represents a braid in $\mathcal{D}$. Since we are dealing with only positive words and the relations of B_4 preserve word length, there are no words of shorter length representing the same braid as u (there are no inverses to cancel, and braid relations cannot make the word shorter). So, u has not yet appeared in any L_i for $i < n+1$. Note that $u = td$ where t is a positive word of length n and $d \in \{a, b, c\}$. Since u represents a braid in $\mathcal{D}$, then so does t by the argument earlier in this paragraph. Since t is a length n word representing a braid in $\mathcal{D}$ and L_n is an exhaustive list of such words with no repeats, then either $t \in L_n$ or t was crossed off because there is exactly one other $t' \in L_n$ with $t \equiv t'$ that appeared earlier in shortlex order. So, either u is not crossed off or we have that $t'd \equiv u$, and $t'd$ is either in L_{n+1} or $t'd$ was crossed off because there is a $u' \in L_{n+1}$ with $u' \equiv t'd \equiv u$ and $u' \prec t'v \prec u$. In any case, the element that u represents is represented somewhere in L_{n+1} (either by t'd or u' or u). So every braid in $\mathcal{D}$ that can be represented by a positive word of length $n+1$ is represented in L_{n+1}.

Note now that L_{n+1} is minimal, since any word that could be reduced to a word already in L_{n+1} was crossed off, so any braid in $\mathcal{D}$ represented by a length $n+1$ word over $\{a, b, c\}$ is represented only once. In the case of $\mathcal{D}$, this brute force procedure eventually yields an empty list, where all potential candidates are crossed off. When this happens, there cannot be any words of higher length in the list, since those words would fail for the same reasons that all words of the preceding length failed. With this procedure, we find that

$$\mathcal{D} = \begin{array}{llllll} a & b & c & & & \\ ab & ac & bc & cb & ba & \\ aba & abc & acb & bac & bcb & cba \\ abac & abcb & acba & bacb & bcba & \\ abacb & abcba & bacba & & & \\ abacba. & & & & & \end{array}$$

The algorithm ensures that the list on the right hand side is indeed an exhaustive list of the elements of $\mathcal{D}$ since we have exhausted the possibilities for elements of $\mathcal{D}$ at each word length. So every element of $\mathcal{D}$ is found somewhere in the list. Conversely, one can check that all elements in the list correspond to positive braids whose strands cross at most once. So the list represents a set of elements of B_4 that is equal to $\mathcal{D}$. Furthermore, one can check that the list contains no duplicates by observing that the function from the list on the right hand side to $S_4 \setminus \{1\}$ induced by $a \mapsto (1\ 2)$, $b \mapsto (2\ 3)$, $c \mapsto (3\ 4)$ is a bijection. We extend this bijection to a bijection from $\mathcal{D} \cup \{1\}$ to S_4 by sending $1 \mapsto 1$. The resulting function is exactly the canonical homomorphism from B_4 to S_4 restricted to $\mathcal{D} \cup \{1\}$. □

In [7, §3.2], Hermiller and Meier produce a set of rewriting rules on the list $\mathcal{D} \cup \{1\}$ from above, where each element is taken as its own letter rather than as a word over $\{a, b, c\}$. This rewriting system together with a particular order relation yields a complete rewriting system that can take a word over $\{a, b, c\}^{\pm 1}$ and output the normal form for the corresponding element of B_4. These rewriting rules for B_4 are listed in Appendix A.

3. Building the Cayley Graph of H

We approach the construction of the Cayley Graph of (B_4) by partitioning the group into cosets of a subgroup

$$H := \langle a^2, b^2, c^2 \rangle$$

which avoids many of the difficulties of B_4. In particular, Droms, Lewin, and Servatius [4, §1 Corollary 3] have shown that

$$H \cong \mathbb{Z} * \mathbb{Z}^2 = \langle x, y, z \mid xz = zx \rangle$$

where the isomorphism is the extension of the mappings

$$a^2 \mapsto x,\ b^2 \mapsto y, \text{ and } c^2 \mapsto z.$$

3.1. **Embedding $\mathbb{Z} * \mathbb{Z}^2$.** In this section, we define an algorithm φ which we show embeds the Cayley graph Γ of $\mathbb{Z} * \mathbb{Z}^2$ within the Euclidean cube $(-1,1)^3 \subset \mathbb{R}^3$.

3.1.1. *The Vector Algorithm.*

Definition 9.

(1) *Let A be a set and consider the word $w = w_1w_2w_3 \cdots w_n$ where each $w_i \in A$. A* **suffix** *of w is any word w' where either $w' = 1$ or*
$$w' = w_i w_{i+1} w_{i+2} \cdots w_n \in A^*, \text{ for some } i \in \{1, \ldots, n\}.$$
(2) *Let A be a set and consider the word $w = w_1w_2w_3 \cdots w_n$ where each $w_i \in A$. A* **prefix** *of w is any word w' where either $w' = 1$ or*
$$w' = w_1 w_2 w_3 \cdots w_i \in A^* \text{ for some } i \in \{1, \ldots, n\}.$$

We will also need to understand some properties of the free product, defined as follows:

Definition 10. *The* **free product** *of two groups G and H, written $G * H$, consists of elements are of the form*
$$w = g_1h_1g_2h_2 \ldots g_nh_n$$
where $g_i \in G$, $h_i \in H$, $n \in \mathbb{N}$, $g_i \neq id_G$ for $i \neq 1$, and $h_i \neq id_H$ for $i \neq n$. Furthermore, the operation on $G * H$ is reduced concatenation, where two words are concatenated and then reduced according the rules of both G and H, eliminating id_G and id_H from the word, except possibly at the first and last factors of the word.

We also introduce some notation for convenience:

Definition 11. *For an element $g \in \mathbb{Z} * \mathbb{Z}^2$ and its normal form $\tilde{g}$ given by Definition 5:*

(1) *Let $\mathbf{p_x(g)}$ denote the absolute value of the power of x in the longest suffix of $\tilde{g}$ which does not contain a $y^{\pm 1}$ letter.*

(2) *Furthermore, let $\mathbf{p_z(g)}$ denote the absolute value of the power of z in the longest suffix of $\tilde{g}$ which does not contain a $y^{\pm 1}$ letter.*

(3) *Let $\gamma(\mathbf{g})$ denote the length of the longest prefix of $\tilde{g}$ which ends in a $y^{\pm 1}$ letter. If no such prefix exists, then define $\gamma(g) = 0$.*

Since $\tilde{g}$ represents g uniquely, we know that p_x, p_z, and γ are well-defined.

We recursively define the vector form of elements of $\mathbb{Z}*\mathbb{Z}^2$ as follows:

(1) First, define the vector form $\overline{1}$ of 1 as $\langle 0,0,0\rangle$ and note that $\gamma(1)=0$.
(2) Assume $\overline{g}$ is defined for some $g\in\mathbb{Z}*\mathbb{Z}^2$. Let $v\in\{x,y,z\}^{\epsilon}$ where $\epsilon=\pm1$, and map gv as follows:

(a) If v is x,
$$\overline{g\cdot v^{\epsilon}} := \overline{g} + \epsilon\cdot\left\langle (0.5)^{\gamma(g)+p_x(g)+1}, 0, 0\right\rangle.$$

(b) If v is z,
$$\overline{g\cdot v^{\epsilon}} := \overline{g} + \epsilon\cdot\left\langle 0, (0.5)^{\gamma(g)+p_z(g)+1}, 0\right\rangle.$$

(c) If v is y,
$$\overline{g\cdot v^{\epsilon}} := \overline{g} + \epsilon\cdot\left\langle 0, 0, (0.5)^{|\tilde{g}|+1}\right\rangle.$$

3.1.2. *Constructing the Cayley Graph of $\mathbb{Z}*\mathbb{Z}^2$.*

Definition 12. *Let*
$$\varphi : \Gamma(\mathbb{Z}*\mathbb{Z}^2)\to\mathbb{R}^3$$
be the function that maps vertices g to $\overline{g}$ and edges $[g, gs, s]$ to the straight line segment connecting $\overline{g}$ to $\overline{g\cdot s}$ labeled by $s\in\{x,y,z\}$. We call φ the **vector algorithm**.

Lemma 2. *The function φ is well-defined:*

Proof. Consider a factor f_i of an arbitrary element g. In other words, if
$$g = g_1h_2g_3h_4\dots g_{n-1}h_n,$$
a factor of g is any g_i or h_j, and $f_i(g)$ is the ith symbol in the above sequence.

We know that f_i is of the form
$$x^{\epsilon n}z^{\eta m} \text{ or } y^{\delta\ell} \text{ for } \ell,m,n\in\mathbb{N}_0 \text{ and } \delta,\epsilon,\eta\in\{\pm1\}.$$

Observe that the definition of the free product yields that if two elements g_1 and g_2 are equal, then they correspond exactly at every factor, up to the commutativity relation $xz = zx$. We claim that φ respects this relation. In other words, we claim

$$\overline{(gx^\epsilon)z^\eta} = \overline{(gz^\eta)x^\epsilon}.$$

To see this, observe that

$$\overline{(gx^\epsilon)} = \overline{g} + \epsilon \cdot \langle (0.5)^{\gamma(g)+p_x(g)+1}, 0, 0\rangle$$

and

$$\overline{(gz^\eta)} = \overline{g} + \eta \cdot \langle 0, (0.5)^{\gamma(g)+p_x(g)+1}, 0\rangle.$$

Therefore,

$$\begin{aligned}
\overline{(gx^\epsilon)z^\eta} &= (\overline{g} + \epsilon \cdot \langle (0.5)^{\gamma(g)+p_x(g)+1}, 0, 0\rangle) + \eta \cdot \langle 0, (0.5)^{\gamma(g)+p_x(g)+1}, 0\rangle \\
&= (\overline{g} + \eta \cdot \langle 0, (0.5)^{\gamma(g)+p_x(g)+1}, 0\rangle) + \epsilon \cdot \langle (0.5)^{\gamma(g)+p_x(g)+1}, 0, 0\rangle \\
&= \overline{(gz^\eta)x^\epsilon}.
\end{aligned}$$

Thus, φ respects the commutativity relation $x^\epsilon z^\eta = z^\eta x^\epsilon$. So the map $g \mapsto \overline{g}$ is well-defined. Moreover, since φ connects edges between well-defined vertices, the edge placement is also well-defined. Therefore, φ is well-defined. □

Definition 13. *Define the restriction subgraph, for each $g \in \mathbb{Z} * \mathbb{Z}^2$,*

$$\Gamma\ |_{g\cdot\mathbb{Z}^2}$$

to be the subgraph of $\Gamma(\mathbb{Z} * \mathbb{Z}^2)$ consisting of vertices of the form

$$g \cdot x^{\epsilon k} \cdot z^{\eta j} \text{ for } k, j \in \mathbb{N}_0 \text{ and } \epsilon, \eta \in \{\pm 1\}$$

along with all edges in $\Gamma(\mathbb{Z} * \mathbb{Z}^2)$ that connect such vertices.

Definition 14. *For each $g \in \mathbb{Z} * \mathbb{Z}^2$, we denote the* **g-plane**

$$\mathbb{Z}^2_g := \varphi(\Gamma\ |_{g\cdot\mathbb{Z}^2}).$$

Note here that $\Gamma\ |_{g\cdot\mathbb{Z}^2}$ (and hence $\mathbb{Z}^2_g$) is uniquely defined by g, up to a suffix $x^{\epsilon k} z^{\eta j}$.

We use φ to map the Cayley graph of $\mathbb{Z}*\mathbb{Z}^2$ into $\mathbb{R}^3$. In Section 3.1.3, we will show that φ is an embedding. We proceed to demonstrate a small portion of $\varphi(\Gamma)$, and then we display a more complete picture of $\varphi(\Gamma)$.

We begin by constructing the plane $\mathbb{Z}_1^2$ of the Cayley graph, which corresponds to the $z = 0$ plane of the cube as follows:

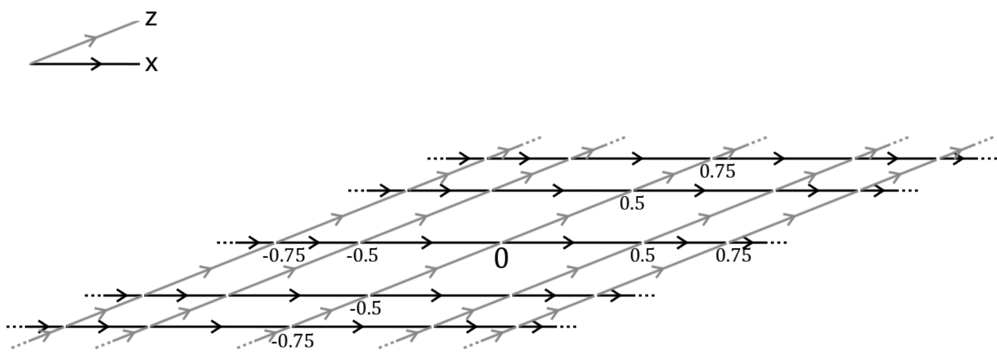

FIGURE 3. Embedding $\mathbb{Z}_1^2$ in $\mathbb{R}^3$.

The mnemonic in the upper left corner of the above figure encodes the directed orientation of the edges in $\mathbb{Z}_1^2$: in particular, the horizontally oriented edges are labeled by "x", and the diagonally oriented edges are labeled by "z". We proceed to show that the above figure is a picture of $\mathbb{Z}_1^2$. Observe that

$$\gamma(x^{\epsilon k} \cdot z^{\eta m}) = 0 \ \forall k, m \in \mathbb{N}_0 \text{ and } \epsilon, \eta \in \{\pm 1\}$$

from the definition of γ. Moreover, recall that for all $i, n \in \mathbb{N}_0$, we have

$$p_x(x^{\epsilon i} \cdot z^{\eta n}) = i \text{ and } p_z(x^{\epsilon i} \cdot z^{\eta n}) = n.$$

Thus,

$$\overline{x^{\epsilon k} \cdot z^{\eta m}} = \left\langle \epsilon \cdot \sum_{i=1}^{k} (0.5)^{i+1}, \eta \cdot \sum_{j=1}^{m} (0.5)^{j+1}, 0 \right\rangle.$$

So, for any vertex $g \in 1 \cdot \mathbb{Z}^2$, $\overline{g}$ is mapped into $(-1,1) \times (-1,1) \times \{0\}$, since

$$0.5 + 0.25 + 0.125 + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i} \text{ converges to } 1.$$

We next define a larger subset of $\varphi(\Gamma)$, which we call $\mathcal{S}$, as the union of $\mathbb{Z}_1^2$ and $\mathbb{Z}_y^2$, pictured in Figure 4 below:

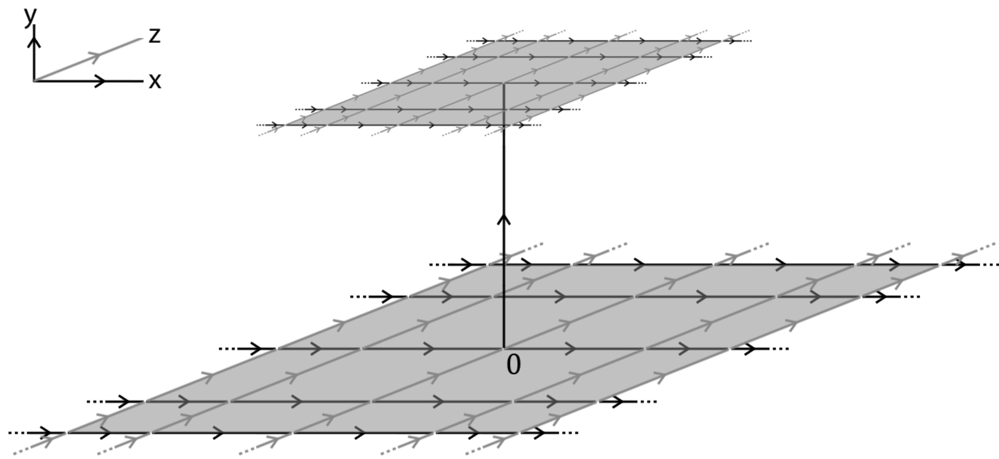

FIGURE 4. Taking the union of $\mathbb{Z}_1^2$ and $\mathbb{Z}_y^2$ to obtain $\mathcal{S}$.

Notice that this portion of the construction now expands to include only elements of the form

$$w := y \cdot x^{\epsilon k} \cdot z^{\eta m}, \text{ where } k, m \in \mathbb{N},\ \epsilon, \eta \in \{\pm 1\}.$$

The order of x, y, z in the above word determines that the vertical edge $y \in \mathcal{S}$ represents the longest prefix of the word $x^{\epsilon k} \cdot z^{\eta m}$ that contains a y. So

$$\gamma(y \cdot x^{\epsilon k} \cdot z^{\eta m}) = 1.$$

Moreover, note that

$$p_x(y \cdot x^{\epsilon k} \cdot z^{\eta m}) = k \text{ and } p_z(y \cdot x^{\epsilon k} \cdot z^{\eta m}) = m.$$

Therefore,

$$\overline{y \cdot x^{\epsilon k} \cdot z^{\eta m}} = \left\langle \epsilon \cdot \sum_{i=1}^{k} (0.5)^{\ell+i+1}, \eta \cdot \sum_{j=1}^{m} (0.5)^{\ell+j+1}, 0.5 \right\rangle.$$

We can perform this same procedure from any point on $\mathbb{Z}_1^2$, not just the origin. For $x^{\epsilon k} z^{\eta j}$, where $\epsilon, \eta \in \{\pm 1\}$ and $k, j \in \mathbb{N}_0$, we can produce a picture analogous to S except with $\mathbb{Z}^2_{x^{\epsilon k} z^{\eta j} y}$ playing the role of $\mathbb{Z}_y^2$ except scaled by a factor of $(0.5)^{k+j}$. For any $g \in \mathbb{Z} * \mathbb{Z}^2$ and its normal form $\tilde{g}$, we can proceed in this way to produce pictures analogous to S starting at any point in $\mathbb{Z}_g^2$ with $\mathbb{Z}_{gy}^2$ playing the role of $\mathbb{Z}_y^2$ except scaled by a factor of $(0.5)^{|\tilde{g}|}$.

Thus, φ fully maps $\Gamma(\mathbb{Z} * \mathbb{Z}^2)$ into $\mathbb{R}^3$. In particular,

$$\bigcup_{w \in \mathbb{Z} * \mathbb{Z}^2} \mathbb{Z}_w^2 = \varphi(\Gamma(\mathbb{Z} * \mathbb{Z}^2)) \subset (-1, 1)^3.$$

Now that we've observed a portion of the construction, we present a more complete Cayley Graph of $\mathbb{Z} * \mathbb{Z}^2 \subseteq (-1,1)^3 \subset \mathbb{R}^3$:

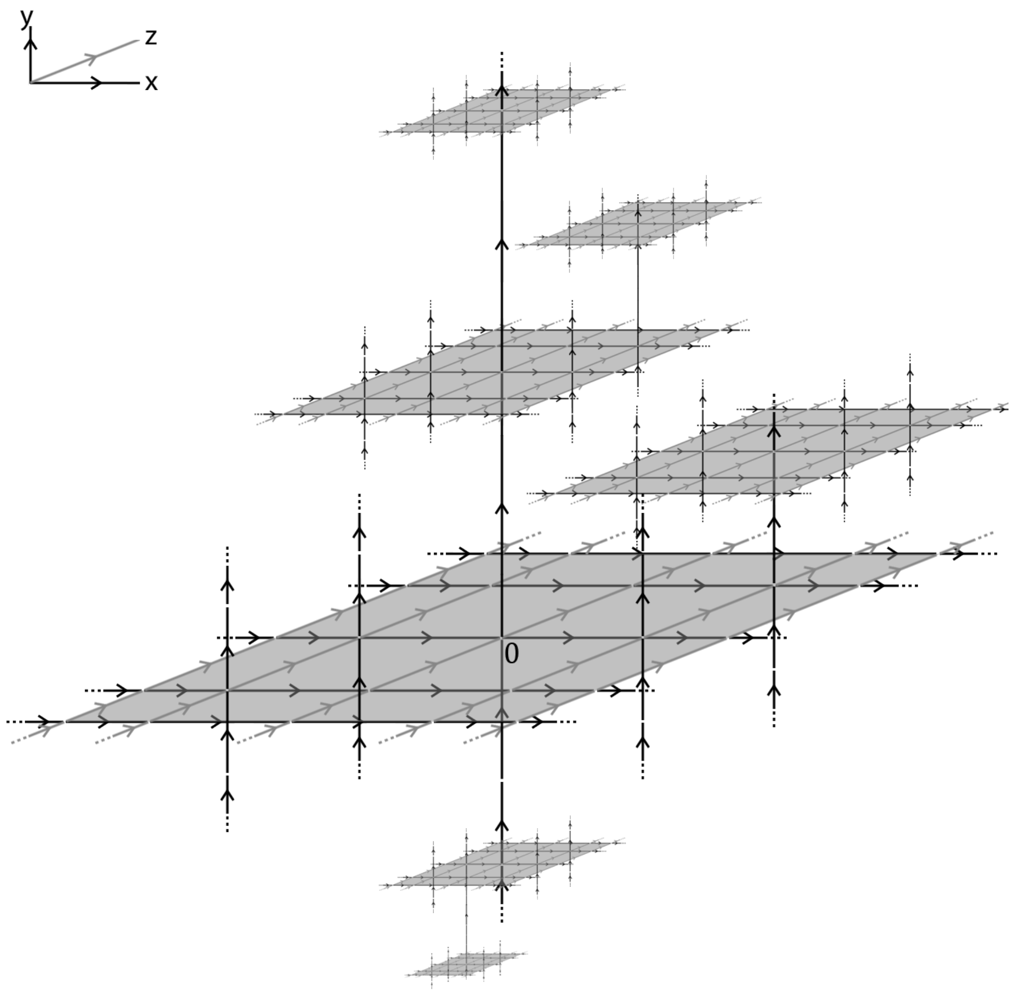

FIGURE 5. The Cayley Graph Γ of $\mathbb{Z} * \mathbb{Z}^2$ mapped into $\mathbb{R}^3$ under φ.

3.1.3. *Proof that φ is an Embedding.*

Theorem 2. *Let Γ be the Cayley graph of $\mathbb{Z} * \mathbb{Z}^2$ with respect to the generating set $\{x, y, z\}$. The map*

$$\varphi : \Gamma \to (-1,1)^3 \subset \mathbb{R}^3$$

is an embedding.

Proof. Since φ maps vertices to vectors and edges to line segments, then the pre-image of any open set in $\mathbb{R}^3$ is open in Γ with its usual graph topology. So φ is continuous. For injectivity, the proof will be split into two smaller steps. First, we will show that the vertices V of Γ map into $(-1,1)^3$. Then we prove that the edges described in the vector algorithm do not contain any points which are the images of the vertices V under the vector algorithm φ, except the edges' endpoints, and we will show that the edges do not intersect one another except at vertices which are their endpoints.

(1) Suppose that $v \in \mathbb{Z} * \mathbb{Z}^2$. Then we can write

$$\varphi(v) = (v_1, v_2, v_3) \text{ with } v_j \in \mathbb{R} \text{ for } j \in \{1,2,3\}.$$

By the construction algorithm, we have that

$$-\sum_{n=1}^{\infty} \frac{1}{2^n} < v_j < \sum_{m=1}^{\infty} \frac{1}{2^m}$$

for all $j \in \{1,2,3\}$. But

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

Hence,

$$-1 < v_j < 1 \text{ for } j \in \{1,2,3\}.$$

Therefore, $\varphi(v) \in (-1,1)^3$ for all $v \in \mathbb{Z} * \mathbb{Z}^2$, since v was arbitrarily chosen.

(2) We will show that in the vector algorithm φ, words that are not equal generate subgraphs that do not intersect.

Definition 15. *We will use the term* **subgraph generated by g** *to mean*

$$\{\varphi(gg') \mid g' \in \mathbb{Z} * \mathbb{Z}^2,\ \tilde{g} \text{ a prefix of } \widetilde{gg'}\}$$

along with the edges connecting these vertices.

Note that for two elements $g_1 \neq g_2$, there are maximal prefixes $\tilde{g}_1$ and $\tilde{g}_2$ of g_1 and g_2, respectively, for which $\tilde{g}_1 = \tilde{g}_2$ (we call this maximal prefix $\tilde{g}$). Now let $\hat{g}_1$ and $\hat{g}_2$ be the corresponding suffixes of g_1 and g_2, respectively. In particular, define $\hat{g}_1$ and $\hat{g}_2$ so that

$$\tilde{g}\hat{g}_1 = g_1 \text{ and } \tilde{g}\hat{g}_2 = g_2.$$

Note that

$$\ell(\hat{g}_1) < \sum_{i=|\tilde{g}_1|}^{\infty} \frac{1}{2^i} \text{ and } \ell(\hat{g}_2) < \sum_{i=|\tilde{g}_2|}^{\infty} \frac{1}{2^i},$$

where ℓ is the physical length of the corresponding path in $\mathbb{R}^3$ defined by the elements g_1 and g_2.

Observe that the first factors of $\hat{g}_1$ and $\hat{g}_2$ must be different, since $\tilde{g}_1$ and $\tilde{g}_2$ were chosen to be maximal. Therefore, WLOG, suppose that the first letter of $\hat{g}_1$ is y and the first letter of $\hat{g}_2$ is x. Note that

$$\varphi(\tilde{g}y) \neq \varphi(\tilde{g}x),$$

and the paths corresponding to appending y and x do not intersect. Call the first edges in these paths e_1 and e_2, respectively. Note that these edges correspond to the edges connecting $\tilde{g}$ to $\tilde{g}y$ and $\tilde{g}$ to $\tilde{g}x$, respectively. Then define ρ_1 and ρ_2 as arbitrary elements of the subgraph generated by $\tilde{g}y$ and $\tilde{g}x$, respectively. Note that

$$\ell(\rho_1) < \sum_{i=|\tilde{g}_1|+1}^{\infty} \frac{1}{2^i},$$

and

$$\ell(\rho_2) < \sum_{i=|\tilde{g}_2|+1}^{\infty} \frac{1}{2^i}.$$

Furthermore,

$$\ell(e_1) = \frac{1}{2^{|\tilde{g}|+1}} = \ell(e_2).$$

In order for the subgraph generated by $\tilde{g}y$ to intersect the subgraph generated by $\tilde{g}x$, the lengths of the paths from $\tilde{g}y$ and $\tilde{g}x$ which intersect must add up to at least the geodesic taxicab distance from $\tilde{g}y$ to $\tilde{g}x$. But the geodesics taxicab distance from $\tilde{g}y$ to $\tilde{g}x$ is exactly

$$\frac{1}{2^{|\tilde{g}|+1}} + \frac{1}{2^{|\tilde{g}|+1}} = 2 \cdot \left(\frac{1}{2^{|\tilde{g}|+1}}\right),$$

the horizontal distance plus the vertical distance. But the length of any path from $\tilde{g}y$ or $\tilde{g}x$ is bounded above by

$$\sum_{i=|\tilde{g}|+2}^{\infty} \frac{1}{2^i}.$$

This follows from the bound on ρ_1 and ρ_2, which are arbitrary words in the subgraphs generated by $\tilde{g}y$ and $\tilde{g}x$, respectively. So then for any two paths p_1 and p_2 in the subgraphs generated by $\tilde{g}y$ and $\tilde{g}x$,

$$\ell(p_1) + \ell(p_2) < 2 \cdot \sum_{i=|\tilde{g_1}|+2}^{\infty} \frac{1}{2^i} = 2 \cdot \frac{1}{2^{|\tilde{g_1}|+1}}.$$

Thus, since the above inequality is strict, it must be that the subgraphs generated by $\tilde{g}y$ and $\tilde{g}x$ do not intersect. In particular, we have shown that the subgraph generated by y does not intersect the subgraph generated by x.

All that is left to show is that the subgraphs generated by x and z are identical, which we know by construction, and that y^{-1} generates a subgraph which does not intersect the subgraphs generated by y or x. But this is true by symmetry in the above argument.

Therefore, φ embeds $\mathbb{Z}*\mathbb{Z}^2$ into $(-1,1)^3$, and the embedding respects the geometric encoding of $\mathbb{Z} * \mathbb{Z}^2$ found in its Cayley Graph $\Gamma(\mathbb{Z} * \mathbb{Z}^2)$. □

3.2. **Discussion.** The subgroup $H = \langle a^2, b^2, c^2 \rangle \leq B_4$ can potentially tell us a great deal about the structure of B_4. Let Λ be the Cayley Graph of B_4 with respect to its natural presentation. From the embedding algorithm in §3.1, we have a complete description of the Cayley Graph of H that sits inside Λ. That is to say, every edge in $\Gamma(H)$ is really two edges in Λ since an edge in $\Gamma(H)$ corresponds to a transition by an element of $\{a^2, b^2, c^2\}^{\pm 1}$ rather than $\{x, y, z\}^{\pm 1}$. So $\Gamma(H)$ corresponds to the subgraph of Λ that looks like $\Gamma(H)$, except with all of its edges subdivided. Similarly, any coset of H is associated with a copy of $\Gamma(H)$ that sits inside Λ that is a translation of the subgraph corresponding to H. Since the cosets of H partition B_4, every vertex in Λ is accounted for when taking the union of all of the subgraphs corresponding to the cosets of H. For any coset, the associated subgraph contains all of the edges in Λ that connect elements of that coset to other elements of the coset. So, when taking the union of all of the subgraphs associated with the cosets of H, there are no vertices missing, and the only edges missing are those that connect elements from one coset to a different coset.

So in order to build Λ, we must understand where an element of a coset goes when a generator is applied that takes it to a different coset. This would tell us exactly what the missing edges we need to characterize Λ are. With the rewriting system from Appendix A, we have a way to compute whether two words represent the same element of B_4. Armed in this way, we may be able to find a pattern to connect the missing edges between the cosets of H.

4. Potential Applications

The results of this paper take great steps towards describing the Cayley Graph of B_4 with respect to its natural presentation. Once found, the Cayley Graph of B_4 can be used to characterize its geodesics:

$$Geo(B_4, \{a, b, c\}) := \left\{ w \in (\{a, b, c\}^{\pm 1})^* \mid |w| = \min\{ |v| \mid v \equiv w\} \right\}.$$

To motivate the importance of this endeavor, we explore a result for B_n taken from Charney and Meier in [2].

We will refer to the language given in [2] as the *Garside* language of B_n. This language is constructed as follows:

Definition 16. *The Garside Language of B_4 is the set of divisors $\mathcal{D}$ of the Garside element Δ, which we describe as a full twist of the braids:*

$$\Delta := abacba$$

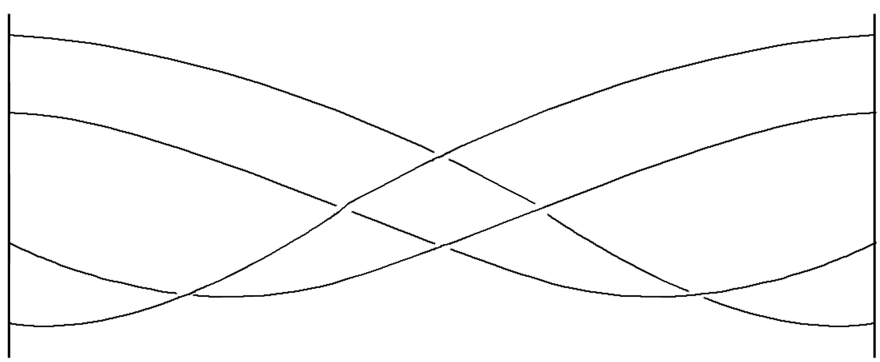

Figure 6. The full twist on 4 strands: $\Delta = abacba$

A simple computation reveals the set of divisors to be the following:

$$\mathcal{D} := \begin{array}{llllll} a & b & c & & & \\ ab & ac & bc & cb & ba & \\ aba & abc & acb & bac & bcb & cba \\ abac & abcb & acba & bacb & bcba & \\ abacb & abcba & bacba & & & \\ abacba. & & & & & \end{array}$$

Note that the set $\mathcal{D}$ is exactly the set $\mathcal{D}$ discussed in §2.4.

In Charney and Meier, the chief result is the following theorem:

Theorem 3 (1, Cor.)**.** *The language of geodesics for the braid group B_n, with respect to the generating set of simple divisors of either Δ or δ, is regular.*

In the above theorem, δ is the braid which crosses the first strand of B_n over the remaining strands. In [2, Question 1.1] the authors ask whether the above theorem holds for B_n with its natural presentation. The only known braid group for which this question has been answered is B_3 as proved in [12]. Sabalka shows in [12, §1, Theorem 1.2] that the set of geodesics for B_3 with respect to the natural generating set is indeed regular, satisfying the conclusion of the above theorem. The constructions developed in this paper will be useful in answering this question for B_4, and perhaps in discovering a pattern that can be generalized to all Braid groups with natural presentation.

Appendix A. Rewriting Rules for B_4

$(abacba)(abacba)^{-1} \to 1$
$ab \to (ab)$
$(ab)a \to (aba)$
$a(ba) \to (aba)$
$ac \to (ac)$
$bc \to (bc)$
$(bc)b \to (bcb)$
$(cb)c \to (bcb)$
$(bacb)a \to (bacba)$
$(ab)c \to (abc)$
$(ac)b \to (acb)$
$(ba)c \to (bac)$
$c(ba) \to (cba)$
$a(bcb) \to (abcb)$
$(acb)a \to (acba)$
$(bcb)a \to (bcba)$
$(abac)b \to (abacb)$
$(abcb)a \to (abcba)$
$(bac)(ba) \to (bacba)$
$(abacba)^{-1}(abacb) \to (bacba)(abacba)^{-1}$
$(bac)(abac) \to (ba)(abcba)$
$(abac)(ba) \to (abacba)$
$(abacba)b \to b(abacba)$
$(ac)(ba) \to (acba)$
$(ac)c \to c(ac)$
$(aba)(ab) \to (ba)(aba)$
$(acba)(bc) \to a(abcba)$
$(aba)(ac) \to b(abac)$
$(ba)(bc) \to (abac)$
$(ac)(aba) \to a(acba)$
$(abacba)^{-1}(abc) \to (cba)(abacba)^{-1}$
$(aba)(ba) \to (ab)(aba)$
$(aba)(bc) \to a(abac)$
$(abc)(aba) \to (abacba)$
$b(acba) \to (bacba)$
$(abac)(aba) \to b(abacba)$
$(bac)(cba) \to (bc)(acba)$
$(ac)a \to a(ac)$
$(bac)c \to (bc)(ac)$

$(abacba)^{-1}(abacba) \to 1$
$ba \to (ba)$
$b(ab) \to (aba)$
$(ba)b \to (aba)$
$ca \to (ac)$
$cb \to (cb)$
$c(bc) \to (bcb)$
$b(cb) \to (bcb)$
$a(bc) \to (abc)$
$a(cb) \to (acb)$
$b(ac) \to (bac)$
$(cb)a \to (cba)$
$(aba)c \to (abac)$
$(abc)b \to (abcb)$
$(bac)b \to (bacb)$
$(aba)(cb) \to (abacb)$
$(ab)(cba) \to (abcba)$
$(ba)(cba) \to (bacba)$
$(abacb)a \to (abacba)$
$(ab)(cb) \to (abcb)$
$(abacba)^{-1}(ab) \to (cb)(abacba)^{-1}$
$(abac)c \to (abc)(ac)$
$(abacba)(bc) \to (ba)(abacba)$
$(acba)b \to a(acba)$
$(aba)a \to b(aba)$
$c(aba) \to (acba)$
$(cba)(bc) \to (abcba)$
$(ba)^2 \to b(aba)$
$(abc)(ba) \to (abcba)$
$(bc)(aba) \to (bacba)$
$(aba)b \to a(aba)$
$(ab)^2 \to a(aba)$
$(abacba)a \to c(abacba)$
$(abac)a \to b(abac)$
$(bacba)(bc) \to (ba)(abcba)$
$(ac)(cba) \to c(acba)$
$(ac)(ab) \to a(acb)$
$(bc)a \to (bac)$
$(ab)(ac) \to (abac)$

$(abacba)(ab) \to (cb)(abacba)$
$(abacba)(ac) \to (ac)(abacba)$
$(abac)(ab) \to b(abacb)$
$(ab)(acba) \to (abacba)$
$b(abc) \to (abac)$
$(bacba)(cba) \to (bc)(abacba)$
$(abacba)(cba) \to (abc)(abacba)$
$(abacba)^{-1}(abcb) \to (acba)(abacba)^{-1}$
$(bacba)b \to (ba)(acba)$
$(acba)(bac) \to (acb)(abac)$
$a(bacba) \to (abacba)$
$(aba)(abc) \to (ba)(abac)$
$(aba)(bac) \to (ab)(abac)$
$(acba)a \to (cb)(aba)$
$(acba)c \to (abcba)$
$(abcba)(ba) \to c(abacba)$
$(aba)(abcba) \to (ba)(abacba)$
$b(acb) \to (bacb)$
$(aba)(acb) \to b(abacb)$
$(cba)(ba) \to (cb)(aba)$
$(bcba)b \to (bacba)$
$(bacba)(ab) \to (cba)(acba)$
$(bacba)(ac) \to c(abacba)$
$(ba)(bcba) \to (abacba)$
$(bcba)(bac) \to c(abacba)$
$(aba)(bcba) \to a(abacba)$
$(bcba)(ba) \to c(bacba)$
$(bcb)(ac) \to b(bcba)$
$(ac)(bc) \to (abcb)$
$(bcb)b \to c(bcb)$
$(bcb)(aba) \to c(bacba)$
$(bacba)(bcba) \to (bac)(abacba)$
$b(cba) \to (bcba)$
$(bcba)(bacba) \to (bcb)(abacba)$
$(abcba)(bacba) \to (abcb)(abacba)$
$(abacba)^{-1}a \to c(abacba)^{-1}$
$c(bac) \to (bcba)$
$(abacb)(ba) \to (bc)(abcba)$
$(abacb)(bc) \to (bacb)(bcb)$
$(abacba)^{-1}(bcba) \to (abac)(abacba)^{-1}$
$(ab)(abcba) \to a(abacba)$
$(abcb)(ab) \to (abacba)$
$(aba)(cba) \to (abacba)$
$(aba)(acba) \to b(abacba)$
$a(cba) \to (acba)$
$a(bac) \to (abac)$
$(bc)(ab) \to (bacb)$
$(cba)b \to (acba)$
$(abcba)b \to (abacba)$
$(bacba)(bac) \to (bacb)(abac)$
$(bac)(aba) \to (ba)(acba)$
$(ab)(abc) \to a(abac)$
$(ba)(bac) \to b(abac)$
$(ac)(bac) \to (abcba)$
$(acba)(ab) \to (cba)(aba)$
$(abacba)^{-1}(abcba) \to (abcba)(abacba)^{-1}$
$(abacba)c \to a(abacba)$
$c(ab) \to (acb)$
$(ab)(acb) \to (abacb)$
$(cb)(ab) \to (acba)$
$(bc)(ba) \to (bcba)$
$(bacba)a \to c(bacba)$
$(bacba)c \to (abacba)$
$(abacba)^{-1}(abac) \to (bcba)(abacba)^{-1}$
$(abacba)(bcba) \to (abac)(abacba)$
$(bcba)^2 \to (bc)(abacba)$
$(bcb)(ab) \to (bacba)$
$(bcb)c \to b(bcb)$
$(acba)(bcba) \to (ac)(abacba)$
$(bc)^2 \to b(bcb)$
$(bcb)(ba) \to c(bcba)$
$(abacba)^{-1}(bac) \to (bac)(abacba)^{-1}$
$(abacb)b \to (bc)(abcb)$
$a(bcba) \to (abcba)$
$c(abac) \to (abcba)$
$b(abcb) \to (abacb)$
$(abacba)^{-1}(bacba) \to (abacb)(abacba)^{-1}$
$(abac)(bac) \to a(abacba)$
$(abacb)(aba) \to (bc)(abacba)$
$(abacba)^{-1}(bc) \to (ba)(abacba)^{-1}$
$b(abcba) \to (abacba)$
$(ac)(bcba) \to c(abcba)$

$(bcb)(abcba) \to (bc)(abacba)$
$(ac)(abac) \to a(abcba)$
$(abacba)(bacba) \to (abacb)(abacba)$
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$(bacb)(ac) \to (abacba)$
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$(abcb)(ac) \to (ab)(bcba)$
$(abcb)(abac) \to (ac)(abacba)$
$(abacba)(acb) \to (acb)(abacba)$
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$(abacba)(bcb) \to (aba)(abacba)$
$(acb)c \to (abcb)$
$(abacba)^{-1}(ba) \to (bc)(abacba)^{-1}$
$(abcb)(aba) \to c(abacba)$
$(abc)(bc) \to (ab)(bcb)$
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$(bacba)(cb) \to b(abacba)$
$(acb)(cb) \to c(abcb)$
$(bac)(ab) \to (ba)(acb)$
$(cba)c \to (bcba)$
$(bc)(bac) \to b(bcba)$
$(bc)(abac) \to (abacba)$
$(abacba)^{-1}b \to b(abacba)^{-1}$
$(abacba)(bac) \to (bac)(abacba)$
$(abc)(bacb) \to (ab)(bacba)$
$(cb)^2 \to c(bcb)$
$(ba)(bcb) \to (abacb)$
$(bcb)(cb) \to (bc)(bcb)$
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$(ba)(bacb) \to b(abacb)$
$(ac)(bacb) \to (abacba)$
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$(abcb)b \to c(abcb)$

$(acb)(cba) \to c(abcba)$
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$(abacba)^{-1}(cba) \to (abc)(abacba)^{-1}$
$(abacb)(bacb) \to (bacb)(bacba)$

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