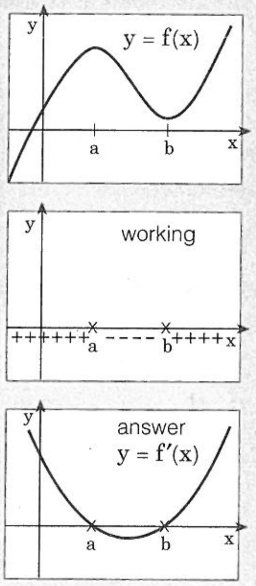
Curves and their Derivatives 1

A Sketching the Gradient Function

Example :
Given a cubic function $y = f(x)$.
Sketch the gradient function $y = f'(x)$.

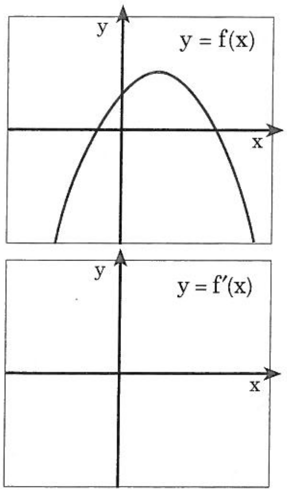
Working :
Look at the turning points. At a and b the gradient is zero, so (a, 0) and (b, 0) are on the gradient function.
For $a < x < b$ the gradient is negative, so the graph must be under the x-axis.
For $x < a$ and $x > b$ the gradient is positive, the graph must be above the x-axis.

Since $f(x)$ is a cubic, $f'(x)$ is a quadratic. So we sketch a parabola with x-intercepts at a and b, the part between a and b must be below the x-axis.

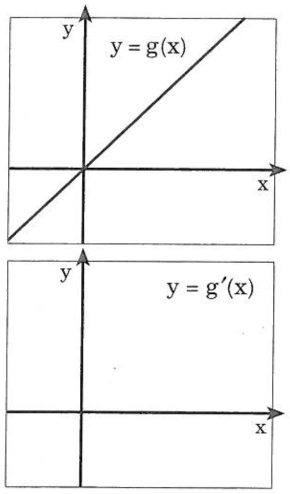


1 Sketch the gradient function of these.

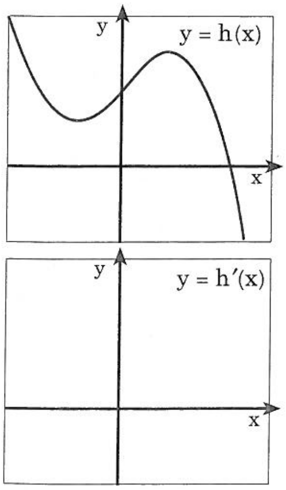
a)



b)



c)



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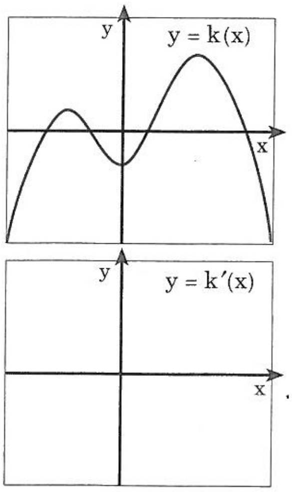
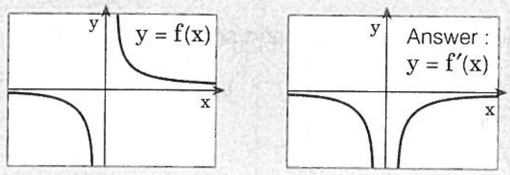


B Holes and Asymptotes

If $f(x)$ is not differentiable at point $x = a$, then the gradient function has a hole or an asymptote at $x = a$.

Example : Sketch the gradient function of $y = f'(x)$.



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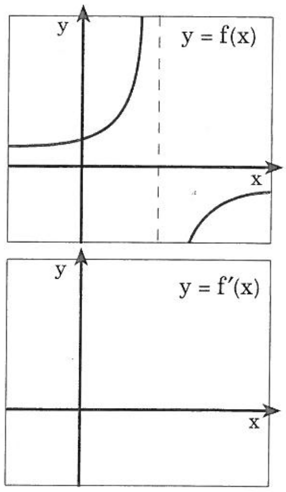

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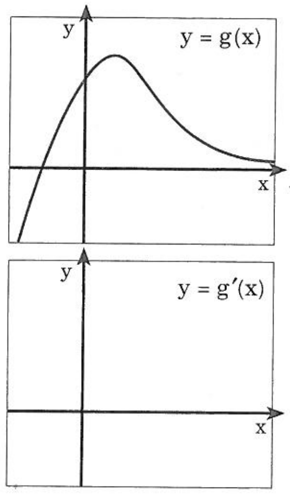
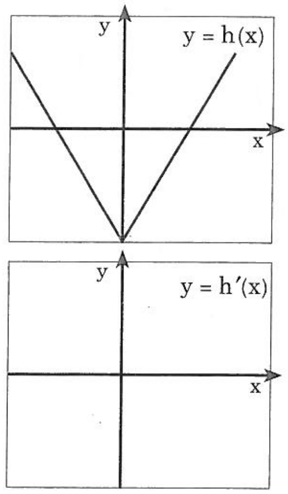


c)



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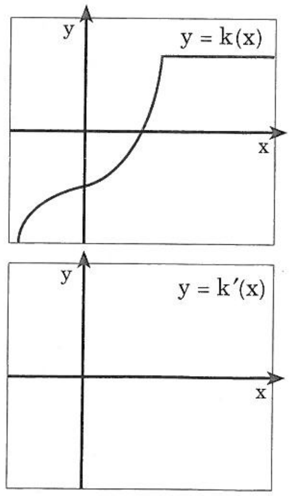
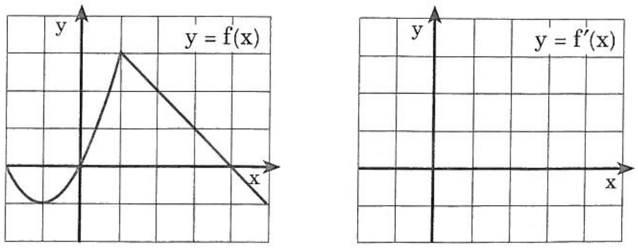


2 Plot a precise graph of $y = f'(x)$.



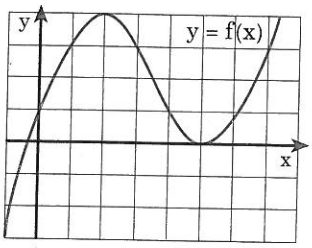
Curves and their Derivatives 2

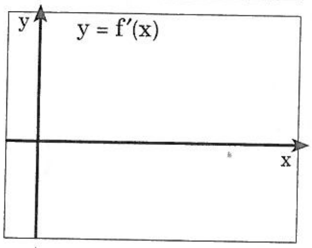
A First and Second Derivatives

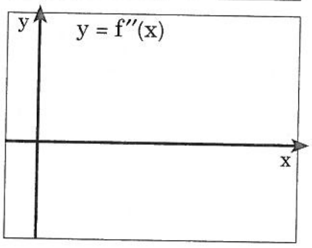
The second derivative $y = f''(x)$ is the gradient function of the first derivative. Therefore to draw the second derivative we must first draw the first derivative and investigate its gradient.

1 Draw the graphs of first and second derivatives.

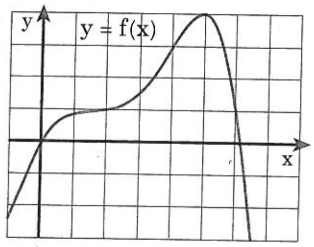
a)

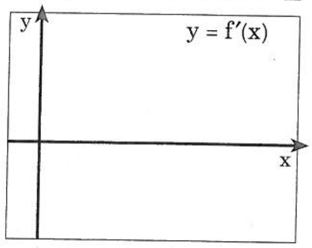


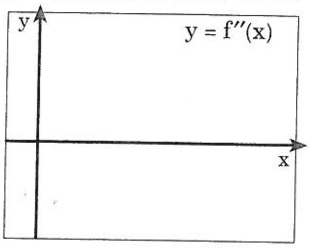




b)







c) Use the graphs above to check these rules :
Point $(a,\ f(a))$ on the graph of $y = f(x)$ is . . .

a maximum turning point	if $f'(a) = 0$ and $f''(a) < 0$
a minimum turning point	if $f'(a) = 0$ and $f''(a) > 0$
a stationary point of inflection	if $f'(a) = 0$ and $f''(a) = 0$

2 Given $f(x) = x^2e^x$
with $f'(x) = xe^x(x + 2)$
and $f''(x) = e^x(x^2 + 4x + 2)$
Find the coordinates of the stationary points and state their nature.

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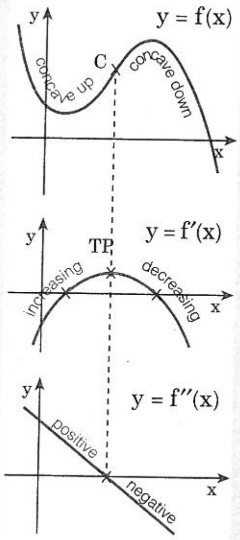
B Inflection and Concavity



A section of the graph $y = f(x)$ is said to be concave up when its *gradient is increasing*, i.e. when $f'(x)$ is increasing, hence, when $f''(x) > 0$.

A section of the graph $y = f(x)$ is concave down when the *gradient is decreasing* i.e. when $y = f'(x)$ is decreasing, hence when $f''(x) < 0$.

Where the graph changes from concave up to concave down (or visa versa) $f(x)$ has a point of inflection, at that point $f'(x)$ has a turning point and $f''(x) = 0$.

Note that if $f''(x) = 0$, but $f'(x) \neq 0$, the point of inflection is not a stationary point of inflection. i.e concavity changes but the tangent is not horizontal.

1 This is the graph of $y = f(x)$, it is a polynomial of degree 5.

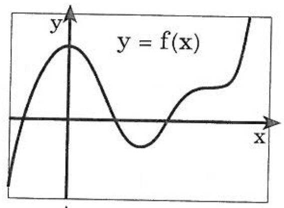
a) Sketch the graph of $y = f'(x)$.

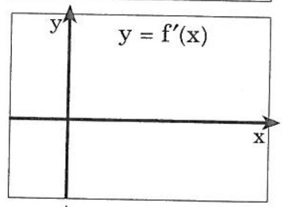
b) Sketch the graph of $y = f''(x)$.

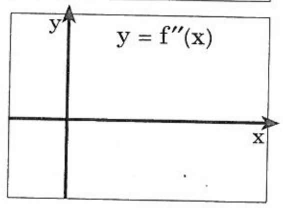
c) On the graph of $y = f(x)$

i) mark three points of inflection. Label them P, Q, R.

ii) colour with red the sections of the curve which are concave up.







2 A curve has equation
$f(x) = x^3 - 3x^2 + 1$.

a) Find $f'(x)$ and $f''(x)$. ....

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b) Is point $(2,\ {}^-3)$ a maximum or is it a minimum? Say why.

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c) Find the coordinates of the point of inflection.

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d) For what values of x is the curve concave up?

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