Vector Calculus Note

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Contents

This note was, again, not in my plan, but rather a tool to develop live tex skills in practice. Together with IA Mathematics B Note in Lent Term. The lecturer on vector calculus (this course) was the famous theoretical physicist Professor David Tong, and the lectures were surely fun. It is a pity that I cannot carry those laughter and happiness in this note, please forgive me.

We won't introduce notations that are included in physics courses such as statistical mechanics or relativity.

Chapter 1

Introduction

We will learn to differentiate and integrate functions of the form

$$f:\ \mathbb{R}^m \longmapsto \mathbb{R}^n \tag{1.1}$$

An element of $\mathbb{R}^m$ or $\mathbb{R}^n$ is a vector.

1.1 Example of Maps

A fuction of $\mathbb{R} \longmapsto \mathbb{R}^n$ defines a curve in $\mathbb{R}^n$. In physics, we might think of $\mathbb{R}$ as time and $\mathbb{R}^n$ as space with $n = 3$ and we write this as

$$f:\ t \longmapsto \vec{x}(t) \text{ with } x \in \mathbb{R}(n) \tag{1.2}$$

Generalising, a map

$$f:\ \mathbb{R} \longmapsto \mathbb{R}^n \tag{1.3}$$

defines a surface in $\mathbb{R}^n$, and so on.

In other applications, the domain $\mathbb{R}^n$ might be viewed as physical space. For example, in physics a scalar field is a map from vector space to scalar space (number line); a vector field is a map from physical space to another space with the same $n = 3$ but more abstract in detail.

Chapter 2

Curves

We consider maps of the form

$$f:\ \mathbb{R} \longmapsto \mathbb{R}^m. \tag{2.1}$$

Assign a coordinate $t \in \mathbb{R}$ and use cartesian coordinates in $\mathbb{R}^n$,

$$\vec{x} = (x^1, \cdots, x^n) = x^i \vec{e}_i,\ \text{with every } x^i \text{ a function of } t. \tag{2.2}$$

Note that the lecturer wrote i on the top right corner of x instead of bottom right corner, which means we have already introduced dual vector space.

The orthonormal basis, $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$, will not be discussed in detail.

The image of the function is a parametrical curve, with t the parameter.

The choice of parameterisation is not unique, for example, consider a simple helix,

$$\vec{x}(t) = (\cos \lambda t, \sin \lambda t, \lambda t), \tag{2.3}$$

this gives the same helix for all $\lambda \neq 0$. Note that this is a 'good' example where the answers are independent of the choice of parameterisation, in real life we should not assume that.

2.1 Differentiating the Curve

A vector function $f(t)$ is differentiable at t if $\delta t \to 0$, we have

$$\vec{x}(t + \delta t) - \vec{x}(t) = \dot{\vec{x}}(t)\delta t + \mathcal{O}(\delta t^2). \tag{2.4}$$

If $\dot{\vec{x}}$ exists everywhere then the curve is said to be **smooth**.

If we are in cartesian coordinates, then we just differentiate vector components

$$\dot{\vec{x}} = \dot{x}^i \hat{e}_i \tag{2.5}$$

Note that if we have a function $f(t)$ and vector $\vec{g}(t)$, $\vec{h}(t)$, the following relations hold

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(f\vec{g}) &= \dot{f}\vec{g} + f\dot{\vec{g}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\vec{g} \cdot \vec{h} &= \dot{\vec{g}} \cdot \vec{h} + \vec{g} \cdot \dot{\vec{h}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\vec{g} \times \vec{h} &= \dot{\vec{g}} \times \vec{h} + \vec{g} \times \dot{\vec{h}} \end{aligned} \tag{2.6}$$

Namely we just apply product rule on components.

2.1.1 Tangent vectors

The derivative $\dot{\vec{x}}(t)$ is the tangent vector to the curve of t. Note that the direction of $\dot{\vec{x}}(t)$ is independent of the choice of parameterisation (up to a sign), but the magnitude will.

We will call a parameterisation **regular** if $\dot{\vec{x}}(t) \neq 0$ for any t.

We will assume that our parameterisation are regular.

The arc length s is just the distance along the curve.

$$\delta s = |\delta \vec{x}| + \mathcal{O}(|\delta \vec{x}^2|), \quad \frac{\mathrm{d}s}{\mathrm{d}t} = \pm|\dot{\vec{x}}| \tag{2.7}$$

The arc length $s(t)$ is defined by the integral

$$s = \int_{t_0}^{t} \mathrm{d}t' \; |\dot{\vec{x}}(t')|. \tag{2.8}$$

As defined, $s > 0$ for $t > t_0$ and $s < 0$ for $t < t_0$. One important property of the arc length is s is independent of the choice of parameterisation and it is obvious to observe. But we will prove it in the following way:

$$\begin{aligned}
&\text{Choose two sets } \vec{x}(t) \text{ and } \vec{x}(\tau)\\
&\text{assume } \frac{\mathrm{d}\tau}{\mathrm{d}t} > 0\\
\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} &= \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\\
s &= \int_{t_0}^{t} \mathrm{d}t' \; \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t'}\right|\\
&= \int_{t_0}^{t} \mathrm{d}t' \; \frac{\mathrm{d}\tau'}{\mathrm{d}t'}\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau'}\right|\\
&= \int_{\tau_0}^{\tau} \mathrm{d}\tau' \; \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau'}\right|
\end{aligned} \tag{2.9}$$

A lot of similar proofs can be done in similar ways, especially when building up a new theory for physics, such as relativity or quantum mechanics. Back to our proof, it means s itself is a natural parameterisation of the curve. So we can think of a $\vec{x}$ as a function of s instead of t, your original parameterisation. Because of the $\mathrm{d}s/\mathrm{d}t$ value, we know that

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}s} = \hat{\tau} \tag{2.10}$$

is always a unit vector in the direction of the curve.

2.1.2 Curvature and Torsion

A curve parametrised by its arc length s. This has tangent vector $\vec{t} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}s}$, with $|t| = 1$, the curvature $\kappa(s)$ is defined as

$$\kappa(s) = \left|\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}s^2}\right|. \tag{2.11}$$

This has a lot of connections to physical curves, the curvature radius of the curve at some point is defined as $\rho = 1/\kappa$.

The principal normal is defined by

$$\hat{n} = \frac{1}{\kappa}\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}s^2} = \frac{\mathrm{d}\hat{\tau}}{\mathrm{d}s} \tag{2.12}$$

which is a unit vector orthogonal to the tangent vector.

Provided that the curvature is not zero, $\hat{n}$ and $\hat{t}$ defines a plane called osculating plane. You can compute curvature by

$$\kappa = \left|\hat{\tau} + \frac{\mathrm{d}\hat{\tau}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\hat{\tau}}{\mathrm{d}s}\right|. \tag{2.13}$$

For curves in $\mathbb{R}^3$, define the binormal

$$\hat{b} = \hat{\tau} \times \hat{n}. \tag{2.14}$$

Hence $\hat{b}$, $\hat{\tau}$ and $\hat{n}$ form an orthonormal basis, which is defined by the curve itself only. This is exactly why we can choose a frame that is moving along a curve. Note that

$$\hat{b} \cdot \frac{\mathrm{d}\hat{b}}{\mathrm{d}s} = 0, \tag{2.15}$$

which can be proved by taking $\hat{\tau} \cdot \hat{b}$. Therefore the modulus of $\hat{b}$ stays the same, it IS a unit vector.

Define the torsion τ as

$$\frac{\mathrm{d}\hat{b}}{\mathrm{d}s} = -\tau\hat{n}. \tag{2.16}$$

The torsion measures how much the curve twists out of the plane. It vanishes for planar curves.

Note that we have

$$\begin{aligned} \frac{\mathrm{d}\hat{\tau}}{\mathrm{d}s} &= \kappa(s)(\hat{b} \times \hat{\tau}) \\ \frac{\mathrm{d}\hat{b}}{\mathrm{d}s} &= \tau(s)(\hat{\tau} \times \hat{b}) \end{aligned} \tag{2.17}$$

These are six first order differential equations (2×3). For fixed κ and τ there is unique set of solutions that gives $\hat{b}$ and $\hat{\tau}$. You cannot specify where the curve sits, but you are able to know how it curves and twists, and the parameterisation you chose decides the scale of the curve.

2.2 Line Integrals

We will talk about two kinds of line integrals, over a scalar field and a vector field.

Definition A scalar field $\phi(\vec{x})$ is a map

$$\phi : \ \mathbb{R}^n \longmapsto \mathbb{R}. \tag{2.18}$$

We would like to integrate ϕ along the curve C, given by some parameterised function, in a way that is independent of the choice of parameterisation.

We work with the arc length s, and we let $\vec{x}(s)$ be a curve that runs from $\vec{a}$ to $\vec{b}$. We define the line integral of the scalar field $\phi(\vec{x})$ from $\vec{a}$ to $\vec{b}$ to be

$$\int_C \phi \ \mathrm{d}s = \int_{s_a}^{s_b} \phi(\vec{x}(s)) \ \mathrm{d}s, \tag{2.19}$$

where, by convention, $s_a < s_b$, to make sure that we have same answer from the opposite direction[1]. In terms of our general parameterisation $\vec{x}(t)$, suppose that $t_a < t_b$,

$$\int_C \phi \ \mathrm{d}s = \int_{t_a}^{t_b} \phi(\vec{x}(t)) \frac{\mathrm{d}s}{\mathrm{d}t} \ \mathrm{d}t = \int_{t_a}^{t_b} \phi(\vec{x}(t)) |\dot{\vec{x}}| \ \mathrm{d}t. \tag{2.20}$$

The factor $\dot{\vec{x}}$ insures the independence of parameterisation.

Definition A vector field $\vec{F}(\vec{x})$ is a map

$$\vec{F} : \ \mathbb{R}^n \longmapsto \mathbb{R}^n \tag{2.21}$$

The line integral of a vector field $\vec{F}(\vec{x})$ along the curve C parameterised by t, from $\vec{x}(t_a)$ to $\vec{x}(t_b)$ is defined as

$$\int_C \vec{F} \cdot \mathrm{d}\vec{x} = \int_{t_a}^{t_b} \vec{F}(\vec{x}(t)) \cdot \dot{\vec{x}}(t) \ \mathrm{d}t \tag{2.22}$$

[1] In this way, physical quantities, or physical "truth", stays the same no matter how you integrate it.

By the dot product we know that we are taking $\vec{F}$ tangent to the curve. Clearly this implies connection with work and energy in physics. Again, this definition is independent of parameters. However, this time the direction of the integral matters. Just add a minus sign if the direction changes[2].

The choice of direction of curve C is called orientation. The line integral of a scalar field does not depend on orientation, but the line integral of a vector field does depend on the orientation. When you are doing a line integral of vector field, it is convention that you must specify the orientation first, as we treat orientation as a natural property of the curve.

Sometimes we will integrate along a **close path** C, with $\vec{a} = \vec{b}$. The integral is called the circulation of $\vec{F}$ around C.

$$\oint \vec{F} \cdot \mathrm{d}\vec{x} \tag{2.23}$$

Sometimes we will have a **piecewise smooth** curve, which is not differentiable at some points. The total integral is just sum of integrals over different parts of the curve.

$$\int_{C_1+C_2} \vec{F} \cdot \mathrm{d}\vec{x} = \int_{C_1} \vec{F} \cdot \mathrm{d}\vec{x} + \int_{C_2} \vec{F} \cdot \mathrm{d}\vec{x} \tag{2.24}$$

In addition, the notation $-C$ means integrating C in the opposite direction:

$$\int_{-C} \vec{F} \cdot \mathrm{d}\vec{x} = \int_{C} \vec{F} \cdot \mathrm{d}\vec{x}. \tag{2.25}$$

There are some tricks that will be useful later. For example, consider a close path to be the addition of two open path, or consider a close path to be the addition of two close path[3].

2.3 Conservative Fields

Question Do there exists $\vec{F}$ such that $\int_C \vec{F} \cdot \mathrm{d}\vec{x}$ is independent of the path taken between $\vec{a}$ and $\vec{b}$ [4]?

2.3.1 The gradient

Consider a scalar field which a map

$$\phi : \mathbb{R}^n \longmapsto \mathbb{R}, \tag{2.26}$$

the partial derivatives are defined to be

$$\frac{\partial \phi}{\partial x} = \lim_{\delta x \to 0} \frac{1}{\delta x}[\phi(x+\delta x, \cdots) - \phi(x, \cdots)]. \tag{2.27}$$

The function is differentiable if all partial differentiation exists.

Let $\{\hat{e}_i\}$ be orthonormal basis of $\mathbb{R}^n$, the gradient of a scalar field ϕ is a vector field, whic is defined as

$$\nabla \phi = \frac{\partial \phi}{\partial x^i} \hat{e}_i = \partial_i \phi. \tag{2.28}$$

If we want to compute how ϕ changes in some direction $\hat{n}$, just compute the directional derivative

$$\hat{n} \cdot \nabla \phi. \tag{2.29}$$

This reaches maximum value if $\hat{n}$ is parallel to $\nabla \phi$, which gives us intuition that gradient of ϕ points to the direction that ϕ in increasing most quickly.

[2]Simple application of this is the work done by a force related to time or position.
[3]Similar to complex analysis, the parallel part vanishes in the middle because $\int_{-C} = -\int_C$.
[4]That leads to the fact that integral over any close path must be zero.

Definition A vector field $\vec{F}$ is called conservative if it can be written as

$$\vec{F} = \nabla\phi, \tag{2.30}$$

in which ϕ is called the potential[5].

Now the question is, how to **prove** that

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{x},\ \forall C \ \Leftrightarrow\ \vec{F} = \nabla\phi. \tag{2.31}$$

If $\vec{F} = \nabla\phi$, then along an open curve C, parameterised by $\vec{x}(t)$,

$$\int_C \vec{F}\cdot \mathrm{d}\vec{x} = \int \nabla\phi\cdot \mathrm{d}\vec{x} = \int_{t_a}^{t_b} \frac{\mathrm{d}}{\mathrm{d}t}\phi(\vec{x}(t))\ \mathrm{d}t = \phi(\vec{x}(t_b)) - \phi(\vec{x}(t_a)). \tag{2.32}$$

Now prove from the opposite direction. Suppose that $\oint_C \vec{F}\cdot \mathrm{d}\vec{x} = 0$ we want to construct ϕ. Note that there is no unique choice of ϕ because you can always add a constant to it and the result also works well. Using that ambiguity, choose $\phi(0) = 0$, and define

$$\phi(\vec{y}) = \int_{C(\vec{y})} \vec{F}\cdot \mathrm{d}\vec{x}. \tag{2.33}$$

Then take the gradient of ϕ,

$$\begin{aligned}
\nabla\phi = \partial_i\phi(\vec{y}) &= \lim_{\delta y\to 0}\frac{1}{\delta y}\left[\int_{C(\vec{y}+\delta\vec{y})} \vec{F}\cdot \mathrm{d}\vec{x} - \int_{C(\vec{y})} \vec{F}\cdot \mathrm{d}\vec{x}\right] \\
&= \lim_{\delta y\to 0}\frac{1}{\delta y}\left[\delta y\vec{F} + \mathcal{O}(\delta y^2)\right] \\
&= F_i(\vec{y}) \\
\Rightarrow\ \vec{F} &= \nabla\phi.
\end{aligned} \tag{2.34}$$

That is the end of our proof.

Another question is that given a function $\vec{F}$, how do we know ϕ? There is a check: if there exists a ϕ, that $F_i = \partial_i\phi$, then

$$\partial_j F_i = \partial_j\partial_i\phi = \partial_i F_j,\ \forall i, j. \tag{2.35}$$

This is a necessary condition of the existence of ϕ. We will prove later that it is also a sufficient condition if $\vec{F}$ is well-defined everywhere.

Definition Given a scalar function $\phi(\vec{x})$, the differential is

$$\mathrm{d}\phi = \partial_i\phi\ \mathrm{d}x^i = \nabla\phi\cdot \mathrm{d}x^i. \tag{2.36}$$

Given a vector field $\vec{F}(\vec{x})$, $\vec{F}\cdot \mathrm{d}\vec{x}$ is exact if it can be written as

$$\vec{F}\cdot \mathrm{d}\vec{x} = \mathrm{d}\phi. \tag{2.37}$$

This $\vec{F}\cdot \mathrm{d}\vec{x}$ is called an exact differential form. It will be important in later courses.

Note Consider

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \tag{2.38}$$

integrate over a circle whose center is the origin, and you will normally find $2\pi \neq 0$. This is because $\vec{F}$ has a singularity at the origin and the $\phi = \arctan(y/x)$ is discontinuous. This can be solved using 'small circle' skills to exclude the singularity, similar to that in complex analysis. The thing that matters here is that

$$0 = \oint \vec{F}\cdot \mathrm{d}\vec{x}\ \Leftrightarrow\ \vec{F} = \nabla\phi \tag{2.39}$$

only holds true if ϕ is continuous so that $\nabla\phi$ is defined all the way along C in $\mathbb{R}^n$.

[5] In classical mechanics there should be a minus sign before the RHS

Chapter 3

Surfaces, Volumes and Fields

3.1 Multiple Integral

Consider a region $D \in \mathbb{R}^2$, we want to integrate a scalar field $\phi(x, y)$ over D. This is written as

$$\int_D \phi \; \mathrm{d}A, \; \mathrm{d}A = \mathrm{d}x\mathrm{d}y, \tag{3.1}$$

in which $\mathrm{d}A$ is called the area element. The basic idea is cut the area into small elements and integrate the function over them. This can be thought as the volume of water with surface at height ϕ. From this point of view, it is obvious that the integration over 1 is just the area.

To evaluate the area integral, we split the region D into strips[1]. This is included in IA NST Math B course. In general,

$$\int_D \phi(x, y) \; \mathrm{d}A = \int_a^b \mathrm{d}y \int_{x_1(y)}^{x_2(y)} \mathrm{d}x \; \phi(x, y) = \int_c^d \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \; \phi(x, y). \tag{3.2}$$

For suitably well-behaved ϕ and D, any way of splitting up $\int \mathrm{d}A$ gives the same answer (Fubini's Theorem, 1907)[2]. It is often useful to evaluate integrals using polar or spherical coordinates. Consider an invertible, smooth change of variables

$$(x, y) \longmapsto (u, v) \tag{3.3}$$

which can be done using some kind of transformations[3], things may be simplified. The integral in (u, v) coordinates is

$$\int_D \phi \; \mathrm{d}A = \int_D \mathrm{d}x\mathrm{d}y \; \phi(x, y) = \int_{D'} \mathrm{d}u\mathrm{d}v \; |J(u, v)|\phi(u, v) \tag{3.4}$$

where D' is the same region as D, but parameterized by (u, v). The new thing $J(u, v)$ is called the **Jacobian**, defined as the determinant

$$J(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial(x, y)}{\partial(u, v)} \tag{3.5}$$

Proof In Cartesian coordinates, we summed over small squares with $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$, now we sum over small parallelgrams between $u, v = \text{const}$ lines.

Let $x = x(u, v)$, $y = y(u, v)$, therefore

$$\begin{aligned} \delta x &= \partial_u x \delta u + \partial_v x \delta v \\ \delta y &= \partial_u y \delta u + \partial_v y \delta v \\ \text{or} \quad \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} &= \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} \end{aligned} \tag{3.6}$$

[1] Though sometimes additional skills are needed.
[2] Clearly as a physicist I couldn't care less.
[3] Usually we use conformal because it reserves an invariant and therefore protects some properties of our field.

Suppose

$$\vec{a} = \begin{pmatrix} \partial_u x \\ \partial_u y \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \partial_v x \\ \partial_v y \end{pmatrix}, \tag{3.7}$$

Therefore the area is just

$$\delta A = |\vec{a} \times \vec{b}| = |J(u,v)|\delta u \delta v. \quad \square \tag{3.8}$$

It is easy to generalise our results to n dimensional space.

We now generalize to integrate over a region $V \in \mathbb{R}^3$, we have

$$\int_V \phi(\vec{x}) \ \mathrm{d}V = \lim_{\delta V \to 0} \sum_n \phi(\vec{x}_n) \ \delta V. \tag{3.9}$$

We again perform the integral one coordinate at a time. We can split the volume into small cubes

$$\int_V \phi(\vec{x}) \ \mathrm{d}V = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} \mathrm{d}z \ \phi(\vec{x}) = \int_c^d \mathrm{d}y \int_{z_1(y)}^{z_2(y)} \mathrm{d}z \int_{x_1(y,z)}^{x_2(y,z)} \mathrm{d}x \ \phi(\vec{x}) = \cdots, \tag{3.10}$$

or into thin layers

$$\int_V \phi(\vec{x}) \ \mathrm{d}V = \int_D \mathrm{d}A \int_{z_1(x,y)}^{z_2(x,y)} \mathrm{d}z \ \phi(\vec{x}) = \int_a^b \mathrm{d}z \int_{D(z)} \mathrm{d}A \ \phi(\vec{x}). \tag{3.11}$$

Consider an invertible, smooth change of coordinates again,

$$(x,y,z) \longmapsto (u,v,w), \quad \mathrm{d}V = |J| \mathrm{d}u \mathrm{d}v \mathrm{d}w, \text{ where } J = \frac{\partial(x,y,z)}{\partial(u,v,w)}. \tag{3.12}$$

Dimensional analysis may be a good choice to check your result.

3.1.1 Examples

Start with a spherically symmetric function $f(r)$ integrated over a ball of radius R gives

$$\int_0^R 4\pi r^2 f(r) \ \mathrm{d}r. \tag{3.13}$$

Again, it is clear that if $f(r) = 1$, the integral just gives us the volume of the ball.

What is the volume of a sphere of radius R with cylinder of radius $s < R$ removed from the middle? We can use spherical coordinates that gives

$$\int_{\theta_0}^{\pi} \pi R^3 \sin^3\theta \ \mathrm{d}\theta, \text{ where } \theta_0 = \arcsin\left(\frac{s}{R}\right), \tag{3.14}$$

or use cylindrical polar coordinates that gives

$$\int_s^R 4\pi\rho\sqrt{R^2 - \rho^2} \ \mathrm{d}\rho = \frac{4}{3}\pi\left(R^2 - s^2\right)^{3/2}. \tag{3.15}$$

A hemisphere H of radius R and $z \geq 0$ has charge density $f(z) = f_0 z/R$, what is the total charge? Convert z into $r\cos\theta$ and it gives

$$\int_0^R \int_0^{\pi/2} 2\pi f_0 r^2 \sin\theta \frac{r\cos\theta}{R} \ \mathrm{d}\theta \mathrm{d}r = \frac{1}{4}\pi R^3 f_0. \tag{3.16}$$

To compute the center of mass of an object, we need vector valued integrals. Let $\rho(\vec{x})$ be the density of a solid object (a scalar field),

$$\vec{x}_c = \frac{\int_V \vec{x}\rho(\vec{x}) \ \mathrm{d}V}{\int_V \rho(\vec{x}) \ \mathrm{d}V}. \tag{3.17}$$

For the solid hemisphere with constant density ρ, by symmetry we can integrate over 0 in the x and y directions,

$$\vec{x}_c = \frac{\int_V \vec{x}\rho \ \mathrm{d}V}{\int_V \rho \ \mathrm{d}V} = \frac{\int_0^R \int_0^{\pi/2} 2\pi(0,0,r\cos\theta) r^2 \sin\theta \ \mathrm{d}\theta \mathrm{d}r}{\frac{2}{3}\pi R^3} = \left(0, 0, \frac{3}{8}R\right). \tag{3.18}$$

3.2 Surface Integral

We can define surfaces in $\mathbb{R}^3$ by a function

$$F(x, y, z) = 0. \tag{3.19}$$

This is one constraint on three variables[4]. 'The surface has codimension 1' is another way to say the surface has dimension $n-1$.

A parameterised surface is defined by

$$\vec{x} : \mathbb{R}^2 \longmapsto \mathbb{R}^3. \tag{3.20}$$

At each point on the surface, the normal vector $\hat{n}$ points away in a perpendicular direction. For the surface $F(\vec{x}) = 0$, then $\hat{n} \parallel \nabla F$.

Proof $\hat{m} \cdot \nabla F$ is the rate of change of $F(\vec{x})$ in the direction $\hat{m}$. There are two linearly independent vectors $\hat{m}_1$ and $\hat{m}_2$ that are tangent to the surface and obey $m_i \cdot \nabla F = 0$ for $i = 1, 2$. The normal vector $\hat{n}$ is normal to $m_{1,2}$ and so $\hat{n} \parallel \nabla F$. $\Box$

Therefore the normal vector can be defined as

$$\hat{n} = \pm \frac{1}{|\nabla F|} \nabla F. \tag{3.21}$$

For a parameterised surface $\vec{x}(u, v)$, the tangent vectors are

$$\partial_u \vec{x} \quad \text{and} \quad \partial_v \vec{x}. \tag{3.22}$$

and the normal vector is $\hat{n} \parallel \partial_u \vec{x} \times \partial_v \vec{x}$.

Definition If $\hat{n} \neq \vec{0}$ at all times, the surface is said to be regular.

A surface can have a boundary. This is a closed curve denoted[5] as $C = \partial S$. The boundary has no end points since it is a closed curve, but the boundary of a boundary is empty, i.e. $\partial C = \partial^2 S = 0$. I asked after class[6], does it work for C with 2 codimensions or 1 dimension only? The answer was 'That's how powerful it is! No matter how many dimensions you have, two, three, or even infinity, or there is a m-dimensional subspace in n-dimensional space. As long as you take the boundary of boundary, it just vanishes. ' Therefore it surly works in all dimensions. Is there any topological reason? 'Maybe, but it is not of our concern. '

Definition A surface is said to be bounded / unbounded if it does not / does stretch into infinity. A bounded surface with no boundary is said to be closed.

There is no canonical way to fix the sign of $\hat{n}$. This is a matter of convention and determines what we mean by 'outside' and 'inside'. A surface is said to be orientable if there is a consistent choice of unit normal $\hat{n}$ which varies smoothly over the surface.

3.2.1 Integrating Scalar Fields

Consider a parameterised surface $\vec{x}(u, v)$ and move a small amount δu or δv. The parallelgram defined by $\partial_u \vec{x}$ and $\partial_v \vec{x}$ has scalar area

$$\delta S = \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| \delta u \delta v, \tag{3.23}$$

therefore the integration is defined by

$$\int_S \phi(\vec{x}) \, \mathrm{d}S = \int_D \mathrm{d}u\mathrm{d}v \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| \phi(\vec{x}(u, v)). \tag{3.24}$$

[4]Think about degree of freedom of a mass point in three dimensions.

[5]This is a reasonable notation. Boundary is defined as $\partial V_r = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{V_{r+\varepsilon}}{V_r}$, see Evans for more information.

[6]It is always enjoyable to discuss with David.

The first advantage of this definition is that it does not depend on the orientation of S. And this definition is independent of parameterisation. To see this, suppose we have $\vec{x}(\tilde{u},\tilde{v})$, then

$$\begin{aligned}\frac{\partial \vec{x}}{\partial u} &= \partial_{\tilde{u}}\vec{x}\partial_u\tilde{u} + \partial_{\tilde{v}}\vec{x}\partial_v\tilde{v}.\\ \partial_u\vec{x}\times\partial_v\vec{x} &= \frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\partial_{\tilde{u}}\vec{x}\times\partial_{\tilde{v}}\vec{x}\\ \mathrm{d}\tilde{u}\mathrm{d}\tilde{v} &= \frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\mathrm{d}u\mathrm{d}v\\ \delta S &= \left|\frac{\partial\vec{x}}{\partial\tilde{u}}\times\frac{\partial\vec{x}}{\partial\tilde{v}}\right|\delta\tilde{u}\delta\tilde{v}\end{aligned} \tag{3.25}$$

Example Let S be the surface of a sphere of radius R subtended by an angle α. What is the area of it?

$$S = \int_0^\alpha 2\pi R^2 \sin\theta \ \mathrm{d}\theta = 2\pi R^2(1-\cos\alpha). \tag{3.26}$$

3.2.2 Integrating Vector Fields

It is often useful to integrate a vector field over a surface to yield a number[7]. We do this by

$$\int_S \vec{F}(\vec{x})\cdot\hat{n}\ \mathrm{d}S = \int_S \vec{F}(\vec{x})\cdot\mathrm{d}\vec{S} = \int_D \mathrm{d}u\mathrm{d}v\ \left(\frac{\partial\vec{x}}{\partial u}\times\frac{\partial\vec{x}}{\partial v}\right)\cdot\vec{F}(\vec{x}(u,v)), \tag{3.27}$$

where $\hat{n}$ is the unit normal to the surface and $\mathrm{d}\vec{S}$ is naturally the vector area element. This is called the **flux** of $\vec{F}$ through the surface S. It is again reparameterisation invariant[8], but it does depend on the orientation of S.

Example Consider a fluid with velocity field $\vec{F}(\vec{x})$. In a small time δt, the amount of fluid flowing through δS is $\vec{F}\delta t\cdot\hat{n}\delta S$, then the flux is[9]

$$\int_S \vec{F}\cdot\mathrm{d}\vec{S} = \text{Fluid crossing } S \text{ per unit time.} \tag{3.28}$$

Here we let $\vec{F} = (-x, 0, z)$. This is easy to plot on paper. Let the surface S be the surface of a sphere of radius R subtended by an angle α. Note that $\vec{F} = (-r\sin\theta\cos\phi, 0, r\cos\theta)$, and $\mathrm{d}\vec{S} = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi\ \hat{r}$,

$$\int_S \vec{F}\cdot\mathrm{d}\vec{S} = \int_0^\alpha \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\ R^3\sin\theta(-\sin^2\theta\cos^2\phi+\cos^2\theta) = \pi R^3\cos\alpha\sin^2\alpha. \tag{3.29}$$

Gauss-Bonnet Theorem Consider a surface S, with normal vector $\hat{n}$ at some point, Draw a plane[10] containing $\hat{n}$, which intersects with S and gives a curve C, with curvature κ at that point. Now rotate the plane containing $\hat{n}$, the curve and κ changes. The Gaussian curvature of S is defined to be

$$K = \kappa_{\min}\kappa_{\max}. \tag{3.30}$$

Pick three points on S and draw a geodesic triangle. The sides are geodesics, meaning curves with the smallest arc length between two points, $\theta_1+\theta_2+\theta_3 = \pi + \int_T K\mathrm{d}S$.

Another version: For a closed surface S, $\int K\mathrm{d}S = 4\pi(1-g)$, $g\in\mathbb{N}$. g is the genus which counts the holes in surface.

[7]Recall that our line integral gives a number too, but here we project $\vec{F}$ into another direction.
[8]The proof should be in the same form as we did in the last subsection.
[9]Choosing the surface to be a small cube, we can prove the Gauss's theorem.
[10]There are infinite planes that can be chosen.

Proof Nope! (Can do in differential geometry for version 1, but version 2 is more like a topological approach. Both have quantum-mechanical proofs, but evolves highly advanced concepts such as supersymmetry.)

3.3 Gradient, Divergence and Curl

3.3.1 The gradient

Given a scalar field $\phi : \mathbb{R}^n \longmapsto \mathbb{R}$ that we define the gradient by

$$\phi(\vec{x} + \vec{h}) = \phi(\vec{x}) + \vec{h} \cdot \nabla\phi + \mathcal{O}(|\vec{h}|^2), \tag{3.31}$$

where $\nabla\phi$ is the gradient.

In Cartesian coordinates it is straightforward to write

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\hat{e}_i. \tag{3.32}$$

Example Consider $\phi : \mathbb{R}^3 \longmapsto \mathbb{R}$ with $\phi(\vec{x}) = -1/r$. Then

$$\nabla\frac{-1}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \frac{\vec{r}}{r^3} = \frac{\hat{r}}{r^2}. \tag{3.33}$$

This will be frequently used in electrodynamics.

Application Let $\vec{x}(t) : \mathbb{R} \longmapsto \mathbb{R}^n$ define a curve in $\mathbb{R}^n$ and $\phi : \mathbb{R}^n \longmapsto \mathbb{R}$ be a scalar field. Then $\phi(\vec{x}(t)) : \mathbb{R} \longmapsto \mathbb{R}$ is the value of the function along the curve. We can differentiate ϕ along the curve, using the chain rule.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial x^i} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t} = \nabla\phi \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{3.34}$$

3.3.2 Divergence and Curl

We define the gradient operator to be

$$\nabla = \hat{e}_i\frac{\partial}{\partial x^i}. \tag{3.35}$$

This is both a vector and an operator[11]. Naturally it can act on other things.

Consider a vector field $\vec{F} : \mathbb{R}^n \longmapsto \mathbb{R}^n$, the divergence of $\vec{F}$ is a scalar field defined by

$$\mathrm{div}\ \vec{F} = \nabla \cdot \vec{F} = \left(\hat{e}_i\frac{\partial}{\partial x^i}\right) \cdot (\hat{e}_j F_j) = \frac{\partial F_i}{\partial x^i}. \tag{3.36}$$

We will later see that the divergence measures the net flow of $\vec{F}$ into or out of the point $\vec{x}$, which is clearly useful in any fields.

For vector field $\vec{F} : \mathbb{R}^n \longmapsto \mathbb{R}^n$ we can define the curl to be

$$\nabla \times \vec{F} = \left(\hat{e}_i\frac{\partial}{\partial x_i}\right) \times (\hat{e}_j F_j) = \epsilon_{ijk}\hat{e}_k\frac{\partial F_j}{\partial x^i}. \tag{3.37}$$

We will see that the curl measures the rotation of $\vec{F}$.

Here are some useful tips when evaluating derivatives of fields with radial symmetry.

$$r^2 = x^i x^i \ \Rightarrow\ 2r\partial_i r = 2x^i \ \Rightarrow\ \partial_i r = x^i/r. \tag{3.38}$$

Then we have

$$\nabla r^p = \hat{e}_i\partial_i r^p = pr^{p-1}\hat{r}. \tag{3.39}$$

[11] It is just waiting for some function to come along and be differentiated.

The vector $\vec{x} = x^i \hat{e}_i$ can also be written as $\vec{r} = r\hat{r}$, to highlight that it points radially outwards.

We have

$$\nabla \cdot \vec{r} = \delta_{ii} = n. \tag{3.40}$$

Also, in $\mathbb{R}^3$,

$$\nabla \times \vec{r} = \epsilon_{ijk} \partial_i x^j \hat{e}_k = 0 \tag{3.41}$$

3.3.3 Some basic properties

For a constant α, scalar field $\phi(\vec{x})$ and $\psi(\vec{x})$, and for vector field $\vec{F}(\vec{x})$ and $\vec{G}(\vec{x})$, we have

$$\begin{aligned} \nabla\left(\alpha\phi + \psi\right) &= \alpha\nabla\phi + \nabla\psi \\ \nabla \cdot \left(\alpha\vec{F} + \vec{G}\right) &= \alpha\nabla \cdot \vec{F} + \nabla \cdot \vec{G} \\ \nabla \times \left(\alpha\vec{F} + \vec{G}\right) &= \alpha\nabla \times \vec{F} + \nabla \times \vec{G} \end{aligned} \tag{3.42}$$

This is the statement that ∇ is a linear operator.

∇ obeys a generalized product rule (Leibniz property),

$$\begin{aligned} \nabla(\phi\psi) &= \phi\nabla\psi + \psi\nabla\phi \\ \nabla \cdot \left(\phi\vec{F}\right) &= (\nabla\phi) \cdot \vec{F} + \phi\nabla \cdot \vec{F} \\ \nabla \times \left(\phi\vec{F}\right) &= (\nabla\phi) \times \vec{F} + \phi\nabla \times \vec{F} \end{aligned} \tag{3.43}$$

The proof of those follow from the definitions. There are further Leibniz properties

$$\begin{aligned} \nabla \cdot \left(\vec{F} \times \vec{G}\right) &= \left(\nabla \times \vec{F}\right) \cdot \vec{G} - \vec{F} \cdot \left(\nabla \times \vec{G}\right) \\ \nabla\left(\vec{F} \cdot \vec{G}\right) &= \vec{F} \times \left(\nabla \times \vec{G}\right) + \vec{G} \times \left(\nabla \times \vec{F}\right) + \left(\vec{F} \cdot \nabla\right)\vec{G} + \left(\vec{G} \cdot \nabla\right)\vec{F} \\ \nabla \times \left(\vec{F} \times \vec{G}\right) &= \left(\nabla \cdot \vec{G}\right)\vec{F} - \left(\nabla \cdot \vec{F}\right)\vec{G} + \left(\vec{G} \cdot \nabla\right)\vec{F} - \left(\vec{F} \cdot \nabla\right)\vec{G} \end{aligned} \tag{3.44}$$

Definitions

- A vector field $\vec{F}$ is conservative if it can be written as $\vec{F} = \nabla\phi$.
- A vector field $\vec{F}$ is irrotational if $\nabla \times \vec{F} = 0$.
- A vector field $\vec{F}$ is divergence free or solenoidal if $\nabla \cdot \vec{F} = 0$.

Theorem For fields defined everywhere on $\mathbb{R}^3$, conservative means irrotational[12], i.e. $\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla\phi$.

Proof [13] If $\vec{F} = \nabla\phi$, then

$$\left(\nabla \times \vec{F}\right)_k = \epsilon_{ijk}\partial_i\partial_j\phi = 0 \tag{3.45}$$

by symmetry. We can prove the other half when we have learned the Stokes' theorem.

Theorem For $\vec{F}$ defined everywhere on $\mathbb{R}^3$, $\nabla \cdot \vec{F} = 0$ if and only if $\vec{F} = \nabla \times \vec{A}$, where $\vec{A}$ is the vector potential.

[12] Baby version of Poincare lemma.
[13] Only half of it.

Proof If $\vec{F} = \nabla \times \vec{A}$, then

$$F_i = \epsilon_{ijk} \partial_j A_k, \tag{3.46}$$

which gives

$$\nabla \cdot \vec{F} = \partial_i \left(\epsilon_{ijk} \partial_j A_k \right) = 0 \tag{3.47}$$

again by antisymmetry. Next, suppose that $\nabla \cdot \vec{F} = 0$, pick a point $\vec{x}_0 = (x_0, y_0, z_0)$ and construct

$$\vec{A}(\vec{x}) = \begin{pmatrix} \int_{z_0}^{z} F_y(x, y, z')\ \mathrm{d}z' \\ \int_{x_0}^{x} F_z(x', y, z)\ \mathrm{d}x' - \int_{z_0}^{z} F_x(x, y, z')\ \mathrm{d}z' \\ 0 \end{pmatrix} \tag{3.48}$$

Since $A_z = 0$, we have

$$\nabla \times \vec{A} = \begin{pmatrix} -\partial_z A_y \\ \partial_z A_x \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \begin{pmatrix} F_x(x, y, z) \\ F_y(x, y, z) \\ F_z(x, y, z) \end{pmatrix} \tag{3.49}$$

Note that

$$\begin{aligned} \left(\nabla \times \vec{A} \right)_z &= F_z(x, y, z_0) - \int_{z_0}^{z} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) \mathrm{d}z' \\ &= F_z(x, y, z_0) - \int_{z_0}^{z} \frac{\partial F_z}{\partial z} \mathrm{d}z' \\ &= F_z(x, y, z_0) + F_z(x, y, z')|_{z_0}^{z} \\ &= F_z(x, y, z) \end{aligned} \tag{3.50}$$

Definition The Laplacian is a second order differential operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x^i \partial x^i} = \partial_i \partial_i. \tag{3.51}$$

If a Laplacian acts on a scalar field ϕ it gives back a scalar field. It acts component-wise on a vector field $\vec{F}$ to give another vector field $\nabla^2 \vec{F}$.

$$\nabla^2 \vec{F} = \nabla \left(\nabla \cdot \vec{F} \right) - \nabla \times \left(\nabla \times \vec{F} \right). \tag{3.52}$$

You can use triple product to prove this.

The Laplacian occurs in many places in math and physics, for example, the heat equation

$$\frac{\partial T}{\partial t} = D \nabla^2 T, \tag{3.53}$$

where D is the diffusion constant.

The linear operator ∇ also appears in many laws of physics, for example, the Maxwell equations for electric field $\vec{E}(\vec{x}, t)$ and magnetic field $\vec{B}(\vec{x}, t)$,

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \end{aligned} \tag{3.54}$$

3.4 Orthogonal Curvilinear Coordinates

We want to find expressions for ∇ in different coordinate systems. Introduce coordinates u, v, w so that

$$\vec{x} = \vec{x}(u, v, w) \tag{3.55}$$

A change in the coordinates u, v, w leads to a change in a point $\vec{x} \to \vec{x} + \mathrm{d}\vec{x}$,

$$\mathrm{d}\vec{x} = \frac{\partial \vec{x}}{\partial u}\mathrm{d}u + \frac{\partial \vec{x}}{\partial v}\mathrm{d}v + \frac{\partial \vec{x}}{\partial w}\mathrm{d}w. \tag{3.56}$$

Remember that $\partial\vec{x}/\partial u$ is the tangent vector to the plane $v, w = \text{const}$. These are good coordinates at a point provided that the tangent vectors being linear independent,

$$\frac{\partial \vec{x}}{\partial u} \cdot \left(\frac{\partial \vec{x}}{\partial v} \times \frac{\partial \vec{x}}{\partial w}\right) \neq 0. \tag{3.57}$$

If the tangent vectors are mutually orthogonal, then the coordinates are said to be orthogonal curvilinear. For such coordinates, we introduce normalized tangent vectors

$$\frac{\partial \vec{x}}{\partial u} = h_u \hat{e}_u, \quad \frac{\partial \vec{x}}{\partial v} = h_v \hat{e}_v, \quad \frac{\partial \vec{x}}{\partial w} = h_w \hat{e}_w \tag{3.58}$$

with $h_{u,v,w} > 0$ and $\hat{e}_{u,v,w}$ are right-handed basis.

$$\mathrm{d}\vec{x} = h_u \hat{e}_u \mathrm{d}u + h_v \hat{e}_v \mathrm{d}v + h_w \hat{e}_w \mathrm{d}w, \tag{3.59}$$

which gives

$$\mathrm{d}\vec{x}^2 = h_u^2 \mathrm{d}u^2 + h_v^2 \mathrm{d}v^2 + h_w^2 \mathrm{d}w^2. \tag{3.60}$$

We can see that $h_{u,v,w}$ tell us changes in lengths.

Example In Cartesian coordinates

$$\mathrm{d}\vec{x}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.61}$$

Example In cylindrical coordinates,

$$\mathrm{d}\vec{x}^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2 + \mathrm{d}z^2. \tag{3.62}$$

Example In polar coordinates[14],

$$\mathrm{d}\vec{x}^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \tag{3.63}$$

3.4.1 Gradient in different coordinate systems

For a scalar function $f(x) : \mathbb{R}^3 \longmapsto \mathbb{R}$,

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{x}. \tag{3.64}$$

Comparing to the chain rule,

$$\frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v + \frac{\partial f}{\partial w}\mathrm{d}w = \nabla f \cdot h_u \hat{e}_u \mathrm{d}u + \nabla f \cdot h_v \hat{e}_v \mathrm{d}v + \nabla f \cdot h_w \hat{e}_w \mathrm{d}w. \tag{3.65}$$

In other words

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w \tag{3.66}$$

using orthonomality.

[14] Not good coordinate systems at the origin and poles.

Example In cylindrical coordinates,

$$\nabla f = \frac{\partial f}{\partial \rho}\mathrm{d}\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathrm{d}\phi + \frac{\partial f}{\partial z}\mathrm{d}z. \tag{3.67}$$

Example In spherical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\mathrm{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathrm{d}\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathrm{d}\phi. \tag{3.68}$$

3.4.2 Divergence, Curl and Laplacian in different coordinate systems

In general, we have

$$\nabla = \frac{1}{h_u}\hat{e}_u\frac{\partial}{\partial u} + \frac{1}{h_v}\hat{e}_v\frac{\partial}{\partial v} + \frac{1}{h_w}\hat{e}_w\frac{\partial}{\partial w} \tag{3.69}$$

They now acts on vector field

$$\vec{F} = F_u\hat{e}_u \tag{3.70}$$

but $\{\hat{e}_u\}$ now depend on (u, v, w) so the derivatives of them is a little bit messy.

$$\begin{aligned} \nabla\cdot\vec{F} &= \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_w h_u F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right) \\ \nabla\times\vec{F} &= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{e}_u & h_v\hat{e}_v & h_w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix} \end{aligned} \tag{3.71}$$

Example In cylindrical coordinates, the divergence of vector field $\vec{F}$ is

$$\nabla\cdot\vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F}{\partial z}, \tag{3.72}$$

the curl is

$$\nabla\times\vec{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{e}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\hat{e}_z, \tag{3.73}$$

the Laplacian is

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{3.74}$$

Example In spherical polar coordinates, the divergence of a vector field $\vec{F}$ is

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}, \tag{3.75}$$

the curl is

$$\nabla\times\vec{F} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right)\hat{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial(rF_\phi)}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{e}_\phi, \tag{3.76}$$

the Laplacian is

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{3.77}$$

3.5 Generalising Wave Equations

The Laplacian is a great way to generalise wave equations (create equations for higher dimensions). Suppose that we have a wave equation for a string

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \tag{3.78}$$

and we want to discuss an elastic plate (think of a drum, for example), the object is in 2D rather than 1D. It is true that we can write the equation in many forms, but considering appropriate symmetry, we find that the equation must be independent of the direction of our chosen axis. Laplacian is the perfect operator that satisfies this equation.

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{3.79}$$

This is the general equation for waves. One can see it at all times, equation for electromagnetic waves, the Schródinger equation, etc.

However, there are equations containing higher-order partial differentials, for example, in laser physics our equation becomes

$$\frac{\partial^4 \psi}{\partial x^4} = \frac{1}{c^4}\frac{\partial^4 \psi}{\partial t^4}. \tag{3.80}$$

The thing you need to do is just adding another Laplacian before an Laplacian, which gives rise to

$$\nabla^2\nabla^2 \psi = \frac{1}{c^4}\frac{\partial^4 \psi}{\partial t^4}. \tag{3.81}$$

Note that, if we expand the LHS (in 2D),

$$\nabla^2\nabla^2 \psi = \frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}, \tag{3.82}$$

the third term is a cross-product. It must exist to ensure the $\nabla^2\nabla^2$ operator is rotational invariant.

Chapter 4

The integral Theorems

The fundamental theorem of calculus says

$$\int_a^b \mathrm{d}x \, \frac{\mathrm{d}f}{\mathrm{d}x} = f(b) - f(a). \tag{4.1}$$

We want to generalise this to higher dimensions.

4.1 The Divergence Theorem

Also known as Gauss's Theorem. Given a smooth vector field $\vec{F}(\vec{x})$ over $\mathbb{R}^3$,

$$\int_V \nabla \cdot \vec{F} \, \mathrm{d}V = \int_S \vec{F} \cdot \mathrm{d}\vec{S}, \quad S = \partial V. \tag{4.2}$$

We should take $\mathrm{d}S$ to be pointing outwards from V.

The divergence theorem gives intuition for the meaning of divergence $\nabla \cdot \vec{F}$. Namely "the divergence is a infinitesimal version of Gauss's Theorem"[1]. Physically, it means divergence measures the net flow of $\vec{F}$ into or out of the region V.

Example Maxwell's equation shows

$$\nabla \cdot \vec{B} = 0, \tag{4.3}$$

which means magnetic field must be continuous. There is no magnets with only one pole. Similarly, the velocity of an incompressible fluid obeys $\nabla \cdot \vec{v} = 0$.

Example Another Maxwell's equation shows

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}. \tag{4.4}$$

The electric field lines are continuous when $\rho(\vec{x}) = 0$. But they can begin and end when there is some charge $\rho(\vec{x}) \neq 0$. Therefore we do have different signs of charge separately.

Proof First, the intuitive idea: Divide V into tiny cubes. Assume cube is small so $\vec{F}$ is roughly constant on each side. Flow of $\vec{F}$ through (y, z) plane is

$$\left[F_x(x + \delta x, z) - F_x(x, z)\right] \delta y \delta z = \frac{\partial F_x}{\partial x} \delta x \delta y \delta z. \tag{4.5}$$

Do the same for other sizes, then sum over all small cubes that make up of V. Things other than the boundary just cancel out.

$$\sum_{\text{cubes}} \int \vec{F} \cdot \mathrm{d}\vec{S} = \int_{\partial V} \vec{F} \cdot \mathrm{d}\vec{S} = \sum_{\text{cube}} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \delta x \delta y \delta z = \int_V \nabla \cdot \vec{F} \, \mathrm{d}V \tag{4.6}$$

[1] Cai Zixing

A concern [2] Can we really approximate the boundary by cubes?[3] A better proof is to say that the Gauss's theorem holds in $\mathbb{R}^n$.

Proof Note that

$$\int_V \nabla \cdot \vec{F} \,\mathrm{d}V = \int_S \vec{F} \cdot \mathrm{d}\vec{S}, \quad S = \partial V \tag{4.7}$$

holds in any dimension. First prove the 2D version. Assume $\vec{F} = F(x,y)\hat{y}$, the proof also holds for $\vec{F} = F(x,y)\hat{x}$ and a general $\vec{F}$ is just the sum of these. Then

$$\int_D \nabla \cdot \vec{F} \,\mathrm{d}A = \int_x \mathrm{d}x \int_{y_-(x)}^{y_+(x)} \mathrm{d}y \,\frac{\partial F}{\partial y}, \tag{4.8}$$

therefore

$$\int_D \nabla \cdot \vec{F} \,\mathrm{d}A = \int_X \mathrm{d}x \,\left(F(x, y_+(x)) - F(x, y_-(x))\right). \tag{4.9}$$

We have succeeded in converting the area integral into an ordinary integral, but it is not quite of the line integral form that we need. By considering the normal vector $\hat{n}$ moving a small distance δs along the curves C_+ and C_-, notice that

$$\begin{aligned} \delta x &= \cos\theta \delta s = \hat{y} \cdot \hat{n} \delta s \text{ along } C_+ \\ \delta x &= \cos\theta \delta s = -\hat{y} \cdot \hat{n} \delta s \text{ along } C_- \end{aligned} \tag{4.10}$$

Therefore the integral is

$$\int_D \nabla \cdot \vec{F} \,\mathrm{d}A = \int_{C_+} \vec{F} \cdot \hat{n} \,\mathrm{d}s + \int_{C_-} \vec{F} \cdot \hat{n} \,\mathrm{d}s = \int_C \vec{F} \cdot \hat{n} \,\mathrm{d}s, \tag{4.11}$$

where $C = C_+ + C_- = \partial D$ is the boundary of the region.

The proof of the 3D case is just as straightforward as this one.

4.1.1 The divergence theorem for scalar fields

Claim For $S = \partial V$, we have

$$\int_V \nabla\phi \,\mathrm{d}V = \int_S \phi \,\mathrm{d}\vec{S} \tag{4.12}$$

Proof Consider the divergence theorem with $\vec{F} = \phi\vec{a}$ with $\vec{a}$ is a constant vector.

4.2 Green's Theorem in the Plane

Let $P(x,y)$ and $Q(x,y)$ be smooth functions on $\mathbb{R}^2$. Then

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \oint_C P \,\mathrm{d}x + \oint_C Q \,\mathrm{d}y, \tag{4.13}$$

where A is a bounded region in the plane and $C = \partial A$ is a piecewise smooth, non-interesting closed curve which is traversed anti-clockwise.

[2] Here is the magic of Professor David Tong, he can always read my mind.

[3] Think about evaluating the length of a circle using the boundary of summed squares, the overall solution is misleading.

Proof Green's theorem is equivalent to the 2D divergence theorem. Construct a vector field $\vec{F} = (Q, -P)$, we have

$$\int_A \nabla \cdot \vec{F} \ \mathrm{d}A = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \ \mathrm{d}A. \tag{4.14}$$

Suppose $\hat{n}$ is the unit normal vector pointing outwards on $C = \partial A$,

$$\vec{F} \cdot \hat{n} = Q \frac{\mathrm{d}y}{\mathrm{d}s} + P \frac{\mathrm{d}x}{\mathrm{d}s}, \tag{4.15}$$

integrating around C,

$$\int_C \vec{F} \cdot \hat{n} \ \mathrm{d}s = \int_C P \ \mathrm{d}x + Q \ \mathrm{d}y. \tag{4.16}$$

We have

$$\int_A \frac{\partial Q}{\partial x} \ \mathrm{d}A = \int \mathrm{d}y \int \mathrm{d}x \ \frac{\partial Q}{\partial y} = \int_C Q \ \mathrm{d}x, \tag{4.17}$$

and a similar equation for P. Therefore we have proved the Green's theorem.

Note Green's theorem for an area with disconnected boundaries, if there is a hole inside, we use an infinitesimal gap to sneak in and out.

4.3 Stokes' Theorem

Stokes' theorem is an extension of Green's theorem, but the surface is no longer restricted to lie in a plane.

Let S be a smooth surface in $\mathbb{R}^3$ with $C = \partial S$ a piecewise smooth curve. For any smooth vector field $\vec{F}(\vec{x})$, we have

$$\int_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{S} = \int_C \vec{F} \cdot \mathrm{d}\vec{x} \tag{4.18}$$

Stokes' theorem implies the meaning of the curl of a vector field. If we integrate $\nabla \times \vec{F}$ over a small surface, we can find that the value of $\nabla \times \vec{F}$ in the direction $\hat{n}$ tells us about the circulation of $\vec{F}$ in the plane normal to $\hat{n}$.

There is another thing to consider before we prove the Stokes' theorem. Earlier we proved that

$$\vec{F} = \nabla \phi \ \Rightarrow \ \nabla \times \vec{F} = \vec{0}, \tag{4.19}$$

but didn't prove the converse. Now it is straightforward from Stokes' theorem because an irrotational vector field obeying $\nabla \times \vec{F} = \vec{0}$, necessarily has

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{x} = 0 \tag{4.20}$$

around any closed curve C. Therefore $\vec{F}$ can be written as $\vec{F} = \nabla \phi$.

Proof As we did before to the divergence theorem, we prove the 2D version first, and there follows trivial steps leading us to higher dimensions.

Consider the vector field $\vec{F} = (P, Q, 0)$ in $\mathbb{R}^3$ and a surface S that lies flat in the $z = 0$ plane. The normal to this surface $\hat{n} = \hat{k}$, and we have

$$\int_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{S} = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}S. \tag{4.21}$$

But Stokes' theorem implies

$$\int_C \vec{F} \cdot \mathrm{d}\vec{x} = \int_C P \ \mathrm{d}x + Q \ \mathrm{d}y. \tag{4.22}$$

The idea is to lift Green's theorem out of the plane to find Stokes' theorem.

Consider a parameterised surface S defined by $\vec{x}(u,v)$ and denote the associated area in the (u,v) plane as A. We parameterise the boundary $C = \partial S$ as $\vec{x}(u(t),v(t))$ and the corresponding boundary ∂A as $(u(t),v(t))$. The key idea is to use Green's theorem in the (u,v) plane for the area A and then uplift this to prove Stokes theorem for the surface S.

We start by looking at the integral around the boundary.

$$\int_C \vec{F}\cdot \mathrm{d}\vec{x} = \int_C \vec{F}\cdot\left(\frac{\partial \vec{x}}{\partial u}\mathrm{d}u + \frac{\partial \vec{x}}{\partial v}\mathrm{d}v\right) = \int_{\partial A} F_u\ \mathrm{d}u + F_v\ \mathrm{d}v. \tag{4.23}$$

Now we are in a position to invoke Green's theorem, in the form

$$\int_{\partial A} F_u\ \mathrm{d}u + F_v\ \mathrm{d}v = \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v}\right)\ \mathrm{d}A. \tag{4.24}$$

The only thing left is to calculate the RHS straight away.

$$\begin{aligned}
\frac{\partial F_v}{\partial u} &= \frac{\partial}{\partial u}\left(\vec{F}\cdot\frac{\partial \vec{x}}{\partial v}\right) = \frac{\partial}{\partial u}\left(F_i\frac{\partial x^i}{\partial v}\right) = \left(\frac{\partial F_i}{\partial x^j}\frac{\partial x^j}{\partial u}\right)\frac{\partial x^i}{\partial v} + F_i\frac{\partial^2 x^i}{\partial u\partial v}\\
\frac{\partial F_u}{\partial v} &= \frac{\partial}{\partial v}\left(\vec{F}\cdot\frac{\partial \vec{x}}{\partial u}\right) = \frac{\partial}{\partial v}\left(F_i\frac{\partial x^i}{\partial u}\right) = \left(\frac{\partial F_i}{\partial x^j}\frac{\partial x^j}{\partial v}\right)\frac{\partial x^i}{\partial u} + F_i\frac{\partial^2 x^i}{\partial v\partial u}
\end{aligned} \tag{4.25}$$

Therefore,

$$\begin{aligned}
\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} &= \frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v}\left(\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i}\right)\\
&= \left(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}\right)\frac{\partial x^k}{\partial u}\frac{\partial x^i}{\partial v}\frac{\partial F_i}{\partial x^j}\\
&= \epsilon_{jip}\epsilon_{pkl}\frac{\partial x^k}{\partial u}\frac{\partial x^i}{\partial v}\frac{\partial F_i}{\partial x^j}\\
&= \left(\nabla\times\vec{F}\right)\cdot\left(\frac{\partial \vec{x}}{\partial u}\times\frac{\partial \vec{x}}{\partial v}\right)
\end{aligned} \tag{4.26}$$

Following the chain of identities above, we have

$$\begin{aligned}
\int_C \vec{F}\cdot\mathrm{d}\vec{x} &= \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v}\right)\ \mathrm{d}u\mathrm{d}v\\
&= \int_A \left(\nabla\times\vec{F}\right))\cdot\left(\frac{\partial \vec{x}}{\partial u}\times\frac{\partial \vec{x}}{\partial v}\right)\ \mathrm{d}u\mathrm{d}v\\
&= \int_S \left(\nabla\times\vec{F}\right)\cdot\mathrm{d}\vec{S}
\end{aligned} \tag{4.27}$$

□

Stokes' theorem is widely used, of course, in electrodynamics to deal with magnetic fields. Naturally it is important in fluids too.

4.4 Coordinate Transformation Again

There are several theorems that we are not going to prove, but quite useful in real exercises.

Theorem The divergence of a vector field $\vec{F}(u,v,w)$ in a general orthogonal, curvilinear coordinate system is given by

$$\nabla\times\vec{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right). \tag{4.28}$$

Theorem The curl of a vector field $\vec{F}(u, v, w)$ in a general orthogonal, curvilinear coordinate system is given by

$$\begin{aligned} \nabla \times F &= \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_w \hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix} \\ &= \frac{1}{h_v h_w} \left(\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v) \right) \hat{e}_u + \cdots + \cdots \end{aligned} \tag{4.29}$$

4.5 The Poisson and Laplace Equation

These are important equations in vector calculus and almost all physical subjects[4]. In this section we will give a brief introduction about some methods to solve the Poisson equation.

The key to solve Poisson equation

$$\nabla^2 \psi(\vec{x}) = -\rho(\vec{x}), \tag{4.30}$$

for a given source $\rho(\vec{x})$ in a variety of cases is to consider the initial and boundary conditions. We shall consider some particular density functions.

4.5.1 Isotropic solutions

If we have some kind of symmetry, the equation can be modified to become simple and straightforward.

For example, we consider spherical symmetry then $\psi(\vec{x}) = \psi(r)$. Using the form of the Laplacian, Laplace equation becomes

$$\nabla^2 \psi = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 \psi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\psi}{\mathrm{d}r} \right) = 0. \tag{4.31}$$

The solution is simply

$$\psi(r) = \frac{A}{r} + B. \tag{4.32}$$

Note that, we cannot say that we have solved the equation in its full domain, because at $r = 0$ the solution diverges if $A \neq 0$[5].

Another example is the cylindrical symmetrical case

$$\nabla^2 \psi = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 \psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}\psi}{\mathrm{d}r} \right) = 0. \tag{4.33}$$

Of course the equation is

$$\psi(r) = A \log r + B, \tag{4.34}$$

here r is $\sqrt{x^2 + y^2}$ and z is another independent variable. Again this solutions diverges at $r = 0$ which correspond to the entire z axis.

Note Note that we are in $\mathbb{R}^3$ just now. If we goes to higher dimensions the exponential factor will be higher.

If $\psi(r)$ is a solution to the Laplace equation, then so too is any derivative of $\psi(r)$. This helps us to construct the multilevel expansion. For example, the dipole term simply has the form

$$\psi(\vec{x}) = \vec{\mathrm{d}} \cdot \nabla \left(\frac{1}{r} \right) = -\frac{\vec{\mathrm{d}} \cdot \vec{x}}{r^3}. \tag{4.35}$$

4.5.2 Boundary conditions and Uniqueness

For boundary conditions, we have completed too many examples in electrodynamics[6]. We will not carry this further.

[4] To learn more advanced topics about this, see Evans: Partial Differential Equations.

[5] In physics, this is where we have charges

[6] Such as Dirichlet and Neumann condition

Theorem Consider the Poisson equation on a bounded region V, with either Dirichlet or Neumann boundary conditions specified on each boundary ∂V. If a solution exists, then it is unique[7].

Proof Let $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$ both satisfy the Poisson equation with the specified boundary conditions. Then $\psi(\vec{x}) = \psi_1 - \psi_2$ must also obey $\nabla^2\psi = 0$ and either $\psi = 0$ or $\hat{n} \cdot \nabla\psi = 0$ on ∂V. Consider

$$\int_V \nabla \cdot (\psi \nabla \psi)\ \mathrm{d}V = \int_V \left(\nabla\psi \cdot \nabla\psi + \psi\nabla^2\psi\right)\ \mathrm{d}V = \int_V |\nabla\psi|^2\ \mathrm{d}V. \tag{4.36}$$

By the divergence theorem, we have

$$\int_V \nabla \cdot (\psi\nabla\psi)\ \mathrm{d}V = \int_{\partial V} \psi\nabla\psi \cdot \mathrm{d}\vec{S} = \int_{\partial V} \psi\left(\hat{n} \cdot \nabla\psi\right)\ \mathrm{d}S = 0, \tag{4.37}$$

where either conditions set the boundary term to zero. Because $|\nabla\psi|^2 \geq 0$, the integral can only vanish if $\nabla\psi = 0$ everywhere, therefore ψ must be constant. If Dirichlet boundary conditions are imposed anywhere, then that constant must be zero.

□

4.5.3 Green's identities

The proof of the uniqueness theorem used a trick known as Green's (first) identity, namely

$$\int_V \phi\nabla^2\psi\ \mathrm{d}V = -\int_V \nabla\phi \cdot \nabla\psi\ \mathrm{d}V + \int_S \phi\nabla\psi \cdot \mathrm{d}\vec{S}. \tag{4.38}$$

This is essentially a 3d version of integration by parts and it follows simply by applying the divergence theorem to $\phi\nabla\psi$.

By anti-symmetrisation we have the Green's second identity.

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\ \mathrm{d}V = \int_S \left(\phi\nabla\psi - \psi\nabla\phi\right) \cdot \mathrm{d}\vec{S}. \tag{4.39}$$

4.5.4 Harmonic functions

Solutions to the Laplace equation

$$\nabla^2\psi = 0 \tag{4.40}$$

is called harmonic functions. Here are two properties that we will not give proofs.

Theorem Suppose that ψ is harmonic in a region V that includes the solid sphere with boundary S_R : $|\vec{x} - \vec{a}| = R$. Then the value of ψ at $\vec{a}$, the centre of the sphere, is given by $\psi(\vec{a}) = \bar{\psi}(R)$ where

$$\bar{\psi}(R) = \frac{1}{4\pi R^2}\int_{S_R} \psi(\vec{x})\ \mathrm{d}S \tag{4.41}$$

is the average of ψ over S_R. This is known as the mean value property.

Theorem A harmonic function can have neither a maximum nor minimum in the interior of a region V. Any maximum of minimum must lie on the boundary ∂V.

4.5.5 Integral Solutions

There is a particularly nice way to write down an expression for the grneral solution to the Poisson equation in $\mathbb{R}^3$, with

$$\nabla^2\psi = -\rho(\vec{x}) \tag{4.42}$$

at least for a localised source $\rho(\vec{x})$ that drops off suitably fast, so $\rho(\vec{x}) \to 0$ as $r \to 0$.

We shall skip the simplest solution for a Dirac delta function source, $\psi = \lambda/4\pi r$, and continue to look at:

[7] This is highly important so we will give a full proof.

The integral solution of the Poisson equation

$$\psi(\vec{x}) = \frac{1}{4\pi} \int_{V'} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}V', \tag{4.43}$$

where the integral is over a region V' parameterised by $\vec{x}'$.

Proof First, a point particle at $\vec{x}'$ gives rise to a potential of the form $\psi = \rho(\vec{x}')/4\pi|\vec{x} - \vec{x}'|$, which is just our simplest solution. The integral solution is just the integral over a volume, this relies on the linear nature of Poisson equation. We shall write

$$\nabla^2 \psi = \frac{1}{4\pi} \int_{V'} \rho(\vec{x}') \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \, \mathrm{d}V', \tag{4.44}$$

then

$$\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi \delta^3(\vec{x} - \vec{x}'). \tag{4.45}$$

Therefore, naturally

$$\nabla^2 \psi = -\int_{V'} \rho(\vec{x}') \delta^3(\vec{x} - \vec{x}') \, \mathrm{d}V' = -\rho(\vec{x}), \tag{4.46}$$

which is just the Poisson equation.

The technique of first solving an equation with a delta function source and subsequently integrating to find the general solution in known as the **Green's function approach**. It is a powerful method to solve differential equation and we will meet it again in many further courses.

The rest of the course discussed fundamental knowledge on tensors, which we shall not repeat. Readers who are interested please refer to the Differential Geometry & General Relativity notes.