向量微積分 恆等式

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[定義] (微分算子). $\nabla := \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

[定義] (梯度). $\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

[定義] (散度). $\operatorname{div} f = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

[定義] (旋度). $\operatorname{curl} f = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

[定義] (拉普拉斯算符). $\Delta f = \nabla^2 f = (\nabla \cdot \nabla) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

定理 1. $\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$

Proof.

$$\begin{aligned}
\nabla(\psi\phi) &= \frac{\partial(\psi\phi)}{\partial x}\mathbf{i} + \frac{\partial(\psi\phi)}{\partial y}\mathbf{j} + \frac{\partial(\psi\phi)}{\partial z}\mathbf{k} \\
&= \left(\frac{\partial\psi}{\partial x}\phi + \frac{\partial\phi}{\partial x}\psi\right)\mathbf{i} + \left(\frac{\partial\psi}{\partial y}\phi + \frac{\partial\phi}{\partial y}\psi\right)\mathbf{j} + \left(\frac{\partial\psi}{\partial z}\phi + \frac{\partial\phi}{\partial z}\psi\right)\mathbf{k} \\
&= \phi\left(\frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k}\right) + \psi\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \\
&= \phi\nabla\psi + \psi\nabla\phi
\end{aligned}$$

□

定理 2. $\nabla \cdot (\psi\mathbf{A}) = \psi\nabla \cdot \mathbf{A} + (\nabla\psi) \cdot \mathbf{A}$

Proof.

$$\begin{aligned}
\nabla \cdot (\psi\mathbf{A}) &= \frac{\partial(\psi A_x)}{\partial x} + \frac{\partial(\psi A_y)}{\partial y} + \frac{\partial(\psi A_z)}{\partial z} \\
&= \frac{\partial\psi}{\partial x}A_x + \frac{\partial A_x}{\partial x}\psi + \frac{\partial\psi}{\partial y}A_y + \frac{\partial A_y}{\partial y}\psi + \frac{\partial\psi}{\partial z}A_z + \frac{\partial A_z}{\partial z}\psi \\
&= \psi\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \left(\frac{\partial\psi}{\partial x}A_x + \frac{\partial\psi}{\partial y}A_y + \frac{\partial\psi}{\partial z}A_z\right) \\
&= \psi\nabla \cdot \mathbf{A} + (\nabla\psi) \cdot \mathbf{A}
\end{aligned}$$

□

定理 3. $\nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} + (\nabla \psi) \times \mathbf{A}$

Proof.

$$\begin{aligned}
\nabla \times (\psi \mathbf{A}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \psi A_x & \psi A_y & \psi A_z \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \psi\frac{\partial}{\partial x} & \psi\frac{\partial}{\partial y} & \psi\frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \psi \nabla \times \mathbf{A} + (\nabla \psi) \times \mathbf{A}
\end{aligned}$$

□

定理 4. $\nabla^2(\psi\phi) = \psi\nabla^2\phi + 2\nabla\psi \cdot \nabla\phi + \phi\nabla^2\psi$

Proof.

$$\begin{aligned}
\nabla^2(\psi\phi) &= \frac{\partial^2(\psi\phi)}{\partial^2 x} + \frac{\partial^2(\psi\phi)}{\partial^2 y} + \frac{\partial^2(\psi\phi)}{\partial^2 z} \\
&= \frac{\partial}{\partial x}\left(\psi\frac{\partial\phi}{\partial x} + \phi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\psi\frac{\partial\phi}{\partial y} + \phi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\psi\frac{\partial\phi}{\partial z} + \phi\frac{\partial\psi}{\partial z}\right) \\
&= 2\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x^2} + \phi\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y} + \psi\frac{\partial^2\phi}{\partial y^2} + \phi\frac{\partial^2\psi}{\partial y^2} + 2\frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial z} + \psi\frac{\partial^2\phi}{\partial z^2} + \phi\frac{\partial^2\psi}{\partial z^2} \\
&= \phi\nabla^2\psi + 2\nabla\psi \cdot \nabla\phi + \psi\nabla^2\phi
\end{aligned}$$

□

定理 5. $\nabla \times (\nabla\psi) = 0$

Proof.

$$\begin{aligned}
\nabla \times (\nabla\psi) &= \nabla \times \left(\frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k}\right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial^2\psi}{\partial z\partial y}\right)\mathbf{i} + \left(\frac{\partial^2\psi}{\partial x\partial z} - \frac{\partial^2\psi}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial^2\psi}{\partial x\partial y}\right)\mathbf{i} \\
&= 0
\end{aligned}$$

□

定理 6. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

Proof.

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = 0$$

□

引理. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Proof. 因爲 $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$，因此 $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \perp \mathbf{B} \times \mathbf{C} \perp \mathbf{B}, \mathbf{C}$，我們可將其表爲：

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = m\mathbf{B} + n\mathbf{C}$$

又 $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \perp \mathbf{A}$：

$$\begin{aligned} &\Rightarrow (\mathbf{A} \times (\mathbf{B} \times \mathbf{C})) \cdot \mathbf{A} = 0 \\ &\Rightarrow (m\mathbf{B} + n\mathbf{C}) \cdot \mathbf{A} = 0 \\ &\Rightarrow m(\mathbf{B} \cdot \mathbf{A}) + n(\mathbf{C} \cdot \mathbf{A}) = 0 \end{aligned}$$

考慮 $\begin{cases} m = k(\mathbf{A} \cdot \mathbf{C}) \\ n = -k(\mathbf{A} \cdot \mathbf{B}) \end{cases}$，並令 $\mathbf{A} = \mathbf{j}$, $\mathbf{B} = \mathbf{i}$, $\mathbf{C} = \mathbf{j}$

$$\begin{aligned} &\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ &k\left[\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})\right] = k\left[\mathbf{i} - \mathbf{0}\right] = k\mathbf{i} \\ &\Rightarrow k = 1 \end{aligned}$$

$$\therefore \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

□

定理 7. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

Proof.

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned}$$

□

定理 8. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

Proof.

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

□

定理 9. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

定理 10. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

定理 11 (散度定理)**.** $\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F})\, dV$

Proof. 以 $S(V)$ 表示 V 的表面，等價於 ∂V

向量場的散度是單位體積中的平均通量，可以寫成：

$$\nabla \cdot \mathbf{F} := \lim_{V \to 0} \frac{1}{|V|} \oiint_{S(V)} \mathbf{F} \cdot d\mathbf{S}$$

由面積分

$$\begin{aligned}
\oiint_{S(V)} \mathbf{F} \cdot d\mathbf{S} &= \lim_{\delta V \to 0} \sum_{\delta V \subset V} \iint_{S(\delta V)} \mathbf{F} \cdot d\mathbf{S} \\
&= \lim_{\delta V \to 0} \sum_{\delta V \subset V} \left(\frac{1}{|\delta V|} \iint_{S(\delta V)} \mathbf{F} \cdot d\mathbf{S} \right) |\delta V| \\
&= \iiint_V \left(\lim_{\delta V \to 0} \frac{1}{|\delta V|} \iint_{S(\delta V)} \mathbf{F} \cdot d\mathbf{S} \right) dV \\
&= \iiint_V (\nabla \cdot \mathbf{F})\, dV
\end{aligned}$$

□

定理 12 (斯托克斯定理)**.** $\oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{\ell} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$

Proof. 以 $C(S)$ 表示 S 的邊界，等價於 ∂S

向量場的旋度平行表面法向量的分量是單位面積中的平均環量，可以寫成：

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} := \lim_{S \to 0} \frac{1}{|S|} \oint_{C(S)} \mathbf{F} \cdot d\boldsymbol{\ell}$$

由環積分

$$\begin{aligned}
\oint_{C(S)} \mathbf{F} \cdot d\boldsymbol{\ell} &= \lim_{\delta S \to 0} \sum_{\delta S \subset S} \int_{C(\delta S)} \mathbf{F} \cdot d\boldsymbol{\ell} \\
&= \lim_{\delta S \to 0} \sum_{\delta S \subset S} \left(\frac{1}{|\delta S|} \int_{C(\delta S)} \mathbf{F} \cdot d\boldsymbol{\ell} \right) |\delta S| \\
&= \iint_S \left(\lim_{\delta S \to 0} \frac{1}{|\delta S|} \int_{C(\delta S)} \mathbf{F} \cdot d\boldsymbol{\ell} \right) dS \\
&= \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \\
&= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
\end{aligned}$$

□

定理 13 (格林第一恆等式)**.** $\oiint_{\partial V} (\psi \nabla \phi) \cdot d\mathbf{S} = \iiint_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi)\, dV$

Proof. 定義 $\mathbf{A} := \psi\nabla\phi$

$$\nabla\cdot\mathbf{A} = \nabla\cdot(\psi\nabla\phi) = \psi\nabla\cdot(\nabla\phi) + (\nabla\psi)\cdot(\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi$$

代入 **定理 10**：

$$\oiint_{\partial V}(\psi\nabla\phi)\cdot d\mathbf{S} = \iiint_V (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\,dV$$

□

定理 14 (格林第二恆等式)**.** $\oiint_{\partial V}(\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S} = \iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dV$

Proof.

$$\nabla\cdot(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi$$
$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

將上兩式相減：

$$\nabla\cdot(\psi\nabla\phi - \phi\nabla\psi) = \psi\nabla^2\phi - \phi\nabla^2\psi$$

代入 **定理 10**：

$$\oiint_{\partial V}(\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S} = \iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dV$$

□