Vector Analysis - Differential Identities

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1 Useful Symbols

Before we started to talk about those identities, here are some useful symbols that will be used at the following section.

1.1 Kronecker Delta

Now, input two integers into the function. If the two are the same, you get 1. Otherwise, i.e., the two are NOT the same, you get 0.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1}$$

1.2 Levi-Civita Symbol

Levi-Civita Symbol is defined as the parity of permutation of $1, 2, \cdots, n$. In 3D space, we can define Levi-Civita Symbol as:

$$\epsilon_{ijk} = \begin{cases} +1 & ijk = 123,\ 231,\ 312 \\ -1 & ijk = 321,\ 213,\ 132 \\ 0 & i = j \text{ or } j = k \text{ or } k = i \end{cases} \tag{2}$$

You can memorize by this picture:

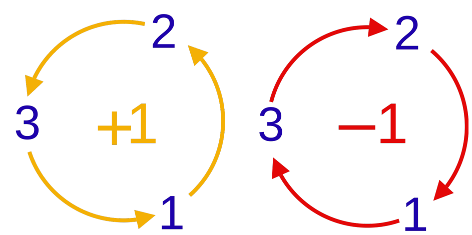

Figure 1: The memorization method of Levi-Civita Symbol

By the circularity of Levi-Civita Symbol, here are some equation:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \tag{3}$$

$$\epsilon_{ijk} = -\epsilon_{kji} \tag{4}$$

1.3 Transformation of LVC & KD

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{5}$$

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km} \tag{6}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{7}$$

1.4 Rewrite the definition of vector operations

First, we simplify the partial differential:

$$\partial_i \equiv \frac{\partial}{\partial x_i} \tag{8}$$

$$\partial_i^2 \equiv \frac{\partial^2}{\partial x_i^2} \tag{9}$$

Now we can rewrite them:

$$\begin{aligned}
\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} &= a_i b_i \\
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \epsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k \\
\boldsymbol{\nabla} f &= \hat{\mathbf{e}}_i (\partial_i f) \\
\boldsymbol{\nabla} \cdot \vec{\mathbf{A}} &= \partial_i A_i \\
\boldsymbol{\nabla} \times \vec{\mathbf{A}} &= \epsilon_{ijk} \partial_i A_j \hat{\mathbf{e}}_k \\
\nabla^2 f &= \partial_i^2 f \\
\nabla^2 \vec{\mathbf{A}} &= \partial_i^2 A_j \\
\vec{\mathbf{A}} \cdot \boldsymbol{\nabla} &= A_i \partial_i
\end{aligned}$$

2 Identities

Theorem 1 (BAC-CAB Law)**.** $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$

Proof.

$$\begin{aligned}
\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) &= \epsilon_{ijk} a_i (\epsilon_{mnl} b_m c_n \hat{\mathbf{e}}_l)_j \hat{\mathbf{e}}_k \\
&= \epsilon_{ijk} \epsilon_{mnj} a_i b_m c_n \hat{\mathbf{e}}_k \\
&= \epsilon_{jki} \epsilon_{jmn} a_i b_m c_n \hat{\mathbf{e}}_k \\
&= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) a_i b_m c_n \hat{\mathbf{e}}_k \\
&= (\delta_{in} a_i c_n)(\delta_{km} b_m \hat{\mathbf{e}}_k) - (\delta_{im} a_i b_m)(\delta_{kn} c_n \hat{\mathbf{e}}_k) \\
&= \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})
\end{aligned}$$

$\square$

Theorem 2. *For any scalar function ϕ:*

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0 \tag{10}$$

Proof.

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) &= \epsilon_{ijk}\partial_i(\partial_n\phi\,\hat{\mathbf{e}}_n)_j\hat{\mathbf{e}}_k \\
&= \epsilon_{ijk}\partial_i\partial_j\phi\,\hat{\mathbf{e}}_k \\
&= \partial_1\partial_2\phi\,\hat{\mathbf{e}}_3 - \partial_2\partial_1\phi\,\hat{\mathbf{e}}_3 + \cdots \\
&= 0
\end{aligned}$$

□

Theorem 3. *For any vector field $\vec{\mathbf{A}}$:*

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) = 0 \tag{11}$$

Proof.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) &= \partial_i(\epsilon_{jkl}\partial_j A_k\hat{\mathbf{e}}_l)_i \\
&= \epsilon_{jki}\partial_i\partial_j A_k \\
&= \partial_1\partial_2 A_3 - \partial_2\partial_1 A_3 + \cdots \\
&= 0
\end{aligned}$$

□

Theorem 4. *For any vector field $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$:*

$$\boldsymbol{\nabla} \times \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \vec{\mathbf{A}}\left(\boldsymbol{\nabla} \cdot \vec{\mathbf{B}}\right) - \vec{\mathbf{B}}\left(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}\right) + \left(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla}\right)\vec{\mathbf{A}} - \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right)\vec{\mathbf{B}} \tag{12}$$

Proof.

$$\begin{aligned}
\boldsymbol{\nabla} \times \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) &= \epsilon_{ijk}\partial_i(\epsilon_{lmn}A_l B_m\hat{\mathbf{e}}_n)_j\hat{\mathbf{e}}_k \\
&= \epsilon_{ijk}\partial_i\,\epsilon_{lmj}A_l B_m\hat{\mathbf{e}}_k \\
&= \epsilon_{jki}\epsilon_{jlm}\,\partial_i(A_l B_m)\hat{\mathbf{e}}_k \\
&= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(\partial_i A_l \cdot B_m + A_l \cdot \partial_i B_m)\hat{\mathbf{e}}_k \\
&= \delta_{kl}\delta_{im}(\partial_i A_l)B_m\hat{\mathbf{e}}_k + \delta_{kl}\delta_{im}A_l(\partial_i B_m)\hat{\mathbf{e}}_k \\
&\quad - \delta_{km}\delta_{il}(\partial_i A_l)B_m\hat{\mathbf{e}}_k - \delta_{km}\delta_{il}A_l(\partial_i B_m)\hat{\mathbf{e}}_k \\
&= (\delta_{im}B_m\partial_i)(\delta_{kl}A_l\hat{\mathbf{e}}_k) + (\delta_{im}\partial_i B_m)(\delta_{kl}A_l\hat{\mathbf{e}}_k) \\
&\quad - (\delta_{il}\partial_i A_l)(\delta_{km}B_m\hat{\mathbf{e}}_k) - (\delta_{il}A_l\partial_i)(\delta_{km}B_m\hat{\mathbf{e}}_k) \\
&= \left(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla}\right)\vec{\mathbf{A}} + \left(\boldsymbol{\nabla} \cdot \vec{\mathbf{B}}\right)\vec{\mathbf{A}} - \left(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}\right)\vec{\mathbf{B}} - \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right)\vec{\mathbf{B}} \\
&= \vec{\mathbf{A}}\left(\boldsymbol{\nabla} \cdot \vec{\mathbf{B}}\right) - \vec{\mathbf{B}}\left(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}\right) + \left(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla}\right)\vec{\mathbf{A}} - \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right)\vec{\mathbf{B}}
\end{aligned}$$

□

Theorem 5. *For any vector field* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$*:*

$$\boldsymbol{\nabla} \cdot \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) \cdot \vec{\mathbf{B}} - \left(\boldsymbol{\nabla} \times \vec{\mathbf{B}}\right) \cdot \vec{\mathbf{A}} \tag{13}$$

Proof.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) &= \partial_n(\epsilon_{ijk} A_i B_j \hat{\mathbf{e}}_k)_n \\
&= \epsilon_{ijk} \partial_k (A_i B_j) \\
&= \epsilon_{kij} (\partial_k A_i) B_j + (-\epsilon_{kji})(\partial_k B_j) A_i \\
&= (\epsilon_{kij} \partial_k A_i \hat{\mathbf{e}}_j)(B_j \hat{\mathbf{e}}_j) - (\epsilon_{kji} \partial_k B_j \hat{\mathbf{e}}_i)(A_i \hat{\mathbf{e}}_i) \\
&= \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) \cdot \vec{\mathbf{B}} - \left(\boldsymbol{\nabla} \times \vec{\mathbf{B}}\right) \cdot \vec{\mathbf{A}}
\end{aligned}$$

□

Theorem 6. *For any vector field* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$*:*

$$\boldsymbol{\nabla}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) = \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right) \vec{\mathbf{B}} + \left(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla}\right) \vec{\mathbf{A}} + \vec{\mathbf{A}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{B}}\right) + \vec{\mathbf{B}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) \tag{14}$$

Proof. Consider $\vec{\mathbf{A}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{B}}\right)$:

$$\begin{aligned}
\vec{\mathbf{A}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{B}}\right) &= \epsilon_{ijk} A_i (\epsilon_{mnl} \partial_m B_n \hat{\mathbf{e}}_l)_j \hat{\mathbf{e}}_k \\
&= \epsilon_{ijk} \epsilon_{mnj} A_i (\partial_m B_n) \hat{\mathbf{e}}_k \\
&= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) \big[A_i (\partial_m B_n) \hat{\mathbf{e}}_k\big] \\
&= \delta_{km}\delta_{in} A_i (\partial_m B_n) \hat{\mathbf{e}}_k - \delta_{kn}\delta_{im} A_i (\partial_m B_n) \hat{\mathbf{e}}_k \\
&= \delta_{km} \left(\delta_{in} A_i \left(\partial_m \vec{\mathbf{B}}\right)_n\right) \hat{\mathbf{e}}_k - \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right) \delta_{kn} B_n \hat{\mathbf{e}}_k \\
&= \left(\partial_m \vec{\mathbf{B}}\right) \cdot \vec{\mathbf{A}}\, \hat{\mathbf{e}}_m - \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right) \vec{\mathbf{B}}
\end{aligned}$$

The same,

$$\vec{\mathbf{B}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) = \left(\partial_m \vec{\mathbf{A}}\right) \cdot \vec{\mathbf{B}}\, \hat{\mathbf{e}}_m - \left(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla}\right) \vec{\mathbf{A}}$$

Then we consider this:

$$\begin{aligned}
\left(\partial_m \vec{\mathbf{B}}\right) \cdot \vec{\mathbf{A}}\, \hat{\mathbf{e}}_m + \left(\partial_m \vec{\mathbf{A}}\right) \cdot \vec{\mathbf{B}}\, \hat{\mathbf{e}}_m &= \left(\partial_m \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \partial_m \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}\right) \hat{\mathbf{e}}_m \\
&= \partial_m \left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) \hat{\mathbf{e}}_m \\
&= \boldsymbol{\nabla}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right)
\end{aligned}$$

Therefore,

$$\boldsymbol{\nabla}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) = \left(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla}\right) \vec{\mathbf{B}} + \left(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla}\right) \vec{\mathbf{A}} + \vec{\mathbf{A}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{B}}\right) + \vec{\mathbf{B}} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right)$$

□