- A couple of these terms can be combined,

$$
\begin{aligned}
\langle \delta\chi_i\chi_j||\chi_i\chi_j\rangle &= \int d\mathrm{x}_1 d\mathrm{x}_2 \left[\delta\chi_i(1)\chi_j(2)\right]^* \frac{1}{r_{12}} \left[\chi_i(1)\chi_j(2) - \chi_j(1)\chi_i(2)\right] \\
&= \int d\mathrm{x}_2 d\mathrm{x}_1 \left[\delta\chi_i(2)\chi_j(1)\right]^* \frac{1}{r_{12}} \left[\chi_i(2)\chi_j(1) - \chi_j(2)\chi_i(1)\right] \qquad \text{flip } 1,2 \\
&= \int d\mathrm{x}_1 d\mathrm{x}_2 \left[\chi_j(1)\delta\chi_i(2)\right]^* \frac{1}{r_{12}} \left[\chi_j(1)\chi_i(2) - \chi_i(1)\chi_j(2)\right] \quad \text{reorder terms} \\
&= \langle \chi_j\delta\chi_i||\chi_j\chi_i\rangle
\end{aligned}
$$

- Now, combine those terms,

$$
\begin{aligned}
\delta\mathcal{L}[\chi_k] = &\sum_i \Big(\langle\delta\chi_i|\hat{h}|\chi_i\rangle + \langle\chi_i|\hat{h}|\delta\chi_i\rangle \Big) \\
&+ \sum_{ij} \left(\langle\delta\chi_i\chi_j||\chi_i\chi_j\rangle + \langle\chi_i\chi_j||\delta\chi_i\chi_j\rangle \right) - \sum_{ij} \epsilon_{ij} \left(\langle\delta\chi_i|\chi_j\rangle + \langle\chi_i|\delta\chi_j\rangle \right) = 0
\end{aligned}
$$

- and seperate out the complex conjugate,

$$
\begin{aligned}
\delta\mathcal{L}[\chi_k] &= \sum_i \langle\delta\chi_i|\hat{h}|\chi_i\rangle + \sum_{ij} \langle\delta\chi_i\chi_j||\chi_i\chi_j\rangle - \sum_{ij} \epsilon_{ij} \langle\delta\chi_i|\chi_j\rangle + \text{C.C.} = 0 \\
&= \sum_i \int d\mathrm{x}_1 \delta\chi_i^*(1)\hat{h}\chi_i(1) + \sum_{ij} \int\int d\mathrm{x}_1 d\mathrm{x}_2 \delta\chi_i^*(1)\chi_j^*(2)\frac{1}{r_{12}} \left(\chi_i(1)\chi_j(2) - \chi_j(1)\chi_i(2)\right) \\
&\quad - \sum_{ij} \epsilon_{ij} \int d\mathrm{x}_1 \delta\chi_i^*(1)\chi_j(1) + \text{C.C.} = 0
\end{aligned}
$$

- Now pull out the integration over electron 1,

$$
\begin{aligned}
= &\sum_i \int d\mathrm{x}_1 \delta\chi_i^*(1) \\
&\times \left[\hat{h}\chi_i(1) + \sum_j \int d\mathrm{x}_2 \chi_j(2)\frac{1}{r_{12}}\chi_j(2)\chi_i(1) - \sum_j \int d\mathrm{x}_2 \chi_j(2)\frac{1}{r_{12}}\chi_i(2)\chi_j(1) - \sum_j \chi_j(1)\epsilon_{ij}\right] \\
&+ \text{C.C.} = 0
\end{aligned}
$$

- The non-trivial solution requires the following expression to hold:

$$
\begin{aligned}
&\hat{h}\chi_i(1) + \sum_j \int d\mathrm{x}_2 \chi_j(2)\frac{1}{r_{12}}\chi_j(2)\chi_i(1) - \sum_j \int d\mathrm{x}_2 \chi_j(2)\frac{1}{r_{12}}\chi_i(2)\chi_j(1) = \sum_j \chi_j(1)\epsilon_{ij} \\
&= \left[\hat{h} + \sum_j \int d\mathrm{x}_2 \chi_j(2)\frac{1}{r_{12}}\chi_j(2) - \sum_j \int d\mathrm{x}_2 \chi_j(2)\frac{1}{r_{12}}\hat{P}(1,2)\chi_j(2)\right] \chi_i(1) = \sum_j \chi_j(1)\epsilon_{ij}
\end{aligned}
$$

- Fock operator