HW5: Digital Integrated Circuit

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1. Introduction to Elmore delay

The Elmore delay has been suggested to estimate time delay of resistor-capacitor (RC) tree, which are generally used to model digital logic circuits. Elmore delay approximates the 50% delay with sufficiently simple calculation. Although higher precision calculation by simulation tool, such as SPICE, could estimate the time delay with higher accuracy, Elmore delay is still useful since it provides the absolute upper bound on delay of RC tree by the simple calculation process.

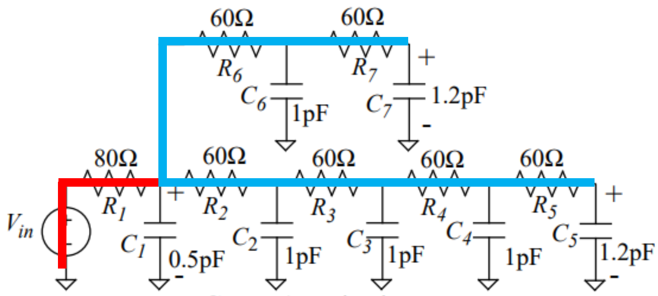

FIGURE 1: A simple RC tree.

Figure 1 shows an example of RC trees. In order to calculate Elmore delay at node *i*, we must find a common path between the input and node *i*, and between the input and node *k*. For example, a red line in figure 1 indicates the common path between node 5 and 7. Then, for every node, its capacitance times resistance of its corresponding common path that we found can be obtained. As a result, Elmore delay is nothing but the sum of multiplication between the resistance and capacitance for every node. In short,

$$T_D = \sum_{k=1}^{N} R_{ki} C_k$$

where R_{ki} is the resistance of the portion of the path between the input and node *i*, that is common with the unique path between the input and node *k*, and C_k is the capacitance at node *k*.

2. Elmore delay: impulse response

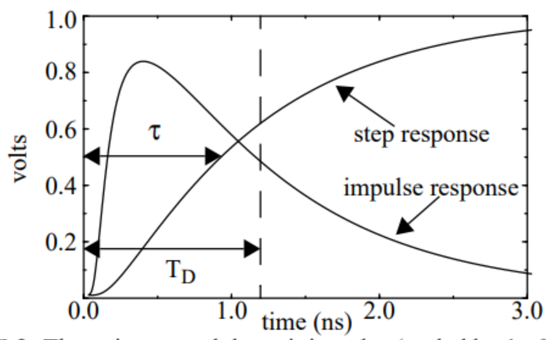

FIGURE 2: The unit step and the unit impulse (scaled by 1e-09) response for the voltage across C_5 in Fig.1.

In order to prove important characteristic of the Elmore delay, there need some definitions to induce them. The first definition, 50%-point delay of the monotonic step response is the time τ, which satisfies following condition,

$$\int_{-\infty}^{\tau} h(t)dt = \int_{\tau}^{\infty} h(t)dt = \frac{1}{2}$$

where $h(t)$ is the unit impulse response function. In this definition, Elmore suggests approximate τ by mean value of the unit impulse response $h(t)$, which can be represented as

$$T_D = \int_0^{\infty} th(t)dt$$

Interestingly, defined Elmore delay is the first coefficient in series expansion of the transfer function for a response node of the RC tree. Finally, the mode of a distribution function, M, is defined as the unimodal solution of

$$f'(x) = 0, f''(x) < 0$$

where the distribution function is continuous and differentiable. With these definitions, we will prove that Elmore delay T_D is always longer than 50%-point delay τ, and, for summary, two lemmas are just given, which are already proven in distributed paper. *Lemma* 1 is that the impulse response at any node of an RC tree is a unimodal and positive function. *Lemma* 2 is that, for the impulse response at any node of an RC tree, the coefficient of skewness is always nonnegative. Then, inequality of three defined quantities for unimodal skewed distribution function is

$$T_D \leq \tau \leq M \text{ or } M \leq \tau \leq T_D$$

However, two *Lemmas* adverted in this paragraph argues that, for which positive skewness, each node in RC tree has a unimodal distribution function. In this condition, assume $T_D \leq \tau \leq M$. Then, since skew is proportional to T_D subtracted by τ, for any node in the RC tree, $T_D \leq \tau \leq M$ induces the negative skewness. However, this is contradiction to *Lemma* 2 that states nonnegative skewness at any node of an RC tree, and $M \leq \tau \leq T_D$ holds. In other words, Elmore delay is upper bound of the 50%-point delay.

The induced *corollary* from this proof argues that a lower bound on the 50% delay for an RC tree is given by subtracting the square root of second momentum from the Elmore delay T_D. In short, the Elmore delay confines not only an upper bound but also lower bound on the 50% delay for an RC tree as

$$\max\left(T_D - \sqrt{\mu_2}, 0\right) \leq \tau$$

3. Elmore delay: general input signals

Although Elmore delay can estimate upper bound of the 50% step response delay, the signal coming out of the digital gate is never a step voltage. Thus, it needs to consider about Elmore delay on general input signals. In fact, Elmore delay on general input signals provides still an upper bound of the signal, when the signal is monotonically increasing, piecewise-smooth, and its time derivative is unimodal function. A proof using Laplace transform and *lemma* 2 justifies this *corollary*.

Moreover, if a monotonically increasing piecewise-smooth input signal satisfies that its time derivative is a symmetric function, as the rise time of the input signal approaching to infinity, the 50% delay of the output signal should approach to Elmore delay. In other words, the skewness, which is proportional to $T_D - \tau$, is approaching to zero. As a result, $\tau \rightarrow T_D$, that is, the area between the input and the output response equals the Elmore delay. The proposition can be proven by Laplace transform, also with symmetric condition of the time derivative of the input.