MTH 100 – 2018 MONSOON.

Notes for solution of linear systems.

EXAMPLES – HOMOGENEOUS SYSTEMS

1.
$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

Coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{R_2 \to} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[R_3 - 5R_2]{R_3 \to}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 + 3R_3]{R_1 \to} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow R$

In R, there are two pivot positions, corresponding to the basic variables x_1 and x_3; x_2 is a free variable. The system $R\bar{x} = \bar{0}$ becomes:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

We now express basic variables in terms of free variables and introduce a dummy equation of the type $x_k = x_k$ for each free variable,

giving the system:
$$\begin{aligned} x_1 &= -2x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \cdot x_2 \end{aligned}$$

(PTO)

Example 1 (cont'd). 2

In vector form, this becomes $\bar{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Since x_3 acts as a parameter, we get <u>infinitely many solutions</u>. The solution set can be concisely described as

$$S = \left\{ t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} ; t \in \mathbb{R} \right\} = \left\{ t\bar{u} : \bar{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Check: $A\bar{u} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -2 \\ 3 & 6 & -4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example 2: $A\bar{x} = \bar{0}$ where $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

We ~~see~~ see that A is already an RREF matrix: x_1 and x_3 are basic variables, x_2 and x_4 are free variables.

This gives the system
$$\begin{aligned} x_1 + 2x_2 \quad + 3x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

Expressing basic variables in terms of free variables and a introducing dummy variables, we get:

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= \quad\quad - x_4 \\ x_4 &= \quad\quad x_4, \end{aligned}$$

which in vector terms becomes:

$$\bar{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\uparrow \bar{u} \qquad\qquad \uparrow \bar{w}$$

So the solution set $S = \{ t\bar{u} + x\bar{w} : t, x \in \mathbb{R} \}$.

Check: $A\bar{u} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A\bar{w} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

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3. An example to illustrate Proposition 3.

$A\bar{x} = \bar{0}$ where $A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1}$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 5 & 8 \end{bmatrix} \xrightarrow[R_3 - 5R_2]{R_3 \to} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3} R_3} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 + R_2]{R_1 \to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\uparrow R$$

Corresponds to the system:

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$

Unique solution $\bar{x} = \bar{0}$, ~~the~~ necessarily the trivial solution.

Examples for Non-Homogeneous Systems

4. Consider $A\bar{x} = \bar{b}$, where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 7 \\ 9 \\ 30 \end{bmatrix}$

Work with the augmented matrix $[A \vdots b] =$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 2 & -1 & 3 & 9 \\ 4 & 1 & 8 & 30 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & -1 & -1 & -5 \\ 0 & 1 & 0 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow[R_3 \to (-1)R_3]{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

RREF matrix

Corresponds to the system: $x_1 = 1$
$x_2 = 2$
$x_3 = 3$

or $\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, unique solution. No free variables.

Check: $A\bar{x} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 30 \end{bmatrix} = \bar{b}$,

as expected.

5. $A\bar{x} = \bar{b}$ where $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 3 \\ 11 \\ 28 \end{bmatrix}$

Work with the augmented matrix $[A:\bar{b}]$ =

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 3 & 8 & 16 & 11 \\ 8 & 20 & 40 & 28 \end{array}\right] \xrightarrow[R_3 \to R_3 - 8R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 4 & 8 & 4 \end{array}\right]$$

$$\xrightarrow[\text{omitted}]{\text{some steps}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF matrix

basic basic

Corresponds to the system:

$$\begin{aligned} x_1 &= 1 + 0x_3 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \end{aligned}$$

In vector terms, this becomes:

$$\bar{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}_{\bar{w}}$$

Check: $A\bar{u} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 28 \end{bmatrix} = \bar{b}$

However, $A\bar{w} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So, we see that the solution set $S = \{\bar{u} + t\bar{w} : t \in \mathbb{R}\}$, where $\bar{u}$ is a solution of the non-homogeneous system, but $\bar{w}$ is a solution of the associated homogeneous system $A\bar{x} = \bar{0}$. Since x_3 (or t) acts as a parameter, we get infinitely many solutions.

6. Consider $A\bar{x} = \bar{b}$ where $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 4 \\ 11 \\ 28 \end{bmatrix}$

Working with the augmented matrix $[A : \bar{b}]$, we get the following RREF matrix

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Here the last row is of the form $[0 \cdots 0 ; b]$, where $b \neq 0$. Thus, by Proposition 4, the system is inconsistent, NO SOLUTION. The RREF matrix corresponds to the system:

$$x_1 = 0$$
$$x_2 + 2x_3 = 0$$
$$0 = 1 \longrightarrow \text{not true.}$$

That is why the system becomes inconsistent if we get a row like this.