NOTES FOR 20180814_TUE [redacted]

(1)

Proof of Observation 6: Suppose the non-homogeneous system $A\bar{x} = \bar{b}$ has at least one solution, say $\bar{u} \neq \bar{0}$.

A vector $\bar{y}$ is a solution of the system if and only if $\bar{y} = \bar{u} + \bar{v}$, where $\bar{v}$ is a solution of the associated homogeneous system $A\bar{x} = \bar{0}$.

[$\Longrightarrow$] Suppose $\bar{y}$ is a ~~so~~ solution of the system. Put $\bar{v} = \bar{y} - \bar{u}$.

Then $A\bar{v} = A(\bar{y} - \bar{u}) = A\bar{y} - A\bar{u}$
$= \bar{b} - \bar{b} = \bar{0}$.

$\therefore$ $\bar{v}$ is a solution of the homogeneous system, and $\bar{y} = \bar{u} + \bar{v}$.

[$\Longleftarrow$] Conversely, suppose $\bar{v}$ is any solution of ~~the~~ homogeneous system.

Then $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v}$
$= \bar{b} + \bar{0} = \bar{b}$.

$\therefore$ $\bar{u} + \bar{v}$ is a solution of the non-homogeneous system.

Summary for Non-Homogeneous System: $A\bar{x} = \bar{b}$

Associated Homogeneous System $A\bar{x} = \bar{0}$		Non-Homogeneous System
Case 1: Unique Solution (trivial) $\Updownarrow$ NO free variable	$\longleftrightarrow$	Inconsistent OR Unique Solution
Case 2: Infinitely many solutions $\Updownarrow$ At least one free variable	$\longleftrightarrow$	Inconsistent OR Infinitely Many solutions

Proof of VIT:

We will proceed as follows:

(a) $\Rightarrow$ (c) $\Rightarrow$ (h) $\Rightarrow$ (a)

[(a) $\Leftrightarrow$ (d), i.e. (a) $\Rightarrow$ (d) $\Rightarrow$ (a) will be done later.]

(a) $\Rightarrow$ (c) Given: A is invertible.

To Prove: $A\bar{x} = \bar{0}$ has only the trivial solution.

Suppose $\bar{y}$ is any solution of $A\bar{x} = \bar{0}$.

$\therefore A\bar{y} = \bar{0}$

Multiply on left by A^{-1}.

$\therefore A^{-1}(A\bar{y}) = A^{-1}\bar{0} = 0$

$LHS = (A^{-1}A)\bar{y} = I\bar{y} = \bar{y}$.

$\therefore \bar{y} = \bar{0}$, as required.

(c) $\Rightarrow$ (h) Given: the homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution.

To prove: A is row-equivalent to I.

Now, if R is the RREF matrix of A, then $R\bar{x} = \bar{0}$ has only the trivial solution

$\Rightarrow$ R has no free variables

$\Rightarrow$ R has only basic variables

$\Rightarrow$ R has a leading 1 in each row (there are m rows)

$\Rightarrow$ R has exactly one 1 in each column (since no. of columns $= m$)

$\Rightarrow$ R is I_m

(h) $\Rightarrow$ (a)

Given: A ~~B~~ is row-equivalent to I.

To prove: A is invertible.

Now, A is row equivalent to I

$\Rightarrow$ There are elementary row operations $e_p, e_{p-1}, \ldots, e_2, e_1$ s.t.

$$e_p(e_{p-1}(\cdots(e_2(e_1)A)\ldots) = I \quad (1)$$

If E_i is the elementary matrix corresponding to e_i, we can write (1) as :-

$$E_p(E_{p-1}(\cdots\cdots(E_2(E_1A))\cdots) = I$$

(using Prop. 5)

$$\therefore (E_p \cdots E_1)A = I \quad (2)$$

Putting $B = E_p \cdots E_1$, we get from Prop. 6 and Observation 4 for Invertible Matrices, that B is invertible.

From (2), $BA = I$.

Multiplying by B^{-1} on the left,

$$B^{-1}(BA) = B^{-1}$$

$$\Rightarrow A = B^{-1}$$

Hence, A, being the inverse of an invertible matrix, is itself invertible (Observation 2).

An example for finding the inverse of a matrix by row-reduction:-

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}.$$

We work with the <u>enlarged</u> <u>matrix</u>

$[A : I]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + R_2]{R_3 \to} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to (-1)R_2]{R_3 \to (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$I \quad : \quad A^{-1}$$

(PTO)

Check:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remark: This method is preferable to the Adjoint / Determinant formula, which requires approx. $n!$ calculations. Gauss-Jordan elimination requires approx. $\frac{3}{2}n^3$ operations.