~~Proof~~ Prop. ~~20~~ 19: If $U$ and $W$ are finite-dimensional subspaces of the vector space $V$, then

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Proof: For convenience, put $K = U \cap W$ and $Z = U + W$, so $K$ and $Z$ are both subspaces of $V$. If either $U$ or $W = \{\bar{0}\}$, the result is obvious.

The main idea of the proof is to use Prop. 15 to construct a basis for $Z$.

Let $B = \{\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m\}$ be a basis for $K$ (it is f-d by Prop. 18). Of course, if $K = \{\bar{0}\}$, this step is not needed.

Since $K \subseteq U$, we expand $B$ to a basis $B_1$ of $U$ by adjoining the vectors $\bar{u}_1, \dots, \bar{u}_n$, i.e. $B_1 = \{\bar{k}_1, \dots, \bar{k}_m, \bar{u}_1, \dots, \bar{u}_n\}$, $m \geq 0, n > 0$.

Similarly, we expand $B$ to a basis of $W$ by adjoining the vectors $\bar{w}_1, \dots, \bar{w}_p$, i.e.

$B_2 = \{\bar{k}_1, \dots, \bar{k}_m, \bar{w}_1, \dots, \bar{w}_p\}$, $p > 0$.

Put $C = B \cup B_1 \cup B_2 = \{\bar{k}_1, \dots, \bar{k}_m, \bar{u}_1, \dots, \bar{u}_n, \bar{w}_1, \dots, \bar{w}_p\}$.

We claim $C$ is a basis of $Z$ (*).

To justify (*), we need to prove that

(i) Span $C = Z$

(ii) $C$ is lin. independent.

(See next page)

Proof of Prop. 19 (cont'd).

(i) Let $\bar{v} = \bar{u} + \bar{w}$ be any vector in $Z$,

where $\bar{u} \in U$ and $\bar{w} \in W$.

$$\therefore \bar{u} = c_1 \bar{k}_1 + \cdots + c_m \bar{k}_m + d_1 \bar{u}_1 + \cdots + d_n \bar{u}_n$$

and $\bar{w} = b_1 \bar{k}_1 + \cdots + b_m \bar{k}_m + g_1 \bar{w}_1 + \cdots + g_p \bar{w}_p$

so that $\bar{v} = (c_1 + b_1)\bar{k}_1 + \cdots + (c_m + b_m)\bar{k}_m +$

$$d_1 \bar{u}_1 + \cdots + d_n \bar{u}_n + g_1 \bar{w}_1 + \cdots + g_p \bar{w}_p,$$

i.e. a lin. comb. of elements of $C$.

(ii) Suppose $c_1 \bar{k}_1 + \cdots c_m \bar{k}_m + d_1 \bar{u}_1 + \cdots + d_n \bar{u}_n + g_1 \bar{w}_1 +$

$$\cdots + g_p \bar{w}_p = \bar{0} \qquad (1)$$

i.e. $c_1 \bar{k}_1 + \cdots + c_m \bar{k}_m + d_1 \bar{u}_1 + \cdots + d_n \bar{u}_n = -g_1 \bar{w}_1 - g_2 \bar{w}_2 - \cdots - g_p \bar{w}_p \qquad (2)$

Now, LHS of (2) is a vector in $U$ and RHS of (2) is a vector in $W$; hence, it is a vector in $K = U \cap W$, i.e. we can re-write (2) as

$$c_1 \bar{k}_1 + \cdots + c_m \bar{k}_m + d_1 \bar{u}_1 + \cdots + d_n \bar{u}_n = b_1 \bar{k}_1 + \cdots + b_m \bar{k}_m \qquad (3)$$

or $h_1 \bar{k}_1 + \cdots + h_m \bar{k}_m + d_1 \bar{u}_1 + \cdots + d_n \bar{u}_n = \bar{0} \qquad (4)$

But now, since $B_1$ is a basis for $U$, hence l.i., we get $d_1 = d_2 = \cdots = d_n = 0 \qquad (5)$

$\therefore$ (1) becomes:

$$c_1 \bar{k}_1 + \cdots + c_m \bar{k}_m + g_1 \bar{w}_1 + \cdots + g_p \bar{w}_p = \bar{0}. \qquad (6)$$

Proof of Prop. 19 (conclusion) (8)

But then again, since $B_2$ is a basis for $W$ and hence l.i., we get:

$$c_1 = c_2 = \cdots = c_m = g_1 = g_2 = \cdots = g_p = 0.$$

So we have proved (i) and (ii), and so $C$ is indeed a basis for $Z = U+W$.

$$\therefore \dim(U+W) = \dim Z = m+n+p \quad (7)$$

~~OTOH, dim U + dim W + dim − dim K~~

~~= (m + p) + s~~

OTOH, $\dim U + \dim W - \dim(U \cap W)$

$$= (m+n) + (m+p) - m$$

$$= m+n+p \quad (8)$$

From (7) and (8),

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W),$$

as required.