SCHOOL OF FINANCE AND APPLIED STATISTICS

FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 11

Question 1

Consider the following assertions about a fixed interest security that pays coupons of 10% at the end of each year and is redeemable at par at the end of the third year.

I The discounted mean term of the cashflows is 2.74
II The volatility of the cashflows is 2.54
III The convexity of the cashflows is 9.11

Which of these assertions are correct, based on an interest rate of 8%.

Solution

The present value of the cashflows for $100 nominal of this stock is:

$$P(i) = 10v + 10v^2 + 110v^3 = \$105.15$$

So the volatility is:

$$\frac{-P'(i)}{P(i)} = \frac{10v^2 + 20v^3 + 330v^4}{105.15} = \frac{267.01}{105.15} = 2.54$$

and the discounted mean term (duration) is:

$2.54(1.08) = 2.74$ years

The convexity is

$$\frac{P''(i)}{P(i)} = \frac{20v^3 + 60v^4 + 1320v^5}{105.15} = \frac{958.35}{105.15} = 9.11$$

So all three assertions are correct.

Question 2

You have entered into a contract to purchase 30,000 AMP shares at $5 each on 30 June 2008 and 20,000 AMP shares at $15 each on 31 December 2013. You wish to set aside an amount of money at 30 June 2005 to be invested in certain assets to ensure you will have enough to fulfil these contracts when they fall due (ie. you wish to immunise your assets you are holding to purchase these shares against interest rate changes).

The assets available for you to purchase are as follows:

Asset A is a two year government bond, paying half yearly coupons of 13% p.a. and redeemable at par.

Asset B is a ten year zero coupon bond.

(a) Calculate the face value of Asset A and Asset B which you will hold in order to immunise your portfolio against interest rate fluctuations. Using convexity, show that these amounts will immunise your portfolio. Assume an interest rate of 8% per annum.

(b) Calculate the present value of the assets and liabilities if the interest rate was to drop to 7.5% for all time periods immediately after purchasing the bonds. Use the Taylor series expansion for both and also check to see how close this is to the actual values.

Solution

(a)

In order to meet the conditions of Redington immunisation:

$V_A(0.08) = V_L(0.08)$	V = Present Value	A = Assets
$\tau_A(0.08) = \tau_L(0.08)$	τ = Duration	L = Liabilities
$c_A(0.08) \geq c_L(0.08)$	c = Convexity	

Let P be the face value of Asset A and Q be the face value of Asset B.

Thus:

$$V_A(0.08) = 0.065Pa_{\overline{4}|j} + P(1.08)^{-2} + Q(1.08)^{-10} \quad \text{where } j = 1.08^{\frac{1}{2}} - 1 = 3.923\%$$
$$= 1.093713P + 0.463193Q$$

$$V_L(0.08) = 30,000(5)v^3 + 20,000(15)v^{8.5} = \$275,037.03$$

Therefore $1.093713P + 0.463193Q = 275,037.03 \qquad (1)$

$$\tau_A(0.08) = \frac{\dfrac{0.065P\left(Ia\right)_{\overline{4}|j}}{2} + 2P(1.08)^{-2} + 10Q(1.08)^{-10}}{1.093713P + 0.463193Q}$$
$$= \frac{2.004501P + 4.631935Q}{1.093713P + 0.463193Q}$$

Note that the first part of the numerator above is divided by 2 to ensure the calculation of duration of the coupons is done in years and not half-years.

$$\tau_L(0.08) = \frac{30,000(5)(3)v^3 + 20,000(15)(8.5)v^{8.5}}{275,037.03} = 6.118824 \text{ years}$$

Therefore $\dfrac{2.004501P+4.631935Q}{1.093713P+0.463193Q}=6.118824$

$4.687736P-1.797739Q=0$ or $Q=2.607573P$ (2)

Solving (1) and (2) simultaneously gives
$P=119,502$
$Q=311,611$

Thus, you need to purchase the two-year government bond with face value $119,502 and the zero coupon bond with face value $311,611

Lastly, we need to ensure that $c_A(0.08)\geq c_L(0.08)$ is true to ensure this combination of assets immunises the portfolio. Note that we use the standard convexity formula for the two year bond to ensure consistency across the calculations.

$$c_A(0.08)=\frac{\sum_{k=1}^{n}C_{t_k}t_k(t_k+1)v_i^{t_k+2}}{PV}$$
$$=\frac{119,502\times 0.065\left(0.5(1.5)v^{2.5}+1(2)v^{3}+1.5(2.5)v^{3.5}+2(3)v^{4}\right)+119,502(2)(3)v^{4}+311,611(10)(11)v^{12}}{275,037.03}$$
$$=\frac{7767.63\left(9.481070\right)+507,025.22+13,611,951.70}{275,037.03}$$
$$=51.6026$$

$$c_L(0.08)=\frac{\sum_{k=1}^{n}C_{t_k}t_k(t_k+1)v_i^{t_k+2}}{PV}$$
$$=\frac{150,000(3)(4)v^{5}+300,000(8.5)(9.5)v^{10.5}}{275,037.03}$$
$$=43.7117$$

Therefore $c_A(0.08)\geq c_L(0.08)$ and the portfolio is immunised against small changes in interest rates if the two year bond is bought at face value $119,502 and the zero coupon bond is bought at face value $311,611.

(b)
Using the Taylor series expansion formula:

$$\frac{PV(i_0+\varepsilon)-PV(i_0)}{PV(i_0)} = -\varepsilon\frac{\tau}{(1+i_0)}+\frac{\varepsilon^2}{2}c$$

Calculating the present value of assets:

$$PV_A(0.075) = PV_A(0.08)\left(0.005\frac{6.118824}{1.08}+\frac{0.005^2}{2}51.6026\right)+PV_A(0.08)$$
$$= 275,037.03\times 0.0289729 + 275,037.03$$
$$= \$283,005.66$$

This compares to the actual figure of:

$$V_A(0.075) = 0.065Pa_{\overline{4}|k} + P(1.075)^{-2} + Q(1.075)^{-10}$$
$$= 1.103067P + 0.485194Q \qquad \text{where } k = 1.075^{\frac{1}{2}} - 1 = 3.682\%$$
$$= \$283,010.50$$

Calculating the present value of liabilities:

$$PV_L(0.075) = PV_L(0.08)\left(0.005\frac{6.118824}{1.08}+\frac{0.005^2}{2}43.7117\right)+PV_A(0.08)$$
$$= 275,037.03\times 0.0288743 + 275,037.03$$
$$= \$282,978.53$$

This compares to the actual figure of:

$$V_L(0.075) = 30,000(5)v^3 + 20,000(15)v^{8.5} = \$282,980.86$$

<u>Question 3</u>

A company has a liability of a capital redemption policy that obligates the payment of \$1,000,000 to a policyholder in exactly 12 years, and requires the policyholder to make annual premium payments (at the start of each of the remaining 12 years) of \$15,000. In addition to the asset income represented by the premiums payable by the policyholder, the company wants to purchase a zero coupon bond with redemption value A_{t_0} and a term of $t_0 > 0$ so that the capital redemption policy will be fully immunised at the current interest rate of 10%. Find t_0 and the face value of the zero coupon bond A_{t_0} that must be purchased, and show that this fully immunises the policy.

Solution

$$PV_A(i) = A_{t_0} v_{0.1}^{t_0} + 15,000\ddot{a}_{\overline{12}|0.1}$$

$$PV_L(i) = 1,000,000 v_{0.1}^{12}$$

For immunisation there are three conditions:

Condition 1: $PV_A(i) = PV_L(i)$

$\Rightarrow A_{t_0} v_{0.1}^{t_0} + 15,000\ddot{a}_{\overline{12}|0.1} = 1,000,000 v_{0.1}^{12}$

$\Rightarrow A_{t_0} v_{0.1}^{t_0} + 112,425.9 = 318,630.8$

$\Rightarrow A_{t_0} v_{0.1}^{t_0} = 206,204.9$ (1)

Condition 2: $PV_A'(i) = PV_L'(i)$.

(this is equivalent to $\upsilon_A(i) = \upsilon_L(i)$ if $PV_A(i) = PV_L(i)$)

Since $\ddot{a}_{\overline{12}|0.1} = \sum_{t=1}^{12} v_{0.1}^{t-1}$

$$PV_A(i) = A_{t_0} v_{0.1}^{t_0} + 15,000 \cdot \sum_{t=1}^{12} v_{0.1}^{t-1}$$

$$\Rightarrow -PV_A'(i) = t_0 \cdot A_{t_0} v_{0.1}^{t_0+1} + 15,000 \cdot \sum_{t=1}^{12} (t-1) \cdot v_{0.1}^{t}$$

$$= t_0 \cdot A_{t_0} v_{0.1}^{t_0+1} + 15,000 \cdot \sum_{t=1}^{12} t v_{0.1}^{t} - v_{0.1}^{t}$$

$$= v \cdot t_0 \cdot A_{t_0} v_{0.1}^{t_0} + 15,000\left((Ia)_{\overline{12}|0.1} - a_{\overline{12}|0.1}\right)$$

(as $v^t = \frac{1}{(1+i)^t} = (1+i)^{-t} \Rightarrow \frac{dv^t}{di} = -t(1+i)^{-t-1} = -t(1+i)^{-(t+1)} = -tv^{t+1}$)

$$-PV_L{}'(i) = 12,000,000 v_{0.1}^{13}$$

$\Rightarrow v \cdot t_0 \cdot A_{t_0} v_{0.1}^{t_0} + 15,000\left((Ia)_{\overline{12}|0.1} - a_{\overline{12}|0.1}\right) = 12,000,000 v_{0.1}^{13}$

$\Rightarrow v \cdot t_0 \cdot A_{t_0} v_{0.1}^{t_0} + (550,723.7 - 102,205.4) = 3,475,973$

$\Rightarrow t_0 \cdot A_{t_0} v_{0.1}^{t_0} = 3,330,200$ (2)

Using equations (1) and (2) we can solve for t_0

$A_{t_0} v_{0.1}^{t_0} = 206,204.9$

$t_0 \cdot A_{t_0} v_{0.1}^{t_0} = 3,330,200$

$\Rightarrow t_0 = 16.15$ and $A_{t_0} = 961,145$

To check if the portfolio is immunised, we need to check if:

Condition 3: $PV_A''(i) \ge PV_L''(i)$.

(this is equivalent to $c_A(i_0) \geq c_L(i_0)$ if $PV_A(i) = PV_L(i)$)

$$PV_A^{"}(i) = t_0(t_0+1)A_{t_0}v_{0.1}^{t_0+2} + 15,000\sum_{t=1}^{12} t(t-1)\cdot v_{0.1}^{t+1}$$
$$= 47,200,687.7 + 3,235,011.3$$
$$= 50,435,699$$

(Note – the second part of the equation above was done in Excel to minimise hand calculation).

$$PV_L^{"}(i) = 1,000,000(12)(13)v_{0.1}^{14} = 41,079,676$$

Thus we have achieved immunisation as $PV_A''(i) \geq PV_L''(i)$.

Past Exam Question – 2005 Final Exam Q6 – ***6(a) was done in Tutorial Week 10***

You inherit an empty block of land from a distant relative and decide to build a house on it with a view to living in it in the future. The house you wish to build costs \$300,000 and is expected to be completed in two years from now. You are required to pay the \$300,000 on the completion of the house.

In the meantime you wish to set aside an amount of money to ensure you will have enough to pay for the house in two years time. You are considering investing in inflation-linked bonds with the following characteristics:

- Term = 3 years
- Indexed Coupon rate = 8% p.a.
- Coupon Frequency = Annual in arrears
- Redemption is at the indexed amount of the initial par value

Assuming an interest rate of 6% p.a. and an expected future inflation rate of 3% p.a.:

a) Show that the initial face value of the indexed linked bond that will need to be purchased to ensure that the present value of the inflation linked bond is equal to the present value of the house payment is \$233,361.25. (4 marks)

b) Calculate the Macaulay duration and convexity of the assets (the bond) and liabilities (the house) at an interest rate of 6% p.a. (6 marks)

c) You are confident that interest rates will not decrease over the next two years, but are unsure if interest rates will rise. Using your answers to b) above, and a Taylor-Series expansion, find an approximate present value of the assets and liabilities if the interest rate was to change to 6.5%. (4 marks)

d) You become aware of a separate investment product with Macaulay duration of 1.5 and convexity of 4.25 at an interest rate of 6% p.a. and decide to immunise your portfolio against interest rate rises and falls. Calculate the purchase price of the inflation-linked bond and the separate investment product, and confirm by calculating convexity that this immunises your portfolio at an interest rate of 6% p.a. (5 marks)

Solution

a) $PV_L = 300,000(1.06^{-2}) = 266,998.93$

$$PV_A = 0.08F\left(\frac{1.03}{1.06}+\left(\frac{1.03}{1.06}\right)^2+\left(\frac{1.03}{1.06}\right)^3\right)+F\left(\frac{1.03}{1.06}\right)^3$$

$= 0.2266696F + 0.9174747F = 1.144144F$

$266,998.93 = 1.144144F \Rightarrow F = \$233,361.25$

b) $\tau_L = \dfrac{2\times 266,998.93}{266,998.93} = 2$

$$c_L = \frac{300,000\times 2\times 3\times v^4}{266,998.93} = 5.33998$$

$$\tau_A = \frac{233,361.25\times 0.08\left(\left(\frac{1.03}{1.06}\right)+2\left(\frac{1.03}{1.06}\right)^2+3\left(\frac{1.03}{1.06}\right)^3\right)+233,361.25\times 3\left(\frac{1.03}{1.06}\right)^3}{266,998.93}$$

$$= \frac{18,668.9\times 5.6125165+642,309.10}{266,998.93} = 2.79810$$

$$c_A = \frac{233.361.25\times\frac{0.08}{1.06^2}\left(2\left(\frac{1.03}{1.06}\right)+6\left(\frac{1.03}{1.06}\right)^2+12\left(\frac{1.03}{1.06}\right)^3\right)+233,361.25\times 12\left(\frac{1.03^3}{1.06^5}\right)}{266,998.93}$$

$$= \frac{16,615.2545\times 18.618275+2,286,611.25}{266,998.93} = 9.72273$$

c) $\dfrac{PV(6.5\%)-PV(6.0\%)}{PV(6.0\%)} = -0.5\%\dfrac{\tau}{1.06}+\dfrac{0.5\%^2}{2}c$

$$\frac{PV_L(6.5\%)-266,998.93}{266,998.93} = -0.5\%\frac{2}{1.06}+\frac{0.5\%^2}{2}5.33998 = -0.0093672$$

$PV_L(6.5\%) = \$264,497.89$

$$\frac{PV_A(6.5\%)-266,998.93}{266,998.93} = -0.5\%\frac{2.79810}{1.06}+\frac{0.5\%^2}{2}9.72273 = -0.013077$$

$PV_A(6.5\%) = \$263,507.37$

d) Let X be the purchase price of the inflation linked bond and Y be the purchase price of the separate investment product.

For immunization:

$PV_A(6\%) = PV_L(6\%)$

$\tau_A(6\%) = \tau_L(6\%)$

$c_A(6\%) \geq c_L(6\%)$

$X+Y=266,998.93 \quad (1)$

Since for any cash flow stream, $\tau \times PV = \sum tC_t v^t$, therefore

$$\frac{2.7981X+1.5Y}{266,998.93}=2 \Rightarrow 2.7981X+1.5Y=533,997.86 \quad (2)$$

Solving (1) and (2) simultaneously:

$$(2.7981-1.5)X=533,997.86-1.5\times 266,998.93$$
$$X=\$102,842.20$$
$$Y=\$164,156.73$$

Check that $c_A(6\%)\geq c_L(6\%)$

$$c_A(6\%)=\frac{9.72273X+4.25Y}{266,998.93}=6.35798\geq c_L(6\%)$$

Therefore portfolio is immunised.