Problem 3

Rewrite the three equations:

$$
\begin{aligned}
t_1 - m_1 t_n &= c_1 v - m_1 g \\
-t_{i-1} + t_i - m_i t_n &= c_i v - m_i g, \text{for } i \in [2, n-1] \\
-t_{n-1} - m_n t_n &= c_n v - m_n g
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & 0 & 0 & \cdots & -m_1 \\ -1 & 1 & 0 & \cdots & -m_2 \\ 0 & -1 & 1 & \cdots & -m_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & -m_n \end{pmatrix}, b = \begin{pmatrix} c_1 v - m_1 g \\ c_2 v - m_2 g \\ c_3 v - m_3 g \\ \vdots \\ c_n v - m_n g \end{pmatrix}
$$

> Note that the direct results of running 3 scripts mentioned below can be found in the appendix at the end of this assignment.

(a) Solution
In case (i), we run the following script:

```
% Take g = 10, v = 6 as constants.
% (a) part i
kappa_vec = [];
t = zeros(32, 4);
ni = [4, 8, 16, 32];
for i = 1:4
    n = ni(i);
    m = linspace(50, 100, n);
    c = 25 - 10*linspace(0, 1, n);
    b = transpose(6.*c -10.*m);
    e = ones(n, 1);
    A = spdiags([-e, e], [-1, 0], n, n);
    A(:, n) = -m;
    tension = A\b;
    display(tension); % output of tension vector
    kappa = condest(A);
    display(kappa); % output of condition number of the matrix
    max_tension = max(tension);
    display(max_tension); % output of maximum tension computed
```

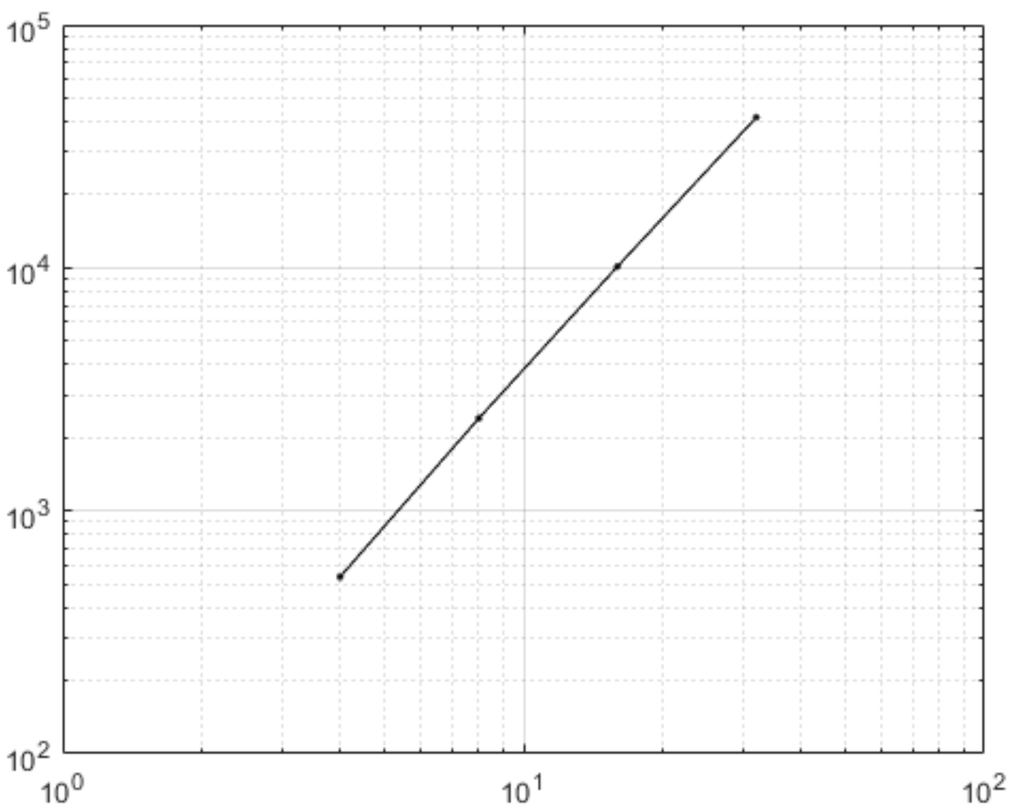
```
    kappa_vec = [kappa_vec, kappa]; % store condtion number
    t(1:n,i) = tension; % store tension vectors
end
figure
loglog([4 8 16 32], kappa_vec, 'k.-') % plot condition number vs. n
grid

figure
plot((1:ni(1)-1)/ni(1), t(1:ni(1)-1, 1), 'r-', ...
   (1:ni(2)-1)/ni(2), t(1:ni(2)-1, 2), 'g--', ...
   (1:ni(3)-1)/ni(3), t(1:ni(3)-1, 3), 'b-.', ...
   (1:ni(4)-1)/ni(4), t(1:ni(4)-1, 4), 'k.');
grid
```

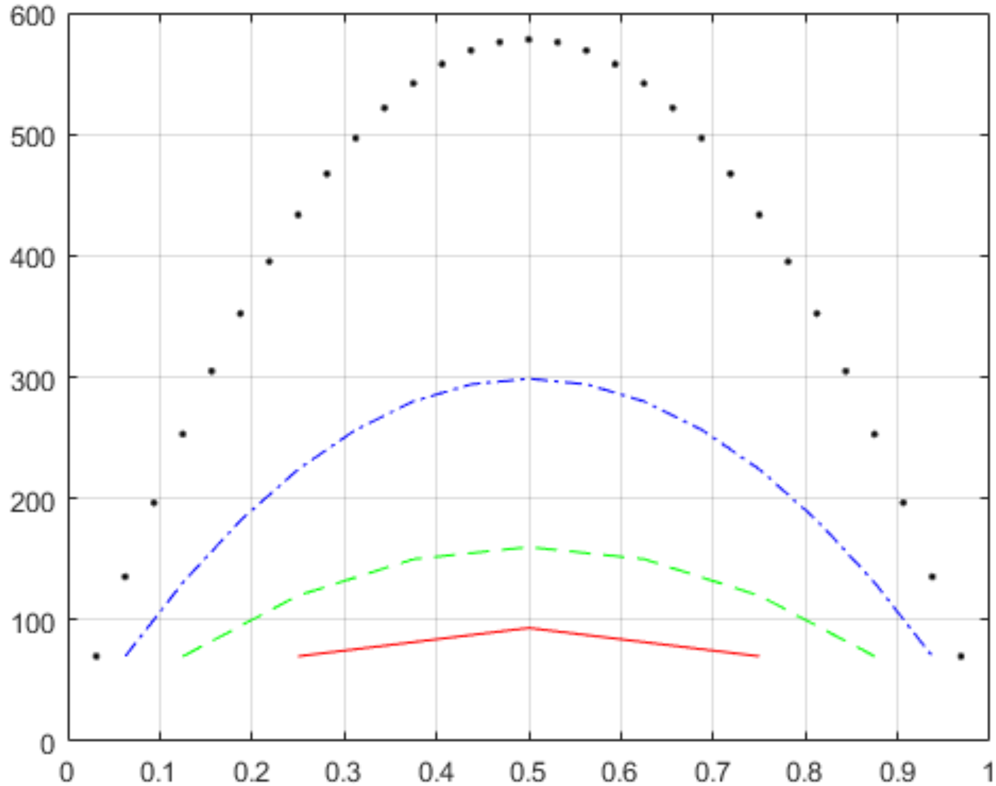
According to the output of this script, we can conclude the condition number of the matrix and maximum tension in one table:

	$n = 4$	$n = 8$	$n = 16$	$n = 32$
condition number	534	2401	10134	41601
maximum tension	93	160	299	578

The plot in log-log scale of condtiion numbers versus n is shown as below:



The plot of tension vectors components versus their normalized index is shown as:



In details,

- red curve (the lowest) indicates the case of $n = 4$,
- green curve (the 2nd lowest) indicates the case of $n = 8$,
- blue curve (the 3rd lowest) indicates the case of $n = 16$,
- black dot curve (the highest) indicates the case of $n = 32$.

In case (ii), we run the modified script instead, which randomly picks m_i and c_i:

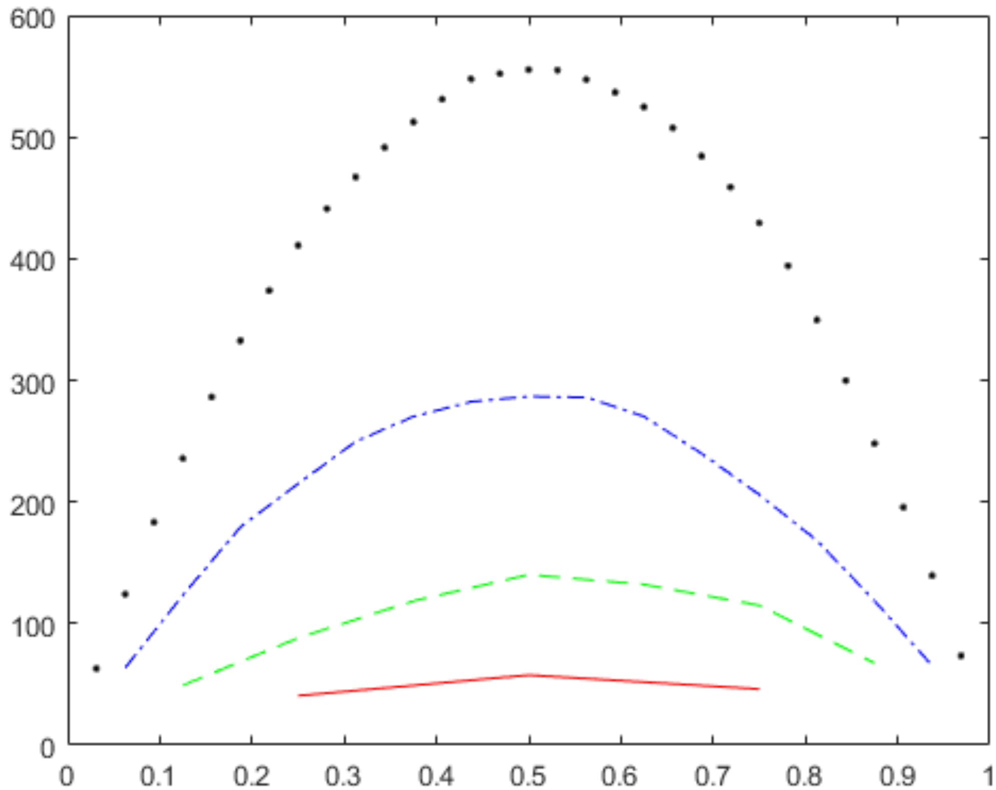
```
% (a) part ii


t = zeros(32, 4);
ni = [4, 8, 16, 32];
for i = 1:4
    n = ni(i);
    m = sort(50 + 50*rand(n, 1), 'ascend');
    c = sort(15 + 10*rand(n, 1), 'descend');
    b = 6.*c -10.*m;
    e = ones(n, 1);
    A = spdiags([-e, e], [-1, 0], n, n);
    A(:, n) = -m;
    tension = A\b;
    display(tension); % output of tension vector
    max_tension = max(tension);
    display(max_tension); % output of maximum tension computed
    t(1:n,i) = tension; % store tension vectors
end

```

```
plot((1:ni(1)-1)/ni(1), t(1:ni(1)-1, 1), 'r-', ...
   (1:ni(2)-1)/ni(2), t(1:ni(2)-1, 2), 'g--', ...
   (1:ni(3)-1)/ni(3), t(1:ni(3)-1, 3), 'b-.', ...
   (1:ni(4)-1)/ni(4), t(1:ni(4)-1, 4), 'k.');
```

This time we only need to show the plot of tension vectors components versus their normalized index:



And we notice that the plot is not as perfectly symmetical as above, but it stays in a very similar shape.

- **Comments on:**
  - **how the acceleration and the maximum tension behave with n.**
    - The acceleration a stays the same as n increases. Note that in this problem we substitute a with t_n, which is the last part of tension, i.e. the last tension vector component.
    - The maximum tension increases as n increases.
  - **how the components of the tension vectors vary with their index.**
    - By observing two "bell-shape" plots we can easily notice that, no matter what n value we pick, the general trend as bell-curve remains. And for larger n, the curve is more concave (with higher top).
  - **where (for which i) the max tension occurs.**
    - The max tension (component) tends to sit in the middle of its tension vector. Specifically, our n is even, index $i = \frac{n}{2}$ indicates max tension.
  - **how the condition numbers behave with n.**
    - The condition numbers increase as n increases. And on log-log scale, such increase is almost proportional.

(b) Solution:

To help us get an understanding of L, U, P, we suppose $n = 4$, and we apply the case (i) script.

```
% (b)
ni = 4;
m = linspace(50, 100, ni);
c = 25 - 10*linspace(0, 1, ni);
b = 6.*c -10.*m;
e = ones(ni, 1);
A = spdiags([-e, e], [-1, 0], ni, ni);
A(:, ni) = -m;
[L, U, P] = lu(A);
display(L);
display(U);
display(P);
```

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 0 & 0 & -50 \\ 0 & 1 & 0 & -116.6667 \\ 0 & 0 & 1 & -200 \\ 0 & 0 & 0 & -300 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we change n to 8, the corresponding L, U, P are:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -107.1429 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -171.4286 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -242.8571 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -321.4286 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -407.1429 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -500.0000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -600.0000 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The form of P would be identity matrix $I_{n \times n}$. This can be explained by the form of matrix A where each row is perfect (no need to reorder for partial pivoting), so the permutation matrix should be identity matrix in fact.
- The form of matrices L and U can be calculated through tedious process. Here we only present the final result:
  - The form of L would be an $n \times n$ matrix with ones on diagonal line and entries $l_{ij} = -1$ for all $i - j = 1$.
  - The form of U would be an $n \times n$ matrix with ones on diagonal line and the last column substituted by a column vector of sums of negative m_i's.
- To represent the three matrices above explicitly and directly, we can write:

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & -m_1 \\ 0 & 1 & \cdots & 0 & -m_1 - m_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\sum_{i=1}^{n-1} m_i \\ 0 & 0 & \cdots & 0 & -\sum_{i=1}^{n} m_i \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

Can also check that $PA = LU$.

(c) Solution:
The acceleration $a = t_n$ is the last component in vector t which satisfies $At = b$.
The tensions are $t_i, i = 1, \dots, n-1$.
So we want closed form formula for all components in t in terms of n, m_i, c_i, v, g. Also find i for max t_i.
So far, we have:

$$\begin{aligned} t_1 - m_1 t_n &= c_1 v - m_1 g \\ -t_1 + t_2 - m_2 t_n &= c_2 v - m_2 g \\ -t_2 + t_3 - m_3 t_n &= c_3 v - m_3 g \\ &\cdots \\ -t_{n-2} + t_{n-1} - m_{n-1} t_n &= c_{n-1} v - m_{n-1} g \\ -t_{n-1} - m_n t_n &= c_n v - m_n g \end{aligned}$$

If we sum up all the equations, we would have this:

$$
\begin{aligned}
LHS = -(m_1 + \cdots + m_n)t_n &= (c_1 + \cdots + c_n)v - (m_1 + \cdots + m_n)g = RHS \\
(m_1 + \cdots + m_n)t_n &= (m_1 + \cdots + m_n)g - (c_1 + \cdots + c_n)v \\
(c_1 + \cdots + c_n)v &= (m_1 + \cdots + m_n)(g - t_n) \\
g - t_n &= \frac{c_1 + \cdots + c_n}{m_1 + \cdots + m_n} v \\
\therefore t_n &= g - \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i} v
\end{aligned}
$$

Then we can plug t_n back into the first equation:

$$
\begin{aligned}
t_1 &= c_1 v - m_1 g + m_1 t_n \\
&= c_1 v + m_1(t_n - g) \\
&= c_1 v + m_1\left(-\frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i} v\right) \\
&= \left[c_1 - m_1 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v.
\end{aligned}
$$

Similarly for t_2 in the second equation, we can plug in t_1 and t_n:

$$
\begin{aligned}
t_2 &= c_2 v - m_2 g + m_2 t_n + t_1 \\
&= c_2 v + m_2(t_n - g) + \left[c_1 - m_1 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v \\
&= \left[c_2 - m_2 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v + \left[c_1 - m_1 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v \\
&= \left[c_1 + c_2 - (m_1 + m_2) \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v.
\end{aligned}
$$

Then for t_3:

$$
\begin{aligned}
t_3 &= c_3 v - m_3 g + m_3 t_n + t_2 \\
&= \left[c_3 - m_3 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v + \left[c_1 + c_2 - (m_1 + m_2) \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v \\
&= \left[c_1 + c_2 + c_3 - (m_1 + m_2 + m_3) \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v
\end{aligned}
$$

For all $2 \leq i \leq n - 1$,

$$t_i = \left\lfloor \sum_{k=1}^{i} c_k - \sum_{k=1}^{i} m_k \frac{\sum_{j=1}^{n} c_j}{\sum_{j=1}^{n} m_j} \right\rfloor v.$$

One of the other task of the problem is to find the index for max tension.
Observe:

$$\begin{aligned} t_1 &= c_1 v - m_1 g + m_1 t_n \\ -t_1 + t_2 &= c_2 v - m_2 g + m_2 t_n \\ -t_2 + t_3 &= c_3 v - m_3 g + m_3 t_n \\ &\dots \\ -t_{n-1} &= c_n v - m_n g + m_n t_n \end{aligned}$$

We know that c_i is monotone decreasing, while m_i is monote increasing. So on the RHS of all these equations, $c_i v - m_i g$ is monotone decreasing, and $m_i t_n$ is monotone increasing. By basic calculus knowledge, the sum of such two parts is concave down, i.e., there exists a local maximum. We also know that t_i are values on this concave curve. And the LHS are differences between each consecutive t_i values.
Therefore we want to find the smallest i such that $t_i - t_{i-1} = c_i v + m_i(t_n - g) < 0$.
Now we take the condition of case (i):

$$\begin{aligned} m_i &= 50 + 50 \frac{i-1}{n-1}, \\ c_i &= 25 - 10 \frac{i-1}{n-1}. \end{aligned}$$

Therefore,

$$t_i - t_{i-1} = (25 - 10\,\frac{i-1}{n-1})v + (50 + 50\,\frac{i-1}{n-1})\,\frac{\sum c}{\sum m}\,v < 0$$

$$5(n-1) - 2(i-1) < \left[10(n-1) + 10(i-1)\right]\frac{\sum c}{\sum m}$$

$$i - 1 > (n-1)\cdot\frac{5 - 10\cdot\frac{\sum c}{\sum m}}{2 + 10\cdot\frac{\sum c}{\sum m}}$$

$$\because \sum c = 25n - 10\cdot\frac{(0+n-1)n}{(n-1)2} = 25n - 5n = 20n$$

$$\because \sum m = 50n + 50\cdot\frac{(0+n-1)n}{(n-1)2} = 50n + 25n = 75n$$

$$\therefore \frac{5 - 10\cdot\frac{\sum c}{\sum m}}{2 + 10\cdot\frac{\sum c}{\sum m}} = \frac{5 - 10\cdot 4/15}{2 + 10\cdot 4/15} = \frac{7/3}{14/3} = \frac{1}{2}$$

$$\therefore i - 1 > \frac{1}{2}\,(n-1)$$

$$\therefore i > \frac{1}{2}\,(n-1) + 1$$

So the largest index we can pick to keep the difference between values t_i, t_{i-1} positive is

$$i = \lceil\frac{1}{2}\,(n-1)\rceil = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{1}{2}\,(n-1), & \text{if } n \text{ is odd} \end{cases}$$

.