STAT2001/6039 Assignment 1 Solutions (2017)

Problem 1 **(a)** Let A_i be the event that the drawn cards contain i Aces, and B the event that they contain at least 1 Ace. Then

$$P(B) = P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$$
$$= \frac{\binom{4}{1}\binom{11}{6}}{\binom{15}{7}} + \frac{\binom{4}{2}\binom{11}{5}}{\binom{15}{7}} + \frac{\binom{4}{3}\binom{11}{4}}{\binom{15}{7}} + \frac{\binom{4}{4}\binom{11}{3}}{\binom{15}{7}} = 0.9487.$$

More simply, observe that $B = \overline{A}_0$. Hence

$$P(B) = 1 - P(A_0) = 1 - \binom{11}{7}\Big/\binom{15}{7} = 1 - \frac{330}{6435} = 0.9487.$$

(b) Let C be the event that the drawn cards contain no Kings or Queens. Then

$$P(A_2 \mid C) = \frac{P(A_2C)}{P(C)} = \frac{\binom{4}{2}\binom{6}{5}\Big/\binom{15}{7}}{\binom{10}{7}\Big/\binom{15}{7}} = 0.3.$$

Problem 2 **(a)** The problem can be solved using a first step analysis. Let H be the event that Homer will win the game, and A the event that 5 or 6 comes up on the first roll. Then by the law of total probability,

$$P(H) = P(A)P(H \mid A) + P(\overline{A})P(H \mid \overline{A}) = \frac{1}{3}(1) + \frac{2}{3}(1 - P(H)),$$

since if Homer doesn't win on the first roll he will be in the same position as Marge was at the beginning of the game. Thus $p = (1/3) + (2/3)(1-p)$, where $p = P(H)$. Hence the probability that Homer will win the game is $p = 3/5$.

Alternatively,

$$P(H) = \frac{1}{3} + \left(\frac{2}{3}\right)^2 \frac{1}{3} + \left(\frac{2}{3}\right)^4 \frac{1}{3} + \cdots = \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots\right) = \frac{1}{3}\left(\frac{1}{1-4/9}\right) = \frac{3}{5}.$$

(b) Let M be the event that Marge will win the game. Also, let:

B = "2, 3 or 4 comes up on the first roll"

C = "1 comes up on the first roll, and 5 or 6 comes up on the second roll"

D = "1 comes up on both of the first two rolls"

E = "1 comes up on the first roll, and 2, 3 or 4 comes up on the second roll".

Then *A*, *B*, *C*, *D* and *E* form a partition of the sample space. (That is, these events are disjoint and one of them must happen.) Hence by the law of total probability,

$$\begin{aligned}P(H) &= P(A)P(H\mid A)+P(B)P(H\mid B)+P(C)P(H\mid C)\\ &\qquad +P(D)P(H\mid D)+P(E)P(H\mid E)\\ &= \frac{1}{3}(1)+\frac{1}{2}P(M)+\frac{1}{6}\left(\frac{1}{3}\right)(0)+\left(\frac{1}{6}\right)^2(0)+\frac{1}{6}\left(\frac{1}{2}\right)P(H).\end{aligned}$$

Similarly,

$$\begin{aligned}P(M) &= P(A)P(M\mid A)+P(B)P(M\mid B)+P(C)P(M\mid C)\\ &\qquad +P(D)P(M\mid D)+P(E)P(M\mid E)\\ &= \frac{1}{3}(0)+\frac{1}{2}P(H)+\frac{1}{6}\left(\frac{1}{3}\right)(1)+\left(\frac{1}{6}\right)^2(0)+\frac{1}{6}\left(\frac{1}{2}\right)P(M).\end{aligned}$$

In summary, $p=(1/3)+(1/2)q+(1/12)p$ and $q=(1/2)p+(1/18)+(1/12)q$, where *p* = *P*(*H*) and *q* = *P*(*M*). Solving these two equations, we find that the probability Homer will win the game is *p* = 48/85 = 0.5647.

Problem 3 **(a)** We will refer to the numbers on the white die as 1, 2, 3, 4, 5a and 5b. Similarly, the numbers on the red die can be labelled 1a, 1b, 2, 3, 4, 5. Let *w* be the number which comes up on the white die and *r* the number on the red die. Then each sample point can be written (*w*,*r*), or more simply *wr*. With this notation, the sample points for the experiment can be listed as follows:

11a, 11b, 12, 13, 14, 15,	21a, 21b, 22, 23, 24, 25,
31a, 31b, 32, 33, 34, 35,	41a, 41b, 42, 43, 44, 45,
5a1a, 5a1b, 5a2, 5a3, 5a4, 5a5,	5b1a, 5b1b, 5b2, 5b3, 5b4, 5b5.

These n_S = 36 sample points are equally likely, so it is reasonable to assign the probability 1/36 to each one of them.

Next let *C* be the event that the total of the two numbers coming up is less than 6. Then, *C* = {11a, 11b, 12, 13, 14, 21a, 21b, 22, 23, 31a, 31b, 32, 41a, 41b}. We see that the number of sample points in *C* is n_C = 14. Therefore, $P(C)=n_C/n_S=7/18$.

(b) Let *A* be the event that the number coming up on the white die is less than 4, and let B_i be the event that the *i*th roll of the red die results in a 1. Then, the required probability is

$$\begin{aligned}P(A\cup(\bar{B}_1\bar{B}_2)) &= 1-P(\overline{A\cup(\bar{B}_1\bar{B}_2)})\\ &= 1-P(\bar{A}\overline{\bar{B}_1\bar{B}_2}) \quad \text{by De Morgan's laws}\end{aligned}$$

$$= 1 - P(\overline{A})P(\overline{\overline{B}_1\overline{B}_2}) \quad \text{by independence of the rolls}$$
$$= 1 - P(\overline{A})P(B_1 \cup B_2) \quad \text{by De Morgan's laws}$$
$$= 1 - \{1 - P(A)\}\{P(B_1) + P(B_2) - P(B_1B_2)\}$$
$$\text{by the additive law of probability}$$
$$= 1 - \{1 - P(A)\}\{P(B_1) + P(B_2) - P(B_1)P(B_2)\} \text{ since } B_1 \perp B_2$$
$$= 1 - \left(1 - \frac{1}{2}\right)\left(\frac{1}{3} + \frac{1}{3} - \frac{1}{3} \times \frac{1}{3}\right) = \frac{13}{18}.$$

More simply, let B be the event that neither of the rolls of the red die results in a 1 (i.e. $B = \overline{B}_1\overline{B}_2$). Then $P(A \cup B) = P(A) + P(B) - P(A)P(B) = \frac{1}{2} + \left(\frac{4}{6}\right)^2 - \frac{1}{2}\left(\frac{4}{6}\right)^2 = \frac{13}{18}$.

(c) Consider the numbers coming up on the final roll of the two dice. The sample space consists of 29 equiprobable sample points, namely all those in S of Part (a) except for 11a, 11b, 22, 33, 44, 5a5 and 5b5. Of these 29 sample points there are 7 for which the total of the two numbers coming up is exactly 6, namely 15, 24, 42, 5a1a, 5a1b, 5b1a and 5b1b. Hence the required probability is 7/29.

Another way to solve the problem is using conditional probability in the context of the two dice being rolled together only *once*. Let D be the event that the total of the numbers coming up on the two dice is exactly 6, and E the event that different numbers come up. Then using the notation in Part (a), DE = {15, 24, 42, 5a1a, 5a1b, 5b1a, 5b1b} and $n_{DE} = 7$. Hence $P(DE) = n_{DE} / n_S = 7/36$. Also, $\overline{E}$ = {11a, 11b, 22, 33, 44, 5a5, 5b5} and $n_{\overline{E}} = 7$, so that $P(\overline{E}) = n_{\overline{E}} / n_S = 7/36$ and $P(E) = 1 - P(\overline{E}) = 29/36$. It follows that the required probability is

$$P(D \mid E) = \frac{P(DE)}{P(E)} = \frac{7/36}{29/36} = \frac{7}{29}.$$

Problem 4 **(a)** Let A_i be the event that i married couples are together on the first dance, and B_i the event that i married couples are together on the second dance. Then the required probability is

$$P(B_2) = P(A_0)P(B_2 \mid A_0) + P(A_1)P(B_2 \mid A_1) + P(A_2)P(B_2 \mid A_2), \qquad (*)$$

where the 6 probabilities on the right hand side are to be determined.

Number the four couples 1, 2, 3 and 4. We will let 2314 denote the event that Husband 2 is dancing with Wife 1, Husband 3 with Wife 2, Husband 1 with Wife 3, and Husband 4 with Wife 4 (his own wife). Similarly, 1234 will denote all four married couples dancing together, and so on. With this notation, the sample space on the first dance consists of 4! = 24 sample points, some of which are 1234, 1243, 1324, 1342, 1423, 1432, 2134, …, 4312, 4321.

Of these 24 sample points, 9 correspond to no couples dancing together, namely 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312 and 4321. Hence $P(A_0) = 9/24 = 3/8$. Also, 8 of the 24 sample points correspond to exactly one married couple being together, namely 1342, 1423, 3241, 4213, 2431, 4132, 2314 and 3124. Hence $P(A_1) = 8/24 = 1/3$. Finally, 6 sample points correspond to exactly 2 married couples being together, namely 1243, 1432, 1324, 4231, 3214 and 2134. Hence $P(A_2) = 6/24 = 1/4$. (Also, $P(A_3) = 0$ and $P(A_4) = 1/24$.)

If no married couples are together on the first dance, then the probability of two married couples being together on the second dance is the corresponding probability on the first dance. That is, $P(B_2 \mid A_0) = P(A_2) = 1/4$.

If one married couple are together on the first dance, then that leaves 3 married couples to be considered for pairing up on the second dance. Numbering these 1, 2 and 3, the sample space now has 3! = 6 elements, namely 123, 132, 213, 231, 312 and 321. Of these, 3 correspond to one married couple being together, namely 132, 213 and 321. Hence the probability that two married couples are together on the second dance, given that one married couple are together on the first dance is $P(B_2 \mid A_1) = 3/6 = 1/2$.

Finally, if two married couples are together on the first dance, then 2 married couples are to be paired up on the second dance. Numbering these 1 and 2, the sample space now consists of 12 and 21. One of these sample points corresponds to no more married couples coming together, namely 21. Hence $P(B_2 \mid A_2) = 1/2$.

By $(*)$, the required probability is $P(B_2) = \frac{3}{8}\left(\frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) = \frac{37}{96} = 0.3854$.

(b) By a logic similar to the one used in Part (a), we find that $P(B_1 \mid A_0) = P(A_1) = 1/3$ and $P(B_1 \mid A_1) = 1/3$. It follows that

$$P(B_1) = P(A_0)P(B_1 \mid A_0) + P(A_1)P(B_1 \mid A_1) = \frac{3}{8}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3}\right) = \frac{17}{72}.$$

Hence the required probability is $P(A_0 \mid B_1) = \frac{P(A_0)P(B_1 \mid A_0)}{P(B_1)} = \frac{(3/8)(1/3)}{17/72} = \frac{9}{17}$.

Problem 5 No. Suppose that two events A and B are *possible*. Then $P(A) \neq 0$ and $P(B) \neq 0$, so that $P(A)P(B) \neq 0$. Now suppose also that A and B are *disjoint*. Then $AB = \varnothing$, so that $P(AB) = 0$. Hence $P(AB) \neq P(A)P(B)$. By definition, this implies $A \not\perp B$. That is, A and B are dependent and cannot be independent.

Alternative solution to Part (b) of Problem 2

Let: A_n = "Game ends on nth roll with 5 or 6", $a_n = P(A_n)$
C_n = "2, 3 or 4 comes up on nth roll", $c_n = P(C_n)$
D_n = "1 comes up on nth roll and game continues", $d_n = P(D_n)$
H = "Homer wins", $h = P(H)$
M = "Marge wins", $m = P(M)$.

Then for $n \geq 2$:

$$a_n = \frac{1}{3}(c_{n-1} + d_{n-1}) \tag{1}$$

$$c_n = \frac{1}{2}(c_{n-1} + d_{n-1}) \tag{2}$$

$$d_n = \frac{1}{6}c_{n-1}. \tag{3}$$

Now (1) & (2) $\Rightarrow a_n = \frac{2}{3}c_n$ (true for all n) (4)

Also, (2) & (3) $\Rightarrow c_n = \frac{1}{2}c_{n-1} + \frac{1}{12}c_{n-2}, \quad n \geq 3$ (5)

Then, (4) & (5) $\Rightarrow a_n = \frac{1}{2}a_{n-1} + \frac{1}{12}a_{n-2}, \quad n \geq 3$. (6)

Hence $\sum_{n=3}^{\infty} a_n = \frac{1}{2}\sum_{n=3}^{\infty} a_{n-1} + \frac{1}{12}\sum_{n=3}^{\infty} a_{n-2}$.

That is, $(a_3 + a_4 + a_5 + \cdots) = \frac{1}{2}(a_2 + a_3 + a_4 + \cdots) + \frac{1}{12}(a_1 + a_2 + a_3 + \cdots)$. (7)

But $h = a_1 + a_3 + a_5 + \cdots$ and $m = a_2 + a_4 + a_6 + \cdots$. Hence $a_1 + a_2 + a_3 + \cdots = h + m$.
Also, $a_1 = 1/3$, and $a_2 = (1/3)(c_1 + d_1) = (1/3)\{(1/2) + (1/6)\} = 2/9$ (by (1)).

It follows from (7) that $h + m - \frac{1}{3} - \frac{2}{9} = \frac{1}{2}(h + m - \frac{1}{3}) + \frac{1}{12}(h + m)$. (8)

Equation (6) also implies that

$$(a_3 + a_5 + a_7 + \cdots) = \frac{1}{2}(a_2 + a_4 + a_6 + \cdots) + \frac{1}{12}(a_1 + a_3 + a_5 + \cdots)$$

$$\Rightarrow h - \frac{1}{3} = \frac{1}{2}m + \frac{1}{12}h. \tag{9}$$

By (8) & (9), the probability that Homer wins is $h = 48/85 = 0.5647$, as before.

Yet another solution to Part (b) of Problem 2

Let H = "Homer wins", M = "Marge wins" and D = "Draw". Also let A_i = "1 on ith roll", B_i = "2, 3 or 4 on ith roll" and C_i ="5 or 6 on ith roll". Then by the LTP:

$$P(H) = P(A_1)P(H \mid A_1) + P(B_1)P(H \mid B_1) + P(C_1)P(H \mid C_1)$$
$$= \frac{1}{6}P(H \mid A_1) + \frac{3}{6}P(M) + \frac{2}{6}\times 1 \quad \textit{(we now have 1 equation in 3 unknowns)}$$
$$P(H \mid A_1) = P(A_2 \mid A_1)P(H \mid A_1A_2) + P(B_2 \mid A_1)P(H \mid A_1B_2) + P(C_2 \mid A_1)P(H \mid A_1C_2)$$
$$= \frac{1}{6}\times 0 + \frac{3}{6}P(H) + \frac{2}{6}\times 0 \quad \textit{(we now have 2 equations in 3 unknowns)}$$
$$P(M) = P(A_1)P(M \mid A_1) + P(B_1)P(M \mid B_1) + P(C_1)P(M \mid C_1)$$
$$= \frac{1}{6}P(M \mid A_1) + \frac{3}{6}P(H) + \frac{2}{6}\times 0 \quad \textit{(we now have 3 equations in 4 unknowns)}$$
$$P(M \mid A_1) = P(A_2 \mid A_1)P(M \mid A_1A_2) + P(B_2 \mid A_1)P(M \mid A_1B_2) + P(C_2 \mid A_1)P(M \mid A_1C_2)$$
$$= \frac{1}{6}\times 0 + \frac{3}{6}P(M) + \frac{2}{6}\times 1 \quad \textit{(we now have 4 equations in 4 unknowns)}$$

Next let $p = P(H)$, $p_1 = P(H \mid A_1)$, $q = P(M)$, $q_1 = P(M \mid A_1)$. Then the above four equations may be written as:

$$6p = p_1 + 3q + 2$$
$$6p_1 = 3p$$
$$6q = q_1 + 3p$$
$$6q_1 = 3q + 2$$

Solving these equations simultaneously, we get p = 0.5647, as before.

Note: One method of solution is to write $Mx = a$, where:

$$M = \begin{pmatrix} 6 & -1 & -3 & 0 \\ -3 & 6 & 0 & 0 \\ 3 & 0 & -6 & 1 \\ 0 & 0 & -3 & 6 \end{pmatrix}, \quad x = \begin{pmatrix} p \\ p_1 \\ q \\ q_1 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}$$

Using the R code below, we find that $x = M^{-1}a = \begin{pmatrix} 0.5647 \\ 0.2824 \\ 0.3686 \\ 0.5176 \end{pmatrix}$

R Code

```
a = c(2,0,0,2)
M = matrix( c(6,-1,-3,0,  -3,6,0,0,  3,0,-6,1,   0,0,-3,6),
      nrow=4, ncol=4, byrow=T)
x = solve(M)%*%a;   t(x)
      # 0.5647059    0.2823529    0.3686275    0.5176471
```

Note that this method of solution and the R code are not assessable.