Map of 1.5

Completenen axiom for $\mathbb{R}$: if $S \neq \emptyset$, bdd Sub-set of $\mathbb{R}$ Then S has lub / glb.

$\downarrow$

ϵ-characterization of $lub(S) = s$

$\forall \epsilon > 0 \; \exists x \in S \quad s - \epsilon < x \leq s$

or $\forall \epsilon > 0 \quad B(\epsilon, s) \cap S \neq \emptyset$

$\downarrow$

$s \in \overline{S}$.

depends on The ordering relation on $\mathbb{R}$, but Such a relation does not exist on $\mathbb{R}^n$

The goal of 1.5 is to extend The notion of completenen to $\mathbb{R}^n$.

(translate Completen to Convergence) 1.16 Monotone Sequence Theorem

any bounded monotone Seq $\{x_k\} \subset \mathbb{R}$ Converges (to its lub if inc or glb if dec)

Pf: $S = \{x_k\}$ is bdd & decreasing $\xrightarrow{\text{Completenen of } \mathbb{R}}$ $glb(S) = d$ exists $\longrightarrow$ $\forall \epsilon > 0 \; \exists x \in S \quad d \leq x < d + \epsilon$

$= x_K$

so $\forall k > K$

$d \leq x_k < x_K < d + \epsilon$

$\therefore |x_k - d| < \epsilon$.

$x_k \to d$ iff $\forall \epsilon > 0 \; \exists K \; \forall k > K \; |x_k - d| < \epsilon$

1.17 Nested Interval Theorem

(translating existence of limit to) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

any nested Seq of intervals $I_n = [a_n, b_n]$ with $\lim_{n \to \infty} b_n - a_n = 0$, Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset = \{x\}$ and has only one pt

Pf: $\{a_n\}$ is monotone inc $\to$ $\lim a_n = a$ exist & $a \leq b$
$\{b_n\}$ is monotone dec $\to$ $\lim b_n = b$

as $b_n - a_n \to 0$ & $b - a < b_n - a_n$ The $b - a = 0$ so $a = b$

If l is some $l \in \bigcap_{n=1}^{\infty} I_n \longrightarrow l \in I_n$ for each n so $a \leq l \leq b$.

1.18 BW1

Every bounded sequence in $\mathbb{R}$ has a convergence subsequence.

Pf: let $\{x_k\} \subset [a_1\ b_1]$. bisect $[a_1 b_1]$ and bick $[a_2\ b_2]$
bdd
to be the half that contains inf many of the x_k's. Continue till we

Construct $I_n = [a_n\ b_n]$ $I_{n+1} \subseteq I_n$

$\bigcap_{n=1}^{\infty} I_n = \{x\}$ apply NIT

pick any $x_{k_j} \in I_j$

$\longrightarrow |x_{k_j} - x| \leq$ length of $I_j = \frac{b_1 - a_1}{2^{j+1}}$

and as $j \to \infty$, $|x_{k_j} - x| \to 0$.

$\frac{b_1 - a_1}{2^{j+1}} \to 0$ so

(key to Completeness of $\mathbb{R}^n$)

1.19 BW2 (extend 1.18 to $\mathbb{R}^n$)

1.20 $\boxed{\text{Cauchy Sequences in } \mathbb{R}^n \text{ Converge}}$ (The new Completeness for $\mathbb{R}^n$)

A sequence $\{x_k\} \subset \mathbb{R}^n$ converges iff it is Cauchy.
$\Longleftrightarrow$

Pf $\Leftarrow$) $\{x_k\}$ is Cauch $\longrightarrow$ $\{x_k\}$ bdd $\xrightarrow[1.19]{}$ a subsequence $\{x_{k_j}\}$ exist. That converges to some x.

Then $\{x_k\} \to x$