General Proofs

1. Let $a, b \in \mathbb{Z}$, let $d(a, b)$ be $\exists n \in \mathbb{N}, a = nb$.

   (a) Using our carefully structured form, prove or disprove:

   $$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, d((yz), x) \to (d(y, x) \wedge d(z, x))$$

   (b) Using our carefully structured form, prove or disprove:

   $$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, d(y, x) \wedge d(z, x) \to d((y + z), x).$$

2. Let n be a positive integer, and let A ben an $n \times n$ array. Let $D = \{0, 1, 2, \ldots, n-2, n-1\}$.

   (a) Consider the following sentence:

   $$S4(A) : \forall i \in D, \forall j \in D, 1 \leq A[i, j] \leq n$$
   $$\wedge \forall i \in D, \forall j \in D, \forall k \in D, A[i, j] = A[i, k] \to j = k$$
   $$\wedge \forall i \in D, \forall j \in D, \forall k \in D, A[j, i] = A[k, i] \to j = k$$

   Write a short description of S4(A) in clear, natural English.

   (b) Consider the following sentence:

   $$S5(A) : \forall i \in D, \forall j \in D, 1 \leq A[i, j] \leq n^2$$
   $$\wedge \exists x \in \mathbb{N}, \forall i \in D, (\sum_{k=0}^{n-1} A[k, i] = x) \wedge (\sum_{k=0}^{n-1} A[i, k]) = x.$$

   (c) Using our carefully structured form, prove or disprove:

      i. S4(A) imples S5(A).

      ii. S5(a) imples S4(A).

3. Consider the graph of $x^2 + y^2 = 1$. Prove that for any point (x, y) on the graph, if (x, y) is in the 1^{st} quadrant, then $x + y \geq 1$.

   - Rewrite the problem using quantifiers by specifying your domain and using the predicate $Q(x, n)$ representing x is in the n^{th} quadrant. You may represent the equations as they are (i.e., say $x + y \geq 1$.)
   - Give a carefully structured proof.

4. Prove or disprove. For all natural numbers a, b, c, if $a|b$ and $b|c$ then $a|(b^2 + c^2)$ where $x|y$ represents x *divides* y.

   - Rewrite the problem using quantifiers . You may represent the equations as they are (i.e., say $a|(b^2 + c^2)$.)
   - Give a carefully structured proof.

5. In this example, we will need two notions. An integer n is called a *perfect square* if there is another integer k such that $n = k^2$. For example, 13689 is a perfect square since 13689 = 117^2.

   The second idea is the *remainder* and *modular arithmetic*. For two integers m and n, $n \ \mathbf{mod}(m) = r$ will be the remainder resulting when we divide m into n. This means that there is an integer q such that $n = mq + r$. For example, 107 mod(29) = 11 since 29 will go into 107 4 times with a remainder of 11 (or, in other words, 107 = (4)(29) + 11). Determining whether or not a positive integer is a perfect square might be difficult. For example, is 82,642,834,671 a perfect square? First we compute 82,642,834,671 mod(4) = 3. Then use this theorem:

THEOREM 1. *If n is a positive integer such that $n \textbf{ mod}(4)$ is 2 or 3, then n is not a perfect square.*

(a) Prove Theorem 1 by first rewriting Theorem 1 in precise symbolic notation using the domain and predicates defined below:

- We denote the positive integers by $\mathbb{Z}^+$.
- $PS(x)$: x is a perfect square.
- $mod(x, y, z)$: $x \textbf{ mod}(y) = z$.

and then constructing a carefully structured proof.

[Hint: *We have learned three* proof methods*: direct proof, indirect proof (contrapositive), proof by cases. Use the method(s) that make(s) the proof the simplest.*]

(b) If $n \textbf{ mod} 4 = 0$ then what does this tell us about n and perfect squares?

6. Let $\mathbb{N}$ be the natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{Z}$ be the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and $\mathbb{R}$ be the real numbers. For $x \in \mathbb{R}$, define $r(x)$ as: $\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, (n > 0) \wedge (x = m/n)$. You may assume $\neg r(\sqrt{2})$.

   Using our structured proof form, prove or disprove the following:

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(x + y)$.

   (b) The converse of (a).

   (c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(xy)$.

   (d) The converse of (c).

   (a) $\forall x_1 \in \mathbb{R}, \forall x_2 \in \mathbb{R}, \forall y_1 \in \mathbb{R}, \forall y_2 \in \mathbb{R}, |x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1 y_1| > |x_2 y_2|$.

7. Let $\mathbb{R}^+$ be the set of positive real numbers. Use our structured proof form to prove or disprove:

   (a)
   $$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

   (b)
   $$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

   (c)
   $$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [-1, 1], \forall y \in [-1, 1], |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

   (d)
   $$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

8. Suppose f and g are functions from $\mathbb{R}$ onto $\mathbb{R}$. Consider the following statements:

   S1 $\quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (f(x) = f(y)) \Rightarrow (x = y)$.

   S2 $\quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (g(x) = g(y)) \Rightarrow (x = y)$.

   S3 $\quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (g(f(x)) = g(f(y))) \Rightarrow (x = y)$.

   Does (S1$\wedge$S2) imply S3? Prove your claim.

Sequences

1. Consider the following sentences about sequences, $a_0, a_1, a_2, \ldots$, of integers:

$$\begin{array}{ll} \text{(S1)} & \exists j \in \mathbb{N}, \forall i \in \mathbb{N}, i \neq j \rightarrow a_i \geq a_j \\ \text{(S2)} & \exists j \in \mathbb{N}, \forall i \in \mathbb{N}, i > j \rightarrow a_i < a_{i+1} \wedge i < j \rightarrow a_i \geq a_{i+1} \\ \text{(S3)} & \forall j \in \mathbb{N}, j > 0 \rightarrow (a_{2j} = -a_{4j} \wedge a_{2j+1} = -a_{2j+3}) \end{array}$$

   (a) For each sentence describe in clear, precise English the property that a sequence must have to satisfy the sentence.

   (b) For each sentence, give the *negation* of the sentence.

   (c) For each of the following sequences, determine whether the sequence satisfies each of the (S1), (S2) and (S3). Justify your claim in English using an example or counter example whenever possible.

   (A1) $10, 9, 8, 7, 9, 11, 13, 15, \ldots$

   (A2) $-1, 2, 1, -2, -1, 2, 1, -2, \ldots$

   (A3) $1, 2, 4, 8, 16, 32, \ldots$

   (d) Is there a sequence that satisfies all three of (S1), (S2) and (S3). If yes, give the sequence, if not, then explain why it is impossible.

2. Consider these sentences about sequence of numbers $a_0, a_1, a_2, \ldots$.

   (S1) $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, (j > i) \rightarrow (a_j \geq 2a_i)$ where each $a_i \in \mathbb{N}$.

   (S2) $\forall i \in \mathbb{N}, (i > 2) \rightarrow (a_i - a_{i-1} \leq a_{i-1} - a_{i-2})$ where each $a_i \in \mathbb{Z}$.

   (S3) $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, (i < j \rightarrow a_i < a_{i+1}) \wedge (j < i \rightarrow a_i > a_{i+1})$ where each $a_i \in \mathbb{Z}$.

   Consider these sequences and answer the questions below.

   (A) -10 -9 -8 -5 -2 0 100 99 97 96 95 94 . . .

   (B) 40 20 10 5 1 0 0 0 0 0 0 0 . . .

   (C) 0 2 5 11 23 47 100 200 400 900. . .

   (a) For each sentence express it's negation moving the negation as far in as possible.

   (b) For each sentence determine whether the sentence is true for each of (A), (B) and (C). If it is true, explain why and if it is not true, show where the sequence fails.

3. Consider the following statements about sequences of natural numbers $a_0, a_1, a_2, \ldots$:

$$(S3) \quad \forall i \in \mathbb{N}\; \exists j \in \mathbb{N}\; \forall k \in \mathbb{N}, k > j \rightarrow a_k \neq a_i$$

$$(S4) \quad \forall i \in \mathbb{N}\; \exists j \in \mathbb{N}, j > i \wedge (a_j > a_i \vee a_j < a_i)$$

$$(S5) \quad \forall i \in \mathbb{N}\; \Big(\big(\exists j \in \mathbb{N}, j > i \wedge a_j > a_i\big) \wedge \big(\exists j \in \mathbb{N}, j > i \wedge a_j < a_i\big) \Big)$$

And the following sequences:

$$(A1) \quad 1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 5, 6, \ldots$$

$$(A2) \quad 1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots$$

$$(A3) \quad 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 1, 1, \ldots$$

For each sequence and each statement, state whether the statement is true or false for the sequence and justify briefly.

4. (a) Prove that

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \rightarrow j < i$$

for the sequence

$$0, 1, 4, 9, 16, 25, \ldots (\forall n \in \mathbb{N}, a_n = n^2)$$

(b) Prove that

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \rightarrow a_j = a_i$$

for the sequence

$$0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, \ldots (\forall n \in \mathbb{N}, a_n = \lfloor n/2 \rfloor)$$

(c) Prove that

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \neq a_i \rightarrow \exists k \in \mathbb{N}, k \neq a_k \wedge a_k > a_j$$

for the sequence

$$0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, \ldots,$$

more formally the sequence is given by:

$$\forall n \in \mathbb{N}, a_n = \begin{cases} 0, & n \text{ is a multiple of 4.} \\ 1, & n - 1 \text{ is a multiple of 4} \\ 2, & n - 2 \text{ is a multiple of 4} \\ 3, & n - 3 \text{ is a multiple of 4} \end{cases}$$