First order systems of ODE's

$x_1, \dots, x_n$ dependent variables

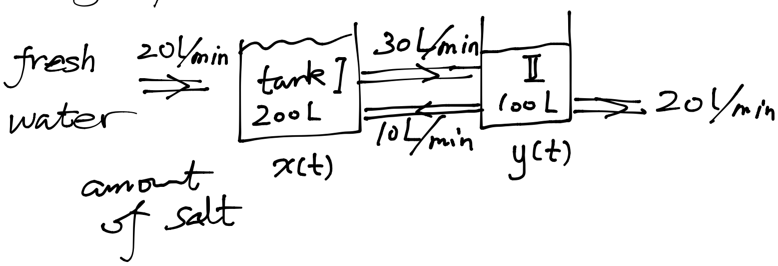
t independent variables

$$\begin{cases} x_1' = F_1(t, x_1, \cdots, x_n) \\ x_2' = F_2(t, x_1, \cdots x_n) \\ \vdots \\ x_n' = F_n(t, x_1, \cdots, x_n) \end{cases}$$

where F_i = functions

Shorthand: $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad \vec{F}(t, \vec{x}) = \begin{pmatrix} F_1(t, x_1, \cdots, x_n) \\ \vdots \\ F_n(t, x_1, \cdots, x_n) \end{pmatrix}$

Eg: Mixing problem



$x(t)$ = amount of salt in tank I $\Rightarrow$ concentration $\frac{x(t)}{200\,L}$

$y(t)$ = amount of salt in tank II $\Rightarrow$ concentration $\frac{y(t)}{100\,L}$

$$\frac{dx}{dt} = 0 \cdot \frac{g}{L} \cdot 200 \frac{L}{min} - \frac{x(t)}{200L} \cdot 30 \frac{L}{min} + \frac{y(t)}{100L} \cdot 10 \frac{L}{min}$$

$$\frac{dy}{dt} = \frac{x(t)}{200L} \cdot 30 \frac{L}{min} - \frac{y(t)}{100L}\left(10 \frac{L}{min} + 20 \frac{L}{min}\right)$$

$$\begin{cases} \frac{dx}{dt} = -\frac{3}{20}x + \frac{1}{10}y \\ \frac{dy}{dt} = \frac{3}{20}x - \frac{3}{10}y \end{cases}$$

In matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{3}{20} & \frac{1}{10} \\ \frac{3}{20} & -\frac{3}{10} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Eg 2: Reduction of higher order ODE's

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})$$

introduce new variables:

$$x_1 = y, \ x_2 = y', \ x_3 = y'', \ \cdots, \ x_n = y^{(n-1)}$$

Then we get a system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f(t, x_1, x_2, \cdots, x_n) \end{cases}$$

Thus, n-th order equations are, in a sense, a special case of systems of 1st order equations.

$$y^{(3)} + 3y'' - 5y' + 6y = \sin(t)$$

becomes a system $(x_1 = y, x_2 = y', x_3 = y'')$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = \sin(t) - 6x_1 + 5x_2 - 3x_3 \end{cases}$$

Example:

$$m\frac{d^2x}{dt^2} = F(t, x) \qquad (*)$$

Introduce $v = \frac{dx}{dt}$ velocity. Then $(*)$ is equivalent to

$$\frac{dx}{dt} = v$$

$$m\frac{dv}{dt} = F(t, x)$$

Theorem: (Existence & Uniqueness) $\qquad \vec{x}' = \vec{F}(t, \vec{x})$

Suppose $F_1, \cdots, F_n$ and its partial derivatives $\frac{\partial F_i}{\partial x_j}$ are continuous near given $(t_0, \vec{x}_0)$

Then the IVP

$$\vec{x}' = \vec{F}(t, \vec{x}), \quad \vec{x}(t_0) = \vec{x}_0$$

has a unique solution, defined for t near t_0.

Linear systems of equation

$$x_1' = P_{11}(t) + P_{12}(t)x_2 + \cdots + P_{1n}(t)x_n + g_1(t)$$

$$\vdots$$

$$x_n' = P_{n1}(t)x_1 + P_{n2}(t)x_2 + \cdots + P_{nn}(t)x_n + g_n(t)$$

Matrix notation :

$$P(t) = \begin{pmatrix} P_{11}(t) & \cdots & P_{1n}(t) \\ \vdots & & \vdots \\ P_{n1}(t) & \cdots & P_{nn}(t) \end{pmatrix}$$

$$g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

$$\boxed{\vec{x}' = P(t)\vec{x} + \vec{g}(t)}$$

Theorem : Suppose $P(t)$ and $\vec{g}(t)$ are continuous on interval $I \subset \mathbb{R}$, and $t_0 \in I$. Then the IVP.

$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$, has a unique solution, defined for $t \in I$.

$\vec{x}(t_0) = \vec{x}_0$