Question One

$$
{}_{0.25}p_0^{ss} = \exp\{-0.25\times(\hat{\mu}^{AB}+\hat{\mu}^{AC}+\hat{\mu}^{AD})\}
$$
$$
= \exp\{-0.25\times(\frac{25}{500}+\frac{50}{500}+\frac{120}{500})\} = 0.907
$$

$$
Var\left(\hat{\mu}^{AB}\right)=\frac{25}{500^2} \quad Var\left(\hat{\mu}^{AC}\right)=\frac{50}{500^2} \quad Var\left(\hat{\mu}^{AD}\right)=\frac{120}{500^2}
$$

$$
Var\left(\hat{\mu}^{AB}+\hat{\mu}^{AC}+\hat{\mu}^{AD}\right)=\frac{195}{500^2}
$$

Let $Y=\hat{\mu}^{AB}+\hat{\mu}^{AC}+\hat{\mu}^{AD}$ and using the delta method,

$$
Var\left(e^{-0.25Y}\right)\approx\left(-0.25e^{-0.25Y}\right)^2 Var\left(Y\right)=0.25^2\left(e^{-0.0975}\right)^2\frac{195}{500^2}
$$

Hence an approximate 95% CI is

$$
0.907\pm2\sqrt{0.25^2\left(e^{-0.0975}\right)^2\frac{195}{500^2}}=0.907\pm2\times0.0063
$$

Question Two

$$
{}_{t+dt}p_x^{12} = {}_t p_x^{11}\,{}_{dt}p_{x+t}^{12} + {}_t p_x^{12}\,{}_{dt}p_{x+t}^{22}
$$

[Looking at state occupied at age x+t]

$$
{}_{t+dt}p_x^{12} = {}_t p_x^{11}(\sigma dt+o(dt)) + {}_t p_x^{12}(1-{}_{dt}p_{x+t}^{21}-{}_{dt}p_{x+t}^{23})
$$

[Okay if you use $\approx$ in the above and ignore the term $o(dt)$]

$$
{}_{t+dt}p_x^{12} = {}_t p_x^{11}\sigma dt-(\rho+\nu)\,{}_t p_x^{12}dt+o(dt)
$$

$$
\Rightarrow \frac{{}_{t+dt}p_x^{12}-{}_t p_x^{12}}{dt} = {}_t p_x^{11}\sigma-(\rho+\nu)\,{}_t p_x^{12}+\frac{o(dt)}{dt}
$$

From here taking the limit as t approaches zero gives the required result.

Question Three

$$ {}_{t+dt}p_x^{\overline{gg}} = {}_{t}p_x^{\overline{gg}}\ {}_{dt}p_x^{\overline{gg}} $$

$$ = {}_{t}p_x^{\overline{gg}}\left(1-\sum_{r\neq g}{}_{dt}p_{x+t}^{gr}\right) $$

$$ = {}_{t}p_x^{\overline{gg}}\left(1-\sum_{r\neq g}\mu_{x+t}^{gr}dt\right) $$

$$ \frac{\partial}{\partial t}{}_{t}p_x^{\overline{gg}} = \lim_{dt->0}\frac{{}_{t}p_x^{\overline{gg}}\left(1-\sum_{r\neq g}\mu_{x+t}^{gr}dt\right)-{}_{t}p_x^{\overline{gg}}}{dt} $$

$$ = -{}_{t}p_x^{\overline{gg}}\sum_{r\neq g}\mu_{x+t}^{gr} $$

Now, using the above result we have:

$$ \frac{\frac{\partial}{\partial t}{}_{t}p_x^{\overline{gg}}}{{}_{t}p_x^{\overline{gg}}} = -\sum_{r\neq g}\mu_{x+t}^{gr} $$

$$ \therefore \frac{\partial}{\partial t}\ln\left({}_{t}p_x^{\overline{gg}}\right) = -\sum_{r\neq g}\mu_{x+t}^{gr} $$

$$ \therefore {}_{t}p_x^{\overline{gg}} = \exp\left(-\int_0^t\sum_{r\neq g}\mu_{x+s}^{gr}ds\right) $$