Worth: 3% **Due:** By 12 noon on Tuesday 7 February.

1. We have

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$P \wedge Q$	$Q \wedge R$	(a)	(b)	(c)	(d)	(e)
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F	F	F	F	F	F
T	F	T	F	T	T	F	F	T	T	F	T	F
T	F	F	F	F	T	F	F	T	T	F	T	F
F	T	T	T	T	T	F	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T	T	T	T	T
F	F	T	T	T	T	F	F	T	T	T	T	T
F	F	F	T	T	T	F	F	T	T	T	T	T

where

(a) $P \Rightarrow (Q \Rightarrow R)$
(b) $Q \Rightarrow (P \Rightarrow R)$
(c) $(P \Rightarrow Q) \wedge (P \Rightarrow R)$
(d) $(P \wedge Q) \Rightarrow R$
(e) $P \Rightarrow (Q \wedge R)$

Comparing columns (a) – (e), we see that

$$(P \Rightarrow (Q \Rightarrow R)) \iff (Q \Rightarrow (P \Rightarrow R)) \iff ((P \wedge Q) \Rightarrow R)$$

and

$$((P \Rightarrow Q) \wedge (P \Rightarrow R)) \iff ((P \Rightarrow (Q \wedge R)).$$

This was a rather tedious and error prone process that I hope motivates the use of derivations that use equivalence rules. You can easily check that each step in a derivation is correct. The number of rows in a truth table is 2^N, where N is the number of truth variables being considered. This number grows quickly with N. The number of steps in a derivation may not grow as quickly.

2.

$$\begin{aligned}
(P \Rightarrow Q) \vee (P \Rightarrow R) &\iff (\neg P \vee Q) \vee (\neg P \vee R) && \text{(implication rule)}\\
&\iff \neg P \vee (Q \vee \neg P) \vee R && \text{(associativity)}\\
&\iff \neg P \vee (\neg P \vee Q) \vee R && \text{(commutivity)}\\
&\iff (\neg P \vee \neg P) \vee (Q \vee R) && \text{(associativity)}\\
&\iff \neg P \vee (Q \vee R) && \text{(idempotency)}\\
&\iff P \Rightarrow (Q \vee R) && \text{(implication rule)}
\end{aligned}$$

3.

$$\begin{aligned}
(P \Rightarrow Q) \vee (Q \Rightarrow R) &\iff (\neg P \vee Q) \vee (\neg Q \vee R) && \text{(implication rule)}\\
&\iff \neg P \vee (Q \vee \neg Q) \vee R && \text{(associativity)}\\
&\iff \neg P \vee R \vee (Q \vee \neg Q) && \text{(commutivity)}\\
&\iff (\neg P \vee R) \vee (Q \vee \neg Q) && \text{(associativity)}\\
&\iff (Q \vee \neg Q) && \text{(an absorption rule)}
\end{aligned}$$

Since $(Q \vee \neg Q)$ is a tautology, and it is equivalent to $(P \Rightarrow Q) \vee (Q \Rightarrow R)$, we know that $(P \Rightarrow Q) \vee (Q \Rightarrow R)$ is a tautology.

4. Note that it was mentioned in lecture that you are allowed to use the "absorption" laws given on the tutorial 3 handout, namely $P \wedge (Q \wedge \neg Q) \iff Q \wedge \neg Q$ and $P \vee (Q \vee \neg Q) \iff Q \vee \neg Q$, but you are not allowed to use Velleman's Absorption laws (pg. 21) $P \iff P \wedge (P \vee Q)$ and $P \iff P \vee (P \wedge Q)$.

   There are many derivations that show that the statement is true. The following is a rather brute force approach that decomposes $\iff$ into two $\Rightarrow$.

$$
\begin{aligned}
P \iff P \wedge (P \vee Q) &\iff (P \Rightarrow (P \wedge (P \vee Q))) \wedge ((P \wedge (P \vee Q)) \Rightarrow P) && \text{(equivalence rule)}\\
&\iff (\neg P \vee (P \wedge (P \vee Q))) \wedge (\neg(P \wedge (P \vee Q)) \vee P) && \text{(implication rule)}\\
&\iff (\neg P \vee ((P \wedge P) \vee (P \wedge Q))) \wedge (\neg P \vee \neg(P \vee Q)) \vee P) && \text{(distributivity,}\\
& && \text{de Morgan's rule)}\\
&\iff (\neg P \vee (P \vee (P \wedge Q))) \wedge (\neg P \vee P \vee \neg(P \vee Q)) && \text{(commutivity)}\\
&\iff ((\neg P \vee P) \vee (P \wedge Q))) \wedge ((\neg P \vee P) \vee \neg(P \vee Q)) && \text{(associativity)}\\
&\iff (\neg P \vee P) \wedge (\neg P \vee P) && \text{(tutorial absorption)}\\
&\iff (\neg P \vee P) && \text{(idempotency)}
\end{aligned}
$$

   Since $(\neg P \vee P)$ is a tautology, and it is equivalent to $P \iff P \wedge (P \vee Q)$, it follows that $P \iff P \wedge (P \vee Q)$ is a true statement.

5. (a) Using words: Given that each student is telling the truth,

   - Xavier's statement tells us that William cheated and that Zachary did not cheat.
   - Since Zachary did not cheat, William's statement tells us that Xavier did not cheat. (Otherwise the implication would be false.)
   - Youssef's statement tells us that Youssef did not cheat. The last part of his statement is consistent with the knowledge that William cheated and Zachary did not.
   - Since William did cheat, Zachary's statement is vaccuously true and does not tell us anything about Youssef.

   We are left with the conclusion that only William cheated on his assignment. Xavier, Youssef and Zachary did not cheat.

   Using logic: We can represent the students' statements as:

$$X \Rightarrow Z, \qquad W \wedge \neg Z, \qquad \neg Y \wedge (W \vee Z), \qquad \neg W \Rightarrow Y$$

   Conjoining the students' statements gives

$$
\begin{aligned}
&(X \Rightarrow Z) \wedge (W \wedge \neg Z) \wedge (\neg Y \wedge (W \vee Z)) \wedge (\neg W \Rightarrow Y)\\
&\iff (\neg X \vee Z) \wedge (W \wedge \neg Z) \wedge (\neg Y \wedge (W \vee Z)) \wedge (W \vee Y) && \text{(implication rule)}\\
&\iff (\neg X \vee Z) \wedge W \wedge \neg Z \wedge \neg Y \wedge (W \vee Z) \wedge (W \vee Y) && \text{(associativity)}
\end{aligned}
$$

   For this statement to be true, we can see immediately that W must be true and Z and Y must be false. For the first clause to be true, X must be false. With this information, the last two clauses are also true.

   We can conclude again that William cheated and the others did not.

(b) In this case, we do not know, at first, which statements are true and which statements are false. We only know that cheaters make false statements and students who did not cheat make true statements.

To decide who cheated and who did not cheat, let's make the assumption that William cheated. Then W is true and the statement $X \Rightarrow Z$ is false. This tells us that X is true (Xavier cheated) and Z is false (Zachary did not cheat). If Xavier cheated, then his statement $W \wedge \neg Z$ must be false. But, we know W is true and Z is false, and so Xavier's statement is true. Our assumption that William cheated has lead us to a contradiction, and so it must be the case that William did not cheat, and W is false.

If William did not cheat, it follows that Xavier's statement is false, from which we can conclude that Xavier cheated. Also, since William did not cheat, his statement is true. And given that we know Xavier cheated, it follows that Zachary cheated too.

Since we know that Zachary cheated, we know that his statement is false. The negation of $\neg W \Rightarrow Y$ is $\neg W \wedge \neg Y$. We can conclude that Youssef did not cheat.

To confirm the consistency of our argument we should check Youssef's statement, $\neg Y \wedge (W \vee Z)$. Since Y is false and Z is true, the statement is true, as it must be, since Youssef told the truth.

We conclude then that William and Youssef did not cheat, but that Xavier and Zachary did cheat.