§3.1 Duality

Definitions: A linear programming problem is in primal standard form provided it is in standard form according to Kolman and Beck, bottom of page 51: a maximization problem with $\leq$ constraints only, except that each decision variable is ≥ 0.

A general linear programming problem (see pg. 51 again) is in dual standard form provided it is a minimization problem with $\leq$ constraints only including: each decision variable is ≥ 0

Main definition:

Given a problem in primal standard form:

Maximize $z = c^T x$ s.t.

$$Ax \leq b,\ x \geq 0 \in \mathbb{R}^n$$

(where $b \in \mathbb{R}^m$, A^T is $n \times m$, $c \in \mathbb{R}^n$, $w \in \mathbb{R}^m$)

Theorem 3.1: Given a primal problem, the dual of its dual problem is again the primal problem.

Proof: in one instance

Let the given primal problem be

$$\text{Maximize } z = x_1 + 2x_2 \text{ s.t.}$$
$$3x_1 - 4x_2 \leq -6$$
$$-5x_1 + 7x_2 \leq 0$$
$$9x_2 \leq 8,\ x_1 \geq 0,\ x_2 \geq 0$$

Its dual is:

$$\text{Minimize } z' = -6w_1 + 8w_3 \text{ s.t.}$$
$$3w_1 - 5w_2 \geq 1$$
$$-4w_1 + 7w_2 + 9w_3 \geq 2 \qquad w_1 \geq 0,\ w_2 \geq 0,\ w_3 \geq 0$$

In primal standard form:

$$\text{Maximize } z'' = 6w_1 - 8w_3 \text{ s.t.}$$
$$-3w_1 + 5w_2 \leq -1$$
$$4w_1 - 7w_2 - 9w_3 \leq -2,\ w_1 \geq 0,\ w_2 \geq 0,\ w_3 \geq 0$$

Again by the definition, the dual of this problem is

Minimize $z''' = -u_1 - 2u_2$ s.t.

$$-3u_1 + 4u_2 \geq 6$$
$$5u_1 - 7u_2 \geq 0$$
$$-9u_3 \geq -8, \quad u_1 \geq 0, u_2 \geq 0, u_3 \geq 0$$

Putting this in primal standard form (with $u_1 = x_1, u_2 = x_2$) yields the original primal problem.

Ex: To find the dual of

Maximize $z = 2x_1 - x_2$ s.t.

$$3x_1 + 4x_2 \geq 5$$
$$6x_1 + 7x_2 \geq 8, \quad x_1 \geq 0, x_2 \geq 0$$

One could put it in dual standard form

Minimize $z' = -2x_1 + x_2$ s.t.

$$3x_1 + 4x_2 \geq 5$$
$$6x_1 + 7x_2 \geq 8, \quad x_1 \geq 0, x_2 \geq 0$$

Then using theorem 3.1 write the problem for which the best problem is the dual.

Maximize $z'' = 5w_1 + 8w_2$ s.t.

$$3w_1 + 6w_2 \leq -2$$
$$4w_1 + 7w_2 \leq 1, \quad w_1 \geq 0, w_2 \geq 0$$

Remark: In a dual pair of problems, each constraint of one problem is associated with a decision variable of the other and vice versa.

Theorem (Theorems 3.2 and theorem 3.3, generalized

In a dual pair of problems, each equality constraint in are problem is associated with a unrestricted variable and vice versa.