Feb 12th

## Diagonalization

Let $F$ be a field

Def: A matrix $X \in M_{n\times n}(F)$ is called diagonalizable if there exists an invertible matrix $A \in M_{n\times n}(F)$, such that $AXA^{-1}$ is a diagonal matrix.

Def: Let $V$ be a finite-dimensional vector space over $F$, and Let $T: V \to V$ be a linear transformation. We call $T$ diagonlizable if there exists a basis $\alpha$ of $V$ s.t. $[T]_\alpha^\alpha$ is a diagonal matrix

Propostion: Let $V$ be a finite-dimensional vector space over $F$, and let $\alpha$ be any basis of $V$. A linear transformation $T: V \to V$ is diagonalizable $\Longleftrightarrow [T]_\alpha^\alpha$ is a diagonalizable Matrix

Proof: ($\Rightarrow$) Assume that $T$ is diagonalizable. There exists a basis $\beta$ of $V$ such that $[T]_\beta^\beta$ is diagonal. Note that $[T]_\beta^\beta = [I_V]_\alpha^\beta [T]_\alpha^\alpha [I_V]_\beta^\alpha = [I_V]_\alpha^\beta [T]_\alpha^\alpha ([I_V]_\alpha^\beta)^{-1} \Rightarrow [T]_\alpha^\alpha$ is a diagonalizable matrix.

Why are $[I_V]_\alpha^\beta$ and $[I_V]_\beta^\alpha$ inverse matrices?

$$[I_V]_\alpha^\beta [I_V]_\beta^\alpha = [I_V \circ I_V]_\beta^\beta = [I_V]_\beta^\beta = I$$

$$[I_V]_\beta^\alpha [I_V]_\alpha^\beta = [I_V \circ I_V]_\alpha^\alpha = [I_V]_\alpha^\alpha = I$$

($\Leftarrow$) Assume that $[T]_\alpha^\alpha$ is a diagonalizable matrix. There exists an invertible matrix $A \in M_{n\times n}(F)$ ($n = \dim V$) such that $A[T]_\alpha^\alpha A^{-1}$ is diagonal

Write $A^{-1} = (C_1, \cdots, C_n)$. Define $V_1, \cdots, V_n \in V$ to be such that $[V_i]_\alpha = C_i$

Consider $\beta = \{V_1, \cdots, V_n\}$

We want to prove that $\beta$ is a basis of $V$. Since $n = \dim V$ and since $\beta$ consists of $n$ vectors, it suggests to prove that $\beta$ is linearly independent.

$$aV_1 + \cdots + a_nV_n = 0$$

Take coordinates of both sides

$$[a_1V_1 + \cdots + a_nV_n]_\alpha = [0]_\alpha = 0$$

$$a_1[V_1]_\alpha + \cdots + a_n[V_n]_\alpha = 0$$

$$a_1C_1 + \cdots + a_nC_n = 0$$

Since $A$ is an invertible matrix, so is $A^{-1}$. So the columns of $A^{-1}$ are lin. indp $\Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow V_1, \cdots, V_n$ are linearly indp. and so $\beta$ is a basis.

By construction, $A^{-1} = [I_V]_\beta^\alpha$ (Reason: $[I_V]_\beta^\alpha = \cdot\ \cdot\ \cdot$ )

$\cdot\ -\ \cdot\ - \Rightarrow$ diagonalizable

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Section 4.1 (a)

$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ Is $A$ diagonalizable? (over $\mathbb{R}$)

$$P_A(t) = \det\left(\begin{pmatrix} 3-t & 1 \\ -1 & 1-t \end{pmatrix}\right) = (3-t)(1-t)+1 = t^2 - 4t + 4 = (t-2)^2$$

$E_2$: $E_2 = \text{null}(A - 2I) = \text{null}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right)$

$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x+y=0 \quad$ i.e. $\; y=-x$

$E_2=\{(x,-x): x\in\mathbb{R}\}=\text{span}\{(1,-1)\}$

So, $E_2$ is 1-dimensional

So the sum of the dimensions of the eigenvalues is less than the dim of the matrix $\Rightarrow A$ is not diagonalizable.