University of Toronto
MAT237Y1Y MIDTERM TEST
Wednesday, Dec. 19, 2012
Duration: 3 hours, No aids allowed

Instructions: There are 17 pages including the cover page. Please answer all the questions. Total mark to be earned is 115 but the test is out of 100. Please note that any proof must be documented by at least roughly quoting any results used in the proof.

NAME: (last, first) MARKING SCHEME

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

CHECK YOUR TUTORIAL:

○ T0101	○ T0201	○ T0301	○ T0401	○ T5101	○ T5201	○ T5301
Mon. 3-4	Mon. 4-5	Tue. 2-3	Wed. 3-4	Tue. 5-6	Wed. 5-6	Thu. 5-6

MARKER'S REPORT:

Question	MARK
Q1	/10
Q2	/14
Q3	/16
Q4	/13
Q5	/19
Q6	/16
Q7	/10
Q8	/17
TOTAL	/115

Ali

1. Topology

a) (3 marks) Show that the open ball $B(r, 0)$ in $\mathbb{R}^n$ is convex.

given two points $a, b \in B(r, 0)$ The line Segment

$|a-0|<r$ & $|b-0|<r$

Connecting a to b is $L: a+t(b-a)$ $t \in [0\ 1]$ ①

or $(1-t)a+tb$.

Now $|(1-t)a+tb| \leq |(1-t)a|+|tb| = (1-t)|a|+t|b| < (1-t)r+tr$

$= r$ ②

so any pt x on The line L Satisfie $|x|<r$ so $x \in B(r, 0)$

b) (2 marks) Show that for any real number v, the set $S = \{x : x \neq v\}$ is an open subset of $\mathbb{R}$.

take $x \neq v$ and let $r = \frac{|x-v|}{2}$ Then $v \notin B(r, x)$, so

$B(r, x) \subset S$ so S is open ②

c) (2 marks) Prove that the intersection of any two open subsets of $\mathbb{R}^n$ is open.

let U and V be open subsets of $\mathbb{R}^n$ and Choose $x \in U \cap V$.

① $x \in U \Rightarrow \exists\, r > 0$ s.t. $B(r,x) \subset U$ } $\Rightarrow B(r_0,x) \subset U \cap V$

$x \in V \Rightarrow \exists\, \delta > 0$ s.t. $B(\delta,x) \subset V$ } let $r_0 = \min\{r,\delta\}$

so $x \in (U \cap V)^{int}$ ∴ $U \cap V$ is open ①

d) (3 marks) Assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and that S is an open subset of $\mathbb{R}^n$. Show that $S_1 = \{x \in S : f(x) \neq v\})$ is also open, where v is any real number.

Note $S_1 = f^{-1}(\{\cancel{y} : y \neq v\}) \cap S$ ①

(0.5) by (b) $\{y : y \neq v\}$ is open

(0.5) Since f is Cont. $f^{-1}(\text{open})$ is open, so $f^{-1}(\{y : y \neq v\})$ is open

Since S is open by (c) $f^{-1}(\{y : y \neq v\}) \cap S$ is open.

①

Yuri

2. Differentiability

a) (5 marks) For a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ give the definition of differentiability at a point $\boldsymbol{a}$. Prove that if f is differentiable at $\boldsymbol{a}$, then it must be continuous there.

f is differentiable at a if $\exists\, \mathbb{C} \in \mathbb{R}^n$ s.t.

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - \mathbb{C}\cdot h}{|h|} = 0$$ ① . This implies

Reason: Either 1. multiply both sides by $|h|$ … or 2. by contradiction, if $\neq 0$ Then $\lim \frac{}{|h|} \to \infty$.

$$\lim_{h\to 0} f(a+h) - f(a) - \mathbb{C}\cdot h = 0$$ ①

but Since $\lim_{h\to 0} \mathbb{C}\cdot h = 0$ (1.5)

Reason: $|\mathbb{C}\cdot h| \leq |\mathbb{C}|\,|h|$ so as $h \to 0$ $|h| \to 0$ and $|\mathbb{C}|\,|h| \to 0$ so $\mathbb{C}\cdot h \to 0$

Cauchy inequality

Then $\lim_{h\to 0} f(a+h) - f(a) = 0$ (1.5) $\therefore \lim_{h\to 0} f(a+h) = f(a)$ $\therefore$ f is Cont at a

b) (4 marks) Suppose that G is a differentiable function on some open set $U \subset \mathbb{R}^3$, and let $S = \{\boldsymbol{x} \in \mathbb{R}^3 : G(\boldsymbol{x}) = 0\}$ be a smooth surface. If $\boldsymbol{a} \in S$, and $\nabla G(\boldsymbol{a}) \neq \boldsymbol{0}$ then show that the vector $\nabla G(\boldsymbol{a})$ is perpendicular to the surface S at $\boldsymbol{a}$.

$\nabla G(a)$ is perpendicular to the surface S ~~at~~ a means $\nabla G(a)$ is orthogonal to every vector that is tangent to S at a. ①

Any vector that is tangent to S at a is tangent to a The graph of smooth curve $g(t)$ that passes through a. ① say $g(t)$ lies on S and passes through a, Then $G(g(t)) = 0$ and $g(t_0) = a$. Then $\frac{d}{dt} G(g(t))\Big|_{t=t_0} = 0$

~~and~~ by chain rule LHS $= \underset{\nabla G(a)}{\nabla G(g(t_0))} \cdot \underset{\text{tangent vector}}{g'(t_0)} = 0$ ①

so $\nabla G(a) \perp g'(t_0)$ ← which is tangent to S at a ①

c) (5 marks) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable at the point $\boldsymbol{a} = (x_0, y_0)$ and $\nabla f(\boldsymbol{a}) = [c_1, c_2]^T$. Express the tangent plane to the graph of f at $\boldsymbol{a}$ in terms of this information. What is the linear approximation of the value of $f(1 + x_0, y_0 - 2)$? What it the linear approximation of $f(1 + x_0, y_0 - 2)$ if the point $\boldsymbol{a} = (x_0, y_0)$ is a critical point of f?

tangent plane to the surface $z = f(\mathbf{x})$ at a is

$\nabla f(a) \cdot (\mathbf{x} - a) = 0$ so $[c_1 \; c_2] \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = 0$ or $c_1(x - x_0) + c_2(y - y_0) = 0$

1.5

linear approx of $f(1 + x_0, y_0 - 2) = f(a) + \nabla f(a) \cdot h =$

$\nearrow$

$(x_0, y_0) + (1, -2)$

$a + h$

$f(x_0, y_0) + [c_1 \; c_2] \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ 1.5

$= f(x_0, y_0) + c_1 - 2c_2$

If a is a critical pt of f, since f is differentiable at a then $\nabla f(a) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so linear approximation 1

to $f(x_0 + 1, y_0 - 2) = f(x_0, y_0) + [0 \; 0] \begin{bmatrix} 1 \\ -2 \end{bmatrix} = f(x_0, y_0)$.

1

3. Optimization

a) (6 marks) State the chain rule for the composite function $f \circ g$ where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^n$, then apply it to show that if the function $f(x, y, z)$ has a local extreme at a point $\boldsymbol{a}$ on the surface determined by $G(x, y, z) = 0$ (both functions differentiable), then $\nabla f(a)$ and $\nabla G(a)$ must be parallel.

Suppose $g(a) = b$ and g is diff at a and f is differentiable at b. ①

Then $\varphi(t) = f(g(t))$ is diff at a and $\varphi'(a) = \nabla f(b) \cdot g'(a)$. ①

assume f has a local extreme at a and f is differentiable There.

Then Consider $h(t)$ a curve on the Surface passing through a. (say $h(0) = a$)

Define $\varphi(t) = f(h(t))$ and note That $\varphi(t)$ has a local extrem ① at 0, and φ is differentiable by Chain rule, so $\varphi'(0) = 0$; but ① $\varphi'(0) = \nabla f(a) \cdot h'(0)$. Since h is arbitrary Then $\nabla f(a)$ is perp to any tangent vector at a. ① By 2(b) $\nabla G(a)$ is also perp to The Surface

Then $\nabla f(a) \parallel \nabla G(a)$ ①

b) (5 marks) Consider the curve of intersection of the plane $x + z = 1/2$ and the cylinder $x^2 + y^2 = 1$. Use Lagrange multipliers (with two constraints) to determine the point(s) on this curve closest to the origin.

two Constraints are $F(x,y,z) = x + z - \frac{1}{2} = 0$ (1.5)

$G(x,y,z) = x^2 + y^2 - 1 = 0$

and $f(x,y,z) = (x-0)^2 + (y-0)^2 + (z-0)^2 = x^2 + y^2 + z^2$

Lagrange multipliers: Solve The System $\begin{cases} \nabla f(x) = \lambda \nabla F(x) + \mu \nabla G(x) \\ F(x) = G(x) = 0 \end{cases}$ ①

That is $\begin{cases} 2x = \lambda + 2x\mu \\ 2y = 2y\mu \\ 2z = \lambda \\ x + z - \frac{1}{2} = 0 \ \& \ x^2 + y^2 - 1 = 0 \end{cases}$

→ if $y \neq 0$ Then $\mu = 1 \Rightarrow \lambda = 0 \Rightarrow z = 0$

① $\Rightarrow x = \frac{1}{2} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

so two points are $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 0)$

but if $y = 0$ Then $x = \pm 1 \Rightarrow z = -\frac{1}{2}$ or $z = \frac{3}{2}$

Four points are $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $(1, 0, -\frac{1}{2})$, $(-1, 0, \frac{3}{2})$

distances to The origine are $= 1$, $= 1$, $= \frac{5}{4}$, $= \frac{13}{4}$

(1.5) ← Shortest distance

6/17

c) (5 marks) Now consider the region of the plane $x + z = 1/2$ inside the cylinder $x^2 + y^2 = 1$ and denote it by S. Find the ~~points~~ on S closest to the origin.

minimize $(\text{distance})^2 = x^2+y^2+z^2 = x^2+y^2+(\frac{1}{2}-x)^2 = 2x^2+y^2-x+\frac{1}{4} = f(x,y)$ ①

$z = \frac{1}{2} - x$

on The region $x^2+y^2 < 1$ ① find critical pts:

$\nabla f(x) = 0$ implies $\begin{bmatrix} 4x-1 \\ 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which implies $\begin{cases} x = \frac{1}{4} \\ y = 0 \end{cases}$ ①

so The pt $(\frac{1}{4}, 0, \frac{1}{4})$ is a candidate for shortest distance.

distance is $\frac{1}{16}+\frac{1}{16} = \frac{1}{8} <$ distance on The boundary (1) from part (b)

②

Louis philippe

4. Chain rule

a) (4 marks) Let f and g be differentiable. Use the chain rule for multivariate functions to find the derivative of $\phi(x) = f(x)^{g(x)}$

Let $h(x_1, x_2) = x_1^{x_2}$ ① $\phi'(x) = \frac{\partial h}{\partial x_1} \cdot \frac{dx_1}{dx} + \frac{\partial h}{\partial x_2} \frac{dx_2}{dx} =$

$x_1 = f(x)$

$x_2 = g(x)$

① $= \left(x_2 x_1^{x_2 - 1}\right) f'(x) + \ln x_1 \, x_1^{x_2} g'(x)$

$= g(x) f(x)^{g(x)-1} f'(x) + \ln \cancel{x} f(x) \, f(x)^{g(x)} \cdot g'(x)$

②

b) (4 marks) Let $w = f(x, y, z, v))$ where $v = g(x, u)$ and $u = h(x)$. Write a formula for $\frac{\partial w}{\partial x}$ in terms of derivatives of f, g and h.

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial f}{\partial x_4} \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_4} \left(\frac{\partial g}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial g}{\partial x_2} \frac{\partial u}{\partial x} \right)$$

① ①

$$= \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_4} \left(\frac{\partial g}{\partial x_1} + \frac{\partial g}{\partial x_2} \frac{dh}{dx} \right)$$

②

c) (5 marks) Two surfaces, defined by $F(x,y,z) = xyz^2 - \log(z-1) - 8 = 0$ and $G(x,y,z) = x - 2y = 0$ intersect in a curve C. At the point $\boldsymbol{a} = (-2,-1,2)$ on this curve determine the equation of plane normal to the curve (that is the plane that is perpendicular to the curve.) (Hint: note that the curve C lies in both surfaces, and now you can use the result in 2(b))

$$\nabla F(a) = \begin{bmatrix} yz^2 \\ xz^2 \\ 2xyz - \frac{1}{z-1} \end{bmatrix} \Bigg|_{\text{at } a} = \begin{bmatrix} -4 \\ -8 \\ 8 - \frac{1}{1} \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 7 \end{bmatrix}$$

①

$$\nabla G(a) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$\nabla F(a)$ $\nabla G(a)$ Span The plane normal to The curve C. ②

normal to The plane can be found by Cross product of $\nabla F(a)$ and $\nabla G(a)$

①

$$\nabla F(a) \times \nabla G(a) = \begin{bmatrix} i & j & k \\ -4 & -8 & 7 \\ 1 & -2 & 0 \end{bmatrix} = 14i + 7j + 16k$$

$$\begin{bmatrix} 14 \\ 7 \\ 16 \end{bmatrix}$$

so equation of The plane is $14(x+2) + 7(y+1) + 16(z-2) = 0$

①

FULGENCIO

5. Taylor polynomials

a) (6 marks) Use the Taylor polynomials for e^x and $\frac{1}{1+x}$ to determine Taylor polynomial of degree 2, near $\boldsymbol{a} = (0,0)$ for the function $\frac{e^{x-2y}}{1+x^2-y}$ (quote any theorem that you are using.)

e^x : $P_{a,2}(x) = 1 + x + \frac{x^2}{2}$

$\frac{1}{1+x}$: $P_{a,2}(x) = 1 - x + x^2$

e^{x-2y} : $P_{a,2}(x,y) = 1 + (x-2y) + \frac{(x-2y)^2}{2}$

(2)

$\frac{1}{1+(x^2-y)}$: $P_{a,2}(x.y) = 1 - (x^2-y) + (x^2-y)^2$

has power 3 & 4 That should be excluded . . .

$\frac{e^{x-2y}}{1+x^2-y}$

multiply

$P_{a,2}(x.y) = 1 - (x^2-y) + (x-2y) + \frac{(x-2y)^2}{2} + (x^2-y)^2$

$-(x^2-y)(x-2y)$

(2)

The rest of The terms are of higher order

$= 1 + (x - 3y - x^2 + \left(\frac{x^2}{2} + \frac{4y^2}{2} - \frac{4xy}{2}\right) + y^2 + \cancel{x^4 - 2x^2y}$

$-2y^2 + xy$

higher order

$-x^3 + 2x^2y \ldots$

$$\boxed{= 1 + x - 3y + \frac{1}{2}x^2 + 8y^2 - 4xy}$$

(2)

b) (5 marks) Use the multi index notation to express the Taylor polynomial of degree 3 with Lagrange remainder for the function $f(x,y,z) = x^2yz$ near the point $(1,2,3)$.

$|\alpha| = 0$ $\alpha = (0,0,0)$ $f(1,2,3) = 6$

$|\alpha| = 1$ $\alpha = (1,0,0), (0,1,0), (0,0,1)$ $\partial^\alpha f(a)$: $\partial_x f = 12$ $\partial_y f = 3$ $\partial_z f = 2$

$|\alpha| = 2$ $\alpha = (2,0,0), \cancel{(0,2,0)}, \cancel{(0,0,2)}$
$(1,1,0), (1,0,1), (0,1,1)$

$\frac{\partial^\alpha f}{\alpha!}$ are $\frac{2yz}{2!}, \frac{2zx}{1!}, \frac{2yx}{1!}, \frac{x^2}{1!}$

(1) (1) which are 6 6 4 1

$|\alpha| = 3$ $\alpha = \cancel{(3,0,0)}, \cancel{(0,3,0)}, \cancel{(0,0,3)}$
$(2,1,0)$ $(2,0,1)$ $\cancel{(1,2,0)}$
$\cancel{(0,2,1)}, \cancel{(0,1,2)}, \cancel{(1,0,2)}$
$(1,1,1)$

$\frac{\partial^\alpha f}{\alpha!}$ are $\frac{2z}{2!}, \frac{2y}{2!}, \frac{2x}{1!}$
$\to 3$ $\to 2$ $\to 2$

$P_{a,3}(h) = 6 + (12h_1 + 3h_2 + 2h_3) +$
(1.5) $(6h_1^2 + 6h_1h_2 + 4h_1h_3 + h_2h_3) +$
$(3h_1^2h_2 + 2h_1^2h_3 + 2h_1h_2h_3)$

$|\alpha| = 4$ the only non-trivial α is :

$(2,1,1)$, for which $\frac{\partial^\alpha f}{\alpha!} = \frac{2}{2!} = 1$

So $R_{a,3}(h) = \frac{\partial^\alpha f(c)}{\alpha!} h_1^2h_2h_3 = h_1^2h_2h_3$ (1.5)

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FULGENCIO

c) (8 marks) Find all the critical points of the function f from part (b). Classify the point $(1,0,0)$. If there are directions along which $(1,0,0)$ is a local max or a local min, exhibit one such direction in each case.

$$\nabla f(\mathbb{x}) = \mathbb{0} \Rightarrow \begin{bmatrix} 2xyz \\ x^2 z \\ x^2 y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x=0 \,,\ y, z \text{ arbitrary} \\ \text{or } x \neq 0 \text{ but } y=0=z \end{array}$$

2

So two groups of critical points: $(0, a, b)$ $a, b \in \mathbb{R}$

and $(a, 0, 0)$ $a \in \mathbb{R}$

at $\mathbb{a} = (1,0,0)$ $\nabla f(\mathbb{a}) = \mathbb{0}$ and $\mathbb{H}(\mathbb{a}) = \begin{bmatrix} 2yz & 2xz & 2xy \\ 2xz & 0 & x^2 \\ 2xy & x^2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ 1

Calculate eigen values $\det \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix} = 0$

$\lambda(\lambda^2 - 1) = 0 \Rightarrow \begin{array}{l} \lambda = 0 \\ \lambda = 1 \\ \lambda = -1 \end{array}$ 1

not all $\lambda \geq 0$
nor all $\lambda \leq 0$
so no local min or max

b/c The necessary condition fails by Thm 2.81 1

to find direction:

eigen vectors

$\lambda = 1$ Solve $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbb{x} = \mathbb{0}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ solution is $\begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ 1

↑ parameter

along $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\mathbb{a}$ is a local min. b/c $[0\ t\ t]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = 2t^2$ so $f(\mathbb{a} + \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}) = f(\mathbb{a}) + 2t^2$ 0.5

$\lambda = -1$ Solve $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \mathbb{x} = 0$ .... Sln is $\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$, so along $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ we have 1

a local max b/c $[0\ -t\ t]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = [0\ t\ -t]\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = -2t^2$. 0.5

Nikita

6. Completeness Axiom

a) (3 marks) State the completeness axiom (for the real numbers.) Also give the ϵ characterization of the *glb*.

Completeness axiom: any bounded above non-empty set S has a lub. ①
(or a version about glb)

a is glb of S if a is a lower bound and $\forall \epsilon > 0 \ \exists s \in S$ st $a \le s < a + \epsilon$. ②

b) (5 marks) Prove that any bounded below monotone decreasing sequence $\{x_k\}$ of reals converges to its *glb*.

let $S = \{x_k\}$ be bdd below, ~~So~~ according to completeness axiom

There is $a \in R$ st a satisfies : $\forall \epsilon > 0 \ \exists s \in S$ st $a \le s < a + \epsilon$.

① given an arbitrary $\epsilon > 0$ choose $s \in S$ ($s = x_K$ for some K) That satisfies $a \le x_K < a + \epsilon$ ②

Since $\{x_k\}$ is monotone decreasing, Then $\forall k > K$ we have $a \le x_k \le x_K < a + \epsilon$

$\therefore \ 0 \le x_k - a < \epsilon \ \therefore$ ②

$\{x_k : n = 1 \dots\} \longrightarrow a$:

Nikita

c) (8 marks) Prove that every Cauchy sequence in $\mathbb{R}^n$ is convergent. Explain how completeness axiom for $\mathbb{R}$ has been fundamentally involved in this process.

0.5 1. Completeness axiom for $\mathbb{R}$ implies Monotone Sequence Theorem for $\mathbb{R}$ (MST)

0.5 2. MST implies nested interval Theorem (NIT) in $\mathbb{R}$ as in (b)

0.5 3. (BWI) NIT was used to show, in $\mathbb{R}$, any bdd Sequence of numbers has a converging Sub-sequence.

1 4. (BWII) This property was extented to $\mathbb{R}^n$: any bdd Sequence $\{x_k\} \subset \mathbb{R}^n$ has a Convergent Subsequence (item 3 (BWI) was used iteratively to The Sequence of $\{x_{1k}\}$, Then $\{x_{2k}\}$ etc of Components of $\{x_k\}$)

5. 0.5 property 4 is now used in The proof of every Cauchy Sequence in $\mathbb{R}^n$ converges as follows:

a) 2 any Cauchy Sequence $\{x_k\}$ is bounded, ~~By~~ $\{x_k\} \subset B(M, 0)$ where $M = \max\{|x_1| \ldots |x_{K-1}|, |x_K| + 1\}$ where K satisfies $\forall_{l,k} > K \; |x_k - x_l| < 1$.

b) 1 by (BWII) $\{x_k\}$ has a Convergent Subsequence $\{x_{k_j}\}$, to l.

c) $\therefore$ $\{x_k\}$ also Converges to l: $\forall \epsilon > 0 \; \exists J$ s.t. $\forall j > J$ $|x_{k_j} - l| < \epsilon/2$

also x_k is Cauchy, so $\exists K$ s.t. $\forall k, m > K \; |x_k - x_m| < \epsilon/2$

2 Then $\forall k > K$ $|x_k - l| \leq |x_k - x_{k_j}| + |x_{k_j} - l|$ $< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

j is Such That $k_j > K$

~~1~~ 2 3 4 5 6

~~0101~~
~~0201~~
~~0301~~
~~5301~~
Assort
S201
~~0401~~
~~S101~~

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Yannis

7. Continuity

a) (1 mark) Give the defintion of continuity for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a point $a \in \mathbb{R}$.

f is Cont. at a if $\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \;\; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

① ①

b) (3 marks) Consider $f(x) = \lfloor x \rfloor$, the greatest ingeger less than or equal to x, and let $S = [0, 2]$. Show that f is not continuous on S by presenting an example of a sequence $\{x_k\}$ in S that convereges to a point a in S, but that the sequence $f(\{x_k\})$ fails to converge to $f(a)$.

let $x_k = 1 - \frac{1}{k}$. Clearly $x_k \longrightarrow 1$ as $k \to \infty$

$f(x_k) = 0$ forall k. but $f(1) = 1$.

or any other example

③

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c) (6 marks) Let $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Assume that for any sequence $\{x_k\}$ in S which converges to a, the sequence $f(\{x_k\})$ also converges to $f(a)$. Show that ~~if~~ f must be continuous at a.

(✯) assume for any Sequence $\{x_k\}$ which Converges to a the Sequence $\{f(x_k)\}$ also Converges to $f(a)$.

(✯✯) Then assume for The sake of a Contradiction That f is NOT Cont. at a, That is ~~①~~ ②

$\exists \epsilon > 0 \ \forall \delta > 0 \ \exists x$ st. $|x-a| < \delta$ but $|f(x)-f(a)| \geq \epsilon$

we use $\forall \delta > 0 \ \exists x$ repeatedly to find a Sequence $\{x_k\}$ which Converges to a but $|f(x_k)-f(a)| \geq \epsilon$ $\forall k$, i.e $f(x_k) \not\to f(a)$ ~~①~~

②

given $\delta = 1$ $\quad \exists x$ (Call it x_1) st. $|x_1 - a| < 1$ & $|f(x_1) - f(a)| \geq \epsilon$

$\vdots$

$\delta = \frac{1}{k}$ $\quad \exists x$ (Call it x_k) st. $|x_k - a| < \frac{1}{k}$ & $f(x_k) - f(a)| \geq \epsilon$

$\vdots$

This Sequence $\{x_k\} \longrightarrow a$ b/c $\forall r > 0 \ \exists K$ st. $\frac{1}{K} < r$, and so for any $k \geq K$ $|x_k - a| < \frac{1}{k} < \frac{1}{K} < r$. ①

$\therefore$ we have a Sequence $\{x_k\}$ That $\longrightarrow a$ but $f(x_k) \not\to f(a)$

This Contradicts The assumption (✯), so our assumption (✯✯) is false. so f is Cont. at a. ①

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8. MVT

a) (6 marks) State and prove the Mean Value Theorem for a function $f : S \longrightarrow \mathbb{R}$ where $S \subset \mathbb{R}^n$. (Note: in your proof you can use the one variable version of the MVT without proof.)

given two points $a, b \in S$ as well as The line Segment L Connecting Them. assume f is diff on L except perhaps at a and b. Then $\exists\, c$ on L st.
$f(b) - f(a) = \nabla f(c) \cdot (b-a)$. ②

proof: let $h = b - a$, so $L = \{a + th : t \in [0,1]\}$. ① Define $\varphi(t) = f(a+th)$, $t \in [0,1]$

φ is cont on $[0\ 1]$ and diff on $(0,1)$. By MVT ✓ $\exists\, u \in (0,1)$ st.

$\varphi'(u) = \varphi(1) - \varphi(0)$ but $LHS = \nabla f(a + uh) \cdot \frac{d}{dt}(a+th)\Big|_{t=u} = \nabla f(c) \cdot h$ ①

and $RHS = f(a+h) - f(a) = f(b) - f(a)$. ①

b) (3 marks) If f is differentiable on ~~a~~ open convex set S and that $|\nabla f(x)| \leq M$ for all $x \in S$, show that for any points a and b in S, $|f(b) - f(a)| \leq M|b-a|$.

given $a, b \in S$, Since S is Convex The line Segment L Connecting a to b falls in S, f is diff on L, so apply M.V.T to get

(2.5) $f(b) - f(a) = \nabla f(c) \cdot h$ $= b - a$ but now by Cauchy inequality

$|f(b) - f(a)| = |\nabla f(c) \cdot h| \leq |\nabla f(c)|\,|h| = M|b-a|$.

(1.[illegible])

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c) (3 marks) Assume f is differentiable on the ball $B(3,0)$, and $|\nabla f(x)| \leq 5$, and that $f(0) = 7$, show that $|f(x)| < 22$ for any $x \in B(3,0)$.

by part (b) $|f(x)-f(0)| \leq M|x-0| = 5\times 3 = 15$ 1.5

$\forall x \in B$

Since $B(3,0)$ is open Convex by 1(a)

so That $||f(x)|-|f(0)|| < 15$ 0.5

$-15 < |f(x)| - |f(0)| < 15$

$|f(x)| < 15 + |f(0)| = 15 + 7 = 22$ 1

d) (5 marks) The following passage is the proof of theorem 2.42 from the textbook. Fill in all the gaps.

Theorem 2.42: Suppose f is differentiable on an <u>*open*</u> <u>*connected*</u> *set S and* $\nabla f(x) = 0$ *for all* $x \in S$*. Then f is constant on S.*

0.5 0.5 **Proof:** *Pick* $a \in S$ *and define* $S_1 = \{x \in S : f(x) = f(a)\}$, *and* $S_2 =$ $\{x \in S : f(x) \neq f(a)\}$. *Obviously* $S_1 \cap S_2 = \emptyset$ *and* $S_1 \cup S_2 =$ S. S_2 *is open because* 1 $= f^{-1}\{y : ~~y~~ \neq f(a)\}$ ← open. S_1 *is also open because*

1.5 given $x \in S_1$, $\exists r > 0$ st. $B(r,x) \subset S$ also B is Convex, so by a

Corrollary $f(x)$ is Const on B, so $f = f(a)$ $\therefore B \subset S_1$.
open

Corrollary $\nabla f = 0$ on a op convex set Th f = Const

Since both S_1 *and* S_2 *are open then* $S_1 \cap \overline{S_2} = \emptyset$ *because* 0.5 S_1 is open and as such all its pts are interior pts & $S_1 \cap S_2 = \emptyset$

Similarly $S_2 \cap \overline{S_1} = \emptyset$. *Now if* $S_2 \neq \emptyset$ *then* (S_1, S_2) *will be* a disconnection *for S which contradicts the assumption that* S is Connected. *This contradiction proves that* S_2 *is* empty *and therefore for all* $x \in S$ $f(x) =$ f(a), *which means f is constant on S.* 1

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