Feb 6th

DEFS: $T: V \to V$, V over F

① $\lambda \in F$ is an eigenvalue of T if there is non-zero vector $v \in V$ st. $Tv = \lambda v$

② The vector v is the eigenector

③ The eigenspace of λ is $E_\lambda = \{v \in V \mid Tv = \lambda v\}$

• Note E_λ subspace of V

$E_\lambda = \mathrm{Ker}(\lambda I - T)$

$$(v \in \mathrm{Ker}(\lambda I - T) \iff (\lambda I - T)v = 0$$
$$\iff \lambda v - Tv = 0$$
$$\iff Tv = \lambda v)$$

$$R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Q: Does $R_{\pi/2}$ have any eigenvalues?

A_1: over $\mathbb{R}$ then no.

i.e. $R_{\pi/2}: \mathbb{R}^2 \to \mathbb{R}^2$

A_2: over $\mathbb{C}$ then yes

$R_{\pi/2}: \mathbb{C}^2 \to \mathbb{C}^2$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}: \mathbb{C}^2 \to \mathbb{C}^2$

1. $p(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1$ ← char. poly

2. eigenvalues are $\pm i$.

3. eigenspaces: $E_i = \mathrm{Ker}(iI - A) = \mathrm{null}\,(iI - A)$

$\mathrm{null}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \mathrm{null}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$

$\Rightarrow E_i = \mathrm{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\},\ E_{-i} = \mathrm{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$

check: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}$

e vector   evalue

Remark: determinant, char. poly ... have the same for $A \in M_n(F)$

Ex: $F = \mathbb{Z}_3$ $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ $\det A = -2 \det \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = (-2)(-2) = 4 = 1$

Thm: $A \in M_n(F)$. Then A invertible $\iff \det A \neq 0$

Q over $\mathbb{Z}_5$ is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ invertible? Yes

$\mathbb{Z}_3$ . . — ? No

over $\mathbb{Z}_3$: $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

If we have $T: V \to V$ how do we compute eigen * ?
- pick a basis α of V
- compute $[T]_\alpha$
- " char poly of $[T]_\alpha$
- get eigenvalues
- translate back to V

Ex: $V = sl_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \operatorname{tr} A = 0\}$

$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in V$

$T(A) = hA - Ah$; $T: V \to V$

[subex: why is $hA - Ah \in V$? i.e. why $\operatorname{tr}(hA - Ah) = 0$?
$\operatorname{tr}(hA - Ah) = \operatorname{tr}(hA) - \operatorname{tr}(Ah) = \operatorname{tr}(hA) - \operatorname{tr}(hA) = 0$]

$\dim V = 3$

$\alpha = \{h, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= e}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{= f}\}$ is a basis

$[T]_\alpha = ?$

$T(h) = 0$; $T(e) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = 2e$

$T(f) = 2f$

h e.vector with evalue 0
e 2
f -2

$[T]_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Ex: ① Suppose $T: V \to V$; $T^2 = I$, what are the possible evalues of T?

$Tv = \lambda v \Rightarrow T^2 v = \lambda T v$
$\Rightarrow v = \lambda^2 v$
$\Rightarrow \lambda^2 = 1$ $\lambda = \pm 1$

② $T: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$
$T(A) = A^T$; note $T^2 = I$

$E_1 = \{A \mid T(A) = A\} = \{A \mid A = A^T\}$ symmetric matrices

E_{-1} = skew-symm matrices = $\{A \mid A^T = -A\}$

Claim: $M_n(\mathbb{R}) = \underbrace{E_1 \oplus E_{-1}}$

"direct sum"

Def of $\oplus$:
$V \supset U, W$
$U \oplus W = V$ means ① $U + W = V$; i.e. any $v \in V$ can expressed as a sum $u + w = v$ for $u \in U, w \in W$
② $U \cap W = \{0\}$

Proof of Claim:
① $E_1 + E_{-1} = M_n$
$A \in M_n$; $A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$
② $E_1 \cap E_{-1} = \{0\}$