Things You Should Know

1 Simple Set Theory

Here are words that you should have heard already. If you don't know any of these words, ask someone!
Here A, B, X, Y are sets. I is an index set, $\{ A_\alpha : \alpha \in I \}$ and $\{ B_\alpha : \alpha \in I \}$ are (indexed) families of sets.

- Empty set: $\emptyset$, the set with no elements.
- Subset: $A \subseteq B$ means "$x \in A \Rightarrow x \in B$"
- Powerset: $\mathcal{P}(X) := \{ A : A \subseteq X \}$
- Union: $A \cup B := \{ x : x \in A \vee x \in B \}$
- Intersection: $A \cap B := \{ x : x \in A \wedge x \in B \}$
- Complement: $X \setminus A := \{ x : x \in X \wedge x \notin A \}$
- Indexed Union: $\bigcup_{\alpha \in I} A_\alpha := \{ x : \exists \alpha \in I, x \in A_\alpha \}$
- Indexed Intersection: $\bigcup_{\alpha \in I} A_\alpha := \{ x : \forall \alpha \in I, x \in A_\alpha \}$
- Cartesian Product: $X \times Y := \{ (x, y) : x \in X, y \in Y \}$

2 Functions

FACT: Let $f : X \longrightarrow Y$ be a function.

- X is the domain, Y is the codomain, $\{ f(x) : x \in X \} \subseteq Y$ is the range;
- f is an injection (or 1-1) if $f(x) = f(a)$ implies $x = a$;
- f is a surjection (or onto) if the range is the entire codomain;
- f is a bijection if it is both an injection and a surjection;
- The composition of two injections is an injection;
- The composition of two surjections is a surjection;
- The composition of two bijections is a bijection;

FACT: Let $A \subseteq X$ and $B \subseteq Y$.

- The preimage of B is: $f^{-1}(B) := \{\, x \in X : f(x) \in B \,\}$.
- If f is an injection with range Y, then the inverse function $f^{-1} : Y \longrightarrow X$ is (1) a function and (2) an injection.

3 DeMorgan's Laws and other interactions

The following are called DeMorgan's laws:

1. $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$
2. $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

FACT: These things are true of all functions $f : X \longrightarrow Y$. Let $A, B \subseteq X$ and $C, D \subseteq Y$.

1. $A \subseteq B$ implies $f(A) \subseteq f(B)$
2. $C \subseteq D$ implies $f^{-1}(C) \subseteq f^{-1}(D)$
3. $f(A \cup B) = f(A) \cup f(B)$
4. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
5. $f(A \cap B) \subseteq f(A) \cap f(B)$
6. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
7. $f(A) \setminus f(B) \subseteq f(A \setminus B)$
8. $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$
9. $f(X \setminus f^{-1}(Y \setminus C)) \subseteq C$
10. $A \subseteq f^{-1}(f(A))$, (with equality if f is an injection)
11. $f(f^{-1}(C)) \subseteq C$, (with equality if f is a surjection)
12. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$

4 Countability

Definition 1. *A set A is said to be countable if there is a bijection $f : \mathbb{N} \longrightarrow A$.*

The following gives equivalent conditions for being countable:

Theorem 2. *For an infinite set A the following are equivalent:*

1. *A is countable;*
2. *There is an injection $f : A \longrightarrow \mathbb{N}$;*
3. *There is a surjection $g : \mathbb{N} \longrightarrow A$.*

FACT: The following sets are countable:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, the algebraic numbers;
- Any infinite subset of a countable set;
- The product of two countable sets;
- The union of finitely many countable sets;
- The countable union of countable sets;
- The countable union of some countable sets and some finite sets;

FACT: The following sets are not countable:

- $\mathbb{R}$, the irrational numbers, the non-algebraic numbers (i.e. the transcendental numbers), $\mathbb{R}^n$;
- Any superset of an uncountable set;
- The powerset of a countable set, e.g. $\mathcal{P}(\mathbb{N})$;
- The set of functions from $\mathbb{N}$ to $\mathbb{N}$.

The following is a combinatorial fact about uncountable sets:

Theorem 3 (Uncountable Pigeonhole Principle)**.** *Let X be an uncountable set. If $\chi : X \longrightarrow \mathbb{N}$ is a function, then there is an $n \in \mathbb{N}$ such that $\chi^{-1}(n)$ is uncountable.*

That can be restated as "*If you try to put uncountably many pigeons into countably many holes, then there is a hole with uncountably many pigeons*".

5 Selected Facts about $\mathbb{R}$

First recall: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. (For us $0 \notin \mathbb{N}$.)
FACT: Between any two distinct real numbers:

- There is a rational number.
- There are infinitely many rational numbers.
- There is an irrational number.
- There are infinitely many irrational numbers.

FACT: Here are some useful facts from calculus:

- $\bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 1] = (0, 1]$;
- $\bigcup_{n\in\mathbb{N}}[0, n] = [0, +\infty)$;
- $\sum_{n\in\mathbb{N}} 2^{-n} = 1$;