Def: The Wronskian of function f, g is the function:

$$W(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t)$$

Sometimes write $W[f,g](t) = W(t)$

$L[y] = y'' + py' + gy$

Suppose y_1, y_2 are two solutions of $L[y] = 0$

$$y = c_1 y_1 + c_2 y_2 \qquad (*)$$

Then the IVP $L[y] = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$ has a solution $(*)$ for suitable c_1, c_2 if $W[y_1, y_2](t_0) \neq 0$.

If $W[y_1, y_2] \neq 0$ then y_1, y_2 are <u>fundamental set of solution</u>.

<u>Example</u> $t^2 y'' - 2ty' - 10y = 0 \ (t > 0)$

has solution $y_1(t) = t^{-2}$, $y_2(t) = t^5$

The solution is $W[y_1, y_2](t) = \begin{vmatrix} t^{-2} & t^5 \\ -2t^{-3} & 5t^4 \end{vmatrix} = 5t^2 - (-2)t^2 = 7t^2 \neq 0$ (since $t > 0$)

$\Rightarrow y_1, y_2$ are fund. set of solution.

<u>General fact</u>: Suppose y_1, y_2 two solutions of $L[y] = 0$ $(L[y] = y'' - py' + gy)$

Then $W[y_1, y_2](t_0) \neq 0$ for some $t_0 \Leftrightarrow W[y_1, y_2](t) \neq 0$ for all t.

This follows from <u>Abel's formula</u>:

We'll derive a formula for $W = W[y_1, y_2] = y_1 y_2' - y_1' y_2$

$$\frac{dW}{dt} = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' = y_1(-py_2' - gy_2) - (-py_1' - gy_1)y_2$$

$$= -y_1 p y_2' + p y_1' y_2$$

$$= -p(y_1 y_2' - y_1' y_2)$$

$$= -pW$$

Thus we see: $\boxed{\frac{dW}{dt} = -p(t)W(t)}$ separable

$\frac{1}{W} dW = -p(t) dt$

Integrate both sides from t_0 to t

$\ln|W(t)| - \ln|W(t_0)| = -\int_{t_0}^{t} p(s)ds$

$\ln\left|\frac{W(t)}{W(t_0)}\right|$

$\frac{W(t)}{W(t_0)} = \exp\left(-\int_{t_0}^{t} p(s)ds\right)$

This shows $W(t) = W(t_0) e^{-\int_{t_0}^{t} p(s)ds}$

Abel's formula

In particular, we see: $W(t_0) \neq 0 \Rightarrow W(t) \neq 0$ for all t.

<u>Example</u>: $t^2y'' - 2ty' - 10y = 0$

$$y_1(t) = t^{-2}, \quad y_2(t) = t^5$$

$$W[y_1, y_2] = 7t^2$$

Consistent with Abel's formula: We have $p(t) = \frac{1}{t^2}(-2t) = -\frac{2}{t}$

$$\int p(t)\,dt = -2\int \frac{1}{t}\,dt = -2\ln(t)$$

$$e^{-\int p(t)dt} = e^{-(-2\ln(t))} = t^2 \quad \checkmark$$

Note: By Abel's formula, the Wronskian $W[y_1, y_2]$ is <u>independent of</u> the solutions y_1, y_2 of $L[y]=0$, <u>up to a constant</u>.

Example: $t^2y'' - 3ty' + 4y = 0 \quad (t > 0)$

Calculate $W[y_1, y_2]$ up to a constant, where y_1, y_2 are solutions of $L[y]=0$. By the formula,
$W[y_1,y_2] = \exp(-\int p(t)dt) = \exp(-\int(-\frac{3}{t})dt) = t^3$

" POPULAR PROBLEM FOR EXAM: CALCULATE WRONSKIAN

<u>Another remark</u>: Sps y_1, y_2 are sol'n of $L[y]=0$. Sps $y_1 \neq 0$.

$$\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \frac{y_2'y_1 - y_2y_1'}{y_1^2} = \frac{W[y_1, y_2]}{y_1^2}$$

Hence $W[y_1,y_2] = 0 \iff \frac{d}{dt}\left(\frac{y_2}{y_1}\right) = 0 \iff \frac{y_2}{y_1} = \text{constant}$

Thus, y_1, y_2 are a fund. set of sol'ns ($W[y_1,y_2] \neq 0$) iff $\frac{y_2}{y_1}$ is constant.

i.e. y_2 is not a multiple of y_1.

<u>Another remark</u>: $\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \frac{W[y_1, y_2]}{y_1^2} \quad (**)$

Sps y_1 is a given solution of $L[y]=0$.

Then we can use $(**)$ together with Abel's formula as an ODE for y_2 (to find second sol'n)

will be covered next class