University of Toronto
FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, DECEMBER 2008

APM 236H1F
Applications of Linear Programming

PLEASE HAND IN

Examiner: P. Kergin
Duration: 3 hours

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NO ______________________________

SIGNATURE ______________________________

INSTRUCTIONS:

NO calculators or other aids allowed. There are 10 questions, each worth 12 marks. Questions 1, 2, 3, 6, 8, and 9 have part-questions, whose values are stated within the part-questions themselves. Total marks = 120.

This exam consists of 16 pages, printed on both sides of the paper. Write solutions in spaces provided. Pages 13, 15, and 16 are blank and may be used for the solution(s) of any of the problems, or for rough work. Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

GRADER'S REPORT	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

1. A note on grading: while diagrams may be useful in this question, a diagram alone does not constitute a proof.

1.(a) (6 marks) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } -1 \leq x+y \leq 2 \ \textbf{ or } \ -2 \leq x-y \leq 1 \right\}$.

Prove that S is **not convex.**

1.(b) (6 marks) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } 2x+y \leq 10 \ \textbf{ and } \ x-y \geq -7 \right\}$.

Prove that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is **not an extreme point** of S. You may assume S is convex.

2.(a) (6 marks) **Find an optimal solution** of the problem: Maximize $z = 4x_1 + 5x_2 + 2x_3$

subject to the constraints $\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 4x_3 & \geq & -1 \\ 2x_1 & + & x_2 & - & 2x_3 & \leq & 4 \end{array}$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

2.(b) (3 marks) **Find a second optimal solution** of the problem of question 2.(a).

2.(c) (3 marks) **Find all optimal solutions** of the problem of question 2.(a).

3. Consider the following linear programming problem (in $\mathbb{R}^4$):

Minimize $z = x_1 + x_2 + x_3 + x_4$ subject to the constraints

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & & & & & = & 4 \\ -3x_1 & + & 6x_2 & + & x_3 & + & 3x_4 & = & -9 \\ & & & & 2x_3 & + & 6x_4 & \geq & 0 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

3.(a) (1 mark) Put the problem in **canonical form.**

3.(b) (7 marks) Find **all basic solutions** (feasible and infeasible) of the **canonical form** of the problem.

3.(c) (2 marks) Find **all extreme points** of the feasible region of the problem **given above** (in $\mathbb{R}^4$).

3.(d) (2 marks) **Solve** the problem **given above** (in $\mathbb{R}^4$). You may assume the problem has an optimal solution.

4. Suppose in solving a linear programming problem by the simplex method we encounter the following tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	0	5	1	−4	3	0	9
x_1	1	−2	0	0	−4	0	7
x_6	0	4	0	−3	−1	1	8
	0	−5	0	−2	0	0	13

Let M be any fixed, but unspecified, non-negative number. Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$ (depending on M), which is **feasible** for the problem, such that, at $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$, the problem has **objective value greater than or equal to** M.

5. **Solve** the problem: Maximize $z = -x_1 - 4x_2 + x_3$ subject to the constraints

$$\begin{array}{rcrcrcr} -x_1 & - & 2x_2 & + & 2x_3 & = & -2 \\ x_1 & + & 2x_2 & + & x_3 & \geq & 5 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

6.(a) (6 marks) **State** the **weak duality theorem**.

6.(b) (6 marks) **State** the **strong duality theorem**.

7. **Solve** the problem: Minimize $z = 7x_1 + 2x_2 + x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} 4x_1 & + & x_2 & + & x_3 & \geq & 8 \\ 2x_1 & + & x_2 & & & \geq & 4 \\ -x_1 & + & x_2 & + & x_3 & \geq & 3 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

8. Consider the primal problem: Maximize $z = -21x_1 + 11x_2 + 7x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} 3x_1 & - & x_2 & + & x_3 & = & 8 \\ -7x_1 & + & 2x_2 & + & x_3 & \leq & 14 \\ -8x_1 & + & 5x_2 & + & 4x_3 & \leq & 37 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

8.(a) (3 marks) **State the dual** of the primal problem.

8.(b) (9 marks) An optimal solution of the primal problem is $x_1 = 0$, $x_2 = \frac{5}{9}$, $x_3 = \frac{77}{9}$.
Solve the dual problem.

9. Consider the problem: Maximize $z = 10x_1 + 15x_2 - 6x_3 + 5x_4$ subject to the constraints

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 3x_2 & - & 3x_3 & + & 2x_4 & \leq & 3 \\ 2x_1 & + & 4x_2 & - & x_3 & + & x_4 & \leq & 5 \\ x_1 & + & 3x_2 & + & 2x_3 & - & x_4 & \leq & 4 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

9.(a) (2 marks) The fourth tableau of the simplex solution of this problem has basic variables $\{x_1, x_6, x_3\}$ (in that order, where x_6 denotes the slack variable for the second constraint). **Use this information** to find the matrix B^{-1} which pertains to the fourth tableau.

9.(b) (10 marks) **Beginning from the fourth tableau**, use the **revised simplex method** to **solve** the above problem.

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10. **Solve** the problem: Minimize

$$\begin{array}{rcrcrcrcrcrcr} z & = & & 2x_{11} & + & 3x_{12} & + & 5x_{13} & + & 6x_{14} & + & 4x_{15} \\ & & + & 3x_{21} & + & x_{22} & + & 2x_{23} & + & 3x_{24} & + & 7x_{25} \\ & & + & x_{31} & + & 4x_{32} & + & 4x_{33} & + & 3x_{34} & + & 6x_{35} \end{array}$$

subject to the constraints

$$\begin{array}{rcrcrcrcrcrcr} x_{11} & + & x_{12} & + & x_{13} & + & x_{14} & + & x_{15} & = & 40 \\ x_{21} & + & x_{22} & + & x_{23} & + & x_{24} & + & x_{25} & = & 60 \\ x_{31} & + & x_{32} & + & x_{33} & + & x_{34} & + & x_{35} & = & 60 \\ & & & & x_{11} & + & x_{21} & + & x_{31} & = & 10 \\ & & & & x_{12} & + & x_{22} & + & x_{32} & = & 20 \\ & & & & x_{13} & + & x_{23} & + & x_{33} & = & 30 \\ & & & & x_{14} & + & x_{24} & + & x_{34} & = & 50 \\ & & & & x_{15} & + & x_{25} & + & x_{35} & = & 50 \end{array}$$

$$x_{ij} \geq 0 \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4, 5.$$

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