UNIVERSITY OF TORONTO MISSISSAUGA

December 2008 Examination
STA257H5F
Instructor: Christine Lim
Duration: 3 hours

Aids Allowed: Non-programmable Calculators
Aids Supplied: Summary of Common Distributions and Standard Normal Table (4 pages)

You may be charged with an academic offence for possessing the following items during the writing of an exam unless otherwise specified: any unauthorized aids, including but not limited to calculators, cell phones, pagers, wristwatch calculators, personal digital assistants (PDAs), iPods, MP3 players, or any other device. If any of these items are in your possession in the area of your desk, please turn them off and put them with your belongings at the front of the room before the examination begins. A penalty may be imposed if any of these items are kept with you during the writing of your exam.

Please note, students are **NOT** allowed to petition to **RE-WRITE** a final examination.

Last Name: ______________________

Given Name: ______________________

Student Number: ______________________

Question Number	Maximum Mark	Earned Mark	Question Number	Maximum Mark	Earned Mark
1	5		8	10	
2	6		9	5	
3	5		10	5	
4	20		11	6	
5	8		12	9	
6	8		13	5	
7	8		**Total**	**100**	

(1) One form of the binomial theorem states that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, n is a postive integer and x, y are real numbers.

5 points

(a) (3 points) Use this theorem to show

$$\sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \theta^x (1-\theta)^{n-x} = n\theta$$

where n is a positive integer and $0 \leq \theta \leq 1$.

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(b) (2 points) X is a binomial random variable with n trials and probability of success θ. Show that

$$E(X) = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \theta^x (1-\theta)^{n-x}$$

and hence show $E(X) = n\theta$.

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6 points

(2) (a) (3 points) State the three axioms of probability.

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(b) (3 points) Use the axioms to show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Continued on page 6

5 points

(3) A shelf contains three identical statistics books, two identical mathematics books, and three identical physics books.

(a) (1 point) How many ways can we arrange these books on the shelf? Simplify your answer.

Two books are now selected at random from the shelf, one at a time.

(b) (1 point) If the selection is *without* replacement, what is the probability that exactly one chosen textbook is a statistics text? Keep your answer in fraction form.

(c) (1 point) If the selection is *without* replacement, what is the probability that the *second* textbook chosen is the first statistics text selected? Keep your answer in fraction form.

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(d) (1 point) If the selection is *with* replacement, what is the probability that exactly one chosen textbook is a statistics text? Keep your answer in fraction form.

(e) (1 point) If the selection is *with* replacement, what is the probability that the *second* textbook chosen is the first statistics text selected? Keep your answer in fraction form.

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20 points

(4) A shelf contains three identical statistics books, two identical mathematics books, and three identical physics books. Two books are selected at random from the shelf at the same time. Let X be the number of statistics books selected and Y be the number of mathematics books selected.

(a) (12 points) Fill in the missing probabilties in the following joint probability distribution of X and Y. There are 12 of them, keep your answers in fraction form.

		x			
		0	1	2	$f_Y(y)$
	0				
y	1			0	
	2		0	0	
	$f_X(x)$				1

(b) (4 points) Compute $Cov(X, Y)$. Round your answer to 4 decimal places.

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(c) (1 point) Are X and Y independent? Justify your answer.

(d) (3 points) It is known that one of the selected books is mathematics. Find the conditional probability distribution of the number of statistics books selected. Present your answer in a tabular form.

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8 points

(5) Shirley is a cashier at a drug mart. It is the time of the year to ask for donations. From experience, 40% of her customers give a donation. Assume all her customers donate independently.

(a) (1 point) How many customers can she expect to give a donation if she has 100 customers tomorrow?

(b) (1 point) What is the probability that her tenth customer is the first to donate tomorrow? Round your answer to 4 decimal places.

(c) (1 point) What is the probability that her tenth customer is the fifth to donate tomorrow? Round your answer to 4 decimal places.

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(d) (5 points) If she has 100 customers tomorrow, what is the probability that more than half of her customers will give a donation? You may want to consider using an approximation. Besure to justify the use of the approximation.

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(6) X is an exponential random variable with mean $\theta > 0$. Its probability density function is

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \text{ for } x > 0.$$

8 points

(a) (5 points) Show that the moment generating function of X is

$$M_X(t) = \frac{1}{1 - \theta t}.$$

Besure to state the range of t values for which the function converges.

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(b) (1 point) Use the moment generating function to show that $E(X) = \theta$.

(c) (2 points) Use the moment generating function to show that $Var(X) = \theta^2$.

Continued on page 14

8 points

(7) Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with mean $\theta > 0$. Let $Y = \sum_{i=1}^{n} X_i$.

(a) (2 points) Show that $E(Y) = n\theta$.

(b) (2 points) Show that $Var(Y) = n\theta^2$.

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(c) (4 points) Use the moment generating function technique to show that $Y \sim \Gamma(n, \theta)$.

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10 points (8) Midterm marks for a class have mean 50 and standard deviation 10.

(a) (3 points) For a randomly selected student from the class, what can you say about the probability of this student scoring at least 90?

(b) (2 points) If it is known that the distribution of the midterm marks is normal, for a randomly selected student from the class, what can you say about the probability of the student scoring at least 90?

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(c) (5 points) 30 students are randomly selected from the class. What can you say about the probability of their average mark falling between 52 and 55 inclusive? You may want to quote a theorem.

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5 points

(9) A software technical support line is open 24 hours a day. The support line receives

- an average of 6 calls per hour between 8am and 10pm, and
- an average of 2 calls per hour between 10pm and 8am.

Number of calls are assumed to follow a Poisson distribution.

(a) (1 point) How many calls do you expect to arrive between 9pm and 11pm?

(b) (1 point) What is the probability that there will be exactly 7 calls between 9pm and 11pm? Round your answer to 4 decimal places.

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