Repeated eigenvalues

Ex $\vec{x}' = \underbrace{\begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}}_{A} \vec{x}$

Eigenvalues $\det(A - rI)$

$$= \begin{vmatrix} 6-r & 3 & -8 \\ 0 & -2-r & 0 \\ 1 & 0 & -3-r \end{vmatrix}$$

Eigenvectors

$r^{(1)} = 5 \quad A - 5I = \begin{pmatrix} 1 & 3 & -8 \\ 0 & -7 & 0 \\ 1 & 0 & -8 \end{pmatrix} \quad \vec{\xi}^{(1)} = \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}$

$r^{(2)} = -2 \quad A - (-2)I = \begin{pmatrix} 8 & 3 & -8 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

only one eigendirection

$(A - (-2)I)^2 = \begin{pmatrix} 56 & 24 & -56 \\ 0 & 0 & 0 \\ 7 & 3 & -7 \end{pmatrix}$

Need $\vec{\eta}$ with $(A-(-2)I)^2 \vec{\eta} = 0$, but not multiple of $\vec{\xi}^{(2)}$. $\quad \vec{\eta} = \begin{pmatrix} -3 \\ 7 \\ 0 \end{pmatrix}$

$(A - (-2)I)\vec{\eta} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

$\vec{x}^{(1)} = e^{5t}\begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}, \ \vec{x}^{(2)} = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \vec{x}^{(3)} = e^{-2t}\left(t\begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \\ 0 \end{pmatrix}\right)$

Linear 2x2 systems

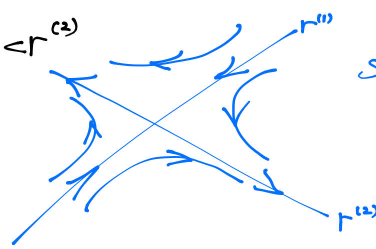
$\vec{x}' = A\vec{x} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $r^{(1)}, r^{(2)}$ be the eigenvalues.

I. Two distinct real eigenvalues

$r^{(1)} < 0 < r^{(2)}$

$\det(A) < 0$

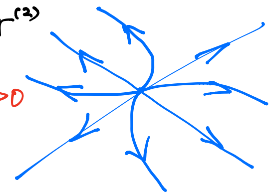


saddle (unstable)

$0<r^{(1)}<r^{(2)}$

$\det(A)>0$

$tr(A)>0$



$r^{(1)}$

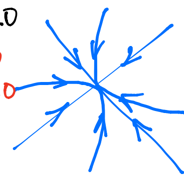
unstable node

$r^{(2)}$

$r^{(1)}<r^{(2)}<0$

$\det(A)>0$

$tr(A)<0$



$r^{(1)}$

stable node

$r^{(2)}$

+ extra case where $r^{(1)}=0$ or $r^{(2)}=0$

Remark. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A)=ad-bc$

$tr(A)=a+d$

The eigenvalues satisfy: $\det(A)=r^{(1)}r^{(2)}$

$tr(A)=r^{(1)}+r^{(2)}$

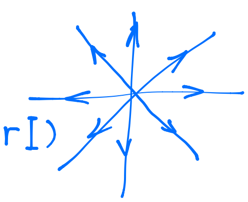
$$r^{(1)},r^{(2)}=\frac{tr(A)}{2}\pm\sqrt{\left(\frac{tr(A)}{2}\right)^2-\det(A)}$$

$$\boxed{tr(A)^2>4\det(A)}$$

II. Repeated real eigenvalues

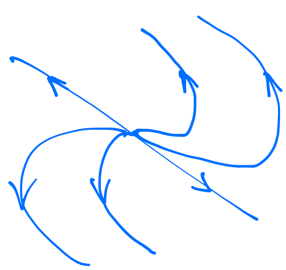
$r>0$, two linearly indep. eigenvalues

$(A=rI)$



unstable node

$r>0$, only one eigendirection



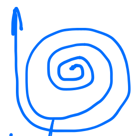
unstable improper node

similar picture for $r<0$.

III. Complex conjugate eigenvalues $r^{(1)}=a+ib$

$tr(A)^2 < 4\det(A)$ $\quad r^{(2)}=a-ib$

IIIa. $a>0$

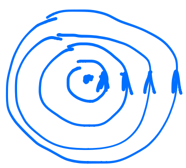
 or 

unstable spiral

How to decide the direction? $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

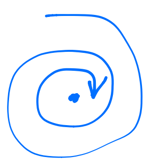
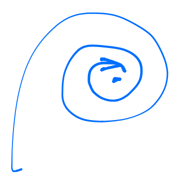
$\vec{x}'=A\vec{x}$ means that, e.g. $A\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}$ direction at $\begin{pmatrix}1\\0\end{pmatrix}$

IIIb. $a=0$

 or 

"center"

IIIc. $a<0$

 or 

stable spiral