Solutions to H/W assignments #'s 1&2

(1)

Ch. 2

46. Using that $X = \sum_{n=1}^{\infty} I_n$, we obtain

$$E[X] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P\{X \geq n\}$$

Making the change of variables $m = n - 1$ gives

$$E[X] = \sum_{m=0}^{\infty} P\{X \geq m+1\} = \sum_{m=0}^{\infty} P\{X > m\}$$

(b) Let

$$I_n = \begin{cases} 1, & \text{if } n \leq X \\ 0, & \text{if } n > X \end{cases}$$

$$J_m = \begin{cases} 1, & \text{if } m \leq Y \\ 0, & \text{if } m > Y \end{cases}$$

Then

$$XY = \sum_{n=1}^{\infty} I_n \sum_{m=1}^{\infty} J_m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} I_n J_m$$

Taking expectations now yields the result

$$E[XY] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E[I_n J_m]$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m)$$

76. $P\left\{\left|\dfrac{X_1 + \cdots + X_n - n\mu}{n}\right| > \epsilon\right\}$

$= P\{|X_1 + \cdots + X_n - n\mu| > n\,\epsilon\}$

$\leq Var\{X_1 + \cdots + X_n\}/n^2\ \epsilon^2$

$= n\sigma^2/n^2\ \epsilon^2$

$\to 0$ as $n \to \infty$

85. (a) Using that $Var\left(\frac{W}{\sigma_W}\right) = 1$ along with the formula for the variance of a sum gives

$$2 + 2\frac{Cov(X, Y)}{\sigma_X \sigma_Y} \geq 0$$

(b) Start with $Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) \geq 0$, and proceed as in part (a).

(c) Squaring both sides yields that the inequality is equivalent to

$$Var(X+Y) \leq Var(X) + Var(Y) + 2\sigma_X\sigma_Y$$

or, using the formula for the variance of a sum

$$Cov(X, Y) \leq \sigma_X\sigma_Y$$

which is part (b).

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87. (a) $P(Z^2 \le x) = P(-\sqrt{x} < Z < \sqrt{x})$
$= F_Z(\sqrt{x}) - F_Z(-\sqrt{x})$

Differentiating yields

$$f_{Z^2}(x) = \frac{1}{2}x^{-1/2}[f_Z(\sqrt{x}) + f_Z(-\sqrt{x})] = \frac{1}{\sqrt{2\pi}}x^{-1/2}e^{-x/2}$$

(b) The sum of n independent gamma random variables with parameters $(1/2, 1/2)$ is gamma with parameters $(n/2, 1/2)$.

Start of Ch. 3

8. (a) $E[X] = E[X|\text{first roll is } 6]\frac{1}{6}$

$$+ E[X|\text{first roll is not } 6]\frac{5}{6}$$

$$= \frac{1}{6} + (1 + E[X])\frac{5}{6}$$

implying that $E[X] = 6$.

(b) $E[X|Y = 1] = 1 + E[X] = 7$

(c) $E[X|Y = 5]$

$$= 1\left[\frac{1}{5}\right] + 2\left[\frac{4}{5}\right]\left[\frac{1}{5}\right] + 3\left[\frac{4}{5}\right]^2\left[\frac{1}{5}\right]$$

$$+4\left[\frac{4}{5}\right]^3\left[\frac{1}{5}\right] + 6\left[\frac{4}{5}\right]^4\left[\frac{1}{6}\right]$$

$$+7\left[\frac{4}{5}\right]^4\left[\frac{5}{6}\right]\left[\frac{1}{6}\right] + \cdots$$

9. $E[X|Y = y] = \sum_x xP\{X = x|Y = y\}$

$$= \sum_x xP\{X = x\} \quad \text{by independence}$$

$$= E[X]$$

10. (Same as in Problem 8.)

Note that #10 is the continuous counterpart of #9.

67. A run of j successive heads can occur in the following mutually exclusive ways: (i) either there is a run of j in the first $n - 1$ flips, or (ii) there is no j-run in the first $n - j - 1$ flips, flip $n - j$ is a tail, and the next j flips are all heads. Consequently, (a) follows. Condition on the time of the first tail:

$$P_j(n) = \sum_{k=1}^{j} P_j(n-k)p^{k-1}(1-p) + p^j, \quad j \le n$$

73. Condition on the value of the sum prior to going over 100. In all cases the most likely value is 101. (For instance, if this sum is 98 then the final sum is equally likely to be either 101, 102, 103, or 104. If the sum prior to going over is 95 then the final sum is 101 with certainty.)

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77. We will prove it when X and Y are discrete.

(a) This part follows from (b) by taking $g(x, y) = xy$.

(b) $$E[g(X, Y)|Y = \bar{y}] = \sum_y \sum_x g(x, y) P\{X = x, Y = y|Y = \bar{y}\}$$

Now,

$$P\{X = x, Y = y|Y = \bar{y}\} = \begin{cases} 0, & \text{if } y \neq \bar{y} \\ P\{X = x, Y = \bar{y}\}, & \text{if } y = \bar{y} \end{cases}$$

So,

$$E[g(X, Y)|Y = \bar{y}] = \sum_k g(x, \bar{y}) P\{X = x|Y = \bar{y}\}$$
$$= E[g(x, \bar{y})|Y = \bar{y}$$

(c) $$E[XY] = E[E[XY|Y]]$$
$$= E[YE[X|Y]] \quad \text{by (a)}$$

92. Let X denote the amount of money Josh picks up when he spots a coin. Then

$$E[X] = (5 + 10 + 25)/4 = 10,$$
$$E[X^2] = (25 + 100 + 625)/4 = 750/4$$

Therefore, the amount he picks up on his way to work is a compound Poisson random variable with mean $10 \cdot 6 = 60$ and variance $6 \cdot 750/4 = 1125$. Because the number of pickup coins that Josh spots is Poisson with mean $6(3/4) = 4.5$, we can also view the amount picked up as a compound Poisson random variable $S = \sum_{i=1}^{N} X_i$ where N is Poisson with mean 4.5, and (with 5 cents as the unit of measurement) the X_i are equally likely to be 1, 2, 3. Either use the recursion developed in the text or condition on the number of pickups to determine $P(S = 5)$. Using the latter approach, with $P(N = i) = e^{-4.5}(4.5)^i/i!$, gives

$$P(S = 5) = (1/3)P(N = 1) + 3(1/3)^3 P(N = 3)$$
$$+ 4(1/3)^4 P(N = 4) + 5(1/3)^5 P(N = 5)$$

96. With $P_j = e^{-\lambda}\lambda^j/j!$, we have that N, the number of children in the family of a randomly chosen family is

$$P(N = j) = \frac{jP_j}{\lambda} = e^{-\lambda}\lambda^{j-1}/(j-1)!, \quad j > 0$$

Hence,

$$P(N - 1 = k) = e^{-\lambda}\lambda^k/k!, \quad k \geq 0$$

Note:
Sol'n to #97 (p. 181)
is on pp. 722-723
Sol'n to #99.b is on p. 723

98. $E[NS] = E[E[NS|N]] = E[NE[S|N]] = E[N^2E[X]] = E[X]E[N^2]$. Hence,

$$Cov(N, S) = E[X]E[N^2] - (E[N])^2E[X] = E[X]Var(N)$$

99. (a) p^k, (b) In order for $N = k + r$ the pattern must not have occurred in the first $r - 1$ trials, trial r must be a failure, and trials $r + 1, \ldots, r + k$ must all be successes.

(c) $1-P(N = k) = \sum_{r=1}^{\infty} P(N = k+r) = \sum_{r=1}^{\infty} P(N > r-1)qp^k = E[N]qp^k$

Sol'n to #16a p. 165 (notation is slightly modified)

Suppose that the pair of r.v.'s (Y_1, Y_2) have a bivariate normal distribution with parameters $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho$, where for each $(y_1, y_2) \in \mathbf{R}^2$,

$$f(y_1, y_2) = \frac{e^{-Q/2}}{2\pi \cdot \sigma_1 \cdot \sigma_2 \cdot \sqrt{1-\rho^2}}.$$

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Here, $$Q = \frac{1}{1-\rho^2} \cdot \left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - 2\rho \cdot \frac{(y_1-\mu_1)\cdot(y_2-\mu_2)}{\sigma_1 \cdot \sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2} \right]$$

Find the marginal probability density function (p.d.f.) of random variable Y_1.

Set $\theta := \mu_2 + \rho\sigma_1^{-1}\sigma_2(y_1-\mu_1)$ & $\tau^2 = \sigma_2^2(1-\rho^2)$

$y_1 \in R^1$ – fixed

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_2 = \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left(\frac{(y_1-\mu_1)^2}{\sigma_1^2}\right)\right\}}{\sqrt{2\pi \cdot \sigma_1^2}}$$

$$\times \frac{\exp\left\{-\frac{1}{2}\left(\frac{(y_2-\theta)^2}{\tau^2}\right)\right\}}{\sqrt{2\pi\tau^2}}\,dy_2 = \frac{\exp\left\{-\frac{1}{2}\left(\frac{(y_1-\mu_1)^2}{\sigma_1^2}\right)\right\}}{\sqrt{2\pi\sigma_1^2}}$$

this factor does not depend on y_2

$$\times \underbrace{\int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left(\frac{(y_2-\theta)^2}{\tau^2}\right)\right\}}{\sqrt{2\pi\tau^2}}\,dy_2}_{=1}$$

$= 1$ (for fixed y_1, θ & τ^2 – also fixed, and $\int = 1$, since it constitutes the total probability

$\Rightarrow f_1(y_1)$ – pdf of $N(\mu_1, \sigma_1^2)$

$\Rightarrow Y_1 \overset{d}{=} N(\mu_1, \sigma_1^2)$

Sol'n to part (b) can be easily found from numerous sources. Pls., search yourselves!