University of Toronto
Department of Mathematics

MAT224H1F
Linear Algebra II

Midterm Examination
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Duration: 1 hour 50 minutes

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

Tutorial Group: ______________________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

[10] **1.** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that has the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

relative to the bases $\alpha = \{(1,-1,1),(0,1,0),(1,0,0)\}$ of $\mathbb{R}^3$ and $\beta = \{(3,2),(2,1)\}$ of $\mathbb{R}^2$. Find $T(x,y,z)$ for any $(x,y,z) \in \mathbb{R}^3$.

Solution: Using the information in the question, we have that:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} = [T]_{\beta\alpha} = \left([T(1,-1,1)]_\beta \;\; [T(0,1,0)]_\beta \;\; [T(1,0,0)]_\beta\right)$$

Therefore, we get:

$$\begin{aligned} T(1,-1,1) &= 2(3,2)+1(2,1) = (8,5) \\ T(0,1,0) &= 3(3,2)+2(2,1) = (13,8) \\ T(1,0,0) &= 1(3,2)+1(2,1) = (5,3) \\ T(0,0,1) &= T(1,-1,1)+T(0,1,0)-T(1,0,0) = (8,5)+(13,8)-(5,3) = (16,10) \\ T(x,y,z) &= xT(1,0,0)+yT(0,1,0)+zT(0,0,1) \\ &= x(5,3)+y(13,8)+z(16,10) = (5x+13y+16z, 3x+8y+10z) \end{aligned}$$

[10] **2.** Let $T\colon P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by

$$T(a + bx + cx^2) = (a + b, b + c, a - c).$$

Find bases for the kernel and image of T.

Solution: Take the bases $\alpha = \{1, x, x^2\}$ of $P_2(\mathbb{R})$ and $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{R}^3$. Then, from the definition of T:

$$\begin{aligned} T(1) &= (1, 0, 1) = (1, 0, 0) + (0, 0, 1) \\ T(x) &= (1, 1, 0) = (1, 0, 0) + (0, 1, 0) \\ T(x^2) &= (0, 1, -1) = (0, 1, 0) - (0, 0, 1) \end{aligned}$$

So, we get the following matrix corresponding to T in bases α and β, and we row-reduce it:

$$[T]_{\beta\alpha} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R3-R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R3+R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The leading ones in the reduced matrix appear in the first and second column, so the first and second column of the original matrix give a basis for its image: $\mathrm{Im}[T]_{\beta\alpha} = \mathrm{span}\{(1, 0, 1), (1, 1, 0)\}$. From this, we can recover a basis $\{v_1, v_2\}$ for the image of T:

$$\begin{aligned} [v_1]_\beta &= (1, 0, 1) \Rightarrow v_1 = 1(1, 0, 1) + 0(0, 1, 0) + 1(0, 0, 1) = (1, 0, 1) \\ [v_2]_\beta &= (1, 1, 0) \Rightarrow v_2 = 1(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1) = (1, 1, 0) \end{aligned}$$

Furthermore, the kernel of the reduced version of $[T]_{\beta\alpha}$ is the same as the kernel of $[T]_{\beta\alpha}$. If (x, y, z) is such a vector, then:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the equations $x - z = 0,\ y + z = 0$ or equivalently $x = z,\ y = -z$. So, the only vectors in the kernel are scalar multiples of $(1, -1, 1)$, which gives a basis for $Ker[T]_{\beta\alpha}$. So, $KerT$ is also one dimensional with basis given by the polynomial $h(x)$ such that $[h(x)]_\alpha = (1, -1, 1)$. Namely, $h(x) = (1)1 + (-1)x + (1)x^2 = x^2 - x + 1$.

To summarize, a basis for the image of T is $\{(1, 1, 0), (1, 0, 1)\}$ and a basis for the kernel of T is $\{x^2 - x + 1\}$.

[10] **3.** Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - 2y + z = 0\}$. Show W is isomorphic to $\mathbb{R}^2$ and find an isomorphism $T\colon W \to \mathbb{R}^2$.

Solution: The equation for W tells us that points in W must satisfy $z = 2y - 3x$ so W consists of vectors $(x, y, 2y - 3x)$ where $x, y \in \mathbb{R}$. We can express this vector $(x, y, 2y - 3x) = x(1, 0, -3) + y(0, 1, 2)$. So, any vector in W is a linear combination of $(1, 0, -3)$ and $(0, 1, 2)$. Furthermore, these two vectors are linearly independent since $a_1(1, 0, 3) + a_2(0, 1, 2) = (0, 0, 0)$ implies $a_1 = 0, a_2 = 0, 3a_1+2a_2 = 0$. Therefore, $\alpha = \{(1, 0, -3), (0, 1, 2)\}$ is a basis for W. So, both W and $\mathbb{R}^2$ are two-dimensional and hence isomorphic. To construct an isomorphism between them, take the standard basis $\beta = \{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$. Then define a map $T\colon W \to \mathbb{R}^2$ by $T(1, 0, -3) = (1, 0)$ and $T(0, 1, 2) = (0, 1)$ and extend T to all of W by requiring it to be a linear transformation, namely, for any $(x, y, z) \in W$, $(x, y, z) = a(1, 0, -3) + b(0, 1, 2)$, then $T(x, y, z) = T(a(1, 0, -3) + b(0, 1, 2)) = aT(1, 0, -3) + bT(0, 1, 2) = a(1, 0) + b(0, 1) = (a, b)$.

We thus have a linear transformation and we need to show that it is injective and surjective, which will mean it is an isomorphism. Since W and $\mathbb{R}^2$ are both of dimension 2, T is surjective if and only if it is injective, so it suffices to prove it is injective. To show this, we will prove that $KerT = \{0\}$. For any vector in the kernel, $(x, y, z) = a(1, 0, -3) + b(0, 1, 2)$, we must have $(0, 0) = T(x, y, z) = (a, b)$. This means $a = b = 0$ and so $(x, y, z) = 0(1, 0, -3) + 0(0, 1, 2) = (0, 0, 0)$. Hence, T is injective and so an isomorphism.

[10] **4.** Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation whose matrix with respect to some basis α for $\mathbb{C}^2$ is

$$\begin{bmatrix} 1+i & 1-i \\ 1-i & 2 \end{bmatrix}.$$

Find the matrix of T^{-1} with respect to α, if possible.

Solution: Given $A = [T]_{\alpha\alpha}$, we need to reduce $[A|I]$ to find A^{-1}:

$$\begin{aligned}
[A|I] = &\left[\begin{array}{cc|cc} 1+i & 1-i & 1 & 0 \\ 1-i & 2 & 0 & 1 \end{array}\right] \xrightarrow{(1-i)/2\, R_1} \left[\begin{array}{cc|cc} 1 & -i & (1-i)/2 & 0 \\ 1-i & 2 & 0 & 1 \end{array}\right] \\
\xrightarrow{R_2-(1-i)R_1} &\left[\begin{array}{cc|cc} 1 & -i & (1-i)/2 & 0 \\ 0 & 3+i & i & 1 \end{array}\right] \xrightarrow{(3-i)/10\, R_2} \left[\begin{array}{cc|cc} 1 & -i & (1-i)/2 & 0 \\ 0 & 1 & (3i+1)/10 & (3-i)/10 \end{array}\right] \\
\xrightarrow{R_1+iR_2} &\left[\begin{array}{cc|cc} 1 & 0 & (1-2i)/5 & (3i+1)/10 \\ 0 & 1 & (3i+1)/10 & (3-i)/10 \end{array}\right]
\end{aligned}$$

So, we get:

$$[T^{-1}]_{\alpha\alpha} = [T]_{\alpha\alpha}^{-1} = \begin{bmatrix} (1-2i)/5 & (3i+1)/10 \\ (3i+1)/10 & (3-i)/10 \end{bmatrix}$$

[10]**5.** Let $T\colon \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$ be defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2, x_1 + x_2 + x_3, x_2 + 2x_3).$$

Show that there is no basis α for $\mathbb{Z}_3^3$ such that $[T]_{\alpha\alpha}$ is diagonal.

Solution: We have that:

$$\begin{aligned} T(x_1, x_2, x_3) &= (2x_1 + x_2, x_1 + x_2 + x_3, x_2 + 2x_3) \\ &= x_1(2, 1, 0) + x_2(1, 1, 1) + x_3(0, 1, 2) \end{aligned}$$

The standard basis for $\mathbb{Z}_3^3$ is $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and by the above we see that:

$$\begin{aligned} T(1, 0, 0) &= (2, 1, 0) \\ T(0, 1, 0) &= (1, 1, 1) \\ T(0, 0, 1) &= (0, 1, 2) \end{aligned}$$

So, the matrix corresponding to T is $A = [T]_{\beta\beta} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

To find the eigenvalues, we compute:

$$\begin{aligned} 0 &= \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda)((1-\lambda)(2-\lambda) - 1) - 1(1(2-\lambda) - 0(1)) = (2-\lambda)\lambda^2 \end{aligned}$$

The two eigenvalues are $\lambda = 2$ and $\lambda = 0$ with multiplicities one and two respectively. To find the eigenspace corresponding to the eigenvalue 0, we need to find the kernel of $A - 0I$. It is easier to do if we reduce the matrix first:

$$\begin{aligned} E_0 &= \text{Ker} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \\ &\xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Now, a vector (x, y, z) is in the kernel if and only if:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This tells us that $x + 2z = y + 2z = 0$ or equivalently $x = y = z$ (in $\mathbb{Z}_3$). This means that the eigenspace E_0 of $A = [T]_{\beta\beta}$ is spanned by the vector $(1, 1, 1)$ so has dimension 1. Therefore, the eigenspace of T for the eigenvalue 0 also has dimension 1, which is smaller than the multiplicity of $\lambda = 0$. So T is not diagonalizable, i.e. there is no basis with respect to which the matrix for T is diagonal.

6. Let V and W be vector spaces over a field F, and $T\colon V \to W$ a linear transformation. Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Prove $\dim(\operatorname{Ker}(T)) = 0$ if and only if $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

Solution: First, assume that $\dim(\operatorname{Ker}(T)) = 0$. We want to show that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent. Since it has dimension zero, $\operatorname{Ker}(T) = \{0\}$. Now if $a_1, \ldots, a_n$ are such that $a_1T(v_1)+\ldots+a_nT(v_n) = 0$, then by linearity of T, $T(a_1v_1+\ldots a_nv_n) = 0$. So, $a_1v_1+\ldots a_nv_n$ is in the kernel of T and therefore $a_1v_1+\ldots a_nv_n = 0$. $\{v_1, \ldots, v_n\}$ is linearly independent so we must have $a_1 = \ldots = a_n = 0$. This implies $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

Now, assume $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent. We want to show that $\dim(\operatorname{Ker}(T)) = 0$. It suffices to show $\operatorname{Ker}(T) = \{0\}$. Take any $v \in V$ which is in the kernel, i.e. $T(v) = 0$. Since $\alpha = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, there are some scalars $a_1, \ldots, a_n$ such that $v = a_1v_1 + \ldots + a_nv_n$. Then, $0 = T(v) = T(a_1v_1 + \ldots a_nv_n) = a_1T(v_1)+\ldots a_nT(v_n)$. We've assumed $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent so we must have $a_1 = \ldots = a_n = 0$. So, $v = 0$ which implies $\operatorname{Ker}(T) = \{0\}$ and so $\dim(\operatorname{Ker}(T)) = 0$.