Faculty of Arts & Science

August 2011 Examinations

MAT224 – Summer 2011 – Final

Thursday, August 18 2011, 7:00 to 10:00 pm.
Time allowed: 3 hours
Instructor: Karene Chu

Family Name: ______________________ Given Name: ______________________

Student Number: ____

Tutorial: ______________________

Signature: ____

1. Before you start, check that this exam has 17 numbered pages.
2. No aids allowed.
3. **For all questions, "justify" means prove or disprove the statement.**
4. **Very little credit will be given if only an answer but no necessary work or proof is shown.**
5. Answer each question in the space provided. You can use the back of each page for rough work. If you want the back of a page marked, clearly indicate this on the front of that page.

Question	Mark
1	/35
2	/35
3	/18
4	/17
5	/15
6	/36
Bonus	/7
Total	/156

1 (35 pts) Given the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$

$$q(\vec{v}) = 3x^2 + 4xy + 3y^2$$

where $[\vec{v}]_\varepsilon = \begin{pmatrix} x \\ y \end{pmatrix}$ are the coordinates of $\vec{v}$ relative to the standard basis ε of $\mathbb{R}^2$.

a. (10 pts) Find an orthonormal basis $\mathcal{A}$ (orthonormal w.r.t. the standard dot product) relative to which the quadratic form is diagonal, i.e. $q(\vec{v}) = a\tilde{x}^2 + b\tilde{y}^2$ where $[\vec{v}]_\mathcal{A} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ are the coordinates of $\vec{v}$ relative to the basis $\mathcal{A}$ and a, b are real numbers.

b. (5 pts) Sketch the locus of $q(\vec{v}) = 1$ on the $xy-$ plane (i.e. with x- and y- coordinate axes and not the $\tilde{x}-$ and $\tilde{y}$- axes). (Make sure to clearly draw the basis elements of $\mathcal{A}$ on the $xy-$ plane.)

c. (5 pts) Find a basis $\mathcal{B}$ relative to which the quadratic form is in normal form, i.e. $q(\vec{v}) = \pm x'^2 \pm y'^2$ where $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ are the coordinates of $\vec{v}$ relative to the basis $\mathcal{B}$.

d (5 pts) Is the basis $\mathcal{B}$ an orthonormal basis? Justify.
What is the shape of the locus of $q(\vec{v}) = 1$ as drawn on the $x'y'$- axes?

e (5 pts) Extend $q(\vec{v})$ to a quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ on the 3-dimensional space $\mathbb{R}^3$ simply by

$$Q(\vec{v}) = 3x^2 + 4xy + 3y^2$$

where $[\vec{v}]_\varepsilon = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are the coordinates of $\vec{v}$ relative to the standard basis ε of $\mathbb{R}^3$.

Describe the shape of the surface defined by $Q(\vec{v}) = 1$. (Hint: Look at slices of the surface at different z-values, e.g. $z = -10000, z = -100, z = 0, z = 0.3, z = 40000$)

f (5 pts) Can there be an inner product on $\mathbb{R}^3$ such that for all $\vec{v}$ in $\mathbb{R}^3$ the norm $\|\vec{v}\| = Q(\vec{v})$? Justify your answer.

2 (35 pts)

a. (7 pts) Verify that the linear function $(\cdot,\cdot):\mathbb{C}^3\times\mathbb{C}^3\to\mathbb{C}$ defined below is an inner product on $\mathbb{C}^3$:

$$(\vec{v},\vec{w}) := [\vec{v}]_\varepsilon{}^* A\ [\vec{w}]_\varepsilon \quad \text{where } A=\begin{pmatrix} 2 & 1-i & 0\\ 1+i & 3 & 0\\ 0 & 0 & 1\end{pmatrix}$$

where $[\vec{v}]_\varepsilon$ are the coordinates of $\vec{v}$ relative to the standard basis ε of $\mathbb{C}^3$.

b. (10 pts) Find a basis $\mathcal{A}$ of $\mathbb{C}^3$ such that relative to it the inner product is just the dot product, i.e.

$$(\vec{v}, \vec{w}) = [\vec{v}]_{\mathcal{A}}^{*} \, [\vec{w}]_{\mathcal{A}}.$$

c. (5 pts) Given a subspace $W = \langle \vec{w} \rangle$ where $[\vec{w}]_{\mathcal{A}} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ are the coordinates of the vector $\vec{w}$ relative to the basis $\mathcal{A}$ from part (b), find an orthonormal basis for the orthogonal complement $W^{\perp}$.

d. (5 pts) Given any vector $\vec{v}$ with coordinates $[\vec{v}]_{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ relative the basis $\mathcal{A}$ from part (b).

Write it as a sum $\vec{v} = \vec{v}_W + \vec{v}_{W^{\perp}}$ where $\vec{v}_W$ and $\vec{v}_{W^{\perp}}$ are vectors in the subspaces W and $W^{\perp}$ respectively. (Clearly indicate in which basis you have written the coordinates of $\vec{v}_W$ and $\vec{v}_{W^{\perp}}$.)

e. (8 pts) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be the linear operator given by

$$[T(\vec{v})]_{\mathcal{A}} = \begin{pmatrix} x+iy \\ x-iy \\ z-x-y \end{pmatrix}_{\mathcal{A}} \quad \text{where } [\vec{v}]_{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Find the adjoint operator T^*.

3 (18 pts)

a. (3 pts) Show that the following matrix A is normal.

$$A = \begin{pmatrix} i & i-1 \\ i+1 & 0 \end{pmatrix}$$

b. (15 pts) Find the spectral decomposition of the matrix A.

(More space for Question 3b)

4 (17 pts)

a. (17 pts) Show that for any Hermitian matrix A, the geometric and algebraic multiplicities of any of its eigenvalues are equal.

bonus (7 pts) Is the statement in part (a) true for normal matrices? If so, modify your proof to show the same statement for normal matrices; if not, give a counter example.

5 (15 pts) In your homework, you have shown that in $\mathbb{R}^n$, a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal*, i.e. satisfies $\|T(\vec{u})\| = \|\vec{u}\|$ for every vector $\vec{u} \in \mathbb{R}^n$, if and only if $T(\vec{u}) \cdot T(\vec{v}) = \vec{u} \cdot \vec{v}$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$.

Show that the same is true in any inner product space, i.e. in any vector space V with an inner product $(\cdot, \cdot)$, a linear operator $T : V \to V$ satisfies $\|T(\vec{u})\| = \|\vec{u}\|$ for every vector $\vec{u} \in V$ if and only if $(T(\vec{u}), T(\vec{v})) = (\vec{u}, \vec{v})$ for all $\vec{u}, \vec{v} \in V$.

6. (36 pts)

a. (10 pts) Give an example matrix for each of the following classes of matrices: (Hint: 2×2 matrices suffice)

i. (2 pts) orthogonal (O)

ii. (2 pts) unitary but not orthogonal (U)

iii. (1 pts) symmetric (S)

iv. (2 pts) Hermitian but not symmetric (H)

v. (3 pts) normal but neither unitary or Hermitian (N)

b. (11 pts) Which of the above classes consist of matrices which all satisfy the following properties? Use the letters in brackets for each of the classes, e.g. U for the set of all unitary matrices that are not orthogonal.

i. (2 pts) unitarily similar to an upper triangular matrix?

ii. (2 pts) unitarily similar to a diagonal matrix?

iii. (2 pts) unitarily similar to a real diagonal matrix?

iv. (3 pts) orthogonally similar to a real diagonal matrix?

v. (2 pts) have only eigenvalues whose absolute values are 1?

c. (5 pts) Give an example of a matrix that is not normal but diagonalizable over $\mathbb{C}$. Justify.

d. (5 pts) Give an example of a matrix that is not diagonalizable over $\mathbb{C}$. Justify.

e. (5 pts) Give an example of a matrix that represents a projection operator. State the overall vector space and the subspace the operator projects into.