Duration: **60 minutes**
Aids Allowed: **none**

Student Number: ___

Family Name(s): ___

Given Name(s): ___

*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 4 questions on 12 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use one of the "blank" pages for rough work. If you need more space for one of your solutions, use a "blank" page and *indicate clearly the part of your work that should be marked*.

In your answers, you may use without proof any theorem or result covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do — part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

MARKING GUIDE

# 1:	___/ 8
# 2:	___/ 8
# 3:	___/10
# 4:	___/10
BONUS MARKS:	___/ 3
TOTAL:	___/36

Good Luck!

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.

Question 1. [8 MARKS]

Part (a) [4 MARKS]

Write a detailed proof *structure* for the following statement.

$$\forall x \in D, \big(P(x) \wedge Q(x) \Rightarrow \exists y \in D, R(x,y)\big)$$

NOTE: You need only design the *structure* of a proof for a general statement of this form.

Part (b) [4 MARKS]

Fill in every missing conclusion in the following proof structure.

Assume $x \in D$ and $y \in D$.
  Assume $P(x,y)$.
    Assume $\neg Q(x) \vee \neg Q(y)$.
      Assume $\neg Q(x)$.
        ...derive a contradiction...
      Assume $\neg Q(y)$.
        ...derive a contradiction...

      Then, in both cases, ______________________________

    Then, ______________________________

  Then, ______________________________

Then, ______________________________

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.

Question 2. [8 MARKS]

Part (a) [3 MARKS]

Recall that for all integers m and n, the notation "$m \mid n$" means that m divides n **exactly**. In that case, we say that m is a *divisor* of n.

Definition: An integer c is a *common divisor* of integers m and n if and only if c is a divisor of m and c is a divisor of n.

Definition: An integer d is called the *greatest common divisor* of integers m and n if and only if d is a common divisor of m and n and d is larger than any other common divisor of m and n. In this case, we write $d = \gcd(m, n)$.

Fill in the blank below to complete the definition of greatest common divisor using the notation of symbolic logic.

$$
\begin{aligned}
& d = \gcd(m, n) \\
\iff\ & d \in \mathbb{Z} \wedge \underline{\hspace{10cm}}
\end{aligned}
$$

Part (b) [5 MARKS]

Prove that for all integers m and n, if d is a *common divisor* of m and n (but d is not necessarily the *greatest* common divisor) then d is a common divisor of n and $m - n$. (HINT: Use proof structures.)

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.

Question 3. [10 MARKS]

Recall that $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ for all numbers x, y.
Consider the following statement:

> For every integer n, the remainder when n^4 is divided by 8 is either 0 or 1.

Part (a) [2 MARKS]

Translate the statement into symbolic notation.

Part (b) [8 MARKS]

Write a detailed proof of the statement. (HINT: Use proof structures.)

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.

Question 4. [10 MARKS]

Consider the following statement:

> For every real number x, there is a **unique** real number y such that $x^2y = x - y$.

Part (a) [2 MARKS]

Translate the statement into symbolic notation.

Part (b) [8 MARKS]

Write a detailed proof of the statement. (HINT: Use proof structures.)

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.

Bonus. [3 marks]

WARNING! This question will be marked harshly: credit will be given only for making *significant* progress toward a correct answer. Please attempt this only *after* you have completed the rest of the test.

Describe *two different approaches* you might use to try and answer the question below. Describe clearly the expected outcome of each of your strategies. (Note that you are **not** being asked to actually answer the question; doing so will earn you **no credit**.)

Question: Determine every integer value of `n` for which the following loop eventually halts.

```
# Assume n is an integer.
while n > 1:
    if n % 2 == 0:  # i.e., if n is even
        n = n / 2
    else:
        n = 3 * n + 1
```

On this page, please write nothing except your name.

Family Name(s): ______________________________

Given Name(s): ______________________________

Total Marks = 36