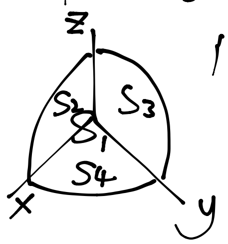
August 7th

① §5.5 #2 change surface S to be the boundary of the portion of the sphere which falls in the Octant $(x, y, z \geq 0)$

S = sphere of radius a, in the first octant



$$\{(x,y,z \mid x,y,z \geq 0,\ x^2+y^2+z^2=a^2\} = S_1$$

$$\iint_S \vec{F}\cdot\hat{n}\, dA = \iint_{S_1} \vec{F}\cdot\hat{n}\, dA + \iint_{S_2} + \iint_{S_3} + \iint_{S_4}$$

Parametrization for S_1

$$\begin{cases} x = a\sin\Phi\cos\theta \quad , y = a\sin\theta\sin\Phi \quad z = a\cos\Phi \\ 0 \leq \theta \leq \pi/2 \quad 0 \leq \Phi \leq \frac{\pi}{2} \end{cases}$$

$$\vec{G}(\theta,\Phi) = a(\sin\Phi\cos\theta,\ \sin\theta\cos\Phi,\ \cos\Phi)$$

$$\vec{N} = \frac{\frac{\partial G}{\partial \Phi}\times\frac{\partial G}{\partial \theta}}{\left|\frac{\partial G}{\partial \theta}\times\frac{\partial G}{\partial \Phi}\right|} \qquad \vec{F} = a^2(a\sin\Phi\cos\theta, a\sin\Phi\sin\theta, a\cos\Phi)$$

$$dA = \left|\frac{\partial G}{\partial \theta}\times\frac{\partial G}{\partial \Phi}\right| d\theta\, d\phi$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, dA = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} a^3 (\sin\Phi\cos\theta, \sin\Phi\sin\theta, \cos\Phi)\cdot \frac{\partial G}{\partial \Phi}\times\frac{\partial G}{\partial \theta}\, d\theta\, d\Phi$$

$$\vec{N} = \frac{\partial G}{\partial \theta}\times\frac{\partial G}{\partial \Phi} = (-a\sin\Phi\sin\theta, a\cos\theta\sin\Phi, 0)\times(a\cos\Phi\cos\theta, a\cos\theta\sin\theta, -a\sin\Phi)$$
$$= -(a^2\sin^2\Phi\cos\theta, a^2\sin^2\Phi\sin\theta, a^2\sin\Phi\cos\Phi)$$

$$\left|\frac{\partial G}{\partial \theta}\times\frac{\partial G}{\partial \Phi}\right| = a^2\sin\Phi$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, dA = \int_0^{\pi/2}\int_0^{\pi/2} d\theta\, d\Phi\, a^5 \left(\sin\Phi\cos\theta, \sin\Phi\sin\theta, \cos\Phi\right)\cdot(\sin^2\Phi\cos\theta, \sin^2\Phi\sin\theta, \sin\Phi\cos\Phi)$$

$$= \iint a^5(\sin^3\Phi\cos^2\theta + \sin^3\Phi\sin^2\theta + \sin\Phi\cos^2\Phi)\, d\theta\, d\Phi$$

$$= \iint a^5(\sin^3\Phi + \sin\Phi\cos^2\Phi)\, d\theta\, d\Phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} a^5\sin\Phi\, d\theta\, d\Phi = \int_0^{\pi/2} \frac{\pi}{2}a^5\sin\Phi\, d\Phi = \frac{\pi}{2}a^5(-\cos\Phi)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}a^5 \qquad \iint_{S_2} \vec{F}\cdot\hat{n}\, dA$$

$\vec{F} = (x^2+y^2+z^2)(x,y,z)$

$\vec{F}\big|_{S_2} = (x^2+y^2\ )(x,y,0) \Longrightarrow \vec{F}\cdot\hat{n}\big|_{S_2} = 0$

Similarly for S_3, S_4.

Conclusion: $S = S_1 \cup S_2 \cup S_3 \cup S_4$

$\iint_S \vec{F}\cdot\hat{n}\,dA = \iint_S \vec{F}\cdot\hat{n}\,dA = \frac{\pi}{2}a^5$

Let E = interior of S ($\partial E = S$)

$\iint_S \vec{F}\cdot\hat{n}\,dA = \iiint_E \nabla\cdot\vec{F}\cdot dV$ (div thm)

$\nabla\cdot\vec{F} = 5(x^2+y^2+z^2)$ Use spherical polar coor: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

$dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

see problem set

$\iint_S \vec{F}\cdot\hat{n}\,dA = \iiint_E \nabla\cdot\vec{F}\cdot dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a 5\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \cdots = \frac{\pi}{2}a^5.$

② $\vec{F}(x,y,z) = \dfrac{(-y,x,0)}{x^2+y^2}$ defined on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$

in cyl. polar coor.

$\vec{F}(\vec{g}(t)) = \dfrac{(-a\sin t, a\cos t, 0)}{a^2}$

(a). $\nabla\times\vec{F} = \vec{0}$

(b).

C. $\vec{g}(t) = (a\cos t, a\sin t, h)$, a = const b = const

$\int_C \vec{F}\cdot d\vec{x} = \int_0^{2\pi} dt\,\dfrac{(-a\sin t, a\cos t, 0)}{a^2}(-a\sin t, a\cos t, 0)$

$= \int_0^{2\pi}\frac{a^2}{a^2}\,dt = 2\pi$

(c). $\int_C \vec{F}\cdot d\vec{x} \underset{\text{Stoke's}}{=} \iint_D \nabla\times\vec{F}\cdot\vec{n}\,dA = \iint (0,0,0)\cdot\vec{n}\,dA = 0$

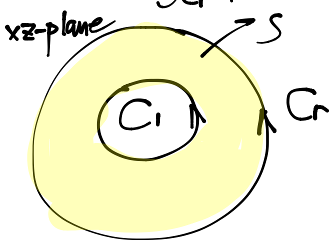
! But $\vec{F}$ wasn't defined along z-axis (it blows up) which passes through D $\longmapsto$ can't apply stokes

③ Let C_r = circle of radius r in the xz-plane around $\vec{0}$

$\vec{F} = C^1$ vector field defined on $\mathbb{R}^3 \setminus \{y\text{-axis}\}$, with

$\int_{C_1} \vec{F}\cdot d\vec{x} = 5$, $\nabla\times\vec{F} = \dfrac{(z,3,-x)}{(x^2+z^2)^2}$

Want $\int_{C_r} \vec{F}\cdot d\vec{x}$



By Stoke's thm

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dA = \int_{C_r}\vec{F}\cdot d\vec{x} - \int_{C_1}\vec{F}\cdot d\vec{x}$$

why? check orientation

$$= \int_{C_r}\vec{F}\cdot d\vec{x} - 5$$

compute this. get $\hat{n} = (0,1,0)$

$$\nabla\times\vec{F}\cdot\hat{n} = \frac{3}{(x^2+z^2)^2} \Rightarrow \iint_S \nabla\times\vec{F}\cdot\vec{n}\,dA$$

$$= 3\pi\left(1-\frac{1}{r^2}\right)$$

$$\int_{C_r}\vec{F}\cdot d\vec{x} = 3\pi\left(1-\frac{1}{r^2}\right)+5$$