Department of Mathematics, University of Toronto

MAT224H1S - Linear Algebra II

Winter 2013

Problem Set 6

- Due Tues. March 26, 6:10pm sharp. Late assignments will not be accepted.
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Suppose V is an inner product space, and $T\colon V \mapsto V$ a linear operator. We already know the dimension theorem:

$$dim(V) = dim(im(T)) + dim(ker(T)).$$

When V is an inner product space, more can be said. The following question describes the relationship among the four subspaces defined by a linear operator T: $ker(T)$, $im(T)$, $ker(T^*)$, $im(T^*)$ - we'll call these four subspaces the *four fundamental subspaces* of T.

Show

(a) $ker(T) = im(T^*)^\perp$, and $ker(T)^\perp = im(T^*)$.

(b) $im(T) = ker(T^*)^\perp$, and $im(T)^\perp = ker(T^*)$.

Notice that all you need to show is $ker(T) = im(T^*)^\perp$, since then $ker(T)^\perp = (im(T^*)^\perp)^\perp = im(T^*)$. Also, part (b) follows by applying (a) to T^* in place of T.

2. Consider $P_2(\mathbb{R})$ together with inner product $< p(x), q(x) >= \int_0^1 p(x)q(x)\,dx$. Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = p'(x)$. Find a basis for each of the *four fundamental subspaces* of T

3. Suppose V is an inner product space, and $T\colon V \mapsto V$ a linear operator. Show that $dim(im(T)) = dim(im(T^*))$.

Hint: Use question 1 together with that fact that for any subspace W of V, $V = W \oplus W^\perp$.

4. Let $N\colon \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$N = \begin{bmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Show that N is nilpotent and find the smallest k such that $N^k = 0$.

(b) Find the canonical form of N and a canonical basis.

5. Suppose A is a real 4×4 matrix that is nilpotent of order (index) k, $1 \leq k \leq 4$. Make a list of all the possible dimensions of $null(A)$, $null(A^2), \ldots, null(A^{k-1})$ (note that $dim(null(A^k)) = 4$) and the corresponding canonical forms of A.

6. Let $T\colon P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear operator defined by $T(p(x)) = p''(x) + p(x)$. Find a basis α for $P_3(\mathbb{R})$ such that $[T]_{\alpha\alpha}$ is in canonical form and determine $[T]_{\alpha\alpha}$.

Suggested Extra Problems (not to be handed in):

- Textbook, Section 6.2 **1-5, 7, 12, 13**
- Let $N\colon \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$N = \begin{bmatrix} 6 & 2 & 1 & -1 \\ -7 & -1 & -1 & 2 \\ -9 & -7 & -2 & -1 \\ 13 & 3 & 2 & -3 \end{bmatrix}$$

(a) Show that N is nilpotent and find the order (index) of N (i.e. the smallest k such that $N^k = 0$).

(b) Find the canonical form of N and a canonical basis.