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MAT337 Homework 1

Page 18, problem B.

10 Solution:

Let $a_n = \sin\frac{n\pi}{2}$, suppose it has limit $L$ s.t.

$\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$.

Set $\varepsilon = \frac{1}{3}$, for $k$ sufficiently large st. $n=4k \geq N$

$|\sin 2k\pi - L| < \varepsilon = \frac{1}{3}$

i.e. $|0 - L| < \varepsilon \Rightarrow |L| < \frac{1}{3}$

Similarly for $k$ sufficiently large st. $n = 4k+1 \geq N$

we have $|\sin(2k+\frac{1}{2})\pi - L| < \varepsilon = \frac{1}{3}$

$|1 - L| < \frac{1}{3}$

So $1 = (1-L) + L$

$|1| = |(1-L)+L| \leq |1-L| + |L| < 2\varepsilon = \frac{2}{3}$ contradiction

So it does not have limit.

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$a_1 = 0$, $a_{n+1} = \sqrt{5+2a_n}$ for $n \geq 1$.

$a_1 = 0$
$a_2 = 2.23607$
$a_3 = 3.07768$
$a_4 = 3.33997$
$a_5 = 3.41759$
$a_6 = 3.44023$
$a_7 = 3.446804$
...

10 Solution:

$a_1 = 0$, $a_2 = \sqrt{5+2\times 0} = \sqrt{5}$, $a_3 = \sqrt{5+2\sqrt{5}}$

By observation, we claim that $0 \leq a_n < a_{n+1} < 3.5$

By induction, $n=1$, $0 \leq a_1 < \sqrt{5} = a_2 < 3.5$

Suppose it holds for $n$.

then $a_{n+2} = \sqrt{5+2a_{n+1}} > \sqrt{5+2a_n} = a_{n+1} \geq 0$

and $a_{n+2} = \sqrt{5+2a_{n+1}} < \sqrt{5+7} = \sqrt{12} < 3.5$

Finished the induction part.

So we have a monotone increasing sequence but which is bounded above!

Then by monotone convergence theorem, it has limit, say it $L$.

So. $\sqrt{5+2L} = L$

$5 + 2L = L^2$

$L^2 - 2L - 5 = 0$

$L = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$

it's impossible for $L = 1 - \sqrt{6} < 0$ since it's increasing and $a_1 = 0$.

Therefore $L = 1 + \sqrt{6}$

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Page 26 problem A

Solution:

$$(a_n) = \left( \frac{n \cos^n(n)}{\sqrt{n^2+2n}} \right)_{n=1}^{\infty}$$

Note that $n^2 < n^2+2n$ $n \geq 1$

so $\frac{n}{\sqrt{n^2+2n}} < 1$

since $\cos(n) \in [-1, 1]$

$|\cos(n)|^n \leq 1$

Therefore $\frac{n\cos^n(n)}{\sqrt{n^2+2n}}$ is bounded above by 1

and bounded below by -1.

By Bolzano-Weierstrass Theorem, it's a bounded sequence of real numbers. so it has a convergent subsequence.

Page 31 problem A

Solution. Let $m = n_k$

so since subsequence $(x_{n_k})$ has $\lim_{k\to\infty} x_{n_k} = a$.

$\forall \varepsilon > 0, \exists N \in \mathbb{Z}$ s.t.

$|x_m - a| < \frac{\varepsilon}{2}$, for all $m \geq N$

and by Cauchy sequence's definition,

$\forall \varepsilon > 0, \exists N' \in \mathbb{Z}$ s.t.

$|x_n - x_m| < \frac{\varepsilon}{2}$ for all $m, n \geq N'$

Therefore,

$\forall \varepsilon > 0, \exists N' \in \mathbb{Z}$ s.t.

$|x_n - a| \leq |x_n - x_m| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

for all $n \geq N'$

Hence $\lim_{n\to\infty} x_n = a$.

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this is too deep abuse of notation. The second line is just not true

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Page 42 H:

Let $\limsup\limits_{n\to\infty} \frac{|a_n|}{b_n} = l < \infty$, then $\exists\, r < l$, suppose $\varepsilon = r - l$, since $\varepsilon > 0$ we can find integer $N > 0$ s.t. $\left|\frac{a_n}{b_n}\right| < r = l + \varepsilon$, $\forall n \geq N$

Therefore $\frac{|a_n|}{b_n} < r$ for all $n \geq N \Rightarrow \sum\limits_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$

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$$< \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} (b_n \cdot r)$$
$$= \sum_{n=1}^{N-1} a_n + r \sum_{n=N}^{\infty} b_n$$

Note that $\sum\limits_{n=1}^{N-1} a_n$ is bounded above (not infinity)

and $\sum_{n=1}^{\infty} b_n$ converges.

By Cauchy Criterion, $\forall \varepsilon > 0$, $\exists N$ s.t. $\left|\sum_{n=N-1}^{\infty} b_n\right| < \varepsilon$

and $\sum_{n=N}^{\infty} b_n \leq \left|\sum_{n=N}^{\infty} b_n\right|$

we can find some $\varepsilon$, $N$ s.t. $\left|\sum_{n=N}^{\infty} b_n\right| < \varepsilon$ for $n > N$

as a result: $r \sum_{n=N}^{\infty} b_n \leq r \left|\sum_{n=N}^{\infty} b_n\right| < r\varepsilon$

as $r \sum_{n=N}^{\infty} b_n$ is also bounded above (not infinity)

so $\sum_{n=1}^{\infty} a_n < \infty$, bounded above $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. ■

Problem I:

$(a_n)_{n=1}^{\infty}$, $a_i > 0$, $\forall i$

Proof: ① If $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\exists\, \varepsilon > 0$ and $N > 1$ s.t.

$\frac{a_{n+1}}{a_n} < 1 - \varepsilon$ for $n > N$ set $1 - \varepsilon = r$

2 (*) $a_n \cdot \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} \cdots \frac{a_N}{a_{N-1}} = a_N < a_n \cdot r^{N-n}$ for $N > n$ ?

on the RHS, by Thm 3.2.2 it's a geometric series with $|r| < 1$. so the constructed geometric series on RHS is summable.

What about $n \leq N$

By Comparison Test, since (*) and RHS is summable then LHS (the actual series) is summable as well

i.e. $\sum_{n=1}^{\infty} a_n$ converges.

5 ② Conversely, $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ (Trivially the same then $\exists\, \varepsilon' > 0$ & $N > 1$ s.t. $\frac{a_{n+1}}{a_n} > 1 + \varepsilon'$ for $n > N$ as above)

set $1 + \varepsilon' = p$.

(**). $a_N > a_n \cdot p^{N-n}$ for $N > n$

on RHS, by Thm 3.2.2, it's a geometric series with $|p| > 1$, not summable.

By comparison test again, LHS is not summable

i.e. $\sum_{n=1}^{\infty} a_n$ diverges. ■