MATH6222 Week 10 Lecture Notes

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1 Monday's Lecture

[15]

Qustion: What's the largest possible subset of [15] that does not have a triple of consecutive elements.

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}$

Claim: 10 is largest. Need to show 11 is impossible.

If we take a subset of size 11 one of these 5 groups must have at least 3 elements.

Pigeonhole Principle: If more than $k \cdot n$ objects are divided into n classes, then one class must have more than k objects.

Proof: Suppose every class had $\leq k$ objects. Then the total number of objects would have to be $\leq n \cdot k$.

Let's construct a large collection of integer points whose midpoints are not integer points (in $\mathbb{R}^2$).

Given (a, b) and (c, d). When is their midpoint an integer point?

$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

get an integer midpoint if (a, c) have same parity, (b, d) have same parity.

Claim: Given 5 integer points on plane, some pair must have an integer midpoint.

Divide integer points into 4 classes. (even, even), (even, odd), (odd, even), (odd, odd).

By PHP, one class must get two points. These two must have integer midpoint.

$$8, 4, 7, 5, 1, 9, 3, 6, 2, 10$$

Can I rearrange to get rid of monotone subsequence of length 5? Yes.
What about 4?

$$3, 2, 1, 6, 5, 4, 9, 8, 7, 10$$

Theorem (Erds, 1935): Given a sequence of $n^2 + 1$ integers, there exists monotone subsequence of length $n + 1$.

Let $a_k : k = 1, 2, \ldots, n^2 + 1$ be our sequence of integers.
Let $x_k :=$ length of the largest increasing subsequence ends at a_k.
Let $y_k :=$ length of the largest decreasing subsequence ends at a_k.

a_k	7	4	8	5	1	9	3	6	2	10
x_k	1	1	2	2	1	3	2	3	2	
y_k	1	2	1	2	3	1	3	2	4	

What I want to show is that one of these numbers x_k, y_k must reach the value $n + 1$.

Suppose this doesn't happen. Then $x_k \leq n, y_k \leq n$, so I have at most n^2 possible pairs (x_k, y_k). Since we have $n^2 + 2$ pairs, some pair must be repeated: $(x_i, y_i) = (x_j, y_j), i < j$. This is impossible.

a_i, a_j
$(x_i, y_i) = (x_j, y_j)$.

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2 Thursday's Lecture

Let$\phi(m) :=$ number of $[m]$ relatively prime to m.
$\phi(6) = 2$, $\{1, 5\}$
$\phi(7) = 6$, $\{1, 2, 3, 4, 5, 6\}$

Question: How can we compute $\phi(m)$ efficiently?

$$\phi(20), 20 = 2^2 \cdot 5$$

$$\phi(20) = 20 - \frac{20}{2} - \frac{20}{5} + \frac{20}{10}$$

10 was excluded twice as both multiple of 2 and multiple of 5, so we have to add it back.

$$m = p_1^{e_1} p_2^{e_2},\ \phi(m) = m - \frac{m}{p_1} - \frac{m}{p_2} + \frac{m}{p_1 p_2}$$

$$30 = 2 \cdot 3 \cdot 5$$

$$\phi(30) = 30 - \frac{30}{2} - \frac{30}{3} - \frac{30}{5} + \frac{30}{2\cdot 3} + \frac{30}{2\cdot 5} + \frac{30}{3\cdot 5} - \frac{30}{2\cdot 3\cdot 5} = 30-15-10-6+5+3+2-1 = 8$$

Suppose $A_1, \dots A_n$ are subsets of a finite set U (Universe). We want to count the number of elements in $U - (A_1 \cup A_2 \cup \cdots A_n)$

$U = [m], m = p_1^{e_1} \cdots p_k^{e_k}$.

A_i = integers $\leq m$ that are multiples of p_i, $\phi(m) = |U - (A_1 \cup \cdots A_k)|$.

Example: $|U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$.

Inclusion-Exclusion Formula: For any subset $S = [n], A_S = \bigcap_{i \in S} A_i$,

$$|U - (A_1 \cup \cdots \cup A_n)| = \sum_{S \subseteq [n]} (-1)^{|S|} |A_S|$$

where $A_S := \cap_{i \in S} A_i$.

Proof: First, consider $x \in U - (A_1 \cup \cdots \cup A_n)$, then x contributes one to the total sum because $x \in A_\varnothing$.

Second, consider $x \in (A_1 \cup \cdots \cup A_n)$.

Define $T \subseteq [n]$ by:

$$i \in T \iff x \in A_i$$

Then $x \in A_S \iff S \subseteq T$.

x contributes $+1$ to total sum for each $S \subseteq T$ such that $|S|$ = even contributes -1 to total sum for each $S \subseteq T$ such that $|S| = $ odd

The total contribution of x to sum is zero.

If |number subsets of T with even size| = |number subsets of T with odd size|. Construct an explicit bijection.

Compute $\phi(m): \ m = p_1^{e_1} \cdots p_k^{e_k}, \ U = [m], \ A_i = $ integers $\leq m$ divisible by p_i $(i = 1, \ldots, k)$.

$$\phi(m) = |U - (A_1 \cup \cdots \cup A_k)|$$

$$A_S = \bigcap_{i \in S} A_i = \text{multiples of } \prod_{i \in S} p_i$$

$$A_{\{1,2\}} = A_1 \cap A_2 = \text{multiples of } p_1 p_2$$

$$\phi(m) = \sum_{S \subseteq [k]} (-1)^{|S|} |A_S| = \sum_{S \subseteq [k]} (-1)^{|S|} \frac{m}{(\prod_{i \in S} p_i)} = m \prod_{i=0}^{k} \left(1 - \frac{1}{p_i}\right)$$

Definition: A derangement is a permutation with no fixed points.

For 1,2,3: only 2,3,1 and 3,1,2 are derangements.

Question: How many derangements of $[n]$?

$U =$permutations of $[n]$, $|U| = n!$.

$A_1 =$permutations fixing 1.

$A_2 =$permutations fixing 2.

$\ldots$

$A_n =$permutations fixing n.

Number of derangements of $[n] = \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.s$

3 Friday's Lecture

Last time: $A_1, \ldots, A_n \subseteq U$

$|U - (A_1 \cup \cdots \cup A_n)| = \sum_{S \subseteq [n]} (-1)^{|S|} |A_S|$

Problem: Roll a die n times. What is the probability that every number 1 to 6 appears at least one?

Counting problem... Then there are 6^n total possibilities. The probability of 1 not appearing is $\left(\frac{5}{6}\right)^n$.

Let $U = \{$ be the all possible rolls of n dice, length in sequences $\{1,2,3,4,5,6\}\}$.

$|U| = 6^n$. Let A_i be the set of rolls where i never appears.

$$|U - (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6)|$$

$|A_i| = S^n, A_{\{1,2\}}, A_S = \bigcap_{i \in S} A_i, |A_S| = (6 - |S|)^n$

$$|U - \cdots| = \sum_{S \subseteq [6]} (-1)^{|S|}(6 - |S|)^n = \sum_{k=0}^{6} (-1) \binom{6}{k} (6-k)^n$$

which is $1 - 6\left(\frac{5}{6}\right)^n + 15\left(\frac{4}{6}\right)^n - 20\left(\frac{3}{6}\right)^n + 15\left(\frac{2}{6}\right)^n - 6\left(\frac{1}{6}\right)^n$.

And the probability is this number divided by 6^n.

(Incorrect) alternative thinking: We have 6! ways to pick out 6 out of n, then we don't care about the rest numbers here. Then we have

$$6! \binom{n}{6} 6^{n-6}$$

ways. But this has "double-counting" issue! Example: 123324415566.

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a	a	b	b	c	d
f	g	e	e	c	d
f	g	i	j	l	l
h	h	i	j	m	m
n	o	q	q	r	r
n	o	p	p	s	s

Table 1: a 6×6 chessboard filled with 2×1 domino tiles

Observation 1: at least one of horizontal or vertical lines is not cut.

Observation 2: a line is cut an even number of times.

Suppose every line is cut, 10 lines, it we need at least 2×10 dominos to cut those lines, but we only have 18. So we are done.

A game: Two choices, can take either Box1 or Box1 and Box2 together. Box2 has \$1,000 dollars. Box1 has a predictor (yesterday) predicted which of these two options you would choose:

- If it thinks you are going to take both boxes, there's nothing in Box1.
- If it thinks you are going to take only Box1, there's \$1,000,000 in Box1.

	Predictor is right	Predictor is wrong
Box 1	1,000,000	0
Box 1&2	1,000	1,001,000

Table 2: A game

Predictor is "always" right (99.99% $\simeq$ = 100%).
expected pay-off of just take Box1: $99.99\% \times 10^6 + .001\% \times 0 = 999900$
expected pay-off of taking Box1&2: $99.99\% \times 10^3 + .001\% \times 1001000 = 1000$

freewill

But the prediction was made yesterday, and we know there are money in two boxes, why don't we just take both boxes?