University of Toronto

Faculty of Arts and Sciences

Sample Final Exam, April-May 2014

MAT 337 H1

Intro Real Analysis

Instructor: Regina Rotman

Duration - 3 hours

No aids allowed

Total marks for this paper is 400

Please write your name in the space provided as well as on the Blue Book

Student Number: ______________________

Last Name: ______________________

Given Name: ______________________

FOR MARKER ONLY	
Question	Mark
1	
2	
3	
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5	
6	
TOTAL	

[90] **Problem 1.**

Is there

[10] (a) a function that is uniformly continuous on the interval $[0,1]$, but is not Lipschitz there, T

[10] (b) a function that is Lipschitz on the interval $[0,\infty)$, but is not uniformly continuous there, F

[10] (c) a differentiable function whose derivative is bounded on the interval $[0,1]$, but the function is not Lipschitz on $[0,1]$,

[10] (d) a function that is continuous on $[0,1]$, but does not attain its minimum value on $[0,1]$, F

[10] (e) a function that is continuous on $\mathbb{R}$, but is nowhere differentiable. T $f(x) = \sum_{n=1}^{\infty} \frac{\{10^n x\}}{10^n}$

[10] (f) a function f that is defined on $[0,1]$, not continuous at any point of $[0,1]$, but f^2 is continuous at every point of $[0,1]$,

[10] (g) a function that is defined on $[0,1]$ and is continuous only at the irrational numbers of $[0,1]$,

[10] (h) a nonconstant continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, that has only irrational numbers in its range,

[10] (i) a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\lim_{n\to\infty} f(\frac{1}{n}) \neq f(0)$,

You may explain your answers either by stating the relevant theorem or by giving an example, when it exists, but you do not have to do it to get a full credit for a correct answer.

[70] **Problem 2.**

[35] (a) A normed vector space V is strictly convex if $||u|| = ||v|| = ||\frac{(u+v)}{2}|| = 1$ for vectors u, v implies that $u = v$. Show that an inner product space is always strictly convex.

[35] (b) Let K be a compact subset of $\mathbb{R}^n$. Let $C(K)$ denote the vector space of all continuous functions on K. For $f \in C(K)$, denote $||f||_\infty = \sup_{x\in K} |f(x)|$. Show that this is a norm on $C(K)$.

[70] **Problem 3.** Prove that a compact subset of a normed vector space is closed and bounded.

[70] **Problem 4.** Prove that an inner product space V satisfies the triangle inequality $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in V$.

[50] **Problem 5.**

Prove that the series $f(x) = \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$.

[50] **Problem 6.**

Find the Fingd the Fourier series for $\sin^3\theta$ on $[-\pi,\pi]$.

Sample.
Final
~~Pr~~ Problem 1.

(a). True . $f(x)=\sqrt{x}$. uniformly continuous but not Lipschitz.

(b). False. Lipschitz $\Rightarrow$ unif. cont.

(c). False.

$$|f(x)-f(y)| = |f'(c)||x-y|$$

$$|f'(c)| \leq M$$

$$\Rightarrow |f(x)-f(y)| \leq M|x-y|$$

$\Rightarrow$ Lipschitz.

(d). False. $[0,1]$ is compact, closed & bdd $\Rightarrow$ EVT

(e). True $f(x) = \sum_{n=1}^{\infty} \frac{\{10^n x\}}{10^n}$

(f). True.

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

(g). True.

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{q}, x=\frac{p}{q}. & \text{if } x \text{ is rational. } \gcd(p,q)=1 \end{cases}$$

(h). False. consider, x, y are irrational #,
~~by IVT,~~ $\exists$ $z \in \mathbb{Q}$ in (x,y) s.t. $x<z<y$.
By IVT, $\exists$ x_0 $f(x_0)=z$.

(i). False. $\frac{1}{n} \to 0$ as $n \to \infty$ f cont. at 0.

$\forall \varepsilon>0, \exists r$ s.t. $|f(x)-f(0)|<\varepsilon$ whenever $|x-0|<r$

$\forall r$ $\exists N$ $|\frac{1}{n}|<r$ whenever $n \geq N$

$\Rightarrow \forall \varepsilon$ $\exists N$ s.t. $|f(\frac{1}{n})-f(0)|<\varepsilon$ whenever $n \geq N$

~~#~~ Problem 2:

(a). inner product space

① $<u,u>=0$ iff $u=0$

② $<x,y>=<y,x>$

③ $<\alpha x+\beta y, z> = \alpha<x,z>+\beta<y,z>$

if $\|u\|=\|\frac{u+v}{2}\|=\|v\|=1$

we have $<u,u>=1$, $<v,v>=1$, $<\frac{u+v}{2},\frac{u+v}{2}>=1$

$\Rightarrow \cancel{\tfrac{1}{2}}<u,\frac{u+v}{2}>+\cancel{\tfrac{1}{2}}<v,\frac{u+v}{2}>=2$

$\frac{1}{2}<u,u>+<u,v>+\frac{1}{2}<v,v>=2$

$\Rightarrow <u,u>+2<u,v>+<v,v>=4$

$<u,v>=1$

but $<u,v>\leq\|u\|\|v\|$

$\Rightarrow u,v$ are colinear

WLOG, sps $u=vt$

$<tv,tv>+2<tv,v>+<v,v>=4$

$t^2<v,v>+2t<v,v>+<v,v>=4$

$(t^2+2t+1)<v,v>=4$

$(t+1)^2 \cdot 1=4$

$t+1=\pm2$

$t=1$ or -3

However $<u,u>=<v,v>$

$\Rightarrow <tv,tv>=<v,v>$

$t^2<v,v>=<v,v>$

so $t^2=1$

$\Rightarrow t=1$

$\Rightarrow u=v$

$\Rightarrow$ inner product space is always strictly convex.

(b). $\|f\|_\infty = \sup_{x\in K} |f(x)|$

① positive definiteness.

$\Rightarrow \sup |f(x)| = 0 \Rightarrow f(x) = 0$ for $\forall x \in K$

$f(x) = 0 \Rightarrow \sup |f(x)| = 0 = \|f\|_\infty$

② homogeneous

$\|\alpha f\|_\infty = \sup |\alpha f(x)| = \sup |\alpha| |f(x)| = |\alpha| \|f\|_\infty$

③ Triangle inequality.

$\|f+g\|_\infty = \sup |f+g| \leq \sup |f| + |g| \leq \sup |f(x)| + \sup |g(x)|$

~~(c)~~ Problem 3.

Show closed:

Compact: $\forall$ seq we have a convergent ~~se~~ subsequence that converges to a point in V.

So for all seq (v_n) in $(V, \|\cdot\|)$, sps. $v_n \to v$. so all subseq $v_{n_i} \to v$ as well.

V is compact, the $v \in V \Rightarrow$ all limits in $V \Rightarrow$ closed.

Show bdd.

consider (v_n), an unbounded seq, s.t. $\|v_n\| \geq n$ for all $\forall n$.

clearly (v_n) does not converge, so any subseq (v_{n_i}) must has $\|v_{n_i}\| \geq n_i \Rightarrow$ subseq does not converge either

$\Rightarrow$ not compact, contradiction

$\Rightarrow$ bdd.

Problem 4.

$$\begin{aligned}\|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \text{ (Cauchy-Schwarz)} \\ &= (\|x\|+\|y\|)^2\end{aligned}$$

Problem 5: $f(x)=\sum_{n=1}^{\infty}\frac{x}{n(1+nx^2)}$

$$a_n' = \frac{n(1+nx^2)+x\,2n^2x}{n^2(1+nx^2)^2} = \frac{n(1-nx^2)}{n^2(1+nx^2)^2}$$

when $x=\sqrt{\frac{1}{n}}$, a_n achieves max.

$$f(\sqrt{\tfrac{1}{n}}) = \sum_{n=1}^{\infty}\frac{\sqrt{\frac{1}{n}}}{n(1+n\frac{1}{n})} = \sum\frac{1}{n\cdot 2\cdot\sqrt{n}} = \sum \frac{1}{n^{3/2}}$$

$$\sup\left|\frac{x}{n(1+nx^2)}\right| = \frac{1}{2n^{3/2}} \leq \frac{1}{n^{3/2}} \text{ for sure}$$

series $\sum\frac{1}{n^{3/2}}$ converges $\Rightarrow$ our ~~fu~~ series of functions cvg. uniformly

Prob 6.

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^3\theta\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}-(1-\cos^2\theta)\, d(\cos\theta)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos^2\theta-1)\, d\cos\theta$$
$$= \frac{1}{2\pi}\,\frac{1}{3}\cos^3\theta - \cos\theta\Big|_{-\pi}^{\pi} = 0$$

$A_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^3\theta\cos n\theta\, d\theta \quad n\geq 1$ is 0 since $\sin^3\theta\cos n\theta$ is odd.

$$B_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^3\theta\,\sin n\theta\, d\theta$$

complex number $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\Rightarrow \sin^3\theta = \frac{(e^{3i\theta}-e^{-3i\theta})-(3e^{i\theta}-3e^{-i\theta})}{8i^3} = -\frac{1}{4}\,\frac{e^{3i\theta}-e^{-3i\theta}}{2i} - \frac{3}{4}\cdot\frac{e^{i\theta}-e^{-i\theta}}{2i}$$
$$= -\frac{1}{4}\sin 3\theta - \frac{3}{4}\sin\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(-\frac{1}{4}\sin 3\theta\sin n\theta - \frac{3}{4}\sin\theta\sin n\theta\right)d\theta = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\cos(n-3)\theta-\cos(n+3)\theta)\,d\theta$$
$$-\frac{3}{4\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\cos(n-1)\theta-\cos(n+1)\theta)\,d\theta$$

$$= -\frac{1}{4\pi}\int_0^{\pi}[\cos(n-3)\theta-\cos(n+3)\theta]\,d\theta - \frac{3}{4\pi}\int_0^{\pi}[\cos(n-1)\theta-\cos(n+1)\theta]\,d\theta$$

$= \ldots$ if $n=3$, $n=1$ ... $\quad f = -\frac{1}{4}\sin 3\theta - \frac{3}{4}\sin\theta$

MAT337 Final Review.

Covered materials:

Ch 2 §2.3 – §2.8

Ch 3, Ch 4, Ch 5

Ch 7, all, except proofs in §7.7

Ch 8, Ch 9: §9.1, §9.2

Suggested problems

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§2.3 D.

For the following sets, find the sup & inf. which have a max or min?

(a). $A = \{a + a^{-1} : a \in \mathbb{Q}, a > 0\}$

(b). $B = \{a + (2a)^{-1} : a \in \mathbb{Q},\ 0.1 \le a \le 5\}$

(c). $C = \{x e^{-x} : x \in \mathbb{R}\}$

(a). $\inf A = 2$, $\min A = 2$

(b). $1 + \frac{1}{2}(-1) \cdot \frac{1}{a^2} = 1 - \frac{1}{2a^2}$

$2a^2 = 1$

$a = \frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2} + (\sqrt{2})^{-1} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$0.1 + (0.2)^{-1} = 0.1 + 5 = 5.1$

$5 + 10^{-1} = 5.1$

$\min B = \sqrt{2}$, $\max B = 5.1$

$\inf B = \sqrt{2}$, $\sup B = 5.1$

(c). $e^{-x} + x \cdot (-e^{-x}) = e^{-x} - x e^{-x}$

$e^{-x}(1 - x) = 0$

$e^{-x} \ne 0$, so $x = 1$,

$1 \cdot e^{-1} = \frac{1}{e}$

so ~~max~~ $= \max C = \sup C = \frac{1}{e}$ when $x = 1$.

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§2.4 A, F, G

(A) compute the limit, using $\varepsilon = 10^{-6}$. find an integer N that satisfies the limit definition.

(a). $\lim\limits_{n \to \infty} \frac{\sin n^2}{\sqrt{n}}$, for $n \ge N$, $\left| \frac{\sin n^2}{\sqrt{n}} - L \right| < \varepsilon = 10^{-6}$

Choose $N = 10000000000^{10}$, $\left| \frac{\sin 10^{200}}{10^{100}} - L \right| < 10^{-6}$ $\max C = \left| \frac{2}{10^{100}} \right| < 10^{-6}$

~~Then $L = 0$, $\left| \frac{\sin 10^{200}}{10^{100}} \right|$~~ let abs be C, so

so $\lim\limits_{n \to \infty} \frac{\sin n^2}{\sqrt{n}} = 0$.

(b) $\lim_{n\to\infty} \frac{1}{\log\log n}$,

$n \geq N \quad \left| \frac{1}{\log\log N} - L \right| < 10^{-6} = \varepsilon$

$N = $

$L = 0$

So $\lim_{n\to\infty} \frac{1}{\log\log n} = 0$

(c). $\lim_{n\to\infty} \frac{3^n}{n!}$

$n \geq N \quad \left| \frac{3^N}{N!} - L \right| < 10^{-6} = \varepsilon$ [redacted] N选的不好但答案应该对!

$N = 100$

$L = 0$ $\quad \left| \frac{3\times3\times\cdots\times3}{1\times2\times3\times\cdots 100} \cancel{= \frac{9}{2} \times \lim_{n\to\infty} \frac{3}{n!}} - L \right| < 10^{-6}$

$\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

(d). $\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2}$

$\cancel{\frac{(n+1)^2}{(2n+1)(n-1)+3}}$

Check by L'Hopital, $\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2} = \lim_{n\to\infty} \frac{2n+2}{4n-1} = \lim_{n\to\infty} \frac{2}{4} = \frac{1}{2}$

$N = 10000000 = 10^7$

$L = \frac{1}{2}$

$\left| \frac{10^{14} + 2\times10^7 + 1}{2\times10^{14} - 10^7 + 2} - \frac{1}{2} \right| < 10^{-6}$ ✓

So $\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2} = \frac{1}{2}$

$\left| \sqrt{(1+10^{-14})\times10^{28}} - 10^{14} \right|$

$= 1+10^{-7}\times10^{14} - 10^{14}$

$= |10^{-7}| < 10^{-6}$ ✓

(e). $\lim_{n\to\infty} \sqrt{n^2+n} - n$

$n \geq N$

$\left| \sqrt{N^2+N} - N - L \right| < 10^{-6}$

$\cancel{N = 10000 = 10^4}$ $\quad N = 10^{14}$ $\quad \left| \sqrt{10^{28} + 10^{14}} - 10^{14} - 0 \right|$

$L = 0$ $\quad \cancel{\left| \sqrt{10^8 + 10^4} - 10^4 - L \right|}$ $\quad \cancel{= \left| \sqrt{1.00000001 \times 10^{14}} - 10^7 - 0 \right|}$

$\cancel{\sqrt{1.0001 \times 10^8} = 1.01 \times 10^4 - 10^4}$

Hence $\lim_{n\to\infty} \sqrt{n^2+n} - n =$

[redacted]

F. Define a sequence $(a_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_{n^2}$ exists but $\lim_{n\to\infty} a_n$ does not exist.

~~Ob~~ =

G. Sps $\lim_{n\to\infty} a_n = L$, $L \neq 0$. Prove $\exists N$ s.t. $a_n \neq 0$ $\forall n \geq N$.

~~$\exists N \ \forall n \geq N, \forall \varepsilon > 0 \ | a_n - L | < \varepsilon$~~

~~when ε~~

Proof: $\lim_{n\to\infty} a_n = L$

$\forall \varepsilon > 0, \exists N$ s.t. $|a_n - L| < \varepsilon$. whenever $n \geq N$

for $\varepsilon = \frac{L}{2}$, $\exists N_1$ s.t. $|a_n - L| < \frac{L}{2}$

$\Rightarrow \frac{L}{2} < a_n < \frac{3L}{2}$ $\forall n \geq N_1$

since $L \neq 0$

~~so~~ since $\varepsilon = \frac{L}{2} > 0$, $L > 0$. $\frac{L}{2} > 0$, so $a_n \neq 0$.

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§2.5 BEI

B. Compute limits.

(a). $\lim_{n\to\infty} \frac{\tan\frac{\pi}{n}}{n\sin^2\frac{2}{n}}$

$\frac{\tan\frac{\pi}{n}}{\frac{1}{n}\sin^2\frac{2}{n}}$ $\frac{1}{n} \to 0$, $\tan\frac{\pi}{n} \to 0$, $\sin^2\frac{2}{n} \to 0$

~~L'Hospital~~

~~$= \frac{0}{\cos\frac{2}{n}}$~~ $\Rightarrow$ ~~$\frac{0 \cdot (-1)n^{-2} \cdot (\sec\frac{2\pi}{n})(-1)n^{-2}}{1 - 2\cos\frac{2}{n}(\sin\frac{2}{n})(-1)n^{-2}}$~~

$= \frac{n^{-4}\sec\frac{2\pi}{n}}{1 - 2\sin\frac{2}{n}\cos\frac{2}{n}n^{-2}} = \frac{0}{1-0} = 0$

~~(b).~~

$\lim_{x\to 0} \frac{x}{\sin x} = 1$

$$\lim_{n\to\infty} \frac{\tan\frac{\pi}{n}}{n\sin^2\frac{2}{n}} = \lim_{n\to\infty} \frac{\frac{2}{n}\tan\frac{\pi}{n}}{2\sin^2\frac{2}{n}} = \lim_{n\to\infty} \frac{\frac{2}{n}}{\sin\frac{2}{n}} \cdot \frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}$$

$$= \lim(\quad)\lim(\quad)$$

$$= 1 \cdot \lim_{n\to\infty} \frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}$$

L'Hopital $\lim_{n\to\infty} \frac{\sec^2(\frac{\pi}{n})\cdot(\frac{\pi}{-n^2})}{2\cos\frac{\pi}{n}(\frac{2}{-n^2})}$ $\qquad \sec\frac{\pi}{n} = \frac{1}{\cos\frac{\pi}{n}}$

$$= \lim_{n\to\infty} \frac{\pi\sec^2(\frac{\pi}{n})}{4\cos\frac{2}{n}}$$

$$= \lim_{n\to\infty} \frac{\pi}{4\cos^3\frac{\pi}{n}} = \frac{\pi}{4} \quad ✓$$

(b). $\lim_{n\to\infty} \frac{2^{100+5n}}{e^{4n-10}}$

L'Hopital $= \lim_{n\to\infty} \frac{(2^5)^n}{(e^4)^n}$ $\lim_{n\to\infty}\left(\frac{2^5}{e^4}\right)^n$

~~$\frac{2^{4n-10+110+n}}{e^{4n-10}}$ $\left(\frac{2}{e}\right)^{4n-10} 2^{n+110}$~~

~~$4\left(\frac{2}{e}\right)^{4n-10} 2^{n+110} + \frac{2}{e}^{4n-10} 2^{n+110}$~~

~~$\left(\frac{2}{e}\right)^{4n-10}\left(2^{n+110+2} + 2^{n+110}\right)$~~

since $2^5 < e^4$

So $\lim = 0$ ✓

(c). $\lim_{n\to\infty} \frac{\csc\frac{1}{n}}{n} + \frac{2\arctan n}{\log n}$ $\qquad \csc\frac{1}{n} = \frac{1}{\sin\frac{1}{n}}$

$\lim_{n\to\infty} \frac{1}{n\sin\frac{1}{n}} + \frac{2\arctan n}{\log n} \to 0$ since $\arctan n \to \frac{\pi}{2}$, $\log n \to \infty$ as $n \to \infty$

$$= \lim_{n\to\infty} \frac{1}{\frac{\sin\frac{1}{n}}{\frac{1}{n}}} \to 1$$

So $\lim(\quad) = 1 + 0 = 1$ ✓

(*) $\lim_{x\to 0} \frac{\sin x}{x} \Rightarrow$ L'Hopital $\lim_{x\to 0}\frac{\cos x}{1}$ $\lim_{x\to 0}\cos x = 1$.

$\lim_{x\to 0} \frac{x}{\sin x} \Rightarrow$ L'Hopital $\lim_{x\to 0} \frac{1}{\cos x} = 1$.

E. Find $\lim_{n\to\infty} \frac{\log(2+3^n)}{2n}$ Hint: $\log(2+3^n) = \log 3^n + \log\frac{2+3^n}{3^n}$

$$= \lim_{n\to\infty} \left(\frac{\log 3^n}{2n} + \frac{\log\frac{2+3^n}{3^n}}{2n}\right)$$

$\downarrow 0$

$$\lim_{n\to\infty} \frac{n\log 3}{2n}$$

$$\lim_{n\to\infty} (\quad) = \frac{\log 3}{2}$$

I. Sps. $\lim_{n\to\infty} a_n = L$. Show $\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = L$

$n\to\infty$, $a_1+\cdots+a_n = nL$

so $\lim(\quad) = L$.

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§2.6.

A. say $\lim_{n\to\infty} a_n = +\infty$ if $\forall R\in\mathbb{R}$, $\exists N$ s.t. $a_n > R$, $\forall n \geq N$. Show that a divergent monotone increasing seq converges to $+\infty$ in this case.

~~$\forall \varepsilon>0, \exists N$ s.t. $|\lim a_n - \lim a_n| < \varepsilon$ whenever $n \geq N$.~~

~~$(a_n)_{n=1}^{\infty}$ divergent~~

~~$\forall R\in\mathbb{R}$, $\exists a_n > R$ and monotone $\uparrow$ $\Rightarrow R < a_n < a_{n+1} < \cdots$~~

~~$\Rightarrow n \geq N$, $a_n > R$~~

~~$\exists \varepsilon>0, \forall N>0$, $|a_n - L| \geq \varepsilon$, $a_n > L+\varepsilon$, $R = L+\varepsilon$.~~

By Monotone convergence Thm for sequence

$a_n \uparrow$. if bdd above, converge to some $L < +\infty$. But we are given ~~a_n~~ is divergent, so a_n cannot be bounded above,

i.e. a_n has no upper bound.

i.e. there is no ~~$R>0$~~ $M>0$ s.t. (a_n) is bdd above by M.

So $\forall M>0, \exists N\in\mathbb{N}$; $a_N > M$.

Since we are given (a_n) is $\nearrow$, $n \geq N \Rightarrow a_n > a_N > M$.

This holds for $M' < M$ too.

so $\forall M>0, \exists N\in\mathbb{N} : n\geq N \to a_n > M$.

hence ~~a~~ $a_n \to +\infty$.

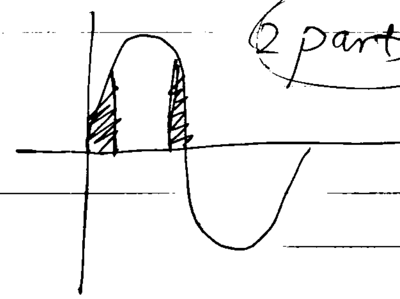
C. Is $S = \{x \in \mathbb{R} : 0 < \sin(\frac{1}{x}) < \frac{1}{2}\}$ bdd above / below? If so, find.

$\cancel{0 < \frac{1}{x} < \frac{\pi}{6}, \ \frac{5}{6}\pi < \frac{1}{x} < \pi}$

$0 < 1 < \frac{\pi}{6}x$

$\cancel{\frac{6}{\pi} < x \quad \frac{5}{6}\pi x < 1 < \pi x}$

$\cancel{\frac{1}{\pi} < x < \frac{6}{5\pi}}$



2 parts

① $\frac{1}{x} \in (\frac{5}{6}\pi + 2k\pi, \pi + 2k\pi)$

let $k = -1$.

$\frac{1}{x} \in (-\frac{7}{6}\pi, -\pi)$

$x \in (-\frac{1}{\pi}, -\frac{6}{7\pi})$

$\cancel{\frac{5}{6}\pi - 2\pi < \frac{1}{x} < \pi - 2\pi}$

$\cancel{-\frac{7}{6}\pi = \frac{5-12}{6}\pi < \frac{1}{x} < -\pi}$

$\cancel{-\frac{7}{6}\pi x > 1 > -\pi x}$

$\cancel{-\frac{1}{\pi} < x < -\frac{6}{7\pi}}$

② $\frac{1}{x} \in (2k\pi, \frac{\pi}{6} + 2k\pi)$

let $k = 0$,

$\frac{1}{x} \in (0, \frac{\pi}{6})$

$\frac{1}{x} \to 0, \ x \to +\infty$ no upper bound

$\frac{1}{x} \in (-2\pi, -\frac{11}{6}\pi)$

$x \in (-\frac{6}{11\pi}, -\frac{1}{2\pi})$

$\inf S = -\frac{6}{11\pi}$.

F. Let a, b be positive real numbers. Set $x_0 = a$, $x_{n+1} = (x_n^{-1} + b)^{-1}$ for $n \geq 0$.

(a). Prove x_n is monotone ↓

(b). Prove limit exists & find it.

$$x_{n+1} = \frac{1}{\frac{1}{x_n} + b}$$

$$x_{n+1}\left(\frac{1}{x_n} + b\right) = 1$$

$$\frac{1}{x_n} + b = \frac{1}{x_{n+1}}$$

$$b = \frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{x_n - x_{n+1}}{x_{n+1}x_n}$$

positive ~~negative~~

positive

positive

So ~~$x_n < x_{n+1}$~~

$x_{n+1} < x_n$

(b). ~~$x_{n+1} = \frac{1}{\frac{1}{x_n}+b}$~~

~~when $n \to \infty$, say $\lim_{n\to\infty} x_n = L$~~

~~then $L = \frac{1}{\frac{1}{L}+b} = L = \frac{1}{1+bL}$~~

~~$L^2(1+bL) = 1$~~

~~$L^2 + bL^3 - 1 = 0$~~

(x_n) is monotone decreasing, & bounded below b/c x_i cannot <0, if so, then the expression of b in part (a) fails.
So, by Monotone sequence thm, (x_n) converges to a number say L which is the limit desired.

~~$L = \frac{1}{\frac{1}{L}+b} \Rightarrow L(\frac{1}{L}+b) = 1$~~

~~$\frac{1}{L}+b = \frac{1}{L}$~~

$L = \frac{1}{\frac{1}{L}+b}$

$L(\frac{1}{L}+b) = 1$

$1 + Lb = 1$

$Lb = 0$

so $L = 0$ ✓ thus $\lim_{n\to\infty} x_n = 0$.

I. (a). let $(a_n)_{n=1}^{\infty}$ be a bdd seq & define a seq $b_n = \sup\{a_k : k \geq n\}$ for $n \geq 1$. Prove (b_n) converges. This is the limit superior of (a_n). denoted by $\limsup a_n$

since (a_n) is bdd seq, then it's both bdd above & below.
~~so $\sup(a_n) =$~~
(b_n) is a subsequence of (a_n), so b_n bounded.
by monotone sequence thm, (b_n) claim monotone decreasing,
then (b_n) converges, converges to $\limsup (a_n)$

(b). without redoing proof, conclude the limit inferior of a bdd sequence (a_n), define $\liminf a_n = \lim_{n\to\infty} (\inf_{k \geq n} a_k)$ always exists

(c) Extend the definitions of $\limsup a_n$ and $\liminf a_n$ to unbounded seq. Provide an example with $\limsup a_n = +\infty$, $\liminf a_n = -\infty$.

J,

① if $(a_n)_{n=1}^{\infty}$ converges to l

$\Rightarrow \forall \varepsilon > 0, \exists N_1 \rightarrow |a_n - l| < \varepsilon$ whenever $n \geq N_1$

$\limsup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$

$\forall n \geq N_1 \ \sup\{a_k : k \geq n\} < l + \varepsilon$, $\inf\{a_k : k \geq n\} > l - \varepsilon$

b/c always have $l - \varepsilon < a_n < l + \varepsilon$

$b_n = \sup\{a_k : k \geq N\}$,

$c_n = \inf\{a_k : k \geq N_1\}$

$\Rightarrow c_{N_1} \leq c_{N_1+1} \leq \cdots \leq b_{N_1-n} \leq \cdots \leq b_{N_1}$

$(b_n) \rightarrow l$, $(c_n) \rightarrow m$

Nested value theorem guarantees that $l = m$. (length is $\frac{2\varepsilon}{2^k}$)

$\lim_{n\to\infty} \sup\{a_k : k \geq N_1\} = l$

$l - \varepsilon < \sup\{a_k : k \geq N_1\} < l + \varepsilon$

$\Rightarrow |\sup\{a_k : k \geq N_1\} - l| < \varepsilon$

② if not, say l'

$\exists$ a subseq. that converges to l' a convergent seq.

See next page for problem

✓J. Show $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ iff $\limsup a_n = \liminf a_n = L$.

P26.

§2.7 A.

Show $(a_n) = \left(\frac{n\cos^n(n)}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$ has a convergent subsequence.

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+2n}} = 1.$ & $\cos^n(n) \in [-1, 1]$ for $n \in \mathbb{R}$

so (a_n) is bounded

By Bolzano-Weierstrass, every bdd seq. has convergent subsequence.

✓

In a neat procedure:

$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = L.$ iff $\lim_{n\to\infty}(a_n) \to L$

① if $a_n \to L$ $\forall \varepsilon > 0, \exists N$ s.t. $|a_n - L| < \varepsilon$ whenever $n \geq N$

$l-\varepsilon < a_n < l+\varepsilon$ $\forall n \geq N$

$\Rightarrow \sup \{a_k : k \geq N\} < L+\varepsilon$

$\sup \{a_k : k \geq N\} > L - \varepsilon$ $\Big\} \Rightarrow$ ~~$\sup\{\ \}$~~ a_n

$\Rightarrow |\sup - L| < \varepsilon$

similarly for inf.

② if $\lim \sup a_n = \lim \inf a_n = L$

$\forall \varepsilon = 0$ $|\sup a_n - L| < \varepsilon$, $n \geq N_1$

$|\inf a_n - l| < \varepsilon$, $n \geq N_2$

$N = \max\{N_1, N_2\}$

$\Rightarrow 1 - \varepsilon < \inf a_n < \sup a_n < L + \varepsilon$

However, $n \geq N$

$\inf a_n \leq a_n \leq \sup a_n$

$\Rightarrow l - \varepsilon < a_n < l + \varepsilon$

$\Rightarrow |l - a_n| < \varepsilon$

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§2.9. C.

Let (a_n) be a sequence such that $\lim_{N\to\infty}\sum_{n=1}^{N}|a_n-a_{n+1}|<\infty$. Show (a_n) is Cauchy.

$\cancel{|a_1-a_2|+|a_2-a_3|+\cdots+|a_n-a_{n+1}|+\cdots+|a_N-a_{N+1}|}$

Want to show. $\left(\begin{array}{l}\text{For } \forall\varepsilon>0,\ \exists M \text{ s.t.}\\ \quad |a_n-a_{n+1}|<\varepsilon \text{ whenever } n+1\geq M\end{array}\right)$

$$\lim_{N\to\infty}\sum_{n=1}^{N}|a_n-a_{n+1}|=L$$

$$\forall\varepsilon>0,\ \exists N,\ \sum_{k=m}^{n}|a_k-a_{k+1}|<\varepsilon \quad \text{whenever } m,n\geq N$$

$$|a_m-a_n|=|a_m-a_{m+1}+a_{m+1}-a_{m+2}+\cdots+a_{n-1}-a_n|$$

$$\leq\sum|a_m-a_n|\leq\varepsilon$$

Cauchy!

✓

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§3.1 ABC.

A Sum the series $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$

"use telescope sum"

$$\frac{2}{n(n+2)}=\frac{1}{n}-\frac{1}{n+2}=\frac{n+2-n}{n(n+2)}. \checkmark$$

$\cancel{\sum_{1}^{\infty} = \frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\frac{1}{4\times6}}+\cdots$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\sum\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\cdots\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{2}\right)$$

$$=\frac{3}{4} \quad \checkmark$$

[redacted]

B. Sum the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+3)(n+4)}$

$$\cancel{= \sum \frac{1}{n(n+1)} \cdot \sum \frac{1}{(n+3)(n+4)}}$$

why this is wrong?

$$\cancel{= \sum \left(\frac{1}{n} - \frac{1}{n+1}\right) \sum \left(\frac{1}{n+3} - \frac{1}{n+4}\right)}$$

$$\cancel{= \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots\right)\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots\right)}$$

$$\cancel{= 1 \cdot \frac{1}{4}}$$

$$\cancel{= \frac{1}{4}}$$

~~Right~~ Correct solution:

know that $\frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+3} - \frac{1}{n+4} = \frac{12}{n(n+1)(n+3)(n+4)}$

$$= \frac{1}{12} \sum \left(\frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+3} - \frac{1}{n+4}\right)$$

$$= \frac{1}{12}\Big(\cancel{1 - \frac{2}{2} + \frac{2}{4} - \frac{1}{5} + \frac{1}{2} - \frac{2}{3} + \frac{2}{5} - \frac{1}{6}}$$

$$\left(\begin{array}{l} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ -\frac{2}{2} - \frac{2}{3} - \cancel{\frac{2}{4}} - \cancel{\frac{2}{5}} - \cdots \\ \cancel{\frac{2}{4}} + \cancel{\frac{2}{5}} + \cancel{\frac{2}{6}} + \cancel{\frac{2}{7}} + \cdots \\ -\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} - \cancel{\frac{1}{7}} - \cancel{\frac{1}{8}} - \cdots \end{array}\right)$$

$$= \frac{1}{12}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{2}{3}\right)$$

$$= \frac{1}{12}\left(\frac{3}{4} + \frac{1}{3} - \frac{2}{3}\right)$$

$$= \frac{1}{12}\left(\frac{9}{12} - \frac{4}{12}\right)$$

$$= \frac{5}{144}$$

?

C. Prove that if $p>1$ and $\sum_{k=1}^{\infty} t_k$ is a convergent series of nonnegative numbers $\sum_{k=1}^{\infty} t_k^p$ converges.

$$\sum_{k=1}^{\infty} t_k = S$$

$$\lim_{n\to\infty} t_n = 0$$

$$\cancel{S^p = \left(\sum_{k=1}^{\infty} t_k\right)^p \geq \sum_{k=1}^{\infty} t_k^p}$$

$$\cancel{S_1 = S^{\frac{1}{p}} = \sum_{k=1}^{\infty} t_k}$$

$\boxed{\text{use Cauchy-Criterion.}}$

since t_k nonnegative,

$$\sum_{k=n+1}^{m} t_k < \varepsilon^{\frac{1}{p}} = \varepsilon_1$$

$$\left(\sum_{k=n+1}^{m} t_k\right)^p < \varepsilon_1^p = \varepsilon$$

$$\Rightarrow \sum_{k=n+1}^{m} t_k^p \leqslant \left(\sum_{k=n+1}^{m} t_k\right)^p < \varepsilon$$

✓

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§ 3.2 P.

(a). $\sum_{n=2}^{\infty} \frac{3n}{n^3+1}$

$$\frac{3n}{n^3+1} < \frac{3n}{n^3} = \frac{3}{n^2} \quad \text{converge}$$

(b). $\sum_{n=1}^{\infty} \frac{n}{2^n}$, $l = \limsup \sqrt[n]{\frac{n}{2^n}} = \frac{\sqrt[n]{n}}{2} < 1$, converge

or

ratio test $\dfrac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n} \longrightarrow \frac{1}{2} < 1$ converge

(c). $\sum_{n=2}^{\infty} \frac{(-1)^n \log n}{n}$

Alternating series test

$$\frac{\log n}{n} \qquad \cancel{\frac{\log(n+1)}{n+1} = \frac{n}{n+1}} \quad \cancel{\log n \log} \log$$

$$\cancel{\frac{\log n}{n}}$$

~~L'Hospital~~

$$\left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0$$

so decreasing monotone.

& $\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{n^{-1}}{1} = 0$ converge (d). $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$

$\sum_{n=1}^{\infty} (\quad) = \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \cdots \quad \lim_{n\to\infty} \sqrt{n+1} - 1$ div.

(e). $\sum_{n=1}^{\infty} e^{-n^2}$ n-th root test

$l = \limsup_{n\to\infty} \sqrt[n]{e^{-n^2}} = \limsup_{n\to\infty} e^{-n} < 1$ converge.

$e^{-n} = 0$

(f) $\sum_{n=1}^{\infty} \sin(\frac{n\pi}{4})$ diverge since $\not\to 0$.

(j) $\sum_{n=2}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} = \lim (\frac{\sqrt{n+1}}{n} - \frac{1}{n})$

$\frac{1}{n} \to 0$

$\frac{\frac{1}{2}(n+1)^{-\frac{1}{2}}}{1} = \frac{1}{2} \frac{1}{\sqrt{n+1}} \to 0$

Converge,

(g). $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{1}{n})$

$\sin\frac{1}{n} < \frac{1}{n}$, so converge

$\downarrow$
0

(h). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4}}$ converge.

(i). $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$ n-th root

$\limsup_{n\to\infty} \sqrt[n]{n} - 1$

(j).

$\limsup_{n\to\infty} (n^{\frac{1}{n}} - 1) < 1$ converge

$\to 0$

(k) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\log n}$ converge.

(l). $\sum_{n=2}^{\infty} \frac{(-n)^n}{\sqrt{n}}$ diverge.

(m). $\sum_{n=2}^{\infty} \frac{1}{(\log n)^k}$ $\begin{cases} \text{case } 1 \\ \text{case } 2 \end{cases}$

(m). $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{\frac{(n+1)!}{(n+1)^{(n+1)}}}{\frac{n!}{n^n}} = \frac{n+1}{(n+1)^{(n+1)}} \cdot n^n = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n < 1$$

decreasing $\Rightarrow$ conv.

(o). $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n}$ conv.

(p). $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$

comverge ~~diverge~~ ?

(p). ~~$= \sum_{n=2k}^{\infty} \frac{1}{\sqrt{n}+1} - \sum_{n=2k+1}^{\infty} \frac{1}{\sqrt{n}-1}$~~

$\frac{(-1)^n}{\sqrt{n}+(-1)^n} < \frac{(-1)^n}{\sqrt{n}-1} \to$ converge

so converges

(q). $\sum_{n=1}^{\infty} (-1)^n (e^{\frac{1}{n}} - 1)$ converge

(r). $\sum_{n=1}^{\infty} (-1)^n \frac{n^{42}}{(n+1)!}$ ratio test + alternative series test

$$\frac{(n+1)^{42}}{(n+2)!} \cdot \frac{(n+1)!}{n^{42}} = \left(\frac{n+1}{n}\right)^{42} \cdot \frac{1}{n+2}$$

$$= \left(1+\frac{1}{n}\right)^{42} \cdot \frac{1}{n+2} \to 0 < 1$$

So converge.

(s). $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ Converge $\frac{1}{(n+1)^2+1} \cdot (n^2+1) = \frac{n^2+1}{n^2+2n+2} = \frac{1}{2} < 1$ converge

(t) $\sum_{n=1}^{\infty} \frac{1}{\log(e^n+e^{-n})}$ $e^n > e^{-n}$ for $n \geq 1$, $\frac{1}{\ln(e^n+e^{-n})} > \frac{1}{\ln(e^n+e^n)} = \frac{1}{\ln(2e^n)} = \frac{1}{\ln 2 + n}$

$\sim \frac{1}{n}$ diverge

~~Since comparison test. since $\log(e^n+e^{-n}) < \log(e^n+e^n) = \log(2e^n)$~~

~~So $\frac{1}{\log(e^n+e^{-n})} > \frac{1}{\log(2e^n)}$.~~

~~$n \to \infty$, $\log 2e^n \to \infty$, so $a_n \to 0$~~

* $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series diverge !!!!

(u). $\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi n}{3})}{n}$ converge

(v). $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ $\frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}} = \frac{1}{10}\left(\frac{1}{n}+1\right)^{10}$

$n \to \infty$, $\boxed{\frac{1}{10}} < 1$ converge.

(w) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ $l = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{(\log n)^n}} = \frac{1}{\log n} < 1 \Rightarrow$ converge.

* (x). $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ Integral test: ~~$\int_2^\infty$~~ ~~Is it necessary?~~

~~Since $a_n \to 0$~~

~~converge~~

$\int_2^\infty \frac{1}{x \ln x} = \int_2^\infty \frac{1}{\ln x} d(\ln x)$

$= \ln(\ln x)\Big|_2^\infty \Rightarrow$ diverge

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§3.3 BC

B. Decide which of the following series converge absolutely, conditionally or not at all.

(a). $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\log(n+1)}$ $\frac{1}{n\log(n+1)}$ ~~converge~~ diverge since ~~$a_n \to 0$~~ a small one $\frac{1}{(n+1)\log(n+1)}$ diverge

~~converge absolutely~~

But by alternating series test, a_n decreasing $\lim a_n = 0$.

So whole series converge.

Hence conditionally converge.

(b). $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2+(-1)^n)n}$

$|\cdots| = \frac{1}{(2+(-1)^n)n} < \frac{1}{3n}$ $\boxed{\text{diverge}}$ ~~but by alternating test.~~

$\frac{1}{n} < \frac{1}{2+(-1)^n n} < \frac{1}{3n}$

~~$a_1 =$~~ ~~$a_2 = \frac{1}{6}$~~ ~~$a_3 =$~~

but $\frac{(-1)^n}{(2+(-1)^n)n} < \frac{(-1)^n}{n}$

converge by alter

So converge

So conv. conditionally.

(e). $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(\frac{1}{n})}{n}$

~~$\frac{\sin\frac{1}{n}}{n}$ $\frac{1}{n}\sin\frac{1}{n}$~~

$\frac{\sin\frac{1}{n}}{n}$ monotone decreasing

$\frac{\sin\frac{1}{n}}{n} \to 0$, so $\sum a_n$ converges

& $\frac{\sin\frac{1}{n}}{n} < \frac{1}{n^2}$ since $\sin\frac{1}{n} < \frac{1}{n}$

$\frac{1}{n^2}$ converges $\quad \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{n^2+2n+1} \Rightarrow \frac{1}{2n+1} < 1$

so converges absolutely.

C. Compute the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)}$ given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Hint $\frac{1}{n^2(2n-1)} = \frac{4}{2n(2n-1)} - \frac{1}{n^2}$

$4\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{n^2}$

$= 4L - \frac{\pi^2}{6}$

$L = \sum \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

is taylor series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \frac{\sum (-1)^{k-1} x^k}{k}$

when $x=1$

So $L = \ln 2$

So $\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)} = 4\ln 2 - \frac{\pi^2}{6}$

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§4.1 A, H, K

A. Establish the Pythagorean formula. If $\vec{x}$ & $\vec{y}$ are orthogonal vectors, prove that $\|\vec{x}+\vec{y}\| = (\|\vec{x}\|^2 + \|\vec{y}\|^2)^{\frac{1}{2}}$

Proof: $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| \cancel{\sin\cos\theta}\cos\theta + \|\vec{y}\|^2 = \|\vec{x}\| + \|\vec{y}\|$

$\theta = 90^\circ$

so $\cos\theta = 0$

$\sqrt{\quad} = \sqrt{\quad}$

so done.

H. For nonzero vectors $\vec{x}$ & $\vec{y}$ in R^n, define θ by $\|\vec{x}\|\|\vec{y}\|\cos\theta = \langle \vec{x}, \vec{y} \rangle$ & call this the angle between them.

(a). Prove the cosine law: if $\vec{x}$ & $\vec{y}$ are vectors & θ is the angle between them, then $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\|\cos\theta + \|\vec{y}\|^2$.

Proof:

$\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\cos\theta\, \|\vec{x}\|\|\vec{y}\|$

(b). Prove that $\langle \vec{x}, \vec{y} \rangle$ can be defined using only the norms of related vectors.

$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\|\cos\theta$

$$\cos\theta = \frac{\|\vec{x}+\vec{y}\|^2 - \|\vec{x}\|^2 - \|\vec{y}\|^2}{2\|\vec{x}\|\|\vec{y}\|}$$

so $\langle \vec{x}, \vec{y} \rangle = \frac{1}{2}\left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}\|^2 - \|\vec{y}\|^2\right)$

K. Let M be a subspace of $\mathbb{R}^n$ with an orthonormal basis $\{\vec{v_1}, \cdots, \vec{v_k}\}$. Define a linear transformation on $\mathbb{R}^n$ by $P\vec{x} = \sum_{i=1}^{k} \langle \vec{x}, \vec{v_i} \rangle \vec{v_i}$

(a). Show that $P\vec{x}$ belongs to M, & $P\vec{y} = \vec{y}$ for all $\vec{y} \in M$. Hence show that $P^2 = P$

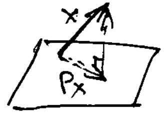
let $\vec{y} = a_1\vec{v_1} + \cdots + a_k\vec{v_k}$

$P\vec{y} = \sum \langle \vec{y}, \vec{v_i} \rangle \vec{v_i}$

$\langle \vec{y}, \vec{v_i} \rangle = \langle a_1\vec{v_1} + \cdots + a_k\vec{v_k}, \vec{v_i} \rangle = \langle a_1\vec{v_1}, \vec{v_i} \rangle + \cdots + \langle a_k\vec{v_k}, \vec{v_i} \rangle = a_i$

so $P\vec{y} = \sum \langle \vec{y}, \vec{v_i} \rangle \vec{v_i} = \sum a_i\vec{v_i} = \vec{y}$.

[redacted]



(b). Show that $\langle P\vec{x}, \vec{x}-P\vec{x}\rangle = 0$.

$\langle P\vec{x}, \vec{x}-P\vec{x}\rangle$ not like this but also easy to show.

$= \langle x, 0\rangle$

$= 0$

(c). Hence show that $\|\vec{x}\|^2 = \|P_x\|^2 + \|x-P_x\|^2$

LHS $= \langle x, x\rangle \cancel{= \langle P_x, P_x\rangle + \langle x-P_x, x-P_x\rangle}$

$= a_1^2 v_1 + \cdots + a_k^2 v_k$

RHS $= \langle P_x, P_x\rangle + \langle x-P_x, x-P_x\rangle$

$= \langle x, x\rangle + \cancel{\langle 0\rangle}\ 0$

$= a_1^2 v_1 + \cdots + a_k^2 v_k$

LHS = RHS

(d). If $\vec{y}$ belongs to M, show that $\|x-y\|^2 = \|y-P_x\|^2 + \|x-P_x\|^2$

LHS $= \langle x-y, x-y\rangle$

RHS $= \langle y-P_x, y-P_x\rangle + \langle x-P_x, x-P_x\rangle$

$= \langle y-x, y-x\rangle + 0$

$= \langle x-y, x-y\rangle$

LHS = RHS

(e). Hence show that P_x is the closest pt in M to $\vec{x}$.

$\|y-x\| = \|y-P_x\|^2 + \|x-P_x\|^2 \geq \|x-P_x\|^2$

equality when $y = P_x$

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§4.2 FH

F. Let $\vec{V_0} = (x_0, y_0)$ with $0 < x_0 < y_0$. Def $\vec{V_{n+1}} = (x_{n+1}, y_{n+1}) = (\sqrt{x_n y_n}, \frac{x_n + y_n}{2})$ $\forall n \geq 0$

a) Show by induction that $0 < x_n < x_{n+1} < y_{n+1} < y_n$.

$V_0 = (x_0, y_0)$

$V_1 = (\sqrt{x_0 y_0}, \frac{x_0 + y_0}{2})$

$\sqrt{x_0 y_0} > x_0$

b/c $x_0 y_0 > x_0 \cdot x_0$ $\quad$ $\frac{x_0 + y_0}{2} < y_n$ b/c $x_0 + y_0 < y_0 + y_0$

$y_0 > x_0$

$\sqrt{x_0 y_0} < \frac{x_0 + y_0}{2}$ b/c

$4x_0 y_0 < x_0^2 + 2x_0 y_0 + y_0^2$

$(x_0 - y_0)^2 > 0$

induction is trivial.

? b) estimate $y_{n+1} - x_{n+1}$ in terms of $y_n - x_n$

$y_{n+1} - x_{n+1} = \frac{x_n + y_n}{2} - \sqrt{x_n y_n}$

(c). Thereby show $\exists c$ s.t. $\lim_{n \to \infty} \vec{V_n} = (c, c)$. c is known as the arithmetic-geometric mean of x_0 & y_0.

(x_n) monotone $\nearrow$

(y_n) $\searrow$

both bdd

$\Rightarrow x_n \to L$

$y_n \to M$

however $\lim (y_n - x_n) = 0$

so $L = M$.

H. Let $T=\begin{bmatrix}5/4 & -1/4\\ 3/4 & 1/4\end{bmatrix}$ Set $x_n=T^n(1,0)$ for $n\geq 1$.

(a). Prove $(\vec{x}_n)$ converges & find limit $\vec{y}$.

1. $\begin{pmatrix}\frac{5}{4} & -\frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{5}{4}\\ \frac{3}{4}\end{pmatrix}$

2. $\begin{pmatrix}\frac{5}{4} & -\frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}\frac{5}{4}\\ \frac{3}{4}\end{pmatrix}=\begin{pmatrix}\frac{25}{16}-\frac{3}{16}\\ \frac{15}{16}+\frac{3}{16}\end{pmatrix}=\begin{pmatrix}\frac{22}{16}\\ \frac{18}{16}\end{pmatrix}$

3. $\begin{pmatrix}\frac{5}{4} & -\frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}\frac{22}{16}\\ \frac{18}{16}\end{pmatrix}=\begin{pmatrix}\frac{110}{64}-\frac{18}{64}\\ \frac{66}{64}+\frac{18}{64}\end{pmatrix}=\begin{pmatrix}\frac{92}{64}\\ \frac{84}{64}\end{pmatrix}$

k. $\frac{3-2^{-k}}{2}, \frac{3(1-2^{-k}\cdot 2)}{2}$

$\lim_{n\to\infty}\left(\frac{3-2^{-n}}{2}, \frac{3(1-2^{-n}\cdot 2)}{2}\right)=\left(\frac{3}{2},\frac{3}{2}\right)=\vec{y}$

b. Find an explicit N s.t. $\|x_n-y\|<\frac{1}{2}10^{-100}$ $\forall n\geq N$

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§ 4.3 A B N

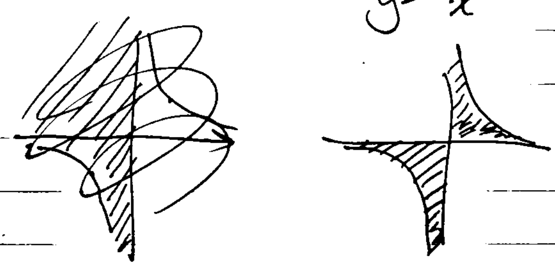
A. Find the closure of the following sets:

(a). $\mathbb{Q}$.

$1.4, 1.41, 1.414, \ldots$ $\sqrt{2}$

so $\overline{\mathbb{Q}} = \mathbb{R}$

(b). $\{(x,y) \in \mathbb{R}^2 : xy < 1\}$ for $x<0$, $xy<1$, $y > \frac{1}{x}$ | $x>0$, $y<\frac{1}{x}$



closure $\Rightarrow \{(x,y) \in \mathbb{R}^2 : xy \le 1\}$

(c). $\{(x, \sin(\frac{1}{x})) : x>0\}$

?

(d). $\{(x,y) \in \mathbb{Q}^2 : x^2+y^2<1\}$ $x^2+y^2 \le 1$

B. Let $(\vec{a_n})_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^k$ with $\lim_{n\to\infty} \vec{a_n} = \vec{a}$

Show that $\{\vec{a_n} : n \ge 1\} \cup \{\vec{a}\}$ is a closure.

By contradiction, suppose $A = \{\vec{a_n} : n \ge 1\} \cup \{\vec{a}\}$

~~$\exists \ldots \in A$, but $\lim \vec{a_p} = \vec{x} \notin A$.~~

Sups $\exists \vec{x}$ is a limit point of A but $\vec{x} \notin A$.

$\exists$ seq. $(b_n) \to \vec{x}$, $(b_n) \in A \ne \vec{a}$

consider 2 cases:

(1). $b_n \in \{a_n : n \ge 1\}$ if $b_n \to \vec{x} \notin A$, $\forall \varepsilon_1 > 0$, $\exists N_1$ s.t. $\|b_n - x\| < \varepsilon_1$ whenever $n \ge N_1$

$\forall \varepsilon_2 > 0, \exists N_2$ s.t. $\|a_n - a\| < \varepsilon_2$ whenever $n \ge N_2$

$\tilde{N} = \max\{N_1, N_2\}$

$a_{n_k} \Rightarrow n_k \ge \tilde{N}$ $\|a_{n_k} - \vec{x}\| < \varepsilon_1$

$\|a_{n_k} - \vec{a}\| < \varepsilon_2$

$\Rightarrow (a_{n_k}) \to \vec{x} \notin A$

$\to \vec{a} \in A$

$(\Rightarrow\Leftarrow)$

② $(b_n) =$ ~~a_{n_k}~~ some pts in A $(a_n) \cup \{\vec{a}\}$

$(b_n) = (a_{n_1}, a_{n_2}, \ldots, \vec{a}) \to \vec{x} \notin A$

$\Rightarrow \|a_{n_k} - x\| < \varepsilon \quad \forall \varepsilon > 0, n_k \geq N$

however, can show $b_n \to \vec{a}$

N. A point $\vec{x}$ is a cluster point of a subset A of $\mathbb{R}^n$ if $\exists$ seq. $(\vec{a}_n)_{n=1}^{\infty}$ with $\vec{a}_n \in A \setminus \{\vec{x}\}$ s.t. $\vec{x} = \lim_{n\to\infty} \vec{a}_n$. Thus every cluster point is a limit pt but not conversely.

a). Show if $\vec{x}$ is a limit pt of A. then either $\vec{x}$ is a cluster pt of A or $\vec{x} \in A$.

b). Hence show that a set is closed if it contains all of its cluster pts.

c). Find all cluster pts of (i) $\mathbb{Q}$

(ii) $\mathbb{Z}$

(iii) $(0,1)$

a. $\lim_{n\to\infty} a_n = \vec{x}$

↘ limit pt.

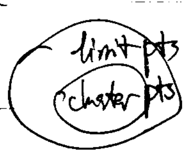
$\vec{x} \in A$ ✓

or

$\vec{x} \notin A \Rightarrow$ cluster point

b). closed means it contains all limit pts.

since



so particularly, it's correct.

c). (i). $\mathbb{R} \setminus \mathbb{Q}$

(ii). $\emptyset$

(iii). $\{0\} \cup \{1\}$

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I. § 4.4.

Let A & B be disjoint closed subsets of $\mathbb{R}^n$. Define

$d(A,B) = \inf\{\|\vec{a}-\vec{b}\| : \vec{a}\in A, \vec{b}\in B\}$

(a). If $A=\{\vec{a}\}$ is a singleton show $d(A,B)>0$.

(b). If A is compact, show $d(A,B)>0$.

(c). Find & e.g. of 2 disjoint closed sets in $\mathbb{R}^2$ with $d(A,B)=0$.

(a). $\boxed{A \text{ is singleton}}$, if $\exists B$ s.t. $d(A,B)=0$

$\Rightarrow \inf\{\|\vec{a}-\vec{b}\| : \vec{a}\in A, \vec{b}\in B\}=0$

$\varepsilon_n = \frac{1}{n}$.

$\varepsilon_1 = 1$, $\exists b_1\in B$ s.t. $\|a-b_1\|<1$

$\varepsilon_2=\frac{1}{2}$, $\exists b_2\in B$ s.t. $\|a-b_2\|<\frac{1}{2}$

$\vdots$

$\varepsilon_n=\frac{1}{n}$, $\exists b_n\in B$ s.t. $\|a-b_n\|<\frac{1}{n}$

so (b_n) is a seq in B s.t. $b_n\to a$

$a\in A$, A,B disjoint

$\Rightarrow a\notin B \Rightarrow B$ not closed ($\Rightarrow\Leftarrow$ by def)

(b). A compact, $\exists B$ st. $d(A,B)=0$

i.e. $\inf = 0$

$\forall \varepsilon>0$, $\exists$ a seq. (d_n) s.t. $d_n\to 0$.

$\{d_n=\|a_n-b_n\| : a_n\in A, b_n\in B\}$

(c) $A=\{0\}$

$B=\{\frac{1}{n} : n\geq 1\}$

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§9.1.
BCE

B. Show that every subset of a discrete metric space is both open & closed.

$\cancel{d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}}$

~~Sps $\forall x \in X$.~~
~~$B_{\frac{1}{2}}(x) = \{x : \rho(x,y) \ldots\}$~~
~~Sps X is a subset,~~
~~$\forall (x,y) \in X$,~~
~~$\exists r > 0$ s.t. $B_r(x,y) \subseteq X$~~

Open:
$\forall x \in X$, $B_{\frac{1}{2}}(x) = \{x : \rho(x,y) < \frac{1}{2}\} = X$

$0 < r < 1$

$\cancel{X = (x=y}$ $X = \{(x,y) \mid x = y\}$

Closed: $\forall$ subset is open $\Rightarrow$ $X \setminus U$, a subset of X is open then its compliment U is closed. & $U \subseteq X$.

C. Prove that U is open in (X, ρ) iff $X \setminus U$ is closed.

~~Proof: if $X \setminus U$ is closed~~
~~(x_n)~~

(⇒) U is open.
Sps $(x_n) \in X \setminus U$ & $x_n \to x$ $\left(\lim_{n\to\infty} \rho(x_n, x) = 0\right)$
but $x \in U$
$\Rightarrow \exists r > 0$ s.t. $B_r(x) \subseteq U$
$\Rightarrow \rho(x_n, x) \geq r$

(⇐) Sps U not open
$\exists x \in U$ s.t. $\exists r$
(contrapositive) $B_r(x) \not\subseteq U$
$\Rightarrow r = \varepsilon_n = \frac{1}{n}$
$B_{\frac{1}{n}}(x) \cap U^c \neq \emptyset$
$x_1 \in (B_{\frac{1}{n}}(x) \cap U^c)$
$\Rightarrow \rho(x_1, x) < 1$
$\rho(x_2, x) < \frac{1}{2}$
$\vdots$
$\rho(x_n, x) < \frac{1}{n}$
$\Rightarrow \lim \rho(x_n, x) = 0$
$\Rightarrow (x_n) \to x \in U$
$\Rightarrow U^c$ not closed

E. Given a metric space (X, ρ), def a new metric on X by $\sigma(x,y) = \min\{\rho(x,y), 1\}$
(a). Show σ is a metric on X. Observe X has finite diameter in the σ metric.
① positive definiteness:
$x = y \Rightarrow \rho(x,y) = 0 \Rightarrow \sigma(x,y) = 0$
$\sigma(x,y) = 0 \Rightarrow \rho(x,y) = 0 \Rightarrow x = y$
② symmetry:
$\sigma(x,y) = \min\{\rho(x,y), 1\} = \min\{\rho(y,x), 1\} = \sigma(y,x)$
③ triangle inequality.
w.t.s $\sigma(x,z) + \sigma(z,y) \geq \sigma(x,y)$
$\min\{\rho(x,z), 1\} + \min\{\rho(z,y), 1\} \geq \min\{\rho(x,y), 1\}$
pick $\rho(x,z) + \rho(z,y) \geq \rho(x,y)$ ✓
~~$\rho(x,z) + 1 \geq 1$~~
~~$\rho(x,z) + 1 \geq \rho(x,y)$~~
$1 + 1 \geq 1$ ✓

9. Show $\lim_{n\to\infty} x_n = x$ in (X, ρ) iff $\lim_{n\to\infty} x_n = x$ in (X, σ).

$\Rightarrow$. $\lim_{n\to\infty} x_n = x$ in (X, ρ)

$\forall \varepsilon > 0, \exists N$ s.t.
$\rho(x_n, x) < \varepsilon$ whenever $n \geq N$

① $\Rightarrow \forall \varepsilon < 1$
$\sigma(x_n, x) = \min\{\rho(x_n, x), 1\} = \rho(x_n, x) < \varepsilon$
s.t. $\sigma(x_n, x) < \varepsilon$ whenever $n \geq N$.

② But $\forall \varepsilon \geq 1$
$\sigma(x_n, x) = \min\{\rho(x_n, x), 1\} = 1 < \varepsilon$
$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\sigma(x_n, x) < \varepsilon$ whenever $n \geq N$. ✓

$\Leftarrow$ $\lim_{n\to\infty} x_n = x$ in (X, σ)

$\forall \varepsilon > 0, \exists N$ s.t. $\sigma(x_n, x) < \varepsilon$ whenever $n \geq N$
clearly $\sigma(x_n, x) \leq 1$ by definition

$\sigma(x_n, x) = \min\{\rho(x, y), 1\} < \varepsilon$
So if $\varepsilon < 1$
$\exists$. — . ✓
if $\varepsilon \geq 1$
. — — . — ✓

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§ 9.2

B. Show that if Y is a subset of a complete metric space X, then Y is compact iff it's closed & totally bdd.

X complete : every Cauchy converges to a pt in X.

Y compact: every open cover of X has a finite subcover.

$\Rightarrow$. Y is compact $\Leftrightarrow$ Y is complete & totally bdd (by B-L thm)

Show closed.

Sps $(x_n) \in Y$, $x_n \to x \notin Y$

(x_n) converges $\Rightarrow (x_n)$ is Cauchy by def of completeness

$x \in Y$

$\Rightarrow\!\Leftarrow$ So $x_n \to x \in Y$ ✓ so it's closed

(Y is closed)

$\Leftarrow$ Y closed & totally bdd

Y closed $\Rightarrow$ $\forall (x_n) \in Y$, $x_n \to x \in Y$ as well.

cvgt $\Rightarrow$ Cauchy

$\Rightarrow \forall$ Cauchy seq $\to$ some pt in Y

$\Rightarrow$ Complete.

C. Show a closed subset of a compact metric space is compact.

$A \subseteq X$

X compact, A closed.

$\forall (x_n) \in A$, $(x_n) \in X$ as well

X compact, $\Rightarrow \exists$ subsequence $(x_{n_k}) \in A$

$\Rightarrow A$ sequentially compact $\Rightarrow A$ compact.

F. Prove Cantor's Intersection Thm: A decreasing sequence of nonempty compact subsets $A_1 \supset A_2 \supset \cdots$ of a metric space (X, ρ) has nonempty intersection

Proof: pick $a_i \in A_i$

$\forall A_i$ compact $\Rightarrow$ sequentially compact

consider (a_i) have a convergent subsequence (a_{i_k}) s.t.

$a_{i_k} \to a \in A_i$

on the other hand

$a_{i_k} \in A_i$

if $k \geq i$ for sure.

$\Rightarrow \forall A_i$ we can just delete first several terms of (a_{i_k}) and say the remaining $a \in A_i \Rightarrow a \in \bigcap A_i$

$\Rightarrow$ nonempty.

J. If f is a continuous one-to-one function of a compact metric space X onto Y, show f^{-1} is continuous. Hint: thm 9.2.4.

By contradiction.

Sps f^{-1} not continuous.

$\Rightarrow \exists b \in Y$ & $(y_n) \in Y$ s.t. $y_n \to b$ but

$x_n = f^{-1}(y_n) \not\to a = f^{-1}(b)$

$\Rightarrow \exists \varepsilon > 0, \forall N$ s.t. $\rho(x_n, a) > \varepsilon$ whenever ~~$\rho(y_n, b)>$~~ $n \geq N$

X compact: $\exists (x_{n_k}) \to a' \in X$

$\Rightarrow \underset{y_{n_k}}{f(x_{n_k})} \to f(a')$

we know $y_n \to f(a)$ & $y_n \to b$

$\Rightarrow y_{n_k} \to b$ as well

$\Rightarrow$ ~~$f(a)$~~ $f(a') = f(a)$

$\Rightarrow a = a'$ (1-1)

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§5.1

B. Let $f(x)=\frac{x}{\sin x}$ for $0<|x|<\frac{\pi}{2}$ and $f(0)=1$. Show that f is continuous at 0. Find an $r>0$ such that $|f(x)-1|<10^{-6}$ for all $|x|<r$.

Hint: use inequalities in e.g. 2.4.7.

Sol: $f(x)=\begin{cases}\frac{x}{\sin x}, & 0<|x|<\frac{\pi}{2}\\ 1, & x=0\end{cases}$

$\forall \varepsilon>0, \exists\delta$ s.t. $\|f(x)-f(0)\|<\varepsilon$ whenever $\|x-0\|<\delta$

w.t.s $\|\frac{x}{\sin x}-1\|<\varepsilon$ whenever $|x|<\delta$

We know $\sin x<x<\cancel{\cos x}\tan x=\frac{\sin x}{\cos x}$

$\Rightarrow \cos x<\frac{\sin x}{x}<1$

$\Rightarrow 1<\frac{x}{\sin x}<\frac{1}{\cos x}$ as $x\to 0$

$\Rightarrow \lim_{x\to 0}\frac{x}{\sin x}=1$

i.e. $\forall\varepsilon>0\ \exists\delta$ s.t. $|\frac{x}{\sin x}-1|<\varepsilon$ whenever $|x|<\delta$ ✓

D. Prove that f is cont. at $(0,y_0)$ where f is defined on R^2 by

$f(x,y)=\begin{cases}(1+xy)^{\frac{1}{x}} & \text{if } x\neq 0\\ e^{y} & \text{if } x=0\end{cases}$

w.t.s. $\forall\varepsilon>0, \exists\delta$ s.t.

$\cancel{|(1+xy_0)^{\frac{1}{x}}-(e^{y_0})|<\varepsilon \text{ when } |x|<\delta}$

know $(1+\frac{1}{x})^{x}\to e^{1}$ when $x\to\infty$

So $(1+\frac{1}{x}y)^{\frac{1}{x}}\to e^{y}$ as $x\to 0$

So $e^{y_0}\to e^{y}$ as $y_0\to y$

take $r=\ln(\varepsilon\cdot e^{-y_0}+1)$

then $|y-y_0|<\ln(\varepsilon\cdot e^{-y_0}+1)$

$e^{y-y_0}=\varepsilon\cdot e^{-y_0}+1$

$e^{y}=\varepsilon+e^{y_0}$

$e^{y}-e^{y_0}=\varepsilon$ ✓

E. $f(x,y)=\begin{cases} 0 & \text{if } y\le 0 \text{ or if } y\ge x^2 \\ \sin(\frac{\pi y}{x^2}) & \text{if } 0<y<x^2 \end{cases}$

(a). Show it's discontinuous at $(0,0)$

(b). Show the restriction of f to any straight line through the origin is continuous.

(a). $f(0,0)=0$

$\sin(\frac{\pi y}{x^2}) \to \sin(\frac{\pi}{2})$.

$y<x^2$ $\quad y=\frac{1}{2}x^2 \to \sin\frac{\pi}{2} \to 1$

$y=\frac{1}{3}x^2 \to \sin\frac{\pi}{3} \to \frac{\sqrt{3}}{2}$

?, (b.) $y=kx - \lim \sin(\frac{k\pi x}{x^2}) = \lim \sin(\frac{k\pi}{x}) \to 0$ $\to$ ~~and~~ should it be undefined?

$\forall \varepsilon>0, \exists \delta$ s.t. $|\sin\frac{k\pi}{x} \to 0| < \varepsilon$ whenever $|x|<\delta$

why?

F. (a). Show that the def of limit can be reformulated using open balls instead of norms as follows: A function f mapping a subset $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ has limit $\vec{v}$ as $\vec{x} \to \vec{a}$ provided that ~~e~~ $\forall$ every $\varepsilon>0$, $\exists r>0$ such that $f(B_r(\vec{a}) \cap S \setminus \{\vec{a}\}) \subset B_\varepsilon(\vec{v})$.

Sol: $\lim_{x\to a} f(x) = v \Rightarrow \forall \varepsilon>0 \exists \delta$ s.t. $|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<r$. and $x\in S$.

~~x∈S~~. $f(B_r(\vec{a}) \cap S \setminus \{\vec{a}\}) \subset B_\varepsilon(\vec{v})$, $0<|x-a|<r$ and $x\in S$.

$\Leftrightarrow x\in B_r(a) \cap S\setminus\{a\}$ clearly.

$f(B_r(a) \cap S \setminus\{a\}) \subseteq B_\varepsilon(\vec{v})$

b. reformulation of f continuous at $\vec{a}$.

$\forall \varepsilon>0, \exists r, |x-a|<r \;\&\; x\in S \Rightarrow |f(x)-v|<\varepsilon$.

~~G. sps ∀r>0, ∃y∈B_r~~

G. Sps $f: \mathbb{R}^n \to \mathbb{R}$ is continuous, if there are $\vec{x} \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that $f(x) < C$, then prove that $\exists\, r>0$ s.t. $\forall y \in B_r(x), f(y) < C$.

(~~Sps $\forall r>0, \exists y \in B_r(x)$ s.t. $f(y) \geq C$.~~)

f continuous at x, $\forall \varepsilon > 0, \exists\, r>0$ s.t. $|f(y) - f(x)| < \varepsilon$ whenever $|y-x| < r$. $\Rightarrow y \in B_r(x)$

let $\varepsilon = C - f(x)$

then $|f(y) - f(x)| < C - f(x)$

$f(y) < C$ ~~contradict~~

J. Show that if: $f: [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \leq M$ on $[a,b]$, then f is Lipschitz. Hint: MVT.

$$MVT: \exists\, c \in [a,b] \text{ s.t. } |f'(c)| = \left|\frac{f(b)-f(a)}{b-a}\right| \leq M$$

So $|f(b) - f(a)| \leq M \|b-a\|$ so f is Lipschitz

L. (a). Show that a linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ ~~with a~~ matrix $[a_{ij}]$ can be written as $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \varepsilon_i \pi_j$

(a). $\pi_j(x_1, \cdots, x_n) = x_j$ --- the jth coordinate

$\varepsilon_i(t) = t\vec{e}_i$ sending $\mathbb{R}$ onto ith coordinate axis.

$\varepsilon_i \pi_j x = x_j e_i$

$= (0, 0, \cdots, x_j, \cdots, 0)$

$\Rightarrow \|\varepsilon_i \pi_j(x) - \varepsilon_i \pi_j(y)\| = \|(0, 0, \cdots, x_j - y_j, \cdots, 0)\|$

$\|x-y\| = \|(x_1-y_1, x_2-y_2, \cdots, x_n-y_n)\|$

clearly $\|\varepsilon_i \pi_j(x) - \varepsilon_i \pi_j(y)\| \leq \|x-y\|$

take equality iff $x_i = y_i$ $(\forall i \neq j)$

[redacted]

(b). Show that $\varepsilon_i \pi_j$ is Lipschitz with constant I.

$a_{ij} \sum_i \pi_j(x) = A(x)$

$\|a_{ij}\varepsilon_i\pi_j(x) - a_{ij}\varepsilon_i\pi_j(y)\| = \|A(x)-A(y)\| \leq |a_{ij}|\,\|x-y\|$ so Lipschitz.

(c). Hence deduce that A is Lipschitz with constant $\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|$.

✓

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§5.2

H. Define f on $\mathbb{R}$ by $f(x) = x\chi_{\mathbb{Q}}(x)$. Show f is continuous at 0 & it's the only continuous point.

① $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

$\forall \varepsilon > 0, \exists r$ s.t. $|f(x) - f(0)| < \varepsilon$ whenever $|x-0| < r$.

$f(0) = 0$

$x \in \mathbb{Q}$ $f(x) = x$

$x \notin \mathbb{Q}$ $f(x) = 0$

so take $\varepsilon = r$

if $|x-0| = |x| < r = \varepsilon$, $|f(x) - 0| < \varepsilon$ for ~~some~~ sure ✓

$\Rightarrow f(x)$ continuous at $x=0$.

② show it not continuous at any other point.

Sps it continuous at $x=a$. $a \neq 0$.

Then ~~$\exists \varepsilon>0$, s.t. $\forall r>0$~~. $\forall \varepsilon > 0$, $\exists r > 0$.

~~$\|f(x)-f(a)\| > \varepsilon$ whenever $|x-a|$~~

$\|$~~$f(x)-f(a)\| < \varepsilon$ whenever $|x-a|$~~ $< r$.

~~i, Suppose $a \in \mathbb{Q}$. then $f(a) = a$. $\|f(x) - a\| < \varepsilon$ whenever~~ $|x-a| < r$

~~pick $\varepsilon = a$. then $\|f(x)-a\| < a \Rightarrow 0 < f(x) < 2a$~~

~~Sps f(x)<ε~~

$\exists \varepsilon>0$ s.t. $\forall r>0$. $|f(x)-f(a)|\geq\varepsilon$ whenever $\exists|x-a|<r$ if $a\in\mathbb{Q}$

$f(a)=a$ pick ε s.t. $\forall r$ $\|f(x)-f(a)\|\geq\varepsilon$

$\exists |x-a|<r$, $\forall \geq 0$, $B_r(a)$ contains irrational number. and $f(x)$ is thus 0.

if $x\notin\mathbb{Q}$. So if pick $\varepsilon=a$?

we have $\forall r>0$, $\exists x\notin\mathbb{Q}$ s.t. $|f(x)-f(a)|=a$

if $a\notin\mathbb{Q}$ $f(a)=0$, pick $\varepsilon>0$ s.t. $\forall r>0$, $\exists x\in B_r(a)$

s.t. $|f(x)-f(a)|\geq\varepsilon$, $|f(x)-0|\geq\varepsilon$.

if $\varepsilon=a$, $\forall r$, $\exists|x-a|<r$, $(a-r<x<a+r)$

Can always pick some rational # x_0 between a & $a+r$

$\Rightarrow f(x_0)=x_0>a$ s.t. $|x_0-0|>\varepsilon=a$

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§5.5

A. Show that $g(x)=\sqrt{x}$ is uniformly continuous on $[0,+\infty)$.

Hint: Show that $\sqrt{a-b}\geq\sqrt{a}-\sqrt{b}$ & $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$.

Uniformly continuous:

A function $f: S\subset\mathbb{R}^n\to\mathbb{R}^m$ is U.C. if $\forall\varepsilon>0, \exists\delta>0$ s.t.

$\|f(x)-f(a)\|<\varepsilon$ whenever $\|x-a\|<\delta$, $x,a\in S$. [Here δ does not depend on a].

Proof: ~~$(\sqrt{a-b})^2=a-b$~~

~~$(\sqrt{a}-\sqrt{b})^2=a^2-2\sqrt{ab}+b^2$~~

~~since $b\geq 0$~~

~~$(a-b)-(a-2\sqrt{ab}+b)=-b+2\sqrt{ab}-b=2(-b)$~~ $2\sqrt{b}(\sqrt{a}-\sqrt{b})$

$\forall\varepsilon>0, \exists\delta$ s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$

$|\sqrt{x}-\sqrt{y}|<\varepsilon$ whenever $|x-y|<\delta$

$|x-y|=|\sqrt{x}+\sqrt{y}||\sqrt{x}-\sqrt{y}|$

Know that $|\sqrt{x}-\sqrt{y}|\leq|\sqrt{x-y}|\leq|\sqrt{x+y}|\leq|\sqrt{x}+\sqrt{y}|$

$\delta>|x-y|=|\sqrt{x}+\sqrt{y}||\sqrt{x}-\sqrt{y}|$

$\geq|\sqrt{x+y}||\sqrt{x}-\sqrt{y}|$

$\geq|\sqrt{x}-\sqrt{y}|^2$

$\Rightarrow |\sqrt{x}-\sqrt{y}|^2\leq|x-y|<\delta$

So $\forall\varepsilon$, pick $\delta=\varepsilon^2$, if $|x-y|<\varepsilon^2$, then $|\sqrt{x}-\sqrt{y}|<\varepsilon$.

∎

D. Show that $f(x)=x^p$ is not uniformly continuous on $\mathbb{R}$ if $p>1$.

$\Rightarrow \exists \varepsilon>0 \ \forall r>0 \ |f(x)-f(y)|\geq \varepsilon$ for some $x,y\in S$ & $|x-y|<r$.

$|f(x)-f(y)|=|x^p-y^p|$

?

H. Sps that f is cont. on (a,c) & $a<b<c$. Show if f is uniformly continuous on both $(a,b]$ & $[b,c)$ then f is uniformly continuous on (a,c).

Proof: given $\varepsilon>0$,

$\exists \delta=\delta_1+\delta_2$

$|f(x)-f(y)|\leq|f(x)-f(b)|+|f(b)-f(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

whenever $|x-y|\leq|x-b+b-y|\leq|x-b|+|b-y|=\delta_1+\delta_2=\delta$

I. $f(x)$ be continuous on $(0,1]$ Show that f is unif. continuous iff $\lim_{x\to 0} f(x)$ exists.

$\Leftarrow$ if $\lim_{x\to 0} f(x)=L$, $\forall \varepsilon>0, \exists r$ s.t. $|f(x)-L|<\frac{\varepsilon}{2}$ whenever $0<x<r$

show unif. cont. on $(0,1]$.

Consider 2 interval $[r,1]$ & $(0,r)$.

① $[r,1]$: compact interval

$\Rightarrow$ uniformly continuous on $[r,1]$

② $(0,r)$

$\lim_{x\to 0^+} f(x)=L \Rightarrow \forall \varepsilon>0, \exists r$ s.t. $|f(x)-L|<\frac{\varepsilon}{2}$, $\forall 0<x<r$.

So, for $\forall \varepsilon>0$, we have $|f(x)-f(y)|<|f(x)-L+L-f(y)|$

$\leq|f(x)-L|+|L-f(y)|$

$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

[redacted]

~~$\Rightarrow \forall \varepsilon>0, \exists \delta$ st~~

this is guranteed that when $0<x<r$ & $0<y<r$.

$\forall \varepsilon>0, \exists \delta=2r>0$ s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta=2r$.

$\Rightarrow f$ is unif. cont. on $(0,1]$.

"$\Rightarrow$" f: unif cont. on $(0,1]$

$\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$ & $x,y \in (0,1]$.

$\Rightarrow \forall \varepsilon>0, \exists \delta$ s.t. $f(x) \in B_\varepsilon(f(y))$ $\forall x \in B_\delta(y) \cap (0,1]$

consider $y_k = \frac{1}{k} \to 0$

?

so $\forall x \in B_\delta(\frac{1}{k}) \cap (0,1]$.

we have $f(x) \in B_\varepsilon(f(y_k))$

$\Rightarrow f(B_\delta(\frac{1}{k}) \cap (0,1]) \subset B_\varepsilon(f(y_k))$

P89

§5.6

A BC.

A. Show $\exists x \in (0, \pi/2)$ s.t. $\cos x = x$ & it's the only solution.

(a). By IVT.

Let $f = \cos x - x$

$f(0) = 1 > 0$

$f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$

So $\exists c \in (0, \frac{\pi}{2})$ st. $f(c) = 0$

(b) $f'(x) = -\sin x - 1 \le 0$ monotone decreasing. so only 1 solution.

B. How many solutions are there to $\tan x = x$ in $[0, 11]$?

~~$2\pi \approx 6.2$ so.~~

T of $\tan x$ is π, so there are 3~4 period in $[0, 11]$.

$$\left.\begin{array}{l} 1 \text{ soln is } x=0 \\ 1 \text{ soln in } \frac{3}{2}\pi, 2\pi \\ 1 \text{ soln in } \frac{5}{2}\pi, 3\pi \\ 1 \text{ soln in } \frac{7}{2}\pi, 11] \end{array}\right\} 4$$

C. Show $2\sin x + 3\cos x = x$ has 3 solutions

$f = 2\sin x + 3\cos x - x = 0$

$-\sqrt{13} \le 2\sin x + 3\cos x \le \sqrt{13}$

so find solutions on $[-\sqrt{13}, \sqrt{13}]$ since $x = 2\sin x + 3\cos x \in [-\sqrt{13}, \sqrt{13}]$

~~$f' = 2\cos x - 3\sin x - 1 = 0$~~

~~$2\cos x - 3\sin x = 1$~~

~~$2\cos x = 1 + 3\sin x$~~

~~let $\sin x = a$~~

~~$1 + 3a = 2\sqrt{1-a^2}$~~

~~$1 + 6a + 9a^2 = 4 - 4a^2$~~

~~$13a^2 + 6a - 3 = 0$~~

~~$a = \frac{-6 \pm \sqrt{36 + 12 \times 13}}{26}$~~

~~$12 \times 13 = 36 + 12 = 156$~~

then should we use calculator?

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§ 5.7 F.

Verify that the formula for the Cantor function in terms of the ~~ternary~~ ternary expansion yields the same answer for both expansions of a point x when 2 expansions exist.

H. For $x \in [0,1]$, express it as a decimal $x = x_0.x_1x_2x_3\cdots$
Use a finite decimal expansion without ~~rea~~ repeating 9's when there is a choice. Then define a function f by
$f(x) = x_0.0x_10x_20x_3\cdots$

(a). Show f is strictly increasing.

$\forall x, y \in [0,1]$. say $x<y$

$$\frac{f(y)-f(x)}{y-x} = \frac{y_0.0y_10y_2\cdots - x_0.0x_10x_2\cdots}{y_0.y_1y_2\cdots - x_0.x_1x_2}$$

$$x = \sum_{i=1}^{\infty} x_i \cdot 10^{-i}$$

$$f(x) = \sum_{i=1}^{\infty} x_i \cdot 100^{-i}$$

P117
§7.1

C. For f in $C'[a,b]$, define $\varphi(f) = \|f'\|_\infty$. Show that φ is nonnegative, homogeneous and satisfies the triangle ineq. Why is it not a norm?

Proof: $\varphi(f) = \|f'\|_\infty = \sup|f'(x)|$

~~since f(x)~~ Since $|f'(x)| \geq 0$ so $\sup|f'(x)| \geq 0$

so $\varphi(f) \geq 0$. (nonnegative)

* if $f=1$, $\varphi(f)=0$ so not positive definite

but NOT POSITIVE DEFINITE

$\cancel{\varphi\|\alpha f\|}$ $\varphi(\alpha f) = \|\alpha f'\|_\infty = \sup|\alpha||f'(x)| = |\alpha|\,\|f'\|_\infty$

(homogeneous)

$\varphi(f+g) = \|f'+g'\|_\infty = \sup|f'(x)+g'(x)|$

$\leq \sup(|f'(x)| + |g'(x)|)$

$= \sup|f'(x)| + \sup|g'(x)|$

(triangle inequality)

So not a norm!

D. If $(V, \|\cdot\|)$ is a normed vector space. Show $\big|\|x\|-\|y\|\big| \leq \|x-y\|$

$\cancel{(\|x\|-\|y\|)^2 \leq \|x-y\|^2}$

$\cancel{\|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \leq \|x\|}$

$\cancel{\|y\| = \|-y\|}$

$\cancel{\|x-y\| \leq \|x\| + \|-y\| - \|x\| \|y\|}$

$\cancel{\|x\| - \|y\| \leq \|x-y\| \|x\| \|y\|}$

$\cancel{\|x-y\| = \|x+(-y)\| \leq \|x\| + \|y\| = \|x\| + \|y\|}$

$\|x\| \leq \|x-y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$

$y = y - x + x$

$\Rightarrow \|y\| \leq \|y-x\| + \|x\|$

$\Rightarrow \|y-x\| \geq \|y\| - \|x\|$

~~Since~~ i.e. $\|x-y\| \geq \|y\| - \|x\| \Rightarrow \|x-y\| \geq \big|\|x\| - \|y\|\big|$

$\cancel{(\|x\|-\|y\|)^2 \le \|x-y\|\|x-y\|}$

$\cancel{\|x\|^2-\|y\|^2}$

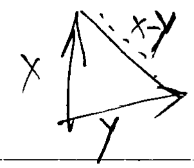
$\cancel{\big|\|x\|-\|y\|\big|\,\big|\|x\|+\|y\|\big| \le \|x-y\|(\|x\|+\|y\|) \ge \|x-y\|\|x+y\|}$ $\cancel{\|x^2-y^2\|}$

$\cancel{\big|\|x\|^2-\|y\|^2\big|}$

E. Show that the unit ball of a normed vector space $(V, \|\cdot\|)$ is convex, meaning that if $\|x\|\le 1$ and $\|y\|\le 1$ then every pt on the line segment b/w x & y has norm at most 1.

$\cancel{\left.\begin{array}{l}\|x\|\le 1\\ \|y\|\le 1\end{array}\right\} \Rightarrow \|k(x-y)\| = |k|\,\|x-y\| \le \|x-y\| \le \|x+(-y)\| = \|x\|+\|y\|}$

$\cancel{\& \ k\in\mathbb{R},\ |k|\le 1}$



$\cancel{\|x\|\le 1,\ \|y\|\le 1 \quad \|k(x-y)\| = |k|\,\|x-y\| \le \|x-y\| \le \|x\|+\|y\|}$

$k\in\mathbb{R}$, $k\in[0,1]$

?

??

line segment b/w x & y is $(1-t)x + ty$ $(0\le t\le 1)$ — should be 1

Show $\|(1-t)x+ty\|\le 1$

$\|(1-t)x + ty\| \le \|(1-t)x\| + \|ty\| = (1-t)\|x\| + t\|y\| \le 1-t+t = 1$ ✓

F. Let K be a compact subset of $\mathbb{R}^n$, and let $C(K,\mathbb{R}^m)$ denote the vector space of all continuous functions from K ~~onto~~ into $\mathbb{R}^m$. Show that for f in $C(K,\mathbb{R}^m)$ the quantity $\|f\|_\infty = \sup_{x\in K}\|f(x)\|_2$ is finite & $\|\cdot\|_\infty$ is a norm on $C(K,\mathbb{R}^m)$.

may be not an Euclidean!

Proof: $\|f\|_\infty = \sup\|f(x)\|_2 = \cancel{\sup\left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac12}} = \sup\left(\int_a^b f(x)^2dx\right)^{\frac12}$

K is closed & bdd so $\exists\, x\in K$ has max, so $\|f_\infty\|$ finite when x is the maximum of K. Then sup ~~is~~ exists.

norm?

Show norm (3 parts!)

I. Let S be any subset of $\mathbb{R}^n$. Let $C_b(S)$ denote the vector space of all bdd continuous functions on S. For $f \in C(S)$, define $\|f\|_\infty = \sup_{x\in S} |f(x)|$

(a). Show that this is a norm on $C_b(S)$

(b). When is this a norm on the vector space of all continuous functions on S?

I. a). $f \in C_b(S)$, $\|f\|_\infty = \sup_{x \in S} |f(x)|$

EVT: finite

① $f=0 \to$ bdd $\sup|f|=0$

if $\sup|f| = \|f\|_\infty = 0 \Rightarrow |f|=0$ for $\forall x \in k$

② $\sup|\alpha f| = |\alpha| \sup(f) = |\alpha| \|f\|_\infty$

③ - —

(b) - ~~compact~~

P119
§7.2

B. Show that every convergent seq. in a normed space is a Cauchy sequence.

Pf: $(x_n)_{n=1}^{\infty}$, so $\lim_{n\to\infty} x_n = L$.

$\forall \varepsilon>0, \exists N$ s.t. $\|x_n - L\| < \frac{\varepsilon}{2}$ whenever $n \geq N$

similarly $\forall \varepsilon>0, \exists N'$ s.t. $\|x_m - L\| < \frac{\varepsilon}{2}$ whenever $m \geq N'$

Let $N'' = \max(N, N')$

Then $\forall \varepsilon>0, \exists N'' = \max(N, N')$

s.t. $\varepsilon > \|x_n - L\| + \|x_m - L\| \geq \|x_n - x_m\|$ whenever $n, m \geq N''$.

Cauchy.

D. Show if A is an arbitrary subset of a normed vector V ~~space~~ and ~~$a \in \mathbb{R}$ then $x+A$~~ U is an open subset, then $A + U = \{a+u : a \in A, u \in U\}$ is open.

~~$\forall$~~ U is open : $\forall u \in U$, $\exists r$ s.t. $B_r(u) \subset U$ ①

wts. $A+U$ is open $\forall a \in A, u \in U$, $\exists r$ s.t. $B_r(a+u) \subset U$ ②

① $\Rightarrow$ for $\forall x \in U$
$\|x - u\| < r$

② $\Rightarrow$ for $\forall x \in U$
$\|x - (a+u)\| < r$

[redacted]

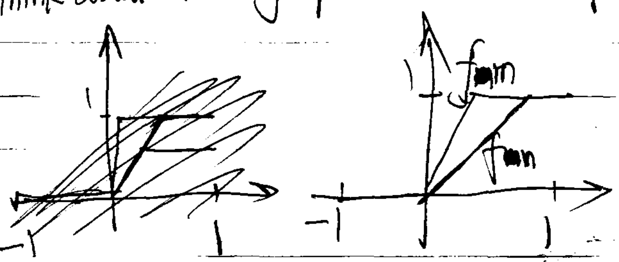
I. Consider piecewise linear functions in $C[-1,1]$ given by $f_n(x)=0$ for $-1\le x\le 0$, $f_n(x)=nx$ for $0\le x\le 1/n$ and $f_n(x)=1$ for $\frac{1}{n}\le x\le 1$.

(a). Show that $\|f_n-f_m\|_\infty \ge \frac{1}{2}$ if $m\ge 2n$.

$$f_n(x)=\begin{cases} 0, & -1\le x\le 0 \\ nx, & 0\le x\le \frac{1}{n} \\ 1, & \frac{1}{n}\le x\le 1 \end{cases}$$

Solution: $\|f_n-f_m\|_\infty = \sup|f_n-f_m| = \begin{cases} 0 & m,n\le 0 \\ \cancel{\ } & \cancel{m>0,\ n<0} \\ |f_m| & \\ & \cancel{m,n>0} \end{cases}$

Think about the graph



the difference exists only when $x\in[0,\frac{1}{n}]$

since $m\ge 2n \Rightarrow \frac{1}{m}\le\frac{1}{2n}$

$$\|f_n-f_m\|_\infty = \sup\left|\frac{1}{m}-\frac{1}{n}\right| \ge \sup\left||\tfrac{1}{m}|-|\tfrac{1}{n}|\right| \ge \sup\left|\tfrac{1}{2n}-\tfrac{1}{n}\right| \ge \sup\left|\tfrac{1}{2n}\right| \ge \tfrac{1}{2}$$ since $n\in[0,1]$

(b). Hence show that no subsequence of $(f_n)_{n=1}^{\infty}$ converges.

Sps $\exists (f_{n_i})$ subsequence of (f_n) that cvg to f.

$\Rightarrow (f_{n_i})$ must be Cauchy

$\Rightarrow$ it satisfies the following properties

$\forall \varepsilon>0, \exists N$ s.t. $\|f_{m_i}-f_{n_i}\|_\infty<\varepsilon$ whenever $m_i,n_i\ge N$

but if we chose $\varepsilon=\frac{1}{2}$, $m_i=2N+1$, $n_i=N$

by a) we have $\|f_{2N+1}-f_N\|_\infty \ge \frac{1}{2}$. $\Rightarrow\!\Leftarrow$!

(c). Conclude that the unit ball of $C[-1,1]$ is not compact

unit ball is $\{f : \|f\|_\infty \le 1\}$ clearly $f_n \in C[-1,1]$

by (b). we know $\nexists$ a cvgt subseq.

~~so not closed~~

so not compact

(d). Show the unit ball of $C[-1,1]$ is closed & bdd & complete.

$\{f : \|f\| \le 1 \ \&\ f \in C[-1,1]\}$

clearly bdd

since $\|f\| \le 1 \ \forall f \in C[-1,1]$ show closed $(f_n) \to f$

① f cont. on $[-1,1]$ ② $\|f\| \le 1$

$\|f(x) - f(a)\|_\infty = \| f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)\|_\infty$

$\le \| \quad \| + \| \quad \| + \| \quad \|$, pick $\|f(x) - f(a)\|_\infty = \frac{\varepsilon}{3}$, $|x-a| < \delta$

$\Rightarrow$ cont. sps $\|f\| > 1 \Rightarrow |f|$ not in the unit ball.

J. Prove that the following are equivalent for a normed vector space $(V, \|\cdot\|)$.

(1). $(V, \|\cdot\|)$ is complete.

(2). Every decreasing seq of closed balls has a nonempty intersection. Note that the balls need not be concentric.

(3). Every decreasing seq of closed balls with radii $r_j \to 0$ has ~~an~~ nonempty intersection.

J:

(1) $\Rightarrow$ (2)

$B_{r_1}(a_1) \supset B_{r_2}(a_2) \supset \cdots \supset B_{r_n}(a_n) \supset \cdot \quad \bigcap_{i=1}^{\infty} B_{r_i}(a_i) \neq \emptyset$

Claim: $(a_n)_{n=1}^{\infty}$ is Cauchy given $\varepsilon > 0$, we pick the largest r_k s.t. $r_k < \varepsilon$ and the corresponding center a_k.

$\Rightarrow B_{r_k}(a_k) \subseteq B_{\varepsilon}(a_k)$ decreasing seq.

$\Rightarrow \forall n > k,\ B_{r_n}(a_n) \subseteq B_{r_k}(a_k) \subseteq B_{\varepsilon}(a_k)$

$\Rightarrow \|a_n - a_k\| \leq r_k < \varepsilon$

Then for $\forall n, n \geq k$, s/s WLOG $n > m$

$\Rightarrow a_n \subseteq \overline{B_{r_m}}(a_m) \subseteq B_{r_k}(a_k) \subseteq B_{\varepsilon}(a_k)$

$\|a_n - a_n\| \leq \|a_n - a_k\| \leq r < \varepsilon$

starting from a_n, subseq. still cvg to a
contained in closed ball

Normed vector space is complete.

$\Rightarrow (a_n)_{n=1}^{\infty} \to a \in V$

$\Rightarrow a \in B_{r_n}(a_n)$ for $\forall n$ since the balls are closed

(2) $\Rightarrow$ (3) true since 3) is a "specific" condition of 2)

3) $\Rightarrow$ 1).

$\bigcap B_{r_i}(a_i) \neq \emptyset$

Show (x_n): Cauchy has a limit

suffices to show a subseq. converge to s.th. in V

can find ~~something~~ subseq. s.t. $\|y_k - y_{k+1}\| < 2^{-k}$

(let $y_k = x_{n_k}$ where $d(x_m, x_n) < 2^{-k}\ \forall\ m, n \geq n_k$)

Consider $\overline{B}_1(y_1)$, $\overline{B}_{2^{-1}}(y_2) \cdots \overline{B}_{2^{-k+1}}(y_k) \cdots$

rad $\to 0$.

check decreasing. if $z \in B_{2^{-(k+1)+1}}(y_{k+1})$

$\|z - y_k\| \leq \|z - y_{k+1}\| + \|y_{k+1} - y_k\| \leq 2^{-k} + 2^{-k} = 2^{-k+1}$

$\Rightarrow z \in B_{2^{-k+1}}(y_k)$. we know $\bigcap \overline{B}_{2^{-k+1}}(y_k) \neq \emptyset$

sps $x \in B_{2^{-k+1}}(y_k) \Rightarrow \|y_k - x\| \le 2^{-k+1}$

$\Rightarrow \lim_{k\to\infty} \|y_k - x\| = 0$ i.e. $\lim_{k\to\infty} y_k = x \Rightarrow (x_n) \to x$ as well.

P123-P124

§7.3.

A. Let V be a finite-dim vector space with 2 norms $\|\cdot\|$ & $|||\cdot|||$. Show that there are constants $0<a<A$ s.t. $a\|v\| \le |||v||| \le A\|v\|$

By linear algebra, know $\forall$ finite dim v-space has a basis

$\{v_1, \cdots, v_n\}$ is basis in $(V, |||\cdot|||)$

$\Rightarrow \forall$ vectors in $(V, |||\cdot|||)$ is a linear combination of basis vectors.

$\Rightarrow \vec{v} \in V$

$\vec{v} = a_1\vec{v_1} + \cdots + a_n\vec{v_n}$

$|||\vec{v}||| = |||a_1\vec{v_1} + \cdots + a_n\vec{v_n}|||$

$\exists\, 0<c<C$ s.t. $c\|\vec{a}\|_2 \le |||\vec{v}||| \le C\|\vec{a}\|_2$

still can pick $0<d<D$

s.t. $d\|\vec{a}\|_2 \le \|\vec{v}\| \le D\|\vec{a}\|_2$

So

$|||\vec{v}||| \le C\|\vec{a}\|_2 = d\|\vec{a}\|_2 \cdot \frac{C}{d} \le \frac{C}{d}\|\vec{v}\| \Rightarrow \boxed{A = \frac{C}{d}}$

$c\|\vec{a}\|_2 \le |||\vec{v}|||$

$\overset{\shortparallel}{D}\|\vec{a}\|_2 \cdot \frac{c}{D}$

$\overset{\vee\!\!/}{\frac{c}{D}}\|v\| \Rightarrow \boxed{a = \frac{c}{D}}$

B) Let T be invertible linear map from Cor. 7.3.2

(a). Use Lipschitz property of T & T^{-1} to show $T(B_r(x))$ contains a ball about Tx in V, and that $T^{-1}(B_r(Tx))$ contains a ball about x in $\mathbb{R}^n$.

(b). Hence show directly that U is open iff $T(U)$ is open.

C. Proof of Cor 7.3.3.

A subset of a finite-dim normed vector space is compact iff it's closed & bdd.

$\Rightarrow$ $\forall$ seq has a convergent subseq. which converge to a pt in V

Sps not.

$(V_n) \to v \notin V$

So $\forall$ subseq of $(V_n) \to v \notin V$ ~~vio~~ violates compactness.

compactness $\Rightarrow$ bdd.

sps not, $\exists$ an unbdd seq. s.t. $\|V_n\| > n$.

$\Rightarrow$ subseq. (V_{n_k}) s.t. $\|V_{n_k}\| > n_k$ which does not converge

E. Show $\forall$ each integer n and each function f in $C[a,b]$, $\exists$ a polynomial of degree at most n ~~&~~ that is closest to f in the max norm on $C[a,b]$.

~~$f \in C[a,b]$ on~~

$\| f \|$ ~~$= \sup_{x \in [a,b]} |f(x)|$~~

$\|f - 0\| = \|f\| \le \inf \{ \|f-g\| : g\text{'s degree} \le n \}$

$\|f\| = \max |f|$

f is cont. ~~on [a,b]~~, $[a,b]$ is compact

EVT $\Rightarrow \max \|f\| = M$ exists

For all g s.t. $\|g-f\| \le M$, $\|g\| \le \|g-f\| + \|f\| \le 2M$

$\Rightarrow \inf \{ \|g-f\| : \forall g \} = \inf \{ \|g-f\| : \forall g \in k \}$

define $f(x) = \|x-f\|$ where $x \in k$,

$\|f(x) - f(y)\| = | \|x-f\| - \|y-f\| | \le \|x-y\|$ $\Rightarrow$ Lipschitz $\Rightarrow$ cont.

k is bdd.

$\forall g_n \to g$, $g_n \in k$, $\|g\| \le \|g_n - g\| + \|g_n\| \le \varepsilon + 2M$, $\forall \varepsilon \Rightarrow \|g\| \le 2M$

$\Rightarrow K$ compact.

P127

§7.4

A. Let $A=\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ Show form $\langle \cdot , \cdot \rangle_A$ is positive definite.

$$\langle x,y\rangle_A = \langle Ax, y\rangle = \sum\sum a_{ij}\, x_i y_j$$

$$\forall \vec{x} \in \mathbb{R}^3,\ \vec{x} = (x_1, x_2, x_3)$$

$$\langle x, x\rangle_A = 3x_1^2 + 1x_1x_2 + 2x_1x_3 + 1x_2x_1 + 2x_2^2 + 1x_2x_3 + 2x_3x_1 + x_3x_2 + \cancel{x_3 x}\, 4x_3^2$$

$$= 3x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_1x_3$$

$$= (\quad)^2 + (\quad)^2 + (\quad)^2 \geq 0$$

So --- positive definite.

C. Minimize the quantity $\|x\|^2 - 2t\langle x,y\rangle + t^2\|y\|^2$ over $t \in \mathbb{R}$.

$\cancel{\|x\|^2 = \langle x,x\rangle}$

$\cancel{\|y\|^2 = \langle y,y\rangle}$

$\cancel{\langle x,x\rangle - 2t\langle x,y\rangle + t^2\langle y,y\rangle}$

$\cancel{= \langle x,x\rangle} - \langle 2tx, y\rangle + \langle ty, ty\rangle$

$=$

$$f(t) = t^2\|y\|^2 - 2t\langle x,y\rangle + \|x\|^2$$

$$f'(t) = 2t\|y\|^2 - 2\langle x,y\rangle = 0 \Rightarrow \quad t = \frac{\langle x,y\rangle}{\|y\|^2}$$

$$f''(t) = 2\|y\|^2 \geq 0$$

~~pick y=0~~

✓ minimizer!

F. Let $w(x)$ be a strictly positive cont. function on $[a,b]$ Define a form on $C[a,b]$ by $\langle f,g\rangle_w = \int_a^b f(x)g(x)w(x)\,dx$ for $f,g \in C[a,b]$. Show that it's an inner product.

2 $\langle f,g\rangle_w = \int_a^b f\cdot g\cdot w\,dx$ w is strictly $\oplus$

$\langle g,f\rangle_w = \int_a^b g\cdot f\cdot w\cdot dx = \int_a^b f g w\,dx$

$\boxed{\text{symmetry}}$

3 bilinear is also trivial

1 <u>positive definiteness</u>

$\langle f,f\rangle_w = \int_a^b f^2 w\,dx$, know $w>0$, $f^2 \geq 0$

so $h = f^2 w \geq 0$

$= \int_a^b h\,dx \quad (h \geq 0)$

so $\langle f,f\rangle_w \geq 0$

if $\langle f,f\rangle_w = 0$

only if $\cdots 0$.

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§ 7.6 Fourier

C. (a). Find the fourier series of $\cos^3\theta$

(b). use trig identities to verify.

[redacted]

(a). $\cos^3\theta = A_0 + \sum_{n=1}^{\infty} A_n\cos n\theta + B_n\sin n\theta$

where $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$

$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$, $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$ for $n \geq 1$.

$\cancel{A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^3\theta\,d\theta}$

know that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$$\cos^3\theta = \frac{e^{3i\theta}+e^{-3i\theta}+3e^{i\theta}+3e^{-i\theta}}{2^3}$$

$$= \frac{1}{4}\left(\frac{e^{3i\theta}+e^{-3i\theta}}{2} + \frac{3(e^{i\theta}+e^{-i\theta})}{2}\right)$$

$$= \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\right]d\theta$$

$$= \frac{1}{2\pi}\left(\frac{1}{12}\sin 3\theta + \frac{3}{4}\sin\theta\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}(0+0-0-0)$$

$$= 0$$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\right)\cdot\left(\frac{e^{in\theta}+e^{-in\theta}}{2}\right)d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{e^{3i\theta}+e^{-3i\theta}}{8} + \frac{3(e^{i\theta}+e^{-i\theta})}{8}\right)\frac{e^{in\theta}+e^{-in\theta}}{2}\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{(3+n)i\theta}+e^{-(3+n)i\theta}}{16} + \frac{3e^{(1+n)i\theta}+3e^{-(1+n)i\theta}}{16}\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{8}\cos(3+n)\theta + \frac{3}{8}\cos(1+n)\theta\,d\theta$$

$$= \frac{1}{\pi}\;\frac{1}{8(3+n)}\sin(3+n)\theta + \frac{3}{8(n+1)}\sin(1+n)\theta\Big|_{-\pi}^{\pi}$$

$$= 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{e^{3i\theta}+e^{-3i\theta}}{8} + \frac{3e^{i\theta}+3e^{-i\theta}}{8}\right)\frac{e^{ni\theta}-e^{-ni\theta}}{2i}\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{(3+n)i\theta}+\cancel{e^{(n-3)i\theta}}-\cancel{e^{(3-n)i\theta}}-e^{-(3+n)i\theta}}{16i} + \frac{3e^{(n+1)i\theta}-3e^{-(n+1)i\theta}}{16i}\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{8}\sin(3+n)\theta + \frac{3}{8}\sin(n+1)\theta\right)d\theta$$

$$= \frac{1}{\pi}\left[\frac{-1}{8(3+n)}\cos(3+n)\theta + \frac{-3}{8(n+1)}\cos(n+1)\theta\right]_{-\pi}^{\pi}$$

$=$ when n is odd, $3+n$ is ~~even~~, $\cos(3+n)\pi$ is 1.

$$\frac{-1}{8(3+n)} - \frac{3}{8(n+1)} -$$

trig property is

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

E. Find the Fourier series

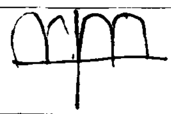
(a). $f(\theta)=|\sin\theta|$

see next page

$A_0+\sum A_n\cos n\theta+B_n\sin n\theta$

~~$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin\theta|d\theta=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin\theta d\theta=\left[-\cos\theta\Big|_{-\pi}^{\pi}\right]\cdot\frac{1}{\pi}=+1-(-1)=0$~~

~~$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin\theta|$~~ $A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin\theta|d\theta$



$=\frac{1}{2\pi}\cdot 2\int_0^{\pi}\sin\theta\, d\theta=\frac{1}{\pi}\left(-\cos\theta\Big|_0^{\pi}\right)=\frac{1}{\pi}(1-(-1))=0$

~~$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}$~~ why always 0?

(b). $f(\theta)=\theta$.

H. $f(\theta)=\begin{cases}1-|\theta| & , -1\le\theta\le 1\\ 0 & \text{o.w.}\end{cases}$

$g(\theta)=\begin{cases}1 & -1\le\theta\le 0\\ -1 & 0<\theta\le 1\\ 0 & \text{o.w.}\end{cases}$

I. show if $f\in C[-\pi,\pi]$ is an odd function, then Fourier series of f involves only of the form $\sin k\theta$.
if even . . . $\cos k\theta$ & constant term.

$I=\int_{-a}^{a}f(x)dx=\int_{-a}^{0}f(x)dx+\int_0^a f(x)dx$

J. $f \in C[-\pi,\pi]$,

$$f_e(\theta) = \frac{1}{2}(f(\theta) + f(-\theta))$$

$$f_o(\theta) = \frac{1}{2}(f(\theta) - f(-\theta))$$

compute f_e & f_o in terms of series of f.

Note $f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) = f(x) = f_e + f_o$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f_e(x) + f_o(x))\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (f_e\,dx)$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (f_e + f_o)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e \cos nx\,dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (f_e + f_o)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o \sin nx\,dx$$

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H. $f(\theta) = \begin{cases} 1-|\theta| & , -1 \le \theta \le 1 \\ 0 & , \text{o.w.} \end{cases}$

even function.

$A_0 = \frac{1}{2}\int_{-1}^{1}(1-|\theta|)d\theta = \int_0^1 1-\theta \, d\theta = \theta - \frac{1}{2}\theta^2 \Big|_0^1 = \frac{1}{2}$

$A_n = 1 \cdot \int_{-1}^{1}(1-|\theta|)\cos n\theta d\theta = 2\int_0^1 \cos n\theta - \theta \sin n\theta \, d\theta$

$= 2\frac{\sin n\theta}{n}\Big|_0^1 - 2\int \theta \cos n\theta \, d\theta$

$= 2\frac{\sin n\theta}{n} - 2\int_0^1 \frac{\theta}{n} \, d(\sin n\theta)$

$= 2\frac{\sin n\theta}{n} - 2\left(\frac{\theta}{n} \cdot \sin n\theta - \int_0^1 \sin n\theta \cdot d\left(\frac{\theta}{n}\right)\right)$

$= \frac{2}{n^2}(1-\cos n)$

$g(\theta) = \begin{cases} 1 & -1 \le \theta < 0 \\ -1 & 0 < \theta \le 1 \\ 0 & \text{o.w.} \end{cases}$

odd:

$B_n = \int_0^1 (-1)\sin n\theta d\theta + \int_{-1}^{0}(+1)\sin n\theta d\theta$

$= -2\int_0^1 \sin n\theta d\theta$

$= \frac{2}{n}\cos n\theta \Big|_0^1$

$= \frac{2}{n}(\cos n - 1)$

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§7.7

B. $\vec{x} = (x_n)_{n=1}^{\infty}$, $\vec{y} = (y_n)_{n=1}^{\infty}$ be elements of l^2.

(a). Show $\sum_{n=1}^{N} |x_n y_n| \leq \|\vec{x}\| \, \|\vec{y}\|$

$$\cancel{\|xy\| = \langle \vec{x}, \vec{y} \rangle = \sum x_n y_n \leq \sum |x_n y_n|}$$

$\sum |x_n y_n| = \langle x, y \rangle$ $\cancel{\|x\| = (\sum x_n^2)^{\frac{1}{2}}}$

$\leq \|x\| \, \|y\|$ $\cancel{\|y\| = (\sum y_n^2)^{\frac{1}{2}}}$

$$\cancel{\|x\| \cdot \|y\| = (\sum x_n^2 \cdot \sum y_n^2)^{\frac{1}{2}} \geq \sum (x_n^2 y_n^2)^{\frac{1}{2}}}$$

$$\cancel{\sum |\langle x, y \rangle| = \|x \, y\| \leq \|x\| \, \|y\|}$$

(b). show $\sum x_n y_n$ converges absolutely.

$\cancel{|x_{n+1} y_{n+1}|}$ $\cancel{|x_n y_n|}$

l^2 ~~space~~ $\cancel{\Rightarrow \|x\|_2 = (\sum x_n^2)^{\frac{1}{2}}}$ ~~is finite~~

$\cancel{\Rightarrow x_n \to 0 \text{ for } \vec{x}}$

~~similarly for~~ $\cancel{\vec{y}}$

~~So~~ $\cancel{\lim_{n \to \infty} |x_n y_n| = 0}$

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B. $f_n(x) = nx(1-x^2)^n$ on $[0,1]$. for $n \geq 1$. find $\lim_{n\to\infty} f_n(x)$.
Is ~~it~~ ~~the~~ convergence uniform?

$\cancel{f_n(x) = nx(1-x^2)}$

Recall: $\lim_{n\to\infty} (1-\frac{h}{n})^n = e^{-h}$

$x=0$, $f_n(0)=0$; $x=1$, $f_n(1)=0$.
$\lim_{n\to\infty} f_n(x) = nx(1-x^2)^n$
know $1-x^2 \in [0,1]$.
so $nx(1-x^2)^n < n(1-x^2)^n$
let $(1-x^2) = t$,
can do nt^n $(0<t<1)$
$\lim_{n\to\infty} \frac{n}{t^{-n}} = (-\frac{1}{\log t}) \cdot t^n$
$t<1$, $t^n \to 0$ as $n\to\infty$
So $f_n(x) \to 0$ as $n\to\infty$. positive lim is
$f=0$, $\|f_n - f\|_\infty = \sup|nx(1-x^2)^n|$
$= \frac{n}{\sqrt{n+1}}(1-\frac{1}{2n+1})^n$
$= \frac{n}{\sqrt{2n+1}} e^{-\frac{1}{2}}$
$\to \infty$
So not unif. conv.

D. $f_n(x) = \frac{x}{1+nx^2}$ on $\mathbb{R}$?

$\forall \varepsilon > 0$, $\exists N$ s.t.
$\| f_n(x) - f \| < \varepsilon$ $\forall n \geq N$.

$\lim_{n\to\infty} f_n(x) = 0$, yes.

Check ~~poit~~ pointwise limit
$\lim_{n\to\infty} \frac{x}{1+nx^2} \to 0$, $\|f_n - 0\|_\infty = \sup|\frac{x}{1+nx^2}|$
$(\frac{x}{1+nx^2})' = \frac{1+nx^2 - x\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$
when $x = \sqrt{\frac{1}{n}}$, $f_n(x) = \frac{\sqrt{\frac{1}{n}}}{1+n\cdot\frac{1}{n}} = \frac{1}{2}\sqrt{\frac{1}{n}}$ $\Rightarrow \|f\|_\infty = \frac{1}{2\sqrt{n}} \to 0$, so cvg. unif.

H. Sps $f_n : [0,1] \to \mathbb{R}$ is a seq. of C^1 functions that conv. pointwise to f.
If there an M s.t. $\|f_n'\|_\infty \leq M$ $\forall n$,
prove (f_n) conv. unif.

$\exists M$ s.t. $\|f_n'\|_\infty \leq M$

$f_n' \leq \sup|f_n'| \leq M$