APM462H1S, Winter 2014 , Assignment 4,

due: Monday March 31, at the beginning of the lecture.

Remark: If you need to repeat some arguments from the lecture and/or notes, it is fine to write "By arguing exactly as in the lecture notes of Evans, page ... , we find that ..." But you should only do this if the argument is really *exactly* the same. Also, whenever you refer to some source, it is best to give a reference that is precise and verifiable, so that a reader can in principle look it up and understand what you are talking about. For this reason, it is better to cite the notes of Evans, for example, than the lecture, since a reader will not know what you are talking about if you write "By arguing exactly as in the lecture, ...".

Exercise 1.
Assume that $x(\cdot)$ is a minimizer of the calculus of variations problem:

$$\text{minimize } I[x(\cdot\ )] = \int_0^1 \left[\frac{1}{2}x'(t)^2 + x'(t)x(t)\right]\ dt \qquad \text{subject to } x(0) = 1.$$

a. Show that if $y(\cdot)$ is any C^2 function such that $y(0) = 0$ (but $(y(1)$ need not equal zero), then

$$\int_0^1 [x'(t)\, y'(t) + x'(t)\, y(t) + x(t)\, y'(t)]\ \ dt = 0$$

b. Explain why $x''(t) = 0$ for all $t \in (0, 1)$.

hint: although $y(1)$ does not need to equal zero, you are certainly free to consider functions $y(\cdot)$ such that $y(1) = 0$, if you wish.

c. Find an additional necessary condition that tells you something about the behaviour of $x(\cdot)$ at the right endpoint $t = 1$.

hint: start from the conclusion of part **(a)** and integrate by parts, but this time consider a function $y(\cdot)$ such that $y(1) \neq 0$.

d. Determine the minimizer $x(\cdot)$.

Exercise 2. Consider the problem:

$$\text{minimize} \quad I[x(\cdot)] = \frac{1}{2}\int_0^\pi x'(t)^2\ dt \tag{1}$$

subject to the conditions $x(0) = x(\pi) = 0$ and the constraint

$$J[x(\cdot)] = \int_0^\pi x(t)^2\ dt\ \ =\ \ 1. \tag{2}$$

Suppose that $x : [0, \pi] \to \mathbb{R}$ is a C^2 function that solves the above problem.
Let $y : [0, \pi] \to \mathbb{R}$ be any other C^2 function such that $y(0) = y(\pi) = 0$.
Define

$$\alpha(s) := \left(\int_0^\pi (x(t) + sy(t))^2\ dt\right)^{1/2}$$

and

$$i(s) := I[\frac{x(\cdot) + sy(\cdot)}{\alpha(s)}].$$

a. Explain why $\alpha(0) = 1$ and $i'(0) = 0$.

b. Show that

$$i'(0) = \int_0^\pi x'(t)\, y'(t)\, dt - \lambda \int_0^\pi x(t)\, y(t)\, dt \tag{3}$$

for some constant λ, and find a formula for λ in terms of $x(t)$.

hint: It may simplify things a little to note that $i(s) = (\alpha(s))^{-2}\ I[x(\cdot) + sy(\cdot)]$.

c. Show that if $x(\cdot)$ solves problem (1), (2), then

$$x''(t) + \lambda x(t) = 0 \qquad \text{for } 0 < t < \pi.$$

Exercise 3. Consider the linear system

$$\begin{cases} \dot{x}(t) &= Mx(t) + N\alpha(t) \qquad\qquad \text{for } t > 0 \\ x(0) &= x_0, \end{cases}$$

where $x(t) \in \mathbb{R}^3$ for all t, M is a 3×3 matrix, N is a 3×1 matrix (ie a column vector), and $\alpha(t) \in [-1, 1]$ for all t. Let G denote the controllability matrix

$$G = [N, MN, M^2N].$$

Find the rank of G in the following examples:

a.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

b.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

c.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

d.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Exercise 4.

a. Consider the single equation

$$x'(t) = mx(t) + n\alpha(t)$$

where $\alpha(t) \in [-1, 1]$ is a control parameter and m, n are positive numbers. Determine *exactly* the reachability set $\mathcal{C}$ for this equation. (Here $n = m = 1$, using notation from the lecture and the online notes of Evans, so the reachability set is a subset of the real line $\mathbb{R}$.)