Lecture week 2

Exercise

1. Express $S(t)$ in terms of $\mu(t)$ $\underline{(0, t)}$

$$\mu(t) = -\frac{S'(t)}{S(t)} \Rightarrow \frac{d}{dt}\log(S(t)) = -\mu(t)$$

$$\Rightarrow \left[\log S(r)\right]_0^t = -\int_0^t \mu(r)\,dr$$

$$\Rightarrow \log(S(t)) = -\int_0^t \mu(r)\,dr$$

$$\Rightarrow S(t) = e^{-\int_0^t \mu(r)\,dr}$$

We see this relationship again

2. $f(t) = \underline{\lambda \exp(-\lambda t)}$. find $\mu(t)$

$$\mu(t) = \frac{f(t)}{S(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \boxed{\lambda}$$

costant dazard at all ages.

Any concerns.

3. $f(t) = \frac{\alpha}{\beta^\alpha} t^{\alpha-1} \exp\left(-\left(\frac{t}{\beta}\right)^\alpha\right)$

This is not a gamma distribution

$$F(t) = 1 - \exp\left(-\left(\frac{t}{\beta}\right)^\alpha\right)$$

since

$$F'(t) = + \frac{t^{\alpha-1}}{\beta^\alpha} \cdot \alpha \cdot \exp\left(-\left(\frac{t}{\beta}\right)^\alpha\right)$$

$$S(t) = \exp\left(-\left(\frac{t}{\beta}\right)^\alpha\right)$$

$$u(t) = \frac{f(t)}{S(t)} = \frac{\frac{\alpha}{\beta^\alpha} t^{\alpha-1} \exp\left(-\left(\frac{t}{\beta}\right)^\alpha\right)}{\exp\left(-\left(\frac{t}{\beta}\right)^\alpha\right)}$$

$$\Rightarrow \quad u(t) = \underline{\frac{\alpha}{\beta^\alpha} \cdot t^{\alpha-1}} \quad \Delta$$

constant ?

when $\alpha > 1$ ↑

e.g. when $\alpha = 2$, $\beta = 1$ $\quad u(t) = 2t$ ↑

Show $\mathring{e}_x = \int_0^\infty {}_tp_x \, dt$.

$\mathring{e}_x = \int_0^\infty t f(t) \, dt$

$f(t)$? From week 1, we have

$_nq_x = \int_0^n {}_tp_x \mu_{x+t} \, dt = P(T_x \le n) = F_x(n)$

$= \int_0^n f(t) \, dt$

$f(t) = {}_tp_x \mu_{x+t}$.

$$\mu_{x+t} = -\frac{1}{l_{x+t}} \frac{d\, l_{x+t}}{dt} = -\frac{\dfrac{d \frac{l_{x+t}}{l_x}}{dt}}{\dfrac{l_{x+t}}{l_x}}$$

$$= -\frac{d\, {}_tp_x}{dt} \Big/ {}_tp_x$$

(or more simply using $f(t) = -S'(t)$)

$$\Rightarrow \mathring{e}_x = \int_0^\infty t \cdot \left(-\frac{d}{dt} {}_tp_x\right) dt$$

(Let $u = t$, $dv = -\frac{d}{dt} {}_tp_x \, dt$

$\Rightarrow v = -{}_tp_x$ $\Rightarrow \int u \, dv = uv - \int v \, du$

$$= -\left[t \; {}_tp_x\right]_0^\infty + \int_0^\infty {}_tp_x \, dt$$

$$= \int_0^\infty {}_tp_x \, dt.$$

(Note : $t \cdot {}_tp_x \to 0$ as $t \to \infty$)

${}_tp_x$ goes to 0 faster than $t \to \infty$

e.g. Exp Dist.

$${}_tp_x = e^{-\lambda t} \qquad t \cdot {}_tp_x = \frac{t}{e^{\lambda t}} \to 0.$$

when $t \to \infty$.

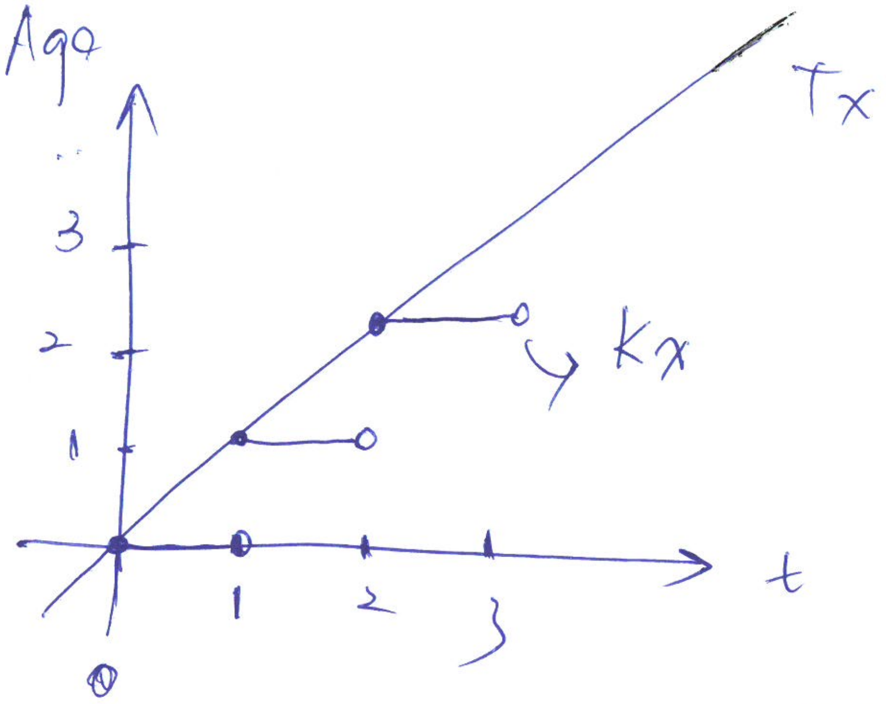
$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

∞

In other stat course, you may have seen a formula for expected value as,

$$E(X) = \int_0^\infty P(X \geq x) dx$$

for non-negative random variable



$${}_k p_x \cdot q_{x+k}$$

$$K_x = k \iff k \le T_x < k+1$$

$$P(K_x = k) = P(k \le T_x < k+1)$$

$$\Rightarrow \quad e_x = \sum_{0}^{\infty} k \cdot P(K_x = k)$$

→ formula for $E(X)$, discrete.

$$= \sum_{0}^{\infty} k \cdot {}_k p_x \cdot q_{x+k}$$

$$= {}_1 p_x \, q_{x+1} + 2 \cdot {}_2 p_x \, q_{x+2} + 3 \cdot {}_3 p_x \, q_{x+3} \cdots$$

$$= {}_1 p_x \, q_{x+1} + {}_2 p_x \, q_{x+2} + {}_3 p_x \, q_{x+3} + \cdots$$
$$+ {}_2 p_x \, q_{x+2} + {}_3 p_x \, q_{x+3} + \cdots$$
$$+ {}_3 p_x \, q_{x+3} + \cdots$$

$$= \sum_{j=1}^{\infty} {}_{j}p_x \, q_{x+j} + \sum_{j=2}^{\infty} {}_{j}p_x \, q_{x+j}$$

$$+ \cdots$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} {}_{j}p_x \, q_{x+j} = \frac{\cancel{l_{x+k}}}{l_x} \cdot \frac{l_{x+k} - \cancel{l_{x+k+1}}}{\cancel{l_{x+k}}}$$

$$+ \frac{\cancel{l_{x+k+1}}}{l_x} \cdot \frac{\cancel{l_{x+k+1}} - l_{x+k+2}}{\cancel{l_{x+k+1}}}$$

$$+ \cdots$$

$$= \frac{l_{x+k}}{l_x} = {}_{k}p_x$$

$$= \sum_{k=1}^{\infty} {}_{k}p_x$$

Alternatively

$$\sum_{j=k}^{\infty} \underline{{}_{j}p_x \, q_{x+j}} = \sum_{j=k}^{\infty} {}_{j|}q_x = \sum_{j=k}^{\infty} P(j \le T_x < j+1)$$

$$= P(k \le T_x < k+1) + P(k+1 \le T_x < k+2) + \cdots$$

$$= P(T_x \ge k) = {}_{k}p_x$$

Exercise week 2

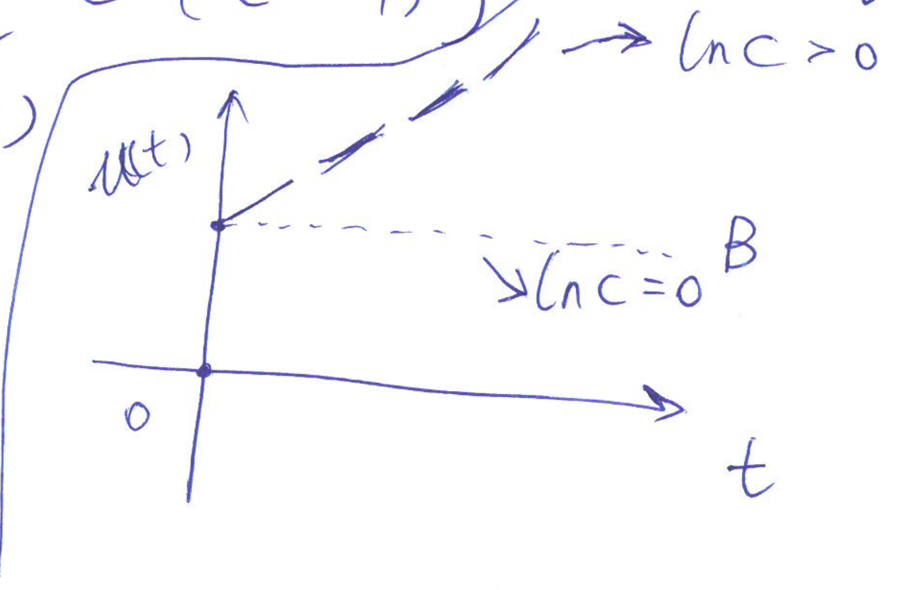
prove $\quad {}_tp_x = g^{c^x(c^t-1)} \quad$ where $\quad g = \exp\left(\frac{-B}{\log c}\right)$

if Gompertz' law holds

$$\mu_x(t) = B \cdot C^{x+t}$$

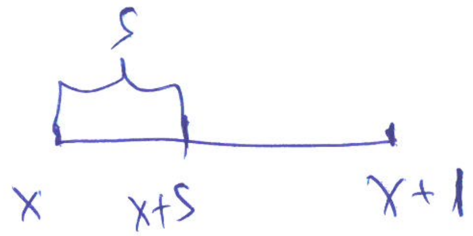
$$
\begin{aligned}
{}_tp_x = S_x(t) &= \exp\left(-\int_0^t \mu_x(s)\,ds\right) \\
&= \exp\left(-\int_0^t B C^{x+s}\,ds\right) \\
&= \exp\left(-B C^x \int_0^t C^s\,ds\right) \\
&= \exp\left(-B C^x \int_0^t e^{s\ln c}\,ds\right) \\
&= \exp\left(\frac{-B C^x}{\ln c}\left[e^{s\ln c}\right]_0^t\right) \\
&= \exp\left(\frac{-B}{\ln c}\, c^x(c^t-1)\right) \\
&= g^{c^x(c^t-1)}
\end{aligned}
$$

Plot: $\log(\mu(t))$
$= \log(B) + \log(c)\, t$



Exercis.

1. UDD: show ${}_sq_x = s\cdot q_x$



$= s\cdot {}_tp_x\,\mu_{x+t}$.

$${}_sq_x = \int_0^s {}_tp_x\,\mu_{x+t}\,dt.$$ from week 1

$$q_x = \int_0^1 {}_tp_x\,\mu_{x+t}\,dt = {}_tp_x\,\mu_{x+t}.$$

$$\boxed{{}_sq_x = s\cdot q_x} \quad \cancel{\Rightarrow \frac{d_{x+s}}{l_x} = \frac{s\cdot d_{x+1}}{l_x}}$$

$$\cancel{\Rightarrow d_{x+s} = s\cdot d}$$

2. UDD: show $\mu(t)\uparrow \quad 0<t<1$

$T \sim U(0,1)$ $\quad a=0,\ b=1$

$$\mu(t) = \frac{f(t)}{S(t)} = \frac{\frac{1}{b-a}}{\frac{b-t}{b-a}} = \frac{1}{1-t}\uparrow \qquad \uparrow$$

UDD $\neq$ constant $\mu(t)$