FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 4

Question 1

12 payments of \$2,000 each are made at 2-year intervals. Find the value of the series

(a) 2 years before the first payment at annual effective interest rate $i = 0.08$,

(b) 8 years before the first payment at nominal annual interest rate $i^{(2)} = 0.08$,

(c) at the time of the final payment at nominal annual discount rate $d^{(4)} = 0.08$,

(d) 18 months after the final payment at nominal annual interest rate $i^{(8)} = 0.08$, and

(e) at the time of the first payment at nominal annual interest rate $i^{(4/3)} = 0.08$.

Solution

Let j be the two-year effective interest rate.

(a) $j = (1.08)^2 - 1$

$$2000a_{\overline{12}|j} = 2000\frac{1-v_j^{12}}{j} = 2000\frac{1-(1+j)^{-12}}{j} = 2000\frac{1-((1.08)^2)^{12}}{(1.08)^2-1} = 10{,}123.81$$

(b) $j = \left(1+\dfrac{0.08}{2}\right)^4 - 1$

$$2000a_{\overline{12}|j}v_j^3 = 2000\frac{1-(1+j)^{-12}}{j}(1+j)^{-3} = 2000\frac{1-((1.04)^4)^{-12}}{(1.04)^4-1}((1.04)^4)^{-3} = 6{,}235.03$$

(c) $j = \left(1-\dfrac{d^{(4)}}{4}\right)^{-4(2)} - 1 = (0.98)^{-8} - 1$

$$2000s_{\overline{12}|j} = 2000\frac{(1+j)^{12}-1}{j} = 2000\frac{(0.98^{-8})^{12}-1}{0.98^{-8}-1} = 67{,}895.89$$

(d) $j = \left(1+\dfrac{i^{(8)}}{8}\right)^{8(2)} - 1 = (1.01)^{16} - 1$

$$2000s_{\overline{12}|j}(1+j)^{0.75} = 2000\frac{(1+j)^{12}-1}{j}(1+j)^{0.75} = 2000\frac{(1.01^{16})^{12}-1}{1.01^{16}-1}(1.01^{16})^{0.75} = 75{,}168.66$$

(e) $j = \left(1 + \frac{i^{(4/3)}}{4/3}\right)^{\frac{4}{3}(2)} - 1 = (1.06)^{8/3} - 1$

$$2000\ddot{a}_{\overline{12}|j} = 2000\frac{1 - v_j^{12}}{d} = 2000\frac{(1+j)\left(1-(1+j)^{-12}\right)}{j} = 2000\frac{1.06^{8/3}\left(1-\left(1.06^{8/3}\right)^{-12}\right)}{1.06^{8/3} - 1} = 11{,}743.76$$

Question 2

Since June 30, 2010 Smith has been making deposits of \$100 each into a bank account on the last day of each month. For all of 2010 and 2011 Smith's account earned nominal interest compounded monthly at an annual rate of 9%. For the first 9 months of 2012 the account earned $i^{(12)} = 0.105$, and since then the account has been earning $i^{(12)} = 0.12$. Find the balance in the account on February 1, 2013.

Solution

For 2010 and 2011 there were (7+12) = 19 deposits. Working with units of time of months, the effective monthly rate of interest is $i = \left(\frac{0.09}{12}\right) = 0.0075$

Therefore, the accumulated value at December 31, 2011 is $100 \cdot s_{\overline{19}|0.0075}$

For the next 9 months the effective monthly rate of interest was: $i = \frac{0.105}{12} = 0.00875$

For the remaining 4 months up to the start of February, 2013 the effective monthly rate of interest was $i = \frac{0.12}{12} = 0.01$

The accumulated amount at December 31, 2011 accumulates to:

$100 \cdot s_{\overline{19}|0.0075}(1.00875)^9(1.01)^4$

The payments for the first 9 months of 2012 accumulate to:

$100 \cdot s_{\overline{9}|0.00875}(1.01)^4$

The last 4 payments accumulate to:

$100 \cdot s_{\overline{4}|0.01}$

Therefore, knowing $s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$ the total accumulated value is:

$$100 \cdot \left(s_{\overline{19}|0.0075}(1.00875)^9(1.01)^4 + s_{\overline{9}|0.00875}(1.01)^4 + s_{\overline{4}|0.01}\right) = 3665.12$$

Question 3
An $m+n$ year annuity of 1 per year has i=7% during the first m years and has i=11% during the remaining n years. If $s_{\overline{m}|0.07} = 34$ and $s_{\overline{n}|0.11} = 128$, first find n and then find the accumulated value of the annuity just after the final payment.

Solution
There are m payments followed by n payments.
The total accumulated value is the value of the m payments accumulated the remaining n years at 11%, plus the accumulated value of the n payments.

$$AV = s_{\overline{n}|0.11} + (1.11)^n s_{\overline{m}|0.07} = 128 + (1.11)^n \cdot 34$$

We need to find n.

$$s_{\overline{n}|0.11} = \frac{(1+i)^n - 1}{i} = 128$$
$$\Rightarrow n = \frac{\ln\left(128 \cdot i + 1\right)}{\ln(1+i)} = \frac{\ln\left(128 \cdot 0.11 + 1\right)}{\ln(1.11)} = 26$$

$$\Rightarrow AV = 128 + (1.11)^{26} \cdot 34 = 640.72$$

Question 4
In return for an investment of $1000 in a fixed interest security, you will receive $40 at the end of each half year plus your money back on redemption in 12 years. You intend to deposit all of the proceeds in a bank account that will pay an effective rate of interest of 8% per annum.

How much money will be in the bank account after 10 years?

Solution
After 10 years you should have:

$$80 s^{(2)}_{\overline{10}|0.08} = 80\left(\frac{1.08^{10} - 1}{i^{(2)}}\right) = 80\left(\frac{1.08^{10} - 1}{2(1.08^{1/2} - 1)}\right) \cong \$1{,}182$$

Alternatively, adjusting the interest to be the effective half-yearly rate:

$$40 s_{\overline{20}|0.03923} = 40\left(\frac{1.03923^{20} - 1}{(1.08^{1/2} - 1)}\right) = \$1{,}182$$

$$i = 1.08^{1/2} - 1 = 0.03923$$

Question 5
Calculate the present value at January 1, 1999 of payments \$100 on the 1st day of each quarter during calendar years 2001 to 2010 inclusive.

Assume effective rates of interest of 8% per annum until December 31, 2005 and 6% thereafter.

Solution
Split the ten-year annuity into two five-year annuities to allow for the change in interest rates. Remember to discount the payments to January 1, 1999.

The present value is:

$$400\left[\ddot{a}^{(4)}_{\overline{5}|0.08}v^{2}_{0.08}+\ddot{a}^{(4)}_{\overline{5}|0.06}v^{7}_{0.08}\right]=400\left[\left(\frac{1-v^{5}_{0.08}}{d^{(4)}_{0.08}}\right)v^{2}_{0.08}+\left(\frac{1-v^{5}_{0.06}}{d^{(4)}_{0.06}}\right)v^{7}_{0.08}\right]=\$2{,}456.80$$

Alternatively, adjusting the interest to be the effective quarterly rate for the annuity payments:

$$100\left[\ddot{a}_{\overline{20}|i}v^{2}_{0.08}+\ddot{a}_{\overline{20}|j}v^{7}_{0.08}\right]=100\left[\left(\frac{1-v^{20}_{i}}{d_{i}}\right)v^{2}_{0.08}+\left(\frac{1-v^{20}_{j}}{d_{j}}\right)v^{7}_{0.08}\right]=\$2{,}456.80$$

where $i=1.08^{1/4}-1$ and $j=1.06^{1/4}-1$

Question 6
State if the following expression is true or false. Show any working used to solve the question.

$$s_{\overline{m-n}|}=v^{n}s_{\overline{m}|}-a_{\overline{n}|} \qquad \text{where } 0<n<m$$

Solution
The expression is TRUE.

$$LHS=s_{\overline{m-n}|}=\frac{(1+i)^{m-n}-1}{i}$$

$$RHS = v^n s_{\overline{m}|} - a_{\overline{n}|} = v^n\left(\frac{(1+i)^m - 1}{i}\right) - \left(\frac{1-v^n}{i}\right)$$

$$= \left(\frac{v^n(1+i)^m - v^n}{i}\right) - \left(\frac{1-v^n}{i}\right)$$

$$= \frac{v^n(1+i)^m - v^n - 1 + v^n}{i}$$

$$= \frac{v^n(1+i)^m - 1}{i}$$

$$= \frac{(1+i)^{-n}(1+i)^m - 1}{i}$$

$$= \frac{(1+i)^{m-n} - 1}{i} = LHS$$

<u>Past Exam Question – 2005 Final Exam Q1(b)(i)</u>

Prove the following equality $a_{\overline{n}|}^{(1/k)} = \dfrac{k}{s_{\overline{k}|}} a_{\overline{n}|}$ (2 marks)

<u>Solution</u>

$$RHS = \frac{k}{s_{\overline{k}|}} a_{\overline{n}|} = \frac{k}{\left(\frac{(1+i)^k - 1}{i}\right)} \times \frac{1-v^n}{i}$$

$$= \frac{k(1-v^n)}{(1+i)^k - 1} = \frac{(1-v^n)}{\frac{1}{k}\left[(1+i)^k - 1\right]} = \frac{(1-v^n)}{i^{(1/k)}} = a_{\overline{n}|}^{(1/k)} = LHS$$