§9 - Separation Axioms

1 Motivation

In this course we have seen the Hausdorff property, and some places where it is useful (like Assignment 4, A.1). Generally the Hausdorff property will be an extra assumption we want our topological spaces to have, but sometimes we will need something a little bit stronger (and very rarely we will only need something weaker). Here we introduce the notions of Regular spaces and Normal spaces, which are stronger versions of Hausdorff spaces. We have already seen these on Assignment 4 (A.3), but we will look at them a bit more closely.

2 Definitions

Recall the definition of a Hausdorff space (which is sometimes called a T_2 space):

Definition. *A topological space* $(X, \mathcal{T})$ *is a* ***Hausdorff Space*** *if for every two distinct* x *and* y *in* X *there are disjoint open sets* U *and* V *such that* $x \in U$ *and* $y \in V$*.*

Recall the definition of a Regular space (which is sometimes called a T_3 space):

Definition. *A topological space* $(X, \mathcal{T})$ *is a* ***Regular Space*** *if for every closed set* C *and* p *in* $X \setminus C$ *there are disjoint open sets* U *and* V *such that* $C \subseteq U$ *and* $p \in V$*.*

Recall the definition of a Normal space (which is sometimes called a T_4 space):

Definition. *A topological space* $(X, \mathcal{T})$ *is a* ***Normal Space*** *if for every two disjoint closed set* C *and* D *there are disjoint open sets* U *and* V *such that* $C \subseteq U$ *and* $D \subseteq V$*.*

The more of these properties that a space has the "nicer" it is. The quintessential example of a nicely behaved space is a discrete space.

Proposition. $X_{discrete}$ *is a Hausdorff, Regular, Normal space.*

Proof. The proofs are all the same. If A and B are disjoint sets in X_{discrete} then they happen to both be open sets that are disjoint and contain A and B respectively. □

The next example we know and love that satisfies all of these is $\mathbb{R}^n$.

Proposition. $\mathbb{R}^n$ *is a Hausdorff, Regular, Normal space.*

Proof. We already know that $\mathbb{R}^n$ is a Hausdorff space, and you may have seen (in an analysis) that $\mathbb{R}^n$ is normal. Here is the sketch of that proof.

Let A, B be disjoint, closed subsets of $\mathbb{R}^n$. For each point $a \in A$ find an $\epsilon_a > 0$ such that $B_{\epsilon_a}(a) \subseteq \mathbb{R}^n \setminus B$. Do the same for the points $b \in B$, finding balls disjoint from A. Now

$$A \subseteq \bigcup_{a \in A} B_{\epsilon_a}(a)$$

and

$$B \subseteq \bigcup_{b \in B} B_{\epsilon_b}(b)$$

but these unions might intersect! Check that

$$U := \bigcup_{a \in A} B_{\frac{\epsilon_a}{2}}(a) \supseteq A$$

and

$$V := \bigcup_{b \in B} B_{\frac{\epsilon_b}{2}}(b) \supseteq B$$

are the required *disjoint* open sets. $\square$

3 Implications

Our next task is to justify the T_i notation, and the first thing we realize is that T_4 does not imply T_2! (Well that is a stupid naming convention then...)

Example. *$X_{indiscrete}$ is a (vacuously) Regular, Normal space that is not Hausdorff.*

Proof. Notice that it is impossible to find any two disjoint closed sets, so the space is vacuously Normal. Similarly, if C is a non-empty closed set in X, then it is impossible to find a point disjoint from C, so the space is vacuously Regular. We have already seen that the space is not Hausdorff. $\square$

To get the implications, we need "points are closed sets". This would definitely yield the following:

Proposition. *Let X be a topological space where each $\{ x \}$ is a closed set. Then*

$$T_4 \Rightarrow T_3 \Rightarrow T_2$$

It turns out that the property "points are closed sets" fits nicely into the T_i hierarchy in another way.

Definition. *A topological space $(X, \mathcal{T})$ is a T_1 **Space** if for every two distinct x and y in X there are open sets U (not containing y) and V (not containing x) such that $x \in U$ and $y \in V$.*

Proposition. *A space X is T_1 iff for all $p \in X$, $\{ p \}$ is closed in X.*

Proof. The [$\Leftarrow$] direction is straightforward; for $x \neq y$, take $X \setminus \{ x \}$ and $X \setminus \{ y \}$ as your open sets.

[$\Rightarrow$] Suppose that X is a T_1 space, and let $p \in X$. For each $x \in X$ (distinct from p), choose an open set U_x that contains x but does not contain p. So $X \setminus U_x$ is a closed set containing p, but not x. Then

$$\{ p \} = \bigcap_{x \in X \setminus \{ p \}} (X \setminus U_x)$$

and we have written $\{ p \}$ as an intersection of closed sets, so it is closed. $\square$

Weaker Still!? Exercise: Define a space to be T_0 if for any two distinct $x \neq y$ there is an open set containing one of the points that does not contain the other. Does the [$\Rightarrow$] direction of the proof above still go through?

Now we can restate our observations (for a general topological space):

Proposition.

$T_1 + T_4 \Rightarrow T_1 + T_3 \Rightarrow T_2 \Rightarrow T_1$

To avoid trivialities, (and having to say $T_1 + T_4$), we define the terms "Normal" and "Regular".

Definition. *A topological space X is **Regular** if it is both T_1 and T_3. A topological space X is **Normal** if it is both T_1 and T_4.*

This notation is easy to forget (the words "normal" and "regular" are very over used in mathematics, and are completely non-descriptive) and is not standard throughout mathematics. Some authors intend Regular spaces to not be T_1, and some do. **We will use the definitions we have just presented**, although, sometimes we will slip. (We are only human.)

4 Some Equivalent Separation Notions

Generally we will use alternate characterizations of normality and regularity. These will be very helpful for us when we look at Urysohn's Lemma.

Proposition. *A topological space X is Regular iff for every open set U and every point $p \in U$ there is an open set V such that $p \in V \subseteq \overline{V} \subseteq U$.*

Proof. The idea here is contained completely within the picture. You should learn the picture, and how to extract the proof from it.

[$\Rightarrow$] Let $p \in U$ an open set. Then p is disjoint from the closed set $X \setminus U$. Thus there are disjoint open sets V and B such that $p \in V$ and $X \setminus U \subseteq B$. Thus

$$p \in V \subseteq X \setminus B \subseteq U$$

and since $X \setminus B$ is closed,

$$p \in V \subseteq \overline{V} \subseteq U$$

[$\Leftarrow$] Let $p \in X$ be a point disjoint from C, a closed set. Then $X \setminus C$ is an open set containing p. Applying the assumption about X twice, we find open sets V and W such that

$$p \in V \subseteq \overline{V} \subseteq W \subseteq \overline{W} \subseteq X \setminus C$$

Thus we see that $p \in V$ and $C \subseteq X \setminus \overline{W}$ are the desired (disjoint) open sets. □

Here's the corresponding proposition for normal spaces, which we leave as an exercise:

Proposition. *A topological space X is Normal iff for every open set U and every closed set $C \subseteq U$ there is an open set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.*

5 Topological Facts

We have looked at topological invariants, hereditary properties and finitely productive properties. Let's see how they interact with the separation axioms.

Proposition. *All of the T_i properties are topological invariants.*

Proof. This is straightforward. □

Proposition. *The T_1, T_2 and T_3 properties are hereditary, but T_4 is not.*

Proof. We have already seen that T_2 and T_3 are hereditary (in the notes, and on Assignment 4). It is straightforward to check that T_1 is a hereditary property. It is challenging to come up with a normal space that isn't hereditarily normal. You can read about the Tychonoff Plank (and its corresponding subspace the deleted Tychonoff Plank) as Example 86 in Counterexamples in Topology. □

All is not lost though! In analogy to how open subspaces of separable spaces are separable, *closed* subspaces of normal spaces are normal.

Proposition. *Closed subspaces of T_4 spaces are T_4.*

Proof. Go through the same proof that subspaces of regular spaces are regular, and notice that the assumption that you have a *closed* subspace eliminates the difficulty you had with generalizing the theorem to normal spaces. (See Assignment 4, A.3.) $\square$

Proposition. *The T_1, T_2 and T_3 properties are finitely productive, but T_4 is not.*

Proof. You showed on Assignment 3 that T_2 was finitely productive, and a similar proof will show that T_1 is finitely productive. The proof that T_3 is productive makes use of the alternate characterization we gave above. (Prove it if you are interested!) Finally we show that T_4 is not productive.

Notice that S, the Sorgenfrey line, is Normal, and that $A := \{ (x, -x) : x \in \mathbb{R} \} \subseteq S \times S$ is a closed, discrete subspace. You can check that $I := \{ (x, -x) : x \in \mathbb{R} \setminus \mathbb{Q} \}$ and $Q := \{ (x, -x) : x \in \mathbb{Q} \}$ are both closed (discrete) subsets of $S \times S$ that cannot be separated by disjoint open sets. Use the fact that $A_x := [x, x+1) \times [-x, -x+1)$ is an open set in $S \times S$ and

$$A_x \cap A = \{ x \}$$

(Note: It is quite challenging to get an air-tight proof that I and Q cannot be separated by disjoint open sets. If you give up, look at Munkres, p.198, example 3. I won't expect you to have an air-tight proof of this fact, but the idea of the anti-diagonal in $S \times S$ is useful.) $\square$

6 A Nice Normal Fact

Finally, we point out a very nice fact that relates normality to second countability.

Theorem. *Every second countable, regular space is normal.*

Proof. Let A, B be closed disjoint sets in X, a regular second countable space. Fix $\mathcal{B}$ a countable basis. For each point $a \in A$ fix a basic open set U_a such that $a \in U_a$ and $\overline{U_a} \cap B = \emptyset$, which we can do since X is regular. Notice that $A \subseteq \bigcup_{a \in A} U_a$, and since we were taking open sets from the *countable* basis $\mathcal{B}$, really there is a countable set $\{ U_n \in \mathcal{B} : n \in \mathbb{N} \}$ such that

$$A \subseteq \bigcup_{a \in A} U_a = \bigcup_{n \in \mathbb{N}} U_n$$

Similarly, find $\{ V_n \in \mathcal{B} : n \in \mathbb{N} \}$ such that

$$B \subseteq \bigcup_{n \in \mathbb{N}} V_n$$

with the additional requirement that $\overline{V_n} \cap A = \emptyset$.

We would *like* to take $\bigcup_{n\in\mathbb{N}} U_n$ and $\bigcup_{n\in\mathbb{N}} V_n$ as our open sets that separate A and B, but these need not be disjoint. We fix this by defining

$$U_n' := U_n \setminus (\bigcup_{i=1}^{n} \overline{V_i}) \text{ and } V_n' := V_n \setminus (\bigcup_{i=1}^{n} \overline{U_i})$$

We notice that each U_n' is open since it is the difference of an open set with a closed set. We also notice that

$$A \subseteq \bigcup_{n\in\mathbb{N}} U_n'$$

Similarly, the V_n' are all open and

$$B \subseteq \bigcup_{n\in\mathbb{N}} V_n'$$

Finally, we observe that the two unions are disjoint.

Claim: $\bigcup_{n\in\mathbb{N}} U_n' \cap \bigcup_{n\in\mathbb{N}} V_n' = \emptyset$.

Suppose $x \in \bigcup_{n\in\mathbb{N}} U_n'$. Then $x \in U_N'$ for some $N \in \mathbb{N}$. Automatically, we see that $x \notin V_i$ for $1 \leq i \leq N$. Moreover, for $i \geq N$ we see that $V_i' \cap U_N = \emptyset$. So $x \notin V_i'$. Thus $x \notin \bigcup_{n\in\mathbb{N}} V_n'$. $\square$

7 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Weaker : Does T_0 imply that points are closed?