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Exercise sheet 5: Solutions

1. Rephrase König's and Dilworth's theorems as pure existence statements, without any inequalities.

   Solution. *König's theorem: For any bipartite graph like G, there exists a matching like $M = \{e_1, e_2, \ldots, e_k\}$ and a set of vertex covering of the graph like $\{v_1, v_2, \ldots, v_k\}$ s.t. $v_i \in e_i$ for all i.*
   Dilworth's theorem: For any finite poset, there exits a set of chains like $\{C_1, C_2, \ldots, C_m\}$ which gives a partition of the poset and an antichain of the poset like $A = \{a_1, a_2, \ldots, a_m\}$ s.t. $a_i \in C_i$ for all i.

2. Let G be an arbitrary d-regular graph with k edges and n vertices. How many edges there may be in G/e?

   Solution. *By contracting, we will remove $2d - 1$ edges of G and add $2d - 2 - l$ edges, where l is the number of the common neighbors of the endvertices of e. So the number of edges of G/e is between $k - 1$ and $k - d$.*

3. Show that a partially ordered set of at least $rs + 1$ elements contains either a chain of size $r + 1$ or an antichain of size $s + 1$.

   Solution. *If the poset has an antichain of size $s + 1$, then we are done, so suppose the largest size of antichains is at most s. Then apply Dilworth theorem, at least one of the chains participating in the partition of the poset to the minimum number of chains must have size at least $r + 1$.*

4. Prove Menger's theorem for the case $k = 2$ using the proposition about 2-connected graphs seen in class (Prop 3.1.1 in the book).

   Solution. *We are going to prove that if the graph G is 2-connected, then it contains 2 independent paths between every two vertices: For this, it is enough to show that every two vertices of the graph lie on a cycle in G. We will prove it by the induction on the number of steps*

used to construct G as mentioned in the Prop 3.1.1. For the induction basis, when G is only a cycle, there is nothing to prove. So consider a graph already constructed in this manner like H and assume the statement is true for H. We are going to prove the statement for the graph G constructed from H by adding a $H-path$ like xPy to H:
Consider two arbitrary vertices of G like u and v. If $u, v \in H$ then it follows form the induction hypothesis and if $u, v \in P - H$ then consider the cycle formed from attaching xPy and a $x - y$ path in H. Now consider the case when $u \in P - H$ and $v \in H$. In this case, it is enough to find a path in H from x to y containing v. By the induction hypothesis, consider two cycles in H like C_x and C_y s.t. $x, v \in C_x$ and $y, v \in C_y$. If $y \in C_x$ we are done. So consider one of the paths from x to v in C_x, let us say P_x, and let z be the first vertex in P_x belonging to C_y. z is well defined since v is common to P_x and C_y. Now consider the path P_y in C_y, from z to y containing v. Then according to the property of z, xP_xzP_yy is a $x - y$ path in H containing v.

5. Let G be a k-connected graph, and let xy be an edge of G. Show that G/xy is k-connected if and only if $G - \{x, y\}$ is $(k-1)$-connected.

   Solution. *Suppose G/xy is k-connected but $G - \{x, y\}$ is not $(k-1)$-connected. Then there exist vertices of G s.t. $\{v_1, v_2, \ldots, v_{k-2}\}$ is a separating set for $G - \{x, y\}$ so $\{x, y, v_1, v_2, \ldots, v_{k-2}\}$ is a separating set for G. Then if v_{xy} is a contraction vertex for G/xy, $\{v_{xy}, v_1, v_2, \ldots, v_{k-2}\}$ is a separating set for G/xy which is a contradiction.*
   For the other direction, suppose $G - \{x, y\}$ is $(k-1)$-connected but G/xy, with v_{xy} as the contraction vertex, is not k-connected. Then there exists a separating vertex set for G/xy like $\{u_1, u_2, \ldots, u_{k-1}\}$. Then one of these vertices must be v_{xy}, otherwise we will have a separating vertex set of size $k-1$ for G which is a contradiction. Assume $u_1 = v_{xy}$, then $\{x, y, u_2, \ldots, u_{k-1}\}$ is a separating set for G and as a result, $\{u_2, \ldots, u_{k-1}\}$ is a separating set for $G - \{x, y\}$, a contradiction.

6. Let G be a graph with and a, b be two vertices of G. Let $X \subseteq V(G) \setminus \{a, b\}$ be an $a - b$ separator in G. Show that X is minimal as an $a - b$ separator if and only if every vertex in X has a neighbour in the component C_a of $G - X$ containing a, and another in the component C_b of $G - X$ containing b.

   Solution. *If the second condition holds, then X is a minimal $a - b$ separator, since for every $x \in X$, there exist its neighbors like $x_a \in C_a$*

and $x_b \in C_b$. Now just consider the path ax_axx_bb.
For the other direction, consider an arbitrary $x \in X$. Then since X is a minimal $a-b$ separator, there exists an $a-b$ path in G with having x as the only member from X, then by considering the graph $G-X$, x must have neighbors in C_a and C_b in order that we can construct the aforementioned path.

7. Find a partially ordered set that has no infinite antichain but is not a union of finitely many chains.

Solution. *For every positive integer i let A_i be a set with i elements and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Consider the poset with elements in $\cup A_i$ and where for two elements if $x \in A_i$ and $y \in A_j$ then x is not comparable to y if $i = j$ and $x \prec y$ if $i < j$.*