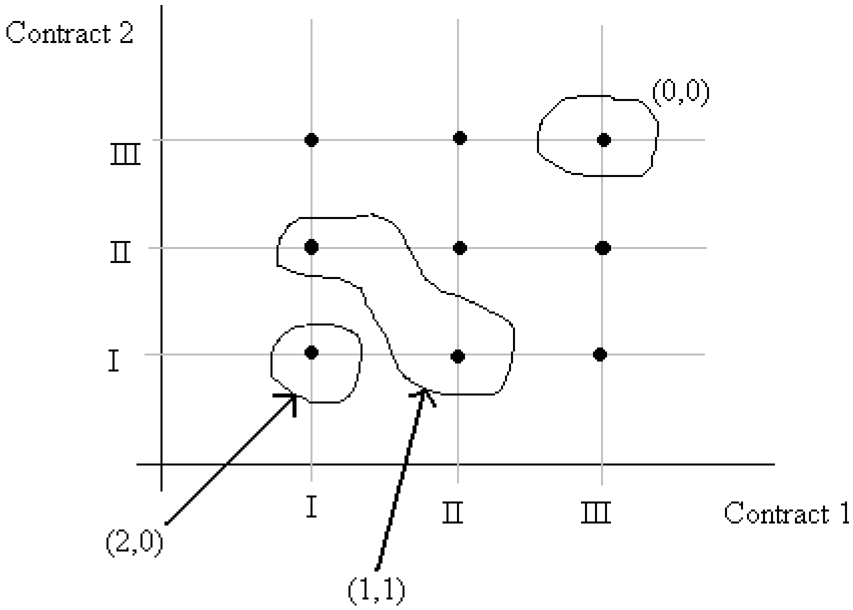
STAT2001 Tutorial 8 Solutions

Problem 1

(a) Some values of (x,y) are shown in the following figure:



We see that $p(0,0) = 1/9$, $p(2,0) = 1/9$, $p(1,1) = 2/9$, etc.

Table of $p(x,y)$:

		y = 0	1	2	$p(x)$ ↓	$p(x\|0)$ ↓
	0	1/9	2/9	1/9	4/9	1/4
x	1	2/9	2/9		4/9	1/2
	2	1/9			1/9	1/4
$p(y)$	→	4/9	4/9	1/9		

(*Check:* 1/9 + 2/9 + 1/9 + 2/9 + 2/9 + 1/9 = 1.)

NB: X and Y have a *multinomial distribution* with parameters 2, 1/3 and 1/3.

We may write $(X,Y) \sim \text{Multi}(2,1/3,1/3)$, and

$$p(x,y) = \frac{2!}{x!\,y!\,(2-x-y)!}\left(\frac{1}{3}\right)^x\left(\frac{1}{3}\right)^y\left(1-\frac{1}{3}-\frac{1}{3}\right)^{2-x-y},$$

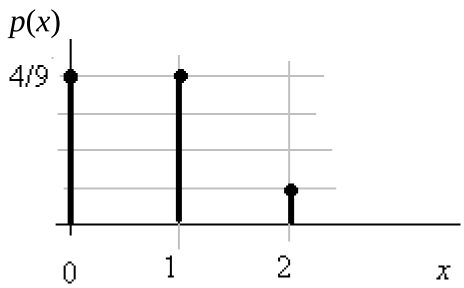
where $x, y \in \{0,1,2\}$ subject to $x+y \le 2$.

Thus, for example,

$$p(1,0) = \frac{2!}{1!\,0!\,(2-1-0)!}\left(\frac{1}{3}\right)^1\left(\frac{1}{3}\right)^0\left(1-\frac{1}{3}-\frac{1}{3}\right)^{2-1-0} = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}.$$

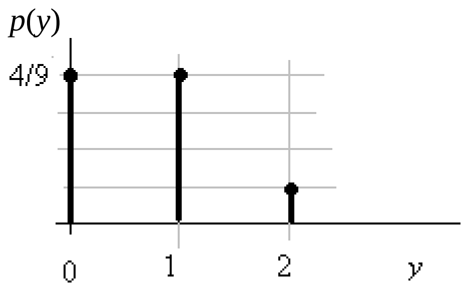
See Section 5.9 in text.

(b) $p(x) = \begin{cases} 4/9, & x = 0,1 \\ 1/9, & x = 2 \end{cases}$



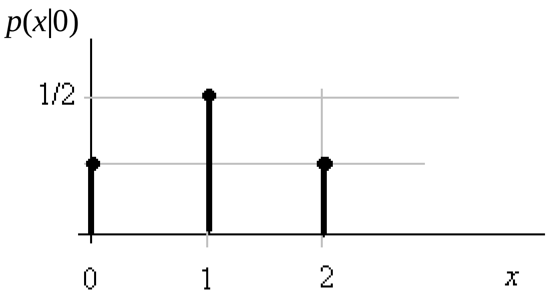
(Thus X ~ Bin(2,1/3). This makes sense because Firm I has a 1/3 chance of getting each of the two contracts.)

$p(y) = \begin{cases} 4/9, & y = 0,1 \\ 1/9, & y = 2 \end{cases}$



(The fact that X and Y have the same distribution makes sense by virtue of the symmetry in the problem.)

(c) $p(x\,|\,0) = \begin{cases} 1/4, & x = 0,1 \\ 1/2, & x = 2 \end{cases}$



(Thus (X | Y = 0) ~ Bin(2,1/2). This makes sense: If Firm II doesn't get a contract, then Firm I has a 50% chance of getting each of the two contracts.)

(d) $P(X \geq 1 | Y = 0) = p_{X|Y}(1|0) + p_{X|Y}(2|0)$

$$= 1/2 + 1/4$$
$$= 3/4.$$

(e) $\mu_X = 2(1/3) = 2/3 = \mu_Y$.

(This is because $X, Y \sim$ Bin(2,1/3). Note that $X \not\perp Y$.)

$$\sigma_X^2 = 2(1/3)(1-1/3) = 4/9 = \sigma_Y^2.$$

$$E(XY) = \sum_{x,y} xyp(x,y) = 0 + \ldots + 0 + 1 \times 1 \times p(1,1) = \frac{2}{9}.$$

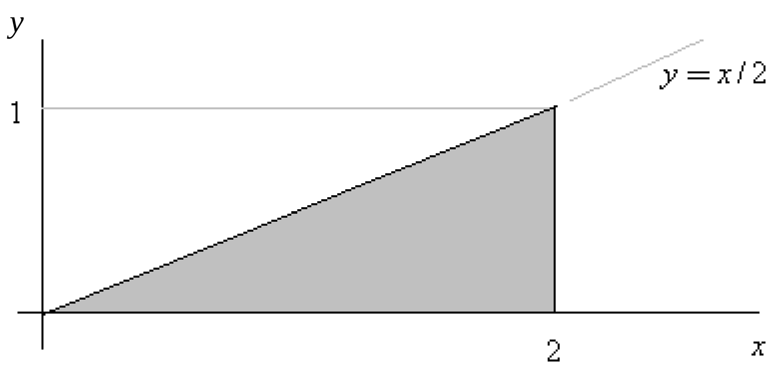
$$Cov(X,Y) = E(XY) - \mu_X \mu_Y = 2/9 - (2/3)^2 = -2/9.$$

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-2/9}{\sqrt{4/9}\sqrt{4/9}} = -\frac{1}{2}.$$

(Thus X and Y are negatively correlated. This makes sense because if Firm I gets a 'large' number of contracts, then Firm II can only get a 'small' number of them, and vice versa.)

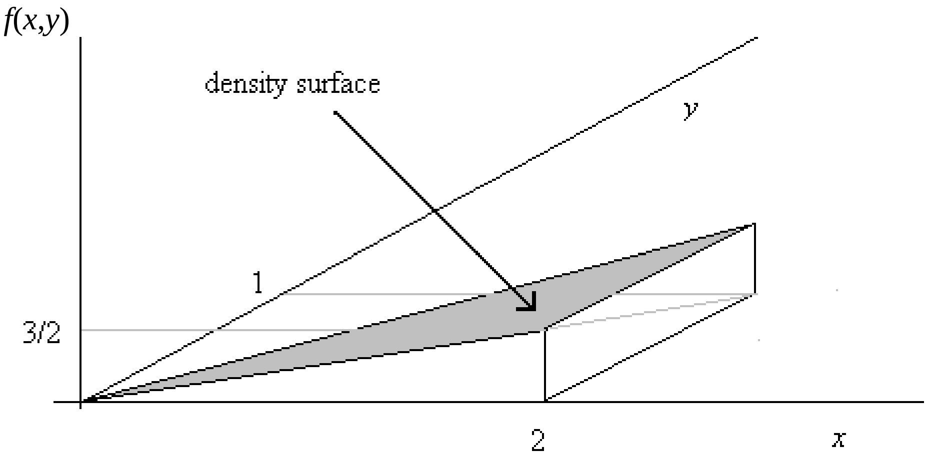
Problem 2

(a) The region where $f(x,y) > 0$ is shown shaded in the following figure.



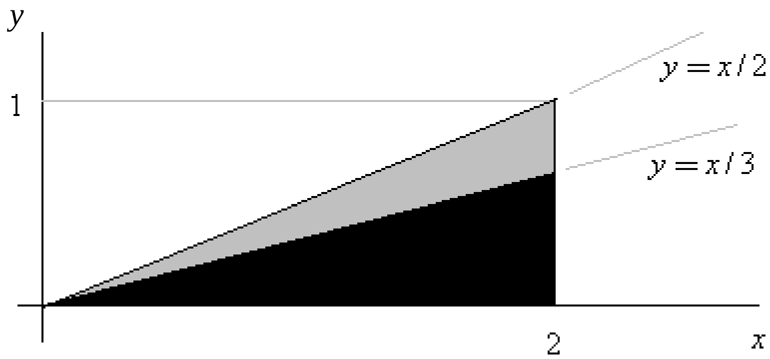
So $1 = \iint f(x,y)dxdy = k\int_{x=0}^{2} x\left(\int_{y=0}^{x/2} dy\right)dx = k\int_{x=0}^{2} x\frac{x}{2}dx = \frac{4}{3}k \Rightarrow k = \frac{3}{4}.$

(The following is a 3-dimensional figure of $f(x,y)$).

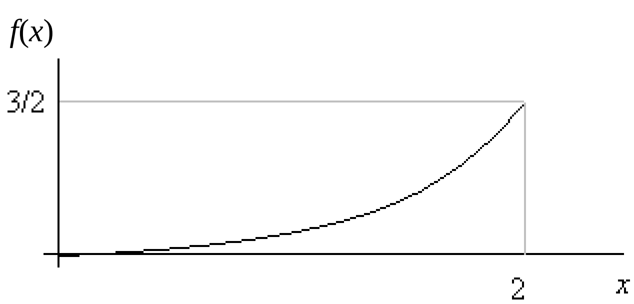


The volume under the density surface equals 1.)

(b) $$P(X > 3Y) = \iint\limits_{x>3y} f(x,y)dxdy = \frac{3}{4}\int_{x=0}^{2} x\left(\int_{y=0}^{x/3} dy\right)dx = \frac{3}{4}\int_{x=0}^{2} x\frac{x}{3}dx = \frac{2}{3}.$$

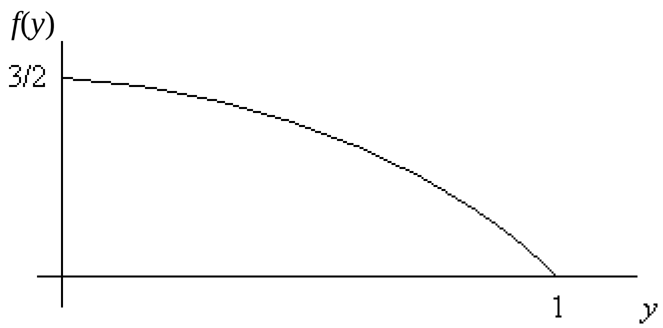


(c) $$f(x) = \int f(x,y)dy = \frac{3}{4}x\int_{0}^{x/2} dy = \frac{3}{8}x^2, 0 < x < 2.$$



(*Check:* $\int f(x)dx = \int_0^2 \frac{3}{8}x^2 dx = 1$.)

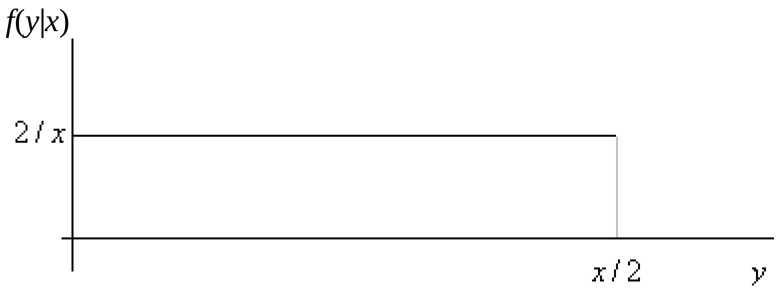
$$f(y) = \int f(x,y)dx = \frac{3}{4}\int_{2y}^{2} xdx = \frac{3}{2}(1-y^2),\ 0<y<1.$$



(*Check:* $\int f(y)dy = \int_0^1 \frac{3}{2}(1-y^2)dy = 1$.)

(d) $f(y\,|\,x) = \dfrac{f(x,y)}{f(x)} = \dfrac{3x/4}{3x^2/8} = \dfrac{2}{x},\ 0<y<\dfrac{x}{2}.$

So (Y | X = x) ~ U(0, x/2).



(e) (Y | X = 1) ~ U(0, 1/2).

So $P(Y > 1/8 \mid X = 1) = 3/4$.