Exerzitien IV

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Oct. 13., in your tutorial.

Reading suggestion: Finish reading **Axler, Chapter 2.**

Exercise 1. *Let*

$$\begin{aligned} v_1 &= (2,3,0,0) \quad v_2 = (0,0,1,-1) \\ v_3 &= (1,0,0,4) \quad v_4 = (0,0,0,2) \end{aligned}$$

and show that (v_1, v_2, v_3, v_4) *form a basis for* $\mathbb{R}^4$*. Find the coordinates of each of the standard basis vectors of* $\mathbb{R}^4$*, in the basis* (v_1, v_2, v_3, v_4)*. Recall that the coordinates of a vector* v *in the basis* $(v_1, \ldots, v_n)$ *are the unique coefficients* $x_1, \ldots, x_n \in \mathbb{F}$ *such that*

$$v = x_1 v_1 + \cdots + x_n v_n.$$

Exercise 2.

1. *Let* V *be a vector space over the field* $\mathbb{F}$*, and let* $f : V \to \mathbb{F}$ *be a linear function, meaning that* $f(v_1 + v_2) = f(v_1) + f(v_2)$ *and* $f(\lambda v_1) = \lambda f(v_1)$ *for all* $\lambda \in \mathbb{F}$ *and* $v_1, v_2 \in V$*. Show that*

   $$H_f = \{v \in V \ : \ f(v) = 0\}$$

   is a linear subspace of V*.*

2. *Show how the above implies the following fact: for any fixed* $(a_1, \ldots, a_n) \in \mathbb{F}^n$*,*

   $$H_{(a_1,\ldots,a_n)} = \{(x_1, \ldots, x_n) \in \mathbb{F}^n \ : \ a_1x_1 + \cdots + a_nx_n = 0\}$$

   is a linear subspace of $\mathbb{F}^n$*.*

3. *Consider the special case of* $H_{(1,2,3)} \subset \mathbb{R}^3$ *defined as above for* $(a_1, a_2, a_3) = (1, 2, 3) \in \mathbb{R}^3$*. Find a basis for* $H_{(1,2,3)}$ *and state the dimension of* $H_{(1,2,3)}$*.*

Exercise 3. *In the previous exercise we defined the concept of a linear function* $f : V \to \mathbb{F}$*. Let* V^* *be the set of all linear functions on the vector space* V*. This is called the "dual vector space" to* V*.*

1. *Prove that* V^* *is a linear subspace of the vector space* $\mathbb{F}^V$ *of all functions from* V *to* $\mathbb{F}$*.*

2. *Suppose that* $(e_1, \ldots, e_n)$ *is a basis for* V*, so that any vector* $v \in V$ *can be written in a unique way as*

   $$v = x_1e_1 + \cdots + x_ne_n, \quad x_i \in \mathbb{F}.$$

   For each $i = 1, \ldots, n$*, define the function* $f_i : V \to \mathbb{F}$ *by* $f_i(v) = x_i$*. Prove that* f_i *is a linear function.*

3. *Use the above to prove that if* V *is finite dimensional, then so is* V^**, and* $\dim V = \dim V^*$*.*

Exercise 4. *Fix a field* $\mathbb{F}$*. Define* $\mathbb{F}^\infty = \{(x_1, x_2, \ldots) \ : \ x_i \in \mathbb{F}\}$*, the vector space of infinite sequences of numbers in* $\mathbb{F}$*. This is a vector space, with operations similar to* $\mathbb{F}^n$*. Prove that* $\mathbb{F}^\infty$ *is infinite dimensional.*