MATH6222 Week 12 Lecture Notes

Rui Qiu

2017-05-22

1 Monday's Lecture

Theorem: G is bipartite $\iff$ G has no odd cycles. ($\chi(G) = 2$)

Definition: G is connected if $\forall\ u, v \in V(G)$, $\exists$ uv path in G.

Proposition: Suppose G satisfies: every vertex $v \in V(G)$ has $d(v) \geq 2$. Then I claim G has a cycle.

Proof: Pick a maximal path P in G. Let v be an endpoint of the path. Since $d(v) \geq 2, \exists$ edge e adjacent to v but not on this path. We claim the other endpoint of e must lie on P.

If not, we could extend P, contradicting maximality. Thus, we get a cycle!

$\implies$ tree has a leaf.

Proposition: Let T be a tree, and let v be a leaf of T ($d(v) = 1$), then $T' := T - \{v\}$ (deleting as well edge connecting v to T).

Then T' is still a tree.

Proof: Need T' connected and no cycles.

1. No cycle:
   A cylce $C_k \subset T'$ would imply $C_k \subset T' \subset T$.
   Contradiction.

2. Connected:
   Need $\forall\ u, v \in T'$, we need a uv path.
   Since T is connected, $\exists$ uv-path in T: since a path uses two edges at every vertex it travels through, it can't possibly travel through a leaf.
   Thus, we have a uv-path in T'.

Corollary: A tree with n vertices has $n - 1$ edges.

Proof: Induction on n.

Suppose T is a tree with n vertices. Let v be a leaf.

Delete v (and connecting edge) from T to T'.

$\implies T'$ is a tree with $n - 1$ vertices.

By inductive hypothesis, has $n - 2$ edges. $\implies$ T has $n - 1$ edges.

Proposition: If T is a tree, then $\chi(T) = 2$.

Proof: Induction on the number of vertices.

Base case: 1, done.

Induction step: Let T be a tree with n vertices, let v be a leaf of T, and T' be the tree obtained by deleting v.

By inductive hypothesis, $\exists$ 2 coloring:

$$f : V(T') \longrightarrow [2]$$

We extend f to a 2-coloring of T by setting:

$$f(v) := \begin{cases} 1 & \text{if } v \text{ is connected to a vertex with color 2} \\ 2 & \text{if } v \text{ is connected to a vertex with color 1} \end{cases}$$

$\implies$ Clearly, f is a 2-coloring.

Theorem: G has no odd cycles $\iff$ G has a 2-coloring. (Take G connected.)

Proof: $\implies$ Induction on number of cycles of G.

Base step: G has no cylces $\implies$ G is tree $\implies$ G has two coloring. Done.

Induction step: Suppose G has k cycles ($k \geq 1$).

Pick v lying on some cycle of G.

Delete v (and adjacent edges to get G').

Now induction hypothesis says G' has 2-coloring.

Let X be vertices in G' with color 1.

Let Y be vertices in G' with color 2.

Need all edges from v to terminate in one of X or Y (not both!)

Suppose v had an edge with endpoint u_1 in X and u_2 in Y.

Consider a u_1, u_2 path in G', since vertices alternate between X and Y, it must have an odd number edges.

Combine the path with $e_1 \& e_2$, then I get an odd cycle.

Contradiction.

1.1 Planar Graph

Definition: A graph is planar if it is possible to draw its vertices and edges in the plane in such a way that edges do not intersect.

K_n indicates complete graph with n vertices.
$K_{r,s}$ denotes a graph with partite sets of size r and s.

We will prove K_5 and $K_{3,3}$ are not planar graphs.

2 Thursday's Lecture

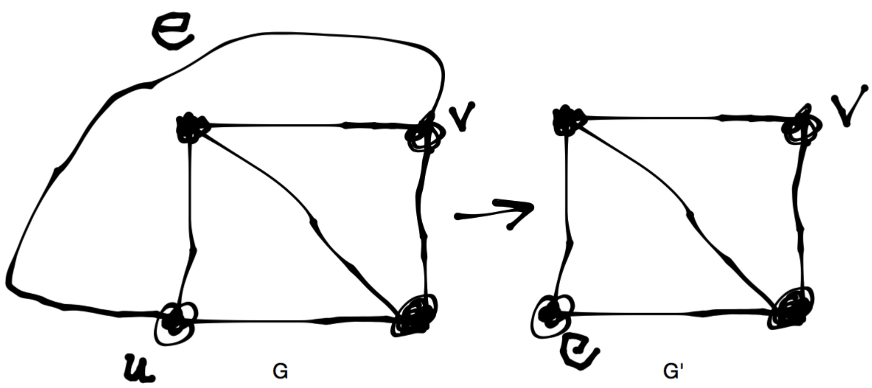
Proposition (Euler's Formula): If G is a connected planar graph, then $v - e + F = 2$ where v is the number of vertices, e is the number of edges, F is the number of faces.

Proof: Induction on the number of cycles.

The base case is just a tree. With n vertices, always $n-1$ edges. 1 face for each tree. Only just one region because no closed area by tree.

Given an arbitrary planar connected graph G, pick an edge e (on a cycle) and delete it. After we delete, we have a connected planar graph with fewer cycles.

v = 4, e = 6, f = 4



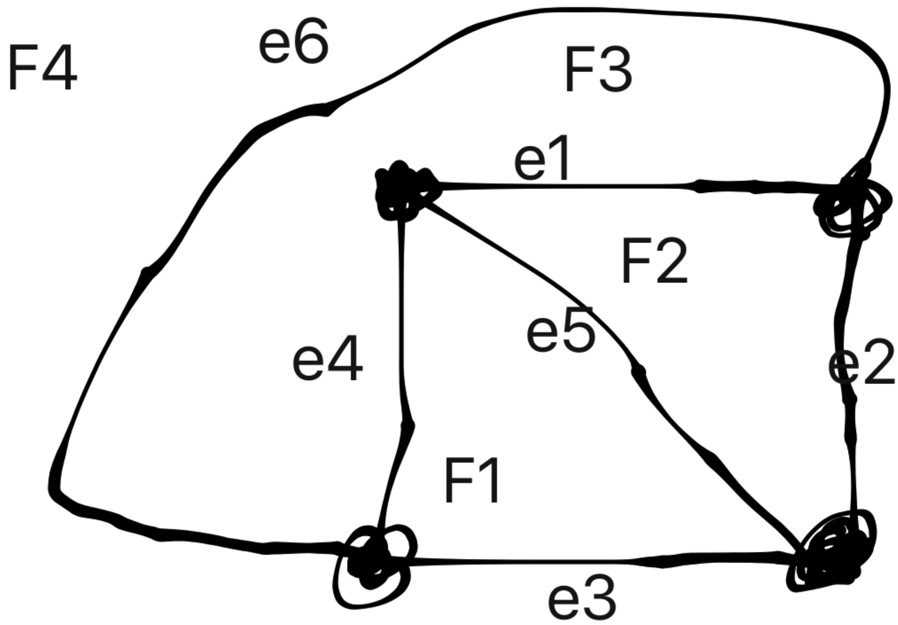
Verify G' connected:

- Given $u, v \in G'$, $\exists$ path connecting uv. We know there exists such a path in G. If this path does not use e, it's still a path in G'. We are done.

- If our path uses e, we argue as follows:

  - Because e is on a cycle, $\exists$ a path in G' from w_1 to w_2. So in G', we have paths u to w_1, w_1 to w_2, w_2 to v.

The number of edges and faces both go down by one.



$F_1 : e_3, e_4, e_5$
$F_2 : e_1, e_2, e_5$
$F_3 : e_1, e_4, e_6$
$F_4 : e_2, e_3, e_6$

- Each edge appears twice.
- If simple, then each face has at least 3 edges along its boundary.

$$3f \leq \sum_{i=1}^{f} \Big(\text{the number of edges on boundary of the } i^{\text{th}} \text{ face}\Big) = 2e$$

For a simple, planar graph: $3f \leq 2e$.

Corollary: For a simple, connected, planar graph:

$$\begin{aligned} f &= 2 - v + e \\ f &\leq \frac{2}{3}e \\ \frac{2}{3}e &\geq f = 2 - v + e \\ \frac{1}{3}e &\leq v - 2 \\ e &\leq 3v - 6 \end{aligned}$$

For $K_5, v = 5, e \leq 3 \cdot 5 - 6 = 9, 4 + 3 + 2 + 1 = 10$. So K_5 is not planar.

For $K_{3,3}, v = 6, e = 9, 9 \leq 3 \cdot 6 - 6 = 12$. This does not suffice that $K_{3,3}$ is not planar.

If G is a simple, planar graph with no 3-cycles then at least 4 edges on boundary of any face.

$$4f \leq 2e \implies f \leq \frac{1}{2}e$$

Therefore, for simple planar, connected graph, with no 3-cycles,

$$\frac{1}{2}e \geq f = 2 - v + e \implies \frac{1}{2}e \leq v - 2 \implies e \leq 2v - 4$$

So $K_{3,3}$ is not planar! ;

Kuratowski's Theorem: A simple, connected graph G is planar if and only if does not contain an expansion of $K_{3,3}$ or K_5 as a subgraph.

Conjecture: Chromatic number of planar graphs should be less or equal to 4.

Let's try to prove that every planar, simple graph has a vertex of small degree...
We know

$$e \leq 3v - 6$$

Note that

$$\sum_{\text{vertices}} d(v_i) = 2e$$

Multiple the above by 2, $\sum_{\text{vertices}} d(v_i) = 2e \leq 6v - 12$
Sort of we "proved" that every planar, simple graph has a vertex of degree ≤ 5...

Proposition: Every simple, planar, connected graph G has $\chi(G) \leq 6$.

3 Friday's Lecture

Continue yesterday's proof of proposition.

Proof: Induction on the number of vertices.

Let G be simple, planar, connected graph with n vertices.

Let v be a vertex of degree ≤ 5.

Consider $G' = G - \{v\}$.

If $G'_1, \ldots, G'_k$ are connected components of G', each can be colored with 6 colours by induction hypothesis.

Now since v is adjacent to ≤ 5 vertices, there are ≤ 5 forbidden colours for v. Thus, we can extend a 6-coloring of G' to a 6-coloring of G.