PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL/MAY 2009 EXAMINATIONS

CSC 165 H1S
Instructor(s): F. Pitt

Duration — 3 hours

PLEASE HAND IN

Examination Aids: *One* 8.5" × 11" sheet of paper, *hand*written on both sides, and one *non-programmable* calculator.

Student Number: _ _ _ _ _ _ _ _ _

Family Name(s): ______________________

Given Name(s): ______________________

*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This final examination paper consists of 8 questions on 13 pages (including this one), printed on one side of each sheet. *When you receive the signal to start, please make sure that your copy of the final examination is complete, fill in the identification section above, and write your student number where indicated at the bottom of every page (except page 1).*

Answer each question directly on this paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or fact covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/fact(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do — part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part), you will get 20% of the marks for that question (or part) if you write "I don't know" and nothing else — you will *not* get those marks if your answer is completely blank, or if it contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

# 1:	____/ 15
# 2:	____/ 10
# 3:	____/ 15
# 4:	____/ 15
# 5:	____/ 20
# 6:	____/ 15
# 7:	____/ 15
# 8:	____/ 5
BONUS MARKS:	____/ 10
TOTAL:	____/110

Good Luck!

Question 1. [15 MARKS]

Let A represent the set of all algorithm and P represent the set of all problems. Let $\mathsf{brunos} \in A$ represent "Bruno's algorithm" and $\mathsf{carols} \in A$ represent "Carol's algorithm" (note that A contains other algorithms). Let $\mathsf{MST} \in P$ represent "the Minimum Spanning Tree problem" and $\mathsf{TSP} \in P$ represent "the Travelling Sales Person problem" (note that P contains other problems). Finally, let $E(x)$ represent the sentence "algorithm x is efficient" and $S(x, y)$ represent the sentence "algorithm x solves problem y".

Part (a) [3 MARKS]

Give a symbolic translation for the sentence: "All algorithms that solve TSP are inefficient."

Part (b) [3 MARKS]

Give a *natural* English translation for the sentence: $\forall y \in P, \exists x \in A, S(x, y) \wedge \neg E(x)$.

Part (c) [3 MARKS]

Give a *natural* English translation for the sentence:
$(\exists x \in A, E(x) \wedge S(x, \mathsf{TSP})) \Rightarrow (\exists x \in A, E(x) \wedge S(x, \mathsf{MST}))$.

Part (d) [3 MARKS]

Give a symbolic translation for the open sentence: "there is an efficient algorithm for problem p".

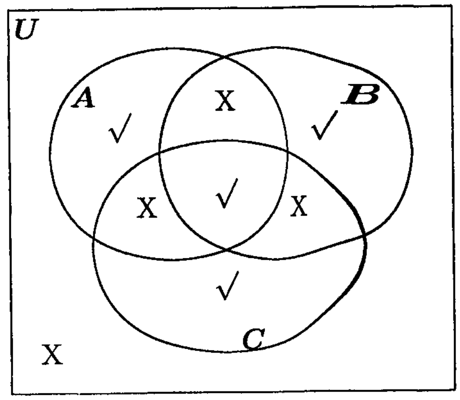
Part (e) [3 MARKS]

Suppose the first statement above is true, and we also know that Bruno's algorithm is efficient. Can we conclude that $S(\mathsf{brunos}, \mathsf{TSP})$? Justify.

Question 2. [10 MARKS]

Part (a) [4 MARKS]

Let A, B, C be three subsets of universe U, and $A(x) : x \in A$, $B(x) : x \in B$, $C(x) : x \in C$ be the corresponding predicates. Write a symbolic statement S such that S is true for every element in a region that contains a check mark "$\surd$", and S is false for every element in a region that contains an "X".



Part (b) [6 MARKS]

Simplify the following symbolic sentence to use *as few connectives* as possible. Show the steps in your derivation, and give a brief (less than one line) justification of each step.

$$(r \wedge \neg s \Rightarrow p) \wedge (q \Rightarrow \neg r \vee s)$$

Question 3. [15 MARKS]

Part (a) [6 MARKS]

Complete each statement below to make it true, by choosing from the following list to fill in the missing expression—use each expression in the list *exactly once* in your answers.

Choose from: $1/n$, 1, n, n^2, 2^n, n^n.

$$2^{2n+2} \in \mathcal{O}(\_\_\_\_\_\_) \qquad 6 + \frac{6}{n} \in \mathcal{O}(\_\_\_\_\_\_)$$

$$\frac{n^2 - 4}{3n^3 + 3} \in \mathcal{O}(\_\_\_\_\_\_) \qquad 8n^8 - 7n^7 + 6n^6 \in \mathcal{O}(\_\_\_\_\_\_)$$

$$\sqrt{5n^2 + 7} \in \mathcal{O}(\_\_\_\_\_\_) \qquad (n + 9)^2 \in \mathcal{O}(\_\_\_\_\_\_)$$

Question 3. (CONTINUED)

Part (b) [9 MARKS]

In your answer to this question, you may refer to any of the following properties without proof:

For all $b, x, y \in \mathbb{R}^+$, $\log_b(x \cdot y) = \log_b x + \log_b y$, $\log_b(x^y) = y \cdot \log_b x$, $b^x \cdot b^y = b^{x+y}$, $b^{x \cdot y} = (b^x)^y$.

Prove that for all functions $f, g : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} g(n) = \infty$, $f \in \mathcal{O}(g) \Rightarrow \log_2 f \in \mathcal{O}(\log_2 g)$, where $(\log_2 f)(n) = \log_2(f(n)), \forall n \in \mathbb{N}$. Give a detailed structured proof.

Question 4. [15 MARKS]

Let S denote the sequence of natural numbers: $0, 0, 1, -1, 2, -2, 3, -3, \ldots; a_n = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (1-n)/2 & \text{if } n \text{ is odd.} \end{cases}$

Let A denote the sentence "$\forall i \in \mathbb{N}, a_i > 0 \Rightarrow \exists j \in \mathbb{N}, j > i \wedge a_j < 0$".

Part (a) [3 MARKS]

Simplify the sentence $\neg A$ by working in the negation as much as possible.

Part (b) [9 MARKS]

Prove or disprove A for sequence S—give a detailed structured proof.

Part (c) [3 MARKS]

Give an explicit sequence A' such that S is true for A' iff S is false for A.

Question 5. [20 MARKS]

Consider the following algorithm, that takes as input a list of integers A.

```
i = 0
while i < len(A):
    if A[i] % 2 == 1:
        j = i
        while j < len(A):
            A[j] = A[j] + 1
            j = j + 1
    i = i + 1
```

Part (a) [5 MARKS]

Compute the **exact** best-case running time of the algorithm—*i.e.*, *without* using $\mathcal{O}$, Ω, or Θ. State **precisely** what you are counting and define clearly all of your variables (in particular, say what n is).

Part (b) [15 MARKS]

Let $T(n)$ be the worst-case running time of the algorithm for inputs of size n. Find a simple function $g(n)$ such that $T(n) \in \Theta(g(n))$, and give a detailed structured proof of this fact. (**Warning!** This is long—keep it in mind when you plan your time.)

[More space on the next page...]

Question 5. (CONTINUED)

Part (b) (CONTINUED)

Question 6. [15 MARKS]

Consider the normalized floating-point system S_0 with $\beta = 2$, $t = 6$, $e_{\min} = -3$, $e_{\max} = +4$, using round-toward-zero (*i.e.*, truncation).

Part (a) [3 MARKS]

Compute the floating-point representation of the number -2.25 in system S_0. Show your work.

Part (b) [3 MARKS]

Give the smallest positive floating-point number in S_0, and compute its decimal value. Show your work.

Part (c) [3 MARKS]

Give the largest positive floating-point number in S_0, and compute its decimal value. Show your work.

Question 6. (CONTINUED)

Part (d) [6 MARKS]

Compute the representation of 1/3 in system S_0. Then, compute the relative rounding error in your representation. Show your work.

Question 7. [15 MARKS]

Consider computing $f(x) = (x-1) \cdot (x-4) + x$ for various values of x.

Part (a) [5 MARKS]

Compute $f(4.044)$ using floating-point operations in the normalized system with $\beta = 10$, $t = 3$, $e_{\text{min}} = -5$, $e_{\text{max}} = +5$, and round-to-nearest. Compute the relative error in your answer. Show your work.

Question 7. (CONTINUED)

Part (b) [5 MARKS]

Does the current expression for $f(x)$ suffer from any problems for values of x close to 4? Explain. Then, describe an alternative formula/algorithm to compute the values of $f(x)$ for values of x close to 4. Explain your reasoning. (HINT: expand and re-factor.)

Part (c) [5 MARKS]

Compute the condition number of $f(4)$. What does this tell you about the computation of $f(x)$ for values of x close to 4?

Question 8. [5 MARKS]

Consider the following problem.

> You are part of a group of friends who choose whom to treat to lunch in the following (unusual) manner:
>
> - They arrange themselves in a circle (approximately).
> - They begin reciting positive integers, in order, in a counter-clockwise direction (viewed from above), starting with the friend at the northern extreme of the circle (who utters "one").
> - Each friend who utters an even number is eliminated from the counting (and consideration for free lunch). The counting "wraps around" so that those who avoided one of the dreaded even numbers on the first round may be exposed on subsequent rounds.
> - The last person left gets free lunch.
>
> For example, if there are friends f_1, f_2, f_3, f_4, and f_5 arranged counter-clockwise, with f_1 at the northern extreme, the first round would eliminate f_2 and f_4 (when they say "two" and "four", respectively). Then f_1 and f_5 would be eliminated in the next round (when they say "six" and "eight", repectively), leaving f_3 to enjoy the free lunch.
>
> If there are n friends, where should you position yourself to get the free lunch?

Describe two different strategies/approaches you could use to attempt to solve this problem. Describe what you hope to achieve with each approach.

Bonus. [10 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer (in particular, "I don't know" will be worth zero). Please attempt this only *after* you have completed the rest of the exam.

Solve the problem from the last question.

Total Marks = 110