Sta347 Probability I

Selected Practice Problems for Final

Dec. 10, 2013

(1) Let X have the density function $f(x) = cxe^{-2x}$, $0 \le x < \infty$. And $f(x) = 0$ otherwise.

(a). Find the value of c.

(b). Give the mean and the variance of X.

(c). Give the moment generating function of X.

(2) Let U_1, U_2 and U_3 be i.i.d. Uniform[0,1] random variables. Find $P(U_1 + U_2 > U_3)$.

(3) The joint density function of X and Y is given by $f(x, y) = 6x^2y$ if $0 \le x \le 1$ and $0 \le y \le 1$. And $f(x, y) = 0$ otherwise.

(a). Find the mean and variance of X.

(b). Find the mean and variance of Y.

(c). Find the conditional expectation of X given Y.

(4) Let the region $A = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1, 0 \le y \le 1, y - x \le 1, y + x \le 1\}$. Let (X, Y) be the random vector which distributes uniformly on A.

(a). Find the marginal density of X.

(b). Find the marginal density of Y.

(c). Find the conditional density function of Y given X.

(d). Find $P(X - Y \ge 0)$.

(5) Let U_1 and U_2 be i.i.d. Uniform[0,1] random variables. Find the density function of $Y = U_1U_2$.

(6) Suppose that $X_1, X_2, \cdots, X_n$ are i.i.d. random variables each with mean μ_1 and variance σ_1^2. Suppose that $Y_1, Y_2, \cdots, Y_n$ are i.i.d. random variables each with mean μ_2 and variance σ_2^2. Show that, as $n \to \infty$, the random variable

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}$$

converges in distribution to a standard normal random variable.

(7) Let X be a chi-squared distribution with n degrees of freedom. In other words, X has density

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}, \quad x \ge 0$$

and $f(x) = 0$ otherwise. Show that the random variable $(X - n)/\sqrt{2n}$ converges in distribution to a standard normal random variable.

(8) For a sequence of vector random variables $(X_i, Y_i)_{i=1}^n$, let $F_i(x, y)$ be the cumulative distribution function of (X_i, Y_i). We say (X_i, Y_i) converges in distribution to a vector random variable (X, Y) with cumulative distribution function $F(x, y)$ if for any $(x_0, y_0) \in \mathcal{R}^2$ such that $F(x, y)$ is continuous at (x_0, y_0), we have $F_i(x_0, y_0) \to F(x_0, y_0)$. Prove that if (X_i, Y_i) converges in distribution to (X, Y), then X_i converges in distribution to X and Y_i converges in distribution to Y.