Some special continuous probability distributions

The uniform distribution

Note that we have already seen an example of this: $f(y) = 0.5$, $0 < y < 2$.
Here, Y has what is called the uniform distribution with parameters 0 and 2.

> A random variable Y has the *uniform distribution* with parameters a and b if its pdf is of the form
>
> $$\boxed{f(y) = \frac{1}{b-a}, \quad a < y < b \quad (a < b).}$$
>
> We write $\boxed{Y \sim U(a,b)}$ and $\boxed{f(y) = f_{U(a,b)}(y)}$.

$f_Y(y) = \frac{1}{b-a}$

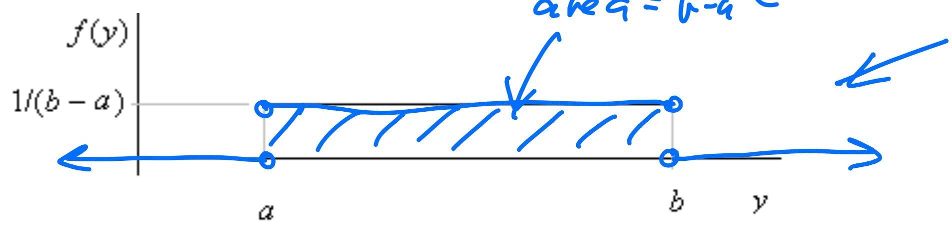
Example 4 Suppose that $Y \sim U(a,b)$. Find Y's cdf.

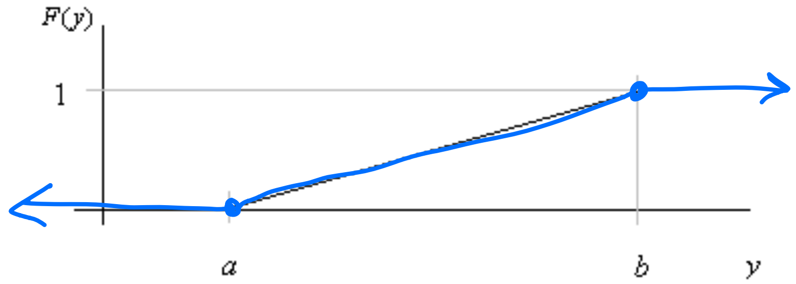
$$F(y) = \int_a^y \underbrace{\frac{1}{b-a}}_{f_Y(t)} dt = \boxed{\frac{y-a}{b-a}}, \quad a < y < b.$$

$$\begin{cases} 0, & y \le a \\ 1, & y \ge b \end{cases}$$

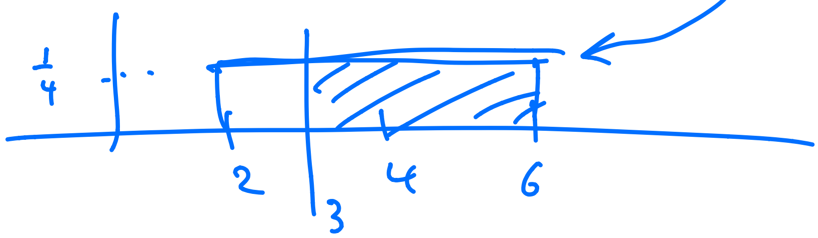
We could also denote this cdf by $F_{U(a,b)}(y)$.

area $= \frac{1}{b-a} \times (b-a) = 1$ ✓





Eg: If $Y \sim U(2,6)$, then $F(y) = (y-2)/4$, $2 < y < 6$.
So $P(Y > 3) = 1 - P(Y < 3) = 1 - F(3) = 1 - (3-2)/4 = 3/4$.



The standard uniform distribution

a b

If $Y \sim U(0,1)$, we say that Y has the *standard uniform distribution*.

Then, $f(y) = 1$, $0 < y < 1$, and $F(y) = y$, $0 < y < 1$.

$\frac{y-0}{1-0}$

The normal distribution

or Gaussian dsn

A random variable Y has the *normal distribution* with parameters a and b^2 if its pdf is of the form

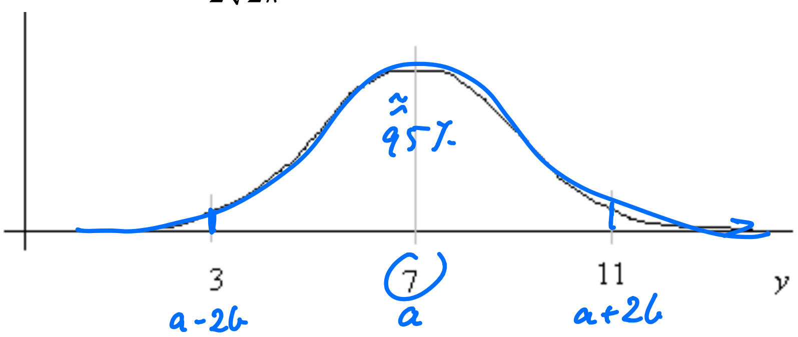
$$f(y) = \boxed{\frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(y-a)^2}}, \quad -\infty < y < \infty \qquad (-\infty < a < \infty,\ b > 0).$$

We write $Y \sim N(a, b^2)$ and $f(y) = \boxed{f_{N(a,b^2)}(y)}$.

2^2

Example 5 Suppose that $Y \sim N(7,4)$. Sketch Y's pdf and cdf. "bell-shaped"

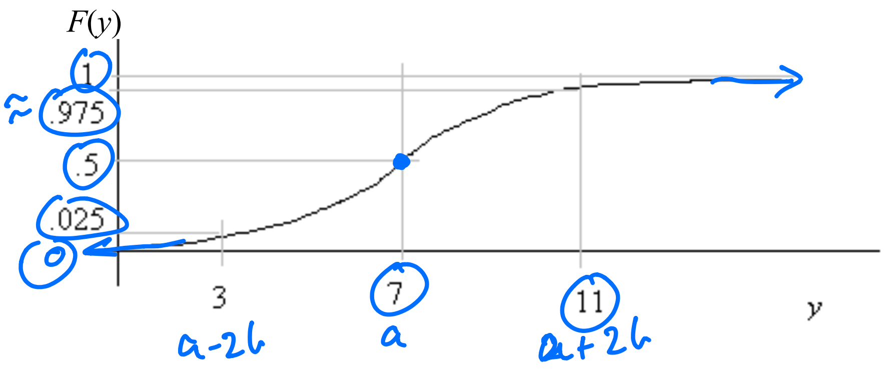
Y's pdf is $f(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2(4)}(y-7)^2}$



Thus $f(y)$ is a smooth and symmetric bell-shaped curve centered at 7, with roughly 95% (exactly 95.45% to 4 significant digits) of the area underneath it between $a - 2b = 7 - 2(2) = 3$ and $a + 2b = 7 + 2(2) = 11$. Note that the total area under the curve is 1.

Y's cdf is $F(y) = \int_{-\infty}^{y} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2(4)}(t-7)^2} dt$ = impossible (analytically)

This is an intractable integral that can however be computed numerically at each y.



NB: The points (3,0.025) and (11,0.975) here are approximate but (7,0.5) is exact.

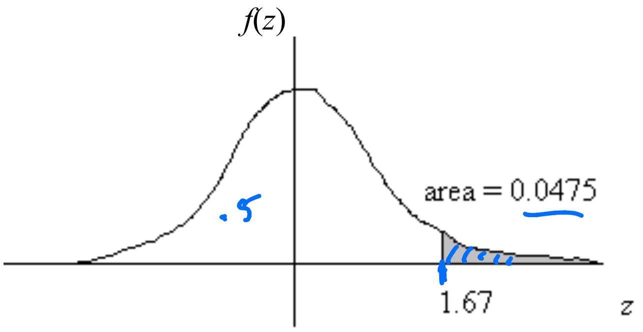
The standard normal distribution

a b or b^2

If $Y \sim N(0,1)$, we say that Y has the *standard normal distribution*.
The letter Z is often used to denote a rv with this dsn.

Values of $P(Z > z)$ are tabulated on the inside front cover of the text (and elsewhere).

For example, $P(Z > 1.67) = 0.0475$.



Also: $P(Z < 1.67) = 1 - 0.0475 = 0.9525$

$P(0 < Z < 1.67) = 0.9525 - 0.5 = 0.4525$

$P(Z < -1.67) = 0.0475$ (by symmetry), etc.

Note: Some books have tables of $P(Z < z)$ or $P(0 < Z < z)$ rather than $P(Z > z)$.

Notation and terminology:

$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

We may write $f_{N(0,1)}(z)$ as $\phi(z)$.

Thus $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$.

We may write $F_{N(0,1)}(z)$ as $\Phi(z)$.

Thus $\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt, \quad -\infty < z < \infty$.

For example: $\Phi(1.67) = 0.9525$.
$\Phi(-1.67) = 0.0475$

$\in (0,1)$

The *(lower) quantile function* of Z is $F_{N(0,1)}^{-1}(p) = \boxed{\Phi^{-1}(p)}$.

For example: $\Phi^{-1}(0.9525) = 1.67$
$\Phi^{-1}(0.0475) = -1.67$.

$\Phi(1.67) = 0.9525$

The *upper quantile function* of Z is $\boxed{z_p} = \boxed{\Phi^{-1}(1-p)}$.

For example: $z_{0.0475} = 1.67$
$z_{0.9525} = -1.67$

$\Phi(1.67) = 1 - .0475$

Other examples: $\Phi(1.96) = 0.975$, $z_{0.025} = 1.96$
$\Phi(2) = 0.97725$, $z_{0.02275} = 2$

$P(-1.96 < Z < 1.96) = \Phi(1.96) - \Phi(-1.96)$
$= 1 - 2\Phi(-1.96) = 1 - 2\times 0.025 = 0.95$.

$P(-2 < Z < 2) = \Phi(2) - \Phi(-2)$
$= 1 - 2\Phi(-2) = 1 - 2\times 0.02275 = 0.9545$.

The standard normal tables can be used to compute probabilities involving *any* normal distribution. For this we require the following result, which will be proved later.

If $Y \sim N(a,b^2)$, then $\boxed{Z = \frac{Y-a}{b}} \sim N(0,1)$.

$a = EY$
$b = SD(Y)$

We say that Y has been *standardised*, and that Z is the *standardised version* of Y.

(Note: *Standardising* a random variable usually means subtracting away its mean and then dividing by the random vriable's standard deviation. It will be shown later that the mean and standard deviation of Y here, i.e. of the $N(a,b^2)$ dsn, are in fact a and b.)

Example 6 Suppose that $Y \sim N(10,16)$. Find $P(Y > 11)$.

$$P(Y > 11) = P\left(\frac{Y-a}{b} > \frac{11-10}{4}\right) = P(Z > 0.25) = 0.4013.$$

(This can be illustrated by two bell shaped curves: (i) the pdf of Y with the region underneath and to the right of 11 shaded, and (ii) the pdf of Z with the region underneath and to the right of 0.25 shaded. Both regions have the same area, 0.4013.)

The gamma distribution

A random variable Y has the *gamma distribution* with parameters a and b if its pdf is of the form

$$f(y) = \boxed{\frac{y^{a-1}e^{-y/b}}{b^a\Gamma(a)}}, \quad y > 0 \quad (a,b > 0).$$

We write $Y \sim Gam(a,b)$ and $f(y) = f_{Gam(a,b)}(y)$.

Note: $\Gamma(\cdot)$ here is the *gamma function*, defined by $\boxed{\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}\,dt}$.

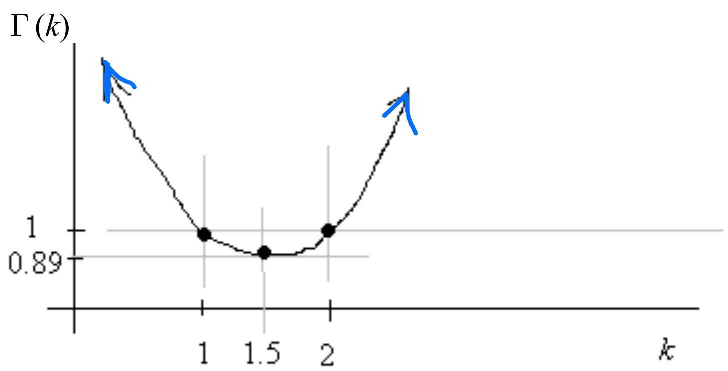
Some of this function's properties are:

$\Gamma(k) = (k-1)\Gamma(k-1)$ if $k > 1$.

$\Gamma(k) = (k-1)!$ if k is a positive integer (eg $\Gamma(4) = 3! = 6$).

Also, $\Gamma(1/2) = \sqrt{\pi}$.

Thus also, for example, $\Gamma(2.5) = 1.5\Gamma(1.5) = 1.5 \times 0.5\Gamma(0.5) = 1.3293$.

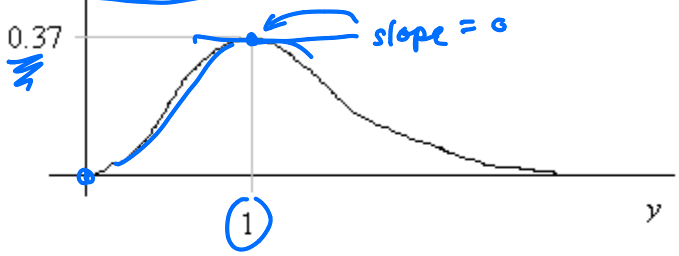


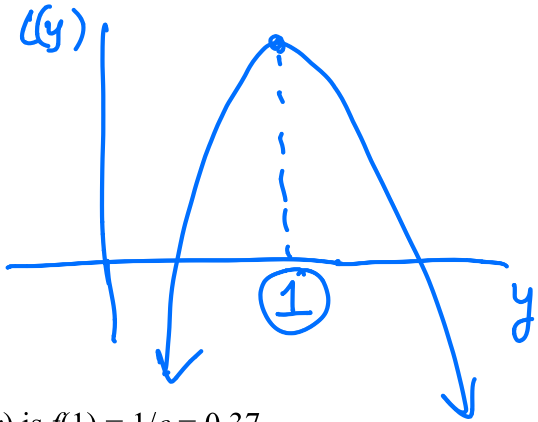
Note: $\Gamma(1.5) = 0.5\,\Gamma(0.5) = 0.5\sqrt{\pi} = 0.8862$ (not exactly the minimum)

$\Gamma(1.46) = 0.8856$ (minimum).

Example 7 Suppose that $Y \sim \text{Gam}(2,1)$. Sketch Y's pdf.

$$f(y) = \boxed{\frac{y^{2-1}e^{-y/1}}{1^2\Gamma(2)}} = \boxed{ye^{-y}},\quad y > 0.$$





Note that the mode of Y is 1, and the maximum value of $f(y)$ is $f(1) = 1/e = 0.37$.

This mode was obtained as follows:

$$f'(y) = y(-e^{-y}) + 1(e^{-y}) \overset{\text{set}}{=} 0 \overset{\text{solve}}{\Rightarrow} y = 1.$$

Equivalently, we could argue that:

$$l(y) = \log f(y) = \log y - y$$

$$l'(y) = \frac{1}{y} - 1 = 0 \Rightarrow y = 1.$$

More generally,

$$l(y) = \log f(y) = (a-1)\log y - y/b + \text{constant}.$$

$$l'(y) = \frac{a-1}{y} - \frac{1}{b} = 0 \Rightarrow y = b(a-1).$$

This assumes that $a \geq 1$. If $a < 1$ then $f(y)$ is maximised at $y = 0$.

Thus generally, $Mode(Y) = \begin{cases} b(a-1) & \text{if } a \geq 1 \\ 0 & \text{if } a < 1. \end{cases}$

Note that $f(0) = 0$ if $a > 1$, $f(0) = 1/b$ if $a = 1$, and $f(0) = \infty$ if $a < 1$.

The chi-square distribution (a special case of the gamma dsn)

If $Y \sim \text{Gam}(n/2, 2)$, we say that Y has the *chi-square distribution* with parameter n.

We call n the *degrees of freedom (DOF).*

We write $Y \sim \chi^2(n)$ and $f(y) = f_{\chi^2(n)}(y)$.

Note: The mode of Y is $n-2$ if $n \geq 2$, and it is 0 if $n < 2$.

$f(0) = 0$ if $n > 2$, $f(0) = 1/2$ if $n = 2$, and $f(0) = \infty$ if $n < 2$.

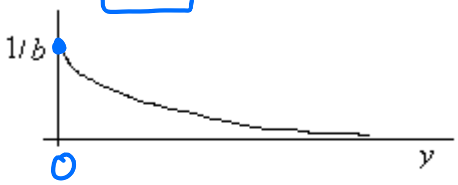
The exponential distribution (another special case of the gamma dsn)

a

If $Y \sim \text{Gam}(1, b)$, then Y has the *exponential distribution* with parameter b.

We write $Y \sim Expo(b)$ and $f(y) = f_{Expo(b)}(y)$.

$$f(y) = \boxed{\frac{1}{b} e^{-y/b}}, \; y > 0$$



n

Note that $Mode(Y) = 0$ for all b.

Also, $Expo(2) = Gam(2/2, 2) = \chi^2(2)$.

Gam(1,2)

Example 8 Find the cdf of the exponential distribution with parameter b.

$$F(y) = \int_0^y \frac{1}{b} e^{-t/b} dt = \left[-e^{-t/b} \Big|_0^y \right] = -e^{-y/b} - (-e^{-0/b}) = \boxed{1 - e^{-y/b}}, \; y > 0.$$

t/b

For example, if $Y \sim \text{Expo}(5)$, then

$$P(Y > 2) = 1 - P(Y < 2) = 1 - F(2) = 1 - (1 - e^{-2/5}) = e^{-2/5} = 0.670.$$

The standard exponential distribution (a special case of the exponential dsn)

b

If $Y \sim Expo(1)$, we say that Y has the *standard exponential distribution*.

$Y \sim Gam(1,1)$