AST121 Tutorial on integration and ordinary differential equations

AST121 TAs

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Outline

- Integration: a summary
- Rules of integration
- Ordinary differential equations
- Worked examples

Integration: computing the area under a curve

- For a function $f(x)$ of a single variable, integration means computing the area under the graph of $f(x)$ in some interval $[a, b]$.
- The simplest way of doing this is to divide the interval into chunks. For each chunk one computes the area of a rectangle that encloses the function. One then sums over all the chunks to get the approximate area.

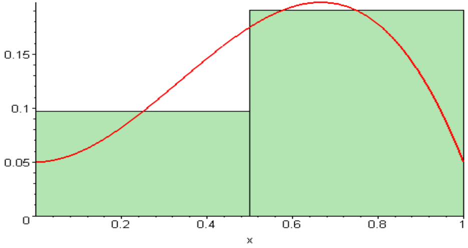

Figure: Taken from Wikipedia

Integration

- It turns out that integration is anti-derivation: it is the "opposite" operation to differentiation.
- Thus the problem of finding the integral of a given function is to find another function whose derivative is equal to the given one.
- How does one actually *compute* an integral (i.e. an area under the curve)?
- The Fundamental Theorem of calculus: Let $f(x)$ be a given function and let $F(x)$ be a function such that $F'(x) = \frac{dF}{dx} = f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Definite and indefinite integrals

- A *definite* integral of a function $f(x)$ over some interval $[a,b]$, is the area under the graph of f between a and b.
- An *indefinite* integral of a function $f(x)$ amounts to finding the function $F(x)$ such that $F'(x) = f(x)$,i.e. finding the antiderivative.
- Notice that if $F(x)$ is a function such that $F'(x) = f(x)$, so is $F(x) + C$ where C is *any* constant, since $\frac{d}{dx}(F(x) + C) = \frac{dF}{dx} + \frac{dC}{dx} = \frac{dF}{dx}$ because the derivative of a constant is zero. This means that if a function has an antiderivative, $F(x)$ it in fact has infinitely many anti-derivatives all differing by a constant.
- This is precisely the origin of *constant of integration*. Practically, this amounts to always remembering a constant of integration: $\int f(x)dx = F(x) + C$.

Rules of integration

- So how does one find the anti-derivative of some function?
- What is the anti-derivative of $f(x) = x^2$? We want some function that when differentiated would give us x^2. Remembering the rules of differentiation for powers: $\frac{d}{dx}x^n = nx^{n-1}$, so if we guess $F(x) = x^3$, we would get $\frac{dF}{dx} = 3x^2$, not quite right. Notice that if we choose $F(x) = \frac{x^3}{3}$, then $\frac{dF}{dx} = \frac{3}{3}x^2 = x^2$
- Generically then, we can guess that the anti-derivative of x^n is

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

  By differentiating you can easily check this is indeed the case

Some other rules of integration

Here are some more anti-derivation rules that are of use in this course:

- $\int A dx = Ax + C$ for some constant A
- $\int \sin(x) dx = -\cos(x) + C$
- $\int \cos(x) dx = \sin(x) + C$
- $\int \frac{1}{x} dx = \ln(x) + C$
- $\int \exp(x) dx = \exp(x) + C$

Differential equations

- In physics, the study of the laws of nature frequently leads to an equation involving the variation in time of some interesting quantity, i.e. the time derivative of this quantity.
- The solution of this *evolution* equation then tells us how the quantity of interest changes with time, which is what allows us to make predictions about natural phenomena
- Intuitively a *differential equation* is simply any equation that includes derivatives of some function.

Example: Newton's law of cooling

- Newton proposed that the rate at which a warm object cools in a colder environment is proportional to the difference between the object's temperature and the temperature of the environment
- This means $\frac{dT}{dt} \propto (T - T_a)$ where T is the temperature of the object and T_a is the ambient temperature.
- Now since the object is cooling, its temperature is *decreasing* and thus the derivative $\frac{dT}{dt}$ should be *negative*. Since $T - T_a > 0$ then we must have

$$\frac{dT}{dt} = -k(T - T_a)$$

with $k > 0$.

Newton's law of cooling (continued)

Question: what is the solution to the differential equation $\frac{dT}{dt} = -k(T - T_a)$ with the *initial condition* $T(t = 0) = T_0$?

- For convenience introduce $u = T - T_a$ as a new variable. Then $\frac{du}{dt} = \frac{dT}{dt}$ so that we can write the equation as

$$\frac{du}{dt} = -ku$$

- This is a *separable* equation: we can put everything with the dependent variable on one side and everything with the independent variable on another: $\frac{1}{u}du = -kdt$.
- The idea is now to integrate the left side with respect to u and right side with respect to t. (Strictly speaking, what we mean to do is integrate both sides of $\frac{1}{u}\frac{du}{dt} = -k$ with respect to t. However, that is equivalent to $\int \frac{1}{u}du$ on the left-hand side.)

Newton's law of cooling: solution

- We start with $\frac{1}{u}du = -kdt$. Integrating: $\int \frac{1}{u}du = \int -kdt$ so that $\ln(u) = -kt + C$ where C is a constant of integration. Exponentiating both sides gives $u = Ae^{-kt}$ so that
  $T = Ae^{-kt} + T_a$
  where $A = e^C$.
- At $t = 0$, $T = T_0$ so that $T_0 = Ae^{-k0} + T_a$. Solving for A, $A = T_0 - T_a$. Thus the final solution is

$$T = (T_0 - T_a)e^{-kt} + T_a$$

- In stead of using the constant of integration we could have made the boundaries of the integrals explicit: $\int_{u_0}^{u} \frac{1}{u'}du' = \int_0^t -kdt'$. This would give $\ln(u) - \ln(u_0) = -kt$. But $\ln(u) - \ln(u_0) = \ln\left(\frac{u}{u_0}\right) = -kt$. Exponentiating both sides and substituting $u_0 = T_0 - T_a$ gives the same results as before

EXAMPLE 2: VISCOUS DRAG

Question: suppose the drag force on an object moving through a fluid is proportional to the square of its velocity. Write down and solve the differential equation describing the variation of the objects velocity with time, using the initial condition $v(t = 0) = v_0$.

- We start with Newton's second law, $F = ma$. Recall that $a = \frac{dv}{dt}$ and we are given that $F_{drag} = -kv^2$ where k is some constant (In reality this is not a constant and depends on fluid density, area of the object, etc but that is irrelevant to our problem).
- Assume the drag force is the only force acting on the object, so that $F_{drag} = ma = m\frac{dv}{dt}$.
- Then equation we want to solve is $\frac{dv}{dt} = -\frac{k}{m}v^2$.

Viscous drag: solution

- Separating variables: $\frac{1}{v^2}dv = -\frac{k}{m}dt$. Integrating both sides, $-\frac{1}{v} = -\frac{k}{m}t + C$. Thus $v = \frac{1}{\frac{k}{m}t - C}$.
- We use the initial condition to find the value of the constant C. At $t = 0$, $v = v_0$, i.e $v_0 = \frac{1}{\frac{k}{m}0 - C}$. Rearranging gives $C = -\frac{1}{v_0}$. Thus the full solution to this problem is

$$v = \frac{1}{\frac{1}{v_0} + \frac{k}{m}t} = \frac{mv_0}{m + kv_0 t}$$

- Does this make sense? We expect that drag makes the body slow down, so that as time goes on, the velocity should decrease. Indeed, as t grows the denominator of the solution increases and thus v decreases as expected.
- Exercise (always do this!): By plugging this result back into the differential equation, check that we have solved the problem correctly.