Homework Assignment #3

MAT 335 – Chaos, Fractals, and Dynamics – Fall 2013

PARTIAL SOLUTION

Chapter 6.1.(g) Let $F_c(x) = x^3 + c$. We want to identify the bifurcation at $c = \frac{2}{3\sqrt{3}}$.
The function is of order 3, so we don't have a formula to find its roots. This means we that we have to be more creative in our approach to the problem.
On both saddle-node and period-doubling bifurcations, at the bifurcation value $c = \frac{2}{3\sqrt{3}}$, there is one fixed point which is neutral. So let us first find the points where the derivative is ± 1:

$$F_c'(x) = \pm 1 \qquad \Leftrightarrow \qquad 3x^2 = \pm 1 \qquad \Leftrightarrow \qquad x^2 = \pm \frac{1}{3}$$

Since the square is never negative, we can only find points with derivative 1:

$$x = \pm \sqrt{\frac{1}{3}}.$$

These points might not be fixed points: they are just the points where F_c has derivative 1. We need to find out when are these points fixed points of F_c:

$$F_c\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \qquad \Leftrightarrow \qquad \frac{1}{3\sqrt{3}} + c = \frac{1}{\sqrt{3}} \qquad \Leftrightarrow c = \frac{2}{3\sqrt{3}}$$
$$F_c\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} \qquad \Leftrightarrow \qquad -\frac{1}{3\sqrt{3}} + c = -\frac{1}{\sqrt{3}} \qquad \Leftrightarrow c = -\frac{2}{3\sqrt{3}}$$

So for $c_0 = \frac{2}{3\sqrt{3}}$ the map has a fixed point $p_{c_0} = \frac{1}{\sqrt{3}}$ which is neutral (and $F_{c_0}'(p_{c_0}) = 1$). Since the derivative is 1 (and not -1), the bifurcation cannot be period-doubling. It can only be saddle node bifurcation.
We need to check that it meets all 3 conditions of a saddle-node bifurcation:

(i) F_{c_0} has one fixed point in I and it is neutral.

Actually, F_{c_0} has another fixed point, which is negative, so we need $I = (0, \infty)$.

(ii) For $c > \frac{2}{3\sqrt{3}}$, The function is above the line $y = x$:

$x^3 - x + c > 0$, since its minimum is at $3x^2 - 1 = 0$, which is at $p = \frac{1}{\sqrt{3}}$ and $p^3 - p + c > 0$ for $c > \frac{2}{3\sqrt{3}}$.

So there are no fixed points in I for $c > \frac{2}{3\sqrt{3}}$.

(iii) For $c < \frac{2}{3\sqrt{3}}$, we need to prove that there are 2 fixed points: one attracting and one repelling.

- Observe that

$$F_c\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} + c < \frac{1}{\sqrt{3}} \qquad \text{(at this point, } F_c(x) \text{ is below the line } y = x)$$

$$F_c(0) = c > 0. \qquad \text{(at this point, } F_c(x) \text{ is above the line } y = x)$$

  For these inequalities to hold, we need $0 < c < \frac{2}{3\sqrt{3}}$, so $\varepsilon \leqslant \frac{2}{3\sqrt{3}}$.
  Then, by the Intermediate Value Theorem, there exists a point $p_- \in \left(0, \frac{1}{\sqrt{3}}\right)$ such that $F_c(p_-) = p_-$.
  So p_- is a fixed point.

  Moreover

$$F_c'(p_-) = 3p_-^2 > 0 \qquad \text{and} \qquad F_c'(p_-) = 3p_-^2 < 3\left(\frac{1}{\sqrt{3}}\right)^2 = 1,$$

  so p_- is an attracting fixed point in I.

- Similarly,

$$F_c\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} + c < \frac{1}{\sqrt{3}} \qquad \text{(at this point, } F_c(x) \text{ is below the line } y = x)$$

$$F_c(2) = 8 + c > 2. \qquad \text{(at this point, } F_c(x) \text{ is above the line } y = x)$$

  For these inequalities to hold, we need $-6 < c < \frac{2}{3\sqrt{3}}$, so $\varepsilon \leqslant \frac{2}{3\sqrt{3}}$ still works.
  Then, by the Intermediate Value Theorem, there exists a point $p_+ \in \left(\frac{1}{\sqrt{3}}, 2\right)$ such that $F_c(p_+) = p_+$.
  So p_+ is a fixed point.

  Moreover

$$F_c'(p_+) = 3p_+^2 > 3\left(\frac{1}{\sqrt{3}}\right)^2 = 1,$$

  so p_+ is a repelling fixed point in I.

We conclude that at $c = \frac{2}{3\sqrt{3}}$, F_c has a saddle-node bifurcation.

Chapter 6.10. Consider the logistic family $F_\lambda(x) = \lambda x(1-x)$. The fixed points are

$$F_\lambda(x) = x \qquad \Leftrightarrow \qquad x = 0 \text{ or } x = 1 - \frac{1}{\lambda}.$$

And $F'_\lambda(x) = \lambda(1-2x)$, so

$$F'_\lambda(0) = \lambda$$
$$F'_\lambda\left(1 - \frac{1}{\lambda}\right) = 2 - \lambda$$

so

- $x = 0$ is an attracting fixed point for $-1 < \lambda < 1$ and repelling for $|\lambda| > 1$
- $x = 1 - \frac{1}{\lambda}$ is attracting for $1 < \lambda < 3$ and repelling for $\lambda < 1$ or $\lambda > 3$

So, if there is a bifurcation, it has to be a period-doubling bifurcation and for the fixed point $p_\lambda = 1 - \frac{1}{\lambda}$. We need to find the 2-cycles, which we computed in lectures:

$$q^1_\lambda = \frac{\lambda + 1 + \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda} = \frac{\lambda + 1 + \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda},$$
$$q^2_\lambda = \frac{\lambda + 1 - \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda} = \frac{\lambda + 1 - \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda},$$

which exist for $\lambda > 3$.
And

$$F'_\lambda(q^1_\lambda)\ F'_\lambda(q^2_\lambda) = 1 - (\lambda+1)(\lambda-3),$$

so the 2-cycle q^1_λ, q^2_λ is attracting if and only if

$$(\lambda+1)(\lambda-3) > 0 \qquad \Leftrightarrow \lambda > 3 \text{ or } \lambda < -1.$$

We now have all the tools we need to show that the 4 conditions of a period-doubling bifurcation hold:

(i) There is a unique fixed point $p_\lambda = 1 - \frac{1}{\lambda}$ in $I = (0, \infty)$ for $\lambda \in (1, 5)$ ($\varepsilon = 2$).

(ii) For $\lambda \leqslant 3$, there are no 2-cycles and p_λ is attracting.

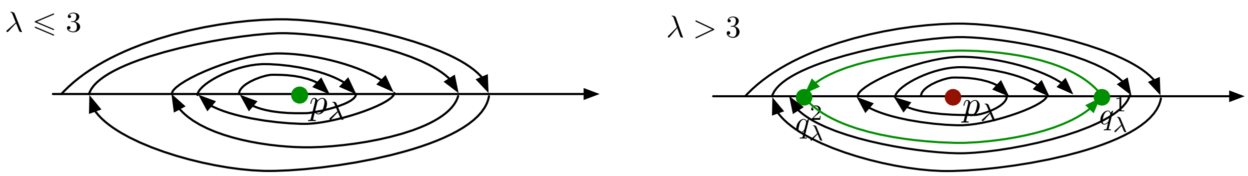
(iii) For $\lambda > 3$, there is a 2-cycle q^1_λ, q^2_λ which is attracting and p_λ is repelling.

(iv) As $\lambda \to 3^+$,

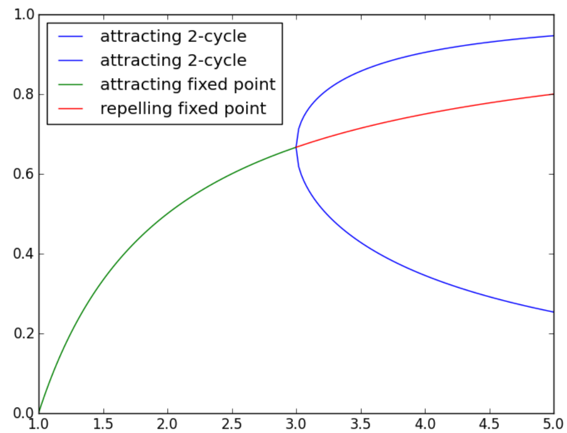
$$\lim_{\lambda \to 3^+} q^i_\lambda = \frac{3+1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3} = p_3.$$

We conclude that at $\lambda = 3$, the logistic family has a period-doubling bifurcation.

Chapter 6.11.



Phase Portrait



Bifurcation Diagram