UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2010 EXAMINATIONS

MAT335H1F
Chaos, Fractals and Dynamics
Examiner: D. Burbulla

PLEASE HAND IN

Duration - 3 hours
Examination Aids: A Scientific Hand Calculator

SURNAME: (as on your T-card) ____________________

GIVEN NAMES: ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

INSTRUCTIONS: Attempt all questions. Present your solutions in the space provided. Use the backs of the sheets if you need more space. Do not tear any pages from this exam. Make sure your exam contains 12 pages.

MARKS: The marks for each question are indicated in parentheses beside the question number.

TOTAL MARKS: 100

QUESTION	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
TOTAL	

1. [50 marks] This question has 10 parts. For the whole question let

$$F_\lambda(x) = x^2 + x + \lambda \quad \text{and} \quad Q_c(x) = x^2 + c.$$

(a) [3 marks] Find the critical point of F_λ and the vertex of the parabola with equation $y = F_\lambda(x)$.

(b) [5 marks] Find the fixed points of F_λ and determine for which values of λ they are repelling or attracting.

(c) [4 marks] Show that the family F_λ has a tangent bifurcation at $\lambda = 0$.

(d) [4 marks] Use graphical analysis to show that if $\lambda > 0$ then for every $x \in \mathbb{R}$

$$\lim_{n \to \infty} F_\lambda^n(x) = \infty.$$

(e) [4 marks] Use graphical analysis to show that if $x \notin [-1-\sqrt{-\lambda}, \sqrt{-\lambda}]$ for $\lambda \leq 0$ then

$$\lim_{n\to\infty} F_\lambda^n(x) = \infty.$$

(f) [4 marks]

Check that $h(x) = x + 1/2$ is a conjugacy between F_λ and $Q_{\lambda+1/4}$. State clearly all the properties that h must satisfy.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F_\lambda} & \mathbb{R} \\ h\downarrow & & \downarrow h \\ \mathbb{R} & \xrightarrow[Q_{\lambda+1/4}]{} & \mathbb{R} \end{array}$$

(g) [8 marks] Are the following statements True or False? Justify your answer.

(*i*) [4 marks] $F_{-9/4} : [-2.5, 1.5] \longrightarrow [-2.5, 1.5]$ is chaotic.

(*ii*) [4 marks] If $\lambda < -9/4$, then $\{x \in [-1 - \sqrt{-\lambda}, \sqrt{-\lambda}] \mid F_\lambda^n(x) \not\to \infty\}$ is a Cantor-like set.

(h) [10 marks] Find both points of prime period 2 for F_λ and determine for which values of λ the 2-cycle is repelling or attracting. What kind of bifurcation, if any, does F_λ have at $\lambda = -1$? Draw the bifurcation diagram of F_λ, for $-3/2 \leq \lambda \leq 1/2$.

(i) [3 marks] Compute $S(F_\lambda)(x)$, the Schwarzian derivative of F_λ.

(j) [5 marks] Explain why the orbit diagram for the family F_λ, $-9/4 \leq \lambda \leq 0$, can be obtained by considering all orbits of $x = -1/2$ under F_λ.

2.(a) [4 marks] Sketch the orbit of 1 in the complex plane under $L_\alpha(z) = \alpha z$, if $\alpha = i/2$.

2.(b) [6 marks] Show that $z_0 = e^{2\pi i/7}$ lies on a cycle of period 3 for $Q_0(z) = z^2$. Is this cycle attracting, repelling or neutral?

3.(a) [5 marks] According to Devaney, a dynamical system $F : X \longrightarrow X$ is chaotic if it has three properties:

1. Density

2. Transitivity

3. Sensitivity

Define precisely, in terms of F and X, what each of these properties is.

3.(b) [5 marks] Prove that the orbit of

$$\hat{\mathrm{s}} = (\underbrace{0\,1}_{\text{all 1 blocks}} \quad \underbrace{00\,01\,10\,11}_{\text{all 2 blocks}} \quad \underbrace{000\,010\,101\,\ldots}_{\text{all 3 blocks}} \ldots)$$

under the shift map σ is dense in Σ.

4. [10 marks] Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(a) [3 marks] Explain why F must have a point of prime period 56 if it has a point of prime period 60.

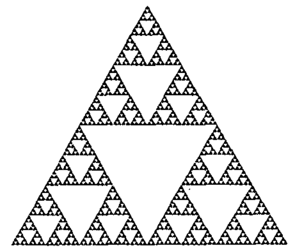
(b) [4 marks] If F is increasing for all x explain why

1. all orbits of x under F are unbounded, ie. $|F^n(x)| \to \infty$,
2. or else F has a fixed point, but no periodic points of any prime period greater than 1.

(c) [3 marks] If F is decreasing for all x explain why F has a fixed point, but no periodic points of any prime period greater than 2.

5. [10 marks]

Describe clearly and briefly two different algorithms that both generate the Sierpinski triangle. What is the fractal dimension of the Sierpinkski triangle?

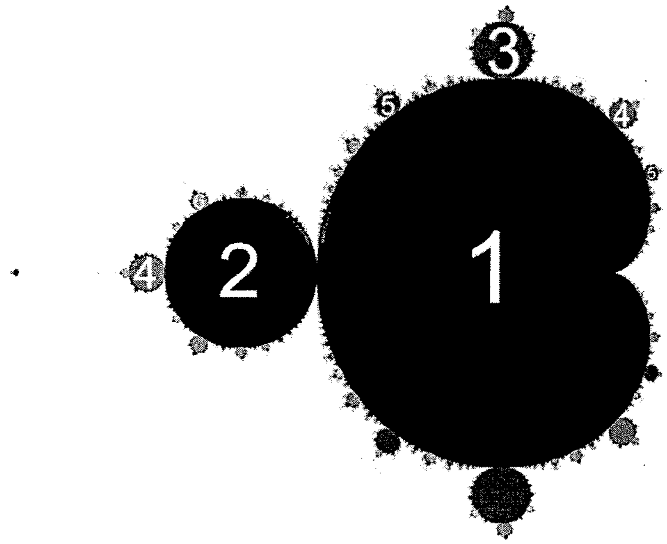


6. [10 marks]

To the right is the Mandelbrot set, with some of its period bulbs labeled. Let

$$Q_c(z) = z^2 + c;$$

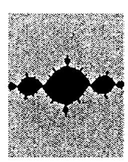
let K_c be the filled Julia set of Q_c; and let J_c be the Julia set of Q_c. Answer the following five questions, 2 marks each:

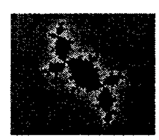


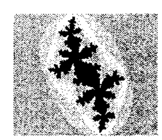
(a) In general, what is the connection between K_c and J_c?

(b) If c is in the period 2 bulb what is the eventual fate of the orbit of 0 under Q_c?

(c) If c is in a period 3 bulb, which one of the following could be K_c?







(d) If $c = -1.8 + 1.8i$, is K_c connected or totally disconnected?

(e) Above the Mandelbrot set, and properly aligned with it, sketch the orbit diagram of $Q_c : \mathbb{R} \longrightarrow \mathbb{R}$ for $-2 \leq c \leq 0.25$.