Unit 3 Part 2: Derivations Exercises Derivations with AND, OR and BICONDITIONAL

SECTION 1: use only MP, MT, R, DN, ADD, S, ADJ, MTP, BC and CB

1. $(P \land Q) \land (R \land S)$. $\therefore (P \land S) \land Q$

1 ;	$\overline{\text{Show}} (P \wedge S) \wedge Q$		show conc
2	$P \wedge Q$	pr1 S	or SL - must use S on the main '^'
3	$R \wedge S$	pr1 S	or SR
4	P	2 S	
5	Q	2 S	
6	S	3 S	
7	$P \wedge S$	4 6 ADJ	must adjoin P ∧ S first
8	$(P \wedge S) \wedge Q$	5 7 ADJ	
9		8 dd	

3. $P \leftrightarrow Q$. $Q \leftrightarrow R$. $\therefore R \leftrightarrow P$

1	Show	$R \leftrightarrow P$		show conc
2		$\overline{\text{Show}} \text{ R} \to \text{P}$		show one direction of the \leftrightarrow
3		R	ass cd	show line 2 is \rightarrow Ass ant., show cons
4		$R \rightarrow Q$	pr2 BC	
5		Q	3 4 mp	
6		$Q \rightarrow P$	pr1 BC	
7		P	5 6 mp, cd	now 2 is shown and available to use
8		$\overline{\text{Show P} \to R}$		show the other direction of the \leftrightarrow
9		P	ass cd	show line 8 is \rightarrow . Ass ant, show cons
10		$P \rightarrow Q$	pr1 BC	
11		Q	9 10 mp	
12		$Q \rightarrow R$	pr2 BC	
13		R	11 12 mp, cd	now 8 is shown and available to use
14		$R \leftrightarrow P$	2 8 CB dd	the derivation is direct (but used two conditional subderivations).

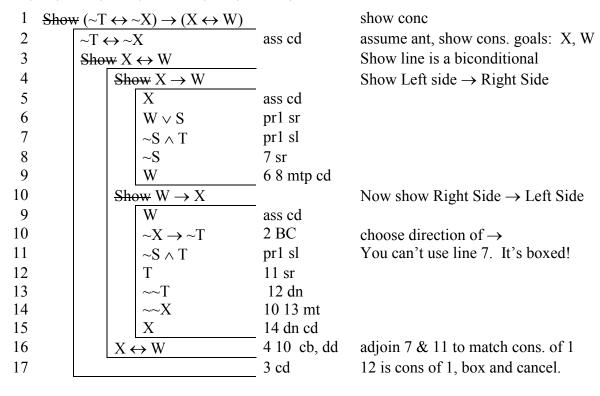
4. $R \lor \sim S$. $S \land \sim T$. $W \leftrightarrow T$. $\therefore R \land \sim W$

1 Show $R \land \sim W$	show conc: goal R, ~W
2 S pr2 s	
$\sim S$ 2 dn	
4 R 3 pr1 n	ntp
5 \sim T pr2 s	
6 $W \rightarrow T$ pr3 bc	
7 \sim W 5 6 mt	
8 $R \land \sim W$ 4 7 adj	, dd

5. $W \vee P$. $P \vee S \rightarrow Q$. $X \leftrightarrow \sim Q$. $\therefore \sim W \rightarrow \sim X$

		_	_	
1	Shov	$*$ \sim $W \rightarrow Q$		show conc
2		~W	ass cd	goal: Q
3		P	2 pr1 mtp	
4		$P \vee S$	3 add	Disjoin 'P' to 'S' so it matches ant. of pr2.
5		Q	4 pr2 mp	
6		$X \rightarrow \sim Q$	pr3 bc	
7		~~Q	5 dn	
8		~X	7 6 mt	
9			_ 8 cd	

6. $(\sim S \wedge T) \wedge (W \vee S)$. $\therefore (\sim T \leftrightarrow \sim X) \rightarrow (X \leftrightarrow W)$



7. $P \leftrightarrow (Q \lor R)$. $R \rightarrow S$. $\sim S \land P$. $W \lor Q \rightarrow R$. $\therefore T$

1	Shov	∀ T		show conc
2		~T	ass id	premises must be inconsistent, since no T in them.
3		~S	pr3 s	
4		~R	pr2 3 mt	
5		\sim (W \vee Q)	pr4 4 mt	
6		P	pr3 s	
7		$P \rightarrow (Q \vee R)$	pr1 bc	
8		$Q \vee R$	6 7 mp	
9		Q	4 8 mtp	
10		$W \vee Q$	9 add	
11			5 10 id	

8.
$$P \lor (Q \lor R)$$
. $Q \leftrightarrow (X \land Z)$. $(W \to Z) \to S$ $\therefore \sim P \to (\sim R \to S)$

9.
$$W \leftrightarrow Q$$
. $(\sim Q \lor S) \land W$. $S \rightarrow T \land U$. $\therefore U \lor \sim T$

	(, ~) , , , , , , , , , , , , , , , , ,		_
1 Shov	¥ U∨~T		show conc
2	W	pr2 s	
3	$W \to Q$	pr1 bc	
4	Q	2 3 mp	
5	~~Q	4 dn	
6	\sim Q \vee S	pr2 s	
7	S	5 6 mtp	
8	$T \wedge U$	pr3 7 mp	
9	U	8 s	
10	$U \lor \sim T$	9 add, dd	

NOTE: out of order to save space.

12.
$$\therefore$$
 $(T \land S) \rightarrow (((T \lor W) \rightarrow \sim S) \rightarrow \sim W)$

2 .	Triangle T	\rightarrow (((T	$\vee W) \rightarrow \sim S) \rightarrow \sim W)$		
	1 Sho	w (T ∧ 🤄	$(((T \vee W) \rightarrow (T \vee W) \rightarrow (T \vee W)))$		show conc
	2	$T \wedge S$		ass id	goal: consequent of 1
	3	Show	$((T \vee W) \to \sim S) \to \sim W$		show consequent of 1
	4	Т	$\vee W \rightarrow \sim S$	ass id	
	5	S	how ~W		goal: consequent of 3
	6		W	ass id	
	7		$T \vee W$	6 add	
	8		~S	7 4 mp	
	9		S	2 s, 8 id	
	10			5 cd	
	11			3 cd	
			<u> </u>		

10. $(\sim P \vee R) \wedge (\sim Q \rightarrow \sim R)$. $Q \leftrightarrow (S \wedge W)$. $S \vee R \rightarrow T$. $T \wedge W \rightarrow P$. $\therefore P \leftrightarrow Q$

Show $P \leftrightarrow Q$ 2 Show $P \rightarrow Q$ 3 Р ass cd $\sim P \vee R$ 4 pr1 s 5 ~~P 3 dn 6 R 4 5 mtp 7 pr1 s $\sim Q \rightarrow \sim R$ 8 ~~R 6 dn 9 ~~Q 7 8 mt 10 Q 9 dn, cd $\textcolor{red}{\textbf{Show}}\: \mathsf{Q} \to \mathsf{P}$ 11 12 Q ass cd 13 $Q \rightarrow (S \wedge W)$ pr2 bc 14 $S \wedge W$ 12 13 mp 15 S 14 s 15 add 16 $S \vee R$ 17 Τ 16 pr3 mp W 18 14 s 19 $\mathsf{T} \wedge \mathsf{W}$ 17 18 adj Ρ 20 19 pr4 mp, cd 21 $P \leftrightarrow Q$ 2 11 cb, dd

show conc

show one direction of \leftrightarrow

Show line 2 is \rightarrow assume antecedent.

show other direction of \leftrightarrow show line 11 is \rightarrow assume antecedent choose direction of \rightarrow to work with 12

11. $\sim (P \land Q)$. $\sim P \rightarrow T$. $\sim T \rightarrow Q$. $\therefore T$

1	Shov	Show T						
2		~T		ass id				
3		Q		2 pr3 mp				
4		~~P		2 pr2 mt				
5		Р		4 dn				
6		$P \wedge Q$		3 5 adj				
7		~(P ∧ Q)		pr1, 6 id				

show conc assume the opposite of show line 1. Goal: contradiction

13. $P \lor Q \to R$. $\sim T \land (P \to S)$. $S \leftrightarrow \sim R$. $\sim P \to T$. $\therefore W$

1	Show -W	
2	~W	ass id
3	~T	pr2 s
4	~~P	3 pr4 mt
5	P	4 dn
6	$P \rightarrow S$	pr2 s
7	S	5 6 mp
8	$P \vee Q$	5 add
9	R	8 pr1 mp
10	$S \rightarrow \sim R$	pr3 bc
11	~~R	9 dn
12	~S	10 11 mt, 7 id

14. $\sim (P \rightarrow Q)$. $P \leftrightarrow R$. $Q \lor S$. $\therefore R \land S$

1	Sho	₩ R	\wedge S		
2		She	w P		_
3			~P		ass id
4			Sho	$P \rightarrow Q$	
5				P	ass cd
6				~P	3 r
7					5 6 id
8			~(P	$\rightarrow Q$)	pr1, 4 id
9		She	₩ ~(Q	_
10			Q		ass id
11			Sho	$P \rightarrow Q$	_
12				P	ass cd
13				Q	10 r
14			~(P	$\rightarrow Q$)	
15		S			9 pr3 mtp
16		P –	→ R		pr2 bc
17		R			2 16 mp
18		R ^	S		15 17 adj, dd

show conc

You need P, and can get it from Pr1.

5 and 6 contradict each other, completing the requirements for ID. Box and cancel.

You need ~Q, and can get it from Pr1

15. \sim (P \wedge R). S \to R. \sim P \to \sim S. \therefore T $\vee \sim$ S

1	Show $T \vee \sim S$	
2	~(T ∨ ~S)	ass id
3	Show S	
4	~S	ass id
5	T ∨ ~S	4 add
6	~(T ∨ ~S)	2 r, 5 id
7	R	3 pr2 mp
8	~~S ~~P	3 dn
9	~~P	8 pr3 mt
10	P	9 dn
11	$P \wedge R$	7 10 adj
12	\sim (P \wedge R)	pr1, 11 id

show conc

Can't show conc. directly, so use id. 2 is a negated v: gives you either disjunct, unnegated. Here, S would be useful. You need it, can show it, so show it!

18. \therefore \sim S \vee Q \rightarrow ((Q \rightarrow \sim (P \rightarrow T)) \rightarrow \sim (T \wedge S))

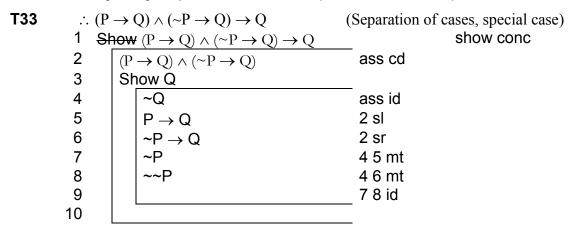
1	Sho	$PW \sim S \vee Q \rightarrow ((Q \rightarrow \sim (P \rightarrow T)) \rightarrow \sim (T \land S))$	
2		~S∨Q	ass cd
3		Show $(Q \rightarrow \sim (P \rightarrow T)) \rightarrow \sim (T \land S)$	
4		$Q \rightarrow \sim (P \rightarrow T)$	ass cd
5		Show \sim (T \wedge S)	
6		$T \wedge S$	ass id
7		S	6 s
8		~~S	7 dn
9		Q	2 8 mtp
10		$\sim (P \to T)$	4 9 mp
11		$\frac{\text{Show P}}{\text{P}} \to \text{T}$	
12		P	ass cd
13		T	6 s, cd
14			11 10 id
15			5 cd
16			3 cd

show conc 1 is \rightarrow , assume ant. show consequent of 1 3 is \rightarrow , assume ant show consequent of 3

19. $\therefore \sim (P \to Q) \to (\sim (R \lor S) \to \sim (S \lor \sim P))$

1	Shov	₩ ~	(P –	→ Q)	$\rightarrow (\sim (R \vee S) \rightarrow \sim (S \vee \sim P))$			show conc
2	,	$\sim (P \rightarrow Q)$			ass o	ed	1 is \rightarrow , assume ant.	
3		Show $\sim (R \vee S) \rightarrow \sim (S \vee \sim P)$					show consequent of 1	
4		$\sim (R \vee S)$			ass c	d	3 is \rightarrow , assume ant	
5		Show $\sim (S \vee \sim P)$					show consequent of 3	
6				S v	~P	ass ic	d	
7				Sho	v w ∼S			
8					S	ass ic	d	
9					$R \vee S$	8 ado	d	
10					\sim (R \vee S)	4 r, 9	id id	
11				~P		6 7 n	ntp	
12				Sho	$P \rightarrow Q$			
13					P	ass c	d	
14					~P	11 r,	13 id	
15				~(P	\rightarrow Q)	2 r, 1	2 id	
16						5 cd		
17						3 cd		

T33 and T49 justify Separation of Cases (Derived rule: SC)

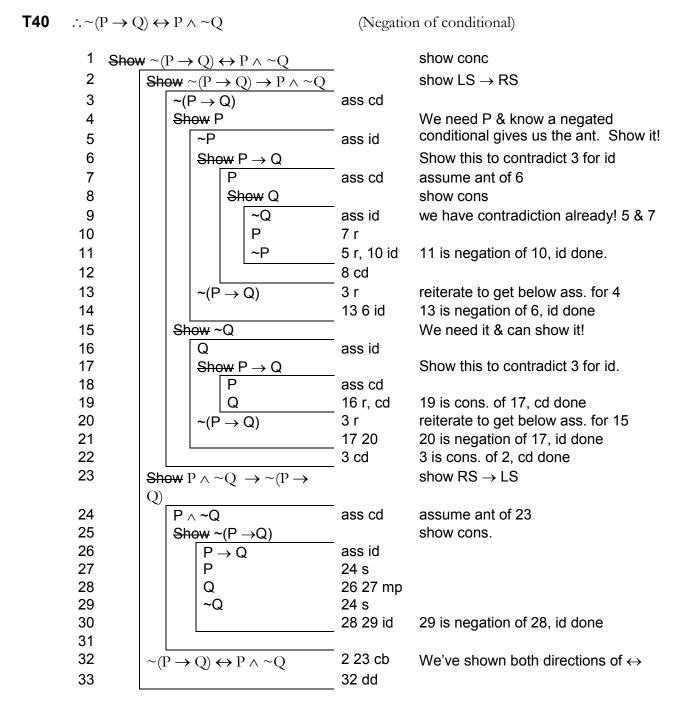


T49 $\therefore (P \lor Q) \land (P \to R) \land (Q \to R) \to R$ (Separation of cases) Show $(P \lor Q) \land (P \to R) \land (Q \to R) \to R$ show conc 2 $(P \lor Q) \land (P \to R) \land (Q \to R)$ ass cd ass ant of 1 3 Show R show cons. ~R 4 ass id $(P \lor Q) \land (P \to R)$ The \wedge on the right is the 5 2 sl main connective in 2 sr 6 $Q \rightarrow R$ informal notation. 7 $P \vee Q$ 4 sl 8 4 sr $P \rightarrow R$ ~P 9 4 8 mt 10 Q 7 9 mtp 11 R 6 10 mp 12 4 11 id 4 is neg. of 11, id done 3 is cons. of 2, cd done

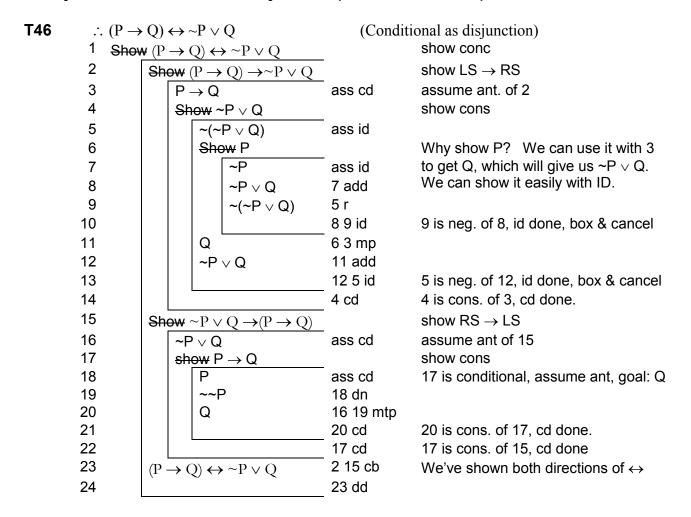
3 cd

13

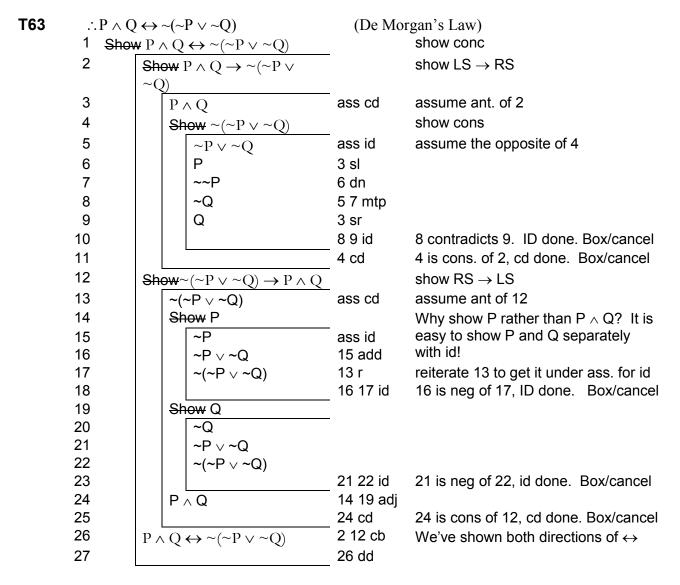
T40 justifies Negation of Conditional (Derived rule: NC)



T46 justifies Conditional as Disjunction (Derived rule: CDJ)

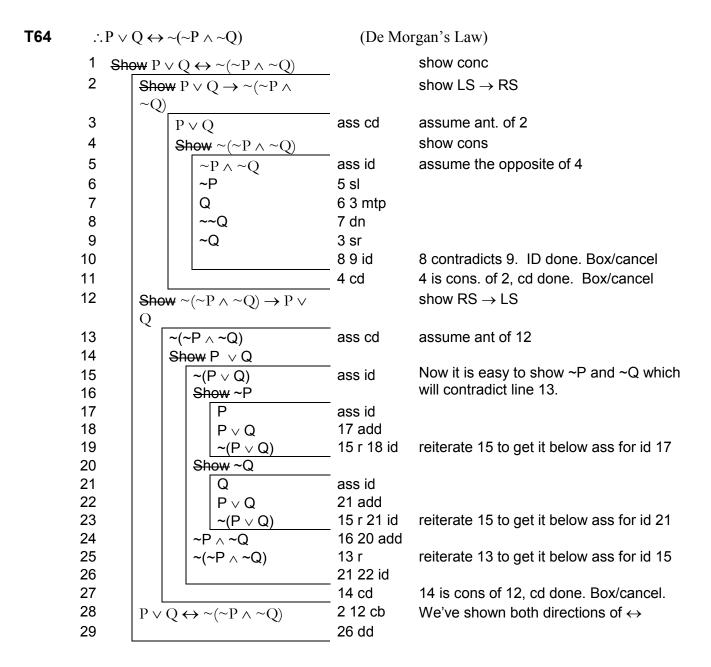


T63, T64, T65, T66 together justify De Morgan's Law (derived rule: DM)



Now it should be easy to prove T66:

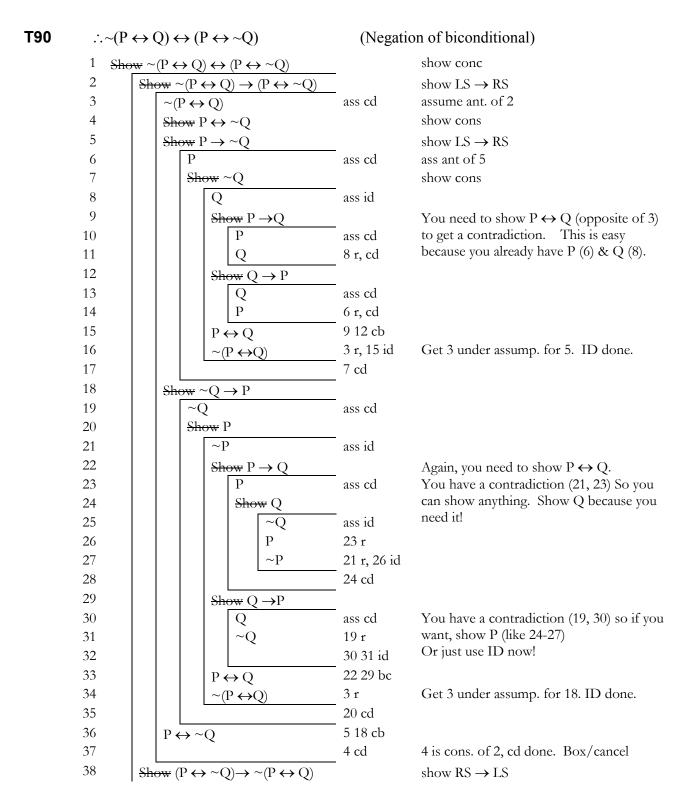
T66
$$\therefore \sim (P \vee Q) \leftrightarrow \sim P \wedge \sim Q$$
 (De Morgan's Law)



Now it should be easy to prove T65.

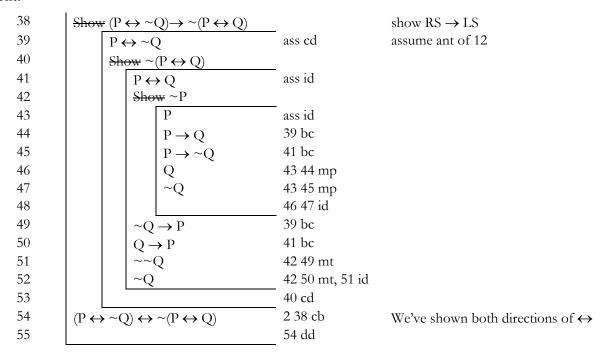
T65
$$\therefore \sim (P \land Q) \leftrightarrow \sim P \lor \sim Q$$
 (De Morgan's Law)

T90 justifies Negation of Biconditional (Derived rule: NB)



CONTINUED ON NEXT PAGE (You would think there would be a faster way to prove this theorem! Good thing for that derived rule!)

T90 cont.



SECTION 3: REMEMBER, YOU CAN'T USE THE DERIVED RULES (DM, NC, ADJ, SC AND NB) UNTIL AFTER YOU HAVE PROVEN THE RELATED THEOREMS!

$21. \quad S \to W. \quad T \vee (\sim\!\! S \to R) \, :: \, (W \vee T) \vee R$

1	Show $(W \lor T) \lor R$	
2	$\sim ((W \lor T) \lor R)$	ASS ID
3	\sim (W \vee T) \wedge \sim R	2 DM
4	~R	3 S
5	\sim (W \vee T)	3 S
6	\sim W $\wedge \sim$ T	5 DM
7	~W	6 S
8	~T	6 S
9	~S	PR1 7 MT
10	$\sim S \rightarrow R$ $\sim \sim S$	8 PR2 MTP
11	~~S	4 10 MT, 9 ID

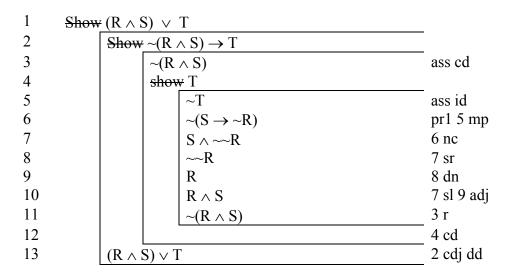
$22. \ \ \, \sim (\sim\!\! P \to Q). \ \ \, \sim (R \vee S \leftrightarrow Q). \quad R \to \sim\!\! T. \quad \sim \!\! S \vee \sim \!\! T. \quad \therefore \sim \!\! (T \vee P)$

1 Show	$F \sim (T \vee P)$	
2	$\sim P \land \sim Q$	PR1 NC
3	~P	2 S
4	~Q	2 S
5	$R \vee S \leftrightarrow \sim Q$	2 NB
6	$\sim Q \rightarrow R \vee S$	5 BC
7	$R \vee S$	4 6 MP
8	$S \rightarrow \sim T$	PR4 CDJ
9	~T	7 PR3 8 SC
10	\sim T $\wedge \sim$ P	3 9 ADJ
11	\sim (T \vee P)	_ 10 DM DD

$24. \quad P\vee Q. \quad {\sim}(P\wedge Q). \quad P\vee {\sim}Q. \quad \therefore {\sim}({\sim}P\vee Q)$

1	Show $\sim (\sim P \vee Q)$	
2	~P ∨ Q	ASS ID
3	$P \rightarrow Q$	2 CDJ
4	$\sim P \rightarrow Q$	PR1 CDJ
5	Q	3 4 SC
6	~~Q	5 DN
7	P	6 PR3 MTP
8	~P ∨ ~Q	PR2 DM
9	~P	6 8 MTP, 7 ID

25. $\sim T \rightarrow \sim (S \rightarrow \sim R)$:: $(R \land S) \lor T$



26.
$$(R \leftrightarrow S) \lor Q$$
. $\sim (Q \rightarrow \sim R) \rightarrow S$. $\therefore R \rightarrow S$

1	Show	2 R \rightarrow S	
2		R	ass cd
3		Show S	
4		~S	ass id
5		$Q \rightarrow \sim R$	4 pr2 mt dn
6		~Q	2 dn 5 mt
7		$R \leftrightarrow S$	pr1 6 mtp
8		S	7 bc 2 mp, 4 id
9			3 cd

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27. \therefore \sim(R \vee S) \wedge \sim(\simP \rightarrow Q) \leftrightarrow \sim((P \vee Q) \vee (R \vee S))
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1
          Show \sim (R \vee S) \wedge \sim (\sim P \rightarrow Q) \leftrightarrow \sim ((P \vee Q) \vee (R \vee S))
2
                     Show \sim (R \vee S) \wedge \sim (\sim P \rightarrow Q) \rightarrow \sim ((P \vee Q) \vee (R \vee S))
3
                               \sim (R \vee S) \wedge \sim (\sim P \rightarrow Q)
                                                                                                                   ass cd
4
                                                                                                                   3 sl
                               \sim (R \vee S)
5
                               \sim (\sim P \rightarrow Q)
                                                                                                                   3 sr
6
                               {\sim}P \wedge {\sim}Q
                                                                                                                   5 nc
7
                               \sim (P \vee Q)
                                                                                                                   6 dm
8
                                                                                                                   4 7 adj
                               \sim (P \vee Q) \wedge \sim (R \vee S)
9
                               \sim ((P \vee Q) \vee (R \vee S))
                                                                                                                   8 dm cd
10
                     Show \sim ((P \lor Q) \lor (R \lor S)) \rightarrow \sim (R \lor S) \land \sim (\sim P \rightarrow Q)
11
                               \sim ((P \lor Q) \lor (R \lor S))
                                                                                                                   ass cd
12
                               \sim (P \vee Q) \wedge \sim (R \vee S)
                                                                                                                    11 dm
13
                                                                                                                    12 sl
                               \sim (P \vee Q)
14
                                                                                                                    13 dm
                               \sim P \land \sim Q
15
                               \sim (\sim P \rightarrow Q)
                                                                                                                    14 nc
16
                               \sim (R \vee S)
                                                                                                                    12 sr
17
                                \sim (R \vee S) \wedge \sim (\sim P \rightarrow Q)
                                                                                                                    16 15 adj
18
                     \sim(R \vee S) \wedge \sim(\simP \rightarrow Q) \leftrightarrow \sim((P \vee Q) \vee (R \vee S))
                                                                                                                   2 10 cb dd
         \sim R \vee W. X \wedge S \rightarrow T. R \vee W. \sim W \vee X. \therefore S \rightarrow T \vee P
30.
1
          \overline{Show} \ S \to T \vee P
                     S
2
                                                                                                                   ass cd
3
                     R \rightarrow W
                                                                                                                   pr1 cdj
4
                     \sim R \rightarrow W
                                                                                                                   pr3 cdj
5
                     W
                                                                                                                   3 4 sc
6
                     {\sim}\!\!\sim\!\! W
                                                                                                                   5 dn
7
                     X
                                                                                                                   pr4 6 mtp
8
                     X \wedge S
                                                                                                                   2 7 adj
9
                     Τ
                                                                                                                   8 pr2 mp
10
                                                                                                                   9 add cd
                     T \vee P
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32. $\sim (P \vee Q)$. $(R \to S) \to \sim (P \to T)$. $(S \vee Q) \vee W$. $\therefore R \vee W$

$\overline{\text{Show}} \; R \vee W$ 1 2 $\sim (R \vee W)$ ass id $\sim R \land \sim W$ $2 \ dm$ 3 ~P ^ ~Q 4 pr1 dm 5 \sim W 3 s 6 $\mathbf{S} \vee \mathbf{Q}$ pr3 mtp 7 4 s ~Q S 6 7 mtp 8 9 8 add $\sim R \vee S$ $R \rightarrow S$ 9 cdj 10 11 $\sim (P \rightarrow T)$ 10 pr2 mp $P \wedge {\sim} T$ 11 nc 12 P 13 12 s 14 \sim P 4 s 13 id

34. $(S \rightarrow (\sim P \lor T)) \rightarrow W$. $(R \leftrightarrow W) \land \sim R$. $\sim (S \rightarrow T) \rightarrow Q$. $\therefore P \land Q$

1 Shov	$\forall P \land Q$	
2	~R	ass id
3	$R \leftrightarrow W$	2 dm
4	$W \to R$	3 bc
5	~W	2 4 mt
6	$\sim (S \to (\sim P \lor T))$	5 pr1 mt
7	$S \wedge \sim (\sim P \vee T)$	6 nc
8	S	7 s
9	$\sim P \land \sim T$	7s, dm
10	~T	9 s
11	$S \wedge \sim T$	8 10 adj
12	$\sim (S \to T)$	11 nc
13	Q	12 pr3 mp
14	P	9 sl dn
15	$P \wedge Q$	_ 13 14 adj