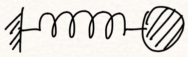
Mass-spring. Forced vibration.

$my''+\gamma y'+ky=F_0\cos(\omega t)$ $\quad \omega_0=\sqrt{\frac{k}{m}}$ characteristic frequency.

If $F_0=0$, $\gamma=0$ then get oscillation with

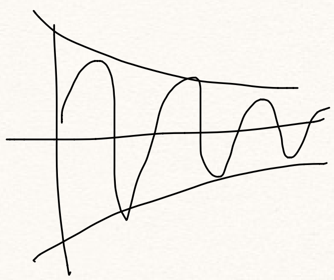
 $\to F(t)$ frequency ω_0.

In general, characteristic equation:

$mr^2+\gamma r+k=0$ has roots $r_1,r_2=-\frac{\gamma}{2m}\pm\frac{1}{2m}\sqrt{\gamma^2-4mk}=-\frac{\gamma}{2m}\pm\sqrt{\left(\frac{\gamma}{2m}\right)^2-\omega_0^2}$

If γ small, get complex conjugate roots: $r_1,r_2=-\frac{\gamma}{2m}\pm i\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}$

$\rightsquigarrow$ hom. equ. has solution: $y(t)=e^{-\frac{\gamma}{2m}t}\left(A\cos\left(\underbrace{\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}}_{\tilde{\omega}_0}t\right)+B\sin(\dots t)\right)$



Inhomogeneous equation (*) has particular solution.

$Y(t)=\frac{F_0}{\Delta}\cos(\omega t-\delta)$ $\quad \Delta=\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

$\tan(\delta)=\frac{\gamma\omega}{m(\omega_0^2-\omega^2)}$ $\quad 0\leq\delta<\pi$ $\quad \cos(\theta)=\frac{m(\omega_0^2-\omega^2)}{\Delta}$

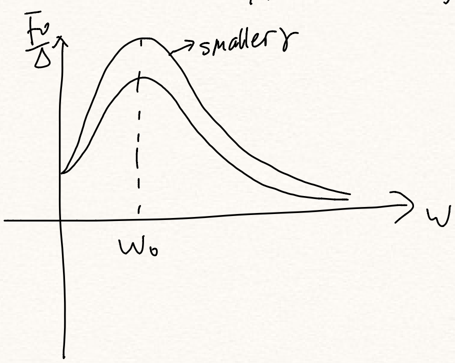
As $\omega\to 0$, $\Delta\to m\omega_0^2$, $\cos(\theta)\to 1$ hence $\theta\to 0$

As $\omega\to\omega_0$, $\Delta\to\gamma\omega_0$, $\cos(\theta)\to 0$, $\theta\to\frac{\pi}{2}$

As $\omega\to\infty$, $\Delta\approx m\omega^2$, $\cos(\theta)\to 1$, $\theta\to\pi$.

Note: If γ is small, then the amplitude $\frac{F_0}{\Delta}$ becomes very large as $\omega\to\omega_0$.

"resonant case" The maximum of $\frac{F_0}{\Delta}$ occurs roughly for $\omega\approx\omega_0$.



- The undamped case $\gamma = 0$

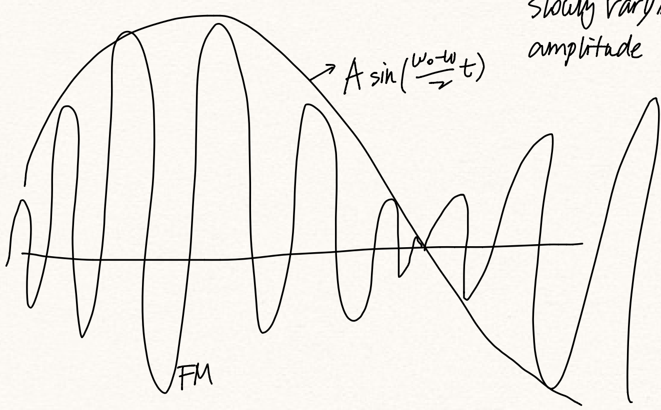
In this case, $\delta = 0$, thus $Y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$

General solution of hom. equation is $R\cos(\omega_0 t - \varphi)$

The initial value problem $my'' + ky = F_0 \cos(\omega t)$ $y(0) = 0$, $y'(0) = 0$. has solution:

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t) \right)$$

Using trig. equations, can write this as: $y(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t\right)}_{\text{slowly varying amplitude}} \underbrace{\sin\left(\frac{\omega + \omega_0}{2} t\right)}_{\text{rapid oscillation.}}$



Application: Radio, tuning fork.

4 Qs.