Please write your family and given names and **underline** your family name on the front page of your paper.

1.

(a) [*10 points*] Find the condition number of $f(x) = (a+x)^{1/4} - a^{1/4}$, for $x > 0$, $a > 0$ and study whether there are ranges of $x \in \mathbb{R}^+$ for which the computation of f is ill-conditioned. (You may need to use de l' Hospital's rule.)

(b) [*10 points*] Consider the (numerical) stability of the computation of the expression $(a+x)^{1/4} - a^{1/4}$, for $x > 0$, $a > 0$, when x is close to 0. Explain what problems the computation of the expression may give rise to. Propose a mathematically equivalent expression that is likely to be more stable for x close to 0, and explain.

(c) [*10 points*] Set $a = 1$. Write a MATLAB script that goes through the values of x in $\{10^{-20}, 10^{-19}, \ldots, 10^{-1}, 1, 10, \cdots, 10^{19}, 10^{20}\}$, and computes and outputs the respective values of f using the original expression, as well as your proposed (more stable) expression, and the respective condition numbers (computed using the values of f with the original expression and the values of f with the proposed expression). Comment on the results. See the course webpage for a template of the script.

2. [*20 points*] Use Taylor's theorem to find a power series expansion about 0 for $\sin(\pi x/2)$. (Essentially, take the $\sin(x)$ expansion and extend it to $\sin(\pi x/2)$.) Give the expression for the remainder, and a bound for the absolute value of the remainder. From this, estimate the number of terms in the series that would be needed to guarantee 6 (significant) decimal digits correct for $\sin(\pi x/2)$, for all x in $[-1, 1]$.

Notes: Recall that an approximation $\hat{x}$ to x is said to be **correct in** r **significant** b**-digits**, if $\left| \frac{x - \hat{x}}{x} \right| \le \frac{1}{2} b^{1-r}$.

In trying to find the number n of terms in the series, you may arrive at an inequality of the form $\frac{(a)^{g(n)}}{f(n)} \le c$, where a and c are constants and $f(n)$ and $g(n)$ are functions of n. Such inequalities are often hard to solve analytically/mathematically for n. When you arrive at such inequality, you can use trial-and-error. But you should get rid of quantities such as x and sin() using analytical/mathematical techniques, before you use trial-and-error.

3. Consider the integrals

$$y_n = \int_0^1 t^n e^{-t} \, dt, \quad n = 0, 1, 2, \cdots$$

(a) [*10 points*] Derive a recurrence relation for y_n relating y_n to y_{n-1}. (You may have to use integration by parts.) Rearrange the formula so that you have a recurrence relation for y_{n-1} relating y_{n-1} to y_n. Name the first recurrence formula (A) and the second one (B).

(b) [*10 points*] With repeated applications of (A), give a formula that gives y_n as a function of y_0, i.e. $y_n = f_n(y_0)$. With repeated applications of (B), give a formula that gives y_n as a function of y_m, for $m > n$, i.e. $y_n = g_{n,m}(y_m)$.

(c) [*10 points*] Find the condition number of the functions

$$f_n(y_0) \text{ and } g_{n,m}(y_m) \text{ for } m > n.$$

(Note: Function f_n has y_0 as the variable, and function $g_{n,m}$ has y_m as the variable.)
Taking into account the condition numbers of the above two functions formulate a stable method for computing $y_0, y_1, \cdots, y_N$, where $N \ge 1$ is given.

(d) [*10 points*] Write and run a MATLAB program that computes and outputs $y_0, y_1, \cdots, y_N$, starting with y_0 and using recursion (A). Explain what happens! (A reasonable N to stop is $N = 20$.)

(e) [*10 points*] Write a MATLAB program that sets y_{N+K} to some appropriate value (which may be approximate), and computes and outputs $y_{N+K-1}, y_{N+K-2}, \cdots, y_N$, starting with y_{N+K} and using recursion (B). Run the program for $K = 3, \cdots, 9$ and $N = 20$ (7 cases). Explain what happens! Comment on how one should compute y_{20}. Also output the error for y_{20}, assuming the exact value is `q = 0.01835046769725620632 6;` in 40 decimal digits precision. Note that, in this case, the code should have a nested loop.

Notes: You should not use any symbolic environment. Use an appropriate format, e.g. `fprintf('%3d %20.16f\n', i-1, y(i));` and `fprintf('%3d %20.16f %20.16f %10.6e\n', 20, y(21), q, q-y(21));`.

For all programming parts of the assignment, submit a hard-copy of your code and results, as well as any hand-written or typed comments on (or explanations of) the results.