CSC336 Tutorial 4 – GE/LU, pivoting, scaling

QUESTION 1 *Let A and b be given by*

$$A = \begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix}, b = \begin{pmatrix} -8 \\ 7 \\ -2 \end{pmatrix}$$

Apply GE without pivoting to A and obtain the L and U factors, such that $A = LU$. Indicate the results of each step of GE. Using L and U, and back and forward substitutions, obtain the solution to $Ax = b$. Indicate any intermediate vector arising, as well as the solution vector x computed.

ANSWER:

$$A = \begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \left(\begin{array}{c|cc} 2 & 3 & -6 \\ \frac{1}{2} & -6-\frac{3}{2} & 8+\frac{6}{2} \\ \frac{3}{2} & -2-\frac{3\cdot 3}{2} & 1+\frac{3\cdot 6}{2} \end{array}\right) = \left(\begin{array}{c|cc} 2 & 3 & -6 \\ \frac{1}{2} & -\frac{15}{2} & 11 \\ \frac{3}{2} & -\frac{13}{2} & 10 \end{array}\right)$$

$$\xrightarrow[k=2]{\text{elim}} \left(\begin{array}{cc|c} 2 & 3 & -6 \\ \frac{1}{2} & -\frac{15}{2} & 11 \\ \frac{3}{2} & \frac{13}{15} & 10-\frac{13}{15}\cdot 11 \end{array}\right) = \left(\begin{array}{cc|c} 2 & 3 & -6 \\ \frac{1}{2} & -\frac{15}{2} & 11 \\ \frac{3}{2} & \frac{13}{15} & \frac{7}{15} \end{array}\right), L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & \frac{13}{15} & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 3 & -6 \\ 0 & -\frac{15}{2} & 11 \\ 0 & 0 & \frac{7}{15} \end{pmatrix}$$

Note: The relation $A = L \cdot U$ can be verified (check it yourself!)

Use the L and U matrices to solve $Ax = b$, for $b = (-8, 7, -2)^T$:

$$Ax = b \Leftrightarrow LUx = b \Leftrightarrow \left\{ \begin{array}{rcl} Ly & = & b \\ Ux & = & y \end{array} \right\}$$

Solve $Ly = b$ (forward substitution)

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & \frac{13}{15} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -8 \\ 7 \\ -2 \end{pmatrix} \Rightarrow \left\{ \begin{array}{rcl} y_1 & = & -8 \\ y_2 & = & 7 - \frac{1}{2}\cdot(-8) = 11 \\ y_3 & = & -2 - \frac{3}{2}\cdot(-8) - \frac{13}{15}\cdot 11 = \frac{7}{15} \end{array} \right.$$

Solve $Ux = y$ (back substitution)

$$\begin{pmatrix} 2 & 3 & -6 \\ 0 & -\frac{15}{2} & 11 \\ 0 & 0 & \frac{7}{15} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -8 \\ 11 \\ \frac{7}{15} \end{pmatrix} \Rightarrow \left\{ \begin{array}{rcl} x_3 & = & 1 \\ x_2 & = & (11 - 11\cdot 1)/-\frac{15}{2} = 0 \\ x_1 & = & (-8 + 6\cdot 1 - 3\cdot 0)/2 = -1 \end{array} \right.$$

Note: The relation $Ax = b$ can be verified (check it yourself!)

QUESTION 2 *Do the same as in Question 1 with partial (row) pivoting. Furthermore, indicate the pivotal vector $ipiv$ at each step of GE, the elementary permutation matrix P_k associated with the kth step of GE, and the total permutation matrix P that reflects all the row interchanges.*

ANSWER: Initial pivotal vector $ipiv = (\cdot, \cdot)$

$$\begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix} \xrightarrow[k=1]{\text{piv}} \begin{pmatrix} 3 & -2 & 1 \\ 1 & -6 & 8 \\ 2 & 3 & -6 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, ipiv = (3, \cdot),$$

$$\xrightarrow[k=1]{\text{elim}} \left(\begin{array}{c|cc} 3 & -2 & 1 \\ \frac{1}{3} & -6+\frac{2}{3} & 8-\frac{1}{3} \\ \frac{2}{3} & 3+\frac{2\cdot 2}{3} & -6-\frac{2}{3} \end{array}\right) = \left(\begin{array}{c|cc} 3 & -2 & 1 \\ \frac{1}{3} & -\frac{16}{3} & \frac{23}{3} \\ \frac{2}{3} & \frac{13}{3} & -\frac{20}{3} \end{array}\right) \xrightarrow[k=2]{\text{piv}} (\text{no changes}), P_2 = \mathbf{I}, ipiv = (3, 2),$$

$$\xrightarrow[k=2]{\text{elim}} \left(\begin{array}{cc|c} 3 & -2 & 1 \\ \frac{1}{3} & -\frac{16}{3} & \frac{23}{3} \\ \frac{2}{3} & -\frac{13}{16} & -\frac{20}{3}+\frac{13\cdot 23}{16\cdot 3} \end{array}\right) = \left(\begin{array}{cc|c} 3 & -2 & 1 \\ \frac{1}{3} & -\frac{16}{3} & \frac{23}{3} \\ \frac{2}{3} & -\frac{13}{16} & -\frac{7}{16} \end{array}\right), L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & -\frac{13}{16} & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & -2 & 1 \\ 0 & -\frac{16}{3} & \frac{23}{3} \\ 0 & 0 & -\frac{7}{16} \end{pmatrix}$$

Note: $PA = LU$ (check it!) where $P = P_2 \cdot P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (permutation matrix)

Use the L, U and P matrices to solve $Ax = b$, for $b = (-8, 7, -2)^T$:

$$Ax = b \Rightarrow PAx = Pb \Rightarrow LUx = Pb \Rightarrow \left\{ \begin{array}{rcl} Ly & = & Pb \\ Ux & = & y \end{array} \right\}, \text{ where } Pb = \begin{pmatrix} -2 \\ 7 \\ -8 \end{pmatrix}$$

Solve $Ly = Pb$ (forward substitution)

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & -\frac{13}{16} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -8 \end{pmatrix} \Rightarrow \left\{ \begin{array}{rcl} y_1 & = & -2 \\ y_2 & = & 7 + 2\cdot\frac{1}{3} = \frac{23}{3} \\ y_3 & = & -8 + 2\cdot\frac{2}{3} + \frac{23\cdot 13}{3\cdot 16} = -\frac{7}{16} \end{array} \right.$$

Solve $Ux = y$ (back substitution)

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & -\frac{16}{3} & \frac{23}{3} \\ 0 & 0 & -\frac{7}{16} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{23}{3} \\ -\frac{7}{16} \end{pmatrix} \Rightarrow \left\{ \begin{array}{rcl} x_3 & = & 1 \\ x_2 & = & (\frac{23}{3} - \frac{23}{3}\cdot 1)/(-\frac{16}{3}) = 0 \\ x_1 & = & (-2 - 1\cdot 1 - 2\cdot 0)/3 = -1 \end{array} \right.$$

In practice, we do not store the permutation matrix P, but the pivotal vector $ipiv$, and from that we can find the permuted vector Pb.

QUESTION 3 *Do the same as in Question 2 with scaled partial (row) pivoting. In addition, give the form of the factorization of A.*

ANSWER: Initial pivotal vector $ipiv = (\cdot, \cdot)$

$$\begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix} \xrightarrow{\text{scale}} \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -1 \\ \frac{1}{8} & -\frac{3}{4} & 1 \\ 1 & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}, D = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \xrightarrow[k=1]{\text{piv}} \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{8} & -\frac{3}{4} & 1 \\ \frac{1}{3} & \frac{1}{2} & -1 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[k=1]{\text{elim}} \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{8} & -\frac{3}{4} + \frac{2}{3\cdot 8} & 1 - \frac{1}{3\cdot 8} \\ \frac{1}{3} & \frac{1}{2} + \frac{2}{3\cdot 3} & -1 - \frac{1}{3\cdot 3} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{8} & -\frac{2}{3} & \frac{23}{24} \\ \frac{1}{3} & \frac{13}{18} & -\frac{10}{9} \end{pmatrix} \xrightarrow[k=2]{\text{piv}} \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{13}{18} & -\frac{10}{9} \\ \frac{1}{8} & -\frac{2}{3} & \frac{23}{24} \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\xrightarrow[k=2]{\text{elim}} \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{13}{18} & -\frac{10}{9} \\ \frac{1}{3} & -\frac{12}{13} & \frac{23}{24} - \frac{12\cdot 10}{13\cdot 9} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{13}{18} & -\frac{10}{9} \\ \frac{1}{8} & -\frac{12}{13} & -\frac{7}{104} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{8} & -\frac{12}{13} & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & \frac{13}{18} & -\frac{10}{9} \\ 0 & 0 & -\frac{7}{104} \end{pmatrix}$$

Note: $PDA = LU$ (check it!) where $P = P_2 \cdot P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (permutation matrix)

and D the diagonal matrix given above, representing the scaling of A.

Use the L, U, P and D matrices to solve $Ax = b$, for $b = (-8, 7, -2)^T$:

$$Ax = b \Rightarrow PDAx = PDb \Rightarrow LUx = PDb \Rightarrow \left\{ \begin{array}{l} Ly = PDb \\ Ux = y \end{array} \right\}$$

We have $PDb = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -8 \\ 7 \\ -2 \end{pmatrix} = P \begin{pmatrix} -\frac{8}{6} \\ \frac{7}{8} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ \frac{7}{8} \end{pmatrix}$.

Solve $Ly = PDb$ (forward substitution):

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{8} & -\frac{12}{13} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ \frac{7}{8} \end{pmatrix} \Rightarrow \left\{ \begin{array}{lcl} y_1 &=& -\frac{2}{3} \\ y_2 &=& -\frac{4}{3} + \frac{1\cdot 2}{3\cdot 3} = -\frac{10}{9} \\ y_3 &=& \frac{7}{8} + \frac{1\cdot 2}{8\cdot 3} - \frac{12\cdot 10}{13\cdot 9} = -\frac{7}{104} \end{array} \right.$$

Solve $Ux = y$ (back substitution):

$$\begin{pmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & \frac{13}{18} & -\frac{10}{9} \\ 0 & 0 & -\frac{7}{104} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{10}{9} \\ -\frac{7}{104} \end{pmatrix} \Rightarrow \left\{ \begin{array}{lcl} x_3 &=& 1 \\ x_2 &=& (-\frac{10}{9} + \frac{10}{9}\cdot 1)/\frac{13}{18} = 0 \\ x_1 &=& (-\frac{2}{3} - \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 0)/1 = -1 \end{array} \right.$$

General note: We obtained the same solution with all three methods, because we used exact (fractional) arithmetic. If we were doing calculations in finite arithmetic, we would not necessarily obtain exactly the same results. In general, among scaled partial pivoting, partial pivoting, and no pivoting, in most (but not necessarily all) cases, the scaled partial pivoting is expected to be the most accurate, followed by partial pivoting.

QUESTION 4 *Assume A is symmetric. Show that the diagonal entries of the D matrix of its LDL^T factorization are positive, iff A is positive definite (p.d.).*

PROOF: Let $A = LDL^T$, with L unit l.t. and $D = diag\{d_{11}, d_{22}, \ldots, d_{nn}\}$.

$\Rightarrow$: Assume $d_{ii} > 0, i = 1, \ldots, n$. Define $D^{1/2} \equiv diag\{d_{11}^{1/2}, d_{22}^{1/2}, \ldots, d_{nn}^{1/2}\}$. Then,

$A = LDL^T = LD^{1/2}D^{1/2}L^T = LD^{1/2}(D^{1/2})^T L^T = LD^{1/2}(LD^{1/2})^T$,

using the fact that the transpose of a diagonal matrix is itself, and the property $(AB)^T = B^T A^T$.

Now define $C \equiv LD^{1/2}$. Then, $A = CC^T$. Since D diagonal with $d_{ii} > 0, i = 1, \ldots, n$, and L unit l.t., we also have that C is l.t. with $c_{ii} = d_{ii}^{1/2} > 0, i = 1, \ldots, n$, i.e. C is non-singular, and so is C^T.

Thus $C^T x = \bar{0}$ has only the trivial solution, and $C^T x \neq \bar{0}$, for any $x \neq \bar{0}$.

Thus, for any $x \neq \bar{0}$, $x^T A x = x^T C C^T x = (C^T x)^T (C^T x) = y^T y$, where $y \neq \bar{0}$. Therefore, $x^T A x = y^T y > 0$, for any $x \neq \bar{0}$, i.e. A is p.d. QED

$\Leftarrow$: Assume A is p.d., i.e. for any $x \neq \bar{0}$, $x^T A x > 0$. Note also that L is unit l.t., thus non-singular, and so L^T is non-singular. Therefore, $\{x \neq \bar{0} \in \mathcal{R}^n\} = \{y = L^T x, x \neq \bar{0} \in \mathcal{R}^n\}$, i.e. L^T and its inverse form one-to-one mappings from the set of non-zero vectors to itself.

Then, with $A = LDL^T$, we have $0 < x^T Ax = x^T LDL^T x = (L^T x)^T DL^T x = y^T Dy$, where $y = L^T x$ is any non-zero vector.
With $y = (0, \ldots, 0, 1, 0, \ldots, 0)^T$, where the "1" is in the ith position, we have $0 < y^T Dy = d_{ii}$. Since i can be any of the $1, 2, \ldots, n$, we have $d_{ii} > 0, i = 1, \ldots, n$. QED

QUESTION 5 *Let*

$$A = \begin{pmatrix} 15 & -1 & 0 \\ -1 & 3 & -6 \\ 0 & -6 & 15 \end{pmatrix}$$

Apply GE without pivoting to A and obtain its L and U factors such that $A = LU$. Using U, obtain its LDL^T factorization. Indicate the results of each step of GE, and the final matrices L and D. Is A positive definite? If yes, using L and D, obtain the Choleski factor C of A.

ANSWER:

$$\begin{pmatrix} 15 & -1 & 0 \\ -1 & 3 & -6 \\ 0 & -6 & 15 \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \left(\begin{array}{c|cc} 15 & -1 & 0 \\ \hline -\frac{1}{15} & 3 - \frac{1}{15} & -6 \\ 0 & -6 & 15 \end{array}\right) \xrightarrow[k=2]{\text{elim}} \left(\begin{array}{cc|c} 15 & -1 & 0 \\ \hline -\frac{1}{15} & \frac{44}{15} & -6 \\ \hline 0 & -6 \cdot \frac{15}{44} & 15 - 6 \cdot \frac{15}{44} \cdot 6 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 15 & -1 & 0 \\ \hline -\frac{1}{15} & \frac{44}{15} & -6 \\ \hline 0 & -\frac{45}{22} & \frac{30}{11} \end{array}\right)$$

Thus

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{15} & 1 & 0 \\ 0 & -\frac{45}{22} & 1 \end{pmatrix}, U = \begin{pmatrix} 15 & -1 & 0 \\ 0 & \frac{44}{15} & -6 \\ 0 & 0 & \frac{30}{11} \end{pmatrix} \text{ and } D = \begin{pmatrix} 15 & 0 & 0 \\ 0 & \frac{44}{15} & 0 \\ 0 & 0 & \frac{30}{11} \end{pmatrix}$$

Using the result of Question 4, the matrix A is positive definite, since it is symmetric and the diagonal entries of the D factor of its LDL^T decomposition are positive.
Since

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{15} & 1 & 0 \\ 0 & -\frac{45}{22} & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 15 & 0 & 0 \\ 0 & \frac{44}{15} & 0 \\ 0 & 0 & \frac{30}{11} \end{pmatrix},$$

we have

$$E = D^{1/2} = \begin{pmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{\frac{44}{15}} & 0 \\ 0 & 0 & \sqrt{\frac{30}{11}} \end{pmatrix} \text{ and } C = L \cdot E = \begin{pmatrix} \sqrt{15} & 0 & 0 \\ -\frac{1}{15}\sqrt{15} & \sqrt{\frac{44}{15}} & 0 \\ 0 & -\frac{45}{22}\sqrt{\frac{44}{15}} & \sqrt{\frac{30}{11}} \end{pmatrix}$$

Then C is the Choleski factor of A and $A = CC^T$ is the Choleski factorization.

General note: At several points, we have used the facts that
• multiplying a matrix A by a diagonal matrix D from the left is equivalent to multiplying each row of A with the respective diagonal entry of D, and
• multiplying a matrix A by a diagonal matrix D from the right is equivalent to multiplying each column of A with the respective diagonal entry of D.

QUESTION 6 *Let A and b be given by $A = \begin{pmatrix} 0.001 & 1 \\ 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Doing all operations in two-decimal-digits floating-point arithmetic with rounding, apply GE to A and obtain the L and U factors, and then f/b/s to obtain the solution to $Ax = b$. Do this three times: (a) without pivoting, (b) with partial (row) pivoting, and (c) with complete pivoting. Also compute the exact L and U factors and the exact solution using exact (fractional) arithmetic and compare the results.*

ANSWER: First compute the exact LU factorization and the exact solution.

LU fact (GE):

$$\begin{pmatrix} 0.001 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \left(\begin{array}{cc} 0.001 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{1000} & 2 - 1000 \times 1 \end{array}\right) = \left(\begin{array}{cc} 0.001 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{1000} & -998 \end{array}\right)$$

Solve $Ly = b$ (f/s):

$$\begin{pmatrix} 1 & 0 \\ 1000 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{l} y_1 = 1 \\ y_2 = 3 - 1000 \times 1 = -997 \end{array}$$

Solve $Ux = y$ (b/s):

$$\begin{pmatrix} 0.001 & 1 \\ 0 & -998 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -997 \end{pmatrix} \Rightarrow \begin{array}{l} x_2 = 997/998 \approx 0.999 \\ x_1 = (1 - 1 \times 997/998)/0.001 = 1000/998 \approx 1.002 \end{array}$$

Note that $x_1 >\sim 1 >\sim x_2$. Also note that, *in exact arithmetic*, as long as we do not run into zeros in the position of the pivots, pivoting would not change the results.

Now use two-decimal-digits floating-point arithmetic with rounding. Note that, initially, there is no representation error.

LU fact (GE):

$$\begin{pmatrix} 0.001 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \left(\begin{array}{cc} 0.001 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{1000} & 2 - 1000 \times 1 \end{array}\right) = \left(\begin{array}{cc} 0.001 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{1000} & -998 \end{array}\right) \xrightarrow[\text{2-dec-dig}]{} \left(\begin{array}{cc} 0.001 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{1000} & -1000 \end{array}\right)$$

Solve $Ly = b$ (f/s):

$$\begin{pmatrix} 1 & 0 \\ 1000 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{l} y_1 = 1 \\ y_2 = 3 - 1000 \times 1 = -997 \xrightarrow[\text{2-dec-dig}]{} -1000 \end{array}$$

Solve $Ux = y$ (b/s):

$$\begin{pmatrix} 0.001 & 1 \\ 0 & -1000 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1000 \end{pmatrix} \Rightarrow \begin{array}{l} x_2 = 1000/1000 = 1 \\ x_1 = (1 - 1 \times 1)/0.001 = 0 \end{array}$$

Note that, for the computed solution, $x_1 = 0 < x_2 \approx 1$, that is, x_1 does not have any correct digits.

LU fact (GE piv):

$$\begin{pmatrix} 0.001 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow[k=1]{\text{piv}} \begin{pmatrix} 1 & 2 \\ 0.001 & 1 \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \left(\begin{array}{cc} 1 & 2 \\ \cline{1-1} \multicolumn{1}{c|}{0.001} & 1 - 0.001 \times 2 \end{array}\right) = \left(\begin{array}{cc} 1 & 2 \\ \cline{1-1} \multicolumn{1}{c|}{0.001} & 0.998 \end{array}\right)$$

$$\xrightarrow[\text{2-dec-dig}]{} \left(\begin{array}{cc} 1 & 2 \\ \cline{1-1} \multicolumn{1}{c|}{0.001} & 1 \end{array}\right)$$

Note: $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $P = P_1$.

Solve $Ly = Pb$ (f/s):

$$\begin{pmatrix} 1 & 0 \\ 0.001 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} y_1 = 3 \\ y_2 = 1 - 0.001 \times 3 = 0.997 \xrightarrow[\text{2-dec-dig}]{} 1 \end{array}$$

Solve $Ux = y$ (b/s):

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} x_2 = 1/1 = 1 \\ x_1 = (3 - 2 \times 1)/1 = 1 \end{array}$$

Both the x_1 and x_2 errors are at the epsilon level.

LU fact (GE complete piv):

$$\begin{pmatrix} 0.001 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow[P_1, Q_1]{\text{compl.piv}} \begin{pmatrix} 2 & 1 \\ 1 & 0.001 \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \left(\begin{array}{cc} 2 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{0.5} & 0.001 - 0.5 \times 1 \end{array}\right) = \left(\begin{array}{cc} 2 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{0.5} & -0.499 \end{array}\right)$$

$$\xrightarrow[\text{2-dec-dig}]{} \left(\begin{array}{cc} 2 & 1 \\ \cline{1-1} \multicolumn{1}{c|}{0.5} & -0.5 \end{array}\right)$$

Note: $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $P = P_1$, $Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q = Q_1$, and $PAQ = LU$.

Solve $Ly = Pb$ (f/s):

$$\begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} y_1 = 3 \\ y_2 = 1 - 0.5 \times 3 = -0.5 \end{array}$$

Let $\hat{x} = Q^T x$. Solve $U\hat{x} = y$ (b/s):

$$\begin{pmatrix} 2 & 1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -0.5 \end{pmatrix} \Rightarrow \begin{array}{l} \hat{x}_2 = -0.5/(-0.5) = 1 \\ \hat{x}_1 = (3 - 1 \times 1)/2 = 1 \end{array}$$

Then, recalling that Q is orthogonal and square, we have $\hat{x} = Q^T x \Rightarrow x = Q\hat{x} \Rightarrow x_1 = \hat{x}_2 = 1, x_2 = \hat{x}_1 = 1$. Both the x_1 and x_2 errors are at the epsilon level, as with partial pivoting.

Notes:

• When Q is orthogonal and square, we have $Q^T Q = QQ^T = \mathbf{I}$ and $Q^{-1} = Q^T$. When Q is orthogonal and not square, we just have $Q^T Q = \mathbf{I}$.

• $PAQ = LU$ and $Ax = b$, imply $PAQQ^T x = Pb \Rightarrow LU\hat{x} = Pb$, where $\hat{x} = Q^T x$. Then, $Ly = Pb$ and $U\hat{x} = y$.

• Not all the details of operations are given above. For each operation on two operands, we should apply two-decimal-digits floating-point arithmetic; e.g. $fl(1 - fl(0.001 \times 3)) = fl(1 - 0.003) = fl(0.997) = 1$.

• There is no guarantee that complete pivoting *always* gives smaller error than partial pivoting.

