Chapter 2

Quantification, Implication, and Symbols

2.1 Universal quantification

Consider the following table that associates employees with properties:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Claims about individual objects can be evaluated immediately (Al is male, Flo makes 20,000). But the tabular form also allows claims about the entire database to be considered. For example:

> Every employee makes less than 70,000.

Is this claim true? So long as we restrict our universe to the six employees, we can determine the answer.[1] When a claim is made about all the objects (in this context, humans are objects!) being considered (*i.e.*, in our "universe"), this is called Universal Quantification. The meaning is that we make explicit the logical quantity (we "quantify") every member of a class or universe. English being the slippery object it is allows several ways to say the same thing:

> Each employee makes less than 70,000.
>
> All employees make less than 70,000.
>
> Employees make less than 70,000.[2]

Our universe (aka "domain") is the given set of six employees. When we say every, we mean every. This is not always true in English, for example "Every day I have homework," probably doesn't consider the days preceding your birth or after your death. Now consider:

> Each employee makes at least 10,000.

Is this claim true? How do you know?[3] A single counter-example is sufficient to refute a universally-quantified claim. What about the following claim:

> All female employees make less than 55,000.

Is this claim true? Restrict the domain and check each case.[4] What about

> Every employee that earns less than 55,000 is female?[5]

How about this claim:

> Every male employee makes less than 55,000.

It worked for females.[6] Notice a pattern. To disprove a universally-quantified statement you need just one counter-example. To prove one you need to consider every element in a domain. A universally-quantified statement of the form

> Every P is a Q

needs a single COUNTER-EXAMPLE to disprove, and verification that every element of the domain is an EXAMPLE to prove.

2.2 EXISTENTIAL QUANTIFICATION

Here's another sort of claim:

> Some employee earns over 57,000.

At first this claim doesn't seem to be about the whole database, but just about an employee who earns over 57,000 (if that employee exists, and Al does exist). But what about:

> There is an employee who earns less than 57,000.

This claim is also true, and it is verified by any of the employees in the set $\{\text{Betty}, \text{Carlos}, \text{Doug}, \text{Ellen}, \text{Flo}\}$. It's not a claim about any particular employee in that five-member set, but rather a claim that the set isn't empty. Although the non-empty set might have many members, one example of a member of the set is enough to show that it's not empty. Now consider:

> Some employee earns over 80,000.

This claim is false. There isn't an employee in the database who earns over 80,000. To show the set of employees earning over 80,000 is empty, you have to consider every employee in the universe and demonstrate that they don't earn over 80,000.

In everyday language existential quantification is expressed as:

> There [is / exists] [a / an / some / at least one] . . . [such that / for which] . . . ,
> or [For] [a / an / some / at least one] . . . , . . .

Note that the English word "some" is always used INCLUSIVELY here, so "some object is a P" is true if every object is a P.

The claims are about the EXISTENCE of one or more elements of a domain with some property, and they are examples of EXISTENTIAL quantification. Existential quantification requires you to exhibit just one EXAMPLE of an element with the property to prove, but it requires you to consider the entire domain to show that every element is a COUNTER-EXAMPLE to disprove.

The anti-symmetry between universal and existential quantification may be better understood by switching our point of view from properties to the sets of elements having those properties.

2.3 Properties, sets, and quantification

Let's look at that table again.

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Saying that Al is male is equivalent to saying Al belongs to the set of males. Symbolically we might write Al $\in M$ or $M(\text{Al})$. It's useful and natural to interchange the ideas of properties and sets. If we denote the set of employees as E, the set of female employees as F, the set of male employees as M, and the set of employees who earn less than 55,000 as L, then we have a notation for concisely (and precisely) evaluating claims such as $M(\text{Flo})$,[7] or $L(\text{Carlos})$.[8] So far the notation doesn't seem to have achieved much, but how about:

Everything in F is also in L (in other notation, $F \subseteq L$)?

So our universally-quantified claim that all females make less than 55,000 turns into a claim about subsets. We already have some intuition about subsets, so let's put it to work by drawing a Venn diagram (see Figure 2.1). Make sure you are solid on the meaning of "subset." Is a set always a subset of itself?[9] Is the empty set (the set with no elements) a subset of any set?[10]

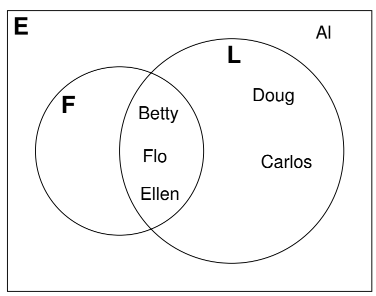

Figure 2.1: The only elements of F are also elements of L, so $F \subseteq L$. In this particular diagram, the maximum number of regions consistent with $F \subseteq L$ are occupied: three out of the four regions are occupied.

Now consider the claim

Something in M is also in $\overline{L}$: there is some male who does not earn less than 55,000

The complement of L is sometimes denoted $\overline{L}$, and means elements that are not in L. One way to denote "something in M is also in $\overline{L}$" in set notation is $M \cap \overline{L} \neq \varnothing$ — saying "something" is in both sets is the same as saying their intersection is non-empty. Now, you should be able to compare this to the definition of a subset to see that this is same as saying that M is not a subset of L, or $M \nsubseteq L$.

The anti-symmetry of universal and existential quantification becomes systematic:

- Every P is a Q means $P \subseteq Q$. To prove this claim you need to consider every element of P and show they are also elements of Q. To disprove this claim, you need to find just one element of P that is not an element of Q.

- Some P is a Q means $P \nsubseteq \overline{Q}$. To prove this you need to find just one P that isn't a non-Q (a roundabout way of saying find just one P that is a Q). To disprove it, you must consider every P and show they are also non-Qs.

2.4 SENTENCES, STATEMENTS, AND PREDICATES

Recall the table of employees with their genders and salaries from above:

EMPLOYEE	GENDER	SALARY
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Now consider the following claims:

CLAIM 2.1: The employee makes less than 55,000.

CLAIM 2.2: Every employee makes less than 55,000.

Can you decide whether both claims are true or false?[11] The basic difference between the two claims is that Claim 2.1 is about a particular employee, and it is true or false depending on the earnings of that employee, whereas Claim 2.2 is about the entire set of employees, E, and it is true or false depending on where that set of employees stands in relation to the set L, those who earn over 55,000.

Claim 2.1 is called a SENTENCE. It may refer to unquantified objects (for example "the employee"). Once the objects are specified (substitutions are made for the variable(s)), a sentence is either true or false (but never both). Claim 2.2 is called a STATEMENT. It doesn't refer to any unquantified variables, and it is either true or false (never both). Every statement is a sentence, but not every sentence is a statement. If you want to make it explicit that a sentence refers to unquantified objects, you may call it an "open sentence." Thus a sentence is a statement if and only if it is not open. Universal quantification transformed Claim 2.1 into Claim 2.2, from an open sentence about an unspecified element of the set of employees, into a statement about the (specified in the database) sets of Employees and those earning over 55,000.

SYMBOLS

Symbols are useful when they make expressions clearer and highlight patterns in similar expressions. We already moved in the direction of making our logical expressions symbolic by naming sets E (employees), F (females), and L (those earning less than 55,000). Naming gives us a concise expression for these sets, and it emphasizes the similar roles these sets play. We introduce more symbolism into our sentences, statements, and predicates now.

As a programmer you create a sentence every time you define a boolean function. In logic, a predicate is a boolean function. For convenience you can name your predicate, and you can define it by showing how it evaluates its input, using a symbol to stand for generic input. For example, if L is the set of employees earning less than 55,000

$L(x)$: $x \in L$.

Notice how similar this is to defining a function in a programming language in terms of how it evaluates its parameters. The symbol x is useful in the definition — it holds the parentheses, "(" and ")", apart so that we can see that exactly one value is needed, and it shows where to plug that value into the definition. Notice that this definition would mean the same things if we replaced the symbol x with the symbol y or the symbol y^3. The symbol x doesn't specify any value that helps determine whether our predicate evaluates to true or false. Our open sentence above, Claim 2.1, is equivalent to $L(x)$ — we can't evaluate it without substituting something from the set E for x. $L(\text{Carlos})$ is true, $L(\text{Al})$ is false.

Claim 2.2 is equivalent to "for all employees x, $L(x)$." The phrase "for all employees x" quantifies the variable x, and changes the claim from an open sentence about unspecified x to a statement about sets E and L, which were specified in the database above. Of course, in this context, "employees" refers to those in our database, and not any other employees.

We can indicate universal quantification symbolically as $\forall$, read as "for all." This makes sense if we specify the universe (domain) from which we are considering "all" objects. With this notation, Claim 2.2 can be written

$\forall$ employees, the employee makes less than 55,000.

Things become clearer if we introduce a name for the unspecified employee:

$\forall$ employees x, x makes less than 55,000.

Since this statement may eventually be embedded in some larger and more complicated structure, we can add to the brevity and clarity by adding a bit more notation. Let E denote the set of employees, and $L(x)$ denote the predicate "x makes less than 55,000." Now Claim 2.2 becomes:

$\forall x \in E, L(x)$.

We can do something similar with existential quantification. We can transform $L(x)$ into a statement by saying there is some element of E that also belongs to L:

There exist employees who earn less than 55,000.

$\exists x \in E, L(x)$.

The symbol we use for "there exists" is $\exists$. This is a statement about the sets E and L (it says they have a common, non-empty subset), and not a statement about individual elements of those sets. The symbol x doesn't stand for a particular element, it rather indicates that there is at least one element common to E and L.

2.5 Implications

Consider a claim of the form

IF an employee is male, THEN he makes less than 55,000.

This is called an IMPLICATION. It says that for employees, being male IMPLIES making less than 55,000.[12] This is universal quantification in disguise, since it could be accurately re-expressed as "Every male employee earns less than 55,000," or $\forall x \in E \cap M, L(x)$. Notice that the implication "males implies less than 55,000" has the same effect as restricting the domain by intersecting E with M in the universally-quantified statement. However, it turns out to be convenient sometimes to keep the implication "male implies less than 55,000" separate from the domain. In this way, we can consider the implication as part of universes other than E (perhaps H, the set of humans, or $X = \{\text{Doug}, \text{Carlos}\}$). Separating the implication from the surrounding universe also means we don't have to define a set for each predicate, so we could have "$M(x)$ implies $L(x)$" without necessarily defining the sets M and L (although we could always come up with suitable definitions if we needed to).

Just as with universal quantification, the only way to disprove the implication "if P then Q" is to show an instance where P is true but Q is false. If, in every possible instance, we have either not-P or Q, then the implication "if P then Q" is true.

In the implication "if P then Q," we call P the ANTECEDENT (sometimes the ASSUMPTION), and Q the CONSEQUENT (sometimes the CONCLUSION).

Since logical implication borrows the English word "if," we need to reject some of the common English uses of "if" that we don't mean when "if" is used in logic. In logic "if... then" tells you nothing about

causality. "If it rained yesterday, then the sun rose today," is a true implication, but the (possible) rain didn't cause the (certain) rising of the sun. Also, when my mother told me "if you eat your vegetables, then you can have dessert," she also meant "otherwise you'll get no dessert." In ordinary English, my mother used "if...then" to mean "if and only if...then." In logic we use the more constrained meaning. We want "If P then Q" to mean "Every P is a Q."
What does "every P is a Q" tell us? In our database example:

CLAIM 2.3: If an employee is female, then she makes less than 55,000.

Claim 2.3 discusses three sets, E, the set of employees, F, the set of female employees, and L, the set of employees making less than 55,000. Claim 2.3 implicitly invokes universal quantification, so it is more than a claim about a particular employee. The Venn diagram Figure 2.1 indicates the situation corresponding to our table. If you had no access to either the table or the Venn diagram, but only knew the Claim 2.3 was true, what would you know about

1. F, the set of female employees? What else does the implication tell you about Ellen if you only know that Ellen is female?

2. L, the set of employees earning less than 55,000? What do you know about Betty (if you only know she's in L) or Carlos (if you only know he's in L)?

3. $\overline{F}$, the set of male employees? Think about both Doug and Al.

4. $\overline{L}$ (the complement of L), the set of employees making 55,000 or more.

Knowing "P implies Q" tells us nothing more about some sets,[13] however it does tell us more about others.[14] Suppose you have a new employee Grnflx (from a domain short of vowels), plus our Venn diagram (2.1). Which region of the Venn diagram would you add Grnflx to in order to make Claim 2.3 false?[15] Once that region is occupied, does it matter whether any of the other regions are occupied or not?[16]

2.6 MORE SYMBOLS

We can write implication symbolically as $\Rightarrow$, read "implies." Now "P implies Q" becomes $P \Rightarrow Q$. Claim 2.3 could now be re-written as

an employee is female $\Rightarrow$ that employee makes less than 55,000.

CONTRAPOSITIVE

The CONTRAPOSITIVE of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$ ($\neg$ is the symbol for negation). In English the contrapositive of "all P is/are Q" is "all non-Q is/are non-P." Put another way, the contrapositive of "P implies Q" is "non-Q implies non-P." The contrapositive of Claim 2.3 is

an employee doesn't make less than 55,000 $\Rightarrow$ that employee is not female.

or, given the structure of the domain E of employees:

an employee makes at least 55,000 $\Rightarrow$ that employee is male.

Does the contrapositive of Claim 2.3 tell us everything that Claim 2.3 itself does? Check the Venn diagram (2.1). Does every Venn diagram that doesn't contradict Claim 2.3 also not contradict the contrapositive of Claim 2.3?[17] Can you apply the contrapositive twice? To do this it helps to know that applying negation ($\neg$) twice toggles the truth value twice (I'm not not going means I'm going). Thus the contrapositive of the contrapositive of $P \Rightarrow Q$ is the contrapositive of $\neg Q \Rightarrow \neg P$, which is $\neg\neg P \Rightarrow \neg\neg Q$, equivalent to $P \Rightarrow Q$.

Converse

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. In words, the converse of "P implies Q" is "Q implies P." An implication and its converse don't mean the same thing. Consider the Venn diagram Figure 2.1. Would it work as a Venn diagram for $L \Rightarrow F$?[18]
Consider an example where the (implicit) domain is the set of pairs of numbers, perhaps $\mathbb{R} \times \mathbb{R}$.

Claim 2.4: $x = 1 \Rightarrow xy = y$

- If we know $x = 1$, then we know $xy = y$.
- If we know $x \neq 1$, then we don't know whether or not $xy = y$.
- If we know $xy = y$, then we don't know whether or not $x = 1$.
- If we know $xy \neq y$, then we know $x \neq 1$.

The contrapositive of Claim 2.4 is:

$$xy \neq y \Rightarrow x \neq 1.$$

Check the four points we knew from Claim 2.4, and see whether we know the same ones from the contrapositive (it may be helpful to read them in reverse order). What about the converse?

$$xy = y \Rightarrow x = 1$$

with equivalent contrapositive

$$x \neq 1 \Rightarrow xy \neq y.$$

The converse of Claim 2.4 is not equivalent to Claim 2.4, for example consider the pair $(5, 0)$, that is $x = 5$ and $y = 0$. Indeed, Claim 2.4 is true, while its converse is false.

2.7 Implication in everyday English

Here are some ways of saying "P implies Q" in everyday language. In each case, try to think about what is being quantified, and what predicates (or perhaps sets) correspond to P and Q.

- If P, [then] Q.
  "If nominated, I will not stand."
  "If you think I'm lying, then you're a liar!"
- When[ever] P, [then] Q.
  "Whenever I hear that song, I think about ice cream."
  "I get heartburn whenever I eat supper too late."
- P is sufficient/enough for Q
  "Differentiability is sufficient for continuity."
  "Matching fingerprints and a motive are enough for guilt."
- Can't have P without Q
  "There are no rights without responsibilities."
  "You can't stay enrolled in csc 165 h without a pulse."
- P requires Q
  "Successful programming requires skill."

- For P to be true, Q must be true / needs to be true / is necessary
  "To pass CSC 165 H, a student needs to get 40% on the final."
- P only if / only when Q
  "I'll go only if you insist."

For the antecedent (P) look for "if," "when," "enough," "sufficient." For the consequent (Q) look for "then," "requires," "must," "need," "necessary," "only if," "when." In all cases, check whether the expected meaning in English matches the meaning of $P \Rightarrow Q$. In other words, you've got an implication if, in every possible instance, either P is false or Q is true.

CHAPTER 2 NOTES

[1]Yes, by verifying the claim for each employee.

[2]But contrast the meaning of "differentiable functions are continuous" (EVERY differentiable function is continuous, no exception) with the meaning of "birds fly" (MOST birds fly, but there are some exceptions).

[3]Betty makes 5,000, which is well-known to be less than 10,000.

[4]Restrict to females, and each one make less than 55,000.

[5]False. Doug and Carlos are counterexamples.

[6]But it is false for males. Al is a counter-example.

[7]False, check the table.

[8]True, check the table.

[9]Yes, since it includes only elements of itself. Don't confuse SUBSET with PROPER SUBSET.

[10]Yes, indeed it is a subset of every set. The reason is that it contains no element that could be outside another set.

[11]Claim 2.1 depends on who you mean by "The employee." If you specify Al, Claim 2.1 is false, but if you specify Ellen, Claim 2.1 is true. Claim 2.2 is quantified, so it depends on the entire universe of employees. Claim 2.2 is false because you can find at least 1 counterexample.

[12]An untrue implication in the universe we're considering, due to the counter-example Al.

[13]$\overline{P}$ (the complement of P), and Q.

[14]P (we know it's a subset of Q) and $\overline{Q}$ (the complement of Q, we know it's a subset of $\overline{P}$).

[15]Add Grnflx to $F - L$ (F outside L). Now Grnflx is a counter-example to the claim that every female employee makes less than 55,000.

[16]No. Counter-example Grnflx makes the implication false, and adding other data doesn't change this.

[17]Yes. The only Venn diagram that contradicts Claim 2.3 or its contrapositive is one that has at least one element in F outside of L.

[18]No, because there are elements in $L - F$ (Doug and Carlos).