## Section 2 QUANTIFIERS

In Section 1 we found that the sentence

$$x^2 - 5x + 6 = 0$$

needed to be considered within a particular context in order to become a statement. When a sentence involves a variable such as $x$, it is customary to use functional notation when referring to it. Thus we write

$$p(x): x^2 - 5x + 6 = 0$$

to indicate that $p(x)$ is the sentence "$x^2 - 5x + 6 = 0$." For a specific value of $x$, $p(x)$ becomes a statement that is either true or false. For example, $p(2)$ is true and $p(4)$ is false.

When a variable is used in an equation or an inequality, we assume that the general context for the variable is the set of real numbers, unless we are told otherwise. Within this context we may remove the ambiguity of $p(x)$ by using a quantifier. The sentence

$$\text{For every } x,\ x^2 - 5x + 6 = 0.$$

is a statement since it is false. In symbols we write

$$\forall\, x,\ p(x),$$

where the **universal quantifier** $\forall$ is read, "For every...," "For all...," "For each...," or a similar equivalent phrase. The sentence

$$\text{There exists an } x \text{ such that } x^2 - 5x + 6 = 0.$$

is also a statement, and it is true. In symbols we write

$$\exists\, x \ni p(x),$$

where the **existential quantifier** $\exists$ is read, "There exists...," "There is at least one...," or something equivalent. The symbol $\ni$ is just a shorthand notation for the phrase "such that."

**2.1 EXAMPLE** The statement

There exists a number less than 7.

can be written

$$\exists\, x \ni x < 7.$$

or in the abbreviated form

$$\exists\, x < 7,$$

where it is understood that $x$ is to represent a number. Sometimes the quantifier is not explicitly written down, as in the statement

If $x$ is greater than 1, then $x^2$ is greater than 1.

The intended meaning is

$$\forall\, x, \text{ if } x > 1, \text{ then } x^2 > 1.$$

In general, if a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.

**2.2 PRACTICE** Rewrite each statement using $\exists$, $\forall$, and $\ni$, as appropriate.

(a) There exists a positive number $x$ such that $x^2 = 5$.
(b) For every positive number $M$ there is a positive number $N$ such that $N < 1/M$.
(c) If $n \geq N$, then $|f_n(x) - f(x)| \leq 3$ for all $x$ in $A$.
(d) No positive number $x$ satisfies the equation $f(x) = 5$.

Having seen several examples of how existential and universal quantifiers are used, let us now consider how quantified statements are negated. Consider the statement

Everyone in the room is awake.

What condition must apply to the people in the room in order for the statement to be false? Must everyone be asleep? No, it is sufficient that at least one person be asleep. On the other hand, in order for the statement

Someone in the room is asleep.

to be false, it must be the case that everyone is awake. Symbolically, if

$$p(x)\text{: } x \text{ is awake},$$

then

$$\sim[\forall\, x,\ p(x)] \Leftrightarrow \exists\, x \ni \sim p(x),$$

where the symbol "~" represents negation. Similarly,

$$\sim[\exists\, x \ni p(x)] \Leftrightarrow \forall\, x,\ \sim p(x).$$

**2.3 EXAMPLE** Let us look at several more quantified statements and derive their negations. Notice in part (b) that the inequality "$0 < g(y) \le 1$" is a conjunction of two inequalities "$0 < g(y)$" and "$g(y) \le 1$." Thus its negation is a disjunction. In a complicated statement like (c), it is helpful to work through the negation one step at a time. Fortunately, (c) is about as messy as it will get.

(a) Statement: For every $x$ in $A$, $f(x) > 5$.

$$\forall\, x \text{ in } A,\ f(x) > 5.$$

Negation: $\exists\, x$ in $A \ni f(x) \le 5$.

There is an $x$ in $A$ such that $f(x) \le 5$.

(b) Statement: There exists a positive number $y$ such that $0 < g(y) \le 1$.

$$\exists\, y > 0 \ni 0 < g(y) \le 1.$$

Negation: $\forall\, y > 0$, $g(y) \le 0$ or $g(y) > 1$.

For every positive number $y$, either $g(y) \le 0$ or $g(y) > 1$.

(c) Statement:

$$\forall\, \varepsilon > 0\ \exists\, N \ni \forall\, n, \text{ if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon.$$

Negation:

$$\exists\, \varepsilon > 0 \ni \sim[\exists\, N \ni \forall\, n, \text{ if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N, \sim[\forall\, n, \text{ if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N\, \exists\, n \ni \sim[\text{if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N\, \exists\, n \ni n \ge N \text{ and } \sim[\forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N\, \exists\, n \ni n \ge N \text{ and } \exists\, x \text{ in } S \ni |f_n(x) - f(x)| \ge \varepsilon.$$

**2.4 PRACTICE** Write the negation of each statement in Practice 2.2.

It is important to realize that the order in which quantifiers are used affects the truth value. For example, when talking about real numbers, the statement

$$\forall\, x\, \exists\, y \ni y > x$$

is true. That is, given any real number $x$ there is always a real number $y$ that is greater than that $x$. But the statement

$$\exists\, y \ni \forall\, x,\ y > x$$

is false, since there is no fixed real number $y$ that is greater than every real number. Thus care must be taken when reading (and writing) quantified statements so that the order of the quantifiers is not inadvertently changed.

**Review of Key Terms in Section 2**

Universal quantifier "$\forall$" Existential quantifier "$\exists$" Such that "$\ni$"

## *ANSWERS TO PRACTICE PROBLEMS*

**2.2** (a) $\exists\, x > 0 \ni x^2 = 5$.
(b) $\forall\, M > 0\ \exists\, N > 0 \ni N < 1/M$.
(c) $\forall\, n$, if $n \geq N$, then $\forall\, x$ in $A$, $|f_n(x) - f(x)| \leq 3$.
(d) The words "no" and "none" are universal quantifiers in a negative sense. In general, the statement "None of them are $P(x)$" is equivalent to "All of them are not $P(x)$." Thus the statement can be written as "$\forall\, x > 0,\ f(x) \neq 5$."

**2.4** (a) $\forall\, x > 0, x^2 \neq 5$.
(b) $\exists\, M > 0 \ni \forall\, N > 0, N \geq 1/M$.
(c) $\exists\, n \ni n \geq N$ and $\exists\, x$ in $A \ni |f_n(x) - f(x)| > 3$.
(d) $\exists\, x > 0 \ni f(x) = 5$.

## *EXERCISES*

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

**2.1** Mark each statement True or False. Justify each answer.
(a) The symbol "$\forall$" means "for every."
(b) The negation of a universal statement is another universal statement.
(c) The symbol "$\ni$" is read "such that."

**2.2** Mark each statement True or False. Justify each answer.

(a) The symbol "$\exists$" means "there exist several."
(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.
(c) The order in which quantifiers are used affects the truth value.

**2.3** Write the negation of each statement. ☆

(a) All the roads in Yellowstone are open.
(b) Some fish are green.
(c) No even integer is prime.
(d) $\exists\, x < 3 \ni x^2 \geq 10$.
(e) $\forall\, x$ in $A$, $\exists\, y < k \ni 0 < f(y) < f(x)$.
(f) If $n > N$, then $\forall\, x$ in $S$, $|f_n(x) - f(x)| < \varepsilon$.

**2.4** Write the negation of each statement.

(a) Some basketball players at Central High are short.
(b) All of the lights are on.
(c) No bounded interval contains infinite many integers.
(d) $\exists\, x$ in $S \ni x \geq 5$.
(e) $\forall\, x \ni 0 < x < 1$, $f(x) < 2$ or $f(x) > 5$.
(f) If $x > 5$, then $\exists\, y > 0 \ni x^2 > 25 + y$.

**2.5** Determine the truth value of each statement, assuming that $x$, $y$, and $z$ are real numbers. ☆

(a) $\exists\, x \ni \forall\, y\, \exists\, z \ni x + y = z$.
(b) $\exists\, x \ni \forall\, y$ and $\forall\, z$, $x + y = z$.
(c) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni y - z = x$.
(d) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni xz = y$.
(e) $\exists\, x \ni \forall\, y$ and $\forall\, z$, $z > y$ implies that $z > x + y$.
(f) $\forall\, x$, $\exists\, y$ and $\exists\, z \ni z > y$ implies that $z > x + y$.

**2.6** Determine the truth value of each statement, assuming that $x$, $y$ and $z$ are real numbers.

(a) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni x + y = z$.
(b) $\forall\, x\, \exists\, y \ni \forall\, z$, $x + y = z$.
(c) $\exists\, x \ni \forall\, y$, $\exists\, z \ni xz = y$.
(d) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni yz = x$.
(e) $\forall\, x\, \exists\, y \ni \forall\, z$, $z > y$ implies that $z > x + y$.
(f) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni z > y$ implies that $z > x + y$.

**2.7** Below are two strategies for determining the truth value of a statement involving a positive number $x$ and another statement $P(x)$.

(*i*) Find some $x > 0$ such that $P(x)$ is true.
(*ii*) Let $x$ be the name for any number greater than 0 and show $P(x)$ is true.

For each statement below, indicate which strategy is more appropriate.

(a) $\forall x > 0, P(x)$. ☆
(b) $\exists x > 0 \ni P(x)$. ☆
(c) $\exists x > 0 \ni \sim P(x)$.
(d) $\forall x > 0, \sim P(x)$.

**2.8** Which of the following best identifies $f$ as a constant function, where $x$ and $y$ are real numbers.

(a) $\exists x \ni \forall y,\ f(x) = y$.
(b) $\forall x \exists y \ni f(x) = y$.
(c) $\exists y \ni \forall x,\ f(x) = y$.
(d) $\forall y \exists x \ni f(x) = y$.

**2.9** Determine the truth value of each statement, assuming $x$ is a real number. ☆

(a) $\exists x \text{ in } [2,4] \ni x < 7$.
(b) $\forall x \text{ in } [2,4], x < 7$.
(c) $\exists x \ni x^2 = 5$.
(d) $\forall x, x^2 = 5$.
(e) $\exists x \ni x^2 \neq -3$.
(f) $\forall x, x^2 \neq -3$.
(g) $\exists x \ni x \div x = 1$
(h) $\forall x, x \div x = 1$.

**2.10** Determine the truth value of each statement, assuming $x$ is a real number.

(a) $\exists x \text{ in } [3,5] \ni x \geq 4$.
(b) $\forall x \text{ in } [3,5], x \geq 4$
(c) $\exists x \ni x^2 \neq 3$.
(d) $\forall x, x^2 \neq 3$.
(e) $\exists x \ni x^2 = -5$.
(f) $\forall x, x^2 = -5$.
(g) $\exists x \ni x - x = 0$.
(h) $\forall x, x - x = 0$.

Exercises 2.11 to 2.19 give certain properties of functions that we shall encounter later in the text. You are to do two things: (a) rewrite the defining conditions in logical symbolism using $\forall$, $\exists$, $\ni$, and $\Rightarrow$, as appropriate; and (b) write the negation of part (a) using the same symbolism. It is not necessary that you understand precisely what each term means.

*Example*: A function $f$ is odd iff for every $x$, $f(-x) = -f(x)$.

(a) defining condition: $\forall x,\ f(-x) = -f(x)$.

(b) negation: $\exists x \ni f(-x) \neq -f(x)$.

**2.11** A function $f$ is *even* iff for every $x$, $f(-x) = f(x)$. ☆

**2.12** A function $f$ is *periodic* iff there exists a $k > 0$ such that for every $x$, $f(x + k) = f(x)$.

**2.13** A function $f$ is *increasing* iff for every $x$ and for every $y$, if $x \leq y$, then $f(x) \leq f(y)$. ☆

**2.14** A function $f$ is *strictly decreasing* iff for every $x$ and for every $y$, if $x < y$, then $f(x) > f(y)$.

**2.15** A function $f: A \to B$ is *injective* iff for every $x$ and $y$ in $A$, if $f(x) = f(y)$, then $x = y$. ☆

**2.16** A function $f: A \to B$ is *surjective* iff for every $y$ in $B$ there exists an $x$ in $A$ such that $f(x) = y$.

**2.17** A function $f: D \to R$ is *continuous* at $c \in D$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in D$. ☆

**2.18** A function $f$ is *uniformly continuous on a set* $S$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x$ and $y$ are in $S$ and $|x - y| < \delta$.

**2.19** The real number $L$ is the *limit* of the function $f: D \to R$ at the point $c$ iff for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$. ☆

**2.20** Consider the following sentences:

(a) The nucleus of a carbon atom consists of protons and neutrons.
(b) Jesus Christ rose from the dead and is alive today.
(c) Every differentiable function is continuous.

Each of these sentences has been affirmed by some people at some time as being "true." Write an essay on the nature of truth, comparing and contrasting its meaning in these (and possibly other) contexts. You might also want to consider some of the following questions: To what extent is truth absolute? To what extent can truth change with time? To what extent is truth based on opinion? To what extent are people free to accept as true anything they wish?

## Section 3 TECHNIQUES OF PROOF: I

In the first two sections we introduced some of the vocabulary of logic and mathematics. Our aim is to be able to read and write mathematics, and this requires more than just vocabulary. It also requires syntax. That is, we need to understand how statements are combined to form the mysterious mathematical entity known as a proof. Since this topic tends to be intimidating to many students, let us ease into it gently by first considering the two main types of logical reasoning: inductive reasoning and deductive reasoning.