Jan 14th

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Outline: 1) Mersenne primes
2) Perfect numbers
3) Induction

Mersenne primes

Def: A prime number is number $p \in \mathbb{N}$ such that the only divisors of p are 1 and p.
convention: 1 is not prime.

Mersenne primes Def: A Mersenne prime is a prime number of the form $2^n - 1$.

n	1	2	3	4	5	6	7	8	9	10	11	...
2^n-1	1	3	7	15	31	63	127	255	511	1023	2047	...
Prime	×	✓	✓	×	✓	×	✓	×	×	×	×	

So if 2^n-1 is prime, n is prime.
composite composite

Prop: Every Mersenne prime is of the form 2^p-1 where p is prime.

Proof: (Contrapositive) Sps n is a composite number. Then $n = ab$, $a, b \geq 2$.

Then $2^n - 1 = (2^a)^b - 1$

$$= \underbrace{(2^a-1)}_{A}\underbrace{((2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + (2^a)^2 + 2^a + 1)}_{B}$$

since $a, b \geq 2$, $A, B \geq 2$ as well.

So, $2^n - 1 = AB$ is composite. ■

Def: A number n is called perfect if it is equal to sum of its proper divisors.

perfect numbers

Ex: 6 is perfect : divisors are 1, 2, 3, 6
proper divisors are 1, 2, 3
$1+2+3 = 6$

Ex: $28 = 1+2+4+7+14$

Ex: $10 \neq 1+2+5$

Prop: If $n = 2^{p-1}(2^p - 1)$ where 2^p-1 is a Mersenne prime, then n is perfect.

Proof: We want to show $n = \sum d$

Since 2^{p-1} is prime, we write $q = 2^p - 1$, $n = 2^{p-1}q$. The divisors of n are $\underbrace{1, 2, 2^2, \cdots, 2^{p-1}}_{A}$, $\underbrace{q, 2q, \cdots, 2^{p-2}q, 2^{p-1}q}_{B}$

The sum of A is

$$1 + 2 + 2^2 + \cdots + 2^{p-1} = \frac{2^p-1}{2-1} = \frac{2^p-1}{1} = q$$

$$q + 2q + \cdots + 2^{p-2}q = q\left(\frac{2^{p-1}-1}{2-1}\right) = 2^{p-1}q - q$$

Then the sum of all proper divisors is $q + 2^{p-1}q - q = n$

$\therefore n$ is perfect. ■

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induction

Prop: For all $n \in \mathbb{N}$, $1+2+3+\cdots+n=\frac{n(n+1)}{2}$

Proof: ______

Prop: $\sqrt{2}$ is irrational

Pf: want to show $\sqrt{2} \neq \frac{p}{q}$ where $q \cdot p \in \mathbb{Z}$ $q \cdot p$ coprime

$\Rightarrow 2=\frac{q^2}{p^2} \Rightarrow 2p^2=q^2$

$\therefore$ q is even so $q=2k$ for some k.

But $2p^2=(2k)^2=4k^2 \Rightarrow p^2=2k^2$

$\therefore$ p is even

But we know $p \cdot q$ are relatively prime, so cannot both be even

Hence, $\sqrt{2}$ is not rational