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Problem Set 3 Solutions

1) $F(x,y,z)=0$ — If $x=x(y,z)$ then $F(x(y,z),y,z)=0$

By Chain Rule, taking $\frac{\partial}{\partial y}$;

$$0=\frac{\partial F}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial F}{\partial y} \Rightarrow \frac{\partial x}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$$

— If $y=y(x,z)$ then

taking $\frac{\partial}{\partial z}$;

$$0=\frac{\partial F}{\partial y}\frac{\partial y}{\partial z}+\frac{\partial F}{\partial z} \Rightarrow \frac{\partial y}{\partial z}=-\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}$$

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— If $z=z(x,y)$, taking $\frac{\partial}{\partial x}$:

$$0=\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial F}{\partial x} \Rightarrow \frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

so $$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=\left(\frac{-\cancel{\partial F/\partial y}}{\cancel{\partial F/\partial x}}\right)\left(-\frac{\cancel{\partial F/\partial z}}{\cancel{\partial F/\partial y}}\right)\left(-\frac{\cancel{\partial F/\partial x}}{\cancel{\partial F/\partial z}}\right)=-1$$

2) a) $u(x,t)=f(x-ct)+g(x+ct)$

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$$\partial_t u(x,t) = -c f'(x-ct)+c g'(x+ct)$$

via chain rule

$$\partial_t^2 u(x,t) = c^2 f''(x-ct)+c^2 g''(x+ct)$$

$$\partial_x u(x,t) = f'(x-ct)+g'(x+ct)$$

$$\partial_x^2 u(x,t) = f''(x-ct)+g''(x+ct)$$

$$\therefore\quad \partial_t^2 u(x,t) = c^2\left[f''(x-ct)+g''(x+ct)\right] = c^2\partial_x^2 u(x,t)$$

b) $u(x,t) = r^{-1} g(ct-r)$ $\qquad r = \sqrt{x^2+y^2+z^2} \Rightarrow \partial_x r = \frac{x}{r}$

$$\partial_t u = c r^{-1} g'(ct-r) \quad , \quad \partial_t^2 u = c^2 r^{-1} g''(ct-r)$$

$$\partial_x u = -r^{-2} \partial_x r \, g(ct-r) - r^{-1} g'(ct-r)\, \partial_x r$$

$$= -\frac{x}{r^3} g(ct-r) - \frac{x}{r^2} g'(ct-r)$$

$$\partial_x^2 u = -\left[(r^{-3} - 3x^2 r^{-5}) g - \frac{x^2}{r^4} g'(ct-r)\right] - \left[(r^{-2} - 2x^2 r^{-4}) g' - \frac{x^2}{r^3} g''\right]$$

$$= (3x^2 r^{-5} - r^{-3}) g(ct-r) + (3x^2 r^{-4} - r^{-2}) g'(ct-r) + \frac{x^2}{r^3} g''(ct-r)$$

Similarly, $\partial_y^2 u = (3y^2 r^{-5} - r^{-3}) g(ct-r) + (3y^2 r^{-4} - r^{-2}) g'(ct-r) + \frac{y^2}{r^3} g''(ct-r)$

8 $\qquad \partial_z^2 u = (3z^2 r^{-5} - r^{-3}) g(ct-r) + (3z^2 r^{-4} - r^{-2}) g'(ct-r) + \frac{z^2}{r^3} g''(ct-r)$

Then

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = \overset{\to 0}{\left(\cancel{3(x^2+y^2+z^2) r^{-5} - 3r^{-3}}\right)} g(ct-r)$$
$$+ \overset{0}{\left(\cancel{3(x^2+y^2+z^2) r^{-4} - 3r^{-2}}\right)} g'(ct-r)$$
$$+ (x^2+y^2+z^2) r^{-3} g''(ct-r)$$

$\nearrow 0$ as $x^2+y^2+z^2 = r^2$

so $= r^{-1} g''(ct-r) \Rightarrow \partial_t^2 u = c^2(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u)$

3) $w = f(x,y,s,t) = x^2y + s^2x + t$, $x = g_1(s,t) = s+t$
$y = g_2(s,t) = t^2$.

$$dw = \partial_1 f\,dx + \partial_2 f\,d\cancel{x}y + \partial_3 f\,ds + \partial_4 f\,dt$$

but $dx = \partial_1 g_1 ds + \partial_2 g_1 dt$

$dy = \partial_1 g_2 ds + \partial_2 g_2 dt$

$$\therefore\ dw = (\partial_1 f\,\partial_1 g_1 + \partial_2 f\,\partial_1 g_2 + \partial_3 f)ds$$
$$+ (\partial_1 f\,\partial_2 g_1 + \partial_2 f\,\partial_2 g_2 + \partial_4 f)dt$$

(19) $\partial_1 f\Big|_{(s,t)=(1,2)} = 2xy + s^2\Big|_{(s,t)=12} = 2(s+t)t^2 + s^2\Big|_{(1,2)} = 25$

$\partial_2 f\Big|_{(s,t)=(1,2)} = x^2\Big|_{(s,t)=(1,2)} = (s+t)^2\Big|_{(1,2)} = 9$

$\partial_3 f\Big|_{(s,t)=(1,2)} = 2sx\Big|_{(s,t)=(1,2)} = 2s(s+t)\Big|_{(1,2)} = 6$, $\partial_4 f = 1$

$\partial_1 g_1\Big|_{(1,2)} = 1$, $\partial_2 g_1\Big|_{(1,2)} = 1$, $\partial_2 g_2\Big|_{(1,2)} = 2t\Big|_{(1,2)} = 4$, $\partial_1 g_2\Big|_{(1,2)} = 0$

Using $dt = 1.98 - 2 = -0.02$ $ds = 1.03 - 1 = 0.03$

$\therefore\ dw = (25 + 9\cdot 0 + 6)\,0.03 + (25 + 9\cdot 4 + 1)(-0.02)$

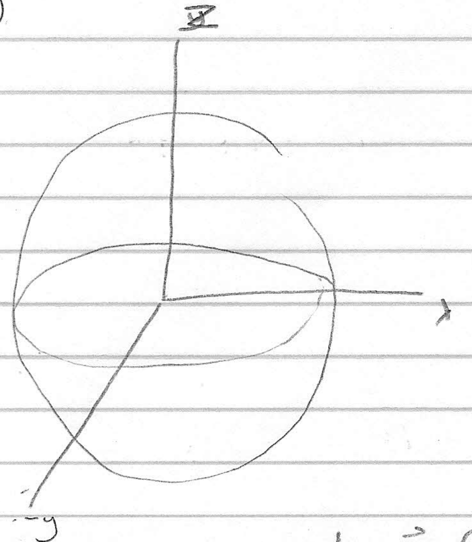
$= 31\cdot 0.03 + 62(-0.02) = -0.31$

- As $62 > 31$, changing t causes bigger change in dw than s.

(29)

4) a) $x^2+y^2+z^2-1=0$ is a sphere;

(1)



b) $\vec{g}_1(t) = (\cos(t), 0, \sin t)$
$\vec{g}_2(t) = (0, \cos t, \sin t)$

has $\cos^2 t + 0^2 + \sin^2 t - 1 = 0$

and $0^2 + \cos^2 t + \sin^2 t - 1 = 0$

as $\cos^2 t + \sin^2 t = 1$

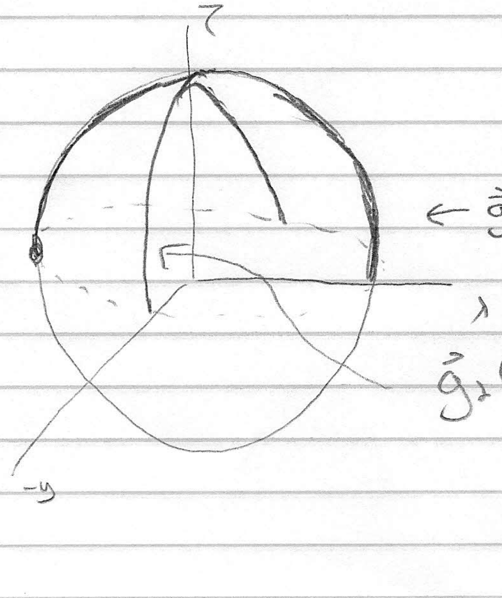
$\therefore$ $\vec{g}_1(t)$ & $\vec{g}_2(t)$ lie in $F(x,y,z) = 0$.

$\vec{g}_1(t^*) = \vec{g}_2(t^*) \Rightarrow (\cos(t^*), 0, \sin t^*) = (0, \cos t^*, \sin t^*)$

$\Rightarrow \cos t^* = 0 \Rightarrow t^* = \frac{\pi}{2} \Rightarrow \sin\frac{\pi}{2} = 1$

$\therefore$ Intersection point is $(0,0,1)$.

(6) $\vec{g}_1(t)$ & $\vec{g}_2(t)$ are semicircles in the x-z plane & y-z plane respectively;



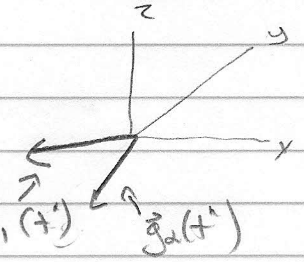
$\leftarrow \vec{g}_1(t),\ t \in [0, \pi]$

$\vec{g}_2(t),\ t \in [0, \pi]$

4c) $\vec{g}_1(t^*) = \vec{g}_1(\frac{\pi}{2}) = \left(-\sin\frac{\pi}{2}, 0, \cos\frac{\pi}{2}\right) = (-1, 0, 0)$

$\vec{g}_2(t^*) = \vec{g}_2(\frac{\pi}{2}) = \left(0, -\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right) = (0, -1, 0)$

From zero:



. Plotting these starting at $(0,0,1)$

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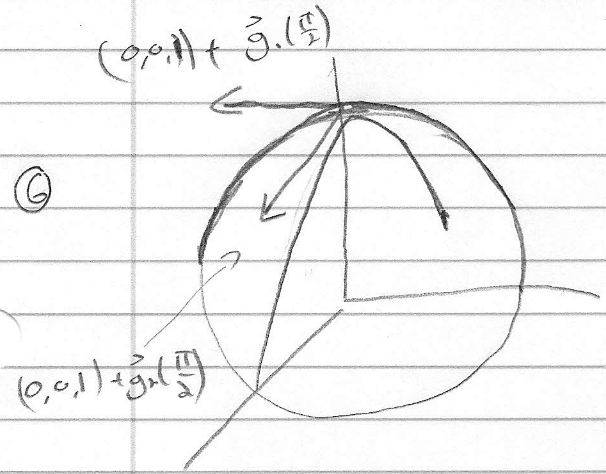

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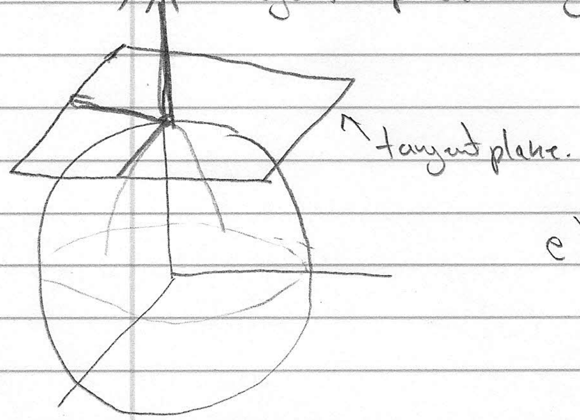
d) $\left.\frac{\partial F}{\partial x}\right|_{(0,0,1)} = \left.\partial x\right|_0 = 0, \quad \left.\frac{\partial F}{\partial y}\right|_{(0,0,1)} = \left.\partial y\right|_0$

$\left.\frac{\partial F}{\partial x}\right|_{(0,0,1)} = \left.\partial z\right|_1 = 1$

so $\nabla F(0,0,1) = (0,0,1)$

$\nabla F(0,0,1) + (0,0,1) \to$ Tangent plane given by $\nabla F(0,0,1) \cdot (\vec{x} - (0,0,1)) = 0$

$\Rightarrow (0,0,1) \cdot (x_1, x_2, x_3 - 1) = 0$



e) $\vec{g}_1(\frac{\pi}{2}) \cdot \nabla F(0,0,1) = (-1,0,0) \cdot (0,0,1) = 0$

$\vec{g}_2(\frac{\pi}{2}) \cdot \nabla F(0,0,1) = (0,-1,0) \cdot (0,0,1) = 0$

$\vec{g}_1(\frac{\pi}{2}) \cdot \vec{g}_2(\frac{\pi}{2}) = (-1,0,0) \cdot (0,-1,0) = 0$

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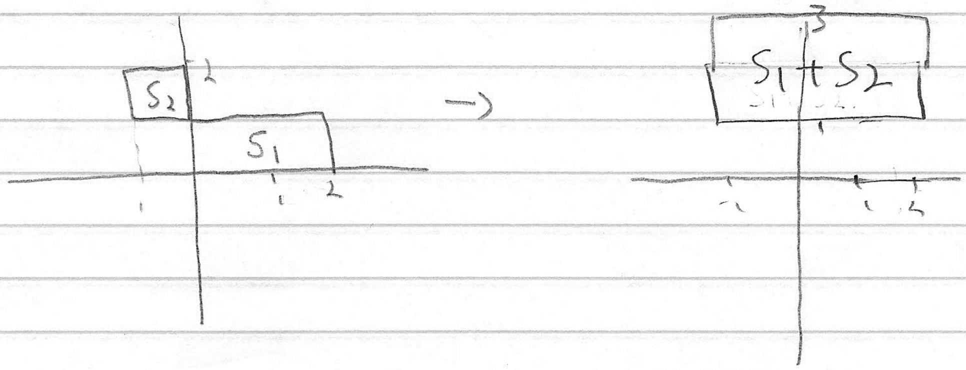
so $\vec{g}_1(\frac{\pi}{2})$, $\vec{g}_2(\frac{\pi}{2})$, $\nabla F(0,0,1)$ mutually orthogonal.

5)



For $S_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le 1\}$

$S_2 = \{(x,y) \in \mathbb{R} \mid -1 \le x \le 0,\ 1 \le y \le 2\}$

Adding vectors from S_1 & S_2 resp. get

$$S_1 + S_2 = \{\vec{a} + \vec{b} \mid a \in S_1,\ b \in S_2\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 2,\ 1 \le y \le 3\}$$

(10)

(to see this, note that for $a \le c \le b$, $d \le f \le e$
$a + d \le c + f \le b + e$)

Now consider $\vec{a}, \vec{b} \in S_1 + S_2$, and consider $\vec{a} + t(\vec{b} - \vec{a}) = (1-t)\vec{a} + t\vec{b}$ for $t \in [0,1]$.

Then for $(1-t)a_1 + tb_1$, as $-1 \le a_1, b_1 \le 2$ and $t, 1-t \ge 0$

$$-1 = -1(1-t) - t \le (1-t)a_1 + tb_1 \le (1-t)2 + 2t = 2$$

Similarly, $1 = (1-t) + t \le (1-t)a_2 + tb_2 \le (1-t)3 + 3t = 3$

$\therefore \vec{a} + t(\vec{b} - \vec{a}) \in S_1 + S_2 \ \forall t \in [0,1] \Rightarrow S_1 + S_2$ convex.

- As a straight path like above is a path, $S_1 + S_2$ convex $\Rightarrow S_1 + S_2$ path connected.
- Path connected $\Rightarrow$ connected $\therefore S_1 + S_2$ connected.

6a) An eigenvalue λ_i & corresponding eigenvector $\vec{x}_{\lambda_i}$ satisfies $(A - I\lambda)\vec{x}_{\lambda_i} = 0$. To ensure non trivial solutions, must have $\det(A - I\lambda) = 0$

For $A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$, $A - I\lambda = \begin{pmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}$

so $\det(A - I\lambda) = (3-\lambda)(1-\lambda) = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 3$

(5)

For $\lambda = 1$: $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda = 3$ $\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b) Normalizing $|\vec{x}_{\lambda_1}| = 1$ so keep $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_1} = \begin{matrix} 0 \\ 1 \end{matrix}$

$|\vec{x}_{\lambda_2}| = \sqrt{5}$ so $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(5)

$P = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} \\ 1 & \frac{1}{\sqrt{5}} \end{pmatrix}$

$P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ -1 & 0 \end{pmatrix} \cdot \frac{-\sqrt{5}}{2}$

$$P^{-1}AP = -\frac{\sqrt{5}}{2}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{2}{\sqrt{5}} \\ 1 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= -\frac{\sqrt{5}}{2}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{2}{\sqrt{5}} \\ 1 & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 \\ 0 & -\frac{6}{\sqrt{5}} \end{pmatrix}\cdot\left(-\frac{\sqrt{5}}{2}\right)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{pmatrix}$$