Lecture 20

Last time: Rouche's Thm

D-domain, γ-simple closed curve in D.

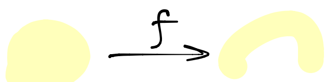
f, g holomorphic on D if $|f(z)+g(z)|<|f(z)|$ on γ

<u>Then</u>: f & g have the same # of zeros on $\mathrm{Int}(\gamma)$

counted w. multiplicity

Maximum Modulus Principle.

<u>Thm</u> (open mapping). Let f be holomorphic on D. $U \subseteq D$ open subset, then $f(U)$ is open.



FALSE for real functions: $f(x)=x^2$ on $\mathbb{R}$:

$f(\mathbb{R})=[0,+\infty)$ NOT OPEN

on $\mathbb{C}$: every # has a square root. $f(z)=z^2$

$\sqrt{re^{i\theta}}=\sqrt{r}\,e^{i\theta/2}$. $f(\mathbb{C})=\mathbb{C}$

Pf (not needed):

<u>WANT</u>: $f(z_0)=0$

(I can assure this after replacing f by f+a, $a\in\mathbb{C}$)

then for any point ε small complex #

$\exists\, z_1$ near z_0 with $f(z_1)=\varepsilon$

i.e.

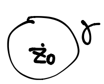
$$z_0 \xrightarrow{f} \gamma$$

i.e. want the function $\varepsilon - f(z)$ to have a zero in a small ball about z_0.

$$|f(z)+(\varepsilon-f(z))|=|\varepsilon| \overset{?}{<} |f(z)| \quad \text{on } \underline{\text{some}} \text{ curve } \gamma, \text{ containing } z_0$$

then: $f(z_0)=0 \Rightarrow \varepsilon - f$ has a zero, we win!

recall: zeros of f are <u>isolated</u>, i.e. $f(z)\neq 0$ for all z near z_0 $(z\neq z_0)$



$\leftarrow$ Let $\gamma = S_r(z_0)$ for r a small positive # such that f has no zeros on γ.

<u>FACT</u>: since γ is a closed compact curve, $f(\gamma)$ has a minimum value, $f(w_0)$, $w_0\in\gamma$)

Since f has no zeros on γ, $|f(w_0)|>0$

Let $|\varepsilon| = \dfrac{|f(w_0)|}{2}$

Then: $|f(z)+(\varepsilon-f(z))|=|\varepsilon|<|f(z)|$ on γ. ∎

<u>Corollary</u> (Maximum Modulus Principle): Let D be an open domain, f is holomorphic on D.

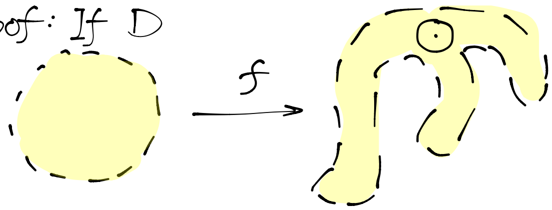
then ① $|f|$ has no max on D.

② $\mathrm{Re}(f)$ has no max on D.

③ $\mathrm{Im}(f)$ has no max on D.

④ ② & ③ but for "min"

Proof: If D



a small ball around $f(z_0)$ is contained in $Im(f)$.

Cor (MMP)

If D is a bounded region, ∂f extends to a continuous function on ∂D, then

① $|f|$ attains its max on ∂D

② $Re f$, $Im f$ attain their max & min on ∂D.

What does "extend" mean?

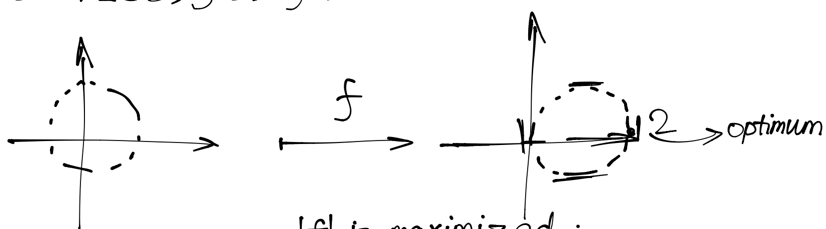
$\exists$ function $g: \overline{D} \to \mathbb{C}$ s.t. $\forall z \in D$, $g(z) = f(z)$

Example:

$f(z) = z+1$

$D = \{z \mid |z| < 1\}$

$\overline{D} = \{z \mid |z| \leq 1\}$



so $f(D)$ = shift right

$|f|$ is maximized:

$|z+1| \leq |z| + 1 \leq 2$

Image $(Re(f)) = (0,2)$

Image $(Im(f)) = (-i, i)$