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UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

DECEMBER 2010 EXAMINATIONS

CSC 165H1F
DURATION — 3 HOURS

ONE 8.5"×11" HANDWRITTEN (BOTH SIDES) AID SHEET ALLOWED

STUDENT NUMBER: |_|_|_|_|_|_|_|_|_|

LAST NAME: _____

FIRST NAME: _____

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This exam consists of 8 questions on 14 pages (including this one). *When you receive the signal to start, please make sure that your copy of the exam is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-14 of this exam.

1: _____/10

2: _____/10

3: _____/10

4: _____/10

5: _____/15

6: _____/10

7: _____/10

8: _____/10

TOTAL: _____/85

Good Luck!

QUESTION 1. [10 MARKS]

Each sentence below is either a universally quantified implication, or a negation of a universally quantified implication. The word "unless" is used to convey "if not" in these sentences. The sentences are labelled s1—s8 to allow you to refer to them conveniently when you answer questions.

s1: I always say, something is not worth doing well unless it is worth doing.

s2: I always say, don't fix it unless it's broken.

s3: I always say, if something is worth doing, then it is worth doing well.

s4: I always say, if it's broken, then fix it.

s5: I have occasionally said, fix it even though it isn't broken.

s6: I have said there is something worth doing well that, however, isn't worth doing.

s7: I have said there is something worth doing that isn't worth doing well.

s8: I have occasionally said, don't fix it, even though it's broken.

PART (A) [4 MARKS]

Write down the labels of four pairs of sentences that are negations of each other.

PART (B) [2 MARKS]

Write down the labels of two pairs of sentences that are converses of each other.

PART (C) [4 MARKS]

Write the contrapositives of four of the sentences. Indicate the original sentence by its label, and then write a full sentence expressing its contrapositive.

QUESTION 2. [10 MARKS]

Use the proof structure from this course to prove, by induction, that for all natural numbers $n > 3$, $3^n > n^3 + 5$.

QUESTION 3. [10 MARKS]

Consider the definitions of $U(m)$, $V(n)$, and $W(o)$:

$$\begin{aligned} U(m) &\Leftrightarrow \exists i \in \mathbb{N}, m = 5i + 2 \\ V(n) &\Leftrightarrow \exists j \in \mathbb{N}, n = 5j + 3 \\ W(o) &\Leftrightarrow \exists k \in \mathbb{N}, o = 5k + 4 \end{aligned}$$

Now prove the following statement, using the proof structure from this course:

$$\forall m, n, o \in \mathbb{N}, (U(m) \wedge V(n) \wedge W(o)) \Rightarrow (V(mo) \wedge U(n + o))$$

QUESTION 4. [10 MARKS]

Consider the definition of $\lfloor x \rfloor$ ($\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integers):

$$\forall x \in \mathbb{R}, y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

Use the definition to prove that $\forall x \in \mathbb{R}, \lfloor x \rfloor + 1 = \lfloor x + 1 \rfloor$. Use the proof structure from class. You may also use the result from class that $\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$.

QUESTION 5. [15 MARKS]

Here is a statement of the Intermediate Value Theorem (IVT):

$$\forall a \in \mathbb{R}, \forall b \in [a, \infty), \forall u \in \mathbb{R},$$
$$f \text{ continuous and real-valued on } [a, b] \text{ and } f(a) < u < f(b) \Rightarrow \exists c \in [a, b], f(c) = u$$

PART (A) [10 MARKS]

Define $f(x) = x^2$. Use the proof structure from this course, the IVT, and the following facts to prove that there is real number whose square is $\pi \in \mathbb{R}$. It's not good enough to state, without proof, that $\sqrt{\pi} \in \mathbb{R}$, although you may use the fact that π itself is a real number.

(i) $f(x)$ is continuous and real-valued on all of $\mathbb{R}$.

(ii) $1 < \pi < 4$.

(iii) $f(1) = 1$ and $f(2) = 4$.

PART (B) [5 MARKS]

What goes wrong if you try to use the approach from the previous page to "prove" that there is a real number whose square is $-\pi$?

QUESTION 6. [10 MARKS]

Consider the definition of $\mathcal{O}(g)$ below:

$$\mathcal{O}(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} : \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)\}$$

Define $f(n) = n^4 - 3n^3 + n^2 + 1$ and $g(n) = 5n^3 + 7n^2 + 9$. Use the proof structure from this course to PROVE $f \notin \mathcal{O}(g)$. You may NOT use the results and techniques of limits from calculus.

QUESTION 7. [10 MARKS]

Use the definition of $\mathcal{O}(g)$ from the preceding page, but now define $f(n) = 5n^3 + 7n^2 + 9$ and $g(n) = n^4 - 3n^3 + n^2 + 1$. Use the proof structure from this course to prove that $f \in \mathcal{O}(g)$. You may NOT use the results and techniques of limits from calculus.

QUESTION 8. [10 MARKS]

Suppose Python supports a floating-point system with base $\beta = 10$, exponents from the set $\{-2, \ldots, 2\}$, $t = 4$ digits in the significand, a single sign symbol + or −, a radix point following the first digit, and the convention that the digit preceding the radix point is non-zero unless we are representing zero itself. In this scheme, 100 is represented by 1.000×10^2 and 0.01 is represented by 1.000×10^{-2}.

What's the problem with the following function definition? Explain how, and why, you would fix it to more accurately perform the operation specified in its docstring.

```
def hundred_plus_n_hundredths(n) :
    """Return the sum of 100 plus n hundredths"""
    sum = 1.000e+2  # 1.000 x 10^2 in Python
    delta = 1.000e-2  # 1.000 x 10^{-2} in Python
    for i in range(n) :
        sum = sum + delta
    return sum
```

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Total Marks = 85