AST121 Tutorial on calculus and its physics applications

AST121 TAs

January 22, 2013

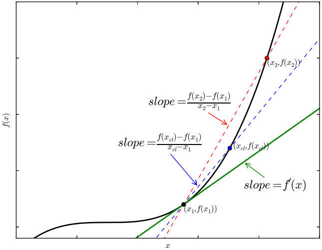
OUTLINE

- Differentiation as a rate of change
- Uniform acceleration
- Rules of differentiation - product rule, chain rule
- Applications
- Worked example:simple harmonic oscillator

DERIVATIVE AS A RATE OF CHANGE

- The average rate of change of a function is the slope of the secant line $m = \frac{rise}{run} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
- The instantaneous rate of change is the slope of the *tangent* line - let the “run” $\Delta x = x_2 - x_1 \to 0$
- This is exactly the notion of the derivative:

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Rules of differentiation

- Derivative of a constant is zero: $\frac{dc}{dx} = 0$
- Derivative of a power: $\frac{d}{dx}x^n = nx^{n-1}$
- Derivative of the exponential: $\frac{d}{dx}\exp(x) = \exp(x)$
- Derivative of the logarithm: $\frac{d}{dx}\log(x) = \frac{1}{x}$
- Derivatives of sine and cosine: $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$

Application: motion due to gravity

- The vertical position of a ball thrown up in the air with initial speed of 20 m/s, can be described by the equation $r(t) = -4.9t^2 + 20t$
- Let's compute the velocity and acceleration of the ball as a function of time
- The velocity is the derivative of the position, $v = \frac{dr}{dt} = \frac{d}{dt}(-4.9t^2 + 20t) = -9.8t + 20$
- The acceleration is the derivative of the velocity, $a = \frac{dv}{dt} = \frac{d}{dt}(-9.8t + 20) = -9.8$
- So in this problem we have assumed that the acceleration due to gravity is constant, which is of course only an approximately true.

Maximum height - 2 ways

Question: what is the maximum height the ball can achieve?

- 2 ways to solve this problem.
  1. At the maximum height the velocity of the ball must be 0, so then $v = -9.8t + 20 = 0$, so this happens at $t_{max} = 2.04$ seconds. The position of the ball at that time is the maximum height, $r_{max} = -4.9(2.04)^2 + 20(2.04) = 20.4$ meters
  2. *Energy conservation*. When we ball is on the ground we can regard it as only having kinetic energy, so that $E_i = K_i = \frac{1}{2}mv_0^2$. At the top, the energy must be purely potential, so that $E_f = U_f = mgr_{max}$. Then we have $r_{max} = \frac{1}{2}v_0^2/g = 20.4$ meters

Rules of differentiation, part II

- *Product rule*. Let $h(x) = f(x)g(x)$. Then

$$\frac{dh}{dx} = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$

- *Chain rule*. Let $h(x) = f(g(x))$. Then

$$\frac{dh}{dx} = \frac{df(g(x))}{dg}\frac{dg}{dx}$$

Formally, the dg's "cancel out"!

Applications: exponential decay

- The radioactive decay of a substance (which powers supernova afterglows!) can be described by the equation

$$A = A_0 e^{-\lambda t}$$

- Question: What is the meaning of the parameter λ? What *units* must it have?
- Consider the amount of time it would take the substance to decay to 1/e of it's starting amount: $\frac{A_0}{e} = A_0 e^{-\lambda t}$ which gives $e^{-1} = e^{-\lambda t}$ and thus $t_{decay} = \frac{1}{\lambda}$. Evidently then, the parameter λ is 1 over the time it takes for the amount to decrease by a factor of e. Now,$\lambda = 1/t_{decay}$ so that λ has units of 1/time. Notice that λt is unitless. This must happen: one can never have any quantity with units inside an exponential (or a log). This is useful check to make sure you have not done anything wrong.

Exponential decay: continued

Question:What is the rate of decay? What does the ratio $|\frac{A}{\frac{dA}{dt}}|$ represent?

- The rate of decay is computed by using the chain rule: $\frac{dA}{dt} = -\lambda A_0 e^{-\lambda t}$
- Notice $|\frac{A}{\frac{dA}{dt}}| = \frac{1}{\lambda} = t_{decay}$. This is a simple example of a *characteristic timescale*
- Suppose now you discover an element with $\lambda = \sqrt{Ct}$. What is the rate of decay in that case?
- Use the chain rule AND the product rule:

$$\frac{dA}{dt} = -\frac{d}{dt}(\lambda(t)t)A_0 e^{-\lambda(t)t}$$

$$\frac{dA}{dt} = -\left(\frac{d\lambda(t)}{dt}t + \lambda(t)\frac{dt}{dt}\right)A = -\left(\frac{1}{2}\sqrt{\frac{C}{t}}t + \sqrt{Ct}\right)A$$

$$\frac{dA}{dt} = -\frac{3}{2}\sqrt{Ct}A$$

SPRINGS!

- Imagine a mass on a stiff spring, which you compress, say by x centimetres. How would you describe the motion of the mass?

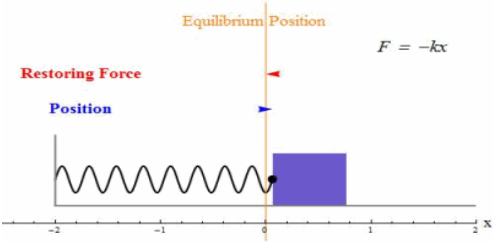


Figure: Taken from http://mathforum.org/mathimages/index.php/Image:SolveHarmonic.gif

The motion is *periodic* with some period T

Simple harmonic motion

Question: What is differential equation describing the motion?

- We know that the force acting on the mass is just the force due to the spring which has the form $F = -kx$ (the minus sign is important!)
- Newton's second law states $F = ma$. Thus

$$ma = -kx$$

- Recall that the velocity $v = \frac{dx}{dt}$, and the acceleration is $a = \frac{dv}{dt}$ so that $a = \frac{d^2x}{dt^2}$. Thus,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

SOLUTION TO THE EQUATIONS OF MOTION

Problem: Show that $x = A\cos(\omega t + \phi)$ is a solution to $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. When is this true? Interpret ω.

- This is an exercise in using chain and product rule. We need $\frac{d^2x}{dt^2}$ so first compute $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{d}{dt}\left(A\cos(\omega t + \phi)\right) = -\omega A \sin(\omega t + phi)$$

- The second derivative is then

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

Solution to the equations of motion II

- Plug into the equation of motion:

$$-\omega^2 A\cos(\omega t+\phi)+\frac{k}{m}A\cos(\omega t+\phi)=\left(-\omega^2+\frac{k}{m}\right)A\cos(\omega t+\phi)$$

- Evidently this equals 0 if $-\omega^2+\frac{k}{m}=0$, so that $\omega=\sqrt{\frac{k}{m}}$
- ω is the *angular frequency* of the periodic motion. The period is given by $T=\frac{2\pi}{\omega}$ (Think about this way: a circle has 2π radians and ω is angular speed so the time to complete 1 revolution is $\frac{2\pi}{\omega}$)
- Incidentally, A, ϕ are the *amplitude* and the *phase* of the motion and essentially describe the freedom of choosing the initial displacement and the initial velocity of the mass.

Simple harmonic motion: Conservation of energy

- When the mass is moving, it has kinetic energy, given as usual by $K = \frac{1}{2}mv^2$. Where is that energy coming from?
- The energy came from the *elastic potential* energy in the spring which has the form $U = \frac{1}{2}kx^2$.
- The total energy is then $E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$
- Based on conservation of energy, at which position is the mass moving the fastest?
- Since the potential energy depends on the distance from the relaxed state, when the mass is at that position (i.e. at $x = 0$) it will not have any potential energy, and thus all the energy must be kinetic. Hence it follows that the mass would be travelling the fastest when $x = 0$.

CHALLENGE

Suppose you have finally fulfilled your childhood dream and have created a tunnel all the way through the Earth to the other side, going right through the centre. Suppose you drop a ball into this tunnel. Can you tell what will happen to this ball?

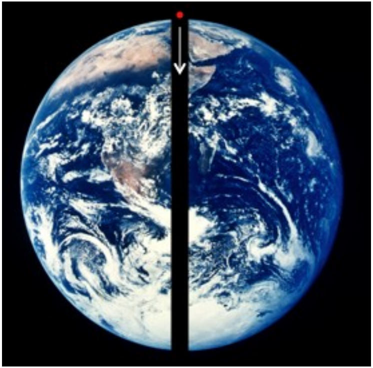


Figure: http://bit.ly/PO5GVm