Worth: 3% **Due:** By 12 noon on Tuesday 28 February.

---

1. (a)
   - *In Symbolic Notation:* $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m^2 - n^2$ is odd $\Rightarrow (m+n)^2$ is odd.
   - Proof structure — an direct proof of a universally-quantified implication.
   - **Proof:**

     Assume $m \in \mathbb{Z}, n \in \mathbb{Z}$.
       Assume $m^2 - n^2$ is odd.
         Then $\exists k \in \mathbb{Z}, m^2 - n^2 = 2k + 1$. # definition of odd
         Let k_0 be such that $m^2 - n^2 = 2k_0 + 1$.
         Then $m^2 = n^2 + 2k_0 + 1$.
         Now $$\begin{aligned}(m+n)^2 &= m^2 + 2mn + n^2 && \text{\# algebra}\\ &= (n^2 + 2k_0 + 1) + 2mn + n^2 && \text{\# substitute earlier result about } m^2.\\ &= 2n^2 + 2k_0 + 2mn + 1\\ &= 2(n^2 + k_0 + mn) + 1.\end{aligned}$$
         Let $k_1 = n^2 + k_0 + mn$.
         Then $k_1 \in \mathbb{Z}$. # since $\mathbb{Z}$ closed under multiplication and addition
         Then $(m+n)^2 = 2k_1 + 1$. # substitution
         Then $\exists k \in \mathbb{Z}, (m+n)^2 = 2k + 1$.
         Then $(m+n)^2$ is odd. # definition of odd
       Then $m^2 - n^2$ is odd $\Rightarrow (m+n)^2$ is odd.
     Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m^2 - n^2$ is odd $\Rightarrow (m+n)^2$ is odd.

   (b)
   - *In Symbolic Notation:* $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m+n)^2$ is odd $\Rightarrow m^2 - n^2$ is odd.
   - Proof structure — an direct proof of a universally-quantified implication.
   - **Proof:**

     Assume $m \in \mathbb{Z}, n \in \mathbb{Z}$.
       Assume $(m+n)^2$ is odd.
         Then $\exists k \in \mathbb{Z}, (m+n)^2 = 2k + 1$. # definition of odd
         Let k_0 be such that $(m+n)^2 = 2k_0 + 1$.
         Then $m^2 + 2mn + n^2 = 2k_0 + 1$.
         Then $m^2 = 2k_0 + 1 - 2mn - n^2$.
         Then $$\begin{aligned}m^2 - n^2 &= 2k_0 + 1 - 2mn - n^2 - n^2 && \text{\# subtract } n^2 \text{ from both sides}\\ &= 2k_0 + 1 - 2mn - 2n^2\\ &= 2(k_0 - mn - n^2) + 1.\end{aligned}$$
         Let $k_1 = k_0 - mn - n^2$.
         Then $k_1 \in \mathbb{Z}$. # since $\mathbb{Z}$ closed under multiplication and subtraction
         Then $m^2 - n^2 = 2k_1 + 1$. # substitution
         Then $\exists k \in \mathbb{Z}, m^2 - n^2 = 2k + 1$.
         Then $m^2 - n^2$ is odd. # definition of odd
       Then $(m+n)^2$ is odd $\Rightarrow m^2 - n^2$ is odd.
     Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m+n)^2$ is odd $\Rightarrow m^2 - n^2$ is odd.

   (c) Since $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m^2 - n^2$ is odd $\Rightarrow (m+n)^2$ is odd and $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m+n)^2$ is odd $\Rightarrow m^2 - n^2$ is odd, it follows from the bi-implication rule that $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m^2 - n^2$ is odd $\iff (m+n)^2$ is odd.

2.
   - *In Symbolic Notation:* $\forall x \in \mathbb{R}, x^4 + 2x^2 - 2x < 0 \Rightarrow 0 < x < 1$.
   - Proof structure — an indirect proof of a universally-quantified implication.
     $\forall x \in \mathbb{R}, x \leqslant 0 \vee x \geqslant 1 \Rightarrow x^4 + 2x^2 - 2x \geqslant 0$.

- **Proof:**

  Assume $x \in \mathbb{R}$.
  Assume $x \leqslant 0 \vee x \geqslant 1$
  Case 1 : Assume $x \leqslant 0$.
  Then $2x \leqslant 0$.
  Then $-2x \geqslant 0$.
  Then $2x^2 \geqslant 0$.
  Then $x^4 \geqslant 0$.
  Then $x^4 + 2x^2 - 2x \geqslant 0$.
  Case 2 : Assume $x \geqslant 1$.
  Then $x^4 \geqslant 0$.
  Then $\begin{aligned} 2x^2 - 2x &= 2x(x-1) \\ &\geqslant 0 \end{aligned}$.
  Then $x^4 + 2x^2 - 2x \geqslant 0$.
  Then $x \leqslant 0 \vee x \geqslant 1 \Rightarrow x^4 + 2x^2 - 2x \geqslant 0$.
  Then $\forall x \in \mathbb{R}, x \leqslant 0 \vee x \geqslant 1 \Rightarrow x^4 + 2x^2 - 2x \geqslant 0$.
  Then $\forall x \in \mathbb{R}, x^4 + 2x^2 - 2x < 0 \Rightarrow 0 < x < 1$.

3. (a)
   - *In Symbolic Notation:* $\forall x \in \mathbb{R}, \lceil -x \rceil = -\lfloor x \rfloor$.
   - Proof structure — an direct proof of a universally-quantified implication.
   - **Proof:**

     Assume $x \in \mathbb{R}$.
     Let $y = \lfloor x \rfloor$.
     Then $y \in \mathbb{Z} \wedge y \leqslant x \wedge (\forall z \in Z, z \leqslant x \Rightarrow z \leqslant y)$. # by definition of floor
     Then $y \in \mathbb{Z}$ # fact (1)
     Then $y \leqslant x$ # fact (2)
     Then $\forall z \in Z, z \leqslant x \Rightarrow z \leqslant y$. # fact (3)
     Then $-y \in \mathbb{Z}$ # from fact (1) and $\mathbb{Z}$ closed under multiplication
     Then $-y \geqslant -x$ # from fact (2) and $a \leqslant b \iff -a \geqslant -b$
     Assume $w \in \mathbb{Z}$
     Assume $w \geqslant -x$
     Then $-w \leqslant x$.
     Then $-w \in \mathbb{Z}$.
     Then $-w \leqslant y$. # from fact (3)
     Then $w \geqslant -y$.
     Then $w \geqslant -x \Rightarrow w \geqslant -y$
     Then $\forall w \in \mathbb{Z}, w \geqslant -x \Rightarrow w \geqslant -y$
     Then $-y \in \mathbb{Z} \wedge -y \geqslant -x \wedge (\forall w \in Z, w \geqslant -x \Rightarrow w \geqslant -y)$
     Then $\lceil -x \rceil = -y$.
     Then $\lceil -x \rceil = -\lfloor x \rfloor$.
     Then $\forall x \in \mathbb{R}, \lceil -x \rceil = -\lfloor x \rfloor$.

   (b) The given statement is false and so we prove that its negation is true.

   - *In Symbolic Notation:* $\neg(\forall x \in \mathbb{R}, \forall n \in \mathbb{N}, \lfloor n \cdot x \rfloor = n \cdot \lfloor x \rfloor)$ or $(\exists x \in \mathbb{R}, \exists n \in \mathbb{N}, \lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor)$.
   - Proof structure — a direct proof of an existential.

- **Proof:**

  Let $x_0 = 1.75$.
  Then $x_0 \in \mathbb{R}$. # well known
  Let $n_0 = 4$.
  Then $n_0 \in \mathbb{Z}$. # well known

  $$\begin{array}{lrcl} \text{Then} & \lfloor n_0 \cdot x_0 \rfloor & = & \lfloor 4 \cdot 1.75 \rfloor \\ & & = & \lfloor 7.0 \rfloor \\ & & = & 7 \\ \text{Then} & n_0 \cdot \lfloor x_0 \rfloor & = & 4 \cdot \lfloor 1.75 \rfloor \\ & & = & 4 \cdot 1\rfloor \\ & & = & 4 \end{array}$$

  Then $\lfloor n_0 \cdot x_0 \rfloor \neq n_0 \cdot \lfloor x_0 \rfloor$. # since $7 \neq 4$
  Then $(\exists x \in \mathbb{R}, \exists n \in \mathbb{N}, \lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor)$.
  Then $\neg\,(\forall x \in \mathbb{R}, \forall n \in \mathbb{N}, \lfloor n \cdot x \rfloor = n \cdot \lfloor x \rfloor)$.
  Then the given statement is false.