5.4 Vector derivatives

∇ **del operator** is the main tool in defining vector derivatives

$\frac{d}{dx}$ is the derivative operator, applied to scalar functions $f(x)$

$\frac{d}{dx} \cdots f(x) \to \frac{d}{dx}f \to$ The result is a new function

$$\nabla(?) = \left(\frac{\partial}{\partial x_1}\boxed{?}, \ldots, \frac{\partial}{\partial x_n}\boxed{?}\right)$$

eg $\underset{\text{grad } f}{\nabla f(x)} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ a vector That captures direction of fastest change in values of $f(x)$

for a v.f. $\mathbb{F} = \langle F_1, \ldots, F_n \rangle$

We can apply ∇ in two ways

$$\nabla \cdot \mathbb{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

dot product, divergence, div $\mathbb{F}$. measures expansion of $\mathbb{F}$

on $\mathbb{R}^3$

$$\underset{\text{Curl } \mathbb{F}}{\nabla \times \mathbb{F}} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

measure rotational component of $\mathbb{F}$

$$= \left\langle \frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Algebraic properties:

$\nabla \cdot (\mathbb{F} \pm \mathbb{G}) = \nabla \cdot \mathbb{F} \pm \nabla \cdot \mathbb{G}$

$\nabla \times (\quad) = \nabla \times \mathbb{F} \pm \nabla \times \mathbb{F}$

etc

Second derivatives:

$\nabla \cdot \nabla f = \nabla^2 f$

Laplacian $\to$ is like acceleration in higher dimension

$\nabla \times \mathbb{F} = 0$ can be generalized to $\mathbb{R}^n$ by assuming

$$\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k} = 0$$

See 5.61

$\nabla \times \nabla f = 0$

$\nabla \cdot \nabla \times \mathbb{F} = 0$

$\nabla \cdot \to \nabla \times \to \nabla$

$0 \quad 0$

not!

grad f: applies on scalar f gives vector field

div $\mathbb{F}$: applies on a v.f $\mathbb{F}$, produces a scalar function

curl $\mathbb{F}$: applies on a v.f. produces a v.f.

Product rules (many possibilities)

$\nabla fg = f\nabla g + g\nabla f$

$\nabla \mathbb{F} \cdot \mathbb{G} = (\mathbb{F} \cdot \nabla)\mathbb{G} + \mathbb{F} \times (\text{Curl } \mathbb{G}) + (\mathbb{G} \cdot \nabla)\mathbb{F} + \mathbb{G} \times (\text{Curl } \mathbb{F})$!

$\nabla \cdot f\mathbb{F} =$ See Pg 237

$\nabla \times f\mathbb{F} =$

$\nabla \times (\mathbb{F} \times \mathbb{G}) = \cdots$

$\nabla \cdot \mathbb{F} \times \mathbb{G} = \cdots$