Alternative Estimate of the Dispersion ϕ

The assumed when fitting a GLM is given at the bottom of page 43 of the lecture notes:

$$\phi_{assumed} = \begin{cases} MSE & \text{for Normal GLMs} \\ 1 & \text{for Binomial/Poisson GLMs} \\ \text{CV estimator} & \text{for Gamma GLMs} \end{cases}$$

An alternative estimate of ϕ_{alt} for Binomial GLMs fitted to aggregate data, Poisson GLMs & even Gamma GLMs can be calculated as follows:

$$\phi_{alt} = \frac{D(Y, \hat{Y})}{n-p} = \frac{\text{Residual deviance}}{\text{Residual df}}$$

If $\phi_{alt} = \phi_{assumed}$ model is "good"

$\phi_{alt} < \phi_{assumed}$ model is under-dispersed

$\phi_{alt} > \phi_{assumed}$ model is over-dispersed

Note for normally distributed models

$$\hat{\phi}_{assumed} = MSE = \hat{\sigma}^2 = S^2 = MS_{Error}$$

$$\hat{\phi}_{alt} = \frac{D(Y, \hat{Y})}{(n-p)} = \frac{\Sigma e_i^2}{(n-p)} = \frac{SS_{Error}}{n-p} = MS_{Error}$$

So under or over-dispersion is NOT an issue with ordinary (normally distributed) linear models

Tests with the Analysis of Deviance (ANODEV) table

1. Nested model test

larger model ← # parameters $= p = k+1$

$$E[g(Y)] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots \beta_l X_l + \underbrace{\dots \beta_k X_k}_{\text{additional terms}}$$

smaller model ← # parameters $= q = l+1$

$$E[g(Y)] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots \beta_l X_l$$

Scaled drop-in-deviance test

$$\frac{D^*(\hat{Y}_s, \hat{Y}_L)}{\hat{\phi}_L} \geq \chi^2_{(p-q)}(1-\alpha)$$

[cf ordinary linear models Nested model F test

$$F = \frac{MS\ \text{additions}}{MS\ \text{error, larger model}} \geq F_{\nu_1,\nu_2}(1-\alpha)$$]

eg to testing $H_0: \beta_{l+1} = \beta_{l+2} = \dots \beta_{k-1} = \beta_k = 0$

vs H_A : at least one $\beta_{l+1}, \dots \beta_k \neq 0$

test statistic is the scaled drop-in-deviance

critical value $\geq \chi^2_{(p-q)}(0.95)$ for $\alpha = 0.05$

This is the test we get when we use

anova (model, test = "Chisq") in R

for Binomial, Poisson models

ANODEV table tests continued

2. Goodness of fit test on the residual deviance

This is a χ^2 test on the scaled residual deviance (which can be viewed as the drop-in-deviance between the current model and a fully saturated model) & therefore can be used as a formal test for over- or under-dispersion

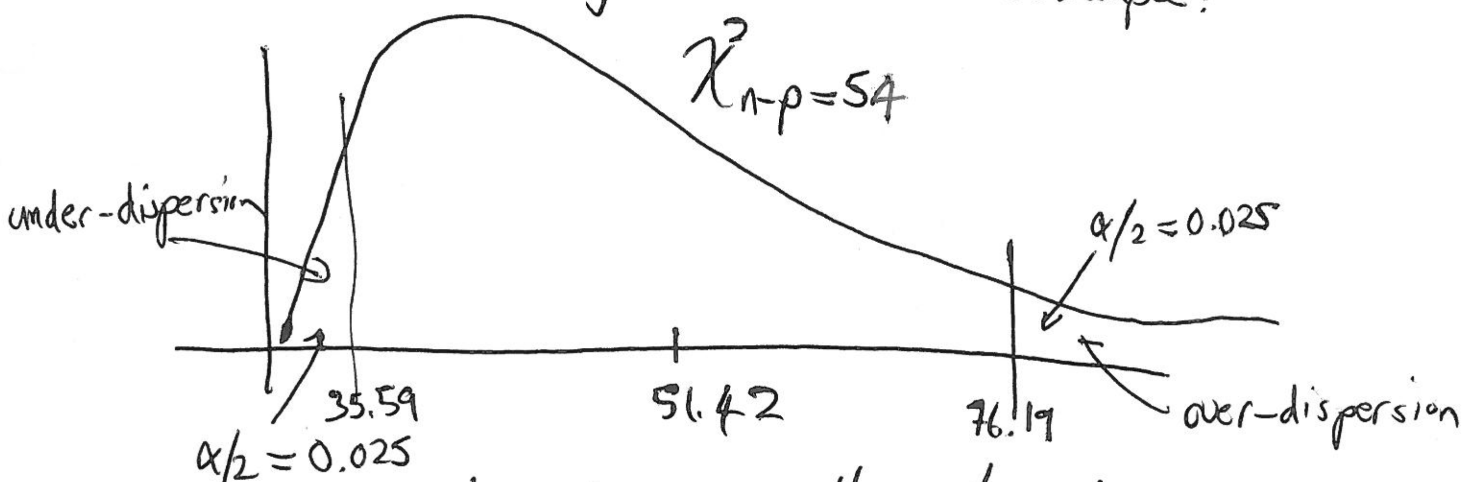
$$H_0 : \phi = \phi_{Assumed}$$

vs
$$\begin{cases} H_A : \phi > \phi_{Assumed} & \text{over-dispersion} \\ H_A : \phi < \phi_{Assumed} & \text{under-dispersion} \\ H_A : \phi \neq \phi_{Assumed} & \text{either of the above} \end{cases}$$

For binomial or Poisson models $\phi_{Assumed} = 1$

We can estimate ϕ using $\hat{\phi}_{Alt} = \dfrac{D(Y, \hat{Y})}{n-p}$

& in the carins.glm1 case in the example:



$$H_0 : \phi = 1 \quad \text{vs} \quad H_A : \phi = 1$$

ie do not reject H_0 in this example as 51.42 lies within (35.59, 76.19)

Note if there were good a priori reasons to suspect under or over-dispersion we could have done a more powerful 1-tailed test

Note on the "rule of thumb" for under/over dispersion on page 57 of the brick

$$E[\chi^2_{n-p}] = n-p \qquad V(\chi^2_{n-p}) = 2(n-p)$$

$D(Y, \hat{Y})$ residual deviance

So $$E\left[\frac{\hat{\phi}_{Alt}}{\phi_{Assumed}}\right] = E\left[\frac{\frac{\sum d_i^2}{n-p}}{1}\right] = E\left[\frac{\sum d_i^2}{n-p}\right]$$

$$= \frac{1}{n-p} E\left[\sum d_i^2\right] = \frac{n-p}{n-p} = 1$$

& $$V\left[\frac{\sum d_i^2}{n-p}\right] = \frac{1}{(n-p)^2} V\left[\sum d_i^2\right] = \frac{2(n-p)}{(n-p)^2} = \frac{2}{(n-p)}$$

So, "rule of thumb" for over-dispersion:

$$\frac{\sum d_i^2}{n-p} > 1 + 3\sqrt{\frac{2}{(n-p)}}$$

$$\equiv \hat{\phi}_{Alt} > E\left[\frac{\hat{\phi}_{Alt}}{\phi_{Assumed}}\right] + 3\sqrt{V\left[\frac{\hat{\phi}_{Alt}}{\phi_{Assumed}}\right]}$$

What about a "rule of thumb" for under-dispersion:

$1 - 3\sqrt{\frac{2}{(n-p)}}$ is likely to go negative

Conclusion: ditch this rule of thumb & use the goodness of fit drop-in-deviance test described in the earlier pages of today's lecture.