Australian National University

STUDENT NUMBER......................

Mid-Semester Examination, First Semester 2016

Financial Mathematics

STAT2032/STAT6046

Writing period: 90 minutes duration

Study period: 0 minutes duration

Permitted materials: Non-programmable calculators

Dictionaries (must be clear of all annotations)

Total Marks Available: 40

Instructions to Candidates:

- ***Please write your student number in the space provided at the top of this page.***
- ***Attempt ALL questions.***
- ***All answers are to be written on the exam paper.***
- ***Please hand in the exam paper before you leave the room.***
- ***A formula sheet and the compound interest tables are attached at the end of the exam paper. You may detach these for your convenience.***
- ***For Questions 2 to 4, you need to show all the working steps in obtaining the solution. Marks may be deducted for failure to show appropriate calculations or formulae.***
- ***If you need additional space, please use the rear of the page and state clearly on the front that you have done so.***

	Q1	Q2	Q3	Q4	Total
Marks	**16**	**9**	**7**	**8**	**40**
Score					

QUESTION 1 (16 marks)
Please write down your answer (either A, B, C, D or E) clearly in the space provided.

(a) **(2 marks) A continuous payment stream is received for a period of T years. The rate of payment at time t is $e^{-0.03t}$ and the force of interest $\delta(t)$ is a constant value of 0.09. Denote $v(t)$ as the present value at time 0 of a \$1 to be payable at time t, which of the following does not represent the present value of this payment stream?**

A. $\int_0^T e^{-0.03t} v(t)\, dt$

B. $\int_0^T e^{-0.03t} e^{-0.09T}\, dt$

C. $e^{-0.09T} \int_0^T \frac{e^{-0.03t}}{v(T-t)}\, dt$

D. $\int_0^T e^{-0.12t}\, dt$

E. $\frac{1}{0.12}\left(1 - e^{-0.12T}\right)$

Answer: (B). It is wrong because of the expression $e^{-0.09T}$, it should be $e^{-0.09t}$.

(b) **(2 marks) Which of the following relationships is wrong?**

A. $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n}|} - v^n$

B. $s_{\overline{n}|} = (1+i)^{n-1} \ddot{a}_{\overline{n}|}$

C. $d^{(p)} = \frac{i^p}{1+\frac{i^{(p)}}{p}}$

D. $(Ia)_{\overline{n}|} + (Da)_{\overline{n-1}|} = n a_{\overline{n}|}$

E. $(Ia)_{\overline{\infty}|} = \frac{1}{i^2 v}$

Answer: (C). It is wrong because the numerator should be $i^{(p)}$.

QUESTION 1 (continued)

(c) (2 marks) A loan of \$50,000 is repayable by annual repayments in arrears for the next 12 years at an effective annual rate of interest i. For the first 4-year period, the payments are K per year; for the second 4-year period, the payments are $2K$ per year; and for the last 4-year period, the payments are $3K$ per year. The expression for K is

A. $\frac{50{,}000}{3a_{\overline{12}|}-a_{\overline{8}|}-a_{\overline{4}|}}$

B. $\frac{50{,}000}{3a_{\overline{12}|}-2a_{\overline{8}|}-a_{\overline{4}|}}$

C. $\frac{50{,}000}{4a_{\overline{12}|}-a_{\overline{8}|}-2a_{\overline{4}|}}$

D. $\frac{50{,}000}{4a_{\overline{12}|}-2a_{\overline{8}|}-a_{\overline{4}|}}$

E. None of the above

Answer: (A). This question involves identifying the cash flows structure using expressions of annuity functions.

(d) (3 marks) A loan of \$20,000 is repayable by level monthly repayments of \$450 made in arrears for as long as necessary. If the nominal rate of interest is 9% per annum compounded monthly, the amount of capital repayment in the 25th repayment is

A. 353.87

B. 356.43

C. 358.92

D. 361.62

E. None of the above

Answer: (C). Only retrospective method is applicable here because the length of repayment is unknown but it is stated as "as long as necessary".

QUESTION 1 (continued)

(e) **(3 marks) Which of the following statements is wrong?**

A. **Under a positive inflation environment, the real interest rate is always less than the money rate.**

B. **The implied constant force of interest for any given period is always less than the effective periodic rate of interest.**

C. **A perpetuity due is a special case of an annuity due with its term n tends to infinity.**

D. **A continuous annuity is a special case of an annuity due payable p times a year when p tends to infinity.**

E. **Under an identical first annual payment of \$1, a geometrically increasing annuity immediate with a constant growth rate of g is always more valuable than an arithmetically increasing annuity immediate with a fixed payment increment of r when $g > r$.**

Answer: (B). It is wrong because for a period of less than a year, the implied constant force of interest can be greater than the effective periodic rate of interest. For instance, if the effective semi-annual rate is 4%, then the implied constant force of interest is $e^{0.5\delta} = 1.04 \Rightarrow \delta = 7.84\%$.

(f) **(4 marks) Consider an increasing annuity immediate that pays \$1 at the end of years 4 to 6, \$2 at the end of years 8 to 10, \$3 at the end of years 12 to 14, ..., \$$k$ at the end of years $4k, 4k+1$ and $4k+2$. Under a constant effective annual rate of interest i, when $k \to \infty$, the present value of this annuity at time 0.**

A. $\frac{a_{\overline{3}|}}{\left((1+i)^4-1\right)}$

B. $\frac{\ddot{a}_{\overline{3}|}}{i^4 d^4}$

C. $\frac{\ddot{a}_{\overline{3}|}}{\left((1+i)^4-1\right)d^4}$

D. $\frac{\ddot{a}_{\overline{3}|}}{(1+i)^4-1+(1+i)^{-4}}$

E. **None of the above**

Answer: (E). The correct answer is $\frac{\ddot{a}_{\overline{3}|}}{\left((1+i)^4-1\right)\left(1-(1+i)^{-4}\right)} = \frac{\ddot{a}_{\overline{3}|}}{(1+i)^4-2+(1+i)^{-4}}$

QUESTION 2 (9 marks)

(a) **(2 marks) Given a force of interest $\delta(t) = 0.04t$ for $0 \le t \le 3$, calculate the value of $s_{\overline{3}|}$.**

$$\begin{aligned} s_{\overline{3}|} &= A(1,3) + A(2,3) + A(3,3) \\ &= e^{\int_1^3 0.04t\,dt} + e^{\int_2^3 0.04t\,dt} + 1 \\ &= e^{0.02t^2\big|_1^3} + e^{0.02t^2\big|_2^3} + 1 \\ &= e^{0.16} + e^{0.1} + 1 \\ &= 3.278681789 = 3.28 \end{aligned}$$

Common mistakes include didn't express the future value as a summation of $A(s,t)$ but instead rely on the formula directly, which is wrong, because the formula is only applicable for constant force of interest.

QUESTION 2 (continued)

(b) **(2 marks) A 9-year deferred annuity-immediate with \$1,700 payable annually will start after a deferred period of 4 years. If the effective quarterly rate of interest in the first 6 years is 2% and the nominal rate of interest afterwards is 10% compounded semi-annually, calculate the present value of this annuity at time 0.**

$$\begin{aligned} PV_0 &= 1700 v_{0.02}^{20}\Big(1 + v_{0.02}^{4}\ddot{a}_{\overline{8}|i=(1+\frac{0.1}{2})^2-1}\Big) \\ &= 1700(1.02^{-20})\Big(1 + 1.02^{-4}\big(\frac{1-1.1025^{-8}}{\frac{0.1025}{1.1025}}\big)\Big) \\ &= 7304.458877 = 7304.46 \end{aligned}$$

Common mistakes include mis-interpreting the 9-year to include the deferred period of 4 years. Other mistakes include the wrong treatment of the effective quarterly rate of 2% in the first 10 (should be 6) years and the nominal semi-annual rate of 10% afterwards.

QUESTION 2 (continued)

(c) **(3 marks) Consider a 10-year continuous annuity that has a payment rate of \$3,500 during the first year, \$4,000 during the second year, \$4,500 during the third year and so on, that is, the payment rate increases by \$500 per annum and will apply throughout every annual period. Given an effective annual rate of interest of 8%, calculate the present value of this annuity at time 0.**

$$\begin{aligned}
PV_0 &= 3500\overline{a}_{\overline{1}|} + 4000v\overline{a}_{\overline{1}|} + 4500v^2\overline{a}_{\overline{1}|} + \cdots + 8000v^9\overline{a}_{\overline{1}|} \\
&= \sum_{t=0}^{9}\big(3500 + 500t\big)v^t\overline{a}_{\overline{1}|} \\
&= \overline{a}_{\overline{1}|}\Big(\sum_{t=0}^{9}3500v^t + \sum_{t=1}^{9}500tv^t\Big) \\
&= \overline{a}_{\overline{1}|}\Big(3500\ddot{a}_{\overline{10}|} + 500(Ia)_{\overline{9}|}\Big) \\
&= \frac{i}{\delta}a_{\overline{1}|}\Big(3500(1+i)a_{\overline{10}|} + 500(Ia)_{\overline{9}|}\Big) @\, i = 0.08 \\
&= 37912.95432 = 37912.95 \\
\textbf{OR} &= 3000\overline{a}_{\overline{10}|} + 500(I\overline{a})_{\overline{10}|} \\
&= 3000\frac{i}{\delta}a_{\overline{10}|} + 500\Big(\frac{(1+i)a_{\overline{10}|} - 10v^{10}}{\delta}\Big) \\
&= 37914.36149 = 37914.36 \\
\textbf{Exact} &= 37913.93681
\end{aligned}$$

Common mistakes include didn't notice this is a continuous annuity and mistakenly identified the second part as $(\bar{I}\bar{a})_{\overline{9}|}$.

QUESTION 2 (continued)

(d) (2 marks) Consider a level (i.e., constant) loan repayment schedule for a fixed rate amortized loan repayable p times a year in arrears for a period of n years, the ratio of the last interest payment to the level repayment can be expressed using only the payment frequency p and the effective annual rate of interest i. In other words, the original loan amount borrowed at time 0 L_0 is irrelevant.

True/False. Write down the expression if it is true, otherwise explain why the statement above is false.

True. Define X as the amount of level repayment,

$$\begin{aligned}
\frac{I_n}{X} &= \frac{\frac{i^{(p)}}{p} L_{n-1}}{X} \\
&= \frac{\frac{i^{(p)}}{p} \frac{X}{1+\frac{i^{(p)}}{p}}}{X} \\
&= \frac{\frac{i^{(p)}}{p}}{1+\frac{i^{(p)}}{p}} \\
&= \frac{(1+i)^{\frac{1}{p}}-1}{(1+i)^{\frac{1}{p}}} \\
&= 1-(1+i)^{-\frac{1}{p}}
\end{aligned}$$

Common mistakes include the wrong interest rate applied to the loan oustanding, it should be $\frac{i^{(p)}}{p}$, not i or $i^{(p)}$. Other mistakes include didn't divide by the level repayment X to obtain the required ratio.

QUESTION 3 (7 marks)

(a) **(1 mark) Consider a loan of \$180,000 to be repayable by level monthly installments of \$2032.46 for a period of n years. Calculate the value of n if the flat rate for this transaction is 4.46%.**

$$\begin{aligned} 0.0446 &= \frac{n(12)(2032.46) - 180000}{n(180000)} \\ \Rightarrow n &= 11 \end{aligned}$$

Common mistakes include mistakenly treated flat rate as APR.

QUESTION 3 (continued)

(b) **(3 marks) Hence, calculate the APR (annual percentage rate of charge) for this loan transaction.**

$$180000 = 12(2032.46)a_{\overline{11}|}^{(12)} = 12(2032.46)\left(\frac{1-(1+i)^{-11}}{12\big((1+i)^{\frac{1}{12}}-1\big)}\right)$$

$$f(i) = 2032.46\left(\frac{1-(1+i)^{-11}}{(1+i)^{\frac{1}{12}}-1}\right) - 180000 = 0$$

$$f(0.08) = 411.456562, \qquad f(0.081) = -383.286384$$

$$i \approx \frac{0.081(411.456562) - 0.08(-383.286384)}{411.456562 - (-383.286384)} = 0.080517722 = 8.05\%$$

Common mistakes include mistakenly treated APR as flat rate. Also, linear interpolation should be performed to obtain more credit.

QUESTION 3 (continued)

(c) (3 marks) Using the APR obtained above, calculate the total interest payments in the first 2 years.

$$\begin{aligned} L_{24} &= \frac{2032.46}{(1.0805)^{\frac{1}{12}} - 1}\left(1 - 1.0805^{-\frac{1}{12}(108)}\right) \\ &= 157573.6408 \\ \sum_{k=1}^{24} I_k &= 24(2032.46) - \left(L_0 - L_{24}\right) \\ &= 26352.68081 = 26352.68 \end{aligned}$$

Generally well done for those of you who managed to solve parts (a) and (b).

QUESTION 4 (8 marks)

(a) (4 marks) Consider a 30-year annuity that has a monthly payment of \$1 payable in advance in the first year, a monthly payment of \$1.05 payable in advance in the second year, a monthly payment of \$$1.05^2$ payable in advance in the third year and so on, that is, the amount of monthly payment grows geometrically at a rate of 5% for each subsequent year. Given a 7% constant effective annual rate of interest, calculate the present value of this annuity at time 0.

$$\begin{aligned}
PV_0 &= 12\ddot{a}^{(12)}_{\overline{1}|i=0.07} + 12(1.05)v\ddot{a}^{(12)}_{\overline{1}|i=0.07} + 12(1.05^2)v^2\ddot{a}^{(12)}_{\overline{1}|i=0.07} + \cdots + 12(1.05^{29})v^{29}\ddot{a}^{(12)}_{\overline{1}|i=0.07} \\
&= 12\ddot{a}^{(12)}_{\overline{1}|i=0.07}\Big(1 + 1.05v + 1.05^2v^2 + \cdots + 1.05^{29}v^{29}\Big) \\
&= 12\ddot{a}^{(12)}_{\overline{1}|i=0.07}\Big(1 + v_{i'} + v_{i'}^2 + \cdots + v_{i'}^{29}\Big) \qquad i' = \frac{1.07}{1.05} - 1 \\
&= 12\ddot{a}^{(12)}_{\overline{1}|i=0.07}\ddot{a}_{\overline{30}|i'=\frac{1.07}{1.05}-1} \\
&= 12\Big(\frac{1 - 1.07^{-1}}{12\big(1 - 1.07^{-\frac{1}{12}}\big)}\Big)\Big(\frac{1 - \big(\frac{1.07}{1.05}\big)^{-30}}{1 - \big(\frac{1.07}{1.05}\big)^{-1}}\Big) \\
&= 269.0751309 = 269.08
\end{aligned}$$

Common mistakes include quoting a wrong formula directly out of nowhere, didn't take into account the monthly cash flows, didn't notice the growth increment only happens once a year.

QUESTION 4 (continued)

(b) (4 marks) Consider an n-year annuity that pays $t(t+1)$ at the end of year t at an effective annual rate of interest i. From first principles, show that its present value at time 0 can be written as

$$\sum_{t=1}^{n} 2tv^{t-1} a_{\overline{n+1-t}|}$$

and subsequently be simplified to

$$\frac{2(I\ddot{a})_{\overline{n}|} - n(n+1)v^n}{i}$$

$$\begin{aligned}
PV_0 &= \sum_{t=1}^{n} t(t+1)v^t \\
&= 2v + 6v^2 + 12v^3 + 20v^4 + \cdots + n(n+1)v^n \\
&= 2\big(v + v^2 + \cdots + v^n\big) + 4v\big(v + v^2 + \cdots + v^{n-1}\big) + 6v^2\big(v + v^2 + \cdots + v^{n-2}\big) + \cdots + 2nv^{n-1}v \\
&= 2a_{\overline{n}|} + 4va_{\overline{n-1}|} + 6v^2 a_{\overline{n-2}|} + \cdots + 2nv^{n-1}a_{\overline{1}|} \\
&= \sum_{t=1}^{n} 2tv^{t-1} a_{\overline{n+1-t}|} = \sum_{t=1}^{n} 2tv^{t-1}\left(\frac{1 - v^{n+1-t}}{i}\right) \\
&= \frac{\sum_{t=1}^{n} 2tv^{t-1} - \sum_{t=1}^{n} 2tv^{n}}{i} \\
&= \frac{2(I\ddot{a})_{\overline{n}|} - 2v^n \frac{n(n+1)}{2}}{i} = \frac{2(I\ddot{a})_{\overline{n}|} - n(n+1)v^n}{i}
\end{aligned}$$

A handful of students managed to show the second part of the proof (the simplification) but not the first part from first principles.

END OF EXAMINATION

Formula Sheet for Mid-Semester Exam

1.
$$A(t_1, t_2) = \exp\Big(\int_{t_1}^{t_2} \delta(t)\, dt \Big)$$

2.
$$1 + i = \Big(1 + \frac{i^{(p)}}{p}\Big)^p = \Big(1 - \frac{d^{(p)}}{p}\Big)^{-p} = (1-d)^{-1}$$

3.
$$PV_t = \sum_{j:t_j \geq t} c_{t_j} v(t, t_j)$$

4.
$$PV(t, T_2) = \int_t^{T_2} \rho(s) \exp\Big(- \int_t^s \delta(u)\, du \Big)\, ds$$

5.
$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

6.
$$a_{\overline{n}|}^{(p)} = \frac{1 - v^n}{i^{(p)}}$$

7.
$$\overline{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$$

8.
$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

9.
$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

10.
$$(\overline{I}\overline{a})_{\overline{n}|} = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta}$$

11.
$$(I\overline{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

12.
$$i \approx \frac{i_2 f(i_1) - i_1 f(i_2)}{f(i_1) - f(i_2)}$$