APM462H1S, Winter 2014 , Assignment 1,

due: Monday February 3

Exercise 1. To approximate a function $g : [0,1] \to R$ by a nth order polynomial, one can minimize the function f defined by

$$f(a) = \int_0^1 (g(x) - p_a(x))^2 \, dx$$

where, for $a = (a_0, \ldots, a_n) \in E^{n+1}$, we use the notation

$$p_a(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k.$$

a. Show that f can be written in the form

$$f(a) = a^T Q a - 2b^T a + c$$

for a $(n+1) \times (n+1)$ matrix Q, a vector $b \in E^{n+1}$, and a number c. Find formulas for Q, b and c. It should be clear from your formula that Q is symmetric.

solution:

$$f(a) = \int_0^1 g(x)^2 dx - 2 \int g(x) p_a(x) dx + \int p_a(x)^2 \, dx.$$

And

$$\begin{aligned}
\int_0^1 p_a(x)^2 &= \int_0^1 (\sum_{j=0}^{n} a_j x^j)(\sum_{k=0}^{n} a_k x^k) dx \\
&= \int_0^1 \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k x^{j+k} dx \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \int_0^1 x^{j+k} dx \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \frac{1}{j+k+1} dx.
\end{aligned}$$

This is the same as $a^T Q a$, where

$$Q = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{pmatrix}.$$

Similarly,

$$\begin{aligned}\int_0^1 g(x)p_a(x)dx &= \int_0^1 g(x)\sum_{k=0}^n a_k x^k \; dx \\ &= \sum_{k=0}^n a_k \int_0^1 g(x)x^k \; dx \\ &= b^T a, \qquad \text{for } b = (\int_0^1 g(x)\; dx, \int_0^1 g(x)x\, dx, \ldots, \int_0^1 g(x)x^n dx).\end{aligned}$$

So it we define $c = \int_0^1 g(x)^2 dx$, then it follows from the above calculations that $f(x) = a^T Qa - 2b^T a + c$.

b. Find the first-order necessary condition for a point $a^* \in E^{n+1}$ to be a minimum point for f.

solution: In general, the first-order condition it that $\nabla f(a^*) = 0$. So we just have to compute ∇f. We have done things like this a number of times in the lectures, and from those computations we know that $\nabla f(a) = 2a^T Q - 2b^T$. It follows that the necessary condition is

$$(a^*)^T Q = b^T,$$

which can be rewritten as

$$Qa^* = b.$$

c. Find *all* minimizing points $a^* \in E^{n+1}$ when g is the constant function $g(x) \equiv 0$. Prove that your answer is correct.

solution: When $g = 0$, the point $a^* = 0$ (this stands for the point $(0, \ldots, 0) \in E^{n+1}$) is the unique minimizer of f.

To see this, note that when $g = 0$,

$$f(a) = \int_0^1 p_a(x)^2 dx.$$

Clearly $f(a) \geq 0$ for all $a \in E^{n+1}$, and $f(0) = 0$. So $a^* = 0$ is a minimizer. Thus, to complete the solution, we have to show that if $a \neq 0$, then $f(a) > 0$. We will do this in two steps:

Step 1: if $a = (a_0, \ldots, a_n)$ and $p_a(x) = 0$ for all $x \in (0, 1)$, then $a_0 = a_1 = \ldots = a_n = 0$.

This is clear, because if $p_a(x) = 0$ for all x, then $p(0) = p'(0) = p''(0) = \ldots = 0$. But $p_a(0) = a_0, p_a'(0) = a_1, p_a''(0) = 2a_2$, and generally $(\frac{d}{dx})^k p_a(x)|_{x=0} = k! a_k$. Thus all coefficients of p_a must equal zero.

Step 2. If follows that if $a \neq 0$, then $p_a(x_0) \neq 0$ for some $x_0 \in (0, 1)$. Since polynomials are continuous, this implies that there exists some interval $(a, b) \subset [0, 1]$ containing the point x_0, such that $p_a(x)^2 \geq \frac{1}{2} p_a(x_0)^2$ for $x \in (a, b)$. Thus

$$f(a) = \int_0^1 p_a(x)^2 dx \geq \int_a^b p_a(x)^2 dx \geq \int_a^b \frac{1}{2} p_a(x_0)^2 dx = \frac{1}{2}(b-a)p_a(x_0)^2 > 0.$$

So we have shown that $f(a) \geq 0$ for all a, and that $f(a^*) = 0$ if and only if $a^* = 0$. Thus $a^* = 0$ is the unique global minimizer.

d. Is Q positive semidefinite? positive definite? justify your answer.

solution

We have seen above that

$$a^T Qa = \int_0^1 p_a(x)^2 dx > 0 \qquad \text{whenever } a \neq (0, \dots, 0).$$

It follows that Q is positive definite (and hence also positive semidefinite).

Exercise 2. Find a global minimum point for the function

$$f(x, y, z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 8y - 8z + 9,$$

and *prove* that your solution really is a global minimum for f.

solution: The first-order necessary conditions are:

$$\begin{aligned} 4x + y - 6 &= 0 \\ x + 2y + z - 8 &= 0 \\ y + 2z - 8 &= 0. \end{aligned}$$

It is a straightforward matter to solve this system and find that the only solution is $(x, y, z) = (1, 2, 3)$. So this is the only possible global minimum point.

To check that it is a global minimum point, it suffices to check that f is convex. We will do this in three ways:

first approach

The standard way to prove convexity is to compute the matrix of second derivatives and check that it is positive definite (or semi-definite). we have:

$$\nabla^2 f = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

The easiest way to prove that this is positive definite is to use the fact that a symmetric $n \times n$ matrix Q is positive definite if and only if for every $k = 1, \dots, n$, the $k \times k$ formed in the upper left corner of Q has positive determinant. (This is often the easiest way to check whether a matrix is positive definite.)

Thus, in this case, we need to check that the matrices

$$\begin{pmatrix} 4 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

all have positive determinant. This is straightforward:

$$\det \begin{pmatrix} 4 \end{pmatrix} = 4, \qquad \det \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = 6, \qquad \det \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 10.$$

So $\nabla^2 f$ is positive definite, and f is convex.

second approach

We can also check that $\nabla^2 f$ is positive definite by directly trying to find all the eigenvalues and showing that they are all positive. From the above formula for

$\nabla^2 f$, we deduce that

$$\nabla^2 f - \lambda I = \begin{pmatrix} 4-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$$

and thus

$$\begin{aligned} \det(\nabla^2 f - \lambda I) &= (4-\lambda)(\lambda^2 - 4\lambda + 2) - 1(2-\lambda) \\ &= -\lambda^3 + 8\lambda^2 - 16\lambda + 6 \end{aligned}$$

The eigenvalues are the roots of this polynomial. In fact it would be very hard to find all the roots by hand. But we can still check that all they are positive, as follows:

First, let us write $p(\lambda)$ to denote the polynomial.

Note that $p(0) = 6$, and

$$p'(\lambda) = -3\lambda^2 + 16\lambda - 16 = -(3\lambda^2 - 16\lambda + 16)$$

which has roots

$$\frac{16 \pm \sqrt{256 - 172}}{6} = \frac{16 \pm 8}{6} \qquad \text{(both positive)}$$

by the quadratic formula. From this it follows that $p'(\lambda)$ does not change from positive to negative when $\lambda \leq 0$. Since $p'(0) < 0$, we conclude that $p'(\lambda) < 0$ for all $\lambda \leq 0$.

But any function that is decreasing for $\lambda \in (-\infty, 0)$ and positive at $\lambda = 0$ can have no negative roots. It follows that all roots of p are positive. Thus Q is positive definite and hence f is convex.

third approach

If we just stare at the formula for f, we see that it can be written as

$$f(x,y,z) = \frac{3}{2}x^2 + \frac{1}{2}(x+y)^2 + \frac{1}{2}(y+z)^2 + \frac{1}{2}z^2 + [9 - 6x - 8y - 8z]$$

which is a sum of convex functions and hence convex. In this way, we can see that f is convex almost without doing any computations.

Exercise 3. Assume that g is a convex function on E^n, that f is a convex function of a single variable, and in addition that f is a nondecreasing function (which means that $f(r) \geq f(s)$ whenever $r \geq s$).

a. Show that $F := f \circ g$ is convex by directly verifying the convexity inequality

$$F(\theta x + (1-\theta)y) \leq \theta F(x) + (1-\theta)F(y).$$

Explain where each hypothesis (convexity of g, convexity of f, and the fact that f is nondecreasing) is used in your reasoning. (The notation $F = f \circ g$ means that $F(x) = f(g(x))$.)

solution: Consider x, y in E^n and $\theta \in (0,1)$.

Since g is convex,

$$g(\theta x + (1-\theta)y) \leq \theta g(x) + (1-\theta)g(y).$$

Since f is nondecreasing, it follows that

$$f(g(\theta x + (1-\theta)y)) \leq f(\theta g(x) + (1-\theta)g(y)).$$

And since f is convex,

$$f(\theta g(x) + (1-\theta)g(y) \leq \theta f(g(x)) + (1-\theta)f(g(y)).$$

Putting together the last two inequalities and rewriting everying in terms of $F = f \circ g$, we get:

$$F(\theta x + (1-\theta)y) \leq \theta F(x) + (1-\theta)F(y).$$

So F is convex.

b. Now assume that f and g are both C^2. Express the matrix of second derivatives $\nabla^2 F(x)$ in terms of f and g, and prove directly (without using part (a)) that $\nabla^2 F$ is positive semidefinite at every x.

solution: By the chain rule,

$$\frac{\partial F}{\partial x_i} = f'(g)\frac{\partial g}{\partial x_i}$$

and again using the chain rule,

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = f''(g)\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} + f'(g)\frac{\partial^2 g}{\partial x_i \partial x_j}.$$

This can be written as:

$$\nabla^2 f = f''(g)\nabla g^T \nabla g + f'(g)\nabla^2 g.$$

To check that this is positive semidefinite, we consider an arbitrary vector v, and we compute

$$\begin{aligned} v^T \nabla^2 F v &= f''(g) v^T \nabla g^T \nabla g\, v + f'(g) v^T \nabla^2 g v \\ &= f''(g)(\nabla g\, v)^2 + f'(g) v^T \nabla^2 g v. \end{aligned}$$

This is nonnegative, since

$$\begin{gathered} f''(g) \geq 0 \text{ by the convexity of } f \\ (\nabla g\, v)^2 \geq 0 \text{ since it's the square of a real number} \\ f'(g) \geq 0 \text{ since } f \text{ is nondecreasing} \\ v^T \nabla^2 g v \geq 0 \text{ since } g \text{ is convex.} \end{gathered}$$

Exercise 4. Verify that if $f_1, \ldots, , f_k$ are convex functions on E^n, then

$$g(x) := \max(f_1(x), \ldots, f_k(x))$$

is also convex.

solution:
First note that

$$\begin{aligned} g(\theta x+(1-\theta)y) &= \max(f_1(\theta x+(1-\theta)y),\ldots,f_k(\theta x+(1-\theta)y)) \\ &\le \max(\theta f_1(x)+(1-\theta)f_1(y),\ldots,\theta f_k(x)+(1-\theta)f_k(y)) \end{aligned}$$

by the convexity of $f_1,\ldots,f_k$. Next, note that for every i, $f_i(x)\le\max(f_1(x),\ldots,f_k(x)) = g(x)$, and similarly for y. So

$$\begin{aligned} &\max(\theta f_1(x)+(1-\theta)f_1(y),\ldots,\theta f_k(x)+(1-\theta)f_k(y)) \\ &\qquad\le \max(\theta g(x)+(1-\theta)g(y),\ldots,\theta g(x)+(1-\theta)g(y)) \\ &\qquad= \theta g(x)+(1-\theta)g(y). \end{aligned}$$

Putting these together, we conclude that

$$g(\theta x+(1-\theta)y)\le\theta g(x)+(1-\theta)g(y).$$

So g is convex.

(We have used several times the fact that if $r_i\le s_i$ for $i=1,\ldots,k$ then

$$\max(r_1,\ldots,r_k)\le\max(s_1,\ldots,s_k)$$

which is obvious, if you think about it.)