Tutorial 3 Solutions

STAT 3013/8027

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1. **Rice Chapter 5 Question 20**: We have $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ uniform$(0,1)$. Let's determine the $Var\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$:

$$
\begin{aligned}
Var\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) &= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} f(x_i)\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(f(x_i)\right) \quad \text{Due to independence we have no cross terms.} \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\left[E(f(x_i)^2) - [E(f(x_i))]^2\right] \\
&= \frac{1}{n^2} n \left[E(f(x_i)^2) - [E(f(x_i))]^2\right] \\
&= \frac{1}{n}\left[\int_0^1 f(x)^2 dx - \left[\int_0^1 f(x) dx\right]^2\right] \\
&= \frac{1}{n}\left[\int_0^1 f(x)^2 dx - [I(f)]^2\right]
\end{aligned}
$$

Let's work out the exact result:

$$
\begin{aligned}
\int_0^1 f(x)dx &= \int_0^1 cos(2\pi x)dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 f(x)^2 dx &= \int_0^1 cos^2(2\pi x)dx \\
&= 1/2
\end{aligned}
$$

$$
\begin{aligned}
Var\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) &= \frac{1}{n}\left[\int_0^1 f(x)^2 dx - [I(f)]^2\right] \\
&= \frac{1}{n}\left[1/2 - [0]^2\right] \\
&= \frac{1}{2n}
\end{aligned}
$$

Now let's use Monte Carlo approximation which leads to:

$$
\begin{aligned}
\frac{1}{n}\left[\int_0^1 f(x)^2dx - [I(f)]^2\right] &= \frac{1}{n}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)^2 - \left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right]^2\right] \\
&= \frac{1}{n}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)^2 - \frac{1}{n^2}\left[\sum_{i=1}^{n} f(x_i)\right]^2\right] \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n} f(x_i)^2 - \frac{1}{n}\left[\sum_{i=1}^{n} f(x_i)\right]^2\right] \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n} cos^2(2\pi x) - \frac{1}{n}\left[\sum_{i=1}^{n} cos(2\pi x)\right]^2\right]
\end{aligned}
$$

- Let's do $n = 100$

```
set.seed(1001)
n <- 100
x <- runif(n)

Var.mc <- (1/n^2)*( sum( cos(2*pi*x)^2 ) - (1/n)*( sum(cos(2*pi*x)) )^2)
Var.mc
```

```
## [1] 0.005351193
```

```
0.5/n
```

```
## [1] 0.005
```

```
abs(0.5/n - Var.mc)
```

```
## [1] 0.0003511931
```

- Let's do $n = 1000$

```
set.seed(1001)
n <- 1000
x <- runif(n)

Var.mc <- (1/n^2)*( sum( cos(2*pi*x)^2 ) - (1/n)*( sum(cos(2*pi*x)) )^2)
Var.mc
```

```
## [1] 0.000498603
```

```
0.5/n
```

```
## [1] 5e-04
```

```
abs(0.5/n - Var.mc)
```

```
## [1] 1.39699e-06
```

- **Rice Chapter 5 Question 21 (a):**

In this question we wish to determine the following integral:

$$I(f) = \int_b^a f(x)dx$$

We note that by multiply and dividing by the density $g(x)$ we have:

$$\begin{aligned}
I(f) &= \int_b^a f(x)dx \\
&= \int_b^a f(x)\frac{g(x)}{g(x)}dx \\
&= \int_b^a \frac{f(x)}{g(x)}g(x)dx \\
&= E\left(\frac{f(x)}{g(x)}\right)
\end{aligned}$$

- Note: $g(x)$ is a density function on $[a, b]$. We can approximate $I(f)$ by the taking random samples from $g(x)$ and calculating:

$$\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n}\frac{f(x)}{g(x)}$$

As in question 20, let's get the $E\left(\hat{I}(f)\right)$:

$$\begin{aligned}
E\left(\hat{I}(f)\right) &= E\left(\frac{1}{n}\sum_{i=1}^{n}\frac{f(x)}{g(x)}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}E\left(\frac{f(x)}{g(x)}\right) \\
&= \frac{1}{n}nE\left(\frac{f(x)}{g(x)}\right) \\
&= E\left(\frac{f(x)}{g(x)}\right) \\
&= I(f) = \int_b^a f(x)dx
\end{aligned}$$

2. Now let's consider a Monte Carlo integration for the following:

$$I = \int_0^\infty 25x^2cos(x^2)exp(-25x)dx$$

Note that if $X \sim$ exponential(25), where the $E(X) = 1/25$, then we have the following density on $[0, \infty)$:

$$g(x) = 25exp(-25x)$$

- Consider the following algorithm:

    a. Generate n random samples from an exponential distribution (we know how to do this based on uniform random variables and the CDF inverse method).

    b. Calculate:

$$\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 cos(x_i^2)$$

```
set.seed(1001)
n <- 10000
x <- rexp(n, rate=1/25)  # the true mean is 25
mean(x)
```

```
## [1] 24.96956
```

```
I.f.samples <- x^2 * cos(x^2)
I.f.hat <- mean(I.f.samples)
I.f.hat
```

```
## [1] -18.90742
```

3. **Answer:** We assume that $U \sim$ uniform$(0, 1)$.

   a.) Let's consider the first case: Let $Y = -log(U)$. To find the density of Y let's use the cdf method:

$$\begin{aligned}
P(Y \leq y) &= P(-log(U) \leq y) \\
&= P(-log(U) \leq y) = P(log(U) > -y) \\
&= P(U > exp(-y)) = 1 - P(U \leq exp(-y)) \\
&= 1 - exp(-y)
\end{aligned}$$

   So we have $F_Y(y) = 1 - exp(-y)$ and $f_Y(y) = exp(-y)$ which is the density for an exponential distribution with $\beta = 1$ for $0 \leq y \leq \infty$.

   - Now let's consider the next case: $Y = -log(1 - U)$. All we need to show is that $V = 1 - U$ is also a uniform $(0, 1)$ random variable and use the previous result. Let's directly use the 'pdf method' (note: V is monotone for $0 \leq v \leq 1$):

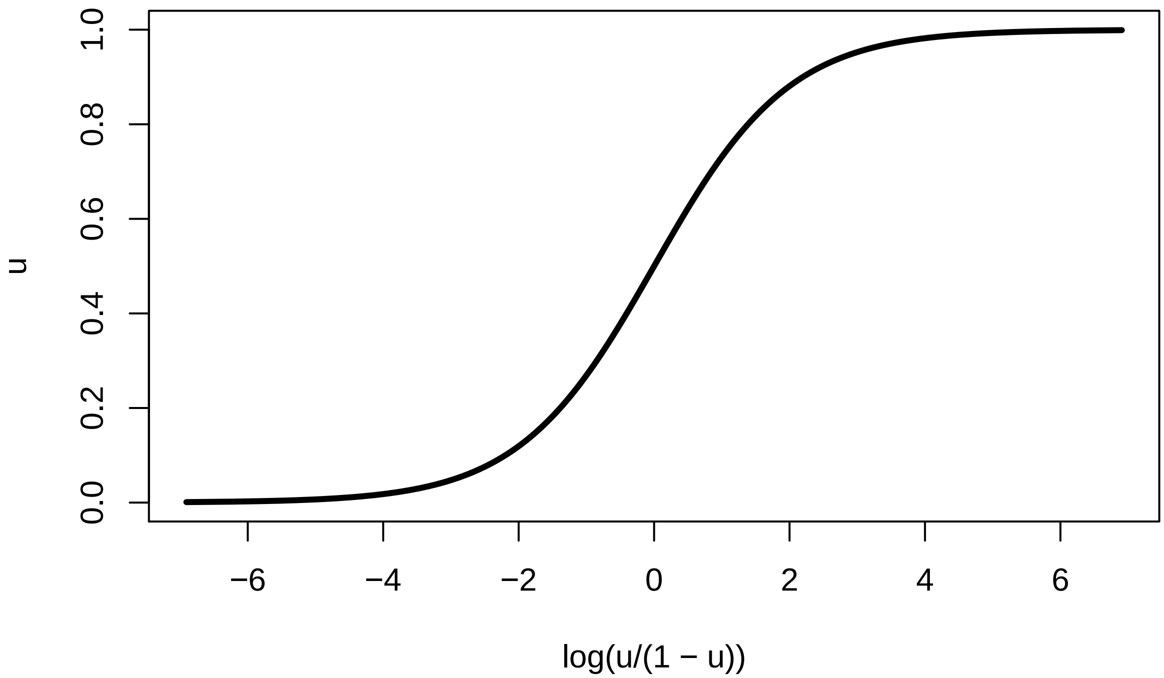
$$V = 1 - U = g(u) \quad \rightarrow \quad U = 1 - V = g^{-1}(v)$$
$$\frac{d}{dv}g^{-1}(v) = -1$$

$$\begin{aligned}
f_V(v) &= f_U\left(g^{-1}(v)\right)\left|\frac{d}{dv}g^{-1}(v)\right| \\
&= 1 \times \Big| -1 \Big| = 1 \text{ for } 0 \leq v \leq 1
\end{aligned}$$

- We can see that $V \sim \text{uniform}(0,1)$, which means $Y \sim \text{exponential}(\beta = 1)$.

b.) Let $X = \log\left(\frac{U}{1-U}\right)$. Let's visually check that X is monotone on $0 \le u \le 1$.

```
u <- seq(0, 1, by=0.001)
plot( log(u/(1-u)), u, type="l", lwd=3)
```



$$x = \log\left(\frac{u}{1-u}\right) = g(u) \quad \to \quad u = \frac{exp(x)}{1+exp(x)} = \frac{1}{1+exp(-x)} = g^{-1}(x)$$
$$\frac{d}{dx}g^{-1}(x) = \frac{exp(-x)}{(1+exp(-x))^2}$$

$$\begin{aligned}
f_X(x) &= f_U\left(g^{-1}(x)\right)\left|\frac{d}{dx}g^{-1}(x)\right| \\
&= 1 \times \left|\frac{exp(-x)}{(1+exp(-x))^2}\right| \\
&= = \frac{exp(-x)}{(1+exp(-x))^2} \text{ for } -\infty \le x \le \infty
\end{aligned}$$

- We can see that X has the density of a logistic distribution with $\mu = 0$ and $\beta = 1$.

c.) Now let's generate from $Y \sim \text{logistic}(\mu = 3, \beta = 2)$.

- We know how to generate $X \sim \text{logistic}(\mu = 0, \beta = 1)$
- Now we want to generate Y which has a pdf:

$$
\begin{aligned}
f_Y(y) &= \frac{1}{\beta}\frac{exp\left(-\frac{(y-\mu)}{\beta}\right)}{\left[1+exp\left(-\frac{(y-\mu)}{\beta}\right)\right]^2} \\
&= \frac{1}{\beta}f_X\left(\frac{(y-\mu)}{\beta}\right)
\end{aligned}
$$

This suggests that the right transformation would be:

$$
Y = \beta X + \mu
$$

$$
\begin{aligned}
Y = \beta X + \mu = g(x) \quad \rightarrow \quad X &= \frac{(Y-\mu)}{\beta} = g^{-1}(y) \\
\frac{d}{dy}g^{-1}(y) &= 1/\beta
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right| \\
&= \frac{exp\left(-\frac{(x-\mu)}{\beta}\right)}{\left[1+exp\left(-\frac{(x-\mu)}{\beta}\right)\right]^2}\frac{1}{\beta}
\end{aligned}
$$

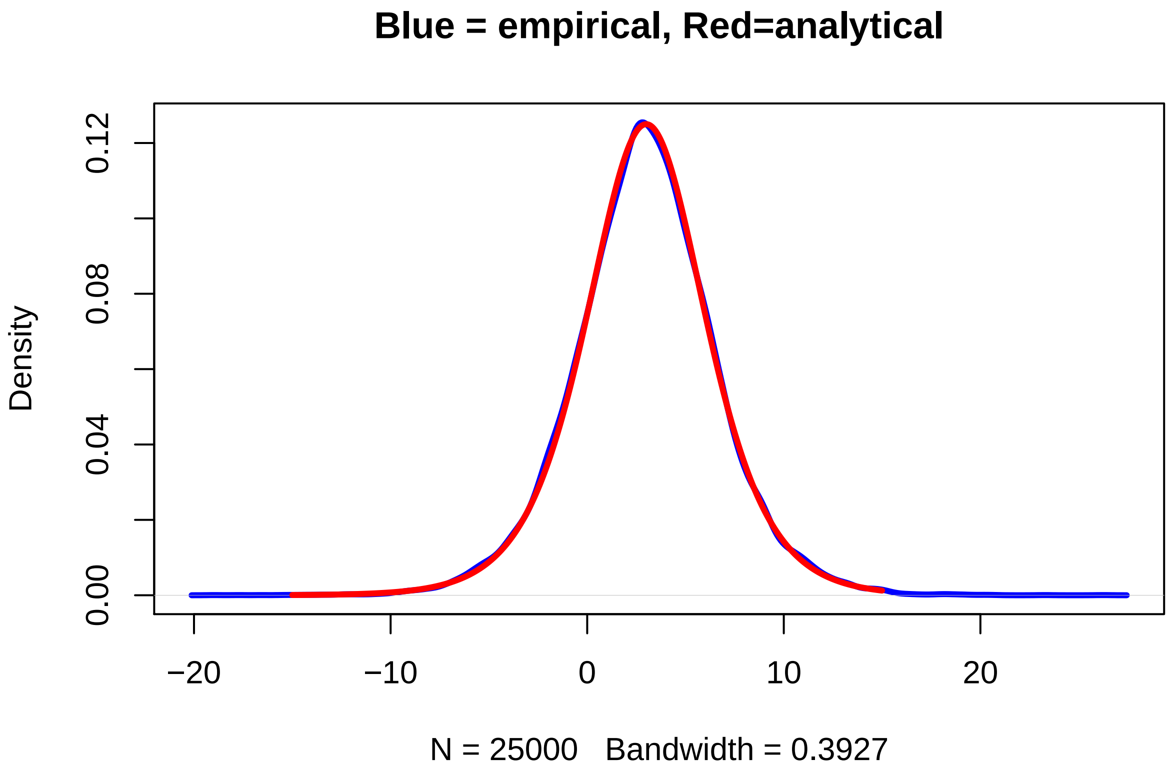
- Generate U from uniform(0,1).
- Generate Y from $\beta\log\left(\frac{U}{1-U}\right)+\mu$

```r
set.seed(1001)
S <- 25000
mu <- 3
beta <- 2



## empirical
u <- runif(S, 0, 1)
y <- beta* log( u/(1-u) ) + mu
plot(density(y), type="l", col="blue", lwd=3,
     main="Blue = empirical, Red=analytical")

## analytical
y.an <- seq(-15, 15 , by=0.01)
f.y.an <- (1/beta)*( exp( - (y.an - mu)/beta )/ ( 1 + exp( - (y.an - mu)/beta ))^2 )
lines(y.an, f.y.an, col="red", lwd=3)
```



Blue = empirical, Red=analytical
Density
0.00
0.04
0.08
0.12
−20
−10
0
10
20
N = 25000 Bandwidth = 0.3927