- Normal model → inference for the mean conditional on the variance.

– If $\theta \sim \text{Norm}(\mu_0, \tau_0^2)$ and $y_i \overset{iid}{\sim} N(\theta, \sigma^2)$ derive $p(\theta \mid y_1 \ldots y_n, \sigma^2)$.

$$p(\theta \mid y_1 \ldots y_n, \sigma^2) \propto p(\theta) \prod_{i=1}^{n} p(y_i \mid \theta, \sigma^2)$$

$$= e^{-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2} \, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2}$$

$$= e^{\overbrace{-\frac{1}{2\tau_0^2}(\theta^2 - 2\theta\mu_0 + \mu_0^2) - \frac{1}{2\sigma^2}(\sum y_i^2 - 2\theta \sum y_i + n\theta^2)}^{\text{quadratic function of } \theta}}$$

$$= e^{-\frac{1}{2}(a\theta^2 - 2b\theta + c)}$$

$$a = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}\right) \qquad b = \left(\frac{\mu_0}{\tau_0^2} + \frac{\sum y_i}{\sigma^2}\right)$$

$$c = \underbrace{c(\mu_0, \tau_0^2, \sigma^2, y_1 \ldots y_n)}_{\text{constant, does not depend on } \theta}$$

(complete the square).

$$p(\theta \mid \sigma^2, y_1 \ldots y_n) \propto e^{-\frac{1}{2}(a\theta^2 - 2b\theta)}$$

$$= e^{-\frac{1}{2}a(\theta^2 - 2b\theta/a + b^2/a^2) + \frac{1}{2}\frac{b^2}{a^2}}$$

$$\propto e^{-\frac{1}{2}a(\theta - b/a)^2}$$

$$= e^{-\frac{1}{2}\left(\frac{\theta - b/a}{1/\sqrt{a}}\right)^2}$$

⇒ Same shape as a normal density
mean $= b/a$
std. dev. $= \frac{1}{\sqrt{a}}$.

$$\therefore \theta \mid \sigma^2, y_1 \ldots y_n \sim N(\mu_n, \tau_n^2)$$

$$\mu_n = \frac{b}{a} = \frac{\frac{\mu_0}{\tau_0^2} + \frac{\sum y_i}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}}$$

$$\tau_n^2 = \frac{1}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}},$$