APM462H1S: Nonlinear optimization, Winter 2014.

Summary of March 3 and 10 lectures.

March 3: Midterm.

March 10: For the rest of the term we are going to be considering topics such as the calculus of variations and optimal control, largely following material in the notes:

An Introduction to Mathematical Optimal Control Theory by L. C. Evans, posted online at `http://math.berkeley.edu/~evans/control.course.pdf`

We began our consideration of this material with

- a brief introduction to the calculus of variations.
- a preliminary discussion of optimal control problems.

Here are more details, in reverse order:

Optimal Control

We went over Examples 1 and 5 in Section 1.2 of Evans' lecture notes, and all of Section 1.3. At the end of the hour we began discussing Chapter 2, defining the key notion of a "reachability set."

Incidentally, the goal for the next lecture (March 17) is to cover as much of Chapter 2 as possible.

I recommend that you read all of Chapter 1 of Evans' lecture notes – it gives a very good introduction to the overall aims of optimal control, together with some nice examples.

the Calculus of Variations. Here we more or less followed Section 4.1.1 of Evans' lecture notes, except that I simplified things a little by considering real-valued functions rather than vector-valued functions, and I discussed a bit of additional material, including the following:

First we considered the example of finding a surface of revolution about the x-axis of minimal area.

That is, suppose we create a surface by rotating the graph of the function $r(x), a \le x \le b$ about the x-axis. (So $r(x)$ is the radius of the surface of revolution at x. We remember from calculus that the area of such a surface is

$$I[r(\cdot)] = 2\pi \int_a^b r(x)(1 + r'(x)^2)^{1/2} dx.$$

So our goal is to minimize this, subject to conditions such as prescribed radii at the ends of the surface: $r(a) = r_0, r(b) = r(1)$.

We can write $I[r(\cdot)]$ as

$$\int_a^b L(r(x), r'(x)) \; dx$$

where

$$L(r, v) = 2\pi r(1 + v^2)^{1/2}.$$

We know from the general theorem we proved that the *Euler-Lagrange equations*, (that is, the first-order necessary conditions) are

$$-\frac{d}{dx}\Big(L_v(r(x), r'(x))\Big) + L_x(r(x), r'(x)) = 0. \tag{1}$$

In this problem, we compute

$$L_v = \frac{\partial L}{\partial v} = 2\pi r(1+v^2)^{-1/2}v, \qquad L_r = \frac{\partial L}{\partial v} = 2\pi(1+v^2)^{1/2}.$$

So, substituting $r(x)$ and $r'(x)$ for r and v respectively, we find that the equation is

$$-\frac{d}{dx}\Big(2\pi r(x)(1+r'(x)^2)^{-1/2}r'(x)\Big) + 2\pi(1+r'(x)^2)^{1/2} = 0. \tag{2}$$

We also verified by direct substitution that

$$r(x) = a\cosh(\frac{x}{a})$$

is a solution of these equations for any nonzero constant a. Thus, these functions give us candidates for area-minimizing surfaces of revolution.

Although we did not do it, one could also derive (2) by repeating the steps we went through in the derivation of the general necessary condition (1). That is, assume that some function $r(x)$ is a minimizer, and let $z(x)$ be any other function such that $z = 0$ at $x = a$ and b. Then define

$$i(s) = I[r(\cdot) + sz(\cdot)].$$

Then (2) can be derived (assuming that r and z are both C^2, for example) by

- noting (exactly as in the proof of the general theorem) that $i'(0) = 0$.
- computing $i'(s)$, setting $s = 0$, and integrating by parts to find that

$$i'(0) = \int_a^b \left[-\frac{d}{dx}\Big(2\pi r(x)(1+r'(x)^2)^{-1/2}r'(x)\Big) + 2\pi(1+r'(x)^2)^{1/2}\right] z(x)\ dx.$$

- using the fact, which we proved, that if w is continuous and

$$\int_a^b w(x)\ z(x)\ dx = 0$$

  for all C^2 functions $z : [a,b] \to \mathbb{R}$ such that $z(a) = z(b) = 0$, then $w(x) = 0$ for all $x \in [a,b]$.

Second, we also discussed the fact that the Euler-Lgrange equation corresponding to the functional

$$I[x(\cdot)] = \frac{1}{2}\int_a^b x'(t)^2\ dt$$

is

$$-x''(t) = 0.$$

We[1] further noted that for this functional, it is not hard to check that if x is *any* C^2 function (not necessarily a minimizer) and z is a C^2 function such that

[1] Everything from here to the end of this summary is just meant to be suggestive, and so is not very precise or rigorous.

$z(a) = z(b) = 0$, then

$$I[x(\cdot) + z(\cdot)] = I[x(\cdot)] + \int_a^b -x''(t)\, z(t) dt + \int_a^b \frac{1}{2} z'(t)^2\, dt$$
$$= I[x(\cdot)] + \int_a^b -x''(t)\, z(t) dt + \text{“}o(z)\text{”}$$

where we write $o(z)$ a bit vaguely to mean "terms that are smaller than linear, as $z \to 0$. (Here, for this to make sense, "$z \to 0$" could mean for example that $z'(x) \to 0$ uniformly in x. But for purposes of this discussion, we just want to point out parallels between the kind of problem considered here and finite-dimensional minimization problems, so these details are not important for now.)

This should be compared to the familiar finite-dimensional formula

$$f(x + z) = f(x) + \nabla f(x) z + o(z)$$

Comparing the two formulas, we can see that the function $-x''(t)$ in the first formula appears to be playing a role like that of $\nabla f(x)$ in the second formula.

Based on this we might be tempted to say that

$$\text{“}\nabla I[x(\cdot)] = -x''(\cdot)\text{ ”} \tag{3}$$

where $x''(\cdot)$ denotes the function $x''(t), a \le t \le b$.

The nice thing about (3) is that, if true (in fact it is justified in more advanced classes) then it says that first-order necessary condition has the same form for this kind of problem as for familiar finite-dimensional problems: if $x(\cdot)$ is a minimizer of I, then $\nabla I[x(\cdot)] = 0$.

Third, continuing with vague considerations, we asserted (without a complete discussion) that for the general problem of minimizing

$$I[x(\cdot)] = \int_a^b L(x(t), x'(t))\ dt,$$

the quantity appearing the in Euler-Lagrange equations:

$$-\frac{d}{dt}\Big(L_v(x(t), x'(t))\Big) + L_x(x(t), x(t)) \tag{4}$$

can be interpreted as "$\nabla I[x(\cdot)]$."

Finally, we mentioned that constrained minimization problems in this more complicated setting are similar, in some ways, to problems we studied earlier.

For example, consider the problem

$$\text{minimize} \quad I[x(\cdot)] = \frac{1}{2}\int_0^\pi x'(t)^2\ dt$$

subject to the conditions $x(0) = x(\pi) = 1$ and the constains

$$J[x(\cdot)] = \frac{1}{2}\int_0^\pi x(t)^2\ dt \ = \ 1.$$

It turns out (this will be a homework exercise, with hints) that the first-order necessary conditions are:

$$-x''(t) + \lambda x(t) = 0 \quad \text{for } 0 < t < \pi, \qquad x(0) = x(\pi) = 0, \qquad \frac{1}{2}\int_0^\pi x'(t)^2\ dt = 1$$

for some $\lambda \in \mathbb{R}$. If we believe that (4) is actually the gradient $\nabla I[x(\cdot)]$, and if we believe a parallel thing about $J[x(\cdot)]$, then this is exactly the familiar Lagrange multiplier equation:

$$\nabla I[x(\cdot)] + \lambda \nabla J[x(\cdot)] = 0.$$