Tutorial 10 Solutions

STAT 3013/4027/8027

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1. **Chapter 9 Question 3:** Here we have the following data: $X \sim \text{binomial}(100, p)$ and the following hypotheses:

$$\begin{aligned} H_0 : \quad & p = 0.5 \\ H_1 : \quad & p \neq 0.5 \end{aligned}$$

Consider the following decision rule: **Reject H_0 if** $\left|X - 50\right| > 10$**.**

a. We want to figure out the α, the probability of a **Type I Error** (reject the null given the null is true).

$$\begin{aligned} P_{H_0}(|X-50| > 10) &= P_{H_0}\left(\frac{|X-50|}{\sqrt{100p(1-p)}} > \frac{10}{\sqrt{np(1-p)}}\right) \\ &= P_{H_0}\left(\frac{|X-50|}{\sqrt{100p(1-p)}} > \frac{10}{\sqrt{100(0.5)(0.5)}}\right) \\ &= P_{H_0}\left(|Z| > \frac{10}{5}\right) \\ &= P_{H_0}\left(|Z| > 2\right) \\ &\approx P_{H_0}\left(Z > 2\right) + P_{H_0}\left(Z < -2\right) = 2P_{H_0}\left(Z < -2\right) \end{aligned}$$

```
alpha <- 2*pnorm(-2)
alpha
```

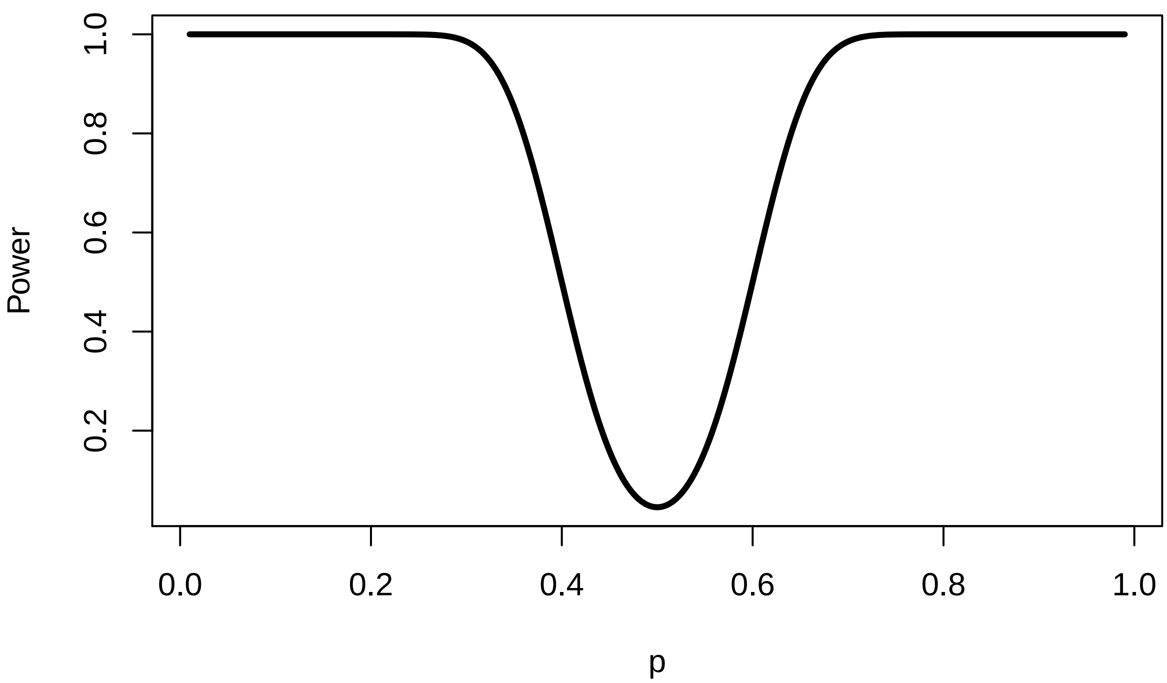
```
## [1] 0.04550026
```

b. Now let's get the power of the test [1- probability (Type II Error)]. Recall the power is the probability that the test reject the null when the alternative is true.

$$\begin{aligned}P_{H_A}(|X-50|>10) &= P_{H_A}(X-50>10)+P_{H_A}(X-50<-10)\\ &= P_{H_A}(X-50>10)+P_{H_A}(X-50<-10)\\ &= P_{H_A}(X>60)+P_{H_A}(X<40)\\ &= [1-P_{H_A}(X<60)]+P_{H_A}(X<40)\end{aligned}$$

We can let R standardize this for use:

```r
p <- seq(0.01, 0.99, by=0.001)
plot(p, 1-pnorm(60, mean=100*p, sd=sqrt(100*p*(1-p))) +
       pnorm(40, mean=100*p, sd=sqrt(100*p*(1-p))), type="l", lwd=3,
     ylab="Power")
```



2. **Chapter 9 Question 4:**

a. The likelihood ratio for each x is as follows:

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)}$$

$$\Lambda = \frac{0.2}{0.1} = 2, \quad \Lambda = \frac{0.3}{0.4} = 0.75, \Lambda = \frac{0.3}{0.1} = 3, \Lambda = \frac{0.2}{0.4} = 0.5.$$

If we rank the xs from smallest to largest for Λ we have: x_4, x_2, x_1, x_3:

$$\Lambda_{x_4} = \frac{0.2}{0.4} = 0.5., \quad \Lambda_{x_2} = \frac{0.3}{0.4} = 0.75, \quad \Lambda_{x_1} = \frac{0.2}{0.1} = 2, \quad \Lambda_{x_3} = \frac{0.3}{0.1} = 3,$$

b. Based on the **Neyman-Pearson lemma** we will reject H_0 for small values of Λ:

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} \leq k$$

To construct an α level test, we need to find a critical value k such that,

$$\begin{aligned} P_{H_0}(\Lambda \leq k) = 0.2 \\ P_{H_0}(X = x_4) = 0.2 \\ P_{H_0}(\Lambda \leq) = 0.2 \end{aligned}$$

Now let's change $\alpha = 0.5$:

$$\begin{aligned} P_{H_0}(\Lambda \leq k) &= \alpha = 0.5 \\ P_{H_0}(X = x_4 \text{ or } X = x_2) &= \alpha = 0.5 \\ P_{H_0}(\Lambda \leq 3/4) &= \alpha = 0.5 \end{aligned}$$

c. If the prior probabilities for H_0 and H_1 are the same (i.e. $P(H_0) = P(H_1) = 1/2$) the we can consider ratio of the posterior probabilities for for the two models:

$$\begin{aligned} \frac{P(H_0|x)}{P(H_1|x)} &= \frac{P(x|H_0)}{P(x|H_1)} \times \frac{P(H_0)}{P(H_1)} \\ &= \frac{P(x|H_0)}{P(x|H_1)} = \Lambda \end{aligned}$$

We can see that $\Lambda < 1$ for x_4, x_2, and so favor H_1. While $\Lambda > 1$ for x_1, x_3, which then favors H_0.

d. We can see that the prior probabilities of $P(H_0) = P(H_1) = 1/2$ correspond to the decision rule based on $\alpha = 0.5$. Let's see if we can extend this idea. We will reject H_0 if $\Lambda > 1$. So we want $\Lambda_{x_2} \leq 1$ and $\Lambda_{x_1} > 1$.

$$\begin{aligned}
\frac{P(H_0|x)}{P(H_1|x)} &= \frac{P(x|H_0)}{P(x|H_1)} \times \frac{P(H_0)}{P(H_1)} \\
\frac{P(H_0|x_2)}{P(H_1|x_2)} &= \frac{P(x|H_0)}{P(x|H_1)} \times \frac{P(H_0)}{P(H_1)} = 0.75 \times \frac{p}{1-p} \leq 1 \\
&\Rightarrow \; p \leq 4/7.
\end{aligned}$$

$$\begin{aligned}
\frac{P(H_0|x_1)}{P(H_1|x_1)} &= \frac{P(x|H_0)}{P(x|H_1)} \times \frac{P(H_0)}{P(H_1)} = 2 \times \frac{p}{1-p} > 1 \\
&\Rightarrow \; p > 1/3.
\end{aligned}$$

$$1/3 < p = P(H_0) \leq 4/7$$

- For the $\alpha = 0.2$ case we would like the following prior probabilities based $\Lambda_{x_2} \leq 1$ and $\Lambda_{x_1} > 1$

$$\begin{aligned}
\frac{P(H_0|x_2)}{P(H_1|x_2)} &= \frac{P(x|H_0)}{P(x|H_1)} \times \frac{P(H_0)}{P(H_1)} = 0.75 \times \frac{p}{1-p} > 1 \\
&\Rightarrow \; p > 4/7.
\end{aligned}$$

$$\begin{aligned}
\frac{P(H_0|x_1)}{P(H_1|x_1)} &= \frac{P(x|H_0)}{P(x|H_1)} \times \frac{P(H_0)}{P(H_1)} = 0.5 \times \frac{p}{1-p} \leq 1 \\
&\Rightarrow \; p \leq 2/3.
\end{aligned}$$

$$4/7 < p = P(H_0) \leq 2/3$$

3. **Chapter 9 Question 9:** Let $X_1, \ldots, X_{25} \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2 = 100)$

a. Let's test the following hypotheses at $\alpha = 0.10$:

$$
\begin{aligned}
H_0 : &\quad \mu = 0 \\
H_1 : &\quad \mu = 1.5
\end{aligned}
$$

This is the standard Neyman-Pearson set-up, so we will reject for small values of k:

$$
\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{exp\left(-\frac{1}{2}\sum_{i=1}^{n} X_i^2\right)}{exp\left(-\frac{1}{2}\sum_{i=1}^{n}(X_i - 1.5)^2\right)} \\
&= exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left[X_i^2 - (X_i - 1.5)^2\right]\right) \\
&= exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left[3X_i - 2.25\right]\right) \\
&= exp\left(\frac{n2.25}{2} - (3/2)\sum_{i=1}^{n} X_i\right)
\end{aligned}
$$

- So we get the rejection region:

$$
\begin{aligned}
R &= \left\{exp\left(\frac{n2.25}{2} - (3/2)\sum_{i=1}^{n} X_i\right) \leq k\right\} \\
&= \left\{\bar{X} > c^*\right\}
\end{aligned}
$$

So under H_0 we have:

$$
\begin{aligned}
P_{H_0}(R) &= P_{H_0}\left(\bar{X} \geq c^*\right) = \alpha \\
&= P_{H_0}\left(\frac{\bar{X} - 0}{10/\sqrt{25}} \geq c^{**}\right) = \alpha \\
&= P_{H_0}\left(Z \geq c^{**}\right) = \alpha \\
&= P_{H_0}\left(Z \geq c^{**}\right) = 0.10
\end{aligned}
$$

```r
qnorm(0.9)
```

```
## [1] 1.281552
```

So $c^* = 1.282$. Or we reject when $\frac{\bar{X}-0}{2} > 1.282 \Rightarrow \bar{X} > 2.56$.

- Now let determine the power for $\mu_1 = 1.5$:

$$
\begin{aligned}
P_{H_1}(R) &= P(\bar{X} > 2.56) \\
&= P_{H_1}\left(\frac{\bar{X}-1.5}{10/\sqrt{25}} > \frac{2.56-1.5}{10/\sqrt{25}}\right) \\
&= 1 - P(Z \leq (2.56-1.5)/2)
\end{aligned}
$$

```r
1 - pnorm((2.56-1.5)/2)
```

```
## [1] 0.298056
```

- Now let's change α to $\alpha = 0.01$:

$$
\begin{aligned}
P_{H_0}(R) &= P_{H_0}\left(\bar{X} \geq c^*\right) = \alpha \\
&= P_{H_0}\left(\frac{\bar{X}-0}{10/\sqrt{25}} \geq c^{**}\right) = \alpha \\
&= P_{H_0}\left(Z \geq c^{**}\right) = \alpha \\
&= P_{H_0}\left(Z \geq c^{**}\right) = 0.01
\end{aligned}
$$

```r
qnorm(0.99)
```

```
## [1] 2.326348
```

So we reject when $\bar{X} > 4.66$.

$$
\begin{aligned}
P_{H_1}(R) &= P(\bar{X} > 4.66) \\
&= P_{H_1}\left(\frac{\bar{X}-1.5}{10/\sqrt{25}} > \frac{4.66-1.5}{10/\sqrt{25}}\right) \\
&= 1 - P(Z \leq (4.66-1.5)/2)
\end{aligned}
$$

```r
1 - pnorm((4.66-1.5)/2)
```

```
## [1] 0.05705343
```

4. **Chapter 9 Question 10:** We know that if T is a sufficient statistic for θ, then we can decompose the likelihood as follows:

$$L(\theta|\boldsymbol{x}) = g(T(\boldsymbol{x})|\theta)h(\boldsymbol{x})$$

This suggests that the likelihood ratio will only be based on $g(\cdot)$:

$$\Lambda = \frac{L(\theta_0|\boldsymbol{x})}{L(\theta_1|\boldsymbol{x})} = \frac{g(T(\boldsymbol{x}|\theta_0))}{g(T(\boldsymbol{x})|\theta_1)}$$

The likelihood ratio rejection is: $\{R : \Lambda < c\}$. If we know the distribution of the sufficient $T(\boldsymbol{x})$ under H_0 then we may be able to determine the distribution of λ under the NULL (as Λ is a function of $T(\boldsymbol{x})$). Perhaps this would have to be done via simulation. Then we can determine c:

$$P_{H_0}(\Lambda < c) = \alpha$$

Once you know the value of c, you look for the values of $T(\boldsymbol{x})$ such that Λ is less than c. This then becomes your rejection region for $T(\boldsymbol{x})$.

5. **Chapter 9 Question 11:** Let $X_1, \ldots, X_{25} \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2 = 100)$

a. Let's test the following hypotheses at $\alpha = 0.10$:

$$\begin{aligned} H_0 : &\quad \mu = 0 \\ H_1 : &\quad \mu \neq 0 \end{aligned}$$

For this type of test, we will consider a **Generalized Likelihood Ratio Test**:

$$\lambda(\boldsymbol{x}) = \frac{\underset{\Theta_0}{sup} L(\theta|\boldsymbol{x})}{\underset{\Theta}{sup} L(\theta|\boldsymbol{x})}$$

$$
\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{(2\pi)^{-n/2} exp[-\sum(x_i-\theta_0)^2/2]}{(2\pi)^{-n/2} exp[-\sum(x_i-\bar{x})^2/2]} \\
&= exp\left[\left(-\sum(x_i-\theta_0)^2 + \sum(x_i-\bar{x})^2\right)/2\right] \\
&= exp\left[\left(-\left[\sum(x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2\right] + \sum(x_i-\bar{x})^2\right)/2\right] \\
&= exp\left[-n(\bar{x}-\theta_0)^2/2\right] \\
&= exp\left[-n(\bar{x}-0)^2/2\right] \\
&= exp\left[-n\bar{x}^2/2\right]
\end{aligned}
$$

$$
\begin{aligned}
R &= \{\lambda(\boldsymbol{x}) \le c\} \\
&= \{exp\left[-n\bar{x}^2/2\right] \le c\} \\
&= \{-n\bar{x}^2/2 \le log(c)\} \\
&= \{\bar{x}^2 > -2log(c)/n\} \\
&= \{\left|\bar{x}\right| > \sqrt{-2log(c)/n}\} \\
&= \{\left|\frac{\bar{x}-0}{2}\right| > \frac{\sqrt{-2log(c)/n}-0}{2}\}
\end{aligned}
$$

- Now we have:

$$
R = \{|Z| > c^*\}
$$

- Under the null hypothesis $\theta = 0$. So $Z \sim$ normal$(0,1)$.

$$
\begin{aligned}
P(|Z| > c^*) &= P(Z > c^*) + P(Z < -c^*) = \alpha \\
&= 2P(Z < -c^*) = \alpha \\
&= P(Z < -c^*) = \alpha/2 \\
&= P(Z < c^{**}) = \alpha/2
\end{aligned}
$$

- Suppose $\alpha = 0.10$, then $c^{**} = 1.64$

```r
qnorm(1-0.10/2)
```

```
## [1] 1.644854
```

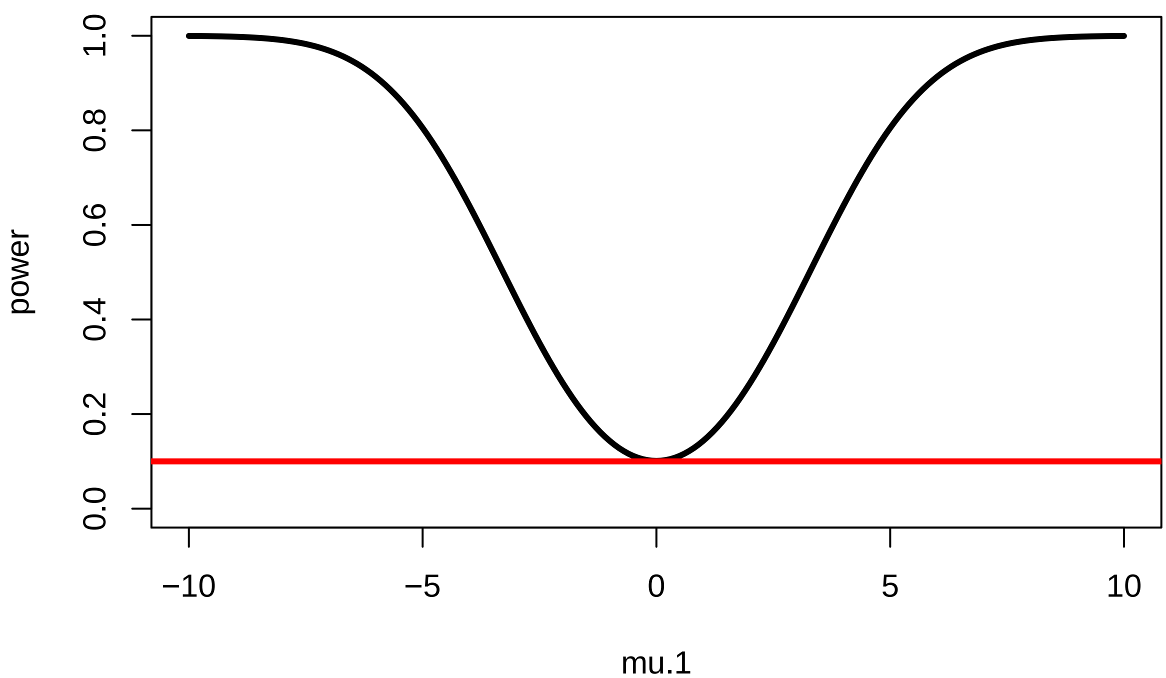
- So we will reject H_0 if:

$$\left\{\left|\frac{(\bar{x}-0)}{2}\right| > 1.64\right\} = |\bar{x}| > 2(1.64)$$

- Now let's get the power:

$$\begin{aligned}
P_{H_1}(R) &= P_{H_1}(\bar{X} > 3.26) + P_{H_1}(\bar{X} < 3.28) \\
&= P_{H_1}((\bar{X} - mu_1)/2 > (3.28 - mu_1)/2) + P_{H_1}(\bar{X} < (-3.28 - \mu_1)/2) \\
&= 1 - P_{H_1}(Z < (3.28 - mu_1)/2) + P_{H_1}(Z < (-3.28 - \mu_1)/2) \\
&= 1 - P_{H_1}(Z < 1.64 - mu_1/2) + P_{H_1}(Z < -1.64 - \mu_1/2)
\end{aligned}$$

```r
mu.1 <- seq(-10,10, by=0.1)
plot(mu.1, 1-pnorm(1.64-mu.1/2) + pnorm(-1.64-mu.1/2), type="l",
     lwd=3, ylab="power", ylim=c(0,1))
abline(h=0.10, col="red", lwd=3)
```



b. You can follow the same procedure for $\alpha = 0.05$. A similar example with $\alpha = 0.05$ was done in class.