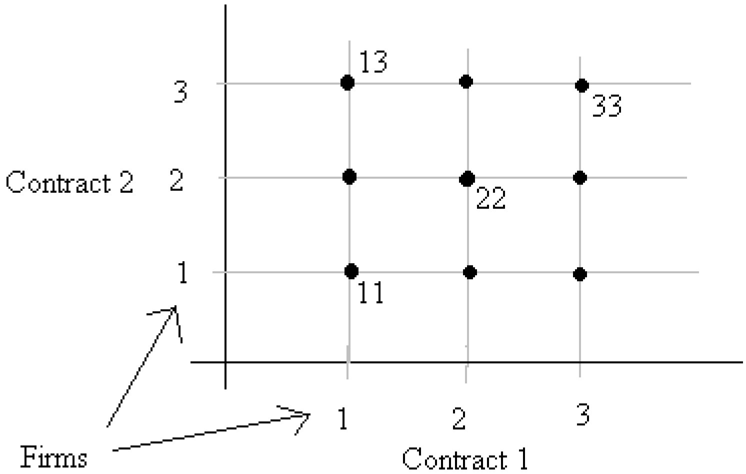
STAT2001 Tutorial 2 Solutions

Problem 1

Let *ij* = "Firm *i* gets Contract 1 and Firm *j* gets Contract 2".
Then *S* = {11,12,13,   21,22,23, 31,32,33}   (9 equally likely sample points).

Venn diagram:



For example, 13 = "Firm 1 gets Contract 1 and Firm 3 gets Contract 2.

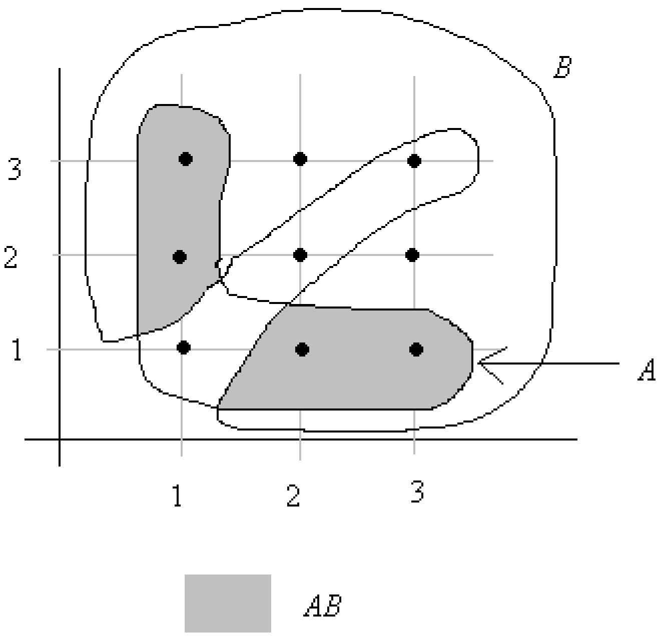
(a) The probability that Firm 1 gets both contracts is *P*(11) = 1/9.

(b) Let *A* = "Firm 1 gets a contract" meaning at least one contract
and *B* = "Both contracts don't go to the same firm".

Then: *A* = {13,12,11,21,31), *P*(*A*) = 5/9
B = {12,13,21,23,31,32}, *P*(*B*) = 6/9
AB = {12,13,21,31}, *P*(*AB*) = 4/9.

So *P*(*A*|*B*) = *P*(*AB*)/*P*(*B*) = (4/9)/(6/9) = 2/3.

Venn diagram:



So another solution is: $P(A \mid B) = \frac{n_{AB}}{n_B} = \frac{4}{6} = \frac{2}{3}$.

Yet another solution is as follows:

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A) - P(A\bar{B})}{1 - P(\bar{B})} = \frac{1 - P(\bar{A}) - P(A\bar{B})}{1 - P(\bar{B})}$$

$$= \frac{1 - (2/3)^2 - (1/3)^2}{1 - 1/3} = \frac{2}{3}.$$

Here, $P(\bar{A})$ is the probability that Firm 1 doesn't get either contract, ie 2/3 times 2/3, $P(A\bar{B})$ is the probability that Firm 1 gets both contracts, ie 1/3 times 1/3, and $P(\bar{B})$ is the probability that Contract 2 goes to the same firm as Contract 1, ie 1/3.

(c) Let C_i = "Firm 1 gets Contract *i*".

Then C_2 = {11,21,31} and $P(C_2)$ = 3/9.

Also, $C_1\bar{C}_2$ = {12,13} and $P(C_1\bar{C}_2)$ = 2/9.

Hence $P(C_1 \mid \bar{C}_2) = P(C_1\bar{C}_2)/P(\bar{C}_2)$ = (2/9)/(1 – 3/9) = 1/3.

Alternatively, $P(C_1 \mid \bar{C}_2) = \frac{n_{C_1\bar{C}_2}}{n_{\bar{C}_2}} = \frac{n_{C_1\bar{C}_2}}{n_S - n_{C_2}} = \frac{2}{9-3} = \frac{1}{3}$.

Problem 2

(a) The number of ways 2 bolts can be selected from 20 is

$$\binom{20}{2} = \frac{20!}{2!18!} = \frac{20(19)}{2} = 190.$$

The number of ways 2 nondefective bolts can be selected from 16 is

$$\binom{16}{2} = \frac{16(15)}{2} = 120.$$

So P(Two nondefectives) = 120/190 = 12/19. Ie, $\binom{16}{2}\binom{4}{0}\Big/\binom{20}{2} = \frac{12}{19}$.

Alternative solution: (16/20)(15/19) = 12/19, by multiplicative law of pr.

(b) Number of ways 2 defectives can be selected from 4 is

$$\binom{4}{2} = \frac{4(3)}{2} = 6.$$

So P(Two defectives) = 6/190.

But "Two defectives" = "No nondefectives".
So P(No nondefectives) = 6/190 also. Ie, $\binom{16}{0}\binom{4}{2}\Big/\binom{20}{2} = \frac{6}{190}$.

It follows that P(At least one nondefective) = 1 – P(No nondefectives)
= 1 – 6/190 = 184/190 = 92/95 = 0.9684.

Alternatively, P(At least one nondefective) = 1 – P(Both defective)
= 1 - P(First defective) P(Second defective | First defective)
= 1 - (4/20)(3/19) = 92/95 = 0.9684.

(c) P(2 nond.| ≥ 1 nond.) = P(2 nond. & ≥ 1 nond.)/P(≥ 1 nond.)
= P(2 nond.)/P(≥ 1 nond.)
= (120/190)/(184/190) by (a) and (b)
= 15/23 = 0.6522.

Problem 3

(a) Let C_i be the event that the *i*th component functions.
Then $P(C_i) = 0.9$, *i* = 1,2,3. Also, $A = C_1 \cup C_2 \cup C_3$.

Therefore $P(A) = 1 - P(\bar{A}) = 1 - P(\overline{C_1 \cup C_2 \cup C_3})$

$= 1 - P(\bar{C}_1\bar{C}_2\bar{C}_3)$ by De Morgan's law (general)

$= 1 - P(\bar{C}_1)P(\bar{C}_2)P(\bar{C}_3)$ by independence

$= 1 - (0.1)^3$

= 0.999 (99.9%).

(b) $P(C_1 | A) = \dfrac{P(C_1 A)}{P(A)} = \dfrac{P(C_1)P(A | C_1)}{P(A)} = \dfrac{0.9(1)}{0.999} = \dfrac{100}{111}$.

So $P(\bar{C}_1 | A) = 1 - P(C_1 | A) = 1 - \dfrac{100}{111} = \dfrac{11}{111}$ = 0.09910 (9.91%).

We have used the result that $P(\bar{B} | D) = 1 - P(B | D)$ (similar to $P(\bar{B}) = 1 - P(B)$).

Proof: LHS = $\dfrac{P(\bar{B}D)}{P(D)} = \dfrac{P(D) - P(BD)}{P(D)}$ by LTP (ie, $P(D) = P(BD) + P(\bar{B}D)$)

$= 1 - \dfrac{P(BD)}{P(D)}$ = RHS.

Problem 4

Let A_i = "The *i*th donor typed is the first donor typed who has the right blood",
and A = "A donor with the right blood is found in the first 8 minutes".

Then $A = A_1 \cup A_2 \cup A_3 \cup A_4$.
Note that the four A_i are disjoint events (they form a partition of *A*).
Therefore $P(A) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$

= .4 + .6(.4) + $.6^2(.4) + .6^3(.4)$

= .4 + .24 + .144 + .0864

= 0.8704.

More simply, $P(A) = 1 - P(\bar{A})$

= 1 – *P*(Four potential donors all have the wrong blood)

= 1 – $.6^4$

= 1 – 0.1296

= 0.8704.

A better solution (better for the accident victim):

If after eight minutes a donor has not been found, a 5th potential donor should be selected and their blood transfused into the victim, without typing beforehand.

(There's nothing to lose at that stage; so this should give the victim a better chance.)

The victim will then die if and only if all of 5 potential donors have the wrong blood.
The probability of this happening is $.6^5$ = 0.0778.
Therefore the probability that the victim will live is 1 – 0.0778 = 0.9222 (> 0.8704).

A related problem:

Suppose the "large number" of untyped donors available is 100, and exactly 40% of these have the right blood. What's the probability the accident victim can be saved?

In this case the probability that the victim will die is the probability that the first donor has the wrong blood times the probability that the second donor has the wrong blood given that the first donor has the wrong blood, and so on, ie

$$\frac{60}{100}\frac{59}{99}\frac{58}{98}\frac{57}{97}\frac{56}{96} = 0.0725 .$$

Therefore the probability that the accident victim will live is 1 – 0.0725 = 0.9275.
This is close to 0.9222 because 100 is 'large'.

If we solve this related problem with "100" changed to "1000" (and 40% kept the same), the result should be even closer to 0.9222. Let's check this:

$$\frac{600}{1000}\frac{599}{999}\frac{598}{998}\frac{597}{997}\frac{596}{996} = 0.07724125 ,\qquad 1 - 0.07724125 = 0.9228.$$