STAT6038 Week 8 Lecture Notes

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1 Wednesday Lecture

1.1 Estimation and Prediction using Multiple (MR) Regression models

Estimate of Y given new values of the X variables

$$\begin{aligned}\hat{Y}|\boldsymbol{x_0} &= \hat{\beta}_0 + \hat{\beta}_1 x_{01} + \hat{\beta}_2 x_{02} + \cdots + \hat{\beta}_k x_{0k} \\ &= \boldsymbol{x}_0^T \hat{\boldsymbol{\beta}}\end{aligned}$$

$$\text{where } \boldsymbol{x_0} = \begin{pmatrix} 1 \\ x_{01} \\ x_{02} \\ x_{03} \\ \vdots \\ x_{0k} \end{pmatrix}, \hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{pmatrix}.$$

$$\begin{aligned}\operatorname{Var}(\hat{Y}|x_0) &= \operatorname{Var}(\boldsymbol{x_0}^T \hat{\boldsymbol{\beta}}) \\ &= \boldsymbol{x_0}^T \operatorname{Var}(\hat{\boldsymbol{\beta}}) (\boldsymbol{x_0}^T)^T \\ &= \sigma^2 \boldsymbol{x_0}^T (X^T X)^{-1} \boldsymbol{x_0}\end{aligned}$$

So, a $100(1-\alpha)\%)$ confidence interval for $E[Y|\boldsymbol{x_0}]$ is $\hat{Y}|\boldsymbol{x_0} \pm t_{n-p}(1-\frac{\alpha}{2}) s \sqrt{\boldsymbol{x_0}^T (X^T X)^{-1} \boldsymbol{x_0}}$ (of SLR $\hat{Y}|x^* \pm t_{n-2}(1-\frac{\alpha}{2}) s \sqrt{\frac{(x^*-\bar{x})^2}{\text{SS}_x}}$)

And, a $100(1-\alpha)\%$ prediction interval for $Y|\boldsymbol{x_0}$ is $\hat{Y}|\boldsymbol{x_0} \pm t_{n-p}(1-\frac{\alpha}{2}) s \sqrt{1 + \boldsymbol{x_0}^T (X^T X)^{-1} \boldsymbol{x_0}}$

$\rightarrow$ again, we leave the implementation of these formulae to R and use the `predict()` function.

1.2 Problem of Multiple Comparisons

Forming a 95% interval estimate (prediction or confidence interval) is directly related to a two-sided hypothesis test.

- both types of inference are forms of comparisons.
  In forming 3 intervals, we have made 3 comparisons, all at the 95% confidence level.
  Is our overall confidence 95%?
  No, with m comparisons, it is closer to

$$\begin{aligned} P(\text{all "tests" accepted})) &= 1 - P(\text{at least one test is rejected}) \\ &\geq 1 - \sum_{i=1}^{m} P(\text{each test is rejected}) \quad \text{(relies on Boole's inequality)} \\ &= 1 - m\alpha \end{aligned}$$

  See Faraway text, pg 87.

In this instance, $m = 3$ comparisons, each at $\alpha = 0.05$.
So, our overall confidence is $\simeq 1 - 3(0.05) = 0.85$ i.e. 85%.
If we know in advance (a priori) that we are going to make $m = 3$ comparisons, we could solve $1 - m\alpha = 0.95 \implies m\alpha = 1 - 0.095 \implies m\alpha = 0.05 \implies \alpha = \frac{0.05}{m} = 0.0167$ i.e. do the 3 "tests", all at the $\alpha = 0.0167$ level of significance or $(1-\alpha)100\% = 0.9833 \times 100\%$, i.e. 98.3% confidence intervals.
This $(1 - \alpha/m)$ correction is called the *Bonferroni* (1936) correction.

2 Thursday Lecture

`predict(..., interval="prediction")` vs. `predict(..., interval="confidence")`

2.1 Residual Diagnostics

Raw residual for hte i^{th} observation

$$e_i = Y_i - \hat{Y}_i$$

$\rightarrow$ these are estimates of the errors ϵ_i i.e. $e_i = \hat{\epsilon}_i$
Note the assumptions about the errors, $\text{Var}(\epsilon) = \sigma^2 I$, but (see earlier) $\text{Var}(e) = \sigma^2(I - H)$ where H is the hat matrix.
So, the **standardized** (internally Studentised) residuals for the i^{th} observation are:

$$r_i = \frac{e_i - 0}{\sqrt{\sigma^2(1 - h_{ii}}} \simeq \frac{e_i}{\sqrt{\text{MS}_{\text{error}}}(1 - h_{ii})}$$

where h_{ii} is the hat value (leverage) of the i^{th} observation, and we use $\hat{\sigma}^2 = s^2 = \text{MS}_{\text{error}}$ to estimate the unknown σ^2.
So

$$r_i = \frac{e_i}{s\sqrt{1-h_{ii}}}$$

As σ^2 is estimated these are approximated distributed as a Student's t distribution.

2.2 Residual Plots (Ian's preferred plots)

1. Main residual plot
   Standardized (internally Studentized) residuals (r_i) against the fitted values ($\hat{Y}_i$)
   $\rightarrow$ we should check this for every model we fit.
   $\rightarrow$ why? checks the key assumptions of independence and constant variance.

2. Normal quantile plot (of the standardized residuals)
   Default `plot(model, which=2)` works fine
   $\rightarrow$ only bother checking once **plot 1** is okay
   $\rightarrow$ checks assumption of normality
   $\rightarrow$ could add 45° line for comparison (`abline(0, 1, lty=2)`)

3. Outlier/Influence plot
   My preference is a bar plot of Cook's distances.
   `plot(model, which=4)`
   $\rightarrow$ only really need if there is some indication that outliers and/or influential points might be a problem on **plots 1 and/or 2**.
   $\rightarrow$ can further investigate - check leverage values (bar plot)
   $\rightarrow$ could also use `plot(model, which=5)` (but ignore the arbitrary cut-offs for Cook's distance)
   $\rightarrow$ also, we could perform a test ...

3 Friday Lecture

3.1 Deletion Residual

Also called **PRESS** residuals, "Prediction Sum of Squares".

$$e_{i,-i} = Y_i - \hat{Y}_{i,-i}$$

where $\hat{Y}_{i,-i}$ is the fitted value for the i^{th} observation based on a model which has been fitted to the data with the i^{th} observation deleted (or excluded).

Surely this involves fitting a model (so we can calculate $e_{i,-i}$ for each $i = 1, 2, \ldots, n$)?

No, as it can be shown that

$$e_{i,-i} = \frac{e_i}{1 - h_{ii}}$$

and these deletion residuals have $\text{Var}(e_{i,-i}) = \frac{\sigma^2}{1-h_{ii}}$.

So, if we standardize the deletion residuals

$$\frac{e_{i,-i} - 0}{\sqrt{\sigma^2/(1 - h_{ii}}} \simeq \frac{e_{i,-i}}{\sqrt{s^2/(1 - h_{ii}}} = \frac{e_i}{(1 - h_{ii})s\sqrt{1/(1 - h_{ii})}} = \frac{e_i}{s\sqrt{1 - h_{ii}}} = r_i$$

Same standardized (internally Studentized) residuals as before!

But the internally Studentized residuals

$$r_i = \frac{e_i}{s\sqrt{1 - h_{ii}}}$$

are called "internally" Studentized as the estimate of σ^2 used is based on a model which uses all data (including the current or i^{th} observation).

Again, we can derive as estimate of σ^2 that excludes the current observation without to fit the entire model to a new reduced data set – it turns out

$$S_{-i} = \sqrt{\frac{(n - p)s^2 - e_i/(1 - h_{ii})}{n - p - 1}}$$

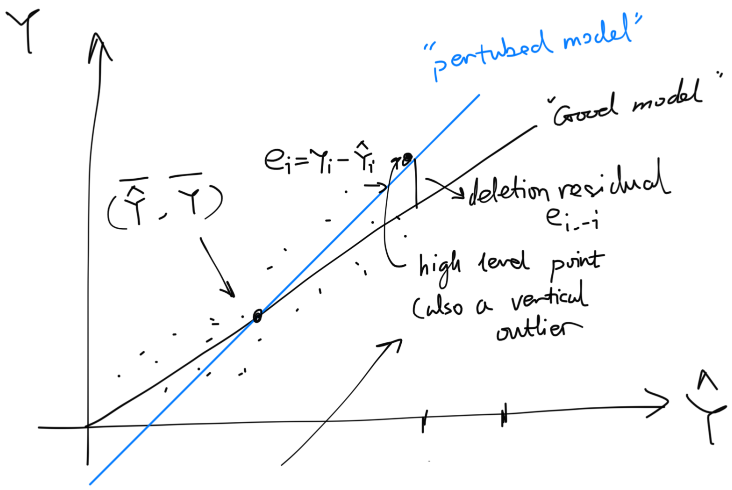
Note the new degrees of freedom used here (based on 1 less observation) as $n - p - 1$.

This gives an alternative type of standardized residuals: **the externally Studentized residuals**.

$$t_i = \frac{e_i}{s_{-i}\sqrt{1 - h_{ii}}} = \frac{e_{i,-i}}{s_{-i}/\sqrt{1 - h_{ii}}}$$

this version clearly shows that both the numerator (the deletion residual) and the denominator (a fixed function of the deletion s estimate) come from a model with the i^{th} observation excluded, hence the name "externally" Studentized residuals.

Back to the Pine example,



Example observation 20 in the full model (`pine.lm`) for the pine data.

indicator variable $I_i = \begin{cases} 0 \text{ if } i = 1, 2, \ldots, 19 \\ 1 \text{ if } i = 20 \end{cases}$

fitted model

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3 + \hat{\beta}_4 I_{20}$$

if $I_{20} = 0 \implies \hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3$

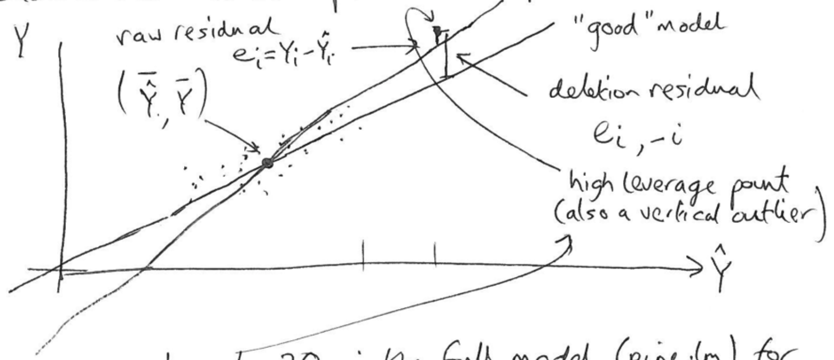
if $I_{20} = 1 \implies \hat{Y} = (\hat{\beta}_0 + \hat{\beta}_4) + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3$

...

another plot

"Observation 20 failed the full model but passed the reduced model."

Back to the Pine example



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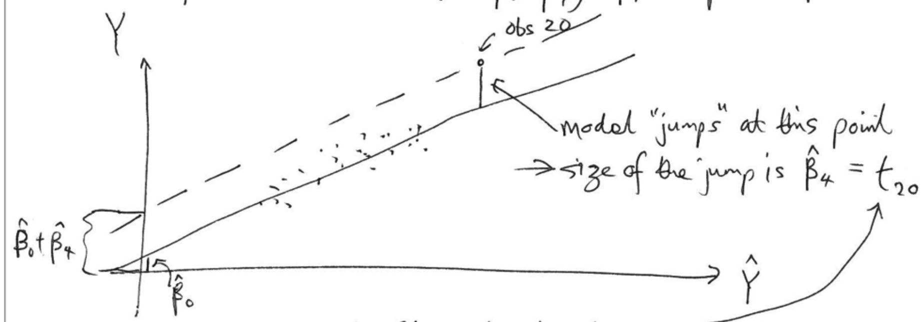

externally Studentised residual value for observation 20