STAT7017 Homework 1

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Q1

Proof: This is proved by mathematical induction.

Basis

By definition, we know that

1. $|\mathbf{A}| = a_{11}$ when $p = 1$.
2. $|\mathbf{A}| = \sum_{j=1}^{p} a_{1j}|\mathbf{A}_{1j}|(-1)^{1+j}$ if $p > 1$ where $\mathbf{A}_{1j}$ is the $(p-1)\times(p-1)$ matrix with the first row and j-th column deleted.

Induction hypothesis

Suppose we have $p = 1$, then our claim that the determinant is equation to the product of diagonal entry is automatically true.

Induction Step

Suppose again it is true that when $p = n$, i.e. $|\mathbf{A}_{n\times n}| = \prod_{j=1}^{n} a_{jj}$.

We want to show it is also true when $p = n+1$, i.e. $|\mathbf{A}_{(n+1)\times(n+1)}| = \prod_{j=1}^{n+1} a_{jj}$.

$$
\begin{aligned}
|\mathbf{A}_{(n+1)\times(n+1)}| &= \sum_{j=1}^{n+1} a_{1j}|\mathbf{A}_{1j}|(-1)^{1+j} \\
&= a_{11}|\mathbf{A}_{-11}|(-1)^{1+1} + a_{12}|\mathbf{A}_{-12}|(-1)^{1+2} + \cdots + a_{1(n+1)}|\mathbf{A}_{-1(n+1)}|(-1)^{1+n+1} \\
&= a_{11}\prod_{j=2}^{n+1} a_{jj} + 0 + \cdots + 0 \\
&= \prod_{j=1}^{n+1} a_{jj}
\end{aligned}
$$

$\therefore$ The determinant of diagonal matrix $\det(\mathbf{A})$ is the product of diagonal elements.

Q2

Proof: Suppose $\lambda_1, \ldots, \lambda_p$ are eigenvalues of a $p\times p$ matrix $\mathbf{A}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = \lambda^p + (-1)\cdot a_1\lambda^{p-1} + \cdots + (-1)^{p-1}\cdot a_{p-1}\lambda + (-1)^p \cdot a_p$$

where a_1 is the trace of $\mathbf{A}$, a_p is $\det(\mathbf{A})$, a_i is the sum of i-rowed diagonal mirros of $\mathbf{A}$.

Since $\lambda_1, \ldots, \lambda_p$ are zeros of $p(\lambda)$,

$$p(\lambda) = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdots (\lambda - \lambda_p)$$

Now we focous on the constant term $(-1)^p \cdot a_p$. We can calcuate it by inserting $\lambda = 0$ in both expressions:

- $|\mathbf{A} - 0 \cdot \mathbf{I}| = |\mathbf{A}| = a_p$
- $(-\lambda_1) \cdot (-\lambda_2) \cdots (-\lambda_p) = (-1)^p \prod_{i=1}^p \lambda_i = (-1)^p \cdot a_p$

Hence

$$(-1)^p \prod_{i=1}^p \lambda_i = (-1)^p \cdot |\mathbf{A}|$$
$$|\mathbf{A}| = \lambda_1 \cdot \lambda_2 \cdots \lambda_p$$

Q3

Solution:

Under the default setup, where $p = 100$, we generate $n = 500$ random normally distributed symmetric $p \times p$ matrices. The eigenvalues are plotted with grey histogram below, and the blue shaded curve indicates its density.

The density function of Wigner semicircle distribution can be written as:

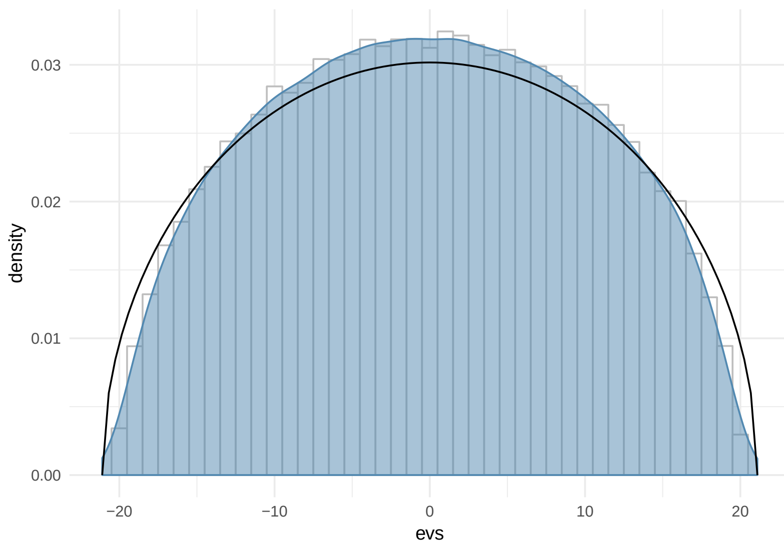
$$f(x) = \frac{2}{\pi R^2} \sqrt{R^2 - x^2}$$

where the parameter R is the interval of possible x value with $x \in [-R, R]$. In our case, we take an upper bound of the maximal absolute values of eigenvalues as R. Specifically, $R = \lceil \max(\lambda_i) \rceil + 0.1$.

The fitted Wigner semicircle distribution is plotted with a black curve.

```
library(ggplot2)
library(VGAM)

set.seed(7017)

wignerplot <- function(p, flag="N") {
    n <- p * 5

    matrices <- list()
    for (i in 1:n) {

        if (flag=="N") {
            A <- matrix(rnorm(p^2, 0, 1), p, p)
        } else if (flag=="T") {
            A <- matrix(rt(p^2, 10), p, p)
        } else if (flag=="inf") {
            # Pareto distribution with shape parameter < 2 has infinite variance
            A <- matrix(rpareto(p^2,shape=1.5), p, p)
        }

        A[lower.tri(A)] <- t(A)[lower.tri(A)]
        matrices[[i]] <- A
    }

    evs <- c()
    for (j in matrices) {
        evs <- c(evs, eigen(j)$values)
    }
```

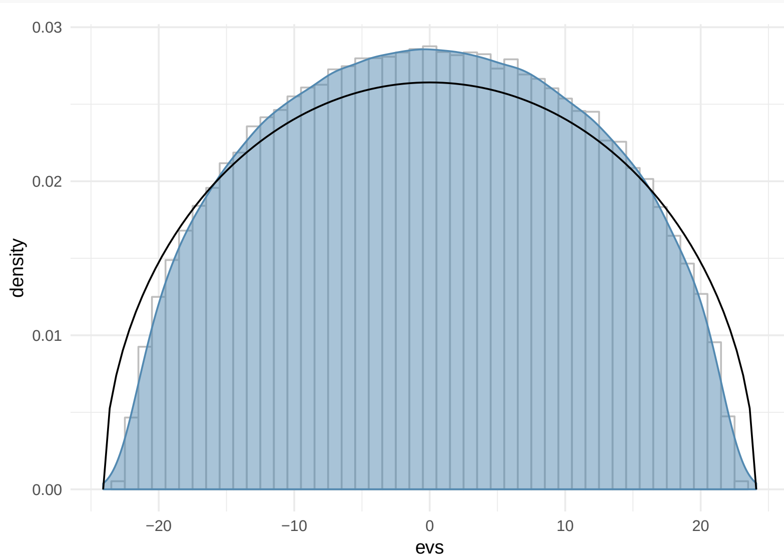
```r
    df <- data.frame(seq_along(evs), evs)

    max(evs)
    min(evs)
    bound <- round(max(evs))+0.1

    g1 <- ggplot(df, aes(evs)) +
        geom_histogram(aes(y=..density..), binwidth=1, colour="grey", fill="white") +
        stat_density(geom="area", alpha=0.5, fill='#5289B1', color="#5289B1") +
        stat_function(fun = function(x) 2/(pi*(bound)^2)*sqrt((bound)^2-x^2)) +
        xlim(-bound,bound) +
        theme_minimal()
    return(g1)
}

wignerplot(100,flag="N")
```



If we keep the n/p ratio constant (5 in this case), then the corresponding plots with $p = 125, 150, 200$ are plotted below:

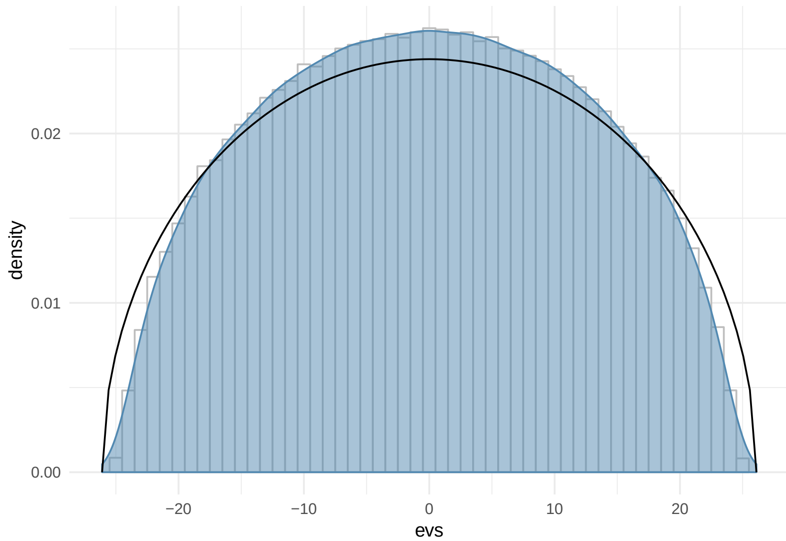
$p = 125$

```r
wignerplot(125)
```



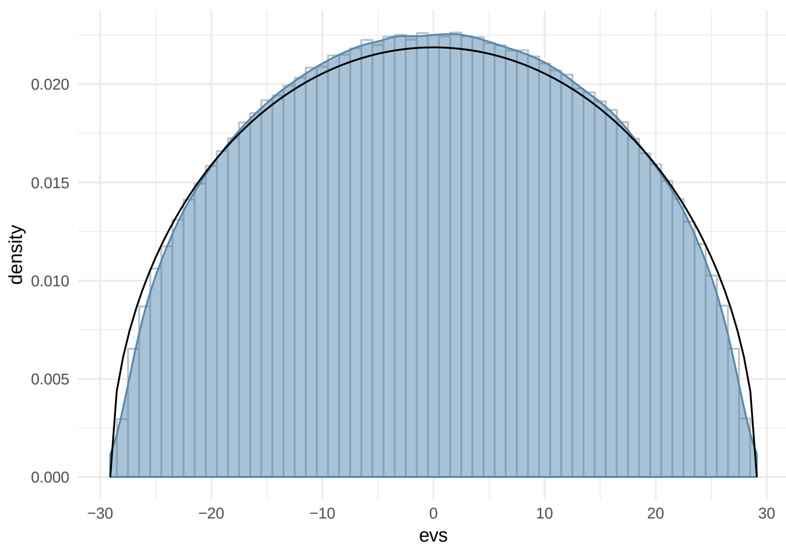
$p = 150$

```r
wignerplot(150)
```



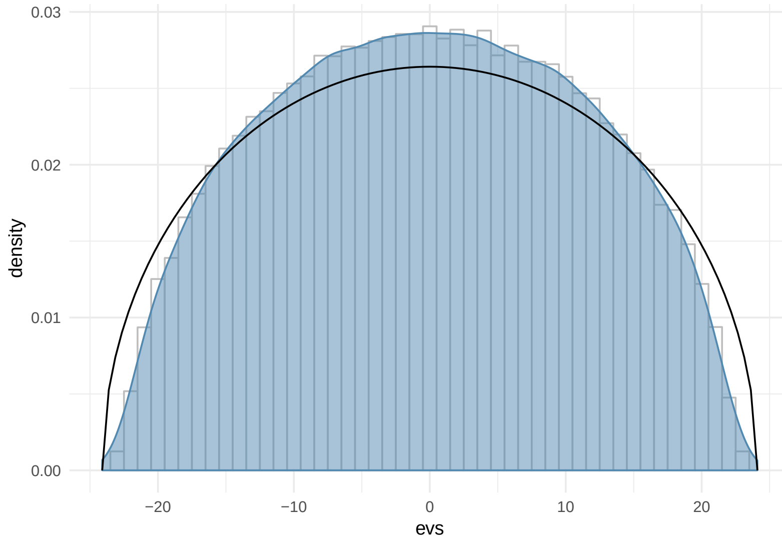
$p = 200$

```r
wignerplot(200)
```

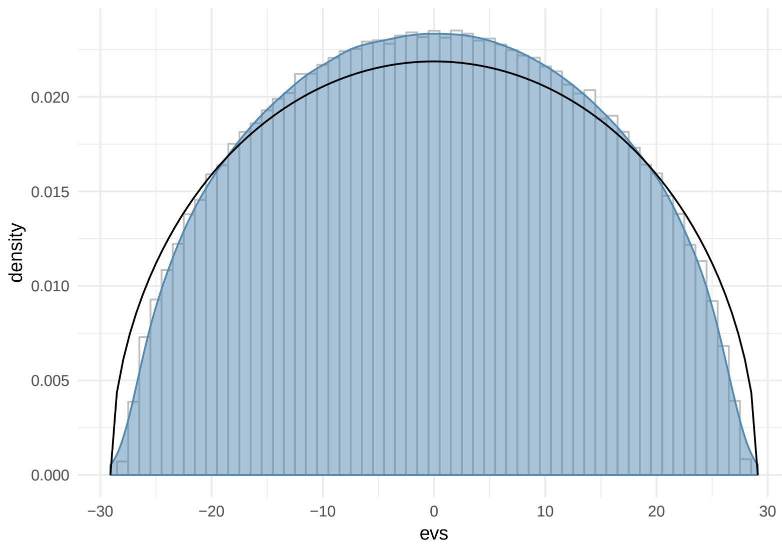


If we let the random matrices follow a Student's T distribution (let `df =10` though not specified) instead of a standard normal, with $p = 100, n = 500$, we have:

```r
wignerplot(100,flag="T")
```




```
wignerplot(150,flag="T")
```



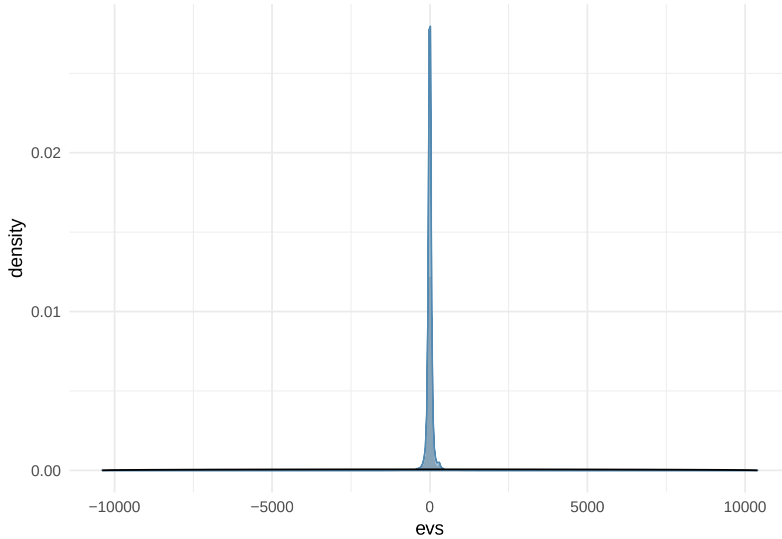
Some observations from Q3:

- Generally, we haved shown the universality conjecture that the spectral density of large random symmetric matrices following a zero mean, finite variance converges to the density of the Wigner Semicircle distribution.
- There are some gaps between the observed density and the theoretical one. However, when n and p get larger (while holding the p/n ratio constant), the gaps seem to be more and more trivial.
- Also, since the scale of y-axes vary in the plots above, one fact could be masked that the "semicircle" are in fact flatter when n and p get larger.
- The last part shows that when sample size is large enough, the Student's T distribution and standard normal distribution have similar effect in this scenario.

Q4

Solution: We need to find a distribution with finite mean but infinite variance. A Pareto distribution with shape parameter $\alpha \leq 2$ has infinite variance. When the distribution is settled, we repeat for $n = 500, p = 100$ setup. But this time, the semicircle law fails.

```
wignerplot(100,flag="inf")
```



References

1. "Wigner semicircle distribution", *Wikipedia*, 2018. [Online]. Available: https://en.wikipedia.org/wiki/Wigner_semicircle_distribution. [Accessed: 04-Aug-2018].

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4. "How can a distribution have infinite mean and variance?," *Cross Validated*. [Online]. Available: https://stats.stackexchange.com/questions/91512/how-can-a-distribution-have-infinite-mean-and-variance/91515. [Accessed: 04-Aug-2018].

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6. F. Benaych-Georges and A. Knowles, "Lectures on the local semicircle law for Wigner matrices", arXiv:1601.04055v3 [math.PR], 18 Oct. 2016.

7. L. Lin, Y. Saad and C. Yang, "Approximating spectral densities of large matrices", arXiv:1308.5467v2 [math.NA], 5 Oct. 2014.