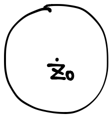
Lecture 12

Singularities

Def'n: We say f has an isolated singularity at z_0 if f is analytic in a punctured disk $0<|z-z_0|<r$



Ex: ① $f(z)=\frac{z^2-1}{z-1}$ has an isolated singularity at $z_0=1$

② $f(z)=\frac{1}{z-1}$ has an isolated singularity at $z_0=1$

③ $f(z)=e^{\frac{1}{z-1}}$ has isolated singularity at $z_0=1$.

FACTS ABOUT ISOLATED SINGULARITIES

① If f has an iso. singularity at z_0 one of the following three things occurs:

(i). $\lim\limits_{z\to z_0} |f(z)|$ is bdd. as $z\to z_0$

(ii). $\lim\limits_{z\to z_0} |f(z)|=\infty$

(iii). Neither (i) nor (ii) occur.

Ex: ① $f(z)=\frac{z^2-1}{z-1}$ $\quad \lim\limits_{z\to 1}\frac{z^2-1}{z-1}=\lim\limits_{z\to 1} z+1=2$ this satisfies (i)

② $f(z)=\frac{1}{z-1}$ satisfies (ii)

③ $f(z)=e^{\frac{1}{z-1}}$ satisfies (iii)

In this class, we mostly talk about (i) & (ii)

Let's look at (i) & (ii) in detail.

Case (i): $|f(z)|\le M$ for all z in $0<|z-z_0|<r$

Define: $g(z)=\begin{cases}(z-z_0)^2\cdot f(z) & \text{if } z\ne z_0\\ 0 & \text{if } z=z_0\end{cases}$

g is analytic for $z\neq z_0$

At $z=z_0$: $g'(z_0)=\lim\limits_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}=\lim\limits_{z\to z_0}\frac{(z-z_0)^2f(z)}{z-z_0}=\lim\limits_{z\to z_0}(z-z_0)f(z)=0$

because f is bounded.

So g is analytic at z_0 too.

Write g as a power series centered at z_0:

$$g(z)=b_0+b_1(z-z_0)+b_2(z-z_0)^2+b_3(z-z_0)^3+\cdots$$

$$=b_2(z-z_0)^2+b_3(z-z_0)^3+\cdots$$

$$\frac{g^{(k)}(z_0)}{k!}=b_k \qquad =(z-z_0)^2(b_2+b_3(z-z_0)+\cdots)$$

$\Rightarrow (z-z_0)^2f(z)=(z-z_0)^2(b_2+b_3(z-z_0)+\cdots)$

if $z\neq z_0$ then $f(z)=b_2+b_3(z-z_0)+\cdots$ (power series for f)

So now we extend f to z_0 by setting $f(z_0)=b_2$

$f(z)=(z^2-1)/(z-1)$ extended to $f(z)$ $\begin{cases} \dfrac{z^2-1}{z-1} & \text{if } z\neq 1 \\ 2 & \text{if } z=1 \end{cases}$

Case (ii)

If $\lim_{z\to z_0}|f(z)|=\infty$ then $\frac{1}{f}$ satisfies case (i).

set $g(z)=\frac{1}{f(z)}$, & extend $g(z)$ to be defined at z_0: $g(z_0)=0$

(should match $\lim_{z\to z_0}\frac{1}{f(z)}=0$)

So g has a zero (say of order m) at z_0.

$g(z)=(z-z_0)^m h(z)$ where $h(z_0)\neq 0$

$\frac{1}{f(z)}=(z-z_0)^m h(z)$

$f(z)=(z-z_0)^{-m}\cdot H(z)$ where $H(z)=\frac{1}{h(z)}$

& $H(z_0)\neq 0$

$$f(z)=\frac{1}{z-1}$$

$$g(z)=\frac{1}{\frac{1}{z-1}}:=\begin{cases} z-1 & \text{if } z\neq 1 \\ 0 & \text{if } z=1 \text{ (extended)} \end{cases}$$

$f(z)=\frac{H(z)}{(z-z_0)^m}$ where $H(z_0)\neq 0$

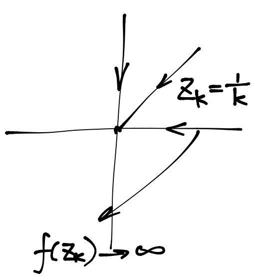
Ex: $f(z)=\frac{z^2-1}{(z-1)^2}=\frac{z+1}{z-1}$ $\frac{}{}z+1 \longrightarrow H(z)$

Terminology: Case (i) $\longrightarrow$ Removable Singularity

Case (ii) $\longrightarrow$ Pole of order m $\qquad f(z)=\frac{H(z)}{(z-z_0)^m}$

Case (iii) $\longrightarrow$ Essential Singularity

Case (iii): $f(z)=e^{\frac{1}{z}}$, $z_0=0$



Approach z using $z_k=\frac{2}{\pi k i}$. $\quad \frac{1}{z_k}=\frac{\pi}{2}ik$

$f(z_k)=e^{i\frac{\pi}{2}k}$

Depending on which direction you approach,

the limits are different,

and the limits could look like any complex number.