Estimation Methods

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Content

1. The method of moments estimation (MOME)
2. the maximum likelihood estimation (MLE)
3. Summary

Method of Moments

Consider a random sample $Y_1, \ldots, Y_n$ from a population with t unknown parameters. The aim is to find point estimators for the t unknown parameters.

Definition (Method of Moments)

Method of moments involve solving the following t equations with respect to the t unknown parameters:

$$\mu_k = m_k, \quad k = 1, \ldots, t,$$

where $\mu_k = \mathbb{E}Y_1^k$ and $m_k = \frac{1}{n}\sum_{j=1}^n y_j^k$ are the kth population moment and kth sample moment respectively.

Maybe more than t equations are needed if for some $1 \le k \le t$, $\mu_k = 0$.

Consistency of MOME

Why MOME performs good?

1. M_k is a consistent estimator of μ_k since $\mathbb{E}M_k' = \frac{1}{n}\sum_{i=1}^n \mathbb{E}Y_i^k = \mu_k$, $Var(M_k) = \frac{1}{n^2}\sum_{i=1}^n Var(Y_i^k) = \frac{Var(Y_1^k)}{n} \to 0$.
2. Because $\mu_k = g(\theta)$ and $m_k = g(\hat{\theta})$, $g(\hat{\theta}) - g(\theta) \xrightarrow{p} 0$. Due to continuity of $g(\cdot)$, we have $\hat{\theta} - \theta \xrightarrow{p} 0$.

Example 4

Question: Consider a random sample of numbers from between 0 and c. Find the method of moments estimate of c.

Analysis:

1. The unknown parameter is c.
2. $\mu_1 = \mathbb{E}Y = \frac{c}{2}$ and $m_1 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.
3. From $\mu_1 = m_1$ we have $c = 2\bar{y}$.

Example 5

Question: Suppose that $Y_1, \dots, Y_n \sim i.i.d. Gam(a, b)$. Find the methods of moments estimates of a and b.

Analysis:

1. Two unknown parameters are a and b.
2. $\mu_1 = \mathbb{E}Y = ab$, $m_1 = \bar{y}$; $\mu_2 = \mathbb{E}Y^2 = Var(Y) + \mu_1^2 = ab^2 + (ab)^2$, $m_2 = \frac{1}{n}\sum_{i=1}^n y_i^2$.
3. $\mu_1 = m_1$ and $\mu_2 = m_2$ derive $ab = \bar{y}$ and $ab^2 + a^2b^2 = \frac{1}{n}\sum_{i=1}^n y_i^2$ respectively.
4. $\hat{a} = \frac{\bar{y}^2}{\frac{1}{n}\sum_{i=1}^n y_i^2 - \bar{y}^2}$ and $\hat{b} = \frac{\bar{y}}{\hat{a}}$.

Exercise

Question: $Y_1, \ldots, Y_n \sim i.i.d.N(a, b^2)$. Find the MOME of a and b^2.
Analysis:

1. The unknown parameters are a and b^2.
2. $\mu_1 = a$, $m_1 = \bar{y}$; $\mu_2 = b^2 + a^2$, $m_2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2$.
3. $\mu_1 = m_1$ and $\mu_2 = m_2$ derive $\hat{a} = \bar{y}$ and $\hat{b^2} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \left(\frac{n-1}{n}\right) s^2$.

Maximum Likelihood Estimation

Suppose that $Y_1, \ldots, Y_n$ is a sample from a population whose pdf $f(y;\theta)$ depends on an unknown parameter θ.

Definition (MLE)

The **maximum likelihood estimate (MLE)** of θ is the value of θ which maximizes **the likelihood function** $L(\theta) = \prod_{i=1}^{n} f(y_i;\theta)$.

The principle is that we should choose the value for the unknown parameter which maximizes the likelihood of what has happened.

An Intuitive Illustration

Suppose that we have a box with two balls in it, each of which is either black or white. But we have no idea how many of these two balls are black. We randomly draw a ball from the box and find that it is black. Is the other ball also black?

If the other ball is black then the probability of us having drawn a black is 100%; if the other ball is white then the probability of us having drawn a black is only 50%.

Because $100\% > 50\%$, it is reasonable to conclude that the other ball is black.

Example 6

Question: We have a box with 2 balls in it, each of which is either black or white. We randomly draw a ball from the box and find that it is black. Find the MLE of the number of black balls originally in the box?

Analysis:

1. The unknown parameter θ is the number of black balls originally in the box; its range is $\theta = 0, 1, 2$.
2. Y_1 is the sample and its value is $y_1 = 1$.
3. $Y_1 \sim Bern(\theta/2)$. The pmf is $p(y) = \frac{\theta}{2}$, if $y = 1$ and $p(y) = 1 - \frac{\theta}{2}$ if $y = 0$.
4. The likelihood function $L(\theta) = p(y_1) = \frac{\theta}{2}$. So $\hat{\theta} = \arg\max_{\theta=0,1,2} L(\theta) = 2$.

Example 7

Question: A bent coin is tosses 5 times and heads come up twice. Find the MLE of the probability of heads coming up on a single toss.

Analysis:

1. The unknown parameter p denotes the probability of interest.
2. The sample $Y_1 \sim Bin(n,p)$ and its value $y_1 = 2$ with $n = 5$ and the pmf is $p(y) = C_n^y p^y (1-p)^{n-y}$, $y = 0, \ldots, 5$.
3. The likelihood function is $L(p) = C_n^{y_1} p^{y_1} (1-p)^{y_1}$.
4. From $L^{'}(p) = 0$ we have $\hat{p} = \frac{y_1}{n} = \frac{2}{5}$.

Remark

The loglikelihood function is $\ell(p) = \log L(p) = \log C_n^y + y \log p + (n-y)\log(1-p)$. Then from $\ell^{'}(p) = 0$ we can derive the same result.

Example 8

Question: Suppose that 1.2, 2.4 and 1.8 are a random sample from an exponential distribution with unknown mean. Find the MLE of that mean.

Analysis:

1. The sample $Y_1, Y_2, Y_3 \sim f(y) = \frac{1}{b}e^{-y/b}$ with b being the unknown parameter.
2. The likelihood function is $L(b) = \prod_{i=1}^{3} f(y_i) = \prod_{i=1}^{3} \frac{1}{b}e^{-y_i/b} = b^{-3}e^{-\frac{1}{b}\sum_{i=1}^{3} y_i} = b^{-3}e^{-\frac{3\bar{y}}{b}}$.
3. The loglikelihood function is $\ell(b) = -3\log(b) - \frac{3\bar{y}}{b}$.
4. From $\ell'(b) = 0$ we have $\hat{b} = \bar{y} = 1.8$.

Example 9

Question: $Y_1, \ldots, Y_n \sim i.i.d N(a, b^2)$. Find the MLE's for a and b^2.
Analysis:

1. The likelihood function is
$L(a, b^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}b} \exp\left\{-\frac{1}{2b^2}(y_i - a)^2\right\} =$
$-b^{-n}(2\pi)^{-n/2} \exp\{-\frac{1}{2b^2}\sum_{i=1}^{n}(y_i - a)^2\}$.
2. The loglikelihood function is
$\ell(a, b^2) = -\frac{n}{2}\log(b^2) - \frac{1}{2b^2}\sum_{i=1}^{n}(y_i - a)^2$.
3. From $\frac{\partial \ell(a,b^2)}{\partial a} = 0$ and $\frac{\partial \ell(a,b^2)}{\partial b^2} = 0$ we have $\hat{a} = \bar{y}$ and
$\hat{b^2} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \left(\frac{n-1}{n}\right)s^2$.

Example 10

Question: A partly melted die is rolled repeatedly until the first 6 comes up. Then it is rolled again the same number of times. Suppose that the first 6 comes up on the third roll, and the numbers which then come up are $6, 2, 6$. We are interested in p, the probability of 6 coming up on a single toss. Find the MLE of p.

Analysis:

1. X: the number of rolls until first 6; Y: the number of 6's on last half of rolls.
2. $X \sim Geo(p)$ and $Y|X = x \sim Bin(x, p)$ with $x = 3$ and $y = 2$.
3. The likelihood function $L(p) = p(x, y) = p(x)p(y|x) = (1-p)^{x-1}p \times C_x^y p^y (1-p)^{x-y} = C_x^y (1-p)^a p^b$ with $a = 2x - y - 1$ and $b = 1 + y$.
4. The loglikelihood function is $\ell(p) = a\log(1-p) + b\log(p)$.
5. From $\ell^{'}(p) = 0$ we have $\hat{p} = \frac{b}{a+b} = \frac{1}{2}$.

Example 11

Question: Suppose that 3.6 and 5.4 are two numbers chosen randomly and independently from between 0 and c. Find the MLE of c.
Analysis:

1. $X, Y \sim U(0, c)$.
2. The likelihood function is $L(c) = p(x, y) = p(x)p(y) = \frac{1}{c^2}$.
3. The loglikelihood function is $\ell(c) = -2log(c)$.
4. From $\ell^{'}(c) = 0$ we have $c = \infty$.
5. However, the maximum value of $L(c)$ is derived when c takes the smallest value in its range. The range of c is $[\max(x, y), \infty)$. So $\hat{c} = \max(x, y)$.

Example 12

Question: Suppose that $Y \sim Bin(n,p)$. What is the MLE of $r = p^2$? Is this MLE unbiased? If not, find an unbiased estimator of r.
Analysis:

1. The MLE of p is $\hat{p} = \bar{Y}$.
2. The MLE of $r = p^2$ is $\hat{r} = \bar{Y}^2$.
3. $\mathbb{E}(\hat{r}) = Var(\hat{p}) + (\mathbb{E}\hat{p})^2 = \frac{p(1-p)}{n} + p^2 = \left(\frac{n-1}{n}\right) r + \frac{p}{n} \neq r$.
4. $Bias(\hat{r}) = \mathbb{E}(\hat{r}) - r = \frac{p-r}{n}$.
5. $\mathbb{E}(\hat{r}) = \frac{n-1}{n} r - \frac{p}{n}$ and then $\mathbb{E}\left(\frac{n}{n-1}\hat{r}\right) = r + \frac{p}{n-1}$.
6. Since $\hat{p}$ is an unbiased estimator for p, we propose the unbiased estimator $\widehat{r} = \frac{n}{n-1}\hat{r} - \frac{\hat{p}}{n-1}$.

Summary

1. The motivation for MOME and MLE;
2. The procedures to find MOME and MLE.