Assignment 8 - MAT 327 - Summer 2014

Due July 28th, 2013 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] On Assignment 5, A.4 you proved that in ω_1 the intersection of two (hence, a finite number of) closed unbounded sets was again closed unbounded, and in particular, nonempty. Does this prove that ω_1 is compact?

[**C.2**] Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Prove that $(X, \mathcal{T})$ is compact if and only if every cover of the space by *basic* open sets has a finite subcover.

[**C.3**] Here's a really cute and useful fact: Let $(X, \mathcal{T})$ be a compact space, let $(Y, \mathcal{U})$ be a Hausdorff space and let $f : X \longrightarrow Y$ be a continuous function. Prove that f is a closed map. Conclude that, if additionally f is a bijection, then f is a homeomorphism.

[**C.4**] Consider $\mathbb{R}^n$ with the usual metric d, and define

$$\rho(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

for $A, B \subseteq \mathbb{R}^n$. Assume that $C \subseteq \mathbb{R}^n$ is closed, and $K \subseteq \mathbb{R}^n$ is compact. Show that they are disjoint if and only if $\rho(C, K) > 0$. Find an example where this fails if both sets are closed, but not compact.

[**C.5**] Prove that the continuous image of a compact set is compact.

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] Let $(X, \leq)$ be a linear order, and let $(X, \mathcal{T})$ be its order topology. Prove that X is compact if and only if every non-empty set in X has a least upper bound (supremum) and a greatest lower bound (infimum).

[**A.2**] Let $2 := \{0,1\}$ be given the discrete topology, and let $\mathbb{N}$ be given the discrete topology, prove that $2^{\mathbb{N}}$, with the product topology is a compact, Hausdorff, metrizable space. You may wish to observe that $2^{\mathbb{N}}$ is a metrizable space. Do not use Tychonoff's theorem.

[**A.3**] Let $(X, \mathcal{T})$ be a compact subpace of $\mathbb{R}^n$, and let $f : X \longrightarrow \mathbb{R}$ be continuous. Prove that f is uniformly continuous.

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] Compactness is a common theme in mathematics, although sometimes it shows up in different guises. Find and state the "compactness theorem from Logic" and the "compactness theorem from Graph Theory about chromatic numbers (the De Bruijn-Erdös Theorem)". Sketch the proof of one of them, and tell me why these are called "compactness" theorems. You don't need to include all of the details, but I will be looking for evidence that you understand the proof. In particular, define (or at least describe) all terms you use. Is there a compact topology hidden somewhere in these proofs? Did we use any version of the axiom of choice in these proofs?

[**NI.2**] Let $\mathcal{U}$ be an open cover of $(X, \mathcal{T})$, a topological space. Prove that if $(X, \mathcal{T})$ is a Hausdorff compact space then it admits a partition of unity subordinate to $\mathcal{U}$. (This question contains some words that we haven't seen before, but mostly uses ideas that we have seen before. This theorem is extremely useful for differential topology, where you do things like cover the sphere with open discs.)

[**NI.3**] This is a question about the Zariski Topology which is of interest to people in Algebra. In $\mathbb{R}^n$ let $\mathcal{B}$ be the family of sets $\{x : p(x) \neq 0\}$ where p is a polynomial in n variables. Show that $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ on $\mathbb{R}^n$ (which is called the Zariski Topology). Show that $(\mathbb{R}^n, \mathcal{T})$ is a compact T_1 space that is not Hausdorff.

Now for $S \subseteq \mathbb{R}^n$ define $I(S) := \{p : p(s) = 0, \forall s \in S\}$, where p is a polynomial in n variables. Sketch a proof that $I(\mathbb{R}^n)$ contains only the zero polynomial. You may wish to look at the Schwartz-Zippel lemma.