SCHOOL OF FINANCE AND APPLIED STATISTICS

FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 8

Question 1

A fund was valued at:

\$5.2m	on 1 January 1999
\$5.35m	on 1 April 1999
\$5.6m	on 1 July 1999
\$5.6m	on 1 October 1999
\$5.6m	on 31 December 1999

The only cashflow during 1999 was an injection of \$100,000 on 30 June. Calculate

(a) the money-weighted rate of return

(b) the time-weighted rate of return

Solution

(a) The money-weighted rate of return is found from the equation:

$$5.2(1+i)+0.1(1+i)^{1/2}=5.6$$

Solving this using the quadratic formula gives: $(1+i)^{1/2}=1.0282 \Rightarrow i=5.72\%$

(b) The time-weighted rate of return is found from the equation:

$$1+i=\left(\frac{5.6-0.1}{5.2}\right)\left(\frac{5.6}{5.6}\right)=1.0577 \Rightarrow i=5.77\%$$

Past Exam Question – 2005 Final Exam Q2(a)
You invest $10,000 in a savings account earning 4.5% p.a. Six months later you invest a further $8,000. Eight months after the initial investment, the interest rate on the savings account changes to 2.5% p.a. Calculate (to 2 decimal places), over the 12 month period after the initial investment, the:

i) Time Weighted Rate of Return (1 mark)

ii) Money Weighted Rate of Return (4 marks)

Solution

i) $\left(1+i\right)=1.045^{\frac{8}{12}}\times1.025^{\frac{4}{12}}\Rightarrow i=3.83\%$

ii) First find the total accumulation over the 12 month period:

AV = $10,000\times1.045^{\frac{8}{12}}\times1.025^{\frac{4}{12}}+8,000\times1.045^{\frac{2}{12}}\times1.025^{\frac{4}{12}}=\$18,508.41$

Now we set up an equation of value to solve for the MWR:

$18,508.41=10,000(1+i)+8,000(1+i)^{\frac{6}{12}}$ Let $x=(1+i)^{\frac{6}{12}}$

$10,000x^2+8,000x-18,508.41=0$

$$x=\frac{-8,000\pm\sqrt{8,000^2+4\times10,000\times18,508.41}}{2\times10,000}=\frac{-8,000\pm28,360.825}{20,000}$$

$x=1.018041$ Ignore negative root

$i=1.018041^2-1=3.64\%$

Question 2
A 25 year bond has a face value of $10,000 and is redeemable at $12,000. The bond pays coupons of 8.0% half-yearly. Find the price of the bond if the yield is 6% p.a. effective.

Solution
Using the formula $P=Fr\cdot a_{\overline{n}|j}+C\cdot v_j^n$

$P=10,000(0.04)\cdot a_{\overline{50}|j}+12,000v_j^{50}$ where $j=1.06^{\frac{1}{2}}-1=0.029563$

$P=\$13,173.87$

Alternatively

$P=2(10,000)(0.04)\cdot a^{(2)}_{\overline{25}|0.06}+12,000v_{0.06}^{25}=\$13,173.87$

Question 3

A $3,000 bond with annual coupons is selling at an annual effective yield equal to twice the annual coupon rate. The present value of the coupons is equal to the present value of the redemption amount, where redemption is at par. What is the selling price?

Solution

Let the annual coupon rate be r, the number of coupon payments (years) be n, and the effective interest rate per coupon period be j.

(Note: in lecture notes these symbols were defined for half-yearly coupon payments, not annual coupon payments as in this question).

We are told that $j = 2r$.

The present value of coupons is, therefore:

$$3000 \cdot r \cdot a_{\overline{n}|j} = 3000 \cdot \frac{j}{2} \cdot a_{\overline{n}|j} = 3000 \cdot \frac{j}{2} \cdot \frac{1-v^n}{j} = 1500(1-v^n)$$

The present value of redemption = $3000v^n$ since redemption is at par.

We are told that this is equal to the present value of the coupons. Therefore,

$$3000v^n = 1500(1-v^n) \Rightarrow v^n = \frac{1}{3}$$

Therefore, as the present value of the coupons is equal to the present value of the redemption amount:

$$\Rightarrow P = 3000\left(\frac{1}{3}\right) \times 2 = \$2,000$$

Question 4

Smith purchased a 20-year, 8% per annum coupon, $1,000 bond redeemable at par with half-yearly coupons for a purchase price that gave a nominal annual yield to maturity, convertible half-yearly, of 10%. After the 20th coupon, Smith sells the bond. At what price did he sell the bond if his actual nominal annual yield over the holding period was 10% convertible half-yearly?

Solution

Half-yearly coupon payments of $1000\left(\frac{0.08}{2}\right) = \$40$ payable for 20 years, ie. $n = 40$.

Half-yearly effective interest rate of $j = \frac{i^{(2)}}{2} = \frac{0.1}{2} = 0.05$.

Therefore, the original purchase price = $40a_{\overline{40}|0.05} + 1000v_{0.05}^{40} = 828.41$

The bond is sold after the 20th coupon. If he purchased the bond for 828.41, then in order to achieve a nominal yield of 10% (convertible half-yearly), he would have had to receive a redemption amount X that satisfies the following equation:

$$828.41 = 40a_{\overline{20}|0.05} + Xv_{0.05}^{20} \Rightarrow X = 875.38$$

Question 5

Two \$1,000 bonds redeemable at par at the end of the same period are bought to yield a nominal rate of 6% convertible half-yearly. One bond costs \$912.93 and has a coupon rate of 5% pa payable half-yearly. The other bond has a coupon rate of 4% pa payable half-yearly. Find the price of the second bond.

Solution

$i^{(2)} = 6\%$

$j = \frac{i^{(2)}}{2} = 3\%$

Use the time period for the first bond to find the time period for both bonds:

$P_1 = 1000(0.025)a_{\overline{n}|j} + 1000v_j^n = 912.93$

$912.93 = 25\left(\frac{1-v^n}{j}\right) + 1000v^n = \frac{25}{0.03} - \frac{25}{0.03}v^n + 1000v^n = 833.33 + 166.67v^n$

$v^n = 0.47758 \Rightarrow n = \frac{\ln(0.47758)}{\ln(v)} = 25$

Calculate the price of the second bond:

$P_2 = 1000(0.02)a_{\overline{25}|j} + 1000v_j^{25} = \$825.87$

Question 6

An investor purchases a \$10,000 bond redeemable at par on 31 December 2018, with 12% p.a. coupons payable half-yearly. Calculate the purchase price of the bond if it was purchased on 30 April 2005 at an effective annual yield of 7.5%.

Solution

$i = 7.5\%$

$j = 1.075^{0.5} - 1 = 0.036822$

Half-yearly coupon payments of $10,000\left(\frac{0.12}{2}\right) = 600$

We first need to calculate the price of the bond just after the 31 December 2004 coupon payment (ie. $n = 14 \times 2 = 28$):

$P_{2004} = 600a_{\overline{28}|j} + 10,000v_j^{28}$

$= 600\left(\frac{1-1.075^{-14}}{1.075^{0.5}-1}\right) + 10,000(1.075^{-14}) = \$14,007.67$

We then accumulate this forward to 30 April 2005:

$P = P_{2004}(1.075^{\frac{4}{12}}) = \$14,349.45$

Question 7

Find the purchase price at 30 June 2005 of a $50,000 nominal bond redeemable at par on 31 December 2010 with a yield of 7.0% p.a. effective (ie. net or gross); under the following conditions:

a) You are not subject to any tax.
b) You are subject to tax on income only of 43.5%.
c) You are subject to tax on income of 43.5% and capital gains of 30%.

Coupons are paid half-yearly at 10% p.a.

Solution

a)

First find effective half-yearly interest rate:

$$j = 1.07^{\frac{1}{2}} - 1 = 0.034408$$

We know that $n = 11$ is the number of coupon payments.

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

$$P = 50,000 \times 0.05 a_{\overline{11}|j} + 50,000 v_j^{11} = \$57,040$$

b)

$$P = Fr(1 - t_I) \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

$$P = 50,000 \times 0.05(1 - 0.435) a_{\overline{11}|j} + 50,000 v_j^{11}$$

$$= \$47,219$$

c)

There is a capital gain as the answer to b) is less than the redemption value of $50,000. Therefore we must calculate a new price with the capital gain.

$$P' = Fr(1 - t_I) \cdot a_{\overline{n}|j} + C \cdot v_j^n - t_C (C - P') v_j^n = P - t_C (C - P') v_j^n$$

$$P' = 47,219 - 0.3(50,000 - P') v_j^{11} = 47,223 - 10,339.93 + 0.2068 P'$$

$$P' = \frac{47,219 - 10,339.05}{1 - 0.20678} = \$46,494$$

Question 8

Find the purchase price at 6 May 2005 of a $10,000 nominal bond redeemable at $15,000 on 31 December 2017 with a net yield of 8.0% p.a. effective. Coupons are paid half-yearly at 5% p.a. You are subject to tax on coupons and capital gains of 33%.

Solution
First find effective half-yearly interest rate:

$$j = 1.08^{\frac{1}{2}} - 1 = 0.03923$$

We know that $n = 26$ is the number of coupon payments.

We first need to calculate the purchase price net of income tax only at the previous coupon date, 31 December 2004:

$$P_{2004} = Fr(1-t_I)\cdot a_{\overline{n}|j} + C\cdot v_j^n$$
$$P_{2004} = 10,000\times 0.025(1-0.33)a_{\overline{26}|j} + 15,000v_j^{26}$$
$$= \$8,215.25$$

We next need to accumulate this forward to a purchase price date of 6 May 2005.

$$P\ = P_{2004}\times 1.08^{\frac{126}{365}} = 8,436.43$$

We know that there is a capital gain on the bond as the price above is less than the redemption value of \$15,000. Therefore we must calculate a new price with the capital gain. In this case the tax on capital gain must be discounted to 6 May 2005 from 31 December 2017; ie. 25 half years and 55 days (the first half of the year has 181 days):

$$P' = P - t_C(C - P')v_j^n$$
$$P' = 8,436.43 - 0.33(15,000 - P')v_j^{\left(25+\frac{55}{181}\right)} = 8,436.43 - 1,869.54 + 0.124636P'$$
$$P' = \frac{8,436.43 - 1,869.54}{1-0.124636} = \$7,502$$

Question 9
A ten-year \$100,000 bond redeemable at par is issued with 6% p.a. coupons payable half-yearly. Income tax is payable at a rate of 15%, with tax payments due 4 months after each coupon is received. Find the purchase price of the bond at a net redemption yield of 9% p.a. effective.

Solution
First find effective half-yearly interest rate:

$$j = 1.09^{\frac{1}{2}} - 1 = 0.04403$$

We know that $n = 20$ is the number of coupon payments.

This question is much the same as any other question with income tax only, with the exception that tax on coupons must be discounted a further 4 months than the coupon payments themselves. This can be done most simply by adjusting the income tax rate to allow for the discounting; ie:

$$t_I^{'} = t_I\left(1.09^{-\frac{4}{12}}\right) = 0.14575$$

Using the standard formula with this adjustment:

$$P = Fr(1-t_I^{'})\cdot a_{\overline{n}|j} + C\cdot v_j^n$$

$$P = 100,000\times 0.03(1-0.14575)a_{\overline{20}|j} + 100,000v_j^{20}$$

$$= \$75,859$$

Past Exam Question – 2005 Final Exam Q2(b)

Using a net effective quarterly yield of 1.5%, calculate the purchase price of a \$100 face value bond, redeemable at \$120 in 7 years. Coupons are payable on a half yearly basis at 9% per annum. Tax on coupons and capital gains is at 33%. (4 marks)

Solution

The half-yearly redemption yield is $j = (1.015)^2 - 1 = 3.0225\%$

We first need to check if capital gains tax is payable or not.

If capital gains tax is not payable, the price will be:

$$P = Fr(1-t_I)\cdot a_{\overline{n}|j} + C\cdot v_j^n = 100(0.045)(0.67)a_{\overline{14}|j} + 120v_j^{14} = \$113.09739$$

Since \$113.10 < \$120 there will be a capital gain equal to $120 - P'$, which will be taxed at 33%, payable when the capital is repaid. The price P' is

$$P' = P - t_C(C-P')v_j^n$$

$$= 113.09739 - 0.33(120-P')0.659099$$

$$= 86.99706 + 0.217503P'$$

$$= \$111.18$$