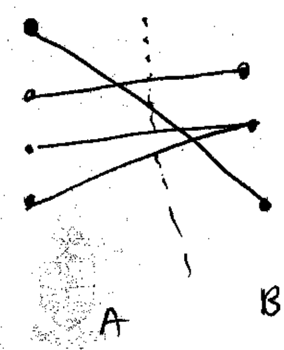
Tutorial 2

Warm Up: True or False ?

1. If every component of a graph is bipartite, then the graph is bipartite.

True. (Recall bipartite means ~~a grap~~ the set V of vertices of a graph can be split into 2 non-empty subsets such that all edges "go from one set to the other." In other words

Ex.



No edges go from vertex "from one set to a vertex from the same set.")

Proof: If all components G_i of a graph G is bipartite, then for each i, the vertices can be split into 2 non-empty subsets A_i and B_i such that no edges go "from A_i to A_i" and no edges go from "B_i to B_i". Then take

$$A = \bigsqcup_{\text{all components } i} A_i \quad ; \quad B = \bigsqcup_{\text{all components } i} B_i$$

Then A, B is a bipartition of all vertices in G with desired property.

Warmup.

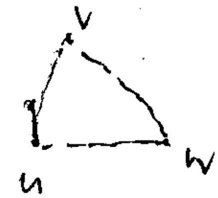
2 True or false?

Define the distance between two connected vertices u, v to be $d(u,v) :=$ length of shortest path connecting them.

does $d(u,v)$ satisfy the triangle inequality?

i.e. $d(u,v) + d(v,w) \geq d(u,w)$

~~when u~~.



True. If there is a path connecting u,v of length $d(u,v)$ and a path connecting w,v of length $d(v,w)$, then there is a path, just take the path from u to v followed by the path from v to w, connecting u and w of length $d(u,v)+d(v,w)$. This has to be at least as big as the length of a shortest path between u and w.

Example.

1/ If u is a vertex with $\deg(u)$ odd, then there is a path from u to another vertex v with $\deg(v)$ odd.

(Recall $\deg(u)$ is the valence of the vertex)
i.e. # of half edges "shooting" out of u.

proof (Thanks to a student)

① Using handshake Lemma. For each connected component of G, the handshake lemma says that there are even number of odd vertices.

(Recall a component containing a vertex v is a subgraph consisting of all vertices connected to v and all edges incident on this set of vertices.

Eg. $G =$ 

2 components. and

but a subgraph is just any subset of both V and E ~~which is a graph~~.

Eg.  is a subgraph of G.)

So it follows that for any odd vertex u, there is another vertex v in the same component, i.e. connected to u

<u>proof 2</u> Elementary w/o using the handshake lemma.

Construct a walk with no repeated edges, then use that all walks from u to v "contains" a path from u to v.

Construction:

⓪ Starting with u, since $\deg(u)$ odd, not all half edges belong to loops, so there is at least 1 edge connecting u to another vertex u_1.

① 2 cases: (i) $\deg(u_1)$ odd $\Rightarrow$ done.

(ii) $\deg(u_1)$ even. For the same reasoning as for u, there has to be an edge "going out", i.e. connecting u_1 with another vertex u_2.

2 cases: $u_2 = u$, do ⓪ again since $\deg(u)$ odd means there is another outgoing edge

$u_2 \neq u$, do ①.

Continue until this walk ends an odd vertex $v \neq u$. This terminates since there are only finitely many edges in the graph.

Example 2.

Every non-empty connected graph G with $\deg(v)$ even for all vertices in G has a closed Eulerian trail.

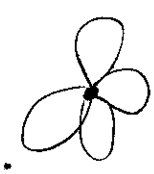
(Recall : (i) walk : no restriction
→ trail : no repeated edge but can repeat vertices
path : no repeated vertices (and so no repeated edges too)
cycle : no repeated vertex except for the starting and finishing vertex.

(ii) an $\boxed{\text{Eulerian}}$ path or closed trail is one that "visits" each edge exactly once. as in the bridge problem in first lecture.

(iii) A $\boxed{\text{Hamiltonian}}$ path or cycle. is one that visits each vertex exactly once.)

proof Use induction on # of vertices. n.

base case: $n = 1$, go through the loops one by one to get an closed Eulerian trail.



Induction step: ~~Consider~~ Suppose all "smaller graphs" has a Eulerian closed trail.

a "smaller graph" here means a non-empty connected graph G with $\deg(v)$ even for all v which has less than n

2 5

"a falling tile can push the next tile"

⟺ Induction step:

If statement true for $n-1$ (or all numbers less than n)

then statement true for n.

Then all tiles fall! ⟺ true for all $n \geq n_0$!

<u>Example</u> ATM machine with only \$2 coins and \$5 bills can handle all amounts $\geq$ \$4

<u>proof</u> base case: \$4 = \$2 × 2.

Induction step: Suppose machine knows how to handle \$$n$ (≥ 4), then we need to show that the machine also knows how to handle \$$n+1$.

Case 1: \$$n$ contains at least 1 \$5

i.e. $n = 2k + 5l$

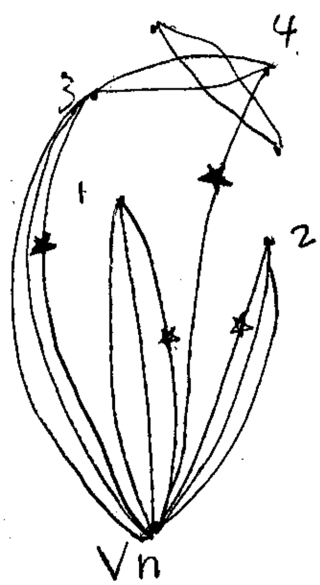
where $l \geq 1$, $k \geq 0$

then replace \$5 bill by 3 \$2.

i.e. \$$n+1$ = (\$2)$\cdot(k+3)$ + \$5$(l-$

Then ~~given any~~ first consider any loopless graph G with n vertices with all the needed properties.

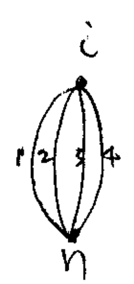
Pick any vertex in G, call it v_n. Since $\deg(v_n)$ even there are even number of other vertices $v_i, i \neq n$ s.t. the number of edges connecting v_n and v_i is odd i.e. a_{ni} is odd, so these can be paired randomly. (Eg. pair up $\{1,2\}$ and $\{3,4\}$ in the diagram).



For each pair $\{i,j\}$, pick an edge from i to n and an edge from j to n (Eg. ~~the~~ for the pair $\{1,2\}$, the edges with the ☆ on them)

~~Now excluding these chosen~~ The remaining edges are such that ~~there~~ for each $i \neq n$, there are even number of these unpicked edges connecting i to n. ~~again~~, pair among the edges connecting i to n.

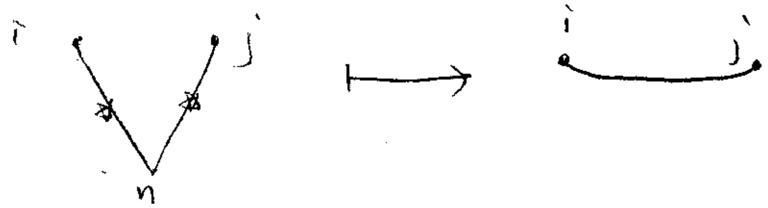
Eg.

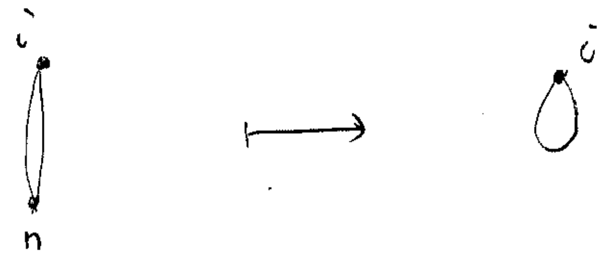


pair edge 1 ad 2
edge 3 and 4.

Now, we obtain a "smaller graph" H by "forgetting the vertex v_n" i.e.

for each pair of paired edges, do the following.





The "smaller graph" ~~has the same valence~~ has vertices 1 to $n-1$ with $\deg_H(i) = \deg_G(i)$ so even

~~Now~~ and is still connected.

Now use the induction hypothesis to get an Eulerian closed trail for the smaller graph.

This trail is still Eulerian once you ~~th~~ "unforget" the vertex n.

Finally, it is easy to show that ~~adding loops~~ ~~does~~ if G is Eulerian then G with loops added