Assignment 5 - MAT 327 - Summer 2014

Due July 7th, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Suppose that $\langle x_n \rangle$ is a (countable) increasing sequence in ω_1. Show that there is a point $p \in \omega_1$ such that $x_n \to p$. (**Hint**: You only know 2 ways to find points in ω_1.)

[**C.2**] Let $(L, \leq)$ be a linear order. Prove that $(L, \leq)$ is a well-order if and only if L does not contain any infinite decreasing chains. (A chain $C = \{ x_n : n \in \mathbb{N} \}$ is decreasing provided that $n < m$ implies $x_m < x_n$.)

[**C.3**] (This is a bit of a trick question, so read it closely...) Suppose that ω_1 is in bijective correspondence with $\mathbb{R}$. Is $\mathbb{R}$ with its usual order a well-order? Why or why not?

The next two exercises make use of the notion of the "Lindelöf property" which is a weaker notion than second countability (as you will show on question C.5). This property is related to compactness (which we will see later) it also has some nice interactions with some topological invariants we have already seen.

Definition. *A space $(X, \mathcal{T})$ is a **Lindelöf** space if every open cover of X has a countable subcover. That is, whenever $X \subseteq \bigcup_{\alpha \in I} U_\alpha$, where I is an index set, and each U_α is open, then there is a countable subset $A \subseteq I$ such that $X \subseteq \bigcup_{\alpha \in A} U_\alpha$.*

[**C.4**] By exhibiting an open cover that doesn't have a countable subcover, prove that ω_1 with the order topology is not a Lindelöf space. (Conclude that first countable does not imply Lindelöf.)

[**C.5**] Prove that every second countable space is a Lindelöf space.

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

This first exercise makes use of two notions specific to linear orders; you have already met the first one in calculus (Dedekind completeness) and the second one is straightforward.

Definition. *A linear order $(L, \leq)$ is **Dedekind complete** if whenever $\emptyset \neq S \subseteq L$ has an upper bound in L then it has a least upper bound in L, called $\sup(S)$, the supremum of S.*

Recall that $\mathbb{R}$ is Dedekind complete (with the usual order). You can also see (on your own) that ω_1 (with its usual ordering) is Dedekind complete.

Definition. *A linear order $(L, \leq)$ has a **gap** provided that there are $x, y \in L$ with $x < y$, but there is no $z \in L$ such that $x < z < y$.*

[**A.1**] Suppose that $(L, \leq)$ is a linear order. Prove that L, with the order topology, has no non-trivial clopen subsets if and only if L is Dedekind complete and has no gaps.

[**A.2**] Prove that $\omega_1 + 1$ is a Lindelöf space. (Conclude that Lindelöf does not imply first countable or second countable.) If you are hungry, then prove that $\omega_1 + 1$ is actually *compact*, (that is, every open cover has a *finite* subcover. Use C.2 for this.).

[**A.3**] Prove that every regular Lindelöf space is normal. (**Hint**: In what other context have we seen regular + something implies normal?)

[**A.4.**] Let $C, D \subseteq \omega_1$ be closed (in the order topology), unbounded (called **club**) subsets of ω_1. Prove that $C \cap D$ is a closed unbounded subset of ω_1. (Recall that $C \subseteq \omega_1$ is unbounded means that for all $\alpha \in \omega_1$, there is a $c \in C$ such that $\alpha < c$.) **Strategy**: Weave the two club sets together and use C.1.

New Ideas

*For this section please work on and sumbit **at least one** of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] Prove that every continuous function $f : \omega_1 \longrightarrow \mathbb{R}$ has a countable range (with both spaces given their order topologies). Conclude that every such function is eventually constant. (There are many different ways to prove this!)

[**NI.2**] Give a clear description of the so called "long-line". Prove that it is "locally Euclidean" and list the topological properties that it has (of the 10 or so that we have seen so far), and prove a couple of them. What are some interesting modifications you can make to the construction?

[**NI.3**] Many students like to claim that $\mathbb{R}$ is in bijection with ω_1. This assertion is called the "Continuum Hypothesis" (CH). Read the wikipedia article for the Continuum Hypothesis and at least one other article about CH. (I might try Gödel's 1947 "What is Cantor's Continuum Problem?" or Juliette Kennedy's 2011 "Can the Continuum Hypothesis be Solved?".)

- In your own words, state the Continuum Hypothesis in language that a student in this course would understand.
- In your own words, state the Continuum Hypothesis in language that a student in MAT 137 would understand.
- State a question in mathematics that is answerable if we assume that CH is true.
- State a question in mathematics that is answerable if we assume that CH is false.
- Who were the important people in proving that CH is independent from the usual axioms of set theory; what were their contributions?
- Ask a working mathematician how they feel about the Continuum Hypothesis; do they "think it's true"?

Note: For the "state a question" parts, please avoid trivial restatements of CH. Also please do not use "ccc is not productive" or something else we have already mentioned in class.