Introduction to Bayesian Data Analysis

Tutorial 3 Solutions

(1) (a) Y is a count of the number of deaths from asthma in a city of size u over a year. Assuming the death rate is small, a Poisson sampling model is appropriate. That is $Y|\theta \sim Pois(u\theta)$, where θ is the number of cases per 100,000 people. The unknown parameter is θ, often called the *rate*.

(b) Now $p(y|\theta) = \frac{(u\theta)^y e^{-u\theta}}{y!} \propto \theta^y e^{-u\theta}$. That is, the likelihood is of the form $\theta^a e^{-b\theta}$ and so the conjuate prior density must be of the form $p(\theta) \propto \theta^A e^{-B\theta}$. In a more conventional parametisation

$$p(\theta) \propto e^{-\beta\theta}\theta^{\alpha-1}$$

which we recognise as a gamma density with parameters α and β. Hence the posterior distribution is

$$p(\theta|y) \propto p(\theta)p(y|\theta) \propto e^{-\beta\theta}\theta^{\alpha-1}e^{-u\theta}\theta^y \propto e^{-(\beta+u)\theta}\theta^{\alpha+y-1}$$

which is the density of a gamma distribution with parameters $\alpha + y$ and $\beta + u$.

(c) 3/200000=1.5 deaths per 100,000 persons per year.

(d) One possibility to formulate the prior is to review previous studies of asthma mortality rates in other parts of the world.

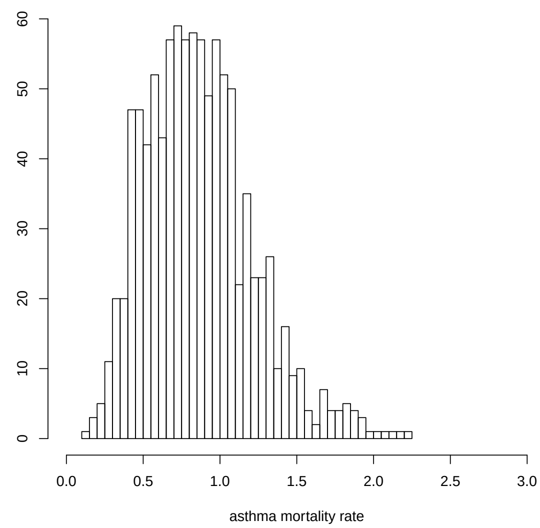
(e) Note the prior has mean 0.6 (with a mode of 0.4) and 97.5% of the mass of the density lies below 1.44. (Compare to a gamma(9,15) prior). The posterior distribution of the asthma mortality rate is

$$\theta|y \sim Gamma(3 + 3, 5 + 2)$$

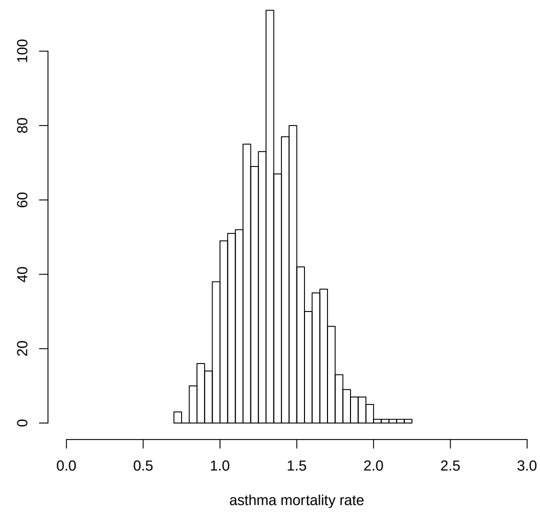
The posterior mean is 6/7=0.86 (the posterior mode is 0.71). Substantial shrinkage has occurred towards the prior distribution. Also, $Pr(\theta > 1.0|y) = 0.3$

```
a<-3
b<-5
y<-3
u<-2
d<-rgamma(1000,a+y,b+u)

> pgamma(1,a+y,b+u,lower.tail=FALSE)
[1] 0.3007083
> sum(d>1)/1000
[1] 0.301
```



(f) Now $\theta|y \sim Gamma(3+30, 5+20)$. The posterior mean is 1.32 (closer to the crude estimate) and the posterior mode is 1.28. Also, $Pr(\theta > 1.0|y) = 0.93$. From the plot below, we see the posterior distribution is much more concentrated than before. In these calculations, we have assumed that the population remains constant at 200,000.



(2) Assume a conjugate prior $p(\theta) \propto \theta^{\alpha} e^{-\beta\theta}$

$$\begin{aligned}
p(\tilde{y}|y_1, ..., y_n) &= \int_0^{\infty} p(\tilde{y}|\theta, y_1, ..., y_n) p(\theta|y_1, ..., y_n) d\theta \\
&= \int p(\tilde{y}|\theta) p(\theta|y_1, ..., y_n) d\theta) \\
&= \int \mathrm{dpois}(\tilde{y}, \theta) \mathrm{dgamma}(\theta, \alpha + \sum_i y_i, \beta + n) d\theta \\
&= \int \left\{ \frac{1}{\tilde{y}!} \theta^{\tilde{y}} e^{-\theta} \right\} \left\{ \frac{(\beta+n)^{\alpha + \sum_i y_i}}{\Gamma(\alpha + \sum_i y_i)} \right\} \theta^{\alpha + \sum_i y_i - 1} e^{-(\beta+n)\theta} d\theta \\
&= \frac{(\beta+n)^{\alpha+\sum_i y_i}}{\Gamma(\tilde{y}+1)\Gamma(\alpha + \sum_i y_i)} \int_0^{\infty} \theta^{\alpha + \sum_i y_i + \tilde{y} - 1} e^{-(\beta+n+1)\theta} d\theta
\end{aligned}$$

To simplify the above integral, note that

$$1 = \int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta$$

for any values $\alpha, \beta > 0$. So $\int_0^{\infty} \theta^{\alpha-1} e^{-\beta\theta} d\theta = \frac{\Gamma(\alpha)}{\beta^{\alpha}}$. Substitute in $\alpha + \sum_i y_i + \tilde{y}$ instead of α and $\beta + n + 1$ instead of β

$$\int_0^\infty \theta^{\alpha+\sum_i y_i+\tilde{y}-1} e^{-(\beta+n+1)\theta} d\theta = \frac{\Gamma(\alpha+\sum_i y_i+\tilde{y})}{(\beta+n+1)^{\alpha+\sum_i y_i+\tilde{y}}}$$

Then we will have:

$$\frac{\Gamma(\alpha+\sum_i y_i+\tilde{y})}{\Gamma(\tilde{y}+1)\Gamma(\alpha+\sum_i y_i)} \left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum_i y_i} \left(\frac{1}{\beta+n+1}\right)^{\tilde{y}}$$

which is a negative binomial density with parameters $(\alpha+\sum_i y_i, \frac{\beta+n}{\beta+n+1})$ $(\tilde{y} \in \{0,1,2,...\})$ where the probability of success on a trial is $\frac{\beta+n}{\beta+n+1}$, and we want to calculate the probability of $\tilde{y}$ failures before $\alpha+\sum_i y_i$ successes.

From the properties of the negative binomial distribution:

$Var[\tilde{Y}|y_1,...,y_n] = \frac{\alpha+\sum_i y_i}{\beta+n}\frac{\beta+n+1}{\beta+n} = Var[\theta|y_1,...,y_n]\times(\beta+n+1) = E[\theta|y_1,...,y_n]\times \frac{(\beta+n+1)}{\beta+n}$

Interpretation: Uncertainty in a new prediction comes from uncertainty about the population and the variability in sampling from the population. Consider when n is large and when n is small. Where does the predictive variability in $\tilde{Y}$ primarily stem from for each case??

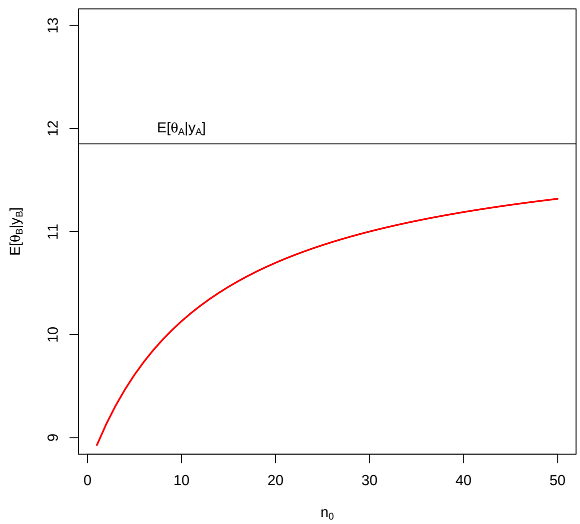
(3) (a) $\sum_i y_{A,i} = 117$; $\sum_i y_{b,i} = 113$.
So $\theta_A|\mathbf{y_A} \sim$ Gamma$(120+117, 10+10)$; $\theta_B|\mathbf{y_B} \sim$ Gamma$(12+113, 1+13)$
$E[\theta_A|\mathbf{y_A}] = 237/20 = 11.9$; $E[\theta_B|\mathbf{y_B}] = 125/14 = 8.9$
$\mathrm{Var}[\theta_A|\mathbf{y_A}] = 237/20^2 = 0.59$; $\mathrm{Var}[\theta_B|\mathbf{y_B}] = 125/14^2 = 0.64$

```
> qgamma(c(0.025,0.975),a1+sum(ya),b1+length(ya))
[1] 10.38924 13.40545
> qgamma(c(0.025,0.975),a2+sum(yb),b2+length(yb))
[1]  7.432064 10.560308
```

95% quantile-based intervals for θ_A and θ_B are (10.4,13.4) and (7.4,10.6) respectively.

In summary, we conclude that expected counts from type B mice are lower given the data.

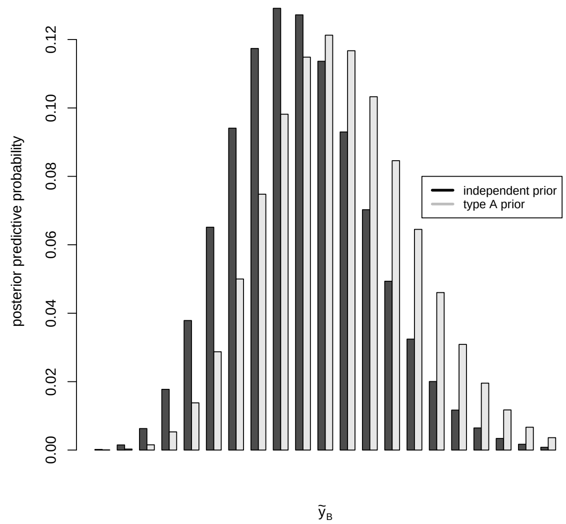
(b) n_0 needs to be greater than 50 in order for the posterior expectation of θ_B to be close to that of θ_A which means we would need to assume that tumor counts from Type B mice are well studied (that is, more prior data).

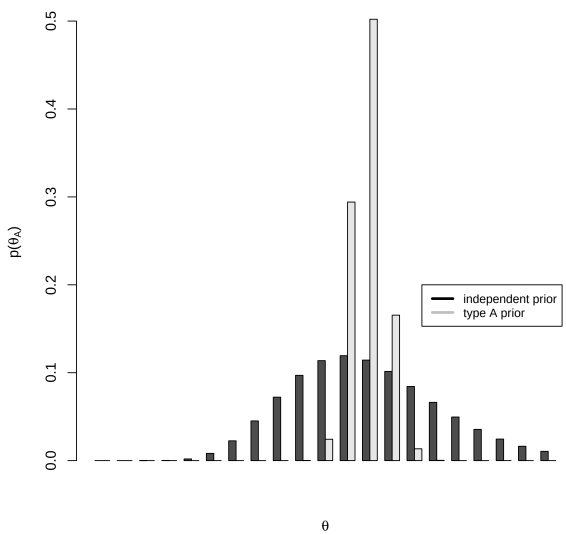


(c) (i) $\sum y_b = 113$. Then $\tilde{y}_B|y_b \sim \text{NegBin}(12+113, 1+13)$. $E[\tilde{y}_B|y_b] = \frac{125}{14} = 8.9$

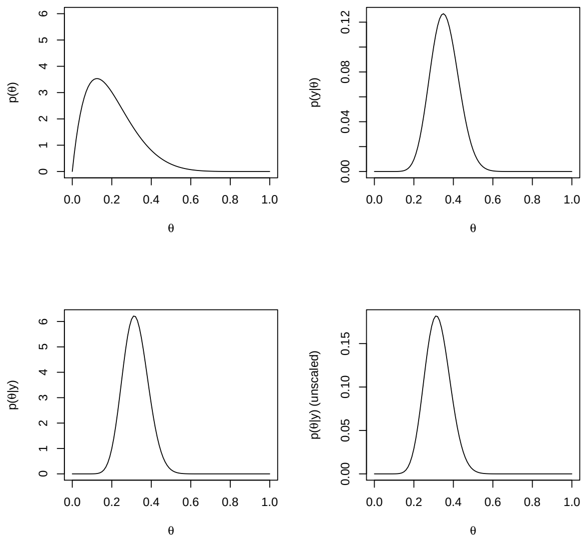
(ii) $\sum y_a = 117$. $p(\theta_b) \propto p(\theta_a|y_a)$; so $\theta_b \sim Gamma(120+117, 10+10)$. Then $\tilde{y}_B|y_b \sim \text{NegBin}(237+113, 20+13)$. $E[\tilde{y}_B|y_b] = \frac{350}{33} = 10.6$
We expect a higher predicted count if we assume the data from mice A form a prior distribution for the posterior of θ_B (as shown by the shift in the location of the posterior predictive distribution (see plot below).

(b) We are told that type B mice are related to type A mice. We are not told the details of the relationship but this is enough information to say that it does not make sense to assume prior independence between θ_a and θ_b. In particular, given that type A mice have been well studied, it makes more sense to use tumor rate data from type A, as a prior guess on the tumor rates of type B mice. The type A prior is a more informative prior.


posterior predictive probability
independent prior
type A prior
$\tilde{y}_B$


$p(\theta_A)$
independent prior
type A prior
θ

(4) (a) $\theta|y \sim \text{Beta}(2+15, 8+43-15)$. $E[\theta|y] = \frac{17}{17+36} = 0.32$. Posterior mode $= \frac{17-1}{17+36-2} = 0.31$. $SD[\theta|y] = \sqrt{\frac{17\times 36}{(17+36)^2(17+36+1)}} = 0.0635$. A 95% posterior interval is (0.20,0.45)

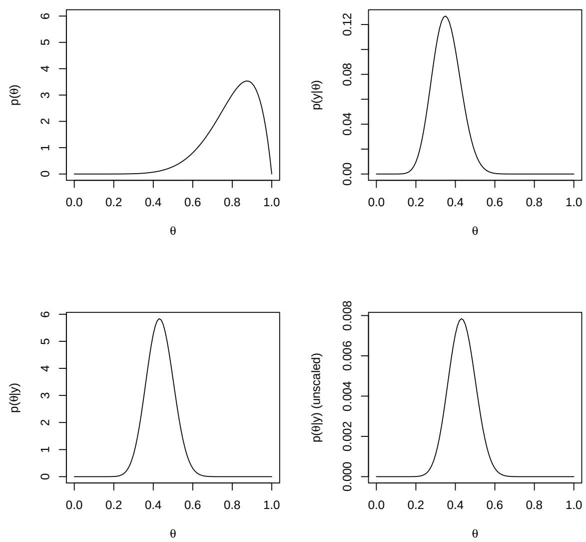



```
> qbeta(0.025,a+y,b+n-y)
[1] 0.2032978
> qbeta(0.975,a+y,b+n-y)
[1] 0.451024
```

(b) $\theta|y \sim \text{Beta}(8+15, 2+43-15)$. $E[\theta|y] = \frac{23}{23+30} = 0.43$. Posterior mode $= \frac{23-1}{23+30-2} = 0.41$. $SD[\theta|y] = \sqrt{\frac{23\times 30}{(23+30)^2(23+30+1)}} = 0.0674$. A 95% posterior interval is (0.30,0.57).

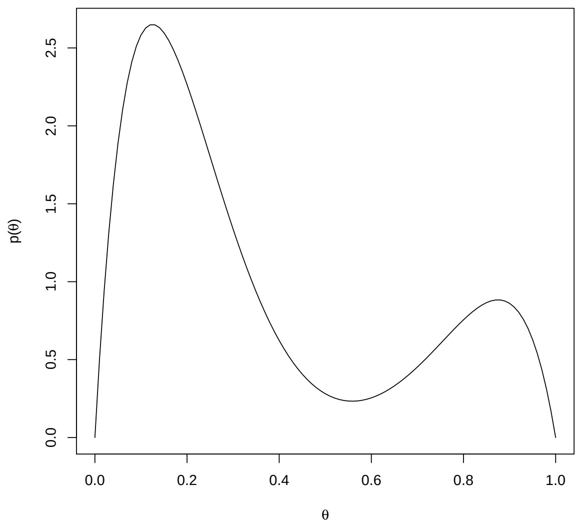
We can see the posterior summaries in (b) are higher in line with a higher prior belief of recidivism, but the uncertainty in our prior belief is similar.

```
> qbeta(0.025,a+y,b+n-y)
[1] 0.3046956
> qbeta(0.975,a+y,b+n-y)
[1] 0.5679528
```



(c)
```
a1<-2
b1<-8
a2<-8
b2<-2
p.theta<-0.75*dbeta(theta,a1,b1)+0.25*dbeta(theta,a2,b2)
plot(theta,p.theta,type="l")
#or
p.theta<-0.25*gamma(10)/(gamma(2)*gamma(8))*(3*theta*(1-theta)^7+
   theta^7*(1-theta))
plot(theta,p.theta,type="l")
```

This prior has two peaks. The prior could be based on reviews of recidivism rates in previous years or other locations, and the prior information could indicate the presence of two subpopulations, one with mean recidivism rate 0.2 and the other with mean recidivism rate 0.8.



(d)

$$
\begin{aligned}
P(\theta|y) &\propto p(y|\theta)p(\theta) \\
&= p(y|\theta)(0.75\text{beta}(2,8) + 0.25\text{beta}(8,2)) \\
&= \theta^y(1-\theta)^{n-y}(0.75\text{beta}(2,8) + 0.25\text{beta}(8,2)) \\
&= \theta^y(1-\theta)^{n-y}\left(0.75\frac{\Gamma(2+8)}{\Gamma(2)\Gamma(8)}\theta^{2-1}(1-\theta)^{8-1} + 0.25\frac{\Gamma(8+2)}{\Gamma(8)\Gamma(2)}\theta^{8-1}(1-\theta)^{2-1}\right) \\
&= 0.75\frac{\Gamma(2+8)}{\Gamma(2)\Gamma(8)}\theta^{y+2-1}(1-\theta)^{n-y+8-1} + 0.25\frac{\Gamma(8+2)}{\Gamma(8)\Gamma(2)}\theta^{y+8-1}(1-\theta)^{n-y+2-1}
\end{aligned}
$$

$$
\begin{aligned}
P(\theta|y) \propto\ & 0.75\frac{\Gamma(2+8)}{\Gamma(2)\Gamma(8)}\frac{\Gamma(y+2)\Gamma(n-y+8)}{\Gamma(2+n+8)}\frac{\Gamma(2+n+8)}{\Gamma(y+2)\Gamma(n-y+8)}\theta^{y+2-1}(1-\theta)^{n-y+8-1} \\
&+ 0.25\frac{\Gamma(8+2)}{\Gamma(8)\Gamma(2)}\frac{\Gamma(y+8)\Gamma(n-y+2)}{\Gamma(8+n+2)}\frac{\Gamma(8+n+2)}{\Gamma(y+8)\Gamma(n-y+2)}\theta^{y+8-1}(1-\theta)^{n-y+2-1}
\end{aligned}
$$

$$
\begin{aligned}
P(\theta|y) \propto\ & 0.75\frac{\Gamma(2+8)}{\Gamma(2)\Gamma(8)}\frac{\Gamma(y+2)\Gamma(n-y+8)}{\Gamma(2+n+8)} \times \text{Beta}(y+2, n-y+8) + \\
& 0.25\frac{\Gamma(8+2)}{\Gamma(8)\Gamma(2)}\frac{\Gamma(y+8)\Gamma(n-y+2)}{\Gamma(8+n+2)}\text{Beta}(y+8, n-y+2)
\end{aligned}
$$

$$w_1 = 0.75 \frac{\Gamma(2+8)}{\Gamma(2)\Gamma(8)} \frac{\Gamma(y+2)\Gamma(n-y+8)}{\Gamma(2+n+8)}$$

$$w_2 = .25 \frac{\Gamma(8+2)}{\Gamma(8)\Gamma(2)} \frac{\Gamma(y+8)\Gamma(n-y+2)}{\Gamma(8+n+2)}$$

$$p_1^* = w_1/(w_1+w_2); p_2^* = 1 - p_1^*$$

$$P(\theta|y) = p_1 \mathrm{Beta}(y+2, n=y+8) + p_2 \mathrm{Beta}(y+8, n-y+2)$$

So a general formula is:

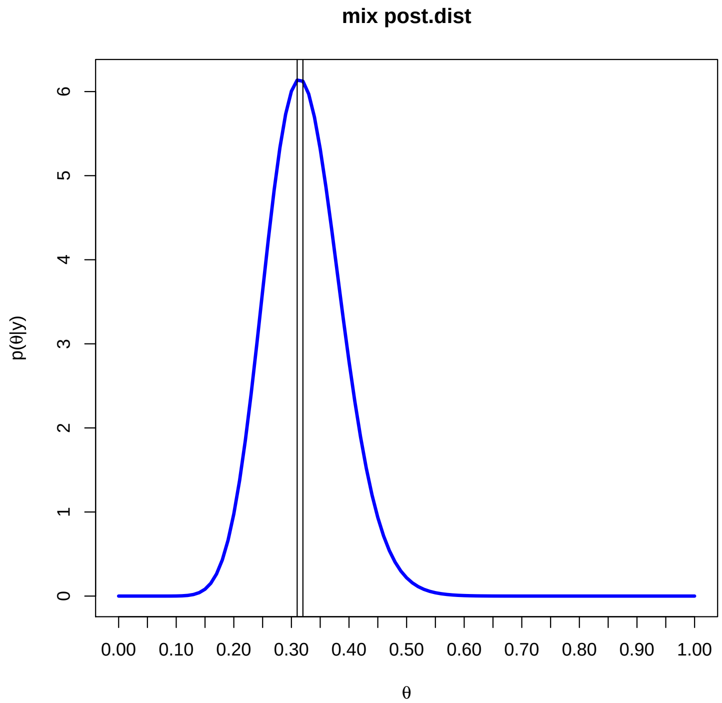
$$w_j = p_j \frac{\Gamma(a_j+b_j)}{\Gamma(a_j)\Gamma(b_j)} \frac{\Gamma(y+a_j)\Gamma(n-y+b_j)}{\Gamma(a_j+n+b_j)}$$

$$p_j^* = w_j / \sum_j w_j$$

$$P(\theta|y) = \sum_j p_j^* P_j(\theta|y)$$

The posterior mode is around 0.315 (similar to the posterior mode in (a), which makes sense given the higher posterior weighting on the prior in (a) in the mixture distribution ($p_1^* = 0.98$).

```
> p1
[1] 0.9849087
> p2
[1] 0.01509134
```



(5) (a)

$$p(y|\theta) = \frac{e^{-\theta}\theta^y}{y!}$$

$$\log p(y|\theta) = -\theta + y\log\theta - \log y!$$

$$\frac{\partial \log p(y|\theta)}{\partial\theta} = -1 + \frac{y}{\theta}$$

$$\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2} = -\frac{y}{\theta^2}$$

$$\mathrm{I}(\theta) = -E\left[\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2}\right] = \frac{1}{\theta}$$

$$p_J(\theta) \propto \theta^{-1/2}$$

Following the distributional form of the family of Gamma distributions, Jeffreys' prior implies a Gamma(1/2,0) distribution which is not a proper distribution.

(b)

$$f(\theta, y) = \theta^{1/2-1}\frac{e^{-\theta}\theta^{y}}{y!} \propto \theta^{1/2-1}e^{-\theta}\theta^{y} = \text{Gamma}(y+1/2, 1)$$

which is a proper posterior density.