Australian National University

RESEARCH SCHOOL OF FINANCE, ACTUARIAL STUDIES AND APPLIED STATISTICS

INTRODUCTORY MATHEMATICAL STATISTICS (STAT2001)
PRINCIPLES OF MATHEMATICAL STATISTICS (STAT6039)

Final Examination – June 2013

Study Period: 15 minutes
Time Allowed: 3 hours
Permitted Material: No restrictions

- ***Undergraduate students (those enrolled in STAT2001) should attempt only the first five problems. For these students the exam is out of 140 marks.***

- ***Graduate students (those enrolled in STAT6039) should attempt all six problems. For these students the exam is out of 150 marks.***

- ***Draw a box around each solution and express each numerical solution in decimal form to at least four significant digits (e.g.*** $\boxed{0.007204}$***). Start your solution to each problem on a new page.***

- ***To ensure full marks, show all the steps in working out your solutions. Marks may be deducted for not showing appropriate calculations or formulae, or for not clearly referencing the results in the text book or course material which you are using.***

Problem 1 ***(20 marks in total)***

A new game at the casino called Luck-Out gives the player a 1% chance of winning \$1000, a 4% chance of winning \$100, and a 95% chance of winning nothing.

(a) You are about to play Luck-Out 250 times at a cost of \$15 for each game. Approximate the probability that you will have more money after playing the 250 games than just before you start playing. ***(10 marks)***

(b) Your friend Jim played Luck-Out exactly eight times and won some money at least twice. How many dollars do you expect that he won? ***(10 marks)***

Problem 2 ***(60 marks in total)***

Consider a random sample, $Y_1,...,Y_n$, from the uniform distribution between *c* and 10, where *c* is an unknown constant which is less than or equal to 10.

(a) Derive the method of moments (MOM) estimator of *c* and its mean squared error (MSE). Calculate the MOM estimate of *c* if we observe $y = (y_1,..., y_n)$ = (2.2, 4.8, 5.0). Also calculate the MSE of the MOM estimator of *c* if *n* = 3 and *c* = 2.0. ***(10 marks)***

(b) Write down $L(c)$, the likelihood function for *c*, and derive the maximum likelihood (ML) estimator of *c*. Then calculate $\hat{c}$, the ML estimate of *c*, if $y = (y_1,..., y_n)$ = (2.2, 4.8, 5.0). Also calculate $r = L(\hat{c}) / L(0)$. ***(10 marks)***

(c) Modify the ML estimator of *c* to obtain an estimator which is unbiased. Then evaluate the modified estimator if $y = (y_1,..., y_n)$ = (2.2, 4.8, 5.0). ***(10 marks)***

(d) Find a central 95% confidence interval (CI) for *c* if $y = y_1$ = 4.0. ***(10 marks)***

(e) Find an approximate central 95% CI for *c* if *n* = 50 and the sample mean is $\bar{y} = (y_1 + ... + y_n) / n$ = 4.0. ***(10 marks)***

(f) Suppose that we wish to test $H_0 : c = 0$ versus $H_1 : c \neq 0$ at the 5% level of significance. Construct a suitable hypothesis test using the sample mean $\bar{Y} = (Y_1 + ... + Y_n) / n$ as the test statistic. Assume that *n* is 'large' (more than 20). Then conduct the hypothesis test for the case where *n* = 50 and $\bar{y}$ = 4.0. Clearly indicate the rejection region. Also calculate the *p*-value. ***(10 marks)***

Problem 3 ***(20 marks in total)***

Suppose that $X, Y \sim iid\ U(0,c)$, where c is an unknown positive constant.

(a) Derive the probability density function of $R = Y/X$.
Then find the mean, mode and median of R if $c = 8$. ***(10 marks)***

(b) Use your solution to (a) to find a central 80% *prediction interval* for Y when the observed value of X is $x = 4.0$. Your answer should be of the form $(g(x), h(x))$, where g and h are two functions which satisfy the equations $P(g(X) < Y < h(X)) = 0.8$ and $P(Y < g(X)) = P(Y > h(X)) = 0.1$.

(10 marks)

Hint: Part (b) involves an adaptation of ideas used for finding confidence intervals.

Problem 4 ***(20 marks in total)***

A standard and fair 6-sided die is about to be rolled 7 times. Let X be the number of times that 1 and 2 will come up in a row (meaning 1 immediately followed by 2).

(a) Find $P(X \geq 1)$. ***(10 marks)***

(b) Find EX and $VarX$. ***(10 marks)***

Problem 5 ***(20 marks in total)***

A fair 6-sided die has the numbers 1, 2, 2, 3, 3, 3 printed on its 6 sides, respectively.

(a) The die is going to be rolled five times. Find the probability that the numbers which come up contain at least one 1 and at least one 3. ***(10 marks)***

(b) The die is going to be rolled repeatedly until a number comes up twice in a row. Find the probability that the last number to come up is 3. ***(10 marks)***

Problem 6 ***(to be done only by STAT6039 students)*** ***(10 marks)***

A fair 6-sided die has the numbers 1, 2, 2, 3, 3, 3 printed on its six sides, respectively.
The die is going to be rolled repeatedly until a 2 comes up.
Find the probability that 3 comes up at least once.

END OF EXAMINATION