$\frac{10}{10}$

MAT335 Assignment 1. Rui Qiu #999292509

5. Solution:

$$F^2(x) = F(F(x)) = (x^2)^2 = x^4 = x^{2^2}$$
$$F^3(x) = F(F^2(x)) = (x^4)^2 = x^8 = x^{2^3}$$
$$F^4(x) = F(F^3(x)) = (x^8)^2 = x^{16} = x^{2^4}$$

...

$F^n(x) = x^{2^n}$ (can prove this by induction)

7. Solution:

a. $F(x_0) = 3x_0 + 2 = x_0$
$$2x_0 = -2$$
$$x_0 = -1$$
The fixed point is -1.

b. $F(x_0) = x_0^2 - 2 = x_0$
$$x_0^2 - x_0 - 2 = 0$$
$$(x_0 - 2)(x_0 + 1) = 0$$
$$x_0 = 2 \text{ or } x_0 = -1$$
The fixed points are 2 and -1.

c. $F(x_0) = x_0^2 + 1 = x_0$
$$x_0^2 - x_0 + 1 = 0$$
Since $(-1)^2 - 4 \times 1 \times 1 < 0$
This function does not have real fixed points.

d. $F(x_0) = x_0^3 - 3x_0 = x_0$
$$x_0^3 - 4x_0 = 0$$
$$x_0(x_0^2 - 4) = 0$$
$$x_0(x_0 + 2)(x_0 - 2) = 0$$
$$x_0 = 0 \text{ or } x_0 = -2 \text{ or } x_0 = 2.$$
The fixed pts are -2, 0 and 2.

e. $F(x_0) = |x_0| = x_0$

when $x_0 \geqslant 0$, $x_0 = x_0$, x_0 can be all real postive numbers,

when $x_0 < 0$, $-x_0 \neq x_0$, no solutions.

Hence, all the points in R^+ are fixed points. ✓

I think $\mathbb{R}^+$ doesn't include zero, though.

f. $F(x_0) = x_0^5 = x_0$

$x_0^5 - x_0 = x_0(x_0^4 - 1) = x_0(x_0^2+1)(x_0^2-1) = x_0(x_0^2+1)(x_0+1)(x_0-1) = 0$

$x_0 = 0$ or $x_0 = -1$ or $x_0 = 1$.

The fixed pts are -1, 0 and 1. ✓

g. $F(x_0) = x_0^6 = x_0$

$x_0 = 0$ or $x_0 = 1$

The fixed points are 0 and 1. ✓

h. $F(x_0) = x_0 \sin x_0 = x_0$

$x_0(\sin x_0 - 1) = 0$

$x_0 = 0$ or $\sin x_0 = 1 \Rightarrow x_0 = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$.

The fixed points are 0 and $\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$. ✓

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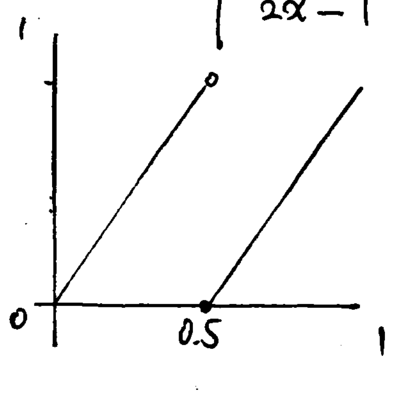
11. Solution: The doubling function D is defined by

$$D(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \leq x < 1 \end{cases}$$



Rui Qiu
#999292509

a. $x_0 = 0.3$

$$D(x_0) = 0.6$$
$$D^2(x_0) = 0.2$$
$$D^3(x_0) = 0.4$$
$$D^4(x_0) = 0.8$$
$$D^5(x_0) = 0.6$$

So. the orbit of 0.3 is eventually periodic with ~~period 4 and~~ period 4.

~~namely, $0.3 \in \mathrm{per}_4^+ D$~~

c. $x_0 = \frac{1}{8}$

$$D(x_0) = \frac{1}{4}$$
$$D^2(x_0) = \frac{1}{2}$$
$$D^3(x_0) = 0$$
$$D^4(x_0) = 0$$
$$\cdots$$

The orbit of $\frac{1}{8}$ is eventually fixed. ~~with a~~

~~So $\frac{1}{8} \in \mathrm{per}_1^3 D \subseteq \mathrm{fix} D$~~

d. $x_0 = \frac{1}{16}$

$$D(x_0) = \frac{1}{8}$$
$$D^2(x_0) = \frac{1}{4}$$
$$D^3(x_0) = \frac{1}{2}$$
$$D^4(x_0) = 0$$
$$D^5(x_0) = 0$$
$$\cdots$$

~~So $\frac{1}{16} \in \mathrm{per}_1^4 D \subseteq \mathrm{fix} D$~~

The orbit of $\frac{1}{16}$ is eventually fixed.

12. Solution:

For $D^2(x)$, we divide $[0,1)$ into 4 subintervals,

(~~why 4 ? b/c~~

$$0 \le x < \tfrac{1}{4} \Rightarrow 0 \le D(x) < \tfrac{1}{2} \Rightarrow D^2(x) = D(D(x)) = D(2x) = 4x$$

$$\tfrac{1}{4} \le x < \tfrac{1}{2} \Rightarrow \tfrac{1}{2} \le D(x) < 1 \Rightarrow D^2(x) = D(D(x)) = 2(2x) - 1 = 4x - 1$$

$$\tfrac{1}{2} \le x < \tfrac{3}{4} \Rightarrow 0 \le D(x) < \tfrac{1}{2} \Rightarrow D^2(x) = D(D(x)) = D(2x-1) = 4x - 2$$

$$\tfrac{3}{4} \le x < 1 \Rightarrow \cancel{0 \le}\ \tfrac{1}{2} \le D(x) < 1 \Rightarrow D^2(x) = D(D(x)) = D(2x-1) = 2(2x-1) - 1 = 4x - 3$$

$$D^2(x) = \begin{cases} 4x, & 0 \le x < \tfrac{1}{4} \\ 4x - 1, & \tfrac{1}{4} \le x < \tfrac{1}{2} \\ 4x - 2, & \tfrac{1}{2} \le x < \tfrac{3}{4} \\ 4x - 3, & \tfrac{3}{4} \le x < 1 \end{cases}$$

✓

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Similarly for $D^3(x)$, we need to divide $[0,1)$ into 8 subintervals.

$0 \le x < \frac{1}{8} \Rightarrow 0 \le D(x) < \frac{1}{4} \Rightarrow 0 \le D^2(x) < \frac{1}{2} \Rightarrow D^3(x) = D(D^2(x))$
$= D(4x)$
$= 8x$

$\frac{1}{8} \le x < \frac{1}{4} \Rightarrow \frac{1}{4} \le D(x) < \frac{1}{2} \Rightarrow \frac{1}{2} \le D^2(x) < 1 \Rightarrow D^3(x) = D(D^2(x))$
$= 2(4x) - 1$
$= 8x - 1$

$\frac{1}{4} \le x < \frac{3}{8} \Rightarrow \frac{1}{2} \le D(x) < \frac{3}{4} \Rightarrow 0 \le D^2(x) < \frac{1}{2} \Rightarrow D^3(x) = D(D^2(x))$
$= 2(4x-1)$
$= 8x - 2$

$\frac{3}{8} \le x < \frac{1}{2} \Rightarrow \frac{3}{4} \le D(x) < 1 \Rightarrow \frac{1}{2} \le D^2(x) < 1 \Rightarrow D^3(x) = D(D^2(x))$
$= 2(4x-1) - 1$
$= 8x - 3$

$\frac{1}{2} \le x < \frac{5}{8} \Rightarrow 0 \le D(x) < \frac{1}{4} \Rightarrow 0 \le D^2(x) < \frac{1}{2} \Rightarrow D^3(x) = 2(2(2x-1))$
$= 8x - 4$

$\frac{5}{8} \le x < \frac{3}{4} \Rightarrow \frac{1}{4} \le D(x) < \frac{1}{2} \Rightarrow \frac{1}{2} \le D^2(x) < 1 \Rightarrow D^3(x) = 2(2(2x-1)) - 1$
$= 8x - 5$

$\frac{3}{4} \le x < \frac{7}{8} \Rightarrow \frac{1}{2} \le D(x) < \frac{3}{4} \Rightarrow 0 \le D^2(x) < \frac{1}{2} \Rightarrow D^3(x) = 8x - 6$

$\frac{7}{8} \le x < 1 \Rightarrow \frac{3}{4} \le D(x) < 1 \Rightarrow \frac{1}{2} \le D^2(x) < 1 \Rightarrow D^3(x) = 8x - 7.$

In a nut shell. $D^3(x) = \begin{cases} 8x, & \text{if } 0 \le x < \frac{1}{8} \\ 8x-1, & \text{if } \frac{1}{8} \le x < \frac{1}{4} \\ 8x-2, & \text{if } \frac{1}{4} \le x < \frac{3}{8} \\ 8x-3, & \text{if } \frac{3}{8} \le x < \frac{1}{2} \\ 8x-4, & \text{if } \frac{1}{2} \le x < \frac{5}{8} \\ 8x-5, & \text{if } \frac{5}{8} \le x < \frac{3}{4} \\ 8x-6, & \text{if } \frac{3}{4} \le x < \frac{7}{8} \\ 8x-7, & \text{if } \frac{7}{8} \le x < 1 \end{cases}$

[redacted]

Inductively, we can write down a general formula for $D^n(x)$.

$$D^n(x) = \begin{cases} 2^n x, & \text{if } 0 \le x < \frac{1}{2^n} \\ 2^n x - 1, & \text{if } \frac{1}{2^n} \le x < \frac{2}{2^n} \\ 2^n x - 2, & \text{if } \frac{2}{2^n} \le x < \frac{3}{2^n} \\ \cdots \\ 2^n x - (2^n x - 1), & \text{if } \frac{2^n - 1}{2^n} \le x < 1 \end{cases}$$

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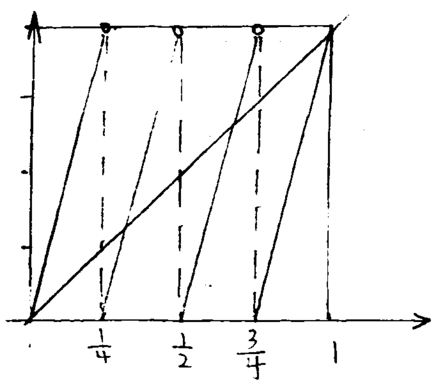
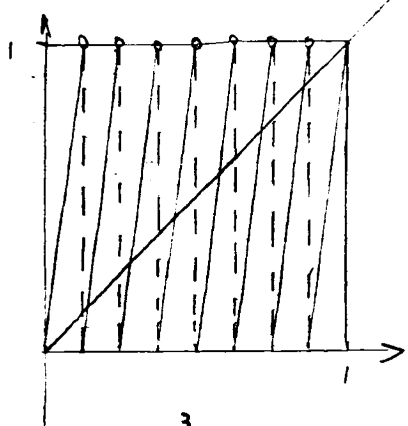
13. Solution:



$D^2(x)$



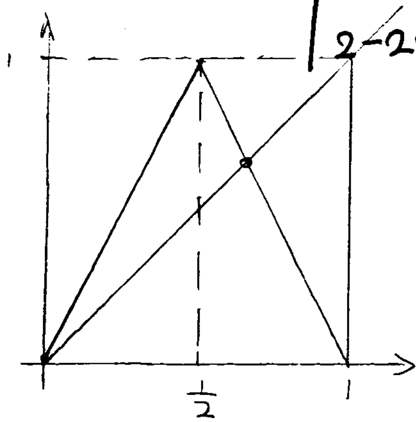
$D^3(x)$

$D^n(x)$ has 2^n pieces of lines with slope 2^n and it is discontinuous at $2^n - 1$ points. And the set of these discontinuities is $\{p \mid p = \frac{k}{2^n},\ k \in \mathbb{Z}^+,\ k \in [1, 2^n - 1]\}$

[redacted]

Rui Qiu
#999292509

15. Solution: $T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$



Set $[0,1]$ into 4 subintervals.

$0 \le x < \frac{1}{4} \Rightarrow 0 \le T(x) \le \frac{1}{2} \Rightarrow T^2(x) = T(T(x))$
$= 4x$

$\frac{1}{4} \le x < \frac{1}{2} \Rightarrow \frac{1}{2} \le T(x) \le 1 \Rightarrow T^2(x) = T(T(x)) = 2-2(2x)$
$= 2-4x$

$\frac{1}{2} \le x < \frac{3}{4} \Rightarrow \frac{1}{2} \le T(x) \le 1 \Rightarrow T^2(x) = T(T(x)) = T(2-2x)$
$= 2-2(2-2x)$
$= 4x-2$

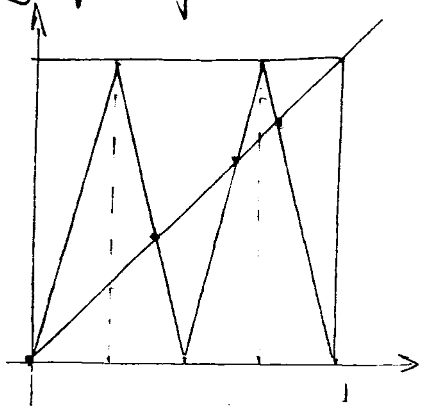
$\frac{3}{4} \le x < 1 \Rightarrow 0 \le T(x) \le 1 \Rightarrow T^2(x) = T(T(x)) = T(2-2x)$
$= 2(2-2x)$
$= 4-4x$

Thus $T^2(x) = \begin{cases} 4x & \text{if } 0 \le x < \frac{1}{4} \\ 2-4x & \text{if } \frac{1}{4} \le x < \frac{1}{2} \\ 4x-2 & \text{if } \frac{1}{2} \le x < \frac{3}{4} \\ 4-4x & \text{if } \frac{3}{4} \le x < 1 \end{cases}$

16. Solution:

The graph of T has been drawn on last page.

The graph of T^2 is:



17. Solution:

For $T(x)$.

Set $2x = x \Rightarrow x = 0$

$2 - 2x = x \Rightarrow x = \frac{2}{3}$

Check $T(x) = T(0) = 0$

$T(x) = T(\frac{2}{3}) = 2 - 2(\frac{2}{3}) = \frac{2}{3}$

For $T^2(x)$

Set $4x = x \Rightarrow x = 0$

$2 - 4x = x \Rightarrow x = \frac{2}{5}$

$4x - 2 = x \Rightarrow x = \frac{2}{3}$

$4 - 4x = x \Rightarrow x = \frac{4}{5}$

Check $T(0) = 0$

$T(\frac{2}{5}) = 2 - 4 \times \frac{2}{5} = \frac{2}{5}$

$T(\frac{2}{3}) = \frac{2}{3}$

$T(\frac{4}{5}) = 4 - 4 \times \frac{4}{5} = \frac{4}{5}$

Hence the fixed points for T are 0 and $\frac{2}{3}$,
fixed points for T^2 are 0, $\frac{2}{5}$, $\frac{2}{3}$ and $\frac{4}{5}$