224 Review

Dimension Theorem

$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$

$[v]_\alpha$ means V vector space over field F with basis $\alpha = \{v_1, \dots, v_n\}$

$v \in V$, the coordinates of v w.r.t. α is

$$[v]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n \quad \text{where } v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

$T: V \to W$ lin. trans

V has basis $\alpha = \{v_1, \dots, v_n\}$

W has basis $\beta = \{w_1, \dots, w_m\}$

The matrix associated to T is $[T]_\alpha^\beta$ or $[T]_{\alpha\beta}$

$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha = [w]_\beta$

V^* "dual"

$T: V \to W$ bijective, then T is invertible

$\exists$ a matrix B s.t. $AB = BA = I$

T^{-1} is unique.

isomophism : $T: V \to W$ if T is invertible

V, W isomorphic if $\exists$ isomorphism $T: V \to W$

$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$

Def: A, B $n \times n$ matrices

A is similar to B if there is an invertible X s.t.

$XAX^{-1} = B$

Def: $T(v) = \lambda v$ & $\vec{v} \neq \vec{0}$ then say λ is an eigenvalue & $\vec{v}$ is an eigenvector.

The eigenspace of λ is $E_\lambda = \{v \in V \mid Tv = \lambda v\}$

$E\lambda = Ker(\lambda I - T)$

$v \in Ker(\lambda I - T)$

$\Leftrightarrow (\lambda I - T)v = 0$

$\Leftrightarrow \lambda I \cancel{v} - Tv = 0$

$\Leftrightarrow Tv = \lambda I$

$V = U \oplus W \Leftrightarrow U + W = V$

$\& \ U \cap W = \{0\}$

Def: $T: V \to V$ is diagonalizable if there is a basis of V which consists of eigenvectors of T.

Prop: $T: V \to V$ is diagonalizable $\overset{?}{\Leftrightarrow}$ for any basis α of V $[T]_\alpha$ is diagonalizable.

Thm 1: $T: V \to V$ $\{v_1, \cdots, v_k\}$ eigenvectors

$T(v_i) = \lambda_i v_i$

Suppose $\lambda_i \neq \lambda_j$ for all $i \neq j$, then $\{v_1, \ldots, v_k\}$ are lin. indpt.

Def: λ is an eigenvalue. The multiplicity of λ is its multiplicity as a root of the characteristic polynomial of T.

Prop: $\dim E_\lambda \leq$ multiplicity of λ.

Def: A (hermitian) inner product on V is a map:
$V \times V \to F$, denoted $\langle v, w \rangle$.

① $\langle a v_1 + b v_2, w \rangle = a_1 \langle v_1, w \rangle + b \langle v_2, w \rangle$

② $\langle v, w \rangle = \overline{\langle w, v \rangle}$ complex conjugate

③ $\langle v, v \rangle \geq 0$ with $\langle v, v \rangle = 0 \Leftrightarrow v = 0$

$V = (a_1, a_2, \cdots, a_n)$

$W = (b_1, b_2, \cdots, b_n)$

$\langle v, w \rangle = a_1 \cdot \overline{b_1} + a_2 \overline{b_2} + \cdots + a_n \cdot \overline{b_n}$.

General properties:

$v \in V$ then $w \in V$ is orthogonal to v. if $\langle v, w \rangle = 0$.

if $W \subseteq V$ is a subspace, $W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}$ orthogonal complement to W.

$V = W \oplus W^{\perp}$

GRam-Schmidt orthogonalization

$\{u_1, \cdots, u_k\}$ lin. ind.

want $\{v_1, \cdots, v_k\}$ s.t. ① $\{v_1, \cdots, v_k\}$ is orthogonal

② $\text{span}\{v_1, \cdots, v_k\} = \text{span}\{u_1, \cdots, u_k\}$

$v_1 = u_1$

$v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

$\cdots$

$$v_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

Then $\left\{ \dfrac{v_1}{\|v_1\|}, \dfrac{v_2}{\|v_2\|}, \cdots, \dfrac{v_k}{\|v_k\|} \right\}$ is such an ORTHO~~GONAL~~ NORMAL? basis

Thm: $U \oplus W = V \Rightarrow \dim U + \dim W = \dim V$

~~P_W~~

$P_W: V \to V$ "projection onto W" is given by $P_W(v)=w$

$v = w + w'$

$w \in W$

$w' \in W^{\perp}$

$V = W \oplus W^{\perp}$

$v \in V$

$$w = P_W(v) = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots + \frac{\langle v, v_k \rangle}{\langle v_k, v_k \rangle} v_k$$

$\{v_1, \dots, v_k\}$ is orth basis of W

$A \in M_n(\mathbb{R})$, A symmetric

$\Leftrightarrow \langle Av, w \rangle = \langle v, Aw \rangle$

why?

$$\langle Av, w \rangle = (Av)^T w = v^T A^T w = \langle v, A^T w \rangle = \langle v, Aw \rangle$$

Def: V inner product space

$T: V \to V$ is symmetric or self-adjoint

if $\forall\, v, w \in V$

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

$A \in M_n(\mathbb{C})$ self adjoint? it is also called "Hermitian"

means : $\langle Av, w \rangle = \langle v, Aw \rangle$

$\Leftrightarrow (Av^*) w = v^* A w$

$\Leftrightarrow v^* A^* w = v^* A w$

$\Leftrightarrow A^* = A$

NOTE: $A^* = \overline{A}^T$

" A Dual" means the transpose of A's complex conjugate.

BTW, P_W is ~~Herm~~ self-adjoint

Thm ①: V ips $T: V \to V$ self-adjoint
$\Rightarrow$ eigenvalues of T are real.

Thm ②: V ips $T: V \to V$ self-adjoint
$\Rightarrow$ If x_1 an eigenvector of T with λ_1
x_2 λ_2

and $\lambda_1 \neq \lambda_2 \Rightarrow x_1, x_2$ orth.

T diagonalizable $\Leftrightarrow$ there is a basis of eigenvectors of T.

$\boxed{\text{SPECTRAL THM}}$:
V ips, $T: V \to V$ self-adjoint operator ($\langle T(v), w\rangle = \langle v, T(w)\rangle$)
Then there is an orthonormal basis of V consisting of eigenvectors of T. In particular, T is diagonalizable.

Claim: $\{x_1, \dots, x_n\}$ is an orthonormal basis of evectors.
- all x_i are evectors
- all unit length
- set is orthogonal since $\{x_2, \dots, x_n\}$ is orth and $x_1 \in W$ while $x_2, \dots, x_n \in W^\perp$.

Let $\lambda_1, \dots, \lambda_k$ distinct eigenvalues. & p = orth. proj. onto E_{λ_i}
Then $= \lambda_1 p_1 + \lambda_2 p_2 + \dots + \lambda_k p_k$

ex: see lecture note #15

Q: How much can spectral thm be generalized?

Def: $T: V \to V$ normal if $TT^* = T^*T$, if T self-adjoint T is normal since $T = T^*$.

Lemma ① $(T^*)^* = T$
② T normal, $\operatorname{Ker} T = \operatorname{Ker} T^*$
③ T normal, $\lambda \in \mathbb{C}$ then $T - \lambda I$ also normal
④ T normal, $Tv = \lambda v$ then $T^* v = \bar{\lambda} v$

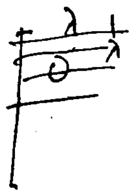
Main property: If $\alpha = \{v_1, \cdots, v_n\}$ is an orthonormal basis of V then

$$[T^*]_\alpha = [T]_\alpha^* \longleftarrow$$ means conjugate transpose

Check orthonormal

Spectral theorem for normal operators
T normal
Then V has an orthonormal basis of eigenvalues of T. In particular, T diagonalizable.

JORDAN CANONICAL FORM



$$\begin{bmatrix} \begin{matrix}\lambda & 1 & * \\ & \lambda & 1 \\ 0 & & \lambda\end{matrix} & 0 \\ 0 & \begin{matrix}\mu & 1 & * \\ & \mu & 1 \\ 0 & & \mu\end{matrix}\end{bmatrix}$$

③ major steps
① Triangularizability
② theorem for "nilpotent" matrices
③ ——— general matrices

Upper triangular matrix is a matrix of the form

$$\begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Def: $T: V \to V$ is triangularizable if there exists a basis α such that $[T]_\alpha$ is upper-triangular.

Def: $T: V \to V$ $W \subseteq V$ then W is invariant under T if $T(w) \subset W$

Prop: $T: V \to V$ T is triangularizable <=> then $\exists$ a sequence of spaces $W_1 \subset W_2 \subset \cdots \subset W_n$ s.t. each W_i is T-invariant and $\dim W_i = i$.

Thm: Every operator is triangularizable.

Nilpotent Matrices

Def: $T: V \to V$ is nilpotent if $T^k = 0$ for some k or equivalently if the only eigenvalue of T is 0.

Def: $\{N^{j-1}(v), N^{j-2}(v), \cdots, v\}$ is called the cycle associated to N and V_j is the length of the cycle & $C(v) = \text{span}\{N^{j-1}(v), N^{j-2}(v), \cdots, v\}$ is called the ~~cyclic of~~ cyclic-subspace associated to N & V.

Properties of $C(v)$

1. $\dim C(v) =$ length of the cycle.
2. $C(v)$ is invariant under N
3. $\alpha = \{N^{j-1}(v), \cdots, v\}$ is a basis of $C(v)$.

$N|_{C(v)}: C(v) \to C(v)$

$$[N|_{C(v)}]_\alpha = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

4. $N^{j-1}(v)$ is an ~~eigenvalue~~ eigenvector of N with eigenvalue 0.

Prop: Let $\{v_1, \dots, v_r\}$ be some vectors in V.
$N: V \to V$ nilpotent operator.
Let $\alpha_1 = \{N^{j_1-1}(v_1), \dots, v_1\}$ be the cycle of v_1

$\vdots$

$\alpha_r = \{N^{j_r-1}(v_r), \dots, v_r\}$ be the cycle of v_r

If $\{N^{j_1-1}(v_1), \dots, N^{j_r-1}(v_r)\}$ are lin. in. then
$\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_r$ is lin ind.

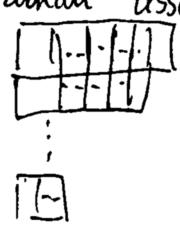
Thm: If $N: V \to V$ has tableau then with respect to any canonical basis α of V for N.

$$[N]_\alpha = J_{k_1} \oplus \cdots \oplus J_{k_r}$$

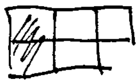
This is called the JCF.

<u>Big Thm</u>: $N: V \to V$ nilpotent, then $\exists$ a canonical basis of V for N.

Given $N: V \to V$ nilpotent
Let $\alpha_1, \dots, \alpha_r$ be non-overlapping cycles s.t.
$\alpha = \alpha_1 \cup \cdots \cup \alpha_r$ is a basis of V, i.e.
α is a canonical basis for N. Let length of
$\alpha_i = k_i$, arranged so that $k_1 \geq k_2 \geq \cdots \geq k_r$
The tableau associated to N is



k_1
k_2
$\vdots$
k_r



dim $N = 2$
dim $N^2 = 4$
dim $N^3 = 5$

JCF of N $\left[\begin{array}{ccc|cc} 0&1&0&& \\ 0&0&1&&0 \\ 0&0&0&& \\ \hline &0&&0&1 \\ &&&0&0 \end{array}\right]$

JCF for nilpotent operator
$T: V \to V$ with one λ.
Given such T, the char. poly of T, $p(x) = (x-\lambda)^n$, $n = \dim V$
Cayley–Hamilton $\Rightarrow p(T) = (T-\lambda I)^n = 0$.
i.e. $N = T - \lambda I$ is nilpotent.
So $\exists$ a canonical basis α, and $k_1 \geq \cdots \geq k_r$ s.t. $[T-\lambda I]_\alpha = J_{k_1} \oplus \cdots \oplus J_{k_r}$

Def'n: The $n \times n$ Jordan matrix with eigenvalue λ is

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

Starting from Lec 24.

Thm: $T: V \to V$ $\exists$ a "canonical basis" γ of V s.t.
$[T]_\gamma = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_k)$
Moreover, the $n_1, \ldots, n_s$ are unique (up to ordering)

To find the $\overset{\text{orthogonal}}{\llcorner}$ projectio matrix.

Let $\{w_1, \ldots, w_k\}$ be an ~~orthog~~ orthonormal basis for the subspace $W \subseteq \mathbb{R}^n$.

a). for $w \in W$, we have $w = \langle w, w_1 \rangle w_1 + \cdots + \langle w, w_k \rangle w_k$

b). for $P_W(x) = \langle x, w_1 \rangle w_1 + \cdots + \langle x, w_k \rangle w_k$

eg. $W = \text{span}\{(1,1,0)\}$.
orthonormal basis has one vector $\{\frac{1}{\sqrt{2}}(1,1,0)\} = w$

$$P_W(x) = \langle x, w \rangle w$$
$$= \frac{1}{\sqrt{2}}(x_1 + x_2)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$
$$= \left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(x_1+x_2), 0\right)$$

Spectral Decompo: eg. find sd of $A = \begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 2 \end{bmatrix}$

① find $\det(A - I\lambda) = \cdots$ find λ

② ~~AS~~ find V_1, V_2, V_3, with $\|V_1\|, \|V_2\|, \|V_3\|$ ...

③ orthonormal basis is $\left\{\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \ldots\right\}$

$P = \begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix}$ $P^{-1} = \begin{pmatrix} \cdots \end{pmatrix}$

s.t. $D = PAP^{-1} = \begin{pmatrix} & \end{pmatrix}$