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$10+10=20$

MAT224 Problem Set 3

#1

(a). Solution: Let $p(x)=a+bx+cx^2 \in P_2(\mathbb{R})$ for $a,b,c\in\mathbb{R}$.

Then $S(p(x)) = x\,p(x) = ax+bx^2+cx^3$

So $T(S(p(x))) = TS(p(x)) = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$

(b). Solution Note that $TS(1) = T(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = (-\frac{1}{2})\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-\frac{1}{2})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$TS(1+x) = T(x+x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$TS(1+x+x^2) = T(x+x^2+x^3) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = (-\frac{1}{2})\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + (-\frac{1}{2})\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-\frac{1}{2})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Hence $[TS]_{\beta\alpha} = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$

(c). Solution: We want $\mathrm{Ker}(TS)$, so we need to do the row reduction to $[TS]_{\beta\alpha}$

$$\begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\mathrm{Ker}(TS) = \{0\}$. $\emptyset$ is a basis for $\mathrm{Ker}(TS)$.

(d). Solution: We want to find a basis of $Im(TS)$

- Since

$$col[TS]_{\beta\alpha} = span\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \right\}$$

Then $Im(TS) = span\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are linearly independent.

Therefore $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is such a basis for $Im(TS)$

#2.

Solution: Since $[I]^{\alpha}_{\alpha'} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix}$

Then $[I]^{\alpha'}_{\alpha} = ([I]^{\alpha}_{\alpha'})^{-1} = \begin{bmatrix} 3 & -4 & 7 \\ 5 & -7 & 11 \\ 1 & -1 & 2 \end{bmatrix}$

So $V_2' = (4,0,7) = -4(-3,5,2) + (-7)(4,1,1) - (V_{31}, V_{32}, V_{33})$

$\Rightarrow V_{31} = -20, \; V_{32} = -27, \; V_{33} = -8$

$V_3 = (-20, -27, -8)$

$V_1' = 3(-3,5,2) + 5(4,1,1) + (-20,-27,-8) = (-9,-7,3)$

$V_3' = 7(-3,5,2) + 11(4,1,1) + 2(-20,-27,-8) = (-17,-8,9)$

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#3.

Solution: Since $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Then the row echelon form of A is:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

So we find that the last column is redundant

Therefore say $\alpha' = (1, x, x^2, x^3)$ and $\beta' = (1, x, x^2)$ for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively.

Therefore $\beta = (1\cdot 1 + 1\cdot x^2,\ 2\cdot 1 + 1\cdot x + 1\cdot x^2,\ 2\cdot x + 1\cdot x^2)$
$= (1+x^2,\ 2+x+x^2,\ 2x+x^2)$

Since $[T]_\beta^\alpha = [I]_\beta^{\beta'} [T]_{\alpha'}^{\beta'} [I]_{\alpha'}^{\alpha}$

For α we just need to take $\alpha = \alpha'$ such that

$$[I]_{\alpha'}^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence $\alpha = \{1, x, x^2, x^3\}$

#4.

Solution:

Since $T(x,y,z)=(x+2y,\ x+y,\ 2x+z)$

Then $T^{-1}(T(x,y,z))=T^{-1}(x+2y,\ x+y,\ 2x+z)=(x,y,z)$

As $-(x+2y)+2(x+y)=x$

$(x+2y)-(x+y)=y$

$2x+z-2(-(x+2y)+2(x+y))=z$

Therefore $T^{-1}(x,y,z)=(-x+2y,\ x-y,\ z-2(-x+2y))$

$=(-x+2y,\ x-y,\ 2x-4y+z)$

#5.

Solution: Since $W=\text{span}\{1+x^2+x^3,\ 1+x+x^2,\ 3+x+3x^2+2x^3,\ -x+x^3\}$

$3+x+3x^2+2x^3=2(1+x^2+x^3)+1+x+x^2$

$=3+x+3x^2+2x^3$

$1+x^2+x^3=(1+x+x^2)+(-x+x^3)$

(*)

So $\{1+x+x^2,\ -x+x^3\}$ is a basis for W.

Thus $\dim W = d = 2$.

We now need an isomorphism T such that $T: W \to \mathbb{R}^2$ is invertible. (Definition 2.6.4)

$\Leftarrow$ T has an inverse T^{-1} (Definition 2.6.3)

$\Leftarrow$ T is bijective (Proposition 2.6.2)

$\Leftarrow$ We only need T is injective/surjective, then T is also surjective/injective automatically since $\dim W=\dim \mathbb{R}^2$ (Proposition 2.4.10)

Suppose $T: W\to\mathbb{R}^2$ by $T(w)=T(a(1+x+x^2)+b(-x+x^3))=(a,b)$ for $w\in W,\ a,b\in\mathbb{R}$.

Note that only $T(0)=(0,0)=0$ since $\{1+x+x^2,\ -x+x^3\}$ is a basis which is linearly independent.

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Thus $Ker(T) = \{0\}$
So $dim(Ker(T)) = 0$
Then T is injective.
By $(*)$, hence T is an isomorphism.

#6.

(a). Solution:

According to the previous reasoning in $(*)$ of Problem #5, we know that $\underbrace{T}_{\text{a linear transformation}}$ is an isomorphism iff T is bijective. (skip the proof of "invertible")

Since $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$

So we need only to prove T injective then it is bijective. $(\dim Ker T + \dim Im T = \dim P_n(\mathbb{R}))$

Suppose $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for all $a_i \in \mathbb{R}$

Then
$$T(p(x)) = p(x) + p'(x)$$
$$= a_0 + a_1x + \cdots + a_nx^n + a_1 + 2a_2x + \cdots + na_nx^{n-1}$$
$$= (a_0 + a_1) + (a_1 + 2a_2)x + \cdots + ((n-1)a_{n-1} + na_n)x^{n-1} + a_nx^n$$

Say $T(p(x)) = 0$

Then $a_0 = -a_1$, $a_1 = -2a_2$, $\cdots a_{n-1} = -\frac{n}{n-1}a_n$, $a_n = 0$

Hence $a_i = 0$ for all i.

Thus $T(p(x))$ only when $p(x) = 0$.

Then $Ker(T) = \{0\} \Rightarrow \dim(Ker T) = 0$

Therefore T is injective $\Rightarrow$ bijective (Why? because $\dim(Im T) = \dim(P_n(\mathbb{R})) - \dim(Ker T) = \dim(P_n(\mathbb{R}))$)

Thus T is isomorphism.

(b). Solution:

Similarly as part (a), also suppose $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for all $a_i \in \mathbb{R}$.

Then $T(p(x)) = x(a_1 + 2a_2x + \cdots + na_nx^{n-1}) = a_1x + 2a_2x^2 + \cdots + na_nx^n$

Say $T(p(x)) = 0$, we only need $a_1 = \cdots = a_n = 0$, there is no restriction about a_0, which means a_0 can be any real numbers such that $p(x)=0$.

So $\mathrm{Ker}(T) \neq \{0\}$

Then $\dim(\mathrm{Ker}\, T) \neq 0 \Rightarrow$ not injective

Similarly, T is not surjective (then not bijective)

Hence T is not invertible, T is not an isomorphism as a result. ■

(c).

Solution:

Suppose also $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for all $a_i \in \mathbb{R}$

Let $T(p(x)) = 0$.

Then $c(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) - (a_1 x + 2a_2 x^2 + \cdots + n a_n x^n)$

$= c a_0 + (c-1)a_1 x + (c-2)a_2 x^2 + \cdots + (c-n)a_n x^n$

$= 0$

According to part (b), if we want T injective

We need $\mathrm{Ker}(T) = \{0\}$, this means only when

$a_0 = a_1 = \cdots = a_n$ that $p(x) = 0$.

Thus we should guarantee that all the

coefficients $(c-i)$, for $0 \leq i \leq n$, $i \in \mathbb{Z}$, that are non-trivial

Therefore c cannot be any i above.

Hence for any real number rather than the integers

in $[0, n]$, c can make T an isomorphism. ■

#7.

Proof: $(\Rightarrow)$ Suppose T is an isomorphism, then T is an invertible transformatio[n]

Let $\underbrace{T^{-1}}_{W \to V}$ denote the inverse of T

Then $T^{-1} \cdot T = I_V$

$T \cdot T^{-1} = I_W$

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So $[I_V]_\alpha^\alpha = [T^{-1}\cdot T]_\alpha^\alpha = [T^{-1}]_\beta^\alpha [T]_\alpha^\beta$

$[I_W]_\beta^\beta = [T\cdot T^{-1}]_\beta^\beta = [T]_\alpha^\beta [T^{-1}]_\beta^\alpha$

Therefore $[T]_\alpha^\beta$ is an invertible matrix.

*(Note that $[T]_{\beta\alpha}$ is $[T]_\alpha^\beta$)

($\Leftarrow$) Suppose $[T]_\alpha^\beta$ is an invertible matrix

Injective: $[T(v)]_\beta = [T]_\alpha^\beta \cdot [v]_\alpha = 0$

$\Rightarrow [v]_\alpha = 0$

$\Rightarrow v = 0$

$\Rightarrow Ker(T) = \{0\}$

$\Rightarrow T$ is injective

Surjective: By rank-nullity theorem:

$\dim(\text{im}\, T) = \dim V - \dim(Ker(T)) = n - 0 = n = \dim V$

$\Rightarrow T$ is surjective

$\Rightarrow T$ is bijective

$\Rightarrow T$ is an isomorphism.