§8 - Finite Products

1 Motivation

In our ongoing efforts to make new topological spaces, we (briefly) revisit the product topology. We have already seen finite topological products, and we have seen how they interact with various topological invariants (we know that the product of two Hausdorff spaces is again Hausdorff from Assignment 3, C.3). We will do a more systematic treatment of properties that are preserved under finite products, we will also mention how continuous functions interact with products. This will be useful in its own right, but it will also foreshadow our study of infinite products later on in the course. So, while reading this, keep in mind which results might generalize to an arbitrary product of topological spaces.

2 Remembering What We Know

Let's dredge up our knowledge of products that we already know:

- For two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, the product space on $X \times Y$ is given by the basis $\mathcal{T} \times \mathcal{U}$.
- If $\mathcal{B}_1$ is a basis for X_1, and $\mathcal{B}_2$ is a basis for X_2, then $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis for $X_1 \times X_2$.
- If $(X_i, \mathcal{T}_i)$, for $1 \le i \le n$ are topological spaces, then the product space on $\prod_{1\le i\le n} X_i$ is given by the basis $\prod_{1\le i\le n} \mathcal{T}_i$.
- We often write $X^2 := X \times X$, and $X^3 := X \times X \times X$, etc..

Here are the facts you have shown on assignment 3:

C.2 : If $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces with $\overline{D} = X$ and $\overline{E} = Y$, then $D \times E$ is a dense subset of $X \times Y$.

C.2 : The (finite) product of separable spaces is separable.

C.3 : The (finite) product of Hausdorff spaces is a Hausdorff space.

3 Properties that are Preserved under Finite Products

Let us expand upon your assignment questions, and look at properties that are preserved under finite products:

Definition. *A property ϕ is said to be* ***finitely productive*** *if whenever $(X_i, \mathcal{T}_i)$ (for $1 \le i \le n$) are each topological space with property ϕ then the product space $\prod_{1\le i\le n} X_i$ has property ϕ.*

Note that by a simple inductive argument, to check that a property is finitely productive, it is enough to check that the property is preserved under the product of any *two* spaces. For example, on Assignment 3, C.3 you showed that the product of any two Hausdorff spaces was Hausdorff, and this is enough (by induction) to show that the Hausdorff property is finitely productive. The proof that each of the following properties is finitely productive is, again, just a matter of unwinding the definitions and doing the obvious thing.

Some examples of finitely productive properties:

- X is a Hausdorff space;
- X is a T_3 space (See the §9 notes for a hint of the proof);
- X has an open point;
- X is finite;
- X is countable;
- X is separable;
- X is first countable;
- X is second countable;

There is only one glaring omission from that list (of the properties we have studied so far): the **countable chain condition**.

It is a rather remarkable fact the assertion that "the countable chain condition is finitely productive" cannot be verified *or* disproved with the usual axioms of set theory. It is a statement whose truth value is independant of the usual axioms of mathematics. (For those of you who are interested, it is a theorem of Rich Laver from the 1970s that if the Continuum Hypothesis is true, then you can construct two topological spaces that are ccc, but whose product is not ccc. You can read more about it in Galvin's article "Chain Conditions and Products".)

Fib Exercise: I told a tiny fib in the previous paragraph. I said that the only non-finitely-productive property we know so far is the ccc. Well on assignment 4 I mentioned the T_4 property. Try to show that the product of the Sorgenfrey line with itself does not have this property.

4 Continuous Functions and Products

Recall that there are natural functions that show up whenever we have a product:

Definition. *For topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, define the projection maps $\pi_1 : X \times Y \longrightarrow X$ by*

$$\pi_1((x, y)) := x$$

and $\pi_2 : X \times Y \longrightarrow Y$ by

$$\pi_2((x, y)) := y.$$

The following facts about projection maps will also be true when we look at infinite products, but we prove them in the special case of the product of two spaces (to avoid abstract notation). For these cases drawing pictures is very helpful! All of these facts use the following identity:

Lemma. *For $A \subseteq X$ and $B \subseteq Y$, on the product $X \times Y$ we have the identities*

$$\pi_1^{-1}(A) = A \times Y$$

and

$$\pi_2^{-1}(B) = X \times B.$$

Proposition. *The projection maps π_1 and π_2 are continuous on the product space $X \times Y$. Moreover, the product topology is the smallest topology where the projection maps are continuous.*

Proof. The proof that they are continuous is straightforward. Let us prove the second fact. Let $\mathcal{V}$ be a topology on the product space where both projection maps are continuous; we want to show that $\mathcal{T} \times \mathcal{U} \subseteq \mathcal{V}$, where $\mathcal{T}$ and $\mathcal{U}$ are the topologies of X and Y (respectively). We will use proposition 9 from §2 of the notes.

Let $A \times B \in \mathcal{T} \times \mathcal{U}$ be a **basic open set**, where A is open in X and B is open in Y. And let $(a, b) \in A \times B$. We need to show that there is an open set $B_2 \in \mathcal{V}$ such that $(a, b) \in B_2 \subseteq A \times B$.

Note that $\pi_1^{-1}(A) \in \mathcal{V}$ and $\pi_2^{-1}(B) \in \mathcal{V}$ since the projection maps are continuous (with $X \times Y$ given the topology $\mathcal{V}$). Finally notice that $(a, b) \in A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \mathcal{V}$. $\square$

Sub Exercise: Rewrite the previous proof using the language of subbases.

The previous proposition tells us that we could take as our definition for the product topology "the smallest topology where each projection map is continuous". This will be useful when we examine infinite products.

We need the next lemma for one thing we will do right away, and one thing we will do much later.

Proposition. *The set* $\mathcal{S} := \{\, \pi_1^{-1}(U) : U$ *is open in* $X \,\} \cup \{\, \pi_2^{-1}(V) : V$ *is open in* $Y \,\}$ *is a subbasis for the product topology on* $X \times Y$*. Moreover,* $\mathcal{S} = \{\, U \times Y : U$ *is open in* $X \,\} \cup \{\, X \times V : V$ *is open in* $Y \,\}$*.*

Proof. The picture says everything. It is enough to notice that any *basic* open sets $A \times B$ in $X \times Y$ can be written as

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) = (A \times Y) \cap (X \times B)$$

So we know that any basic open set can be written as the intersection of two elements from the subbasis.

Strictly speaking, we also need to observe that each element of $\mathcal{S}$ is open in the product, but this is because if U is open in X, then $\pi_1^{-1}(U) = U \times Y$ which is open in the product. Similarly, if V is open in Y then $\pi_2^{-1}(V) = X \times V$, which is open in the product. $\square$

The next proposition is a useful way for checking continuity of a function that is being mapped into a product. For example, is the function $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$f(x) := (x^2, \arctan(2x), x - 7)$$

a continuous function?

Proposition. *Let* $(\, X, \mathcal{T}\,), (\, Y_1, \mathcal{U}_1\,)$ *and* $(\, Y_2, \mathcal{U}_2\,)$ *be topological spaces and let* $f : X \longrightarrow Y_1 \times Y_2$ *be a function. TFAE:*

- f *is continuous;*
- $\pi_1 \circ f$ *and* $\pi_2 \circ f$ *are each continuous.*

Proof. The [$\Rightarrow$] direction is clear, since we know that the projection maps are continuous, and the composition of continuous maps is again continuous.

[$\Leftarrow$] Suppose that $\pi_1 \circ f$ and $\pi_2 \circ f$ are each continuous. By a previous (prescient) theorem about continuity, we only need to check that the preimage of *subbasic* open sets in the product are open. Let $U \times Y_2$ be a subbasic open set in the product (the $Y_1 \times V$ case is analogous). Then

$$f^{-1}(U \times Y_2) = f^{-1}(\pi_1^{-1}(U)) = (\pi_1 \circ f)^{-1}(U)$$

which is an open set, since $\pi_1 \circ f$ is continuous. $\square$

Socks and Shoes Exercise: In class I mentioned something cryptic about the "Socks and Shoes principle" for dealing with compostions and inverses. Take a look online and see if you can figure out what I was talking about.

Now we know that the function we stated before this proposition is continuous, since each composition with the projection functions is continuous (by first-year calculus):

- $\pi_1 \circ f(x) = x^2$
- $\pi_2 \circ f(x) = \arctan(2x)$
- $\pi_3 \circ f(x) = x - 7$

5 Two Amazing Facts

Before we (temporarily) leave products, let us mention two remarkable facts which stand in sharp contrast to the fact from Assignment 4 (A.2) that $\mathbb{R} \not\cong \mathbb{R}^2$:

Theorem (Cantor)**.** *For $n \in \mathbb{N}$, $\mathbb{Q} \cong \mathbb{Q}^n$ (where they have the usual subspace topologies).*

Let $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$, the irrationals, with the subspace topology.

Theorem. *For $n \in \mathbb{N}$, $\mathbb{I} \cong \mathbb{I}^n$ (where they have the usual subspace topologies).*

We will not present the proofs, but you may find them online if you are interested.

What on Earth... Exercise: The previous two theorems may seem very surprising at first because in some sense we think of $\mathbb{Q}$ and $\mathbb{I}$ as being good approximations to $\mathbb{R}$. List out the properties the have in common, and find out which properties they don't share. You may wish to investigate the dimension of these spaces (as defined on Assignment 4 (NI.2)).

6 Other Facts

Here's a dump of facts that we probably won't need but are worth observing:

Proposition. *For topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, we have that $X \times Y \cong Y \times X$.*

Proposition. *For $A \subseteq X$ and $B \subseteq Y$, then $\overline{A \times B} = \overline{A} \times \overline{B}$.*

The next exercise is one that every student of topology should do at least once in their life:

Proposition. *X is Hausdorff if and only if $\Delta := \{ (x,x) \in X^2 : x \in X \}$ is closed in X^2.*

Some of you (at some point in the distant future) might wonder how taking subspaces interacts with taking products. Well fear not keen student, it doesn't matter what order you take subspaces and products:

Proposition (Theorem 16.3 in Munkres)**.** *If A is a subspace of X, and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. A basis element in $X \times Y$ is of the form $U \times V$, where U is open in X and V is open in Y. So

$$(U \times V) \cap (A \times B)$$

is a general basic open set in $A \times B$ (as a subspace). Notice that

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

with $U \cap A$ open in the subspace A, and $V \cap B$ open in the subspace B. Since every basic open set in the product topology $A \times B$ looks like

$$(U \cap A) \times (V \cap B)$$

we can conclude that the basis for $A \times B$ as a product and the basis for $A \times B$ as a subspace, are identical. So the topologies are identical. $\square$

7 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Fib : Show that the square of the Sorgenfrey line is not T_4.

Sub : Show that the product topology is the smallest topology where the projection maps are continuous, using the language of subbases.

Socks-Shoes : In class I mentioned something cryptic about the "Socks and Shoes principle" for dealing with compostions and inverses. Take a look online and see if you can figure out what I was talking about.

WoE... : Investigate the properties that $\mathbb{R}, \mathbb{Q}$ and $\mathbb{I}$ share, and which they don't. Look at their (respective) topological dimensions.

8 References

Galvin, Fred. "**Chain conditions and products.**" *Fundamenta Mathematicae* 108.1 (1980): 33-48. http://eudml.org/doc/211145