UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2009 EXAMINATIONS
MAT244H1F
Duration - 3 hours
Examiners: Dilip Raghavan, Hiro Oh (Circle one)
No Aids Allowed

PLEASE HAND IN

Name: ______________________________

Student ID #: ______________________

Instruction: You must justify your answer for full credit by showing all the necessary steps. If you do not have enough space, you may use the back of the page. Make sure to write that you are continuing your solution to the back of the page.

1.	/ 10
2.	/ 23
3.	/ 17
4.	/ 8
5.	/ 7
6.	/ 20
7.	/ 15
Total	/ 100

Total of 17 pages

1. (a)(7 pts.) Use the method of successive approximation to find $\phi_0(t), \phi_1(t), \cdots, \phi_n(t)$ for the following initial value problem (IVP).

$$y' = y + 1 - t, \quad y(0) = 0.$$

cont'd...

(b)(3 pts.) What do these approximations converge to?

cont'd...

2. (a)(5 pts.) Consider the following ordinary differential equation (ODE):

$$(t-1)y^{(4)} + (t+1)y^{(2)} + (\tan t)y = 1.$$

Transform the ODE into a system of first order linear ODEs of the form

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t).$$

Find the matrix $\mathbf{P}(t)$ and the vector $\mathbf{g}(t)$. Also, find the intervals where it is guaranteed to have a unique solution. Do **not** solve the equations.

cont'd...

(b)(9 pts.) Solve the following IVP:

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Describe the behaviour of the solution as $t \to \infty$.

cont'd...

(c)(9 pts.) Find the general solution to:

$$\mathbf{x}' = \begin{pmatrix} 7 & -3 & 0 \\ 10 & -4 & 0 \\ 12 & -6 & 1 \end{pmatrix} \mathbf{x}.$$

cont'd...

3. (a)(9 pts.) Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}.$$

cont'd...

(b)(8 pts.) Suppose that $\mathbf{\Psi}(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$ is the fundamental matrix of the following system:

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sec t \\ \csc t \end{pmatrix}.$$

such that $\mathbf{\Psi}(0) = \mathbf{I}$. Use the variation of parameters and find a particular solution.

cont'd...

4. Consider the following system of ODEs:

$$\begin{cases} x' = y - (x^2 + y^2)(x^3 + xy^2) \\ y' = -x - (x^2 + y^2)(x^2 y + y^3). \end{cases}$$

(a)(5 pts.) Let $r(t) = \sqrt{x^2(t) + y^2(t)}$. Use the expressions given for x' and y' to find an ODE for $r(t)$ of the form $r' = f(r)$.

(b)(3 pts.) Find the equilibrium points for $r(t)$ and classify each one as stable, unstable, or semistable.

cont'd...

5. (7 pts.) Consider the following IVP

$$\begin{cases} y^{(4)}(t) - 3y^{(3)}(t) + 4y''(t) - y(t) = e^{3t} + 1 + t^4 \\ y(0) = -2,\ y'(0) = 3,\ y''(0) = 0,\ y^{(3)}(0) = -1. \end{cases}$$

Take the Laplace transforms of the equation and solve for $Y(s) = \mathcal{L}\{y\}(s)$. Do **not** solve for $y(t)$. Also, do **not** simply the expression for $Y(s)$.

cont'd...

6. (a)(4 pts. each) Find the inverse Laplace transforms of the following functions. i.e. find $f(t)$ such that $\mathcal{L}\{f\}(s) = F(s)$.

(i)

$$F(s) = \frac{2s-5}{s^2-2s+5}.$$

(ii)

$$F(s) = \frac{e^{-3s}}{s^2-16}$$

cont'd...

(iii)

$$F(s) = \frac{s^3 + 4}{s^2(s^2 + 4)}.$$

cont'd...

(b)(4 pts. each) Find the Laplace transforms of the following functions. Do *not* simplify your solutions. i.e. you do not need to add fractions in your solutions.

(i) $f(t) = \begin{cases} 0, & t < 3 \\ (3t-9)^3, & t \geq 3 \end{cases}$

(ii) $f(t) = \begin{cases} 0, & t < 1 \\ t^2, & t \geq 1 \end{cases}$

cont'd...

7. In this problem, we will solve the following ODE:

$$t^2y'' + 2ty' - 2y = 0, \quad \text{for } t > 0.$$

(a)(4 pts.) Substitute $y = t^r$ in the ODE and derive an equation for r Then, solve the equation and determine the values of r.

cont'd...

(b)(4 pts.) Let r_1 and r_2 be the values of r found in part (a) and let $y_1(t) = t^{r_1}$ and $y_2(t) = t^{r_2}$. (i.e. $y_1(t)$ and $y_2(t)$ are solutions to the ODE above.)
Suppose that we know that y_1 and y_2 form a fundamental set of solutions to the homogeneous ODE. Solve the following IVP:

$$t^2y'' + 2ty' - 2y = 0, \ y(1) = 5, \ y'(1) = 2, \ \text{for } t > 0.$$

cont'd...

(c) Consider

$$t^2y'' + (1-2\alpha)ty' + \alpha^2 = 0. \tag{1}$$

(c.i)(4 pts.) Substitute $y = t^r$ in the ODE and derive an equation for r Then, solve the equation and determine the values of r.

(c.ii)(3 pts.) By solving the characteristic equation for

$$y'' - (r_1 + r_2)y' + r_1 r_2 y = 0, \quad r_1 \neq r_2, \tag{2}$$

one immediately finds two linearly independent solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.

Now, consider the limit $r_2 \to r_1$ with r_1 *fixed.* In the limit, r_1 is a double root for the characteristic equation. As a linear combination of y_1 and y_2, $\frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = -\frac{1}{r_2 - r_1} e^{r_1 t} + \frac{1}{r_2 - r_1} e^{r_2 t}$ is a solution to (2) for as long as $r_2 \neq r_1$. In the limit as $r_2 \to r_1$, we have

$$\lim_{r_2 \to r_1} \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = te^{r_1 t}.$$

Thus, we just found the second solution to (2), which is independent from $y_1(t) = e^{r_1 t}$.

Use this idea to solve (1). i.e. Find two solutions for (1).

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
14. $e^{ct} f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c} F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\displaystyle\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s) G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$