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A function $f : A \longrightarrow \mathbb{R}^m$ (where A is a subset of $\mathbb{R}^n$) is said to be Continuous at a point a point $\boldsymbol{a} \in A$ if

$$\forall \epsilon \; \exists \delta \text{ such that } \forall x \in S \; |\boldsymbol{x} - \boldsymbol{a}| < \delta \Longrightarrow |f(\boldsymbol{x} - f(\boldsymbol{a})| < \epsilon$$

This statement can be translated to

$$\forall \epsilon \; \exists \delta \text{ such that } \forall x \in A \; \boldsymbol{x} \in B(\delta, \boldsymbol{a}) \Longrightarrow f(\boldsymbol{x} \in B(\epsilon, f(\boldsymbol{a}))$$

which can be understood as follows: as long as $\boldsymbol{x}$ remains close enough(1) to $\boldsymbol{a}$ the image $f(\boldsymbol{x}$ will be close(2) to $f(\boldsymbol{a})$. Here we have two notions of close (numbered 1 and 2), and notice how the two notions of closedness are related to choice of the quantifiers in the statement of continuity: it looks like the second notion of closeness appears first and is relevant to the universal quantifier and the arbitrariness of choice of ϵ, while the first notion of closeness is related to the seond quantifier, existential quantifier and it appears after the ϵ is selected. Reflect on this relationship. We will soon see what this relationship means. Let's continue translating the definition of continuity. Carefully read and understand Theorem 1.13. The set S is known as the *inverse image of the set U under f*. Let's denote this by $f^{\leftarrow}(U)$. We shall study this concept later, but for now just take it as a piece of language; a convention. For example $f^{\leftarrow}(\mathbb{R}^m) = A$ and $f^{\leftarrow}(\emptyset) = \emptyset$ (why?) Now using this notation we can see that the definition of continuity translates to:

$$\forall \epsilon \; \exists \delta \text{ such that } \forall x \in A \; \boldsymbol{x} \in B(\delta, \boldsymbol{a}) \Longrightarrow \boldsymbol{x} \in f^{\leftarrow}(B(\epsilon, f(\boldsymbol{a})))$$

which can further translate to

$$\forall \epsilon \; \exists \delta \text{ such that } B(\delta, \boldsymbol{a}) \subseteq f^{\leftarrow}(B(\epsilon, f(\boldsymbol{a})))$$

This statement is saying that the inverse image of any open ball is an open set in $\mathbb{R}^n$. This is the meaning of continuity: it is always possible to find a collection of interior points near $\boldsymbol{a}$ which are mapped to a collection of interior points near $f(\boldsymbol{a})$. Even though this does not mean that interior points remain interior under the continuous functions, it rather means that if there are some interior points near $f(\boldsymbol{a})$ there will also be some interior points near $\boldsymbol{a}$ which correspond to those interior points (via f).

Continuity has serval important theoretical consequences:

1. theorems 1.13 (under continuous maps inverse image of an open set is open and inverse image of a closed set is close.) it is not true that in general the image of an open set should be open, not should the image of a closed set shouldbe closed.

2. theorem 1.15 (equivalent definition of continuity in the language of sequences.)

3. theorem 1.22 (if a set is both closed and bounded then the image of it under continuous maps remains both closed and bounded.)

4. corollary 1.23 (the extreme value theorem)

5. theorem 1.26 (continuous functions respect connectedness) however the inverse image of connected sets may not be connected.

6. corollary 1.27 (the intermediate value theorem)
7. definition of a arc or path (as in arc connected in section 1.7)
8. page 58, the concept of class C^1, all the functions which who are differentiable and their derivative is continuous.
9. theorems 2.45, 2.46: class C^2 and equality of mixed partial derivatives
10. theorem 2.83 (extreme points of a continuous function on a set S)
11. theorem 4.11 (continuity implies integrability)
12. theorem 4.12, 4.13 (integrability of almost continous functions)
13. theorem 4.15 (Fundamental Theorem of Calculus)