MAP of 5.8

When do we have anti-derivative for a v.f. $\mathbb{G}$?

<u>Question</u> of existence of anti derivative for derivative operations grad, curl and div

to say given $\mathbb{G}$ There is a C^1 function f such that $\mathbb{G} = \nabla f$ means f is anti gradient (anti derivative) of $\mathbb{G}$. or a potential function for $\mathbb{G}$

answer 1:

5.60 : if $\mathbb{G}$ is <u>Cont.</u> on Open (+ Connected) $\mathbb{R}$ Then

$\mathbb{G}$ is Conservative on $\mathbb{R}$ $\Longleftrightarrow$ $\mathbb{G} = \nabla f$ for some C^1 f

line integrals are independent of path

See 5.59

See 5.59 & independence of path

answer 2:

5.62 : if $\mathbb{G}$ is C^1 and $R \subset \mathbb{R}^3$ is Convex, open Then

Curl $\mathbb{G} = \mathbb{0}$ $\Longrightarrow$ $\mathbb{G} = \nabla f$ for some C^2 function f

f is anti-gradient of $\mathbb{G}$ or potential function for $\mathbb{G}$

This condition means $\mathbb{G}$ has no rotational component or effect

more <u>practical</u>

but 1. more strict conditions on $\mathbb{R}$: Convex open (almost like a ball)

2. good for $\mathbb{R}^3$ only

3. $\Rightarrow$ not $\Longleftrightarrow$

but the pay off is that f is C^2

See also Simple regions in proof of Green's & Stokes' Theorems

answer 3:

5.63 : if $\mathbb{G}$ is C^1 on a Convex open $R \subset \mathbb{R}^3$, Then

div $\mathbb{G} = 0$ $\Longrightarrow$ $\mathbb{G} = \nabla \times \mathbb{F}$ for some C^2 vector field $\mathbb{F}$

$\mathbb{F}$ is anti-curl of $\mathbb{G}$, or vector potential for $\mathbb{G}$

This condition means the expanding component of $\mathbb{G}$ is nil, That is, $\mathbb{G}$ is only rotating

See also 5.64 & 5.65 for more answers

C is a closed curve : $g(a)=g(b)$

proof : assumption : $G=\nabla f$ C is a cont curve $g:[a,b]\to\mathbb{R}^n$ piecewise diff

$$\int_C G\cdot dx = \int_C \nabla f\cdot dx = \int_a^b \nabla f(g(t))\cdot g'(t)\,dt = \int_a^b \frac{d}{dt} f(g(t))\,dt = f(g(b))-f(g(a)) = 0$$

Chainrule 2.26

FTC

proof

$\boxed{5.60}$

let G be cont. on $R\subset\mathbb{R}^n$, R is open & connected

$G=\nabla f \Longrightarrow G$ is conservative (i.e. $\int_C G\cdot dx$ depends ONLY on the begining and end points of C, for any path C)

$\Longleftarrow$ proof

assume G is conservative, fix a pt $a\in R$ and define $f:R\to\mathbb{R}$: $f(x)=\int_C G\cdot dx$

f is well-defined

Now read the rest of the proof that shows $\nabla f = G$. C is $\boxed{\text{any}}$ path that connects a to x

See also[4.15] FTC and $F(x)=\int_a^x f(t)\,dt$

1.30 : open + connected $\Rightarrow$ path connected $\Longrightarrow$ any two pts of R can be connected by a path C.

Pf of 1.30 : Define $S_1=\{x\in R : x$ can be joined to a via a curve$\}$

$S_2=\{x\in R : x$ can not " " " $\}$

we prove (S_1,S_2) is a disconnection of R, $S_1\neq\emptyset$ & R is connected ($a\in S_1$)

so $S_2=\emptyset$, so $R=S_1$, so R is path connected.

- $S_1\cup S_2=R$, $S_1\cap S_2=\emptyset$. Note $\bar{S_1}\subseteq S_1$ b/c if $x\in\bar{S_1}$ & R

$\exists r>0$ $B(r,x)\subset R$ & $B\cap S_1\neq\emptyset$, so let $y\in B\cap S_1$, so

$a \rightsquigarrow y \rightsquigarrow x$ b/c B is convex. Similarly $\bar{S_2}\subseteq S_2$, so $\bar{S_1}\cap S_2=\emptyset$ & $S_1\cap\bar{S_2}=\emptyset$

Open ball in $\mathbb{R}^n$ is convex : any two pt can be connected by a straight line

in particular any interior pt of Ball in $\mathbb{R}^2$ has a horizontal and a vertical pt through it