MAT335 - Chaos, Fractals, and Dynamics - Fall 2013

Solution of Term Test - October 21, 2013

Time allotted: 50 minutes. Aids permitted: None.

1. Consider the function

$$F(x) = \begin{cases} 1 & \text{if } x \leqslant \frac{1}{2} \\ 2x & \text{if } \frac{1}{2} < x \leqslant \frac{3}{2} \\ 6(2-x) & \text{if } x > \frac{3}{2} \end{cases}$$

(a) Find the fixed points of F and determine whether they are attracting, repelling, or neutral.

Solution. To find the fixed points of F, we need to solve the equation

$$F(x) = x. \tag{FP}$$

Since F is defined in 3 branches, we need to solve in 3 parts.

If $x \leqslant \frac{1}{2}$. Then $F(x) = 1$, so only $x = 1$ will solve (FP). Since $x = 1$ is not in this interval, there are no fixed points for $x \leqslant \frac{1}{2}$.

If $\frac{1}{2} < x \leqslant \frac{3}{2}$. Then $F(x) = 2x$, so only $x = 0$ will solve (FP). Since $x = -$ is not in this interval, there are no fixed points for $\frac{1}{2} < x \leqslant \frac{3}{2}$.

If $x > \frac{3}{2}$. Then $F(x) = 6(2-x)$ which means that (FP) becomes

$$12 - 6x = x \qquad \Leftrightarrow \qquad x = \frac{12}{7}.$$

In this case, $\frac{12}{7} > \frac{3}{2}$, so the function F has a fixed point $p = \frac{12}{7}$.

To find whether this fixed point is attracting, neutral, or repelling, we compute $F'(x) = -6 < -1$ for $x > \frac{3}{2}$, so the fixed point $p = \frac{12}{7}$ is repelling. $\square$

(b) Show that the orbit of $x_0 = 1$ is periodic. What is its prime period? Is it attracting, repelling, or neutral?

Solution. The orbit of $x_0 = 1$ is

$$\begin{aligned} x_0 &= 1 \\ x_1 &= F(x_0) = F(1) = 2 \\ x_2 &= F(x_1) = F(2) = 0 \\ x_3 &= F(x_2) = F(0) = 1 \\ &\vdots \end{aligned}$$

So this proves that the orbit is periodic and its prime period is 3.

We now check

$$F'(0)\,F'(1)\,F'(2) = 0 \cdot 2 \cdot (-6) = 0,$$

so the cycle is attracting. □

(c) Show that any point $x_0 < \frac{1}{2}$ such that $x_0 \neq 0$, is eventually periodic.

Solution. If $x_0 < \frac{1}{2}$, then $x_1 = F(x_0) = 1$. Since 1 is a periodic point, the orbit of x_0 is eventually periodic for $x_0 \neq 0$. □

(d) Show that any point $x_0 > 3$ is eventually periodic.

Solution. If $x_0 > 3$, then $x_1 = F(x_0) = 6(2 - x_0) < 6(2 - 3) = -6$. We can apply the part (c) with $x_1 < -6 < \frac{1}{2}$ instead of x_0 to deduce that x_0 is eventually periodic. □

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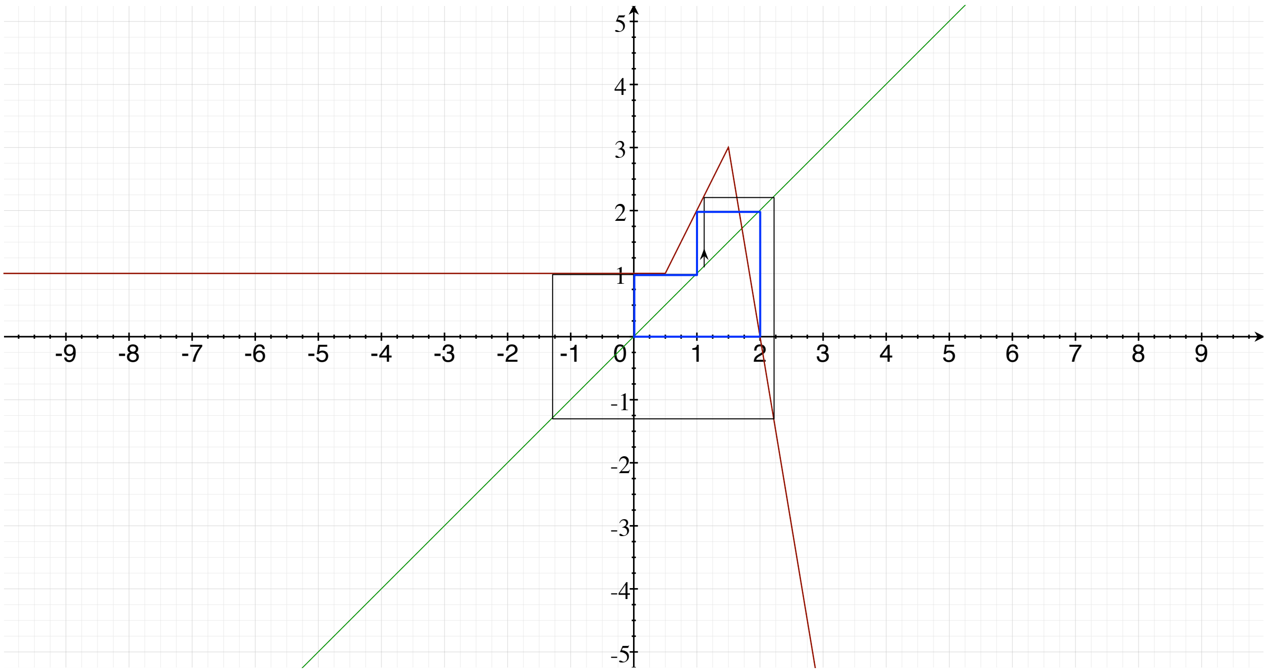
(e) Plot the graphs $y = f(x)$ and $y = x$. What happens to the orbit of x_0 under F if $\frac{1}{2} < x_0 < 3$?



Solution. Because the 3-cycle is the only attracting cycle, it seems that the orbit of a typical $x_0 \in \left(\frac{1}{2}, 3\right)$ under F is like the one sketched on the graph, which merges with the periodic orbit $(0, 1, 2)$ marked in blue. So it is eventually periodic.

In fact there are periodic orbits with all prime periods. For example, the point $x_0 = \frac{12}{13}$ is periodic with prime period 2.

Note. The answer was considered correct if the student only identified the eventually periodic orbit. $\square$

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2. Let $F(x)$ be an odd function: $F(-x) = -F(x)$ for all x.

Show that if $F(x_0) = -x_0$, then x_0 lies on a 2-cycle of $F(x)$.

Solution. Let x_0 be such that $F(x_0) = -x_0$.

Then we need to prove that x_0 lies on a 2-cycle, which means that it satisfies the equation

$$F^2(x_0) = x_0.$$

We verify this:

$$F^2(x_0) = F\big(F(x_0)\big) = F\big(-x_0\big),$$

and we know that F is odd, so

$$F^2(x_0) = F\big(-x_0\big) = -F(x_0) = -(-x_0) = x_0,$$

so we proved that x_0 is periodic with period 2. $\square$

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3. Consider the family of functions $F_\lambda(x) = \lambda x \cos x$ for $\lambda \neq 0$.

(a) Show that there is one unique fixed point for F_λ when $-1 < \lambda < 1$. Is it attracting, repelling, or neutral?

Solution. To find the fixed points of F_λ, we need to solve the equation

$$\begin{aligned} F_\lambda(x) &= x \\ \lambda x \cos x &= x \\ x\,(\lambda \cos x - 1) &= 0 \\ x = 0 \quad &\text{or} \quad \cos x = \frac{1}{\lambda} \end{aligned}$$

For $-1 < \lambda < 1$, the second equation has no solutions, since $\cos x \in [-1, 1]$ and $\frac{1}{\lambda} \in (-\infty, -1) \cup (1, \infty)$.

We conclude that for $-1 < \lambda < 1$, there is one unique fixed point $x = 0$ for F_λ.

We now compute

$$F'_\lambda(x) = \lambda\,(\cos x - x \sin x)\,,$$

so $F'_\lambda(0) = \lambda$. This implies that $x = 0$ is an attracting fixed point for $-1 < \lambda < 1$. □

(b) When $\lambda < -1$, is the fixed point from (a) attracting, repelling, or neutral?

Solution. From the previous part, $F'_\lambda(0) = \lambda$, so for $\lambda < -1$, the fixed point $x = 0$ is repelling. □

Continues on page 6. . .

(c) Find the two periodic points q_1 and q_2 of prime period 2 that have the smallest absolute value. Are they attracting, repelling, or neutral?

(**Hint 1.** $F_\lambda(x)$ is an odd function)

(**Hint 2.** You can use arccos in your answer and remember that $\arccos : [-1,1] \to [0,\pi]$)

Solution. The function is odd, so we can find a 2-cycle by solving

$$F_\lambda(x) = -x,$$

and disregarding the solution $x = 0$ (because it is a fixed point).

We obtain

$$\begin{aligned} F_\lambda(x) &= -x \\ \lambda x \cos x &= -x \\ x\,(\lambda \cos x + 1) &= 0 \\ x = 0 \quad \text{or} \quad \cos x &= -\frac{1}{\lambda} \end{aligned}$$

For $|\lambda| > 1$, there are two solutions to the second equation $\cos x = -\frac{1}{\lambda}$, which are

$$q_\pm = \pm \arccos\left(-\frac{1}{\lambda}\right).$$

Observe that $q_- = -q_+$ and $q_+ \in [0,\pi]$. We now compute

$$\begin{aligned} F'_\lambda(q_-)\ F'_\lambda(q_+) &= \lambda^2(\cos q_- - q_- \sin q_-)(\cos q_+ - q_+ \sin q_+) \\ &= \lambda^2(\cos q_+ - q_+ \sin q_+)(\cos q_+ - q_+ \sin q_+) \\ &= \lambda^2 \left(-\frac{1}{\lambda} - q_+ \sin q_+\right)^2 \end{aligned}$$

If $\boldsymbol{\lambda < -1}$, then

$$F'_\lambda(q_-)\ F'_\lambda(q_+) = (-\lambda)^2\left(-\frac{1}{\lambda} - q_+ \sin q_+\right)^2 = \big(1 + \underbrace{\lambda}_{<0}\,\underbrace{q_+}_{\geqslant 0}\,\underbrace{\sin q_+}_{\geqslant 0})\big)^2 \leqslant 1$$

The term inside the square is smaller than 1, since $q_+ \in [0,\pi]$ and $\lambda < 0$.

We conclude that the 2-cycle is attracting.

If $\boldsymbol{\lambda > 1}$, then

$$F'_\lambda(q_-)\ F'_\lambda(q_+) = \lambda^2\left(-\frac{1}{\lambda} - q_+ \sin q_+\right)^2 = \big(-1 - \underbrace{\lambda}_{>0}\,\underbrace{q_+}_{\geqslant 0}\,\underbrace{\sin q_+}_{\geqslant 0})\big)^2 \geqslant 1$$

The term inside the square is smaller than -1, since $q_+ \in [0,\pi]$ and $\lambda > 0$.

We conclude that the 2-cycle is repelling. $\square$

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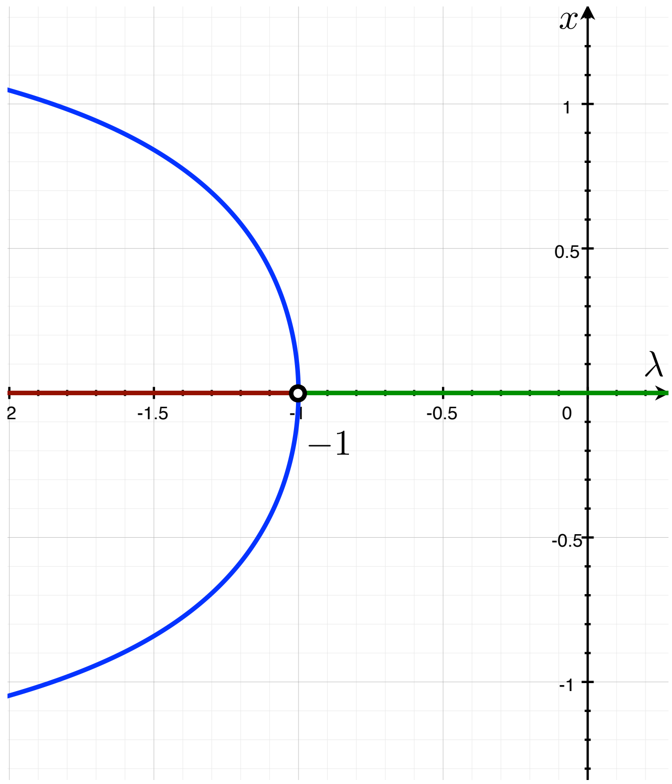
(d) Based on your results in the previous parts, sketch the bifurcation diagram for $F_\lambda(x)$ for $-2 < \lambda < 0$. Label the nodes and indicate if each node is a saddle-node bifurcation, a period-doubling bifurcation, or neither.

Solution.



- The green line is the fixed point $x = 0$, where it is attracting.
- The red line is the fixed point $x = 0$, where it is repelling.
- The blue curve are the periodic points $q_\pm$ which form an attracting 2-cycle.

This means that at $\lambda = -1$, the family F_λ undergoes a period-doubling bifurcation. $\square$

The end.