chapter 2

Q1. Let M be a matching in a bipartite graph G. Show that if M is suboptimal, i.e. contains fewer edges than some other matching in G, then G contains an augmenting path with respect to M. Does this fact generalize to matchings in non-bipartite graphs?

Hint: Recall how an augmenting path turns a given matching into a larger one. Can you reverse this process to obtain an augmenting path from the two matchings?

page 1 question 3

Q5. Derive the marriage theorem from K¨onig's theorem.

Hint: If there is no matching of A, then by K¨onig's theorem few vertices cover all the edges. How can this assumption help you to find a large subset of A with few neighbours?

problem 2

Q7. Find an infinite counterexample to the statement of the marriage theorem.

Hint: If you have S proper subset in S' ⊊ A with |S| = |N(S)| in the finite case, the marriage condition ensures that N(S) proper subset N(S'): increasing S makes more neighbours available. Use the fact that this fails when S is infinite.

the first problem is a solution

Q8. Let k be an integer. Show that any two partitions of a finite set into k-sets admit a common choice of representatives.

Hint: Apply the marriage theorem.

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Q11. Let G be a bipartite graph with bipartition {A, B}. Assume that $\delta(G) >= 1$, and that $d(a) >= d(b)$ for every edge ab with $a \in A$. Show that G contains a matching of A.

Hint: Intuitively, the edges between a set $S \subseteq A$ and N(S) create larger degrees in S than in N(S), so they must be spread over more vertices of N(S) than of S. To make this precise, count both S and N(S) as a sum indexed by those edges. Alternatively, consider a minimal set S violating the marriage condition, and count the edges between S and N(S) in two ways.

page 1, first problem

Q12. Find a bipartite graph with a set of preferences such that no matching of maximum size is stable and no stable matching has maximum size. Find a non-bipartite graph with a set of preferences that has no stable matching.

Hint: For the second task, remember that change occurs most likely if unhappy vertices can bring it about without having to ask the happy ones. If philosophy does not help, try K^3.

[DO IT YOURSELF]

Q13. Consider the algorithm described in the proof of the stable marriage theorem. Observe that once a vertex of B is matched, she remains matched and gets happier with every change of her matching edge. On the other hand, show that the sequence of matching edges incident with a given vertex of A makes this vertex unhappier with every change (disregarding the interim periods when he is unmatched).

Hint: Consider the transition from a matching edge ab to a later matching edge ab' . Suppose a prefers b' to b. Why did he not marry b' in the first place?

page2, first theorem proof?

Q14. Show that all stable matchings of a given graph cover the same vertices. (In particular, they have the same size.)

Hint: Alternating paths.

page 1, second problem

Q19. Find a cubic graph without a 1-factor.

Hint: Corollary 2.2.2.

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chapter 3

Q1. let G be a graph with vertices a and b, and let $X \subseteq V(G) \setminus \{a, b\}$ be an a–b separator in G. Show that X is minimal as an a–b separator if and only if every vertex in X has a neighbour in the component C_a of G – X containing a, and another in the component C_b of G – X containing b.

Hint: Recall the definitions of ‘separate’ and ‘component’.

Q7. Show that the block graph of any connected graph is a tree.

Hint: Deduce the connectedness of the block graph from that of the graph itself, and its acyclicity from the maximality of each block.

Q8. Let G be a k-connected graph, and let xy be an edge of G. Show that G/xy is k-connected if and only if G – {x, y} is (k – 1)-connected.

Hint: Assuming that G/xy is not k-connected, distinguish the cases when v_{xy} lies inside or outside a separator of at most k – 1 vertices.

Q10. Let e be an edge in a 3-connected graph G != K^4 . Show that either G – e (there is a dot on the -) or . G/e is again 3-connected.

Hint: Suppose that both after contracting e and after deleting e there is a 2-separator in the resulting graph. How are these separators arranged with respect to each other?

page 5, problem 6

Q15. Find the error in the following ‘simple proof’ of Menger’s theorem (3.3.1). Let X be an A–B separator of minimum size. Denote by G_A the subgraph of G induced by X and all the components of G – X that meet A, and define G_B correspondingly. By the minimality of X, there can be no A–X separator in G_A with fewer than |X| vertices, so G_A contains k disjoint A–X paths by induction. Similarly, G_B contains k disjoint X–B paths. Together, all these paths form the desired A–B paths in G.

Hint: Check the induction.

Q16. Prove Menger's theorem by induction on ||G||, as follows. Given an edge e = xy, consider a smallest A–B separator S in G – e. Show that the induction hypothesis implies a solution for G unless $S \cup \{x\}$ and $S \cup \{y\}$ are smallest A–B separators in G. Then show that if choosing neither of these separators as X in the previous exercise gives a valid proof, there is only one easy case left to do.

Hint: How big is S? To recognize the easy remaining case, it helps to have solved the previous exercise first.

Q19. Let k >= 2. Show that in a k-connected graph any k vertices lie on a common cycle.

Hint: Consider a cycle through as many of the k given vertices as possible. If one them is missed, can you re-route the cycle through it?