STAT2001 Tutorial 10 Solutions

Problem 1

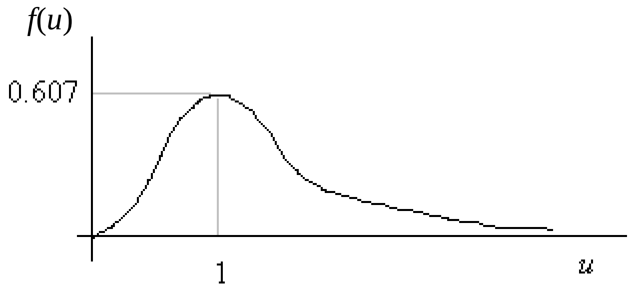
(a) Y has pdf $f(y) = \frac{1}{2}e^{-y/2}$, $y > 0$.

Now $u = \sqrt{y}$ is a strictly increasing function for all $y > 0$.

So we can use the transformation method.

$u = y^{1/2} \Rightarrow y = u^2, \frac{dy}{du} = 2u$.

Therefore $f(u) = f(y)\left|\frac{dy}{du}\right| = \frac{1}{2}e^{-u^2/2}|2u| = ue^{-u^2/2}$, $u > 0$.



Note: $l(u) = \log f(u) = \log u - \frac{1}{2}u^2$.

$l'(u) = \frac{1}{u} - u$. $l'(u) = 0 \Rightarrow u = 1$.   Thus $Mode(U) = 1$.

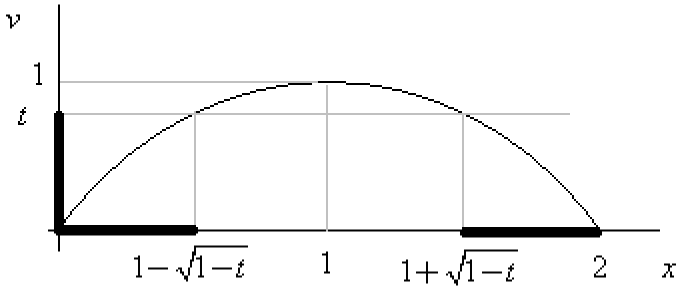
(b) $v = x(2 - x)$ is neither strictly increasing nor strictly decreasing over (0,2).
So the transformation method cannot be used. We will use the cdf method.

V has cdf $F_V(t) = P(V < t) = P(X(2-X) < t)$

$= P(X < 1-\sqrt{1-t}) + P(X > 1+\sqrt{1-t})$ (see below)

$= 2P(X < 1-\sqrt{1-t})$ by symmetry

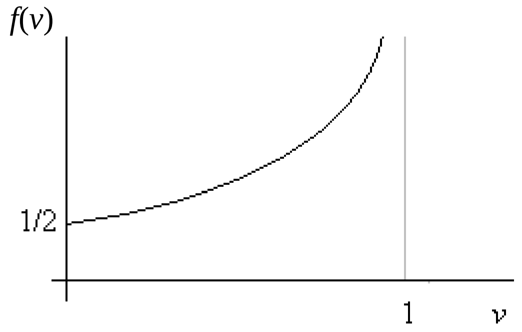
$= 2 \times \frac{1}{2}(1-\sqrt{1-t})$, since $X \sim U(0,2)$.



Working: $x(2 - x) = t \Rightarrow x^2 - 2x + t = 0 \Rightarrow x = \frac{2 \pm \sqrt{4-4t}}{2} = 1 \pm \sqrt{1-t}$.

We have shown that $F(v) = 1-(1-v)^{1/2}$, $0 < v < 1$.

It follows that V's pdf is $f(v) = -\frac{1}{2}(1-v)^{-1/2}(-1) = \frac{1}{2\sqrt{1-v}}$, $0 < v < 1$.



Check: $\int_0^1 \frac{1}{2\sqrt{1-v}}dv = -\frac{1}{2}\int_1^0 r^{-1/2}dr$ after substituting $r = 1-v$

$$= \frac{1}{2}\int_0^1 r^{-1/2}dr = \left(r^{1/2}\Big|_0^1 \right) = 1 \text{ (correct).}$$

$$EV = \int_0^1 v\frac{1}{2\sqrt{1-v}}dv = -\frac{1}{2}\int_1^0 (1-r)r^{-1/2}dr = \frac{1}{2}\int_0^1 (r^{-1/2} - r^{1/2})dr$$

$$= \left(r^{1/2} - \frac{r^{3/2}}{3}\Big|_0^1 \right) = 1 - \frac{1}{3} = \frac{2}{3}.$$

Alternatively,

$$EV = 2EX - EX^2 = 2EX - \{VarX + (EX)^2\} = 2(1) - \left(\frac{(2-0)^2}{12} + 1^2 \right) = \frac{2}{3}.$$

Problem 2

$f(x,y) = f(x)f(y) = 1\times e^{-y}$, $0 < x < 1$, $y > 0$.

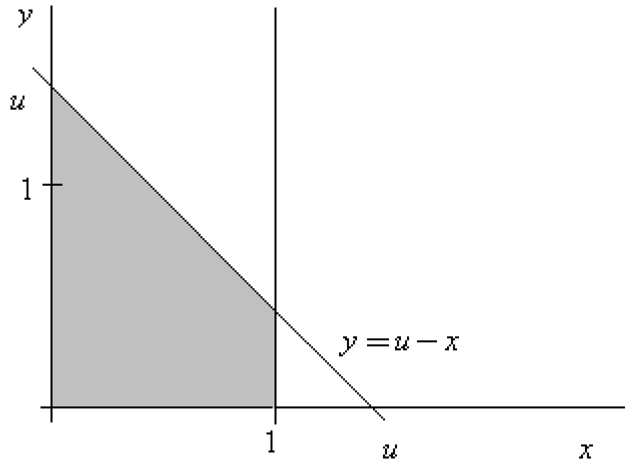
$$F(u) = P(U < u) = P(X+Y < u) = P(Y < u - X)$$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{u-x} e^{-y}dy \right)dx$$

$$= \int_{x=0}^{1} (1 - e^{-(u-x)})\,dx$$

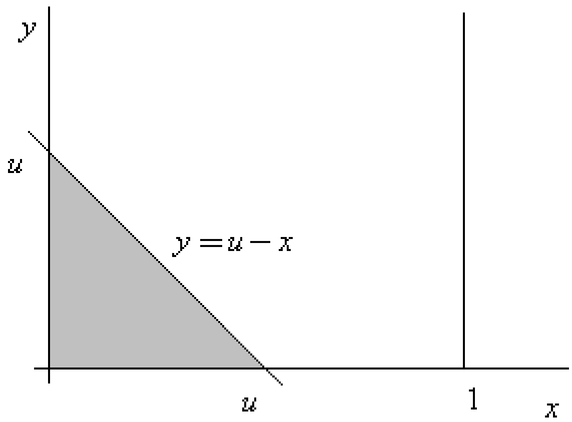
$$= 1 - (e-1)e^{-u}, \quad u > 1.$$

Graph for the case $u > 1$:



For $u < 1$, we find that $F(u) = \int_{x=0}^{u} \left(\int_{y=0}^{u-x} e^{-y} dy \right) dx = u - 1 + e^{-u}$.

Graph for the case $u < 1$:



In summary so far, $F(u) = \begin{cases} u - 1 + e^{-u}, & 0 < u < 1 \\ 1 - (e-1)e^{-u}, & u > 1 \end{cases}$

Therefore U has pdf $f(u) = F'(u) = \begin{cases} 1 - e^{-u}, & 0 < u < 1 \\ (e-1)e^{-u}, & u > 1 \end{cases}$

Note: It can be shown that the slope of $f(u)$ is:

1 at 0,   0.368 at 1 on the left,   and –0.632 at 1 on the right.

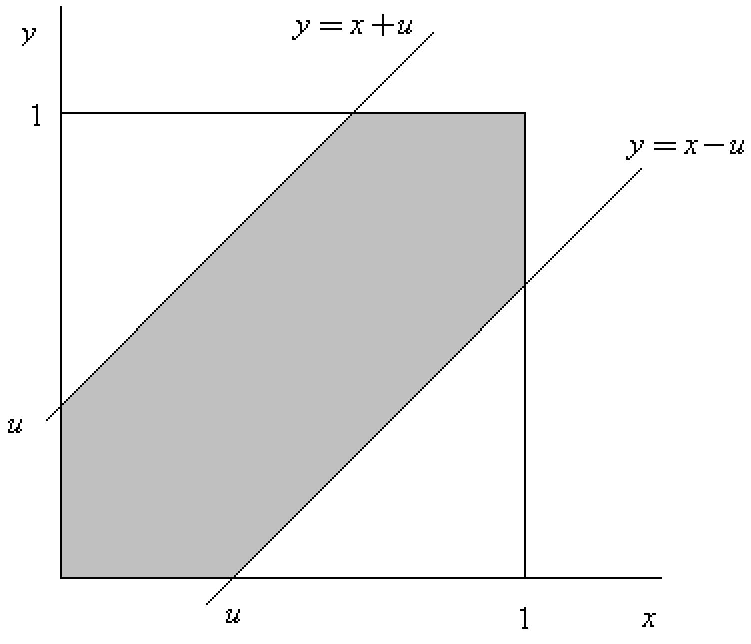
Problem 3

(a) Let X and Y be the distances from the left end of the stick to the two points. Then the distance between the two points is $U = |X - Y|$.

Now $X,Y \sim$ iid U(0,1),
so that $f(x,y) = f(x)\,f(y) = 1, \quad 0 < x < 1, \quad 0 < y < 1$.

It follows that U has cdf

$$\begin{aligned}
F(u) = P(U < u) = P(|Y - X| < u) &= \iint_{|y-x|<u} f(x,y)\,dxdy \\
&= P(-u < Y - X < u) \\
&= P(-u < Y - X, Y - X < u) \\
&= P(X - u < Y, Y < X + u) \\
&= P(X - u < Y < X + u) \\
&= \text{ area of shaded region below} \\
&= 1 - (1-u)^2,\ 0 < u < 1.
\end{aligned}$$



It follows that U has pdf $f(u) = F'(u) = 2(1-u)$, $0 < u < 1$.

Therefore $EU = \int_0^1 u2(1-u)du = \dfrac{1}{3}$.

Another solution:

$$E\,|Y-X| = \int_0^1\int_0^1 |y-x|\,dxdy = \int_{y=0}^{1}\left(\int_{x=0}^{y}(y-x)dx\right)dy + \int_{x=0}^{1}\left(\int_{y=0}^{x}(x-y)dy\right)dx$$
$$= 2\int_{y=0}^{1}\left(\int_{x=0}^{y}(y-x)dx\right)dy \quad \text{by symmetry.}$$

The last inner integral equals $\left(yx-\frac{x^2}{2}\Big|_{x=0}^{y}\right) = y^2-\frac{y^2}{2}=\frac{y^2}{2}$.

Therefore $E\,|Y-X| = 2\int_{y=0}^{1}\frac{y^2}{2}dy=\frac{1}{3}$.

Note: There are also solutions to this problem which do not involve integration.

(b) The distance from the left end of the stick to the nearest point is the 1st order statistic, $V=\min(X,Y)$. This random variable has cdf

$$F(v) = P(V<v) = 1-P(V>v) = 1-P(X>v, Y>v)$$
$$= 1-P(X>v)P(Y>v)$$
$$= 1-(1-v)^2,\quad 0<v<1.$$

We see that V has the same distribution as U in Part (a).
Therefore $EV = EU = 1/3$.

(c) Let $W = (X \mid X < 1/2)$.
Then

$$F(w) = P(W<w) = P(X<w \mid X<1/2) = \frac{P(X<w, X<1/2)}{P(X<1/2)}$$
$$= \frac{P(X<w)}{1/2} \quad \text{assuming } w<1/2$$
$$= 2w,\quad 0<w<1/2.$$

So $f(w)=F'(w)=2$, $0<w<1/2$.
Thus $(X \mid X < 1/2) \sim U(0,1/2)$,
and so $E(X \mid X < 1/2) = 1/4$.

Problem 4

(a) The total number of accidents $U = X + Y$ has mgf

$$m_U(t) = m_X(t)m_Y(t) = e^{a(e^t-1)}e^{b(e^t-1)} = e^{(a+b)(e^t-1)}.$$

Therefore U has the Poisson distribution with mean $a + b$, and its pdf is

$$p(u) = \frac{e^{-(a+b)}(a+b)^u}{u!}, \quad u = 0,1,2,3,...$$

(b) $$p(x\,|\,u) = \frac{p(x,u)}{p(u)}.$$

Now $p(x,u) = P(X = x, U = u) = P(X = x, X + Y = u)$

$$\begin{aligned} &= P(X = x, x + Y = u) \\ &= P(X = x, Y = u - x) \\ &= P(X = x)P(Y = u - x) \quad \text{since } X \perp Y \\ &= p_X(x)\,p_Y(u - x). \end{aligned}$$

So $$p(x\,|\,u) = \frac{\left(\dfrac{\cancel{e^{-a}}a^x}{x!}\right)\left(\dfrac{\cancel{e^{-b}}b^{u-x}}{(u-x)!}\right)}{\left(\dfrac{\cancel{e^{-(a+b)}}(a+b)^u}{u!}\right)} = \frac{u!}{x!(u-x)!}\frac{a^x}{(a+b)^x}\frac{b^{u-x}}{(a+b)^{u-x}}$$

$$= \binom{u}{x}\left(\frac{a}{a+b}\right)^x\left(1-\frac{a}{a+b}\right)^{u-x}, \quad x = 0,...,u.$$

We see that $(X\,|\,U = u) \sim \text{Bin}(u,c)$, where $c = a/(a + b)$.