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A function $\boldsymbol{f} : S \longrightarrow \mathbb{R}^n$ is said to be uniformly continuous on the set S whenever

$$\forall\epsilon\exists\delta \text{ such that } \forall\boldsymbol{x},\boldsymbol{y} \in S\ |\boldsymbol{x}-\boldsymbol{y}| < \delta \Longrightarrow |\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{y})| < \epsilon$$

Whereas the usual continuity (as we studied it in section 1.3) is continuity at a given point $\boldsymbol{a} \in S$. So, to say that the function $\boldsymbol{f} : S \longrightarrow \mathbb{R}^n$ is continuous on S mean that $\boldsymbol{f}$ is continuous at every point of S, that is

$$\forall\boldsymbol{a} \in S\ \forall\epsilon\exists\delta \text{ such that } \forall\boldsymbol{x} \in S\ |\boldsymbol{x}-\boldsymbol{a}| < \delta \Longrightarrow |\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{a})| < \epsilon$$

Which is the same as

$$\forall\epsilon\forall\boldsymbol{a} \in S\ \exists\delta \text{ such that } \forall\boldsymbol{x} \in S\ |\boldsymbol{x}-\boldsymbol{a}| < \delta \Longrightarrow |\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{a})| < \epsilon$$

The difference between the statements of continuity and uniform continuity seems to be the order of quantifiers dealing with ϵ and δ: in continuity the choice of δ usually depends on the choice of the points $\boldsymbol{a}$ and ϵ, that is, the value of δ may change from point to point. Whereas in uniform continuity the choice of $delta$ depends only on the choice of ϵ; that is, there is a δ which works for all the points in S. Continuity is said to be a local property, while the uniform continuity is a global property:

- in continuity for each $\boldsymbol{a}$ has a δ which works for $\boldsymbol{a}$, whereas
- in uniform continuity there is δ which works for all the points in S.

The most important applications of uniform continuity is that no matter which two points $\boldsymbol{x}$ and $\boldsymbol{y}$ of S that we select, as long as the two points are close enough to each other (in the sense of δ) then the difference $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|$ will be bounded (that is, under control) on a set S. See exercise 1, Holder continuity.

Uniform continuity is continuity without the complications discussed in the proof of theorem 1.10.

Here is a list of places where uniform continuity is essential for the discussions of our textbook.

- theorem 1.33 is a tool to achieve uniform continuity: if $\boldsymbol{f}$ is continuous on S and S is compact then $\boldsymbol{f}$ is uniformly continuous on S.
- Exercise 5 (of section 1.8) is an important consequence
- Proof of 4.11, to show that any continuous function is integrable
- Proof of 4.46 and 4.47