PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 4
DUE FRIDAY, MARCH 24, 4PM.

Warm-up problems. These are completely optional.

(1) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective, but not surjective. Give an example of a function $g : \mathbb{N} \to \mathbb{N}$ which is surjective, but not injective.

(2) Let $a, b \in \mathbb{R}$ with $a \neq 0$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is a bijection. Compute a formula for f^{-1}.

Problems to be handed in. Solve four of the following five problems. One of the four must be Problem (1).

(1) Let $f : A \to B$ and let $g : B \to C$ be functions, and let $h = g \circ f$. Determine which of the following statements are true. Give proofs of the true statements and counterexamples for the false statements.
   (a) If h is injective, then f is injective.
   (b) If h is injective, then g is injective.
   (c) If h is surjective, then f is surjective.
   (d) If h is surjective, then g is surjective.

a) TRUE. Let h be injective. We want to show that f is injective. That is, we want to prove that if $f(x) = f(y)$ in B, then $x = y$ in A. Suppose we have $x, y \in A$ such that $f(x) = f(y)$. Then, since g is a function[1], we must have $g(f(x)) = g(f(y))$. That is, $h(x) = h(y)$. Since h is injective, this implies $x = y$, and we are done.

b) and c) FALSE. Take $A = \mathbb{Z}$, $B = \mathbb{R}$, and $C = \mathbb{Z}$, and define functions $f : \mathbb{Z} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{Z}$ by

$$f(n) = n$$
$$g(x) = \lfloor x \rfloor$$

The function g is known as the floor function. If x is a real number, then $\lfloor x \rfloor$ is simply x rounded down to the nearest integer. So $\lfloor 2.59 \rfloor$ is just 2.
The composition of these functions, $g \circ f$, is a map $h : \mathbb{Z} \to \mathbb{Z}$ which is the identity map. That is, $h(n) = n$ for all $n \in \mathbb{Z}$. It is both surjective and injective.
The map g is not injective, however, since $\lfloor 1 \rfloor = \lfloor 1.5 \rfloor$. This gives a counterexample to b).
We also see that the map f is not surjective. There is, for instance, no natural number n such that $f(n) = \pi$. This gives a counterexample to c).

d) TRUE. Let h be surjective. We wish to show that g is surjective. That is, we want to prove that for all $c \in C$, there is some $b \in B$ such that $g(b) = c$. Fix an arbitrary $c \in C$. Since h is surjective, we know there is some $a \in A$ such that $h(a) = g(f(a)) = c$.

[1]Recall that a function takes an element of the domain to a *unique* element of the codomain

We know that $f(a)$ must be in B, we simply label $b = f(a)$, and this is b is the element we are looking for.

(2) Let $f : A \to B$ be any function. A function $g : B \to A$ is called a *left-inverse for f* if it satisfies

$$(g \circ f)(a) = a \text{ for all } a \in A.$$

It is called a *right-inverse for f* if it satisfies

$$(f \circ g)(b) = b \text{ for all } b \in B.$$

It is called *two-sided inverse for f* if it satisfies both these conditions.

(a) Prove that if f has a two-sided inverse, then f is a bijection.
(b) Give an example of a function f which has a left-inverse, but is not a bijection.
(c) Give an example of a function f which has a right-inverse, but is not a bijection.

a) Suppose that f has a two-sided inverse.

The composition $g \circ f$ is injective, since $g(f(x)) = g(f(y)) \implies x = y$, since $g(f(a)) = a$ for all $a \in A$. By part a) of Question 2, f must therefore be injective.

The composition $f \circ g$ is surjective, since for all $b \in B$, there is some element (in this case the same one!) such that $(g \circ f)(b) = b$. By part d) of Question 2, f must therefore be surjective. It follows that f is a bijection.

b) Consider the functions defined in Question 2 b) and c). We see that $g(f(n)) = n$ for all $n \in \mathbb{Z}$, so that g is a left-inverse for f. On the other hand, f is not surjective, so it cannot be a bijection.

c) Consider the following functions: $F : \mathbb{N} \to \mathbb{N}$ and $G : \mathbb{N} \to \mathbb{N}$ defined by

$$\begin{aligned} &F(1) = 1 \\ &F(n) = n - 1 \qquad \forall n \geq 2 \end{aligned}$$

and

$$G(n) = n + 1 \qquad \forall n \geq 1$$

Then $F(G(n)) = F(n+1) = n$ for all $n \in \mathbb{N}$, but F is not injective,[2] and therefore not a bijection.

(3) Recall that $[n] = \{1, 2, \dots, n\}$. Let A denote set of subsets of $[n]$ with an even number of elements, and let B denote the set of subsets of $[n]$ with an odd number of elements. Prove that $|A| = |B|$ by constructing an explicit bijection from A to B.

Let $f : A \to B$ be the map defined as follows[3]

$$f(a) = \begin{cases} a \cup \{1\} & \text{if } 1 \notin a \\ a \setminus \{1\} & \text{if } 1 \in a \end{cases}$$

To see that this map is surjective, take some b in B. If b contains 1, then $a = b \setminus \{1\}$ will satisfy $f(a) = b$. If b does not contain 1, then $a = b \cup \{1\}$ will satisfy $f(a) = b$. Either way, for all $b \in B$, there is some $a \in A$ such that $f(a) = b$.

[2] Can you see why?

[3] recall that if $a \in A$, then $a \subset [n]$

To see that the map f is injective, suppose we have $a, a' \in A$ such that $f(a) = f(a')$ in B. If both a and a' contain 1, then $f(a) = a \setminus \{1\} = a' \setminus \{1\} = f(a')$, which implies $a = a'$. If neither a nor a' contain 1, then $f(a) = a \cup \{1\} = a' \cup \{1\} = f(a')$, which again implies $a = a'$. Either way, f is injective.

Since f is both injective and surjective, it is a bijection, and therefore $|A| = |B|$.

(4) Construct explicit bijections $f : (0, 1) \to [0, 1)$ and $g : (0, 1) \to [0, 1]$.

Consider the function $f : (0, 1) \to [0, 1)$ given by

$$f(x) = \begin{cases} 0 & x = \frac{1}{2} \\ \frac{1}{n-1} & x = \frac{1}{n}, \text{ for } n \geq 3 \\ x & \text{otherwise} \end{cases}$$

and $g : (0, 1) \to [0, 1]$ given by

$$g(x) = \begin{cases} 0 & x = \frac{1}{2} \\ \frac{1}{n-2} & x = \frac{1}{n}, \text{ for } n \geq 3 \\ x & \text{otherwise} \end{cases}$$

You should be able to explicitly show that these are bijections.

(5) Let L be the set of all sentences of the English language. Prove that L is countable. (For the purpose of this exercise, a sentence of the English language is any finite sequence of characters chosen from the set of characters visible on your computer's keyboard.)

A set is countable if we can put it in a bijection with the natural numbers. This means we can write the elements of our set as a list, where we don't write any element more than once, and we don't miss out on any elements. Let's describe how to do this for the set L.

Take all the characters visible on your computers keyboard, and arrange them in an ordered (finite) list: $\{a, b, c, ...\}$. Denote the length[4] of this list by n. The actual order of the list isn't important, as long as we keep it fixed. This list contains all the one-character sentences in the English language. We now add to the list all of the two-character sentences, ordered lexicographically[5], so our list becomes $\{a, b, c, ..., aa, ab, ac, ..., ba, bb, bc, ...\}$. We now add all the three-character sentences, then four-character sentences, and k-character sentences, for all $k \in \mathbb{N}$. With this procedure, we obtain a list (that is, a bijection from $\mathbb{N}$ to L) of all the sentences in the English language.

Why is this a bijection? Well, it is a surjection because every sentence in L is somewhere on the list. If x is a sentence in L, then it is a finite string of letters, of length, say, k. By construction we have included every k-character sentence, *for all* $k \in \mathbb{N}$, so x must be in our list somewhere. It is also an injection because we haven't repeated any sentences. It follows that the set of all sentences of the English language are countable.

[4]because I can't really be bothered counting how many keys there are on my keyboard

[5]This is just a fancy way of saying that they are ordered the same way as words in a dictionary are ordered. To see how two strings are ordered compared to one another, look at the first letter and order them based on which letter comes first. If the first letters are the same, look at the next letter, and so on. So $aa < ab < cc < dg$