UNIVERSITY OF TORONTO
Faculty of Arts and Science
APRIL EXAMINATIONS 2013
CSC336H1 LEC 5101 S (20131)
Duration - 2 hours
Aids allowed: pocket calculators

Last (family) name :

First (given) name :

Student id :

Question	Marks
1	/ 15
2	/ 30
3	/ 15
4	/ 25
5	/ 15
Total	/ 100

As noted in the syllabus of the course, you must achieve at least 33% in the final exam in order to pass the course.

Please write legibly. Unreadable answers are worthless.

You can use both sides of all seven (7) sheets for your answers, except the front (cover) page.
You must return all 7 sheets.

1. [*15 points*] Consider the following data:

t	-2	-1	0	1	2
f	0	0	1	0	0

Obtain the least squares fit to this data (i.e. to $f(t)$) by a quadratic polynomial. More specifically, formulate the problem into a (non-square) linear system, indicate the size of the (non-square) matrix arising, then solve the system by the normal equations method.

2. Consider the function $f(x) = \cos(x) - \frac{x^2}{2} + \frac{1}{2}$.

(a) [*3 points*] Using appropriate graphs, locate approximately the positive root of $f(x)$. (Choose functions easy to graph, so that you can draw the graphs by hand.)

(b) [*4 points*] Using mathematical arguments, show that there exists exactly one positive root of $f(x)$, and indicate an interval $[a, b]$ of length not greater than $\frac{\pi}{4}$, where the positive root lies.
If you don't have a calculator, you can use the following approximate values, if you need them: $\sqrt{2} \approx 1.4$, $\pi^2 \approx 10$, $\cos(1) \approx 0.54$.

(c) [*7 points*] Using $x^{(0)} = \frac{\pi}{2}$ as initial guess, apply one Newton iteration to compute an approximation $x^{(1)}$ to the root. While indicating how $x^{(1)}$ is computed, simplify as much as you can, but you do not need to do all calculations; just indicate the values of any trigonometric functions arising, and leave the result in terms of π.

(d) [*16 points*] Consider the equation $f(x) = 0$ written as $x = \sqrt{2\cos(x) + 1}$, and the associated fixed-point iteration method, with $\phi(x) = \sqrt{2\cos(x) + 1}$.
Apply one fixed-point iteration with initial guess $x^{(0)} = \frac{\pi}{2}$, and indicate the value of $x^{(1)}$.
(Note: this $x^{(1)}$ is different than the one in (c).)
Let $\delta = 0.01$. Show that if $x^{(0)} \in [\frac{\pi}{4}, \frac{\pi}{2} - \delta]$, the fixed-point iteration converges to the positive root of f.
What is the order of convergence of the fixed-point iteration? Explain.
If you don't have a calculator, you can use the following approximate values, if you need them: $\frac{\pi}{4} \approx 0.785$, $\phi(\frac{\pi}{2} - \delta) \approx 1.01$, $\phi(\frac{\pi}{4}) \approx 1.55$, $\frac{\pi}{2} - \delta \approx 1.56$.

3. Consider the system of two nonlinear equations with respect to the two unknowns θ and ϕ

$$\cos(\theta) + \cos(\theta + \phi) = x$$

$$\sin(\theta) + \sin(\theta + \phi) = y$$

where x and y are given.

(a) [*8 points*] Formulate the Jacobian matrix for the above nonlinear system. The entries of the Jacobian should be given in terms of trigonometric functions of θ and ϕ.

(b) [*7 points*] Let $x = 2$ and $y = 0$. Is Newton's method applicable to the above system with $(\theta^{(0)}, \phi^{(0)}) = (\frac{\pi}{2}, 0)$? If yes, apply one Newton iteration and indicate $(\theta^{(1)}, \phi^{(1)})$. Otherwise, explain why Newton's is not applicable.

4. Consider the function $f(x) = \sqrt{x}$, and the data $(\frac{1}{4}, \frac{1}{2})$, $(1, 1)$ and $(4, 2)$ arising from f.

(a) [*10 points*] Using the basis functions of your choice (monomial, Lagrange or NDD), construct the quadratic polynomial $p_2(x)$ interpolating $f(x)$ at the above data.

(b) [*5 points*] Give the error formula for this interpolation problem (i.e. for *this* f and *these* data points). The formula should involve an unknown point ξ. Any other functions involved in the formula should be given explicitly in terms of x.

(c) [*10 points*] Assume we evaluate p_2 at some $x \in [\frac{1}{16}, 4]$. Indicate the interval where ξ belongs to and explain. Describe how you can find an upper bound (as sharp as you can) for the error $|f(x) - p_2(x)|$ when $x \in [\frac{1}{16}, 4]$. You do not have to give a numerical value to the bound. You may specify it as the maximum of a fixed number of quantities simplified as much as possible.

Excerpt from notes:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j),$$

where ξ is an unknown point in $\mathrm{ospr}\{x_0, x_1, \cdots, x_n, x\}$, that depends on x.

5. Assume we interpolate a function $f \in C^4$ in the interval $[1,2]$, by linear splines and by cubic splines. Assume $\max_{1 \le x \le 2} |f''''(x)| = 3 \max_{1 \le x \le 2} |f''(x)|$. Let x_i, $i = 0, \cdots, n$, be some knots. Let

$$e = \frac{1}{8} \max_{x \in [1,2]} |f''(x)| \max_{i=1,\cdots,n} (x_i - x_{i-1})^2. \tag{1}$$

(a) [*6 points*] Assume the knots x_i, $i = 0, \cdots, n$, are not necessarily equidistant, and (1) holds. How would you pick some new knots and how many would you pick to guarantee a maximum error of $\frac{e}{10^4}$ with the linear spline interpolant? Explain. The answer to "how many" is to be given in terms of n.

(b) [*9 points*] Assume the knots, $i = 0, \cdots, n$, are equidistant, and (1) holds. How would you pick some new knots and how many would you pick to guarantee a maximum error of $\frac{e}{10^4}$ with the cubic spline interpolant? Explain. The answer to "how many" is to be given in terms of n.

Excerpt from notes:

$$|f(x) - L(x)| \le \frac{1}{8} \max_{x \in [a,b]} |f''(x)| \max_{i=1,\cdots,n} (x_i - x_{i-1})^2$$

$$|f(x) - C(x)| \le \frac{5}{384} \max_{x \in [a,b]} |f^{(4)}(x)| \max_{i=1,\cdots,n} (x_i - x_{i-1})^4$$

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