Assignment IX: week of Mar. 18th

This is the 9th assignment. You are encouraged to work on this by coming to the help sessions (Thursday 12-1 MP202, Friday 1-2 at MP102) and **grouping** *up with a few other students. Teaching assistants will be at hand to help. You* **DO** *have to hand this one in by 5PM Wed. March 27th.*

1. We can not see directly into the big bang. The Cosmic Microwave Background acts as a layer of 'fog'. But we can stare at the CMB. Summarize, qualitatively, what one can learn by staring at the Cosmic Microwave Background.

2. Static universe. A universe that lives forever saves us the trouble of explaining where it comes from and where it is going. However, such a universe is inconsistent with the observed Hubble expansion. Moreover, it predicts a night sky that is as bright as day. Lastly, it says that we should all be toasted to vapor. In the following exercise, we assume all stars in the universe are identical (with a luminosity of L and a radius R) and are uniformly distributed with a number density of n stars per cubic meter. Here, n is a tiny number and its value does not affect our conclusions.

   (a) If the universe has been around forever, we ought to see stars out to an infinite distance. The energy flux received by a unit area on Earth, from a star at distance r, is $L/(4\pi r^2)$. Consider a spherical shell at distance r with a thickness dr. Let the number of stars in this shell be dN: $dN = n 4\pi r^2 dr$. The total energy flux received from all these stars is

   $$dF = \frac{L}{4\pi r^2} dN. \tag{1}$$

   Integrate this flux over all possible r, with limits from $r = 0$ to $r = \infty$, and show that the total flux diverges. This would mean we have an infinitely bright sky.

   (b) In the above calculation, we ignore the fact that a star lying right in front of another star can block the latter's light from us. For instance, we can't see stars behind the Sun. As a result, the brightness at every patch of the sky is now finite, and interestingly, identical – independent of how far the closest star in that patch is. Viewed on Earth, the solid angle occupied by a star at distance r is $\pi R^2/r^2$. If all stars are at distance r, we need a certain number of them to cover up the sky without a hole. Show that the total energy flux we receive from these stars is independent of r. This allows us to conclude that, brightness from any patch of the sky is the same regardless of how far the closest stars in that path are. Or, night is as bright as day. This is also called the 'Olbers' paradox'.

   (c) The Sun is our closest star and it occupies a solid angle of 0.2 **square** degree. Now, how much higher energy flux does the Earth receive if for every 0.2 square degree on the sky, we receive the same amount of radiation as that from the Sun? *(Hint: be careful when converting from square degree to ster-radian.)*

(d) The amount of energy received by the Earth has to be exactly radiated away, otherwise the Earth will heat up. Assume the Earth radiates like a blackbody, namely, every square metre of the ground gives off an energy flux of σT^4, where σ is the Stefan-Boltzman constant. The amount of energy received from the Sun alone is able to heat the Earth up to ~ 270 K. With the above enhanced energy flux, what temperature would the Earth be radiating in? How does this compare to the surface temperature of the Sun?

(e) Fortunately our sky is not covered up by stars without a hole. However, we are heated by CMB photons coming from every direction on the sky. Short of any other energy source in the universe (the Sun, Earth's own energy source, other stars...), what temperature will we be heated to? *(Hint: give a guess. No real calculation needed)*

Now reflect back on what you have achieved in this problem. It is interesting to wonder why the static universe idea was taken seriously for a long time.

3. Formation of "Structure". Galaxies and stars form out the primordial fluctuations that we observe on the CMB. Here, we investigate a few key processes during structure formation.

(a) CMB hot spots have a physical size that corresponds, roughly, to the size of the horizon at age $300,000$ yrs.[1] Assume one such hot spot region has been expanding together with the universe, how big in physical size (express in unit of light-year) has it become today? For this exercise, use a model of the universe that is matter-dominated and assume $\Omega = \Omega_m = 1$, so $a \propto t^{2/3}$.[2] For comparison, the Virgo super-cluster currently has a size of ~ 100 million light years, and it is marginally expanding with the Hubble flow. There are no structure larger than a super-cluster in our universe today.

(b) CMB hot spots have matter density that is $\sim 10^{-5}$ higher than the background average. Let us consider the evolution of such a hot spot with the opposite assumption as above. Assume the matter density in the hot spot is sufficient to resist the cosmological expansion and assume that this keeps the density constant, while the rest of the universe expands as $a \propto t^{2/3}$. Compared to today's mean matter density in the universe ($\sim 30\%$ of the critical density), how much higher is the matter density in these hot spots today? In contrast, the average density in a super-cluster is only a few times of the average density in the universe. What can you conclude from this?

(c) Typical spacing between galaxies and the distance to Andromeda. Take the average density in the universe today ($\sim 30\%$ of the critical density) and calculate the volumn needed if one wants to collect enough mass to build the Milky Way (a total mass of $\sim 10^{42}$ kg). This would also yields the typical distance between Milky-way type galaxies, if galaxies are roughly equally spaced in the universe. Our closest twin, the Andromeda galaxy, is at 2.5 million light-years away from us. What can you infer by comparing these two distances?

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[1] It is actually the sound horizon but we ignore that fine detail here.

[2] This is not strictly correct but it suffices for our rough estimates here.