UNIVERSITY OF TORONTO
Faculty of Arts and Sciences

AUGUST EXAMINATIONS

MAT237Y1Y

Duration - 3 hours

No Aids Allowed

Instructions: There are 8 questions and 13 pages including the cover page. There is a total of 110 marks which include 10 bonus marks. Please provide your answers within the spaces provided, and clearly specify if you use back of a sheet to answer a question.

NAME: (last, first) ______________________

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

MARKER'S REPORT:

Question	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
TOTAL	

1.

a) (5 marks) First present Green's theorem, and then use it to derive the divergence theorem in the plane.

b) (5 marks) Assume $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is of class C^1 and that $\boldsymbol{c} : [a, b] \longrightarrow \mathbb{R}^3$ is a piecewise C^1 path. Show that $\int_{\boldsymbol{c}} \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{c}(b)) - f(\boldsymbol{c}(a))$

2. Conservative vector fields

a) (10 marks) Suppose that the C^1 vector field $\boldsymbol{F}$ defined on $\mathbb{R}^3$ has the independence of path property, that is the line integral of $\boldsymbol{F}$ along any path depends only on the initial and the terminal points of the path. Prove that $\boldsymbol{F}$ must be the gradient of some scalar function f.

b) (6 marks) First determine whether $\boldsymbol{F}$ has the independence of path property. If not explain why, and if so determine a potential function for $\boldsymbol{F}(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$

c) (4 marks) Calculate the line integral: $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ where $\boldsymbol{F}$ is as in part (b) and C is the circle of radius 3 centered at the origin.

3. Consider the vector field $\boldsymbol{F}(x, y, z) = (x, -y + \ln(1 + \cos^2(x^2)), 3)$ and let S be portion of the graph of the paraboloid $z = 9 - x^2 - y^2$ above the xy plane, oriented out and upward.

a) (8 marks) First state Gauss' theorem, and then (without direct calculation) use Gauss' theorem to determine the Flux of $\boldsymbol{F}$ across S.

b) (6 marks) First present the statement of Stokes' theorem, then state a theorem that enables you to use Stokes' theorem to compute the Flux of $\boldsymbol{F}$ across S. Use this theorem and the Stokes' theorem to calculate the flux of $\boldsymbol{F}$ across S.

4.

a) (6 marks) Assume f is a real valued C^1 defined on $U \subset \mathbb{R}^3$. Prove that if f, restricted to a surface S, has a local maximum or local minimum at $\boldsymbol{x}_0$, then $\nabla f(\boldsymbol{x}_0)$ is perpendicular to S at $\boldsymbol{x}_0$.

b) (10 marks) Use method of Lagrange multiplier to determine the extrema of $f(x, y, z) = x + 2y + 3z$ subject to $x^2 + y^2 = 1$ and $x - y + z = 1$. Then apply the second derivative test to study the nature of the extrema.

5. Implicit function theorem

a) (3 marks) Present the general implicit function theorem for $m = 2$, that is for two equations and two unknowns. Also for simplicity assume $n = 2$.

b) (3 marks) Present the inverse function theorem as a special case of the general implicit function theorem.

c) (5 marks) Apply the inverse function theorem to decide near which points (x, y) the function has a local inverse: $f(x, y) = (\frac{x^4+y^4}{x}, \sin x + \cos y)$.

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6.

a) (8 marks) Let $\omega = xydy+(x+y)^2dx$ and find $d\omega$. (Please demonstrate the use of rules of differentiation for forms.)

b) (4 marks) Demonstrate Green's theorem in the language of differential forms. Present all the calculations involved.

c) (7 marks) Show that for a differential form $\eta = Pdxdy + Qdydz + Rdzdx$ we have

$$d\eta = \nabla \cdot (P, Q, R)\ dxdydz.$$

7. (10 marks) First state the change of variable formula for double integrals, and then use a suitable change of variable to integrate $\iint_R x^2y^2\,dy\,dx$ where R is the region in the first quadrant bounded by the parabolas $y = x^2, y = 2x^2$ and the hyperbolas $xy = 1$ and $xy = 2$.

8. (10 marks) With D the region bounded by the line $y = -x$ and y-axis and the line $y = 1$. Determine whether the improper integral $\iint_D \frac{-dxdy}{x(x+y)}$ exists; explain why, and if it exists calculate its value (justify each step please.)