Assignment 7 - MAT 327 - Summer 2014

Due July 21, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Determine which of the following sequences converge to $\langle 0, 0, 0, \ldots \rangle$ when $\mathbb{R}^{\mathbb{N}}$ is given (1) the product topology, (2) the uniform topology, (3) the box topology:

1. $\langle 1, 1, 1, 1, \ldots \rangle$, $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle$, $\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots \right\rangle$, $\ldots$;
2. $\langle 1, 1, 1, 1, \ldots \rangle$, $\langle 0, 1, 1, 1, \ldots \rangle$, $\langle 0, 0, 1, 1, \ldots \rangle$, $\ldots$;
3. $\langle 1, 1, 1, 1, \ldots \rangle$, $\left\langle 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle$, $\left\langle 0, 0, \frac{1}{3}, \frac{1}{3}, \ldots \right\rangle$, $\ldots$;
4. $\langle 0, 1, 0, 0, \ldots \rangle$, $\left\langle 0, \frac{1}{2}, 0, 0, \ldots \right\rangle$, $\left\langle 0, \frac{1}{3}, 0, 0, \ldots \right\rangle$, $\ldots$;

[**C.2**] Let X be a topological space, and let I be a non-empty set. Write down a natural subbasis for the product topology on X^I. Prove that it is a subbasis and that it generates the product topology.

[**C.3**] Let X be a topological space, and let I, J be non-empty sets. Show that if $|I| = |J|$, then $X^I \cong X^J$, both with the product topology. Show that the converse is false.

[**C.4**] With the product topology, is $\mathbb{N}^{\mathbb{N}}$ discrete? What about $\{0,1\}^{\mathbb{N}}$? What about $\mathbb{N}^{\{0,1\}}$? Are these spaces countable or uncountable?

Definition. *A T_1 space $(X, \mathcal{T})$ is said to be **completely regular** if whenever $C \subseteq X$ is a closed set, and $p \in X \setminus C$ then there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(c) = 0$ $(\forall c \in C)$ and $f(p) = 1$.*

[**C.5**] Let X be a T_1 space. Prove that if X is normal, then it is completely regular. Prove that if X is completely regular, then it is regular. (Both of those should be one or two line proofs). Conclude that calling "complete regularity" the $T_{3\frac{1}{2}}$ property, makes sense.

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] We saw in class that metrizability is a countably productive property. Prove that metrizability is *not* preserved under arbitrary products. (One approach is to show that $\mathbb{R}^{\mathbb{R}}$ is not first countable.)

[**A.2**] Let X be an infinite set, and we will see how to make $\mathcal{P}(X)$ into a topological space. We already know that $\{0,1\}^X$ can be given the product topology, and there is a natural correspondence between subsets of X and elements of $\{0,1\}^X$. Identify $A \subseteq X$ with its characteristic function χ_A, which is defined by $\chi_A(x) = 1$ iff $x \in A$. Check that this is a bijection, and then apply C.1 (from Assignment 1!) to get a topology on $\mathcal{P}(X)$. Write down a natural basis for this topology on $\mathcal{P}(X)$. Explain what it means for a sequence of subsets of $\mathbb{N}$, $\langle A_n \rangle$ to converge to a subset A, when $\mathcal{P}(\mathbb{N})$ is given this topology. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. In this topology is $\mathcal{U}$ a closed subset of $\mathcal{P}(\mathbb{N})$?

[**A.3**] Let's expand upon C.5. prove directly (without using Urysohn's Lemma) that every metrizable space is completely regular. Use the idea of the distance from a point to a closed set: $d(C,x) := \inf_{c \in C} d(c,x)$. (You may need to assume that the metric that generates your metrizable space is bounded.) Does this argument *easily* extend to showing that every metrizable space is "completely normal", (where that has the obvious definition)?

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] The following is an open question: "Is $\mathbb{R}^{\mathbb{N}}$ with the box topology a normal space?". Investigate this problem and provide some evidence for why this problem is difficult. "Investigate" is a word open to interpretation, but I want to be convinced that you understand the problem, why it's difficult, and what are some possible attacks against the problem. Perhaps you want to give me some interesting pairs of disjoint closed sets? It's up to you! (Also, if you do happen to solve this problem, then you will get an A+ in this course.)

[**NI.2**] List the topological properties (of the ones we've studied so far) of $\mathbb{N}^\mathbb{N}$ and $\{0,1\}^\mathbb{N}$ (with the product topologies). Prove that they are both separable by exhibiting explicit countable dense sets. Find subspaces $A, B \subseteq \mathbb{R}_{\text{usual}}$ such that $A \cong \mathbb{N}^\mathbb{N}$ and $B \cong \{0,1\}^\mathbb{N}$.

[**NI.3**] State and prove the two common versions of the Tietze extension theorem. Find (or invent!) a question that uses the Tietze extension theorem that could show up on a problem set. Find an application for the Tietze Extension Theorem or Urysohn's Lemma.