UNIT 7

INTERPRETATIONS AND MODELS: SEMANTICS FOR PREDICATE LOGIC

7.3 EG1 On the interpretations below, is the following true or false?

a) $\sim Fa \wedge \forall x(Fx \rightarrow G(xa))$

Universe: positive integers — F^1: *a* is a multiple of 4.

a^0: 2 — G^2: *a* is divisible by *b*

The first conjunct can be interpreted: 2 is not a multiple of 4.
The second conjunct can be interpreted: Every multiple of 4 is divisible by 2.
Since both conjuncts are true, the sentence is true on this interpretation.

b) $\sim Fa \wedge \forall x(Fx \rightarrow G(ax))$

Universe: people — F^1: *a* is Canadian

a^0: Bill Gates — G^2: *a* is richer than *b*

The first conjunct can be interpreted: Bill Gates is not Canadian.
The second conjunct can be interpreted: Bill Gates is richer than all Canadians.
Since both conjuncts are true, the sentence is true on this interpretation.

c) $\sim Fa \wedge \forall x(Fx \rightarrow G(ax))$

Universe: positive integers — F^1: *a* is odd

a^0: 2 — G^2: *a* is smaller than *b*

The first conjunct can be interpreted: 2 is not odd.
The second conjunct can be interpreted: 2 is smaller than all odd numbers.
Since the second conjunct is false, the sentence is false on this interpretation.

7.3 EG2 On the interpretations below, is the following true or false?

a) $\forall x(Fx \rightarrow \forall y(G(xy) \rightarrow H(yx))$

Universe: People — F1: a is a politician

H2: a believes b — G2: a makes a promise to b

Every politician is such that if he or she makes a promise to somebody, then that person believes the politician.

OR … Everybody believes every politician that makes them a promise.

This sentence is false!

b) $\forall x(Fx \rightarrow \forall y(G(xy) \rightarrow H(yx))$

Universe: Positive Integers

F^1: a is even

H^2: a is less than or equal to b

G^2: a is a factor of b

The sentence can be interpreted:

All even numbers are such that if the even number is a factor of any number then that number is less then or equal to the even number. This is false.

c) $\forall x(Fx \rightarrow \forall y(G(xy) \rightarrow H(yx))$

Universe: positive integers

F^1: a is purple

G^2: a is less than b

H^2: a is a multiple of b

The sentence can be interpreted: All purple positive integers are such that if the purple integer is less than any number then that number is a multiple of it. It is true because there are no purple positive integers.

7.4 E1:

Show that the following sentences are contingent by providing, for each, an interpretation on which it is true and one on which it is false:

NOTE: THERE ARE AN INFINITE NUMBER OF RIGHT ANSWERS!
After each example answer (for the first few questions) are the conditions that any right answer must meet. Check yours against the conditions.

a) $Fa \leftrightarrow \sim Ga$

True: U: unrestricted F^1: a is female. G^1: a is male a: Anna

'a' has to have property F or property G but not both.

False: U: positive integers F^1: a is even. G^1: a is divisible by two. a: 4

'a' has to have, or fail to have, both properties F and G.

b) $(Fa \rightarrow \sim Ga) \wedge (Fb \rightarrow \sim Gb)$

True: U: unrestricted F^1: *a* is female. G^1: *a* is male. a: Anna b: Betty

Both 'a' and 'b' must fail to have at least one of the two properties, F and G.

False: U: positive integers F^1: *a* is even. G^1: *a* is divisible by three. a: 4 b: 12

Either 'a' or 'b' must have both properties, F and G.

c) $L(bc) \wedge L(cb) \rightarrow \sim L(bb)$

True: U: positive integers — L^2: *a* is larger than *b* — b: 3 — c: 2

Either b does not stand in relation L to itself, or (b,c) and (c,b) aren't both in the extension of L.

False: U: positive integers — L^2: *a* is equal to *b* — b: 3 — c: 3

(b,c), (c,b) and (b,b) are all in the extension of L.

d) $\exists x(Gx \wedge \sim Fx)$

True: U: unrestricted — F^1: *a* is a dog. — G^1: *a* is a mammal.

Something has property G but not F.

False: U: positive integers — F^1: *a* is greater or equal to 1 — G^1: *a* is even

All things with property G also have property F.

e) $\forall x(Fx \rightarrow \sim Gx)$

True: U: positive integers — F^1: *a* is odd — G^1: *a* is even

Either nothing has property F, or nothing with property F has property G.

False: U: unrestricted — F^1: *a* is human. — G^1: *a* is male.

Something has property F and property G.

f) $\forall x\forall y\forall z(L(xy) \wedge L(yz) \rightarrow L(xz))$

True: U: positive integers — L^2: *a* is smaller or equal to *b*

L is transitive.

False: U: positive integers — L^2: *a* is two less than *b*

L is not transitive.

g) $\forall x(Fx \rightarrow \exists y(Gy \wedge M(xy)))$

True: U: positive integers — F^1: *a* is odd — G^1: *a* is even — M^2: *a* is less than *b*

Either nothing has property F, or everything that has property F and stands in the M relation to at least one thing with property G.

False: U: positive integers — F^1: *a* is odd — G^1: *a* is even — M^2: *a* is larger than *b*

Something with property F doesn't stand in the M relation to anything with property G.

7.4 E2 Show that the following sentences are not logical truths:

a) $\forall x\exists yF(xy) \rightarrow \exists y\forall xF(xy)$

U: positive integers F^2: *a* is less than *b*.

All things stand in the F relation to something or another AND nothing is such that everything stands in the F relation to it.

b) $\exists x(Fx \rightarrow \exists y(Fy \wedge L(xy)) \rightarrow \forall x(Fx \rightarrow \exists y(Fy \wedge L(xy))$

U: positive integers F^1: *a* is odd L^2: *a* is larger than than *b*

Something is F and stands in the L relation to something that is F AND something that is F doesn't stand in the L relation to anything that is F.

c) $\sim\exists xGx \rightarrow \forall y(F(yy) \rightarrow Gy)$

U: positive integers G^1: *a* is negative F^2: *a* is equal to *b*

Nothing is G AND something that is not G stands in the F relation to itself.

d) $\exists x(Bx \wedge \forall y{\sim}L(xy)) \vee {\sim}\forall x\exists y(L(xy) \rightarrow {\sim}Bx)$

U: positive integers B^1: *a* is odd. L: *a* is smaller than *b*.

All B's stand in the L relation to something AND all thing that stand in the L relation to everything fail to be B.

7.4 E3:

a) Show that the following sentences are not logical falsehoods:

a) $\forall x\exists yF(xy) \rightarrow \forall x\exists y{\sim}F(xy)$

U: positive integers F^2: *a* is equal to *b*.

Either nothing is in the F relation to anything OR everything is in the F relation to something and fails to be in the F relation to something.

b) $\exists x({\sim}Fx \wedge Gx) \wedge \forall x(Fx \rightarrow Gx)$

U: positive integers F^1: *a* is a multiple of 4. G^1: *a* is even.

Something is G but not F, but all F's are G.

c) $\forall x(Gx \rightarrow {\sim}\exists yL(xy)) \rightarrow \forall x({\sim}Gx \wedge \exists yL(xy))$

U: positive integers G^1: *a* is negative L^2: *a* is smaller than *b*

Either not all G's are in the L relation to nothing OR all non-G's are in the L relation to something. (If nothing is G, then all things must be in the L relation to something.)

d) $\forall x(Bx \rightarrow \exists yC(xy)) \wedge \forall y\forall x(C(yx) \rightarrow \sim By)$

U: Positive Integers. B^1: *a* is negative. C^2: *a* is greater than *b*.

All B's are in the C relation to something AND nothing that stands in the C relation to anything is B.

7.4 E4

Provide an interpretation that shows that the following arguments are invalid:

a) $\forall x(Fx \rightarrow \sim Gx)$. $\forall y(Hy \leftrightarrow \sim Gy)$. $\therefore \forall z(Hz \rightarrow Fz)$

U: integers F^1: *a* is purple H^1: *a* is odd G^1: *a* is divisible by two
U: animals F^1: *a* is a mammal H^1: *a* is a snake. G^1: *a* is a bird.
No F's are G. Nothing H is G and nothing G is H. Something is H and not F.

b) $\forall x\exists y(Fx \rightarrow (Gy \wedge L(xy)))$. $\forall x\exists y(Gx \rightarrow (Fy \wedge L(xy)))$. $\exists x\forall y(Fx \wedge (Gy \rightarrow \sim L(yx)))$
$\therefore \exists xL(xx)$

U: positive integers F^1: *a* is odd G^1: *a* is even L^2*: a* is smaller than *b*

All F's are in the L relation to some G. All G's are in the L relation to some L. Some F is such that no G is in the L relation to it. Nothing is in the L relation to itself.

c) $\exists x\forall y(Bx \wedge A(xy))$. $\forall x\exists y\ (Fx \rightarrow \sim A(xy))$. $\therefore \forall x(Fx \rightarrow \sim A(xx))$

U: Positive Integers. B^1: *a* is odd. F^1: *a* is even. A^2: *a* is smaller or equal to *b*.
Some B is in the A relation to everything. All F's fail to be in the A relation to something. Some F is in the A relation to itself.

7.4 E5

Provide an interpretation that shows that the set of sentences tautological implies the final sentence

i) $\{\ \forall x(Fx \rightarrow Gx),\ \ \forall y(Hy \rightarrow \sim Fy)\ \}$

$\exists z(Gz \wedge \sim Hz)$

U: Positive integers F^1: *a* is female $G^{1:}$ *a* is negative H^1: *a* is purple

All female positive integers are negative. No purple positive integer is female. (Both true, since no numbers are female or purple.) Therefore some positive integer is both negative and is not purple (false!)

ii) $\{ \exists x \forall y(H(xy) \vee J(xy)), \quad \exists x \forall y{\sim}H(xy) \}$

$\exists x \forall y J(xy)$

U: positive integers H^2: *a* times *b* is *b*. J^2: *a* is larger than *b*

Some positive integer, x, is such that any number, y, is such that either x times y is y or x is larger than y. (True: 1 satisfies this). Some number is not such that that number times any other number is that number (true: 2 satisfies this.) Therefore, some number is larger than all numbers. (False.)

7.4 E6

Provide an interpretation that shows that the following sets of sentences are consistent:

a) $\forall x({\sim}Fx \to Gx)$. $\forall y(Hy \leftrightarrow Gy)$. $\exists z(Hz \wedge Fz)$ ${\sim}\forall xFx$

U: positive integers F^1: *a* is odd G^1: *a* is even H^2: *a* is divisible by 2.

b) $\exists x(Fx \wedge \forall y(Gy \to L(xy)))$. $\forall y(Gy \vee Fy \to {\sim}L(yy))$. ${\sim}\exists z(Gz \wedge \forall y(Fy \to L(zy))$

U: positive integers F^1: *a* is odd G^1: *a* is even L^2: *a* is smaller than *b*

c) $\forall x(Fx \to Gx)$. $\forall x(Gx \to Hx \vee Jx)$. ${\sim}\exists x(Hx \wedge {\sim}Kx)$ $\forall x(Fx \to ({\sim}Kx \wedge {\sim}Hx))$

U: positive integers. F^1: a is negative. G^1: a is divisible by 3. H^1: a is even.
K^2: a is divisible by 2. J^1: a is odd.

d) $\exists x(Fx \wedge \forall y{\sim}H(xy))$. $\exists x(Gx \wedge \exists yH(xy))$. $\exists x(Fx \wedge Gx)$.
U: Positive Integers. F^1: a is odd. G^1: a is prime. H^2: a is greater than b.

7.4 E7

Provide English language interpretations that show that the following pairs of sentences are not equivalent:

a) $\forall x(Fx \wedge Gx \to \exists yH(yx))$ $\forall x(Fx \wedge (Gx \to \exists yH(yx)))$

U: positive integers F: *a* is even G: *a* is positive. H: *a* is larger than *b*.

The first sentence is true: For each positive even positive integer there is some number that is larger than it.

The second sentence is false: not all positive integers are even (but the second conjunct is true: if it is positive then some number is larger.)

b) $\forall x(Fx \rightarrow \sim\forall y(Gy \rightarrow H(xy)))$ $\sim\forall y(Gy \rightarrow \forall x(Fx \rightarrow H(xy)))$

U: positive integers F^1: *a* is odd G^1: *a* is even. H^1: *a* is less than *b*.

False: Every odd number is such that it is not the case that it is less than all even numbers.

(One is less than all even numbers.)

True: Not all even numbers are such that all odd numbers are less than it.

(Actually, there is no even number that all odd numbers are less than.)

c) $\exists x(Fx \wedge \forall y(Gy \rightarrow \exists z(Hz \wedge B(xyz))))$ $\forall y(Gy \rightarrow \exists x(Fx \wedge \exists z(Hz \wedge B(xyz))))$

U: integers F^1: *a* is even. G^1: *a* is odd. H^1: *a* is positive
B^3: *a* minus *b* equals *c*

The first sentence is false: there is no even number such that when you subtract any odd number from it, the result is a positive number. This is false because there is always an odd number that is bigger than the even number.

The second sentence is true: every odd number is such that there is some even number that you can subtract from it such that the result is a positive number. (This is true because there is always a negative even number whose absolute value is greater than that of the odd number, so when you subtract the negative even number, the result will be greater than 0.)

7.5 E1 On the interpretations below, is the following sentence true or false?

$\forall x(Fx \rightarrow \exists yG(xy))$

a)	UD: {0}	F^1: {0}	G^2: {(0,0)}	true
b)	UD: {0,1}	F^1: {0,1}	G^2: {(1,0), (1,1)}	false
c)	UD: {0,1}	F^1: {0,1}	G^2: {(0,0), (1,1)}	true
d)	UD: {0,1,2}	F^1: {1,2}	G^2: {(1,0), (1,2), (2,2)}	true

7.5 E2 Construct a finite model that shows that these sentences are not logical truths.

All you need is the finite model itself – but there are infinitely many possible answers.
I have also put a truth-functional expansion and one line truth table that proves it.

a) $\forall x(Fx \rightarrow \forall xFx)$

U:{0,1} F^1: {0} This is all that is necessary (the finite model).

(F0	→	(F0	∧	F1))	∧	(F1	→	(F0	∧	F1))
T	F	T	F	F	F	F	T	T	F	F

b) $\exists x(\sim Fx \wedge Gx) \wedge \forall x(Fx \rightarrow Gx)$

U:{0} F^1: {0} G^1: {0}

(~	F0	∧	G0)	∧	(F0	→	G0)
F	T	F	T	F	T	T	T

c) $\forall x\exists yF(xy) \rightarrow \forall x\exists y\sim F(xy)$

U:{0} F^2: {(0,0)}

F(00)	→	~	F(00)
T	F	F	T

7.5 E3 Construct a finite model that shows that these sentences are not logical falsehoods.

All you need is the finite model itself – but there are infinitely many possible answers.
I have also put a truth-functional expansion and one line truth table that proves it.

a) $\forall x(Fx \rightarrow \forall xFx)$

U:{0,1} F^1: {0,1}

(F0	$\rightarrow$	(F0	$\wedge$	F1))	$\wedge$	(F1	$\rightarrow$	(F0	$\wedge$	F1))
T	T	T	T	T	T	T	T	T	T	T

or with a smaller UD: U:{0} F^1: { }

b) $\exists x(\sim Fx \wedge Gx) \wedge \forall x(Fx \rightarrow Gx)$

U:{0,1} F^1: { } G^1: {0}

((~	F0	$\wedge$	G0)	$\vee$	(~	F1	$\wedge$	G1))	$\wedge$	((F0	$\rightarrow$	G0)	$\wedge$	(F1	$\rightarrow$	G1))
T	F	T	T	T	T	F	F	F	T	F	T	T	T	F	T	F

or with a smaller UD: U:{0} F^1: { } G^1: {0}

c) $\forall x\exists yF(xy) \rightarrow \forall x\exists y\sim F(xy)$

U:{0,1} F^2: { (10), (11) }

(((F(00)	$\vee$	F(01))	$\wedge$	(F(10)	$\vee$	F(11)))	$\rightarrow$	(((~	F(00)	$\vee$	~	F(01))	$\wedge$	((~	F(10)	$\vee$	~	F(11)))
F	F	F	F	T	T	T	T	T	F	T	T	F	F	F	T	F	F	T

or with a smaller UD: U:{0} F^2: { }

7.6 E1 Provide a truth-functional expansion for each of the following sentences, using the specified universe of discourse.

a) $\exists xB(xx) \leftrightarrow \forall y(Gy \rightarrow Hy)$ UD: {0}

$B(00) \leftrightarrow (G0 \rightarrow H0)$

b) $\exists xB(xx) \leftrightarrow \forall y(Gy \rightarrow Hy)$ UD: {0,1}

$(B(00) \vee B(11)) \leftrightarrow ((G0 \rightarrow H0)) \wedge (G1 \rightarrow H1))$

c) $\exists xB(xx) \leftrightarrow \forall y(Gy \rightarrow Hy)$ UD: {0,1,2}

$(B(00) \vee B(11) \vee B(22)) \leftrightarrow ((G0 \rightarrow H0)) \wedge (G1 \rightarrow H1) \wedge (G1 \rightarrow H1))$

d) $\forall x(Gx \rightarrow \exists y(Hy \wedge K(yx)))$ UD: {0,1}

$(G0 \rightarrow ((H0 \wedge K(00)) \vee (H1 \wedge K(10)))) \wedge (G1 \rightarrow ((H0 \wedge K(01)) \vee (H1 \wedge K(11)))$

e) $\forall x\exists y(Gx \rightarrow (Hy \wedge K(yx)))$ UD: {0,1}

$((G0 \rightarrow (H0 \wedge K(00))) \vee (G0 \rightarrow (H1 \wedge K(10)))) \wedge ((G1 \rightarrow (H0 \wedge K(01))) \vee (G1 \rightarrow (H1 \wedge K(11))))$

f) $\sim\exists x(Bx \wedge \forall y(D(xy) \leftrightarrow \sim D(yx)))$ UD: {0,1}

$\sim ((B0 \wedge ((D(00) \leftrightarrow \sim D(00)) \wedge (D(01) \leftrightarrow \sim D(10)))) \vee (B1 \wedge ((D(10) \leftrightarrow \sim D(01)) \wedge (D(11) \leftrightarrow \sim D(11))))$

7.6 E2 Provide a truth-functional expansion for each of the following sentences, using the specified universe of discourse.

a) $\forall xFx \leftrightarrow \exists yGy$ UD: {0}

$F0 \leftrightarrow G0$

b) $\exists x(Bx \wedge \forall y(Cy \rightarrow D(yx)))$ UD: {0}

$B0 \wedge (C0 \rightarrow D(00)))$

c) $\forall x(Bx \rightarrow \sim Cx)$ UD: {0,1}

$(B0 \rightarrow \sim C0) \wedge (B1 \rightarrow \sim C1)$

d) $\exists x(\sim Dx \wedge Ex)$ UD: {0,1}

$(\sim D0 \wedge E0) \vee (\sim D1 \wedge E1)$

e) $\sim\forall x(Fx \rightarrow L(xx)$ UD: {0,1}

$\sim((F0 \rightarrow L(00)) \wedge (F1 \rightarrow L(11)))$

f) $\exists x\exists yL(xy)$ UD: {0,1}

$(L(00) \vee L(01)) \vee (L(10) \vee L(11))$

g) $\forall x\exists yL(xy)$ UD: {0,1}

$(L(00) \vee L(01)) \wedge (L(10) \vee L(11))$

h) $\exists x\forall y(Fx \wedge L(xy))$ UD: {0,1}

$((F0 \wedge L(00)) \wedge (F0 \wedge L(01))) \vee ((F1 \wedge L(10)) \wedge (F1 \wedge L(11)))$

i) $\exists x (Fx \wedge \forall yL(xy))$ UD: {0,1}

$((F0 \wedge (L(00) \wedge L(01))) \vee ((F1 \wedge (L(10)) \wedge L(11)))$

j) $\sim\forall x\exists y(Gx \rightarrow \sim F(yx))$ UD: {0,1}

$\sim(((G0 \rightarrow \sim F(00)) \vee (G0 \rightarrow \sim F(10))) \wedge ((G1 \rightarrow \sim F(01)) \vee (G1 \rightarrow \sim F(11))))$

k) $\sim\forall x(Gx \rightarrow \sim\exists yF(yx))$ UD: {0,1}

$\sim((G0 \rightarrow \sim(F(00) \vee F(10))) \wedge (G1 \rightarrow \sim(F(01) \vee F(11))))$

l) $\forall x(Gx \rightarrow \sim L(xx))$ UD: {0,1,2}

$(G0 \rightarrow \sim L(00)) \wedge (G1 \rightarrow \sim L(11)) \wedge (G2 \rightarrow \sim L(22))$

m) $\forall xGx \rightarrow \exists yL(yy)$ UD: {0,1,2}

$(G0 \wedge G1 \wedge G2) \rightarrow (L(00) \vee L(11) \vee L(22))$

7.7 E1:

Provide a truth-functional expansion for the following arguments, and use it to create a finite model that shows that the following arguments are invalid:

NOTE: THERE ARE MANY CORRECT ANSWERS TO EACH ONE.

a) ∀x(Fx → (Gx ∧ Hx)) . ∃x(~Fx ∧ Gx) ∧ ∃yFy. ∴ ∀x(~Hx ∨ Fx)

U: {0,1} **F^1:{1}** **G^1:{0,1}** **H^1: {0,1}**

(F0	→	(G0	∧	H0))	∧	(F1	→	(G1	∧	H1))
F	T	T	T	T	**T**	T	T	T	T	T

((~	F0	∧	G0)	∨	(~	F1	∧	G1))	∧	(F0	∨	F1)
T	F	T	T	T	F	T	F	T	**T**	F	T	T

(~	H0	∨	F0)	∧	(~	H1	∨	F1)
F	T	F	F	**F**	F	T	T	T

b) ∃xFx. ∃xGx. ∃xHx. ∀x((Fx ∧ Gx) ∨ (Fx ∧ Hx) ∨ (Gx ∧ Hx) → Jx). ∴ ∃xJx

U:{0,1,2} **F^1:{0}** **G^1:{1}** **H^1:{2}** **J:{ }**

F0	∨	F1	∨	F2	G0	∨	G1	∨	G2	H0	∨	H1	∨	H2
T							T							T

((F0	∧	G0)	∨	(F0	∧	H0)	∨	(G0	∧	H0)	→	J0)	∧
T	F	F	F	T	F	F	F	F	F	F	T	F	T

((F1	∧	G1)	∨	(F1	∧	H1)	∨	(G1	∧	H1)	→	J1)	∧
F	F	T	F	F	F	F	F	T	F	F	T	F	T

((F2	∧	G2)	∨	(F2	∧	H2)	∨	(G2	∧	H2)	→	J2)
F	F	F	F	F	F	T	F	F	F	T	T	F

c) $\exists x(Fx \land Kx)$. $\forall x(Jx \leftrightarrow Kx)$. $\sim\forall x(\sim Fx \land Kx)$. $\therefore \forall x(Jx \to Fx)$

U: {0,1} F^1:{0} J^1: {0,1} K^1: {0,1}

(F0	$\land$	K0)	$\lor$	(F1	$\land$	K1)
T	T	T	**T**	F	F	T

(J0	$\leftrightarrow$	K0)	$\land$	(J1	$\leftrightarrow$	K1)
T	T	T	**T**	T	T	T

~	((~	F0	$\land$	K0)	$\land$	(~	F1	$\land$	K1))
T	F	T	F	T	F	T	F	T	T

$\therefore$	(J0	$\to$	F0)	$\land$	(J1	$\to$	F1)
	T	T	T	**F**	T	F	F

d) $(\sim\exists yFy \to \exists yFy) \lor \sim Fa$. $\therefore \exists xFx$.

U:{0}   F1:{ } a:0

(~	F0	$\to$	F0)	$\lor$	~	F0
T	F	F	F	**T**	T	F

$\therefore$	**F0**
	F

with a larger UD: **U:{0,1}   F1:{ } a:0**

(~	(F0	$\lor$	F1)	$\to$	(F0	$\lor$	F1))	$\lor$	~	F0
T	F	F	F	F	F	F	F	**T**	T	F

$\therefore$	(F0	$\lor$	F1)
	F	**F**	F

e) $Fa \land \exists yG(ya)$. $Fb \leftrightarrow \exists y\sim G(yb)$. $\therefore \exists yG(by)$

UD: {0,1} F^1: {0,1} G^2: {(00)}   a: 0   b: 1

F0	$\land$	(G(00)	$\lor$	G(10))
T	T	T		F

F1	$\leftrightarrow$	(~	G(01)	$\lor$	~	G(11))
T	T	T	F	T	T	F

$\therefore$	(G(10)	$\lor$	G(11))
	F	F	F

f) $\forall x \forall y(L(xy) \rightarrow H(xy))$. $\therefore \forall x \forall y(L(xy) \rightarrow (H(xy) \wedge H(yx))$

U:{0,1} **L^2: {(00), (01), (11)}** **H^2: {(00), (01), (11)}**

((L(00)	→	H(00))	∧	(L(01)	→	H(01)))	∧	((L(10)	→	H(10))	∧	(L(11)	→	H(11)))
T	T	T	T	T	T	T	T	F	T	F	T	T	T	T

∴	(L(00)	→	H(00)	∧	H(00))	∧	(L(01)	→	H(01)	∧	H(10))	∧
	T	T	T	T	T	F	T	F	T	F	F	F

(L(10)	→	H(10)	∧	H(01))	∧	(L(11)	→	H(11)	∧	H(11))
F	T	F	F	T	T	T	T	T	T	T

g) $\exists x \forall y(Gx \wedge L(xy))$. $\exists y L(yy) \rightarrow \forall x(Gx \rightarrow Hx)$ $\therefore \exists x(Hx \rightarrow \forall y L(yx))$

U: {0,1} **G^1:{0}** **H^1: {0,1}** **L^2:{(00), (01)}**

((G0	∧	L(00))	∧	(G0	∧	L(01)))	∨	((G1	∧	L(10))	∧	(G1	∧	L(11)))
T	T	T	T	T	T	T	T	F	F	F	F	F	F	F

(L(00)	∨	L(11))	→	((G0	→	H0)	∧	(G1	→	H1))
T	T	F	T	T	T	T	T	F	T	T

∴	(H0	→	(L(00)	∧	L(10)))	∨	(H1	→	(L(01)	∧	L(11)))
	T	F	T	F	F	F	T	F	T	F	F

h) $\forall x \exists y \forall z(F(zy) \leftrightarrow Gz \wedge F(zx))$. $\therefore \exists x \forall y(F(yx) \leftrightarrow Gy)$

U: {0,1} **G^1:{0, 1}** **F^2:{(1,0), (1,1)}**

$((F(00) \leftrightarrow G0 \wedge F(00)) \wedge (F(10) \leftrightarrow G1 \wedge F(10))) \vee ((F(01) \leftrightarrow G0 \wedge F(00)) \wedge (F(11) \leftrightarrow G1 \wedge F(10)))$

F T T F F T T T T T T T F T T F F T T T T T T

$\wedge$

THIS IS ALL ONE SENTENCE …

T

$((F(00) \leftrightarrow G0 \wedge F(01)) \wedge (F(10) \leftrightarrow G1 \wedge F(11))) \vee ((F(01) \leftrightarrow G0 \wedge F(01)) \wedge (F(11) \leftrightarrow G1 \wedge F(11)))$

F T T F F T T T T T T T F T T F F T T T T T T

$\therefore \exists x \forall y(F(yx) \leftrightarrow Gy)$

$((F(00) \leftrightarrow G0) \wedge (F(10) \leftrightarrow G1)) \vee ((F(01) \leftrightarrow G0) \wedge (F(11) \leftrightarrow G1))$

F F T F T T T F F F T F T T T

7.7 E2

7.7 E2

Provide a truth-functional expansion for the following sentences and use it to create a finite model that show the property indicated:

a) $\exists x \exists y(Fx \wedge Gy) \wedge \sim\exists x(Fx \leftrightarrow Gx)$ Show this is not a logical falsehood.

$((F0 \wedge G0) \vee (F0 \wedge G1)) \vee ((F1 \wedge G0) \vee (F1 \wedge G1)) \wedge \sim((F0 \leftrightarrow G0) \vee (F1 \leftrightarrow G1))$

T F F T T T T T F F F F F F T T T T F F F F F T

U: {0,1} **F^1:{0}** **G^1:{ 1}**

b) $\forall x\ (Fx \wedge \exists yL(xy)) \wedge \sim\exists y \forall xL(yx))$ Show this is not a logical falsehood.

$((F0 \wedge (L(0,0) \vee L(0,1))) \wedge (F1 \wedge (L(1,0) \vee L(1,1)))) \wedge \sim((L(0,0) \wedge L(1,0)) \vee (L(0,1) \wedge L(1,1)))$

T T T T F T T T F T T T T T F F F F F T

U: {0,1} **F^1: {0,1}** **L^2: {(0,0, (1,1)}**

c) $\forall x(Fx \rightarrow \exists yL(xy)) \leftrightarrow \exists y \forall x(\sim Fx \vee L(xy))$ Show this is not a logical truth.

$((F0 \rightarrow (L(0,0) \vee L(0,1)) \wedge (F1 \rightarrow (L(1,0) \vee L(1,1))) \leftrightarrow$

T F F T T T T T T T F F

$(((\sim F0 \vee L(0,0)) \wedge (\sim F1 \vee L(1,0)) \vee (((\sim F0 \vee L(0,1)) \wedge (\sim F1 \vee L(1,1)))$

F T F F F F T T F F T T T F F T F F

U: {0,1} **F^1: {0,1}** **L^2: {(0,1), (1,0)}**

d) $\forall x(Fx \rightarrow \exists y(Gy \wedge A(xy))) \rightarrow \forall x(Gx \rightarrow \exists y(Fy \wedge A(yx)))$ Show this is not a logical truth.

$((F0 \rightarrow ((G0 \wedge A(0,0)) \vee (G1 \wedge A(0,1))) \wedge (F1 \rightarrow ((G0 \wedge A(1,0)) \vee (G1 \wedge A(1,1)))) \rightarrow$

F T T F F F F F F T F T T F F F F F F F

$((G0 \rightarrow (F0 \wedge A(0,0)) \vee (F1 \wedge A(1,0))) \wedge (G1 \rightarrow (F0 \wedge A(0,1)) \vee (F1 \wedge A(1,1))))$

T F F F F F F F F F F T F F F F F F F

U: {0,1} **F^1: { }** **G^1: {0}** **A^2: { }**

e) $\forall xF(xa(x)) \rightarrow \exists xF(a(x)x)$ Show this is not a logical truth.

$F(0,1) \wedge F(1,2) \wedge F(2,0)) \rightarrow F(1,0) \vee F(2,1) \vee F(0,2)$

T T T T T **F** F F F F F

U: {0,1,2} $\mathbf{F^1}$: {(0,1) (1,2) (2,0)} $\mathbf{a^1}$: a(0)=1 a(1)=2 a(2)=0

f) $\forall x(\sim Gx \rightarrow Fx)$. $\forall y(Hy \leftrightarrow Gy)$.

$\exists z(Hz \wedge Fz)$ $\sim\forall xFx$ Show that these are consistent.

U:{0, 1} F^1: {0} G^1: {0,1} H^1: {0,1}

$(\sim G0 \rightarrow F0) \wedge (\sim G1 \rightarrow F1)$.

F T T T T F T T F

$(H0 \leftrightarrow G0) \wedge (H1 \leftrightarrow G1)$

T T T T T T T

$(H0 \wedge F0) \vee (H1 \wedge F1)$

T T T T T F F

$\sim(F0 \wedge F1)$

T T F F

g) $\exists x(Fx \wedge \forall y(Gy \rightarrow L(xy)))$. $\forall y(Gy \vee Fy \rightarrow \sim L(yy))$.

$\sim\exists z(Gz \wedge \forall y(Fy \rightarrow L(zy))$ Show that these are consistent.

U:{0, 1} F^1: {0} G^1: {1} L^2: {(0,1)}

$(F0 \wedge (G0 \rightarrow L(0,0)) \wedge (G1 \rightarrow L(0,1)))) \vee (F1 \wedge (G0 \rightarrow L(1,0)) \wedge (G1 \rightarrow L(1,1)))$.

T T F T F T T T T **T** F F F T F F T F F

$(G0 \vee F0 \rightarrow \sim L(0,0)) \wedge (G1 \vee F1 \rightarrow \sim L(1,1))$.

F T T T T F **T** T T T T T F

$\sim((G0 \wedge ((F0 \rightarrow L(0,0)) \wedge (F1 \rightarrow L(0,1))) \vee (G1 \wedge ((F0 \rightarrow L(1,0)) \wedge (F1 \rightarrow L(1,1))))$

T F F T F F F F T T F T F T F F F F T F

7.7 E3:

Use an abstract finite model to demonstrate the invalidity of each of the following arguments:

i) provide a truth-functional expansion (to two individuals) for each sentence in this argument with respect to the model

ii) define a model with a universe of two individuals that shows that this argument is invalid.

a) $\forall x(Fx \rightarrow \exists yG(xy))$. $\exists x\forall y{\sim}G(yx)$. $\therefore \exists x{\sim}Fx$

$((F0 \rightarrow (G(0,0) \vee G(0,1))) \wedge (F1 \rightarrow (G(1,0) \vee G(1,1)))$.

$({\sim}G(0,0) \wedge {\sim}G(1,0)) \vee ({\sim}G(0,1) \wedge {\sim}G(1,1))$

$\therefore\ {\sim}F0 \vee {\sim}F1$

Now set the conclusion F and the premises T and do a shortened truth table to generate the model.

Since the conclusion is F, F0 and F1 are both true. Then, setting the premises T, you will find that either: G(0,1)=T, G(1,1)=T, G(0,0)=F & G(1,0)=F OR G(0,0)=T, G(1,0)=T, G(0,1)=F & G(1,1)=F

So either of the following models are correct:

U:{0,1} F1:{0,1} G1:{(0,1), (1,1)}

or **U:{0,1} F1:{0,1} G1:{(0,0), (1,0)}**

b) $\forall x\exists y(F(xy) \rightarrow {\sim}F(yx))$. $\exists x(Gx \wedge \exists yF(xy))$ $\therefore\ {\sim}\exists x(Gx \wedge F(xx))$

$((F(0,0) \rightarrow {\sim}F(0,0)) \vee (F(0,1) \rightarrow {\sim}F(1,0))) \wedge ((F(1,0) \rightarrow {\sim}F(0,1)) \vee (F(1,1) \rightarrow {\sim}F(1,1)))$

$(G0 \wedge (F(0,0)) \vee F(0,1))) \vee\ (G1 \wedge (F(1,0)) \vee F(1,1)))$

$\therefore {\sim} ((G0 \wedge F(0,0)) \vee (G1 \wedge F(1,1)))$

Now make the two premises T and the conclusion false …and generate the model
(There are different correct solutions… here is one.)

U: {1,0}. G1:{0} F1:{(0,0),(0,1)}

another: **U: {1,0}. G1:{1} F1:{(1,1),(1,0)}**

c) $\sim\forall x\exists y\,(Fx \rightarrow \sim G(xy))$. $\forall x(\sim Hx \vee \exists y\sim G(xy))$. $\therefore\ \forall x(G(xx) \rightarrow \sim Hx)$

$\sim (((F0 \rightarrow \sim G(0,0)) \vee (F0 \rightarrow \sim G(0,1))) \wedge ((F1 \rightarrow \sim G(1,0)) \vee (F1 \rightarrow \sim G(1,1)))$

$((\sim H0 \vee (\sim G(0,0) \vee \sim G(0,1))) \wedge (\sim H1 \vee (\sim G(1,0) \vee \sim G(1,1)))$

$\therefore\ (G(0,0) \rightarrow \sim H0) \wedge (G(1,1) \rightarrow \sim H1)$

Now make the two premises T and the conclusion false …and generate the model
There are different correct solutions…

U: {0,1}. F1:{0} H1:{1} G1:{(0,0),(0,1),(1,1)}
U: {0,1}. F1:{1} H1:{0} G1:{(1,1),(1,0),(0,0)}

d) $\exists x\forall y(Gx \wedge L(xy))$. $\forall x(L(xx) \rightarrow \exists y\sim H(xy))$ $\therefore \exists x(Gx \wedge \sim\exists yH(yx))$

$(((G0 \wedge L(0,0)) \wedge (G0 \wedge L(0,1))) \vee ((G1 \wedge L(1,0)) \wedge (G1 \wedge L(1,1)))$

$(L(0,0) \rightarrow (\sim H(0,0) \vee \sim H(0,1))) \wedge (L(1,1) \rightarrow (\sim H(1,0) \vee \sim H(1,1)))$

$\therefore (G0 \wedge \sim(H(0,0) \vee H(1,0)) \vee (G1 \wedge \sim(H(0,1) \vee H(1,1))$

Now make the two premises T and the conclusion false …and generate the model
There are different correct solutions…

U: {0,1}. G1:{0} L2:{(0,0),(0,1)} H2:{(0,0)}
U: {0,1}. G1:{1} L2:{(1,1),(1,0)} H2:{(1,1)} (1,0)
U: {0,1}. G1:{0} L2:{(0,0),(0,1)} H2:{(0,0), (1,0), (1,1)} and more

e) $\exists x\forall y(Bx \wedge C(xy))$. $\forall x(Ax \rightarrow \exists y\sim C(xy))$. $\therefore\ \forall x(Ax \rightarrow \sim C(xx))$

$((B0 \wedge C(0,0)) \wedge (B0 \wedge C(0,1))) \vee ((B1 \wedge C(1,0)) \wedge (B1 \wedge C(1,1)))$

$(A0 \rightarrow (\sim C(0,0) \vee \sim C(0,1))) \wedge (A1 \rightarrow (\sim C(1,0) \vee \sim C(1,1)))$

$\therefore\ (A0 \rightarrow \sim C(0,0)) \wedge (A1 \rightarrow \sim C(1,1))$

Now make the two premises T and the conclusion false …and generate the model
There are different correct solutions…

U: {0,1}. A:{0} B:{1} C:{(0,0) (1,0) (1,1)}

U: {0,1}. A:{1} B:{0} C:{(0,0) (0,1) (1,1)}

f) $\sim\forall x\,(Gx \rightarrow \exists y B(xy))$. $\exists x \forall y (Hx \rightarrow B(xy))$. $\therefore\ \sim\exists x(Gx \wedge \exists y(Hy \wedge B(xy))$

$\sim((G0 \rightarrow (B(00) \vee B(01))) \wedge (G1 \rightarrow (B(10) \vee B(11)))$.
$((H0 \rightarrow B(00)) \wedge (H0 \rightarrow B(01))) \vee ((H1 \rightarrow B(10)) \wedge (H1 \rightarrow B(11)))$

$\therefore\ \sim(((G0 \wedge ((H0 \wedge B(00)) \vee (H1 \wedge B(01))) \vee ((G1 \wedge ((H0 \wedge B(10)) \vee (H1 \wedge B(11))))$

Now set the premises true and the conclusion false. Many solutions!

UD: {0,1} **G^1: {0}** **H^1: {1}** **B^2: {(10)}**
UD: {0,1} **G^1: {0,1}** **H^1: {0}** **B^2: {(01),(00)}**

7.8 E1: Explain why the following sentences are logically true:

a) $\forall xFx \vee \exists x{\sim}Fx$

The sentence is a disjunction, thus it is true if at least one disjunct is true. Consider any interpretation: the first disjunct must be true or false. If it is true the sentence is true on that interpretation. If it is false, then it is not the case that every member of the universe is in the extension of F, thus there is some member of the universe that is not in the extension of F – and that is what the second disjunct says. Therefore, on any interpretation that the first disjunct is false, the second disjunct is true. Thus the sentence is true on any interpretation and thus is a logical truth.

b) $\forall x(Fx \rightarrow \exists yGy) \leftrightarrow (\exists xFx \rightarrow \exists yGy)$

This sentence is a biconditional, thus it is true if both sides have the same truth-value. Consider any interpretation: the left side must be true or false on that interpretation. If the left side is true then for every member of the universe, if it is in the extension of F then some member of U must be in the extension of G. Thus, if some member of U is in the extension of F it follows that some member of U must be in the extension of G – which is what the right side says. Thus the right side must be true if the left side is. The left side is a universal, so if it is false, the conditional $(Fx \rightarrow \exists yGy)$ must be false for at least one member of U. Thus, that member of U must be in the extension of F and there must be no member of U in the extension of G. Thus the right side must be false as well – some member of U is in the extension of F but no member of U is in the extension of G. Thus, on any interpretation the sentence is true – both sides have the same truth-value.

c) $\exists x(Fx \rightarrow \forall yGy) \leftrightarrow (\forall xFx \rightarrow \forall yGy)$

This sentence is a biconditional, thus it is true if both sides have the same truth-value. Consider any interpretation: the left side must be true or false on that interpretation. If the left side is true then the conditional $(Fx \rightarrow \forall yGy)$ is true for some member of the universe. The conditional is true if the antecedent is false or the consequent is true: so it is true if that member of U is not in the extension F or if every member of U is in the extension of G. If so, the right side is also true: for it says that if every member of U is in the extension of F then every member of U is in the extension of G. If some member of U is not in the extension of F, then the antecedent on the right is false and the conditional on the right is true. If every member of U is in the extension of G then the consequent on the right is true and the sentence on the right is true. Now consider the case in which the sentence on the left is false. If so, there is no member of U for which the conditional $(Fx \rightarrow \forall yGy)$ is true. So it is false for every member of U: everything is in the extension of F but not everything is in the extension of G. This makes the right side false as well: for the right side says that if everything is in the extension of F then every member of U is in the extension of G. Thus, on any interpretation the sentence is true – both sides have the same truth-value. Thus it is a logical truth.

d) $\forall x\forall yF(xy) \rightarrow \forall xF(xx)$

This sentence is a conditional, thus it is true if the antecedent is false or the consequent is true. Consider any interpretation: either it is the case that the antecedent is false or it is the case that the antecedent is true. In the first case, the whole sentence is true. In the second case, then it is true that every member of U is in the F relation to every member of U. Thus, there every member of U must stand in the F relation to itself, which is what the right side says. Thus, if the left side is true, the right side must be true as well. Thus, the sentence is true on every interpretation and so is logically true.

e) $\exists x(Fx \rightarrow \forall y{\sim}Gy) \rightarrow (\forall xFx \rightarrow {\sim}\exists yGy)$

A logical truth is true on every interpretation. This sentence is a conditional, so on every interpretation the antecedent must be false or the consequent must true. Suppose the antecedent is true for some interpretation: in that case, there is some member of the universe of discourse such that if it is in the extension of F then all members of the universe fail to be in the extension of G. So, there is some member of the universe that is not in the extension of F or nothing is in the extension of G. So, if every member of the universe is in the extension of F then nothing is in the extension of G. But, that is what the consequent says. Thus, if the antecedent is true for some interpretation, then consequent must be true as well. Therefore, it is logically true since it is true on every interpretation.

Explain why the following sentences are logically false:

f) $\forall x{\sim}(Fx \vee Gx) \wedge \exists x({\sim}Fx \rightarrow Gx)$

This sentence is a conjunction, thus it is true if both conjuncts are true. Consider any interpretation: the first conjunct must be true or false on that interpretation. If it is false, the whole sentence is false. Now, consider the case in which the conjunct on the left is true: every member of U is such that it is neither in the extension of F nor is it in the extension of G. If nothing is in the extension of F or G, then the second conjunct must be false. The second conjunct states that there is some member of U such that if it is not in the extension of F then it is in the extension of G. Every member of U makes the antecedent on the right true (it is not in the extension of F) and the consequent on the right false (it is not in the extension of G.) Thus, in this case, there is no member of U for which the second conjunct is true. So if the first conjunct is true, the second must be false. Thus, the sentence is false on every interpretation and is logically false.

g) $\forall x(Fx \rightarrow Gx) \wedge \exists x(Fx \wedge {\sim}Gx)$

This sentence is a conjunction, thus it is true if both conjuncts are true. Consider any interpretation on which the first conjunct is true: every member of the U is such that if it is in the extension of F then it is also in the extension of G. But, if that is the case the second conjunct is false, since no member of the UD is in the extension of F but not in the extension of G. Since the second conjunct is false on any interpretation on which the first conjunct is true, the sentence is false on every interpretation and thus it is logically false.

h) $\forall x\exists yF(xy) \leftrightarrow \exists x\forall y{\sim}F(xy)$

This sentence is a biconditional. It is false if the two sides have different truth-values. Consider any interpretation: the left side is true or false on that interpretation. If the left side is true, then every member or U stands in the F relation to some member of U (or every member of U is in the first place of an ordered pair that is in the extension of F.) In this case, there is not some member of U that fails to stand in the F relation to every member of U (or there is no member of U that fails to be in the first place of an ordered pair in the extension of F, with any member of U in the second place.) Thus, the right side is false if the left side is true. Now consider the case in which the left side is false. It is not the case that every member of U stands in the F relation to some member of U (it is not the case that every member of U is in the first place of an ordered pair that is in the extension of F). Thus, there is some member of U that fails to stand in the F relation to every member of U (or there is some member of U that is not in the first place of any ordered pair in the extension of F) – which is what the right side says. Thus, if the left side is false, the right side is true. Thus, this biconditional is false on any interpretation and so is logically false.

i) $\exists x \forall y(Fx \wedge \sim G(xy)) \wedge \forall x(Fx \rightarrow \exists y G(xy))$

If the sentence is a logical falsehood, then it is false on any interpretation. Since it is a conjunction, at least one conjunct must be false on each interpretation. Consider an interpretation on which the first conjunct is true. Thus, there is some member of the universe of discourse that is in the extension of F and that stands in the G relation to no member of the universe. If that is true, then it is false that all things in the extension of F stand in the G relation to some member of the universe. Thus, the second conjunct is false if the first conjunct is true. Hence, the sentence is false for every interpretation.

Explain why the following arguments are valid:

j) $\forall x(Fx \rightarrow Gx)$. $\exists x \sim Gx$. $\therefore \sim \forall x Fx$.

Consider any interpretation on which the two premises are true. The second premise is an existential statement, so if it is true, there is some member of U that fails to be in the extension of G. The first premise states that every member of U in the extension of F is also in the extension of G. Thus, the member of U that is not in the extension of G must also fail to be in the extension of F (or else the first premise would not be true!) Thus, not all members of U are in the extension of F – that is what the conclusion states. Thus, on any interpretation on which the premises are true, so is the conclusion. Thus, this argument is valid.

k) $\forall x \sim (Fx \wedge Gx)$. $\therefore \exists x \sim Fx \vee \exists x \sim Gx$

Consider any interpretation on which the premise is true. Every member of U is not both in the extension of F and in the extension of G. Thus, every member of U either fails to be in the extension of F or it fails to be in the extension of G. Thus, there is at least one member of U that is not in the extension of F or there is at least one member of U that is not in the extension of G. That what the conclusion states. Thus, on any interpretation on which the premise is true, so is the conclusion. Thus, this argument is valid.

l) $\exists x \forall y L(xy)$ $\therefore \forall y \exists x L(xy)$

Consider any interpretation on which the premise is true. There is some member of U that stands in the L relation to every member of U. (Thus, there is some member of U such that in the extension of L is every ordered pairs in which that member of U is in the first place and any member of U is in the second place.) Thus, every member of U is such that some member of U stands in the L relation to it. (Thus, every member of U is in the second place of an ordered pair within the extension of L.) But, that what the conclusion states. Thus, on any interpretation on which the premise is true, so is the conclusion. Thus, this argument is valid.

m) $\exists x(Fx \wedge \forall y L(xy))$ $\therefore \forall x \exists y(Fy \wedge L(yx))$

If the argument is valid, then on any interpretation on which the premise is true, the conclusion must be true as well. Consider any interpretation on which the premise is true. Then, there is some member of the universe of discourse that is in the extension of F and that stands in the L relation to all members of the universe. If that is true, then every member of the universe is such that there is something in the extension of F that stands in the L relation to it (namely, that which stands in the L relation to all!) Thus, the conclusion is true on any interpretation on which the premise is true. The argument is valid.