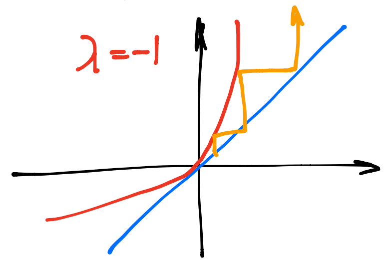
Lecture 11

Example: Let $E_\lambda(x) = e^x + \lambda$ (exponential family)

To find the fixed points, we solve
$E_\lambda(x) = x \iff e^x + \lambda = x$

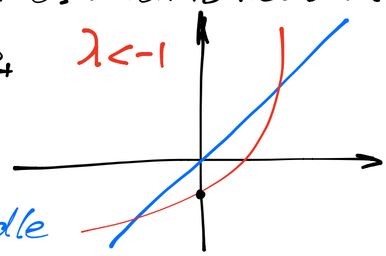
- when $x=0$, $(e^x)' = 1$. Which means that if $x=0$ is a fixed point, it is neutral.



- Find λ such that $x=0$ is a fixed point
  $\lambda = -1$

- For $\lambda < -1$, $e^x + \lambda > x$
  so there are no fixed points

- For $\lambda = -1$, there is one fixed points at $x=0$, which is neutral.



- For $\lambda < -1$, there are 2 fixed pts. $P_- < 0 < P_+$
  $\Rightarrow E'_\lambda(P_-) = e^{P_-} < 1$ attracting
  $\Rightarrow E'_\lambda(P_+) = e^{P_+} > 1$ repelling

So The Exponential family $E_\lambda(x)$ has a saddle-node bifurcation at $\lambda_0 = -1$.

Example: let $F_\lambda(x) = \lambda x(1-x)$ called the logistic family.
We find the fixed points:
$$\lambda x(1-x) = x \Rightarrow x(\lambda(1-x)-1) = 0$$
$$x(\lambda - 1 - \lambda x) = 0 \Rightarrow x = 0 \text{ or } x = 1 - 1/\lambda$$
- when $\lambda = 1$, we have 1 fixed point
- when $\lambda \neq 1$, there are 2 fixed pts. $\Rightarrow F_\lambda$ does not have a bifurcation.

§ 6.3 Period-doubling Bifurcation

Definition: A one parameter family $F_\lambda(x)$ has period-doubling bifurcation at λ_0 in the open interval I if there exists $\varepsilon > 0$ s.t.

(i) for each $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, there is a unique fixed point P_λ for F_λ in I

(ii) For all λ in one half of the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ including λ_0, F_λ has no 2-cycles and P_λ is attracting (RESP. repelling)

(iii) For all λ in the other half of $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ excluding λ_0, F_λ has a unique 2-cycle, $q^1_\lambda, q^2_\lambda \in I$ with $F_\lambda(q^1_\lambda) = q^2_\lambda$ and $F_\lambda(q^2_\lambda) = q^1_\lambda$
This cycle is attracting (RESP. repelling)
And P_λ is repelling (Resp. attracting)

(iv) As $\lambda \to \lambda_0$ (from the "other" half)

$$q^1_\lambda \longrightarrow P_{\lambda_0}$$