CSC336 Tutorial 7 – Interpolation

QUESTION 1 *Construct a polynomial of degree at most* 2 *that interpolates* $(0, 1)$, $(1, 3)$, $(3, 13)$. *Is it unique?*

Solution: Let us use monomial basis functions. We write the polynomial of degree at most 2 as

$$p_2(x) = a_0 + a_1x + a_2x^2.$$

The interpolating conditions are

$p_2(0) = 1 \Rightarrow a_0 + a_1 \times 0 + a_2 \times 0^2 = 1 \Rightarrow a_0 = 1$

$p_2(1) = 3 \Rightarrow a_0 + a_1 \times 1 + a_2 \times 1^2 = 3 \Rightarrow a_0 + a_1 + a_2 = 3$

$p_2(3) = 13 \Rightarrow a_0 + a_1 \times 3 + a_2 \times 3^2 = 13 \Rightarrow a_0 + 3a_1 + 9a_2 = 13$

or, in matrix form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 13 \end{bmatrix}.$$

The solution to this system is $a_0 = 1, a_1 = 1, a_2 = 1$, i.e.

$$p_2(x) = 1 + x + x^2.$$

The polynomial $p_2(x)$ is the unique polynomial of degree at most 2 that interpolates

the (three) given data, because, according to the uniqueness of polynomial interpolant theorem, there is a unique polynomial of degree at most n which interpolates $n + 1$ data with distinct abscissae.

QUESTION 2 *Construct a polynomial of degree at most* 1 *that interpolates* $(0, 1)$, $(1, 3)$, $(3, 13)$ *(same data with previous question 1), if it exists.*

Solution: Again, we use monomial basis functions. We write the polynomial of degree at most 1 as

$$p_1(x) = a_0 + a_1x.$$

The interpolating conditions are

$p_1(0) = 1 \Rightarrow a_0 + a_1 \times 0 = 1$

$p_1(1) = 3 \Rightarrow a_0 + a_1 \times 1 = 3$

$p_1(3) = 13 \Rightarrow a_0 + a_1 \times 3 = 13$

or, in matrix form,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 13 \end{bmatrix}.$$

This is an overdetermined linear system that has no solution. Notice that the equations are inconsistent (e.g. $a_0 = 1, a_1 = 2, 1+2\times 3 \neq 13$). Therefore, there is no polynomial of degree 1 or less that interpolates the above data.

(We can find the linear polynomial that fits the above data in the least squares sense, but techniques to construct such a polynomial are not taught in this course. Furthermore, this polynomial will *not interpolate* the data.)

QUESTION 3 *Construct a polynomial of degree at most* 3 *that interpolates* $(0, 1)$, $(1, 3)$, $(3, 13)$. *Is it unique?*

Solution: Using monomial basis functions, we write a polynomial of degree at most 3 as

$$p_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

The interpolating conditions are

$p_3(0) = 1 \Rightarrow a_0 + a_1 \times 0 + a_2 \times 0^2 + a_3 \times 0^3 = 1 \Rightarrow a_0 = 1$

$p_3(1) = 3 \Rightarrow a_0 + a_1 \times 1 + a_2 \times 1^2 + a_3 \times 1^3 = 3 \Rightarrow a_0 + a_1 + a_2 + a_3 = 3$

$p_3(3) = 13 \Rightarrow a_0 + a_1 \times 3 + a_2 \times 3^2 + a_3 \times 3^3 = 13 \Rightarrow a_0 + 3a_1 + 9a_2 + 27a_3 = 13$

or, in matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 13 \end{bmatrix}.$$

This is an underdetermined linear system that has infinitely many solutions. Although $a_0 = 1$, the remaining 2 equations for the 3 unknowns have infinitely many solutions.

We can write them in parametric form.

Let $\alpha = a_3$ be the (free) parameter. Note that $a_0 = 1$. Then we have

$$\begin{cases} a_1 + a_2 = 3 - \alpha - a_0 = 2 - \alpha \\ 3a_1 + 9a_2 = 13 - 27\alpha - a_0 = 12 - 27\alpha \end{cases}$$

$$\Rightarrow \begin{cases} 3a_1 + 3a_2 = 6 - 3\alpha \\ 3a_1 + 9a_2 = 12 - 27\alpha \end{cases}$$

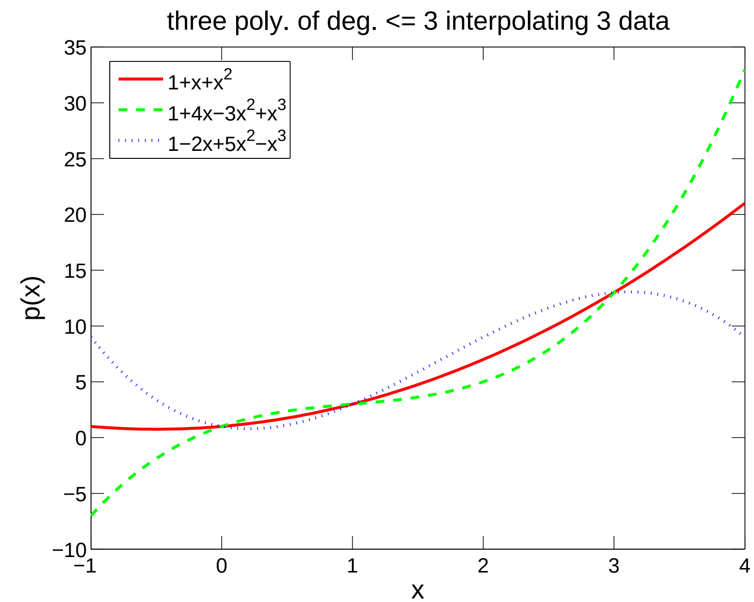
$$\Rightarrow \begin{cases} a_1 = 1 + 3\alpha \\ a_2 = 1 - 4\alpha \end{cases}$$

For each choice of α, we get a polynomial of degree at most 3 that interpolates the given data. For example:

Choosing $a_3 = \alpha = 0$ gives $a_1 = 1, a_2 = 1$. The polynomial $p_3(x) = 1 + x + x^2$ is of degree $2 < 3$. (This is the polynomial we obtained from the previous question).

Choosing $a_3 = \alpha = 1$ gives $a_1 = 4, a_2 = -3$. The polynomial $p_3(x) = 1 + 4x - 3x^2 + x^3$ is of degree 3.

Choosing $a_3 = \alpha = -1$ gives $a_1 = -2, a_2 = 5$. The polynomial $p_3(x) = 1 - 2x + 5x^2 - x^3$ is of degree 3.



Three different polynomials of degree ≤ 3 interpolating the same 3 data.

QUESTION 4 *Using (a) monomial, (b) Lagrange, and (c) Newton's Divided Differences (NDD) bases, construct a polynomial interpolant for the data*

x	y
0	*1*
1	*2*
3	*28*

Solution: Since $n = 2$, we choose the degree of the polynomial interpolant to be at most 2, so that we have a unique polynomial.

(a) Let

$$p_2(x) = a_0 + a_1 x + a_2 x^2.$$

The interpolating conditions are

$p_2(0) = 1 \Rightarrow a_0 + a_1 \times 0 + a_2 \times 0^2 = 1 \Rightarrow a_0 = 1$

$p_2(1) = 2 \Rightarrow a_0 + a_1 \times 1 + a_2 \times 1^2 = 2 \Rightarrow a_1 + a_2 = 1$

$p_2(3) = 28 \Rightarrow a_0 + a_1 \times 3 + a_2 \times 3^2 = 28 \Rightarrow 3a_1 + 9a_2 = 27$

or, in matrix form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 28 \end{bmatrix}.$$

The solution to this system is $a_0 = 1, a_1 = -3, a_2 = 4$, i.e.

$$p_2(x) = 1 - 3x + 4x^2.$$

(b) We have

$$p_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) = l_0(x) + 2l_1(x) + 28l_2(x),$$

where

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3),$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3),$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1).$$

Thus

$$\begin{aligned} p_2(x) &= 1 \times \frac{1}{3}(x-1)(x-3) - 2 \times \frac{1}{2}x(x-3) + 28 \times \frac{1}{6}x(x-1) \\ &= \frac{1}{3}(x^2-4x+3) - (x^2-3x) + \frac{14}{3}(x^2-x) \\ &= \frac{1}{3}(x^2 - 3x^2 + 14x^2 - 4x + 9x - 14x + 3) \\ &= \frac{1}{3}(12x^2 - 9x + 3) = 1 - 3x + 4x^2, \end{aligned}$$

same as in (a).

(c) Construct the NDD table for the data

x	y		
0	1		
		1	
1	2		4
		13	
3	28		

Thus

$$\begin{aligned} p_2(x) &= 1 + x + 4x(x-1) \\ &= 1 + x + 4x^2 - 4x \\ &= 1 - 3x + 4x^2 \end{aligned}$$

same as in (a) and (b).

Check:

$p_2(0) = 1 - 3 \times 0 + 4 \times 0^2 = 1$ (correct)

$p_2(1) = 1 - 3 \times 1 + 4 \times 1^2 = 2$ (correct)

$p_2(3) = 1 - 3 \times 3 + 4 \times 3^2 = 28$ (correct).

If we add one more data point, say $(4, 65)$, then the updated NDD table becomes

x	y			
0	1			
		1		
1	2		4	
		13		1
3	28		8	
		37		
4	65			

In this case, we aim to get a polynomial of degree ≤ 3:

$$\begin{aligned} p_3(x) &= 1 + x + 4x(x-1) + x(x-1)(x-3) \\ &= 1 + x + 4x^2 - 4x + x^3 - 4x^2 + 3x = x^3 + 1. \end{aligned}$$

Notice that the updated NDD table differs from the previous one in the lowest diagonal only. Also note that

$$\begin{aligned} p_3(x) &= p_2(x) + x(x-1)(x-3) \\ &= 1 - 3x + 4x^2 + x^3 - 4x^2 + 3x = x^3 + 1. \end{aligned}$$

If we consider the above data in a different order such as

x	y			
0	1			
		1		
1	2		5	
		21		1
4	65		8	
		37		
3	28			

we will have

$$\begin{aligned} p_3(x) &= 1 + x + 5x(x-1) + x(x-1)(x-4) \\ &= 1 + x + 5x^2 - 5x + x^3 - 5x^2 + 4x \\ &= x^3 + 1. \end{aligned}$$

same as before and this agrees with theory.

Moral: The x_i's can be in any order.

Important note: To check if the interpolating polynomial is correct, we just need to check if $p_*(x_i) = y_i$.

QUESTION 5 *Construct the least degree polynomial that interpolates the data*

x	*0*	*1*	*3*	*4*
y	*1*	*2*	*10*	*17*

Solution: The NDD table for the data is

x	y			
0	1			
		1		
1	2		1	
		4		0
3	10		1	
		7		
4	17			

In this case, we aim to get a polynomial of deg. at most 3:

$$p_3(x) = 1 + x + x(x-1) + 0 \times x(x-1)(x-3) = 1 + x^2.$$

So the polynomial is of degree $2 < 3$.
Moral: The degree of the interpolant does not always turn out to be exactly n when $n+1$ data are given. It is $\leq n$.

QUESTION 6 *Assume that we are given the values of* $\ln(x)$ *at points* $x_0 = 0.4$ *and* $x_1 = 0.7$ *and wish to approximate* $\ln(0.6)$ *using a polynomial interpolant. We also wish to obtain an upper bound on the error at* $x = 0.6$*, and at other points.*
Solution:
The number of data is 2, so we consider a polynomial interpolant of degree at most $n = 1$

$$p_1(x) = a_0 + a_1 x$$

that passes through $(0.4, \ln(0.4))$ and $(0.7, \ln(0.7))$.
The Lagrange form of $p_1(x)$ is

$$p_1(x) = \ln(0.4)\frac{x - 0.7}{0.4 - 0.7} + \ln(0.7)\frac{x - 0.4}{0.7 - 0.4}.$$

Evaluating $p_1(x)$ at $x = 0.6$ (in 5 digits) gives

$$\begin{aligned} p_1(0.6) &= \ln(0.4)\frac{0.6 - 0.7}{0.4 - 0.7} + \ln(0.7)\frac{0.6 - 0.4}{0.7 - 0.4} \\ &= \frac{1}{3}\ln(0.4) + \frac{2}{3}\ln(0.7) = \boxed{-0.54321}. \end{aligned}$$

We take the value $p_1(0.6) = -0.54321$ as approximation to $\ln(0.6)$.

To find an upper bound for the error, first consider the error formula

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x - x_j),$$

where $\xi \in ospr\{x, x_0, x_1, \ldots, x_n\}$, and $ospr$ is the open spread of the points x_i, $i = 0, \ldots, n$, and x. Since, in our case, $n = 1, x_0 = 0.4, x_1 = 0.7$ and $f(x) = \ln(x)$, the error formula becomes

$$\ln(x) - p_1(x) = \frac{\ln''(x)|_{x=\xi}}{2!}(x - 0.4)(x - 0.7).$$

We have that $f'(x) = [\ln(x)]' = \frac{1}{x}$ and $f''(x) = [\ln(x)]'' = -\frac{1}{x^2}$ and thus the error formula is

$$\ln(x) - p_1(x) = -\frac{1}{2!\xi^2}(x - 0.4)(x - 0.7). \quad (1)$$

For $x = 0.6$ the error bound is

$$\ln(0.6) - p_1(0.6) = -\frac{1}{2!\xi^2}(0.6 - 0.4)(0.6 - 0.7) = -\frac{1}{2\xi^2}0.2(-0.1) = \frac{10^{-2}}{\xi^2}.$$

Since $x = 0.6$ is between the interpolating points 0.4 and 0.7, we have $ospr\{0.6, 0.4, 0.7\} = ospr\{0.4, 0.7\} = (0.4, 0.7)$, thus $0.4 < \xi < 0.7$, and it follows that

$$|\ln(0.6) - p_1(0.6)| < \frac{10^{-2}}{0.4 \times 0.4} = \frac{10^{-2}}{0.16} = \boxed{0.0625}.$$

Using MATLAB, we can get the value of $\ln(0.6)$ with about 15 decimal digits accuracy: $\ln(0.6) \approx -0.510825623765990$. In 5 digits (just for simplicity), $\ln(0.6) \approx \boxed{-0.51083}$. Assuming this is the exact value of $\ln(0.6)$, the actual error is

$$|\ln(0.6) - p_1(0.6)| = \boxed{0.03238} < 0.0625.$$

Note that the error is less than the mathematical bound and no theory is violated.

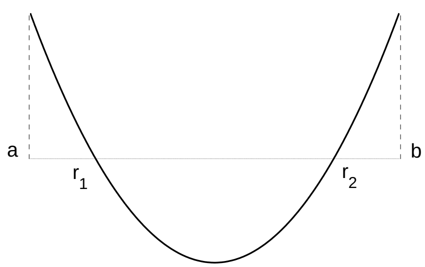
To get a bound for $|\ln(x) - p_1(x)|$ for any x in $[0.4, 0.7]$, we consider the error formula (1), and first maximize $|W(x)| = |(x - 0.4)(x - 0.7)|$.
Note that $W(x) = (x - 0.4)(x - 0.7)$ is a quadratic function and $W'(x) = 2x - 1.1$, $W''(x) = 2 > 0$. Solve $W'(x) = 0$.
$W'(x) = 0 \Rightarrow x = \frac{1.1}{2} = 0.55$. Thus, $W(x)$ reaches the minimal value at $x = 0.55$ and $W_{\min}(x) = W(0.55) = (0.55 - 0.4)(0.55 - 0.7) = -0.15^2 = -0.0225$. Therefore,

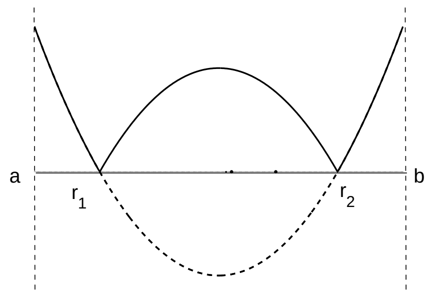
$\max_{0.4 \le x \le 0.7} |W(x)| =$
$= \max\{|W(x)| x = 0.4, x = 0.7, x = 0.55\} = \max\{0, 0.0225\} = 0.0225$. Then, for $0.4 \le x \le 0.7$,

$$|\ln(x) - p_1(x)| \le \frac{0.0225}{2 \times \xi^2} < \frac{0.0225}{2 \times 0.4^2} = \boxed{0.07031}$$

taking into account that, as before, we still have $0.4 < \xi < 0.7$. Naturally, we expected the error bound for $0.4 \le x \le 0.7$ to be at least as large as the error bound for $x = 0.6$.
If we want to get a bound for the interpolation error at some point x outside the interval of interpolation, besides computing $|W(x)|$, we would have to consider that, in general, $\xi \in ospr\{x, x_0, x_1, \ldots, x_n\}$. For example, with $n = 1, x_0 = 0.4, x_1 = 0.7$, if $x = 0.3$, then $\xi \in (0.3, 0.7)$, and, if $x = 0.8$, then $\xi \in (0.4, 0.8)$.
If we want to get a bound for the interpolation error at some interval for x larger than the interval of interpolation, again, we would have to consider that, in general, $\xi \in ospr\{x, x_0, x_1, \ldots, x_n\}$, and we would also have to compute the maximum of $|W(x)|$ in the extended interval.

Important note: If $W(x)$ is a function having the type of graph to the left, then $|W(x)|$ has the type of graph to the right.





So $\max_{a \le x \le b} |W(x)| = \max\{|W(a)|, |W(b)|, |\min_{a \le x \le b} W(x)|\}$