# MATH6222 week 5 lecture 13

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**Chapter 5: Combinatorial Reasoning**

**Question:** Let $A$ be a set with $n$ elements, let $k \leq n$ be an integer. How many size $k$ subsets $B \subseteq A$ are there?

**Example:** {chocolate, vanilla, rainbow} $(n = 3)$

- $k = 0, \emptyset$
- $k = 1$, {c}, {v}, {r}.
- $k = 2$, {c,v}, {v,r }, {r,c}.
- $k = 3$, {c,v,r}.

**Example:** $\{c, v, r, s\}$.

- $k = 0, \emptyset$
- $k = 1$, {c}, {v}, {r, {s}.
- $k = 2$, {c,v}, {v,r }, {r,c}, {c, s}, {v, s}, {r, s}.
- $k = 3$, {c,v,r}, {c,v,s}, {c,r, s}, {s,v,r}.
- $k = 4$, {c,v,r,s}.

**Definition:** If $n$ integer, $n! = n(n-1)(n-2)\cdots\cdot 2\cdot 1$.

This is the number of bijections from $[n] \to [n]$.

How many ordered $k$-tuples of elements from $A$?

$$|\{(a_1, \ldots, a_k) : a_i \in A\}| = n^k$$

How many ordered $k$-tuples with no repeats?

$$|\{(a_1,\ldots,a_k) : a_i \in A,\ \text{all } a_i \text{ distinct } a_i \neq a_j| = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

**Proposition:** If $|A| = n$, let $\binom{n}{k}$ (read as $n$ choose $k$) denote the number of subsets $B \subseteq A$ of size $k$. Then

$$\binom{n}{k} = \frac{n!}{(n-k)!}$$

**Proof:** The number of ordered $k$-tuples from $A$ with no repeats is: $n(n-1)(n-2)\cdots(n-k+1)$. Each subset of $k$ can be listed as an ordered $k$-tuple in $k!$ ways.

$$k!\binom{n}{k} = n(n-1)(n-2)\cdots(n-k+1)$$

**Properties:**

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. $\sum_{k=0, k \text{ even}}^{n} \binom{n}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = \sum_{k=0, k \text{ odd}}^{n} \binom{n}{k}$
3. $\binom{n}{k} = \binom{n}{n-k}$
   hint: to prove $\{\text{subset of size } k\} \Longleftrightarrow \{\text{subset of size } n-k\}$

Reminder: If $A$ is finite set, $2^A$ (power set of $A$) has size $2^{|A|}$.
If $|A| = n$, subsets of $A$ correspond to $0-1$ strings of length $n$.

**Different Interpretations of $\binom{n}{k}$**

$\binom{n}{k}$ as the number of binary strings of length $n$ with exactly $k$ 1's.

Suppose a bug moves only up and to the right. How many paths can the bug take to reach point $(a, b)$?

$$\binom{a+b}{a} = \binom{a+b}{b}$$

**Proof:** Any path requires $a+b$ steps, and is uniquely determined by choosing a size of subset of steps in which to move "right".

Therefore, $\binom{n}{k}$ can be interpreted as the number of bug-paths to the point $(k, n-k)$.