MAT 337 HW3 Rui Qiu #999292509 44/60

P179.

6.

(a). Suppose X_1 is an open set in metric space (X, ρ) (WLOG)

Since X_1 is open, then $\forall x \in X_1$, $\exists r>0$, s.t. $B_r^{\rho}(x) \subseteq X_1$

we know $\forall x \in X$ and $r>0$, $\exists s=s(r,x)>0$,

(we can say for our X_1, $\exists s_1 = s_1(r,x)>0$) such that

$B_{s_1}^{\sigma}(x) \subset B_r^{\rho}(x)$

Since $B_r^{\rho}(x) \subseteq X_1$

so $B_{s_1}^{\sigma}(x) \subset X_1$

so $\forall x \in X_1$ under the metric σ, $\exists s_1 = s_1(x,r)>0$ s.t. $B_{s_1}^{\sigma}(x) \subseteq X_1$

i.e. if a set X_1 is open in (X,ρ), it's also open in (X, σ).

Now we can suppose X_2 as an open set in metric space (X, σ)

Similarly, since X_2 is open, $\forall x \in X_2$, $\exists r>0$ s.t. $B_r^{\sigma}(x) \subseteq X_2$

$\forall x \in X$ and $r>0$, $\exists s = s(r,x)>0$

(we ~~define~~ apply it our X_2, $\exists s_2 = s_2(r,x)>0$) s.t.

$B_{s_2}^{\rho}(x) \subset B_r^{\sigma}(x)$

Since $B_r^{\sigma}(x) \subseteq X_2$

So $B_{s_2}^{\rho}(x) \subseteq X_2$

So $\forall x \in X_2$ under ρ, $\exists s_2 = s_2(x,r)>0$ s.t. $B_{s_2}^{\rho}(x) \subseteq X_1$

i.e. if a set X_2 ~~under metric~~ is open in (X,σ), it's also open in (X,ρ).

So far, we proved that topologically equivalent metrics have the same open sets.

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~~Now suppose X_1 is an open set in (X,ρ), we know it's also open in (X,σ) $X \setminus X_1$ is closed in (X,ρ), then $X \setminus X_1$ is closed in (X,σ) as well.~~

Now suppose $X \setminus X_1$ is a closed set in (X,ρ)

$\Rightarrow X_1$ is an open set in (X,ρ)

$\Rightarrow X_1$ is an open set in (X,σ)

$\Rightarrow X \setminus X_1$ is a closed set in (X,σ)

Similarly, suppose $X \setminus X_2$ is a closed set in (X, σ)

$\Rightarrow X_2$ is a n ~~clo~~ open set in (X, σ)

$\Rightarrow X_2$ is an open set in (X, φ)

$\Rightarrow X \setminus X_2$ is a closed set in (X, φ).

Hence (X, φ) and (X, σ) have the same closed sets.

(b). Suppose a sequence (x_n) is convergent in (X, φ),

ie $x_n \to x$, $\lim_{n\to\infty} \varphi(x, x_n) = 0$

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$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\varphi(x, x_n) < \varepsilon$ whenever $n \geq N$

We can rewrite $\varphi(x, x_n) < \varepsilon$ to $x_n \in B^{\varphi}_{\varepsilon}(x)$

We assume $\varepsilon = s = s(r, x) > 0$, you haven't introduced r yet so nonsense

~~$\exists r > 0$ s.t. $\varepsilon =$~~

Since φ, σ are topologically equivalent on X,

then $B^{\varphi}_{s}(x) = B^{\varphi}_{\varepsilon}(x) \subseteq B^{\sigma}_{r}(x)$

so $x_n \in B^{\varphi}_{\varepsilon}(x) \Rightarrow x_n \in B^{\sigma}_{r}(x)$

$\Rightarrow \sigma(x, x_n) < r$

$\Rightarrow \forall r > 0, \exists N$ s.t. $\sigma(x_n, x) < r$ whenever $n \geq N$

$\Rightarrow (x_n)$ converges to x in (X, σ) as well.

Similarly, suppose (y_n) is convergent in (X, σ)

i.e. $y_n \to x$, $\lim_{n\to\infty} \sigma(y_n, x) = 0$

$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\sigma(x, y_n) < \varepsilon$ whenever $n \geq N$

rewrite $\sigma(y, x_n) < \varepsilon$ to $y_n \in B^{\sigma}_{\varepsilon}(x)$

assume $\varepsilon = s = s(r, x) > 0$

since φ, σ are topologically equivalent on X

then $B^{\sigma}_{s}(x) = B^{\sigma}_{\varepsilon}(x) \subseteq B^{\varphi}_{r}(x)$

So $y_n \in B^{\sigma}_{\varepsilon}(x) \Rightarrow y_n \in B^{\varphi}_{r}(x)$

$\Rightarrow \sigma(x, y_n) < r$

$\Rightarrow \forall r > 0, \exists N$ s.t. $\varphi(y_n, x) < r$ whenever $n \geq N$

$\Rightarrow (y_n)$ converges to x in (X, φ) as well.

Hence, they have the same convergent sequences.

$\rho(x,y)=|x-y|$

(C). Choose ρ to be "normal standard example" (aka. Euclidean metric ~~space~~)
define $\sigma(x,y)=|\frac{1}{x}-\frac{1}{y}|, \forall x,y \in X$, X is positive real number.

① First need to show σ is a metric.

(a). positive definite: if $x=y$, $\sigma(x,y)=|\frac{1}{x}-\frac{1}{x}|=0$
if $\sigma(x,y)=0$, then $\frac{1}{x}=\frac{1}{y} \Rightarrow x=y$.

(b). symmetry: $\sigma(x,y)=|\frac{1}{x}-\frac{1}{y}|=|\frac{1}{y}-\frac{1}{x}|=\sigma(y,x)$

(c). triangle inequality: $\sigma(x,y)=|\frac{1}{x}-\frac{1}{y}|=|\frac{1}{x}-\frac{1}{z}+\frac{1}{z}-\frac{1}{y}|$
$\leq |\frac{1}{x}-\frac{1}{z}|+|\frac{1}{z}-\frac{1}{y}|=\sigma(x,z)+\sigma(z,y)$

Hence $\sigma(x,y)$ is a metric on X.

② Secondly need to show ρ and σ are topologically equivalent.

$\forall x\in X, r>0$, let $\boxed{s=\min\{\frac{x}{2},\frac{x^2}{2}r,\frac{r}{2x^2},\frac{1}{2x}\}}$

$\forall y \in B^{\rho}_{s}(x)$, $\rho(x,y)<s$

particularly, $\rho(x,y)<\frac{x}{2}$,

The selection of s is important

$\Rightarrow |x-y|<\frac{x}{2}$

$\Rightarrow \frac{x}{2}<y<\frac{3}{2}x$

Since $x>0$ then $\frac{x^2}{2}<xy<\frac{3}{2}x^2$

For $\sigma(x,y)=|\frac{1}{x}-\frac{1}{y}|=|\frac{y-x}{xy}|<\frac{s}{\frac{x^2}{2}}<\frac{\frac{x^2}{2}r}{\frac{x^2}{2}}=r$

So $y\in B^{\sigma}_{r}(x)$

Therefore $B^{\rho}_{s}(x)\subseteq B^{\sigma}_{r}(x)$

Similarly, $\forall y\in B^{\sigma}_{s}(x)$, $\sigma(x,y)<s$

particularly, $|\frac{1}{x}-\frac{1}{y}|<\frac{1}{2x}$

$\Rightarrow \frac{1}{2x}<\frac{1}{y}<\frac{3}{2x}$

Since $x>0, y>0$

then $\frac{2x}{3}<y<2x$

then

$|\frac{1}{x}-\frac{1}{y}|=|\frac{y-x}{xy}|<s$

$\Rightarrow |y-x|<s|xy|<s|2x^2|<\frac{r}{2x^2}\cdot 2x^2$
$=r$

$\Rightarrow y\in B^{\rho}_{r}(x)$

~~Hence $B^{\rho}_{s}(x) \supseteq B^{\sigma}_{r}(x)$~~

Hence $B^{\sigma}_{s}(x)\subseteq B^{\rho}_{r}(x)$

So far, we proved that ρ and σ are topologically equivalent.

③ Give a counter example Cauchy sequences. (where n is a natural number)

Say, $x_n=\frac{1}{n}$, So (x_n) is such a sequence we are looking for.

~~For~~

It's Cauchy in (X,ρ)

why?

$\boxed{\forall \varepsilon>0, \exists N \text{ s.t. } |x_m-x_n|<\varepsilon \text{ whenever } m,n\geq N.}$

b/c $\lim_{n\to\infty} x_n=\lim_{n\to\infty}\frac{1}{n}=0$ (convergent)

But for $\sigma(x,y)$, pick $\varepsilon=\frac{1}{2}$.

$\sigma(x_m,x_n)=|\frac{1}{\frac{1}{m}}-\frac{1}{\frac{1}{n}}|$
$=|m-n|$
$\geq 1 > \varepsilon=\frac{1}{2}$

Hence (x_n) is not Cauchy in (X,σ).

We're done.

H. Solution:

(a). Suppose ρ, σ are topologically equivalent

4 by definition, for $0<c<C$, we have

$\forall x,y \in X,\ c\rho(x,y) \le \sigma(x,y) \le C\rho(x,y)$

$\forall x \in X, r>0$, let $s = \min\{rc, \frac{r}{C}\}$

① $\Rightarrow \forall y \in B^{\rho}_{s}(x) \Rightarrow \rho(x,y) < s$

In particular $\rho(x,y) < \frac{r}{C}$

$C\rho(x,y) < r$

since $\sigma(x,y) < C\rho(x,y)$

so $\sigma(x,y) < r$, $y \in B^{\sigma}_{r}(x)$

hence $B^{\rho}_{s}(x) \subseteq B^{\sigma}_{r}(x)$

② $\Rightarrow \forall y \in B^{\sigma}_{s}(x) \Rightarrow \sigma(x,y) < s$

in particular $\sigma(x,y) < rc$

since $c\rho(x,y) \le \sigma(x,y)$

so $c\rho(x,y) < rc$

so $\rho(x,y) < r$, $y \in B^{\rho}_{r}(x)$

hence $B^{\sigma}_{s}(x) \subseteq B^{\rho}_{r}(x)$

Therefore, by ① & ②, ρ & σ are topologically equivalent.

(b). Suppose (x_n) is Cauchy in (X, ρ)

$\Rightarrow \forall \frac{\varepsilon}{C} > 0, \exists N$ s.t. $\rho(x_m, x_n) < \frac{\varepsilon}{C}$

4 wherever $m, n \ge N$

$\Rightarrow \sigma(x_m, x_n) < C \cdot \rho(x_m, x_n) < C \cdot \frac{\varepsilon}{C} = \varepsilon$

$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\sigma(x_m, x_n) < \varepsilon$

wherever $m, n \ge N$

$\Rightarrow (x_n)$ is Cauchy in (X, σ).

Similarly, suppose (y_n) is Cauchy in (X, σ)

$\Rightarrow \forall c \cdot \varepsilon > 0, \exists N$ s.t. $\sigma(y_m, y_n) < c \cdot \varepsilon$

whenever $m, n \ge N$

$\Rightarrow c \cdot \rho(y_m, y_n) < \sigma(y_m, y_n) < c \cdot \varepsilon$

$\Rightarrow \rho(y_m, y_n) < \varepsilon$

$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\rho(y_m, y_n) < \varepsilon$

wherever $m, n \ge N$

therefore, (y_n) is also Cauchy in (X, ρ).

So it's the same for both metrics when we refer to Cauchy sequences.

(c). Recall the sequence we used in problem G(c). for $\rho(x,y) = |x-y|$, $\sigma(x,y) = |\frac{1}{x} - \frac{1}{y}|$ $\forall x,y \in \mathbb{R}^{+}$, and ρ and σ are topologically equivalent. 2

But they do not have the same Cauchy sequences.

Then we can use the fact that equivalent metrics have the same Cauchy sequences.

Take its contrapositive which is, if they don't have the same Cauchy sequences, then they are not two equivalent metrics.

So we are done here, such ρ and σ althought are topologically equivalent, still not equivalent.

I. Solution:

(A) prove $\varphi(A,B) = \text{rank}(A-B)$ is a metric.

① positive definite:

$A=B \Rightarrow \varphi(A,B) = \text{rank}(A-A) = 0$

$\varphi(A,B)=0 \Rightarrow \text{rank}(A-B)=0$

rank is the number of linearly independent rows or columns in a matrix.

So $\text{rank}(A-B)=0 \Rightarrow A-B=0 \Rightarrow A=B$ (the only option is ~~that~~ $A-B$ is ~~zero~~ matrix)

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② symmetry:

$\varphi(A,B) = \text{rank}(A-B)$

$\varphi(B,A) = \text{rank}(B-A)$

since $(-1)(A-B) = (B-A)$.

this does not affect the linear independence between rows & columns of matrix $A-B$.

So $\text{rank}(A-B) = \text{rank}(B-A) \Rightarrow \cancel{A=B}\ \varphi(A,B) = \varphi(B,A)$

③ triangle inequality:

Suppose $\text{rank} A = a \leq n$, $\text{rank} B = b \leq n$, $A, B \in M_n$

so $\text{rank}(A+B) \leq \min\{a+b, n\}$

Particularly, $\text{rank}(A+B) \leq \cancel{\min}\ a+b = \text{rank}A + \text{rank}B$

so $\varphi(A,B) = \text{rank}(A-B) = \text{rank}(A-C+C-B)$

$\leq \text{rank}(A-C) + \text{rank}(C-B)$

$= \varphi(A,C) + \varphi(C,B)$

Hence φ is a metric.

(B). Now we want to see φ and discrete metric ~~space~~ $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ are topologically equivalent, we are about to show

0

$\forall x \in X, r > 0\ \exists\ S$ s.t. $S = S(r,x)$,

$B^d_s(x) \subseteq B^\varphi_r(x)$ and $B^\varphi_s(x) \subseteq B^d_r(x)$

① Show $B^d_s(x) \subseteq B^\varphi_r(x)$

Let $s=0$, then $B^d_s(x) = X$, $\forall r = \varphi(A,B) = \text{rank}(A-B)$

we all have, $B^\varphi_r(X) \supseteq X$ b/c $X =$ $B^d_s(x) \subseteq B^\varphi_r(x)\ \forall r \geq 0$

② Show $B^\varphi_s(x) \subseteq B^d_r(x)$

If $r \geq 1$, $B_r^d(x)$ is metric space X, then we can chose s as we like (s is positive)

If $0<r<1$, then $B_r^d(x)$ is point x.

Let $B_s^\rho(x) \subseteq X$

or we can let $B_s^\rho(x) = x$

$\text{rank}(X-X) = \rho(x, x) = 0 \Rightarrow s = 0$ X

$\Rightarrow B_s^\rho(x) \subseteq B_r^d(x)$

So far we have proved that ρ is a metric, and ρ and d are topologically equivalent on X. Then we are done.

P71. M.

(a). Solution:

$A = [a_{ij}]$, $1 \leq i \leq 4$, $1 \leq j \leq 4$.

$Ax = (\sum_{j=1}^4 a_{1j} x_j, \cdots, \sum_{j=1}^4 a_{4j} x_j)$

$\|Ax - Ay\| = \|A(x-y)\| = (\sum_{i=1}^4 (\sum_{j=1}^4 a_{ij}(x_j - y_j))^2)^{\frac{1}{2}}$

Apply Cauchy-schwartz inequality.

$$\|A(x-y)\| \leq |\sum_{i=1}^4 \sum_{j=1}^4 (a_{ij})^2| \cdot |\sum_{j=1}^4 (x_j - y_j)|^2$$

since $\sum_{j=1}^4 |x_j - y_j|^2 = \|x-y\|^2$

then

$$\|A(x-y)\|^2 \leq |\sum_{i=1}^4 \sum_{j=1}^4 (a_{ij})^2| \cdot \|x-y\|^2$$

$$\|A(x-y)\| \leq |\sum_{i=1}^4 \sum_{j=1}^4 (a_{ij})^2|^{\frac{1}{2}} \cdot \|x-y\|$$

So the Lipchitz constant defined by Cor. 5.1.7 is

$$C = (\sum_{i=1}^4 \sum_{j=1}^4 |a_{ij}|^2)^{\frac{1}{2}}$$

$$= (\tfrac{1}{4} \times 16)^{\frac{1}{2}}$$

$$= 2$$

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(b). Let $x=(a,b,c,d)\in\mathbb{R}^4$

$$\|Ax\|=\tfrac{1}{2}(a+b+c+d,\ a-b+c-d,\ a+b-c-d,\ a-b-c+d)$$

$$=\tfrac{1}{2}(a^2+b^2+c^2+d^2+2ab+2ac+\overset{+2ad}{2bc}+2bd+2cd$$
$$+a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$$
$$+a^2+b^2+c^2+d^2+2ab-2ac-2ad-2bc-2bd+2cd$$
$$+a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd)^{\frac{1}{2}}$$

$$=\tfrac{1}{2}\cdot 2(a^2+b^2+c^2+d^2)^{\frac{1}{2}}$$
$$=\|x\|$$

$\Rightarrow \|Ax-Ay\|=\|A(x-y)\|=\|x-y\|$

so 1 is a Lipschitz constant.

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Claim that ~~Assume~~ it's the optimal Lipschitz constant.

(we are going to construct a contradiction)

Suppose $\exists\, c<1$, it's also a Lipschitz constant.

then $\forall x,y\in\mathbb{R}^4$,

~~$\|A(x-y)\|$~~ $\|Ax-Ay\|\leq c\|x-y\|$

and we know

$\|Ax-Ay\|=\|A(x-y)\|=\|x-y\|$

so $c\geq 1$

(contradiction)

Hence 1 is the optimal Lipschitz constant.

P76.

A. $\chi_A=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases}$

① For $a\in \text{int}A$, we know $\text{int}A$ is open, $\text{int}A\subseteq A$

Say, $x,a\in \text{int}A$ — r doesn't depend on x

$\forall \varepsilon>0,\ \exists r>0$ s.t.

~~$\|f(x)-f(a)\|$~~ $=\|\chi_A(x)-\chi_A(a)\|=\|1-1\|=0<\varepsilon$

whenever $\|x-a\|<r$

so it's continuous in $\text{int}A$.

② For $b \in int(A^c)$, and we know $int(A^c) \subseteq A^c$
similarly, say $x, b \in int(A^c)$
$\forall \varepsilon > 0, \exists r > 0$ s.t.
$\|\chi_A(x) - \chi_A(b)\| = \|0-0\| = 0 < \varepsilon$ whenever $\|x-b\| < r$
So it's continuous on $int(A^c)$

③ For $c \in \partial A = \overline{A} \cap \overline{A^c}$
$\partial A = \overline{A} \cap \overline{A^c} \Rightarrow c \in \overline{A}$ and $c \in \overline{A^c}$
~~So c∉~~
$\Rightarrow c \notin int A^c$ and $c \notin int A$
$\Rightarrow \forall r > 0, B_r(c) \cap A^c \neq \emptyset$,
$B_r(c) \cap A \neq \emptyset$
$\|x-c\| < r \Leftrightarrow x \in B_r(c)$
Choose $x_1 \in B_r(c) \cap A^c$
$x_2 \in B_r(c) \cap A$
7 Then
$$\begin{aligned} 1 = \|\chi_A(x_2) - \chi_A(x_1)\| &= \|\chi_A(x_2) - \chi_A(c) + \chi_A(c) - \chi_A(x_1)\| \\ &\leq \|\chi_A(x_2) - \chi_A(c)\| + \|\chi_A(c) - \chi_A(x_1)\| \\ &= \|1 - \chi_A(c)\| + \|\chi_A(c) - 0\| \end{aligned}$$
Therefore $\max\{\|1-\chi_A(c)\|, \|\chi_A(c)\|\} \geq \frac{1}{2}$
Let $\varepsilon = \frac{1}{2}$
$\forall r > 0, \|x-c\| < r \Rightarrow \|\chi_A(x) - \chi_A(c)\| \geq \frac{1}{2}$
So χ_A is ~~not~~ discontinuous on ∂A by the definition of discontinuity.

B. Solution:
$f(x) = x\log x^2 = 2x\log x$ $\times$ $x < 0$?

3 $\lim_{x\to 0^-} 2x\log x = \lim_{x\to 0^-} \frac{2\log x}{\frac{1}{x}} = \lim_{x\to 0^-} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^-} (-2x) = 0$ (by L'Hopital Rule)

$\lim_{x\to 0^+} 2x\log x = 0$ as well.
But $f(x)$ is undefined when $x=0$, thus it's a removable singularity.