Assignment 2 - Solutions - MAT 327 - Summer 2014

Comprehension

[**C.1**] Let $A \subseteq X$, a topological space. Prove that

$$\overline{A} = \bigcap \{ C \subseteq X : C \text{ is closed and } A \subseteq C \}$$

Solution for C.1. We show both containments:
[$\supseteq$] Let $p \notin \overline{A}$. Then $\overline{A}$ is a closed set containing A that does not contain p. So

$$p \notin \bigcap \{ C \subseteq X : C \text{ is closed and } A \subseteq C \}$$

[$\subseteq$] Let

$$p \notin \bigcap \{ C \subseteq X : C \text{ is closed and } A \subseteq C \}$$

Then there is a closed set C containing A, such that $p \notin C$. Thus $X \setminus C$ is an open set containing p, that is disjoint from A. $\square$

[**C.2**] Find the interior and closure of the following sets (no proof needed):

1. $(3, 4]$ in $\mathbb{R}$ with the usual topology;
2. $(3, 4]$ in $\mathbb{R}$ with the Sorgenfrey Line;
3. $(3, 4]$ in $\mathbb{R}$ with the discrete topology;
4. $B_{77}(0, 0, 0) \setminus B_{13}(0, 0, 0)$ in $\mathbb{R}^3$ with the usual topology;
5. $\mathbb{Q}$ (as a subset of $\mathbb{R}$ with the usual topology).

Solution for C.2.

	Interior	Closure
$(3, 4]$ in $\mathbb{R}_{\text{usual}}$	(3,4)	[3,4]
$(3, 4]$ in $\mathbb{R}$ as the Sorgenfrey Line	(3,4)	[3,4]
$(3, 4]$ in $\mathbb{R}_{\text{discrete}}$	(3,4]	(3,4]
$B_{77}(0, 0, 0) \setminus B_{13}(0, 0, 0)$ in $\mathbb{R}^3_{\text{usual}}$	$\{ p \in \mathbb{R}^3 : 13 < \lvert p\rvert < 77 \}$	$\{ p \in \mathbb{R}^3 : 13 \leq \lvert p\rvert \leq 77 \}$
$\mathbb{Q}$ (as a subset of $\mathbb{R}_{\text{usual}}$)	$\emptyset$	$\mathbb{R}$

$\square$

[**C.3**] Let $A \subseteq X$ a topological space. Prove that A is open iff $A = \text{int}(A)$. Conclude that

$$\text{int}(\text{int}(A)) = \text{int}(A)$$

Solution for C.3. First remark that $\text{int}(A) \subseteq A$ is always true.
[$\Rightarrow$] Suppose that A is open. Then $A \subseteq \text{int}(A)$, by definition of the interior. So then together with our previous observation, $A = \text{int}(A)$.
[$\Leftarrow$] Now, suppose that $A = \text{int}(A)$. It is clear that $\text{int}(A)$ is an open set as it is the union of open sets. So A must be an open set.
The second part follows since $\text{int}(A)$ is an open set. $\square$

[**C.4**] Is it true $\text{int}(\overline{A}) = \overline{(\text{int}(A))}$? Is $\text{int}(\overline{A}) \subseteq \overline{(\text{int}(A))}$? What about $\text{int}(\overline{A}) \supseteq \overline{(\text{int}(A))}$?

Solution for C.4. In general, none of these is true. Taking $A = [2,3]$ in $\mathbb{R}_{\text{usual}}$ shows that

$$\text{int}(\overline{A}) = \text{int}([2,3]) = (2,3) \neq [2,3] = \overline{(2,3)} = \overline{(\text{int}(A))}$$

The same $A = [2,3]$ also shows that

$$\text{int}(\overline{A}) \not\supseteq \overline{(\text{int}(A))}$$

Finally, $A = \mathbb{Q}$ shows

$$\text{int}(\overline{A}) = \text{int}(\mathbb{R}) = \mathbb{R} \not\subseteq \emptyset = \overline{\emptyset} = \overline{(\text{int}(A))}$$

$\square$

[**C.5**] Is a dense subset of the $\mathbb{R}_{\text{cofinite}}$ always dense in $\mathbb{R}_{\text{usual}}$? Is a dense subset of $\mathbb{R}_{\text{usual}}$ always dense in $\mathbb{R}_{\text{cofinite}}$? State your own proposition that captures the general idea here and provide a short proof.

Solution for C.5. The following fact is quite helpful:
Fact: A subset $D \subseteq \mathbb{R}_{\text{cofinite}}$ is dense if and only if D is infinite.
The proof of this fact is straightforward (and we mentioned it in lecture). So $\mathbb{N}$ is an example of a set that is dense in $\mathbb{R}_{\text{cofinite}}$, but not dense in $\mathbb{R}_{\text{usual}}$. For the other question, we note the following proposition:
Proposition. Suppose that $(X, \mathcal{U})$ refines $(X, \mathcal{T})$, and $D \subseteq X$. If D is dense in $\mathcal{U}$ then D is dense in $\mathcal{T}$.

Since D intersects every nonempty element of $\mathcal{U}$ it necessarily intersects every nonempty element of $\mathcal{T}$.

$\square$

Application

[**A.1**] For $A \subseteq X$ a topological space, prove that $X = \text{int}(A) \sqcup \partial(A) \sqcup \text{int}(X \setminus A)$, (where $D = B \sqcup C$' means that $D = B \cup C$ and $B \cap C = \emptyset$. Give an example of a set A where both $\text{int}(A) \neq \emptyset, \text{int}(X \setminus A) \neq \emptyset$ but $\partial(A) = \emptyset$. What properties must such an A have?

Solution for A.1. We prove this in three parts:
Claim 1: $X \subseteq \text{int}(A) \cup \partial(A) \cup \text{int}(X \setminus A)$, with no claim that the sets are mutually disjoint.
Suppose that $x \in X$, but $x \notin \text{int}(A)$ and $x \notin \text{int}(X \setminus A)$. First we see that $x \in X \setminus \text{int}(A)$ is a closed set containing $X \setminus A$. So we see that $x \in \overline{X \setminus A}$. Analogously, since $x \in X \setminus \text{int}(X \setminus A)$ is a closed set containing A we get that $x \in \overline{A}$. Thus $x \in \overline{A} \cap \overline{X \setminus A} = \partial(A)$.
Claim 2: $\text{int}(A) \cap \text{int}(X \setminus A) = \emptyset$.
This follows from $A \cap (X \setminus A) = \emptyset$ and $\text{int}(A) \subseteq A$ and $\text{int}(X \setminus A) \subseteq X \setminus A$.
Claim 3: $\text{int}(A) \cap \partial(A) = \emptyset$.
Let $x \in \text{int}(A)$, an open set. Since $\text{int}(A) \cap (X \setminus A) = \emptyset$ we see that $x \notin \overline{X \setminus A}$. So $x \notin \partial(A)$. Completely analogously, we see that $\text{int}(X \setminus A) \cap \partial(X \setminus A) = \emptyset$, and since $\partial(A) = \partial(X \setminus A)$ we are finished.
Moreover...
In the Sorgenfrey line, let $A = [4, 7)$. We can see that $\text{int}(A) = [4, 7), \partial(A) = \emptyset$ and $\text{int}(X \setminus A) = \text{int}((-\infty, 4) \cup [7, +\infty)) = (-\infty, 4) \cup [7, +\infty)$.
In general, we can see that a set has the desired property if and only if it is closed, open and nonempty. This is from $X = \text{int}(A) \sqcup \text{int}(X \setminus A)$. $\square$

[**A.2**] Is it true that the intersection of two dense sets is again dense? What about the intersection of a dense set with an open set? What if both sets are dense and open? What about the intersection of finitely many such sets? Let's now shift our focus to $\mathbb{R}$ (with the usual topology). Is the only open set that contains $\mathbb{Q}$ all of $\mathbb{R}$? Is it possible to make such an open set that has an infinite complement? What about an uncountable complement? What about a dense complement?

Sketch of A.2. A set that is both dense and open is called "dense open".

Is it true that the intersection of two dense sets is again dense?

No, for example, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}_{\text{usual}}$, but they are clearly disjoint.

What about the intersection of a dense set with an open set?

Again no, taking $\mathbb{Q}$ as the dense set, and $(2,3)$ as the open set, we see that there is no way that the intersection is dense in $\mathbb{R}$. However, we can see that $\overline{\mathbb{Q} \cap (2,3)} = \overline{(2,3)}$, and this isn't true for closed sets. For example,

$$\overline{\mathbb{Q} \cap \{\, \pi \,\}} = \emptyset \neq \{\, \pi \,\} = \overline{\{\, \pi \,\}}$$

What if both sets are dense and open?

Yes! Let A, B be sets that are both dense open in some topological space X. It is clear that the intersection is again open, so we only need to show the following claim:

Claim: $A \cap B$ is dense.

Let U be a non-empty open set in X, we will show that $U \cap A \cap B \neq \emptyset$. Since A is dense, $U \cap A \neq \emptyset$. Moreover, this is a non-empty open set, and since B is dense, $U \cap A \cap B \neq \emptyset$, as required.

What about the intersection of finitely many such sets?

Since the intersection of finitely many open sets is open, the previous exercise, together with induction, gives us that the intersection of finitely many dense open sets is again dense open.

Is the only open set that contains $\mathbb{Q}$ all of $\mathbb{R}$?

No! For example $\mathbb{R} \setminus \{\pi\}$ is such an example.

Is it possible to make such an open set that has an infinite complement?

Yes, for example $\mathbb{R} \setminus \{\pi + n : n \in \mathbb{N}\}$.

What about an uncountable complement?

Yes! There is a very counter-intuitive open set that contains all of $\mathbb{Q}$, but avoids many real numbers. Start by enumerating the rational numbers as

$$q_1, q_2, q_3, q_4, q_5, \ldots$$

Then the desired set is

$$A := \bigcup_{n \in \mathbb{N}} B_{2^{-n}}(q_n)$$

This is clearly open, and we can see that it has "length" less than or equal to

$$\sum_{n\in\mathbb{N}} 2 \cdot 2^{-n} = 2 \cdot 1 = 2$$

Since $\mathbb{R}$ has "length" greater than 2, it cannot be that A covers $\mathbb{R}$. In fact, the "length" of what remains is infinite, so there must be uncountably many points not covered.

The notion of "length" on $\mathbb{R}$ is covered in more depth in a course on measure theory.

What about a dense complement?

This is impossible because we know that every nonempty open set must intersect every dense set. $\square$

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] Prove that the intersection of countably many sets that are dense and open in $\mathbb{R}$ is again dense. Does the intersection have to be open? What if we allow uncountable intersections? Write a couple sentences explaining your thoughts on the following assertion: "There is an uncountable collection of mutually different dense and open sets whose intersection is dense." *After* you've thought about this on your own, read the Wikipedia article on Martin's Axiom and explain in your own words (in a way that a student in this course would understand) what Martin's Axiom says about that assertion.

Proof. What can we say about the intersection of infinitely many sets that are dense and open in $\mathbb{R}$?

These might be empty! For example, each set $A_x := \mathbb{R} \setminus \{ x \}$ is dense open, for $x \in \mathbb{R}$, but

$$\bigcap_{x\in\mathbb{R}} (\mathbb{R} \setminus \{ x \}) = \mathbb{R} \setminus \mathbb{R} = \emptyset$$

Does it matter if we only intersect countably many such sets?

Theorem (Baire Category Theorem)**.** *The intersection of countably many dense open sets in $\mathbb{R}$ is again dense.*

The proof of this can be found in most analysis textbooks. Note that the intersection of countably many dense open sets might not be open. For example, $\mathbb{R} \setminus \{ q \}$, where $q \in \mathbb{Q}$, is a dense open set, and

$$\bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{ q \}) = \mathbb{R} \setminus \mathbb{Q}$$

which is not open.

<u>What if we allow uncountable intersections?</u>

We already saw that they could be empty. On the other hand, the intersection might be a dense set! For example, each set $A_x := \mathbb{R} \setminus \{ x \}$ is dense open, for $x \in \mathbb{R}$, but

$$\bigcap_{x \in \mathbb{R} \setminus \mathbb{Q}} (\mathbb{R} \setminus \{ x \}) = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{Q}$$

<u>What does Martin's Axiom say about this situation?</u>

Martin's Axiom is an axiom in mathematical logic or set-theoretic topology that yields many topological results about $\mathbb{R}$ that are of a combinatorial nature. For example the following is true.

Fact: Suppose Martin's Axiom is true. Suppose I is an index set with $|I| < |\mathbb{R}|$, and D_α is a dense open set for each $\alpha \in I$. Then

$$\bigcap_{\alpha \in I} D_\alpha \neq \emptyset.$$

In English,"Martin's Axiom says that an intersection of a small number of dense open sets is nonempty". If there are no sets I such that $|\mathbb{N}| < |I| < |\mathbb{R}|$ (which is what the Continuum Hypothesis asserts), then the fact above just follows from the Baire Category theorem. The interesting case is where there *are* such uncountable sets I, in which case Martin's Axiom says "they behave like countable sets". $\square$

[**NI.2**] Write a program that:

1. Allows a user to input a collection of subsets of $X = \{0, 1, 2, 3, 4\}$;
2. Generates the smallest topology $\mathcal{T}$ that contains those sets (always include the full space X);
3. Allows the user to input a subset of $A \subseteq X$, and your program returns the interior, closure and boundary of A (in the topology $\mathcal{T}$).

NI.2 solutions. Soon (with permission) I will post some of the solutions I received for this question. $\square$

[**NI.3**] This is a very famous (and fun!) problem called the **Kuratowski 14-set problem**. Let $A \subseteq X$ a topological space.

1. Prove that there are at most 14 different subsets of X that can be obtained from A by applying the operations of closures and complements successively. (There is a tutorial problem that you will probably find useful!)

2. Find a subset A of $\mathbb{R}$ (with the usual topology) such that the 14 subsets of $\mathbb{R}$ can be obtained from A by applying the operations of closure and complements successively.

Sketch of NI.3. Like I said, this is a very famous problem. Very good write-ups are available online, and the wikipedia article:

http://en.wikipedia.org/wiki/Kuratowski%27s_closure-complement_problem

should give you a good idea of what is happening here. $\square$