Sta 347 Practice Final Solutions

Dec. 13 / 2013

(1). (a). $\int_0^\infty f(x)\,dx = 1 \Rightarrow c\int_0^\infty x e^{-2x}\,dx = 1$

Let $2x = t$

$$\int_0^\infty x e^{-2x}\,dx = \frac{1}{4}\int_0^\infty t e^{-t}\,dt$$

$$= \frac{1}{4}\Gamma(2) = \frac{1}{4}.$$

Recall that $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\,dt$.

$\Rightarrow c = 4.$

(b). $$\int_0^\infty x f(x)\,dx = 4\int_0^\infty x^2 e^{-2x}\,dx$$

$$= \frac{4}{8}\int_0^\infty t^2 e^{-t}\,dt$$

$$= \frac{1}{2}\Gamma(3) = 1.$$

$\Rightarrow EX = 1$

Similarly. $EX^2 = \int_0^\infty x^2 f(x)\,dx = \frac{1}{4}\Gamma(4) = \frac{3}{2}$

$\Rightarrow V(X) = EX^2 - (EX)^2 = 0.5$

(c)

$$E(e^{tX}) = 4\int_0^\infty e^{tx}\, x e^{-2x}\, dx$$

$$= 4\int_0^\infty x e^{(t-2)x}\, dx \quad \text{for } t < 2.$$

$$\overset{(t-2)x=-z}{=\!=} 4\int_0^\infty \frac{-z}{(t-2)} e^{-z}\, d\frac{-z}{t-2}$$

$$= \frac{4}{(t-2)^2}\int_0^\infty z e^{-z}\, dz$$

$$= \frac{4}{(t-2)^2}.$$

(2).

$$P(U_1+U_2 > U_3) = E\{P(U_1+U_2>U_3 \mid U_1, U_2)\}$$

$$\overset{\circledast}{=} E\{ I\{U_1+U_2>1\} + (U_1+U_2)\, I\{U_1+U_2\le 1\}\}.$$

$\circledast$ is because of independence of U_3 & (U_1, U_2).

$$= P(U_1+U_2>1) + E(U_1 I\{U_1+U_2\le 1\}) + E(U_2 I\{U_1+U_2\le 1\}).$$

$$:= A + B + C.$$

$$A = E[P(U_1+U_2>1 \mid U_1)] = E[U_1] = \frac{1}{2}.$$

$$B = E(E(U_1 I\{U_1+U_2\le 1\} \mid U_1)) = E(U_1(1-U_1)) = \frac{1}{6}.$$

Similarly $C = \frac{1}{6}$

$$\Rightarrow P(U_1+U_2 > U_3) = \frac{1}{2}+\frac{1}{6}+\frac{1}{6} = \frac{5}{6}.$$

(3).

(a). $f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 6x^2y\,dy$

$= 3x^2 \quad (0 \leq x \leq 1)$

$\therefore \ EX = \int_0^1 x f(x)\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}$.

$EX^2 = \int_0^1 x^2 f(x)\,dx = \int_0^1 3x^4\,dx = \frac{3}{5}$

$V(X) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$.

(b) $f(y) = \int_0^1 f(x,y)\,dx = 2y$.

$EY = \int_0^1 y f(y)\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}$

$EY^2 = \int_0^1 y^2 f(y)\,dy = \int_0^1 2y^3\,dy = \frac{1}{2}$

$\Rightarrow V[Y] = EY^2 - (EY)^2 = \frac{1}{18}$.

(c) Conditional density

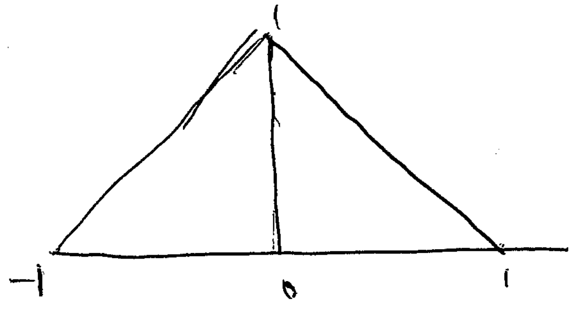
$\cancel{f(x|y) =}\ f(x|y) = \frac{f(x,y)}{f_Y(y)}$

$= \frac{6x^2y}{2y}$

$= 3x^2$ for $\forall\, y \in [0,1]$.

(4).

(a). The region A is plotted below:



For fixed x.

$y \le 1-|x|$.

$\therefore f(x) = \int_0^{1-|x|} 1 \, dy = 1-|x|$. $\quad |x| \le 1$.

(Note that A has area 1. Hence the joint density is 1 on A)

(b). For fixed y. x can take

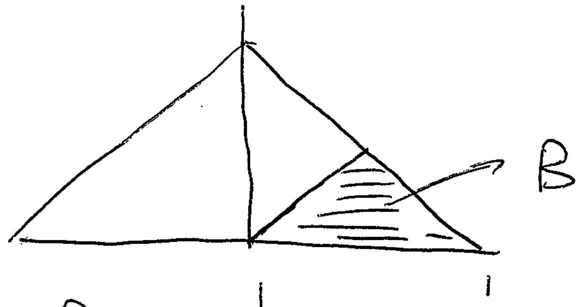
$y-1 \le x \le 1-y$

$\therefore f(y) = \int_{y-1}^{1-y} 1 \cdot dx = 2(1-y) \quad 0 \le y \le 1$

(c). $f(Y|x) = \dfrac{f(x,y)}{f(x)} = \dfrac{I\{(x,y) \in A\}}{1-|x|}$.

(d).

$p(X-Y \geq 0)$



$= p((x,y) \in B) =$ area of $B = \frac{1}{4}$.

(5).

Let $X = U_1$

Then $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}$ is one to one with inverse

$\cancel{X = U_1}$ $U_1 = X$ $\quad \begin{pmatrix} X \\ Y \end{pmatrix} \in \{(x,y) : 0 \le y \le x \le 1\}$

$U_2 = \frac{Y}{X}$.

The Jacobian

$$= \left| \det \begin{pmatrix} 1 & 0 \\ \frac{-Y}{X^2} & \frac{1}{X} \end{pmatrix} \right| = \frac{1}{X}.$$

$\therefore$ The joint density of $\begin{pmatrix} X \\ Y \end{pmatrix}$

$$= \frac{1}{x} I\{0 \le y \le x \le 1\}.$$

$\Rightarrow$ density of Y

$$= \int_y^1 \frac{1}{x} dx = \begin{cases} -\ln y & (0 \le y \le 1) \\ 0 & \text{o.w} \end{cases}.$$

(6). Let $Z_i = X_i - Y_i$

Then Z_i's are i.i.d. with $EZ_i = \mu_1 - \mu_2$

& $V(Z_i) = \sigma_1^2 + \sigma_2^2$.

According to the CLT.

$$\frac{\frac{\sum_{i=1}^{n} Z_i}{n} - EZ_i}{\sqrt{V(Z_i)/n}} \xrightarrow{D} N(0,1)$$

$$\Rightarrow \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}} \xrightarrow{D} N(0,1).$$

(7).

The ~~MGF~~ C.F. of X

$\Rightarrow \varphi(t) = E e^{itX}$

$$= \int_0^\infty e^{itx} \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} dx$$

$$= \left(\int_0^\infty x^{n/2-1} e^{-(\frac{1}{2}-it)x} dx\right) \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}$$

$\Rightarrow$ Let $y = (\frac{1}{2}-it)x$

$$\Rightarrow \varphi(t) = \int_0^\infty \left(\frac{y}{\frac{1}{2}-it}\right)^{\frac{n}{2}-1} e^{-y} d\frac{y}{(\frac{1}{2}-it)} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \cdot \frac{1}{(\frac{1}{2}-it)^{\frac{n}{2}}} \int_0^\infty y^{\frac{n}{2}-1} e^{-y} dy$$

$$= \frac{1}{(1-2it)^{\frac{n}{2}}}.$$

Let $Y_n = (X-n)/\sqrt{2n}$.

The C.F. of $Y_n \Rightarrow \varphi_n(t) = e^{-\sqrt{\frac{n}{2}}it}\varphi(\frac{t}{\sqrt{2n}})$

$$= e^{-\sqrt{\frac{n}{2}}it} \cdot \frac{1}{(1-\frac{2it}{\sqrt{2n}})^{\frac{n}{2}}}$$

Note

$$\log(\varphi_n(t)) = -\sqrt{\frac{n}{2}}\,it - \frac{n}{2}\log\left(1-\frac{2it}{\sqrt{2n}}\right)$$

Note $\log(1-x) = -x - \frac{x^2}{2} + o(x^2)$ for $|x| \leq 1$.

$$\Rightarrow \log(\varphi_n(t)) = -\sqrt{\frac{n}{2}}\,it - \frac{n}{2}\left(-\frac{2it}{\sqrt{2n}} - \left(\frac{2it}{\sqrt{2n}}\right)^2 + o(n)\right)$$

$$= -\frac{t^2}{2} + o(1).$$

$$\Rightarrow \varphi_n(t) \longrightarrow e^{-\frac{t^2}{2}} \text{ as } n \to \infty.$$

$$\Rightarrow \cancel{X_n - \sqrt{\ }}\ \frac{X-n}{\sqrt{2n}} \xrightarrow{D} N(0,1).$$

(8). I will only show that

$$F_n(X) \longrightarrow F(X) \quad \text{for } \forall \text{ cts point } x \text{ of } F(x)$$

where $F_n(X)$ & $F(X)$ are the CDF of X_n & X, respectively. as results for y holds similarly.

For $\forall\, \varepsilon > 0$, $\exists\, C_\varepsilon > 0$. s.t.

$$| F(x, y) - F(x) | \leq \frac{\varepsilon}{4}$$

for $\forall\, y > C_\varepsilon$ & (x, y) is cts point of $F(\cdot,\cdot)$

$(\because F(x, y) \to F(x) \text{ as } y \to \infty)$.

Furthermore, $\because F_n(x, y) \to F(x, y)$.

$\Longrightarrow \exists\, N_y$. s.t.

$$| F_n(x, y) - F(x, y) | \leq \frac{\varepsilon}{4}$$

for $\forall\, n \geq N_y$.

For each n. $|F_n(x, y) - F_n(x)| \leq \frac{\varepsilon}{4}$

for ~~n≥~~ $y \geq$ ~~X_n~~. N_ε.

Let $y \to \infty$. We have $| F_n(x) - F(x) | < \varepsilon$.