Generalisations to several events

Pairwise independence

Events $A_1,\dots,A_n$ are said to be *pairwise independent*
if each pair of them are independent (ie, $P(A_iA_j)=P(A_i)P(A_j)\ \forall\, i<j$).

For example A_1, A_2, A_3 are pairwise independent if
$P(A_1A_2)=P(A_1)P(A_2)$, $P(A_1A_3)=P(A_1)P(A_3)$ and $P(A_2A_3)=P(A_2)P(A_3)$. (*)

Mutual independence

Events $A_1,\dots,A_n$ are said to be *mutually* (or *totally*) *independent* if, for any collection of them, say $B_1,\dots,B_m$ $(2\le m\le n)$, it is true that $P(B_1\dots B_m)=P(B_1)\dots P(B_m)$.

Eg, A_1, A_2, A_3 are mutually independent if (*) *and* $P(A_1A_2A_3)=P(A_1)P(A_2)P(A_3)$.

We see that mutual independence implies pairwise independence, but not the other way around. An example illustrating this point is given in Exercise 2.175 of the text.

Note: Unless otherwise specified, "independent", means "mutually independent".

The multiplicative law of probability for several events

$$P(A_1\dots A_n)=P(A_1)P(A_2\mid A_1)P(A_3\mid A_1A_2)\dots P(A_n\mid A_1\dots A_{n-1}).$$

Pf of law for n = 3: $P(ABC) = P((AB)C) = P(AB)P(C|AB) = P(A)P(B|A)P(C|AB)$.

Example 20 Three cards are drawn randomly from a stack containing 4 aces, 2 kings and 1 queen.
Find the pr that an ace, a king and another ace are drawn, in that order.

Let: A = "An ace is obtained on the 1st draw"
B = "A king is obtained on the 2nd draw"
C = "An ace is obtained on the 3rd draw".
Then $P(ABC) = P(A)P(B|A)P(C|AB) = (4/7)(2/6)(3/5) = 4/35 = 0.114$.

Alternatively, $P(BCA)=P(B)P(C\mid B)P(A\mid BC)=(2/7)(4/6)(3/5)=4/35$, etc.

What about the probability of 2 aces & a king being drawn *in any order*?

Call this probability p and Let A_i = "Ace on ith draw" and K_i = "King on ith draw".
Then $p = P(K_1A_2A_3) + P(A_1K_2A_3) + P(A_1A_2K_3)$

$$= \left(\frac{2}{7}\times\frac{4}{6}\times\frac{3}{5}\right) + \left(\frac{4}{7}\times\frac{2}{6}\times\frac{3}{5}\right) + \left(\frac{4}{7}\times\frac{3}{6}\times\frac{1}{5}\right) = 3\times\frac{4}{35} = \frac{12}{35}.$$

Alternatively, there are $\binom{4}{2} = 6$ ways to select 2 aces from the 4 aces,

and $\binom{2}{1} = 2$ ways to select one king from the 2 kings.

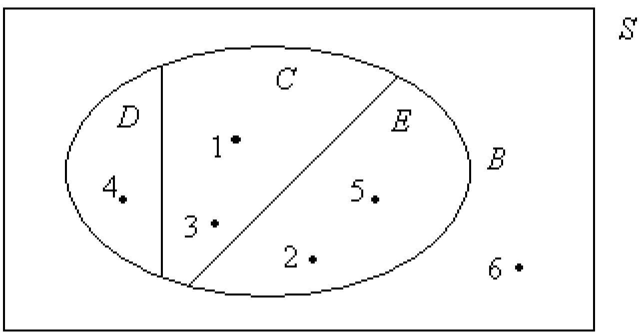
Also, there are $\binom{7}{3} = 35$ ways to select 3 cards from the 7 cards, all equally likely.

So the required probability is $\binom{4}{2}\binom{2}{1}\Big/\binom{7}{3} = \frac{6\times 2}{35} = \frac{12}{35}$.

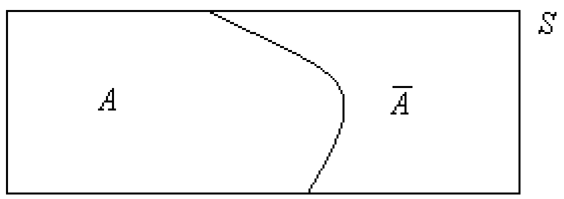
Partition of an event (different to the concept of partitions in combinatorics)

Suppose that $B_1,\ldots,B_n$ are disjoint events whose union is B. (That is, suppose that $B_iB_j = \varnothing$ for all $i < j$, and $B_1 \cup \ldots \cup B_n = B$.) Then we say that $B_1,\ldots,B_n$ form a *partition* of B (or that they *partition B*).

Eg, suppose S = {1,2,3,4,5,6}, B = {1,2,3,4,5}, C = {1,3}, D = {4} and E = {5,2}.
Then C, D and E form a partition of B:



Note: For any event A, it is true that A and $\bar{A}$ form a partition of S:



Also, B and $\bar{B} = \{6\}$ form a partition of S.

The law of total probability (LTP) for several events

Suppose that $B_1,\ldots,B_n$ form a partition of *S*. Then for any event *A*:

$$P(A) = P(AB_1) + \ldots + P(AB_n)$$
$$= P(B_1)P(A \mid B_1) + \ldots + P(B_n)P(A \mid B_n) .$$

(This is also true if we replace "form a partition of *S*" by "form a partition of *A*.")

Example 21   (n = 3)

There are three stacks of cards on a table, each well shuffled and face down.
Stack A contains 5 white cards and 5 red cards.
Stack B contains 6 white cards and 4 red cards.
Stack C contains 7 white cards and 3 red cards.

Your friend Joe is going to roll a standard 6-sided die.
If 1, 2 or 3 comes up, he will select Stack A.
If 4 or 5 comes up, he will select Stack B.
If 6 comes up, he will select Stack C.

Joe will then randomly draw a card from the selected stack.

What is the probability that Joe will draw a white card?

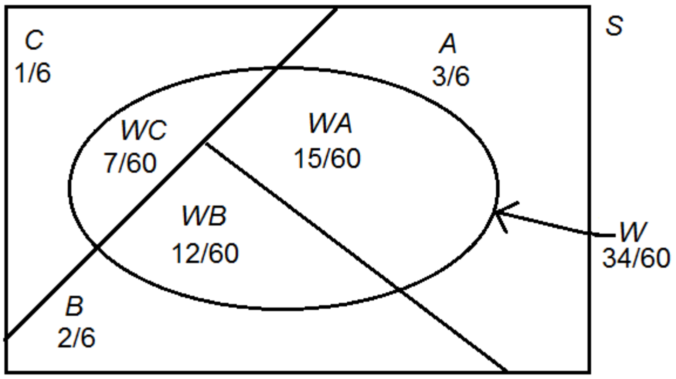
Let: *A* = “Stack A is selected”, *B* = “Stack B is selected”, *C* = “Stack C is selected”
W = “The drawn card is white".

Then: $P(A) = 3/6,\quad P(B) = 2/6,\quad P(C) = 1/6$
$P(W \mid A) = 5/10,\ P(W \mid B) = 6/10,\ P(W \mid C) = 7/10 .$

So: $$P(WA) = P(A)P(W \mid A) = \frac{3}{6}\times\frac{5}{10} = \frac{15}{60}$$
$$P(WB) = P(B)P(W \mid B) = \frac{2}{6}\times\frac{6}{10} = \frac{12}{60}$$
$$P(WC) = P(C)P(W \mid C) = \frac{1}{6}\times\frac{7}{10} = \frac{7}{60} .$$

It follows that $P(W) = P(WA) + P(WB) + P(WC) = \frac{15}{60} + \frac{12}{60} + \frac{7}{60} = \frac{34}{60}$.

Venn diagram



Think of W as a 'pie' that is sliced up into three bits, namely WA, WB and WC.

Check: Using the LTP again, the probability of a *red* card coming up is

(3/6)(5/10) + (2/6)(4/10) + (1/6)(3/10) = (15 + 8 + 3)/60 = 26/60.

So the probability of a white card must be 1 - 26/60 = 34/60.

Bayes' rule for several events

Suppose that $B_1,\ldots,B_n$ form a partition of S. Then for any event A, and any i = 1,...,n:

$$P(B_i \mid A) = \frac{P(B_i)P(A \mid B_i)}{P(A)}$$
$$= \frac{P(B_i)P(A \mid B_i)}{P(B_1)P(A \mid B_1) + \ldots + P(B_n)P(A \mid B_n)}.$$

(This follows directly from the LTP for several events.
It is also true if we replace "form a partition of S" by "form a partition of A.")

Example 22 Refer to Ex. 21, and suppose that the drawn card is white.
Find the probability that Stack A was selected.

The required (posterior) probability is $P(A \mid W) = \dfrac{P(WA)}{P(W)} = \dfrac{15/60}{34/60} = \dfrac{15}{34}$.

(The *prior* probability of Stack A being selected is 3/6.)

What's the probability that Stack B was selected? 12/34.
What's the probability that Stack C was selected? 7/34.
What's the probability that a 3 was rolled? (15/34)/3 = 5/34.

The additive law of probability for several events

$$P(A_1 \cup \ldots \cup A_n) = \{P(A_1) + P(A_2) + \ldots + P(A_n)\}$$

$$-\{P(A_1A_2) + P(A_1A_3) + \ldots + P(A_{n-1}A_n)\}$$

(all possible combinations of 2 events)

$$+\{P(A_1A_2A_3) + P(A_1A_2A_4) + \ldots + P(A_{n-2}A_{n-1}A_n)\}$$

(all possible combinations of 3 events)

$$-\ldots$$

$$+(-1)^{n+1}P(A_1 \ldots A_n) \qquad \text{(all } n \text{ events).}$$

This can be proved via induction and the additive law of probability for two events. See Tutorial 3.

Example 23 The inefficient secretary problem

An inefficient secretary has 3 letters and 3 matching envelopes.
He randomly puts the letters into the envelopes (one each).

Find the pr that *at least one* of the 3 letters is correctly addressed.

Let A_i = "The *i*th letter is in the correct envelope"
and A = "At least one letter is in the correct envelope".

Then $P(A) = P(A_1 \cup A_2 \cup A_3) = \{P(A_1) + P(A_2) + P(A_3)\}$
$-\{P(A_1A_2) + P(A_1A_3) + P(A_2A_3)\}$
$+P(A_1A_2A_3)$.

Now: $P(A_1) = P(A_2) = P(A_3) = 1/3$
$P(A_1A_2) = P(A_1)P(A_2 \mid A_1) = (1/3)(1/2) = 1/6 = P(A_1A_3) = P(A_2A_3)$
(since the numbering of the letters is arbitrary)
$P(A_1A_2A_3) = (1/3)(1/2)(1/1) = 1/6$.

Therefore $P(A) = \{1/3 + 1/3 + 1/3\} - \{1/6 + 1/6 + 1/6\} + 1/6$
= 1 – 1/2 + 1/6 (= 1 – 1/2! + 1/3!)
= 2/3 (~ 67%).

Check via the sample point method

Let the number combination "213" represent the event that letters 2, 1 and 3 go into envelopes 1, 2 and 3, respectively, etc.Then there are 3! = 6 possible outcomes, under each of which we may write "*A*" or "not *A*", as appropriate.

$\underline{123}$	$\underline{1}32$	$21\underline{3}$	231	312	$3\underline{2}1$
A	*A*	*A*	not *A*	not *A*	*A*

(We have underlined each number indicating a correct placement.)

Since all 6 outcomes are equally likely, $P(A)$ = 4/6 = 2/3, as before.

Generalisation

Suppose *n* different letters are randomly placed into *n* matching envelopes (one per envelope). What then is the probability of at least one letter being correctly placed?

First,
$$\begin{aligned} P(A) &= P(A_1 \cup \ldots \cup A_n) \\ &= \{P(A_1) + \ldots + P(A_n)\} && (n = C(n,1) \text{ terms}) \\ &\quad -\{P(A_1A_2) + P(A_1A_3) + \ldots + P(A_{n-1}A_n)\} && (C(n,2) \text{ terms}) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad +(-1)^{n+1}P(A_1 \ldots A_n) && (1 = C(n,n) \text{ term}). \end{aligned}$$

Now:
$$P(A_1) = \frac{1}{n} = P(A_2) = \ldots = P(A_n)$$
$$P(A_1A_2) = P(A_1)P(A_2 \mid A_1) = \frac{1}{n}\frac{1}{n-1} = P(A_1A_3) = P(A_2A_3) = \ldots = P(A_{n-1}A_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P(A_1 \ldots A_n) = \frac{1}{n}\frac{1}{n-1}\ldots\frac{1}{1} = \frac{1}{n!}.$$

So
$$\begin{aligned} P(A) &= \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{1}{n}\frac{1}{n-1} + \binom{n}{3}\frac{1}{n}\frac{1}{n-1}\frac{1}{n-2} - \ldots + (-1)^{n+1}\binom{n}{n}\frac{1}{n!} \\ &= 1 - \frac{n!}{2!(n-2)!}\frac{(n-2)!}{n!} + \frac{n!}{3!(n-3)!}\frac{(n-3)!}{n!} - \ldots + (-1)^{n+1}\frac{1}{n!} \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n+1}\frac{1}{n!}. \end{aligned}$$

Observation

What happens to $P(A)$ as n increases?

n	$P(A)$
1	1.0000
2	0.5000
3	0.6667
4	0.6250
5	0.6333
6	0.6319
7	0.6321
8	0.6321
100	0.6321
10000	0.6321

We observe a dampening up-down oscillation and a convergence that appears to be fairly rapid, with the probability being 63% to two significant digits for all $n > 3$.

In fact, it turns out that $P(A) \to 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots = 1 - \frac{1}{e}$ = 0.6321206,

where e is the transcendental number equal to $\lim_{r\to\infty}\left(1+\frac{1}{r}\right)^r$ = 2.71828......

Proof: By Taylor's theorem in calculus, e^x may be written $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Thus $e^{-1} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \ldots = \frac{1}{2!} - \frac{1}{3!} + \ldots$

So $1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots$

Further extension

Suppose that n different letters are randomly placed into n matching envelopes (one per envelope). What is the probability that exactly k letters are correctly placed?

(Do this as an exercise and check your formula by applying the sample point method to a few simple cases, eg n = 2 and n = 3. Then see the solution below.)

Solution: Let p_k^n be the pr that *n* letter-envelope pairs will result in *k* correct.

Then, as already shown, $p_0^n = 1 - P(A) = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}$.

So $p_k^n = P(A_1 \ldots A_k \bar{A}_{k+1} \ldots \bar{A}_n) + \ldots + P(\bar{A}_1 \ldots \bar{A}_{n-k} A_{n-k+1} \ldots A_n)$

($C(n,k)$ identical terms)

$$= \binom{n}{k} \times P(A_1 \ldots A_k) \times P(\bar{A}_{k+1} \ldots \bar{A}_n \mid A_1 \ldots A_k)$$

$$= \frac{n!}{k!(n-k)!} \times \left\{ \frac{1}{n} \frac{1}{n-1} \cdots \frac{1}{n-k+1} \right\} \times p_0^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} p_0^{n-k} = \frac{1}{k!} p_0^{n-k}$$

$$= \frac{1}{k!} \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-k} \frac{1}{(n-k)!} \right\} \quad (n = 1,2,3,\ldots,\ k = 0,1,2,\ldots,n).$$

Observe that $p_k^n \approx \frac{1}{ek!}$ if $n - k$ is large.

Example with $n = 3$:

$$p_0^3 = \frac{1}{0!}\left\{\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right\} = \frac{1}{3}, \qquad p_1^3 = \frac{1}{1!}\left\{\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!}\right\} = \frac{1}{2},$$

$$p_2^3 = \frac{1}{2!}\left\{\frac{1}{0!} - \frac{1}{1!}\right\} = 0, \qquad p_3^3 = \frac{1}{3!}\left\{\frac{1}{0!}\right\} = \frac{1}{6}.$$

Sample point:	<u>123</u>	<u>1</u>32	21<u>3</u>	231	312	3<u>2</u>1
Number correct:	3	1	1	0	0	1

So $p_0^3 = 2/6$, $p_1^3 = 3/6$, $p_2^3 = 0$, $p_3^3 = 1/6$ (all in agreement with the above).
(That $p_2^3 = 0$ also follows from the fact that if any specific two letters are correctly placed, the remaining third letter must also be in its matching envelope.)

Example with $n = 2$:

$$p_0^2 = \frac{1}{0!}\left\{\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!}\right\} = \frac{1}{2},\ p_1^2 = \frac{1}{1!}\left\{\frac{1}{0!} - \frac{1}{1!}\right\} = 0,\ p_2^2 = \frac{1}{2!}\left\{\frac{1}{0!}\right\} = \frac{1}{2}.$$

These values are correct since the two letters must either be both correctly placed or both incorrectly placed, with both of these two possibilities equally likely.

Note: In both of the above examples, another check is that $p_0^n + \ldots + p_n^n = 1$.

De Morgan's laws for several events

$$\overline{A_1 \cup \ldots \cup A_n} = \overline{A}_1 \ldots \overline{A}_n$$
$$\overline{A_1 \ldots A_n} = \overline{A}_1 \cup \ldots \cup \overline{A}_n .$$

Proof of De Morgan's first law for the case n = 3:

$\overline{A \cup B \cup C} = \overline{A \cup (B \cup C)} = \overline{A}\,\overline{(B \cup C)}$ by De Morgan's laws for 2 events

$= \overline{A}\overline{B}\overline{C}$ again by De Morgan's laws for 2 events.

Proof of De Morgan's first law for all n = 3,4,5,..... via induction:

Suppose the 1st law is true for $n = m$, for some positive integer m greater than 1.
Then $\overline{A_1 \cup \ldots \cup A_{m+1}} = \overline{(A_1 \cup \ldots \cup A_m) \cup A_{m+1}} = \overline{(A_1 \cup \ldots \cup A_m)}\,\overline{A}_{m+1} = (\overline{A}_1 \ldots \overline{A}_m)\overline{A}_{m+1}$.
Thus the law is true for $n = m + 1$. But we know the law is true for n = 2.
It follows by induction that the law is true for all n = 3, 4, 5,....

Example 24 A die is rolled four times. Find the pr that at least one 6 comes up.

Let A_i = "6 comes up on ith roll" and A = "At least one 6 comes up".

Then $P(A) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$= 1 - P(\overline{A_1 \cup A_2 \cup A_3 \cup A_4})$ (by Theorem 3)

$= 1 - P(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4)$ (by De Morgan's 1st law)

$= 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)P(\overline{A}_4)$ (by independence of rolls)

= 1 – (5/6)(5/6)(5/6)(5/6) (1,2,3,4 or 5 on each roll)

$= 1 - (5/6)^4$

= 0.518.

(Note that in practice one would just write $P(A) = 1 - P(\overline{A}) = 1 - (5/6)^4 = 0.518$.)

Example 25 The coin tossing problem

John and Kate take turns tossing a coin, starting with John.
The first person to get a head wins.

Find the probability that John wins the game.

Let J = "John wins". Then J = {H, TTH, TTTTH, ...}

(ie J gets a H straightaway; *or* J gets a T, K gets a T, and then J gets a H; *or* etc)

So
$$\begin{aligned} P(J) &= P(H) + P(TTH) + P(TTTTH) + \dots \\ &= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \dots \\ &= \frac{1}{2}\left\{1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots\right\} \\ &= \frac{1}{2}\left(\frac{1}{1-1/4}\right) \\ &= \frac{2}{3}. \end{aligned}$$

Note that we have here used the result $1 + r + r^2 + \dots = 1/(1-r)$, $-1 < r < 1$,

(sum of an infinite geometric progression) with r = 1/4. Proof (for completeness):

Let $s = 1 + r + r^2 + \dots + r^n$, where n is a positive integer.
Then $rs = r + r^2 + \dots + r^{n+1} = (1 + r + r^2 + \dots + r^n) + r^{n+1} - 1$.
Thus $rs = s + r^{n+1} - 1$, which implies that $s = \dfrac{1 - r^{n+1}}{1 - r}$

(sum of a *finite* geometric progression).

This formula is true for all finite n, and all r except 1, in which case $s = n + 1$.
If $-1 < r < 1$, then $s \to 1/(1-r)$ as $n \to \infty$.
So in that case, $1 + r + r^2 + \dots = 1/(1-r)$.

Another solution

Let K = "Kate wins".

Then K = {TH, TTTH, TTTTTH,..., and hence $P(K) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \dots$

Observe that $P(K) = \frac{1}{2}\left\{\frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \dots\right\} = \frac{1}{2}P(J)$.

Now $P(J) + P(K) = 1$ (since *someone* must win).

Therefore $P(J) + \frac{1}{2}P(J) = 1$, which implies that $P(J) = 2/3$, as before.

This solution is simpler than the first since it avoids having to *sum* an infinite series.

Yet another solution

Let A = "Heads come up on the 1st toss".
Then $P(J) = P(A)P(J \mid A) + P(\bar{A})P(J \mid \bar{A})$ (by the LTP).
So $P(J) = \frac{1}{2}(1) + \frac{1}{2}\{1 - P(J)\}$.
Solving this (single equation in a single unknown) we get $P(J) = 2/3$, as before.

Note: If a H comes up on the 1st toss, then J definitely wins. Thus $P(J \mid A) = 1$.
If a T comes up on the 1st toss, then J is in the same position that K was in at the beginning of the game (ie *she* is now the one to be rolling next). So his chance of winning in that case and at that point in the game is $P(J \mid \bar{A}) = P(K) = 1 - P(J)$.

This third solution avoids having to even *consider* an infinite series. The method used here is called *first step analysis*, a powerful tool for computing probabilities.

Another example of first step analysis

Two dice are rolled together repeatedly. Find the probability that a total of 12 comes up before a total of 11.

Let A = "12 comes up before 11",
B = "10 or less on 1st roll", C = "11 on 1st roll" and D = "12 on first roll".

Then B, C and D form a partition of the sample space. So

$P(A) = P(B)P(A|B) + P(C)P(A|C) + P(D)P(A|D)$, which implies that
$P(A) = (33/36)P(A) + (2/36)0 + (1/36)1$, which then yields $P(A) = 1/3$.

Alternatively, $P(A) = 1/36 + (33/36)1/36 + (33/36)(33/36)1/36 + ...$
$= (1/36)\{1 + 33/36 + (33/36)(33/36) + ...\}$
$= (1/36)1/(1 - 33/36) = 1/3$.

Another solution is as follows. Consider the experiment of observing the *last* total in a sequence which ends with the *first* 11 or 12. This last total must be 11 or 12. But 12 is half as likely as 11 on any given roll of the two dice together (the probabilities are 1/36 and 2/36, respectively). So the probability of the last total being 12 must be half of the probability that the last total is 11, and the only way that this can happen is if those two probabilities are 1/3 and 2/3, respectively. Hence the answer again, 1/3.

Some handy notation

Sums:	$\sum_{i=a}^{b} x_i = x_a + x_{a+1} + \ldots + x_b$	(Eg	$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 14$)
Products:	$\prod_{i=a}^{b} x_i = x_a x_{a+1} \ldots x_b$	(Eg	$\prod_{i=1}^{3} i^2 = 1^2 2^2 3^2 = 36$)
Intersections:	$\bigcap_{i=a}^{b} A_i = A_a A_{a+1} \ldots A_b$	(Eg	$\bigcap_{i=1}^{3} A_i = A_1 A_2 A_3$)
Unions:	$\bigcup_{i=a}^{b} A_i = A_a \cup A_{a+1} \cup \ldots \cup A_b$.	(Eg	$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3$)

Thus, for example, we may write De Morgan's laws as $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A}_i$, $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A}_i$.

Stirling's formula

If n is large then $n! \approx n^n e^{-n} \sqrt{2\pi n}$.

Example 1: $50! \approx 50^{50} e^{-50} \sqrt{2 \times 3.14159 \times 50} = 3.036 \times 10^{64}$.
Check: Using a calculator we get 50! = 3.041×10^{64} (exact to 4 significant digits).

Example 2: Suppose we want to evaluate 1000! Most calculators can't do this.
However, using Stirling's formula we may write $1000! \approx \sqrt{2\pi 1000} \times 1000^{1000} e^{-1000}$.

Now, $\sqrt{2\pi 1000} = 79.26655$, and $1000^{1000} = 10^{3000}$. Also,

$$e^{-1000} = 10^{\log_{10}(e^{-1000})} = 10^{-1000\log_{10} e} = 10^{-434.2945} = 10^{-434} 10^{-0.2945} = 0.5076 \times 10^{-434}.$$

So $1000! \approx 79.26655 \times 10^{3000} \times 0.5076 \times 10^{-434} = 4.02 \times 10^{2567}$.

The exact value can in fact be obtained, using a computer (see R code below):

$$1000! = e^{5912.128} = 10^{\log_{10}(e^{5912.128})} = 10^{5912.128\log_{10} e} = 10^{2567.605} = 10^{2567} 10^{0.605} = 4.03 \times 10^{2567}.$$

Example 3: $\binom{1000}{400} = \frac{1000!}{400!600!} \approx \frac{\sqrt{2\pi 1000} \times 1000^{1000} e^{-1000}}{\sqrt{2\pi 400} \times 400^{400} e^{-400} \sqrt{2\pi 600} \times 600^{600} e^{-600}}$

$$= \sqrt{\frac{1}{2\pi 4 \times 60}} \times \frac{1000^{1000}}{400^{400} \times 600^{600}} = 0.0257516 \times 10^{3000} \times 10^{-400\log_{10} 400} \times 10^{-600\log_{10} 600}$$

$$= 0.0257516 \times 10^{292} \times 10^{0.2853} = 4.967 \times 10^{290}.$$

The exact value is 4.965×10^{290}, obtained using R code (see below).

R Code (non-assessible)

```
lgamma(1001) # 5912.128    (this is the natural logarithm of 1000!)
choose(1000,400) #  4.965272e+290
```