Assignment 6 - Solutions - MAT 327 - Summer 2014

Due July 14th, 2013 at 4:10 PM

Comprehension

[**C.1**] Let $p \in \mathbb{R}$, and let $\mathcal{T}$ be the usual topology on $\mathbb{R}$. Let $\mathcal{B}_p := \{ U \in \mathcal{T} : p \in U \}$. Show that this is not a filter, but there is a filter $\mathcal{F}$ such that $\mathcal{B}_p \subseteq \mathcal{F}$. Moreover, make sure that $\{ p \} \notin \mathcal{F}$. (Hint: Take $\mathcal{F}_p := \{A \subseteq \mathbb{R} : \exists U \in B_p$ such that $U \subseteq A\}$.)

Solution for C.1. Since $[p-1, p+1]$ is not an open set in $\mathbb{R}_{\text{usual}}$, $[p-1, p+1] \notin \mathcal{B}_p$, but $(p-1, p+1) \in \mathcal{B}_p$ and $(p-1, p+1) \subseteq [p-1, p+1]$. So $\mathcal{B}_p$ is not a filter.

Define $\mathcal{F}_p := \{A \subseteq \mathbb{R} : \exists U \in \mathcal{B}_p$ such that $U \subseteq A\}$. We claim that this is a filter.

[$\emptyset, \mathbb{R}$] Clearly $\mathbb{R} \in \mathcal{B}_p \subseteq \mathcal{F}_p$, and $p \notin \emptyset$, so $\emptyset \notin \mathcal{F}_p$.

[Finite Intersection Property] Let $A_1, \ldots, A_N \in \mathcal{F}_p$. So there are open $U_i \in \mathcal{B}_p$ such that $U_i \subseteq A_i$ (for $1 \leq i \leq N$). Each U_i is an open set that contains p, so $U_1 \cap \ldots \cap U_N$ is an open set that contains p. Since $U_1 \cap \ldots \cap U_N \subseteq A_1 \cap \ldots \cap A_N$ we get that $A_1 \cap \ldots \cap A_N \in \mathcal{F}_p$, as desired.

[Closed Upwards] This is immediate from our definition of $\mathcal{F}_p$: Let $A \in \mathcal{F}_p$ and let $A \subseteq B$. Then there is a $U \in \mathcal{B}_p$ such that $U \subseteq A$. Hence $U \subseteq B$. So $B \in \mathcal{F}_p$.

Claim: $\{ p \} \notin \mathcal{F}_p$.

The only open subset of $\{ p \}$ is the empty set, and we know that $\emptyset \notin \mathcal{B}_p$. So we have the claim. $\square$

[**C.2**] Use Zorn's Lemma to show that every filter $\mathcal{F}$ on a set X is contained in a maximal filter $\mathcal{U}$ (i.e. $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U}$ is a maximal element in the partial order of all filters on X that contain $\mathcal{F}$, ordered by "$\subseteq$"). Maximal filters are called **Ultrafilters**.

Solution to C.2. Let

$$\mathbb{P} := \{ \mathcal{G} : \mathcal{F} \subseteq \mathcal{G}, \text{ and } \mathcal{G} \text{ is a filter} \}$$

ordered under "$\subseteq$". We note that this is non-empty since $\mathcal{F} \subseteq \mathcal{F}$, so $\mathcal{F} \in \mathbb{P}$.

[Chains are bounded] Let $\mathcal{C} \subseteq \mathbb{P}$ be a non-empty chain (of filters). We see that $\mathcal{G} \subseteq \bigcup \mathcal{C}$ for each $\mathcal{G} \in \mathcal{C}$. Now we just check that $\bigcup \mathcal{C} \in \mathbb{P}$, that is, we check that $\bigcup \mathcal{C}$ is a filter. (Clearly, $\mathcal{F} \subseteq \bigcup \mathcal{G}$.)

Claim 1: $\emptyset \notin \bigcup \mathcal{C}$ and $X \in \bigcup \mathcal{C}$.

Since each $\mathcal{G} \in \mathcal{C}$ is a filter, and C is non-empty, we have this claim.

Claim 2: $\bigcup \mathcal{C}$ has the finite intersection property.

Let $F_1, F_2, \cdots, F_n \in \bigcup \mathcal{C}$, then there are $\mathcal{F}_1, \cdots, \mathcal{F}_n$ such that $F_i \in \mathcal{F}_i$ for $1 \leq i \leq n$. Since $\mathcal{C}$ is a chain, there is a $1 \leq N \leq n$ such that $\mathcal{F}_i \subseteq \mathcal{F}_N$ for all $1 \leq i \leq n$. So then $F_i \in \mathcal{F}_N$ for all $1 \leq i \leq n$ and because $\mathcal{F}_N$ has the FIP,

$$\bigcap_{i=1}^{n} F_i \in \mathcal{F}_N \subseteq \bigcup \mathcal{C}$$

Claim 3: $\bigcup \mathcal{C}$ is closed upwards.

Let $A \in \bigcup \mathcal{C}$ and $A \subseteq B$. There is some $\mathcal{G} \in \mathcal{C}$ such that $A \in \mathcal{G}$. Since $\mathcal{G}$ is closed upwards, $B \in \mathcal{G} \subseteq \bigcup \mathcal{C}$.

Now we have checked all of the hypotheses of Zorn's Lemma, so there is a maximal element $\mathcal{U}$ of this partial order, which is exactly what we wanted to construct. $\square$

[**C.3**] Go through our Zorn's Lemma proof in §11 and answer the following two questions:

1. Why did we define $\mathbb{P}$ to be the set of all pairwise disjoint open, **countable** subsets of ω_1? What happens if we drop the countable condition? Does the proof still go through?
2. What "type" of set is the chain $\mathcal{C}$? I mean is $\mathcal{C}$ an element of ω_1? A subset of ω_1? A collection of subsets of ω_1? A collection of collections of subsets of ω_1? In that proof find examples of sets of the other types.

Solution to C.3. The collections needed to only contain countable sets so that maximal elements of the partial order were *uncountable* collections. If we drop this requirement we see that $\{\omega_1\}$ is a maximal collection of pairwise disjoint open sets that has cardinality one.

- $\mathcal{C}$ is a collection of collections of open subsets of ω_1;

- $\mathbb{P}$ is a collection of collections of open subsets of ω_1;
- $\mathcal{A}$ is a collection of open subsets of ω_1;
- A is an open subset of ω_1;
- α is an element of ω_1.

$\square$

[**C.4**] Let C^0 be the collection of all continuous, real-valued functions on the interval $[a, b]$. Define a distance function on C^0 by

$$d(f, g) = \int_a^b |f(x) - g(x)| \; dx$$

Prove that (C^0, d) is a metric space. (For this question you may wish to consult your second-year calculus notes.) If $[a, b] = [0, 1]$ describe the ball of radius 1 around the function $f(x) = x$ for all $x \in [0, 1]$.

Solution to C.4. Let us check the 4 properties of being a metric space.

$\boxed{d(f, g) = 0 \text{ iff } f = g}$ Let $f, g \in C^0$. If $f = g$, then $f(x) = g(x)$, for all $x \in [a, b]$, hence $|f(x) - g(x)| = 0$, for all $x \in [a, b]$, so

$$d(f, g) = \int_a^b |f(x) - g(x)| \; dx = 0.$$

Suppose that $f \neq g$. So there is a point $x_0 \in [a, b]$ such that $f(x_0) \neq g(x_0)$. Hence $|f(x_0) - g(x_0)| > 0$. Since $f - g$ is a continuous function, we see that there is an interval $I \subseteq [a, b]$ with non-empty interior such that $x_0 \in I$, and $|f(x) - g(x)| > 0$ for all $x \in I$. Hence

$$d(f, g) = \int_a^b |f(x) - g(x)| \; dx > 0.$$

$\boxed{d(f, g) \geq 0}$ This is obvious, since we are always integrating over a non-negative function.

$\boxed{d(f, g) = d(g, f)}$ Observe:

$$d(f, g) = |f(x) - g(x)| = |-(g(x) - f(x))| = |-1| \cdot |g(x) - f(x)| = |g(x) - f(x)| = d(g, f)$$

$\boxed{d(f, h) \leq d(f, g) + d(g, h)}$ This follows from the linearity of the integral, and the triangle inequality for $\mathbb{R}$. Observe that

$$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|, \forall x \in [a, b]$$

Hence, by linearity of the integral:

$$\begin{aligned} d(f,h) &= \int_a^b |f(x) - h(x)| \ dx \\ &\le \int_a^b |f(x) - g(x)| + |g(x) - h(x)| \ dx \\ &= \int_a^b |f(x) - g(x)| \ dx + \int_a^b |g(x) - h(x)| \ dx \\ &= d(f,g) + d(g,h) \end{aligned}$$

Now if $[a,b] = [0,1]$, then the ball of radius 1 around $f(x) = x$ contains functions g such that

$$\int_0^1 |g(x) - x| \ dx < 1.$$

□

[**C.5**] Let ρ be the usual distance function on $\mathbb{R}$, and let (C^0, d) be defined as above. Define a function

$$F : (C^0, d) \longrightarrow (\mathbb{R}, \rho)$$

by

$$F(f) = \int_a^b f(x) \ dx.$$

Prove that F is continuous.

Solution to C.5. We note that (C^0, d) is a metric space, so to check continuity it suffices to check that whenever $\langle f_n \rangle \to f$ that $\langle F(f_n) \rangle \to F(f)$.

Suppose that $\langle f_n \rangle \to f$. That is,

$$\forall \epsilon, \exists N \in \mathbb{N}, \text{ such that } n \ge N \Rightarrow \int_a^b |f_n(x) - f(x)| \ dx < \epsilon.$$

Fix an $\epsilon > 0$ and choose a corresponding $N \in \mathbb{N}$. Note that for all $n \ge N$:

$$\begin{aligned} \rho(F(f_n), F(f)) &= \left| \int_a^b f_n(x) \ dx - \int_a^b f(x) \ dx \right| && (1) \\ &= \left| \int_a^b f_n(x) - f(x) \ dx \right| && (2) \\ &\le \int_a^b |f_n(x) - f(x)| \ dx < \epsilon && (3) \end{aligned}$$

Line (2) is the linearity of the integral, and line (3) is because of how absolute values interact with integrals. Hence $\langle F(f_n)\rangle \to F(f)$. $\square$

Application

[**A.1**] Show that there is a metric space (X, d), with $y \in X$, such that the ball $B_{1.001}(y)$ contains 100 pairwise disjoint open balls of radius 1.

Solution to A.1. The point here is that the notion of distance we have doesn't always line up with our notion of the geometry of a space. Let (X, d) be any discrete metric space where X has at least 100 points. We readily see that $B_{1.001}(y) = X$ for any $y \in X$, but $B_1(x) = \{x\}$ for any $x \in X$, so these unit balls will all be pairwise disjoint. $\square$

[**A.2**] Let (X, d) be a metric space. Prove that X is separable iff X is second countable iff X is ccc. Give an example of a topological invariant that is not equivalent to these in a metric space.

Solution to A.2. We have already seen that in any topological space, separable implies ccc (Assignment 3, C.5) and it is very easy to show that second countable implies separable (you just pick an element from each of the elements of the basis!). As a result, we only need to show that in a metrizable space, ccc implies second countable. To be a little slower we will show that in a metrizable space ccc implies separable, and then separable implies second countable.

[ccc $\Rightarrow$ separable] Let (X, d) be a metric space. For $n \in \mathbb{N}$ let $\mathcal{B}_n$ be a maximal family of pairwise disjoint open balls of radius $\frac{1}{n}$. (These of course exist by Zorn's Lemma.) Now since our space is ccc, each $\mathcal{B}_n$ is countable, so let I_n be the family of centers of the balls in $\mathcal{B}_n$. Note that $I := \bigcup_{n\in\mathbb{N}} I_n$ is countable, and we claim that it is also a dense set.

Suppose that it isn't, so there is some open U such that $U \cap I = \emptyset$. Since U is open there is some point p, and $\frac{1}{k}$ such that $B_{\frac{1}{k}}(p) \subseteq U$. Now observe that $B_{\frac{1}{2k}}(p) \subseteq U$ and so is still disjoint from I. Now however, we have (by the triangle inequality) that

$$B_{\frac{1}{2k}}(p) \cap B_{\frac{1}{2k}}(i) = \emptyset, \forall i \in I$$

In particular this shows that B_{2k} was not a maximal collection of pairwise disjoint open sets, a contradiction.

[Separable $\Rightarrow$ Second Countable] This is fairly quick. Let D be a countable dense subset of (X, d), a metric space. Consider

$$\mathcal{B} := \{\, B_{\frac{1}{n}}(d) : d \in D \,\}$$

which is clearly a countabe collection of open sets. We claim that it is a basis.

Let $U \subseteq X$ be open, and let $p \in U$. There is an $\epsilon > 0$ such that $B_\epsilon(p) \subseteq U$. Find a $\frac{1}{k} < \epsilon$. Since D is dense, let $d \in B_{\frac{1}{2k}}(p)$. By symmetry, $p \in B_{\frac{1}{2k}}(d)$, and by the triangle inequality, $B_{\frac{1}{2k}}(d) \subseteq B_{\frac{1}{k}}(p) \subseteq U$. We finally note that $B_{\frac{1}{2k}}(d) \in \mathcal{B}$. $\square$

[**A.3**] Using Zorn's Lemma, prove that every uncountable set $A \subseteq \mathbb{R}$ contains an uncountable distance special set.

Solution to A.3. Let A be an uncountable subset of $\mathbb{R}$. Let

$$\mathbb{P} := \{\, B \subseteq A : B \text{ is a distance special set} \,\}$$

ordered under "$\subseteq$".

Clearly $\mathbb{P}$ is non-empty.

Let us show that every chain has an upper bound. Let $\mathcal{C} \subseteq \mathbb{P}$ be a chain. Clearly, $\bigcup \mathcal{C} \subseteq A$, we only need to check that it is distance special. Suppose that $a, b \in \bigcup \mathcal{C}$ are such that $d(a, b) = M > 0$ and suppose that $c, d \in \bigcup \mathcal{C}$ are such that $d(c, d) = M$. Since $a, b, c, d \in \bigcup \mathcal{C}$, there are $B_a, B_b, B_c, B_d \in \mathcal{C}$ such that $a \in B_a, b \in B_b, c \in B_c, d \in B_d$. Since $\mathcal{C}$ is a chain, one of these B_α is a superset of all of the B_a, B_b, B_c, B_d. Hence $a, b, c, d \in B_\alpha$, and since B_α is a distance special set, $\{a, b\} = \{c, d\}$. Hence $\bigcup \mathcal{C}$ is distance special.

Your solution to Assignment 3, A.1, shows that a maximal element in $\mathbb{P}$ is not countable. Hence, a maximal element in this partial order is an uncountable distance special set. $\square$

New Ideas

[**NI**] Let's investigate the notion of filters and ultrafilters on a set X, which we saw in question C.2. Prove the following facts:

1. If $x \in X$, then $\mathcal{U}_x := \{\, A \subseteq X : x \in A \,\}$ is an ultrafilter.
2. A filter $\mathcal{F}$ is an ultrafilter iff for every $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ (but not both!).
3. Let $\mathcal{U}$ be an ultrafilter and let $A \subseteq X$. Prove that $A \in \mathcal{U}$ iff $U \cap A \neq \emptyset$ for all $U \in \mathcal{U}$.

4. Let $\mathcal{F}$ be a collection of subsets of X with the **first two properties** of being a filter, which is called "having the finite intersection property (FIP)". Prove that if $\mathcal{F}$ is maximal (with respect to the FIP) then (1) $\mathcal{F}$ is a filter and (2) $\mathcal{F}$ is an ultrafilter.

The strategy is to prove [3] $\Rightarrow$ [2] $\Rightarrow$ [1] then to prove 4.

Solution to 1. From [2] this is immediate. $\mathcal{U}_x$ is clearly a filter, and if $A \subseteq X$ then $X \in \mathcal{U}_x$ iff $x \in A$. $\square$

Solution to 2. Suppose that $\mathcal{F}$ is an ultrafilter, and let $A \subseteq X$ with $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$. By [3], there are $U \in \mathcal{F}$ and $V \in \mathcal{F}$ such that $U \cap A = \emptyset$ and $V \cap (X \setminus A) = \emptyset$. Thus $U \subseteq X \setminus A$ and $V \subseteq A$. Thus $U \cap V = \emptyset$, which contradicts the fact that $\mathcal{F}$ is a filter.

Now suppose that for every $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. Let $\mathcal{F} \subsetneq \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter. Choose some $A \in \mathcal{U}$ that isn't in $\mathcal{F}$. Since $A \notin \mathcal{F}$, then $X \setminus A \in \mathcal{F}$, so $X \setminus A \in \mathcal{U}$. This can't happen though, since $A, X \setminus A \in \mathcal{U}$, but $A \cap (X \setminus A) = \emptyset$. $\square$

Solution to 3. Suppose that $A \in \mathcal{U}$. Then since $\mathcal{U}$ has the FIP, this is clear. Conversely, suppose that $\forall U \in \mathcal{U}, A \cap U \neq \emptyset$. We show that $\{A\} \cup \mathcal{U}$ has the FIP, in which case the result follows by maximality of $\mathcal{U}$. Let $U_1, \cdots, U_n \in \mathcal{U}$. Since $\mathcal{U}$ is a filter we have that $\emptyset \neq \bigcap_{i=1}^{n} U_i \in \mathcal{U}$. So then $(\bigcap_{i=1}^{n} U_i) \cap A \neq \emptyset$, by assumption. $\square$

Solution to 4. First we show that $\mathcal{F}$ is a filter.

<u>FIP</u>: We need to show that finite intersections are in $\mathcal{F}$. Let $F_1, \ldots, F_n \in \mathcal{F}$. Then $\bigcap_{i=1}^{n} F_i \neq \emptyset$ by the FIP. Now it is clear that $\{\bigcap_{i=1}^{n} F_i\} \cup \mathcal{F}$ has the FIP. So by maximality of $\mathcal{F}$ we have $\bigcap_{i=1}^{n} F_i \in \mathcal{F} = \{\bigcap_{i=1}^{n} F_i\} \cup \mathcal{F}$.

<u>$\emptyset, X$</u>: Clearly $\emptyset \notin \mathcal{F}$ as $\mathcal{F}$ has the FIP. Thus $X \cap F = F \neq \emptyset$ for each $F \in \mathcal{F}$. So $\mathcal{F} \cup \{X\}$ has the FIP. By maximality of $\mathcal{F}$, $X \in \mathcal{F} = \mathcal{F} \cup \{X\}$.

<u>Closed Upwards</u>: This is analogous to how we showed $X \in \mathcal{F}$.

Now we show that $\mathcal{F}$ is an ultrafilter. This means we need to show that $\mathcal{F}$ is maximal with respect to ultrafilters, right now we only know that it is maximal with respect to families with the FIP. But every ultrafilter has the FIP, so if $\mathcal{F} \subseteq \mathcal{U}$, an ultrafilter, then since $\mathcal{F}$ is maximal with respect to the FIP, we get that $\mathcal{F} = \mathcal{U}$, as desired. $\square$