Today – prove the duality theorem (Theorem 3.7)
– not complemenary slackness (Monday)
– § 3.3 (example)

§ 3.2

Ingredients for the strong duality theorem

① weak duality theorem optimal criterion (Theorem 3.6)

② B^{-1}

③ C_B or C_B^T (reconstructing the objective row of a later tableau.

Eg. (from "A Simplex Optimization")

Given $z = 3x_1 + 7x_2$ (maximize)

$(+0x_3 + 0x_4 + 0x_5)$

The constraint part of a later tableau is

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{1}{6}$	$\frac{5}{6}$	0	9
x_5	0	0	$-\frac{1}{6}$	$\frac{7}{6}$	1	7
x_2	0	1	$\frac{1}{6}$	$-\frac{1}{6}$	0	2

← Tableau ① objectiveness

To apply the optimality criterion, we eliminate the objective row coefficients of the variables, as in the 2-phase method.

Replace the tableau ① objective row with

Tableau ① objective row $+ 3 \cdot x_1$ row
$+ 0 \cdot x_5$ row
$+ 7 \cdot x_2$ row

(To get

x_1	x_2	x_3	x_4	x_5	
0	0	$\frac{5}{3}$	$\frac{4}{3}$	0	41

, an opstimal tableau (④ in "A Simplex Optimization")

Here $C_B = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$, the coefficients of the basic variable $\left(\begin{bmatrix} x_1 \\ x_5 \\ x_2 \end{bmatrix}\right)$ in the original objective function.

So $C_B^T = [3, 0, 7]$ and the tableau ④ objective row is $\underbrace{[-3\ \ 7\ \ 0\ \ 0\ \ 0 | 0]}_{\text{tableau ① objective row}}$
$+\ C_B^T \cdot$ (the constraint part of tableau ④)

Strong Duality Theorem

(Notation) Let I_m denote the $m \times m$ identity matrix. If A is a matrix having m rows, $A_1, A_2, \cdots$ denote the columns of A

Consider the problem :

Maximize $z = c^T x$ s.t.

$$Ax \leq b$$

$$x \geq 0 \in R^n$$

where $b \geq 0$ in $\mathbb{R}^m$ and A is $m \times n$

If this problem has an optional solution, so does its dual. Moreover, the optimal objective values of the 2 problems are equal.

Proof of the duality theorem :

Tableau ①

	$x_1 \cdots x_n$	$x_{n+1} \cdots x_{n+m}$	
x_{n+1} $\vdots$ x_{n+m}	A	I_m	b

(objective row omitted

A later tableau

Table Ⓟ

	$x_1 \cdots x_n$	$x_{n+1} \cdots x_{n+m}$	
x_{i_1} $\vdots$ x_{i_m}	$B^{-1}A$	B^{-1}	$B^{-1}b$

To constraint the objective row of tableau Ⓟ start with the tableau ① objective row :

$x_1 \cdots x_n$	$x_{n+1} \cdots x_{n+m}$	
$-C^T$	0	0

Then let $C_B^T = [\, c_{i_1}, \cdots, c_{i_m}]$

and add $C_B^T [B^{-1}A \mid B^{-1} \mid B^{-1}b]$

to get

$$\begin{array}{c|c|c|c}
 & x_1 \cdots x_n & x_{n+1} \cdots x_{n+m} & \\
\hline
 & C_B^T B^{-1} A - C^T & C_B^T B^{-1} & C_B^T B^{-1} b
\end{array}$$

Let $w^T = C_B^T B^{-1}$

This is optimal provided

$$C_B^T B^{-1} A \geq C^T$$

$$C_B^T B^{-1} \geq 0 \in \mathbb{R}^m$$

That is, W_B^T is feasible for the dual problem $A^T w \geq C$, $W_B \geq 0 \in \mathbb{R}^m$

And the objective values are equal $C_B^T (B^{-1} b)$

$$= W^T b$$

$$= b^T w$$