STAT2001/6039 Mid-Semester Exam 1st Semester 2016 Solutions

Solution to Problem 1

Let R = "At least one red", B = "At least blue" and K = "At least one black".

Then we wish to calculate $P(RB \mid K) = P(RBK)/P(K)$. Now,

$$P(K) = 1 - P(\bar{K}) = 1 - \binom{3}{0}\binom{8}{4}\Big/\binom{11}{4} = \frac{26}{33} = 0.787879.$$

Also, $P(RBK) = 1 - P(\overline{RBK}) = 1 - P(\bar{R} \cup \bar{B} \cup \bar{K})$ by De Morgan's laws

$$= 1 - \{P(\bar{R}) + P(\bar{B}) + P(\bar{K}) - P(\bar{R}\bar{B}) - P(\bar{R}\bar{K}) - P(\bar{B}\bar{K}) + P(\bar{R}\bar{B}\bar{K})\}$$

$$= 1 - \left\{ \frac{\binom{2}{0}\binom{9}{4}}{\binom{11}{4}} + \frac{\binom{6}{0}\binom{5}{4}}{\binom{11}{4}} + \frac{\binom{3}{0}\binom{8}{4}}{\binom{11}{4}} - 0 - \frac{\binom{5}{0}\binom{6}{4}}{\binom{11}{4}} - 0 + 0 \right\} = \frac{24}{55} = 0.436364.$$

It follows that the required probability is $\dfrac{24/55}{26/33} = \dfrac{36}{65} = \boxed{0.5538}$.

Alternative working: RBK may also be expressed as: “Two cards have the same colour and the other two cards have the other two colours, one each”. This logic implies that

$$P(RBK) = \frac{\binom{2}{2}\binom{6}{1}\binom{3}{1}}{\binom{11}{4}} + \frac{\binom{2}{1}\binom{6}{2}\binom{3}{1}}{\binom{11}{4}} + \frac{\binom{2}{1}\binom{6}{1}\binom{3}{2}}{\binom{11}{4}} = \frac{24}{55}.$$

Solution to Problem 2

We wish to find $P(A \mid B)$, where A = "At least one die shows 6 after the initial roll" and B = "Exactly one die shows 6 at the end". Now, $P(\bar{A} \mid B) = P(\bar{A}B)/P(B)$, where

$$P(\bar{A}B) = P(\bar{A})P(B \mid \bar{A}) = \left(\frac{5}{6}\right)^4 \times \binom{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 = \frac{4 \times 5^7}{6^8} = 0.186054.$$

Next, consider the fact that each of the four dice initially had an overall chance of showing 6 at the end equal to $(1/6)\times 1+(5/6)\times 1/6=11/36$, or $1-(5/6)^2=11/36$.

This implies that $P(B)=\binom{4}{1}\left(\frac{11}{36}\right)^1\left(\frac{25}{36}\right)^3=\frac{4\times 11\times 5^6}{6^8}$ = 0.409320.

Thus $P(\bar{A}\,|\,B)=\frac{4\times 5^7/6^8}{4\times 11\times 5^6/6^8}=\frac{5}{11}$ = 0.4545,

and therefore $P(A\,|\,B)=1-\frac{5}{11}=\frac{6}{11}$ = $\boxed{0.5455}$.

Alternative working: Let A_i = "Exactly *i* of the dice show 6 after the initial roll".

$$\text{Then } P(A\,|\,B)=\frac{P(AB)}{P(B)}=\frac{P(A_1B)}{P(A_0B)+P(A_1B)}=\frac{P(A_1)P(B\,|\,A_1)}{P(A_0)P(B\,|\,A_0)+P(A_1)P(B\,|\,A_1)}$$

$$=\frac{4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3\times\left(\frac{5}{6}\right)^3}{\left(\frac{5}{6}\right)^4\times 4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3+4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3\times\left(\frac{5}{6}\right)^3}=\frac{1}{\frac{5}{6}+1}=\frac{6}{11}.$$

Note: If exactly one die shows 6 at the end (B), then either none or one of the dice can show 6 after the initial roll. This fact implies that $AB=A_1B$ and $B=(A_0B)\cup(A_1B)$.

Solution to Problem 3

We wish to find p_0, where $p_y\equiv P(Y=y)$ and:

$$\mu=EY=0p_0+1p_1+2p_2 = 1.2$$

$$\sigma^2=VY=EY^2-(EY)^2=\left(0^2p_0+1^2p_1+2^2p_2\right)-\mu^2 = 0.25.$$

Now, $\sigma^2=\mu+2p_2-\mu^2$, and so $p_2=\frac{1}{2}\left(\sigma^2-\mu+\mu^2\right)=\frac{1}{2}\left(0.25-1.2+1.2^2\right)=0.245$.

It then follows that $p_1=\mu-2p_2=1.2-2\times 0.245$ = 0.71,

and therefore $p_0=1-p_1-p_2=1-0.71-0.245$ = $\boxed{0.045}$.

Solution to Problem 4

Observe that $m_Y(t) = \dfrac{1}{2e^{-t}-1} = \dfrac{e^t}{2-e^t} = \dfrac{(1/2)e^t}{1-(1-1/2)e^t}$.

By comparison with a table of mgfs, this implies that $Y \sim Geometric\left(\dfrac{1}{2}\right)$,

with pdf $f(y) = P(Y = y) = \dfrac{1}{2}\left(1-\dfrac{1}{2}\right)^{y-1} = \left(\dfrac{1}{2}\right)^y$, $y = 1, 2, 3, \ldots$

So $P(Y > 10) = \sum_{y=11}^{\infty}\left(\dfrac{1}{2}\right)^y = \left(\dfrac{1}{2}\right)^{11}\sum_{y=11}^{\infty}\left(\dfrac{1}{2}\right)^{y-11} = \left(\dfrac{1}{2}\right)^{11}\sum_{x=0}^{\infty}\left(\dfrac{1}{2}\right)^x$

$$= \left(\frac{1}{2}\right)^{11}\frac{1}{1-1/2} = \frac{1}{2^{10}} = \frac{1}{1024} = \boxed{0.0009766.}$$

Alternative working: $P(Y > 10)$ = P(No failures in first 10 trials) = $(1/2)^{10}$.

Solution to Problem 5

Let A_k be the event that Ann wins the game with rules as described but more generally with k red cards and three green cards originally in the box, and with the winner being the person on whose turn the *last* red car is burnt. We wish to find $P(A_2)$. Next, let R be the event that the first card drawn is red and let $G = \overline{R}$ be the event that the first card drawn is green.

Then, as part of a first-step analysis, we have that

$$P(A_2) = P(R)P(A_2 \mid R) + P(G)P(A_2 \mid G) = \frac{2}{5}\big(1-P(A_1)\big) + \frac{3}{5}\big(1-P(A_2)\big). \quad (1)$$

Note: If the first card drawn is red then Ann’s chance of winning becomes the same as Bob’s initial chance of winning in a game starting with only one red card, which is one minus Ann's initial chance of winning in such a game. Conversely, if the first card drawn is green then Ann’s chance of winning becomes the same as Bob’s chance was initially in the original game, i.e. one minus Ann's chance in the original game.

We now consider Ann's probability of winning the game but starting with only one red card in the box (not two). By a logic similar to that used before, this probability is

$$P(A_1) = P(R)P(A_1 \mid R) + P(G)P(A_1 \mid G) \; = \frac{1}{4}\times 1 + \frac{3}{4}\left(1 - P(A_1)\right). \tag{2}$$

With $a_k = P(A_k)$ equations (1) and (2) may also be written as:

$$a_2 = \frac{2}{5}(1-a_1) + \frac{3}{5}(1-a_2) \tag{3}$$

$$a_1 = \frac{1}{4} + \frac{3}{4}(1-a_1). \tag{4}$$

Now, (4) implies $a_1 = 4/7$, and then (3) yields $a_2 = P(A_2) = 27/56 = \boxed{0.4821}$.

Alternative working: Let O be the event that the first red card burnt is drawn on an *odd*-numbered draw (i.e. drawn by Ann), and let $E = \bar{O}$ be the event that the first red card burnt is drawn on an *even*-numbered roll, (i.e. drawn by Bob). Also let $B = \bar{A}$ be the event that Bob will win the game. Then:

$$P(O) = \frac{2}{5} + \left(\frac{3}{5}\right)^2 \frac{2}{5} + \left(\frac{3}{5}\right)^4 \frac{2}{5} + \ldots = \frac{2}{5}\left\{1 + \frac{9}{25} + \left(\frac{9}{25}\right)^2 + \ldots\right\} = \frac{2}{5}\left(\frac{1}{1-9/25}\right) = \frac{5}{8}$$

$$P(E) = 1 - P(O) = \frac{3}{8}$$

$$P(A \mid E) = \frac{1}{4} + \left(\frac{3}{4}\right)^2 \frac{1}{4} + \left(\frac{3}{4}\right)^4 \frac{1}{4} + \ldots = \frac{1}{4}\left\{1 + \frac{9}{16} + \left(\frac{9}{16}\right)^2 + \ldots\right\} = \frac{1}{4}\left(\frac{1}{1-9/16}\right) = \frac{4}{7}$$

$$P(A \mid O) = \left(\frac{3}{4}\right)\frac{1}{4} + \left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{3}{4}\right)^5 \frac{1}{4} + \ldots = \frac{3}{16}\left\{1 + \frac{9}{16} + \left(\frac{9}{16}\right)^2 + \ldots\right\} = \frac{3}{16}\left(\frac{1}{1-9/16}\right) = \frac{3}{7}$$

or, using another logic, $P(A \mid O) = 1 - P(B \mid O) = 1 - P(A \mid E) = 1 - 4/7 = 3/7$.

It follows by the law of total probability (and again a first-step analysis) that

$$P(A) = P(O)P(A \mid O) + P(E)P(A \mid E)$$

$$= \frac{5}{8}\times\frac{3}{7} + \frac{3}{8}\times\frac{4}{7} = \frac{27}{56}.$$