Australian National University

Student Number: |___|___|___|___|___|___|___|___|

Research School of Finance, Actuarial Studies and Statistics
Examinatation

Semester 1 2018

STAT3013/4027/8027 Statistical Inference

Examination/Writing Time Duration: 180 minutes
Reading Time: 15 minutes minutes

Exam Conditions:
Central Examination
Students must return the examination paper at the end of the examination
This examination paper is not available to the ANU Library archives

Materials Permitted In The Exam Venue:
(No electronic aids are permitted e.g. laptops, phones)
A4 pages (Two sheets) with handwritten notes on both sides
Paper-based Dictionary, no approval required (must be clear of ALL annotations)
Calculator (non - programmable)

Materials to Be Supplied To Students:
Scribble Paper

For any question, if a closed form analytical solution is not available then provide pseudo-code for an appropriate computational approach to the problem.

The real exam will have roughly five to seven questions. I will not provide solutions to the exam. Please consult your textbook, notes, or see either myself of Mr. Souveek Halder for hints.

Question: Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{uniform}(0, \theta)$. Consider the following three estimators for θ:

$$\begin{aligned}
&\text{Estimator 1}: && 2\bar{X} \\
&\text{Estimator 2}: && \frac{(n+1)X_{(n)}}{n} \\
&\text{Estimator 3}: && \text{the maximum likelihood estimator}
\end{aligned}$$

where $X_{(n)}$ is the largest order statistic.

a. Determine the maximum likelihood estimator (MLE) for θ.

b. For each of the three estimators, determine whether they are unbiased.

c. Which of the three estimators has the smallest mean squared error (MSE)?

d. For each of the three estimators, determine whether they are consistent.

Question: If θ is the frequency of an allele causing a mendelian recessive disease, then the probability that an individual is affected is θ^2. A random sample of size n individuals is taken form a very large population, and x individuals are observed to be affected with the disease.

a. What is the maximum likelihood estimator of θ, and what is its approximate distribution when the sample size is large?

b. In small samples, is the estimator for θ an UMVUE (uniform minimum variance unbiased estimator)?

c. Use two approaches to construct a 95% confidence interval for θ.

d. Use two approaches to construct a 95% confidence interval for θ^2.

Question: Find the form of the critical region for the uniformly most powerful test of H_0 against H_1 when (if needed you may consider large sample approximations):

a. $x_1, \ldots, x_n$ are a random sample from a Poisson distribution with mean θ and $H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1, \quad \theta_1 > \theta_0$.

b. $x_{1,1}, x_{1,2}, \ldots, x_{1,n} \sim$ normal(μ_1, σ_1^2). And $x_{2,1}, x_{2,2}, \ldots, x_{2,n} \sim$ normal(μ_2, σ_2^2). All the $x_{i,j}$'s are independent of each other and consider $H_0 : \mu_2 = \mu_1, \quad H_1 : \mu_2 = \mu_1 + \delta$, with $\delta > 0$. (δ, σ_1^2, σ_2^2 are known constants).

Question 3 [**20 marks**]: Let $X = U^{1/\lambda}$, where $\lambda > 0$ and $U \sim \text{uniform}(0, 1)$.

a. [**5 marks**] Find the pdf of X.

b. [**15 marks**] Suppose you observe $X_1, \ldots, X_n$, where $X_i = U_i^{1/\lambda}$ and $\lambda > 0$ is an unknown parameter. Additionally, $U_1, \ldots, U_n \overset{\text{iid}}{\sim} \text{uniform}(0, 1)$. Note that $U_1, \ldots, U_n$ are unobservable. Based on $X_1, \ldots, X_n$ derive a uniformly most powerful test (UMP) of size α for testing:

$$\begin{aligned} H_0 : &\quad \lambda = 1 \\ H_1 : &\quad \lambda > 1 \end{aligned}$$

If possible determine a closed form analytical solution for the critical region, otherwise consider a computational approach. Derive the power function of the test.

Question: Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{beta}(\theta, 1)$. Consider the following questions.

a. Assume $n = 1$. Determine the uniformly most power test for testing the following hypotheses:

$$\begin{aligned} H_0 : &\quad \theta = 1 \\ H_1 : &\quad \theta = \theta_1, \text{ where } \theta_1 > 1. \end{aligned}$$

Make sure to clearly derive the rejection region. Provide a numeric answer at the $\alpha = 0.05$ significance level.

b. Assume $n = 1$. Determine the power function for this test. What is the power of the test if we assume that under the alternative $\theta = 10$?

c. Consider the full data set $(X_1, \ldots, X_n)$ and construct a maximum likelihood ratio test for testing:

$$\begin{aligned} H_0 : &\quad \theta = \theta_0 \\ H_1 : &\quad \theta \neq \theta_0 \end{aligned}$$

Make sure to clearly derive the rejection region as precisely as possible.

Question: Let $X_1, \dots, X_n \overset{\text{iid}}{\sim}$ geometric(θ), where we model the number of failures until the first success:

$$P(X = x|\theta) = \theta(1-\theta)^x, \quad \text{for } x = 1, 2, 3, \dots$$

Consider the following questions:

a. Determine the family of conjugate priors for θ. How do you know that family is conjugate?

b. Assuming a uniform$(0, 1)$ prior distribution for θ, derive the posterior mode as a point estimator for θ. Additionally, determine the variance of the posterior distribution.

c. Consider the following hypotheses, again assume a uniform$(0, 1)$ prior distribution for θ:

$$\begin{array}{ll} H_0 & \theta = 1/2 \\ H_1 & \theta \neq 1/2 \end{array}$$

Determine the Bayes' factor for the test.

d. Let $n = 1$ and $x_1 = 2$. Consider the following prior for θ:

$$p(\theta) = \theta^{-1}(1-\theta)^{-1}.$$

Determine each of the following: a 95% **highest posterior density (HPD)** credible interval, and a 95% **equal tailed** credible interval. Hint: graph the posterior distribution.

END OF EXAMINATION