20.09.11

Lecture 3 handout

- Thursday lecture moved to Tuesday - Starting next week T2-5.

<u>Definition</u>: A subgraph of a graph $G = (V_G, E_G, \psi_G)$ is $H = (V_H, E_H, \psi_H)$ such that $V_G \supseteq V_H$, $E_G \supseteq E_H$, and $\psi_G|_H = \psi_H$.

 $\subset$  but  $\not\subset$ 

A <u>path</u> is a subgraph $A_n \subseteq G$

path

maximal path

A <u>maximal path</u> cannot be extended at either end.

- Length of the <u>longest path</u> is a graph invariant.

 $\not\cong$ 

<u>Skipped topic</u>: Dijkstra's algorithm.

A <u>walk</u> of <u>length $n \geq 0$</u> is

$$W: v_0, e_1, v_1, e_2, \ldots, e_n, v_n \quad \text{with } \psi(e_i) = \{v_{i-1}, v_i\}$$

In simple graphs, we write just

$$W: v_0, v_1, \ldots, v_n$$

<u>Prop</u>: Every u-v walk contains a u-v path.

<u>Proof</u>: Induction.

<u>Prop</u>: If exactly 2 vertices have odd degree, they are connected by a path.

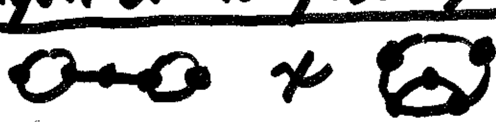
<u>Defn</u>: A <u>connected component</u> of a graph is a maximal connected subgraph $G' \subseteq G$.

- <u>Number of connected components</u> is a graph invariant.

<u>Defn</u>: A <u>cycle</u> is a subgraph $C_n \subseteq G$.

A graph without cycles is a <u>tree</u>

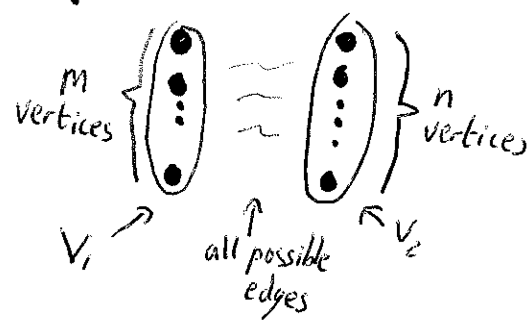
- <u>Length of longest cycle</u> is a graph invariant.



Bipartite graphs

A **bipartite graph** is $G=(V,E,\Psi)$ such that V can be split into non-empty V_1, V_2, and every edge connects a vertex in V_1 to a vertex in V_2.

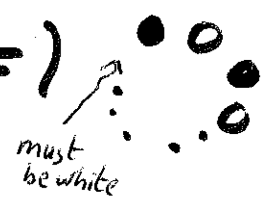
Complete bipartite graph $K_{m,n}$:



$K_{1,n}$ is a **star graph**

Theorem: A graph is bipartite iff it has no odd cycles.

Proof: ($\Leftarrow$)



$\rightsquigarrow$ cycle must be even

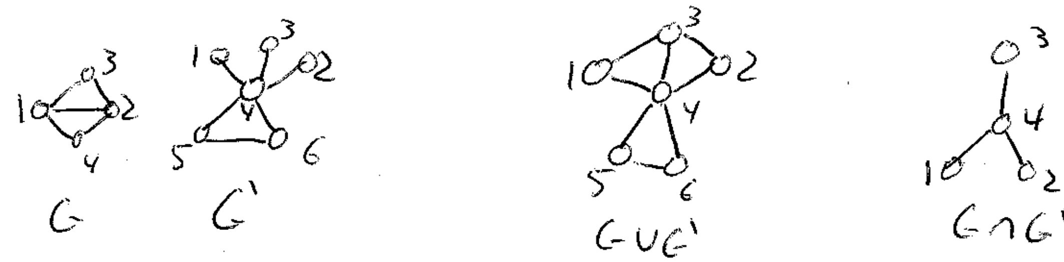
($\Rightarrow$) Colour one vertex •, all its neighbours o, and so on. Only possible obstruction is an odd cycle.

Beautiful, constructive proof !!!

Invariant: Is the graph bipartite Y/N.

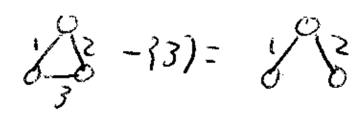
New graphs from old

Union: $G=(V,E)$, $G'=(V',E')$, $G\cup G'=(V\cup V', E\cup E')$

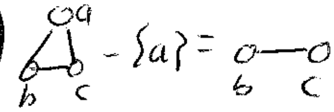


Intersection: $G\cap G'=(V\cap V', E\cap E')$

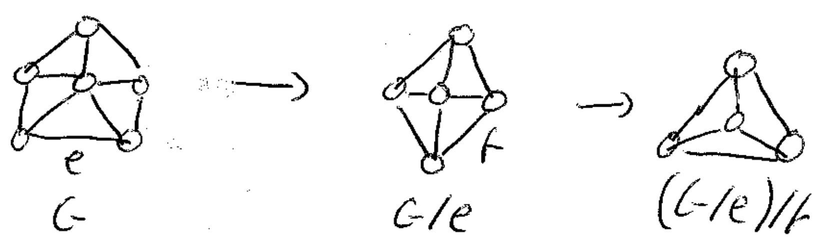
Edge deletion: $G-\{e\}=(V, E-\{e\})$



Vertex deletion: $G-\{v\}=(V-\{v\}, E-\{e \mid v\in\psi(e)\})$

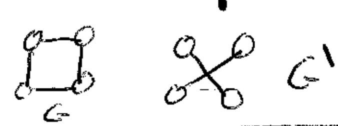


Contraction: G/e is:
- Delete e
- Identify $v_1, v_2 \in \psi(e)$
- Delete parallel edges



Complement: A simple graph G has complement

$$\bar{G}=K_{|G|}-E$$



Next time: Directed graphs, matrices of graphs (bonus