MATH6222 week 3 lecture 9

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Yesterday, L_n be arrangement of $3n^2$ tiles obtained by removing the top right quadrant from a $2^n \times 2^n$ square.

Let $P(n)$ be it is possible to L-tile L_n.

We prove this statement by induction on n.

It suffices to prove that:

1. $P(1), P(2)$ are both true.
2. $P(n-2) \implies P(n)$ for all $n \in \mathbb{N}$.

$P(1)$ is trivial, $P(2)$ is not hard as well. Base step checked.

Just need to check $P(n-2) \implies P(n)$

Observe that L_n is obtained from L_{n-2} by adding a band of width 2 as illustrated below...

By induction hypothesis, we may assume that L_{n-2} admits an L-tiling. So it suffices to prove (for any $n \geq 3$) that this width 2 band admits an l-tiling.

We consider three cases when n divisible by 3, $n-1$ divisible by 3, $n-2$ divisible by 3.

Since every integer n satisfies one of these three conditions, this is sufficient to solve our problem.

We make one preliminary observation:

If we let R_n denote a $2 \times n$ rectangle of squares, then R_n admits an L-tiling whenever n is divisible by 3.

Case 1: Note n divisible by 3 so that $2n-6$ divisible by 3. So we may tile the band as follows...

Case 2: Note $n-1$ divisible by 3 so that $n-4$ divisible by 3, $2n-8$ divisible by 3. The band could be tiled as follows:...

Case 3: Note $n-2$ divisible by 3 so that $2n-4$ divisible by 3

Principles of Strong Induction: We want to prove $\{P(k) : k \in \mathbb{N}\}$. It suffices to prove:

1. $P(1)$
2. If $P(i)$ is true for all $i < n$, then $P(n)$ is true.

Proof:

Suppose not all $P(k)$ are true. $P(1), P(2), P(3), \ldots,$
Look at the minimal k such that $P(k)$ is false.
Note by 1. that $k \neq 1$.
By our choice of k we know $P(1), P(2), \cdots, P(k-1)$ true.
By 2. knowing that $P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k-1)$ implies $P(k)$. Contradiction!

Game of Nim:

Each player takes a turn by removing some positive number of coins from some piles. The person who takes the last coin wins.
Let $P(n)$ be the statement that player 2 has a winning strategy for this game, when the starting configuration consists of 2 piles of equal size n.
$P(1)$ player 1 remove 1 pile, then player 2 win by taking the second pile.
Assume $P(1), \ldots, P(n-1)$ true, must prove $P(n)$.
Player 1 must start by taking m coins from one pile ($m \leq n$). Player 2 can respond by taking m coins from other pile. (If $m \leq n$, player 2 wins.)
Now player 1 and player 2 face the same game with a starting size of $n-m$. But buy induction hypothesis, $P(n-m)$ is true, so player 2 can win.

Let $f : A \rightarrow B$ be a function. We say f is **injective** for each $b \in B$ there is at most one $a \in A$ such that $f(a) = b$. For all $a_1, a_2 \in A.(a_1 \neq a_2), f(a_1) \neq f(a_2)$.

We say f is **surjective** if for each $b \in B$, there is at least one $a \in A$ usch that $f(a) = b$.

We say f is **bijective** if it is both injective and surjective (also say f is a one-to-one correspondence).

If f is bijective, we may define $f^{-1} : B \rightarrow A$ by setting $f^{-1}(b)$ to be the unique $a \in A$ such that $f(a) = b$.

Let's consider $f(x) = x^2$ as a function from $\mathbb{R} \rightarrow \mathbb{R}$.
Is it injective? $f(1) = f(-1) = 1$. Not injective.
Is it surjective? $\nexists x \in \mathbb{R}$, such that $f(x) = -1$. Not surjective.
$\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$

Now consider $f(x) = x^2$ as a function from $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Now injective and surjective.

Just need to say for any $y \in \mathbb{R}_{\geq 0}$, $\sqrt{y}$ is the unique positive real number such that