33. When studying the spread of an epidemic, we assume that the probability that an infected individual will spread the disease to an uninfected individual is a function of the distance between them. Consider a circular city of radius 10 mi in which the population is uniformly distributed. For an uninfected individual at a fixed point $A(x_0, y_0)$, assume that the probability function is given by

$$f(P) = \tfrac{1}{20}[20 - d(P, A)]$$

where $d(P, A)$ denotes the distance between P and A.

(a) Suppose the exposure of a person to the disease is the sum of the probabilities of catching the disease from all members of the population. Assume that the infected people are uniformly distributed throughout the city, with k infected individuals per square mile. Find a double integral that represents the exposure of a person residing at A.

(b) Evaluate the integral for the case in which A is the center of the city and for the case in which A is located on the edge of the city. Where would you prefer to live?

15.6 TRIPLE INTEGRALS

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

1 $$B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x\, \Delta y\, \Delta z$.

Then we form the **triple Riemann sum**

2 $$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}. By analogy with the definition of a double integral (15.1.5), we define the triple integral as the limit of the triple Riemann sums in (2).

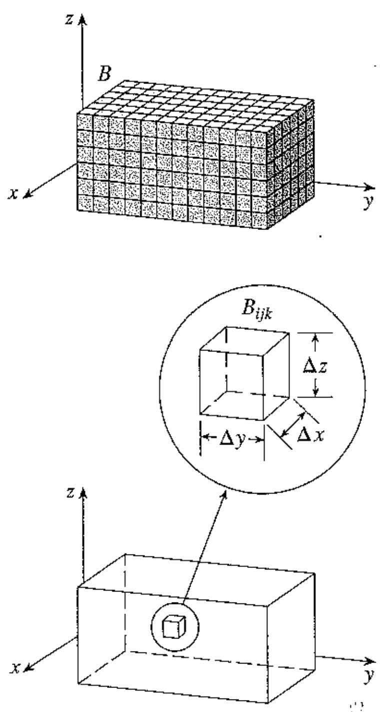


FIGURE 1

3 DEFINITION The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\boxed{\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V}$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 FUBINI'S THEOREM FOR TRIPLE INTEGRALS If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4}$$ □

Now we define the **triple integral over a general bounded region E** in three-dimensional space (a solid) by much the same procedure that we used for double integrals (15.3.2). We enclose E in a box B of the type given by Equation 1. Then we define a function F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth." The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 15.3).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\} \tag{5}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

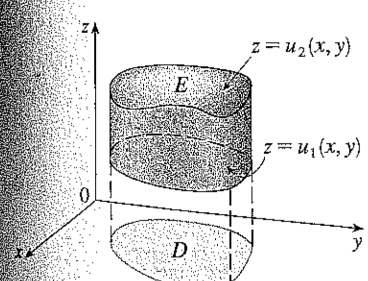


FIGURE 2
A type 1 solid region

By the same sort of argument that led to Formula 15.3.3, it can be shown that if E is a type 1 region given by Equation 5, then

$$\boxed{6} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

$$\boxed{7} \qquad \iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

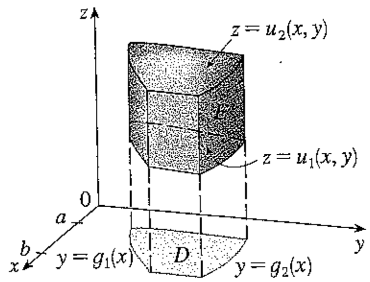


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

If, on the other hand, D is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

$$\boxed{8} \qquad \iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

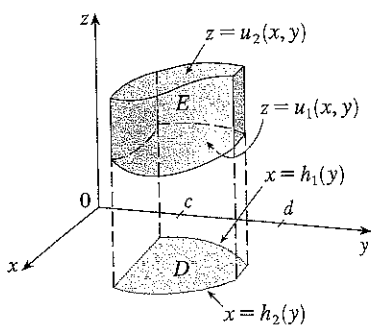


FIGURE 4
A type 1 solid region with a type II projection

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 5) and one of its projection D on the xy-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper

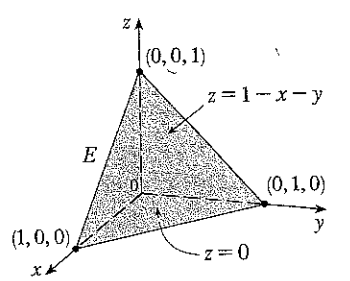


FIGURE 5

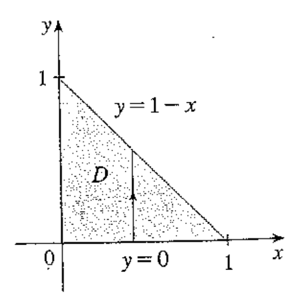


FIGURE 6

boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 6, and we have

$$\boxed{9} \qquad E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$

□

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

$$\boxed{10} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx\right] dA$$

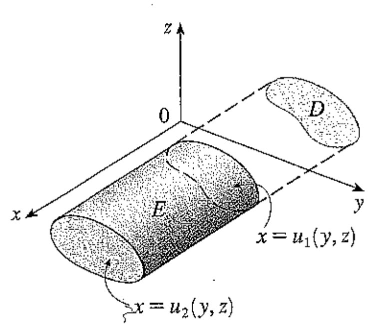


FIGURE 7 A type 2 region

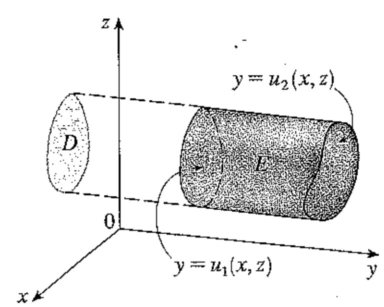


FIGURE 8 A type 3 region

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8). For this type of region we have

$$\boxed{11} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy\right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

SOLUTION The solid E is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

TEC Visual 15.6 illustrates how solid regions (including the one in Figure 9) project onto coordinate planes.

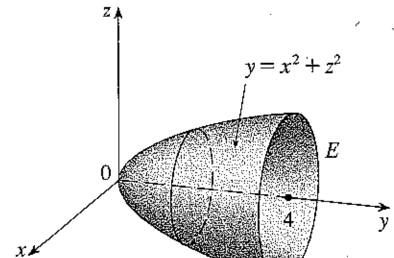


FIGURE 9
Region of integration

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

and so we obtain

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx$$

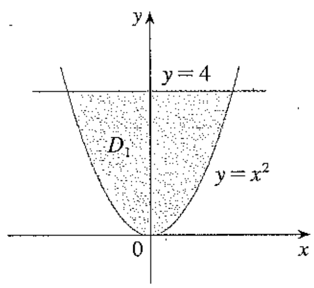


FIGURE 10
Projection on xy-plane

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz-plane is the disk $x^2 + z^2 \leqslant 4$ shown in Figure 11.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}\left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] dA = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA$$

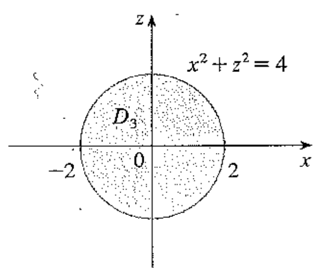


FIGURE 11
Projection on xz-plane

Although this integral could be written as

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dz\,dx$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 (4 - r^2)r\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\,dr$$

$$= 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$

The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

APPLICATIONS OF TRIPLE INTEGRALS

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y)\,dA$ represents the volume under the surface $z = f(x, y)$ and above D. The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\,dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\,dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12
$$\boxed{V(E) = \iiint_E dV}$$

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz\right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\,dA$$

and from Section 15.3 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

EXAMPLE 4 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D on the xy-plane are shown in Figures 12 and 13. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

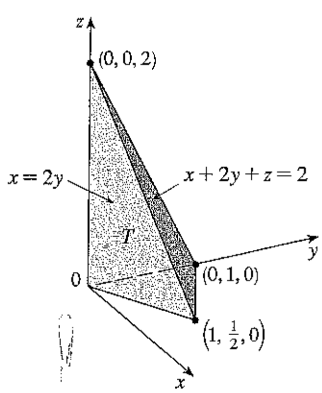


FIGURE 12

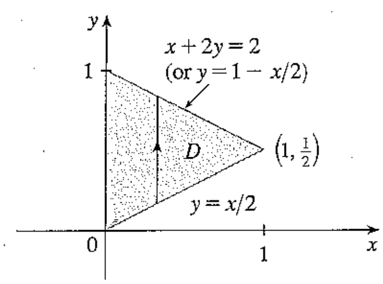


FIGURE 13

Therefore we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx = \tfrac{1}{3}$$

by the same calculation as in Example 4 in Section 15.3.

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.) ▫

All the applications of double integrals in Section 15.5 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

13
$$m = \iiint_E \rho(x, y, z)\, dV$$

and its **moments** about the three coordinate planes are

14
$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

15
$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

16
$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$$

As in Section 15.5, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\, dV$$

In particular,

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d,\ r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

EXAMPLE 5 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

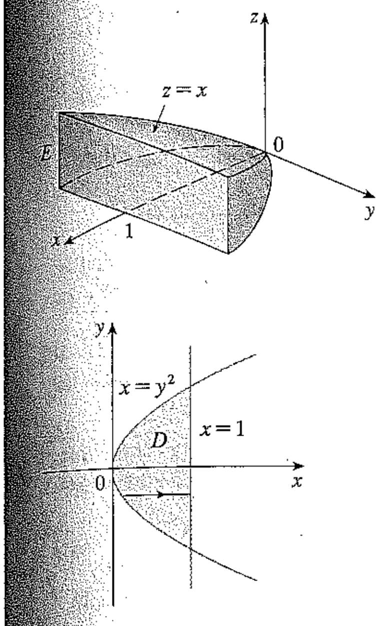


FIGURE 14

SOLUTION The solid E and its projection onto the xy-plane are shown in Figure 14. The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant x\}$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint_E \rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} \rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy$$

$$= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\, dy = \rho \int_{0}^{1} (1 - y^4)\, dy$$

$$= \rho \left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$M_{yz} = \iiint_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} x\rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy$$

$$= \frac{2\rho}{3} \int_{0}^{1} (1 - y^6)\, dy = \frac{2\rho}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \frac{4\rho}{7}$$

$$M_{xy} = \iiint_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z\rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2}\right]_{z=0}^{z=x} dx\, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy$$

$$= \frac{\rho}{3} \int_{0}^{1} (1 - y^6)\, dy = \frac{2\rho}{7}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14}\right) \qquad \square$$

15.6 EXERCISES

1. Evaluate the integral in Example 1, integrating first with respect to y, then z, and then x.

2. Evaluate the integral $\iiint_E (xz - y^3)\,dV$, where

$$E = \{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 1\}$$

using three different orders of integration.

3–8 Evaluate the iterated integral.

3. $\int_0^1 \int_0^z \int_0^{x+z} 6xz\,dy\,dx\,dz$

4. $\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx$

5. $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy$

6. $\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz$

7. $\int_0^{\pi/2} \int_0^y \int_0^x \cos(x + y + z)\,dz\,dx\,dy$

8. $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\,dy\,dz\,dx$

9–18 Evaluate the triple integral.

9. $\iiint_E 2x\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant \sqrt{4 - y^2},\ 0 \leqslant z \leqslant y\}$

10. $\iiint_E yz\cos(x^5)\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x,\ x \leqslant z \leqslant 2x\}$

11. $\iiint_E 6xy\,dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

12. $\iiint_E y\,dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$

13. $\iiint_E x^2 e^y\,dV$, where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$

14. $\iiint_E xy\,dV$, where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$

15. $\iiint_T x^2\,dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

16. $\iiint_T xyz\,dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 0, 1)$

17. $\iiint_E x\,dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

18. $\iiint_E z\,dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

19–22 Use a triple integral to find the volume of the given solid.

19. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

20. The solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$, and $y = 9$

21. The solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$

22. The solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 16$

23. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.

CAS (b) Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to find the exact value of the triple integral in part (a).

24. (a) In the **Midpoint Rule for triple integrals** we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate $\iiint_B \sqrt{x^2 + y^2 + z^2}\,dV$, where B is the cube defined by $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 4$, $0 \leqslant z \leqslant 4$. Divide B into eight cubes of equal size.

CAS (b) Use a computer algebra system to approximate the integral in part (a) correct to the nearest integer. Compare with the answer to part (a).

25–26 Use the Midpoint Rule for triple integrals (Exercise 24) to estimate the value of the integral. Divide B into eight sub-boxes of equal size.

25. $\iiint_B \dfrac{1}{\ln(1 + x + y + z)}\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 8,\ 0 \leqslant z \leqslant 4\}$

26. $\iiint_B \sin(xy^2z^3)\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 1\}$

27–28 Sketch the solid whose volume is given by the iterated integral.

27. $\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx$

28. $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\,dz\,dy$

29–32 Express the integral $\iiint_E f(x, y, z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces.

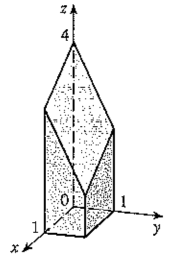
29. $y = 4 - x^2 - 4z^2$, $\quad y = 0$

30. $y^2 + z^2 = 9$, $\quad x = -2$, $\quad x = 2$

31. $y = x^2$, $\quad z = 0$, $\quad y + 2z = 4$

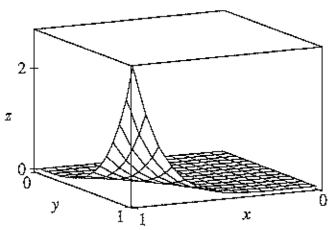
32. $x = 2$, $\quad y = 2$, $\quad z = 0$, $\quad x + y - 2z = 2$

23.



25. 47.5 **27.** $\frac{166}{27}$ **29.** 2 **31.** $\frac{64}{3}$

33. $21e - 57$



35. $\frac{5}{6}$

37. Fubini's Theorem does not apply. The integrand has an infinite discontinuity at the origin.

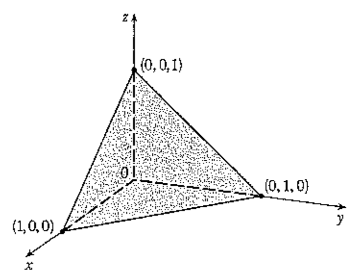
EXERCISES 15.3 ▪ PAGE 972

1. 32 **3.** $\frac{3}{10}$ **5.** $e - 1$ **7.** $\frac{4}{3}$ **9.** π **11.** $\frac{1}{2}e^{16} - \frac{17}{2}$

13. $\frac{1}{2}(1 - \cos 1)$ **15.** $\frac{147}{20}$ **17.** 0 **19.** $\frac{7}{18}$ **21.** $\frac{31}{8}$

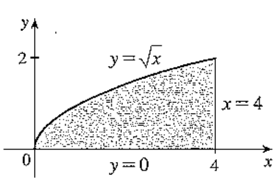
23. 6 **25.** $\frac{128}{15}$ **27.** $\frac{1}{3}$ **29.** 0, 1.213, 0.713 **31.** $\frac{64}{3}$

33.

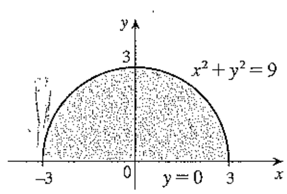


35. 13,984,735,616/14,549,535 **37.** $\pi/2$

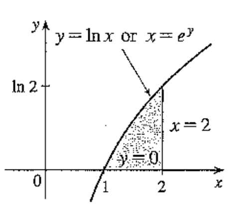
39. $\int_0^2 \int_{y^2}^4 f(x, y)\, dx\, dy$



41. $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x, y)\, dy\, dx$



43. $\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\, dx\, dy$



45. $\frac{1}{6}(e^9 - 1)$ **47.** $\frac{1}{3}\ln 9$ **49.** $\frac{1}{3}(2\sqrt{2} - 1)$ **51.** 1

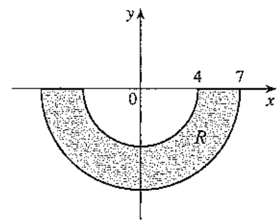
53. $(\pi/16)e^{-1/16} \leqslant \iint_Q e^{-(x^2+y^2)^2}\, dA \leqslant \pi/16$ **55.** $\frac{3}{4}$

59. 8π **61.** $2\pi/3$

EXERCISES 15.4 ▪ PAGE 978

1. $\int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$ **3.** $\int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\, dy\, dx$

5.



$33\pi/2$

7. 0 **9.** $\frac{1}{2}\pi \sin 9$ **11.** $(\pi/2)(1 - e^{-4})$ **13.** $\frac{3}{64}\pi^2$

15. $\pi/12$ **17.** $\frac{1}{8}(\pi - 2)$ **19.** $\frac{16}{3}\pi$ **21.** $\frac{4}{3}\pi$

23. $\frac{4}{3}\pi a^3$ **25.** $(2\pi/3)\left[1 - (1/\sqrt{2})\right]$

27. $(8\pi/3)(64 - 24\sqrt{3})$

29. $\frac{1}{2}\pi(1 - \cos 9)$ **31.** $2\sqrt{2}/3$

33. 1800π ft^3 **35.** $\frac{15}{16}$ **37.** (a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/2$

EXERCISES 15.5 ▪ PAGE 988

1. $\frac{64}{3}$ C **3.** $\frac{4}{3}$, $\left(\frac{4}{3}, 0\right)$ **5.** 6, $\left(\frac{3}{4}, \frac{3}{2}\right)$

7. $\frac{1}{4}(e^2 - 1), \left(\dfrac{e^2 + 1}{2(e^2 - 1)}, \dfrac{4(e^3 - 1)}{9(e^2 - 1)}\right)$

9. $L/4$, $(L/2, 16/(9\pi))$ **11.** $\left(\frac{3}{8}, 3\pi/16\right)$ **13.** $(0, 45/(14\pi))$

15. $(2a/5, 2a/5)$ if vertex is (0, 0) and sides are along positive axes

17. $\frac{1}{16}(e^4 - 1), \frac{1}{8}(e^2 - 1), \frac{1}{16}(e^4 + 2e^2 - 3)$

19. $7ka^6/180$, $7ka^6/180$, $7ka^6/90$ if vertex is (0, 0) and sides are along positive axes

21. $m = \pi^2/8$, $(\bar{x}, \bar{y}) = \left(\dfrac{2\pi}{3} - \dfrac{1}{\pi}, \dfrac{16}{9\pi}\right)$, $I_x = 3\pi^2/64$,
$I_y = \frac{1}{16}(\pi^4 - 3\pi^2)$, $I_0 = \pi^4/16 - 9\pi^2/64$

23. $\rho bh^3/3$, $\rho b^3h/3$; $b/\sqrt{3}$, $h/\sqrt{3}$

25. $\rho a^4\pi/16$, $\rho a^4\pi/16$; $a/2$, $a/2$

27. (a) $\frac{1}{2}$ (b) 0.375 (c) $\frac{5}{48} \approx 0.1042$

29. (b) (i) $e^{-0.2} \approx 0.8187$
(ii) $1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$ (c) 2, 5

31. (a) $\approx$0.500 (b) $\approx$0.632

33. (a) $\iint_D (k/20)\left[20 - \sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$, where D is the disk with radius 10 mi centered at the center of the city
(b) $200\pi k/3 \approx 209k$, $200(\pi/2 - \frac{8}{9})k \approx 136k$, on the edge

EXERCISES 15.6 ▪ PAGE 998

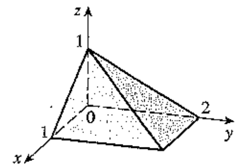
1. $\frac{27}{4}$ **3.** 1 **5.** $\frac{1}{3}(e^3 - 1)$ **7.** $-\frac{1}{3}$ **9.** 4 **11.** $\frac{65}{28}$

13. $8/(3e)$ **15.** $\frac{1}{60}$ **17.** $16\pi/3$ **19.** $\frac{16}{3}$ **21.** 36π

23. (a) $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$ (b) $\frac{1}{4}\pi - \frac{1}{3}$

25. 60.533

27.



29. $\int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dy\, dx$
$= \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dx\, dy$
$= \int_{-1}^{1}\int_{0}^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dy\, dz$
$= \int_{0}^{4}\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dz\, dy$
$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{0}^{4-x^2-4z^2} f(x, y, z)\, dy\, dz\, dx$
$= \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_{0}^{4-x^2-4z^2} f(x, y, z)\, dy\, dx\, dz$

31. $\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-y/2} f(x, y, z)\, dz\, dy\, dx$
$= \int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{2-y/2} f(x, y, z)\, dz\, dx\, dy$
$= \int_{0}^{2}\int_{0}^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$
$= \int_{0}^{4}\int_{0}^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dz\, dy$
$= \int_{-2}^{2}\int_{0}^{2-x^2/2}\int_{x^2}^{4-2z} f(x, y, z)\, dy\, dz\, dx$
$= \int_{0}^{2}\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x, y, z)\, dy\, dx\, dz$

33. $\int_{0}^{1}\int_{\sqrt{x}}^{1}\int_{0}^{1-y} f(x, y, z)\, dz\, dy\, dx$
$= \int_{0}^{1}\int_{0}^{y^2}\int_{0}^{1-y} f(x, y, z)\, dz\, dx\, dy$
$= \int_{0}^{1}\int_{0}^{1-z}\int_{0}^{y^2} f(x, y, z)\, dx\, dy\, dz$
$= \int_{0}^{1}\int_{0}^{1-y}\int_{0}^{y^2} f(x, y, z)\, dx\, dz\, dy$
$= \int_{0}^{1}\int_{0}^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx$
$= \int_{0}^{1}\int_{0}^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz$

35. $\int_{0}^{1}\int_{y}^{1}\int_{0}^{y} f(x, y, z)\, dz\, dx\, dy = \int_{0}^{1}\int_{0}^{x}\int_{0}^{y} f(x, y, z)\, dz\, dy\, dx$
$= \int_{0}^{1}\int_{z}^{1}\int_{y}^{1} f(x, y, z)\, dx\, dy\, dz = \int_{0}^{1}\int_{0}^{y}\int_{y}^{1} f(x, y, z)\, dx\, dz\, dy$
$= \int_{0}^{1}\int_{0}^{x}\int_{z}^{x} f(x, y, z)\, dy\, dz\, dx = \int_{0}^{1}\int_{z}^{1}\int_{z}^{x} f(x, y, z)\, dy\, dx\, dz$

37. $\frac{79}{30}, \left(\frac{358}{553}, \frac{33}{79}, \frac{571}{553}\right)$ **39.** $a^5, (7a/12, 7a/12, 7a/12)$

41. $I_x = I_y = I_z = \frac{2}{3}kL^5$ **43.** $\frac{1}{2}\pi kha^4$

45. (a) $m = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} \sqrt{x^2 + y^2}\, dz\, dy\, dx$
(b) $(\bar{x}, \bar{y}, \bar{z})$, where
$\bar{x} = (1/m)\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} x\sqrt{x^2 + y^2}\, dz\, dy\, dx$
$\bar{y} = (1/m)\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} y\sqrt{x^2 + y^2}\, dz\, dy\, dx$
$\bar{z} = (1/m)\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} z\sqrt{x^2 + y^2}\, dz\, dy\, dx$
(c) $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} (x^2 + y^2)^{3/2}\, dz\, dy\, dx$

47. (a) $\frac{3}{32}\pi + \frac{11}{24}$
(b) $(\bar{x}, \bar{y}, \bar{z}) = \left(\dfrac{28}{9\pi + 44}, \dfrac{30\pi + 128}{45\pi + 220}, \dfrac{45\pi + 208}{135\pi + 660}\right)$
(c) $\frac{1}{240}(68 + 15\pi)$

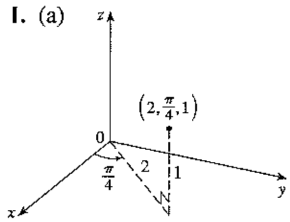
49. (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$

51. $L^3/8$

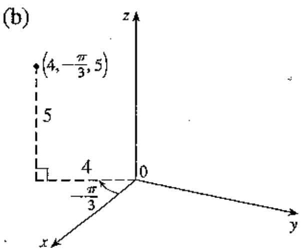
53. The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$

EXERCISES 15.7 ▪ PAGE 1004

1. (a)



(b)



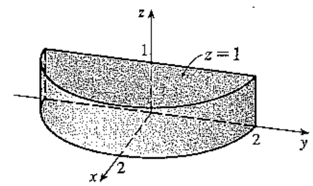
$(\sqrt{2}, \sqrt{2}, 1)$ $(2, -2\sqrt{3}, 5)$

3. (a) $(\sqrt{2}, 7\pi/4, 4)$ (b) $(2, 4\pi/3, 2)$

5. Vertical half-plane through the z-axis **7.** Circular paraboloid

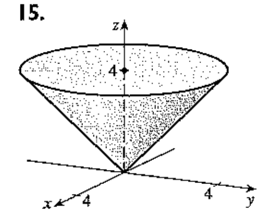
9. (a) $z = r^2$ (b) $r = 2\sin\theta$

11.



13. Cylindrical coordinates: $6 \leq r \leq 7$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 20$

15.



$64\pi/3$

17. 384π **19.** $\pi(e^6 - e - 5)$ **21.** $2\pi/5$

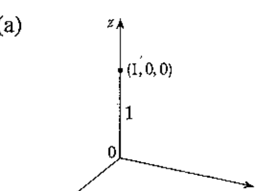
23. (a) 162π (b) $(0, 0, 15)$

25. $\pi Ka^2/8, (0, 0, 2a/3)$ **27.** 0

29. (a) $\iiint_C h(P)g(P)\, dV$, where C is the cone
(b) $\approx 3.1 \times 10^{19}$ ft-lb

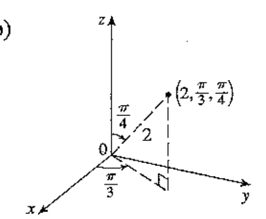
EXERCISES 15.8 ▪ PAGE 1010

1. (a)



$(0, 0, 1)$

(b)



$(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{6}, \sqrt{2})$