Slns ps 8

1.

a) $$\int_C \nabla f \cdot dx = \int_a^b \underbrace{\nabla f\,(g(t))\cdot g'(t)\,dt}$$

$$\int_a^b \frac{d}{dt} f(g(t))\;dt \underset{FTC}{=} f(g(t))\Big|_a^b$$

$$= f(g(b)) - f(g(a)) = f(B) - f(A)$$

b) $$\int_C H \cdot dx = \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt = 2\pi \neq 0$$

let C be The unit Circle $g(t) = (\cos t, \sin t)$
$g'(t) = (-\sin t, \cos t)$

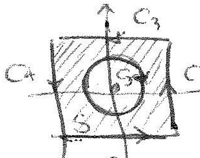
so $H \neq \nabla f$ for any C' function f.

c) $$\nabla \times H = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = 2k \neq 0 \text{ so } H \neq \nabla f \text{ as } \nabla\times\nabla f = 0$$

d) $$P(x,y) = \frac{-y}{x^2+y^2} \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = 0$$

$$Q(x,y) = \frac{x}{x^2+y^2} \qquad \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$$

$$\int_{\partial S} P\,dx + Q\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5} = \int_{C_1} P\,dx + \int_{C_2} Q\,dy + \int_{C_3} P\,dx + \int_{C_4} Q\,dy + \boxed{2\pi}$$



C_1: $dy = 0$, $y = -2$, $-2 \le x \le 2$

C_2: $dx = 0$, $x = 2$, $-2 \le y \le 2$

C_3: $dy = 0$, $y = 2$, $-2 \le x \le 2$

C_4: $dx = 0$, $x = -2$

2π: Similar to (b)

$$= \int_{-2}^{2} \frac{2}{x^2+4}\,dx + \int_{-2}^{2} \frac{2\,dy}{4+y^2} + \int_{t=-2}^{2} \frac{-2}{t^2+4}(-dt) + \int_{t=-2}^{2} \frac{-2}{4+t^2}(-dt) + 2\pi$$

$t = -x \qquad t = -y$

$-2 \le t \le 2$

$= 0$

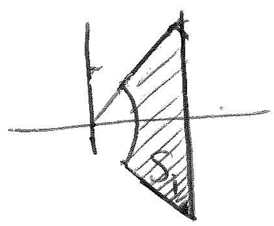
$$= 4\int_{-2}^{2} \frac{2\,dt}{4+t^2} - 2\pi = 4\int_{u=-1}^{1} \frac{4\,du}{4+4u^2} - 2\pi = 4\int_{-1}^{1} \frac{du}{1+u^2} = 4\tan^{-1}u\Big|_{-1}^{1} = 4\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$u = \frac{t}{2} \qquad du = \frac{dt}{2}$

$= 2\pi - 2\pi$

$= 0$

e)

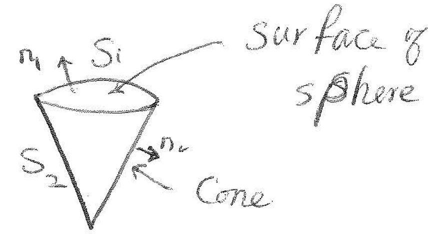


$$\nabla \tan^{-1}\frac{y}{x} = \begin{bmatrix} \dfrac{-y/x^2}{1+(y/x)^2} \\ \dfrac{1/x}{1+(y/x)^2} \end{bmatrix} = \begin{bmatrix} \dfrac{-y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \end{bmatrix}$$

Then $\int_{\partial S} H \cdot dx = \int_{\partial S} \nabla f \cdot dx = 0$ b/c ∂S lies within the region where $\tan^{-1}\frac{y}{x}$ is defined $x > 0$.

2.

a)



n_1 is (x, y, z) → $|n_1| = 1$

n_2 is ?

S_2: $G(u,v) = (u, v, 2\sqrt{u^2+v^2})$

$\frac{\partial G}{\partial u} = (1, 0, \frac{2u}{\sqrt{u^2+v^2}})$

$\frac{\partial G}{\partial v} = (0, 1, \frac{2v}{\sqrt{u^2+v^2}})$

$\frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} = \langle \frac{-2u}{\sqrt{\ }}, -\frac{2v}{\sqrt{\ }}, 1 \rangle$

$\left| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right| = \sqrt{\frac{4u^2+4v^2}{u^2+v^2}+1}$
$= \sqrt{5}$

Surface area $= \iint_{S_1} dS + \iint_{S_2} dS = \iint_{u^2+v^2 \le 1} \sqrt{5}\, dA + \iint_{x^2+y^2 \le 1} 1\, dA = \sqrt{5}\pi + \pi$

b) S: $G(x,y) = \langle x, y, 4-x^2-y^2 \rangle$ $\quad x^2+y^2 \le 4$

$\partial_x G \times \partial_y G = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$
$= \langle 2x, 2y, 1 \rangle$

$|\partial_x G \times \partial_y G| = \sqrt{1+4x^2+4y^2}$

$\iint_S f(x)\, dA = \iint_{x^2+y^2 \le 4} f(G(x,y)) \sqrt{1+4(x^2+y^2)}\, dx\, dy =$

Change to polar $\int_{\theta=0}^{2\pi} \int_{r=0}^{4} \underbrace{[4-\underbrace{(4-x^2-y^2)}_{z}]}_{r^2} \sqrt{1+4r^2}\, r\, dr\, d\theta$

$= \left[\int_0^{2\pi} d\theta\right] \left[\int_0^4 r^3 \sqrt{1+4r^2}\, dr\right] = (2\pi) \int_{w=1}^{65} \frac{w-1}{4 \times 8}\, dw = \frac{\pi}{16} \left[\frac{w^2}{2} - w\right]_1^{65}$

$1+4r^2 = w \quad dw = 8r\,dr$

so $r^2 = \frac{w-1}{4}$

$= \frac{\pi}{16} \left[\frac{65^2}{2} - 65 - \frac{1}{2} + 1\right]$

$= 128\pi$

c) $\mathbb{G}(u,v) = (u\cos v,\ u\sin v,\ v)$ $\qquad 0 \le v \le 2\pi$
$\qquad 0 \le u \le 1$

$$\partial_u \mathbb{G} \times \partial_v \mathbb{G} = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle$$

$$\mathbb{F}(\mathbb{G}(u,v)) = \langle \underbrace{u\cos v}_{x},\ \underbrace{u\sin v}_{y},\ \underbrace{v - 2u\sin v}_{z-2y} \rangle$$

$$\mathbb{F}(\mathbb{G}(u,v)) \cdot \partial_u\mathbb{G} \times \partial_v \mathbb{G} = u\cos v\sin v - u\sin v\cos v + uv - 2u^2\sin v$$

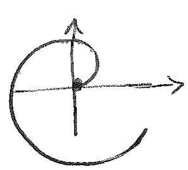
$$\iint_S \mathbb{F}\cdot n\,dA = \int_{v=0}^{2\pi}\int_{u=0}^{1} (\ \text{〃}\ )\,du\,dv = \int_{v=0}^{2\pi}\int_{u=0}^{1} [uv - 2u^2\sin v]\,du\,dv$$

$$= \left(\int_0^{2\pi} v\,dv\right)\left(\int_0^1 u\,du\right) - \left(2\int_0^1 u^2\,du\right)\underbrace{\left(\int_0^{2\pi}\sin v\,dv\right)}_{=0}$$

$$\frac{4\pi^2}{2} \times \frac{1}{2} = \pi^2$$

3 (a) $$g'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt},\ \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt} \right\rangle$$

$$\Rightarrow |g'(t)| = \cdots = \cdots = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}$$

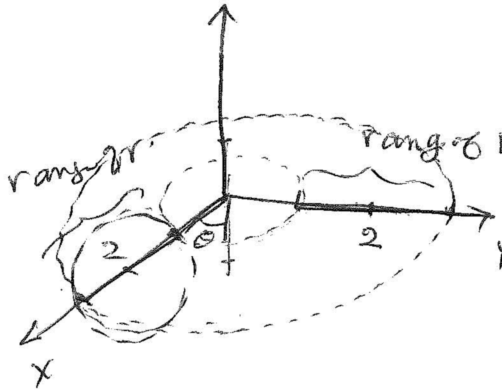
b) 

$$\text{length} = \int_0^c g'(t)\,dt = \int_0^c \sqrt{1+t^2}\,dt$$

$r = t \quad \frac{dr}{dt} = 1$

$\theta = t \quad \frac{d\theta}{dt} = 1$

c)



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$0 \le \theta \le 2\pi$

$x = r\cos\theta$

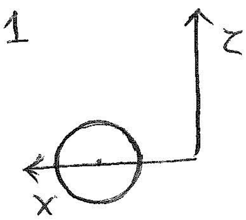
$z = \cos u \quad y = r\sin\theta$

On The xz plane $\quad \theta = 0 \quad x = r$

$x = \sin u + 2$

$z = \cos u$

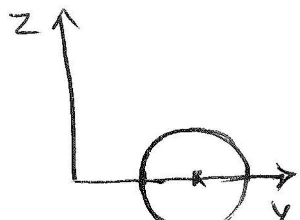
or $\quad (x-2)^2 + z^2 = 1$



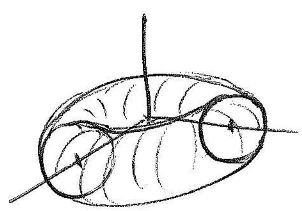
Similarly on The yz plane

$\theta = \frac{\pi}{2}$ so $y = r$

$= \sin u + 2$



To put This all together we get



and for any fixed $\theta = \theta_0$ (it is a plane That includes The z-axis)

$r =$

d)

$$\iint_T \left| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right| du\,dv \qquad \frac{\partial G}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle =$$

$$= \Big\langle \underbrace{\frac{dr}{du}}_{\cos u} \cos\theta - \underbrace{r\sin\theta}_{v} \underbrace{\frac{d\theta}{du}}_{=0},\ \underbrace{\frac{dr}{du}}_{\cos u} \sin\theta + \underbrace{r\cos\theta}_{\sin u + 2} \underbrace{\frac{d\theta}{du}}_{=0},\ -\cos u \Big\rangle$$

$$= \langle \cos u \cos v,\ \cos u \sin v,\ -\cos u \rangle \qquad \text{Similarly } \frac{\partial G}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$= \Big\langle \underbrace{\frac{dr}{dv}}_{0} \cos\theta - r\sin\theta \underbrace{\frac{d\theta}{dv}}_{1},\ \underbrace{\frac{dr}{dv}}_{0} \sin\theta + r\cos\theta \underbrace{\frac{d\theta}{dv}}_{1},\ 0 \Big\rangle = \langle -(2+\sin u)\sin v,\ (2+\sin u)\cos v,\ 0 \rangle$$

$=$

$$\frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} = \begin{vmatrix} i & j & k \\ \cos u \cos v & \cos u \sin v & -\cos u \\ -(2+\sin u)\sin v & (2+\sin u)\cos v & 0 \end{vmatrix} = 2\cos u \begin{vmatrix} i & j & k \\ \cos v & \sin v & -1 \\ -\sin v & \cos v & 0 \end{vmatrix} +$$

$$+ \sin u \cos u \begin{vmatrix} i & j & k \\ \cos v & \sin v & -1 \\ -\sin v & \cos v & 0 \end{vmatrix} =$$

$$2\cos u \langle +\cos v, \sin v, 1 \rangle + \sin u \cos u \langle \cos v, \sin v, 1 \rangle$$

$$= \cos u (2+\sin u) \langle \cos v, \sin v, 1 \rangle$$

$$|\partial_u G \times \partial_v G| = |\cos u (2+\sin u)| \, |\langle \cos v, \sin v, 1 \rangle| =$$

$$= |\cos u (2+\sin u)| \sqrt{\cos^2 v + \sin^2 v + 1} = \sqrt{2} |\cos u| \underbrace{(2+\sin u)}_{>0}$$

$$\text{Surface area} = \int_{v\,0}^{2\pi} \int_{u=0}^{2\pi} \sqrt{2}\, |\cos u| (2+\sin u)\, du\, dv$$

$$= 2 \int_{v=0}^{2\pi} \int_{u=\pi/2}^{3\pi/2} -\sqrt{2} \cos u (2+\sin u)\, du\, dv =$$

$$-2\sqrt{2} \left(\int_0^{2\pi} dv \right) \left(\int_{\pi/2}^{3\pi/2} \underbrace{(2+\sin u)}_{w} \underbrace{\cos u\, du}_{dw} \right)$$

$$= -2\sqrt{2}\,(2\pi) \left(\int_{w=3}^{1} w\, dw \right) = 2\sqrt{2}\pi \left(\frac{w^2}{2} \Big|_1^3 \right) = 2\sqrt{2}\pi (9-1)$$

$$= 16\sqrt{2}\pi$$