Australian National University
Research School of Finance, Actuarial Studies and Applied Statistics

STAT2032/6046: Financial Mathematics

Review Questions (Week 1 – Week 3)

WEEK 1

Question 1

At a certain rate of simple interest \$1100 will accumulate to \$1250 after a certain period of time. Find the accumulated value of \$500 at a rate of simple interest three fifths as great over twice as long a period of time.

Solution

$$1100(1+it)=1250 \Rightarrow (1+it)=\frac{1250}{1100} \Rightarrow it=\frac{150}{1100}$$

$$500\left(1+\frac{3}{5}2it\right)=X \Rightarrow X=500\left(1+\frac{6}{5}it\right) \Rightarrow X=500\left(1+\frac{6}{5}.\frac{150}{1100}\right)=581.82$$

Question 2

Find the total present value as at 1 June 1999 of payments of \$100 on 1 January 2000 and \$200 on 1 May 2000, assuming a rate of interest of 10% pa convertible quarterly.

Solution

$i^{(4)}=10\%$

The effective interest rate is $\frac{i^{(4)}}{4}=2.5\%$

There are four quarters in a year. The first payment of \$100 is 7 months after 1 June 1999 and the second payment of \$200 is 11 months after 1 June 1999. Working in units of a quarter this is 7/3 and 11/3 quarters respectively.

$$PV = 100v^{7/3} + 200v^{11/3} = 277.09$$

Question 3

i. Find the effective annual rate of interest corresponding to:
   a. a nominal rate of 13% convertible half-yearly
   b. a nominal rate of interest of 10% convertible monthly

ii. Find the rate of interest convertible monthly corresponding to:
   a. an effective rate of 5% per annum
   b. a nominal rate of 21% convertible five times a year

Solution

i. Effective annual rate of interest

   a. $i = \left(1 + \frac{i^{(\mathrm{m})}}{m}\right)^m - 1 = \left(1 + \frac{0.13}{2}\right)^2 - 1 = 13.42\%$

   b. $i = \left(1 + \frac{i^{(\mathrm{m})}}{m}\right)^m - 1 = \left(1 + \frac{0.10}{12}\right)^{12} - 1 = 10.47\%$

ii. Nominal rate convertible monthly

   a. $i^{(12)} = m\left((1+i)^{1/m} - 1\right) = 12\left((1.05)^{1/12} - 1\right) = 4.89\%$

   b. $i^{(12)} = m\left((1+i)^{1/m} - 1\right) = m\left(\left(\left(1 + \frac{i^{(\mathrm{k})}}{k}\right)^k\right)^{1/m} - 1\right) = 12\left(\left(1 + \frac{0.21}{5}\right)^{5/12} - 1\right) = 20.75\%$

Question 4

At a certain interest rate the present value of the following two payment patterns are equal:

i. \$150 at the end of 5 years plus \$450 at the end of 10 years.
ii. \$400 at the end of 5 years.

At the same interest rate \$80 invested now plus \$100 invested at the end of 5 years will accumulate to P at the end of 10 years. Calculate P.

Solution

$PV_1 = 150v_i^5 + 450v_i^{10}$
$PV_2 = 400v_i^5$
$PV = PV_2 \Rightarrow 150v_i^5 + 450v_i^{10} = 400v_i^5$

Multiply both sides by $(1+i)^{10}$

$$\Rightarrow 150(1+i)^5 + 450 = 400(1+i)^5 \Rightarrow (1+i)^5 = \frac{450}{250} = 1.8$$

$$P = 80(1+i)^{10} + 100(1+i)^5 = 80 \times 1.8^2 + 100 \times 1.8 = 439.20$$

Question 5

Fund A accumulates at 6% p.a. effective and Fund B accumulates at 8% p.a. effective. At the end of 18 years the total of the two funds is $3000. At the end of 10 years the amount in Fund A is half that in Fund B. What is the total of the two funds at the end of 7 years?

Solution

Let an amount of A be invested at time 0 in Fund A and B be invested in Fund B at time 0.

$$A(1.06)^{18} + B(1.08)^{18} = 3000$$
$$A(1.06)^{10} = 0.5B(1.08)^{10} \Rightarrow B = \frac{2A(1.06)^{10}}{(1.08)^{10}}$$
$$\Rightarrow A(1.06)^{18} + 2A(1.06)^{10}(1.08)^8 = 3000$$
$$\Rightarrow A = \frac{3000}{(1.06)^{18} + 2(1.06)^{10}(1.08)^8} = 316.33$$
$$\Rightarrow B = \frac{2(316.33)(1.06)^{10}}{(1.06)^{10}} = 524.80$$

After 7 years the accumulated value is:

$$A(1.06)^7 + B(1.08)^7 = 316.33(1.06)^7 + 524.80(1.08)^7 = \$1,375.06$$

WEEK 2

Question 6

A rate of interest of 8% pa convertible weekly is equivalent to an annual effective rate of discount of *d*. Find *d*.

Solution

$$1-d=(1+i)^{-1}=\left(1+\frac{i^{(\mathrm{m})}}{m}\right)^{-m}=\left(1+\frac{0.08}{52}\right)^{-52}=0.92317$$

$$d=1-0.92317=7.68\%$$

Question 7

Find the accumulated value of $5.50, assuming a force of interest of 4% per annum, after:

a. 1 month
b. 3 years and 12 days

Solution

a. $5.50e^{0.04/12}=\$5.52$
b. $5.50e^{0.04(3+12/365)}=\$6.21$

Question 8

$780 is invested for 13 months at i = 9.00%, then the investment is switched to one that pays interest at a force of δ = 8.00% for 10 months and then δ = 10.00% for five years. What is the final accumulated value?

Solution

$$780(1.09)^{13/12}\,e^{0.08(10/12)}e^{0.1(5)}=\$1{,}509.18$$

Question 9

When i = 8.00% find:

a. $i^{(1/2)}$
b. $d^{(5.5)}$
c. $i^{(4)}$
d. $d^{(2)}$
e. δ

Solution

a. $i^{(1/2)} = 0.5\left(1.08^2 - 1\right) = 0.0832$

b. $d = 1-(1+i)^{-1} = 0.074074 \Rightarrow d^{(5.5)} = 5.5\left(1-(1-0.074074)^{1/5.5}\right) = 0.076425$

c. $i^{(4)} = 4\left(1.08^{0.25} - 1\right) = 0.077706$

d. $d^{(2)} = 2\left(1-(1-0.074074)^{0.5}\right) = 0.075499$

e. $\delta = \ln(1.08) = 0.076961$

WEEK 3

Question 10

Find the present value of an annuity-due (ie. payable at the start of each period) of \$300 per annum payable half-yearly for 10 years if $d^{(12)}$ = 9.00%.

Solution

$$300\ddot{a}_{\overline{10|}}^{(2)} = 300\frac{1-v_i^{10}}{d^{(2)}}$$

$$1-d = \left(1-\frac{d^{(\mathrm{m})}}{m}\right)^m \Rightarrow d = 1-\left(1-\frac{d^{(12)}}{12}\right)^{12} = 0.086379$$

$$i = \frac{d}{1-d}$$

$$d^{(\mathrm{m})} = m\left(1-(1-d)^{1/\mathrm{m}}\right) \Rightarrow d^{(2)} = 2\left(1-(1-d)^{1/2}\right) = 0.088329$$

$$\Rightarrow 300\frac{1-v_i^{10}}{d^{(2)}} = 300\frac{1-1.094545^{-10}}{0.088329} = 2020.19$$

Question 11

Show that $a_{\overline{n}|}^{(m)} = \frac{1}{m} \ddot{a}_{\overline{n}|} \sum_{t=1}^{m} v^{t/m}$

Solution

$$a_{\overline{n}|}^{(m)} = \left[\frac{1}{m}v^{1/m} + \frac{1}{m}v^{2/m} + \ldots + \frac{1}{m}v^{m/m}\right] + \left[\frac{1}{m}v^{(1+m)/m} + \frac{1}{m}v^{(2+m)/m} + \ldots + \frac{1}{m}v^{2m/m}\right]$$
$$+ \ldots + \left[\frac{1}{m}v^{(1+m(n-1))/m} + \frac{1}{m}v^{(2+m(n-1))/m} + \ldots + \frac{1}{m}v^{nm/m}\right]$$

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left[v^{1/m} + v^{2/m} + \ldots + v^{m/m}\right] + \frac{1}{m}v\left[v^{1/m} + v^{2/m} + \ldots + v^{m/m}\right]$$
$$+ \ldots + \frac{1}{m}v^{n-1}\left[v^{1/m} + v^{2/m} + \ldots + v^{m/m}\right]$$

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left[v^{1/m} + v^{2/m} + \ldots + v^{m/m}\right] \times \left[1 + v + v^2 + \ldots + v^{n-1}\right] = \frac{1}{m}\ddot{a}_{\overline{n}|}\sum_{t=1}^{m} v^{t/m}$$

Question 12

If $\bar{a}_{\overline{n}|} = 3$ and $\bar{s}_{\overline{n}|} = 6$, find δ, where δ > 0

Solution

$$\bar{s}_{\overline{n}|} = \bar{a}_{\overline{n}|}(1+i)^n \Rightarrow (1+i)^n = \frac{\bar{s}_{\overline{n}|}}{\bar{a}_{\overline{n}|}} = 2$$

$$\bar{s}_{\overline{n}|} = 6 = \frac{(1+i)^n - 1}{\delta} = \frac{1}{\delta}$$

$$\delta = \frac{1}{6}$$

Question 13

If $a_{\overline{n}|} = x$ and $a_{\overline{2n}|} = y$, express *i* as a function of *x* and *y*.

Solution

$$a_{\overline{n}|} = x \Rightarrow \frac{1-v^n}{i} = x \Rightarrow v^n = 1 - ix \Rightarrow v^{2n} = (1-ix)^2 = 1 - 2ix + i^2x^2$$

$$a_{\overline{2n}|} = y \Rightarrow \frac{1-v^{2n}}{i} = y \Rightarrow v^{2n} = 1 - iy$$

$$1 - iy = 1 - 2ix + i^2x^2 \Rightarrow i\left(y - 2x + ix^2\right) = 0 \Rightarrow \left(y - 2x + ix^2\right) = 0$$
$$\Rightarrow i = \frac{2x-y}{x^2}$$

Question 14

Show that:

i. $\ddot{a}_{\overline{n}|} = a_{\overline{n}|} + 1 - v^n$

ii. $\ddot{s}_{\overline{n}|} = s_{\overline{n}|} - 1 + (1+i)^n$

Solution

i. $\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1} = 1 + \left[v + v^2 + \ldots + v^{n-1} + v^n\right] - v^n = a_{\overline{n}|} + 1 - v^n$

ii.

$$\begin{aligned}\ddot{s}_{\overline{n}|} &= (1+i)^n + (1+i)^{n-1} + \ldots + (1+i)^2 + (1+i) \\ &= (1+i)^n + \left[(1+i)^{n-1} + \ldots + (1+i)^2 + (1+i) + 1\right] - 1 \\ &= (1+i)^n + s_{\overline{n}|} - 1\end{aligned}$$

Question 15

Find the present value to the nearest dollar on January 1 of an annuity which pays $3000 every 6 months for 9 years. The first payment is due on the next April 1 and the rate of interest is 9% convertible half-yearly.

Solution

Work in time periods of 6-months, n = 18, half-yearly effective interest rate i = 4.5%

The present value at April 1 is:

$$3000\ddot{a}_{\overline{18}|0.045} = 3000\frac{1-v_{0.045}^{18}}{d} = 3000\frac{1-v_{0.045}^{18}}{0.045/1.045} = 38121.57$$

Therefore, the present value at January 1 (3 months prior to April 1) is:

$$(38121.57)v_{0.045}^{0.5} \cong \$37{,}292$$

Question 16

Evaluate the following at δ = 1.5%

i. $s_{\overline{5}|}$

ii. $\ddot{s}_{\overline{10}|}$

iii. $\bar{s}_{\overline{4.5}|}$

Solution

$$i = e^{\delta} - 1 = e^{0.015} - 1 = 1.51113\%$$
$$d = 1 - e^{\delta} = 1 - e^{-0.015} = 1.4888\%$$

i. $s_{\overline{5}|} = \dfrac{(1+i)^5 - 1}{i} = 5.15343$

ii. $\ddot{s}_{\overline{10}|} = \dfrac{(1+i)^{10} - 1}{d} = 10.87006$

iii. $\bar{s}_{\overline{4.5}|} = \dfrac{(1+i)^{4.5} - 1}{\delta} = 4.65533$

Question 17

Evaluate the following functions at i = 4.5%.

i. $a_{\overline{10}|}^{(12)}$

ii. $\ddot{a}_{\overline{20}|}^{(2)}$

iii. $\bar{s}_{\overline{15}|}$

Solution

i. $a^{(12)}_{\overline{10|}} = \dfrac{1-v^{10}}{i^{(12)}} = \dfrac{1-1.045^{-10}}{12\left(1.045^{1/12}-1\right)} = 8.07462$

ii. $\ddot{a}^{(2)}_{\overline{20|}} = \dfrac{1-v^{20}}{d^{(2)}}$

$$\frac{1}{1.045} = \left(1-\frac{d^{(2)}}{2}\right)^2 \Rightarrow d^{(2)} = 0.043536$$

$$\ddot{a}^{(2)}_{\overline{20|}} = \frac{1-1.045^{-20}}{0.043536} = 13.44536$$

iii. $\bar{s}_{\overline{15|}} = \dfrac{\left(1+i\right)^{15}-1}{\delta} = \dfrac{1.045^{15}-1}{\ln\left(1.045\right)} = 21.24826$

Question 18

Find the present value as at 1 June 2000 of 50 monthly payments each of \$200 commencing on 1 January 2001, assuming a rate of interest of 12% pa convertible half yearly.

Solution

Working in units of months, we have $n = 50$.

$i^{(2)} = 0.12 \Rightarrow$ half-yearly effective interest rate = 6%

$\Rightarrow$ monthly effective interest rate = $(1.06)^{1/6} - 1 = 0.9759\%$

$$200\ddot{a}_{\overline{50|}}v^7 = 200a_{\overline{50|}}v^6 = \$7,437$$

Question 19

i. Find the combined present value of an immediate annuity payable monthly in arrears such that payments are \$500 pa for the first 5 years and \$350 pa for the next 2 years.
ii. Calculate the amount of the level annuity payable continuously for 5 years having the same present value as the payments in (i)
iii. Calculate the accumulated values of the first 4 years' payments at the end of the 4th year for the payments in (i) and (ii).

Assume an interest rate of 24% pa convertible monthly.

Solution

i. Working in monthly time periods, using a rate of interest of 2% per month, the required value is:

$$PV = \frac{500}{12} a_{\overline{60}|} + \frac{350}{12} a_{\overline{24}|} v^{60} = \$1,616.51$$

ii. If the annual rate of payment is X then:

$$1616.51 = \frac{X}{12} \bar{a}_{\overline{60}|} = \frac{X}{12} \frac{1-v^{60}}{\delta} = \frac{X}{12} 35.10735$$

$$X = \$552.20$$

iii. The accumulated value of the first 4 years' payments in (i) is:

$$AV = \frac{500}{12} s_{\overline{48}|} = \$3,306.40$$

The accumulated value for (ii) is:

$$AV = \frac{552.20}{12} \bar{s}_{\overline{48}|} = \$3,687.98$$

Question 20

Find the accumulated value 32 years after the first payment is made of an annuity on which there are 8 payments of \$1000 each made at two-year intervals. The nominal rate of interest convertible half-yearly is 7%.

Solution

Work in intervals of 2-years. The annual effective interest rate is $\left(1+\frac{0.07}{2}\right)^2 - 1$ so the two-year effective rate is $i = \left(\left(1+\frac{0.07}{2}\right)^2\right)^2 - 1 = (1.035)^4 - 1 = 14.7523\%$

The accumulated value at the date of the last payment (at year 14) is:

$$1000 s_{\overline{8}|i} = 1000 \frac{(1+i)^8 - 1}{i} = 13,602.68$$

We need to accumulate for another 9 periods of two-years: $13,602.68(1+i)^9 = \$46,932.85$