STAT7017 Big Data Statistics **Semester 2 2018**

Lecture 6: 27 August

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Central Limit Theorem

Recall that Central limit theorems (CLTs) describe how the sum of random variables fluctuates around some quantity (e.g. the mean).

The classic CLT case is to consider a sequence $X_1, X_2, \dots$ of I.I.D. random variables with $\mathbb{E}[X_i] = \mu$ and $Var[X_i] = \sigma^2 < \infty$, then the (Lindeberg-Levy) CLT says if $S_n := \sum_{k=1}^n X_k$ then

$$\sqrt{n}(S_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$

This lecture we will look at some equivalent statements in our random matrix setting. In particular of linear spectral statistics of the form

$$T_n = \frac{1}{p}\sum_{k=1}^{p} \phi(\lambda_k) = \int \phi(x) dF^{\mathbf{A}_n}(x) := F^{\mathbf{A}_n}(\phi).$$

of some sample matrix $\mathbf{A}_n$, e.g.

$$\mathbf{A}_n = \begin{cases} \mathbf{S}_n, & \text{sample covariance matrix.} \\ \mathbf{F}_n, & \text{Fisher matrix.} \end{cases}$$

Some examples that we will see later in the course are:

Example 1: The generalized variance is

$$T_n = \frac{1}{p}\log|S_n| = \frac{1}{p}\sum_{k=1}^{p}\log(\lambda_k).$$

$$\phi(x) = \log(x)$$

Example 2: Later in the course, we shall look at testing equality of sample covariance matrices. To test the hypothesis $H_0 : \sum = \mathbf{I}_p$, we shall look at the log-likelihood ratio statistic

$$LRT_1 = tr\mathbf{S}_n - \log|\mathbf{S}_n| - p = \sum_{k=1}^{p}(\lambda_k - \log(\lambda_k) - 1).$$

i.e. $\phi(x) = x - \log(x) - 1$.

Example 3: We shall also look at the two-sample test of the hypothesis $H_0 : \sum_1 = \sum_2$ that two populations have a common covariance matrix

$$LRT_2 = -\log|\mathbf{I}_p + \alpha_n \mathbf{F}_n| = -\sum_{k=1}^{p}(1 + \alpha_n \log(\lambda_k))$$

where α_n is same constant.

$$\phi(x) = -\log(1 - \alpha_n x).$$

CLT for Linear Spectral Statistics of $\mathbf{S}_n$

We shall consider simple case

$$\mathbf{S}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^*$$

where these are "independent vectors without cross-correlation".

In other words, the data matrix $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = (x_{ij})$ of size $p \times n$ has IID entries with $\mathbb{E}[x_{ij}] = 0, \mathbb{E}|x_{ij}|^2 = 1$.

$$\mathbf{S}_n = \frac{1}{n}\mathbf{X}\mathbf{X}^*.$$

The LSD of $\mathbf{S}_n$ is the Marchenko-Pastur law F_y where $y = \lim \frac{p}{n}$: This means, $F^{\mathbf{S}_n}(\phi) \to F_y(\phi)$ for any continuous function ϕ.

Making an analogy to the class CLT, we would like to understand how $F^{\mathbf{S}_n}(\phi)$ fluctuates around $F_y(\phi)$ as $n \to \infty (p \to \infty)$.

From RMT, we know that $F^{\mathbf{S}_n}(\phi)$ fluctuates around its mean in such a way that $P[F^{\mathbf{S}_n}(\phi) - \mathbb{E}(F^{\mathbf{S}_n}(\phi))] \sim$ Normal.

We can decompose

$$P[F^{\mathbf{S}_n}(\phi) - F_y(\phi)] = P[F^{\mathbf{S}_n}(\phi) - \mathbb{E}F^{\mathbf{S}_n}(\phi)] + P[\mathbb{E}[F^{\mathbf{S}_n}(\phi)] - F_y(\phi)] = \text{Normal} + \text{Bias}$$

The "bias" term is often a function of $y_n - y = \frac{p}{n} - y$.

y_n is called the dimension-to-sample ratio and the difference to y can be of any order. For example, if

$$y_n - y \approx p^{-\alpha}, \alpha > 0$$

then the bias term behaves like $p^{-1-\alpha}$ and the value depends on α. If α small then $p^{1-\alpha}$ can blow-up and if α large then $p^{1-\alpha}$ converges to zero or constant, as $p \to \infty$.

We need more restrictions on $y_n - y$.

We also need to accurately estimate $\mathbb{E}F^{\mathbf{S}_n}(\phi)$. One way is to estimate $\mathbb{E}F^{\mathbf{S}_n}(\phi) \approx F_{y_n}(\phi)$. "finite horizon proxy".

We saw last week that the ST $\underline{S}$ of $\underline{F}_y := (1-y)\delta_0 + yF_y$ satisfies the equation that we found for the Generalized MP ($H = \delta_1$):

$$Z = -\frac{1}{\underline{S}} + \frac{y}{1+\underline{S}}, Z \in \mathbb{C}.$$

Let $\beta = \mathbb{E}|x_{ij}|^4 - 1 - k, h = \sqrt{y}$. (????)

Set $k = 2$ if entries of $\mathbf{X}$ are real and $k = 1$ if complex values.

If entries are Gaussian, $\beta = 0$.

The following theorem quantifies the fluctuations of

$$P(F^{\mathbf{S}_n}(\phi) - F_{y_n}(\phi)).$$

Theorem 6.1 (Bai & Silverstein; 2004) *Assume $p \times n$ data matrix $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ has IID entires $\mathbb{E}x_{ij} = 0, \mathbb{E}|x_{ij}|^2 = 1, \mathbb{E}|x_{ij}|^4 = \beta + 1 + k < \infty$.*

Also, $p \to \infty, n \to \infty, p/n \to y > 0$.

Let $f_1, f_2, \ldots, f_k$ be analytic functions on an open region containing support of F_y.

The random vector $(X_n(f_1), X_n(f_2), \ldots, X_n(f_k))$ where

$$X_n(f) := P(F^{\mathbf{S}_n}(f) - \mathbf{F}_{y_n}(f))$$

converges weakly to a Gaussian vector

$$(X_{f_1}, \ldots, X_{f_k})$$

with mean

$$\mathbb{E}X_f = (k-1)I_1(f) - \beta I_2(f)$$

and

$$Cov(X_f, X_g) = kJ_1(f,g) + \beta J_2(f,g).$$

where

$$I_1(f) = -\frac{1}{2\pi i}\oint \frac{y(\underline{S}/(1+\underline{S})^3(z)f(z)}{[\mathbf{1} - y(\underline{S}/(1+\underline{S}))^2]^2}dz.$$

$$I_2(f) = -\frac{1}{2\pi i}\oint \frac{y(\underline{S}/(1+\underline{S}))^3(z)f(z)}{\mathbf{1} - y(\underline{S}/(1+\underline{S}))}dz.$$

and

$$J_1(f,g) = -\frac{1}{4\pi^2}\oint\oint \frac{f(z_1)f(z_2)}{(\underline{S}(z_1)-\underline{S}(z_2))^2}\underline{S}'(z_1)\underline{S}'(z_2)dz_1dz_2.$$

$$J_2(f,g) = -\frac{y}{4\pi^2}\oint f(z_1)\frac{\partial}{\partial z_1}\left(\frac{\underline{S}}{1+\underline{S}}(z_1)\right)dz_1 \times \oint g(z_2)\frac{\partial}{\partial z_2}\left(\frac{\underline{S}}{1+\underline{S}}(z_2)\right)dz_2$$

where the integrals are over contours enclosing the support of F_y.

Remarks:

- The asymptotic mean $\mathbb{E}[X_f]$ is non-null and depends on fourth moment.
- This theorem is difficult to use in practice because the limiting parameters are integrals on contours that are not given explicitly.
- This theorem, from 2004, was a big breakthrough as it gave explicit formulas for the limiting mean and covariance.

A more explicit version of this theorem can be obtained:

Proposition 6.2 *We have*

$$I_1(f) = \lim_{r\downarrow 1} I_1(f,r)$$

$$I_2(f) = \frac{1}{2\pi i}\oint_{|\xi|=1} f(|1+h\xi|^2)\frac{1}{\xi^3}d\xi$$

$$J_1(f,g) = \lim_{r\downarrow 1} J_1(f,g,r)$$

$$J_2(f,g) = -\frac{1}{4\pi^2}\oint_{|\xi_1|=1}\frac{f(|1+h\xi_1|^2)}{\xi_1^2}d\xi_1\oint_{|\xi_2|=1}\frac{g(|1+h\xi_2|^2)}{\xi_2^2}d\xi_2$$

with

$$I_1(f,r) = \frac{1}{2\pi i}\oint_{|\xi|=1} f(|1+h\xi|^2)\left[\frac{\xi}{\xi^2-r^{-2}}-\frac{1}{\xi}\right]d\xi$$

$$I_2(f,g,r) = -\frac{1}{4\pi^2}\oint_{|\xi_1|=1}\oint_{|\xi_2|=1}\frac{f(|1+h\xi_1|^2)g(|1+h\xi_2|^2)}{(\xi_1-r\xi_2)^2}d\xi_1 d\xi_2$$

Proof: We are just going to look at the simplest case of $I_2(f)$.

The idea is to perform change of variable $z = 1+hr\xi_h r^{-1}\bar{\xi}+h^2$ with $r>1$ but close to 1, and $|\xi|=1, h=\sqrt{y}$.

As ξ runs anticlockwise, z runs on contour c encloses support $[a,b]=[(1\pm h)^2]$.

Since $z = -\frac{1}{\underline{S}} + \frac{y}{1+\underline{S}}, z \in \mathbb{C}^+$. We have $\underline{S} = -\frac{1}{1+hr\xi}$ and $dz = h(r - r^{-1}\xi^{-2})d\xi$.

Applying this to $I_2(f)$ in theorem:

$$I_2(f) = \lim_{r \downarrow 1} \frac{1}{2\pi i} \oint_{|\xi|=1} f(z) \frac{1}{\xi^3} \frac{r\xi^2 - r^{-1}}{r(r^2\xi^2 - 1)} d\xi = \frac{1}{2\pi i} \oint_{|\xi|=1} f(|1 + h\xi|^2) \frac{1}{\xi^3} d\xi.$$

as

$$\begin{aligned} |1 + h\xi|^2 &= (1 + h\xi)\overline{(1 + h\xi)} \\ &= (1 + h\xi)(1 + h\bar{\xi}) \\ &= 1 + h\xi + h\bar{\xi} + h^2|\xi| \\ &= 1 + h\xi + h\bar{\xi} + h^2. \end{aligned} \tag{6.1}$$

■

An example application of CLT

Proposition 6.3 *Consider two linear spectral statistics*

$$\sum_{i=1}^p \log(\lambda_i), \sum_{i=1}^p \lambda_i$$

where (λ_i) are eigenvalues of sample covariance $\mathbf{S}_n$. Then, under assumptions of Theorem, the vector

$$\begin{pmatrix} \sum_{i=1}^p \log(\lambda_i) - pF_{y_n}(\log x) \\ \sum_{i=1}^p \lambda_i - pF_{y_n}(x) \end{pmatrix} \xrightarrow{d} N(\mu_1, \mathbf{Q}_1)$$

$$\mu_1 = \begin{pmatrix} \frac{k-1}{2}\log(1-y) - \frac{1}{2}\beta y \\ 0 \end{pmatrix}$$

$$\mathbf{Q}_1 = \begin{pmatrix} -k\log(1-y) + \beta y & (\beta + k)y \\ (\beta + k)y & (\beta + k)y \end{pmatrix}$$

$$F_{y_n}(x) = 1, F_{y_n}(\log x) = \frac{y_n - 1}{y_n}\log(1) - y_n - 1.$$

Proof: In the Theorem, take $k = 2$ with

$$f(x) = \log(x), g(x) = x, x > 0.$$

and we are going to consider the vector (X_f, X_g).

$$\mathbb{E}[X_f] = (k-1)I_1(f) + \beta I_2(f), \mathbb{E}[X_g] = (k-1)I_1(g) + \beta I_2(g)$$

etc. We shall use the proposition to calculate

$$
\begin{aligned}
I_1(f,r) &= \frac{1}{2\pi i}\oint_{|\xi|=1} f(|1+h\xi|^2)\left[\frac{\xi}{\xi^2-r^{-2}}-\frac{1}{\xi}\right]d\xi \\
&= \frac{1}{2\pi i}\oint_{|\xi|=1} \log(|1+h\xi|^2)\left[\frac{\xi}{\xi^2-r^{-2}}-\frac{1}{\xi}\right]d\xi
\end{aligned} \tag{6.2}
$$

Recall

$$
|1+h\xi|^2 = (1+h\xi)(1+h\bar{\xi}) = (1+h\xi)(1+h\frac{1}{\xi})
$$

$$
|\xi| = 1 \implies \bar{\xi} = e^{-i\theta} = \frac{1}{\xi}.
$$

continued (6.2)

$$
\begin{aligned}
&= \frac{1}{2\pi i}\oint_{|\xi|=1}[\log(1+h\xi)+\log(1+h/\xi)]\left[\frac{\xi}{\xi^2-r^{-2}}-\frac{1}{\xi}\right]d\xi \\
&= \frac{1}{2\pi i}\left[\oint_{|\xi|=1}\log(1+h\xi)\frac{\xi}{\xi^2-r^{-2}}d\xi - \oint_{|\xi|=1}\log(1+h\xi)\frac{1}{\xi}d\xi + \oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{\xi}{\xi^2-r^{-2}}d\xi - \oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{1}{\xi}d\xi\right]
\end{aligned} \tag{6.3}
$$

For the first integral, the poles are $\pm\frac{1}{r}$.

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{|\xi|=1}\log(1+h\xi)\frac{\xi}{\xi^2-r^{-2}}d\xi &= \left.\frac{\log(1+h\xi)\xi}{\xi-r^{-1}}\right|_{\xi=-r^{-1}} + \left.\frac{\log(1+h\xi)\xi}{\xi+r^{-1}}\right|_{\xi=r^{-1}} \\
&= \frac{1}{2}\log\left(1-\frac{h^2}{r^2}\right).
\end{aligned} \tag{6.4}
$$

For the second integral, singularity at $\xi = 0$.

$$
\frac{1}{2\pi i}\oint_{|\xi|=1}\log(1+h\xi)\frac{1}{\xi}d\xi = \log(1+h\xi)\Big|_{\xi=0} = 0.
$$

For third integral, we perform a change of variable $z = \frac{1}{\xi}$, so $d\xi = -z^{-2}dz$.

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{\xi}{\xi^2-r^{-2}}d\xi &= -\frac{1}{2\pi i}\oint_{|z|=1}\log(1+hz)\frac{z^{-1}}{z^{-2}-r^{-2}}\frac{-1}{z^2}dz \\
&= \frac{1}{2\pi i}\oint_{|z|=1}\frac{\log(1+hz)r^2}{z(z+r)(z-r)}dz \\
&= \left.\frac{\log(1+hz)r^2}{(z+r)(z-r)}\right|_{z=0} \\
&= 0
\end{aligned} \tag{6.5}
$$

Fourth integral: $z = \xi^{-1}, d\xi = -z^{-2}dz$.

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{1}{\xi}d\xi &= -\frac{1}{2\pi i}\oint_{|z|=1}\log(1+hz)\frac{-z}{z^2}dz \\ &= \log(1+hz)\Big|_{z=0} = 0\end{aligned} \tag{6.6}$$

Collecting all terms gives $I_1(f,r) = \frac{1}{2}\log(1-h^2/r^2)$.

$$I_1(g,r) = \frac{1}{2\pi i}\oint_{|\xi|=1} g(1|1+h\xi|^2)\cdot\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right]d\xi = \frac{1}{2\pi i}\oint_{|\xi|=1}|1+h\xi|^2\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right]d\xi$$

and

$$|1+h\xi|^2 = (1+h\xi)(1+h\bar{\xi}) = 1+h\xi^{-1}+h\xi+h^2 = \frac{\xi+h+h\xi^2+h^2\xi}{\xi}$$

so (continued)

$$= \frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\xi+h+h\xi^2+h^2\xi}{\xi}\times\frac{\xi}{\xi^2-r^{-2}}d\xi - \frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\xi+h+h\xi^2+h^2\xi}{\xi}\frac{1}{\xi}d\xi.$$

The first integral

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\xi+h+h\xi^2+h^2\xi}{(\xi-r)(\xi+r)}d\xi &= \frac{\xi+h+h\xi^2+h^2\xi}{\xi-r}\Big|_{\xi=-r^{-1}} + \frac{\xi+h+h\xi^2+h^2\xi}{\xi+r}\Big|_{\xi=r^{-1}} \\ &= 1+h^2\end{aligned} \tag{6.7}$$

and second integral (2nd order pole at $\xi = 0$.)

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\xi+h+h\xi^2+h^2\xi}{\xi^2}d\xi &= \frac{\partial}{\partial\xi}(\xi+h+h\xi^2+h\xi)\Big|_{\xi=0} \\ &= 1+h^2\end{aligned} \tag{6.8}$$

Hence $I_1(g,r) = 0$.

$$\begin{aligned}I_2(f) &= \frac{1}{2\pi i}\oint_{|\xi|=1}\log(|1+h\xi|^2)\frac{1}{\xi^3}d\xi \\ &= \frac{1}{2\pi i}\left[\oint_{|\xi|=1}\frac{\log(1+h\xi)}{\xi^3}d\xi + \oint_{|\xi|=1}\frac{\log(1+h\xi^{-1}}{\xi^3}d\xi\right].\end{aligned} \tag{6.9}$$

First integral (3rd order pole):

$$\frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\log(1+h\xi)}{\xi^3}d\xi = \frac{1}{2}\frac{\partial^2}{\partial\xi^2}\log(1+h\xi)\bigg|_{\xi=0} = -\frac{1}{2}h^2.$$

Second integral: $z=\xi^{-1}, d\xi = -z^{-2}dz$.

$$\frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\log(1+h\xi^{-1})}{\xi^3}d\xi = -\frac{1}{2\pi i}\oint_{|z|=1}\frac{\log(1+hz)}{z^{-3}}\frac{-1}{z^2}dz = \log(1+hz)\bigg|_{z=0} = 0.$$

Now for the covariance terms:

$$\begin{aligned}J_1(f,g,r) &= -\frac{1}{4\pi^2}\oint_{|\xi_1|=1}\oint_{|\xi_2|=1}\frac{\log(|1+h\xi_1|^2)|1+h\xi_2|^2}{(\xi_1-r\xi_2)^2}d\xi_1 d\xi_2 \\ &= \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(|1+h\xi_1|^2)}{(\xi_1-r\xi_2)^2}d\xi_1\cdot\frac{1}{2\pi i}\oint_{|\xi_2|=1}|1+h\xi_2|^2 d\xi_2.\end{aligned} \tag{6.10}$$

First integral,

$$\frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(|1+h\xi_1|^2)}{(\xi_1-r\xi_2)^2}d\xi_1 = \frac{1}{2\pi i}\left[\oint_{|\xi_1|=1}\frac{\log(1+h\xi)}{(\xi_1-r\xi_2)^2}d\xi_1 + \oint_{|\xi_1|=1}\frac{\log(1+h\xi^{-1})}{(\xi_1-r\xi_2)^2}d\xi_1\right] = \frac{1}{2\pi i}[A+B].$$

Notice for A, for $|\xi_2|=1$ fixed, $|r\xi_2|>1$ so $r\xi_2$ not a pole.

$$A=0, z=\frac{1}{\xi_1}, d\xi_1 = -z^{-2}dz.$$

$$\begin{aligned}B &= \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1^{-1}}{(\xi_1-r\xi_2)^2}d\xi_1 \\ &= -\frac{1}{2\pi i}\oint_{|z|=1}\frac{\log(1+hz)}{(z^{-1}-r\xi_2)^2}\frac{-1}{z^2}dz \\ &= \frac{1}{2\pi i}\frac{1}{(r\xi_2)^2}\oint_{|z|=1}\frac{\log(1+hz)}{(z-\frac{1}{r\xi_2})^2}dz \qquad \text{2nd order at } z=\frac{1}{r\xi_2} \\ &= \frac{1}{(r\xi_2)^2}\frac{\partial}{\partial z}(\log(1+hz))\bigg|_{z=\frac{1}{r\xi_2}} \\ &= \frac{h}{r\xi_2(r\xi_2+h)}\end{aligned} \tag{6.11}$$

Now,

$$
\begin{aligned}
J_1(f,g,r) &= \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(|1+h\xi_1|^2)}{(\xi_1-r\xi_2)^2}d\xi_1\cdot\frac{1}{2\pi i}\oint_{|\xi_2|=1}|1+h\xi_2|^2 d\xi_2\\
&= \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{(1+h\xi_2)(1+h\bar{\xi}_2)}{\xi_2(\xi_2+hr^{-1})}d\xi_2\\
&= \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{\xi_2+h\xi_2^2+h+h^2\xi_2}{\xi_2^2(\xi_2+hr^{-1})}d\xi_2\\
&= \frac{h}{2\pi i r^2}\left[\oint_{|\xi_2|=1}\frac{1+h^2}{\xi_2(\xi_2+hr^{-1})}d\xi_2+\oint_{|\xi_2|=1}\frac{h}{(\xi_2+hr^{-1})}d\xi_2+\oint_{|\xi_2|=1}\frac{h}{\xi_2^2(\xi_2+hr^{-1})}d\xi_2\right]\\
&= \frac{h}{2\pi i r^2}\left[0+2\pi i h+0\right]\\
&= \frac{h^2}{r^2}.\\
J_1(f,f,r) &= \frac{1}{2\pi i}\oint_{|\xi_2|=1}f(|1+h\xi_2|^2)\cdot\frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{f(|1+h\xi_2|^2)}{(\xi_1-r\xi_2)^2}d\xi_1 d\xi_2\\
&= \frac{1}{2\pi i}\oint_{|\xi_2|=1}f(|1+h\xi_2|^2)\frac{h}{r\xi_2(r\xi_2+h)}d\xi_2\\
&= \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{\log(1+h\xi_2)}{\xi_2\left(\frac{h}{r}+\xi_2\right)}d\xi_2+\frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{\log(1+h\xi_2^{-1})}{\xi_2\left(\frac{h}{r}+\xi_2\right)}d\xi_2\\
&= A+B.\\
A &= \frac{h}{r^2}\left[\left.\frac{\log(1+h\xi_2)}{\frac{h}{r}+\xi_2}\right|_{\xi_2=0}+\left.\frac{\log(1+h\xi_2)}{\xi_2}\right|_{\xi_2=-\frac{h}{r}}\right]\\
&= -\frac{1}{r^2}\log\left(1-\frac{h^2}{r}\right).\\
B &= \frac{-h}{2\pi i r^2}\oint_{|z|=1}\frac{\log(1+hz)}{z^{-1}\left(\frac{h}{r}+z^{-1}\right)}\frac{-1}{z^2}dz\\
&= \frac{1}{2\pi i r}\oint_{|z|=1}\frac{\log(1+hz)}{z+rh^{-1}}dz = 0 \text{ not a pole since } |r/h|>1
\end{aligned}
\tag{6.12}
$$

Hence, $J_1(f,f,r) = -\frac{1}{r}\log(1-\frac{h^2}{r})$.

$$
J_1(g,g,r) = \frac{1}{2\pi i}\oint_{|\xi_2|=1}|1+h\xi_2|^2\cdot\frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{|1+h\xi|^2}{(\xi_1-r\xi_2)^2}d\xi_1 d\xi_2.
$$

and

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{|1+h\xi_1|^2}{(\xi_1-r\xi_2)^2}d\xi_1 &= \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\xi_1+h\xi_1^2+h+h^2\xi_1}{\xi_1(\xi_1-r\xi_2)}d\xi_1\\
&= \frac{1}{2\pi i}\left[\oint_{|\xi_1|=1}\frac{1+h^2}{(\xi_1-r\xi_2)^2}d\xi_1+\oint_{|\xi_1|=1}\frac{h\xi_1}{(\xi_1-r\xi_2)^2}d\xi_1+\oint_{|\xi_1|=1}\frac{h}{\xi_1(\xi_1-r\xi_2)^2}d\xi_1\right]\\
&= \frac{1}{2\pi i}\left[0+0+\frac{2\pi i h}{r^2\xi_2^2}\right]\\
&= \frac{h}{r^2\xi_2^2}.
\end{aligned} \tag{6.13}$$

since $2\pi i\frac{h}{(\xi_1-r\xi_2)^2}\Big|_{\xi_1=0} = \frac{2\pi i h}{r^2\xi_2^2}$.

Therefore,

$$\begin{aligned}
J_1(g,g,r) &= \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{\xi_2+h\xi_2+h+h^2\xi_2}{\xi_2^2}d\xi_2\\
&= \frac{h}{2\pi i r^2}\left[\oint_{|\xi_2|=1}\frac{1+h^2}{\xi_2^2}d\xi_2+\oint_{|\xi_2|=1}\frac{h}{\xi_2}d\xi_2+\oint_{|\xi_2|=1}\frac{h}{\xi_2^3}d\xi_2\right]\\
&= \frac{h^2}{r^2}.
\end{aligned} \tag{6.14}$$

Now we have to calculate all the J_2 terms:

$$J_2(f,g), J_2(f,f), J_2(g,g).$$

$$J_2(F,G) = -\frac{1}{4\pi^2}\oint_{|\xi_1|=1}\frac{F(|1+h\xi_1|^2)}{\xi_1^2}d\xi_1\oint_{|\xi_2|=1}\frac{G(|1+h\xi_2|^2)}{\xi_2^2}d\xi_2$$

First notice that

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1|^2)}{\xi_1^2}d\xi_1 &= \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{\log(1+h\xi_1)+\log(1+h\xi_1^{-1})}{\xi_1^2}d\xi_1\\
&= \frac{1}{2\pi i}\left[2\pi i\left[\frac{\partial}{\partial\xi_1}\log(1+h\xi_1)\right]\Big|_{\xi_1=0}-\oint_{|z|=1}\frac{\log(1+hz)}{z^{-2}}\frac{-1}{z^2}dz\right]\\
&= h-0\\
&= h
\end{aligned} \tag{6.15}$$

And we have

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{g(|1+h\xi_2|^2)}{\xi_2^2}d\xi_2 = \frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{\xi_2+h\xi_2^2+h+h^2\xi_2}{\xi_2^3}d\xi_2 = h \\
J_2(f,g) = \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{f(|1+h\xi_2|^2)}{\xi_1^2}d\xi_1\cdot\frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{g(|1+h\xi_2|^2)}{\xi_2^2}d\xi_2 = h^2 \\
J_2(f,f) = \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{f(|1+h\xi_2|^2)}{\xi_1^2}d\xi_1\cdot\oint_{|\xi_2|=1}\frac{f(|1+h\xi_2|^2)}{\xi_2^2}d\xi_2 = h^2 \\
J_2(g,g,) = \frac{1}{2\pi i}\oint_{|\xi_1|=1}\frac{g(|1+h\xi_1|^2)}{\xi_1^2}d\xi_1\cdot\frac{1}{2\pi i}\oint_{|\xi_2|=1}\frac{g(|1+h\xi_2|^2}{\xi_2^2}d\xi_2 = h^2
\end{aligned} \tag{6.16}$$

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