Statistical Inference

Lecture 05a

ANU - RSFAS

Last Updated: Sun Mar 18 16:19:54 2018

Exponential Families

Definition 2.7: We say that a random variable belongs to the k-parameter exponential family of distributions if its pdf can be written in the following form:

$$f(x;\boldsymbol{\theta}) = exp\left(\sum_{j=1}^{k} A_j(\boldsymbol{\theta})B_j(x) + C(x) + D(\boldsymbol{\theta})\right)$$

or

$$f(x;\boldsymbol{\theta}) = C^*(x)D^*(\boldsymbol{\theta})exp\left(\sum_{j=1}^{k} A_j(\boldsymbol{\theta})B_j(x)\right)$$

Exponential Families

Eg: Poisson distribution.

$$X \sim \mathrm{Poisson}(\lambda), \qquad x = 0, 1, 2, 3, \ldots$$

$$\begin{aligned} f(x;\lambda) &= \frac{\lambda^x exp(-\lambda)}{x!} \\ &= exp\{x\ ln(\lambda) - \lambda - ln(x!)\} \end{aligned}$$

- The Poisson family is a one-dimensional exponential family with functions:

$$\begin{aligned} A_1(\lambda) &= ln(\lambda) \\ B_1(x) &= x \\ C(x) &= -ln(x!) \\ D(\lambda) &= -\lambda \end{aligned}$$

Exponential Families - Canonical Form

- If we define:

$$\phi = (\phi_1, \ldots, \phi_k) = A(\boldsymbol{\theta}) = \{A_1(\theta), \ldots, A_k(\theta)\}$$

then ϕ is referred to as the canonical parameter for the exponential family and the density function can be written in the form:

$$f(x; \theta) = exp\left\{\sum_{j=1}^{k} \phi_i B_i(x) + C(x) + D(\phi)\right\}$$

- Note:

$$\theta = A^{-1}(\phi)$$

$$D(\phi) = D\{A^{-1}(\phi)\}$$

Exponential Families - Canonical Form

Eg: Poisson distribution.

$$X \sim \mathrm{Poisson}(\lambda), \qquad x = 0, 1, 2, 3, \ldots$$

$$\begin{aligned} f(x;\lambda) &= \frac{\lambda^x exp(-\lambda)}{x!} \\ &= exp\{xln(\lambda) - \lambda - ln(x!)\} \end{aligned}$$

$\lambda = exp(\phi)$

- The canonical parameter is $\phi = ln(\lambda)$. So based on the inverse relationship we have:

$\Downarrow$ plug in.

$$f_X(x;\lambda) = exp\{x\phi - exp(\phi) - ln(x!)\}$$

Poisson Regression - Canonical Link Function

- In generalized linear models, one of the 'link' functions (the main one) is the canonical link function.

$(Y_i, X_{1i}, X_{2i}) \mid Y_i \sim N(\mu_i, \sigma^2)$?

- The canonical link function is from the canonical form of an exponential family.
- Suppose we have data that may reasonably be considered from a Poisson distribution:

$$Y_1, \ldots, Y_n \overset{\text{indep.}}{\sim} \text{Poisson}(\lambda_i)$$

  - Now we want to relate the mean of Y_i to a linear function of covariates $(x_1, \ldots, x_p)$:

$$E[Y_i] = \lambda_i = exp(\beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{pi}) = \exp(\phi_i)$$

  - So we link the mean of the response (Y) to a linear function of the covariates (ϕ) via the link function.

Sufficiency

Lemma 2.4: If the usual regularity condition hold, then a vector of k sufficient statistics $\boldsymbol{T}$ exists for a vector or parameters $\boldsymbol{\theta}$ if and only if the distribution of $\boldsymbol{X}$ belong to the k-parameter exponential family.

Proof:

$$f(\boldsymbol{x};\boldsymbol{\theta}) = exp\left\{\sum_{j=1}^{k} A_j(\boldsymbol{\theta})\left(\sum_{i=1}^{n} B_j(x_i)\right) + nD(\boldsymbol{\theta}) + \sum_{i=1}^{n} C(x_i)\right\}$$

vectors

- Let $\boldsymbol{t} = (\sum_{i=1}^{n} B_1(x_i), \ldots, \sum_{i=1}^{n} B_k(x_i))$.
- $K_1 = exp\left\{\sum_{j=1}^{k} A_j(\boldsymbol{\theta})t_j + nD(\boldsymbol{\theta})\right\}$
- $K_2 = exp\left\{\sum_{i=1}^{n} C(x_i)\right\}$

Minimal Sufficient

Lemma 2.5: Under the same conditions as Lemma 2.4, $\boldsymbol{T}$ is also miniminal sufficient.

Proof: Use the ratio approach. page 30, P. Garthwaite.

$$\frac{L(\vec{\theta};\vec{x})}{L(\vec{\theta};\vec{y})}$$

Complete Statistic

Lemma 2.8: Under the same conditions as Lemma 2.4, $\boldsymbol{T}$ is also complete.*

~~**Proof:** Beyond the scope of the course.~~

MVUE - Approach 2

$T = \sum_{i=1}^{n} X_i \Rightarrow E(T) = n\lambda$

$\therefore h(T) = \frac{\sum X_i}{n}$

$E(h(T)) = \lambda \quad \therefore$ MVUE

- For exponential families, we can easily find complete and sufficient statistics.
- All that is needed to find $h(T)$ such that $E[h(T)] = \tau(\theta)$, then we have the unique MVUE!

$\boxed{X} \sim Pois(\theta)$

$\uparrow$

a single observation

$$\tau = \tau[\theta] = e^{-2\theta} = P(X_1=0, X_2=0)$$

$$= P(X_1=0)\,P(X_2=0)$$

$$= \frac{e^{-\theta}\theta^0}{0!} \cdot \frac{e^{-\theta}\cdot\theta^0}{0!}$$

MVUE

$$\boxed{= e^{-2\theta}}$$

$T(X) = X$ is suff & complete

b/c exp. family theories

· Consider $Y = (-1)^x$

$$E(Y) = \sum_{x=0}^{\infty} (-1)^x \frac{\theta^x e^{-\theta}}{x!}$$

$$= e^{-\theta} \underbrace{\sum_{i=0}^{\infty} \frac{-\theta^x}{x!}}_{e^{-\theta}}$$

$$= e^{-2\theta}$$

math fact:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$\therefore$ Y is MVUE

$Y = (-1)^x = \pm 1$ But $0 < e^{-2\theta} < 1$!!!

may not be necessarily done