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How to decide if a function of several variables is continuous?

As in the case of functions of one variable, a function $\boldsymbol{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous at a point $\boldsymbol{a} \in \mathbb{R}^n$ if $\lim_{\boldsymbol{x} \longrightarrow \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a})$. In the case of $n = 2$ this means a pair of variables (x_1, x_2) must be approaching the point (a_1, a_2) in any possible manner: along a straight line, along a parabola, or any other well known curve, or even through a completely random path like the path of the price of a stock, or even through some random jumps. As such it is impossible to actually trace the approach of more than one variable toward their destination.

- It is easier to determine that the function is not continuous at a give point based on only two different paths along which the function may tend to two distinct values. This contrdicts the fact that the limit, if it exists, must be unique. (See examples 1 and 2 of section 1.3)

- by converting the variable (x_1, x_2) into polar coordinates (that is by letting $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$) we can decompose the movement of the variable (x_1, x_2) into two components of *distance*, r and rotation about a center (θ). As such we have isolated the 'approaching to $(0, 0)$' and have translated it to $r \to 0$. This translates the question of limit of a two variable function into the limit of a one variable function. For example in example 1 we see that the function f in polar coordinates will look like $\dfrac{r^2\cos\theta\sin\theta}{r^2}$ which is really $\frac{1}{2}\sin 2\theta$. But since the variable r has disappeared from the picture then clearly the function f need not approach 0. Indeed for different values of θ the value of f as $r \to 0$ is different. So the limit does not exist.

- The textbook uses a series of continuity theorems and corollaries as well as some well known functions of calculus of one variable to establish the continuity of a function of several variables. The idea is the most natural: functions of several variables are made of a series of primitive operations on the variables. Each primitive operation is continuous and the composition of these operations is also continuous. For example $f(x, y) = \sin(x + y)$ is a function of two variables which is made of composing two functions: $\sin$ function with the function $f_1(x, y)$ presented in theorem 1.10. Similarly the function $f(x, y) = ((tan(1+x^2y), \sqrt{1 + e^x}, 2xy)$ is a continuous function because each component of it is made up of continuous functions applied on continuous operations.

  The way this idea is organized is as follows: first 1.10 establishes that the three primary operations of addition, multiplication and inversion are continuous. They are captured in functions f_1, f_2 and g and it is proved that they are continuous. Theorem 1.9 claims that composition of two continuous functions gives an continuous function. This theorem is used together with 1.10 to prove (corollary 1.11) that $f_3(x, y) = x - y$ and $f_4(x, y) = x/y$ are continuous (see the proofs.)