STAT2032/6046: Financial Mathematics

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Mid-semester Revision

Revision - basics

Interest: - compensation borrower of money pays the lender of money for its use.

The value of an investment changes over time due to interest - this is the **time value of money**.

Valuation of cashflows - must incorporate the effect of interest.

Revision - basics

For a single cashflow K at time t, and effective rate of interest i per period

	Compound Interest	**Simple Interest**
Accum. Val (at $t = n$)	$K(1+i)^{n-t}$	$K(1+i(n-t))$
Present Value (at t=0)	$K(1+i)^{-t} = Kv^t$	$\frac{K}{1+it}$

Accumulation - value cashflows at a future date in time, apply an accumulation factor.

Present value - value cashflows at t=0, apply discount factor to cashflows.

Revision - types of interest rates

1. **Effective interest rates** - interest is paid **once** in the specified period

   Example: Effective 6-month interest rate is 6% → \$100 invested now, will be worth \$106 in 6 months time

   Example: The 2-year interest rate is 6% effective → \$100 invested now, will be worth \$106 in 2 years time

   KEY LEARNING OBJECTIVE: CALCULATE EQUIVALENT EFFECTIVE INTEREST RATES

   Equivalent effective rates produce the same accumulated amounts over the same time period (given the same initial investment)

Q: if the effective 2-year interest rate is 6%, what is the equivalent effective 6-month interest rate??

Revision - types of interest rates

1. **Effective interest rates** - interest is paid **once** in the specified period

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   KEY LEARNING OBJECTIVE: CALCULATE EQUIVALENT EFFECTIVE INTEREST RATES

   Equivalent effective rates produce the same accumulated amounts over the same time period (given the same initial investment)

Q: if the effective 2-year interest rate is 6%, what is the equivalent effective 6-month interest rate??

Let j be the equivalent 6-month interest rate. Then $(1 + j)^4 = 1.06$. Therefore, $j = 1.06^{1/4} - 1 = 1.47\%$.

Revision - types of interest rates

2. **Nominal interest rates**: Interest is paid **more** or **less** frequently than once per measurement period.

   Example: $i^{(12)}$=18% pa - nominal rate of interest per annum convertible/payable 12 times *per year* (that is, m=12)

   Equivalent to $\frac{i^{(12)}}{12} = 1.5\%$ *effective* per $\frac{1}{12}^{th}$ of a year (that is, effective per month)

   Equivalence relationship between i and $i^{(m)}$

$$\boxed{1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m}$$

   where i is the effective rate per annum

Revision - types of interest rates

2. **Nominal interest rates**:
   Example: $i^{(2)} = 5\%$ pa

Time	Interest	Balance
0		100
0.5	$2.5\% \times 100 = 2.5$	102.5
1	$2.5\% \times 102.5 = 2.5625$	105.0625

   The equivalent annual effective interest rate is

Revision - types of interest rates

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0.5	$2.5\% \times 100 = 2.5$	102.5
1	$2.5\% \times 102.5 = 2.5625$	105.0625

   The equivalent annual effective interest rate is 5.0625%
   $(= 1.025^2 - 1)$

Revision - types of interest rates

2. **Nominal interest rates**:

   What is the limiting behaviour of $i^{(m)}$?? (that is, as $m \to \infty$?).

   $$\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m = e^{\delta}$$

   where δ is the force of interest per annum, paid out continuously

   $$\lim_{m\to\infty} i^{(m)} = \delta$$

   .

   $\boxed{\text{REMEMBER: } i > i^{(2)} > i^{(3)} > .... > \delta}$

Revision - types of interest rates

3. **Discount rates**:
   (don't confuse with discount factor v)
   Interest - paid at the **end** of the period on the balance at the **beginning** of the period

$$i = \frac{d}{1-d}$$

   Discount - paid at the **beginning** of the period on the balance at the **end** of the period

$$d = \frac{i}{1+i}$$

Revision - types of interest rates

3. **Discount rates**:
   Suppose i = 10%; n=3; Balance at t=3 is 1000.

Time	Interest	Balance
0		751.3148
1	75.13148	826.4463
2	82.64463	909.0909
3	90.90909	1000

   Suppose d = 10%; n=3; Balance at t=3 is 1000.

Time	Discount	Balance
0	81	729
1	90	810
2	100	900
3		1000

Revision - types of interest rates

3. **Discount rates**: Nominal discount rate:

$$\boxed{1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m}$$

Example: $d^{(2)} = 5\%$ pa; Balance at t=1 is 100

Time	Discount	Balance
0	$2.5\% \times 97.5 = 2.4375$	97.5-2.4375=95.0625
0.5	$2.5\% \times 100 = 2.5$	100-2.5=97.5
1		100

The equivalent annual effective discount rate is

Revision - types of interest rates

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1		100

The equivalent annual effective discount rate is $d = 4.9375\%$
$= \left(1 - \frac{95.0625}{100}\right) = \left(1 - (1 - \frac{0.05}{2})^2\right)$

$\boxed{d < d^{(2)} < d^{(3)} < .... < \delta}$.

Revision - types of interest rates

4. **Force of interest**:
   For an effective annual rate of interest, the equivalent nominal rate of interest as the number of compounding periods m approaches infinity is called the **force of interest** δ. (In other words, the instantaneous rate of change of the investment value S(t)).

   Derive from first principles $\lim_{m\to\infty} d^{(m)} = \lim_{m\to\infty} i^{(m)} = \delta$

Revision - types of interest rates

4. **Force of interest**:
   $\delta_t = \frac{S'(t)}{S(t)} = \frac{\partial}{\partial t} \ln S(t)$.

   Show under compound interest

$$\delta_t = \ln(1+i)$$

$$S(n) = S(0) \cdot \exp\left(\int_0^n \delta_t dt\right)$$

$$S(0) = S(n) \cdot \exp\left(-\int_0^n \delta_t dt\right)$$

Q: what is the present value of $1000 due in four years time if the force of interest is $\delta_t = 0.05 - 0.01t$

Revision - types of interest rates

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Q: what is the present value of \$1000 due in four years time if the force of interest is $\delta_t = 0.05 - 0.01t$

A: $1000 \exp\left(-\int_0^4 0.05 - 0.01t \; dt\right) = \$886.92$

Revision - Annuities

Distinguish between and value the cashflows of different types of annuities

- Annuity in arrears
- Annuity-due
- Deferred annuities
- Annuities payable more (or less frequently) than annually
- Perpetuities
- Continuous annuities
- Increasing annuities
- Decreasing annuities
- Indexed annuities
- Continuous varying annuities

Revision - Annuities

Formulas to value these annuities are in the formula sheet.

You should be able to derive all formulas from first principles as discussed in class.

Understand the relationship between the value of different types of annuities.

Revision - Annuities

Q: Show that $a_{\overline{m+n}|} = a_{\overline{n}|} + v^n a_{\overline{m}|} = a_{\overline{m}|} + v^m a_{\overline{n}|}$. Provide a written explanation for this relationship.

Revision - Annuities

Q: Show that $a_{\overline{m+n|}} = a_{\overline{n|}} + v^n a_{\overline{m|}} = a_{\overline{m|}} + v^m a_{\overline{n|}}$. Provide a written explanation for this relationship.

$$
\begin{aligned}
RHS &= a_{\overline{m|}} + v^m a_{\overline{n|}} \\
&= \frac{1 - v^m}{i} + v^m \cdot \frac{1 - v^n}{i} \\
&= \frac{1 - v^m + v^m - v^{m+n}}{i} \\
&= a_{\overline{m+n|}} \\
&= LHS
\end{aligned}
$$

Similarly show $a_{\overline{m+n|}} = a_{\overline{n|}} + v^n a_{\overline{m|}}$.

Written explanation: Consider an annuity that pays \$1 at the end of each year for $(m + n)$ years. The present value of the first m payments is $a_{\overline{m|}}$. The remaining n payments have value $a_{\overline{n|}}$ at time $t = m$, discounted to the present these are worth $v^m a_{\overline{n|}}$ at time $t = 0$.

Revision - Equations of value

Equations of value set the value of all cash inflows equal to the value of all cash outflows at a common time point.

Analysing financial transactions involves constructing and solving equations of value.

Key step: Select a common reference time point t.

Example: If t=0, the equation of value is PV inflows=PV outflows.

Revision - Equations of value

Question:

A debt of 7,000 is due at the end of 5 years. If 2,000 is paid at the end of 1 year, what single payment should be made at the end of the 2nd year to liquidate the debt, assuming interest at the rate of 6.5% per year, compounded quarterly.

Revision - Equations of value

(1) Select reference time point, say t=5.

(2) Set up equation of value

$$2000(1+j)^4 + X(1+j)^3 = 7000$$

where $j = \left(1 + \frac{0.065}{4}\right)^4 - 1 = 6.660\%$

(3) Solving for X, we have

$$X = \$3,653.67$$

Interpretation: If \$2000 is paid back at t=1, and \$3,653.67 at time t=2, then the lender can invest these repayments at 6.5% interest compounded quarterly to have the required \$7,000 at time t=5.

Note: *Does your equation of value make sense?? You should be able to tell a story to describe the equation of value.*

Revision - Linear interpolation

$$x_0 \simeq x_1 + \frac{f(x_0) - f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1)$$

Approximation method when an analytic solution to the equation of value cannot be found.

Revision - Actuarial tables

Be familiar with what's available.

It may be more efficient to lookup pre-calculated values in the tables.

Revision - Inflation

$$(1 + i) = (1 + i_{real})(1 + r)$$

$$1 + i_{real} = \frac{1 + i}{1 + r}$$

$$i_{real} = \frac{i - r}{1 + r}$$

Real interest rate: the growth in investment value due to interest only, (that is, remove the effect of inflation).

Real cashflows (that is, with inflation effect removed, so all cashflows are in same units of purchasing power), should be valued at the real interest rate.

Mid-Term Exam - Reminder

- When: 9:30am, Tuesday 18 April 2017
- Where: ANU College Room 08, ANU College Room 09
- Duration: 15 minutes reading time; 1.5 hours writing time
- The mid-term is worth 20% of your final grade.
- The mid-term is redeemable.
- Formula sheet and actuarial tables will be provided.
- Bring your own non-programmable calculator, unmarked dictionaries also permitted. Personal copies of actuarial tables are not allowed.
- **SHOW ALL WORKING!** (credit may be given for partial solutions)
- I will hold normal consultation hours + open-door policy during the teaching break.

Final Words

You can do it as long as you really want to do it.

Good luck and all the best to everyone!