STA261H1S: Solution to second quiz

Question 1: *(10 marks)* Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution having variance 4. Consider two simple hypothesis:

$$H_0 : \mu = 1 \text{ v.s. } H_1 : \mu = \mu_1$$

where $\mu_1 < 1$. Let the significance level α be 0.1.

a. *(6 marks)* Find the most powerful test using Neyman-Pearson Lemma.

The Neyman-Pearson Lemma states that among all tests with significance level α, the test that rejects for small values of the **likelihood ratio** is most powerful (1 mark). We thus calculate the likelihood ratio statistic, which is

$$\frac{L_0(\mu)}{L_1(\mu)} = \frac{\exp\left[-\frac{1}{8}\sum_{i=1}^n (X_i - 1)^2\right]}{\exp\left[-\frac{1}{8}\sum_{i=1}^n (X_i - \mu_1)^2\right]} \leq k \quad \text{(1mark)}$$

This inequality holds if and only if

$$\sum_{i=1}^n X_i \leq \left[4\log k - n(\mu_1^2 - 1)/2\right]/(1 - \mu_1) \quad \text{(1mark)}$$

In this case, the rejection region is the set $\sum_{i=1}^n X_i \leq c$ where c is a constant that can be determined so that the probability of Type I error is significance level α.

$$\begin{aligned} 0.1 = \alpha &= P(\sum_{i=1}^n X_i \leq c | \mu = 1) \\ &= P(\sum_{i=1}^n (X_i - 1)/(2\sqrt{n}) \leq (c - n)/(2\sqrt{n}) | \mu = 1) \end{aligned}$$

$$(c - n)/(2\sqrt{n}) = -z_\alpha = -1.28, c = -2.56\sqrt{n} + n \quad \text{(2marks)}$$

If $\sum_{i=1}^n X_i \leq -2.56\sqrt{n} + n$, then reject H_0, otherwise accept H_0. (1mark)

b. *(4 marks)* Given the sample size $n = 9$, compute the powers when $\mu_1 = 0.5$ and $\mu_1 = 0$.

When $\mu_1 = 0.5$, the power is

$$\begin{aligned} & P(\sum_{i=1}^n X_i \leq -2.56\sqrt{n} + n | \mu_1 = 0.5) \quad \text{(1mark)} \\ = & P(\sum_{i=1}^n (X_i - 0.5)/(2\sqrt{n}) \leq (-2.56\sqrt{n} + n - 0.5n)/(2\sqrt{n}) | \mu_1 = 0.5) \\ = & P(Z \leq -0.53) \\ = & 0.2981 \quad \text{(1mark)} \end{aligned}$$

where $Z \sim N(0,1)$.

When $\mu_1 = 0$, the power is

$$\begin{aligned}
& P(\sum_{i=1}^{n} X_i \leq -2.56\sqrt{n} + n | \mu_1 = 0) \quad \text{(1mark)} \\
= \; & P(\sum_{i=1}^{n} X_i/(2\sqrt{n}) \leq (-2.56\sqrt{n} + n)/(2\sqrt{n}) | \mu_1 = 0) \\
= \; & P(\sum_{i=1}^{n} X_i/(2\sqrt{n}) \leq 0.22 | \mu_1 = 0) \\
= \; & 1 - P(Z \leq -0.22) \\
= \; & 1 - 0.4129 \\
= \; & 0.5871 \quad \text{(1mark)}
\end{aligned}$$