University of Toronto
FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, DECEMBER 2008

MAT 240H1F - ALGEBRA I

Instructor: F. Murnaghan
Duration - 3 hours

Total marks: 100

No calculators or other aids allowed.

Notation:

If m and n are positive integers, $M_{m\times n}(F)$ is the vector space of $m \times n$ matrices with entries in the field F.
$P(F)$ is the vector space of polynomials in one variable with coefficients in the field F.
If n is a nonnegative integer, $P_n(F)$ is the subspace of $P(F)$ consisting of polynomials of degree at most n.
If V and W are vector spaces over a field F, $\mathcal{L}(V,W)$ denotes the vector space of linear transformations from V to W. If $V=W$, $\mathcal{L}(V)=\mathcal{L}(V,V)$.

[14] 1. In each case below, determine whether the subset W of the vector space V is a subspace of V. If W is a subspace of V, prove it. If W is not a subspace of V, demonstrate how one of the properties of subspace fails to hold.

a) Let $V=P(\mathbb{R})$ and let $W=\{\, f\in V \mid f(0)f(-1)=0\,\}$.

b) Let $V=\mathcal{L}(F^3,F^2)$, where F is a field. Let β be an ordered basis for F^3 and let γ be an ordered basis for F^2. Let

$$W=\left\{\, T\in V \mid\ [T]_\beta^\gamma=\begin{pmatrix} a & b & 0\\ -b & 0 & a\end{pmatrix}\ \text{ for some } a \text{ and } b\in F \,\right\}.$$

[15] 2. In each case below, determine whether the function T is a linear transformation.

a) Define $T: P_4(\mathbb{R})\to P_6(\mathbb{R})$ by $T(f)(x)=x^2f(x-1)+f(0)(x^5+x),\quad f\in P_4(\mathbb{R})$.

b) Let n be a positive integer and let V be an n-dimensional vector space over a field F. Let β and γ be ordered bases for V. Let $A\in M_{n\times n}(F)$. Define $T:\mathcal{L}(V)\to M_{n\times n}(F)$ by $T(U)=[U]_\beta^\gamma - A[U]_\beta^\beta,\quad U\in\mathcal{L}(V)$.

[15] 3. Determine whether V and W are isomorphic vector spaces. Please justify your answers.

a) Let $V=\mathcal{L}(M_{2\times 2}(\mathbb{C}),P_2(\mathbb{C}))$ and $W=M_{6\times 2}(\mathbb{C})$.

b) Let

$$V=\{\,(\,a,\,b,\,c,\,d\,)\in\mathbb{C}^4 \mid -ia+d=0\,\}$$
$$\text{and}\quad W=\{\,T\in\mathcal{L}(\mathbb{C}^3)\mid R(T)\subset \operatorname{span}\{\,(\,1,\,i,\,0\,)\,\}\,\}.$$

[23] 4. Let $T \in \mathcal{L}(P_3(\mathbb{C}))$ be defined by:

$$T(ax^3 + bx^2 + cx + d) = idx^3 + ax^2 - bx + ic, \qquad a,\ b,\ c,\ d \in \mathbb{C}.$$

a) Find $T^{-1}(ax^3 + bx^2 + cx + d)$ for all a, b, c and $d \in \mathbb{C}$.

b) Find the characteristic polynomial and all of the eigenvalues of T.

[8] 5. Let V be a 4-dimensional vector space over a field F. Prove that there exists $T \in \mathcal{L}(V)$ such that $\operatorname{rank}(T) = 2$ and $\dim(N(T) \cap R(T)) = 1$.

[25] 6. Let V be a finite-dimensional vector space over a field F and let $T \in \mathcal{L}(V)$. Let $T^2 = T \circ T$.

a) Prove that $N(T) \subset N(T^2)$.

b) Prove that $\operatorname{nullity}(T) = \operatorname{nullity}(T^2)$ if and only if $N(T) \cap R(T) = \{\mathbf{0}\}$.
(*Hint*: It may be easier to prove the following equivalent statement: $\operatorname{nullity}(T) \neq \operatorname{nullity}(T^2)$ if and only if $N(T) \cap R(T) \neq \{\mathbf{0}\}$.)

c) Suppose that T is diagonalizable. Prove that $\operatorname{nullity}(T) = \operatorname{nullity}(T^2)$.

d) Suppose that $\dim(V) \geq 2$ and m is an integer such that $1 \leq m < \dim(V)$. Suppose that $A \in M_{m\times m}(F)$ and there exists an ordered basis β for V such that

$$[T]_\beta = [T]_\beta^\beta = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

(Here, each 0 is a zero matrix of the appropriate size.)
Prove that if A is invertible, then $\operatorname{nullity}(T) = \operatorname{nullity}(T^2) = n - m$.