Today – §3.2 – Theorem 3.5
Theorem 3.6
Statement of the strong Duality Theorem
(proof mostly on Wednesday)

Corollary 3.5 If a problem is unbounded its dual is infeasible.

Proof: If w_0 were feasible, then $b^T w_0 \geq$ every feasible primal objective, contradicting the assumption the primal problem is unbounded.

Eg. The converse is false. Some infeasible problems have unbounded duals. Some infeasible problems have infeasible duals.

Maximize $z = x_1 - x_2$ s.t.

$$x_2 \leq -1$$
$$-x_1 \leq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

is infeasible and its dual:

Maximize $z' = -w_1 + w_2$ s.t.

$$-w_2 \geq 1$$
$$w_1 \geq -1$$
$$w_1 \geq 0,\ w_2 \geq 0$$

which with $x_1 = w_1, x_2 = w_2$ is the same problem.

Corollary 3.6

If a dual pair of problems has feasible solutions x_0 and w_0 (respectively) with $c^T x_0 = b^T w_0$, then x_0 and w_0 are optimal for their respective problems. (Weak Duality Theorem)

Proof $b^T w_0$ is an upper bound for all feasible z-value

$c^T x \leqslant b^T w_0 = c^T x_0$ so z is maximized optimality of w_0 is similar.

Eg. (See "A Simplex Optimization" and "A Dual Simplex Solution")

The primal problem

$$\text{Maximize } z = 3x_1 + 7x_2 \text{ s.t.}$$
$$x_1 + 5x_2 \leqslant 19$$
$$x_1 - x_2 \leqslant 7$$
$$-x_1 + 2x_2 \leqslant 2$$
$$x_1 \geqslant 0, x_2 \geqslant 0$$

has dual

$$\text{Maximize } z' = 19w_1 + 7w_2 + 2w_3 \text{ s.t.}$$
$$w_1 + w_2 - w_3 \geqslant 3$$
$$5w_1 - w_2 + 2w_3 \geqslant 7$$
$$w_1 \geqslant 0, w_2 \geqslant 0, w_3 \geqslant 0$$

Proposed feasible solutions are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ 0 \end{bmatrix}$$

Feasibility check

$$x_1 + 5x_2 = 9 + 5 \cdot 2 = 19 \checkmark$$
$$x_1 - x_2 = 9 - 2 \leqslant 7 \checkmark$$
$$-x_1 + 2x_2 = -9 + 2 \cdot 2 = -5 \leqslant 2 \checkmark$$

(Note that $x_1 \geqslant 0, x_2 \geqslant 0$.)

Dual feasibility check:

$$w_1 + w_2 - w_3 = \frac{5}{3} + \frac{4}{3} - 0 \geqslant 3 \checkmark$$
$$5w_1 - w_2 + 2w_3 = 5 \times \frac{5}{3} - \frac{4}{3} + 2 \cdot 0 \geqslant 7 \checkmark$$

(Note that $w_1 \geqslant 0, w_2 \geqslant 0, w_3 \geqslant 0$)

Computation of objective values

$$z = 3x + 7x_2 = 3 \cdot 9 + 7 \cdot 2 = 41$$

$$z' = 19w_1 + 7w_2 + 2w_3 = 19 \cdot \frac{5}{3} + 7 \cdot \frac{4}{3} + 2 \cdot 0 = \frac{95+28}{3} = \frac{123}{3} = 41$$

since $z = z'$, both proposed solutions are optimal.