MARKING SCHEME

50

MAT 237, Quiz 1 Name **TUT0101**

Part A: (2 marks) What does it mean for the point $\boldsymbol{a}$ to be a boundary point of the set S.

$a \in \partial S$ iff $\forall r>0$ (1) $B(r,a) \cap S \neq \emptyset$ & (0.5)

$B(r,a) \cap S^c \neq \emptyset$ (0.5)

Part B: (3 marks) Find a boundary point of the set $S = (0,1)$ and prove your claim.

$0 \in \partial S$ b/c (1) $\forall r>0$ $B(r,0) = (-r,r)$ and $(-r,r) \cap (0,1)$ (1)

$= (0,r) \neq \emptyset$ as $r>0$

also $(-r,r) \cap S^c \supseteq (-r,0) \neq \emptyset$ (1)

Part C: (5 marks) Prove that for any set $S \subseteq \mathbb{R}^n$, $\partial S = \partial S^c$.

$a \in \partial S \iff \forall r>0$ $B(r,a) \cap S \neq \emptyset$ & $B(r,a) \cap S^c \neq \emptyset$ $\iff$

$\iff \begin{cases} B(r,a) \cap S^{c^c} \neq \emptyset \\ B(r,a) \cap S^c \neq \emptyset \end{cases} \iff \begin{cases} B(r,a) \cap S^c \neq \emptyset \\ B(r,a) \cap S^{c^c} \neq \emptyset \end{cases}$

$\iff a \in \partial S^c$

One direction $\partial S \subseteq \partial S^c$ (3)

so $\partial S = \partial S^c$

another direction (2)

MAT 237, Quiz 1 **Name** MARKING SCHEME **TUT0201**

Part A: (2 marks) What does it mean for the point $\boldsymbol{a}$ to be an interior point of the set S.

$a \in S^{int}$ iff $\exists r > 0$ $B(r, a) \subset S$.

1 1

Part B: (3 marks) Find an interior point of the set $S = (0, 1)$ and prove your claim.

$0.5 \in S^{int}$ b/c $B(\frac{1}{4}, 0.5) \overset{1}{\subset} (0, 1)$.

$= (0.25, 0.75)$

1 1

Part C: (5 marks) Prove that for any sets $A \subseteq B \subseteq \mathbb{R}^n$, $A^{int} \subseteq B^{int}$.

let $a \in A^{int}$, Then $\exists r > 0$ $B(r, a) \subset A$, but $A \subseteq B$, so

$B(r, a) \subset B$ $\therefore$ $\exists r > 0$ $B(r, a) \subset B$ $\therefore$ $a \in B^{int}$

MAT 237, Quiz 1 Name ________________ Tutorial TUT0301

Part A: (2 marks) What does it mean for a set S to be open? (please be carefule not mistake the definition of closed and open with the equivalent statement 1.4)

S is open if $\partial S \subset S^c$

2

Part B: (4 points) Show that for the set $S = \{(x, y) : y = x^2, 1 < x < 2\}$, we have $S \subset \partial S$.

pick $(x,y) \in S$ and Consider $B(r,(x,y))$. Note That

$B(r,(x,y)) \cap S \neq \emptyset$ as $(x,y) \in S \cap B(r,(x,y))$ 1 and

$B(\quad) \cap S^c \neq \emptyset$ b/c $(x+\frac{r}{2}, x^2) \in S^c$ b/c $(x+\frac{r}{2})^2 \neq x^2$

2

$(\quad)^2 \neq y$

and $(x+\frac{r}{2}, x^2) \in B(r,(x,y))$ b/c

$y = x^2$

1

$\sqrt{(x+\frac{r}{2}, x^2)-(x,x^2)} = \sqrt{(\frac{r}{2})^2} = \frac{r}{2} < r$.

Part C: (4 points) Prove that a set S is open if $S \subseteq S^{int}$.

1

assume $S \subseteq S^{int}$, and let $x \in \partial S$, Then $x \notin S^{int}$, so $x \notin S$

1

$\forall r>0$ $B(r,x) \cap S \neq \emptyset$

1

$B(r,x) \cap S^c \neq \emptyset$

Therefore $x \in S^c$, so $\partial S \subset S^c$.

1

MAT 237, Quiz 1 Name Marking Scheme **TUT0401**

Part A: (2 marks) For a set $S \subset \mathbb{R}^n$ What does it mean for a point x to be a boundary point of S?

$\forall r > 0 \quad B(r, x) \cap S \neq \emptyset$ (1)

& $B(r, x) \cap S^c \neq \emptyset$ (1)

Part B: (4 marks) Show that any point on the unit circle is a boundary point of $B(1, (0, 0))$.

let $x = (x, y)$ be on the unit circle: so $x^2 + y^2 = 1$.

let $r > 0$ be given, design $y_1 = (1 - \frac{r}{2})x$ and $y_2 = (1 + \frac{r}{2})x$ and

note $y_1 \in B(1, (0,0)) \cap B(r, x)$ (1) --- (1)

show why ...

& $y_2 \in B^c(1, (0,0)) \cap B(r, x)$ (1) --- (1)

Part C: (4 marks) If $r < s$ are real numbers, show that $\overline{B(r, \boldsymbol{a})} \subset B(s, \boldsymbol{a})$. (Hint: you may assume that the circle of radius r is exactly the boundary of the ball of radius r.)

if $x \in \overline{B(r, a)}$ then $|x - a| \leq r$, but $r < s$, so

$|x - a| < s$ so $x \in B(s, a)$.

(2) for the trick

(2) for the method

Part A: (2 marks) What is S^{int}?

$S^{int} = \{ x \in S : \exists r > 0 \quad B(r, a) \subset S \}$

1 1

Part B: (5 marks) Determine S^{int} for the set $S = \{(x,y) : x^2 + y^2 = 1\}$

1 $S^{int} = \emptyset$ b/c for any $(x,y) \in S$ and any $r > 0$ the two points

$(x \pm \frac{r}{2}, y) \in B(r, (x,y))$ b/c $|(x \pm \frac{r}{2}, y) - (x,y)| = \sqrt{(x \pm \frac{r}{2} - x)^2 + (y-y)^2}$

1

$= \sqrt{(\frac{r}{2})^2} = \frac{r}{2} < r$ 1 But at least one of the two points

is not in S (So $B(r,(x,y)) \not\subset S$) This is b/c

$(x \pm \frac{r}{2})^2 + y^2 = x^2 \pm rx + \frac{r^2}{4} + y^2 = x^2 + y^2 \pm rx + \frac{r^2}{4} = 1 \pm rx + \frac{r^2}{4}$.

2 Now either $1 + rx + \frac{r^2}{4} \neq 1$ (i.e $(x + \frac{r}{2}, y) \notin S$) or else

if $1 + rx + \frac{r^2}{4} = 1$ Then $rx + \frac{r^2}{4} = 0$ so $rx = -\frac{r^2}{4}$, in which

Case $1 - rx + \frac{r^2}{4} = 1 + \frac{r^2}{4} + \frac{r^2}{4} = 1 + \frac{r^2}{2} \neq 1$.

Part C: (3 marks) Prove that for any set S, if a given point $x \notin S^{int}$ then $\forall r > 0 \ B(r, \mathbf{a}) \cap S^c \neq \emptyset$.

if $x \notin S^{int}$ Then $\sim [\exists r > 0 \quad B(r, a) \subset S]$, so That

1 1 $\forall r > 0 \quad B(r, a) \not\subset S$ so that

1 $\forall r > 0 \quad B(r, a) \cap S^c \neq \emptyset$

Part A: (2 marks) Present Cauchy's inequality for two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$

1 $|a \circ b| \le |a|\,|b|$

absolute value 1 norm in $\mathbb{R}^n$

Part B:(4 marks) Use Cauchy's inequality to to prove the inequality $|a+b+c| \le \sqrt{3}\sqrt{a^2+b^2+c^2}$.

Apply Cauchy ineq to the vectors $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $Y = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and note

$X \circ Y = a+b+c$ and $|X| = \sqrt{3}$ $|Y| = \sqrt{a^2+b^2+c^2}$

so $|a+b+c| \le \sqrt{3}\,\sqrt{a^2+b^2+c^2}$ 4

Part C: (4 marks) Use Cauchy's inequality to prove triangle inequality.

$|X+Y|^2 = (X+Y) \overset{1}{\circ} (X+Y) = X \circ X + X \circ Y + Y \circ X + Y \circ Y$

$= |X|^2 + 2X \circ Y + |Y|^2 \underset{1}{\le} |X|^2 + 2|X||Y| + |Y|^2 \overset{1}{=}$

$(|X|+|Y|)^2$, Therefore $|X+Y|^2 \le (|X|+|Y|)^2$, and

$|X+Y| \le |X|+|Y|$ 1 by square rooting both sides.

Part A: (2 mark) What does it mean for a set $S \subset \mathbb{R}^n$ to be closed? what does it mean for S to be open?

S is closed if $\partial S \subset S$. S is open if $\partial S \subset S^c$

1 1

Part B: (4 marks) Prove that the set $S = \{\boldsymbol{a}\}$, that is a set consisting of a single point is closed. (Please present all the necessary details.)

note That $\partial S = S$ b/c $a \in \partial S$: $\forall r>0$ $y = (1+\frac{r}{2})a \in S^c \cap B(r,a)$

1.5 & $a \in B(r,a) \cap S$.

also if $b \notin S^c$, i.e. $b \neq a$ Then let $r = \frac{|b-a|}{2}$, Then

$a \notin B(r,b)$ b/c $|a-b| \nless r$, so $S \cap B(r,b) = \emptyset$ so $b \notin \partial S$.

$\therefore$ $\partial S = \{a\}$, and $\therefore$ $\partial S \subset S$, so S is closed. 2.5

Part C:(4 marks) Prove that if S is open then $S \subseteq S^{int}$.

given $x \in S$ 1 We know $x \notin \partial S$ b/c $\partial S \subset S^c$. so

$\sim [\forall r>0$ $B(r,x) \cap S \neq \emptyset$ & $B(r,x) \cap S^c \neq \emptyset]$ i.e $\exists r>0$ 2 $B(r,x) \cap S = \emptyset$ or $B(r,x) \cap S^c = \emptyset$

but Since $x \in S$ $B(r,x) \cap S \neq \emptyset$, so $B(r,x) \cap S^c = \emptyset$, so 1

$x \in S^{int}$.