Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 4

- Due Tues. March 12, 6:10pm sharp, in class . Late assignments will not be accepted - even if it's one minute late!
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Consider $\mathbb{C}^3$ together with the standard inner product.

(a) Find an orthonormal basis for $W = Span\{(0, -i, 1), (1 + i, 2, 1)\}$.

(b) Find the matrix of the orthogonal projection onto W with respect to the standard basis of $\mathbb{C}^3$.

2. Let $p(x), q(x) \in P_n(\mathbb{C})$, and suppose $x_0, x_1, \ldots, x_n$ are $n + 1$ distinct complex numbers. Show that

$$< p(x), q(x) >= p(x_0)\overline{q(x_0)} + p(x_1)\overline{q(x_1)} + \cdots + p(x_n)\overline{q(x_n)}$$

defines an inner product on $P_n(\mathbb{C})$. That is, satisfies the conditions of Definition 5.3.1, page 242 of the textbook.

Hint: The first two conditions in the definition should be fairly straightforward to show, they follow from properties of complex numbers. For the third, you may need the Fundamental Theorem of Algebra: Every polynomial of degree n has exaclty n complex roots (counting multiplicities).

3. Consider $P_2(\mathbb{R})$ together with inner product

$$< p(x), q(x) >= p(-1)q(-1) + p(0)q(0) + p(2)q(2).$$

For what value of c will the set $S = \{3x^2 - 2x - 1, cx^2 + x - 1, 5x^2 + cx - 9\}$ be orthogonal?

4. Find an orthonormal basis for $P_2(\mathbb{C})$ with respect to the inner product

$$< p(x), q(x) >= p(0)\overline{q(0)} + p(i)\overline{q(i)} + p(2i)\overline{q(2i)}.$$

5. If W_1 and W_2 are subspaces of a vector space V, we say V is the direct sum of W_1 and W_2, in symbols $V = W_1 \oplus W_2$, if

$$V = W_1 + W_2 \text{ and } W_1 \cap W_2 = \{0\}.$$

Show that the following statements are equivalent.

(i) $V = W_1 \oplus W_2$.

(ii) Every vector $v \in V$ can be written uniquely as $v = w_1 + w_2$ where $w_1 \in W_1$, $w_2 \in W_2$.

(iii) $V = W_1 + W_2$, and for vectors $w_1 \in W_1, w_2 \in W_2$, if $w_1 + w_2 = 0$ then $w_1 = w_2$.

(iv) If α_1 is a basis for W_1 and α_2 is a basis for W_2, then $\alpha = \alpha_1 \cup \alpha_2$ is a basis for V.

6. Let $W_1 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = A^T\}$, and $W_2 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = -A^T\}$. Show that $M_{n\times n}(\mathbb{R}) = W_1 \oplus W_2$.

7. Suppose $T\colon V \to V$ has only two distinct eigenvalues λ_1 and λ_2. Show that T is diagonalizable iff $V = E_{\lambda_1} \oplus E_{\lambda_2}$.

Suggested Extra Problems (not to be handed in):

- Textbook, Section 4.3 **1-13**
- Textbook, Section 4.4 **1-8, 13, 15, 17**
- Let W be the subspace of $\mathbb{C}^3$ spanned by $(i, 1, 0)$ and $(-1, 1+i, 1)$. Find $P_W((2-i, 1, i))$, the orthogonal projection of $(2-i, 1, i)$ onto W.
- Consider $V = P_5(\mathbb{R})$ together with inner product $< p, q >= \int_{-1}^{1} p(x)q(x)\,dx$. Find $W^\perp$ if $W = \{p(x) \in P_5(\mathbb{R}) \mid p(0) = p'(0) = p''(0) = 0\}$.
- An extension of question 5. If $W_1, W_2, \ldots, W_k$ are subspaces of a vector space V, say V is the direct sum of $W_1, W_2, \ldots, W_k$, in symbols $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$, if

$$V = W_1 + W_2 + \cdots + W_k \text{ and } W_j \cap \sum_{i\neq j} W_i = \{0\} \text{ for each } j = 1, 2, \ldots, k$$

  Write out a corresponding set of four equivalent statements as in question 5.
- Let $W_1 = Span\{(1, 0, 0, 0), (0, 1, 0, 0)\}$, $W_2 = Span\{(0, 0, 1, 0), (0, 0, 0, 1)\}$, and $W_3 = Span\{(1, 1, 1, 1), (1, 1, 1, -1)\}$. Show that $\mathbb{R}^4 = W_1 \oplus W_2$ and $\mathbb{R}^4 = W_1 \oplus W_3$.
- Note the above question shows that in general that if $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$, it does **not** imply that $W_2 = W_3$. However, show that if $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$, then $dim(W_2) = dim(W_3)$.
- Show that a linear transformation $T\colon V \to V$ is diagonalizable iff V is the direct sum of the eigenspaces of T.