Exerzitien VII

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Nov. 10., in your tutorial.

Reading suggestion: Axler Chapter 3 and 4.

Exercise 1. *Let $\theta \in \mathbb{R}$, $0 < \theta < \pi$. Consider the 2-dimensional rotation matrix*

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1. *Is there a vector $u \in \mathbb{R}^2$ such that $Tu = \lambda u$, for some nonzero $\lambda \in \mathbb{R}$? Justify your answer.*
2. *Now view the same T as a linear operator on $\mathbb{C}^2$. Find all vectors $v \in \mathbb{C}^2$ such that $Tv = \lambda v$, for some nonzero $\lambda \in \mathbb{C}$.*
3. *Let γ be a basis of $\mathbb{C}^2$ consisting of vectors found in part 2. Write the change of basis matrices $[I]_\gamma^e$ and $[I]_e^\gamma$ (where e is the standard basis). Finally, compute $[T]_\gamma^\gamma$.*

Exercise 2. *Let $A = [a_{ij}] \in \mathbb{F}^{n\times n}$ be a square $n \times n$ matrix. We define the* trace *of A to be the number*

$$\mathrm{T}r(A) := \sum_{k=1}^{n} a_{kk}.$$

1. *Prove that $\mathrm{T}r : \mathbb{F}^{n\times n} \to \mathbb{F}$ is a linear function.*
2. *Let $A, B \in \mathbb{F}^{n\times n}$. Prove that $\mathrm{T}r(AB) = \mathrm{T}r(BA)$.*
3. *Prove that $\mathrm{T}r(PAP^{-1}) = \mathrm{T}r(A)$ for any invertible matrix P in $\mathbb{F}^{n\times n}$.*

Exercise 3. *For each of the following matrices, compute the rank (i.e. the dimension of the range of the associated linear map) and the inverse if it exists. Use the row reduction procedure to find the inverse, and show steps (make it neat, please).*

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$$