PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

AUGUST 2011 EXAMINATIONS

FINAL EXAM

CSC 165H1Y
DURATION — 2 HOURS

NO AIDS ALLOWED

PLEASE HAND IN

LAST NAME: ______________________

FIRST NAME: ______________________

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 8 questions on 18 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question,". You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-18 of this test.

Marking breakdown (Total = 100 marks).

Question 1	20 marks	Question 5	16 marks
Question 2	16 marks	Question 6	12 marks
Question 3	6 marks	Question 7	8 marks
Question 4	10 marks	Question 8	12 marks

Good Luck!

Here are some definitions and results that will be useful throughout the exam. (note: you may also use any result from X = {notes, assignments, tutorials, lectures} by saying "from the x", where $x \in X$)

1. Let $\mathbb{N}$= the set of natural numbers (i.e {0, 1, 2, 3, ...})
2. Let $\mathbb{R}$= the set of real numbers and $\mathbb{R}^+$ = the set of positive real numbers
3. Let $\mathbb{F} = \{\mathbb{N} \rightarrow \mathbb{R}^{\geq 0}\}$
4. $\forall f, g \in \mathbb{F}$: $f \in \mathcal{O}(g) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c * g(n)$
5. $\forall f, g \in \mathbb{F}$: $f \in \Omega(g) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq c * g(n)$
6. $\forall f, g \in \mathbb{F}$: $f \in \Theta(g) \Leftrightarrow \exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 * g(n) \leq f(n) \leq c_2 * g(n)$
7. $\forall f, g \in \mathbb{F}, \forall z \in \mathbb{R}^+, f = z * g \Rightarrow f \in \Theta(g)$
8. $\forall m, n, r \in \mathbb{N}, r = m\%n \Leftrightarrow (0 \leq r < n) \wedge (\exists q \in \mathbb{N}, m = q * n + r)$
9. $W_P(n) \in \Omega(f(n)) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, \exists x \in I, size(x) = n \wedge n \geq B \Rightarrow t_P(x) \geq c * f(n)$
10. $y = log_b(x) \Leftrightarrow b^y = x$
11. $log_b(xy) = log_b(x) + log_b(y)$
12. $log_b(x/y) = log_b(x) - log_b(y)$

commutative laws	$P \wedge Q$	$\Leftrightarrow$	$Q \wedge P$
	$P \vee Q$	$\Leftrightarrow$	$Q \vee P$
	$(P \Leftrightarrow Q)$	$\Leftrightarrow$	$(Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R$	$\Leftrightarrow$	$P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R$	$\Leftrightarrow$	$P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R)$	$\Leftrightarrow$	$(P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R)$	$\Leftrightarrow$	$(P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q$	$\Leftrightarrow$	$\neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q$	$\Leftrightarrow$	$\neg P \vee Q$
equivalence	$(P \Leftrightarrow Q)$	$\Leftrightarrow$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
double negation	$\neg(\neg P)$	$\Leftrightarrow$	P
DeMorgan's laws	$\neg(P \wedge Q)$	$\Leftrightarrow$	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	$\Leftrightarrow$	$\neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q)$	$\Leftrightarrow$	$P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q)$	$\Leftrightarrow$	$\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x))$	$\Leftrightarrow$	$\exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x))$	$\Leftrightarrow$	$\forall x \in D, \neg P(x)$
identity	$P \vee (Q \wedge \neg Q)$	$\Leftrightarrow$	P
	$P \wedge (Q \vee \neg Q)$	$\Leftrightarrow$	P
idempotence	$P \vee P$	$\Leftrightarrow$	P
	$P \wedge P$	$\Leftrightarrow$	P
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x)$	$\Leftrightarrow$	$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
	$\exists x \in D, P(x) \vee Q(x)$	$\Leftrightarrow$	$(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

QUESTION 1. [20 MARKS]

PART (A) [5 MARKS]

Consider the predicate

$$\text{Prime}(x) : \text{x is prime}.$$

Using the above predicate, provide an equivalent symbolic statement for the statement below:

(S1) A natural number n is called a prime number if it is bigger than one and has no divisors other than 1 and itself.

(*Reminder: We say that n divides m or n is a divisor of m, and write $n|m$. For example 4 is a divisor of 12.*)

PART (B) [5 MARKS]

What is the negation of S1?

PART (C) [5 MARKS]

What is the converse of S1? Is it true? Can we write S1 as a bi-implication? Explain your answer.

PART (D) [5 MARKS]

Write S1 as a disjunction. Simplify the statement and make comments about what manipulation rules are used at each step.

QUESTION 2. [16 MARKS]

Equivalence

PART (A) [6 MARKS]

Use the manipulation rules to prove S2A and S2B are equivalent. What rules are you using at each step?

(S2A) $((A \wedge B) \Rightarrow (\neg C \vee D))$

(S2B) $((A \wedge \neg D) \Rightarrow (\neg B \vee \neg C))$

PART (B) [10 MARKS]

Let

(S2c) $A \Rightarrow (B \vee C)$

1. What is the contrapositive of S2C? [2 MARKS]

2. Is S2c equivalent to its contrapositive? Justify your answer by using truth table and Venn diagram <u>both</u>. [8 MARKS]

QUESTION 3. [6 MARKS]

Suppose E is a set of natural numbers, and consider the statement

S3: Every element of E is an even number.

Which of the following statements are necessary conditions for S3 (in other words they are implied by S3)? Which are sufficient for S3 (in other words they imply S3)? Explain your answers carefully in precise, grammatical English.

PART (A) [3 MARKS]

E is empty set.

PART (B) [3 MARKS]

5 is not an element of E, or 42 is an element of E.

QUESTION 4. [10 MARKS]

Prove S4.

S4: For every positive integer n, $3^{3n-2} + 2^{3n+1}$ is divisible by 19.

Include comments justifying steps.

(Hint: 1. First write the statement symbolically and then use induction.
2. Integer a is divisible by integer b iff $\exists k \in \mathbb{N}, a = bk$.*)*

QUESTION 5. [16 MARKS]

Prove or disprove

Let

$$\text{S5: } \forall a \in \mathbb{Z}, \text{ If } a^2 - 2a + 7 \text{ is even, then } a \text{ is odd.}$$

PROVE OR DISPROVE S5, using the following methods:
(*Note: You must use the proof structure from this course and include comments justifying each step.*)

PART (A) [8 MARKS]

Use the contrapositive of S5.

PART (B) [8 MARKS]

Prove S5 by contradiction.

QUESTION 6. [12 MARKS]

In this question, use the definition of big-Oh and the proof structure of this course. Include comments justifying your step, when necessary.

PART (A) [6 MARKS]

Let $f(n) = 12^n$ and $g(n) = 9^n$. Prove or disprove that $f(n) \in \mathcal{O}(g(n))$.

(You may not use the technique of limits from calculus. You may, however, consult the cheat sheet for logarithm rules, i.e. rules 10 and 11.)

PART (B) [6 MARKS]

Define $\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$, and define $(f + g)(n) = f(n) + g(n)$. Prove or disprove

$$\forall f, g, h \in \mathcal{F}, f \in \Omega(h) \wedge g \in \Omega(h) \Rightarrow (f + g) \in \Omega(h)$$

QUESTION 7. [8 MARKS]

Consider the following Python program for finding the index of the first even element in a list.

```
def FEV(A):
 """ Return index i if A[i] is even. Otherwise, return -1. """
i = 0
while i < len(A):
    if A[i] % 2 == 0:
        return i
    i = i + 1
return -1\\
```

PART (A) [4 MARKS]

What is the time complexity tFEV(A), if the first index where A has an even element is j? Explain your answer.

PART (B) [4 MARKS]

What is the WORST CASE complexity? Explain your answer.

QUESTION 8. [12 MARKS]

Floating points

PART (A) [2 MARKS]

Convert $(1001101)_2$ from binary to decimal. Include the details.

PART (B) [4 MARKS]

What is the condition number of $f = x^2 - 3$? What does the condition number show? When do we get a large condition number?

PART (C) [2 MARKS]

Suppose you have a normalized floating-point representation that has $\beta = 10$ (base, or radix, 10), $t = 5$ digits, and $e \in \{-5, 5\}$. How do you represent $1/16$ in this system?

PART (D) [4 MARKS]

Consider the <u>normalized</u> floating point system with base $\beta = 2$, number of digits $t = 4$, and a range $[-3, 3]$ of exponents (integers).

1. What are the smallest and largest positive (and non-zero) numbers representable in this system?

2. When do we encounter underflow or overflow in this system? Explain your answer.

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# 2:	______/ 16
# 3:	______/ 6
# 4:	______/ 10
# 5:	______/ 16
# 6:	______/ 12
# 7:	______/ 8
# 8:	______/ 12
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