1. **Predicate:** Define $P(n)$ to be the statement: "playing the game with a single group of n coins generates exactly $n(n-1)/2$ dollars, no matter how the game is played."

   We prove $\forall n \geqslant 1, P(n)$ by complete induction.

   Base Case: Consider playing the game with a single group of 1 coin. The game is immediately over — no group can be split — and the total gain is 0 dollars $= 1(1-1)/2$ dollars.

   Hence, $P(1)$.

   Ind. Hyp.: Assume $n > 1$ and $\forall k \in \{1, 2, \dots, n-1\}, P(k)$.

   Ind. Step: Consider playing the game with a single group of n coins. During the first round, n will be split into two groups.

   Let a, b be the number of coins in each group. (We make no further assumption about a and b so that the rest of the proof applies to all possible ways to split up n into a, b.) Then we win ab dollars for the first round.

   By the Ind. Hyp., the group of a coins generates exactly $a(a-1)/2$ dollars and the group of b coins generates exactly $b(b-1)/2$ dollars, no matter how these groups are split up. (This is because $a \geqslant 1$ and $b \geqslant 1$ and $n = a + b$, so $a < n$ and $b < n$.)

   This means the total amount gained is equal to

$$
\begin{aligned}
ab + a(a-1)/2 + b(b-1)/2 &= \big(2ab + (a^2 - a) + (b^2 - b)\big)/2 \\
&= \big(a^2 + 2ab + b^2 - a - b\big)/2 \\
&= \big((a+b)^2 - (a+b)\big)/2 \\
&= (a+b)(a+b-1)/2 \\
&= n(n-1)/2
\end{aligned}
$$

   Because this applies no matter how n is initially split up into a, b, we have that $P(n)$.

   Conclusion: By induction, $\forall n \geqslant 1, P(n)$, *i.e.*, playing the game with n coins always generates exactly $n(n-1)/2$ dollars.

2. **Predicate:** Define $P(x, y)$ to be the statement: "$\exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$."

   We prove $\forall (x, y) \in M, P(x, y)$ by structural induction on M.

   Base Case: $(3, 2) = (2^1 + 1, 2^0 + 1)$ so $P(3, 2)$ — just pick $k = 0$.

   Ind. Hyp.: Assume $(x, y) \in M$ and $P(x, y)$.

   Ind. Step: Let $k_0 \in \mathbb{N}$ be such that $(x, y) = (2^{k_0+1} + 1, 2^{k_0} + 1)$ — by the Ind. Hyp.

$$
\begin{aligned}
\text{Then } (3x - 2y, x) &= \big(3(2^{k_0+1} + 1) - 2(2^{k_0} + 1), 2^{k_0+1} + 1\big) \\
&= \big(6 \cdot 2^{k_0} + 3 - 2 \cdot 2^{k_0} - 2, 2^{k_0+1} + 1\big) \\
&= \big(4 \cdot 2^{k_0} + 1, 2^{k_0+1} + 1\big) \\
&= \big(2^{k_0+2} + 1, 2^{k_0+1} + 1\big)
\end{aligned}
$$

   So $P(3x - 2y, x)$ — just pick $k = k_0 + 1$.

   Conclusion: By structural induction on M, $\forall (x, y) \in M, \exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$.