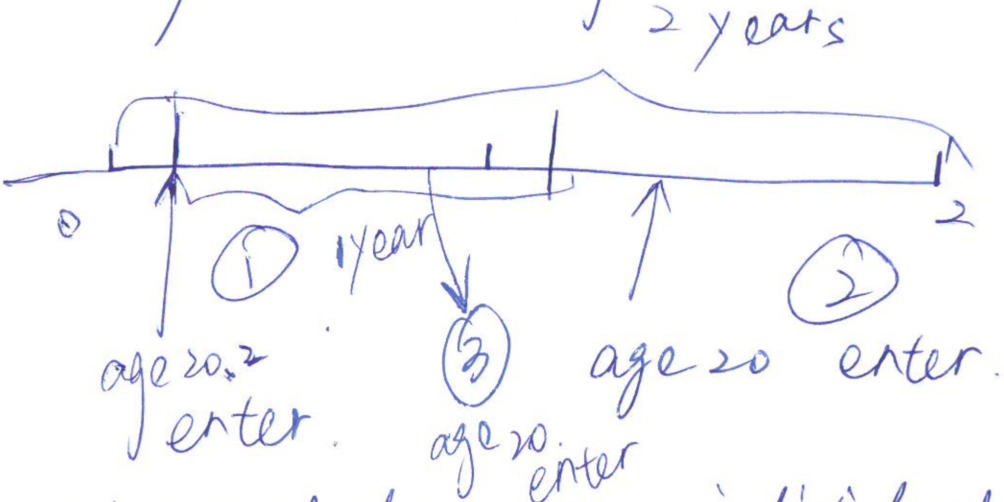
Lecture week 7.

Example :

study individuals aged 20.

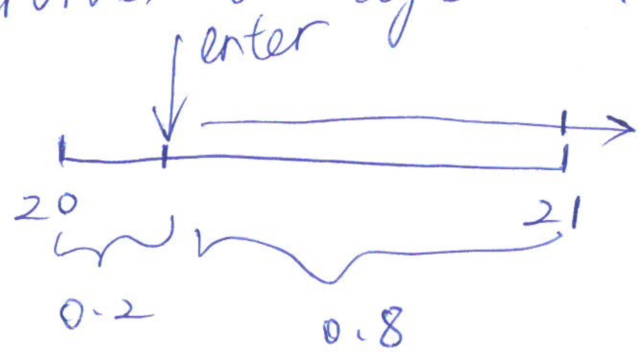
aim: estimate μ_{20}. $(20, 21)$

2-years investigation



N individuals. Some individual will turn to 20 during this 2-year period. some individuals enter the study after 20, due to some reasons.

Person 1 : enter the study at age 20.2, survives to age 21. : enough time remaing in this study to observe until 21



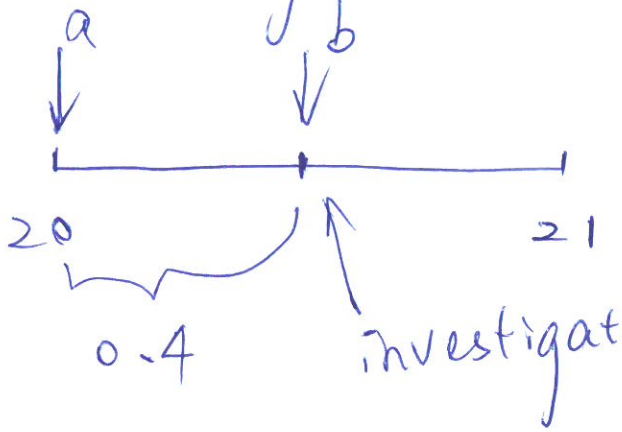
$a = 0.2$

$b = 1$

$V = b - a = 0.8$

$\delta = 0$ $\quad T = 1$ (time lived)

Person 2 : enter the study at age 20, but investigation ends at age 20.4.



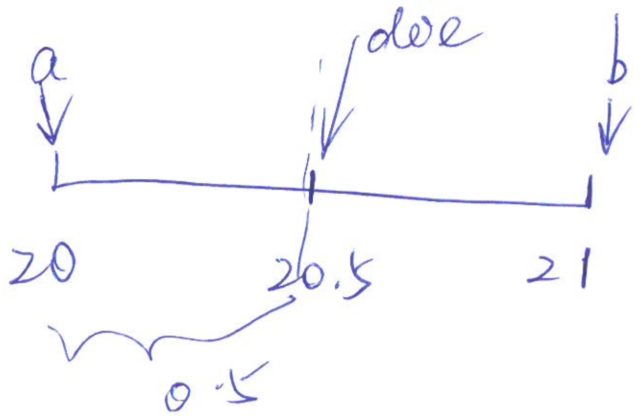
: not enough time remaining to observe until 21

investigation ends. → censored.

$a = 0$ $\quad T = 0.4$ $\quad \delta = 0$

$b = 0.4$ $\quad V = 0.4$

Person 3 : enter the study at age 20. dies at age 20.5.



$a = 0$ $\quad T = 0.5$ $\quad \delta = 1$

$b = 1$ $\quad V = 0.5$

MLE - censoring

$L = \prod f(t) \prod S(c)$

$\downarrow$ complete

$\downarrow$ censoring $P(T_i \geq C_i)$

$\delta_i = 1$: $\qquad \delta_i = 0 \qquad v_i \, p_{x+a_i}$

$\downarrow$

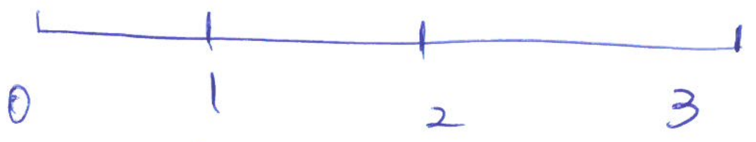
$v_i \, p_{x+a_i} \, \mu(x+a_i+v_i)$

conclude: i^{th} contribution to the likelihood

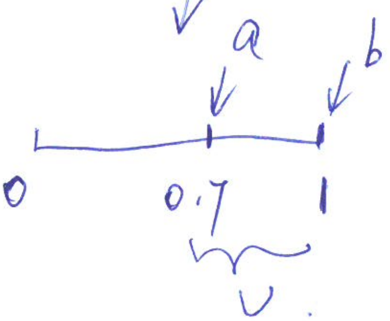
$v_i \, p_{x+a_i} \, \mu(x+a_i+v_i)^{\delta_i}$

Example :

3 years investigation

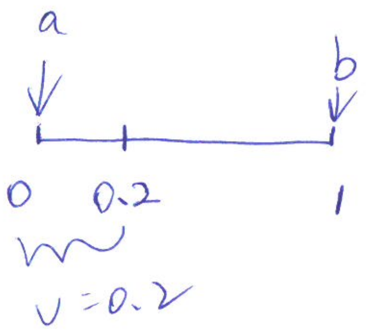


only interested in 2^{nd} year of life.

P1 : 

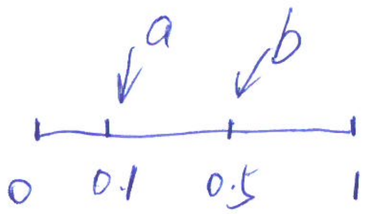
$a = 0.7 \quad b = 1$

$T = 1 \quad \delta = 0 \quad v = 0.3$.

P2 : 

$a = 0, \quad b = 1$

$T = 0.2 \quad \delta = 1 \quad v = 0.2$

P3 : 

$a = 0.1, \quad b = 0.5,$

$T = 0.5, \quad \delta = 0, \quad v = 0.4$

P4 : 

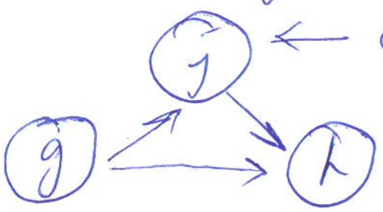
X omit. dies before second year.

Nothing observed.

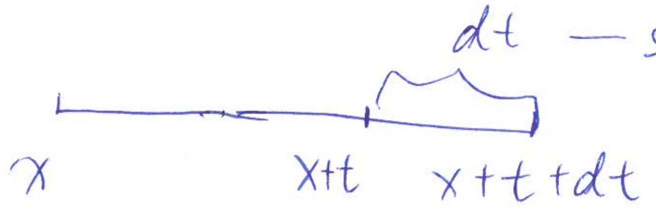
Proof of Kolmogorov Forward equation:

By law of total probability:

$$ {}_{t+dt}p_x^{gh} = \sum_{j\neq h} {}_tp_x^{gj}\; {}_{dt}p_{x+t}^{jh} + {}_tp_x^{gh}\; {}_{dt}p_{x+t}^{hh} $$



← other state than h, include g



j includes g

$$ g \longrightarrow j \longrightarrow h $$

$$ g \longrightarrow h \longrightarrow h $$

complement.

We also have $\quad {}_{dt}p_{x+t}^{hh} = 1 - \sum_{j\neq h} {}_{dt}p_{x+t}^{hj}$

$\Rightarrow$

$$ {}_{t+dt}p_x^{gh} = \sum_{j\neq h} {}_tp_x^{gj}\; {}_{dt}p_{x+t}^{jh} + {}_tp_x^{gh} \Big(1 - \sum_{j\neq h} {}_{dt}p_{x+t}^{hj}\Big) $$

Using assumption $\quad {}_{dt}p_{x+t}^{jh} \approx \mu_{x+t}^{jh}\cdot dt.$

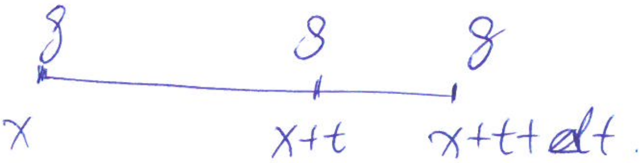
$${}_{t+dt}p_x^{gh} - {}_tp_x^{gh}$$

$$\approx \sum_{j\neq h} {}_tp_x^{gj}\,\mu^{jh}\,dt - \sum_{j\neq h} {}_tp_x^{gh}\,\mu^{hj}\,dt.$$

$$\Rightarrow \underline{\lim_{dt\to 0} \frac{{}_{t+dt}p_x^{gh} - {}_tp_x^{gh}}{dt}} \approx \sum_{j\neq h}\left({}_tp_x^{gj}\,\mu^{jh} - {}_tp_x^{gh}\,\mu^{hj}\right)$$

$$\frac{d\,{}_tp_x^{gh}}{dt}.$$

Similarly

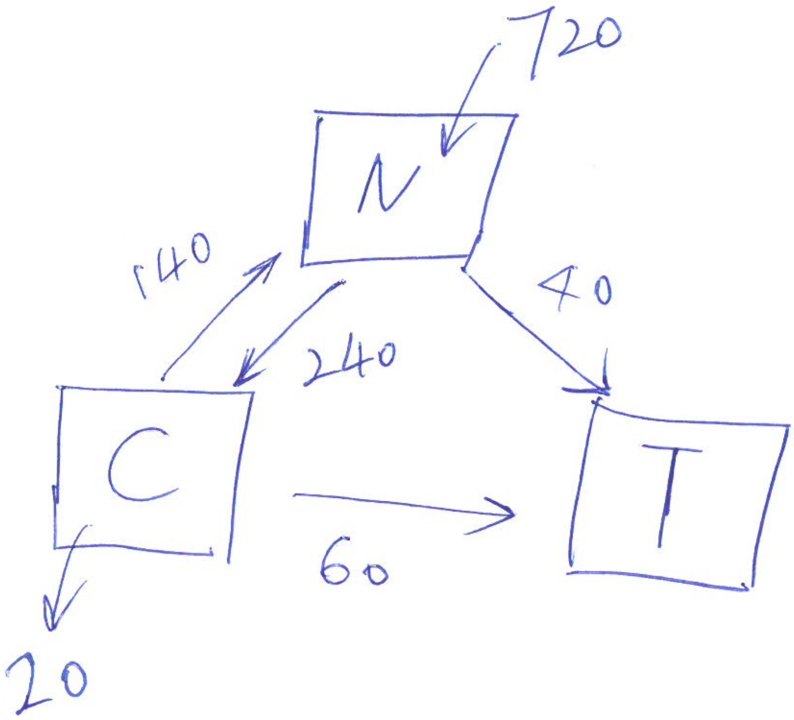


$${}_{t+dt}p_x^{\overline{gg}} = {}_tp_x^{\overline{gg}}\;{}_{dt}p_{x+t}^{\overline{gg}}$$

$$\approx {}_tp_x^{\overline{gg}}\Big(1-\sum_{j\neq g}\mu^{gj}\cdot dt\Big)$$

$$\Rightarrow \frac{d\,{}_tp_x^{\overline{gg}}}{dt} = -{}_tp_x^{\overline{gg}}\sum_{j\neq g}\mu^{gj}$$

Example: multistate



$$\nu^N = 720$$

$$\nu^C = 20$$

$$\delta^{NC} = 240$$

$$\delta^{CN} = 140$$

$$\delta^{NT} = 40$$

$$\delta^{CT} = 60$$

$$\hat{\mu}^{NC} = \frac{240}{720}$$

$$\hat{\mu}^{CN} = \frac{140}{20}$$

$$\hat{\mu}^{NT} = \frac{40}{720}$$

$$\hat{\mu}^{CT} = \frac{60}{20}.$$

$$\mathrm{Var}\left(\hat{p}^{\overline{NN}}\right) = \mathrm{Var}\left(\hat{\mu}^{NC} + \hat{\mu}^{NT}\right) \cdot \left(\hat{p}^{\overline{NN}}\right)^2$$

Delta method for multivariate.

(O'Neil's note, page 12)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \quad p \times 1 \qquad u_x = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix}$$

$$\Sigma = \left[\begin{pmatrix} \sigma_1^2 & \sigma_{21} & \cdots & \cdots \\ \sigma_{12} & \sigma_2^2 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \cdots & \cdots & \cdots & \sigma_p^2 \end{pmatrix} \right]$$

$$Var[g(X)] = \underbrace{\left(\frac{\partial g(X)}{\partial u_x} \right)^T}_{1 \times p} \underbrace{\Sigma}_{p \times p} \underbrace{\left(\frac{\partial g(X)}{\partial u_x} \right)}_{p \times 1}$$

In this example.

$$\underbrace{\hat{p}^{\overline{NN}}}_{g(X)} = e^{-\underbrace{\hat{u}^{NC}}_{x_1} - \underbrace{\hat{u}^{NT}}_{x_2}}$$

$$Var(\hat{p}^{\overline{NN}}) = \begin{pmatrix} -e^{-\hat{u}^{NC} - \hat{u}^{NT}} \\ -e^{-\hat{u}^{NC} - \hat{u}^{NT}} \end{pmatrix}^T \begin{bmatrix} Var(\hat{u}^{NC}) & 0 \\ 0 & Var(\hat{u}^{NT}) \end{bmatrix} \cdot$$

$$= \left[-e^{-\hat{u}^{NC} - \hat{u}^{NT}} \cdot Var(\hat{u}^{NC}) \right.$$

$$\left. , -e^{-\hat{u}^{NC} - \hat{u}^{NT}} \cdot Var(\hat{u}^{NT}) \right] \cdot \begin{bmatrix} -e^{-\hat{u}^{NC} - \hat{u}^{NT}} \\ -e^{-\hat{u}^{NC} - \hat{u}^{NT}} \end{bmatrix}$$

$$= \left(\hat{p}^{\overline{NN}}\right)^2 \cdot \left[Var(\hat{u}^{NC}) + Var(\hat{u}^{NT}) \right]$$