Introduction to Bayesian Data Analysis

Tutorial 3

(1) Let Y be a random variable denoting the number of deaths from asthma in a US city in a given year. Let u be a measure of the population size of the city.

(a) What would be an appropriate sampling model for the data? Write down your likelihood function. Be sure to clearly define all your parameters.

(b) What is the natural conjugate prior distribution for your sampling model? Hence, derive the posterior distribution of your parameters given a data point y.

(c) Suppose it is found that 3 persons out of a population of 200,000 died of asthma. What is the crude estimated mortality rate in the city?

(d) What information can be used to set up a prior distribution for the asthma mortality rate?

(e) Suppose a Gamma(3.0,5.0) is decided upon as a plausible prior density for the asthma mortality rate. (Note in practice, specifying the prior mean sets the ratio of the two gamma parameters, and then the shape parameter can be altered by trial and error to match the prior knowledge about the tail of the distribution) What is the posterior distribution for the asthma mortality rate given the observed data? Plot the density. What is the posterior mean of the asthma mortality rate? How does the posterior mean compare to the prior mean and the crude estimated mortality rate in (c)? What is the posterior probability that the long-term death rate from asthma in the city is more than 1.0 per 100,000 per year?

(f) Now suppose that ten years of data are obtained for the city. It is found that the mortality rate of 1.5 per 100,000 is maintained; y=30 deaths are found over the ten years. Assuming the outcomes in the ten years are independent with constant long term rate, what is the posterior distribution of the asthma mortality rate, given the ten years of data? Plot the density and compare to the density in (f). What is the posterior mean of the asthma mortality rate and what is the posterior probability that the asthma mortality rate exceeds 1.0 given the ten years of data? State any additional assumptions you make

(2) Assume $y_1, ..., y_n|\theta \overset{\text{iid}}{\sim} Pois(\theta)$. Assume a congugate prior for θ with parameters α and β. Let $\tilde{y}$ be an unobserved value of y. Derive the posterior predictive distribution $p(\tilde{y}|y_1, ..., y_n)$. Show that $Var(\tilde{Y}|y_1, ..., y_n) = E[\theta|y_1, ..., y_n] \times \frac{\beta+n+1}{\beta+n}$ and interpret this result.

(3) Problem 3.3 (Hoff).
Tumor counts: A cancer laboratory is estimating the rate of tumorigenesis in two strains of mice, A and B. They have tumor count data for 10 mice in strain A and 13 mice in strain B. Type A mice have been well studied, and information from other laboratories suggests that type A mice have tumor counts that are approximately Poisson-distributed with a mean of 12. Tumor count rates for type B mice are unknown, but type B mice are related to type A mice. The observed tumor counts are:

$$y_A = (12, 9, 12, 14, 13, 13, 15, 8, 15, 6)$$

$$y_B = (11, 11, 10, 9, 9, 8, 7, 10, 6, 8, 8, 9, 7)$$

(a) Find the posterior distributions, means, variances and 95% quantile-based confidence intervals for θ_A and θ_B, assuming a Poisson sampling distribution for each group and the following prior distribution
$\theta_A \sim Gamma(120, 10)$, $\theta_B \sim Gamma(12, 1)$, $p(\theta_A, \theta_B) = p(\theta_A) \times p(\theta_B)$

(b) Compute and plot the posterior expectation of θ_B under the prior distributions $\theta_B \sim \text{Gamma}(12 \times n_0, n_0)$ for each value of $n_0 \in \{1, 2, ..., 50\}$. Describe what sort of prior beliefs about θ_B would be necessary in order for the posterior expectation of θ_B to be close to that of θ_A.

(c) A new mouse of type B is delivered to the lab. Predict the expected tumor counts for the new mouse assuming:

(i) Independent priors for θ_A and θ_B.

(ii) The data from mice A form a prior distribution for the posterior of θ_B.

(d) Should knowledge about population A tell us anything about population B? Discuss whether or not it makes sense to have $p(\theta_A, \theta_B) = p(\theta_A) \times p(\theta_B)$.

(4) Problem 3.4 (Hoff) Estimate the probability θ of teen recidivism based on a study in which there were n=43 individuals released from incarceration and y=15 re-offenders within 36 months.

(a) Using a beta(2,8) prior for θ, plot $p(\theta)$, $p(y|\theta)$ and $p(\theta|y)$ as functions of θ. Find the posterior mean, mode and standard deviation of θ. Find a 95% quantile-based confidence interval.

(b) Repeat (a), but using a beta(8,2) prior for θ.

(c) Consider the following prior distribution for θ:

$$p(\theta) = \frac{1}{4}\frac{\Gamma(10)}{\Gamma(2)\Gamma(8)}[3\theta(1-\theta)^7 + \theta^7(1-\theta)]$$

which is a 75-25% mixture of a beta(2,8) and a beta (8,2) prior distribution. Plot this prior distribution and compare it to the priors in (a) and (b). Describe what sort of prior opinion this may represent.

(d) For the prior in (c):

(i) Write out mathematically $p(\theta) \times p(y|\theta)$ and simplify as much as possible.

(ii) The posterior distribution is a mixture of two distributions you know. Identify these distributions.

(iii) On a computer, calculate and plot $p(\theta) \times p(y|\theta)$ for a variety of θ values. Also find (approximately) the posterior mode, and discuss its relation to the modes in (a) and (b).

(e) Find a general formula for the weights in the mixture distribution in (d)(ii), and provide an interpretation of their values.

(5) Problem 3.13 (Hoff)

Improper Jefferys' prior: Let $Y \sim \text{Poisson}(\theta)$.

(a) Apply Jeffrey's procedure to this model, and compare the result to the family of gamma densities. Does Jeffreys' procedure produce an actual probability density for θ? In other words, can $\sqrt{I(\theta)}$ be proportional to an actual probability density for $\theta \in (0, \infty)$?

(b) Obtain the form of the function $f(\theta, y) = \sqrt{I(\theta)} \times p(y|\theta)$. What probability density for θ is $f(\theta, y)$ proportional to? Can we think of $f(\theta, y)/\int f(\theta, y)d\theta$ as a posterior density of θ given Y=y?