Lecture 17

Improper Integral

Ex: $\displaystyle\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx$

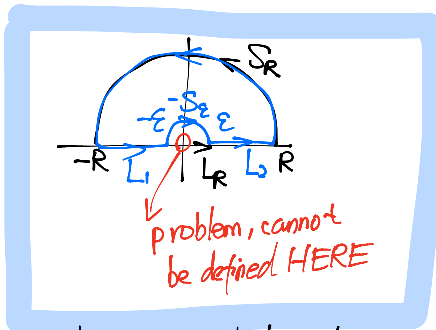
Idea: similar to last class

$f(z) = ?$

$\gamma = ?$

$$f(z) = \frac{\log z}{(z^2+1)^2}$$

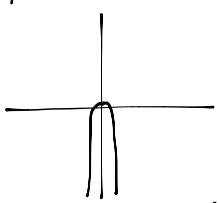
$$\gamma_{R,\varepsilon} = S_R \cup L_1 \cup -S_\varepsilon \cup L_2$$



Idea: Let $\left.\begin{array}{l} R \to \infty \\ \varepsilon \to 0 \end{array}\right\} \Rightarrow \displaystyle\int_{L_2} \longrightarrow \int_0^\infty \frac{\log x}{(x^2+1)^2}$

f has poles at $z = \pm i$

Since L_1 includes $-$ve reals, can't use $\operatorname{Log} z$, we need a different branch.



Let's take $\log z = \ln|z| + i\arg z$, where $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

(delete $-$ve imaginary axis)

On S_R: $\displaystyle |f(z)| = \left|\frac{\log z}{(z^2+1)^2}\right| \le \frac{|\log z|}{\frac{1}{2}\, |\cdot (R^2)^2} = \frac{2\,|\ln|z| + i\arg z|}{R^4} \le \frac{2}{R^4}\big|\ln|z|\big| + |\arg z|$

$$\le \frac{2}{R^4}(R+\pi)$$

$$\le \frac{2}{R^3} + \frac{\pi}{R^4}$$

$\displaystyle \Rightarrow \left|\int_{S_R} \frac{\log z}{(z^2+1)^2}\,dz\right| \le \text{length}\,\gamma_R \cdot \max|f(z)| \le \pi R \cdot \left(\frac{2}{R^3} + \frac{\pi}{R^4}\right) = \frac{2\pi}{R^2} + \frac{\pi^2}{R^3} \longrightarrow 0$ as $R \to \infty$

On S_ε:

$$|f(z)| = \left|\frac{\log z}{(z^2+1)^2}\right| \le \frac{-2\ln z}{(1-\varepsilon^2)^2}$$

$$\left|\int_{-S_\varepsilon} f(z)\,dz\right| \le \text{length}(-S_\varepsilon) \cdot \max|f(z)| \le \frac{\pi\varepsilon \cdot (-2)\ln\varepsilon}{(1-\varepsilon^2)^2}$$

$$= \frac{-2\pi\varepsilon\ln\varepsilon}{(1-\varepsilon^2)^2} \longrightarrow 0 \text{ as } \varepsilon \to 0$$

$$\boxed{\lim_{\varepsilon\to 0} \varepsilon\ln\varepsilon = 0}$$

On L_1: $\displaystyle \int_{L_1} \frac{\log z}{(z^2+1)^2}\,dz = \int_{-R}^{-\varepsilon} \frac{\ln|z| + i\arg z}{(z^2+1)^2}\,dz = \int_{-R}^{-\varepsilon} \frac{\ln(-x) + i\pi}{(x^2+1)^2}\,dx$

$$= \int_\varepsilon^R \frac{\ln x + i\pi}{(x^2+1)^2}\,dx = \int_\varepsilon^R \frac{\ln x}{(x^2+1)^2}\,dx + i\pi\int_\varepsilon^R \frac{1}{(x^2+1)^2}\,dx$$

On L_2: $\displaystyle \int_{L_2} \frac{\log z}{(z^2+1)^2}\,dz = \int_\varepsilon^R \frac{\ln x + i\cdot 0}{(x^2+1)^2}\,dx = \int_\varepsilon^R \frac{\ln x}{(x^2+1)^2}\,dx$

Put this together:

$$\int_{\gamma_{R,\varepsilon}} f(z)dz = \int_{S_R} f(z)dz + \int_{-S_\varepsilon} f(z)dz + \int_{L_1} f(z)dz + \int_{L_2} f(z)dz$$

$$= \int_{S_R} + \int_{-S_\varepsilon} + \int_\varepsilon^R \frac{\ln x}{(x^2+1)^2}dx + \int_\varepsilon^R \frac{\ln x}{(x^2+1)^2}dx + i\pi\int_\varepsilon^R \frac{dx}{(x^2+1)^2}$$

$$2\pi i \cdot Res(f:i) = \int_{S_R} + \int_{-S_\varepsilon} + 2\int_\varepsilon^R \frac{\ln x}{(x^2+1)^2}dx + i\pi\int_\varepsilon^R \frac{dx}{(x^2+1)^2}$$

Now let $\varepsilon \to 0$, $R \to \infty$

$$2\pi i\, Res(f:i) = 0 + 0 + 2\int_0^\infty \frac{\ln x}{x^2+1}dx + i\pi\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$-\pi + \frac{i\pi^2}{2} = 2\int_0^\infty \frac{\ln x}{x^2+1} + i\pi\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$Res(f:i) = \frac{\pi^2}{8} + \frac{i\pi}{2}$$

Take real parts gives: $-\pi = 2\int_0^\infty \frac{\ln x}{(x^2+1)^2}dx$

$$\boxed{\int_0^\infty \frac{\ln x}{(x^2+1)^2}dx = -\frac{\pi}{2}}$$ the right solution is

$$\Rightarrow \int_0^\infty \frac{1}{(x^2+1)^2}dx = \frac{\pi}{4} \quad \text{(not asked)}$$