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MAT337 Homework 1

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10 Solution:

Let $a_n = \sin\frac{n\pi}{2}$, suppose it has limit L s.t.

$\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$.

Set $\varepsilon = \frac{1}{3}$, for k sufficiently large s.t. $n = 4k \geq N$

$|\sin 2k\pi - L| < \varepsilon = \frac{1}{3}$

i.e. $|0 - L| < \varepsilon \Rightarrow |L| < \frac{1}{3}$

Similarly for k sufficiently large s.t. $n = 4k+1 \geq N$

we have $|\sin(2k+\frac{1}{2})\pi - L| < \varepsilon = \frac{1}{3}$

$|1 - L| < \frac{1}{3}$

So $1 = (1-L) + L$

$|1| = |(1-L)+L| \leq |1-L| + |L| < 2\varepsilon = \frac{2}{3}$ contradiction

So it does not have limit

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10 $a_1 = 0$, $a_{n+1} = \sqrt{5 + 2a_n}$ for $n \geq 1$.

$a_1 = 0$
$a_2 = 2.23607$
$a_3 = 3.07768$
$a_4 = 3.33997$
$a_5 = 3.41759$
$a_6 = 3.44023$
$a_7 = 3.446804$
...

Solution:

$a_1 = 0$, $a_2 = \sqrt{5 + 2\times 0} = \sqrt{5}$, $a_3 = \sqrt{5 + 2\sqrt{5}}$

By observation, we claim that $0 \leq a_n < a_{n+1} < 3.5$

By induction, $n=1$, $0 \leq a_1 < \sqrt{5} = a_2 < 3.5$

Suppose it holds for n.

then $a_{n+2} = \sqrt{5 + 2a_{n+1}} > \sqrt{5 + 2a_n} = a_{n+1} \geq 0$

and $a_{n+2} = \sqrt{5 + 2a_{n+1}} < \sqrt{5+7} = \sqrt{12} < 3.5$

Finished the induction part.

So we have a monotone increasing sequence but which is bounded above

Then by monotone convergence theorem, it has limit, say it L.

So. $\sqrt{5 + 2L} = L$

$5 + 2L = L^2$

$L^2 - 2L - 5 = 0$

$L = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$

it's impossible for $L = 1 - \sqrt{6} < 0$ since it's increasing and $a_1 = 0$.

Therefore $L = 1 + \sqrt{6}$

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Solution:
$$(a_n) = \left(\frac{n\cos^n(n)}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$$

Note that $n^2 < n^2+2n$

so $\frac{n}{\sqrt{n^2+2n}} < 1$

since $\cos(n) \in [-1, 1]$

$|\cos(n)|^n \leq 1$ ≤ 1

Therefore $\frac{n\cos^n(n)}{\sqrt{n^2+2n}}$ is bounded above by 1

and bounded below by -1.

By Bolzano-Weierstrass Theorem, it's a bounded sequence of real numbers, so it has a convergent subsequence.

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Solution. Let $m = n_k$.

so since subsequence (x_{n_k}) has $\lim_{k\to\infty} x_{n_k} = a$.

$\forall \varepsilon > 0, \exists N \in \mathbb{Z}$ s.t.

$|x_m - a| < \frac{\varepsilon}{2}$, for all $m \geq N$

and by Cauchy sequence's definition,

$\forall \varepsilon > 0, \exists N' \in \mathbb{Z}$ s.t.

$|x_n - x_m| < \frac{\varepsilon}{2}$ for all $m, n \geq N'$

Therefore,

$\forall \varepsilon > 0, \exists N'' \in \mathbb{Z}$ s.t.

$|x_n - a| \leq |x_n - x_m| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

for all $n \geq N''$

Hence $\lim_{n\to\infty} x_n = a$.

this is too deep abuse of notation. The second line is just not true

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Let $\limsup_{n\to\infty} \frac{|a_n|}{b_n} = l < \infty$, then $\exists$ ~~$r<l$~~, suppose $\varepsilon = r - l$, since $\varepsilon > 0$ we can find integer $N>0$ s.t. $|\frac{a_n}{b_n}| < r = l + \varepsilon$, $\forall n \geq N$

Therefore $\frac{|a_n|}{b_n} < r$ for all $n \geq N \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$

$$< \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} (b_n \cdot r)$$

$$= \sum_{n=1}^{N-1} a_n + r \sum_{n=N}^{\infty} b_n$$

Note that $\sum_{n=1}^{N-1} a_n$ is bounded above (not infinity) and $\sum_{n=1}^{\infty} b_n$ converges.

By Cauchy Criterion, $\forall \varepsilon > 0$, $\exists N$ s.t. $|\sum_{n=N-1}^{\infty} b_n| < \varepsilon$

and $\sum_{n=N}^{\infty} b_n \leq |\sum_{n=N}^{\infty} b_n|$

we can find some ε, N s.t. $|\sum_{n=N}^{\infty} b_n| < \varepsilon$ for $n \geq N$

as a result: $r \sum_{n=N}^{\infty} b_n \leq r |\sum_{n=N}^{\infty} b_n| < r\varepsilon$

as $r \sum_{n=N}^{\infty} b_n$ is also bounded above (not infinity)

so $\sum_{n=1}^{\infty} a_n < \infty$, bounded above $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Problem I:

$(a_n)_{n=1}^{\infty}$, $a_i > 0$, $\forall i$

Proof: ① If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\exists \varepsilon > 0$ and $N > 1$ s.t.

$\frac{a_{n+1}}{a_n} < 1 - \varepsilon$ for $n > N$ set $1 - \varepsilon = r$

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(*) $a_n \cdot \frac{a_{n+1}}{a_n} \frac{a_{n+2}}{a_{n+1}} \cdots \frac{a_N}{a_{N-1}} = a_N < a_n \cdot r^{N-n}$ for $N > n$?

on the RHS, by Thm 3.2.2, it's a geometric series with $|r| < 1$, so the constructed geometric series on RHS is summable.

what about $n \leq N$

By Comparison Test, since (*) and RHS is summable then LHS (the actual series) is summable as well

i.e. $\sum_{n=1}^{\infty} a_n$ converges.

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② Conversely, $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ (Trivially the same as above)

then $\exists \varepsilon' > 0$ & $N > 1$ s.t. $\frac{a_{n+1}}{a_n} > 1 + \varepsilon'$ for $n > N$

set $1 + \varepsilon' = p$

(**). $a_N > a_n \cdot p^{N-n}$ for $N > n$

on RHS, by Thm 3.2.2, it's a geometric series with $|p| > 1$, not summable.

By comparison test again, LHS is not summable

i.e. $\sum_{n=1}^{\infty} a_n$ diverges