STAT6046 Formulas Lists

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1 Week 1

1.1 Effective rate of interest

$$\text{Effective rate of interest for a specified period} = \frac{\text{amount of interest for the period}}{\text{amount at the start of the period}}$$

$S(t)$ represents the value of an investment at time t. Then the annual rate of interest for a period from year u to $u+1$ is

$$i_{u+1} = \frac{S(u+1) - S(u)}{S(u)}$$

1.2 Simple interest

$$S(t) = S(0) + iS(0) + iS(0) + \cdots + i(S(0) = S(0) \cdot (1 + ti)$$

1.3 Compound interest

$$S(t) = S(0) \cdot (1+i)^t$$

1.4 Accumulation factor

$$A(t_1, t_2) = \frac{S(t_2)}{S(t_1)} = (1+i)^{t_2 - t_1}$$

1.5 The principle of consistency

$$S(t_2) = S(t_1) \cdot (1+i)^{t_2-t_1} = S(0) \cdot (1+i)^{t_1}(1+i)^{t_2-t_1} = S(0) \cdot (1+i)^{t_2}$$

$$A(0, t_n) = A(0, t_1)A(t_1, t_2) \cdots A(t_{n-1}, t_n)$$

1.6 Present values

The amount that should be put aside *now* to provide for payments in the future is **the present value (PV)** or **discounted value** of the payments.

The **discount factor** equals the amount that must be invested at the start of the period to accumulate to 1 at the end of the period.

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

Under compound interest,

$Kv^t = K(1+i)^{-t} =$ the present value (at time 0) of an amount K due at time t.

Under simple interest,

$K(1+it)^{-t} =$ the present value (at time 0) of an amount K due at time t.

More generally,

$Kv^{t_2-t_1} = K(1+i)^{-(t_2-t_1)} =$ the present value at time t_1 of an amount K due at time t_2.

1.7 Rounding

Intermediate steps: at least 5 significant digits.

Final step: round to nearest cent and interest rates to one decimal place.

1.8 Investing with different interest rates

The present value at time $t = 0$, of an amount K payable in t years is

$$K(1+i_1)^{-1}(1+i_2)^{-1}(1+i_3)^{-1}\cdots(1+i_t)^{-1} = \frac{K}{(1+i_1)(1+i_2)(1+i_3)\cdots(1+i_t)}.$$

1.9 Converting between effective rates of interest

Always remember: **Equivalent rates produces the same accumulated amounts over the same time period.**

2 Week 2

2.1 Nominal rates of interest

We define $i^{(m)}$ as the nominal rate of interest per annum convertible m times per year, $i^{(m)}$ is payable in equal installments of $\frac{i^{(m)}}{m}$ at the **end** of each subinterval of length $\frac{1}{m}$ years (i.e. at times $\frac{1}{m}, \frac{2}{m}, \ldots, 1$.

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = (1+i)$$

2.2 Converting between interest rates

Nominal and effective annual rates of interest are convertible:

$$(1+i)^t = \left(1 + \frac{i^{(m)}}{m}\right)^{mt}.$$

2.3 Present values with nominal rate of interest

The present value at time 0 of an amount K due at time t (in years), when nominal rate of interest of $i^{(m)}$ apply is

$$K \cdot \left(1 + \frac{i^{(m)}}{m}\right)^{-mt}.$$

2.4 Effective and nominal rates of discount

The interest paid at the end of an interest compounding period is **interest payable in arrears.**

The interest payable at the start of an interest compounding period is **interest payable in advance.**

i paid at the **end** of the period on the balance at the **beginning** of the period.

$$d = \frac{\text{amount of interest for the period}}{\text{balance at the end of the period}}$$

d paid at the **beginning** of the period on the balance at the **end** of the period.

$$i = \frac{\text{amount of interest for the period}}{\text{balance at the start of the period}}$$

Note that

$$d = \frac{i}{1+i}.$$

Also

$$i = \frac{d}{1-d}.$$

And

$$v = 1 - d.$$

since $v = \frac{1}{1+i}$, as v is discount factor.

2.5 Nominal discount rates

Define $d^{(m)}$ to be the total amount of interest, payable in equal installments at the **start** of each subinterval (i.e. at time $0, 1/m, 2/m, \ldots, (m-1)/m$.

$d^{(m)}$ implies a $\frac{1}{m}$-year compound discount rate of $\frac{d^{(m)}}{m}$.

Nominal and effective annual rates of discount are convertible:

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m.$$

So the present value (at time 0) of 1 payable at time t is

$$v^t = (1-d)^t = \left(1 - \frac{d^{(m)}}{m}\right)^{mt}.$$

Similarly, the accumulated value of 1 from time 0 to time t is

$$(1+i)^t = (1-d)^{-t} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}.$$

Present values in this context have been expressed in the form of compound discount.

If with simple discount, the present value is expressed as

$$(1 - d \cdot t).$$

For a fixed nominal rate of discount $d^{(m)}$, the effective annual discount rate d decreases as m increases.

2.6 Force of interest

For an effective annual rate of interest, the equivalent nominal rate of interest as the number of compounding periods m approaches infinity is called the **force of interest.**

$$\lim_{m\to\infty} i^{(m)} = \delta.$$

We use δ_t to denote the **force of interest at time** t or the **instantaneous rate of growth at time** t**.**

If δ_t is constant, it is written as δ.

Similarly,

$$\lim_{m\to\infty} d^{(m)} = \delta.$$

Therefore, under compound interest at an annual effective rate i, the equivalent force of interest is

$$\delta_t = \ln(1+i) \text{ or } i = e^{\delta_t} - 1.$$

Note:

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i.$$

Also

$$S(n) = S(0)\cdot \exp\left(\int_0^n \delta_t dt\right).$$
$$S(0) = S(n)\cdot \exp\left(-\int_0^n \delta_t dt\right).$$

More generally,

$$S(t_2) = S(t_1)\exp\left(\int_{t_1}^{t_2} \delta_t dt\right), \text{ or } A(t_1,t_2) = \exp\left(\int_{t_1}^{t_2} \delta_t dt\right).$$

For present value:

$$S(t_1) = S(t_2)\cdot \exp\left(-\int_{t_1}^{t_2} \delta_t dt\right).$$

When $\delta_t = \delta$:

$$S(n) = S(0) \cdot \exp\left(\int_0^n \delta dt\right) = S(0) \cdot e^{\delta n}.$$
$$S(0) = S(n) \cdot e^{-\delta n}$$

3 Week 3

3.1 The valuation of periodic payments – annuities

Geometric series:

$$1 + x + x^2 + x^3 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x} = \frac{x^{k+1} - 1}{x - 1}.$$

3.2 Accumulated value of an immediate annuity

The accumulated value of a series of periodic payments is **annuity.**

Consider a series of n payments of 1 unit made at the **end** of equally spaced time intervals, where each payment is invested at an effective interest rate of i per time interval, and where interest is credited on payment dates.

The accumulated value of these payments at time n, where the final payment is made at time n, can be found by noting the following:

The first payment accumulates from time 1 to time n, i.e. $n-1$ periods of time, or: $(1 + i)^{n-1}$.

The second payment accumulates from time 2 to time n, i.e. $n - 2$ periods of time, or: $(1 + i)^{n-2}$.

…

The second-last payment accumulates from time $n - 1$ to time n, i.e. 1 period of time, or: $(1 + i)$.

The last payment of 1 made at time n.

Therefore, using the geometric series expansion, the summation of these accumulated payments is:

$$s_{\overline{n}|} = s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \cdots + (1+i) + 1 = \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}$$

In summary, the accumulated value at the end of n periods of an **immediate annuity** of 1 unit per period payable at the end of each period for a total of n period is:

$$s_{\overline{n}|} = \sum_{t=0}^{n-1} (1+i)^t = \frac{(1+i)^n - 1}{i}$$

Since the payment is made at the end of each period, it's also referred to as the accumulated value of an annuity certain payable in **arrears.**

In the case of accumulated value, when the annuity is valued at the time of the final payment this is referred to as an **immediate** annuity.

3.3 Present value of an immediate annuity

The present value at time 0 of an immediate annuity of 1 unit per period payable at the end of each period for n period is:

$$a_{\overline{n}|} = s_{\overline{n}|} \cdot v^n = \frac{1 - v^n}{i}.$$

value	$a_{\overline{n}\|}$							$s_{\overline{n}\|}$
time	0	1	2	3	4	...	$n-1$	n
amount		1	1	1	1	...	1	1

Table 1: Visualization of immediate annuity

3.4 Annuities due

An annuity payable in advance (i.e. payment at the beginning of each period) is called an **annuity due.**

The accumulated value at the end of n periods of an annuity of 1 unit per period payable at the **beginning** of each period for n periods is:

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \frac{i}{d} s_{\overline{n}|}.$$

The present value of 1 unit per period payable at the **beginning** of each period for n periods is:

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d} = \frac{i}{d} a_{\overline{n}|}.$$

value	$a_{\overline{n}\rceil}$	$\ddot{a}_{\overline{n}\rceil}$					$s_{\overline{n}\rceil}$	$\ddot{s}_{\overline{n}\rceil}$
time	0	1	2	3	4	…	n	$n+1$
amount		1	1	1	1	…	1	

Table 2: Visualization of annuity due

3.5 Deferred annuities

If an annuity is to be valued more than 1 unit of time before commencement of the stream of payments, we call this a **deferred annuity.**

Suppose k, n non-negative integers, the value at time 0 of a series of n payments, each of amount 1, commencing at $k+1$, is denoted by ${}_{k}\big|a_{\overline{n}\rceil}$, as **$n$-payment immediate annuity deferred for k payment periods.**

$$ {}_{k}\big|a_{\overline{n}\rceil} = v^{k+1} + v^{k+2} + \cdots + v^{k+n} = v^{k}\left[v^{1} + v^{2} + \cdots + v^{n}\right] = v^{k} \cdot a_{\overline{n}\rceil} = a_{\overline{n+k}\rceil} - a_{\overline{k}\rceil} . $$

Similarly, the equivalent **n-payment annuity-due deferred for k payment period** is:

$$ {}_{k}\big|\ddot{a}_{\overline{n}\rceil} = v^{k} \cdot \ddot{a}_{\overline{n}\rceil} = \ddot{a}_{\overline{n+k}\rceil} - \ddot{a}_{\overline{k}\rceil} . $$

3.6 Valuing annuities with more than one interest rate

3.7 Annuities payable more frequently than annually

3.8