Department of Mathematics	**Problem Set #3**
University of Toronto	**Deadline: Tuesday November 15, 3:00 p.m.**
MAT332F, 2011	Assignment Posted/Revised: October 31, 2011

Read the following instructions carefully! In contains guidelines on handing in assignments for this course.

For solutions to all problem sets, please remember that when you are asked to find or calculate something you must justify that what you have found is correct and complete. You may use results from your lecture to help in your justification. You are learning to present your results in a clear and convincing manner. Thus, you will be graded on your presentation and justification; we are not simply verifying whether you know the answer.

Present your solutions with complete sentences. Pretend the marker does **not** know how to solve the question.

Required Information. The front page must include your name and student number. *Failure to put your name and/or your student number on your problem set will result in a zero on your assignment.* A cover page is not required as long as the necessary information is on the top of the first page.

Submitting your assignment. You must hand your assignment to your instructor before the beginning of lecture, or deposit the instructor's personal mailbox on the 6th floor of the Bahen Centre.

If you are unable to complete homework or if you miss a term test due to illness or other circumstances outside of your control, please contact your instructor immediately in order to receive special consideration. Note that special consideration will be given on an individual basis and will not be given automatically. In other words, you risk getting a mark of zero for missed work unless you contact the instructor promptly.

In the case of illness, medical documentation must be supplied on the standard University of Toronto Student Medical Certificate. You can also obtain a paper copy of this certificate from your college registrar or in your registration handbook. (A simple "note" from your doctor is not acceptable.)

Late submission. Late assignments will be accepted up to 25 hours after their deadlines with the following penalties.

Submission time	Penalty
by 3pm on Tuesday	none
by 10am on Wednesday	-10%
by 4pm on Wednesday	-25%

Note that lateness penalties will be computed as a percentage of the total marks on the assignment, not of the mark you obtain. Late assignments must be submitted directly into the instructor's personal mailbox on the 6th floor of Bahen Center (in the Math Department office), unless you require special consideration (see the section above for details). Please write the *exact* submission time on your assignment if you are submitting late.

Policy on Plagiarism on Assignments. Plagiarism is a form of academic fraud and is treated very seriously by the Faculty. *The assignments you hand in must not contain anyone else's work or ideas without proper attribution.* A working definition of plagiarism suitable for this course may be found at `http://www.northwestern.edu/provost/students/integrity/plagiarism.html`.

In science, collaboration is the norm, and in this course student collaboration is permitted to an extent. Namely, you are permitted to abstractly discuss possible solutions to a problem with other students. However, a student is forbidden from guiding another student through a solution step by step.

You are permitted to submit a joint answer to a problem set question. If two students have contributed to the solution of the problem, please write both names and student numbers near the problem, and you may share the marks.

Core Problems.

(1) Let $\mathbf{p} \stackrel{\text{def}}{=} (p_1, p_2, \ldots, p_m)$ and $\mathbf{q} \stackrel{\text{def}}{=} (q_1, q_2, \ldots, q_n)$ be two sequences of nonnegative integers. The pair $(\mathbf{p}, \mathbf{q})$ is said to be *realizable* by a simple bipartite graph if there exists a simple bipartite graph G and a bipartition $(\{x_1, x_2, \ldots, x_m\}, \{y_1, y_2, \ldots, y_n\})$ such that $\deg(x_i) = p_i$ for $1 \leq i \leq m$, and $\deg(y_j) = q_j$ for $1 \leq j \leq n$.
   (a) Formulate as a network flow problem the problem of determining whether a given pair $(\mathbf{p}, \mathbf{q})$ is realizable by a simple bipartite graph. (*10pt*)
   (b) Suppose that $q_1 \geq q_2 \geq \cdots \geq q_n$. Deduce from the Max-Flow Min-Cut Theorem that $(\mathbf{p}, \mathbf{q})$ is realizable by a simple bipartite graph if and only if

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j \quad \text{and} \quad \sum_{i=1}^{m} \min(p_i, k) \geq \sum_{j=1}^{k} q_j \quad \text{for } 1 \leq k \leq n. \qquad (10pt)$$

(2) Prove that every network has a maximal flow in which no current enters the source and no current leaves the sink. (*15*pt)

(3) True or false: If G has a perfect matching then G has a spanning tree that also has a perfect matching. Prove your answer. (*15pt*)

(4) Prove that a tree T has a perfect matching if and only if $T - \{v\}$ has a single component of odd order for each $v \in V(T)$. (*15pt*)

(5) Let $B = \{b_1, \ldots, b_k\}$ be a set of men and $G = \{g_1, \ldots, g_n\}$ be a set of women, and suppose that each man wants to marry up to four of the women whom he fancies and who fancy him back (*i.e.* polygamy up to four wives is allowed, and man b_i wants $0 \leq n_i \leq 4$ wives). Find necessary and sufficient graph theoretical conditions for this marriage problem to have a solution. (*15pt*)

(6) Let G be a simple bipartite graph with bipartition $\{X, Y\}$. Suppose the Marriage Condition is satisfied, and that each $x \in X$ has valence at least d. Prove that G contains at least $d!$ perfect matchings if $d \leq |X|$ and $d(d-1)\cdots(d-|X|+1)$ perfect matchings if $d > |X|$. (*20pt*)

Bonus Problems.

(1) How many permutations are there of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in which none of the patterns 12, 34, 56, or 78 appears? (*5 bonus points*)

(2) Cuthred is having difficulty remembering the ten digit phone number of his new cell phone. All he remembers about it is that the first, fourth, and fifth digits are either 7 or 9, that the third and tenth digits are either 2 or 4, that there are two zeros in the number, and that the sum of the digits is 42. Given this information, how many possibilities are there for Cuthred's phone number? (*7 bonus points*)