UNIVERSITY OF TORONTO
Faculty of Arts and Science

December Final Examination, 2012

STA347H1F

Duration - 2 hours

Examination Aids: Non-Programmable Calculators.

Name (Print Clearly!):______________________ Student ID:__________________

- Do not turn the page until told to do so.
- This is a closed book examination. You should have your hand calculator. You should have no written material with you during the exam.
- If a question asks you do some calculations, you must show your work to receive full credit. In particular, if you are basing your calculations on a formula, write down that formula before you substitute numbers into it.
- If a later part of a question depends on an earlier part, the later part will be graded conditionally on how you answered the earlier part, so that a mistake on the earlier part will not cost you points on the later part (unless perhaps the previous mistake was absolutely horrendous). If you can't work out the actual answer to an earlier part, put down your best guess and proceed.
- Write your answers in the spaces provided. If you do not have enough room to show all your work in the space provided, use the back of a nearby page; in such cases write something like "see also the back of page n" and be sure to mark clearly which problem the material on the back of any page refers to. If you pull the pages apart, sign all the pages.
- Use your time wisely, taking note of the points assigned to the various parts of the questions. There are a total of 100 points. It would not be wise to spend, say, 15 minutes on a part worth 5 points before you have tried to work the other parts of the exam.
- If you don't understand a question, or are having some other difficulty, see your instructor or TA.

1. **[32 points]** Consider a random variable X with the following density function

$$f(x) = cx(1-x) \quad 0 \leq x \leq 1$$

and $f(x) = 0$ otherwise.

(a). **[8 points]** Find the value of c such that $f(x)$ is an appropriate density function.

(b). **[8 points]** Find the mean $E(X)$ and the variance $V[X]$.

CONT'D.

(c). **[8 points]** Let $Y = X^2$. Find the density function of Y.

(d). **[8 points]** Let $X_1, X_2, \cdots, X_7$ be random variables with the same density $f(x)$. Assume that $X_1, \cdots, X_7$ are independent. Let Z be the median of $X_1, \cdots, X_7$. Find the density function of Z.

2. **[30 points]** Suppose random variables X and Y have the following joint density function

$$f(x,y) = x + y \quad 0 \le x \le 1, 0 \le y \le 1$$

and $f(x,y) = 0$ otherwise.

(a). **[7 points]** Find the marginal density function $f_X(x)$ of X.

(b). **[8 points]** Find the covariance $Cov(X,Y)$.

CONT'D.

(c). [**7 points**] Find $E(X|Y = 0.6)$.

(d). [**8 points**] Let $Z = X + Y$. Find the density function of Z.

CONT'D.

3. [**30 points**] Prove the following results.

(a). [**8 points**] Suppose that X, Y are **discrete** random variables and $E(|X|) < \infty$. Prove that $E[E[X|Y]] = E[X]$.

(b). [**8 points**] Suppose that $X_1, X_2, \cdots, X_n$ are independent exponential random variables with mean 1. Prove that $Y = X_1 + X_2 + \cdots + X_n$ is a Gamma$(n, 1)$ random variable. Reminder: An exponential random variable with mean θ has density $f(x) = \frac{1}{\theta}e^{x/\theta}$, $x \geq 0$. A Gamma(α, β) distribution has density $g(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$, $x \geq 0$.

(c). [**7 points**] Let X and Y be independent standard normal random variables. Let $Z = X + Y$ and $W = X - Y$. Prove that Z and W are independent. Reminder: A standard normal random variable has density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in (-\infty, \infty)$.

(d). [**7 points**] Suppose that $X_1, X_2, \cdots, X_n$ are independent exponential random variables with mean 1. Let $Y_n = \min[X_1, \cdots, X_n]$. Prove that Y_n converges in probability to 0 as $n \to \infty$. Reminder: An exponential random variable with mean θ has density $f(x) = \frac{1}{\theta} e^{x/\theta}$, $x \geq 0$.

CONT'D.

4. [**8 points**]+ [**5 bonus points**]

(a). [**8 points**] There are three brands of milk for sale in a supermarket. Let's call them brand A, B and C. A customer who purchases milk will choose brands A, B and C with probabilities 0.6, 0.3 and 0.1, respectively. Each day, the total number of customers who purchase milk at this market follows a Binomial(200, 0.5) distribution. Let X be the daily number of sales of brand A milk. Find $E(X)$ and $V(X)$. Reminder: A Binomial(n,p) random variable Z has probability mass function $P(Z=k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,\cdots,n$.

(b). **Bonus** [**5 points**]. Suppose that $X_1, X_2, \cdots, X_n$ are independent Uniform[0,1] random variables. Let

$$Y_n = n(1-\max[X_1, X_2, \cdots, X_n]) \text{ and } Z_n = n\min[X_1, X_2, \cdots, X_n].$$

Prove that the vector (Y_n, Z_n) converges in distribution to some two-dimensional distribution (Y, Z) as $n \to \infty$. Prove that Y and Z are independent. Reminder: The Uniform$[0,1]$ distribution has density $f(x)=1$ if $0 \le x \le 1$ and $f(x)=0$ otherwise.

CONT'D.

Total Pages = 9.

Total Marks = 100+5.