§4 - Countability Review

1 Topological Motivation

In many branches of mathematics, the notion of infinity arises. As is turns out, not all infinities are created equal. For us countable sets will often be thought of as quite small (and infinite), whereas uncountable sets will be thought of as large (and infinite). Most often we will be thinking about $\mathbb{R}$, and countable subsets of it. The other reason we will look at countability is to say that a topological space is small in some sense. Having a *countable* basis will mean that a space is "small". The existence of a countable set that is dense in the whole space will also mean that the space is "small".

We will have many different notions of small in this course, and we will spend some time investigating how these different notions of smallness interact and relate to each other.

Scale Exercise: Think of two different notions of "small" in math, and find an object that is small with respect to one of those notions and large with respect to the other.

2 Learning to Count

Questions. *A large group of people show up outside a movie theatre. How do we know if there are the same amount of seats and people? Are there more people or seats?*

The answer to this question that you have used all of your life is: "Count the number of people and count the number of seats. If it is the same number then there are the same amount of seats and people." This method works great ... if you are really good at counting, and have a lot of time to count people and seats. But what if you are really bad at counting? Let's say you only know how to count to 4; can you still figure out if there is the same amount of seats as people?

Yes! Just let people into the theatre, one at a time, and ask each of them to sit in an empty seat. Eventually, one (or more) of two things will happen:

1. All of the seats are full; or
2. Everyone is sitting down.

The first case tells us that the amount of people is greater or equal to the amount of seats. The second case tells us that the amount of seats is greater than or equal to the number of people. If *both* cases happen simultaneously, then we know that there is the same amount of seats and people.

Great! Notice how we didn't use anything about numbers or counting. We never know (numerically) *how many* people there are, nor do we know how many seats there are, but we do know which one is bigger! This is kind of like comparing weights with a scale; we don't know how much the things weigh, but we do know which one is heavier.

This will allow us to compare the sizes of any sets, regardless of whether we "know their sizes" or not. The *idea* is that even if we have a (possibly infinite) amount of people and a (possibly infinite) amount of seats, we will say that those amounts are the same if we can sit *each* person in a *different* seat in such a way that *each* seat is filled.

3 The Definition

Definition. *A set A is said to be* ***countable*** *if there is a bijection $f : \mathbb{N} \longrightarrow A$.*

This means that A **has the same cardinality** as $\mathbb{N}$.

Definition. *A set A is said to be* ***at most countable*** *if there is a surjection $f : \mathbb{N} \longrightarrow A$.*

Sometimes we are a bit sloppy and say that finite sets are countable. This is mostly so that we can say things like "the co-countable topology on X" rather than saying "the co-(countable or finite) topology" or, even worse, "the co-at-most-countable topology". We will try to stay consistent, but be aware that some other people use countable to mean "at most countable". Stay on your toes!

The intuition is that a set A is countable if we can "list it out", like

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots$$

"forever" without repeating. For example, the even numbers are countable, and that is witnessed by this list:

$$2, 4, 6, 8, 10, 12, 14, \ldots$$

We talk about bijections instead of "listing things out forever" because a bijection is a precise mathematical object, but "listing something out forever" is not a well defined notion.

4 Examples and Basic Facts

Let's look at a whole slew of sets that are countable. Also, remember that for us, $0 \notin \mathbb{N}$.

1. The set $A := \{ n \in \mathbb{N} : n \geq 800 \}$ is countable because the function $f : \mathbb{N} \longrightarrow A$ defined by $f(n) := n + 799$ is a bijection.

2. The set $A := \{ x \in \mathbb{Z} : x \geq -10 \}$ is countable because the function $f : \mathbb{N} \longrightarrow A$ defined by $f(n) := n - 11$ is a bijection.

3. The set $\mathbb{E} := \{ n \in \mathbb{N} : n \text{ is even } \}$ is countable because the function $f : \mathbb{N} \longrightarrow \mathbb{E}$ defined by $f(n) := 2 \cdot n$ is a bijection.

4. The set $F := \{ 1, 4, 9, 100 \}$ is *at most* countable. Define $f : \mathbb{N} \longrightarrow F$ (explicitly) by $f(1) = 9, f(2) = 4, f(3) = 100$ and $f(n) = 1$ for any $n \geq 4$. This is certainly a surjection onto F.

This hints at the first basic fact about countable sets:

Proposition. *If A is a countable set and F is a finite set, then $A \cup F$ is countable.*

Proof. The idea here is to first list the finitely many elements of F, then list the elements of A.

Let $F \backslash A = \{x_1, x_2, \ldots, x_n\}$, and let $f : \mathbb{N} \longrightarrow A$ be a bijection. Then define $g : \mathbb{N} \longrightarrow A \cup F$ by $g(i) := x_i$ for $i \leq n$ and $g(i) = f(i - n)$ for $i > n$. You can check that this is a bijection.

The picture here is:

$$\underbrace{x_1, x_2, \ldots, x_n}_{F \backslash A}, \underbrace{a_1, a_2, \ldots}_{A}$$

$\square$

Finite + Countable Exercise: Convince yourself that this is a bijection. Why did we use $F \setminus A$ here and not just A?

We can also see that any infinite subset of a countable set is countable. This says that "countable is the least size of infinity".

Proposition. *If B is an infinite subset of A, a countable set, then B is also countable.*

Proof. The idea here is that since A is countable, then "listing out" the elements of A will allow us to pick out the elements of B one by one by looking at the indices of the elements of A.

Let $f : \mathbb{N} \longrightarrow A$ be a bijection. We need to construct a bijection $g : \mathbb{N} \longrightarrow B$.

Let $g(1)$ be $f(i_1)$, where

$$i_1 := \min\{\, i \in \mathbb{N} : f(i) \in B \,\}$$

Now, suppose we have definied $g(1), g(2), \ldots, g(n)$. Let $g(n+1)$ be $f(i_{n+1})$ where

$$i_{n+1} := \min\{\, i \in \mathbb{N} : f(i) \in B \setminus \{\, g(1), g(2), \ldots, g(n) \,\} \,\}$$

we notice that the minimum exists because B is infinite, so $B \setminus \{\, g(1), g(2), \ldots, g(n) \,\} \neq \emptyset$.

The picture here is:

$$a_1, a_2, \ldots, a_n, \underbrace{a_{i_1}}_{\in B}, a_{n+2}, \ldots, a_m, \underbrace{a_{i_2}}_{\in B}, a_{m+2}$$

By construction this is clearly injective, and so we only check surjectivity. Let $b \in B$. Since f is a bijection, there is an n such that $f(n) = b$. It is easy to see that b will be in the range of $g \restriction \{\, 1, 2, \ldots, n \,\}$. (A technical proof of this fact involves looking at the least n for which $f(n) \in B$, but $f(n)$ is not in the range of g.) $\square$

Sometimes it is easier to map *into* the natural numbers rather than *from* the natural numbers. We record this useful fact:

Proposition. *A set A is countable iff there is a bijection $g : A \longrightarrow \mathbb{N}$.*

Proof. Take g to be the inverse function defined by $g(a) = n$ iff $a = f(n)$. This is exactly the desired bijection. $\square$

Sometimes it is easier to put a set A in bijection with a set we already know is countable (rather than put it in bijection with $\mathbb{N}$).

Proposition. *Let C be a countable set.*

A set A is countable iff there is a bijection $g : C \longrightarrow A$.

Proof. Since C is countable, there is a bijection $f : \mathbb{N} \longrightarrow C$. If there is a bijection $g : C \longrightarrow A$, then $g \circ f : \mathbb{N} \longrightarrow A$ is a bijection, so A is countable. On the other hand, if we assume A is countable, then there is a bijection $h : \mathbb{N} \to A$. So the desired bijection is $h \circ f^{-1} : C \longrightarrow A$. $\square$

Finally, we give a set of conditions for checking countability:

Theorem. *For an infinite set A the following are equivalent:*

1. *A is countable;*
2. *There is an injection $f : A \longrightarrow \mathbb{N}$;*
3. *There is a surjection $g : \mathbb{N} \longrightarrow A$.*

Proof. Since we know that infinite subsets of countable sets are countable, we can see that [i] iff [ii]. We also know that [i] implies [iii], so it is enough to check that [iii] implies [ii].

Let $g : \mathbb{N} \longrightarrow A$ be a surjection. Now for each $a \in A$, we know that

$$g^{-1}(a) := \{ n \in \mathbb{N} : g(n) = a \}$$

is a non-empty set. So we define an injection $f : A \longrightarrow \mathbb{N}$ by letting $f(a) := \min g^{-1}(a)$.

We see that if $f(a) = f(b)$, then $\min g^{-1}(a) = \min g^{-1}(b)$. "Taking g on both sides" gives us that $a = b$. $\square$

I Choose You! Exercise: Is this theorem still true if we replace all instances of "$\mathbb{N}$" with an arbitrary countable set? What part of the proof might someone be worried about?

5 Some Useful Countable Sets

The next example of a countable set is $\mathbb{Z}$, the set of integers.

Proposition. $\mathbb{Z}$ *is countable.*

Proof. The idea is to describe a function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ that "counts" the integers like this:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

If we can find a function such that $f(1) = 0, f(2) = 1, f(3) = -1, f(4) = 2, \ldots$ then this will be the desired bijection. It turns out that by defining $f(2n) := n$ and $f(2n+1) = -n$, we get such a a function. $\square$

The proof that $\mathbb{Z}$ is countable should suggest to you that a general fact is true:

Proposition. *If A and B be (disjoint) countable sets, then $A \cup B$ is countable.*

Proof. Since A is countable there is a bijection $f : \mathbb{N} \longrightarrow A$, and since B is countable there is a bijection $g : \mathbb{N} \longrightarrow B$.

Define $h : \mathbb{N} \longrightarrow A \cup B$ by $h(2n) := f(n)$ and $h(2n-1) = g(n)$. It is straightforward that h is a bijection (but you should prove this if you are sceptical). Basically h inherits being a bijection from f and g. □

Why did we assume that the two sets were disjoint? Well if A and B both contain the same element x, then the function h we described would not be a bijection (as there would be two different natural numbers, one even and one odd, that we mapped to x). So let's fix this!

Proposition. *If A and B are countable sets, then $A \cup B$ is countable.*

Proof. Note that $A \cup B = (A \setminus B) \sqcup B$. We know B is countable, and the other set is finite or countable. If $A \setminus B$ is countable, we know that the disjoint union of two countable sets is countable. If $A \setminus B$ is finite, a previous proposition gives us what we want. □

We now extend this by induction to the following important fact, which says "**A finite union of countable sets is countable**":

Theorem. *Let $A_1, A_2, \ldots, A_N$ each be countable sets. Then $\bigcup\limits_{1\leq i\leq N} A_i$ is countable.*

Proof. We have just seen that the union of *two* countable sets is countable. Now let our induction hypothesis be "the union of k many countable sets is countable". Then

$$\bigcup_{1\leq i\leq k+1} A_i = (\bigcup_{1\leq i\leq k} A_i) \cup A_{k+1}$$

we know by the induction hypothesis that $\bigcup_{1\leq i\leq k} A_i$ is countable, and A_{k+1} is countable by assumption, so the total union must be countable as we have written it as the union of two countable sets. □

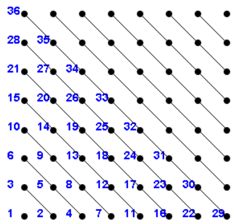

Figure 1: “Proof” that the product of two countable sets is countable. http://ocw.mit.edu/ans7870/18/18.013a/textbook/HTML/chapter01/section02.html

6 Cartesian Products

Now we investigate how the product of two countable sets is countable. This will tell us that the following sets are countable: $\mathbb{N}\times\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}\times\mathbb{Q}$ and the collection of all open intervals in $\mathbb{R}$ with rational numbers as endpoints.

Theorem. *If A and B are countable sets, then so is $A \times B$.*

Proof. Behold!

From the picture you should be able to extract a proof.

□

Now we show that $\mathbb{Q}$ is countable. In this course, this fact will be one of the most useful facts from this section.

Proposition. $\mathbb{Q}$ *is countable.*

Proof. We first note that $\mathbb{Q}$ is infinite, since $\mathbb{N} \subseteq \mathbb{Q}$. Next we observe that $\mathbb{Q} = \{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \}$, and the map $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$ defined by $f(n, m) := \frac{n}{m}$ is onto. So $\mathbb{Q}$ is an infinite set that is at most countable. Therefore it is countable. □

Proposition. $\mathcal{B} := \{ (a, b) \subseteq \mathbb{R} : a \in \mathbb{Q}, b \in \mathbb{Q} \}$ *is a countable set.*

Proof. We observe that $\mathbb{Q} \times \mathbb{Q}$ is countable, $\mathcal{B}$ is infinite, and the map $f : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathcal{B}$ defined by $f(p, q) :=$ the interval (p, q), is onto. So $\mathcal{B}$ is countable. □

7 Countable Unions of Countable Sets

The last thing we will show about countability is that "**a countable union of countable sets is countable**":

Theorem. *Let* $\{ A_n : n \in \mathbb{N} \}$ *be a family of countable sets. THEN* $\bigcup_{n\in\mathbb{N}} A_n$ *is countable.*

Proof. This should just follow from the fact that the product of two countable sets is countable. (Try to draw the picture!)

We will assume that each of the A_n are pairwise disjoint. We are allowed to do this because we are only making it *harder* to prove that the union is countable.

For each A_n, fix a bijection $f_n : \mathbb{N} \longrightarrow A_n$.

Then we can check that the function $g : \mathbb{N} \times \mathbb{N} \longrightarrow \bigcup_{n\in\mathbb{N}} A_n$, defined by $g(n, i) = f_n(i)$ (which goes to the ith element of A_n) is a bijection.

Claim 1: g is an injection.

Suppose that $g(n, i) = g(m, j)$. This means that $f_n(i) = f_m(j)$. This means that the ith element of A_n is the same as the jth element of A_m. Since the A_n are pairwise disjoint, that mean that $A_n = A_m$, and so $f_n(i) = f_m(j) = f_n(j)$. Since f_n is a bijection, we get that $i = j$. Hence $(n, i) = (m, j)$.

Claim 2: g is a surjection.

Let $y \in \bigcup_{n\in\mathbb{N}} A_n$. Then there is an $n \in \mathbb{N}$ such that $y = A_n$. Now since f_n is a bijection, there is an $i \in \mathbb{N}$ such that $f_n(i) = y$. □

This previous proposition will be extremely useful for us.

8 Uncountable Sets

You may have gotten the impression that *all* sets are at most countable. So what are some examples of sets that are *not* at most countable?

Definition. *If A is a set that is not at most countable, then we say that A is* ***uncountable****.*

We can first observe the following fact:

Proposition. *Let R and A be sets, and let $f : A \longrightarrow R$ be a bijection. Then R is uncountable iff A is uncountable.*

Proof. The idea here is to prove the contrapositive of both directions. So we need to establish that if R is at most countable, then so is A, and vice versa. But we have already seen this! $\square$

For us, the most important uncountable sets will be the following:

- $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^n$;
- Any (non-empty) open set in $\mathbb{R}$, with the usual topology.
- A very special set called ω_1 which will be covered in a later class.

We will give a very standard proof which contains a very beautiful idea called "diagonalizing".

Theorem (Cantor, 1880s)**.** *$(0,1) \subseteq \mathbb{R}$ is uncountable.*

Proof. Assume that there is a injection $f : \mathbb{N} \longrightarrow (0,1)$. We will show that it cannot be a surjection, which will be enough to show the theorem.

Write out the digits of each $f(n)$ as

$$f(n) = 0.x_{n,1}x_{n,2}x_{n,3}x_{n,4}x_{n,5}x_{n,6}x_{n,7}\cdots$$

We construct a real number $y \in (0,1)$ that is not in the range of f. Notice that it will be enough to show that $y \neq f(n)$ for each $n \in \mathbb{N}$. Well, let's do that!

$$\begin{array}{cccccc}
f(1) = & 0. & x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\
f(2) = & 0. & x_{2,1} & x_{2,2} & x_{3,3} & \cdots \\
f(3) = & 0. & x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We'll construct a $y := 0.y_1y_2y_3y_4y_5y_6\ldots.$ Since y needs to be different from $f(1)$, let's make sure that $y_1 \neq x_{1,1}$. Can we do that? Sure we can! We have ten choices $\{0,1,2,3,4,5,6,7,8,9\}$ for y_1, and $x_{1,1}$ is only one of those ten. This is true of every digit y_n though. We can just make sure that $y_n \neq x_{n,n}$.

$$\begin{array}{cccccc}
f(1) = & 0. & \boxed{x_{1,1}} & x_{1,2} & x_{1,3} & \cdots \\
f(2) = & 0. & x_{2,1} & \boxed{x_{2,2}} & x_{3,3} & \cdots \\
f(3) = & 0. & x_{3,1} & x_{3,2} & \boxed{x_{3,3}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Also, to make sure that we don't accidentally pick $y = 0$ or $y = 1$, just make sure that one of the y_n we pick is not 0 or 9.

Now we see that $y \in (0, 1)$ is different from each $f(n)$ in at least one digit! So f cannot be a surjection. $\square$

It will really pay to understand that proof very well. It comes up time and time again in mathematics. Diagonalization is a very powerful tool in mathematics!

Now here's a fact we will use often:

Theorem. $\mathbb{R}$ *is uncountable*

proof 1. By the previous theorem, it is enough to put $(0, 1)$ into bijection with $\mathbb{R}$. It is an exercise to show that $(0, 1)$ is in bijection (by a linear function) with the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. And $\arctan : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ is a bijection. So the composition of these two bijections is the desired bijection. $\square$

proof 2. Since $(0, 1) \subseteq \mathbb{R}$, and $(0, 1)$ is uncountable, $\mathbb{R}$ cannot be countable. $\square$

Finally, I would be thrown in jail if I talked about uncountable sets and didn't present the following proof, which exploits the idea behind Russel's Paradox:

Theorem. *There is no surjection from* $\mathbb{N}$ *onto* $\mathcal{P}(\mathbb{N})$*, the set of all subsets of* $\mathbb{N}$*. As a result,* $\mathcal{P}(\mathbb{N})$ *is uncountable.*

Proof. Let $f : \mathbb{N} \longrightarrow \mathcal{P}(\mathbb{N})$ be a function. So to each natural number we associate a subset of $\mathbb{N}$. Let

$$X := \{n \in \mathbb{N} : n \notin f(n)\}$$

which is a subset of $\mathbb{N}$. So $X \in \mathcal{P}(\mathbb{N})$.

Claim: There is no $N \in \mathbb{N}$ such that $f(N) = X$.

Suppose for the sake of contradiction that there is an N such that $f(N) = X$.

Question: Is $N \in X$?

If the answer is "Yes" ($N \in X = f(N)$), then we see that N must satisfy the defining property of X; that is: $N \notin f(N)$. Oops!

If the answer is "No" ($N \notin X = f(N)$), then we see that N cannot satisfy the defining property of X; that is: $N \in f(N)$. Oops!

Either way we get a contradiction! So the claim (and theorem) is proved. $\square$

We can distill the Cantor Diagonalization argument to the bare minimum of what we need. We never needed to use all ten digits, and in fact, we only needed two digits.

Binary Strings Exercise: Prove using Cantor's diagonaliztion technique that the set of all real numbers in the interval $(0, 1)$ written using only zeros and ones, is an uncountable set.

When I was an undergrad it took me until my third year to believe the previous exercise. My confusion came from the following exercise which I understood very well.

Binary Numers Exercise: Prove (directly) that the set of all binary numbers is countable.

Summary of Important Facts

Let's amass the important facts that we have seen. These are the facts I included in the "Things you should know" notes:

Definition. *A set A is said to be countable if there is a bijection $f : \mathbb{N} \longrightarrow A$.*

The following gives equivalent conditions for being countable:

Theorem. *For an infinite set A the following are equivalent:*

1. *A is countable;*
2. *There is an injection $f : A \longrightarrow \mathbb{N}$;*
3. *There is a surjection $g : \mathbb{N} \longrightarrow A$.*

FACT: The following sets are countable:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, the algebraic numbers;
- Any infinite subset of a countable set;
- The product of two countable sets;
- The union of finitely many countable sets;
- The countable union of countable sets;

- The countable union of some countable sets and some finite sets;

FACT: The following sets are not countable:

- $\mathbb{R}$, the irrational numbers, the non-algebraic numbers (i.e. the transcendental numbers), $\mathbb{R}^n$;
- Any superset of an uncountable set;
- The powerset of a countable set, e.g. $\mathcal{P}(\mathbb{N})$;
- The set of functions from $\mathbb{N}$ to $\mathbb{N}$.

The following is a combinatorial fact about uncountable sets:

Theorem (Uncountable Pigeonhole Principle)**.** *Let X be an uncountable set. If $\chi : X \longrightarrow \mathbb{N}$ is a function, then there is an $n \in \mathbb{N}$ such that $\chi^{-1}(n)$ is uncountable.*

That can be restated as "*If you try to put uncountably many pigeons into countably many holes, then there is a hole with uncountably many pigeons*".

Pigeon Exercise: Prove the uncountable pigeonhole principle.

Applications to Our Course

Ok, great, now you know how to count things, and some things are countable while others are not. Who cares?

Well... we, as topologists, will care about this stuff. Topologies are somewhat complicated infinite structures that are closed under *arbitrary* unions, and arbitrary can be very big. Sometimes we really only care about a small, but infinite, amount of things, and in that case the notion of countability really helps. Here we will hint at some things we will investigate further later.

One way to use this is to look at spaces with a small basis. We know that we can use a basis to describe all of the open sets in a topology, so if we know that a topological space has a *countable* basis then we know that "we can describe the topology using a small amount of information". We often use this idea when we prove facts about the reals, because it has a countable basis.

Definition. *A topological space with a countable basis is said to be* ***second countable****.*

Notice that this does not say that *every* basis for a second countable space is countable, just that it has at least one countable basis. The usual basis for the reals is certainly not countable. The reals with the discrete topology is a good example of a space that is not second-countable.

A word about the language used: it is pretty bad. Second countable is something you will probably forget and you will confuse it with the next two notions we talk about. Oh, and we will learn what "first"-countable is later.

Another important fact about the reals is that it has a countable dense subset, namely $\mathbb{Q}$. This property is very important in the study of mathematics.

Definition. *A topological space with a countable dense subset is said to be* ***separable****.*

Again, this word isn't so evocative, but it is the word that stuck.

9 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Scale Exercise : Think of two different notions of "small" in math, and find an object that is small with respect to one of those notions and large with respect to the other.

Finite + Ctble : In the proof that the union of a finite set F and a countable set A is countable we used $F \setminus A$ in the proof. Why?

I Choose You! : For the theorem that characterizes countability in three ways, can we replace all instances of "$\mathbb{N}$" with an arbitrary countable set?

Binary Strings : Prove using Cantor's diagonaliztion technique that the set of all real numbers in the interval $(0, 1)$ written using only zeros and ones, is an uncountable set.

Binary Numers : Prove (directly) that the set of all binary numbers is countable.

Pigeons! : Prove the uncountable pigeonhole principle.