University of Toronto
MAT237Y1Y TERM TEST 3
Thursday, Feb.9, 2011
Duration: 90 minutes

No aids allowed

Instructions: There are 12 pages including the cover page and the extra sheet at the back. Please answer all questions in the spaces provided (if you use back of a sheet or the extra sheet, please clearly specify that.) Document your arguments by briefly stating the results you use (in most cases you may find relevant definitions or results in an earlier question of this test.) The value for each question is indicated in parentheses beside the question number. The test is out of 100 but there is a total of 110 marks available of which 10 marks are bonus marks.

NAME: (last, first) Marking Scheme

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

CHECK YOUR TUTORIAL:

○ TUT01 M3-4		○ TUT03 T2-3
○ TUT04 W3-4	○ TUT05 W5-6	○ TUT06 R5-6

MARKER'S REPORT:

Question	MARK
Q1	/16
Q2	/25
Q3	/20
Q4	/22
Q5	/27
TOTAL	/110

1. Differentiability

a) (5 marks) Give definition of differentiability of a function $\boldsymbol{f} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ at a given point $\boldsymbol{a} \in \mathbb{R}^m$, and present the Frechet derivative $D\boldsymbol{f}(\boldsymbol{a})$.

b) (5 marks) Demonstrate how you would apply Chain rule III to determine the derivative of the composition of functions $\boldsymbol{g} : \mathbb{R}^1 \longrightarrow \mathbb{R}^3$ and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^1$, defined by $\boldsymbol{g}(t) = (1-t, t^2, \sqrt{t})$ and $f(x,y,z) = \dfrac{y-z}{xz}$, at $t = 4$.

let $H(t) = f(g(t))$. $\quad DH(t) = Df(g(t))\, Dg(t)$ 1

$$= \left[\frac{-z(y-z)}{x^2z^2},\ \frac{1}{xz},\ \frac{-xz - x(y-z)}{(xz)^2} \right] \begin{bmatrix} -1 \\ 2t \\ \frac{1}{2\sqrt{t}} \end{bmatrix} \quad \text{at } t=4$$ 2

$x = 1 - t = -3$

$y = t^2 = 16$ 1

$z = \sqrt{t} = 2$

$$= \begin{bmatrix} \frac{-14}{18} & \frac{1}{-6} & \frac{-2-14}{-12} \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{7}{9} & -\frac{1}{6} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1/4 \end{bmatrix} = \frac{7}{9} - \frac{4}{6} + \frac{1}{3}$$

1

$$= \frac{14 - 12 + 6}{18} = \frac{8}{18}$$

$$= \frac{4}{9}$$

c) (6 marks) In case of $\boldsymbol{f}$ being a transformation of $\mathbb{R}^n$, and invertible near a point $\boldsymbol{a}$, present an argument, using Chain Rule, to show that the Frechet derivative of the inverse transformation $(\boldsymbol{f}^{-1})$ at $\boldsymbol{f}(\boldsymbol{a})$ is the (matrix) inverse of the Frechet derivative of $\boldsymbol{f}$ at the point $\boldsymbol{a}$.

$f^{-1}(f(x)) = x$ 1

$Df^{-1}(f(x))\,Df(x) = Dx$

$$Dx = \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} = I_{n\times n}$$

2

So at $f(a)$ $\quad Df^{-1}(f(a))\,Df(a) = I$ 1.5

so $\quad Df^{-1}(f(a)) = [Df(a)]^{-1}$.

1.5

2. Curves in space

a) (8 marks) Give the three representations of a curve in space (in the order of appearance as in the textbook.) Then show how representation (i) (the graph version) can be obviously translated to versions (ii) (Locus version) and (iii) (parametric version.)

graph
i) $y = f(x)$ & $z = g(x)$ $\quad f, g$ are C^1 (Similar expressions with Coordinates permuted)

1.5

ii) locus : $F(x,y,z) = G(x,y,z) = 0 \quad F, G \;\; C^1$

1.5

iii) parametric : range of $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{f}$ is C^1 1

$i \Rightarrow ii \quad F(x,y,z) = y - f(x) = 0$ and $G(x,y,z) = z - g(x)$ 2

$i \Rightarrow iii \quad$ let $\mathbf{f}(t) = \begin{bmatrix} t \\ f(t) \\ g(t) \end{bmatrix}$ 2

b) (10 marks) Prove, using the system version of the Implicit Function Theorem, that version (ii) can be locally translated to graph representation (i) under the appropriate regularity condition.

Let $\mathbb{F}(x,y,z) = \begin{bmatrix} F(x,y,z) \\ G(x,y,z) \end{bmatrix} = \mathbb{0}$ $\quad$ assume $\nabla F \times \nabla G = \mathbb{0}$ at some pt

2

Then $\begin{vmatrix} i & j & k \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} \neq \mathbb{0}$ $\quad$ Then $i \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} - j \left| \frac{\partial(F,G)}{\partial(x,z)} \right| + k \left| \frac{\partial(F,G)}{\partial(x,y)} \right| \neq \mathbb{0}$

3

so at least one of the Components is not Zero,

say $\left| \frac{\partial(F,G)}{\partial(y,z)} \right| \neq 0$; but this is the Condition for Solvability of the System by 3.9 1

2

$\mathbb{F}(x,y,z) = \mathbb{0}$ for y, z in terms of x, That is exists C^1 functions f and g

3

S.t. $\quad y = f(x)$ & $z = g(x)$

c) (7 mark) Investigate solvability of the following system for variables u and v near the point $\boldsymbol{a} = (x, y, u, v) = (0, 1, 1, 5)$

$$\begin{cases} u = xu/y + xy + v - 4 & \Rightarrow F = u - \frac{xu}{y} - xy - v + 4 = 0 \\ x^2 = uv - 2vy + 5 & \Rightarrow G = x^2 - uv + 2vy \cancel{+} - 5 = 0 \end{cases}$$

Then determine the partial derivative $\dfrac{\partial u}{\partial x}$ at the point $\boldsymbol{a}$.

2 (2)

$$\left|\frac{\partial(F,G)}{\partial(u,v)}\right| = \begin{vmatrix} 1-\frac{x}{y} & -1 \\ -v & -u+2y \end{vmatrix} \xrightarrow{\text{at } (0,1,1,5)} = \begin{vmatrix} 1-0 & -1 \\ -1 & -1+2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & \cancel{-3}\ 1 \end{vmatrix} = \cancel{-4}\ 2 \neq 0$$

(1) So yes Solvable. to find $\frac{\partial u}{\partial x}$ we diff $F(x, y, u(x,y), v(x,y))$ & $G(\quad " \quad)$

$$\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial G}{\partial x} = 0 \end{cases} \rightsquigarrow \begin{cases} u_x - \frac{u}{y} - \frac{x u_x}{y} - y - v_x = 0 \\ 2x - u_x v + u v_x + 2 v_x y = 0 \end{cases}$$

(2) at a we get

$$\begin{cases} u_x - 1 - 0 - 1 - v_x = 0 \\ 0 - u_x + v_x + 2v_x = 0 \end{cases} \rightsquigarrow \begin{cases} u_x - v_x + 2 = 0 \\ -u_x + 3v_x = 0 \end{cases}$$

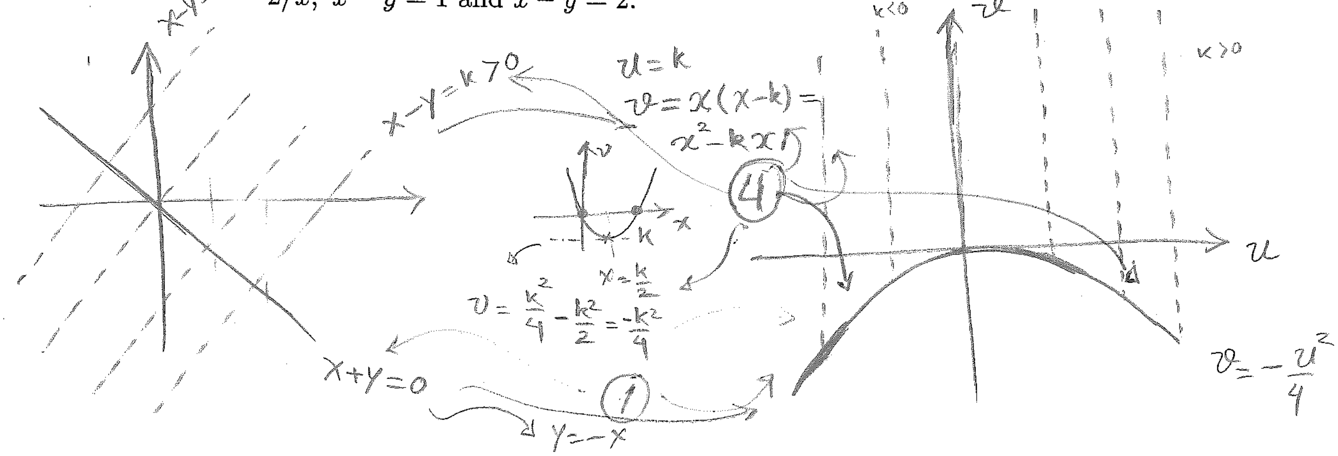
(2) $\Longrightarrow -4u_x = -6 \Rightarrow \boxed{u_x = \frac{3}{2}}$

eliminate v_x

$v_x = 1 \Rightarrow u_x = 3v_x = 3$

3. transformations of $\mathbb{R}^2$

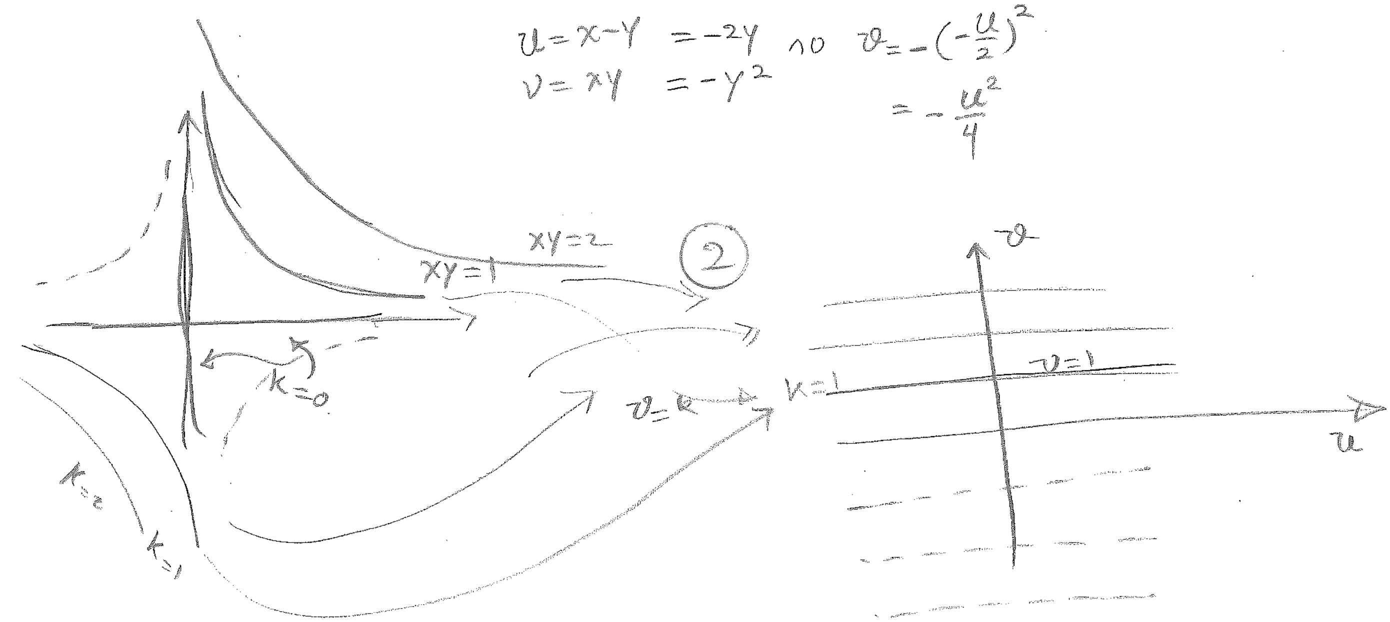
a) (10 marks) Consider the transformation $(u, v) = \boldsymbol{f}(x, y) = (x - y, xy)$. Demonstrate the effect of this transformation on the lines $x - y =$ constant, $x + y = 0$, and the curves $xy =$ constant. In particular demonstrate the effect of this transformation of the region of the x, y-plane bounded by the curves $y = 1/x$, $y = 2/x$, $x - y = 1$ and $x - y = 2$.

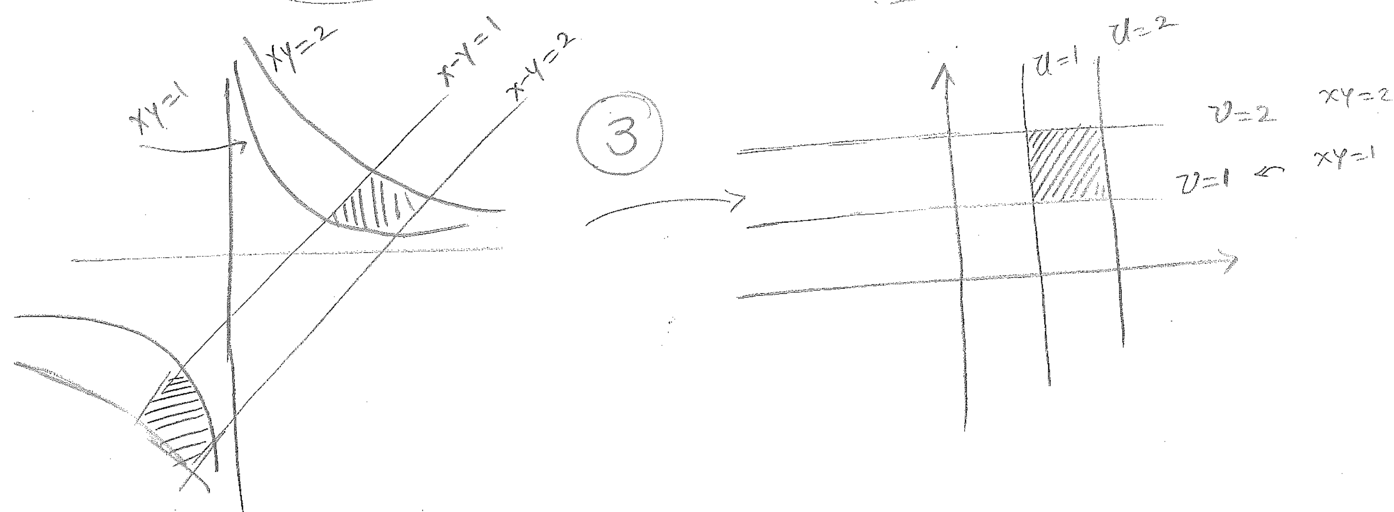


$u = x - y = -2y$ so $v = -\left(-\frac{u}{2}\right)^2$

$v = xy = -y^2$

$= -\frac{u^2}{4}$





b) (10 marks) Investigate the possibility of finding an inverse for this transformation near the generic point $\boldsymbol{x} = (x, y)$. Determine the points (u, v) where the conditions of inverstibility fail. Continue to determine the Frechet derivative of the (local) inverse of the transformation $\boldsymbol{f}$ near the point $(u, v) = (1, 2)$.

2

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} 1 & -1 \\ y & x \end{vmatrix} = x+y$$ so at $x+y \neq 0$ we have the possibility of finding the inverse (locally).

1

$(u,v) = (1,2) \Rightarrow$ $x - y = 1$, $xy = 2$ so $y = x-1$, $x(x-1) = 2$

2

$x^2 - x - 2 = 0$ $x = \frac{1 \pm \sqrt{1+8}}{2}$

$x = \frac{1 \pm \sqrt{9}}{2}$ $\frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$

$y = \begin{cases} 2-1 = 1 \\ -2 \end{cases}$

$$Df^{-1}(1,2) = [Df(\ \ )]^{-1}$$

1

$(x,y) =$

two pts $(2,1)$ and $(-1,-2)$

both map to $(1,2) = (u,v)$ 1

So let N and M be near $(u,v) = (1,2)$ and $(x,y) = (2,1)$ respectively.

$$Df(2,1) = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$ and

$$[Df(2,1)]^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = Df^{-1}(1,2).$$

2 1

$$[Df(x,y)]^{-1} = \frac{1}{x+y}\begin{bmatrix} x & 1 \\ -y & 1 \end{bmatrix}$$

4. Smooth curves and surfaces

a) (6 marks) Find the parametric description of the intersection of the plane $x + z = 1$ with the cone $z^2 = x^2 + y^2$. Explain what the curve of the intersection is (you may look at the projection of the curve in the x, y plane.)

$$\begin{cases} x+z=1 \Rightarrow F(x,y,z)=0 \\ z^2 = x^2+y^2 \Rightarrow G(x,y,z)=0 \end{cases} \quad \left|\frac{\partial(F,G)}{\partial(y,z)}\right| = \begin{vmatrix} 0 & 1 \\ 2y & 2z \end{vmatrix} = -2y \neq 0 \text{ if } y \neq 0$$

So, can solve for y, z in terms of x

$$\begin{cases} z = 1-x \\ y^2 = z^2 - x^2 = (1-x)^2 - x^2 = 1-2x \end{cases} \quad \xleftrightarrow{2} \quad \begin{cases} z = 1-x \\ x = \frac{1-y^2}{2} \end{cases}$$

let y be t, Then

$x = \frac{1-t^2}{2}$ and $z = 1-x = 1 - \frac{1-t^2}{2} = \frac{1+t^2}{2}$

2

$\mathbf{f}(t) = \left(\frac{1-t^2}{2}, t, \frac{1+t^2}{2}\right)$

Curve is on The plane $x+z=1$ (planar Curve)

and on The xy plane it look like a parabola

$x = \frac{1-y^2}{2}$

1

b) (9 marks) Determine whether the parametric surface $\boldsymbol{f}(u, v) = (u \cos v, u \sin v, u^2)$, with $-\pi \leq v \leq \pi$ and $u \in \mathbb{R}$ satisfies regularity condition condition at all its points. If not, determine the point(s) at which the condition fails. Explain what happens to the surface at that (those) points.

$$f_u \times f_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = i(2u^2\cos v) - (2u^2 \sin v)j + uk$$

so at $u=0$, and any v 2

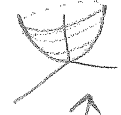
3 or

or $f_v = 0$ at $u=0$, $v =$ any thing

$f_u \times f_v = 0$ and The sufficient Condition of regularity fails at $(0, v)$ 1 any v

That is at The pt 1 $(0,0,0)$ on The surface. However The Surface

satisfies $z = x^2+y^2$ which is



level curves, when $z=k$ are Circles of radius k, and when $y=0$ $z=x^2$, $x=0$ $z=y^2$

at $(0,0,0)$ The Surface has tangent plane 2 (xy plane) so The Surface remains Smooth 1 although regularity Condition fails there.

c) (7 marks) At the point $(2, 0, 4)$ determine the tangent plane to the surface.

$(2,0,4)$ corresponds to $u=2$ and $v=0$ 2

$$f_u(2,0) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \text{ and } f_v(2,0) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad f_u \times f_v(2,0) = \begin{bmatrix} -8 \\ 0 \\ 2 \end{bmatrix}$$

3 normal to the tangent plane

2

so $\begin{bmatrix} -8 & 0 & 2 \end{bmatrix} \begin{bmatrix} x-2 \\ y-0 \\ z-4 \end{bmatrix} = 0$ is the equation of the plane

or $-8(x-2) + 2(z-4) = 0$ 7

5. Integration

a) (8 marks) Let f be a bounded function on the interval $[a, b]$. Give definition of upper and lower Riemann sum with respect to a partition $\mathcal{P}$ of the interval $[a, b]$; also give the definition of integrability of f on the interval $[a, b]$. (Please make sure not to mistake this definition with the ϵ-characterization of integrability, lemma 4.5)

upper sum

$S_{\mathcal{P}} f = \sum_{i=1}^{n} M_i \Delta x_i$ where $M_i = \max\{f(x) : x \in [x_{i-1}, x_i]\}$

(2) (1)

$m_i = \min\{f(x) : x \in [x_{i-1}, x_i]\}$

$s_{\mathcal{P}} f = \sum m_i \Delta x$

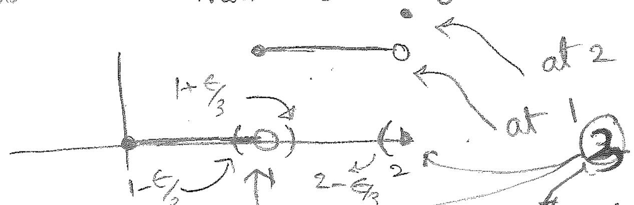
f is integ on $[a\ b]$ iff $\text{lub}\{s_{\mathcal{P}} f : \mathcal{P}$ is a partition of $[a\ b]\} = \text{glb}\{S_{\mathcal{P}} f \ldots\}$

5

b) (10 marks) State and use the ϵ characteristic of integrability (lemma 4.5) to prove that the function $f(x) = [x]$ (this is the integer part of the number x, otherwise known as the floor function,) on the interval $[0, 2]$ is integrable. Show how your proof actually leads you to the value of the integral.

4.5 If f is bdd on $[a\ b]$. Then f is integrable over $[a\ b]$ iff $\forall \epsilon > 0, \exists \mathcal{P}$ (4) partition of $[a\ b]$ $S_{\mathcal{P}} f - s_{\mathcal{P}} f < \epsilon$.

$f(x)$ is



given $\epsilon > 0$ (1) let $\mathcal{P} = \{0, 1-\epsilon/3, 1+\epsilon/3, 2-\epsilon/3, 2\}$ Then $S_{\mathcal{P}} f - s_{\mathcal{P}} f =$

$$\Big[0 + 1\times[(1+\tfrac{\epsilon}{3}) - (1-\tfrac{\epsilon}{3})] + 1(2-\tfrac{\epsilon}{3}-1-\tfrac{\epsilon}{3}) + 2(2-(2-\tfrac{\epsilon}{3}))\Big] - \Big(0 + 0\times\tfrac{2\epsilon}{3} + 1\times(1-\tfrac{2\epsilon}{3}) + 1\,\tfrac{\epsilon}{3}\Big)$$

$$= \Big(\underbrace{2\tfrac{\epsilon}{3}}_{}+1-2\tfrac{\epsilon}{3}+2\tfrac{\epsilon}{3}\Big) - \Big(1-\tfrac{2\epsilon}{3}+\tfrac{\epsilon}{3}\Big) = \Big(1+\tfrac{2\epsilon}{3}\Big) - \Big(1-\tfrac{\epsilon}{3}\Big) = \tfrac{\epsilon}{3} < \epsilon$$

(2)

1 1

$I_a^b f = 1$ = actual value of integ

9

c) (9 marks) Prove if a function f is monotone increasing on the interval $[0,1]$ then f is integrable there.

1

Note : if We Select a partition with equal intervals, that is $\Delta x_i = x_i - x_{i-1} = \frac{1-0}{k} = \frac{1}{k}$ 1 Then b/c f is increasing

$P_k = \{x_0, \dots, x_n\} = \{0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k}{k}\}$

$M_i = \max f(x)$ on $[x_{i-1}, x_i] = f(x_i)$ 2 so

$m_i = \min f(x)$ on $[x_{i-1}, x_i] = f(x_{i-1})$

$$S_{P_k} f - s_{P_k} f = \left[\frac{f(x_1)}{k} + \frac{f(x_2)}{k} + \dots + \frac{f(x_k)}{k} \right] - \left[\frac{f(x_0)}{k} + \frac{f(x_1)}{k} + \dots + \frac{f(x_{k-1})}{k} \right]$$

$$= \frac{f(x_k)}{k} - \frac{f(x_0)}{k} = \frac{f(1) - f(0)}{k} \quad 2$$

so apply lemma 4.5, 1 given $\epsilon > 0$ 1 Choose k Such That

$\frac{f(1) - f(0)}{k} < \epsilon$, so $S_{P_k} f - s_{P_k} f < \epsilon$.

1

extra +1 for showing bdd