-

The concept of open ball is a generalization of the interval $(a-\delta, a+\delta)$, which is also denoted by $\{x : |x-a| < \delta\}$. To say that $\boldsymbol{x} \in B(\delta, \boldsymbol{a})$ is to say that $|\boldsymbol{x}-\boldsymbol{a}| < \delta$, or that $\boldsymbol{x}$ is in a delta neighborhood of the point $\boldsymbol{a}$. Using this notation the continuity condition for a function f at a point $\boldsymbol{a}$ can be written as

$$\forall\epsilon\exists\delta \text{ such that } \forall\boldsymbol{x}\ \boldsymbol{x} \in B(\delta, \boldsymbol{a}) \Longrightarrow f(\boldsymbol{x}) \in B(\epsilon, f(\boldsymbol{a}))$$

Open balls are the main tool for determining if a point $\boldsymbol{a}$ of a set S is an *interior point.* to say that a point $\boldsymbol{a}$ is an interior point of a set S is to say that the point $\boldsymbol{a}$ can be placed within an open ball which is completely contained in S; that is

$$\boldsymbol{a} \text{ is an interior point of } S \text{ iff } \exists r.0 \text{ such that } B(r, \boldsymbol{a}) \subset S$$

An interior point of a set S is a special point of S: it is surrounded by other interior points of S! This sounds like a puzzle: there is no interior point that is an immediate neighbor of a boundary point! This is a mystic idea and is suitable for studying the limit: remember when a variable $\boldsymbol{x}$ approaches a point $\boldsymbol{a}$ it gets closer to it but it can never reach it. The idea of interior point also should remind us of inequalities like $x > a$ where there is no point next to the point a. Various operations in Calculus involve limit, and in the multivariate sense one needs to be able to approach a point from any direction from within the set S. This is where the idea of interior point becomes important. Differentiability of a function at a point $\boldsymbol{a}$ is a good example of such operation that requires limit of an expression near the point. As such we discuss differentiability of a function at the interior points of the domain of the function only.

An open ball provides a private space around a point $\boldsymbol{a}$ in which Calculus about $\boldsymbol{a}$ takes place. Whatever activity that can be done within this open ball are often referred to as *local* activities. In the statement of many theorems we focus on an open set as a domain of a function, and the reason is that much of our analysis can only be performed on open sets (about the interior points) See the remark made on page 137 just after the proof of theorem 3.18.

Open balls are convex, that is any two points of an open ball can be connected by a straight line, and this is a huge advantage because using the techniques of linear algebra much of the calculus of one variable can be discussed in higher dimensions (within an open ball.) A large collection of proofs of our textbook use Mean Value Theorem, which is not meaningful in higher dimension (due to lack of ordering of $\mathbb{R}^n$,) but a variation of it holds for convex sets. See section 2.4 on MVT, and see how the assumption of convexity is crucial in the applications of MVT (see statements and proofs of 2.39-2.41.) Convexity continues to be important in the course until the proof of theorem 5.62 (the last theorem in the course.)