UNIVERSITY OF TORONTO
Faculty of Arts and Sciences

APRIL EXAMINATIONS

MAT237Y1Y

Duration - 3 hours

No Aids Allowed

PLEASE HAND IN

Instructions: There are 9 questions and 16 pages including the cover page. There is a total of 130 marks which include 30 bonus marks. Try to answer as many questions as you can. Note that the number of questions is more than you are expected to answer in a 3 hour exam, so please make a careful selection and answer those questions whose answers you are more confident about, within the space provided; (please clearly specify if you use back of a sheet to answer a question.)

NAME: (last, first) ______________________

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

MARKER'S REPORT:

Question	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
Q9	
TOTAL	

1. (10 marks) Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are both continuous in a neighborhood of a point (a, b). Show the equation $y = b + \int_a^x f(t, y)dt$ defines y as a function of x, say $y = g(x)$, in a neighborhood of $x = a$. Find $g'(x)$ in this neighborhood. (Note: if you use any version of Implicit Function Theorem, please state the theorem carefully, and be sure to explain why the conditions of the theorem are satisfied.)

2.

a) (4 marks) Present the statement of Green's theorem.

b) (8 marks) Consider C to be an arbitrary (not any specific) C^1 curve that joins the points $(0,0)$ and (a,b), and let $\boldsymbol{F}(x,y) = (e^x \sin y, e^x \cos y)$. Calculate $\int_C \boldsymbol{F} \cdot d\boldsymbol{x}$. Present your reasoning.

3. Let S denote the piece of the surface $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$ oriented downward (that is the normal vector to the plane is pointing down; in other word the face of the surface is the convex side.)

   a) (6 marks) Present a parametric representation $\boldsymbol{G}$ for this surface as well as the unit normal vector to the surface at a point (x, y) on S.

   b) (4 marks) Present the equation of the tangent plane to S at the point (a, b, c) on S.

c) (6 marks) Let P be the tangent plane to S at $(\frac{1}{2}, 0, \frac{1}{4})$. Use lagrange multiplier mehod to determine the point on the plane P that is closest to the point $A(0, 0, 1)$.

d) (4 marks) Calculate the surface integral of the vector valued function $\boldsymbol{F}(x, y, z) = (x, y, 2z)$ on S.

e) (5 marks) Calculate the surface integral of the scalar valued function $f(x, y, z) = z$ on S.

4.

a) (3 marks) State Stokes' theorem.

b) (7 marks) Recall the surface S of question 3: the portion of $z = x^2+y^2$ inside the cylinder $x^2+y^2 = 1$, with the downward orientation. With the help of Stokes' theorem and other simpler surfaces that may be relevant to S calculate the line integral $\int_{\partial S} \boldsymbol{G}(\boldsymbol{x})\cdot d\boldsymbol{x}$ where $\boldsymbol{G}(x, y, z) = (\cos z, zx+\log(y^2+1), yz^2)$.

5.

a) (4 marks) Define what it means for a vector field to be conservative.

b) (5 marks) Prove that if a vector field is gradiant of a C^1 scalar function on an open set, then the vector field is conservative there.

c) (5 marks) Prove if a vector field $\boldsymbol{F}(x, y)$ is conservative on an open connected region S then $\boldsymbol{F}$ must be the gradient of a scalar valued function $f(x, y)$.

d) (6 marks) Show that the vector field $\boldsymbol{F}(x, y, z) = (2xy, x^2 - \log z, \frac{1-y}{z})$ is conservative and find a potential function for $\boldsymbol{F}$.

6.

a) (2 marks) Define what it means for a subset S of $\mathbb{R}^n$ to be of content zero.

b) (6 marks) Assume $f : [a, b] \longrightarrow \mathbb{R}$ is C^1. Prove that the graph of f, that is $\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$, is of content zero. (Either direct prove or proper set up and application of theorems and definitions.)

c) (8 marks) Define what it means for a set S to be Jordan measurable. If f is positive on $[a.b]$ then show that the set $S = \{(x, y) : x \in [a, b], 0 \leq y \leq f(x)\}$ is measurable.

7.

a) (3 marks) Let $\phi(t) = f(\boldsymbol{g}(t))$, with $\boldsymbol{g} : \mathbb{R} \longrightarrow \mathbb{R}^3$. Use chain rule to express $\frac{d\phi}{dt}$.

b) (7 marks) Use Chain rule, as well as the theorem about differentiation under the integral sign, to determine $\frac{\partial F}{\partial x}(2,1)$ where $F(x,y) = \int_{x^2-xy}^{e^x-1} \frac{\ln(t+x-y)}{t}\,dt$

8.

a) (3 marks) State the divergence theorem.

b) (2 marks) Determine $f(x, y, z) = \nabla \cdot \boldsymbol{F}$, where $\boldsymbol{F}(x, y, z) = (x^3, y^3, z^3 + z)$.

c) (6 marks) Use Spherical coordinates to calculate the triple integral:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x, y, z) dz dy dx$$

d) (6 marks) Consider S to be the surface of the solid described in the bounds of the triple integral, minus the area on the xy plane. Use divergence theorem to evaluate the surface integral $\iint_S \boldsymbol{F}(x, y, z) \cdot d\boldsymbol{S}$.

9.

a) (5 marks) Prove for any scalar valued C^1 function f and any C^1 vector field $\boldsymbol{G}$ we have the equality $div(f\boldsymbol{G}) = f\ div\boldsymbol{G} + (grad\ f) \cdot \boldsymbol{G}$

b) (5 marks) Prove that if f and g are two scalar valued C^1 functions defined on the closure of a simple region S of the plane, then

$$\iint_S f\nabla^2 g dA = \int_{\partial S} (f\nabla g) \cdot \boldsymbol{n} ds - \iint_S \nabla f \cdot \nabla g dA$$