STA457/2202H1S PRACTICE QUESTIONS

A sequence or collection of random variables $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is called a stochastic process and serves as a model for an observed time series.

The mean function of a stochastic process $\{X_t\}$ is the expected value of the process at time t and given by

$$\mu_t = E(X_t) \text{ for } t = 0, \pm 1, \pm 2, \pm 3, \dots.$$

The second moments of a stochastic process $\{X_t\}$ are its autocovariance functions $\gamma(t,s)$ or autocorrelation functions $\rho(t,s)$, and given by

$$\gamma(t,s) = Cov(X_t, X_s) = E(X_t - \mu_t)(X_s - \mu_s)$$

$$\rho(t,s) = Corr(X_t, X) = \frac{Cov(X_t, X_s)}{\sqrt{Var(X_t)Var(Xs)}} = \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}}$$

for $t, s = 0, \pm 1, \pm 2, \pm 3, \dots$. If $\{X_t\}$ is a stationary time series, $\gamma(t,s)$ can be expressed as $\gamma(h)$, where $h = t - s$. It is easy to show $\rho(0) = 1, \rho(h) = \rho(-h),$ and $|\rho(h)| \leq 1$.

The following result is useful to investigate the second moments of a time series model and answer the practice below.

If $c_1, c_2, \dots, c_m$ and $d_1, d_2, \dots, d_n$ are constants and $t_1, t_2, \dots, t_m$ and $s_1, s_2, \dots, s_n$ are time indices, then

$$Cov\left(\sum_{i=1}^{m} c_i X_{t_i}, \sum_{j=1}^{n} d_j X_{s_j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_i d_j \, Cov(X_{t_i}, X_{s_j})$$

$$Var\left(\sum_{i=1}^{n} c_i X_{t_i}\right) = \sum_{i=1}^{n} c_i^2 Var(X_{t_i}) + 2\sum_{i=2}^{n}\sum_{j=1}^{i=1} c_i c_j \, Cov(X_{t_i}, X_{t_j})$$

Question 1 (Random walk): $Y_t = Y_{t-1} + e_t$, $a_t \sim NID(0, \sigma_e^2)$. Show that for $1 \le t \le s$

(1) $E(Y_t) = 0$
(2) $var(Y_t) = t\sigma_e^2$
(3) $\gamma(t,s) = t\sigma_e^2$
(4) $\rho(t,s) = \sqrt{t/s}$

Question 2 (Moving average of order 2): $Y_t = 0.5\, e_t + 0.5\, e_{t-1}$, $e_t \sim NID(0, \sigma_e^2)$. Show that

(1) $E(Y_t) = 0$
(2) $var(Y_t) = 0.5\, \sigma_e^2$
(3) $\gamma(t,s) = \begin{cases} 0.5\, \sigma_e^2, & t = 0 \\ 0.25\, \sigma_e^2, & |t-s| = 1 \\ 0, & |t-s| > 1 \end{cases}$
(4) $\rho(t,s) = \begin{cases} 1, & t = 0 \\ 0.5, & |t-s| = 1 \\ 0, & |t-s| > 1 \end{cases}$

Question 3 (General linear process): A general linear process, or $MA(\infty)$ process in class, is a weighted linear combination of present and past white noise terms as

$$Y_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \sum_{j=0}^{\infty} \psi_j a_{t-j},$$

where $\psi_0 = 1$, $\sum_{j=1}^{\infty} |\psi_j| < \infty$, and $a_t \sim NID(0, \sigma^2)$. Show that

(1) $E(Y_t) = 0$,
(2) $\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+|h|}$ for $|h| = 0,1,2, \ldots$.

Question 4 Consider a MA(1) process as

$$X_t = a_t + \theta a_{t-1}, \qquad a_t \sim NID(0, \sigma^2).$$

Calculate $var(X_1 + X_2 + X_3)$.

$X_t = a_t + \theta a_{t-1}$
$\gamma(0) = (1+\theta^2)\sigma^2$
$\gamma(1) = \theta\sigma^2$
$\gamma(h) = 0,\ h = 2$

$Var(X_1 + X_2 + X_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} Cov(X_i, X_j)$
$= Var(X_1) + Var(X_2) + Var(X_3) + 2Cov(X_1, X_2) + 2Cov(X_2, X_3) + 2Cov(X_1, X_3)$
$= \gamma(0) + \gamma(0) + \gamma(0) + 2\gamma(1) + 2\gamma(1) + 2\gamma(2)$
$= 3\gamma(0) + 4\gamma(1) + 2\gamma(2)$
$= 3(1+\theta^2)\sigma^2 + 4\theta\sigma^2 + 2 \times 0$
$= \ldots$

Sps $X_t=\phi_1 X_{t-1}+\cdots+\phi_p X_{t-p}+\mu+a_t$ stationary:

then $\underline{E(X_t)}=\phi_1 \underline{E(X_{t-1})}+\cdots+\phi_p \underline{E(X_{t-p})}+\mu$

all the same

$(1-\phi_1-\cdots-\phi_p)E(X_t)=\mu$. $E(X_t)=\dfrac{\mu}{1-\phi_1-\cdots-\phi_p}$

Question 5 (MA(q) processes): $Y_t=\mu+a_t-\theta_1 a_{t-1}-\cdots-\theta_q a_{t-q}$, $a_t \sim NID(0,\sigma^2)$.

Show that

(1) $E(Y_t)=\mu$, Take expectation on both sides: $E(Y_t)=\overset{\mu}{\overset{\parallel}{E(\mu)}}+\overset{WN}{\overbrace{0+0+\cdots}}$

(2) $\gamma(0)=\sigma^2(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)$,

(3) $\gamma(h)=Cov(Y_t,Y_{t+h})=\begin{cases}\sigma^2(-\theta_h+\theta_1\theta_{h+1}+\cdots+\theta_{q-h}\theta_q, & h=1,2,\ldots,q \\ 0 & ,h>q\end{cases}$.

(4) Suppose that $\{Y_t\}$ is invertible and can be expressed as $Y_t=\sum_1^\infty \pi_j Y_{t-j}+a_t$. Find π_j for $j=0,1,2,3,4,5$.

Question 6 (Stationary AR(2) processes):

$Y_t-\phi_1 X_{t-1}-\phi_2 X_{t-2}=\mu+a_t, a_t \sim NID(0,\sigma^2)$. Show that

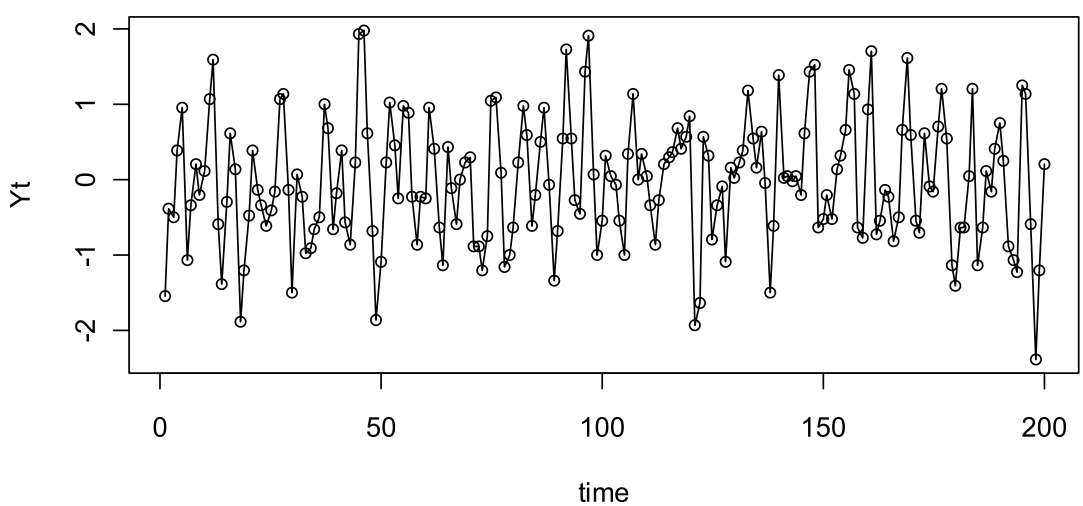
(1) $E(Y_t)=\mu/(1-\phi_1-\phi_2)$,

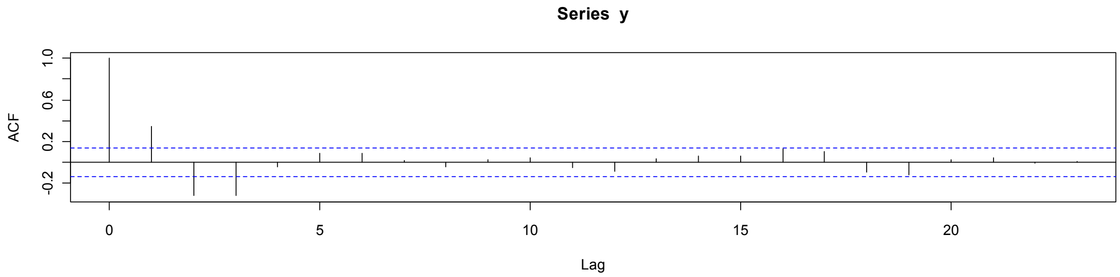
(2) Write down the corresponding Yule-Walker equations.

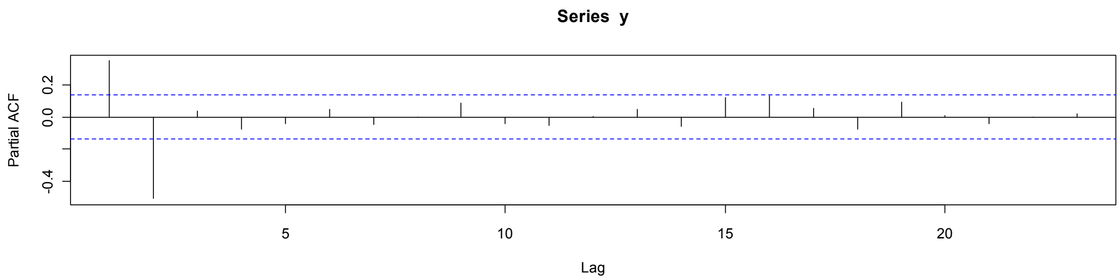
(3) Calculate the partial autocorrelation functions of $\{Y_t\}$ for lag=1,2,3, ...

(4) Suppose that the casual representation of $\{Y_t\}$ is given by $Y_t=\sum_0^\infty \psi_j a_{t-j}$. Find ψ_j for $j=0,1,2,3,4,5$.

Question 7 (The method of moment estimation): An analyst decides to find an AR(2) model for this time series by observing the time series plot and correlogram of $\{Y_t\}$ below.







The analyst calculated the sample autocorrelation functions of $\{Y_t\}$ for $\hat{\rho}(h), h = 1,2,3, \dots 10$ and the results are listed below.

lag	1	2	3	4	5	6	7	8	9	10
rho	-0.78	0.64	-0.53	0.43	-0.36	0.29	-0.24	0.20	-0.16	0.13

(1) Does the analyst make the correct decision to fit an $AR(2)$ model? Why and why not?

(2) Estimate the autoregressive parameters, i.e., ϕ_1 and ϕ_2, using the method of moments. (Hint: Yule-Walker equations)

(3) Is the model stationary?

(4) Suppose the residual autocorrelations functions for lag $1,2,3, \dots 10$ are

$$\{0.030\ -0.072\ \ 0.013\ \ 0.020\ -0.131\ \ 0.036\ \ 0.057\ -0.063\ \ 0.019\ \ 0.054\}$$

Check the model adequacy using the Ljung-box test for $m = 5, 10$.

Question 8 (Definition):

(1) Define strictly and weakly time series. What is the relationship between them?

(2) Describe the general approach to time series modeling.

(3) Define an autoregressive moving average model of order p and q ($ARMA(p, q)$).

(4) What is the dual relationship between AR and MA models.

(5) Define *Wold Decomposition*. How does this method provide support to the use of $ARMA$ models.

(6) Derive the Yule-Walker equations for an $AR(p)$ process.

(7) Define partial autocorrelation functions.

(8) Describe two methods of model selection that were introduced in class.

Question 9 (Causal and invertible process): Determine which of the following processes are causal and/or invertible. Assume that $a_t \sim NID(0,1)$.

(1) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = a_t$

(2) $X_t + 1.9X_{t-1} + 0.88X_{t-2} = a_t + 0.2a_{t-1} + 0.7_{t-2}$

(3) $X_t + 0.6X_{t-2} = a_t + 1.2a_{t-1}$

(4) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = a_t$

(5) $X_t + 1.6X_{t-1} = a_t - 0.4a_{t-1} + 0.04a_{t-2}$