Improper priors

- $k_0, \nu_0 \to 0 \Rightarrow$ No prior information
- $\mu_n \to \bar{y} \quad \sigma_n^2 \to \frac{(n-1)s^2}{n}.$

(show $p(\sigma^2) = \frac{1}{\sigma^2} \equiv p(\log\sigma) \propto 1.$)

$$p(\theta, \sigma^2 | y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \theta)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \theta)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \theta)^2\right]\right).$$

$$p(\sigma^2 | y) \propto \int_{-\infty}^{\infty} \sigma^{-n-2} \exp\left[-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\theta)^2\right]\right] d\theta.$$

$$\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \int_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{2\sigma^2/n}(\bar{y}-\theta)^2}}_{\propto \text{ Normal distribution}} d\theta.$$

$$\propto \sigma^{-n-2} \exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right] \times \sqrt{\frac{\sigma^2}{n}}.$$

$$\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right).$$

$$= (\sigma^2)^{-\left(\left(\frac{n-1}{2}\right)+1\right)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)$$

$$\therefore \sigma^2 | y \sim \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$p(\theta|y) = \int_0^\infty p(\theta, \sigma^2|y)\, d\sigma^2$$

$$= \int_0^\infty \sigma^{-n-2} \exp\left[-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\theta)^2\right]\right] d\sigma^2$$

Let $A = (n-1)s^2 + n(\theta - \bar{y})^2$ and $z = \frac{A}{2\sigma^2}$.

$$\left|\frac{dz}{d\sigma^2}\right| = A(\sigma^2)^{-2}$$

$$d\sigma^2 = A^{-1}(\sigma^2)^2 dz.$$

apply integration by substitution

$$= A^{-n/2} \int_0^\infty z^{(n-2)/2} e^{-z}\, dz$$

$$\propto \left[(n-1)s^2 + n(\theta - \bar{y})^2\right]^{-n/2}$$

$$= \left[1 + \frac{n(\theta - \bar{y})}{(n-1)s^2}\right]^{-n/2} \sim t_{n-1}\left(\bar{y}, \frac{s^2}{n}\right).$$

or

$$\left.\frac{\theta - \bar{y}}{s/\sqrt{n}}\right| y \sim t_{n-1}$$ does not depend on data.

Compare to result from sampling theory

pivotal quantity $$\left.\frac{\bar{y} - \theta}{s/\sqrt{n}}\right| \theta, \sigma^2 \sim t_{n-1}$$