t distribution

$Z \sim N(0,1)$ independent of $X \sim \chi^2_{(n)}$

Fact: $X_i \sim \chi^2_{(1)}$, $i = 1, \dots, n$

$$\sum_{i=1}^{n} X_i = X_1 + \dots + X_n \sim \chi^2_{(n)} = \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

$$T = \frac{Z}{\sqrt{X/n}}$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad f_X(x) = \frac{x^{n/2-1} e^{-\frac{x}{2}}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$$

$$f_{Z,X}(z,x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \cdot \frac{x^{n/2-1} e^{-x/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$$

$$u_1 = \frac{z}{\sqrt{x/n}} = h_1(z,x), \quad u_2 = x = h_2(z,x)$$

$$z = \frac{u_1\sqrt{u_2}}{\sqrt{n}} = h_1^{-1}(u_1,u_2), \quad x = u_2 = h_2^{-1}(u_1,u_2)$$

$$J = \det\begin{bmatrix} \sqrt{\frac{u_2}{n}} & \frac{u_1}{2\sqrt{n u_2}} \\ 0 & 1 \end{bmatrix} = \sqrt{\frac{u_2}{n}}$$

$$f_{U_1,U_2}(u_1,u_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u_1^2 u_2}{2n}} \frac{u_2^{n/2-1} e^{-\frac{u_2}{2}}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \sqrt{\frac{u_2}{n}}$$

$$-\infty < u_1 < \infty$$
$$0 < u_2 < \infty$$

$$f_{U_1}(u_1) = \int_0^\infty \frac{1}{\sqrt{2\pi n}} \frac{u_2^{\frac{n}{2}-\frac{1}{2}} \exp\left[-\frac{u_2}{2}\left(1+\frac{u_1^2}{2}\right)\right]}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} du_2$$

$$w = \frac{u_2}{2}\left(1+\frac{u_1^2}{2}\right)$$

$$dw = \frac{1}{2}\left(1+\frac{u_1^2}{2}\right) du_2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \left[\frac{2w}{1+\frac{u_1^2}{2}}\right]^{\frac{n}{2}-\frac{1}{2}} e^{-w} \frac{2}{\left[1+\frac{u_1^2}{2}\right]} dw$$

$$= \frac{1}{\sqrt{2\pi n}} \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \left[\frac{2}{1+\frac{u_1^2}{2}}\right]^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \underbrace{\int_0^\infty \frac{w^{\frac{n+1}{2}-1} e^{-w}}{\Gamma\left(\frac{n+1}{2}\right)} dw}_{=1} \sim \text{Gamma}\left(\tfrac{n+1}{2}, 1\right)$$

$$\boxed{f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \left[1+\frac{t^2}{2}\right]^{-\frac{n+1}{2}}}$$

$T \sim t_{(n)}$

t distribution with n df

F distribution

Let $X \sim \chi^2_{(n)}$ be independent of $Y \sim \chi^2_{(m)}$

$$Z = \frac{X/n}{Y/m}$$

$$f_{X,Y}(x,y) = \frac{x^{\frac{n}{2}-1} y^{\frac{m}{2}-1} e^{-\frac{(x+y)}{2}}}{2^{\frac{n+m}{2}} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})}$$

$$u_1 = \frac{x/n}{y/m} = h_1(x,y), \quad u_2 = y = h_2(x,y)$$

$$x = \frac{n}{m} u_1 u_2 = h_1^{-1}(u_1,u_2), \quad y = u_2 = h_2^{-1}(u_1,u_2)$$

$$J = \det \begin{bmatrix} \frac{n}{m}u_2 & \frac{n}{m}u_1 \\ 0 & \end{bmatrix} = \frac{n}{m} u_2$$

$$f_{U_1,U_2}(u_1,u_2) = \frac{(\frac{n}{m}u_1 u_2)^{\frac{n}{2}-1} \cdot u_2^{\frac{m}{2}-1} e^{-\frac{(\frac{n}{m}u_1 u_2 + u_2)}{2}}}{2^{\frac{n+m}{2}} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} \frac{n}{m}|u_2|$$

$0 < u_1 < \infty$

$0 < u_2 < \infty$

$$f_{U_1}(u_1) = \int_0^\infty \frac{(\frac{n}{m})^{\frac{n}{2}} u_1^{\frac{n}{2}-1} u_2^{\frac{n+m}{2}-1}}{2^{\frac{n+m}{2}} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} \exp\left[-\frac{u_2}{2}\left(\frac{n}{m}u_1 + 1\right)\right] du_2$$

$$w = \frac{u_2}{2}\left(\frac{n}{m}u_1 + 1\right)$$

$$dw = \frac{1}{2}\left(\frac{n}{m}u_1 + 1\right) du_2$$

$$= \int_0^\infty \frac{(\frac{n}{m})^{n/2} u_1^{\frac{n}{2}-1}}{2^{\frac{n+m}{2}} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} \left[\frac{2w}{\frac{n}{m}u_1 + 1}\right]^{\frac{n+m}{2}-1} e^{-w} \frac{2}{\frac{n}{m}u_1 + 1} dw$$

$$\frac{\left(\frac{n}{m}\right)^{n/2} u^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(1+\frac{n}{m}u\right)^{\frac{n+m}{2}}} \Gamma\left(\frac{n+m}{2}\right) \underbrace{\int_0^\infty \frac{w^{\frac{n+m}{2}-1} e^{-w}}{\Gamma\left(\frac{n+m}{2}\right)} dw}_{=1}$$

$$Z = \frac{X/n}{Y/n}$$

$$\boxed{f_Z(z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right)^{n/2} z^{\frac{n}{2}-1} \left(1+\frac{n}{m}z\right)^{-\frac{n+m}{2}}}$$

$$Z \sim F(n,m) \text{ or } F_{n,m}$$

n is numerator degrees of freedom

m is denominator df

Beta distribution

$Y \sim Beta(\alpha, \beta)$, $\alpha, \beta > 0$ iff

$$f_Y(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, & 0 \le y \le 1 \\ 0, & ow \end{cases}$$

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy \overset{?}{=} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

→ Beta function

$u = y^{\alpha-1}$ $\quad du = (\alpha-1)y^{\alpha-2}dy$

$dv = (1-y)^{\beta-1}dy$ $\quad v = -\frac{(1-y)^{\beta}}{\beta}$

$$B(\alpha,\beta) = \underbrace{y^{\alpha-1}\left(-\frac{(1-y)^{\beta}}{\beta}\right)\Big|_0^1}_{=0} + \frac{\alpha-1}{\beta}\int_0^1 y^{\alpha-2}(1-y)^{\beta}dy$$

$$= \frac{\alpha-1}{\beta}B(\alpha-1, \beta+1)$$

$$= \underbrace{\frac{(\alpha-1)(\alpha-2)\cdots 1}{\beta(\beta+1)\cdots(\beta+\alpha-2)}}\ \underbrace{B(1, \beta+\alpha-1)} =$$

$$B(1, \beta+\alpha-1) = \int_0^1 (1-y)^{\alpha+\beta-2}dy = \frac{-(1-y)^{\alpha+\beta-1}}{\alpha+\beta-1}\Big|_0^1$$

$$= \frac{(\alpha-1)(\alpha-2)\cdots 1}{\beta(\beta+1)\cdots(\beta+\alpha-2)(\beta+\alpha-1)} \frac{(\beta-1)(\beta-2)\cdots 1}{(\beta-1)(\beta-2)\cdots 1}$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \blacksquare$$

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$