UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL 2013 EXAMINATIONS

STA 447H1 S - STA 2006H S

Duration: 3 hours

No aids allowed

Monday, April 29, 2013 – 2:00 - 5:00 - Room EX 320

Content: 6 pages, 6 exercises

Indications

- Write your **name** and **student number** on top of each of your booklets.
- Indicate **how many booklets you used**, by writing e.g. 1/3, 2/3 and 3/3 on top of them.
- Write with a pencil or a pen, but assignments written in pencil will not be regraded.
- Try to be clear, precise and concise. A good explanation is not necessarily a long explanation (and in general, quite the opposite).
- If you use a result, tell **what assumptions it requires to be applied**.
- **Show your computations** (for instance, to compute an invariant distribution, show how you solve the system).
- **Read the questions well**. Do not miss the point, or forget to answer half of a question. Explain your reasonings, especially when it is explicitly asked.
- The total is **100 marks**.

Exercise 1: Hammy the hamster [15]

Hammy the hamster does only three things: sleeping, eating, or running in a wheel. Every activity lasts 10 minutes, and he switches from one to another as follows.

- When he sleeps, he has probability $3/4$ to keep sleeping, and $1/4$ to wake up and eat.
- After eating, he has probability $2/3$ to go to sleep, and $1/3$ to go run on his wheel.
- After running, he has probability $1/2$ to go to sleep or to go eat.

1. Write the transition matrix corresponding to this Markov chain. Is it irreducible? Aperiodic? (no need to explain) **[2]**
2. Compute its invariant distribution. **[5]**
3. On average, what is the fraction of time he spends on each activity? **[2]**
4. After running in his wheel, how long is there, on average, until he runs again? **[2]**
5. Assume that, whenever he eats, it costs you 5 cents. But you attached a generator to his wheel, which provides energy to your lamp and makes you save 1 cent every time he runs. It does not cost you anything when he sleeps. What is the approximate amount of money you spend every day to take care of him? Give a rough numerical approximation. **[4]**

Exercise 2: Doubly stochastic matrices [4]

Consider a transition matrix P on a finite state space S, and assume that it is doubly stochastic, i.e. that the sum of the entries in every column is 1: for all $x \in S$,

$$\sum_{y \in S} P(x, y) = 1.$$

Prove that the uniform distribution on S,

$$\pi(x) = 1/\#S, \quad x \in S,$$

is invariant for P (here, $\#S$ is the cardinality of S).

Exercise 3: Limiting behavior [20]

Consider a Markov chain on $S = \{A, B, C, D, E, F, G, H\}$ with transition matrix

$$P = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.4 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}.$$

1. Draw the graph of this chain. Deduce the irreducible classes (no need to explain). [4]
2. What are the recurrent classes? And the transient classes? Give a very short explanation for each. [3]
3. Give the period of each class, with a very short explanation. [1]
4. Starting from E, explain the long-term behavior of the chain, that is, compute the limit as $n \to +\infty$ of $\mathbb{P}_E(X_n = x)$ for every $x \in S$. Explain your reasoning. You may want to use the result of Exercise 2, even if you did not prove it. [12]

CONT'D.

Exercise 4: Branching process with immigration [16]

Consider a branching process with immigration: at each generation, a random number of additional individuals arrive, and they then behave like any other individual. In symbols,

- let μ be a distribution on $\mathbb{N}$, and let m be its mean;
- let $(X_{n,i})_{n\geq 0,\, i\geq 1}$ be i.i.d. random variables with law μ (think of $X_{n,i}$ as the number of children of the i-th individual at generation n);
- let $(Y_n)_{n\geq 1}$ be i.i.d. random variables, independent from the $X_{n,i}$ (think of Y_n as the number of additional individuals arriving at generation n).

Let $m = \mathbb{E}(X_{n,i})$, and assume that $m \neq 1$. Let also $\theta = \mathbb{E}(Y_n)$. Then the branching process with immigration $(Z_n)_{n\geq 0}$ is defined by

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_{n+1}.$$

1. Prove that
$$M_n = \frac{1}{m^n}\left(Z_n - \theta\frac{1-m^n}{1-m}\right)$$
is a martingale in the filtration $\mathcal{F}_n = \{Z_1, \ldots, Z_n\}$. **[10]**
2. Compute $\mathbb{E}(Z_n)$. **[3]**
3. What is $\mathbb{E}(Z_n)$ when there is no migration? [1]
4. Assume now that $m > 1$. Compute
$$\lim_{n\to+\infty} \frac{\mathbb{E}(Z_n)}{m^n}$$
with and without migration. Does migration have a dramatic effect on the average growth of the population? **[2]**

CONT'D.

Exercise 5: Shopping spree [17]

Three friends, A, B and C, go on a shopping spree from 10 am to 6 pm. We can assume that the number of items A buys is a Poisson process $(N^A(t))$ on $[0, 8]$, with rate $1/3$, and the same for B and C, with rate $1/2$ and $1/6$. We moreover assume that these processes are independent.

You should obviously simplify your formulas as much as possible, but you do not need to give numerical approximations.

1. What is the probability that A buys nothing before 1 pm? Explain. **[2]**
2. What is the distribution of the time before he buys anything? And the average time before he does? Explain. **[2]**
3. What is the probability that C goes home with nothing, but A has bought something? **[2]**
4. What is the process of the total number of items bought by A, B and C? Explain. **[2]**
5. What is the distribution of the total number of items they bought that day? What is its mean and variance? **[2]**
6. What is the probability that no item was bought in the morning (before 12), and at least one was bought in the afternoon? Justify your reasoning. **[3]**
7. What is the probability that exactly one item was bought in the morning, but nothing else was bought by 3 pm? Explain your reasoning. **[4]**

CONT'D.

Exercise 6: Area under the Brownian curve [28]

In this exercise, feel free to use Fubini's theorem, namely that "expectations and integrals can be interchanged", i.e., if (X_s) is some process, then

$$\mathbb{E}\left(\int_a^b X_s \,\mathrm{d}s\right) = \int_a^b \mathbb{E}(X_s)\,\mathrm{d}s$$

and the same holds for conditional expectations:

$$\mathbb{E}\left(\int_a^b X_s \,\mathrm{d}s \middle| \mathcal{F}\right) = \int_a^b \mathbb{E}(X_s|\mathcal{F})\,\mathrm{d}s.$$

It can be proved that the assumptions of this result hold here.

1. Consider a Brownian motion (B_t), and the reflected Brownian motion $(|B_t|)$.
   (a) Compute $E(|B_t|)$. **[3]**
   (b) Compute
   $$\mathbb{E}\left(\int_0^1 |B_t|\,\mathrm{d}t\right),$$
   the expected area under the reflected Brownian curve on $[0,1]$. **[3]**
2. (a) Define
   $$M_t = B_t^3 - 3\int_0^t B_u\,\mathrm{d}u.$$
   Prove that (M_t) is a martingale in the Brownian filtration $(\mathcal{F}_s)$. **[10]**
   (b) Take $0 \le x \le a$, and assume that (B_t) starts at x. Define the stopping time $T = \inf\{t \ge 0,\ B_t = 0 \text{ or } B_t = a\}$. Apply (the first part of) the optional stopping theorem to M and T to compute
   $$\mathbb{E}\left(\int_0^T B_u\,\mathrm{d}u\right),$$
   the expected area under the Brownian curve before hitting 0 or a. Justify everything that you do. You may use the fact that
   $$\mathbb{P}(B_T = a) = \frac{x}{a}, \quad \mathbb{P}(B_T = 0) = \frac{a-x}{a},$$
   which we proved in class. **[10]**
   (c) For what x is this area maximal? Briefly explain why it is reasonable that $a/2 < x < a$. **[2]**