Exerzitien VI

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Nov. 3., in your tutorial.

Reading suggestion: Axler Chapter 3.

Exercise 1. *Let* $L : \mathbb{R}^5 \to \mathbb{R}^5$ *be the left-shift linear operator defined by*

$$L : (x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_3, x_4, x_5, 0).$$

1. *What is the matrix of* L *using the standard basis for* $\mathbb{R}^5$*?*
2. *Compute the matrices of* L^k *for* $k = 1, 2, \ldots$.
3. *Determine* $\dim \mathrm{null}(L^k)$ *and* $\dim \mathrm{range}(L^k)$ *for* $k = 1, 2, \ldots$.

Exercise 2. *Let* $S : U \to V$ *and* $T : V \to U$ *be linear maps. Recall that* I_U *means the identity map on* U.

1. *If* $TS = I_U$*, prove that* S *is injective and* T *is surjective.*
2. *If* $TS = I_U$*, must* S, T *both be isomorphisms? Justify your answer by giving a proof or a counterexample.*
3. *If* $TS = I_U$*, prove that* $V = \mathrm{range}(S) \oplus \mathrm{null}(T)$ *[Can you apply Assignment 5, Exercise 2, to* ST*?]*
4. *If* ST *and* TS *are isomorphisms from* $V \to V$ *and* $U \to U$*, respectively, prove that* S *and* T *are themselves isomorphisms.*

Exercise 3. *Let* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Define a map* $T : \mathrm{Mat}(2, 2, \mathbb{Q}) \to \mathrm{Mat}(2, 2, \mathbb{Q})$ *via*

$$T(X) = AX - XA.$$

1. *Prove that* T *is a linear map.*
2. *Find a basis for* $\mathrm{null}(T)$ *and* $\mathrm{range}(T)$.

Exercise 4.

1. *Let* $S : W \to W$ *be a linear operator and suppose that* $S^n = 0$. *Show that* $S - I$ *is an isomorphism. [Hint: since* $S^n = 0$*, it is also true that* $I - S^n = I$*.]*
2. *Use the result to find the inverse of* $L - I$*, where* L *is from Exercise 1.*
3. *Generalize the result in the following way: Show that if* $(T - aI)^n = 0$ *for some scalar* a *and operator* $T : W \to W$*, then* $T - bI$ *is invertible (for a scalar* b*) if and only if* $a \neq b$*. [Hint: consider* $S = (b - a)^{-1}(T - aI)$*.]*