# STAT2001 Mid-Semester Examination – 1st Sem. 2015 - Solutions

**(Note: The STAT2001 and STAT6039 exams are identical, as are their solutions.)**

## Solution to Problem 1

Here: $P(X=2)=P(RR)=\frac{3}{7}\times\frac{2}{6}=\frac{5}{35}$ (where $RR$ is the event "2 red balls selected")

$$P(X=3)=P(WRR)+P(RWR)=\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}+\frac{3}{7}\times\frac{4}{6}\times\frac{2}{5}=\frac{8}{35}$$

$$P(X=4)=P(WWRR)+P(WRWR)+P(RWWR)=\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times 3=\frac{9}{35}$$

$$P(X=5)=P(WWWRR)+P(WWRWR)+P(WRWWR)+P(RWWWR)$$

$$=\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}\times 4=\frac{8}{35}$$

$$P(X=6)=P(WWWWRR)+P(WWWRWR)+P(WWRWWR)$$

$$+P(WRWWWR)+P(RWWWWR)\;=\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}\times\frac{2}{2}\times 5=\frac{5}{35}.$$

So the distribution of $X$ is given by the pmf $f(x)=\begin{cases}5/35=0.1429, & x=2,6\\ 8/35=0.2286, & x=3,5\\ 9/35=0.2571, & x=4.\end{cases}$

**Note:** $P(X=6)$ could also have been obtained by calculating $1-\sum_{x=2}^{5}P(X=x)$.

But working out $P(X=6)$ separately allows us to check that $\sum_x f(x) = 1$.

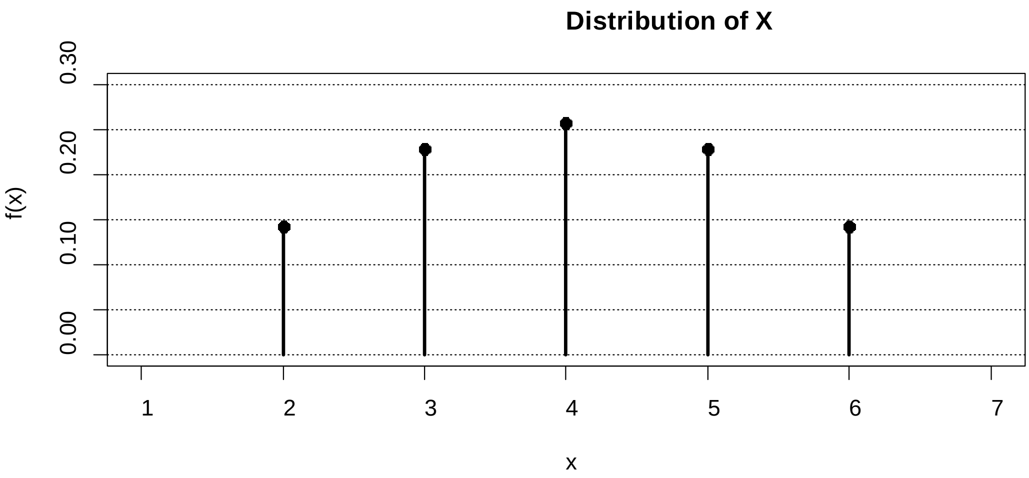
Below is a sketch of $X$'s pmf. By symmetry, $EX$ = **4**, which can also be confirmed by

the calculation $EX=\sum_{x=2}^{6}xf(x)=2\times\frac{5}{35}+3\times\frac{8}{35}+4\times\frac{9}{35}+5\times\frac{8}{35}+6\times\frac{5}{35}=\frac{140}{35}=4.$

Also, the variance of $X$ is $VX=E(X-4)^2=\sum_{x=2}^{6}(x-4)^2 f(x)$

$$=(2-4)^2\frac{5}{35}+(3-4)^2\frac{8}{35}+(4-4)^2\frac{9}{35}+(5-4)^2\frac{8}{35}+(6-4)^2\frac{5}{35}$$

$$=\frac{1}{35}\{20+8+0+8+20\}=\frac{56}{35}=\underline{\mathbf{1.6}}.$$




**Alternative working**

Consider the general situation where a box contains $N$ balls, of which $r$ are red and $N-r$ are white, and where balls are drawn without replacement until $n$ reds have been selected. We wish to find $p_x$, the probability that exactly $x$ balls are drawn.

Now suppose that all $N$ balls are drawn, one by one, and the entire sequence of red and white balls is observed. Then $p_x = m_x / m$, where $m$ is the total number of arrangements ($N$-choose-$r$), and where $m_x$ is the number of arrangements with:

**(a)** $n-1$ red balls amongst the first $x-1$ positions

**(b)** a red ball in position $x$

**(c)** $r-n$ red balls in the last $N-x$ positions.

We see that

$$p_x = \binom{x-1}{n-1}\binom{1}{1}\binom{N-x}{r-n}\Big/\binom{N}{r}, \ x = n, ..., N-r+n.$$

Thus, for the case where $N$ = 7, $r$ = 3 and $n$ = 2, we have that

$$p_x = \binom{x-1}{2-1}\binom{1}{1}\binom{7-x}{3-2}\Big/\binom{7}{3} = \frac{(x-1)(7-x)}{35} = \begin{cases} 5/35, & x = 2,6 \\ 8/35, & x = 3,5 \\ 9/35, & x = 4. \end{cases}$$

## R Code for Problem 1

```
X11(w=8,h=4)

plot(c(1,7),c(0,0.3),type="n",xlab="x",ylab="f(x)",main="Distribution of X")

xvec=c(2,3,4,5,6); fvec= c(5,8,9,8,5)/35

points(xvec,fvec,pch=16,cex=1.2)

for(i in 1:5) lines( rep(xvec[i],2), c(0,fvec[i]), lwd=3)

abline(h=seq(0,0.3,0.05), lty=3)


pfun = function(x=2,N=7,r=3,n=2){  choose(x-1,n-1)*choose(N-x,r-n)/choose(N,r)   }

xvec=2:6; pvec=pfun(xvec); sum(pvec) # 1 OK

round( rbind(xvec,pvec), 4)

# xvec 2.0000 3.0000 4.0000 5.0000 6.0000

# pvec 0.1429 0.2286 0.2571 0.2286 0.1429  OK


(xvec-1)*(7-xvec)/35 # 0.1428571 0.2285714 0.2571429 0.2285714 0.1428571  OK


N=10; r=4; n=3; minx=n; maxx=N-r+n   # Another example (for checking function pfun)

xvec=minx:maxx; pvec=pfun(xvec,N=N,r=r,n=n); sum(pvec) # 1 OK

round( rbind(xvec,pvec), 4)

# xvec 3.0000 4.0000 5.0000 6.0000 7.0000  8.0 9.0000

# pvec 0.0333 0.0857 0.1429 0.1905 0.2143  0.2 0.1333  OK
```

**Solution to Problem 2**

Let $A_i$ be the event that outcome *i* comes up at least once. Also let $B_i = \overline{A_i}$ and *n* = 12.

Then the event of interest is $A_1 \cdots A_6$, and the required probability is $p = 1 - q$, where

$$q = P(\overline{A_1 \cdots A_6}) = P(\overline{A_1} \cup \cdots \cup \overline{A_6}) = P(B_1 \cup \cdots \cup B_6) \quad \text{by De Morgan's laws}$$

$$= \{P(B_1) + \cdots + P(B_6)\} \quad - \quad \{P(B_1B_2) + \cdots + P(B_5B_6)\}$$

$$+ \quad \{P(B_1B_2B_3) + \cdots + P(B_4B_5B_6)\} \quad - \quad \{P(B_1B_2B_3B_4) + \cdots + P(B_3B_4B_5B_6)\}$$

$$+ \quad \{P(B_1B_2B_3B_4B_5) + \cdots + P(B_2B_3B_4B_5B_6)\} \quad - \quad P(B_1B_2B_3B_4B_5B_6)$$

by the general additive law of probability.

Now, $P(B_1B_2B_3B_4B_5B_6) = 0$. Also, for all $i < j < k < l < m$: $P(B_i) = \left(\frac{5}{6}\right)^n$,

$$P(B_iB_j) = \left(\frac{4}{6}\right)^n, \; P(B_iB_jB_k) = \left(\frac{3}{6}\right)^n, \; P(B_iB_jB_kB_l) = \left(\frac{2}{6}\right)^n, \; P(B_iB_jB_kB_lB_m) = \left(\frac{1}{6}\right)^n.$$

Therefore $q = 6\left(\frac{5}{6}\right)^n - \binom{6}{2}\left(\frac{4}{6}\right)^n + \binom{6}{3}\left(\frac{3}{6}\right)^n - \binom{6}{4}\left(\frac{2}{6}\right)^n + 6\left(\frac{1}{6}\right)^n$ = 0.5622.

So the required probability is $p = 1 - q$ = **<u>0.4378</u>**.

**R Code for Problem 2**

```
n=12; q=6*(5/6)^n-choose(6,2)*(4/6)^n+choose(6,3)*(3/6)^n-choose(6,4)*(2/6)^n+6/6^n

p=1-q; c(q,p) # 0.5621843 0.4378157


# Alternative code

1  -  sum(   ((-1)^(0:4))   *  choose(6,1:5)   *  (1-(1:5)/6)^12   )  #   0.4378157
```

**Solution to Problem 3**

Let $c$ = 9, $m$ = 7, $w$ = 20 and $s$ = 4. Also let $A$ be the event that the committee has at least two men, and let $A_i$ be the event that the committee has exactly $i$ men. Finally, let $B$ be the event that the sub-committee has at least one man. Then the required probability is $P(A\,|\,B) = P(AB)/P(B)$,

where $P(B) = 1 - P(\overline{B}) = 1 - \dfrac{\binom{m}{0}\binom{w}{s}}{\binom{m+w}{s}} = 1 - \dfrac{\binom{20}{4}}{\binom{27}{4}}$ = 1 – 0.27607 = 0.72393

and $P(AB) = 1 - P(\overline{AB}) = 1 - P(\overline{A} \cup \overline{B})$ by De Morgan's laws

$$= 1 - \{P(\overline{A}) + P(\overline{B}) - P(\overline{A}\overline{B}) \;\; = P(B) - P(\overline{A}) + P(\overline{A}\overline{B}).$$

Now, $P(\overline{A}) = P(A_0) + P(A_1) = \dfrac{\binom{m}{0}\binom{w}{c}}{\binom{m+w}{c}} + \dfrac{\binom{m}{1}\binom{w}{c-1}}{\binom{m+w}{c}} = \dfrac{\binom{20}{4}}{\binom{27}{4}} + \dfrac{7\binom{20}{3}}{\binom{27}{4}}$

= 0.03584 + 0.18814 = 0.22398.

Also, $P(\overline{A}\overline{B}) = P((A_0 \cup A_1) \cap \overline{B}) = P(A_0\overline{B}) + P(A_1\overline{B})$,

where: $P(A_0\overline{B}) = P(A_0)P(\overline{B}\,|\,A_0) = P(A_0) \times 1 = P(A_0)$ = 0.03584

and $P(A_1\overline{B}) = P(A_1)P(\overline{B}\,|\,A_1) = \dfrac{\binom{m}{1}\binom{w}{c-1}}{\binom{m+w}{c}} \times \dfrac{\binom{1}{0}\binom{c-1}{s}}{\binom{c}{s}} = \dfrac{7\binom{20}{3}}{\binom{27}{4}} \times \dfrac{\binom{8}{4}}{\binom{9}{4}}$ = 0.10452.

Thereby we obtain $P(\overline{A}\overline{B}) = P(A_0\overline{B}) + P(A_1\overline{B})$ = 0.03584 + 0.10452 = 0.14036.

So $P(AB) = P(B) - P(\overline{A}) + P(\overline{A}\overline{B})$ = 0.72393 – 0.22398 + 0.14036 = 0.64031.

It follows that the required probability is $P(A\,|\,B) = \dfrac{P(AB)}{P(B)} = \dfrac{0.64031}{0.72393}$ = **0.8845**.

**Alternative working**

Observe that $P(A \mid B) = 1 - P(A_0 \mid B) - P(A_1 \mid B)$

$$= 1 - P(A_1 B) / P(B), \quad \text{since } P(A_0 \mid B) = 0 .$$

Next, $P(A_1 B) = P(A_1 B_1)$, where $B_i$ = "Subcommittee has exactly *i* men".

So $P(A_1 B) = P(A_1)P(B_1 \mid A_1)$

$$= \frac{\binom{m}{1}\binom{w}{c-1}}{\binom{m+w}{c}} \times \frac{\binom{1}{1}\binom{c-1}{s-1}}{\binom{c}{s}} = \frac{7\binom{20}{8}}{\binom{27}{9}} \times \frac{\binom{8}{3}}{\binom{9}{4}} = 0.083602.$$

It follows that $P(A \mid B) = 1 - \dfrac{P(A_1 B)}{P(B)} = 1 - \dfrac{0.083602}{0.72393}$ = 1 – 0.1155 = 0.8845.

**R Code for Problem 3**

```
m=7; w=20; c=9; s=4; options(digits=5);

PBbar=choose(w,s)/choose(m+w,s); PB=1-PBbar

c(PBbar,PB) # 0.27607 0.72393

PA0=choose(w,c)/choose(m+w,c);

PA1=m*choose(w,c-1)/choose(m+w,c)

PAbar=PA0+PA1

c(PA0,PA1,PAbar) # 0.035837 0.188142 0.223979


PA0Bbar=PA0

PA1Bbar=(m*choose(w,c-1)/choose(m+w,c)) * choose(c-1,s)/choose(c,s)

PAbarBbar= PA0Bbar+ PA1Bbar;

c(PA0Bbar, PA1Bbar, PAbarBbar) # 0.035837 0.104523 0.140360
```

```
PAB=PB-PAbar+PAbarBbar; PAB # 0.64031

p=PAB/PB; p # 0.88449


# Alternative working

PA1B=m*choose(w,c-1)*choose(c-1,s-1)/(choose(m+w,c)*choose(c,s))

c(  PA1B,   PA1B/PB  ,  1 - PA1B/PB  )   # 0.083619 0.115506 0.884494


# Check via Monte Carlo (not assessable material, only for interest)

set.seed(179); com=sample(x=c(rep(0,20),rep(1,7)), size=9)

menC=sum(com) # men on committee

subcom=sample(c(rep(0,9-menC),rep(1,menC)), size=4)

menSC=sum(subcom); c(menC,menSC) # 3 2   OK


top=0; bot=0; J=160000; set.seed(348); date();

for(j in 1:J){    # Start of simulations

        com=sample(x=c(rep(0,20),rep(1,7)), size=9);    menC=sum(com);

        subcom=sample(c(rep(0,9-menC),rep(1,menC)), size=4);   menSC=sum(subcom);

        if(menSC>0){      bot=bot+1;      if(menC>1) top=top+1    }

        }; date()      # The simulations too less than 4 seconds


p=top/bot; ci=p+c(-1,1)*qnorm(0.975)*sqrt(p*(1-p)/J)

c(p,ci) # 0.88426 0.88269 0.88583


# So we estimate the required probability as 0.8843,

# with the 95% confidence interval (0.8827, 0.8858).
```

**Solution to Problem 4**

Let $A$ be the event that Ann wins, and let $B$ be the event that Bob wins. Also, let 0 stand for an odd number (1, 3 or 5), and let 600644 (for instance) denote a sequence of numbers coming up in a game where Bob wins after 4 has come up twice in a row.

Then, applying a first-step analysis, we have that

$$P(A) = P(0)P(A|0) + P(2)P(A|2) + P(4)P(A|4) + P(6)P(A|6)$$

by the law of total probability with partition $S = \{0, 2, 4, 6\}$

$$= \frac{3}{6}P(B) + \frac{1}{6}P(A|2) + \frac{1}{6}P(A|4) + \frac{1}{6}P(A|6).$$

$$= \frac{3}{6}\{1 - P(A)\} + \frac{3}{6}P(A|6). \qquad (1)$$

**Note:** If an odd number comes up on the first roll, Ann will be in the same situation as Bob was when the game started. Also, $P(A|2) = P(A|4) = P(A|6)$, by symmetry.

Next, $P(A|6) = P(60|6)P(A|6,60)$

$$+P(62|6)P(A|6,62) + P(64|6)P(A|6,64) + P(66|6)P(A|6,66)$$

by the LTP with partition $\{6\} = \{60, 62, 64, 66\}$

$$= \frac{3}{6}P(A) + \frac{2}{6}\{1 - P(A|6)\} + \frac{1}{6} \times 0. \qquad (2)$$

**Note:** Here, $P(A|6,60)$ (for example) is the same as $P(A|60)$. If Bob rolls 2 on the second roll after Ann has rolled 6, Ann will roll next and be in the same situation Bob was in after Ann rolled 6. So $P(A|62) = 1 - P(A|6)$. Also, $P(A|62) = P(A|64)$, by symmetry. Also, if 6 comes up on the first two rolls then Bob wins. So $P(A|66) = 0$.

With $p = P(A)$ and $q = P(A|6)$, equations (1) and (2) may be written as:

$$p = \frac{3}{6}(1 - p) + \frac{3}{6}q$$

$$q = \frac{3}{6}p + \frac{2}{6}(1 - q).$$

Solving these two equations, we obtain $q$ = 3/7 = 0.4286 and $p$ = 10/21 = **<u>0.4762</u>**.

## Brief Solutions – Same as above but with less detail and adequate for full marks

### Solution to Problem 1 (Brief)

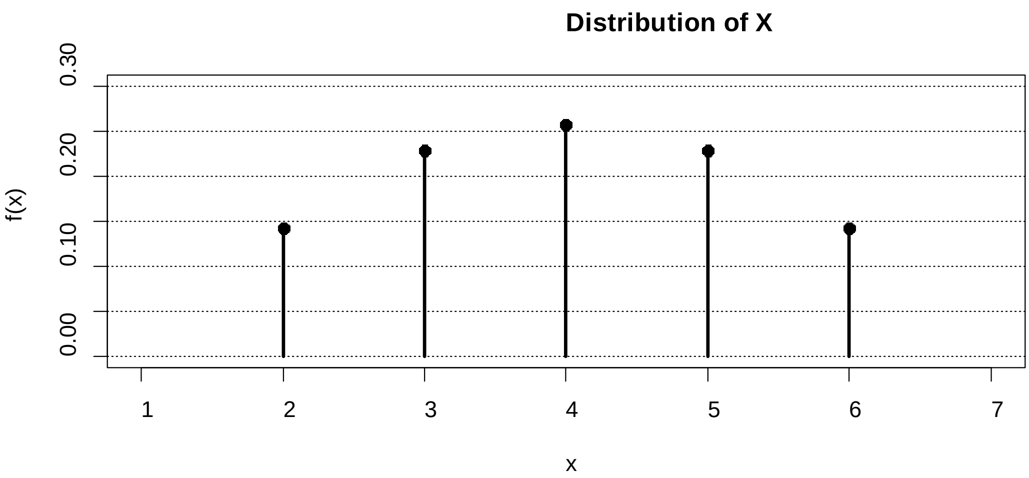
$$f(2) = P(RR) = \frac{3}{7}\times\frac{2}{6} = \frac{5}{35}\ ,\quad f(3) = P(WRR) + P(RWR) = \frac{4}{7}\times\frac{3}{6}\times\frac{2}{5} + \frac{3}{7}\times\frac{4}{6}\times\frac{2}{5} = \frac{8}{35}$$

$$f(4) = P(WWRR) + P(WRWR) + P(RWWR) = \frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times 3 = \frac{9}{35}$$

$$f(5) = P(WWWRR) + P(WWRWR) + P(WRWWR) + P(RWWWR)$$

$$= \frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}\times 4 = \frac{8}{35},\qquad f(6) = 1 - f(2) - \ldots - f(5) = \frac{5}{35}.$$

$EX$ = **4**. $VX = \frac{1}{35}\left\{(2-4)^2 5 + (3-4)^2 8 + (4-4)^2 9 + (5-4)^2 8 + (6-4)^2 5\right\} =$ **1.6**.




### Solution to Problem 2 (Brief)

Let $A_i$ be the event that outcome *i* comes up at least once. Then

$$P(A_1 \cdots A_6) = 1 - P(\overline{A_1 \cdots A_6}) = 1 - P(B_1 \cup \cdots \cup B_6) \quad \text{where } B_i = \overline{A}_i$$

$$= 1 - \left\{\sum_{i=1}^{6} P(B_i) - \sum_{i<j} P(B_i B_j) + \ldots + \sum_{i<j<k<l<m} P(B_i B_j B_k B_l B_m)\right\}$$

$$= 1 - \left\{6\left(\frac{5}{6}\right)^{12} - \binom{6}{2}\left(\frac{4}{6}\right)^{12} + \binom{6}{3}\left(\frac{3}{6}\right)^{12} - \binom{6}{4}\left(\frac{2}{6}\right)^{12} + 6\left(\frac{1}{6}\right)^{12}\right\} = \mathbf{\underline{0.4378}}.$$

**Solution to Problem 3 (Brief)**

Let: $A_i$ = "Committee has $i$ men", $A$ = "Committee has at least 2 men"

$B_i$ = "Subcommittee has $i$ men", $B$ = "Subcom. has at least one man".

Then: $P(B) = 1 - P(B_0) = 1 - \dfrac{\binom{7}{0}\binom{20}{4}}{\binom{27}{4}} = 0.72393$

$$P(\bar{A}B) = P(A_1B_1) = P(A_1)P(B_1 \mid A_1) = \frac{\binom{7}{1}\binom{20}{8}}{\binom{27}{9}} \times \frac{\binom{1}{1}\binom{8}{3}}{\binom{9}{4}} = 0.083602.$$

So $P(A \mid B) = 1 - P(\bar{A} \mid B) = 1 - \dfrac{P(\bar{A}B)}{P(B)}$ = **<u>0.8845</u>**.

**Solution to Problem 4 (Brief)**

Define $A$ = "Ann wins", and let 0 denote an odd number. Then:

$$P(A) = P(0)P(A \mid 0) + P(2)P(A \mid 2) + P(4)P(A \mid 4) + P(6)P(A \mid 6)$$

$$= \frac{3}{6}\{1 - P(A)\} + \frac{3}{6}P(A \mid 6) \quad \text{since } P(A \mid 2) = P(A \mid 4) = P(A \mid 6)$$

$$P(A \mid 6) = P(60 \mid 6)P(A \mid 60) + P(62 \mid 6)P(A \mid 62) + P(64 \mid 6)P(A \mid 64) + P(66 \mid 6)P(A \mid 66)$$

$$= \frac{3}{6}P(A) + \frac{2}{6}\{1 - P(A \mid 6)\} + \frac{1}{6} \times 0.$$

Solving, we obtain $P(A)$ = 10/21 = **<u>0.4762</u>**.