STAT2001 Tutorial 9 Solutions

Problem 1

(a) $EU = EX - 3EY = 10 - 3(-5) = 25$.

$VarU = (1)^2 VarX + (-3)^2 VarY = 16 + 9(4) = 52$.

(b) EU = 25 (same as in (a)).

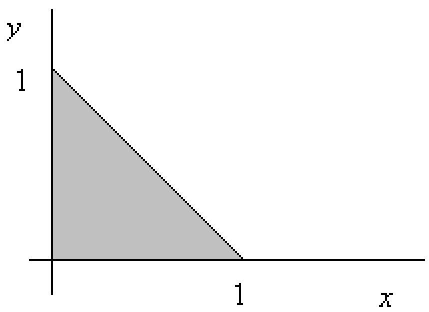
$\rho = \dfrac{Cov(X,Y)}{SD(X)SD(Y)} \Rightarrow Cov(X,Y) = \rho SD(X)SD(Y) = 0.65\sqrt{16}\sqrt{4} = 5.2$.

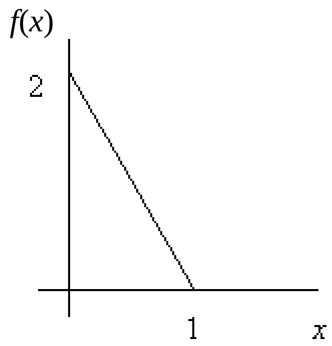
So $VarU = (1)^2 VarX + (-3)^2 VarY + 2(1)(-3)Cov(X,Y)$

$= 16 + 9(4) - 6(5.2) = 20.8$.

Problem 2

(a) $f(x) = \int_0^{1-x} 2dy = 2(1-x)$, $0 < x < 1$.





So: $EX = \int_0^1 x2(1-x)dx = \dfrac{1}{3}$

$EX^2 = \int_0^1 x^2 2(1-x)dx = \dfrac{1}{6}$, $\qquad VarX = \dfrac{1}{6} - \left(\dfrac{1}{3}\right)^2 = \dfrac{1}{18}$.

By symmetry, EY = 1/3 and $VarY$ = 1/18 also.

$$E(XY) = \int_{x=0}^{1}\int_{y=0}^{1-x} xy2dy\,dx = 2\int_{x=0}^{1} x\left(\int_{y=0}^{1-x} ydy\right)dx = 2\int_{x=0}^{1} x\frac{1}{2}(1-x)^2\,dx = \frac{1}{12}.$$

So $Cov(X,Y) = E(XY) - (EX)EY = \dfrac{1}{12} - \dfrac{1}{3}\times\dfrac{1}{3} = -\dfrac{1}{36}$.

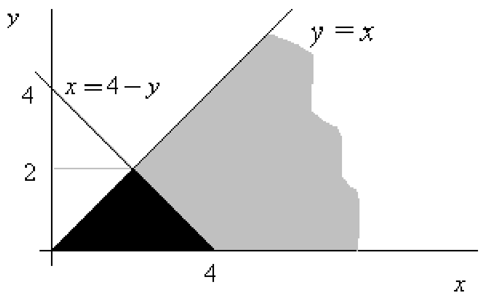
So $\rho = \dfrac{Cov(X,Y)}{SD(X)SD(Y)} = \dfrac{-1/36}{\sqrt{1/18}\sqrt{1/18}} = -\dfrac{1}{2}$.

(b) $EU = EX - 2EY = 1/3 - 2(1/3) = -1/3.$

$$VarU = (1)^2 VarX + (-2)^2 VarY + 2(1)(-2)Cov(X,Y)$$
$$= 1/18 + 4(1/18) - 4(-1/36) = 7/18.$$

Problem 3

(a) $$P(X+Y>4) = 1 - P(X+Y<4) = 1 - \int_{y=0}^{2}\left(\int_{x=y}^{4-y} e^{-x}dx\right)dy.$$



The inner integral above equals $= \left[-e^{-x}\Big|_{x=y}^{4-y}\right] = -e^{-(4-y)} + e^{-y}$.

So $P(X+Y>4) = 1 - \int_{y=0}^{2}(e^{-y} - e^{-4+y})\,dy$

$$= 1 - \left[-e^{-y} - e^{-4}e^{y}\Big|_0^2\right] = 1 + (e^{-2} - e^{-4}e^{2}) - (e^{0} - e^{-4}e^{0})$$
$$= 1 + 2e^{-2} - 1 - e^{-4} = 2e^{-2} - e^{-4} = 0.2524.$$

(b) $P(X+Y>4\,|\,Y=2) = P(X+2>4\,|\,Y=2) = P(X>2\,|\,Y=2) = 1.$

(This is because Y = 2 implies that X > 2 (see figure in (a)) and $f(x|2)$ in (d).)

(c) $$E(Ye^{-X}) = \iint ye^{-x}f(x,y)dxdy = \int_{x=0}^{\infty} e^{-2x}\left(\int_{y=0}^{x} ydy\right)dx = \int_{x=0}^{\infty} e^{-2x}\frac{1}{2}x^2dx$$
$$= \frac{(1/2)^3\Gamma(3)}{2}\int_{x=0}^{\infty}\frac{x^{3-1}e^{-x/(1/2)}}{(1/2)^3\Gamma(3)}dx$$
$$= \frac{(1/2)^3\Gamma(3)}{2} = \frac{(1/8)2}{2} = \frac{1}{8}.$$

(d) $$f(y) = \int_{y}^{\infty} e^{-x}dx = \left[-e^{-x}\Big|_y^{\infty}\right] = -e^{-\infty} + e^{-y} = e^{-y},\ y>0.$$

So $f(x\,|\,y) = \dfrac{f(x,y)}{f(y)} = \dfrac{e^{-x}}{e^{-y}} = e^{-(x-y)},\ x > y$.

($(X|Y = y)$ has the standard exponential distribution shifted to the right by y.
We could also write $(X\,|\,Y = y) \sim \text{Expo}(1) + y$, or $(X - y\,|\,Y = y) \sim \text{Expo}(1)$.)

In particular, $f(x\,|\,2) = e^{-(x-2)},\ x > 2$.

So $E(Ye^{-X}\,|\,Y = 2) = \int_2^\infty 2e^{-x}e^{-(x-2)}dx = e^2\int_2^\infty 2e^{-2x}dx = e^2e^{-2(2)} = e^{-2} = 0.1353$.

Problem 4

(a) Observe that: $X \sim Bin(2,1/2)$
$(Y \mid X = x) \sim Bin(x,1/2)$.

Therefore:

$P(X = 0, Y = 0) = P(X = 0)P(Y = 0 \mid X = 0) = (1/4)(1) = 4/16$
$P(X = 1, Y = 0) = P(X = 1)P(Y = 0 \mid X = 1) = (1/2)(1/2) = 4/16$
$P(X = 1, Y = 1) = P(X = 1)P(Y = 1 \mid X = 1) = (1/2)(1/2) = 4/16$
$P(X = 2, Y = 0) = P(X = 2)P(Y = 0 \mid X = 2) = (1/4)(1/4) = 1/16$
$P(X = 2, Y = 1) = P(X = 2)P(Y = 1 \mid X = 2) = (1/4)(1/2) = 2/16$
$P(X = 2, Y = 2) = P(X = 2)P(Y = 2 \mid X = 2) = (1/4)(1/4) = 1/16$.

Table of $p(x,y)$:

		y		
		0	1	2
	0	4/16	0	0
x	1	4/16	4/16	0
	2	1/16	2/16	1/16
$p(y)$	→	9/16	6/16	1/16

$p(x,y) = p(x)p(y\,|\,x) = \binom{2}{x}\dfrac{1}{2^2} \times \binom{x}{y}\dfrac{1}{2^x} = \binom{2}{x}\binom{x}{y}\dfrac{1}{2^{2+x}}$; $x = 0,1,2$; $y = 0,...,x$.

(b) $EY = 0(9/16) + 1(6/16) + 2(1/16) = 1/2$.
$EY^2 = 0^2(9/16) + 1^2(6/16) + 2^2(1/16) = 5/8$.
$VarY = (5/8) - (1/2)^2 = 3/8$.

Alternative working

Observe that: $EX = 2(1/2) = 1$

$E(Y \mid X = x) = x(1/2)$, or equivalently, $E(Y \mid x) = x/2$

$E(Y \mid X) = X/2$.

So by the law of iterated expectation,

$$EY = EE(Y \mid X) = E(X/2) = (1/2)EX = (1/2)(1) = 1/2.$$

Similarly: $VarX = 2(1/2)(1 - 1/2) = 1/2$

$Var(Y \mid X = x) = x(1/2)(1 - 1/2) = x/4$

$Var(Y \mid X) = X/4$.

Hence
$$\begin{aligned} VarY = EVar(Y \mid X) + VarE(Y \mid X) &= E(X/4) + Var(X/2) \\ &= (1/4)EX + (1/4)VarX \\ &= (1/4)(1) + (1/4)(1/2) \\ &= 3/8. \end{aligned}$$

(c) $E(XY) = 1(1)(4/16) + 2(1)(2/16) + 2(2)(1/16) = 3/4$.

So $Cov(X,Y) = E(XY) - (EX)EY = 3/4 - 1(1/2) = 1/4$.

Hence $\rho = \dfrac{Cov(X,Y)}{SD(X)SD(Y)} = \dfrac{1/4}{\sqrt{1/2}\sqrt{3/8}} = \dfrac{1}{\sqrt{3}} = 0.5774$.

Alternative working

$$\begin{aligned} E(XY) = EE(XY \mid X) &= E\{XE(Y \mid X)\} \\ &= E\{X(X/2)\} \\ &= (1/2)\{VarX + (EX)^2\} \\ &= (1/2)\{(1/2) + 1^2\} = 3/4. \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= ECov(X,Y \mid X) + Cov\{E(X \mid X), E(Y \mid X)\} \\ &= E0 + Cov(X, X/2) \\ &= 0 + (1/2)VarX \\ &= (1/2)(1/2) = 1/4. \end{aligned}$$

(d) In this case $p(x,y)=\binom{20}{x}\frac{1}{2^{20}}\times\binom{x}{y}\frac{1}{2^x}$; $x = 0,...,20$; $y = 0,...,x$.

So $$P(Y=0)=\sum_x p(x,0)=\frac{1}{2^{20}}\sum_{x=0}^{20}\binom{20}{x}\binom{x}{0}\frac{1}{2^x}$$
$$=\frac{1}{2^{20}}\sum_{x=0}^{20}\binom{20}{x}\left(\frac{1}{2}\right)^x 1^{20-x}$$
$$=\frac{1}{2^{20}}\left(\frac{1}{2}+1\right)^{20} \quad \text{by the binomial theorem}$$
$$=\left(\frac{3}{4}\right)^{20} = 0.00317.$$

Alternative working

Observe that $P(Y=0\,|\,X=x)=(1/2)^x=2^{-x}$. Therefore $P(Y=0\,|\,X)=2^{-X}$.
It follows that $P(Y=0)=EP(Y=0\,|\,X)=E2^{-X}=Ee^{-X\log 2}=m_X(-\log 2)$
$$=\left(1-\frac{1}{2}+\frac{1}{2}e^{-\log 2}\right)^{20}=\left(1-\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\right)^{20}=\left(\frac{3}{4}\right)^{20}.$$

We have here used the fact that for a *Binomial*(n, p) random variable R, the moment generating function is given by $m_R(t)=Ee^{Rt}=\left(1-p+pe^t\right)^n$.

We have also used the fact that, for any event A and any random variable W,
$$P(A)=EP(A|W)=\begin{Bmatrix}\int P(A|W=w)f(w)dw & \text{if } W \text{ is continuous}\\ \sum_x P(A|W=w)f(w) & \text{if } W \text{ is discrete}\end{Bmatrix} \qquad (*)$$

This follows directly from the *law of iterated expectation*, $EZ=EE(Z\,|\,W)$, after substituting $Z=I(A)=\begin{Bmatrix}1, & \text{if } A\\ 0, & \text{if } \bar{A}\end{Bmatrix}$ (the indicator variable for event A).

Proof of (*): $EZ=\sum_{z=0}^{1}zP(Z=z)=0P(Z=0)+1P(Z=1)=P(Z=1)=P(A)$

and $E(Z\,|\,W=w)=\sum_{z=0}^{1}zP(Z=z\,|\,W=w)$
$$=0P(Z=0\,|\,W=w)+1P(Z=1\,|\,W=w)=P(Z=1\,|\,W=w)=P(A|W=w)$$
so that $E(Z\,|\,W)=P(A|W)$ and therefore $P(A)=EP(A|W)$, as required.