THE FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL-MAY 2012

MAT246H1S
Concepts in Abstract Mathematics

Examiners: V. Kapovitch and J.Korman
Duration: 3 hours

NO AIDS ALLOWED. Total: 100 marks

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Write your answer in the space provided. Use the back sides of the pages for scrap work DO NOT tear any pages from this test.

FOR MARKER'S USE ONLY	
Problem 1:	/10
Problem 2:	/10
Problem 3:	/10
Problem 4:	/10
Problem 5:	/10
Problem 6:	/10
Problem 7:	/10
Problem 8:	/10
Problem 9:	/10
Problem 10:	/10
TOTAL:	/100

1. (10 pts) The Fibonacci sequence is the sequence of numbers $F(1), F(2), \ldots$ defined by the following recurrence relations:
   $F(1) = 1, F(2) = 1,\ F(n) = F(n-1) + F(n-2)$ for all $n > 2$.
   For example, the first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \ldots$

   (a) Prove by induction that for any $n \geq 1$ the consequtive Fibonacci numbers $F(n)$ and $F(n+1)$ are relatively prime.

   (b) Prove by induction that for any $n \geq 1$ the following identity holds

$$F(2) + F(4) + \ldots F(2n) = F(2n+1) - 1$$

2. (10 pts)

   (a) Find the remainder when $7^{3^{100}}$ is divided by 20.

   (b) Find $2^{p!} (\mod p)$ where p is an odd prime.

3. (10 pts) Prove that $q_1\sqrt{2} + q_2\sqrt{6}$ is irrational for any rational q_1, q_2 unless $q_1 = q_2 = 0$.

4. (10 pts) Suppose $(\phi(m), m) = 1$. Here m is a natural number and ϕ is the Euler function. Prove that $\sqrt{m}$ is irrational.

5. (10 pts) Let $p = 11, q = 5$ and $E = 11$. Let $N = 11 \cdot 5 = 55$. The receiver broadcasts the numbers $N = 15, E = 23$. The sender sends a secret message M to the receiver using RSA encryption. What is sent is the number $R = 2$.

   Decode the original message M.

6. (10 pts)

   (a) Find all complex roots of the equation

$$z^6 + (1-i)z^3 - i = 0$$

   (b) Express as $a + bi$ for some real a, b:

$$\frac{6^{100}}{(3+\sqrt{3}i)^{103}}$$

7. (10 pts) A complex number is called *algebraic* if it is a root of a polynomial with integer coefficients. Prove that the set of algebraic numbers is countable.

8. (10 pts) Suppose $0 < \alpha < \pi/2$ satisfies $\cos\alpha = \frac{2}{3}$. Prove that the angle α can not be trisected with a ruler and a compass.

9. (10 pts) Let S be that set of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

   Prove that $|S| > |\mathbb{R}|$.

10. (10 pts) For each of the following answer "true" or "false". Justify your answer.

a) $\sqrt{\frac{\sqrt{5}}{\sqrt[3]{2}+\sqrt{11}}}$ is constructible.

b) If x is not constructible then $\sqrt{x}$ is also not constructible.

c) If x is constructible then $\sqrt[8]{x}$ is also constructible.