MATH6222: Homework #4

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Problem 1

Let $f : A \to B$ and let $g : B \to C$ be functions, and let $h = g \circ f$. Determine which of the following statements are true. Give proofs of the true statements and counterexamples for the false statements.
(a) If h is injective, then f is injective.
(b) If h is injective, then g is injective.
(c) If h is surjective, then f is surjective.
(d) If h is surjective, then g is surjective.

Proof:

Consider h is injective, for any $a_i \neq a_j \in A, h(a_i) \neq h(a_j)$

(a) **TRUE**. **Proof by contradiction:** Suppose f is not injective, then $\exists\ a_i \neq a_j$ but $f(a_i) = f(a_j)$, then

$$h(a_i) = g(f(a_i)) = g(f(a_j)) = h(a_j).$$

which contradicts the fact that h is injective.
So statement (a) is true. ■

(b) **FALSE**. **Counterexample:** Suppose $A = \{a\}, B = \{b_1, b_2\}, C = \{c\}$, then $f(a) = b_1$, but $g(b_1) = g(b_2) = c$. Clearly, f, h are injective, but g is not injective as $b_1 \neq b_2$.
So statement (b) is false.

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Consider h is surjective now, $\forall c \in C, \exists\ a \in A$ such that $h(a) = c$.

(c) **FALSE**. **Counterexample:** Suppose $A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c\}$, then $f(a_1) = f(a_2) = b_1, g(b_1) = g(b_2) = c$. Then $h(a_1) = h(a_2) = c$, h is surjective. But f is not surjective, since there is not an $a \in A$ such that $f(a) = b_2$.
So statement (c) is false.

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(d) **TRUE**. Suppose $a \in A, h(a) \in C$, then $h(a) = g(f(a))$. Since h is surjective, the image of h covers everything in C. Besides, the image of h is a subset of the image of g, so the image of g contains everything in C as well, i.e. g is surjective.

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Problem 3

Recall that $[n] = \{1, 2, \ldots, n\}$. Let A denote set of subsets of $[n]$ with an even number of elements, and let B denote the set of subsets of $[n]$ with an odd number of elements. Prove that $|A| = |B|$ by constructing an explicit bijection from A to B.

Proof:

First we have to claim that $n \neq 0$ since if $[0] = \{\emptyset\}$ then $A = \{\emptyset\}, B = \emptyset$, the statement is just trivially false.

Then, $\forall n \in \mathbb{N}, n \geq 1$, we construct the following interesting function: $\forall a \in A$,

$$f(a) = \begin{cases} f(a)\backslash\{1\} & \text{when } 1 \in a, \\ f(a) \cup \{1\} & \text{when } 1 \notin a. \end{cases}$$

This means, if a is an element of A, i.e. a is a subset of $[n]$ with even number of elements, then we can find a way to map it to a unique element of B (with odd number of elements).

The easiest way is just add/remove an element to/from a, so that the total number of elements becomes odd. And we can set up a criteria for this:

- If a has a certain element, then we remove it from a.
- If a does not have such certain element, then we append it to a.

Actually, this certain element could be any element in the powerset of $[n]$. For simplicity, we such select $\{1\}$.

Now we claim $f : A \to B$ is a bijection.

- Suppose $x \neq y, x, y \in A$.
  - If only one of x, y contains $\{1\}$. WLOG, say $\{1\} \subseteq x$, then $\{1\} \not\subseteq f(x), \{1\} \subseteq f(y)$. Therefore, $f(x) \neq f(y)$, f is injective in this case.
  - If x, y both contain $\{1\}$, then neither of $f(x), f(y)$ has $\{1\}$, but still the rest part $f(x) - \{1\} \neq f(y) - \{1\}$. f is injective in this case as well.
  - Similarly, if x, y both don't have $\{1\}$, then both $f(x), f(y)$ do have $\{1\}$, but stil $f(x) \cup \{1\} \neq f(y) \cup \{1\}$. f is injective.
- $\forall\ b \in B$, we can always go backward and find an a such that $f(a) = b$. The idea is intuitive: just check if b contains $\{1\}$. In this way, f is surjective.

Hence, $f : A \to B$ is a bijection.

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Problem 4

Construct explicit bijections: $f : (0, 1) \to [0, 1)$ and $g : (0, 1) \to [0, 1]$.

Solution:

Consider the function:

$$f(x) = \begin{cases} 0, & x = \frac{1}{2}; \\ \frac{1}{x^{-1}-1}, & x = \frac{1}{n} \text{ for n=3,4,5,\ldots;} \\ x, & \text{otherwise.} \end{cases}$$

Again, consider the function:

$$g(x) = \begin{cases} 0, & x = \frac{1}{2}; \\ 1, & x = \frac{1}{3}; \\ \frac{1}{x^{-1}-2}, & x = \frac{1}{n} \text{ for n=4,5,6,\ldots;} \\ x, & \text{otherwise.} \end{cases}$$

The idea of constructing these two bijections is to use some fixed point value in the domain to map to the boundary value in the image (in f, we use $f(\frac{1}{2}) = 0$; in g, we use $g(\frac{1}{2}) = 0, g(\frac{1}{3}) = 1$. Then the values of $f(\frac{1}{n})$ and $g(\frac{1}{n})$ are just shifted to $\frac{1}{n-1}$ and $\frac{1}{n-2}$. And the rest part of the function remains as $f(x) = x$ (or $g(x) = x$).

f and g are obviously injective, since not a number in the image was hit twice.

They are also surjective, since every number in the image is covered.

(*Recall:* These two functions are just the variations of an example in week 4 tutorial.)

Problem 5

Let L be the set of all sentences of the English language. Prove that L is countable. (For the purpose of this exercise, a sentence of the English language is any finite sequence of characters chosen from the set of characters visible on your computer's keyboard.)

Proof:

Consider $A = \{\text{All characters visible on the computer's keyboard}\}$, which is trivially countable.

Now we claim the set of all finite sequences of elements of A, which is the set of all sentences of the English language i.e. L, is also countable.

Suppose $\forall n \in \mathbb{N}, L_n = \{\text{All sequences of length } n \text{ of elements of } A\}$, we can prove this by induction.

Base step: When $n = 0$, $L_0 = \{\emptyset\}$. Obviously, L_0 is countable.

Inductive step: Suppose $k \in \mathbb{N}^+$ and L_k is countable, we want to show

$$L_{k+1} = \{\text{All sequences of length } k+1 \text{ of elements of } A\}$$

is also countable.

Consider the function $F : L_k \times A \to L_{k+1}$ as

$$F(f, a) = f \cup \{a\},$$

where $f \in L_k$ is a sequence of k elements of A, and $a \in A$ is an element of A. In fact, $F(f, a)$ is a sequence of length $k+1$ starting with sequence f and end with a as its $(k+1)$th term in the sequence.

And such F is a bijection, because

- If $(f_1, a_1) \neq (f_2, a_2)$, then $f_1 \cup \{a_1\} \neq f_2 \cup \{a_2\}$. F is injective.
- For all element in L_{k+1}, it can be decomposed into two parts: a sentence of length k as f, which is in L_k, and an element of A as a. F is surjective.

Therefore, the following two sets should have the same cardinality as:

Problem 5 continued on next page. . .

$$|L_k \times A| = |L_{k+1}|$$

By theorem that **the cartesian product of 2 countable sets is also countable,** here both L_k and A are countable, so L_{k+1} should his countable too.

Therefore, for any finite length n, L_n is a countable set. Hence L is countable.

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