MATH6222 week 4 lecture 10

Rui Qiu

2017-03-16

If A is a finite set, let $|A|$ denotes the number of elements in A.
Claim that

$$\begin{aligned}
|A| = |B| &\iff \exists \text{ a bijection } f : A \to B. \\
|A| \leq |B| &\iff \exists \text{ a injection } f : A \to B. \\
|A| \geq |B| &\iff \exists \text{ a surjection } f : A \to B.
\end{aligned}$$

Proof of the first:

Suppose $|A| = |B|$, then $A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_n\}$.
Then define $f : A \to B$, i.e. $f(a_i) = b_i$. Clearly f is a bijection (which maps every a_i to b_i).
On the other hand, if $f : A \to B$ is a bijection.
Then I claim A and B have the same number of elements.
Let's write $A = \{a_1, \dots, a_n\}$.
Then $B = \{b_1, \dots, b_n\}$.

What does it mean to say A *has n elements*?
It means we can write A as $\{a_1, a_2, \dots, a_n\}$

Given 2 arbitrary sets A, B, we'll say that A and B have the same cardinality (yeah i'm fancy) and write $|A| = |B|$ if $\exists$ bijection $f : A \to B$.

Proposition:

1. For any set $A, |A| = |A|$. (id. $A \to A$)
2. For any set $A, B, |A| = |B| \iff |B| = |A|$. ($f : A \to B, f^{-1} : B \to A$)
3. For any set A, B, C, if $|A| = |B|, |B| = |C| \implies |A| = |C|$. ($f : A \to B, g : B \to C, g \circ f : A \to C$)

Remark:

Given $f : A \to B, g : B \to C$, we can define

$$g \circ f : A \to C$$

Proposition: If $f : A \to B$ and $g : B \to C$ are injective/surjective, then $g \circ f$ is injective/surjective.

Proof:
$g \circ f : A \to C$ Given $a_1, a_2 \in A$ with $a_1 \neq a_2$, must show $(g \circ f)(a_1) \neq (g \circ f)(a_2)$.

Since f injective, $f(a_1) \neq f(a_2)$ (as elements of B).

Since g injective, $g(f(a_1)) \neq g(f(a_2))$.

For surjections...

Example:
$\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{N} \setminus \{1\} = \{2, 3, 4, \dots\}$
$|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$

Next example:
$\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{E} = \{2, 4, 6, \dots\}$
$\exists$ a bijection $\mathbb{N} \to \mathbb{E}$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
Example:
$\exists$ bijection $\mathbb{N} \to \mathbb{Z}, n \to \begin{cases} \frac{n}{2}, & \text{even} \\ -\frac{n-1}{2}, & \text{odd} \end{cases}$

Definition: We say a set A is **countably infinite** if $|A| = |\mathbb{N}|$.

$\mathbb{Q}^+ = \{\frac{a}{b} : a, b \in \mathbb{N}$ st. a and b have no common factors

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \dots$$
$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \dots$$
$$\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \dots$$
$$\frac{1}{4}, \frac{3}{4}, \dots$$

Then ... zig-zag...

What we really need to show here is to show $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
Think about this: $(i, j) \to 2^{i-1}(2j - 1)$

$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$
do $\{0, \frac{1}{1}, -\frac{1}{1}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{2}, -\frac{1}{2}, \dots\}$