NAME (PRINT): ______________________________
Last/Surname First /Given Name

STUDENT #: ____________________ **SIGNATURE:** ____________________

UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2009 FINAL EXAMINATION
STA257H5F
Probability and Statistics I
Alison Weir
Duration - 3 hours
Aids: Statistical Calculators; Statistical Formulas Table

The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of an exam, including but not limited to any electronic devices with storage, such as cell phones, pagers, personal digital assistants (PDAs), iPods, and MP3 players. Unauthorized calculators and notes are also not permitted. Do not have any of these items in your possession in the area of your desk. Please turn the electronics off and put all unauthorized aids with your belongings at the front of the room before the examination begins. If any of these items are kept with you during the writing of your exam, you may be charged with an academic offence. A typical penalty may cause you to fail the course.

Please note, you ***CANNOT*** *petition to* ***RE-WRITE*** *an examination once you have begun writing.*

1. (17 marks) The length of time required by students to complete a one-hour test is a random variable with density function

$$f(x) = \begin{cases} kx^2 + x & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. (3 marks) Find the constant k.

b. (3 marks) Find F(x)

c. (3 marks) What is the probability a randomly selected student will finish in less than half an hour?

d. (1 mark) What is the probability a randomly selected student will finish in exactly half an hour?

e. (4 marks) Given that a particular student needs at least 15 minutes to complete the test, what is the probability she needs at least 30 minutes to finish the test?

f. (3 marks) What is the mean length of time it takes to complete the test?

2. (17 marks) Suppose X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 6(1-y) & 0 \le x \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. (2 marks) Sketch the region of positive density.

b. (4 marks) Find $P(X + Y > 1/2)$

c. (3 marks) Find the marginal distribution of Y.

d. (3 marks) Find the conditional distribution of X given $Y = 1/2$

e. (4 marks) What is the covariance of X and Y?

f. (1 mark) Are X and Y independent? Justify your answer.

3. (4 marks) A, B, and C are events such that $P(A) > P(B)$ and $P(C) > 0$. Construct an example that shows it possible to have $P(A|C) < P(B|C)$.

4. (4 marks) A and B are independent events with $P(A) = 0.5$ and $P(B) = 0.2$. What is $P(A^c \cup B^c)$?

5. (4 marks) If A and B are events and B $\subset$ A, why is it "obvious" that $P(B) \le P(A)$?

6. (4 marks) Explain why $F(x) = \begin{cases} 0 & x \le 0 \\ 1 & \cancel{x < 0 < 1} \\ 1 & x \ge 1 \end{cases}$ is not a valid cumulative distribution function.

$0 < x < 1$

7. (6 marks) The moment generating function of random variable X is $m(t) = (1/6)e^{t} + (1/3)e^{2t} + (1/2)e^{3t}$.

   a. (3 marks) Find E(X)

   b. (3 marks) If Y = 2-X, what is the moment generating function of Y?

8. (7 marks) The number of a certain type of bacteria colonies in samples of polluted water has a Poisson distribution with a mean of 2 per cubic centimetre.

   a. (3 marks) If four cm^3 samples are independently selected from the water, what is the probability that at least one sample contains one or more bacteria colonies?

   b. (4 marks) How many cm^3 samples should be selected in order to have probability of approximately 0.95 of seeing at least one bacteria colony?

9. (11 marks) The telephone lines serving an airline reservation office are all busy 60% of the time.

   a. (3 marks) If you're calling the office what is the probability you complete your call on the third try?

   b. (3 marks) You and four friends must all complete calls to the office. What is the probability that a total of 20 calls are necessary for all five calls to get through?

   c. (3 marks) You call the office a total of 20 times. What is the probability you get through on five of your calls?

   d. (2 marks) How many calls do you expect will get through if the office is called 20 times?

10. (7 marks) A student takes a multiple choice test where each question has four possible answers. Suppose the probability a person knows the answer to a question is 0.8 and the probability the student guesses is 0.2. If the student guesses, the probability they select the correct answer is 0.25. If the student correctly answers a question, what is the probability the student really knew the correct answer?

11. (9 marks) Three beer drinkers (say I, II, and III) are asked to rank four brands of beer (say A, B, C, and D) in a blindfold taste-test. Each drinker ranks one of the four beers as 1 (for the beer they liked the best), and another of the four beers as 2 (for the beer they liked second best), 3, or 4.

   a. (3 marks) Describe the sample space. Note that we need to specify the ranking of all four beers for all three drinkers.

   b. (2 marks) How many points are in the sample space?

   c. (4 marks) Suppose the drinkers can't taste any differences between the four brands of beer, and that the drinkers assign the ranks at random. After all beers have been ranked by all drinkers, the ranks are summed. What is the probability that at least one beer receives total rank of 4 or less?

Total Marks 90

Distribution	pmf	Support	Mean	Variance	mgf	Θ
Bernoulli	$f(x)=\theta^x(1-\theta)^{1-x}$	$x=0,1$	θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$	$0\leq\theta\leq 1$
Binomial	$f(x)=\binom{n}{x}\theta^x(1-\theta)^{n-x}$	$x=0,1,2,\ldots,n$	$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$	$0\leq\theta\leq 1$
Hypergeometric	$f(x)=\frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}}$	$x=0,1,2,\ldots,n$	$\frac{nK}{M}$	$n\frac{K}{M}\frac{M-K}{M}\frac{M-n}{M-1}$	Not Useful	$M=1,2,\ldots$ $K=0,1,2,\ldots,M$ $n=1,2,\ldots,M$
Poisson	$f(x)=\frac{e^{-\lambda}\lambda^x}{x!}$	$x=0,1,2,\ldots$	λ	λ	$exp(\lambda(e^t-1)$	$0<\lambda<\infty$
Geometric	$f(x)=\theta(1-\theta)^x$	$x=1,2,3\ldots$	$\frac{1}{\theta}$	$\frac{1-\theta}{\theta^2}$	$\frac{\theta e^t}{1-(1-\theta)e^t}$	$0<\theta\leq 1$
Negative Binomial	$f(x)=\binom{x-1}{k-1}\theta^k(1-\theta)^{x-k}$	$x=k,k+1,k+2,\ldots$	$\frac{k}{\theta}$	$\frac{k(1-\theta)}{\theta^2}$	$\left(\frac{\theta e^t}{1-(1-\theta)e^t}\right)^k$	$0<\theta\leq 1$
Discrete Uniform	$f(x)=\frac{1}{n}$	$x=x_1,x_2,\ldots,x_n$	$\frac{1}{n}\sum x_i$	$\frac{1}{n}\sum(x_i-\mu)^2$	$\frac{1}{n}\sum e^{tx_i}$	Not Applicable