BDA Ch 3

Exercise 6: Non informative prior distribution.

$p\left(\log\left(\frac{\theta}{1-\theta}\right)\right) \propto 1$

(proper prior on logit(θ))

find $p(\theta)$. → use Method of Transformation.

Define $X = \log\left(\frac{\theta}{1-\theta}\right)$ $p(x) \propto 1$.

$= \log\theta - \log(1-\theta)$

$\theta = \frac{e^x}{1+e^x} = g(x)$ $\qquad$ $\frac{dx}{d\theta} = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)}$.

$\therefore \quad p(\theta) = P_x(g^{-1}(\theta)) \times \left|\frac{dx}{d\theta}\right|$

$= 1 \times \theta^{-1}(1-\theta^{-1}) \sim Beta(0,0)$

$\Rightarrow$ improper prior on θ.

R packages implement bayesian models have non informative priors as the default setting.

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Exercise 7 Show that Jeffreys prior is invariant to parametrization.

Let $\phi = h(\theta)$.

Suppose $p(\theta) = J(\theta) = -E\left[\frac{d^2 \log p(y|\theta)}{d\theta^2} \middle| \theta\right]$

(that is, assume Jeffrey's prior)

What is $J(\phi) = ?$

$$J(\phi) = -E\left[\frac{d^2 \log p(y|\phi)}{d\phi^2}\right] \quad \text{(by definition)}$$

$$= -E\left[\frac{d^2 \log p(y|\theta = h^{-1}(\phi))}{d\theta^2} \left|\frac{d\theta}{d\phi}\right|^2\right]$$

$$= J(\theta)\left|\frac{d\theta}{d\phi}\right|^2. \quad \text{(method of transformation)}$$

$$J(\phi)^{1/2} = J(\theta)^{1/2}\left|\frac{d\theta}{d\phi}\right|$$

(so. Jeffrey's prior satisfies the invariance principle!!).

Exercise 8 Jeffrey's prior for the binomial model.

$$Y|\theta \sim Bin(n, \theta)$$

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$\log p(y|\theta) = \log \binom{n}{y} + y \log\theta + (n-y)\log(1-\theta).$$

$$\frac{\partial \log p(y|\theta)}{\partial\theta} = y/\theta - \frac{n-y}{1-\theta}.$$

$$\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2} = -y/\theta^2 - \frac{(n-y)}{(1-\theta)^2}.$$

$$-E\left[\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2}\right] = \frac{n\theta}{\theta^2} + \frac{(\cancel{1}-\theta)n}{(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}$$

$$J(\theta) = \frac{n}{\theta(1-\theta)}.$$

$$\underbrace{J(\theta)^{1/2} \propto \theta^{-1/2}(1-\theta)^{-1/2}}_{\text{Jeffreys prior.}} = Beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Exercise 8.

Now suppose $\phi = \log\left(\frac{\theta}{1-\theta}\right) \Rightarrow \theta = \frac{e^{\phi}}{1+e^{\phi}}$

Find Jeffreys prior $p(\phi)$.

$$p(y|\phi) = \binom{n}{y}\left(\frac{e^{\phi}}{1+e^{\phi}}\right)^{y}\left(\frac{1}{1+e^{\phi}}\right)^{n-y}.$$

$$= \binom{n}{y} e^{\phi y}(1+e^{\phi})^{-n}.$$

$$\log p(y|\phi) = \phi y - n\log(1+e^{\phi})$$

$$\frac{\partial \log p(y|\phi)}{\partial \phi} = y - \frac{ne^{\phi}}{(1+e^{\phi})}$$

$$\frac{\partial^2 \log p(y|\phi)}{\partial \phi^2} = \frac{-ne^{\phi}}{(1+e^{\phi})^2}.$$

$$J(\phi) = -E\left[\frac{\partial^2 \log p(y|\phi)}{\partial \phi^2}\right] = \frac{ne^{\phi}}{(1+e^{\phi})^2}.$$

$$J(\phi) \propto \frac{e^{\phi/2}}{(1+e^{\phi})}$$

Exercise 8

Alternatively, take ~~from~~ $p_J(\theta)$ and apply method of transformations to find $p_{\phi}(\phi)$.

$$p_J(\phi) = p_J(h^{-1}(\phi)) \times \left|\frac{d\theta}{d\phi}\right| \qquad \left\{ \begin{array}{l} \phi = \log\frac{\theta}{1-\theta} \\ \quad = h(\theta) \\ h^{-1}(\phi) = \frac{e^{\phi}}{1+e^{\phi}} \end{array} \right\}$$

$$= \left(\frac{e^{\phi}}{1+e^{\phi}}\right)^{-1/2} \left(\frac{1}{1+e^{\phi}}\right)^{-1/2} \times \left|\frac{d\theta}{d\phi}\right|$$

$$= e^{-\phi/2} (1+e^{\phi})^{+1} \frac{e^{\phi}}{(1+e^{\phi})^2}$$

$$= \frac{e^{\phi/2}}{(1+e^{\phi})}.$$

→ same as derived by directly working from $p(y|\phi)$.

This consistency under reparametrization is the defining characteristic of Jeffrey's prior.