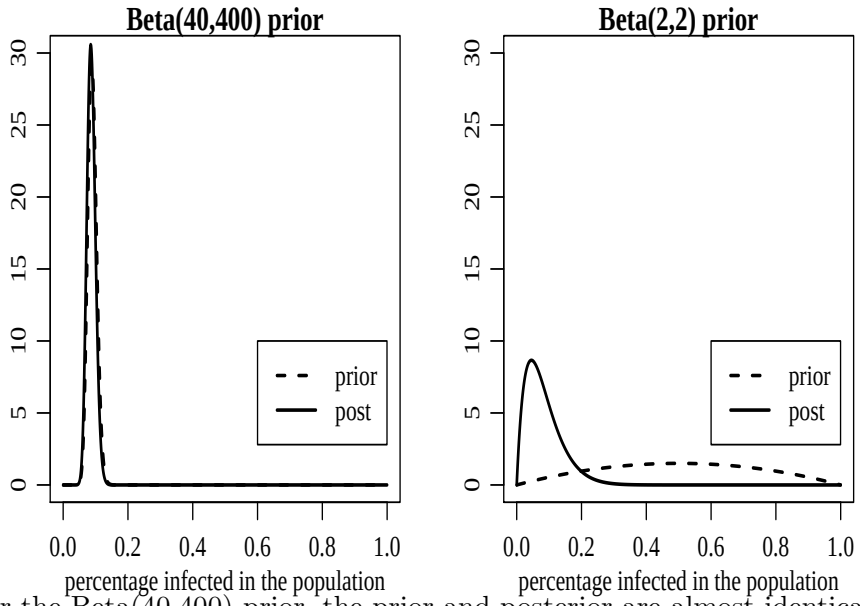
Introduction to Bayesian Data Analysis

Tutorial 1 - Solutions

[see `"Tut1.R"` for R code].

(1)



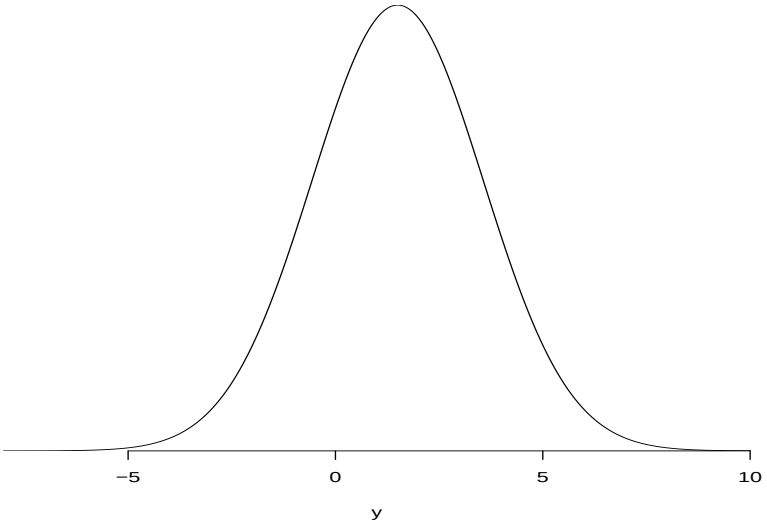
Under the Beta(40,400) prior, the prior and posterior are almost identical, with the mode at approximately 10%. This reflects the strong prior belief that θ=40/(40+400). In contrast, we have collected data on only n=20 units, and this sample size is small in comparison to the amount of prior information, hence the prior dominates the posterior.

Under the Beta(2,2) prior, we see the prior distribution is quite flat. We have some prior information that the infection rate is around 50% but we are quite uncertain about this guess (a weak prior). The data provides us with a lot more information, and we note the considerable change in our beliefs looking at the posterior distribution, which puts approximately zero probability on the infection rate being 50%, and the posterior mode is much less at $\approx$ 5%. Here, the likelihood dominates the posterior.

(2) Conditional probability: suppose that if $\theta = 1$, the y has a normal distribution with mean 1 and standard deviation σ, and if $\theta = 2$, then y has a normal distribution with mean 2 and standard deviation σ. Also, suppose Pr($\theta = 1$)=0.5 and Pr($\theta = 2$)=0.5

(a)

$$\begin{aligned} p(y|\sigma^2 = 2^2) &= \sum_\theta p(y, \theta|\sigma^2 = 2^2) \\ &= \sum_\theta p(y|\theta, \sigma^2 = 2^2)p(\theta) \\ &= Pr(\theta = 1)p(y|\theta = 1, \sigma^2 = 2^2) + Pr(\theta = 2)p(y|\theta = 2, \sigma^2 = 2^2) \\ &= 0.5N(y|\theta = 1, \sigma^2 = 2^2) + 0.5N(y|\theta = 2, \sigma^2 = 2^2) \end{aligned}$$

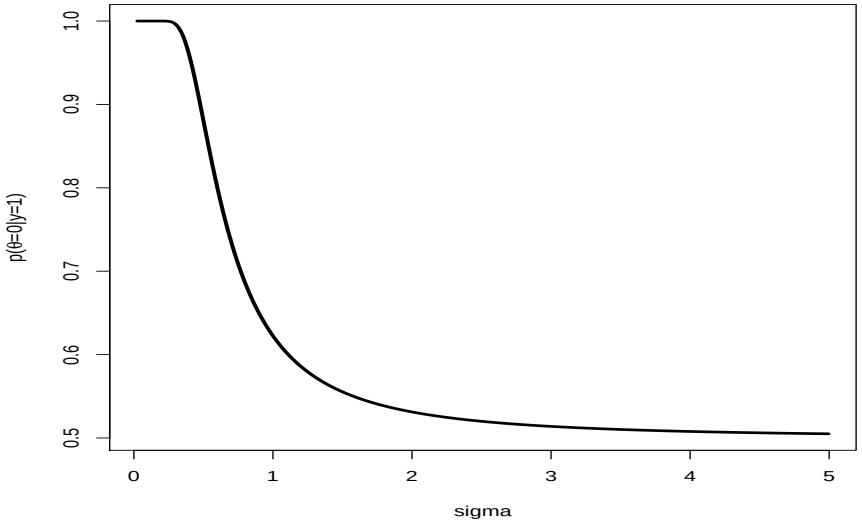


(b)

$$\begin{aligned} p(\theta = 1|y = 1) &= \frac{p(y = 1, \theta = 1)}{p(y = 1)} \\ &= \frac{0.5N(y = 1|\theta = 1, \sigma^2 = 2^2)}{0.5N(y|\theta = 1, \sigma^2 = 2^2) + 0.5N(y|\theta = 2, \sigma^2 = 2^2)} \\ &= 0.5312094 \end{aligned}$$

Note that the posterior probability that $\theta = 1$ is slightly higher than 0.5, given that we observed $y = 1$.

(c) The posterior density $Pr(\theta = 1|y = 1)$ decreases towards the prior probability ($Pr(\theta = 1) = 0.5$) as σ increases. That is, as the uncertainty in the observed data increases, this reduces the evidence to support the value $\theta = 1$ even though we observed $y = 1$, because we are assuming more variability in the sampling model.



(3) Index the boxes by 1,2, and 3. Without loss of generality, lets assume the contestant selects the Box labelled 1. The sample space consists of elements $\{XYZ, I\}$ where X=1 if Box 1 contains the fabulous prize and X=0 otherwise, and Y and Z are defined similarly for Box 2 and Box 3, and $I \neq 1$ denotes the box that Monty opens. We assume that Monty does not discriminate between the boxes with the two smaller prizes. Also suppose that Monty opens the Box labelled 2. Enumerate the following conditional probabilities:

$$Pr(X = 1|I = 2) = \frac{Pr(I = 2|X = 1)Pr(X = 1)}{Pr(I = 2)} = \frac{1/2 \times 1/3}{1/2} = 1/3$$

$$Pr(Y = 1|I = 2) = \frac{Pr(I = 2|Y = 1)Pr(Y = 1)}{Pr(I = 2)} = \frac{0 \times 1/3}{1/2} = 0$$

$$Pr(Z = 1|I = 2) = \frac{Pr(I = 2|Z = 1)Pr(Z = 1)}{Pr(I = 2)} = \frac{1 \times 1/3}{1/2} = 2/3$$

The conditional probability that the big prize is behind Box 1 (the box the contestant had originally chosen) is only 1/3. Therefore, the contestant should switch.

(nb: we assume that Monty does not choose a box which contains a smaller prize by chance but does know which box contains the big prize and purposely avoids opening this box always).

(4) (a) The posterior distribution is $g(p=k|y) = \frac{g(y|p=k)g(p=k)}{g(y)}$.
We have:

$$\begin{aligned} g(y) &= \sum_p g(y|p)g(p) \\ &= \binom{60}{55} 1^{55}0^5 \times 0.8 + \binom{60}{55} 0.975^{55}0.025^5 \times 0.1 + \binom{60}{55} 0.95^{55}0.05^5 \times 0.05 + \\ &\quad + \binom{60}{55} 0.925^{55}0.075^5 \times 0.035 + \binom{60}{55} 0.90^{55}0.10^5 \times 0.015 \end{aligned}$$

So $g(p=0|y) = \frac{\binom{60}{55}1^{55}0^5 \times 0.8}{g(y)} = 0$ and $g(p=0.10|y) = \frac{\binom{60}{55}0.9^{55}0.1^5 \times 0.015}{g(y)} = 0.165$.
That is, the posterior probability that all passengers show up is zero, and the posterior probability that 10% of passengers do not show up is 0.165.

(b) We need to sequentially update the posterior after the data from each flight. (Run a loop function in R). Each time, the prior is updated to be the posterior from the previous cycle, as in the following code:

```
g_p<-c(0.80, 0.10,0.05,0.035,0.015)
p<-c(0,0.025,0.05,0.075,0.10)

for (i in 1:10){
  g.post<-dbinom(y[i],n,1-p)*g_p
  g.post<-g.post/sum(g.post)
  g_p<-g.post #updated posterior becomes new prior
> g.post
[1] 0.000000e+00 1.261763e-07 1.407355e-01 8.445976e-01 1.466673e-02
}
```

The posterior mode is 7.5% of passengers do not show up. (note: check that your code gives you the same answer, regardless of the ordering of your data vector y. This shows that y is an *exchangeable* sequence of events).

(c) If the airline company maintains the same prior distribution $g(p)$, and bases their profit forecasts on this prior distribution, then the company would be overestimating the probability that all customers show up (p=0). The company

could potentially overbook a higher proportion of seats to allow for the higher than assumed no-show rate which would boost profits. In other words, based on current assumptions, the airline is under estimating its profits based on assumed seat occupancy rates. In a very competitive industry, obtaining an accurate estimate of the passenger no-show rate is important. The airline earns money on each ticket sold. If too few tickets are sold, then seats are wasted. If too many tickets are sold (that is, the flight is overbooked and passengers need to be 'bumped' off), additional costs are incurred to compensate the 'bumped' off passengers, plus there is the potential loss of airline brand loyalty. If past data is used to continuously update the probability of passengers showing up for particular flights allowing for more accurate predictions, such losses can be reduced.