May 28 th

Suppose S is bounded set. $a = \sup S$
if $x \leq a$ $\forall x \in S$ $\forall \varepsilon > 0$ $\exists x \in S$ s.t. $a - \varepsilon < x$

supremum
(always exist)
by completeness

$a = \max\{S\}$, $a \in S$, $\forall x \in S$, $x \leq a$

maximum
(may not exist)
if exists, must be the boundary point

P33. #6.
distance between two sets $U, V \subset \mathbb{R}^n$ is defined to be

$$d(U, V) = \inf\{|\vec{x} - \vec{y}| : \vec{x} \in U, \vec{y} \in V\}$$

e.g. $d(\{0\}, \{\frac{1}{n}\}) = 0$
$d((1,2), (3,4)) = 1$

(a). Show that $d(U, V) = 0$ if either of the sets U, V contains a point in the closure of the other one.
*(b). Show if U is compact, V is closed, $U \cap V = \emptyset$, then $d(U, V) > 0$.
(c). Give an eg. of two closed sets U & V in $\mathbb{R}^2$ that no point in common but satisfy $d(U, V) = 0$.

(a). Suppose $a \in \overline{U}$, $a \in V$ WLOG
We can choose $\{x_n\} \in U$, $|x_n - a| \to 0$
$0 \leq d(U, V) \leq |x_n - a| \to 0$
hence $d(U, V) = 0$.

∎

(b). Proof by contradiction.
Suppose not, in other word,
i.e. U compact, V closed, $U \cap V = \emptyset$ but $d(U, V) = 0$.

def of inf : $\forall \varepsilon > 0, \exists x \in S$, s.t. $\inf + \varepsilon > x$

Then $\forall \varepsilon > 0, \exists x \in U, y \in V$, s.t. $|x - y| < \varepsilon$

Take $\varepsilon = \frac{1}{n}$, find a sequence of points $\{x_n\}$ in U and $\{y_n\}$ in V s.t.
$|x_n - y_n| < \frac{1}{n}$

Since U is compact, $\exists$ a subsequence $\{x_{n_j}\}$ in x_n and $a \in U$ s.t. $|x_{n_j} - a| \to 0$

$$|y_{n_j} - a| \leq \underbrace{|y_{n_j} - x_{n_j}|}_{\text{approaches } 0} + \underbrace{|x_{n_j} - a|}_{\text{also approaches } 0}$$

two properties:
① def
② sequential compact

hence $|y_{n_j} - a| \to 0$ $a \in V$

so $a \in U \cap V$ contradicts the condition $U \cap V = \emptyset$.

∎

(c). consider graph of $f(x)=\frac{1}{x}$

① Show U, V closed.

$U=\{(x,y) \mid y=\frac{1}{x}\}$

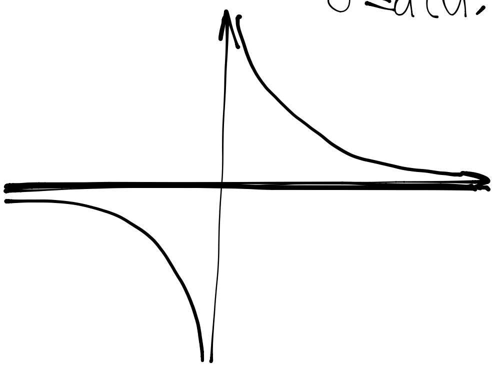
$V=\{x\text{-axis}\}$

② Choose y_n in V, $y_n=n$

$y_n=(n,0)$

$x_n=(n,\frac{1}{n})$

$0\leq d(U,V)\leq |y_n-x_n| = \sqrt{0^2+(\frac{1}{n})^2}=\frac{1}{n}\to 0$



(a). $S\subset\mathbb{R}^n$ closed $f:\mathbb{R}^n\to\mathbb{R}^m$, find a $f(S)$ not closed.

S has to be unbounded!
Why? b/c S compact, f cont. then f(S) compact.

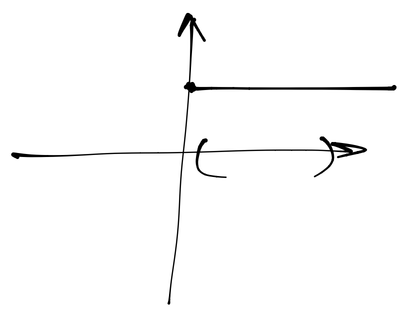
like

$(-\infty,0]\longrightarrow(\cdots]$

e.g. $f(x)=-\frac{1}{x}$, $x\in S=(-\infty,-1]$

(b). $S\subset\mathbb{R}^n$ open, f cont. $f(S)$ not open?

e.g. constant function (a "point" is closed)



e.g. sine function $f(S)\in[-1,1]$