MATH6222: Homework #11

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Problem 1

Let G be a simple graph with n vertices.

(a) Let x and y be nonadjacent vertices of degree at least $(n+k-2)/2$. Prove that x and y have at least k common neighbors.

Proof: Suppose the set of adjacent vertices of x is X, similarly Y is the set the of adjacent vertices of y. Then $|X \cup Y| \leq n-2$ since x and y are not adjacent, so there are at most $n-2$ vertices (x, y excluded) such that they are adjacent to either x or y.

We are interested in $|X \cap Y|$, which is equal to

$$|X \cap Y| = |X| + |Y| - |X \cup Y| \geq \frac{n+k-2}{2} + \frac{n+k-2}{2} - (n-2) = k$$

Then we are done. ■

(b) Prove that if every vertex has degree at leaset $\lfloor n/2 \rfloor$, then G is connected. Show that this bound is the best possible whenever $n \geq 2$ by exhibiting a disconnected n-vertex graph where every vertex has at least $\lfloor n/2 \rfloor - 1$ neighbors.

Proof: Again we use the assumption of X, Y as sets of adjacent vertices of x, y. Since the $|X| \geq \lfloor n/2 \rfloor$, $|Y| \geq \lfloor n/2 \rfloor$,

$$|X| + |Y| = \lfloor n/2 \rfloor + \lfloor n/2 \rfloor \geq n-1$$

(For example, $\lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor = 7 - 1 = 6$.)
Similarly, we follow the idea in part (a):

$$|X \cap Y| = |X| + |Y| - |X \cup Y| \geq (n-1) - (n-2) = 1$$

This can be interpreted as every nonadjacent vertices have at least one common neighbor, i.e. G is connected.

If we have a disconnected graph G' with one part of $\lfloor n/2 \rfloor$ vertices and one part of $\lceil n/2 \rceil$ vertices, for $n \geq 2$, then although each vertex has at least $\lfloor n/2 \rfloor - 1$ neighbors, it is still disconnected. (The arithmetic reasoning here is $\lfloor n/2 \rfloor - 1 < \lfloor n/2 \rfloor \leq \lceil n/2 \rceil$.) ■

Problem 2

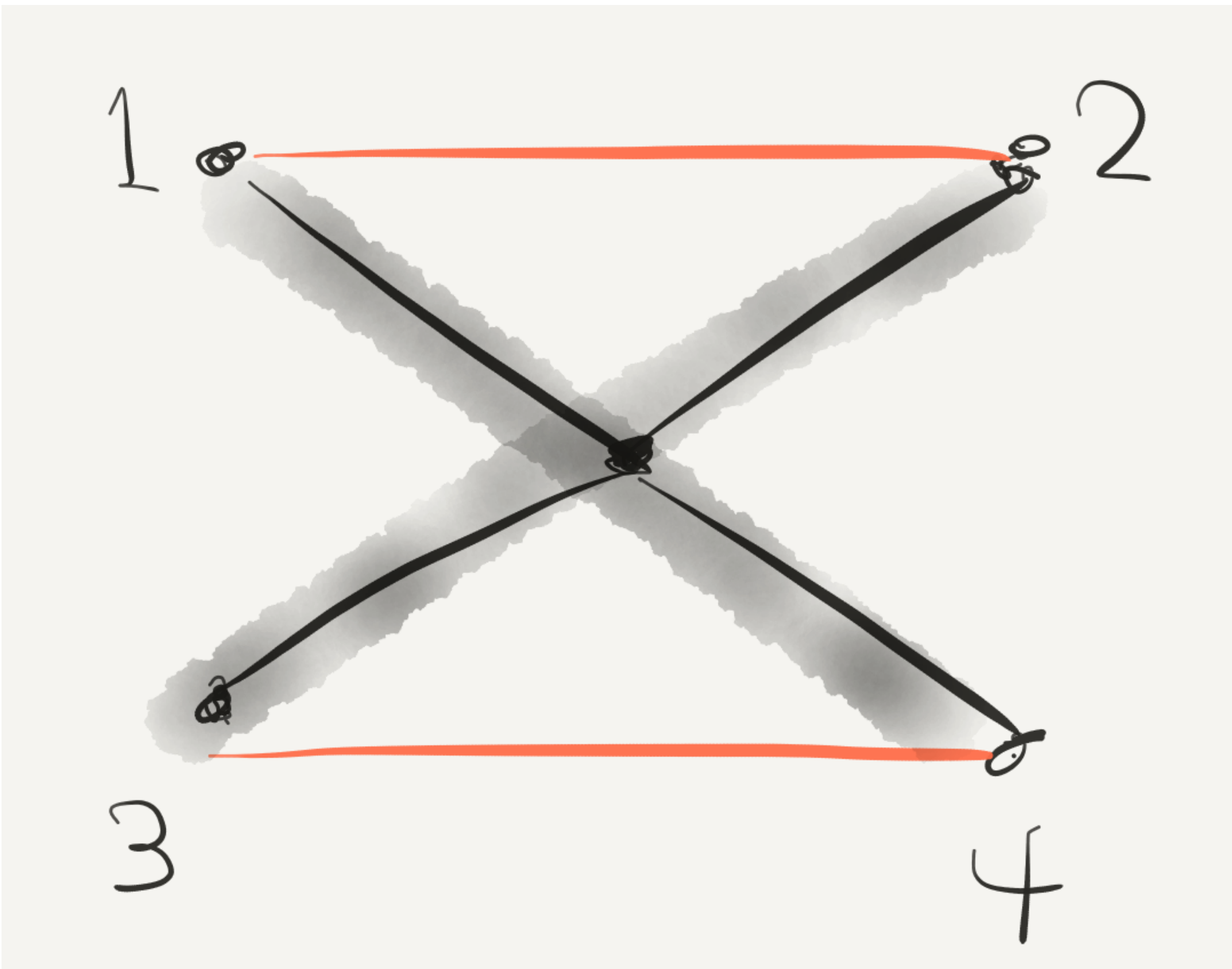
Let G be a connected graph with $m \geq 2$ vertices of odd degrees. (Recall from the previous tutorial that m is even). Prove that the minimum number of trails that together traverse

Problem 2 continued on next page...

each edge of G exactly once is $m/2$. (Hint: Transform G into a new graph G' by adding edges and/or vertices.)

Proof: By theorem, a graph G is Eulerian if and only if each vertex has even degree and each edge is reachable from every other. So the G in our problem must be non-Eulerian. Also by corollary, G has an even number of odd-degree vertices, say the number is $m = 2n, n \in \mathbb{N}$.

Suppose $v_1, v_2, \ldots, v_m$ are those vertices with odd degrees, we pair them up to n pairs as $(v_1, v_2), (v_3, v_4), \ldots, (v_{m-1}, v_m)$, such that a new graph is formed, and we call it G'.



G' is connected (because G was), and each vertex has an even degrees, by the theorem we used at beginning, G' is Eulerian.

Now here comes the tricky part, G' is Eulerian, so whenever we traverse it, we will at some point traverse some edges we added additionally, i.e. not in $E(G)$, but in $G' - E(G)$. So whenever this happens when traversing G' , we count 1, because if we are now really traversing G, we have to "start" another trail (or you can consider it as "jump" through the artificially added edges).

In this way, we have to make $n = m/2$ trails to fully traverse the original graph G. ■

Problem 3

Let G be a graph with n vertices and no cycles of length three. Prove that G has at most $n^2/4$ edges. (Hint: Consider the subgraph consisting of neighbors of a vertex of maximum degree and the edges among them.)

Proof: Suppose G is a graph with n vertices and no cycles of length 3. Suppose again that $v_0 \in G$ is the vertex with maximum degree k. We call the neighbors of v_0, $\{v_1, v_2, \ldots, v_k\}$. We have to understand that the adjacent vertices of v_0 are not adjacent, since otherwise two of those and v_0 would form a cycle of length 3. There are other $n-k-1$ vertices which are not neighbour of v_0 (nor itself), if we sum up all the degrees of $n-k-1$ non-neighbors of v_0 and the degree of v_0, we will have a sum greater or equal to the number of total edges of G, which is $e(G)$. In other words,

$$d(v_0) + d(v_{k+1}) + \cdots + d(v_{n-1}) \geq e(G)$$

Also, for $v_0, v_{k+1}, v_{k+2}, \ldots, v_{n-1}$, these $n-k$ vertices, each has at most k degrees. So

$$\begin{aligned} d(v_0) + d(v_{k+1}) + \cdots + d(v_{n-1}) &\leq k \cdot (n-k) \\ e(G) &\leq k(n-k) \\ e(G) \leq \frac{k+n-k}{2} \cdot \frac{k+n-k}{2} &= \frac{n^2}{4} \end{aligned}$$

Therefore, G has at most $\frac{n^2}{4}$ edges. ■

Problem 4

Suppose that every vertex of a graph G has degree at most k. Prove that $\chi(G) \leq k+1$. Show that this bound is the best possible by exhibiting (for every k) a graph with maximum degree k and chromatic number $k+1$.

Proof: We can prove this by induction.

Base step: suppose $k = 1$, i.e. the maximum degree of a vertex in G is 1. So G is a path. For a path, $\chi(G) = 2 \leq 1 + 1$. So we are done.

Problem 4 continued on next page. . .

Inductive hypothesis: suppose $k = n, n \in \mathbb{N}$, the maximum degree of a vertex in G is n, and $\chi(G) \leq n + 1$.

Suppose G is such a graph with maximum degree n, now we add a new vertex v_a into G, and connect it to the vertex v_0 with maximum degree. We consider the worst case, that this v_a is also connected with all other adjacent vertices of v_0. In order to differ it from all other adjacent vertices of v_0, we cannot use the colors of those adjacent vertices, and we also cannot use the color of v_0 because v_a, v_0 are adjacent. So we have to use a totally new color for v_a, thus increasing the bound of $\chi(G)$ by 1. Therefore, $\chi(G) \leq n + 2$ for $k = n + 1$.

Hence, for every vertex of a graph G with degree at most k, $\chi(G) \leq k + 1$.

Also, if every vertex in G is the "worst case" we were talking about above, then G is a complete graph, as K_{k+1}, i.e. each vertex is adjacent to all other vertices, all vertices have the same degree k, and $\chi(G) = k + 1$. And naturally, for any case "better" than the worst case, $\chi(G) \leq k + 1$. ■