Lecture 20 § 8.2 Uniform Convergence & Continuity.

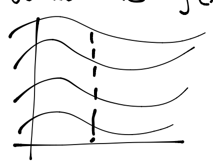
C stands for continuous func

Theorem (Completeness Theorem for $C(K, \mathbb{R}^m)$)
If $K \subset \mathbb{R}^n$ is a compact set, the space $C(K, \mathbb{R}^m)$ of all continuous $\mathbb{R}^m$-valued functions on K with the sup norm is complete.

Proof: Let $(f_k) \in C(K, \mathbb{R}^m)$ be Cauchy $\Rightarrow \forall \varepsilon > 0\ \exists N$ s.t. $\|f_k - f_l\|_\infty < \varepsilon$ $\forall\ k, l \geq N$

We must show that (f_k) has a uniform limit f $\Rightarrow$ we will know that $f \in C(K, \mathbb{R}^n)$

What is $f(x)$?



Fix x Euclidean norm

$\|f_k(x) - f_l(x)\| \leq \|f_k - f_l\|_\infty < \varepsilon$ for all $k, l \geq N$

$\Rightarrow f_k(x)$ is Cauchy $\Rightarrow$ It converges to $f(x)$.

So $f_k(x)$ converges to $f(x)$ pointwise. Let ε, N be as above.

$\|f(x) - f_m(x)\| = \lim\limits_{n\to\infty} \|f_n(x) - f_m(x)\| \leq \varepsilon$ for all $m \geq N$, this is true $\forall x \in K$.

$\|f - f_m\|_\infty \leq \varepsilon \Rightarrow f(x)$ is continuous $f \in C(K, \mathbb{R}^n)$.

∎

Thm. Let (f_n) be a sequence of continuous functions on $[a, b]$. Converging uniformly to $f(x)$, let $c \in [a, b]$. Then $F_n(x) = \int_c^x f_n(t)\,dt$ $n \geq 1$ converge uniformly on $[a, b]$ to $F(x) = \int_c^x f(t)\,dt$

Proof: $|F_n(x) - F(x)| = \left| \int_c^x (f_n(t) - f(t))\,dt \right| \leq \int_c^x |f_n(t) - f(t)|\,dt \leq \int_c^x \|f_n - f\|_\infty\,dt$

$= (x - c)\|f_n - f\|_\infty \leq |b - a|\,\|f_n - f\|_\infty$

$\|f_n - f\| < \dfrac{\varepsilon}{|b-a|}$ when $n \geq N$, so $|F_n - F| < \varepsilon$.

§ 8.4 Series of functions

$\sum_{n=1}^\infty f_n(x)$

we can talk about pointwise & uniform convergence.

Ex: $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ whether converge?

Take k be $\geq l$

We will show that it's Cauchy

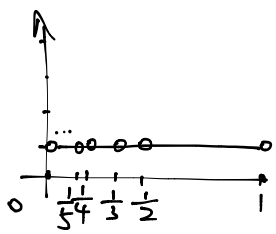
$$\left|\sum_{n=1}^{k} f_n - \sum_{n=1}^{l} f_n\right| = \left|\sum_{n=l+1}^{k} f_n\right| \leq \sum_{n=l+1}^{k} |f_n| \leq \sum_{n=l+1}^{k} \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$ cauchy

Ex.

f_n on $[0,1]$

$f_n = \chi_{(0,\frac{1}{n})}$



$f(0) = 0$

$f(1) = 0$

$f(\frac{3}{4}) = 1$

$x \in (\frac{1}{2}, 1) \Rightarrow f(x) = 1$

$x \in [\frac{1}{3}, \frac{1}{2}) \Rightarrow f(x) = 2$

$x \in [\frac{1}{4}, \frac{1}{3}) \Rightarrow f(x) = 3$

$x \in [\frac{1}{n+1}, \frac{1}{n}) \Rightarrow f(x) = n$

Doesn't converge uniformly

$$\left|\sum_{n=1}^{k} f_n(x) - \sum_{n=1}^{l} f_n(x)\right| \geq f_{l+1}(x) = 1 \quad \text{for all } x \in (0, \tfrac{1}{l+1})$$

Power series

$\sum_{n=0}^{\infty} a_n x^n$ $\qquad$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges ?

Def: Let $S \subset \mathbb{R}^n$, a series of functions form S to $\mathbb{R}^m$ is uniformly Cauchy on S if for every $\varepsilon > 0$, $\exists N$ s.t. $\|\sum_{i=k+1}^{\infty} f_i(x)\|_\infty < \varepsilon$, $\forall\ l \geq k \geq N$, $x \in S$.

Thm: A series of functions converges uniformly iff it is uniformly Cauchy.

Proof: Let S_k be the k-th partial sum.

Suppose S_k converges to S uniformly.

$\Rightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|S_k - S\|_\infty < \varepsilon/2$, $\forall\ k \geq N$.

$\|S_k - S_l\| = \|S_k - S + S - S_l\| \leq \|S_k - S\| + \|S - S_l\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, $\quad k, l \geq N$

Suppose S_k is uniformly Cauchy. Need to define $S(x)$.

Fix $x \Rightarrow S_k(x)$ is a seq. of pts in a Euclidean space.

It is Cauchy $\Rightarrow$ converg to $S(x)$

$\|S - S_k\| = \lim_{l \to \infty} \|S_l - S_k| < \varepsilon$

$\|S - S_k\| \leq \varepsilon$ ■