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1	17
2	28
3	30
4	13
T	88

STA 410/2102 — Second Test — 2014-11-27

For all questions, show enough of your work to indicate how you obtained your answer. No books, notes, or calculators are allowed. You have 110 minutes to write this test. The total number of marks for all questions is 100.

Question 1: [20 Marks] Consider using the Trapezoid Rule or Simpson's Rule to approximate the integral $\int_0^1 f(x)dx$ for various choices of the function $f(x)$. For each of the choices of $f(x)$ below, and for both of these rules, say whether, for any number, n, of sub-intervals over the range 0 to 1, the result of using the rule will be exactly correct, or that it will be greater than the true value, or that it will be less than the true value, or that it is difficult to tell whether the result will be greater, less, or equal to the true value. Briefly explain each of your answers.

a) $f(x) = 1 + 2x$

Trapezoid Rule: the result will be exactly correct. Since each small interval, the area we use to approximate is the exact area we are integrating.

Simpson's Rule: the result will be exactly correct Since the quadratic function we fit by $f(a+ih), f(a+\frac{i}{2}h)$, ~~f(a+..)~~ will be a straight line passing $f(a+ih)$, $f(a+(k+1)h)$ which is the exact function we are integrating

b) $f(x) = 2 + 3x^2$

Trapezoid Rule: it will be greater than the true value, since for each sub-interval, the side of trapezoid is higher than ~~the~~ function thus have a larger area.

Simpson's Rule: result will be exact, since it's a quadratic function the qudratic function we fit is exactly the function we want to integrate

$f' = -2x^2$

c) $f(x) = 7 - x^3$

Trapezoid Rule: it will be smaller than the true value, since $f'(x) < 0$ $f''(x) < 0$. ~~The~~ upper side of ~~each~~ trapezoid is below the function.

Simpson's Rule: can't tell,

d) $f(x) = 1 + 2x + (x - 1/2)^3 = \frac{7}{8} + \frac{11}{4}x - \frac{3}{2}x^2 + x^3$

$f' \ \frac{11}{4} - 3x + 3x^2 \quad f'' = -3 + 6x$

$1 + 2x + (x^2 + \frac{1}{4} - x)(x - \frac{1}{2})$

$1 + 2x + x^3 + \frac{1}{4}x - x^2 - \frac{1}{2}x^2 - \frac{1}{8} + \frac{1}{2}x$

Trapezoid Rule: Can't tell, since on $(0, \frac{1}{2})$ f is concave down and $(\frac{1}{2}, 1)$ concave down.

Simpson's Rule: can't tell either.

Question 2: [30 Marks] Suppose that we model a single real-valued data point, x, as coming from a normal distribution with mean μ and variance one. Suppose also that we use a prior distribution for μ has the following density function over the reals:

$h()\ \frac{f()}{g()}$

$$f(\mu) = (1/2)\exp(-|\mu|) \qquad N(\mu, 1) \qquad \pi$$

a) Write an R function called `met` that samples from the posterior distribution for μ using the Metropolis algorithm, with the proposal distribution for μ being normal with mean equal to the current value and variance one. Your function should take as arguments an initial value for μ and the number of transitions to do, and return the vector of values for μ after each transition. and data x

$\min(1, \frac{\pi(x^*)}{\pi(mu)})$

```
Met = function(mu, n, x) {
  res= numeric(n)
  for (i in 1:n) {

      fmu = 1/2 * exp(-1 * abs(mu)) * ~~log~~ ( dnorm(x, mu, 1))
      fmu_new = rnorm(fmu, 1) need prior here too
      if (runif(1) < (fmu_new / fmu))
              mu = fmu
      else   mu = mu
    res[i] = mu }
  res

}
```

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b) Write R commands to use the `met` function from part (a) to find the posterior expected value of $\boxed{\mu^2}$ given the observation $\boxed{x = 1.5}$. Use a starting value of zero for the Markov chain, and assume that the first 100 iterations should be discarded as "burn-in" (not necessarily close to having the desired distribution). You should then estimate the expected value of μ^2 using 1000 iterations after the burn-in iterations.

$E\ \mu^2$

$\mu^2 = E(x^2) - E(x)$

$V = \boxed{E(x^2)} - \boxed{E(x)}^2$

μ

```
x=1.5
Res = met( 0, 1100, x)[100:1000]
Exp= mean(Res)
VAR= var(Res)
output = Exp^2 + VAR
```

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mean(res^2)

is easier

Question 3: [30 Marks] Suppose we wish to sample from the distribution on R^2 that is uniform on the half circle above the horizontal axis with centre at the origin and radius one. In other words, the distribution is uniform over the region $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1 \text{ and } x_2 \geq 0\}$.

a) Write an R function called **gibbs** that does Gibbs sampling for this distribution. Your function should take as arguments an initial point (a vector of length two, which you can assume is inside the half-circle) and the number of Gibbs sampling transitions to do (with each transition updating first x_1 and then x_2). It should return a matrix with two columns and one row for each transition.

$x^2+y^2=1$

```
gibbs = function(X, n)
  x= numeric(n); px = x[1]
  y= numeric(n); py = X[2]
  for (i in 1:n){
    px= runif(1,-sqrt(1-py^2), sqrt(1-py^2))
    py= runif(1, 0, sqrt(1-px^2))

    x[i] = px
    y[i] = py

  }

  as.matrix(cbind(x, y))

}
```

0.81
0.19

b) Suppose that the initial state for Gibbs sampling from this disribution is (0.1, 0.9). Answer the following questions (no explanation required):

After one Gibbs sampling transition, is it possible that the state will be (0.2, 0.1)?

Yes

Is it possible that after one transition, the state will be (0.5, 0.4)?

No

After two Gibbs sampling transitions, is it possible that the state will be (0.2, 0.1)?

Yes

Is it possible that after two transitions, the state will be (0.5, 0.4)?

Yes

4

Question 4: [20 Marks] Let π be the distribution on the space $\{1, 2, 3, 4\}$ in which 1 and 2 each have probability 1/3 and 3 and 4 each have probability 1/6. Suppose we define a Markov chain to sample from this distribution using the Metropolis method, with the proposal probabilities being given by

$$g(x^*|x) = \begin{cases} 1/2 & \text{if } x^* = x+1 \text{ or } x = 4 \text{ and } x^* = 1 \\ 1/2 & \text{if } x^* = x-1 \text{ or } x = 1 \text{ and } x^* = 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Write down the 4×4 matrix of transition probabilities for the Metropolis method using this proposal distribution and with π as the distribution that should be left invariant. (The entry in row i column j of the matrix T should be the probability that the next state will be j if the current state is i.)

$$T = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 2 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 3 & 0 & & 0 & \frac{1}{3} \\ 4 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{array}$$

$\frac{2}{9} + \frac{2}{12}$

5/12

$\frac{2}{3} \times \frac{1}{4}$ $\quad \frac{1}{4} \times \frac{1}{3} = \frac{2}{12}$

$\frac{2}{12}$

b) Suppose that the initial state of the Markov chain is randomly drawn from the uniform distribution on $\{1, 2, 3, 4\}$. What will be the distribution of the next state of the Markov chain after this initial state? Show how you obtained your answer.

if initial state is uniform distributed on $\{1,2,3,4\}$

$\pi_1 = (\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4})$

$$\pi_2 = \pi_1 T = \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

$= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$ 8/8