MAT337 Review Term 1

Coverage: 2.3–2.8 ; 3.1–3.3 ; 4.1–4.4 ; 9.1

§2.3 The least upper bound principle

~~2.3.1 Definition~~: bounded above/below, upper/lower bound, bounded.

Supremum/least upper bound, infimum/greatest lower bound.

maximum, minimum.

$\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$

Least upper bound principle: nonempty subset $S \subset \mathbb{R}$ that bdd above has sup, bdd below has inf.

§2.4 Limits

L is the limit of a seq. of real numbers $(a_n)_{n=1}^{\infty}$ if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon) > 0$ st. $|a_n - L| < \varepsilon$ $\forall n \geq N$.

converge

The squeeze theorem: 3 seq. $a_n \leq b_n \leq c_n$ & $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ then $\lim\limits_{n\to\infty} b_n = L$

§2.5 Basic properties of limits

prop: (a_n) convergent seq. of real #. then $\{a_n\}$ is bdd.

Thm: $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = M$

$\lim\limits_{n\to\infty} a_n + b_n = M + L$ $\lim\limits_{n\to\infty} \alpha a_n = \alpha L$

$\lim\limits_{n\to\infty} a_n b_n = LM$ $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if 若 $M \neq 0$.

§2.6 Monotone sequences

(strictly) monotone ..

Monotone convergence theorem: $\nearrow$ converges

$\searrow$ converges

Nested Intervals Theorem:

$I_n=[a_n,b_n]$, $a_n<b_n$ & $I_{n+1}\subseteq I_n$. $\forall n\geq 1$. Then $\bigcap_{n\geq 1} I_n\neq\emptyset$

§2.7 subsequences.

subsequence

Bolzano-weierstrass thm:

Every bdd sequence of real numbers has a convergent subsequence

Thm: every sequence has a monotone subsequence.

§2.8 Cauchy sequences.

prop: $(a_n)\to L$, $\forall \varepsilon>0$, $\exists N\in \mathbb{Z}_+$ s.t. $|a_n-a_m|<\varepsilon$ $\forall$ $n, m\geq N$.

Cauchy if $\forall \varepsilon>0$, $\exists N$ s.t. $|a_n-a_m|<\varepsilon$, $\forall \underset{m}{n}\geq N$.

prop: every cauchy seq. is bdd.

def: a subset S of $\mathbb{R}$ is said to be complete if every Cauchy seq. in S $\to$ to a point in S.

completeness thm: every Cauchy sequence of real numbers converges, so $\mathbb{R}$ is complete.

[redacted]

§ 3.1 convergent series

summable, convergent, divergent.

thm. Σa_n convergent, $\lim_{n\to\infty} a_n = 0$. (nth term test)

Cauchy criterion for series:

following are equivalent: ① Series converges

② $\forall \varepsilon>0, \exists N\in\mathbb{N}, \forall n\geq N, \left|\sum_{n+1}^{\infty} a_k\right| < \varepsilon.$

③ $\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t. $\forall n,m\geq N, \left|\sum_{n+1}^{m} a_k\right| < \varepsilon.$

§ 3.2 Convergence Tests for series.

prop. $a_k \geq 0$ $k\geq 1$, $S_n = \sum_{k=1}^{\infty} a_k$, either

① $(S_n)_{n=1}^{\infty}$ is bdd above, in which case, $\sum_{n=1}^{\infty} a_n$ converges.

② $(S_n)_{n=1}^{\infty}$ is unbounded $\Rightarrow \sum a_n$ diverges.

The comparison test

~~consider two~~ $(a_n)(b_n)$, $|a_n| \leq b_n$ $\forall n\geq 1$.

- if (b_n) summable, then (a_n) summable &.

  $\left|\sum a_n\right| \leq \sum b_n$
- if (a_n) not, then (b_n) not.

The n-th root test.

Sps $a_n \geq 0$, $l = \limsup \sqrt[n]{a_n}$. if $l<1$ $\sum a_n$ converges

$l>1$ diverges

$=1$ nothing

alternating series

Leibniz alternating series test.

$(a_n)_{n=1}^{\infty}$ is a monotone decreasing $a_1\geq a_2\geq\cdots\geq 0$

$\lim_{n\to\infty} a_n = 0$. then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. conditionally

§3.3 ~~Abosute Abosul~~ Absolute and conditional convergence.

Prop: ~~An absolutely convergent series is~~

$\sum_{n=1}^{\infty} a_n$ converges ab. if $\sum_{n=1}^{\infty} |a_n|$ converges.

rearrangement : π: a permutation of $\mathbb{N}$.

$\sum_{n=1}^{\infty} a_{\pi(n)}$

Thm: Every ~~reag~~ rearrangement of an absolutely convergent series converges to the same limit.

rearrangement thm:

if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series then $\forall L \in \mathbb{R}$, $\exists$ a rearrangement that converges to L.

§4.1 n-dimensional space.

norm, dot product / inner product

Schwarz inequality: $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$

Triangle inequality $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$, $\forall\, \vec{x}, \vec{y} \in \mathbb{R}^n$

lemma: $\{\vec{v_1}, \dots, \vec{v_m}\}$ orthonormal set in $\mathbb{R}^n$. Then $\longrightarrow$ $\langle v_i, v_j \rangle = \begin{cases} 1 & \text{, when } i=j \\ 0 & \text{, o.w.} \end{cases}$

$$\left\| \sum_{i=1}^{m} a_i \vec{v_i} \right\| = \left(\sum_{i=1}^{m} |a_i|^2 \right)^{\frac{1}{2}}$$

An orthonormal set in $\mathbb{R}^n$ is linearly independent. So an orthonormal basis for $\mathbb{R}^n$ is a basis and has exactly n elements.

§4.2 Convergence & Completeness in $\mathbb{R}^n$

seq. of points converges to $\vec{a}$ if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ s.t.
$\|\vec{x_k} - \vec{a}\| < \varepsilon$ for all $k \geq N$.

$\lim_{k\to\infty} \vec{x_k} = \vec{a}$

lemma: $\lim_{k\to\infty} \vec{x_k} = \vec{a}$ iff $\lim_{k\to\infty} \|\vec{x_k} - \vec{a}\| = 0$

lemma: $\vec{x_k} \to \vec{a}$ iff every $x_1 \to a_1$, $x_2 \to a_2$, ...

def: sequence Cauchy:
set complete: every Cauchy seq. of pts in S $\to$ to a pt in S.

Completeness theorem for $\mathbb{R}^n$
every Cauchy seq. in $\mathbb{R}^n$ converges, $\mathbb{R}^n$ is complete.

§4.3 Closed & open subsets of $\mathbb{R}^n$.

limit point

closed (contains all of its limit points)

prop: ~~$A,B\in\mathbb{R}^n$~~ $A, B \subset \mathbb{R}^n$, ~~the closure~~ both closed $\Rightarrow A\cup B \subset \mathbb{R}^n$ is closed. If $\{A_i : i\in I\}$ is a family of closed sets in $\mathbb{R}^n$, then ~~$A\cap B$~~ $\bigcap A_i$ is closed.

closure $\overline{A}$ is the smallest set contains all limit pts of A.

$\overline{\overline{A}} = \overline{A}$.

Ball: $B_r(\vec{a}) = \{\vec{x}\in\mathbb{R}^n \mid \|\vec{x}-\vec{a}\| < r\}$

~~#~~ $U \subseteq \mathbb{R}^n$ is open if $\forall \vec{a}\in U$, $\exists$ ~~r=~~ $r(\vec{a}) > 0$ s.t. $B_r(\vec{a}) \subset U$.

Thm. A set $A\subset\mathbb{R}^n$ is open iff A^c is closed.

Prop: U, V open in $\mathbb{R}^n$ $\Rightarrow U\cap V$ open.

If $\{U_i : i\in I\}$ is a family of open subsets of $\mathbb{R}^n$, then ~~$\cap$~~ $\bigcup_{i\in I} U_i$ is open.

§4.4 Compact Sets & the Heine–Borel Theorem

Compact: if every seq. $(\vec{a}_k)_{k=1}^{\infty}$ of pts in A has a convergent subsequence $(\vec{a}_{k_i})_{i=1}^{\infty}$ ~~which~~ with limit $\vec{a} = \lim_{i\to\infty} \vec{a}_{k_i}$ in A.

Thm: compact $\Leftrightarrow$ closed & bounded.

Lemma

Subset $S\subset\mathbb{R}^n$ is bdd if $\exists R\in\mathbb{R}$ s.t. $S\subset B_R(0) \Leftrightarrow \sup_{\vec{x}\in S}\|x\| < \infty$

Lemma: C is closed subset of a compact subset of $\mathbb{R}^n$, then C is compact.

Thm: cube $[a,b]^n$ is a compact subset of $\mathbb{R}^n$.

HB Theorem: $S \subset R^n$ is compact $\Leftrightarrow$ closed & bdd.

The Cantor's intersection thm

$A_1 \supset A_2 \supset \cdots$ decreasing seq. of nonempty compact subsets of $\mathbb{R}^n$

$\Rightarrow \bigcap_{k \geq 1} A_k \neq \emptyset$ ~~then~~.

Cantor set: fractal set. (delete every middle one third)

~~tern~~ ternary expansion.

$x = (x_0 . x_1 x_2 x_3 \ldots)_{base\ 3} = \sum_{k \geq 0} 3^{-k} x_k$

nowhere dense: A set whose closure has no interior.

isolated: if $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \{x\}$, $x \in A$, x is isolated.

perfect set: no isolated pts.

Metric spaces.

~~§~~ §9.1 Definitions & Examples

~~#~~ metric:

a metric on a set X is a func. ρ defined on $X \times X$ taking values in $[0, \infty)$ with the following properties:

① positive definiteness $\rho(x,y) = 0$ iff $x = y$

② symmetry $\rho(x,y) = \rho(y,x)$ $\forall x,y \in X$

③ triangle ineq. $\rho(x,z) \le \rho(x,y) + \rho(y,z)$ $\forall x,y,z \in X$

(1). $\rho(x,y) = \|x-y\|$ standard example

(2). geodesic

(3). discrete metric on a set X is given by $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$

(4). define metric on $\mathbb{Z}$ by $\rho_2(m,n) = 2^{-d}$, where d is the largest power of 2 dividing $m-n \neq 0$.

2-adic metric.

↓ prime

p-adic metric.

Ball. $B_r(x) = \{y \in X : \rho(x,y) < r\}$

write $B_r^{\rho}(x)$ if metric is ambiguous.

- If a subset U is open if $\forall x \in U$, $\exists r > 0$ s.t. $B_r(x) \subset U$ and $\text{int} A$, $\int A$ is the largest open set contained in A.
  (x_n) is said to converge to x if $\lim_{n \to \infty} \rho(x, x_n) = 0$.
- A set C is closed if it contains all limit pts of sequences of pts in C and the closure of a set A, $\overline{A}$, is the set of all limit pts of A.
- A seq. $(x_n)_{n=1}^{\infty}$ ~~is~~ in a metric space (X, ρ) is a Cauchy seq. if $\forall \varepsilon > 0$, $\exists N$ s.t. $\rho(x_i, x_j) < \varepsilon$ $\forall i,j \ge N$
- Metric space X is complete if every Cauchy sequence converges (in X).

def: f from metric space (X,ρ) into a metric space (Y,σ) is continuous if for $\forall x_0 \in X$ & $\varepsilon>0$, $\exists \delta>0$ s.t $\sigma(f(x),f(x_0))<\varepsilon$ whenever $\rho(x,x_0)<\delta$.

Thm: f map $(X,\rho)\to(Y,\sigma)$.
then

① f is continuous ~~iff~~ on X

$\Leftrightarrow$

② $\forall (x_n)$ with $\lim_{n\to\infty} x_n = a \in X$, we have $\lim_{n\to\infty} f(x_n)=f(a)$

$\Leftrightarrow$

③ $f^{-1}(U)=\{x\in X : f(x)\in U\}$ is open in X $\forall$ open set U in Y.

Thm: The space $C_b(X,\mathbb{R}^m)$ of all bounded continuous functions on a metric space X with sup norm $\|f\|=\sup\{\|f(x)\| : x\in X\}$ is complete.