Lecture 3

Office Hours: You are welcome to talk to the instructor after class, or any time you find him in his office, or you can e-mail him to arrange another time to meet. In addition, we have arranged special office hours with a TA before the midterm and homeworks are due, as follows: Tues Feb 9, 12:10-2:00, RW141; Tues Feb 23, 12:10-2:00, RW141; Tues Mar 15, 12:10-2:00, RW141; Wed Apr 6, 11:10-1:00, RW141; with four more hours coming before the final exam.

STRONG RECURRENCE THEOREM:

If chain irreducible, then the following are equivalent:

1. There are $k, l \in S$ with $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$
2. For all $i, j \in S$ we have $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$
3. There is $k \in S$ with $f_{kk} = 1$, i.e. with k recurrent
4. For all $j \in S$, we have $f_{jj} = 1$, i.e. all states are recurrent
5. For all $i, j \in S$, we have $f_{ij} = 1$.

Proof: (1) $\Rightarrow$ (2) Sum Lemma

(2) $\Rightarrow$ (3) Recurrence Thm (with $i=j=k$)

(3) $\Rightarrow$ (1) Recurrence Thm (with $l=k$)

(2) $\Rightarrow$ (4) Recurrence Thm (with $i=j$)

(4) $\Rightarrow$ (5) F-lemma

(5) $\Rightarrow$ (3) Immediate.

QED.

Simple random walk with $p = \frac{1}{2}$. $P(\exists n \geq 1 \text{ with } X_n = 1000000 \mid X_0 = 0) = 1$ etc.

Example: $S = \{1, 2, 3\}$, and $(p_{ij}) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- Then $\sum_{n=1}^{\infty} p_{12}^{(n)} = \sum_{n=1}^{\infty} (\frac{1}{2}) = \infty$
- $f_{22} = 1$. Recurrent
- $f_{11} = 0 < 1$. Transient!
- $f_{12} = \frac{1}{2} < 1$
- Not irreducible!

Example: Simple random walk with $p > \frac{1}{2}$

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