Worth: 3% **Due:** By 12 noon on Tuesday 27 March.

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1. (a) **Assume $a \in \mathbb{R}, b \in \mathbb{R}$**
   Assume $a \leqslant b$
   Then $b - a \geqslant 0$.

   Let $c_0 = 1$ and $B_0 = 0$.
   Then $c_0 \in \mathbb{R}^+$.
   Then $B_0 \in \mathbb{N}$.

   Assume $n \in \mathbb{N}, n \geqslant B_0$

   $$\begin{aligned} \textbf{Then } n^a &\leqslant n^a \cdot n^{b-a} \quad \# \text{ since } b-a \geqslant 0, n^{b-a} \geqslant 1 \\ &= n^{a+b-a} \\ &= n^b \\ &= c_0 \cdot n^b \end{aligned}$$

   Then $n^a \leqslant c_0 \cdot n^b$
   Then $\forall n \in \mathbb{N}, n \geqslant B_0 \Rightarrow n^a \leqslant c_0 \cdot n^b$
   Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow n^a \leqslant c \cdot n^b$.
   Then $n^a \in \mathcal{O}(n^b)$.

   Then $a \leqslant b \Rightarrow n^a \in \mathcal{O}(n^b)$.
   Then $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, a \leqslant b \Rightarrow n^a \in \mathcal{O}(n^b)$.

   (b) **Assume $a \in \mathbb{R}, b \in \mathbb{R}$**
   Assume $1 < a \leqslant b$
   Then $1 < a$.
   Then $a \leqslant b$.
   Then $\ln(a) \leqslant \ln(b)$. # natural logarithm is monotone increasing

   Let $c_0 = 1$ and $B_0 = 0$.
   Then $c_0 \in \mathbb{R}^+$.
   Then $B_0 \in \mathbb{N}$.

   Assume $n \in \mathbb{N}, n \geqslant B_0$
   Since $\ln(a) \leqslant \ln(b)$,
   Then $n \cdot \ln(a) \leqslant n \cdot \ln(b)$. # $n \geqslant 0$
   Then $\ln(a^n) \leqslant \ln(b^n)$.
   Then $a^n \leqslant b^n$.
   Then $a^n \leqslant c_0 \cdot b^n$.

   Then $\forall n \in \mathbb{N}, n \geqslant B_0 \Rightarrow a^n \leqslant c_0 \cdot b^n$
   Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow a^n \leqslant c \cdot b^n$.
   Then $a^n \in \mathcal{O}(b^n)$.

   Then $1 < a \leqslant b \Rightarrow a^n \in \mathcal{O}(b^n)$.
   Then $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, 1 < a \leqslant b \Rightarrow a^n \in \mathcal{O}(b^n)$.

   (c) **It turns out that this question is a little trickier than intended if you allow all non-negative logarithm bases that are not 1. This is because for logarithm base a with $0 < a < 1$, $\log_a(x)$ is monotone decreasing and is negative for $x > 1$. While for logarithm base a with $1 < a$, $\log_a(x)$ is monotone increasing and is positive for $x > 1$.**

The means that for $a < 1, b > 1$, and $n \in \mathbb{N}$ with $n > 1$, $\log_a(n)$ is negative and $\log_b(n)$ is positive. We can get $\log_a(n) \in \mathcal{O}(\log_b(n))$ but not $\log_a(n) \in \Omega(\log_b(n))$. Similarly (but $\mathcal{O}, \Omega$ reversed) for $a > 1, b < 1$.

But the Θ result does hold for $a < 1, b < 1$ and $a > 1, b > 1$. It boils down to the observation that $\log_a(n) = \dfrac{1}{\log_b(a)} \cdot \log_b(n)$. We have a nice relationship between $\log_a(n)$ and $\log_b(n)$, but the constant $\dfrac{1}{\log_b(a)}$ will only be positive when $a < 1, b < 1$ or $a > 1, b > 1$. And we need the constant to be positive in Θ.

Since it is most common for logarithm bases to be greater than 1, let's prove the result for $a > 1, b > 1$.

Assume $a \in \mathbb{R}^{>1}, b \in \mathbb{R}^{>1}$
    Let $c_0 = 1/\log_b(a)$, $c_1 = c_0$ and $B_0 = 1$.
    # need $B_0 \geqslant 1$ since log taken
    Then $c_0 \in \mathbb{R}^+$.
    Then $c_1 \in \mathbb{R}^+$.
    Then $B_0 \in \mathbb{N}$.

    Assume $n \in \mathbb{N}, n \geqslant B_0$
        Then $\log_a(n) = \dfrac{1}{\log_b(a)} \cdot \log_b(n)$
        Then $c_0 \log_b(n) \leqslant \log_a(n) \leqslant c_1 \log_b(n)$

    Then $\forall n \in \mathbb{N}, n \geqslant B_0 \Rightarrow c_0 \log_b(n) \leqslant \log_a(n) \leqslant c_1 \log_b(n)$
    Then $\exists c_0 \in \mathbb{R}^+, \exists c_1 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow c_0 \log_b(n) \leqslant \log_a(n) \leqslant c_1 \log_b(n)$.
    Then $\log_a(n) \in \Theta(\log_b(n))$.

Then $\forall a \in \mathbb{R}^{>1}, \forall b \in \mathbb{R}^{>1}, \log_a(n) \in \Theta(\log_b(n))$.

2. Let us start by defining the predicate $P(n)$: "$\sum_{j=0}^{n} t_j = n(n+1)(n+2)/6$", where we have $\forall k \in \mathbb{N}, t_k = k(k+1)/2$.

   We need to prove that $\forall n \in \mathbb{N}, P(n)$.

   Prove $P(0)$:

   Let $n_0 = 0$.
   Then
$$\begin{aligned} \textstyle\sum_{j=0}^{n_0} t_j &= \textstyle\sum_{j=0}^{0} t_j \\ &= t_0 \\ &= 0(0+1)/2 \\ &= 0 \\ &= 0(0+1)(0+2)/6 \\ &= n_0(n_0+1)(n_0+2)/6. \end{aligned}$$
   Then $P(0)$.

   Prove $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$:

   Assume $n \in \mathbb{N}$
       Assume $P(n)$
           Then $\sum_{j=0}^{n} t_j = n(n+1)(n+2)/6$.

$$\begin{aligned}\textbf{Then } \sum_{j=0}^{n+1} t_j &= \left(\sum_{j=0}^{n} t_j\right) + t_{n+1} \\ &= n(n+1)(n+2)/6 + (n+1)((n+1)+1)/2 \\ &= (n(n+1)(n+2) + 3(n+1)(n+2))/6 \\ &= (n+1)(n+2)(n+3)/6 \\ &= (n+1)((n+1)+1)((n+1)+2)/6.\end{aligned}$$

Then $P(n+1)$.

Then $P(n) \Rightarrow P(n+1)$.

Then $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$.

Then $P(0) \wedge \forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$.

Then, by the Principle of Simple Induction, $\forall n \in \mathbb{N}, P(n)$.

Then $\sum_{j=0}^{n} t_j = n(n+1)(n+2)/6$.