PROBLEM-SOLVING AND PROOFS
ASSIGNMENT 3 SOLUTIONS

(1) Find and prove a formula for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

First solution. Each summand can be written as a difference of fractions:

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

When taking the sum of the series, the negative portion of one summand cancels with the positive portion of the subsequent summand, resulting in nearly complete cancellation:

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i(i+1)} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

Remark: a *telescoping series* is a series with a large amount of cancellation. In fact, the Wikipedia page on telescoping series has this series in its introduction!

Second solution. Suppose we didn't see the partial fraction decomposition of the summand. Let $S(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}$. Trying small values of n for the sum tells us that $S(1) = \frac{1}{2}$, $S(2) = \frac{2}{3}$, $S(3) = \frac{3}{4}$, $S(4) = \frac{4}{5}$, etc. From this information, we conjecture that

$$S(n) = \frac{n}{n+1}.$$

To prove our hunch, we use induction. The base case, for $n = 1$, is

$$S(1) = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}.$$

Superb. Assume for the sake of induction that $S(k-1) = \frac{k-1}{k}$. Then

$$\begin{aligned}
S(k) = \sum_{i=1}^{k} \frac{1}{i(i+1)} &= \sum_{i=1}^{k-1} \frac{1}{i(i+1)} + \frac{1}{k(k+1)} \\
&= S(k) + \frac{1}{k(k+1)} \\
&= \frac{k-1}{k} + \frac{1}{k(k+1)} \qquad \text{(from the inductive assumption)} \\
&= \frac{(k-1)(k+1)}{k(k+1)} + \frac{1}{k(k+1)} \\
&= \frac{k^2}{k(k+1)} \\
&= \frac{k}{k+1}.
\end{aligned}$$

Excellent! Therefore by induction, we have reaffirmed that $S(n) = \frac{n}{n+1}$.

(2) Determine (with proof) the set of natural numbers for which the following inequalities hold.

(a) $3^{n+1} > n^4$.
(b) $n^3 + (n+1)^3 > (n+2)^3$.

Solution.

(a) By substituting small numbers for n, it can be checked that $3^{n+1} > n^4$ is false for $n = 3, 4$, but true for $n = 1, 2, 5, 6, 7$. This suggests that we could prove $3^{n+1} > n^4$ when $n \geq 5$. We will do so by induction.
The base case is $n = 5$, which is true because $3^{5+1} = 729 > 625 = 5^4$. Assume the inequality for $n = k \geq 5$ is true, i.e. that $3^{k+1} > k^4$. We want to show the inequality is true for $n = k+1$. Start by

$$\begin{aligned}
3^{k+2} &= 3 \times 3^{k+1} \\
&> 3 \times k^4 \qquad \text{(by the inductive hypothesis).}
\end{aligned}$$

On the other hand, we assumed that $k \geq 5$ (because we start the induction from $n = 5$). Therefore, we have the inequalities

$$\begin{aligned}
k^4 &\geq 5 \cdot k^3 > 4k^3; \\
\tfrac{1}{3}k^4 &\geq 5^2 \cdot \tfrac{1}{3}k^2 = \tfrac{25}{3}k^2 > 6k^2; \\
\tfrac{1}{3}k^4 &\geq 5^3 \cdot \tfrac{1}{3}k = \tfrac{125}{3}k > 4k; \\
\tfrac{1}{3}k^4 &\geq 5^4 \cdot \tfrac{1}{3} = \tfrac{625}{3} > 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
3^{k+2} &> 3k^4 \\
&= k^4 + k^4 + \tfrac{1}{3}k^4 + \tfrac{1}{3}k^4 + \tfrac{1}{3}k^4 \\
&> k^4 + 4k^3 + 6k^2 + 4k + 1 \\
&= (k+1)^4.
\end{aligned}$$

Via induction, we have proved the inequality $3^{n+1} > n^4$ holds for $n \geq 5$. Checking when $n < 5$ shows that the inequality holds for all natural numbers n other than 3 or 4.

(b) We can substitute small numbers for n and come to the conclusion that $n^3 + (n+1)^3 > (n+2)^3$ is not true for $n = 1, 2, 3, 4, 5$. We will show, however, that $n^3 + (n+1)^3 > (n+2)^3$ is true for $n \geq 6$. Instead of induction, we will use a more direct approach. Computing the difference gives

$$\begin{aligned} n^3 + (n+1)^3 - (n+2)^3 &= n^3 + (n^3 + 3n^2 + 3n + 1) - (n^3 + 6n^2 + 12n + 8) \\ &= n^3 - 3n^2 - 9n - 7. \end{aligned}$$

Since $n \geq 6$, we can use the n^3 term to balance out the negative terms, namely

$$\begin{aligned} &\tfrac{1}{2}n^3 \geq 3n^2; \\ &\tfrac{1}{4}n^3 \geq 9n; \\ &\tfrac{1}{4}n^3 \geq 54 > 7. \end{aligned}$$

Combining these three inequalities gives $n^3 - 3n^2 - 9n - 7 > 0$. The difference is positive so we can conclude that $n^3 + (n+1)^3 > (n+2)^3$ for all $n \geq 6$.

(3) Determine the set of positive real numbers x such that

$$x^n + x < x^{n+1}$$

for all $n = 1, 2, 3, \ldots$

Solution. Firstly, if $n = 1$, then the inequality becomes $2x < x^2$. This holds when $x(x-2) > 0$. Because x is positive, $(x-2)$ must also be positive, so $x > 2$ is a necessary condition.

It turns out this is also a sufficient condition. That is, if $x > 2$, then for all $n = 1, 2, \ldots$, $x^n + x < x^{n+1}$. Like with question 2b, a more direct approach works here. Note that $x^n \geq x$ when $x > 2$ and $n \geq 1$. Then

$$x^n + x \leq x^n + x^n = 2x^n < x(x^n) = x^{n+1},$$

proving sufficiency.

In conclusion, the set of x where $x^n + x < x^{n+1}$ holds for all natural n, is all $x > 2$.

(4) Starting from 0, two players take turns adding 1, 2, or 3 to a single running total. The first player who brings the total to 1,000 or more wins. Prove that the second player has a winning strategy for this game.

Solution. Consider the exact same game where the target total is $4n$ instead of 1000 (so whoever brings the total to $\geq 4n$ wins). We shall prove by induction that when $n \geq 1$, Player 2 (the second player) always has a winning strategy.

We induct on n. When $n = 1$, let Player 1's first move be add k to the total, where $k \in \{1, 2, 3\}$. The running total becomes k. As $k < 4$, the total does not reach the target total of 4, so Player 1 does not win that turn. The value $4 - k$, no matter what k is, will always be either 1, 2 or 3. Player 2 can reply by adding $4 - k$ to the total, so the running total becomes $k + (4 - k) = 4$, reaching the target. Player 2 wins.

Assume that Player 2 has a winning strategy for the $4n$ game (where the target total is $4n$). We want to show that Player 2 also has a winning strategy for the $4n + 4$ game.

In the $4n + 4$ game, let Player 1's first move be add k to the total. The running total is $k < 4n + 4$, so Player 1 does not win that turn. As above, Player 2 can then make the move of adding $4 - k$ to the total, making the running total 4.

From this point onwards, Player 2 can put on his/her special "subtract-four glasses", which filters numbers Player 2 sees by subtracting 4. As a result, Player 2, viewing the game through his/her "subtract-four glasses", sees the current total as 0 and the target as $4n$. However, subtracting 4 does not fundamentally change the game: all possible moves by both players remain functionally the same, with the same win conditions. He/she can then use the winning strategy from the $4n$ game (which exists via the inductive hypothesis) to reach the target of $4n$ (whilst wearing the glasses). Then after removing the "subtract-four glasses", Player 2 can see he/she actually reached $4n + 4$ first, and wins. Hence Player 2 has a winning strategy for the $4n + 4$ game.

Therefore, we have proved (via induction) that Player 2 has a winning strategy for the $4n$ game. In particular, Player 2 has a winning strategy for this game, by setting $n = 250$.

(5) Recall that an L-tile is just a tile with three squares shaped like an L. We say a board admits an L-tiling if it is possible to completely cover it with L-tiles, such that each tile lies completely on the board, and no two tiles overlap.

  (a) Prove that a $2^k \times 2^k$ chessboard with a single square in the lower left corner deleted admits an L-tiling, for any $k \in \mathbb{N}$.
  (b) Prove that a $2^k \times 2^k$ chessboard with *any* single square deleted admits an L-tiling, for any $k \in \mathbb{N}$.

Solution. Part (a) of this problem is a consequence of part (b), so we will prove part (b) here. This is (yet) another induction argument. The base case is $k = 1$. In this situation, removing any one square from the 2×2 chessboard leaves a single L-tile shaped board, which can be trivially tiled by a single L-tile.

Assume for our inductive hypothesis that the $2^{k-1} \times 2^{k-1}$ chessboard, with any one square removed, can be tiled by L-tiles. Consider any $2^k \times 2^k$ chessboard with any single square removed.

The missing square must appear in one of the four $2^{k-1} \times 2^{k-1}$ sub-chessboards, so without loss of generality assume the missing square is in the top-left $2^{k-1} \times 2^{k-1}$ sub-chessboard. This sub-chessboard is missing a square, so by the inductive hypothesis, can be tiled by L-tiles.

Afterwards, we can fit a single L-tile in the centre of the main chessboard which occupies a square in all three of the remaining untiled sub-chessboards. If we consider these three sub-chessboards as one corner piece missing, the inductive hypothesis can be used again to tile each sub-chessboard with L-tiles.[1] Hence every square of the original chessboard can be occupied with a square of a L-tiling. Thus the $2^k \times 2^k$ chessboard, with a missing square, can be L-tiled, completing the induction.

[1] Alternatively, the three other sub-chessboards form a giant L. Solution 3.27 from the prescribed text "Mathematical Thinking: Problem Solving and Proofs" shows this giant L can be tiled with L-tiles.

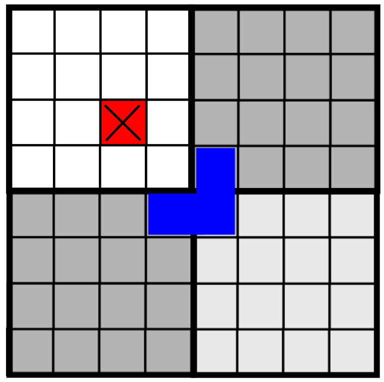

Figure: The chessboard is divided into four sub-chessboards. The red square with the cross is the removed square. The central blue L-tile occupies one square in the remaining three sub-chessboards.