MAT135H1S Calculus I(A)

Solution to even-numbered problem in Section 4.1 and 4.3

(Section 4.1, Q34)
Given that $g(t) = |3t - 4|$. In other words, we have

$$g(t) = \begin{cases} 3t - 4 & \text{if } t \geq 4/3, \\ 4 - 3t & \text{if } t < 4/3. \end{cases}$$

Therefore,

$$g'(t) = \begin{cases} 3 & \text{if } t > 4/3, \\ -3 & \text{if } t < 4/3. \end{cases}$$

and $g'(t)$ does not exist at $t = 4/3$. Hence the critical number is $t = 4/3$.

(Section 4.1, Q44)
Given that $f(x) = x^{-2} \ln x$. Then we have

$$f'(x) = \frac{x^2\left(\frac{1}{x}\right) - \ln x(2x)}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

Solving $f'(x) = 0$, we obtain $1 - 2\ln x = 0$ which implies that $\ln x = 1/2$, or $x = e^{1/2}$.
On the other hand, $f'(x)$ is not defined for $x < 0$. However, the domain of f does not contain these points. Therefore, the only critical number is $x = e^{1/2}$.

(Section 4.3, Q12)
Given that $f(x) = \dfrac{x}{x^2 + 1}$. Differentiating, we obtain

$$f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

(a) Rewriting, we have $f'(x) = \dfrac{(1 + x)(1 - x)}{(x^2 + 1)^2}$.

Interval	$1 + x$	$1 - x$	$(x^2 + 1)^2$	$f'(x)$
$x < -1$	$-$	$+$	$+$	$-$
$-1 < x < 1$	$+$	$+$	$+$	$+$
$x > 1$	$+$	$-$	$+$	$-$

Therefore, from the above chart, we have that $f(x)$ is increasing in the interval $(-1, 1)$, and decreasing in the intervals $(-\infty, -1)$ and $(1, \infty)$.

(b) Since $f'(x)$ changes from negative to positive at $x = -1$, there is a local minimum at $x = -1$, and the local minimum value is

$$f(-1) = \frac{-1}{(-1)^2 + 1} = -\frac{1}{2}.$$

Since $f'(x)$ changes from positive to negative at $x = -1$, there is a local maximum at $x = 1$, and the local maximum value is

$$f(1) = \frac{1}{(1)^2 + 1} = \frac{1}{2}.$$

(c) Differentiating, we obtain

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2) \cdot 2(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x(x^2+1)(x^2-3)}{(x^2+1)^4} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Rewriting, we have

$$f''(x) = \frac{2x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}.$$

Interval	$2x$	$x+\sqrt{3}$	$x-\sqrt{3}$	$(x^2+1)^3$	$f''(x)$
$x < -\sqrt{3}$	$-$	$-$	$-$	$+$	$-$
$-\sqrt{3} < x < 0$	$-$	$+$	$-$	$+$	$+$
$0 < x < \sqrt{3}$	$+$	$+$	$-$	$+$	$-$
$x > \sqrt{3}$	$+$	$+$	$+$	$+$	$+$

Therefore, $f(x)$ is concave up in the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$, and concave down in the intervals $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.
The points of inflection are $(-\sqrt{3}, -\sqrt{3}/4)$, $(0, 0)$ and $(\sqrt{3}, \sqrt{3}/4)$.