FUNCTIONS OF RANDOM VARIABLES (Chapter 6)

The discrete case

Example 1 A coin is tossed twice. Let Y be the number heads that come up. Find the dsn of $X = 3Y - 1$.

Here, $Y \sim$ Bin(2,1/2). So $p(y) = \begin{cases} 1/4, & y=0 \\ 1/2, & y=1 \\ 1/4, & y=2 \end{cases}$

If $y = 0$ then $x = 3(0) - 1 = -1$.
If $y = 1$ then $x = 3(1) - 1 = 2$.
If $y = 2$ then $x = 3(2) - 1 = 5$.

Therefore $p(x) = \begin{cases} 1/4, & x=-1 \\ 1/2, & x=2 \\ 1/4, & x=5 \end{cases}$ (same probabilities but different values)

Note that there is a *one-to-one correspondence* here between x and y values. This made the solution easy.

In general, if Y is a discrete random variable, then $X = g(Y)$ has pdf
$$p(x) = \sum_{y:g(y)=x} p(y).$$

Example 2 $Y \sim$ Bin(2,1/2). Find the dsn of $U = (Y-1)^2$.

In this case there are two possible values of u: 0 (if $y = 1$), and 1 (if $y = 0$ or 2).

$p_U(0) = \sum_{y:(y-1)^2=0} p(y) = p(1) = 1/2$.
$p_U(1) = \sum_{y:(y-1)^2=1} p(y) = p(0) + p(2) = 1/4 + 1/4 = 1/2$.

(Note: The second 1/2 could have been obtained by subtracting the first 1/2 from 1.)

Thus $p(u) = 1/2$, $u = 0,1$. (Ie, $U \sim$ Bern(1/2).)

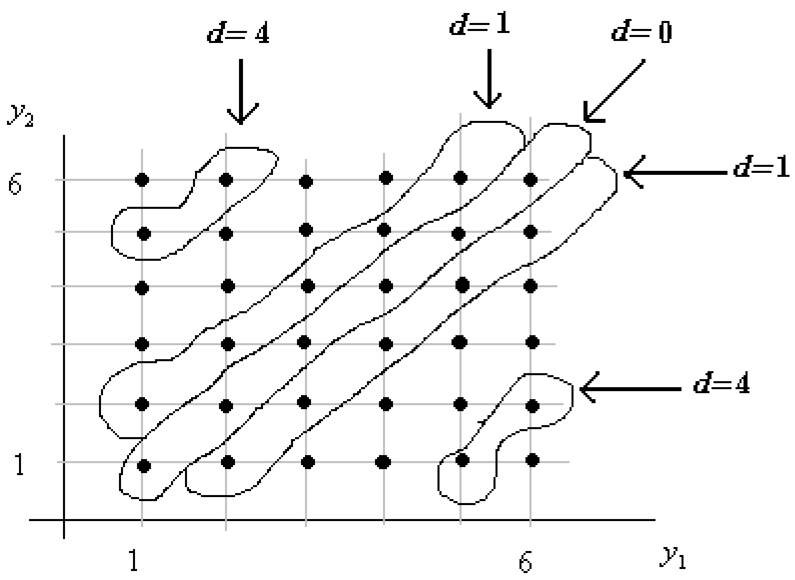
What if we want to find the dsn of a function of *two* rv's?
Then we use the same formula as above, interpreting *y* as a vector quantity.

Example 3 If we roll two dice, what is the expected difference between the two numbers that come up?

Let Y_i be the number on the *i*th die.
We wish to find the expected value of $D = |Y_1 - Y_2|$.

We will first obtain the pdf of *D*, according to $p(d) = \sum_{y_1, y_2 : |y_1 - y_2| = d} p(y_1, y_2)$.

This is best done graphically.



We see that: $p(d) = \begin{cases} 6/36, & d = 0 \\ 10/36, & d = 1 \\ 8/36, & d = 2 \\ 6/36, & d = 3 \\ 4/36, & d = 4 \\ 2/36, & d = 5 \end{cases}$ (note that these pr's sum to 1)

It follows that $ED = \sum_{d=0}^{5} dp(d) = 0 \times \frac{6}{36} + 1 \times \frac{10}{36} + \ldots + 5 \times \frac{2}{36} = \frac{35}{18}$.

Alternatively,

$$ED = \sum_{y_1, y_2} |y_1 - y_2| \, f(y_1, y_2) = |1-1| \frac{1}{36} + |1-2| \frac{1}{36} + \ldots + |6-6| \frac{1}{36} = \frac{35}{18}.$$

The continuous case

There are three main strategies we'll look at:

the *cdf method*, the *transformation method (or rule)*, the *mgf method*.

1. The cdf method

This consists of two steps:

1. Find the cdf of the rv of interest.
2. Differentiate this cdf to obtain the required pdf.

Example 4 Suppose that $Y \sim U(0,2)$. Find the pdf of $X = 3Y - 1$.

1. X has cdf $F(x) = P(X < x)$ (since X is cts, we may write < instead of $\leq$)

$$= P(3Y - 1 < x)$$

$$= P\left(Y < \frac{x+1}{3}\right)$$

$$= \int_0^{(x+1)/3} \frac{1}{2} dy \qquad \text{(since } f(y) = 1/2,\ 0 < y < 2)$$

$$= \frac{x+1}{6},\ -1 < x < 5 \quad \text{(since } 3(0) - 1 = -1 \text{ and } 3(2) - 1 = 5).$$

2. So X has pdf $f(x) = F'(x) = \frac{1}{6}$, $-1 < x < 5$. (Ie, $X \sim U(-1, 5)$.)

Example 5 Suppose that $X, Y \sim \text{iid}\, U(0,1)$. Find the pdf of $U = X + Y$.

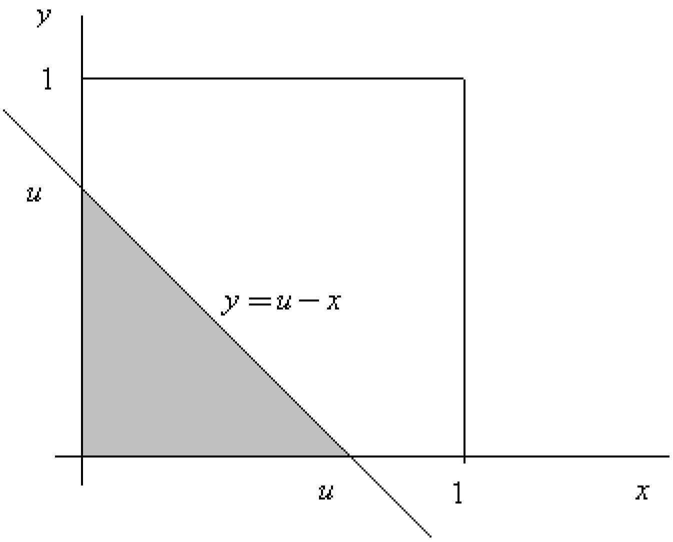
First observe that $f(x,y) = 1$, $0 < x < 1$, $0 < y < 1$.

1. So U has cdf $F(u) = P(U < u)$

$$= P(X + Y < u)$$

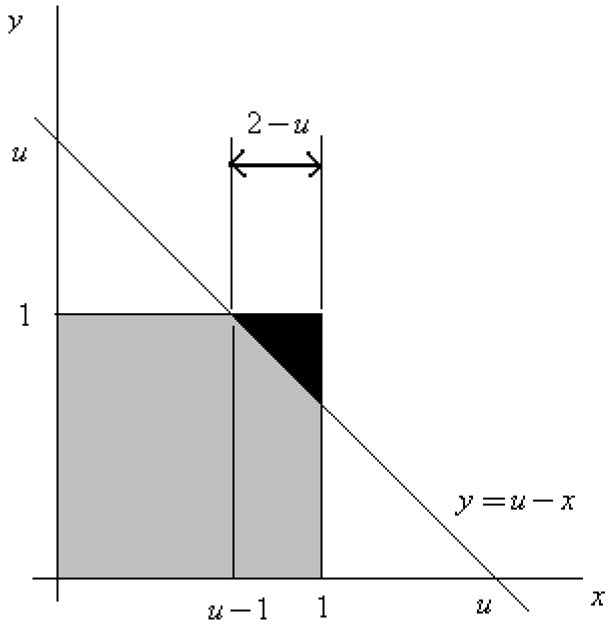
$$= P(Y < u - X)$$

$$= \frac{1}{2}u^2 \qquad \text{(area of shaded region below).}$$



But this is true only if $u < 1$.

For $u > 1$, we need to draw another diagram.



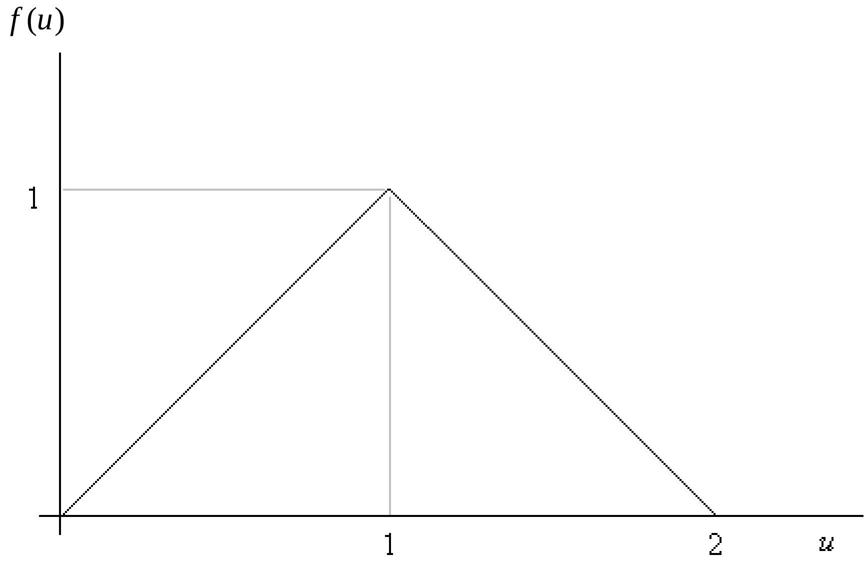
We see that $F(u) = P(Y < u - X)$ (area of grey region)

$= 1 - P(Y > u - X)$ (1 minus area of black region)

$= 1 - \frac{1}{2}(2-u)^2$.

In summary, $U = X + Y$ has cdf $F(u) = \begin{cases} \frac{1}{2}u^2, & 0 < u < 1 \\ 1 - \frac{1}{2}(2-u)^2, & 1 < u < 2 \end{cases}$

2. Therefore U has pdf $f(u) = F'(u) = \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 < u < 2 \end{cases}$



2. The transformation method

This is a shortcut version of the cdf method.

Suppose that Y is a cts rv with pdf $f(y)$, and $x = g(y)$ is a function which is either

(a) strictly increasing

or **(b)** strictly decreasing,

for all possible values y of Y.

Then $X = g(Y)$ has pdf

$$f(x) = f(y)\left|\frac{dy}{dx}\right|,$$

where $y = g^{-1}(x)$. (This is the inverse function of g.)

Example 6 Suppose that $Y \sim U(0,2)$.
Find the pdf of $X = 3Y - 1$. (This is the same as Example 4.)

Here: $x = 3y - 1$ ($x = g(y)$ is a strictly increasing function)

$y = \dfrac{x+1}{3}$ (the inverse function of g)

$\dfrac{dy}{dx} = \dfrac{1}{3}$

$f(y) = 1/2,\ 0 < y < 2.$

So: $f(x) = f(y)\left|\dfrac{dy}{dx}\right| = \dfrac{1}{2}\left|\dfrac{1}{3}\right| = \dfrac{1}{6},\ -1 < x < 5$ (as before).

Example 7 $Y \sim N(a, b^2)$. Find the dsn of $Z = \dfrac{Y-a}{b}$.

Here: $z = \dfrac{y-a}{b}$ (a strictly increasing function of y)

$y = a + bz$

$\dfrac{dy}{dz} = b$

$f(y) = \dfrac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(y-a)^2},\ -\infty < y < \infty$.

So $f(z) = f(y)\left|\dfrac{dy}{dz}\right| = \dfrac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}((a+bz)-a)^2}|b| = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2},\ -\infty < z < \infty$.

Thus $Z \sim N(0,1)$.

Exercise: $Z \sim N(0,1)$. Find the dsn of $Y = a + bZ$ (very similar to above).

Example 8 $Z \sim N(0,1)$. Find the dsn of $X = Z^2$.

In this case, $x = z^2$ is neither strictly increasing nor strictly decreasing.
So the transformation method cannot be used (at least not directly).

We could find the pdf of X using the cdf method (do this as an exercise).

Another way to proceed is via the mgf method.