9+7 16/20

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Q1

Solution: $\beta = \{(1,1,0), (1,0,-1), (2,1,0)\}$ for $x, y \in \mathbb{R}^3$

if $\langle T(x), y\rangle = \langle x, T(y)\rangle$ then T is symmetric

For a). $T(1,1,0) = (2,1,-1)$

a). $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T(1,0,-1) = (2,1,-1)$

$T(2,1,0) = (2,1,0)$

Let $y = (1,0,0)$ then $T(y) = T(1,0,0)$

$= (-2,-1,1) + (2,1,0)$

$= (0,0,1)$

$\langle (1,1,0), (0,0,1)\rangle = 0$

$\langle (2,1,-1), (1,0,0)\rangle = 2$

$0 \neq 2$

So a) is not such a matrix.

For b). $T(1,1,0) = (2,2,0)$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$T(1,0,-1) = (2,0,-1)$

$T(2,1,0) = (4,2,0)$

Let $y = (1,0,0)$ then $T(y) = T(1,0,0)$

$= (-2,-2,0) + (4,2,0)$

$= (2,0,0)$

$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$\langle (1,1,0), (2,0,0)\rangle = 2 = \langle (2,2,0), (1,0,0)\rangle$

$\langle (1,0,-1), (2,0,0)\rangle = 2 = \langle (2,0,-1), (1,0,0)\rangle$

$\langle (2,1,0), (2,0,0)\rangle = 4 = \langle (4,2,0), (1,0,0)\rangle$

For c). $T(1,1,0) = (4,1,-1)$

$T(1,0,-1) = (5,1,-4)$

$T(2,1,0) = (4,3,0)$

Let $y=(1,0,0)$ then

$$T(y) = T(1,0,0)$$
$$= (-4,-1,1)+(4,3,0)$$
$$= (0,2,1)$$

$\langle(1,1,0),(0,2,1)\rangle = 0$

$\langle(4,1,-1),(1,0,0)\rangle = 4$

$2 \neq 4$

So c). is not such a matrix.

Then only $A=\begin{bmatrix} 2&0&0 \\ 0&2&0 \\ 0&0&2 \end{bmatrix}$ is a matrix $A=[T]_\beta^\beta$ that defines symmetric ~~s~~mappings of $\mathbb{R}^3$.

d). Solution.

$T(1,0,0) = 2(1,0,0)+3(0,1,0)+(-1)(0,0,1) = (2,3,-1)$

$T(0,1,0) = 3(1,0,0)+0(0,1,0)+1(0,0,1) = (3,0,1)$

$T(0,0,1) = (-1)(1,0,0)+1(0,1,0)+1(0,0,1) = (-1,1,1)$

Therefore $A=\begin{bmatrix} 2&3&-1 \\ 3&0&1 \\ -1&1&1 \end{bmatrix}$

Since $A=A^T$, A is symmetric

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Q2

Find the spectral decomposition of matrix $A=\begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 2 \end{bmatrix}$

Solution

$$\det(A-\lambda I)=\det\begin{vmatrix} 1-\lambda & 0 & -i \\ 0 & 2-\lambda & 0 \\ i & 0 & 2-\lambda \end{vmatrix}$$

$$=(1-\lambda)(2-\lambda)^2+(-i)(0-i(2-\lambda))$$

$$=(1-\lambda)(2-\lambda)^2+(i)^2(2-\lambda)$$

$$=(1-\lambda)(2-\lambda)^2-(2-\lambda)$$

$$=(2-3\lambda+\lambda^2-1)(2-\lambda)$$

$$=(1-3\lambda+\lambda^2)(2-\lambda)$$

So eigenvalues are $\lambda_1=2$, $\lambda_2=\frac{3+\sqrt{9-4}}{2}=\frac{3+\sqrt{5}}{2}$, $\lambda_3=\frac{3-\sqrt{5}}{2}$

For $\lambda=2$

$$E_2=\text{null}\begin{bmatrix} -1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}=\text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Apply Gram-Schmidt process to this

$V_1=(0,1,0)$

$\|V_1\|=\sqrt{\langle V,V\rangle}=1$

For $\lambda=\frac{3+\sqrt{5}}{2}$

$$E_{\frac{3+\sqrt{5}}{2}}=\text{null}\begin{bmatrix} \frac{-1-\sqrt{5}}{2} & 0 & -i \\ 0 & \frac{1-\sqrt{5}}{2} & 0 \\ i & 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}=\text{span}\left\{\begin{bmatrix} 2i \\ 0 \\ -1-\sqrt{5} \end{bmatrix}\right\}$$

Let $V_2=(2i, 0, -1-\sqrt{5})$

$\|V_2\|=\sqrt{\langle V_2, V_2\rangle}$

$=\sqrt{2i\cdot(\overline{2i})+(-1-\sqrt{5})^2}$

$\|V_2\|=\sqrt{10+2\sqrt{5}}$

For $\lambda=\frac{3-\sqrt{5}}{2}$

$$E_{\frac{3-\sqrt{5}}{2}}=\text{null}\begin{bmatrix} \frac{\sqrt{5}-1}{2} & 0 & -i \\ 0 & \frac{1+\sqrt{5}}{2} & 0 \\ i & 0 & \frac{1+\sqrt{5}}{2} \end{bmatrix}=\text{span}\left\{\begin{bmatrix} 2i \\ 0 \\ \sqrt{5}-1 \end{bmatrix}\right\}$$

the same as previous one.

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$$\|V_3\| = \sqrt{\langle V_3, V_3 \rangle}$$

$$= \sqrt{(2i)(-2i)+(\sqrt{5}-1)^2}$$

$$= \sqrt{4+5+1-2\sqrt{5}}$$

$$= \sqrt{10-2\sqrt{5}}$$

Then the orthonormal basis is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{10+2\sqrt{5}}}\begin{pmatrix} 2i \\ 0 \\ -1-\sqrt{5} \end{pmatrix}, \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{pmatrix} 2i \\ 0 \\ \sqrt{5}-1 \end{pmatrix} \right\}$

Suppose $P = \begin{pmatrix} 0 & \frac{2i}{\sqrt{10+2\sqrt{5}}} & \frac{2i}{\sqrt{10-2\sqrt{5}}} \\ 1 & 0 & 0 \\ 0 & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \end{pmatrix}$ $\quad P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-2i}{\sqrt{10+2\sqrt{5}}} & 0 & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \\ \frac{-2i}{\sqrt{10-2\sqrt{5}}} & 0 & \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \end{pmatrix}$

s.t. $D = P \cdot A \cdot P^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}$

Want spectral decomposition of $A = 2P_{E_2} + \frac{3+\sqrt{5}}{2} P_{E_{(\frac{3+\sqrt{5}}{2})}} + \frac{3-\sqrt{5}}{2} P_{E_{(\frac{3-\sqrt{5}}{2})}}$

$$= 2P\begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}P^{-1} + \frac{3+\sqrt{5}}{2}P\begin{pmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{pmatrix}P^{-1} + \frac{3-\sqrt{5}}{2}P\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix}P^{-1}$$

$$P\begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{pmatrix}$$

$$P\begin{pmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & \frac{2i}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \end{pmatrix}P^{-1} = \begin{pmatrix} \frac{+4}{10+2\sqrt{5}} & 0 & \frac{-(2-2\sqrt{5})i}{10+2\sqrt{5}} \\ 0 & 0 & 0 \\ \frac{(-2-2\sqrt{5})i}{10+2\sqrt{5}} & 0 & \frac{-(-1-\sqrt{5})^2}{10+2\sqrt{5}} \end{pmatrix}$$

$$P\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & 0 & \frac{2i}{\sqrt{10-2\sqrt{5}}} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \end{pmatrix}P^{-1} = \begin{pmatrix} \frac{+4}{10-2\sqrt{5}} & 0 & \frac{-(2\sqrt{5}-2)i}{10-2\sqrt{5}} \\ 0 & 0 & 0 \\ \frac{-(2\sqrt{5}-2)i}{10-2\sqrt{5}} & 0 & \frac{-(\sqrt{5}-1)^2}{10-2\sqrt{5}} \end{pmatrix}$$

$\frac{(2-2\sqrt{5})i}{10-2\sqrt{5}}$

So $A = 2\begin{pmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{pmatrix} + \frac{3+\sqrt{5}}{2}\begin{pmatrix} \frac{4}{10+2\sqrt{5}} & 0 & \frac{(2+2\sqrt{5})i}{10+2\sqrt{5}} \\ 0 & 0 & 0 \\ \frac{(2+2\sqrt{5})i}{10+2\sqrt{5}} & 0 & \frac{-(1+\sqrt{5})^2}{10+2\sqrt{5}} \end{pmatrix} + \frac{3-\sqrt{5}}{2}\begin{pmatrix} \frac{4}{10-2\sqrt{5}} & 0 & \\ 0 & 0 & 0 \\ \frac{(2-2\sqrt{5})i}{10-2\sqrt{5}} & 0 & \frac{-(\sqrt{5}-1)^2}{10-2\sqrt{5}} \end{pmatrix}$

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Q3. Rewrite $(a_1x_1+\cdots+a_nx_n)^2$ in the form x^TAx.

Solution: $(a_1x_1+\cdots+a_nx_n)^2 = a_1^2x_1^2+2a_1a_2x_1x_2+2a_1a_3x_1x_3+\cdots+2a_1a_nx_1x_n+a_2^2x_2^2+\cdots+a_n^2x_n^2$

$$A=\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & a_3^2 & & \vdots \\ \vdots & \vdots & & \ddots & \\ a_na_1 & a_na_2 & \cdots & & a_n^2 \end{bmatrix}$$

$$(a_1x_1+\cdots+a_nx_n)^2=[x_1,x_2,\cdots,x_n]\begin{bmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & \cdots & a_2a_n \\ \vdots & & \ddots & \vdots \\ a_na_1 & \cdots & \cdots & a_n^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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Q4.

Conic section $7x^2+2\sqrt{3}xy+5y^2=1$

Solution: Write it in form of $Ax^2+2Bxy+Cy^2=[x\ y]\begin{bmatrix}A & B\\ B & C\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=1$

$$M=\begin{bmatrix}A & B\\ B & C\end{bmatrix}=\begin{bmatrix}7 & \sqrt{3}\\ \sqrt{3} & 5\end{bmatrix} \text{ in this case}$$

$$\begin{aligned}P(\lambda)&=(7-\lambda)(5-\lambda)-(\sqrt{3})^2=(7-\lambda)(5-\lambda)-3\\ &=\lambda^2-12\lambda+32\\ &=(\lambda-4)(\lambda-8)\end{aligned}$$

We have two eigenvalues, $\lambda_1=4$ and $\lambda_2=8$.

$$E_4=\text{null}\begin{bmatrix}7-4 & \sqrt{3}\\ \sqrt{3} & 5-4\end{bmatrix}=\text{null}\begin{bmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}=\text{span}\left\{\begin{bmatrix}1\\ -\sqrt{3}\end{bmatrix}\right\}$$

and $$E_8=\text{null}\begin{bmatrix}7-8 & \sqrt{3}\\ \sqrt{3} & 5-8\end{bmatrix}=\text{null}\begin{bmatrix}-1 & \sqrt{3}\\ \sqrt{3} & -3\end{bmatrix}=\text{span}\left\{\begin{bmatrix}\sqrt{3}\\ 1\end{bmatrix}\right\}$$

Hence $\alpha=\left\{\frac{1}{2}\begin{bmatrix}1\\ -\sqrt{3}\end{bmatrix}, \frac{1}{2}\begin{bmatrix}\sqrt{3}\\ 1\end{bmatrix}\right\}$ is an orthonormal basis of $\mathbb{R}^2$

Let $Q=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{-\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$, the change of basis matrix from α to the standard basis.

$$\begin{aligned}\text{Then } Q^{-1}MQ&=\frac{1}{\frac{1}{2}\cdot\left(\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)^2}\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}7 & \sqrt{3}\\ \sqrt{3} & 5\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\\ &=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}7 & \sqrt{3}\\ \sqrt{3} & 5\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\\ &=\begin{bmatrix}4 & 0\\ 0 & 8\end{bmatrix}\end{aligned}$$

Now introduce new coordinates in $\mathbb{R}^2$ by setting $\begin{bmatrix}x_1'\\ x_2'\end{bmatrix}=Q^{-1}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

~~note that~~ $\cancel{QQ^T=\left(\frac{1}{2}\right)\begin{bmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{bmatrix}\left(\frac{1}{2}\right)\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}}$

$\cancel{=\begin{bmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & \end{bmatrix}}$

[redacted]

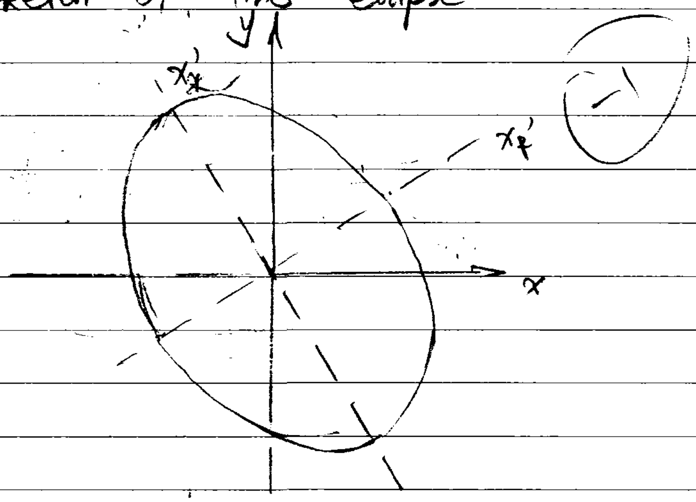
The equation of the conic is

$$[x_1', x_2'] \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$= [4x_1' \quad 8x_2'] \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$= 4x_1'^2 + 8x_2'^2 = 1 \qquad \frac{x_1'^2}{2} + x_2'^2 = \frac{1}{8}$$

The sketch of this ellipse



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Q5

Solution.

Suppose $p(x)=a+bx+cx^2$,

$q(x)=m+nx+tx^2$

$T^*(q(x))=T^*(m+nx+tx^2)=\alpha+\beta x+\gamma x^2$ for arbitrary $a,b,c,m,n,t,\alpha,\beta,\gamma \in \mathbb{R}$

We want to find T^* such that

$\langle T(p(x)), q(x)\rangle = \langle p(x), T^*(q(x))\rangle$

$$\begin{aligned}
LHS &= \langle b+2cx, m+nx+tx^2\rangle \\
&= (b-2c)(m-n+t)+b\cdot m+(b+2c)(m+n+t) \\
&= bm-2cm-bn+2cn+bt-2ct \\
&\quad +bm \\
&\quad +bm+2cm+bn+2cn+bt+2ct \\
&= 3bm+4cn+2bt \\
&= b(3m+2t)+c(4n)
\end{aligned}$$

$$\begin{aligned}
RHS &= \langle a+bx+cx^2, \alpha+\beta x+\gamma x^2\rangle \\
&= (a-b+c)(\alpha-\beta+\gamma)+a\cdot\alpha+(a+b+c)(\alpha+\beta+\gamma) \\
&= a\alpha-b\alpha+c\alpha-a\beta+b\beta-c\beta+a\gamma-b\gamma+c\gamma \\
&\quad +a\alpha \\
&\quad +a\alpha+b\alpha+c\alpha+a\beta+b\beta+c\beta+a\gamma+b\gamma+c\gamma \\
&= 3a\alpha+2c\alpha+2b\beta+2a\gamma+2c\gamma \\
&= a(3\alpha+2\gamma)+b(2\beta)+c(2\alpha+2\gamma)
\end{aligned}$$

Since $LHS=RHS$

$$b(3m+2t)+c(4n)=a(3\alpha+2\gamma)+b(2\beta)+c(2\alpha+2\gamma)$$

So

$$3\alpha+2\gamma=0$$
$$2\beta=3m+2t$$
$$2\alpha+2\gamma=4n$$

Solve this we get

$$\alpha=-4n$$
$$\beta=\tfrac{3}{2}m+t$$
$$\gamma=6n$$

Therefore $T^*(q(x))=T^*(m+nx+tx^2)=-4n+(\tfrac{3}{2}m+t)x+6nx^2$

i.e. $T^*(p(x))=T^*(a+bx+cx^2)=-4b+(\tfrac{3}{2}a+c)x+6bx^2$

for arbitrary $p(x)=a+bx+cx^2$.

[redacted]

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Q6

Proof: ($\Rightarrow$)

Say $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

Let $T_k = A \cdot E_k A^{-1}$ where $E_k = e_k e_k^T$, for $k=1,2,3$.

Then $T = aT_1 + bT_2 + cT_3$

~~and $T=T^*$ iff $aT_1+bT_2+cT_3 = aT_1^*+bT_2^*+cT_3^*$~~

We just simply the very complex calculation

$T_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $T_2 = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$T_3 = \frac{1}{3}\begin{bmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

~~we find that $T_1 = T_1^*$ but $T_2 = T_2^*$, $T_3 = T_2^*$~~

Therefore if $b=c$ then $T=T^*$ i.e. T is self-adjoint.

$\Downarrow$ $aT_1+bT_2+cT_3 = aT_1^*+bT_2^*+cT_3^*$

($\Leftarrow$). Suppose $T=T^*$

then $aT_1+bT_2+cT_3 = aT_1^*+bT_2^*+cT_3^*$

$\frac{1}{3}a\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3}b\begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3}c\begin{bmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$= \frac{1}{3}a\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3}b\begin{bmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} + \frac{1}{3}c\begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence $b=c$.

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Q7

(a). Proof: ~~(b)~~

Suppose $w_1 = a_{11}v_1 + \cdots + a_{1n}v_n$

$w_2 = a_{21}v_1 + \cdots + a_{2n}v_n$

$\cdots$

$w_k = a_{k1}v_1 + \cdots + a_{kn}v_n$

Then $A = ([w_1]_\alpha \, [w_2]_\alpha \cdots [w_k]_\alpha) = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & & & \vdots \\ \vdots & & & \vdots \\ a_{1n} & \cdots & \cdots & a_{kn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$

c_i means the ith column

$A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{k1} & \cdots & \cdots & a_{kn} \end{bmatrix} = \begin{bmatrix} — c_1^T — \\ — c_2^T — \\ \vdots \\ — c_n^T — \end{bmatrix}$

this is A^TA

Then $AA^T = \begin{bmatrix} — c_1^T — \\ — c_2^T — \\ \vdots \\ — c_n^T — \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} c_1 \cdot c_1 & \cdots & c_1 \cdot c_n \\ \vdots & & \vdots \\ c_n \cdot c_1 & \cdots & c_n \cdot c_n \end{bmatrix}$

$P_W(v_1) = \frac{\langle v_1, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots \frac{\langle v_1, w_k \rangle}{\|w_k\|^2} w_k$ since β is orthonormal

$\|w_i\|^2 = 1$ for $i = 1, \ldots k$

$= \langle v_1, w_1 \rangle w_1 + \cdots + \langle v_1, w_k \rangle w_k$

$= \langle v_1, (a_{11}v_1 + \cdots + a_{1n}v_n) \rangle w_1 + \cdots + \langle v_1, (a_{k1}v_1 + \cdots + a_{kn}v_n) \rangle w_k$

$= (w_1 + w_2 + \cdots + w_k)\, c_1$

$$\boxed{\text{Hence } P_W(v_i) = \frac{\langle v_i, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v_i, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \cdots + \frac{\langle v_i, w_k \rangle}{\langle w_k, w_k \rangle} w_k}$$

$[P_W(v_1)]_\alpha = a_{11}c_1 + a_{21}c_2 + \cdots + a_{k1}c_1$ [illegible] is some form of [illegible]

$\to [P_W(v_i)]_{\alpha\alpha} = \begin{pmatrix} c_1^T c_i \\ \\ c_n^T c_i \end{pmatrix}$

So $[P_W]_{\alpha\alpha} = AA^T$

(b). ~~Proof: Want $[P_W]^T_{\alpha\alpha} = (AA^T)^T = A^TA = [P_W]_{\alpha\alpha}$.~~

~~So it means we need to prove AA^T is symmetric.~~

~~And this is proved in part (a).~~

~~Then want again $[P_W]^2_{\alpha\alpha} = (AA^T)^2 = AA^TAA^T = AA^T = [P_W]_{\alpha\alpha}$~~

~~namely, we need $AA^T = I$~~

~~By theorem 4.6.3 that~~

~~$I = P_1 + P_2 + \cdots + P_k$~~

~~Since in this problem P_i = (the part I don't know how to write in part (a))~~

~~then $AA^T = [P_W]_{\alpha\alpha} = P_1 + P_2 + \cdots + P_k = I$~~

~~Hence $[P_W]^2_{\alpha\alpha} = [P_W]_{\alpha\alpha}$ and $[P_W]^T_{\alpha\alpha} = [P_W]_{\alpha\alpha}$ proved~~

(b). Solution: $[P_W]_{\alpha\alpha} = AA^T$

$$([P_W]_{\alpha\alpha})^2 = (AA^T)^2 = (AA^T)(AA^T) = AIA^T = AA^T = [P_W]_{\alpha\alpha}$$ 5

$$([P_W]_{\alpha\alpha})^T = (AA^T)^T = (A^T)^TA^T = AA^T = [P_W]_{\alpha\alpha}.$$