DERIVATIONS: NATURAL DEDUCTION
Part 1

3.2 E1
Which inference rule justifies the following arguments? (mp, mt, dn or none)

a) $\sim R \rightarrow P$
$\sim R$
$\therefore P$

MP

b) $\sim\sim S \rightarrow T$
$\therefore S \rightarrow T$

NONE!
DN cannot be used on a sentential part

c) $P \rightarrow \sim Q$
Q
$\therefore \sim P$

NONE!
First you must use DN, then you can use MT

d) $(P \rightarrow \sim R) \rightarrow \sim S$
$\sim\sim S$
$\therefore \sim(P \rightarrow \sim R)$

MT

e) $\sim(\sim P \rightarrow Q)$
$\therefore P \rightarrow Q$

None!

f) $P \rightarrow (P \rightarrow \sim P)$
P
$\therefore P \rightarrow \sim P$

MP

g) $S \rightarrow R$
$\sim P$
$\therefore \sim S$

NONE!

h) $Q \rightarrow (S \rightarrow P)$
$\sim (S \rightarrow P)$
$\therefore \sim Q$

MT

3.2 E2

What can you infer (if anything) in one step from the following? What rule of inference are you using? (mp, mt, dn)

a) $P \rightarrow R$
$\sim P$
$\therefore ?$

nothing with MP/MT

b) $\sim\sim(V \rightarrow W)$
$\sim W$
$\therefore ?$

nothing in one step with MP/MT.
After DN on the first premise, MT yields ~V.

c) $\sim S \rightarrow \sim\sim T$
$\sim S$
$\therefore ?$

$\sim\sim T$ MP

d) $\sim Y \rightarrow \sim Z$
$\sim Z$
$\therefore ?$

nothing with MP/MT

e) $P \rightarrow (Q \rightarrow R)$
$\sim Q \rightarrow R$
$\therefore ?$
nothing with MP/MT

f) $P \rightarrow (Q \rightarrow R)$
$\sim(Q \rightarrow R)$
$\therefore ?$

$\sim P$ MT

g) $\sim\sim(\sim P \rightarrow \sim\sim\sim Q)$
$\therefore ?$

$(\sim P \rightarrow \sim\sim\sim Q)$
DN

h) $\sim Z \rightarrow \sim X$
$\therefore ?$

$\sim\sim(\sim Z \rightarrow \sim X)$
DN

i) $(P \rightarrow Q) \rightarrow R$
$P \rightarrow Q$
$\therefore ?$

R MP

j) $X \rightarrow \sim Y$
Y
$\therefore ?$
nothing in one step.
After DN on the second premise, MT gets you ~X.

k) $\sim W \rightarrow (Z \rightarrow \sim X)$
$\sim\sim X$
$\therefore ?$

nothing

l) $(\sim P \rightarrow R) \rightarrow \sim Q$
$\sim\sim Q$
$\therefore ?$

$\sim(\sim P \rightarrow R)$
MT

In all of these, you can infer the double negated premises (premise with two ~ in front) with DN. For example, a) ~~(~S → ~~T) dn, or ~~~S dn.

3.3 E1:

Check the work in the following derivations. Does each line follow from available lines using the rule cited?

(a) $\sim T \rightarrow \sim S$. $R \rightarrow \sim\sim T$. S. $\therefore R$

1	~~Show~~ $\sim R$		ERROR. show line incorrect
2	S	pr3	
3	$\sim T \rightarrow \sim S$	pr1	
4	T	2 3 mt	ERROR. You need the negated consequent to use MT.
5	$R \rightarrow \sim\sim T$	pr2	
6	$R \rightarrow T$	5 dn	ERROR. dn cannot be used on a sentential component.
7	R	4 6 mp	ERROR. mp moves from a conditional and the antecedent to the consquent.
8		7 dd	

This cannot be fixed. It is not valid, so the conclusion cannot be derived from the premises.

(b) $\sim(P \rightarrow \sim Q) \rightarrow \sim\sim S$. $Q \rightarrow \sim S$. Q $\therefore \sim P$

1	~~Show~~ $\sim P$		
2	$\sim S$	pr2 pr3 mp	
3	$\sim(P \rightarrow \sim Q) \rightarrow S$	pr1 dn	ERROR. dn cannot be used on a sentential component.
4	$\sim\sim(P \rightarrow \sim Q)$	2 3 mt	
5	$P \rightarrow \sim Q$	4 dn	
6	$\sim\sim Q$	pr3 dn	
7	$\sim P$	5 6 mt	
8		7 dd	

To fix this one: $\sim(P \rightarrow \sim Q) \rightarrow \sim\sim S$. $Q \rightarrow \sim S$. Q $\therefore \sim P$

1	~~Show~~ $\sim P$	
2	$\sim S$	pr2 pr3 mp
3	$\sim\sim\sim S$	2 dn
4	$\sim\sim(P \rightarrow \sim Q)$	pr1 3 mt
5	$P \rightarrow \sim Q$	4 dn
6	$\sim\sim Q$	pr3 dn
7	$\sim P$	5 6 mt
8		7 dd

(c) Z→ (X →~W). ~Z→ ~X. X. ∴~W

1	~~Show~~ ~W		ALL CORRECT!
2	X	pr3	
3	~~X	2 dn	
4	~Z→~X	pr2	
5	~~Z	3 4 mt	
6	Z	5 dn	
7	Z→ (X →~W).	pr1	
8	X →~W	6 7 mp	
9	~W	2 8 mp dd	

(d) ~P → S. ~S. Q → (P → Q). ~(~S → Q) → ~P. ∴P → Q

1	~~Show~~ P → Q		
2	~P → S	pr1	
3	~S	pr2	
4	~~P	2 3 mt	
5	P	4 dn	
6	Q	1 5 mp	**VERY BIG ERROR!** The show line cannot be used in your proof. After all, it is what you are trying to show! You *can* use *canceled* show lines, but those would be in a previous subderivation (section 3.6).
7	Q → (P → Q)	pr3	
8	P → Q	6 7 mp dd	

3.3 E2

Construct direct derivations for the following, showing that the conclusion can be validly inferred from the premises.

(a) P → Q. R → ~Q. ~S → R. P. ∴S

1	~~Show~~ S	show conclusion
2	P → Q	pr1
3	P	pr4
4	Q	2 3 mp (or pr1 pr4 mp)
5	~~Q	4 dn
6	R → ~Q	pr2
7	~R	5 6 mt (or 5 pr2 mt)
8	~S → R	pr3
9	~~S	7 8 mt (or 7 pr3 mt)
10	S	9 dn
11		10 dd

(b) Y. $X \to (Y \to Z)$. $\sim X \to \sim W$. W. $\therefore \sim\sim Z$

1	~~Show~~ $\sim\sim Z$	show conclusion
2	W	pr4
3	$\sim\sim W$	2 dn
4	$\sim X \to \sim W$	pr3
5	$\sim\sim X$	3 4 mt (or 3 pr3 mt)
6	X	5 dn
7	$X \to (Y \to Z)$	pr2
8	$Y \to Z$	6 7 mp (or 6 pr2 mp)
9	Y	pr1
10	Z	8 9 mp (or 9 pr1 mp)
11	$\sim\sim Z$	10 dn
12		11 dd

(c) $P \to (Q \to (R \to S))$. $\sim\sim P$. $(R \to S) \to \sim P$. $\therefore \sim Q$

1	~~Show~~ $\sim Q$	show conclusion
2	$\sim\sim P$	pr2
3	P	2 dn
4	$P \to (Q \to (R \to S))$	pr1
5	$Q \to (R \to S)$	3 4 mp (or 3 pr1 mp)
6	$(R \to S) \to \sim P$	pr3
7	$\sim(R \to S)$	2 6 mt (or pr2 pr3 mt)
8	$\sim Q$	5 7 mt
9		8 dd

(d) $(P \to \sim Q) \to (\sim R \to S)$. $\sim S$. $\sim(P \to \sim Q) \to T$. $T \to S$. $\therefore R$

1	~~Show~~ R	show conclusion
2	$\sim S$	pr2
3	$T \to S$	pr4
4	$\sim T$	2 3 mp (or pr2 pr4 mp)
5	$\sim(P \to \sim Q) \to T$	pr3
6	$\sim\sim(P \to \sim Q)$	5 4 mt (or 4 pr3 mt)
7	$P \to \sim Q$	6 dn
8	$(P \to \sim Q) \to (\sim R \to S)$	pr1
9	$\sim R \to S$	7 8 mp (or 7 pr1 mp)
10	$\sim\sim R$	2 9 mt (or pr2 9 mt)
11	R	10 dn
12		11 dd

3.4 E1 Now, try a few:

(a) W → (X → ~Y). Z → Y. Z. ∴ W→~X

1	~~Show~~ W→~X	show conclusion
2	W	ass CD
3	X → ~Y	2 pr1 mp
4	Y	pr2 pr3 mp
5	~~Y	4 dn
6	~X	3 5 mt
7		5 cd

(b) P→(R→T). P→~T. W→R. ∴ P→~W

1	~~Show~~ P→~W	SHOW CONC.
2	P	ASS CD
3	P→(R→T)	PR1
4	P→~T	PR2
5	W→R	PR3
6	R → T	2 3 MP
7	~T	2 4 MP
8	~R	6 7 MT
9	~W	8 5 MT CD

3.5 E1 Now, try a few:

(a) P→R. Q →~R. P. ∴~ (P→Q)

1	~~Show~~ ~ (P→Q)	show conc.
2	P → Q	ass ID
3	P → R	pr1
4	Q → ~R	pr2
5	P	pr3
6	Q	2 5 mp
7	R	3 5 mp
8	~~R	7 dn
9	~Q	8 4 mt
10		6 9 ID

(b) ~(~Y → Y) → Y. ∴Y

1	~~Show~~ Y	Show conc.
2	~Y	ass ID
3	~~(~Y → Y)	2 pr1 mt
4	~Y → Y	3 dn
5	Y	2 4 mp
6		2 5 id

(c) P → R. R→S. P. S → ~P. ∴ Z

1	~~Show~~ Z	show conc.	
2	~Z	ass id	
3	P →R	pr1	
4	R → S	pr2	
5	P	pr3	
6	S → ~P	pr4	
7	R	3 5 mp	
8	S	4 7 mp	
9	~P	6 8 mp	
10		5 9 id	

3.6 EG1 Let's try one:

(a) P → Q. S → ~Q. (P→~S) → T. ∴ P → T

1	~~Show~~ P → T		show conclusion
2	P	ass cd	
3	~~show~~ P → ~S		show ant. pr3
4	P	ass cd	
5	Q	4 pr1 mp	alternately, 2 pr1 mp
6	~~Q	5 dn	
7	~S	6 pr2 mt	
8		7 cd	
9	T	3 pr3 mp	
10		9 cd	

3.6 EG2 Let's try one:

P → ~Q. S → P. Q → S. ∴ Q → ~R

1	~~Show~~ Q → ~R	show conc.	
2	Q	ass cd	
3	~~show~~ ~R	show cons.	
4	R	ass id	alternatively ~~R
5	S	pr3 2 mp	
6	P	5 pr2 mp	
7	~Q	6 pr1 mp	
8	Q	2 r	reiterate to get it under ass. for id.
9		7 8 id	
10		3 cd	

3.6 E1

~P→R. P→S. T→~S. (~R→~~S)→(T→P). T. ∴ U

1	~~SHOW~~ U	SHOW CONC
2	~U	ASS ID
3	~S	PR5 PR3 MP
4	~P	PR2 3 MT
5	R	PR1 4 MP
6	~~SHOW~~ ~R→~~S	SHOW ANT. PR4
7	~R	ASS CD
8	~~SHOW~~ ~~S	
9	~S	ASS ID
10	R	5 R
11	~R	7 R 10 ID
12		8 CD
13	T→P	6 PR4 MP
14	P	PR5 13 MP
15		4 14 ID

3.6 E 2 Provide a derivation to prove that the following arguments are valid.

a) S → ~P. P →R. R → (~S → Q). ∴ P → Q

1	~~SHOW~~ P → Q	SHOW CONC
2	P	ass cd
3	R	pr2 2 mp
4	~~P	2 dn
5	~S	4 pr1 mt
6	~S → Q	pr3 3 mp
7	Q	5 6 mp
8		7 cd

b) $(P \to Q) \to R$. $S \to {\sim}P$. ${\sim}S \to (T \to Q)$. $\therefore T \to (P \to R)$

1	~~SHOW~~ $T \to (P \to R)$	SHOW CONC
2	T	ass cd
3	~~show~~ $P \to R$	
4	P	ass cd
5	${\sim\sim}P$	4 dn
6	${\sim}S$	pr2 5 mt
7	$T \to Q$	pr3 6 mp
8	~~Show~~ $P \to Q$	
9	P	ass cd
10	Q	2 7 mp
11		10 cd
12	R	8 pr1 mp
13		12 cd
14		3 cd

c) $P \to {\sim}W$. $S \to W$. $P \to (T \to R)$. $({\sim}R \to {\sim}T) \to S$. $\therefore {\sim}P$

1	~~SHOW~~ ${\sim}P$	SHOW CONC
2	P	ass id
3	${\sim}W$	2 pr1 mp
4	${\sim}S$	3 pr2 mt
5	~~Show~~ ${\sim}R \to {\sim}T$	Show pr4 ant
6	${\sim}R$	ass cd
7	$T \to R$	2 pr3 mp
8	${\sim}T$	6 7 mt
9		8 cd
10	S	5 pr4 mp
11		4 10 id

d) $\sim(P \to S)$. $R \to \sim P$. $\sim R \to (T \to S)$. $P \to T$. $\therefore$ W

1	~~SHOW~~ W	SHOW CONC
2	$\sim W$	ass id
3	$\sim(P \to S)$	pr1
4	$R \to \sim P$	pr2
5	$\sim R \to (T \to S)$	pr3
6	$P \to T$	pr4
7	~~Show~~ $P \to S$	show opposite of line 3/pr1. Why? Because it will give you a contradiction! Plus, it should be easy to show. If you need it and can show it, put in a show line!
8	P	ass cd
9	T	8 6 mp
10	$\sim\sim P$	8 dn
11	$\sim R$	4 10 mt
12	$T \to S$	5 11 mp
13	S	9 12 mp
14		13 cd
15		3 7 id

3.8 EG1 Let's try another.

T4 $(P\to Q) \to ((Q\to R) \to (P\to R))$

1	~~Show~~ $(P\to Q) \to ((Q\to R) \to (P\to R))$	show conc.
2	$P \to Q$	ass cd
3	~~show~~ $(Q\to R) \to (P \to R)$	show cons.
4	$Q \to R$	ass cd
5	~~show~~ $P \to R$	show cons.
6	P	ass cd
7	Q	2 6 mp
8	R	4 7 mp
9		8 cd
10		5 cd
11		3 cd

3.8 E 1 Prove the following theorems:

a) $\therefore ((P\to Q)\to(P\to R))\to(P\to(Q\to R))$

1	~~SHOW~~ $((P\to Q)\to(P\to R))\to(P\to(Q\to R))$	SHOW CONC
2	$(P\to Q)\to(P\to R)$	ass cd
3	~~Show~~ $P\to(Q\to R)$	show cons
4	P	ass cd
5	~~show~~ $Q\to R$	show cons
6	Q	
7	~~show~~ $P\to Q$	show ant. of 2
8	P	ass cd
9	Q	6 r
10		9 cd
11	$P\to R$	7 2 mp
12	R	4 11 mp, cd
13		5 cd
14		3 cd

b) $\therefore\ {\sim}(P\to Q)\to{\sim}Q$

1	~~SHOW~~ ${\sim}(P\to Q)\to{\sim}Q$	SHOW CONC
2	${\sim}(P\to Q)$	ass cd
3	~~Show~~ ${\sim}Q$	show cons
4	Q	ass id
5	~~show~~ $P\to Q$	show opp of 2
6	P	ass cd
7	Q	4 r
8		7 cd
9	${\sim}(P\to Q)$	3 r
10		5 9 id
11		3 cd

c) $\therefore (P\to S)\to((T\to P)\to({\sim}S\to{\sim}T))$

1	~~SHOW~~ $(P\to S)\to((T\to P)\to({\sim}S\to{\sim}T))$	SHOW CONC
2	$P\to S$	ass cd
3	~~show~~ $(T\to P)\to({\sim}S\to{\sim}T)$	show cons of 1
4	$T\to P$	ass cd
5	~~Show~~ ${\sim}S\to{\sim}T$	show cons of 3
6	${\sim}S$	ass cd
7	${\sim}P$	2 6 mt
8	${\sim}T$	7 4 mt
9		8 cd
13		5 cd
14		3 cd