Assignment 4 - Solutions - MAT 327 - Summer 2014

Comprehension

[**C.1**] Let $A := (-\infty, 0) \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2\} \cup (3, 4] \subseteq \mathbb{R}$, and give A the induced topology from $\mathbb{R}_{\text{usual}}$.

1. What is the interior (calculated in A) of $\{2\} \cup (3, 4]$?

2. Is $\{\frac{1}{2n} : n \in \mathbb{N}\}$ a convergent sequence in the space $(A, \mathcal{T}_{\text{subspace}})$?

3. What are the non-trival clopen sets in this subspace?

Sketch of (i). Since

$$\{2\} \cup (3, 4] = (\frac{3}{2}, 5) \cap A$$

we see that the set is its own interior. □

Sketch of (ii). It is not a convergent sequence. Since A is a subspace of a Hausdorff space, A must also be a Hausdroff space, so if the sequence does converge, it converges to only one point. It is clear that the only *plausible* point that the sequence can converge to is the point 0, but $0 \notin A$, so the sequence does not converge. □

Sketch of (iii). The sets $\{2\}$, $(3, 4]$ and $(-\infty, 0)$ are each clopen in A. Moreover, each singleton $\{\frac{1}{n}\}$ is clopen in A. We can also take any unions of these sets, and they will still be clopen. (In general we should be careful about taking arbitrary unions of closed sets, but here we don't need to worry.) □

[**C.2**] Using the list of topological invariants we gave at the end of the notes in §6, distinguish the following spaces: $\mathbb{R}_{\text{usual}}, \mathbb{R}_{\text{co-countable}}, \mathbb{R}_{\text{Sorgenfrey}}$ and $\mathbb{R}_{\text{discrete}}$.

Proof. There are many ways of doing this and here is one way.

- $\mathbb{R}_{\text{discrete}}$ is the only one of the 4 spaces that has an open singleton;

- $\mathbb{R}_{\text{co-countable}}$ is the only one of the 4 spaces that is not first countable;

- Finally, $\mathbb{R}_{\text{usual}}$ is second countable, but $\mathbb{R}_{\text{Sorgenfrey}}$ is not.

□

[**C.3**] Prove that "X is separable" is a topological invariant.

Proof. We will show something a little bit stronger: Let $f : X \longrightarrow Y$ be a continuous surjection, and let $D \subseteq X$ be a countable dense set.
<u>Claim</u>: $E := \{ f(d) : d \in D \}$ is a countable dense subset of Y.
Let $V \subseteq Y$ be a non-empty open subset of Y. Then $f^{-1}(V)$ is a non-empty (*because f is onto*) open (*because f is continuous*) subset of X. So there is a $d \in D \cap f^{-1}(V)$. Thus $f(d) \in V \cap E$. [End of Claim]
Now if $f : X \longrightarrow Y$ is a homeomorphism, then both $f : X \longrightarrow Y$ and $f^{-1} : Y \longrightarrow X$ are continuous surjections. Thus we see that separability is a topological invariant. $\square$

[**C.4**] Show that any two (n-dimensional) epsilon balls $B_{\epsilon_1}(x_1)$ and $B_{\epsilon_2}(x_2)$ are homeomorphic, when thought of as subspaces of $\mathbb{R}^n$. Does this imply that the 2 dimensional unit ball $B_1(0)$ is homeomorphic to $\mathbb{R}^2$?

Sketch 1 of C.4. We will show that any $B_\epsilon(\vec{x})$ is homeomorphic to $B_2(\vec{0})$, (for $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$). (The "2" is there just to help "follow the ϵ" through the proof. Using the unit ball would make the proof (technically) simpler.) Rather than thinking about $\mathbb{R}^n$ we will be "thinking about $\mathbb{R}^2$". This will make the visualization easier.
Consider $f : B_2(\vec{0}) \longrightarrow B_\epsilon(\vec{x})$ given by sending

$$f(\vec{y}) = \frac{\epsilon}{2} \cdot \vec{y} + \vec{x}$$

which is clearly a bijection. All that remains is to check that this is an open, continuous function (hence a homeomorphism by C.5.). The symbols about to be presented just code a very simple idea: "open disks get sent to open disks".
Notice that for an open ball $B_\delta(\vec{y}) \subseteq B_2(\vec{0})$ we get

$$f(B_\delta(\vec{y})) = \{ f(\vec{z}) : \vec{z} \in B_\delta \} = \{ \frac{\epsilon}{2} \cdot \vec{z} + \vec{x} : \vec{z} \in B_\delta \}$$

And we notice that

$$f(B_\delta(\vec{y})) = B_{\frac{\epsilon}{\delta}}(\frac{\epsilon}{2} \cdot \vec{z} + \vec{x}) \subseteq B_\epsilon(\vec{x})$$

So f is an open map. Continuity is similar. $\square$

Sketch 1 of C.4. Note that any translation map is clearly a homeomorphism. Also, dilation/contraction maps (by a positive non-zero constant) are homeomorphisms. (Showing this requires a little bit of though, but is presented above). Since the composition of homeomorphisms is again a homeomorphism, we have the result. □

[**C.5**] Prove the proposition at the end of §6.6:

Proposition. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$ be a bijection. TFAE:*

(i) f is a homeomorphism

(ii) f is continuous and an open function

(iii) f is continuous and a closed function

Proof. Notice that $i \Leftrightarrow ii$ is immediate, just by observing that a bijection f is continuous iff f^{-1} is open. This is just unwinding definitions.
[$ii \Rightarrow iii$] Assume ii. Let $C \subseteq X$ be a closed subset of X. Then $X \setminus C$ is open, and so $f(X \setminus C)$ is an open subset of Y. Since Y is a bijection (see Things You Should Know), we get

$$f(X \setminus C) = f(X) \setminus f(C) = Y \setminus f(C)$$

Hence $f(C)$ is a closed subset of Y.
[$iii \Rightarrow ii$] This proof is completely analogous to the previous direction. □

Application

[**A.1**] Let X be a topological space, with D a dense subset of X, and let Y be a Hausdorff space. Prove that, if $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are continuous functions that agree on D (i.e. $f(d) = g(d)$ for all $d \in D$) then f and g identical functions.

Proof. Suppose for the sake of contradiction that $x \in X$ is a point such that $f(x) \neq g(x)$. Since Y is a Hausdorff space, find U, V disjoint open sets in Y such that $f(x) \in U$ and $g(x) \in V$. Notice that by continuity, $f^{-1}(U)$ and $g^{-1}(V)$ are open sets in X and both contain the element x. Thus

$$x \in f^{-1}(U) \cap g^{-1}(V)$$

which is a non-empty open set in X. By density of D, there is a $d \in D$ such that

$$d \in f^{-1}(U) \cap g^{-1}(V) \cap D$$

Thus $f(d) \in U$ and $g(d) \in V$. Since f and g agree on D, we have that

$$f(d) = g(d) \in U \cap V$$

contradicting the fact that U and V are disjoint.

Note: It doesn't make sense to talk about sequences here, because we are not assuming that X or Y is first countable. So don't talk about sequences here! $\square$

[**A.2**] Show that "having a cut-point" is a topological invariant. Moreover, show that "cut-points get sent to cut-points", that is under a homeomorphism $f : X \longrightarrow Y$, $f(p)$ is a cut-point of Y iff p is a cut-point of X. Use this to distinguish the following spaces:

1. $(0, 1)$ as a subspace of $\mathbb{R}_{\text{usual}}$;
2. $[0, 1)$ as a subspace of $\mathbb{R}_{\text{usual}}$;
3. $[0, 1]$ as a subspace of $\mathbb{R}_{\text{usual}}$;
4. $\mathbb{R}^n$, for $n \geq 2$.

Conclude that $\mathbb{R}^1 \ncong \mathbb{R}^2$. (You may assume that $\mathbb{R}^n$ has no non-trivial clopen subsets.) Does this argument also show that $\mathbb{R}^2 \ncong \mathbb{R}^3$?

Lemma. *"Having a non-trivial clopen subset" is a topological invariant. Moreover, "clopen subsets get sent to clopen subsets".*

Proof. Let $f : X \longrightarrow Y$ be a homeomorphism, and let $K \subseteq X$ be a non-empty clopen subset with non-empty complement. By C.5 we know that f is an open and closed map. Thus $f(K)$ is a clopen subset of Y. Clearly, it is non-empty and its complement is non-empty. If $A \subseteq Y$ is a clopen subset of Y, then by continuity, $f^{-1}(A)$ is a clopen subset of X. $\square$

Proposition. *"Having a cut-point" is a topological invariant. Moreover, "cut-points get sent to cut-points".*

Proof. Let $f : X \longrightarrow Y$ be a homeomorphism, and let $\{ p \}$ be a cut-point of X. The previous lemma asserts that Y has no nontrivial clopen subsets. Let us check that $Y \setminus \{ f(p) \}$ contains a non-trivial clopen subset.

Notice that the restricted function $f : X \setminus \{ p \} \longrightarrow Y \setminus \{ f(p) \}$ is a homeomorphism (bijection is obvious, and continuity/openness follows from our notes on subspaces.) Thus the previous lemma tells us that the non-trivial clopen subset of $X \setminus \{ p \}$ gets mapped to a non-trivial clopen subset of $Y \setminus \{ f(p) \}$. □

Proposition. $\mathbb{R}^n$ *has no non-trivial clopen subsets.*

Proof. We will be assuming that this is true for $\mathbb{R}$, and we will prove this later in the course. Suppose for the sake of contradiction that $\mathbb{R}^n$ (for $n \geq 2$) contains a non-trivial clopen subset A. Find a line L that passes through A and its complement. (This can be done since A is a *non-trivial* clopen set.)

Notice that L, as a subspace of $\mathbb{R}^n$, is homeomorphic to $\mathbb{R}_{\text{usual}}$. We also see that $L \cap A$ and $L \cap (\mathbb{R}_n \setminus A)$ are (non-trivial) clopen subsets of L. This contradicts our assumption about R. □

Proposition. *Fix* $p \in \mathbb{R}^n$*, and* $n \geq 2$*.* $\mathbb{R}^n \setminus \{ p \}$ *has no non-trivial clopen subsets.*

Proof. This is very similar to the previous argument. Suppose for the sake of contradiction that A is a non-trivial clopen subset of $\mathbb{R}^n \setminus \{ p \}$. Find a line L that goes through A and its complement (here we use that $n \geq 2$). A little bit of drawing (and a tiny bit of case analysis) will convince you that such a line exists that does not pass through the point p. The same arugment as before produces a contradiction. □

Corollary. *For* $n \geq 2$*,* R^n *does not have a cut-point.*

The following facts are stated without proof, and their proof is straightforward:

- Every point in $\mathbb{R}$ is a cut-point;
- Every point in $(0, 1)$ is a cut-point;
- The only non-cut-point in $[0, 1)$ is 0;

- The only non-cut-points in $[0,1]$ are 0 and 1;

Thus we conclude that since cut-points get sent to cut-points, for a fixed $n \geq 2$, each of $\mathbb{R}, (0,1), [0,1), [0,1]$ and $\mathbb{R}^n$ are (topologically) different. Note that this does not distinguish $\mathbb{R}^2$ and $\mathbb{R}^3$ since neither of those spaces have cut-points. We could try using "cut-lines" but that doesn't work; can you see why? We could also try "cut-circles", but that is not nearly as straightforward as cut-points.

[**A.3**] Show that the T_3 property is hereditary. Why doesn't this argument also show that the T_4 property ("you can separate any two disjoint closed sets by disjoint open sets") is hereditary?

Proof. Let $(X, \mathcal{T})$ be a T_3 space, and let $A \subseteq X$, with subspace topology $(A, \mathcal{T}_A)$. Let $C \subseteq A$ be a closed subset of A, and let $p \in A \setminus C$. Since C is closed in A, there is $D \subseteq X$ that is closed in X such that

$$C = D \cap A$$

(This is an easy consequence of the definition of the subspace topology.) Notice that $p \notin D$. So there are disjoint open sets U, V (in X) such that $p \in U$ and $D \subseteq V$. Thus $p \in U \cap A$, an open set in A, and $C = D \cap A \subseteq V \cap A$ an open set in A.

This does *not* happen in general for T_4 spaces because when we extend C, D from closed subsets of A to closed subsets of X, the corresponding closed sets need not be disjoint any more.

You can find an example of a T_4, but not hereditarily T_4 space in Counterexamples in Topology, example 86. $\square$

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] Let's go further than A.1, with a fun problem. Suppose that $f : \mathbb{R}_{\text{usual}} \longrightarrow \mathbb{R}_{\text{usual}}$ is a continuous function that satisfies the following additive identity:

$$f(x+y) = f(x) + f(y), \forall x, y \in \mathbb{R}$$

Show that f is completely described by the value it takes on 1.

If you have taken a group theory course, conclude that

$C^1_+(\mathbb{R}) := \{ f : \mathbb{R} \longrightarrow \mathbb{R} | f$ is continuous and additive as above and f is not the 0 function $\}$

is a group (under what operation?) that is isomorphic to $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$.

What happens if we look at $C_+(\mathbb{R})$, the collection of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ (with *no* assumption about continuity)? How is this related to $\mathbb{R}$ as a vector space over $\mathbb{Q}$?

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such a function.

Claim 1: $f(0) = 0$.
Notice that

$$f(0) = f(0+0) = f(0) + f(0) = 2f(0)$$

which is only true if $f(0) = 0$.

Claim 2: $f(n) = nf(1)$ for each $n \in \mathbb{N}$.
Notice that

$$f(n) = f(\underbrace{1 + \ldots + 1}_{\text{n-times}}) = nf(1)$$

as desired.

Claim 3: $f(\frac{1}{n}) = \frac{1}{n} f(1)$ for each $n \in \mathbb{N}$.
Notice that

$$f(1) = f(\underbrace{\frac{1}{n} + \ldots + \frac{1}{n}}_{\text{n-times}}) = nf(\frac{1}{n})$$

and rearranging gives the claim.

Claim 4: $f(-x) = -f(x)$ for each $x \in \mathbb{R}$.
Notice that

$$f(-x) = f(0) - f(x) = -f(x)$$

Claim 5: $f(x) = xf(1)$ for all $x \in \mathbb{Q}$.
This follows from claims 2, 3 and 4 since we can write fractions as $\frac{p}{q}$.

Claim 6: $f(x) = xf(1)$ for all $x \in \mathbb{R}$.
Since f is continuous, and (on the dense set $\mathbb{Q}$) agrees with with the continuous (linear) function $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(x) = xf(1)$, by A.1 we have that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Claim 7: For each $\alpha \in \mathbb{R}$ define the function f_α where f_α is the (unique) continuous additive function that sends 1 to α. THEN the family $C^1_+(\mathbb{R})$ is a group, under the operation $f_\alpha \oplus f_\beta = f_{\alpha\cdot\beta}$, that is isomorphic to $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$.
We note that f_1 is the identity; the operation is clearly associative (and even Abelian!); and the inverse of f_α is $f_{-\alpha}$. This is an easy exercise in group theory to show that the map that sends f_α to α is an isomorphism.

What happens without continuity?
Without any assumptions about continuity we get all the way to claim 5. We no longer get claim 6. In fact, the assertition that "There is an additive function from $\mathbb{R}$ to $\mathbb{R}$ is not a line through the origin" is equivalent to a weak form of axiom of choice. The existence of such a function follows from the assertition that "As a vector space over $\mathbb{Q}$, $\mathbb{R}$ has a basis" which follows from the axiom of choice. $\square$

[**NI.2**] Show that "having topological dimension n" is a topological invariant, then show that $\mathbb{R}^n$ has dimension n. Conclude that you can distinguish $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$. (See the problem set for the definitions.)

Sketch of NI.2. It is easy to check that $\dim(\mathbb{R}) = 1$ since any basic open set (a, b) has boundary $\{a\} \cup \{b\}$ which is a discrete (zero-dimensional) subspace of $\mathbb{R}$.

In general, you can check that the boundary of an n-dimensional $B_\epsilon(x)$ is the $n-1$ dimensional sphere S^{n-1}, of radius ϵ. This is hard to visualize, but easy to figure out using the metric definitions of an ϵ-ball and a sphere. $\square$

[**NI.3**] This is a question for people interested in "functional analysis", or "topological dynamics", which also makes use of A.1. In §6 we saw the definition of Homeo(X) the (auto)-homeomorphisms of X onto itself. Prove that

$$\text{Homeo}([0,1]) \cong \text{Homeo}((0,1)) \cong \text{Homeo}(\mathbb{R}) \cong \text{Homeo}(\mathbb{Q})$$

where each space (the "X"s) has its usual subspace topology inherited from $\mathbb{R}_{\text{usual}}$.
The topology on Homeo(X), where $X \subseteq \mathbb{R}$ is given by the basis

$$\mathcal{B} := \{\, V(\epsilon, F, g) : \epsilon > 0, F \text{ is a finite subset of } X, g \in \text{Homeo}(X)\,\}$$

where

$$V(\epsilon, F, g) := \{\, f \in \text{Homeo}(X) : d(g(a), f(a)) < \epsilon, \forall a \in F\,\}$$

Note. Here I am also assuming that the homeomorphism are orientation preserving.

Sketch of NI.3. Claim 1: Homeo($\mathbb{R}$) $\cong$ Homeo($(0,1)$)
Since the is a homeomorphism $T : \mathbb{R} \longrightarrow (0,1)$ it is just a matter of unwinding definitions to see that the map $\Phi : \text{Homeo}(\mathbb{R}) \longrightarrow \text{Homeo}((0,1))$ defined by

$$\Phi(f)(x) := T \circ f \circ T^{-1}(x)$$

is a homeomorphism. The first thing to realize is that $\Phi(f)$ really is an (auto)-homeomorphism of $(0,1)$.

Claim 2: Homeo($(0,1)$) $\cong$ Homeo($[0,1]$).
The key here is to realize that by question A.2, the points $\{0,1\}$ get sent to $\{0,1\}$, and any auto-homeomorphism on $[0,1]$ restricts to an auto-homeomorphism on $(0,1)$.
Claim 3: Homeo($\mathbb{R}$) $\cong$ Homeo($(0,1)$)
This is similar to the previous claims, but here we also use A.1 and the observation that every continuous monotonic function on $\mathbb{Q}$ extends to a continuous monotonic function on $\mathbb{R}$. $\square$