4

Sequences

Having laid a solid foundation by looking carefully at the properties of real numbers, we now move to a more dynamic topic: the study of sequences. We shall find that sequences play a crucial role throughout analysis, so it is important to gain a thorough understanding of what they are and how they may be used. After discussing the covergence of sequences in Section 16, we devote Section 17 to several theorems that enable us to find the limit of a sequence more easily. In Section 18 we develop some of the properties of monotone sequences and Cauchy sequences, and in the final section we look at subsequences.

Section 16 CONVERGENCE

A **sequence** is a function whose domain is the set $\mathbb{N}$ of natural numbers. If s is a sequence, we usually denote its value at n by s_n instead of $s(n)$. We may refer to the sequence s as (s_n) or by listing the elements $(s_1, s_2, s_3, \ldots)$. We call s_n the nth term of the sequence and we often describe a sequence by giving a formula for the nth term. Thus $(1/n)$ is an abbreviation for the sequence

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right).$$

Sometimes we may wish to change the domain of a sequence from $\mathbb{N}$ to $\mathbb{N} \cup \{0\}$ or $\{n \in \mathbb{N} : n \geqslant m\}$. That is, we may want to start with s_0 or s_m, for some $m \in \mathbb{N}$. In this case we write $(s_n)_{n=0}^{\infty}$ or $(s_n)_{n=m}^{\infty}$, respectively. If no mention is made to the contrary, we assume that the domain is just $\mathbb{N}$.

16.1 EXAMPLES (a) Consider the sequence (s_n) given by $s_n = 1 + (-1)^n$. Writing out the first few terms of the sequence, we obtain $(0, 2, 0, 2, 0, \ldots)$, and the pattern to be followed for the rest of the terms is clear. Formally, this sequence is a function

$$s(n) = 1 + (-1)^n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even,} \end{cases}$$

but it is often more helpful to visualize the sequence as a listing $(0, 2, 0, 2, \ldots)$. Notice that the terms in a sequence do not have to be distinct. We consider s_2 and s_4 to be different terms even though their values are both equal to 2. The range of the sequence is just the set of values obtained, $\{0, 2\}$. Thus, while a sequence will always have infinitely many terms, the set of values in the sequence may be finite.

(b) For any denumerable set S, there exists a bijection from $\mathbb{N}$ onto S. This bijection may be thought of as a sequence that lists the members of S in a particular order. For example, the sequence given by $s_n = 2n$; that is,

$$(2, 4, 6, 8, 10, \ldots),$$

is precisely the function we used in Example 8.6 to show that the set of positive even integers is denumerable. Since $s_1 = 2$, we think of 2 as the "first" even number. Since $s_2 = 4$, 4 is the "second" even number, and so on. Since this function is injective, the terms are all distinct. Thus the range of the sequence is the set $\{2, 4, 6, 8, \ldots\}$. In general, we may think of any denumerable set as the range of a sequence of distinct terms. This is what we mean when we say that the elements of a denumerable set can be listed in a sequence.

(c) The sequence given by $s_n = 1 + 1/2^n$ can be written as

$$\left(\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \ldots\right).$$

The graph of this sequence (thinking of it as a function) is shown in Figure 16.1. Sometimes we reduce the graph by displaying only the range, as in Figure 16.2. This can be helpful when the terms of the sequence are distinct, but it can be misleading when they repeat. Notice that the "farther" we go in the sequence, the "closer" the terms appear to get to 1. This prompts us to say that the limit of the sequence is equal to 1. We make this more precise in the following definition.

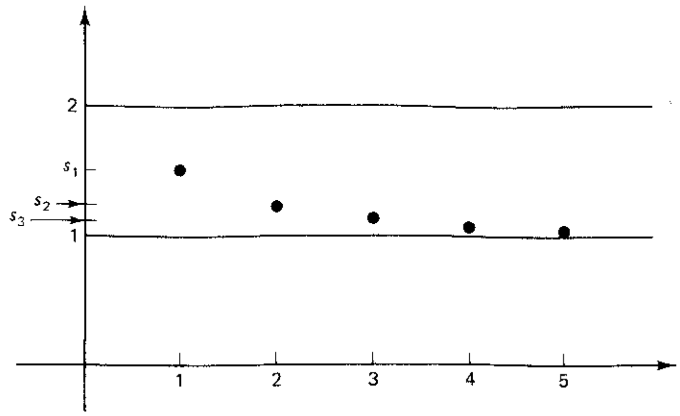


Figure 16.1 $s_n = 1 + \frac{1}{2}^n$

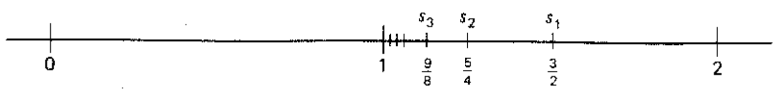


Figure 16.2 $s_n = 1 + \frac{1}{2}^n$

16.2 DEFINITION A sequence (s_n) is said to **converge** to the real number s provided that

> for each $\varepsilon > 0$ there exists a real number N such that $n > N$ implies that $|s_n - s| < \varepsilon$.

If (s_n) converges to s, then s is called the **limit** of the sequence (s_n), and we write $\lim_{n\to\infty} s_n = s$, $\lim s_n = s$, or $s_n \to s$. If a sequence does not converge to a real number, it is said to **diverge**.

It is important to note the order of the quantifiers in Definition 16.2. In trying to show that $s_n \to s$, the N that must be found may depend on the positive number ε. For each ε there must exist an N, but it is not necessary to find one N that works for all ε. We illustrate this in the following examples.

16.3 EXAMPLE Let us show that $\lim 1/n = 0$. Given any particular $\varepsilon > 0$, we want to make $|1/n - 0| < \varepsilon$. Now $|1/n - 0| = 1/n$, and $1/n < \varepsilon$ whenever $n > 1/\varepsilon$. Thus it suffices to let $N = 1/\varepsilon$. We can organize this in a formal proof as follows:

> Given $\varepsilon > 0$, let $N = 1/\varepsilon$. Then for any $n > N$ we have $|1/n - 0| = 1/n < 1/N = \varepsilon$. Thus $\lim 1/n = 0$. ∎

16.4 PRACTICE To show that $\lim 1/\sqrt{n} = 0$, given any $\varepsilon > 0$ we have to find N such that $n > N$ implies that $1/\sqrt{n} < \varepsilon$. What can we take for N?

16.5 EXAMPLE In Example 16.1(c) we observed that $1 + 1/2^n$ seemed to approach 1 as n got large. We can now prove that conjecture. Given any $\varepsilon > 0$, we want $|(1 + 1/2^n) - 1| = 1/2^n < \varepsilon$. Instead of solving directly for n this time, we observe that $1/2^n < 1/n$ for all $n \in \mathbb{N}$. (This can easily be proved using induction, but we omit the details.) Thus once again it suffices to let $N = 1/\varepsilon$. Here is the formal argument:

> Given $\varepsilon > 0$, let $N = 1/\varepsilon$. Then for any $n > N$ we have $|(1 + 1/2^n) - 1| = 1/2^n < 1/n < 1/N = \varepsilon$. Thus $\lim (1 + 1/2^n) = 1$. ▮

16.6 EXAMPLE For a more complicated example let us show that $\lim (n^2 + 2n)/(n^3 - 5) = 0$. Given any $\varepsilon > 0$, we want to make $|(n^2 + 2n)/(n^3 - 5)| < \varepsilon$. By considering only $n \geqslant 2$, we can remove the absolute value signs since $n^3 - 5$ will be positive. Thus we want to know how big n has to be in order to make $(n^2 + 2n)/(n^3 - 5) < \varepsilon$. Since this inequality would be very messy to solve for n, we shall try to find some estimate of how large the left side can be. To do this we seek an upper bound for the numerator and a lower bound for the denominator. Since $n^2 + 2n$ behaves like n^2 for large values of n, we shall try to find an upper bound on $n^2 + 2n$ of the sort bn^2. Similarly, we seek a lower bound for $n^3 - 5$ that is a multiple of n^3, say cn^3. Then we have

$$\frac{n^2 + 2n}{n^3 - 5} \leqslant \frac{bn^2}{cn^3} = \frac{b}{c}\left(\frac{1}{n}\right),$$

and it is relatively easy to make the latter expression small.

Now $n^2 + 2n \leqslant n^2 + n^2 = 2n^2$ when $n \geqslant 2$. And $n^3 - 5 \geqslant n^3/2$ when $n^3/2 \geqslant 5$ or $n^3 \geqslant 10$ or $n \geqslant 3$. Thus for $n \geqslant 3$ we have

$$\frac{n^2 + 2n}{n^3 - 5} \leqslant \frac{2n^2}{n^3 - 5} \leqslant \frac{2n^2}{\frac{1}{2}n^3} = \frac{4}{n}.$$

To make this less than ε, we want $n > 4/\varepsilon$. Thus there are two conditions to be satisfied: we want $n \geqslant 3$ and $n > 4/\varepsilon$. We can accomplish this by letting $N = \max\{3, 4/\varepsilon\}$. We are now ready to organize this into a formal proof.

> Given $\varepsilon > 0$, let $N = \max\{3, 4/\varepsilon\}$. Then $n > N$ implies that $n > 3$ and $n > 4/\varepsilon$. Since $n > 3$ we have $n^2 + 2n \leqslant 2n^2$ and $n^3 - 5 \geqslant n^3/2$. Thus for $n > N$ we have
>
> $$\left|\frac{n^2 + 2n}{n^3 - 5} - 0\right| = \frac{n^2 + 2n}{n^3 - 5} \leqslant \frac{2n^2}{\frac{1}{2}n^3} = \frac{4}{n} < \varepsilon.$$
>
> Hence $\lim (n^2 + 2n)/(n^3 - 5) = 0$. ▮

cn^3

16.7 PRACTICE Find $k > 0$ and $m \in \mathbb{N}$ so that $5n^3 + 7n \leqslant kn^3$ for all $n \geqslant m$.

The technique involved in our last example can be used in many settings. The amount of work involved can be reduced somewhat by means of the following general theorem.

16.8 THEOREM Let (s_n) and (a_n) be sequences of real numbers and let $s \in \mathbb{R}$. If for some $k > 0$ and some $m \in \mathbb{N}$, we have

$$|s_n - s| \leqslant k|a_n|, \qquad \text{for all } n > m,$$

and if $\lim a_n = 0$, then it follows that $\lim s_n = s$.

Proof: Given any $\varepsilon > 0$, since $\lim a_n = 0$ there exists $N_1 \in \mathbb{R}$ such that $n > N_1$ implies that $|a_n| < \varepsilon/k$. Now let $N = \max \{m, N_1\}$. Then for $n > N$ we have $n > m$ and $n > N_1$, so that

$$|s_n - s| \leqslant k|a_n| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$$

Thus $\lim s_n = s$. ∎

16.9 EXAMPLE To illustrate the use of Theorem 16.8, we shall prove that $\lim (4n^2 - 3)/(5n^2 - 2n) = 4/5$. To apply the theorem, we need to find an upper bound for

$$\left|\frac{4n^2 - 3}{5n^2 - 2n} - \frac{4}{5}\right| = \left|\frac{8n - 15}{5(5n^2 - 2n)}\right|$$

when n is sufficiently large. The numerator is easy since $|8n - 15| < 8n$ for all n. For the denominator we want to make $5n^2 - 2n \geqslant kn^2$ for some $k > 0$. If we try $k = 4$, then $5n^2 - 2n \geqslant 4n^2$ or $n^2 \geqslant 2n$ or $n \geqslant 2$. Writing this as a formal proof, we have the following:

If $n \geqslant 2$, then $n^2 \geqslant 2n$ and $5n^2 - 2n \geqslant 4n^2$, so that

$$\left|\frac{4n^2 - 3}{5n^2 - 2n} - \frac{4}{5}\right| = \left|\frac{8n - 15}{5(5n^2 - 2n)}\right| < \frac{8n}{5(4n^2)} = \frac{2}{5}\left(\frac{1}{n}\right).$$

Since $\lim (1/n) = 0$, Theorem 16.8 implies that

$$\lim \frac{4n^2 - 3}{5n^2 - 2n} = \frac{4}{5}. \;∎$$

16.10 PRACTICE Find $k > 0$ and $m \in \mathbb{N}$ so that $n^3 - 7n \geqslant kn^3$ for all $n \geqslant m$.

16.11 EXAMPLE Let us prove that $\lim n^{1/n} = 1$. Since $n^{1/n} \geqslant 1$ for all n, the number $b_n = n^{1/n} - 1$ is nonnegative. Since $1 + b_n = n^{1/n}$, we have $n = (1 + b_n)^n$.

From the binomial theorem (Exercise 10.14), when $n \geqslant 2$ we obtain

$$n = (1 + b_n)^n = 1 + nb_n + \frac{1}{2}\,n(n-1)b_n^2 + \cdots + b_n^n \geqslant 1 + \frac{1}{2}\,n(n-1)b_n^2.$$

It follows that $n - 1 \geqslant \frac{1}{2}n(n-1)b_n^2$, so that $b_n^2 \leqslant 2/n$ and $b_n \leqslant \sqrt{2/n}$. Hence for $n \geqslant 2$ we have

$$|n^{1/n} - 1| = b_n \leqslant \sqrt{2}\left(\frac{1}{\sqrt{n}}\right)$$

Since $\lim\,(1/\sqrt{n}) = 0$ by Practice 16.4, Theorem 16.8 implies that $\lim n^{1/n} = 1$.

In our next example we show that the sequence given by $s_n = 1 + (-1)^n$ as in Example 16.1(a) is not convergent. Since $(s_n) = (0, 2, 0, 2, \ldots)$, if the limit existed it would have to be close to both 0 and 2. Since no number is less than 1 away from both 0 and 2, we can use $\varepsilon = 1$ in the definition of convergence to obtain a contradiction. This is the reasoning behind our argument.

16.12 EXAMPLE To prove that the sequence $s_n = 1 + (-1)^n$ is divergent, let us suppose that s_n converges to some real number s. Letting $\varepsilon = 1$ in the definition of convergence, we find that there exists N such that $n > N$ implies that $|1 + (-1)^n - s| < 1$. If $n > N$ and n is odd, then we obtain $|s| < 1$ so that $-1 < s < 1$. On the other hand, if $n > N$ and n is even, then $|2 - s| < 1$ and we must have $1 < s < 3$. Since s cannot satisfy both inequalities, we have reached a contradiction. Thus the sequence (s_n) is divergent.

We conclude this section by deriving two important properties of convergent sequences. A sequence (s_n) is said to be **bounded** if the range $\{s_n\colon n \in \mathbb{N}\}$ is a bounded set, that is, if there exists $M \geqslant 0$ such that $|s_n| \leqslant M$ for all $n \in \mathbb{N}$.

16.13 THEOREM Every convergent sequence is bounded.

Proof: Let (s_n) be a convergent sequence and let $\lim s_n = s$. From the definition of convergence with $\varepsilon = 1$, we obtain $N \in \mathbb{R}$ such that $|s_n - s| < 1$ whenever $n > N$. Thus for $n > N$ the triangle inequality [11.8(d)] implies that $|s_n| < |s| + 1$. If we let

$$M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s| + 1\}.$$

then we have $|s_n| \leqslant M$ for all $n \in \mathbb{N}$, so (s_n) is bounded. ∎

16.14 THEOREM If a sequence converges, its limit is unique.

Proof: Let (s_n) be a sequence and suppose that (s_n) converges to both s and t. Then, given any $\varepsilon > 0$, there exists $N_1 \in \mathbb{R}$ such that

$$|s_n - s| < \frac{\varepsilon}{2}, \qquad \text{for every } n > N_1.$$

Similarly, there exists $N_2 \in \mathbb{R}$ such that

$$|s_n - t| < \frac{\varepsilon}{2}, \qquad \text{for every } n > N_2.$$

Therefore, if $n > \max\{N_1, N_2\}$, then from the triangle inequality [Theorem 11.8(d)] we have

$$\begin{aligned} |s - t| &= |s - s_n + s_n - t| \\ &\leqslant |s - s_n| + |s_n - t| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we must have $s = t$. (See Theorem 11.6.) ∎

ANSWERS TO PRACTICE PROBLEMS

16.4 We can take $N = 1/\varepsilon^2$. Any large N will also work.

16.7 There are many possible answers. For example, take $k = 6$ and $m = 3$. Then for $n \geqslant m$ we have $n^2 \geqslant 7$, so that $5n^3 + 7n \leqslant 5n^3 + n^2 n = 6n^3$. As another example, take $k = 12$ and $m = 1$. Then for $n \geqslant m$ we have $n^3 \geqslant n$, so that $5n^3 + 7n \leqslant 5n^3 + 7n^3 = 12n^3$.

16.10 If $7n \leqslant \frac{1}{2}n^3$, then $n^3 - 7n \geqslant \frac{1}{2}n^3$. Now $7n \leqslant \frac{1}{2}n^3$ when $n^2 \geqslant 14$ or $n \geqslant 4$. Thus we can take $k = \frac{1}{2}$ and $m = 4$. Then for $n \geqslant m$ we have $n^2 \geqslant 14$, so that $\frac{1}{2}n^3 - 7n \geqslant 0$. It follows that $n^3 - 7n = \frac{1}{2}n^3 + (\frac{1}{2}n^3 - 7n) \geqslant \frac{1}{2}n^3$. Once again, other estimates are also possible.

EXERCISES

16.1 Write out the first seven terms of each sequence.

(a) $a_n = n^2$ (b) $b_n = \dfrac{(-1)^n}{n}$

(c) $c_n = \cos \dfrac{n\pi}{3}$ (d) $d_n = \dfrac{2n + 1}{3n - 1}$

16.2 Using only Definition 16.2, prove the following.

(a) For any real number k, $\lim_{n\to\infty} (k/n) = 0$.

*(b) For any real number $k > 0$, $\lim_{n\to\infty} (1/n^k) = 0$.

(c) $\lim \dfrac{3n+1}{n+2} = 3$

(d) $\lim \dfrac{\sin n}{n} = 0$

(e) $\lim \dfrac{n+2}{n^2-3} = 0$

16.3 Using any of the results in this section, prove the following.

(a) $\lim \dfrac{1}{1+3n} = 0$ (b) $\lim \dfrac{4n^2-7}{2n^3-5} = 0$

(c) $\lim \dfrac{6n^2+5}{2n^2-3n} = 3$ (d) $\lim \dfrac{\sqrt{n}}{n+1} = 0$

(e) $\lim \dfrac{n^2}{n!} = 0$ *(f) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

16.4 Show that each of the following sequences is divergent.

(a) $a_n = 2n$ (b) $b_n = (-1)^n$

(c) $c_n = \cos \dfrac{n\pi}{3}$ (d) $d_n = (-n)^2$

16.5 For each of the following, prove or give a counterexample.

*(a) If (s_n) converges to s, then $(|s_n|)$ converges to $|s|$.

(b) If $(|s_n|)$ is convergent, then (s_n) is convergent.

(c) $\lim s_n = 0$ iff $\lim |s_n| = 0$.

16.6 Find an example of each of the following.

(a) A convergent sequence of rational numbers having an irrational limit.

(b) A convergent sequence of irrational numbers having a rational limit.

***16.7** Given a sequence (s_n) and given $k \in \mathbb{N}$, let (t_n) be the sequence defined by $t_n = s_{n+k}$. That is, the terms in (t_n) are the same as the terms in (s_n) after the first k terms have been skipped. Prove that (t_n) converges iff (s_n) converges, and if they converge, show that $\lim t_n = \lim s_n$. Thus the convergence of a sequence is not affected by omitting (or changing) a finite number of terms.

***16.8** Suppose that $\lim s_n = 0$. If (t_n) is a bounded sequence, prove that $\lim (s_n t_n) = 0$.

16.9 Suppose that (a_n), (b_n), and (c_n) are sequences such that $a_n \leqslant b_n \leqslant c_n$ for all $n \in \mathbb{N}$ and such that $\lim a_n = \lim c_n = b$. Prove that $\lim b_n = b$.

16.10 Suppose that $\lim s_n = s$, with $s > 0$. Prove that there exists $N \in \mathbb{R}$ such that $s_n > 0$ for all $n > N$.

***16.11** (a) Prove that x is an accumulation point of a set S iff there exists a sequence (s_n) of points in $S \backslash \{x\}$ such that (s_n) converges to x.

(b) Prove that a set S is closed iff whenever (s_n) is a convergent sequence of points in S, it follows that $\lim s_n$ is in S.

*16.12 Recall that $N(s;\varepsilon) = \{x : |x - s| < \varepsilon\}$ is the neighborhood of x of radius ε. Prove the following.

(a) $s_n \to s$ iff for each $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $n \geqslant M$ implies that $s_n \in N(s;\varepsilon)$.

(b) $s_n \to s$ iff for each $\varepsilon > 0$, all but finitely many s_n are in $N(s;\varepsilon)$.

(c) $s_n \to s$ iff given any open set U with $s \in U$, all but finitely many s_n are in U.

Section 17 LIMIT THEOREMS

In Section 16 we saw that the definition of convergence can sometimes be messy to use even for sequences given by relatively simple formulas (see Example 16.6). In this section we derive some basic results that will greatly simplify our work. We also introduce the notion of an infinite limit. Our first theorem is a very important result showing that algebraic operations are compatible with taking limits.

17.1 THEOREM Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. Then

(a) $\lim (s_n + t_n) = s + t$.

(b) $\lim (ks_n) = ks$ and $\lim (k + s_n) = k + s$, for any $k \in \mathbb{R}$.

(c) $\lim (s_n t_n) = st$.

(d) $\lim (s_n/t_n) = s/t$, provided that $t_n \neq 0$ for all n and $t \neq 0$.

Proof: (a) To show that $\lim (s_n + t_n) = s + t$, we need to make the difference $|(s_n + t_n) - (s + t)|$ small. Using the triangle inequality [Theorem 11.8(d)], we have

$$\begin{aligned}|(s_n + t_n) - (s + t)| &= |(s_n - s) + (t_n - t)| \\ &\leqslant |s_n - s| + |t_n - t|.\end{aligned}$$

Now given any $\varepsilon > 0$, since $s_n \to s$, there exists N_1 such that $n > N_1$ implies that $|s_n - s| < \varepsilon/2$. Similarly, since $t_n \to t$, there exists N_2 such that $n > N_2$ implies that $|t_n - t| < \varepsilon/2$. Thus, if we let $N = \max\{N_1, N_2\}$, then $n > N$ implies that

$$|(s_n + t_n) - (s + t)| \leqslant |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, we conclude that $\lim (s_n + t_n) = s + t$.

(b) Exercise 17.2(a).

(c) This time we use the inequality

$$\begin{aligned}|s_n t_n - st| &= |(s_n t_n - s_n t) + (s_n t - st)| \\ &\leqslant |s_n t_n - s_n t| + |s_n t - st| \\ &= |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|.\end{aligned}$$

17.:

We know from Theorem 16.13 that the convergent sequence (s_n) is bounded. Thus there exists $M_1 > 0$ such that $|s_n| \leqslant M_1$ for all n. Letting $M = \max\{M_1, |t|\}$, we obtain the inequality

$$|s_n t_n - st| \leqslant M|t_n - t| + M|s_n - s|.$$

Now, given any $\varepsilon > 0$, there exists N_1 and N_2 such that

$$|t_n - t| < \frac{\varepsilon}{2M} \text{ when } n > N_1 \quad \text{and} \quad |s_n - s| < \frac{\varepsilon}{2M} \text{ when } n > N_2.$$

Let $N = \max\{N_1, N_2\}$. Then $n > N$ implies that

$$\begin{aligned} |s_n t_n - st| &\leqslant M|t_n - t| + M|s_n - s| \\ &< M\left(\frac{\varepsilon}{2M}\right) + M\left(\frac{\varepsilon}{2M}\right) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $\lim (s_n t_n) = st$.

(d) Since $s_n/t_n = s_n(1/t_n)$, it suffices from part (c) to show that $\lim (1/t_n) = 1/t$. That is, given $\varepsilon > 0$ we must make

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right| < \varepsilon$$

for all n sufficiently large. To get a lower bound on how small the denominator can be, we note that since $t \neq 0$ there exists N_1 such that $n > N_1$ implies that $|t_n - t| < |t|/2$. Thus for $n > N_1$ we have

$$|t_n| = |t - (t - t_n)| \geqslant |t| - |t - t_n| > |t| - \frac{|t|}{2} = \frac{|t|}{2}$$

by Exercise 11.4. There also exists N_2 such that $n > N_2$ implies that $|t_n - t| < \frac{1}{2}\varepsilon|t|^2$. Let $N = \max\{N_1, N_2\}$. Then $n > N$ implies that

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right| < \frac{2}{|t|^2}|t - t_n| < \varepsilon.$$

Hence $\lim (1/t_n) = 1/t$. ∎

To illustrate the usefulness of Theorem 17.1, let us return to the sequence used in Example 16.9.

17.2 EXAMPLE To prove that $\lim (4n^2 - 3)/(5n^2 - 2n) = 4/5$, we note that

$$s_n = \frac{4n^2 - 3}{5n^2 - 2n} = \frac{4 - 3/n^2}{5 - 2/n}.$$

Now $\lim (1/n^2) = 0$ by Exercise 16.2(b), so $\lim [(-3)/n^2] = 0$ by Theorem 17.1(b). Thus $\lim [4 - (3/n^2)] = 4$ by 17.1(b). Similarly,

$$\lim \left(5 - \frac{2}{n}\right) = 5 - 2\left(\lim \frac{1}{n}\right) = 5 - 2(0) = 5.$$

Finally, from 17.1(d) we conclude that $\lim s_n = \frac{4}{5}$.

17.3 PRACTICE Show that $\left(\dfrac{n+3}{n^2 - 5n}\right)$ converges and find its limit.

Another useful fact is that the order relation "$\leqslant$" is preserved when taking limits.

17.4 THEOREM Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. If $s_n \leqslant t_n$ for all $n \in \mathbb{N}$, then $s \leqslant t$.

Proof: Suppose that $s > t$. Then $\varepsilon = (s - t)/2 > 0$. Thus there exists N_1 such that $n > N_1$ implies that

$$s - \varepsilon < s_n < s + \varepsilon.$$

Similarly, there exists N_2 such that $n > N_2$ implies that

$$t - \varepsilon < t_n < t + \varepsilon.$$

Let $N = \max \{N_1, N_2\}$. Then for $n > N$ we have

$$t_n < t + \varepsilon = s - \varepsilon < s_n,$$

which contradicts the assumption that $s_n \leqslant t_n$ for all n. Thus we conclude that $s \leqslant t$. ▌

17.5 COROLLARY If (t_n) converges to t and $t_n \geqslant 0$ for all $n \in \mathbb{N}$, then $t \geqslant 0$.

Proof: Exercise 17.2(b). ▌

17.6 EXAMPLE Suppose that (t_n) converges to t and that $t_n \geqslant 0$ for all $n \in \mathbb{N}$. To illustrate how algebraic manipulations can be useful in evaluating limits, let us show that $\lim (\sqrt{t_n}) = \sqrt{t}$. First, we note that Corollary 17.5 implies that $t \geqslant 0$, so that $\sqrt{t}$ is defined. Our argument consists of two cases, depending on whether t is positive or zero.

Suppose that $t > 0$. To get a bound on the difference $|\sqrt{t_n} - \sqrt{t}|$ in terms of $|t_n - t|$, we multiply and divide by the conjugate $|\sqrt{t_n} + \sqrt{t}|$. Thus

$$|\sqrt{t_n} - \sqrt{t}| = \frac{|\sqrt{t_n} - \sqrt{t}| \cdot |\sqrt{t_n} + \sqrt{t}|}{|\sqrt{t_n} + \sqrt{t}|} = \frac{|t_n - t|}{|\sqrt{t_n} + \sqrt{t}|}.$$

Since $\sqrt{t_n} + \sqrt{t} \geqslant \sqrt{t} > 0$, we obtain

$$|\sqrt{t_n} - \sqrt{t}| \leqslant \frac{1}{\sqrt{t}}|t_n - t|.$$

Now $\lim (t_n - t) = 0$ since $\lim t_n = t$. Thus from Theorem 16.8 we may conclude that $\lim \sqrt{t_n} = \sqrt{t}$.

The proof of the case when $t = 0$ is similar to Practice 16.4 and is left to the reader.

Our next theorem gives a "ratio test" that can be used to show that certain sequences converge to zero.

17.7 THEOREM Suppose that (s_n) is a sequence of positive terms and that the limit $L = \lim (s_{n+1}/s_n)$ exists. If $L < 1$, then $\lim s_n = 0$.

Proof: Since $L < 1$, there exists a real number c such that $L < c < 1$. Let $\varepsilon = c - L$ so that $\varepsilon > 0$. Then there exists an integer N such that $n > N$ implies that

$$\left|\frac{s_{n+1}}{s_n} - L\right| < \varepsilon.$$

Let $k = N + 1$. Then for all $n > k$ we have $n - 1 > N$, so that

$$\frac{s_n}{s_{n-1}} < L + \varepsilon = L + (c - L) = c.$$

It follows that, for all $n > k$,

$$0 < s_n < s_{n-1}c < s_{n-2}c^2 < \cdots < s_k c^{n-k}.$$

Letting $M = s_k/c^k$, we obtain $0 < s_n < Mc^n$ for all $n > k$. Since $0 < c < 1$, Exercise 16.3(f) implies that $\lim c^n = 0$. Thus $\lim s_n = 0$ by Theorem 16.8. ∎

17.8 PRACTICE Suppose that $0 < x < 1$. Apply Theorem 17.7 to the sequence given by $s_n = nx^n$.

Infinite Limits

The sequence given by $s_n = n$ is certainly not convergent since it is not bounded (Theorem 16.13). But its behavior is not the least erratic: the terms get larger and larger. Although there is no real number that the terms "approach," we would like to be able to say that s_n "goes to ∞." We make this precise in the following definition.

17.9 DEFINITION A sequence (s_n) is said to **diverge to $+\infty$**, and we write $\lim s_n = +\infty$ provided that

> for every $M \in \mathbb{R}$ there exists a number N such that $n > N$ implies that $s_n > M$.

Similarly, (s_n) is said to **diverge to $-\infty$**, and we write $\lim s_n = -\infty$, provided that

> for every $M \in \mathbb{R}$ there exists a number N such that $n > N$ implies that $s_n < M$.

It is important to note that the symbols $+\infty$ and $-\infty$ do not represent real numbers. They are simply part of the notation that is used to describe the behavior of certain sequences. When $\lim s_n = +\infty$ (or $-\infty$), we shall say that the limit exists, but this does not mean that the sequence converges; in fact, it diverges. Thus a sequence converges iff its limit exists *as a real number*. Since Theorems 17.1 and 17.4 refer to convergent sequences, they cannot be used with infinite limits.

17.10 PRACTICE Show that $\lim n^2 = +\infty$.

17.11 EXAMPLE The technique of developing proofs for infinite limits is similar to that for finite limits. To illustrate, let us show that $\lim (4n^2 - 3)/(n + 2) = +\infty$. This time we want to get a *lower* bound on the numerator. We find that

$$4n^2 - 3 \geqslant 4n^2 - n^2 = 3n^2, \qquad \text{when } n > 1.$$

For an *upper* bound on the denominator, we have

$$n + 2 \leqslant n + n = 2n, \qquad \text{when } n > 1.$$

Thus for $n > 1$ we obtain

$$\frac{4n^2 - 3}{n + 2} \geqslant \frac{3n^2}{2n} = \frac{3n}{2}.$$

To make this greater than any particular M, we want $n > 2M/3$. Thus there are two conditions to be satisfied: $n > 1$ and $n > 2M/3$. Here is the proof written out formally:

> Given any $M \in \mathbb{R}$, let $N = \max\{1, 2M/3\}$. Then $n > N$ implies that $n > 1$ and $n > 2M/3$. Since $n > 1$ we have $4n^2 - 3 \geqslant 4n^2 - n^2 = 3n^2$ and $n + 2 \leqslant n + n = 2n$. Thus for $n > N$ we have
>
> $$\frac{4n^2 - 3}{n + 2} \geqslant \frac{3n^2}{2n} = \frac{3n}{2} > M.$$
>
> Hence $\lim (4n^2 - 3)/(n + 2) = +\infty$. ∎

As an analog of Theorem 17.4, we have the following result for infinite limits.

17.12 THEOREM Suppose that (s_n) and (t_n) are sequences such that $s_n \leqslant t_n$ for all $n \in \mathbb{N}$.

(a) If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.
(b) If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

Proof: Exercise 17.7. ▮

For our final theorem in this section we show the relationship between infinite limits and zero limits.

17.13 THEOREM Let (s_n) be a sequence of positive numbers. Then $\lim s_n = +\infty$ iff $\lim (1/s_n) = 0$.

Proof: Suppose that $\lim s_n = +\infty$. Given any $\varepsilon > 0$, let $M = 1/\varepsilon$. Then there exists N such that $n > N$ implies that $s_n > M = 1/\varepsilon$. Since each s_n is positive we have

$$\left|\frac{1}{s_n} - 0\right| < \varepsilon, \qquad \text{whenever } n > N.$$

Thus $\lim (1/s_n) = 0$.

The converse is analogous and is left to the reader (Exercise 17.8). ▮

ANSWERS TO PRACTICE PROBLEMS

17.3 We have

$$\lim\left(\frac{n+3}{n^2-5n}\right) = \lim\left(\frac{1/n + 3/n^2}{1 - 5/n}\right)$$
$$= \frac{(\lim 1/n) + 3(\lim 1/n^2)}{1 - 5(\lim 1/n)} = \frac{0 + 3(0)}{1 - 5(0)} = 0.$$

17.8 $\dfrac{s_{n+1}}{s_n} = \dfrac{(n+1)x^{n+1}}{nx^n} = x\left(1 + \dfrac{1}{n}\right) \to x < 1$. Hence $\lim nx^n = 0$.

17.10 Given $M \in \mathbb{R}$, let $N = |M|$. Then for $n > N$ we have $n^2 \geqslant n > N \geqslant M$. Thus $\lim n^2 = +\infty$.

EXERCISES

17.1 Use Theorem 17.1 to find the following limits. Justify your answers.

(a) $\lim \dfrac{3n^2 + 4n}{7n^2 - 5n}$ (b) $\lim \dfrac{n^4 + 13}{2n^5 + 3}$

17.2 (a) Prove Theorem 17.1(b).
(b) Prove Corollary 17.5.

17.3 For s_n given by the following formulas, determine the convergence or divergence of the sequence (s_n). Find any limits that exist.

(a) $s_n = \dfrac{3 - 2n}{1 + n}$ (b) $s_n = \dfrac{(-1)^n}{n + 3}$

(c) $s_n = \dfrac{(-1)^n n}{2n - 1}$ (d) $s_n = \dfrac{2^{3n}}{3^{2n}}$

(e) $s_n = \dfrac{n^2 - 2}{n + 1}$ (f) $s_n = \dfrac{3 + n - n^2}{1 + 2n}$

(g) $s_n = \dfrac{1 - n}{2^n}$ (h) $s_n = \dfrac{3^n}{n^3 + 5}$

(i) $s_n = \dfrac{n!}{2^n}$ (j) $s_n = \dfrac{n!}{n^n}$

(k) $s_n = \dfrac{n^2}{2^n}$ (l) $s_n = \dfrac{n^2}{n!}$

17.4 For each of the following, prove or give a counterexample.
(a) If (s_n) and (t_n) are divergent sequences, then $(s_n + t_n)$ diverges.
(b) If (s_n) and (t_n) are divergent sequences, then $(s_n t_n)$ diverges.
(c) If (s_n) and $(s_n + t_n)$ are convergent sequences, then (t_n) converges.
(d) If (s_n) and $(s_n t_n)$ are convergent sequences, then (t_n) converges.

17.5 Give an example of an unbounded sequence that does not diverge to $+\infty$ or to $-\infty$.

17.6 (a) Give an example of a convergent sequence (s_n) of positive numbers such that $\lim (s_{n+1}/s_n) = 1$.
(b) Give an example of a divergent sequence (t_n) of positive numbers such that $\lim (t_{n+1}/t_n) = 1$.

17.7 Prove Theorem 17.12.

17.8 Prove the converse part of Theorem 17.13.

17.9 Prove: If $\lim s_n = 0$, then for any $k > 0$, $\lim_{n\to\infty} s_n^k = 0$. This finishes the proof in Example 17.6.

17.10 Suppose that (s_n) converges to s. Prove that (s_n^2) converges to s^2 directly without using the product formula of Theorem 17.1(c).

17.11 Write an alternative proof of Theorem 17.1(c) that does not use Theorem 16.13 by using the identity $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$.

17.12 Prove that $\lim \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 0$.

17.13 Prove the following.

(a) $\lim(\sqrt{n+1}-\sqrt{n})=0$

(b) $\lim(\sqrt{n^2+1}-n)=0$

(c) $\lim(\sqrt{n^2+n}-n)=\frac{1}{2}$

17.14 Let (s_n) be a sequence of positive terms such that $L=\lim(s_{n+1}/s_n)$ exists. Prove that if $L>1$ then $\lim s_n=+\infty$.

***17.15** (a) Show that $\lim_{n\to\infty} k^n/n! = 0$ for all $k\in\mathbb{R}$.

(b) What can be said about $\lim_{n\to\infty} n!/k^n$?

***17.16** Suppose that (s_n) is a convergent sequence with $a\leqslant s_n\leqslant b$ for all $n\in\mathbb{N}$. Prove that $a\leqslant \lim s_n\leqslant b$.

17.17 Prove the following.

(a) If $\lim s_n=+\infty$ and $k>0$, then $\lim ks_n=+\infty$.

(b) If $\lim s_n=+\infty$ and $k<0$, then $\lim ks_n=-\infty$.

(c) $\lim s_n=+\infty$ iff $\lim(-s_n)=-\infty$.

(d) If $\lim s_n=+\infty$ and if (t_n) is a bounded sequence, then $\lim(s_n+t_n)=+\infty$.

17.18 Let (s_n), (t_n), and (u_n) be sequences such that $s_n\leqslant t_n\leqslant u_n$ for all $n\in\mathbb{N}$. Suppose that (s_n) and (u_n) both converge to the real number s. Prove that (t_n) also converges to s.

Section 18 MONOTONE SEQUENCES AND CAUCHY SEQUENCES

In the preceding two sections we have seen a number of results that enable us to show that a sequence converges. Unfortunately, most of these techniques depend on our knowing (or guessing) what the limit of the sequence is before we begin. Often in applications it is desirable to be able to show that a given sequence is convergent without knowing precisely the value of the limit. In this section we obtain two important theorems (18.3 and 18.12) that enable us to do just that.

Monotone Sequences

18.1 DEFINITION A sequence (s_n) of real numbers is **increasing** if $s_n\leqslant s_{n+1}$ for all $n\in\mathbb{N}$ and is **decreasing** if $s_n\geqslant s_{n+1}$ for all $n\in\mathbb{N}$. A sequence is **monotone** if it is either increasing or decreasing.†

18.2 EXAMPLE The sequences given by $a_n=n$, $b_n=2^n$, and $c_n=2-1/n$ are all increasing. The sequence $(d_n)=(1,1,2,2,3,3,\ldots)$ is also called increasing even

† Some authors refer to an increasing sequence as "nondecreasing" and reserve the term "increasing" to apply to a "strictly increasing" sequence: $s_n<s_{n+1}$ for all $n\in\mathbb{N}$.

though some adjacent terms are equal. The sequences given by $s_n = 2/n$ and $t_n = -3n$ are decreasing. A constant sequence $(u_n) = (1, 1, 1, 1, \ldots)$ is both increasing and decreasing. The sequences given by $x_n = (-1)^n/n$ and $y_n = \cos(n\pi/3)$ are not monotone.

Of the monotone examples given above, the sequences (c_n), (s_n), and (u_n) are bounded, while (a_n), (b_n), (d_n), and (t_n) are not bounded. We also note that (c_n), (s_n), and (u_n) are convergent, while the unbounded monotone sequences diverge. It turns out that this is not just a coincidence.

18.3 THEOREM (Monotone Convergence Theorem) A monotone sequence is convergent iff it is bounded.

Proof: Suppose that (s_n) is a bounded increasing sequence. Let S denote the nonempty bounded set $\{s_n : n \in \mathbb{N}\}$. By the completeness axiom (see Section 12) S has a least upper bound, and we let $s = \sup S$. We claim that $\lim s_n = s$. Given any $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound for S. Thus there exists N such that $s_N > s - \varepsilon$. Furthermore, since (s_n) is increasing and s is an upper bound for S, we have

$$s - \varepsilon < s_N \leqslant s_n \leqslant s$$

for all $n > N$. Hence (s_n) converges to s.

In the case when the sequence is decreasing, let $s = \inf S$ and proceed in a similar manner. (See Exercise 18.5.)

The converse implication has already been proved as Theorem 16.13. ∎

18.4 EXAMPLE Let (S_n) be the sequence defined by $s_1 = 1$ and $s_{n+1} = \sqrt{1 + s_n}$ for $n \geqslant 1$. We shall show that (s_n) is a bounded increasing sequence. Computing the next three terms of the sequence, we find

$$\begin{aligned} s_2 &= \sqrt{2} && \approx 1.414 \\ s_3 &= \sqrt{1 + \sqrt{2}} && \approx 1.554 \\ s_4 &= \sqrt{1 + \sqrt{1 + \sqrt{2}}} && \approx 1.598, \end{aligned}$$

where the decimals have been rounded off. It appears that the sequence is bounded above by 2. To see if this conjecture is true, let us try to prove it using induction. Certainly, $s_1 = 1 < 2$. Now suppose that $s_k < 2$ for some $k \in \mathbb{N}$. Then $s_{k+1} = \sqrt{1 + s_k} < \sqrt{1 + 2} = \sqrt{3} < 2$. Thus we may conclude by induction that $s_n < 2$ for all $n \in \mathbb{N}$.

To verify that (s_n) is an increasing sequence, we also argue by induction. Since $s_1 = 1$ and $s_2 = 2$, we have $s_1 < s_2$, which establishes the basis for induction. Now suppose that $s_k < s_{k+1}$ for some $k \in \mathbb{N}$. Then we have

$$s_{k+1} = \sqrt{1 + s_k} < \sqrt{1 + s_{k+1}} = s_{k+2}.$$

Thus the induction step holds and we conclude that $s_n < s_{n+1}$ for all $n \in \mathbb{N}$.

Thus (s_n) is an increasing sequence and it is bounded by the interval $[1, 2]$. We concluded from the monotone convergence theorem (18.3) that (s_n) is convergent. The only question that remains is to find the value s to which it converges. Since $\lim s_{n+1} = \lim s_n$ (Exercise 16.7), we see that s must satisfy the equation

$$s = \sqrt{1 + s}.$$

(Here we have used Theorem 17.1 and Example 17.6.) Solving algebraically for s, we obtain $s = (1 \pm \sqrt{5})/2$. Since $s_n \geqslant 1$ for all n, $(1 - \sqrt{5})/2$ cannot be the limit. We conclude that $\lim s_n = s = (1 + \sqrt{5})/2$.

18.5 EXAMPLE Consider the sequence (t_n) defined by $t_1 = 1$ and $t_{n+1} = (t_n + 1)/4$. The first four terms are $t_1 = 1$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{8}$, and $t_4 = \frac{11}{32}$. In Practice 18.6 you are asked to show that the sequence is decreasing. Assuming this to be true, we have $t_n \leqslant t_1 = 1$ for all n. Since each t_n is clearly positive, we see that (t_n) is a bounded monotone sequence, and hence is convergent. In Practice 18.7 you are asked to find the value t of the limit.

18.6 PRACTICE Use induction to show $t_n > t_{n+1}$ for all n. We have already established the basis for induction: $t_1 > t_2$. The induction step remains.

18.7 PRACTICE Use the fact that $t = \lim t_n = \lim t_{n+1}$ to find t.

While unbounded monotone sequences do not converge, they do have limits.

18.8 THEOREM (a) If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.
(b) If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Proof: (a) Let (s_n) be an increasing sequence and suppose that the set $S = \{s_n\colon n \in \mathbb{N}\}$ is unbounded. Since (s_n) is increasing, S is bounded below by s_1. Hence S must be unbounded above. Thus, given any $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $s_N > M$. But then for any $n > N$ we have $s_n \geqslant s_N > M$, so $\lim s_n = +\infty$. The proof of (b) is similar (Exercise 18.6). ∎

Cauchy Sequences

When a sequence (s_n) is convergent, the terms all get close to the value of the limit for large n. By so doing, they also get close to each other. It turns out that the latter property (called the Cauchy property) is actually sufficient to imply convergence. We prove this after a preliminary definition and two lemmas.

18.9 DEFINITION A sequence (s_n) of real numbers is said to be a **Cauchy sequence** if

for each $\varepsilon > 0$ there exists a number N such that $m, n > N$ implies that $|s_n - s_m| < \varepsilon$.

18.10 LEMMA Every convergent sequence is a Cauchy sequence.

Proof: Suppose that (s_n) converges to s. To show that s_n is close to s_m, we use the fact that they are both close to s. A clever use of the triangle inequality gives us the following estimate:

$$|s_n - s_m| = |s_n - s + s - s_m| \leqslant |s_n - s| + |s - s_m|.$$

Thus, given any $\varepsilon > 0$, we choose N so that $k > N$ implies that $|s_k - s| < \varepsilon/2$. (We can do this since $\lim s_n = s$.) Then for $m, n > N$ we have

$$|s_n - s_m| \leqslant |s_n - s| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (s_n) is a Cauchy sequence. ▮

18.11 LEMMA Every Cauchy sequence is bounded.

Proof: The proof is similar to that of Theorem 16.13 and is included as Exercise 18.7. ▮

18.12 THEOREM (Cauchy Convergence Criterion) A sequence of real numbers is convergent iff it is a Cauchy sequence.

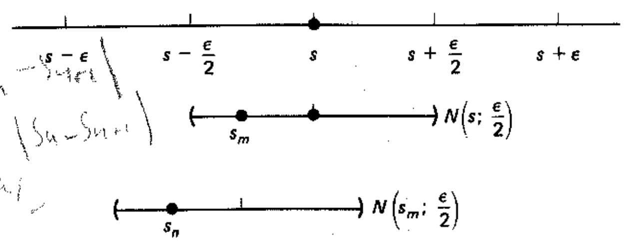
Proof: We have already shown (Lemma 18.10) that a convergent sequence is a Cauchy sequence. For the converse we suppose that (s_n) is a Cauchy sequence and let $S = \{s_n : n \in \mathbb{N}\}$ be the range of the sequence. We consider two cases, depending on whether S is finite or infinite. 18.13

If S is finite, then the minimum distance ε between distinct points of S is positive. Since (s_n) is Cauchy, there exists N such that $m, n > N$ implies that $|s_n - s_m| < \varepsilon$. Let n_0 be the smallest integer greater than N. Given any $m > N$, s_m and s_{n_0} are both in S, so if the distance between them is less than ε, it must be zero (since ε is the minimum distance between *distinct* points in S). Thus $s_m = s_{n_0}$ for all $n > N$. It follows that $\lim s_n = s_{n_0}$.

Now suppose that S is infinite. From Lemma 18.11 we know that S is bounded. Thus from the Bolzano–Weierstrass theorem (14.6) there exists a point s in $\mathbb{R}$ that is an accumulation point of S. We claim that (s_n) converges to s. Given any $\varepsilon > 0$, there exists N such that $|x_n - x_m| < \varepsilon/2$ whenever $m, n > N$. Since s is an accumulation point of S, the neighborhood $N(s; \varepsilon/2) = (s - \varepsilon/2, s + \varepsilon/2)$ contains infinitely many points of S. (See Exercise 13.9.) Thus in particular there exists $m > N$ such that $s_m \in N(s; \varepsilon/2)$. (See Figure 18.1.) Hence for any $n > N$ we have

$$|s_n - s| = |s_n - s_m + s_m - s|$$
$$\leqslant |s_n - s_m| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim s_n = s$. ▮



$|S_{n+2} - S_{n+1}| = |S_{n+2} - S_n + S_n - S_{n+1}|$
$\leq |S_{n+2} - S_n| + |S_n - S_{n+1}|$
$|S_k - S| < \varepsilon$

Figure 18.1 $|s_n - s| \leqslant |s_n - s_m| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

It is important to note that the Cauchy convergence criterion depends on the completeness of $\mathbb{R}$ since the proof uses the Bolzano–Weierstrass theorem. In fact, it can be shown that an Archimedean ordered field is complete iff the Cauchy convergence criterion holds. [See Olmsted (1962), page 203.] The property of being a Cauchy sequence can be defined in any setting in which there is a notion of distance. (See Sections 15 and 24.) In this more general setting, a Cauchy sequence may not necessarily converge, although it will be bounded.

18.13 EXAMPLE We illustrate the use of the Cauchy criterion by showing that the sequence given by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is divergent. If $m > n$, then

$$s_m - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

$$> \underbrace{\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{m-n \text{ terms}} = \frac{m-n}{m} = 1 - \frac{n}{m}.$$

In particular, when $m = 2n$ we have $s_{2n} - s_n > \frac{1}{2}$. Thus the sequence (s_n) cannot be Cauchy and hence it is not convergent.

ANSWERS TO PRACTICE PROBLEMS

18.6 Suppose that $t_k < t_{k+1}$ for some $k \in \mathbb{N}$. Then $t_{k+1} = (t_k + 1)/4 > (t_{k+1} + 1)/4 = t_{k+2}$.

18.7 Since $t = (t + 1)/4$, we obtain $t = \frac{1}{3}$.

EXERCISES

18.1 Prove that each sequence is monotone and bounded. Then find the limit.

(a) $s_1 = 1$ and $s_{n+1} = \frac{1}{4}(s_n + 5)$ for $n \in \mathbb{N}$.

(b) $s_1 = 2$ and $s_{n+1} = \frac{1}{4}(s_n + 5)$ for $n \in \mathbb{N}$.

(c) $s_1 = 1$ and $s_{n+1} = \frac{1}{4}(2s_n + 5)$ for $n \in \mathbb{N}$.

(d) $s_1 = 2$ and $s_{n+1} = \sqrt{2s_n + 1}$ for $n \in \mathbb{N}$.

(e) $s_1 = 3$ and $s_{n+1} = \sqrt{10s_n - 17}$ for $n \in \mathbb{N}$.

18.2 Find an example of a sequence of real numbers satisfying each set of properties.

(a) Cauchy, but not monotone

(b) Monotone, but not Cauchy

(c) Bounded, but not Cauchy

18.3 Suppose that $x > 0$. Define a sequence (s_n) by $s_1 = k$ and $s_{n+1} = (s_n^2 + x)/(2s_n)$ for $n \in \mathbb{N}$. Prove that, for any $k > 0$, $\lim s_n = \sqrt{x}$.

18.4 (a) Suppose that $|r| < 1$. Recall from Exercise 10.5 that $1 + r + r^2 + \cdots + r^n = (1 - r^{n+1})/(1 - r)$. Find $\lim_{n\to\infty}(1 + r + r^2 + \cdots + r^n)$.

(b) If we let the infinite repeating decimal $0.9999\cdots$ stand for the limit

$$\lim_{n\to\infty}\left(\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n}\right),$$

show that $0.9999\cdots = 1$.

18.5 Finish the proof of Theorem 18.3 for a bounded decreasing sequence.

18.6 Prove Theorem 18.8(b).

18.7 Prove Lemma 18.11.

***18.8** Let (s_n) be the sequence defined by $s_n = (1 + 1/n)^n$. Use the binomial theorem (Exercise 10.14) to show that (s_n) is an increasing sequence with $s_n < 3$ for all n. Conclude that (s_n) is convergent. The limit of (s_n) is referred to as e and is used as the base for natural logarithms. The approximate value of e is 2.71828.

18.9 A sequence (s_n) is said to be **contractive** if there exists a constant k, $0 < k < 1$, such that $|s_{n+2} - s_{n+1}| \leqslant k|s_{n+1} - s_n|$ for all $n \in \mathbb{N}$. Prove that every contractive sequence is a Cauchy sequence, and hence is convergent.

Section 19 SUBSEQUENCES

19.1 DEFINITION Let $(s_n)_{n=1}^{\infty}$ be a sequence and let $(n_k)_{k=1}^{\infty}$ be any sequence of natural numbers such that $n_1 < n_2 < n_3 < \cdots$. The sequence $(s_{n_k})_{k=1}^{\infty}$ is called a **subsequence** of $(s_n)_{n=1}^{\infty}$.

If we delete a finite number of the terms of a sequence and renumber the remaining ones in the same order, we obtain a subsequence. In fact, we may delete infinitely many of the terms in the original sequence as long as there are still infinitely many terms left. Thus the sequence

$$(s_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

has, for example,

$$(t_k) = \left(\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots\right) \quad \text{and} \quad (u_k) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right)$$

as subsequences. Of course, it has many other subsequences, including (s_n) itself. We note, however, that

1+4 =

$$(v_n) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{4}, \frac{1}{7}, \frac{1}{6}, \ldots\right)$$ 5 = 1+x

is not a subsequence of (s_n) since the order of the terms is not preserved. If we use the notation of Definition 19.1 and write $t_k = s_{n_k}$, then we have $n_k = k + 4$. Sometimes we shorten the notation and simply refer to the subsequence (t_k) as (s_{n+4}).

19.2 PRACTICE If $u_k = s_{n_k}$ as given above, what is n_k?

If a sequence is convergent, we would expect that any subsequence is also convergent. This is easy to prove once we have the following simple result.

19.3 PRACTICE Let $(n_k)_{k=1}^{\infty}$ be a sequence of natural numbers such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Use induction to show that $n_k \geqslant k$ for all $k \in \mathbb{N}$.

19.4 THEOREM If a sequence (s_n) converges to a real number s, then every subsequence of (s_n) also converges to s.

Proof: Let (s_{n_k}) be any subsequence of (s_n). Given any $\varepsilon > 0$, there exists N such that $n > N$ implies that $|s_n - s| < \varepsilon$. Thus when $k > N$, we apply Practice 19.3 to obtain $n_k \geqslant k > N$, so that $|s_{n_k} - s| < \varepsilon$. Hence $\lim_{k\to\infty} s_{n_k} = s$. ∎

19.5 EXAMPLE One application of Theorem 19.4 is in finding the value of the limit of a convergent sequence. Suppose that $0 < x < 1$ and consider the sequence (s_n) defined by $s_n = x^{1/n}$. Since $0 < x^{1/n} < 1$ for all n, (s_n) is bounded. Since

$$x^{1/(n+1)} - x^{1/n} = x^{1/(n+1)}(1 - x^{1/[n(n+1)]}) > 0, \quad \text{for all } n,$$

(s_n) is an increasing sequence. Thus, by the monotone convergence theorem (18.3), (s_n) converges to some number, say s. Now for each n, $s_{2n} = x^{1/(2n)} = (x^{1/n})^{1/2} = \sqrt{s_n}$. But by Theorem 19.4, $\lim s_{2n} = \lim s_n$ and by Example 17.6, $\lim \sqrt{s_n} = \sqrt{\lim s_n}$. Thus we have

$$s = \lim s_n = \lim s_{2n} = \lim \sqrt{s_n} = \sqrt{\lim s_n} = \sqrt{s}.$$

It follows that $s^2 = s$, so that $s = 0$ or $s = 1$. But $s_1 = x > 0$ and the sequence is increasing, so $s \neq 0$. Hence $s = 1$ and $\lim x^{1/n} = 1$.

19.6 EXAMPLE Theorem 19.4 can also be useful in showing that a sequence is divergent. For example, if the sequence $s_n = (-1)^n$ were convergent to some number s, then every subsequence would also converge to s. But (s_{2n}) converges to $+1$ and (s_{2n-1}) converges to -1. We conclude that (s_n) is not convergent.

If a sequence is divergent, the behavior of its subsequences can be quite varied. For example, we just saw that $s_n = (-1)^n$ has subsequences converging to two different numbers. On the other hand, none of the subsequences of $(1, 2, 3, 4, \ldots)$ are convergent. If, however, a given sequence is bounded, it will have at least one convergent subsequence. This result is sometimes known as the Bolzano–Weierstrass theorem for sequences.

19.7 THEOREM Every bounded sequence has a convergent subsequence.

Proof: Let (s_n) be a sequence whose range $S = \{s_n : n \in \mathbb{N}\}$ is bounded. Suppose first that S is finite. Then there is some number x in S that is equal to s_n for infinitely many values of n. That is, there

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exists $n_1 < n_2 < \cdots < n_k < \cdots$ such that $s_{n_k} = x$ for all $k \in \mathbb{N}$. It follows that the subsequence (s_{n_k}) converges to x.

On the other hand, suppose that S is infinite. Then the Bolzano–Weierstrass theorem (14.6) implies that S has an accumulation point, say y, in $\mathbb{R}$. We now construct a subsequence of (s_n) that converges to y. For each $k \in \mathbb{N}$, let $A_k = (y - 1/k, y + 1/k)$ be the neighborhood about y of radius $1/k$. Since y is an accumulation point of S, given any $k \in \mathbb{N}$, there are infinitely many values of n such that $s_n \in A_k$. Thus we can pick $s_{n_1} \in A_1$. Then we can choose $n_2 > n_1$ with $s_{n_2} \in A_2$. In general we choose $s_{n_k} \in A_k$ with $n_k > n_{k-1}$. By so doing we obtain a subsequence (s_{n_k}) of (s_n) for which $|s_{n_k} - y| < 1/k$ for all $k \in \mathbb{N}$. It follows from Theorem 16.8 that $\lim_{k\to\infty} s_{n_k} = y$. ▮

While an unbounded sequence may not have any convergent subsequence, it will contain a subsequence that has an infinite limit. In fact, we can prove the following slightly stronger result.

19.8 THEOREM Every unbounded sequence contains a monotone subsequence that has either $+\infty$ or $-\infty$ as a limit.

Proof: Suppose that (s_n) is unbounded above. We shall construct an unbounded increasing subsequence of (s_n). Given any $M \in \mathbb{R}$, there must be infinitely many terms of (s_n) larger than M. (Otherwise, the maximum of the finite number of terms would be an upper bound.) In particular, there exists $n_1 \in \mathbb{N}$ such that $s_{n_1} > 1$. Then there exists $n_2 > n_1$ such that $s_{n_2} > \max\{2, s_{n_1}\}$. In general, given $n_1, \ldots, n_k$ there exists $n_{k+1} > n_k$ such that $s_{n_{k+1}} > \max\{k, s_{n_k}\}$. It follows that the subsequence (s_{n_k}) is unbounded and increasing. By Theorem 18.8, $\lim_{k\to\infty} s_{n_k} = +\infty$.

Finally, if (s_n) is not unbounded above, then it must be unbounded below and a similar argument produces an unbounded decreasing subsequence having limit $-\infty$. ▮

lim sup and lim inf

19.9 DEFINITION Let (s_n) be a bounded sequence. A **subsequential limit** of (s_n) is any real number that is the limit of some subsequence of (s_n). If S is the set of all subsequential limits of (s_n), then we define the **limit superior** (or **upper limit**) of (s_n) to be

$$\limsup s_n = \sup S.$$

Similarly, we define the **limit inferior** (or **lower limit**) of (s_n) to be

$$\liminf s_n = \inf S.$$

We should note that in Definition 19.9 we require (s_n) to be bounded. Thus Theorem 19.7 implies that (s_n) contains a convergent subsequence, so the set S of subsequential limits will be nonempty. It will also be bounded since (s_n) is bounded. The completeness axiom then implies that sup S and inf S both exist as real numbers.

It should be clear that we always have $\liminf s_n \leqslant \limsup s_n$. Now, if (s_n) is convergent to some number s, then all its subsequences converge to s, so we have $\liminf s_n = \limsup s_n = s$. The converse of this is also true (Exercise 19.5). If it happens that $\liminf s_n < \limsup s_n$, then we say that (s_n) **oscillates.**

19.10 EXAMPLE Let $s_n = (-1)^n + 1/n$. We see that $|s_n| \leqslant |(-1)^n| + |1/n| \leqslant 2$ for all n, so the sequence (s_n) is bounded. The first few terms are

$$0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, -\frac{6}{7}, \ldots.$$

The subsequence (s_{2n}) is seen to converge to 1 and the subsequence (s_{2n-1}) converges to -1. Since these are the only possible subsequential limits, we have $\limsup s_n = 1$ and $\liminf s_n = -1$.

There are some occasions when we wish to generalize the notion of the limit superior and the limit inferior to apply to unbounded sequences. There are two cases to consider for the lim sup, with analogous definitions applying to the lim inf.

1. Suppose that (s_n) is unbounded above. Then the proof of Theorem 19.1 implies that there exists a subsequence having $+\infty$ as its limit. This prompts us to define $\limsup s_n = +\infty$.
2. Suppose that (s_n) is bounded above but not bounded below. If some subsequence converges to a finite number, we define $\limsup s_n$ to be the supremum of the set of subsequential limits. Essentially, this coincides with Definition 19.9. If no subsequence converges to a finite number, we must have $\lim s_n = -\infty$, so we define $\limsup s_n = -\infty$.

Thus for any sequence (s_n), $\limsup s_n$ always exists as either a real number or $+\infty$ or $-\infty$. When $k \in \mathbb{R}$ and $\alpha = \limsup s_n$, then writing $\alpha > k$ means that α is a real number greater than k or that $\alpha = +\infty$. Similarly, $\alpha < k$ means α is a real number less than k or $\alpha = -\infty$. Sometimes we write $k < \alpha < +\infty$ to indicate that α is a real number greater than k and thereby explicitly rule out the possibility that $\alpha = +\infty$. The only times we shall use this extended meaning for the inequality sign is when we are referring to the value of a limit, a lim sup, or a lim inf. In all other cases, the use of an inequality implies a comparison of real numbers.

If $m = \limsup s_n$ is a real number, then some special properties apply. In particular, no number larger than m can be a subsequential limit of (s_n). Thus, given any $\varepsilon > 0$, there can only be finitely many terms s_n as large as $m + \varepsilon$. (If there were infinitely many terms as large as $m + \varepsilon$, then a subsequence of these terms would have a limit greater than m.) On the other hand, if we consider $m - \varepsilon$, then there must be infinitely many terms greater than $m - \varepsilon$. (For otherwise no subsequence could have a limit greater than $m - \varepsilon$, and $m - \varepsilon$ would be an upper bound for the set of subsequential limits.) We summarize these results in our next theorem.

19.11 THEOREM Let (s_n) be a sequence and suppose that $m = \limsup s_n$ is a real number. Then the following properties hold:

(a) For every $\varepsilon > 0$ there exists N such that $n > N$ implies that $s_n < m + \varepsilon$.
(b) For every $\varepsilon > 0$ and for every $i \in \mathbb{N}$, there exists an integer $k > i$ such that $s_k > m - \varepsilon$.

Furthermore, if m is a real number satisfying properties (a) and (b), then $m = \limsup s_n$.

Proof: The only thing left to prove is the final statement. If m satisfies (a), then (s_n) is bounded above and no number larger than m can be a subsequential limit. If m satisfies (b), then no number smaller than m is an upper bound for the set S of subsequential limits. Hence $m = \sup S = \limsup s_n$. ▮

19.12 COROLLARY Let (s_n) be a sequence and suppose that $m = \limsup s_n$ is a real number. Then $m \in S$, where S is the set of subsequential limits of (s_n). That is, there exists a subsequence of (s_n) that converges to m.

Proof: Taken together, parts (a) and (b) of Theorem 19.11 imply the existence of a subsequence (s_{n_k}) of (s_n) such that

$$m - \frac{1}{k} < s_{n_k} < m + \frac{1}{k}.$$

Clearly, (s_{n_k}) converges to m. ▮

19.13 PRACTICE Let $s_n = n \sin^2 (n\pi/2)$. Find the set S of subsequential limits, the lim sup, and the lim inf of (s_n).

We conclude this section with a particular result that will be useful later in working with power series (see Theorems 34.3 and 37.2).

19.14 THEOREM Suppose that (r_n) converges to a positive number r and (s_n) is a bounded sequence. Then

$$\limsup r_n s_n = r \cdot \limsup s_n.$$

Proof: Let $s = \limsup s_n$ and $t = \limsup r_n s_n$. By Corollary 19.12 there exists a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = s$. Now $\lim_{k\to\infty} r_{n_k} = r$ by Theorem 19.4, so $\lim_{k\to\infty} r_{n_k}s_{n_k} = rs$. Thus $rs \leqslant \limsup r_n s_n = t$.

Similarly, let $(r_{n_k}s_{n_k})$ be a subsequence of $(r_n s_n)$ that converges to t. Then since $t > 0$,

$$\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} \frac{r_{n_k}s_{n_k}}{r_{n_k}} = \frac{t}{r},$$

so that $t/r \leqslant s$. That is, $t \leqslant rs$. Since $rs \leqslant t$ and $t \leqslant rs$, we conclude that $t = rs$. ∎

ANSWERS TO PRACTICE PROBLEMS

19.2 $n_k = 2^k$

19.3 Since $n_1 \in \mathbb{N}$, $n_1 \geqslant 1$. Now suppose that $n_k \geqslant k$ for some $k \in \mathbb{N}$. Then $n_{k+1} > n_k \geqslant k$, so that $n_{k+1} \geqslant k + 1$. Thus $n_k \geqslant k$ for all $k \in \mathbb{N}$.

19.13 $S = \{0, +\infty\}$, $\limsup s_n = +\infty$, $\liminf s_n = 0$

EXERCISES

19.1 For each sequence, find the set S of subsequential limits, the lim sup, and the lim inf.

(a) $s_n = (-1)^n$

(b) $(t_n) = \left(\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \ldots\right)$

(c) $u_n = n^2[-1 + (-1)^n]$

(d) $v_n = n \sin \frac{n\pi}{2}$

19.2 For each sequence, find the set S of subsequential limits, the lim sup, and the lim inf.

(a) $w_n = \frac{(-1)^n}{n}$

(b) $(x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \ldots)$
(c) $y_n = n[2 + (-1)^n]$
(d) $z_n = (-n)^n$

5 **19.3** Use Exercise 18.8 to find the limit of each sequence.

(a) $s_n = \left(1 + \frac{1}{2n}\right)^{2n}$ (b) $s_n = \left(1 + \frac{1}{n}\right)^{2n}$

(c) $s_n = \left(1 + \frac{1}{n}\right)^{n-1}$ (d) $s_n = \left(1 + \frac{1}{n}\right)^{-n}$

(e) $s_n = \left(1 + \frac{1}{2n}\right)^{n}$ (f) $s_n = \left(\frac{n+2}{n+1}\right)^{n+3}$

19.4 If (s_n) is a subsequence of (t_n) and (t_n) is a subsequence of (s_n), can we conclude that $(s_n) = (t_n)$? Prove or give a counterexample.

7 **19.5** Let (s_n) be a bounded sequence and suppose that $\liminf s_n = \limsup s_n = s$. Prove that (s_n) is convergent and that $\lim s_n = s$.

***19.6** Suppose that $x > 1$. Prove that $\lim x^{1/n} = 1$.

9 **19.7** Let (s_n) be a bounded sequence and let S denote the set of subsequential limits of (s_n). Prove that S is closed.

19.8 Let $A = \{x \in \mathbb{Q}: 0 \leqslant x < 2\}$. Since A is denumerable, there exists a bijection $s: \mathbb{N} \to A$. Letting $s(n) = s_n$, find the set of subsequential limits of the sequence (s_n).

11 **19.9** Let (s_n) and (t_n) be bounded sequences.
(a) Prove that $\limsup (s_n + t_n) \leqslant \limsup s_n + \limsup t_n$.
(b) Find an example to show that equality may not hold in part (a).

19.10 State and prove the analog of Theorem 19.11 for lim inf.

19.11 Let (s_n) and (t_n) be bounded sequences.
(a) Prove that $\liminf s_n + \liminf t_n \leqslant \liminf (s_n + t_n)$.
(b) Find an example to show that equality may not hold in part (a).

19.12 Let (s_n) be a bounded sequence.
(a) Prove that $\limsup s_n = \lim_{N\to\infty} \sup \{s_n: n > N\}$.
(b) Prove that $\liminf s_n = \lim_{N\to\infty} \inf \{s_n: n > N\}$.

***19.13** Prove that if $\limsup s_n = +\infty$ and $k > 0$ then $\limsup (ks_n) = +\infty$.

19.14 Let C be a nonempty subset of $\mathbb{R}$. Prove that C is compact iff every sequence in C has a subsequence that converges to a point in C.