MAT135H1S Calculus I(A)

Solution to even-numbered problems in Section 3.5 and 3.6

(Section 3.5, Q10)

Given that $xe^y = x - y$, we differentiate implicitly with respect to x and obtain

$$\frac{d}{dx}\left(xe^y\right) = \frac{d}{dx}(x-y)$$
$$e^y + xe^y\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{1-e^y}{xe^y+1}.$$

(Section 3.5, Q54)

Given that $y = \tan^{-1} x - \sqrt{1+x^2}$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1+(x-\sqrt{1+x^2})^2}\left(1 - \frac{x}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{1+x^2-2x\sqrt{1+x^2}+1+x^2}\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{2(1+x^2)-2x\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{2(\sqrt{1+x^2})(\sqrt{1+x^2}-x)}\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right) \\
&= \frac{1}{2(1+x^2)}
\end{aligned}$$

(Section 3.5, Q60)

Given that $y = \tan^{-1} \sqrt{\dfrac{1-x}{1+x}}$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1+(\sqrt{\frac{1-x}{1+x}})^2} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right) \cdot \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} \\
&= \frac{1}{1+\frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right) \cdot \frac{-2}{(1+x)^2} \\
&= \frac{1}{\frac{1+x}{1+x}+\frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right) \cdot \frac{-2}{(1+x)^2} \\
&= \frac{1+x}{2} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right) \cdot \frac{-2}{(1+x)^2} \\
&= \frac{-1}{2\sqrt{1-x^2}}
\end{aligned}$$

(Section 3.5, Q78)

(a) Since e^x is an increasing function, it follows that whenever $a < b$, we have $e^a < e^b$. Hence $a + e^a < b + e^b$. Thus, $f(x)$ is an increasing function and therefore one-to-one.

(b) By the definition of inverse function, we have $f^{-1}(1) = a$ if and only if $f(a) = 1$. Therefore, we need to find a such that $f(a) = 1$. By inspection, we see that $f(0) = 0 + e^0 = 1$. Therefore, $a = 0$ and $f^{-1}(1) = 0$.

(c) By Exercise 77, we have $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$. By part (b), this last quantity is $\dfrac{1}{f'(0)}$. Now $f'(x) = 1 + e^x$, so $f'(0) = 1 + e^0 = 2$. Thus, $(f^{-1})'(1) = 1/2$.

(Section 3.6, Q12)

Given that $h(x) = \ln(x + \sqrt{x^2-1})$. Then

$$\begin{aligned}
h'(x) &= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}$$

(Section 3.6, Q16)

Given that $y = \ln|1 + t - t^3|$. Then

$$\begin{aligned}
\frac{dy}{dt} &= \frac{1}{1+t-t^3} \cdot \left(1-3t^2\right) \\
&= \frac{1-3t^2}{1+t-t^3}
\end{aligned}$$

(Section 3.6, Q32)
Given that $f(x) = \ln(1 + e^{2x})$. Then

$$f'(x) = \frac{1}{1 + e^{2x}}\big(2e^{2x}\big) = \frac{2e^{2x}}{1 + e^{2x}}.$$

$$\text{Therefore, } f'(0) = \frac{2e^0}{1 + e^0} = \frac{2(1)}{1 + 1} = 1.$$

(Section 3.6, Q44)
Given that $y = x^{\cos x}$. Taking logarithm and simplifying, we have

$$\ln y = \ln(x^{\cos x}) = \cos x \ln x.$$

Differentiate implicitly with respect to x, we obtain

$$y'y = (\cos x)(\frac{1}{x}) + (\ln x)(-sinx) = \frac{\cos x}{x} - \ln x \sin x.$$

Therefore, we have

$$y' = y\Big(\frac{\cos x}{x} - \ln x \sin x\Big) = x^{\cos x}\Big(\frac{\cos x}{x} - \ln x \sin x\Big).$$