

- (1) Prove that

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

- (2) Prove that

$$1 + 2q + 3q^2 + \dots + nq^{n-1} = \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)^2}$$

- (3) The Fibonacci sequence is the sequence of numbers $F(1), F(2), \dots$ defined by the following recurrence relations:

$$F(1) = 1, F(2) = 1, F(n) = F(n-1) + F(n-2) \text{ for all } n > 2.$$

For example, the first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, \dots

Use induction to prove the identity:

$$F(n+1)F(n-1) - F(n)^2 = (-1)^n$$

for all natural numbers n .

- (4) Let $x_1 > 2$. Define x_n by the formula $x_{n+1} = \frac{3x_n+2}{x_n+2}$

Prove that $x_n > 2$ for all n .

- (5) Using the method from class find the formula for the sum

$$1^3 + 2^3 + \dots + n^3$$

Then prove the formula you've found by mathematical induction.

- (6) Find a mistake in the following "proof".

Claim. Any two natural numbers are equal.

We'll prove the following statement by induction in n : Any two natural numbers $\leq n$ are equal.

We prove it by induction in n .

- a) The statement is trivially true for $n = 1$.
b) Suppose it's true for $n \geq 1$. Let a, b be two natural numbers $\leq n + 1$. Then $a - 1 \leq n$ and $b - 1 \leq n$. Therefore, by the induction assumption

$$a - 1 = b - 1$$

Adding 1 to both sides of the above equality we get that $a = b$. Thus the statement is true for $n + 1$. By the principle of mathematical induction this means that it's true for all natural n . \square .

- (7) Using the method from class write a table of all prime numbers ≤ 100 . Explain why you only need to cross out the numbers divisible by 2, 3, 5 and 7.