TORONTO *LIFE SCIENCES*

Study Package Solutions: PART2

Term
TEST 2

DEC

2008

TERM TEST II

Your Key to Success

Our Instructor: **James T. Nguyen**, B.Sc., M.Sc., CFA (IPR)., MBA (IPR)., DDS (IPR)

SOLUTIONS: PART 2

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Solution's: MAT135
TEST 2 Study Package

§ Optimization Problems:

1) $xy = 25$ want $F = x + y$

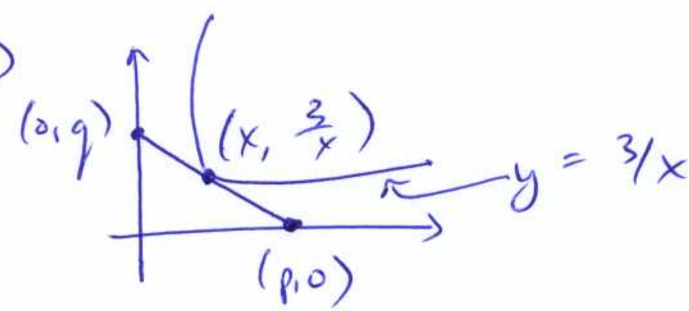
$\Rightarrow y = \frac{25}{x} \Rightarrow F = x + \frac{25}{x}$

$F' = 1 - \frac{25}{x^2} \Rightarrow F' = 0 \Rightarrow 1 = \frac{25}{x^2}$

$\Rightarrow \boxed{x = 5}$

$\Rightarrow y = 5$ $\quad \therefore F = x + y = 5 + 5 = 10$

3)



note: we can form 3 slopes, $y_1' = -\frac{3}{x^2}$

and $y_2' = \frac{q - \frac{3}{x}}{0 - x}$ and $y_3' = \frac{\frac{3}{x}}{x - p}$

Now, $y_1' = y_2' = y_3'$;

$$\frac{q - \frac{3}{x}}{-x} = \frac{-3}{x^2} = \frac{\frac{3}{x}}{x - p}$$

We want to minimize, $q^2 + p^2 = d^2$

$$\Rightarrow \frac{q - \frac{3}{x}}{-x} = \frac{-3}{x^2} \Rightarrow q - \frac{3}{x} = \frac{3}{x}$$

$$\Rightarrow \boxed{q = \frac{6}{x}}$$

and $$-\frac{3}{x^2} = \frac{\frac{3}{x}}{x - p} \Rightarrow \frac{-3}{x^2}(x-p) = \frac{3}{x}$$

$$\Rightarrow x - p = -x$$

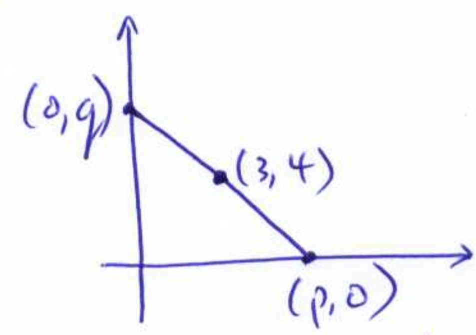
$$\Rightarrow p = 2x$$

$$\therefore d^2 = \frac{36}{x^2} + 4x^2 = F(x)$$

$$F' = 0 \Rightarrow \frac{-72}{x^2} + 8x = 0 \Rightarrow x^2 = 3$$

$$\therefore d^2 = \frac{36}{3} + 4(3) = 24 \Rightarrow d = \sqrt{24} = 2\sqrt{6}$$

4)



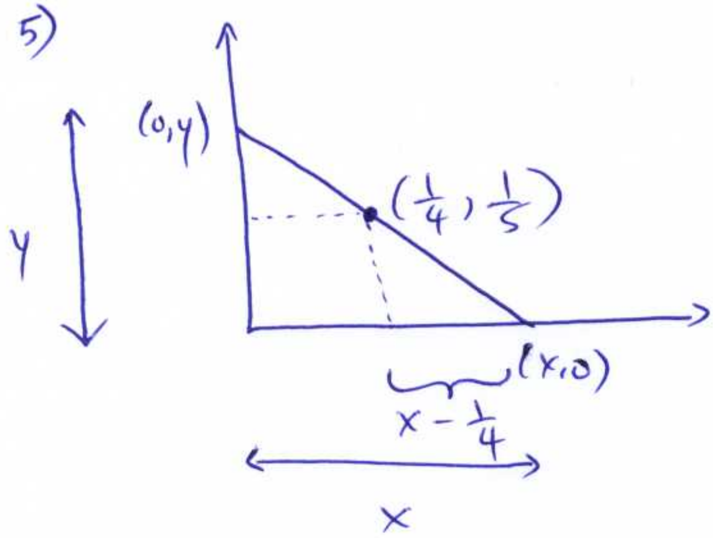
Similar to above problem:

<u>Want</u> $F(x) = p + q$

<u>note</u>: we can find 2 slopes as follows,

$$\underbrace{\frac{q-4}{0-3}}_{\text{Slope 1}} = \underbrace{\frac{4-0}{3-p}}_{\text{Slope 2}}$$

5)



By similar Δ's:

$$\frac{y}{x} = \frac{1/5}{x-1/4}$$

$$\Rightarrow \quad y = \frac{1}{5}\,\frac{x}{x-1/4}$$

Now, $A = \frac{1}{2}xy = \frac{1}{10}\,\frac{x^2}{x-1/4}$

$$A' = \frac{1}{10}\left[\frac{(x-1/4)2x - x^2}{(x-1/4)^2}\right]$$

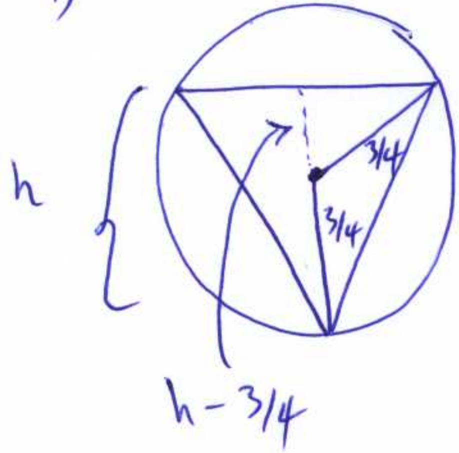
$$A'(x) = 0 \Rightarrow 2x^2 - \frac{x}{2} - x^2 = 0$$

$$\Rightarrow x^2 - \frac{x}{2} = 0$$

$$\Rightarrow x(x-1/2) = 0 \Rightarrow \boxed{x = 1/2}$$

$$A = \frac{1}{10}\,\frac{x^2}{x-1/4} = \frac{1}{10}\,\frac{1/4}{\frac{1}{2}-\frac{1}{4}} = 1/10$$

9)



note: $r^2 + (h-\frac{3}{4})^2 = (\frac{3}{4})^2$

$V = \frac{\pi r^2 h}{3}$, now $r^2 = \frac{9}{16} - (h-\frac{3}{4})^2$

$$r^2 = \frac{9}{16} - \left(h^2 - \frac{3h}{2} + \frac{9}{16}\right)$$

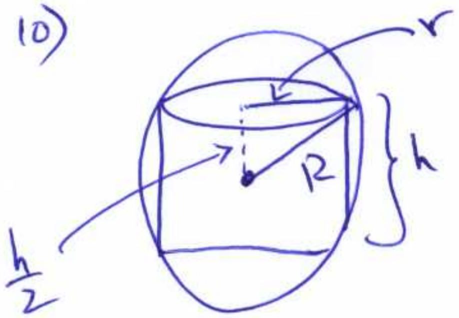
$$\Rightarrow r^2 = -h^2 + \frac{3h}{2}$$

$$V = \frac{\pi}{3}\left(-h^2 + \frac{3h}{2}\right)h$$
$$V = \frac{\pi}{3}\left(-h^3 + \frac{3}{2}h^2\right)$$
$$V' = \frac{\pi}{3}\left[-3h^2 + 3h\right]$$

and $V' = 0$
$$-3h^2 + 3h = 0$$
$$-h^2 + h = 0$$
$$-h + 1 = 0 \Rightarrow \boxed{h = 1}$$

and $V = \frac{\pi}{3}\left(-h^3 + \frac{3}{2}h^2\right)$
$$V(1) = \frac{\pi}{3}\left(-1 + \frac{3}{2}\right) = \pi/3 \cdot \frac{1}{2} = \frac{\pi}{6}$$

10)



note: $R^2 = r^2 + \frac{h^2}{4}$
$$V_{cly} = \pi r^2 h$$
$$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$$
$$V = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

and $V' = \pi\left(R^2 - \frac{3h^2}{4}\right) \Rightarrow V' = 0 \Rightarrow h^2 = \frac{4R^2}{3}$
$$\Rightarrow h = \frac{2R}{\sqrt{3}}$$

Now, the largest cylinder, $V_{max} = \frac{3\sqrt{3}}{16}\pi$ and
$$V = \pi\left(R^2 h - \frac{h^3}{4}\right)$$
$$V = \pi\left(R^2 \cdot \frac{2R}{\sqrt{3}} - \frac{\left(\frac{2R}{\sqrt{3}}\right)^3}{4}\right)$$
$$V = \pi\left[\frac{2R^3}{\sqrt{3}} - \frac{8R^3}{4 \cdot 3\sqrt{3}}\right]$$

$$V = \pi\left[\frac{2R^3}{\sqrt{3}} - \frac{2R^3}{3\sqrt{3}}\right]$$

now, $$\frac{3\sqrt{3}\pi}{16} = \pi\left[\frac{2R^3}{\sqrt{3}} - \frac{2R^2}{3\sqrt{3}}\right]$$

$$\frac{3\cdot 3\pi}{16} = \pi\left(2R^3 - \frac{2R^3}{3}\right)$$

$$\frac{9}{32} = R^3 - \frac{R^3}{3}$$

$$\frac{27}{32} = 3R^3 - R^3 \Rightarrow \frac{27}{64} = R^3$$

$$\Rightarrow \boxed{R = \frac{3}{4}}$$

12) Given: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and $A = \pi ab$ ← please note that this will be our formula in this case

* at $(5, 6)$:

$$\frac{25}{a^2} + \frac{36}{b^2} = 1$$

$$\frac{25}{a^2} = 1 - \frac{36}{b^2} \Rightarrow \frac{25}{1 - \frac{36}{b^2}} = a^2$$

$$\Rightarrow a = \frac{5}{\sqrt{1 - \frac{36}{b^2}}}$$

Now, $$A = \frac{\pi \; 5b}{\sqrt{1-\frac{36}{b^2}}} = 5\pi b\left(1-\frac{36}{b^2}\right)^{-1/2}$$

$$A' = 5\pi\left(1-\frac{36}{b^2}\right)^{-1/2} + 5\pi b\left(-\tfrac{1}{2}\right)\left(1-\frac{36}{b^2}\right)^{-3/2}\left(\frac{72}{b^3}\right)$$

$$A' = \frac{5\pi}{\sqrt{1-\frac{36}{b^2}}} - \frac{5\pi \cdot 36}{b^2}\cdot\frac{1}{\left(\sqrt{1-\frac{36}{b^2}}\right)^3}$$

and $A'=0$ gives: $$\frac{1}{\sqrt{1-\frac{36}{b^2}}} = \frac{36}{b^2}\,\frac{1}{\left(\sqrt{1-\frac{36}{b^2}}\right)^3}$$

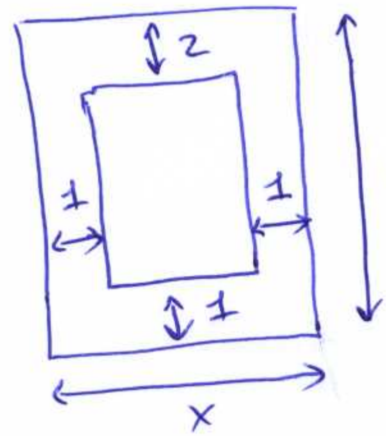
$$\frac{\left(\sqrt{1-\frac{36}{b^2}}\right)^3}{\sqrt{1-\frac{36}{b^2}}} = \frac{36}{b^2}$$

$$1-\frac{36}{b^2} = \frac{36}{b^2} \Rightarrow b^2-36 = 36$$
$$\Rightarrow \boxed{b^2 = 72}$$
$$\Rightarrow \boxed{b = 6\sqrt{2}}$$

Now, $$A = \frac{5\pi b}{\sqrt{1-\frac{36}{b^2}}}$$

$$A = \frac{5\pi \cdot 6\sqrt{2}}{\sqrt{1-\frac{36}{72}}} = \frac{30\pi\sqrt{2}}{\frac{1}{\sqrt{2}}} = 60\pi \; //$$

ex13)



Given: $xy = 180$

Let A be printed area

$$A = (x-2)(y-3)$$
$$= xy - 2y - 3x + 6$$
$$= 186 - \frac{360}{x} - 3x$$

$$A' = \frac{360}{x^2} - 3 \Rightarrow A' = 0 \Rightarrow x = \sqrt{120}$$
$$x = 2\sqrt{30} \quad (x>0)$$

(and then you can solve for y easily)

Horizontal Asymptotes:

ex1) answer is $1/5$

ex2) $$f(x) = \begin{cases} \frac{2-3x}{1+3x} & \text{if } x>0 \\ x + \sqrt{x^2+x+4} & \text{if } x \le 0 \end{cases}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2-3x}{1+3x} = -1 \longleftarrow \text{H.A at } y=-1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x + \sqrt{x^2+x+4} \cdot \left(\frac{x-\sqrt{x^2+x+4}}{x-\sqrt{x^2+x+4}}\right)$$
$$= \lim_{x\to-\infty} \frac{x^2-(x^2+x+4)}{x-\sqrt{x^2+x+4}}$$

$$= \lim_{x\to-\infty} \frac{-x-4}{x-\sqrt{x^2+x+4}} = \lim_{x\to-\infty} \frac{-1-\frac{4}{x}}{1-\sqrt{x^2+x+4}}$$

$$= \lim_{x\to-\infty} \frac{-1-\frac{4}{x}}{1+\frac{\sqrt{x^2+x+4}}{\sqrt{x^2}}}$$

* since $x \to -\infty$

$|x| = -x = -\sqrt{x^2}$

$$= \lim_{x\to-\infty} \frac{-1-\frac{4}{x}}{1+\sqrt{1+\frac{1}{x}+\frac{4}{x^2}}} = -\frac{1}{2} \leftarrow \text{H.A}$$

at $y = -1/2$

ex3) The line $y = 3$ is a horizontal Asymptote.

ex4) The $y = -e$ is a horizontal asymptote of $f(x)$

Vertical Asymptotes

ex1) $$f(x) = \frac{2x^2-3x-2}{(x^2-x-2)}$$

$$= \frac{2x^2-3x-2}{(x-2)(x+1)} = \frac{2x^2-4x+x-2}{(x-2)(x+1)}$$

$$= \frac{(2x+1)(x-2)}{(x-2)(x+1)} = \frac{2x+1}{x+1}$$

Check at $x = -1$

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \frac{2x+1}{x+1} = \frac{-1}{(0.00001)} = -\infty$$

$\Rightarrow$ V.A

ex2) Check for H.A. by $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \frac{|x|-2}{x^2-4} = \lim_{x\to\infty} \frac{x-2}{(x-2)(x+2)} = 0 \longleftarrow \text{H.A}$$

Check for Vertical Asy:

$$f(x) = \frac{|x|-2}{x^2-4} = \frac{|x|-2}{(x-2)(x+2)}$$

Consider $x=-2, 2$

Check at $x=2$:

$$\left.\begin{aligned} \lim_{x\to2^+} \frac{|x|-2}{(x-2)(x+2)} &= \lim_{x\to2^+} \frac{x-2}{(x-2)(x+2)} = 1/4 \\ \lim_{x\to2^-} \frac{|x|-2}{(x-2)(x+2)} &= \lim_{x\to2^-} \frac{x-2}{(x-2)(x+2)} = 1/4 \end{aligned}\right\} \text{No V.A.}$$

Check at $x=-2$

$$\left.\begin{aligned} \lim_{x\to-2^+} \frac{|x|-2}{(x-2)(x+2)} &= \lim_{x\to-2^+} \frac{-x-2}{(x-2)(x+2)} = 1/4 \\ \lim_{x\to-2^-} \frac{|x|-2}{(x-2)(x+2)} &= 1/4 \end{aligned}\right\} \text{No V.A.}$$

$\therefore$ The answer must be choice A

ex3) The answer must be $y=2$

§ THEOREMS

ex2) $f(x) = x^3 - x + 1\ ;\ [0, 2]$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{7 - 1}{2} = 3$$

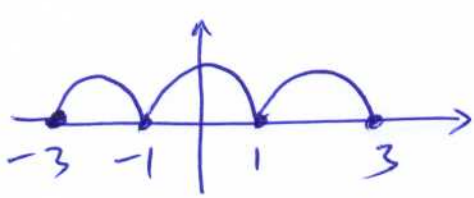
now, $f'(x) = 3x^2 - 1 \Rightarrow f'(c) = 3c^2 - 1$

$$3c^2 - 1 = 3 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3}$$

$$\Rightarrow c = \frac{2}{\sqrt{3}}\ ;\ \frac{2}{\sqrt{3}} \in [0, 2]$$

ex3) Rolle's Theorem

Should be able to draw this graph from the clues given:



Clue's:

1) even polynomials $\Rightarrow$ symmetric around origin
2) $f(1) = 0 = f(-1)$
   $f(-3) = 0 = f(3)$

Base on the picture and Rolle's Theorem, we are guaranteed at <u>Least</u> 3 root's ex4) I and II are easy to show that they are True! III is true, but you need to show it by the Mean Value Theorem. (Answer: all statements are <u>true</u>!)

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§ <u>Continuity</u>:

ex1) Let $f(x) = \begin{cases} k^2 + 4x + 1 & , \text{if } x < 2 \\ -2 + (1-3k)x & , \text{if } x \geq 2 \end{cases}$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

$$\lim_{x \to 2^+} -2 + (1-3k)x = \lim_{x \to 2^-} k^2 + 4x + 1$$

$$-2 + (1-3k)(2) = k^2 + 8 + 1$$

$$-2 + 2 - 6k = k^2 + 9$$

$$\Rightarrow k^2 + 6k + 9 = 0 \Rightarrow (k+3)^2 = 0$$

$$\Rightarrow k = -3$$

ex2) $f(x) = kx(x+1)(x+2)$, find $f'(1) = 22$, find $k = ?$

$$f(x) = k(x^2+x)(x+2)$$

$$f(x) = k(x^3 + 2x^2 + x^2 + 2x)$$

$f'(x) = k\left(3x^2 + 4x + 2x + 2\right)$

$\underline{f'(1) = 22}$: $\quad 22 = k(3 + 4 + 2 + 2)$

$22 = k \cdot 11 \Rightarrow \boxed{k = 2}$

ex3) $f(x) = \begin{cases} e^{-x} & , x \le 0 \\ ax + b & , x > 0 \end{cases}$

$f'(x) = \begin{cases} -e^{-x} & , x < 0 \\ a & , x > 0 \end{cases}$ now; f is diff $\Rightarrow f$ cont at $x = 0$

$\underline{\text{at } x = 0}$

$-e^{-x} = a$

$\Rightarrow \boxed{-1 = a}$

<u>and by continuity</u>:

$\lim_{x \to 0^-} e^{-x} = \lim_{x \to 0^+} -x + b$

$\boxed{1 = b}$

Now, $f(x) = \begin{cases} e^{-x} & , x \le 0 \\ -x + 1 & , x > 0 \end{cases}$

$f(x) = 0 \Rightarrow -x + 1 = 0 \Rightarrow \boxed{x = 1}$

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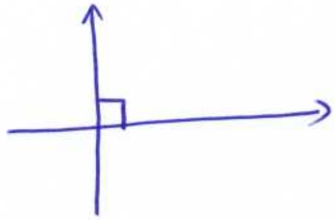
§ Orthogonality

ex 1) Answer is D $(xy = c)$

ex2) $y = ax^5$ & $x^2 + k^2 y = b$



note: there is no need for a graph.

* $y_1' = 5ax^4$
* $2x + 2kyy' = 0 \Rightarrow y_2' = \dfrac{-x}{ky}$

We have the property that:

$$\boxed{y_1' y_2' = -1} \Rightarrow 5ax^4 \cdot \left(\frac{-x}{ky}\right) = -1$$

$$\frac{5ax^5}{ky} = 1 \Rightarrow \frac{5ax^5}{kax^5} = 1 \Rightarrow \boxed{k = 5}$$

ex3) Answer choice c)

$(3y^2 + x^2 = c)$

ex4) Answer is $\sqrt[3]{3}$ choice B

§ Lines and tangents

ex2) $y^2 + xy = 5 \Rightarrow 2yy' + y + xy' = 5$ at $(4,-5)$

$-10y' - 5 + 4y' = 5$

$-6y' = 10 \Rightarrow y' = \dfrac{10}{-6} = \dfrac{-5}{3}$

$\therefore\ y - (-5) = \dfrac{-5}{3}(x - 4)$ is the line we want.

ex3) <u>we have</u>: $y = e^{3x}$ and parallel to $y = 6x+1$

$\Rightarrow y' = 3e^{3x}$ and we get:

$$3e^{3x} = 6 \Rightarrow e^{3x} = 2$$

$$\Rightarrow \ln e^{3x} = \ln 2$$

$$\Rightarrow 3x = \ln 2$$

$$\Rightarrow \boxed{x = \frac{\ln 2}{3}}$$

Horizontal tangent:

ex3) $f(x) = 5^x$, $g(x) = e^{4x} - 4e^{2x} - 12x + 3$

$$h = f(g(x))$$

$$h = f(e^{4x} - 4e^{2x} - 12x + 3)$$

$$h = 5^{e^{4x} - 4e^{2x} - 12x + 3}$$

$$h' = \underset{\neq 0}{5^{e^{4x} - 4e^{2x} - 12x + 3}} \cdot \underset{\neq 0}{\ln 5} \underset{\overset{!!!}{0}}{\left[4e^{4x} - 8e^{2x} - 12\right]}$$

<u>note</u>: How we reason which pice must be equal to zero.

$$4e^{4x} - 8e^{2x} - 12 = 0$$

$$\Rightarrow e^{4x} - 2e^{2x} - 3 = 0$$

$$\Rightarrow \left(e^{2x}-3\right)\underbrace{\left(e^{2x}+1\right)}_{\neq 0}=0$$

$$\therefore\ e^{2x}-3=0 \Rightarrow e^{2x}=3$$

$$\ln e^{2x}=\ln 3$$

$$2x=\ln 3 \Rightarrow x=\frac{1}{2}\ln 3$$

§ Question's that require Manipulation:

ex 4) $f(x)=axe^{bx^2}$, max at $x=3$ with value 2

note: from the clues given, we get $f'(3)=0$

$$f'(x)=ae^{bx^2}+2ax^2e^{bx^2}$$

$$\Rightarrow ae^{9b}+18ae^{9b}\cdot b=0 \quad (1)$$

and also $f(3)=2$ (at the maximum

$$2=3ae^{9b} \Rightarrow \frac{2}{3}=ae^{9b} \quad (2)\ )$$

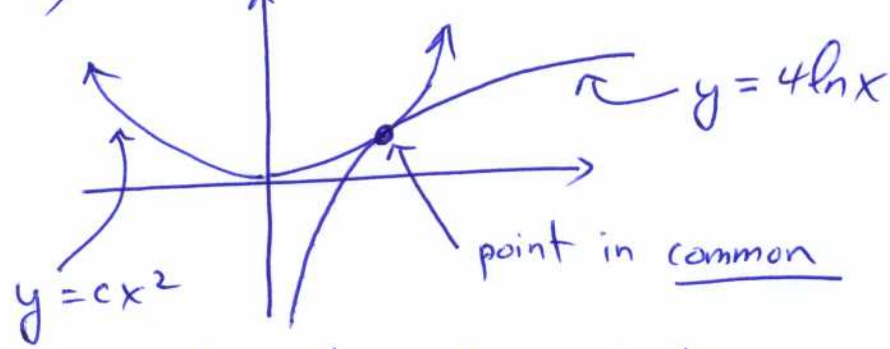
Using (1) & (2): $\frac{2}{3}+18\cdot\frac{2}{3}b=0 \Rightarrow \boxed{b=-\frac{1}{18}}$

(i.e. use (2) and sub into (1))

Now, $2=3ae^{-1/2} \Rightarrow \frac{2}{3}e^{1/2}=a$

$$ab=\frac{2}{3}\sqrt{e}\cdot\left(-\frac{1}{18}\right)=\frac{-\sqrt{e}}{27}$$

ex5)



note: here there is two very special pieces of information given to us.

1) at the point in common they have the same tangent line.

2) at the point in common, they have the same y-value.

$\therefore\ y_1' = \frac{4}{x} \qquad y_2' = 2cx$

and $\frac{4}{x} = 2cx \Rightarrow \boxed{2 = cx^2}$ (1)

$\therefore\ \underbrace{\boxed{4\ln x = cx^2}}_{\text{same y-value}}$ (2)

Now, $2 = 4\ln x \Rightarrow e^{1/2} = e^{\ln x}$

$\Rightarrow \sqrt{e} = x$

hence, $2 = cx^2 \Rightarrow 2 = c(\sqrt{e})^2$

$\Rightarrow \boxed{c = \frac{2}{e}}$

§ Implicit Differentiation

ex1) $-5/6$ ex2) $y' = -7/17$ ex3) $y' = 0$

ex4) 87π ex5) $y' = -1/6$ ex6) $y''' = 44$

§ Differential Calculus

ex1) $y = \ln(x \ln x)$, $f'(e) = ?$

$$y' = \frac{1}{x \ln x}[\ln x + 1]$$

$$y'(e) = \frac{1}{e \ln e}[\ln e + 1] = \frac{1}{e}(2) = \frac{2}{e}$$

ex2) $y = \frac{\ln x}{x} \Rightarrow y' = \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$

$$y'\left(\frac{1}{e}\right) = \frac{1 - \ln\left(\frac{1}{e}\right)}{1/e^2} = (1 - \ln e^{-1})e^2$$

$$= (1 + \ln e)e^2 = 2e^2$$

ex3) $y = x^{2x}$, $f'(e) = ?$

$\ln y = \ln x \Leftrightarrow \ln y = 2x \ln x$

$\Leftrightarrow \frac{1}{y} y' = 2\ln x + 2x \cdot \frac{1}{x}$

$y' = y[2\ln x + 2]$

$= x^{2x}[2\ln x + 2]$

$y' = e^{2e}[2\ln e + 2] = 4e^{2e}$

ex 4) $y = \arctan x$, $f''(\sqrt{3}) = ?$

$y' = \frac{1}{1+x^2} \Rightarrow y'' = -\frac{1}{(1+x^2)^2} \cdot 2x$

$\Rightarrow y''(\sqrt{3}) = \frac{-2 \cdot \sqrt{3}}{(1+(\sqrt{3})^2)^2}$

$y'' = \frac{-2 \cdot \sqrt{3}}{16} = \frac{-\sqrt{3}}{8}$

ex5) note: $y' = (\ln x)^x [\ln(\ln x) + \frac{1}{\ln x}]$

$y'(e) = (\ln e)^e [\ln(\ln e) + \frac{1}{\ln e}]$

$= 1$

ex7) Answer is 11