UNIVERSITY OF TORONTO
Faculty of Arts and Science

STA447/2006H1 (Stochastic Processes)
MIDTERM TEST
February 25, 2016, 6:10 p.m.

Duration: 120 minutes. Total points: 60.

**** SOLUTIONS ****

1. Consider a Markov chain on the state space $S = \{1, 2, 3, 4\}$ with the following transition matrix:

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.2 & 0.3 \end{pmatrix}$$

Let π be the uniform distribution on S, so $\pi_i = 1/4$ for all $i \in S$.

(a) [2] Compute $p_{14}^{(2)}$.

Solution: $p_{14}^{(2)} = \sum_{k \in S} p_{1k} p_{k4} = p_{11}p_{14} + p_{12}p_{24} + p_{13}p_{34} + p_{14}p_{44} = (0.1)(0.2) + (0.2)(0.1) + (0.5)(0.4) + (0.2)(0.3) = 0.02 + 0.02 + 0.20 + 0.06 = 0.30 = 0.3.$

(b) [2] Is this Markov chain reversible with respect to π?

Solution: *No. For example, if $i = 1$ and $j = 2$, then $\pi_i p_{ij} = (1/4)(0.2) = 1/20$, while $\pi_j p_{ji} = (1/4)(0.4) = 1/10$, so $\pi_i p_{ij} \neq \pi_j p_{ji}$.*

(c) [3] Is π a stationary distribution for this Markov chain?

Solution: *Yes. Indeed, the matrix P has each column-sum equal to 1 (as well as, of course, having each row-sum equal to 1), i.e. it is doubly-stochastic. Hence, for any $j \in S$, we have that $\sum_{i \in S} \pi_i p_{ij} = \sum_{i \in S} (1/4) p_{ij} = (1/4) \sum_{i \in S} p_{ij} = (1/4)(1) = 1/4 = \pi_j$. So, π is a stationarity distribution.*

(d) [3] Does $\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$? Why or why not?

Solution: *Yes. Indeed, P is irreducible (since $p_{ij} > 0$ for all $i, j \in S$), and aperiodic (since $p_{ii} > 0$ for some, in fact all, $i \in S$), and π is stationary (by part c above), so by the Markov Chain Convergence Theorem, we have $\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$.*

2. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) [3] The chain is irreducible and periodic (i.e., not aperiodic), and has a stationary probability distribution.

Solution: *Yes. For example, let $S = \{1, 2\}$, with $p_{12} = p_{21} = 1$ (and $p_{11} = p_{22} = 0$). Then the chain is irreducible (since it can get from each i to $3-i$ in one step, and from i to i in two steps), and periodic with period 2 (since it only returns to each i in even numbers of steps). Furthermore, if $\pi_1 = \pi_2 = 1/2$, then for $i \neq j$, we have $\pi_i p_{ij} = (1/2)(1) = \pi_j p_{ji}$. Hence, the chain is reversible with respect to π, so π is a stationarity distribution.*

(b) [3] The chain is irreducible, and there are states $k \in S$ having period 2, and $\ell \in S$ having period 4.

Solution: *Does not exist. By the Equal Periods Lemma, since the chain is irreducible, all states must have the same period.*

(c) [3] There are distinct states $k, \ell \in S$ such that if the chain is started at k, then there is a positive probability that the chain will visit ℓ exactly five times (and then never again).

Solution: *Yes. For example, let $S = \{1, 2, 3\}$, with $p_{12} = 1$, $p_{22} = 1/3$, $p_{23} = 2/3$, and $p_{33} = 1$ (with $p_{ij} = 0$ otherwise). Then if the chain is started at $k = 1$, then it will initially follow the path $1 \to 2 \to 2 \to 2 \to 2 \to 2 \to 3$ with probability $(1)(1/3)(1/3)(1/3)(1/3)(2/3) > 0$, after which it will remain in the state 3 forever.*

(d) [3] The chain is irreducible and transient, and there are $k, \ell \in S$ with $f_{k\ell} = 1$.

Solution: *Yes. For example, consider simple random walk with $p = 3/4$, so $S = \mathbf{Z}$ and $p_{i,i+1} = 3/4$ and $p_{i,i-1} = 1/4$ for all $i \in S$ (with $p_{ij} = 0$ otherwise). Let $k = 0$ and $\ell = 5$. Then as shown in class, $f_{05} = 1$, and the chain is irreducible and transient. (Of course, S is infinite here; if S is finite then all irreducible chains are recurrent.)*

(e) [3] The chain is irreducible and transient, and is reversible with respect to some probability distribution π.

Solution: *Does not exist. Indeed, if the chain is reversible with respect to π, then π is a stationarity distribution. Then if it is also irreducible, then by the Stationarity Recurrence Lemma, it is recurrent, i.e. it is not transient.*

(f) [3] The chain is irreducible and has a stationary probability distribution π, and $p_{ij} < 1$ for all $i, j \in S$, but the chain is not reversible with respect to π.

Solution: *Yes. For example, let $S = \{1, 2, 3\}$, with $p_{12} = p_{23} = p_{31} = 1/3$, and $p_{21} = p_{32} = p_{13} = 2/3$ (with $p_{ij} = 0$ otherwise). And let $\pi_1 = \pi_2 = \pi_3 = 1/3$, so π is a probability distribution on S. Then $\pi_1 p_{12} = (1/3)(1/3) \neq (1/3)(2/3) = \pi_2 p_{21}$, so the chain is not reversible with respect to π. On the other hand, for any $j \in S$, we have $\sum_i \pi_i p_{ij} = (1/3)(1/3 + 2/3) = 1/3 = \pi_j$, so π is a stationary distribution.*

(g) [3] The chain is irreducible and transient, and there are $k, \ell \in S$ with $p_{k\ell}^{(n)} \geq 1/3$ for all $n \in \mathbf{N}$.

Solution: *Does not exist. Indeed, we know from the Cases Theorem that for any irreducible transient Markov chain,* $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} < \infty$ *for all* $k, \ell \in S$. *In particular, we must have* $\lim_{n\to\infty} p_{k\ell}^{(n)} = 0$. *So, it is impossible that* $p_{k\ell}^{(n)} \geq 1/3$ *for all* $n \in \mathbf{N}$.

(h) [3] The chain is irreducible, and there are distinct states $i, j, k, \ell \in S$ such that $f_{ij} < 1$, and $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$.

Solution: *Does not exist. Indeed, we know from the Stronger Recurrence Theorem that for any irreducible Markov chain, if* $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$ *for any one pair* $k, \ell \in S$, *then the chain is recurrent, and* $f_{ij} = 1$ *for all* $i, j \in S$.

(i) [3] There are states $i, j, k \in S$ with $p_{ij} > 0$, $p_{jk}^{(2)} > 0$, and $p_{ki}^{(3)} > 0$, and the state i is periodic (i.e., has period > 1).

Solution: *Yes. For example, let* $S = \{1, 2, 3, 4, 5, 6\}$, *with* $p_{12} = p_{23} = p_{34} = p_{56} = p_{61} = 1$ *(with* $p_{ij} = 0$ *otherwise). Let* $i = 1$, *and* $j = 2$, *and* $k = 4$. *Then* $p_{ij} = p_{12} = 1 > 0$, *and* $p_{jk}^{(2)} = p_{23}p_{34} = 1 > 0$, *and* $p_{ki}^{(3)} = p_{45}p_{56}p_{61} = 1 > 0$, *but state* i *has period 6 since it is only possible to return from* i *to* i *in multiples of six steps.*

3. [6] Let $S = \{1, 2, 3\}$, with $\pi_1 = 1/2$ and $\pi_2 = 1/3$ and $\pi_3 = 1/6$. Find (with proof) irreducible transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that π is a stationarity distribution. [Hint: Don't forget the Metropolis (MCMC) algorithm.]

   Solution: *The Metropolis algorithm says that for* $i \neq j$ *we want* $p_{ij} = (1/2)\min(1, \pi_j/\pi_i)$. *So, we set* $p_{21} = p_{32} = (1/2)(1) = 1/2$, *and* $p_{12} = (1/2)(2/3) = 1/3$ *and* $p_{23} = (1/2)(3/6) = 1/4$. *Then to make* $\sum_j p_{ij} = 1$ *for all* $i \in S$, *we set* $p_{11} = 2/3$, *and* $p_{22} = 1/4$, *and* $p_{33} = 1/2$. *Then by construction,* $\pi_i p_{ij} = \pi_j p_{ji}$ *for all* $i, j \in S$. *Hence, the chain is reversible with respect to* π. *Hence,* π *is a stationary distribution.*

4. [6] Consider the undirected graph with vertex set $V = \{1, 2, 3, 4\}$, and an undirected edge (of weight 1) between each of the following four pairs of edges (and no other edges): (1,2), (2,3), (3,4), and (2,4). Let $\{p_{ij}\}_{i,j \in V}$ be the transition probabilities for random walk on this graph. Compute (with full explanation) $\lim_{n\to\infty} p_{21}^{(n)}$, or prove that this limit does not exist.

   Solution: *The graph is connected (since we can get from* $1 \to 2 \to 3 \to 4$ *and back), so the walk is irreducible. Also, the walk is aperiodic since e.g. we can get from 2 to 2 in 2 steps by* $2 \to 3 \to 2$, *or in 3 steps by* $2 \to 3 \to 4 \to 2$, *and* $gcd(2, 3) = 1$. *And, as shown in class, if* $\pi_u = d(u)/Z = d(u)/2|E| = d(u)/8$, *then the walk is reversible with respect to* π, *so* π *is a stationary distribution. Hence,* $\lim_{n\to\infty} p_{21}^{(n)} = \pi_1 = d(1)/8 = 1/8$, *since* $d(1) = 1$ *because there is only one edge originating from the vertex 1.*

5. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4\}$, with $X_0 = 2$, and with transition probabilities satisfying that $p_{11} = p_{44} = 1$, $p_{21} = 1/4$, $p_{34} = 1/5$, and $p_{23} = p_{31} = p_{12} = p_{13} = p_{14} = p_{41} = p_{42} = p_{43} = 0$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } 4\}$.

   (a) [5] Find (with explanation) non-negative values of p_{22}, p_{24}, p_{32}, and p_{33}, such that $\sum_{j \in S} p_{ij} = 1$ for all $i \in S$ (as it must), and also $\{X_n\}$ is a <u>martingale</u>.

   Solution: *For a Markov chain to be a martingale, we need that $\sum_{j \in S} j\, p_{ij} = i$ for all $i \in S$. With $i = 2$, we need that $(1/4)(1) + p_{22}(2) + p_{24}(4) = 2$. But we must have $\sum_{j \in S} p_{ij} = 1$, i.e. $(1/4) + p_{22} + p_{24} = 1$, i.e. $p_{22} = 3/4 - p_{24}$, so we must have $(1/4)(1) + (3/4 - p_{24})(2) + p_{24}(4) = 2$, or $p_{24}(4 - 2) = 2 - 1/4 - 3/2 = 1/4$, so $p_{24} = (1/4)/2 = 1/8$. (Or, more simply, from 2 the chain has probability 1/4 of decreasing by 1, so to preserve expectations it must have probability 1/8 of increasing by 2.) Then $p_{22} = 3/4 - p_{24} = 3/4 - 1/8 = 5/8$. Similarly, with $i = 3$, we need $p_{32}(2) + p_{33}(3) + (1/5)(4) = 3$. For simplicity, since we must have $p_{32} + p_{33} + (1/5) = 1$, we can subtract 3 from each term, to get that $p_{32}(-1) + p_{33}(0) + (1/5)(1) = 0$, so $p_{32} = 1/5$, and then $p_{33} = 1 - 1/5 - p_{32} = 3/5$. In summary, if $p_{24} = 1/8$, $p_{22} = 5/8$, $p_{32} = 1/5$, and $p_{33} = 3/5$, then we have valid Markov chain transitions which make it a martingale.*

   (b) [3] For the values found in part (a), compute with justification $\mathbf{E}(X_T)$.

   Solution: *Clearly the chain is bounded up to time T, indeed we always have $|X_n| \leq 4$. Hence, by the Optional Stopping Theorem, $\mathbf{E}(X_T) = \mathbf{E}(X_0) = 2$.*

   (c) [3] For the values found in part (a), compute with justification $\mathbf{P}(X_T = 1)$.

   Solution: *Let $p = \mathbf{P}(X_T = 1)$. Then since we must have $X_T = 1$ or 4, therefore $\mathbf{P}(X_T = 4) = 1 - p$, and $\mathbf{E}(X_T) = p(1) + (1 - p)(4) = 4 - 3p$. Solving and using part (b), we must have that $2 = 4 - 3p$, so $3p = 4 - 2 = 2$, whence $p = 2/3$.*