Lecture 26

$\boxed{\text{10. CHAOS}}$

Density: we say that $Y \subseteq X$ is dense in X if
① for any open set $A \subseteq X$, $A \cap Y \neq \emptyset$
or
② for any $x \in X$, there is $y_n \in Y$ s.t. $d(x, y_n) \longrightarrow 0$
or
③ $\overline{Y} = X$

Example: ① The set of periodic points in Σ is dense in Σ.

Take $s \in \Sigma$, $s = (s_0 s_1 s_2 \cdots)$

Define
$t_0 = (\overline{s_0})$
$t_1 = (\overline{s_0 s_1})$
$t_2 = (\overline{s_0 s_1 s_2})$
$\cdots$
$t_n = (\overline{s_0 s_1 \cdots s_n})$

$d[s, t_n] \leq \frac{1}{2^n}$ By the Proximity thm, since s and t have the same first $n+1$ entries
so $d[s, t_n] \longrightarrow 0$

② The orbit of

$$\hat{s} = (\underbrace{0\ 1}_{1 \text{ blocks}}\ \underbrace{00\ \ 01\ \ 10\ \ 11}_{2 \text{ blocks}}\ \underbrace{000\ \ 001\ \ 010\ \ 011\ \ 100\ \ 101\ \ 110\ \ 111}_{3 \text{ blocks}}\ \cdots)$$

The orbit of $\hat{s}$ under σ is dense in Σ.

Proof: Let $s < \Sigma$, and $\varepsilon > 0$ then there is $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \varepsilon$
at some point n, the $(n+1)$-block region or $\hat{s}$ there is a sequence which matches $s_0 s_1 \cdots s_n$.

That means there is $k \in \mathbb{N}$ s.t. $\sigma^k(\hat{s}) = (\underbrace{s_0 s_1 \cdots s_n}$ other terms$)$

By the proximity thm, $d[s, \sigma^k(\hat{s})] \leq \frac{1}{2^n} < \varepsilon$

∎

Transitivity: $f: X \longrightarrow X$ is transitive if for any pair of points, $x, y \in X$. And any $\varepsilon > 0$, there is a third point $z \in X$ s.t. the orbit of z under F passes within distance ε of both X and Y.
or
Equivalently, for all $x, y \in X$, $\varepsilon > 0$, there is $z \in X$ and $k \in \mathbb{N}$ s.t.
$d(x, z) < \varepsilon$ and $d(F^k(z), y) < \varepsilon$

Example: The shift map is transitive.

$x = (1 0 \cdots)$
$y = (0 1 \cdots)$

$$z = \sigma^6(g) \sim x$$

$$\sigma^8(z) = \sigma^{14}(g) \sim y$$

Proof: we have seen that the orbit of $\hat{s}$ is dense in Σ.
so for any $t, s \in \Sigma$, $\varepsilon > 0$.

- there is $j \in \mathbb{N}$ s.t. $d[\sigma^j(\hat{s}), s] < \varepsilon$
- there is $k \in \mathbb{N}$ s.t. $d[\sigma^{j+k}(\hat{s}), t] < \varepsilon$

$\blacksquare$

Proposition: Any dynamic system with a dense orbit is transitive.
The converse is also true: A transitive dynamic system must have a dense orbit.

Sensitivity: $F: X \to Y$ is sensitive if there is $\beta > 0$ s.t. for any $x \in X$ and $\varepsilon > 0$, there exists $y \in X$ within distance ε of x and there is $k \in \mathbb{N}$ s.t. $d(F^k(x), F^k(y)) \geq \beta$.
or equivalently, there is $\beta > 0$ s.t. for any $x \in X$, there exists $y \in X$ and $k \in \mathbb{N}$ s.t.

$$d(x, y) < \varepsilon \quad \text{BUT} \quad d(F^k(x), F^k(y)) \geq \beta$$

Remarks: ① roughly, it means that we can find y as close to x as we want, and the orbits will eventually be separated by β.

② Numerically, it implies that small errors can lead to completely different orbits.