Knowledge Representation and Reasoning: Pure logical reasoning

Chapter 7.5, 7.6, Chapter 9

Recall propositional logic

$\neg$	
0	1
1	0

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	1	1
1	0	1

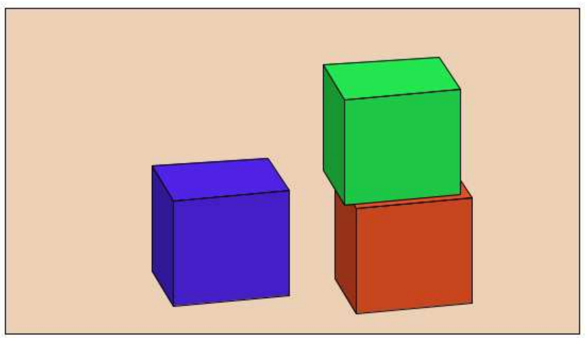
$\leftrightarrow$	0	1
0	1	0
1	0	1

- ◊ Truth value of any propositional formula can be computed given an assignment of the values 1 (true) and 0 (false) to the atoms
- ◊ This computation is entirely deterministic and easy (linear time)
- ◊ Gives mechanical test for validity of inferences

SAT problems: examples 1

propositional satisfiability problem

SAT representations of discrete problems
— Any case expressed as a set of yes/no decisions



Atomic propositions to describe state
greenOnRed, blueOnGreen, etc.

More to describe possible moves
GreenToTable, blueToGreen, etc.

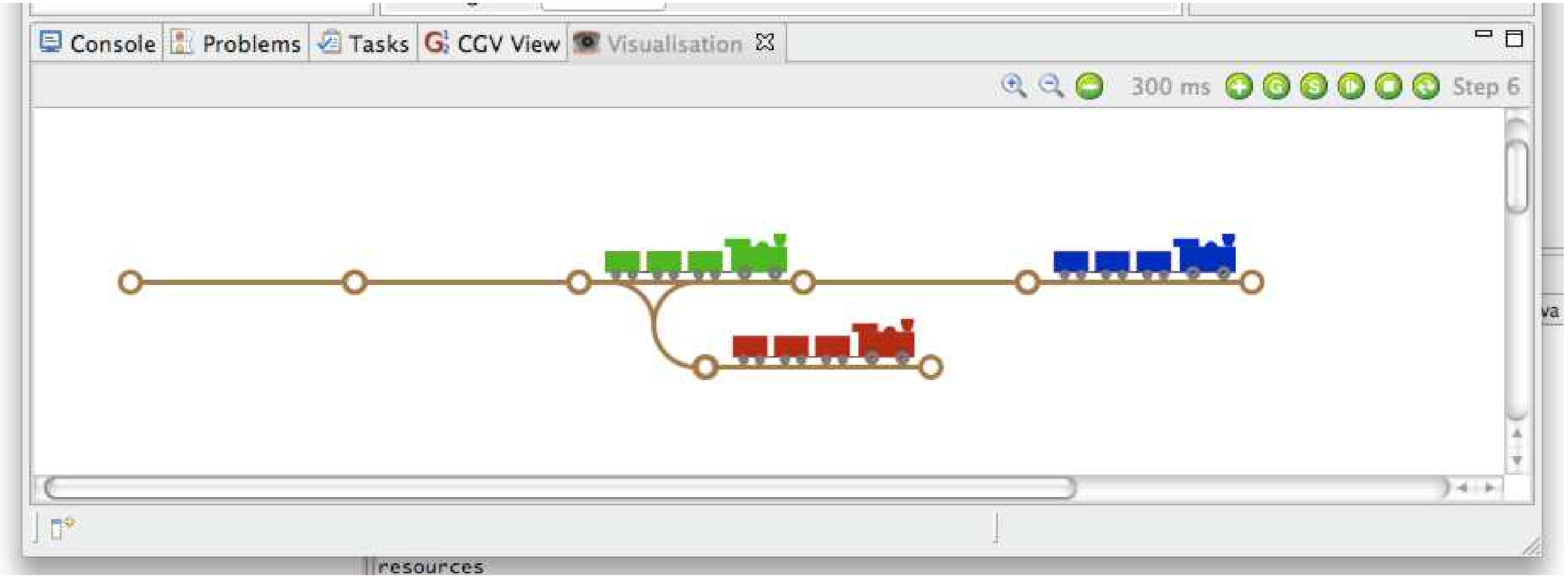
finite states

discrete changes

not so dependent on numbers

Can encode sequences of moves (e.g. plans) in this vocabulary

SAT problems: examples 2



- Meet-pass planning problems: getting the trains past each other using the given track sectors and the siding, obeying safety conditions
- First order problem representation is quite easy
- Reduces to SAT because everything is finite
- Still a "toy" example (270 atomic formulae) but closer to reality

SAT applications

◇ Industrial scale problems with thousands of variables (or more)

◇ Some obviously discrete problems
— circuit analysis
— model checking for hardware / software verification
— classical planning
— diagnosis
— combinatorial design (experiments, cryptography, drug design, etc)

◇ Often used for sub-problems
— Generating test patterns
— Scheduling (applied in many domains)
— Design and analysis of protocols

SAT: the bad news

- Number of possible truth-value assignments grows exponentially
  — with n atoms, 2^n assignments of values (possible worlds)
  — 2^{2^n} sets of possible worlds (truth functions / propositions)
  — Testing for satisfiability (SAT) is (probably) hard in the worst case
- Important SAT problems have thousands of variables – even millions
- Brute force is hopeless!
- SAT is the classic NP-complete problem
  — All known solution methods require exponential time
  — Generally taken to be intractable

SAT: the better news

- Work towards intelligent search for solutions
- First step: simplify the structure of formulae
- $A \leftrightarrow B$ equivalent to $(A \land B) \lor (\neg A \land \neg B)$
- $A \rightarrow B$ equivalent to $\neg A \lor B$
- Every formula has an equivalent using $\land$, $\lor$ and $\neg$ only

reduce them for faster computation

Example:

$(p \rightarrow q) \rightarrow (r \land \neg(p \lor \neg s))$

$\neg(p \rightarrow q) \lor (r \land \neg(p \lor \neg s))$

$\neg(\neg p \lor q) \lor (r \land \neg(p \lor \neg s))$

SAT: better news continues

- $\neg(A \wedge B)$ equivalent to $\neg A \vee \neg B$
- $\neg(A \vee B)$ equivalent to $\neg A \wedge \neg B$

De Morgan's law

- $\neg\neg A$ equivalent to A
- Every formula has an equivalent using ① $\wedge$, $\vee$ and $\neg$ only, in which negation ($\neg$) applies only to atoms

called

- This is ② **Negation Normal Form** (NNF)

$$\neg(\neg p \vee q) \vee (r \wedge \neg(p \wedge \neg s))$$

$$(\neg\neg p \wedge \neg q) \vee (r \wedge \neg(p \wedge \neg s))$$

$$(p \wedge \neg q) \vee (r \wedge \neg(p \wedge \neg s))$$

$$(p \wedge \neg q) \vee (r \wedge (\neg p \vee \neg\neg s))$$

$$(p \wedge \neg q) \vee (r \wedge (\neg p \vee s))$$

Simplify

SAT: better and better news

- $A \wedge B$ equivalent to $B \wedge A$
- $A \vee B$ equivalent to $B \vee A$
- $(A \wedge B) \vee C$ equivalent to $(A \vee C) \wedge (B \vee C)$
- $(A \vee B) \wedge C$ equivalent to $(A \wedge C) \vee (B \wedge C)$

distribution laws

- Every formula has an equivalent which is 1 a conjunction ($\wedge$) of disjunctions ($\vee$) of literals 2 (atoms and negated atoms)
- This is **Conjunctive Normal Form** (CNF), aka **Clause Form**

called

- So any technique for reasoning with clauses can do propositional logic

"whole thing True iff each clause is true"

Reduction to CNF: example

$(p \to q) \to (r \wedge \neg(p \vee \neg s))$

reduces to NNF

$(p \wedge \neg q) \vee (r \wedge (\neg p \vee s))$

then moving conjunction outside disjunction:

$(p \vee (r \wedge (\neg p \vee s))) \wedge (\neg q \vee (r \wedge (\neg p \vee s)))$

$(p \vee r) \wedge \underline{(p \vee \neg p} \vee s) \wedge (\neg q \vee r) \wedge (\neg q \vee \neg p \vee s)$

Second conjunct is a tautology, so can be deleted without loss, giving a set of clauses equivalent to the original formula:

$p \vee r$
$\neg q \vee r$
$\neg q \vee \neg p \vee s$

Resolution

Resolution is a logical inference rule which operates on clauses:

$$\frac{p_1 \vee \ \ldots \ \vee p_n \vee \underline{q} \qquad\qquad \underline{\neg q} \vee r_1 \vee \ \ldots \ \vee r_m}{p_1 \vee \ \ldots \ \vee p_n \vee r_1 \vee \ \ldots \ \vee r_m}$$ // q & $\neg q$ cancelled out

Alternatively, looking at a clause as a set of literals:

$$\frac{\overset{\text{[Suppose True]}}{\Gamma} \qquad\qquad \overset{\text{[Suppose True]}}{\Delta}}{(\Gamma \underbrace{\setminus \{q\}}_{\text{exclude}}) \cup (\Delta \underline{\setminus \{\neg q\}})}$$

Resolution derivation (example)

Show $\{p \vee q,\ p \vee \neg q,\ \neg p \vee r,\ \neg r \vee s,\ \neg r \vee \neg s\}$ unsatisfiable

1.	$p \vee q$	given	
2.	$p \vee \neg q$	given	
3.	$\neg p \vee r$	given	
4.	$\neg r \vee s$	given	
5.	$\neg r \vee \neg s$	given	
6.	p	1, 2	(with factoring to reduce $p \vee p$ to p)
7.	$\neg r$	4, 5	(with factoring)
8.	r	3, 6	
9.	$\perp$	7, 8	

Davis – Putnam – Logemann – Loveland

A better idea: DPLL Algorithm

◇ Any assignment satisfying a set Γ of clauses must make any specific atom p that occurs in Γ either true or false.

◇ Therefore Γ is satisfiable iff either $\Gamma \cup \{p\}$ is satisfiable or else $\Gamma \cup \{\neg p\}$ is satisfiable.

◇ Let Γ' be Γ with all clauses containing literal p deleted, and with $\neg p$ removed from all clauses in which it occurs. Then Γ' is satisfiable iff $\Gamma \cup \{p\}$ is satisfiable. Note that Γ' contains

— fewer clauses than Γ
— shorter clauses than Γ
— fewer atoms than Γ

◇ The same holds for Γ'', defined similarly using $\neg p$ instead of p.

◇ Therefore the problem of deciding whether Γ is satisfiable can be replaced by the two strictly simpler problems of deciding satisfiability of Γ' and Γ''.

Unit propagation

- A pure literal is one whose complement does not appear anywhere
- Obviously any pure literal can be made true without bad consequences
- Therefore any clause containing a pure literal may be deleted
- A unit clause is a clause consisting of only one literal
- Obviously this literal has to be set to true
- Therefore its complement can be deleted from all clauses
  — Literal is then pure and triggers purity deletion
- Iterating these inference moves is unit propagation
- DPLL amounts to splitting plus unit propagation

until $\perp$ or satisfying assignment

Improving DPLL

◇ Clause learning
— The search backtracks when it runs into a contradiction
— The decisions determining the branch can't all be right
— Add complements of [a subset of] the chosen literals as a new clause [learning]
— So we never backtrack twice for the same reason

◇ Choosing good atoms for branching
— E.g. one that occurs most often in shortest clauses (MOMS)
— Or one that occurs most often in currently satisfied clauses

◇ Intelligent backtracking
— Can obviously jump back to a variable in the latest nogood
— May pay to jump back further

◇ Restarts
— Can jump right back to the root of the search tree and probe it
— Depends heavily on learned clauses to prevent repeated work

What about quantifiers?

- Sometimes need to reason about large or unspecified domains
- Reduction to SAT not possible in such cases
- Trivial example: subset transitivity:

  $\forall x \forall y (\mathsf{sub}(x, y) \leftrightarrow \forall z (\mathsf{in}(z, x) \rightarrow \mathsf{in}(z, y)))$

  therefore

  $\forall x \forall y \forall z ((\mathsf{sub}(x, y) \wedge \mathsf{sub}(y, z)) \rightarrow \mathsf{sub}(x, z))$

  _ is a sub of _

Prenex normal form

- First problem: get all quantifiers to the front
  Assume $\rightarrow$ and $\leftrightarrow$ rewritten using $\wedge$, $\vee$ and $\neg$

- Moving quantifiers outside negation
  $\neg\forall x A$ equivalent to $\exists x \neg A$
  $\neg\exists x A$ equivalent to $\forall x \neg A$
  $\forall$ $\exists$
  So quantifiers may switch between universal and existential

- Moving quantifier binding x outside another one. E.g.:
  $\forall x A(x) \vee \forall x B(x)$ goes to $\forall x(A(x) \vee \forall x B(x))$
  Solution: rewrite variables:
  $\forall x(A(x) \vee \forall y B(y))$
  $\forall x \forall y(A(x) \vee B(y))$

Removing the quantifiers

- Existential quantifiers removed by Skolemisation
  — Variable replaced by a new name or function
  — Then quantifier deleted

  E.g. $\exists x \forall y \exists z R(x, y, z)$

  goes to $\forall y \exists z R(a, y, z)$

  then to $\forall y R(a, y, f(y))$

- All quantifiers are now universal. They can be removed
  — Free variables are implicitly universal

- Note: Skolemised formula not equivalent to the original, but they are satisfiable if and only if the original is.

- Quantifier-free formula can be put into clause form

First order resolution

- Resolution applies to first order clauses too
- Usually requires **unification**: substituting terms for variables in order to make literals match

E.g.	$P(x,a) \vee \neg Q(x)$ and $\neg P(b,y) \vee R(y)$
unifier	$[x \leftarrow b, y \leftarrow a]$
gives	$P(b,a) \vee \neg Q(b)$ and $\neg P(b,a) \vee R(a)$
Resolvent:	$\neg Q(b) \vee R(a)$

general principle

[complete proof technique]

Example: subset transitivity (1)

$\forall x \forall y(\mathsf{sub}(x,y) \leftrightarrow \forall z(\mathsf{in}(z,x) \rightarrow \mathsf{in}(z,y)))$

$\neg\forall x \forall y \forall z((\mathsf{sub}(x,y) \wedge \mathsf{sub}(y,z)) \rightarrow \mathsf{sub}(x,z))$

clausifies to

$\neg\mathsf{sub}(x,y) \vee \neg\mathsf{in}(z,x) \vee \mathsf{in}(z,y)$

$\mathsf{in}(f(x,y),x) \vee \mathsf{sub}(x,y)$

$\neg\mathsf{in}(f(x,y),y) \vee \mathsf{sub}(x,y)$

$\mathsf{sub}(a,b)$

$\mathsf{sub}(b,c)$

$\neg\mathsf{sub}(a,c)$

Example: subset transitivity (2)

1.	$\neg\mathsf{sub}(x, y) \lor \neg\mathsf{in}(z, x) \lor \mathsf{in}(z, y)$	given
2.	$\mathsf{in}(f(x, y), x) \lor \mathsf{sub}(x, y)$	given
3.	$\neg\mathsf{in}(f(x, y), y) \lor \mathsf{sub}(x, y)$	given
4.	$\mathsf{sub}(a, b)$	given
5.	$\mathsf{sub}(b, c)$	given
6.	$\neg\mathsf{sub}(a, c)$	given
7.	$\neg\mathsf{in}(z, a) \lor \mathsf{in}(z, b)$	from 1, 4 $[x \leftarrow a, y \leftarrow b]$
8.	$\neg\mathsf{in}(z, b) \lor \mathsf{in}(z, c)$	from 1, 5 $[x \leftarrow b, y \leftarrow c]$
9.	$\mathsf{in}(f(a, c), a)$	from 2, 6 $[x \leftarrow a, y \leftarrow c]$
10.	$\neg\mathsf{in}(f(a, c), c)$	from 3, 6 $[x \leftarrow a, y \leftarrow c]$
11.	$\neg\mathsf{in}(f(a, c), b)$	from 7, 9 $[z \leftarrow f(a, c)]$
12.	$\mathsf{in}(f(a, c), c)$	from 8, 11 $[z \leftarrow f(a, c)]$
13.	$\perp$	from 10, 12

unifiers

Summary

◇ Problems from many domains can be coded as SAT
— Discrete, finite, not too much arithmetic

◇ Intelligent solution methods dominate brute force

◇ Reduction to clause form
— Apply logical equivalences: DeMorgan's laws, distribution

◇ Simple inference rules operate on clauses
— Resolution (not much used for pure SAT problems)
— DPLL and its variants generally preferred
— SAT solvers now useful for real industrial problems / "machine-friendly technique"
— easy to implement

◇ Normal forms also for first order logic
— Prenex, skolem, clause form

◇ Resolution is more useful at the first order level — Resolution-like methods are the state of the art