NOTE TO STUDENTS: This file contains sample solutions to the term test together with the marking scheme and comments for each question. Please read the solutions and the marking schemes and comments carefully. Make sure that you understand why the solutions given here are correct, that you understand the mistakes that you made (if any), and that you understand *why* your mistakes were mistakes.

Remember that although you may not agree completely with the marking scheme given here it was followed the same way for all students. We will remark your test only if you clearly demonstrate that the marking scheme was not followed correctly.

For all remarking requests, please submit your request **in writing** directly to your instructor. For all other questions, please don't hesitate to ask your instructor during office hours or by e-mail.

Question 1. [12 MARKS]

Part (a) [1 MARK]

Write the *converse* of the following statement: "If I am tired, then I don't drive."

SAMPLE SOLUTION:

If I don't drive, then I am tired.

MARKING SCHEME:

- Correct statement (either in English or symbolically).

Part (b) [1 MARK]

Write the *contrapositive* of the following statement: "If I am tired, then I don't drive."

SAMPLE SOLUTION:

If I drive, then I am not tired.

MARKING SCHEME:

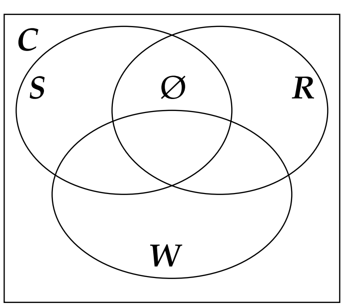
- Correct statement (either in English or symbolically).

Part (c) [4 MARKS]

Draw a Venn diagram showing a situation where the following statement is true. Clearly label each part of your diagram and explain what each part represents.

"Every student who studied and is rested will do well on this test."

SAMPLE SOLUTION:



C: set of all students (domain)
S: set of all students who studied
R: set of all students who are rested
W: set of all students who will do well on this test

The region marked "Ø" must be empty for the statement to be true; all others may be empty or not.

MARKING SCHEME:

- **Diagram** [2 marks]: 1 mark for a clear diagram (even if incorrect); 1 mark for a correct diagram (even if poorly drawn)
- **Labels** [2 marks]

MARKER'S COMMENTS:

- common error: not drawing the set of all students (the domain)
- common error: not considering cases where students did well but did not study or rest
- common error: not indicating which areas are empty/non-empty

Part (d) [3 MARKS]

Is the statement below True or False? Justify your response.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$$

SAMPLE SOLUTION:

FALSE: Let $x = 0$; then $xy = 0 \neq 1$ for all $y \in \mathbb{R}$.

MARKING SCHEME:

- **Answer** [1 mark]: correct answer (even if not justified)
- **Justification** [2 marks]: good justification — give marks for good attempts to justify an incorrect answer

MARKER'S COMMENTS:

- error code $\boldsymbol{W_1}$ [0–0.5 marks in total]: incorrect justification
- error code $\boldsymbol{W_2}$ [1 mark in total]: incorrect answer (forgetting the case $x = 0$)
- error code $\boldsymbol{W_3}$: not making it clear whether the statement is true of false
- common error [1 mark in total]: answering: "false because $xy = 1$ only when $x = y = 1$"

Part (e) [3 MARKS]

Is the statement below True or False? Justify your response.

$$\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n > 1 \Rightarrow (m < n \wedge m \mid n)$$

(The notation "$x \mid y$" means "x exactly divides y" — or equivalently, "y is an integer multiple of x".)

SAMPLE SOLUTION:

TRUE: Let $m = 1$; then for all natural numbers $n > 1$, $1 < n$ (obviously) and $1 \mid n$ ($n = 1 \times n$).

MARKING SCHEME:

- **Answer** [1 mark]: correct answer (even if not justified)
- **Justification** [2 marks]: good justification — give marks for good attempts to justify an incorrect answer

MARKER'S COMMENTS:

- error code $\boldsymbol{W_2}$ [0–0.5 marks in total]: incorrect justification
- error code $\boldsymbol{W_3}$: vague, unclear
- error code $\boldsymbol{W_4}$: switching the order of $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}$
- error code $\boldsymbol{W_5}$ [−0.5 marks]: incorrect argument: "when $n = 0$, $m \mid n$ implies division by 0"
- common error [1 mark in total]: arguing the statement is false for one specific value of m
- common error [−1 mark]: some students misunderstood "$m \mid n$" as "$n \mid m$"

Question 2. [12 MARKS]

Consider domain $D = \{\text{all programs and all programmers}\}$, and predicate symbols $C(x)$: "x is a program", $P(x)$: "x is a programmer", $K(x, y)$: "x knows y", $W(x, y)$: "x wrote y", and $G(x)$: "x is good".

Using only these symbols (in addition to appropriate connectives and quantifiers), translate each sentence below. That is, give a natural English sentence that corresponds to each given symbolic sentence, and give a clear symbolic sentence that corresponds to each given English sentence.

Part (a) [2 MARKS]

Every program was written by some programmer.

SAMPLE SOLUTION:

$$\forall x \in D, C(x) \Rightarrow \exists y \in D, P(y) \wedge W(y, x)$$

MARKING SCHEME:

- **Meaning** [1 mark]: answer clearly has the correct meaning (even if poorly written up)
- **Notation** [1 mark]: correct use of symbolic notation (be picky on this one)

MARKER'S COMMENTS:

- common error [−1 mark]: $\forall x \in D, \exists y \in D, W(y, x)$
- common error [−1 mark]: $\forall x \in D, \exists y \in D, C(x) \wedge P(y) \Rightarrow W(y, x)$
- common error [−1 mark]: swapping quantifiers
- common error [−2 marks]: $\forall x \in D, \exists y \in D, C(x) \wedge P(y) \wedge W(y, x)$

Part (b) [2 MARKS]

$\exists x \in D, \neg G(x) \wedge (P(x) \vee C(x))$

SAMPLE SOLUTION:

There is at least one bad programmer or one bad program.

MARKING SCHEME:

- **Meaning** [1 mark]: answer clearly has the correct meaning (even if poorly written up)
- **Naturalness** [1 mark]: English is fairly natural (be lenient on this one)

MARKER'S COMMENTS:

- common error [−1 mark]: "Some programs **and** programmers are not good."

Part (c) [2 MARKS]

Every programmer knows some programmer who has written a bad program.

SAMPLE SOLUTION:

$$\forall x \in D, P(x) \Rightarrow \exists y \in D, P(y) \wedge K(x, y) \wedge \exists z \in D, C(z) \wedge \neg G(z) \wedge W(y, z)$$

MARKING SCHEME:

- **Meaning** [1 mark]: answer clearly has the correct meaning (even if poorly written up)
- **Notation** [1 mark]: correct use of symbolic notation (be picky on this one)

MARKER'S COMMENTS:

- common error [−1 mark]: $\forall x \in D, \exists y \in D, \exists z \in D, P(x) \wedge P(y) \wedge C(z) \wedge \neg G(z) \Rightarrow K(x, y) \wedge W(y, z)$

Part (d) [2 MARKS]
$\forall x \in D, P(x) \Rightarrow \exists y \in D, C(y) \wedge W(x, y)$
SAMPLE SOLUTION:

Every programmer has written some program.

MARKING SCHEME:

- **Meaning** [1 mark]: answer clearly has the correct meaning (even if poorly written up)
- **Naturalness** [1 mark]: English is fairly natural (be lenient on this one)

Part (e) [2 MARKS]
Some good program was written by an unknown programmer (one known by no programmer).
SAMPLE SOLUTION:

$$\exists x \in D, C(x) \wedge G(x) \wedge \exists y \in D, P(y) \wedge W(y, x) \wedge \forall z \in D, P(z) \Rightarrow \neg K(z, y)$$

MARKING SCHEME:

- **Meaning** [1 mark]: answer clearly has the correct meaning (even if poorly written up)
- **Notation** [1 mark]: correct use of symbolic notation (be picky on this one)

Part (f) [2 MARKS]
$\exists x \in D, P(x) \wedge G(x) \wedge \neg\exists y \in D, C(y) \wedge W(x, y)$
SAMPLE SOLUTION:

Some good programmer has written no program.

MARKING SCHEME:

- **Meaning** [1 mark]: answer clearly has the correct meaning (even if poorly written up)
- **Naturalness** [1 mark]: English is fairly natural (be lenient on this one)

Question 3. [12 MARKS]

For each equivalence below:

- if the equivalence is always true, provide a derivation from one side to the other (justify each step of your derivation from the list of standard equivalences given at the end of this question);
- otherwise, provide an interpretation that makes the equivalence false (along with a brief explanation/ justification that the equivalence is false under your interpretation).

NOTE: In this question, we use a simplified quantifier notation where we omit the domain — *e.g.*, we write "$\forall x$" instead of "$\forall x \in D$". You should also use this simplified form in your answers to this question.

Part (a) [4 MARKS]
$\forall x, P(x) \iff \exists y, P(y)$
SAMPLE SOLUTION:

THE EQUIVALENCE DOES NOT ALWAYS HOLD:

Let $D = \mathbb{N}$ and $P(x)$ mean "x is even".

Then $\forall x, P(x)$ means "every natural number is even" (which is clearly false) and $\exists y, P(y)$ means "some natural number is even" (which is clearly true).

MARKING SCHEME:

- **Answer** [1 mark]: clear attempt to disprove the equivalence (even if incorrect)
- **Interpretation** [2 marks]: correct interpretation to make the equivalence false
- **Justification** [1 mark]: clear explanation of why the equivalence is false under the given interpretation

EXPECTED ERRORS:

- **Wrong Answer** [up to 2 marks]: good attempt to prove the equivalence holds by a well-written derivation

MARKER'S COMMENTS:

- common error [1 mark in total]: stating that "for all" is different from "there exists" with no other justification

Part (b) [4 MARKS]

$P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow R$

SAMPLE SOLUTION:

THE EQUIVALENCE DOES NOT ALWAYS HOLD:

Let $P = \mathsf{False}$, $Q = \mathsf{True}$, $R = \mathsf{False}$.

Then $P \Rightarrow (Q \Rightarrow R) = \mathsf{False} \Rightarrow (\mathsf{True} \Rightarrow \mathsf{False}) = \mathsf{False} \Rightarrow \mathsf{False} = \mathsf{True}$ and $(P \Rightarrow Q) \Rightarrow R = (\mathsf{False} \Rightarrow \mathsf{True}) \Rightarrow \mathsf{False} = \mathsf{True} \Rightarrow \mathsf{False} = \mathsf{False}$.

MARKING SCHEME:

- **Answer** [1 mark]: clear attempt to disprove the equivalence (even if incorrect)
- **Interpretation** [2 marks]: correct interpretation to make the equivalence false
- **Justification** [1 mark]: clear explanation of why the equivalence is false under the given interpretation

EXPECTED ERRORS:

- **Wrong Answer** [up to 2 marks]: good attempt to prove the equivalence holds by a well-written derivation

MARKER'S COMMENTS:

- common error [1 mark in total]: doing some derivation and concluding left-hand-side $\neq$ right-hand-side with no further justification

Part (c) [4 MARKS]

$(\forall x, P(x)) \Rightarrow (\exists x, Q(x)) \iff \exists x, P(x) \Rightarrow Q(x)$

SAMPLE SOLUTION:

THE EQUIVALENCE ALWAYS HOLDS:

$$\begin{aligned}
(\forall x, P(x)) \Rightarrow (\exists x, Q(x)) &\iff \neg(\forall x, P(x)) \vee (\exists x, Q(x)) && \text{(implication)} \\
&\iff (\exists x, \neg P(x)) \vee (\exists x, Q(x)) && \text{(quantifier negation)} \\
&\iff \exists x, \neg P(x) \vee Q(x) && \text{(quantifier distributivity)} \\
&\iff \exists x, P(x) \Rightarrow Q(x) && \text{(implication)}
\end{aligned}$$

MARKING SCHEME:

- **Answer** [1 mark]: clear attempt to prove the equivalence (even if incorrect)
- **Derivation** [2 marks]: correct derivation to show the equivalence holds
- **Justification** [1 mark]: clear justifications for each step of the derivation

EXPECTED ERRORS:

- **Wrong Answer** [up to 2 marks]: good attempt to prove the equivalence is false by providing an explicit interpretation

MARKER'S COMMENTS:

- common error [1.5 marks in total]: attempting to show right-hand-side true **and** left-hand-side false by looking at only one value of x

Bonus. [3 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer. Please attempt this only *after* you have completed the rest of the test.

Give a derivation to show that

$$P \iff P \wedge (P \vee Q)$$

using the equivalences at the end of question 3 to justify **each** step of your derivation — every change in the expression must be justified. *You may use no other equivalence as justification, and you will receive no credit for using truth-tables.*

SAMPLE SOLUTION: (Not provided for bonus...)

MARKING SCHEME: Be particularly picky when marking the bonus. In particular, no credit is assigned to structure only, and the details are more important than for regular questions (where the structure and main idea are more important).

- Use negative marking (-1 per error or missing step or missing justification).

MARKER'S COMMENTS:

- a few students did get full marks
- many students started a derivation that did not lead to a solution; they were awarded 0.5 marks [*Instructor's Comment*: Note that this is more generous than I would have liked for the bonus!]