March 07th

PS4

Q2: $p(x), q(x) \in P_n(\mathbb{C})$. sps $x_0, \dots, x_n$ are n+1 distinct complex numbers

Show that $\langle p(x), q(x)\rangle = p(x_0)\overline{q(x_0)} + p(x_1)\overline{q(x_1)} + \cdots + p(x_n)\overline{q(x_n)}$

Prove that this defines an inner product on $P_n(\mathbb{C})$

Claim: If $p(x) \in P_n(\mathbb{C})$ and $\langle p(x), p(x)\rangle = 0$ then $p(x) = 0$

$0 = \langle p(x), p(x)\rangle = p(x_0)\overline{p(x_0)} + \cdots + p(x_n)\overline{q(x_n)} = |p(x_0)|^2 + \cdots + |p(x_n)|^2$

$\Rightarrow |p(x_0)| = 0, \dots, |p(x_n)| = 0$ (Sps $z \in \mathbb{C}$

$\Rightarrow p(x_0) = 0, \dots, p(x_n) = 0$ $z = 0 \Leftrightarrow |z| = 0$)

We want to prove that $p(x) = 0$

Assume that $p(x) \neq 0$

Let $m = \deg(p(x))$ ($m \leq n$ since $p(x) \in P_n(\mathbb{C})$)

FTA $p(x) = a(x - z_1)(x - z_2)\cdots(x - z_n)$ $z_1, \dots z_n \in \mathbb{C}$ $a \in \mathbb{C}$ $a \neq 0$

$\Rightarrow p(x)$ vanishes at $\leq m$ points

Q3: $P_2(\mathbb{R})$

$\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(2)q(2)$

For what value of c will be the set

$S = \{\underbrace{3x^2 - 2x - 1}_{f(x)}, \underbrace{cx^2 + x - 1}_{g(x)}, \underbrace{5x^2 + cx - 9}_{h(x)}\}$ be orthogonal?

$\forall c \in \mathbb{R}$ s.t. $\langle f(x), g(x)\rangle = 0$

$\langle f(x), h(x)\rangle = 0$

$\langle g(x), h(x)\rangle = 0$

Actually this problem has no solutions!!!

$$\begin{aligned}\langle f(x), g(x)\rangle &= f(-1)g(-1) + f(0)g(0) + f(2)g(2)\\ &= 4(c-2) + (-1)(-1) + 7(4c+1)\\ &= 32c - 8 + 1 + 7\\ &= 32c = 0\end{aligned}$$

Set $c = 0$ $\langle f(x)\, h(x)\rangle = 4(-4) + (-1)(-9) + 7(11)$ no c!!!

Q6 : Let $W_1 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = A^T\}$ (symmetric) and $W_2 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = -A^T\}$ (skew-symmetric) Show that $M_{n\times n}(\mathbb{R}) = W_1 \oplus W_2$ i.e. $W_1 \cap W_2 = \{0\}$ & $M_{n\times n}(\mathbb{R}) = W_1 + W_2$

Def: Let V be a vector space with subspace W_1 & W_2. We say that V is a direct sum of W_1 & W_2 if

- $W_1 \cap W_2 = \{0\}$
- $V = W_1 + W_2$ (Recall that $W_1 + W_2 = \{w_1 + w_2 \in V : w_1 \in W_1, w_2 \in W_2\}$ = all vectors in V that can be written as a sum of a vector in W_1 and a vector in W_2.

$A = -A^T \Rightarrow -A = A^T$

$A = A^T$

$(A - A^T)^T = A^T - A = -(A - A^T) \Rightarrow A - A^T \in W_2$