Homework Assignment #4

MAT 335 – Chaos, Fractals, and Dynamics – Fall 2013

PARTIAL SOLUTION

In the following two solutions, I use the notation $x \equiv (0.a_1a_2a_3\ldots)$ to write x has the ternary expansion $(0.a_1a_2a_3\ldots)$.

Chapter 7.14. We want to prove that the Cantor set $K = \{x \equiv (0.a_1a_2\ldots) : a_i \in \{0,2\}\}$ is equal to $\Gamma = \{x \in [0,1] : T^n(x) \not\to -\infty\}$.
Let $x \in \Gamma$. Then we can write $x \equiv (0.a_1a_2a_3\ldots)$, for $a_i \in \{0,1,2\}$. To remove the ambiguity that some numbers can have two different ternary expansions, assume that if it is possible to write a number with only 0's and 2's, we will do so*.
We want to prove that if $a_i = 1$ for some i, then $T^n(x) \to -\infty$, and thus $x \notin \Gamma$.
From questions 7.12 and 7.13, observe that T transforms

$$(\tfrac{1}{3}, \tfrac{2}{3}) \to (1, \tfrac{3}{2})$$
$$(\tfrac{1}{9}, \tfrac{2}{9}) \to (\tfrac{1}{3}, \tfrac{2}{3})$$
$$(\tfrac{7}{9}, \tfrac{8}{9}) \to (\tfrac{1}{3}, \tfrac{2}{3})$$

We can state this result in terms of the ternary expansion:

$$a_1 = 1 \quad \Rightarrow \quad T^n(x) \to -\infty$$
$$a_2 = 1 \quad \Rightarrow \quad T(x) \equiv (0.b_1b_2\ldots) \text{ with } b_1 = 1$$

So, consider $x \notin K$, i.e. $x \in [0,1]$ is such that its ternary expansion has at least one 1. Let $a_k = 1$ be the first digit equal to 1. Then the ternary expansion of $T(x)$ has 1 on the $(k-1)^{\text{th}}$digit, and $T^k(x) \equiv (0.1b_2b_3\ldots) \in (\frac{1}{3}, \frac{2}{3})$, which means that $T^n(x) \to -\infty$. Thus $x \notin \Gamma$.

Conversely, if $x \in K$, then $x \equiv 0.a_1a_2a_3\ldots$ with $a_i \in \{0,2\}$, so by the next question 7.15†, we know that the ternary expansion of $T^n(x)$ will always have only 0's and 2's, which means that $T^n(x) \notin (\frac{1}{3}, \frac{2}{3})$ for all n, thus the orbit never leaves the interval $[0,1]$. This proves that $x \in \Gamma$.
We conclude that $\Gamma = K$.

*For example $(0.020210000\ldots) = (0.0202022222\ldots)$. In fact the "problematic" numbers are the ones that end with $100000\ldots$.

†On the solution of question 7.15, we don't actually need the fact that $\Gamma = K$, we only need the fact that the ternary expansion of x doesn't contain any 1's.

Chapter 7.15. Let $x \in \Gamma = K$, which has ternary expansion $x \equiv 0.a_1a_2a_3\dots$ where $a_i \in \{0, 2\}$. Then, we have two cases:

Case 1. If $x \leqslant \frac{1}{2}$, then from the previous question, x is in the first third, i.e., $x \in [0, \frac{1}{3}]$ and $a_1 = 0$:

$$T(x) = 3x \equiv 0.b_1b_2b_3\dots,$$

where $b_i = a_{i+1}$, since multiplying a ternary expansion by 3 is the same as shifting all the digits to the left by one position.

Case 2. If $x > \frac{1}{2}$, then from the previous question, x is in the last third, i.e. $x \in [\frac{2}{3}, 1]$ and $a_1 = 2$:

$$T(x) = 3 - 3x = 3(1 - x)$$

which means that first we transform take x and write $1 - x$:

$$1 - x = 0.b_1b_2b_3\dots,$$

where $b_i = 2 - a_i$ (invert all the digits $0 \leftrightarrow 2$). Then we multiply by 3, which is equivalent to a shift:

$$T(x) = 3 - 3x = 3(1 - x) \equiv 0.c_1c_2c_3\dots,$$

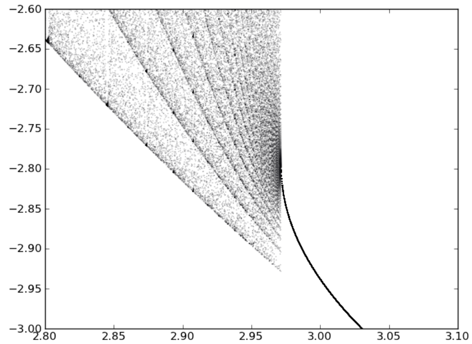
where $c_i = b_{i+1} = 2 - a_{i+1}$.
Concluding, if $x \equiv 0.a_1a_2a_3\dots$, then

$$T(x) \equiv 0.b_1b_2b_3\dots,$$

where

$$b_i = \begin{cases} a_{i+1} & \text{if } a_1 = 0 \\ 2 - a_{i+1} & \text{if } a_1 = 2. \end{cases}$$

Chapter 8.5. Below is a zoomed version of the orbit diagram for this question near $\lambda = 2.96$



For $\lambda > 2.96\ldots$, there seems to be one attracting fixed point, and notice that as $\lambda \to 2.96\ldots^{+}$, the graph becomes vertical, which hints at the existence of a repelling fixed point for $\lambda > 2.96\ldots$. On the other hand, for $\lambda < 2.96\ldots$, there are no fixed points, although there seems to be many periodic points.

This orbit diagram hints at a saddle node bifurcation at $\lambda = 2.96\ldots$.