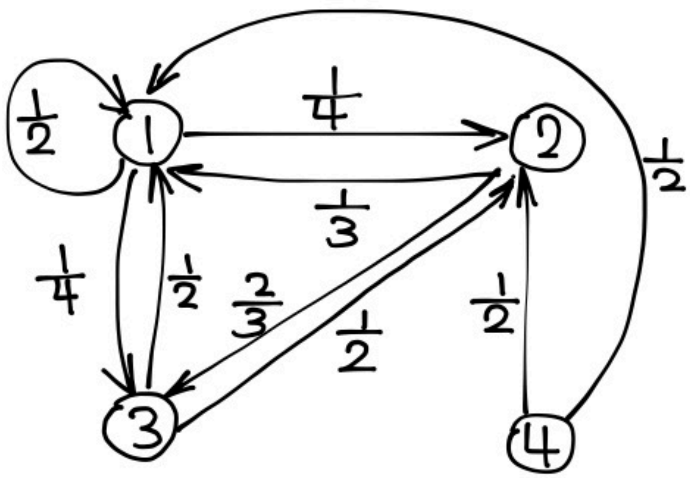
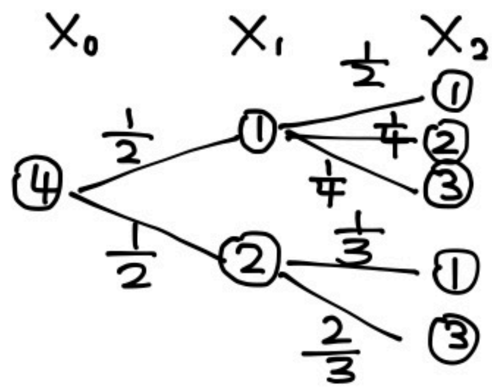
STA447 Homework 1

Rui Qiu #999292509
Enrolled in: STA447
2016-02-09

Problem 1

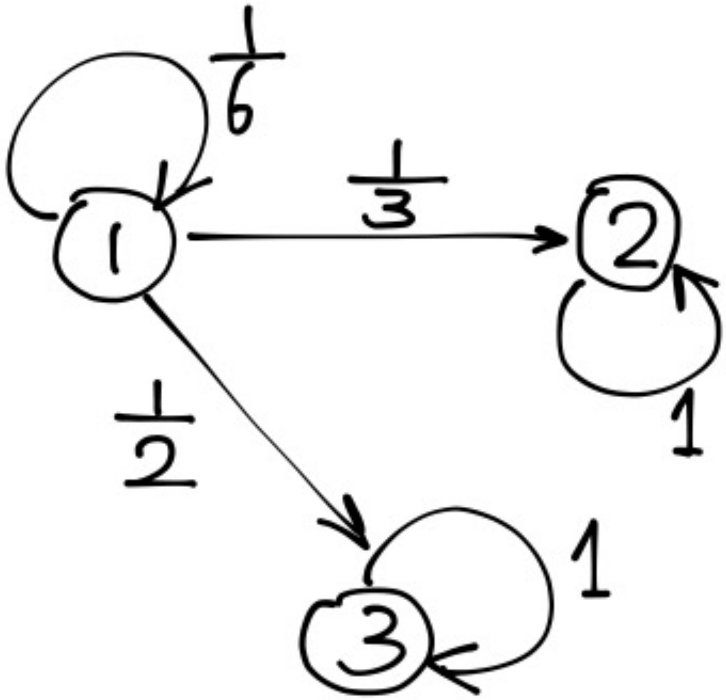




Solution:
If the initial state is given as 4, there are only two possibilities: a '4-2-3' path or a '4-1-3' path.

$$\begin{aligned} P(X_2 = 3|X_0 = 4) &= P(X_2 = 3|X_1 = 1)P(X_1 = 1|X_0 = 4) + P(X_2 = 3|X_1 = 2)P(X_1 = 2|X_0 = 4) \\ &= p_{13}p_{41} + p_{23}p_{42} \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} \\ &= \frac{1}{8} + \frac{1}{3} \\ &= \frac{11}{24} \end{aligned}$$

Problem 2



(a) Solution:

f_{12} is the probability of starting from 1 and eventually hitting 2. Note that if we are at 3, we are staying at 3 forever because $p_{33} = 1$, therefore, $f_{31} = f_{32} = 0$.

$$\begin{aligned} f_{12} &= P_1(X_n = 2 \text{ for some } n \geq 1) \\ &= P(X_n = 2, n \geq 1 | X_0 = 1) \\ &= p_{12} + p_{13}f_{32} + p_{11}f_{12} \\ (1 - p_{11})f_{12} &= p_{12} + p_{13}f_{32} \\ f_{32} &= 0 \\ f_{12} &= \frac{p_{12}}{1 - p_{11}} = \frac{\frac{1}{3}}{\frac{5}{6}} = \frac{1}{3} \cdot \frac{6}{5} = \frac{2}{5} \end{aligned}$$

(b) Proof:

Since $p_{21} = p_{31} = p_{23} = p_{32} = 0$ and $p_{22} = p_{33} = 1$, the only possible state before hitting state 2 is staying at state 1. And $1 - (\frac{1}{6})^n \geq 1 - \frac{1}{6}$ for $n \geq 1$.

$$
\begin{aligned}
p_{12}^{(n)} &= P(X_n = 2|X_0 = 1) \\
&= p_{12} + p_{11}p_{12} + p_{11}^2 p_{12} + \cdots + p_{11}^{n-1} p_{12} \\
&= p_{12} \left(1 + p_{11} + \cdots + p_{11}^{n-1}\right) \\
&= p_{12} \cdot \frac{1 - p_{11}^n}{1 - p_{11}} \\
&= \frac{1}{3} \cdot (1 - p_{11}^n) \cdot \frac{6}{5} \\
&\geq \frac{1}{3} \cdot (1 - \frac{1}{6}) \cdot \frac{6}{5} \\
&\geq \frac{1}{3}
\end{aligned}
$$

(c) Solution:

$$
\begin{aligned}
\sum_{n=1}^{\infty} p_{12}^{(n)} &= p_{12}^{(1)} + p_{12}^{(2)} + p_{12}^{(3)} + \cdots \\
&= \frac{1}{3} + \left(\frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times 1\right) + \left(\left(\frac{1}{6}\right)^2 \times \frac{1}{3} + \frac{1}{3} \times 1^2\right) + \cdots \\
&\geq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots \\
&= \infty
\end{aligned}
$$

Note that for each $p_{12}^{(n)}$ term, it is no less than $\frac{1}{3}$, and there are infinitely many such terms, so the sum of these term is infinity.

(d) Solution:

In part (a) we have $f_{12} = \frac{2}{5}$.
Since $f_{12} \neq 1$, by the contrapositive of F-lemma:

> If $f_{ij} \neq 1$, then $j \nrightarrow 1$ or $f_{jj} \neq 1$.

In fact, we already have $f_{22} = 1$ and we do have $2 \nrightarrow 1$ because $f_{21} = 0$.

In part (c) we have $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$.
Apply sum lemma we can get a trivial result:

> Since 1 communicates with 1, 2 communicates with 2, and $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$, so $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$.

Problem 3

(a) Solution:

> If the conditional probability distribution of future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it, we call such stochastic process has Markov property. A sequence of random variables $X_1, \dots$ with Markov property, given the present state is a Markov Chain.

In this case,

- The state space is finite and countable, $S = \{1, 2, 3, 4, 5, 6\}$,
- Initial probabilities $\{\nu_i\}_{i \in S} = \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$, since every value in a fair dice rolling has the same probability to be the initial value.
- The transition probabilities can be shown in a matrix

$$\{p_{ij}\}_{i,j \in S} = P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\{X_n\}$ follows Markov property because the current X_n is only determined by the number we roll now and the previous largest value X_{n-1}. The future X_{n+1} are the past X_{n-1} are independent.

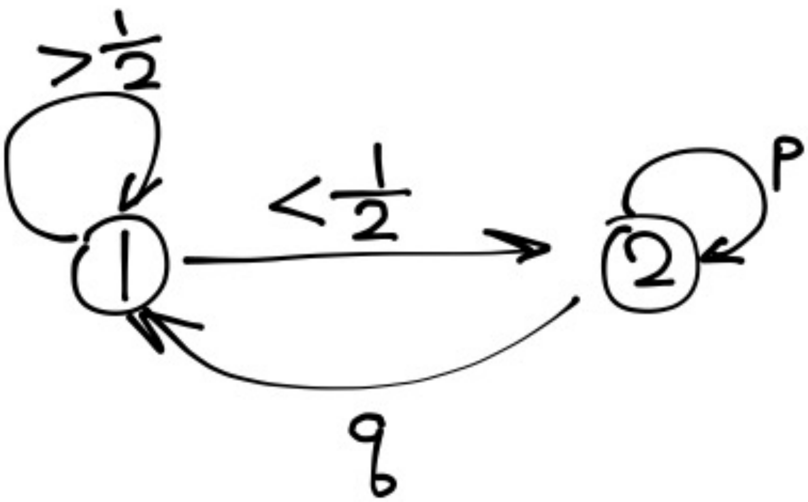
(b) Solution:

$\because p_{ij}^{(2)} = P(X_2 = j | X_0 = i) = \sum_{k \in S} P(X_2 = j, X_1 = k) | X_0 = i) = \sum_{k \in S} p_{ik} p_{kj}$.

$$\therefore \{p_{ij}^{(2)}\} = PP = P^2 = \begin{pmatrix} \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & \frac{1}{9} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & 0 & \frac{1}{4} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & 0 & 0 & \frac{4}{9} & \frac{1}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 4

(a) Solution:
EXISTS.



$p + q = 1, p_{11} > \frac{1}{2}$
Let $p = 1, q = 0$, state 1 is trasient.
Such Markov Chain can exist with:

- $S = \{1, 2\}$,
- $P = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ 0 & 1 \end{pmatrix}$.

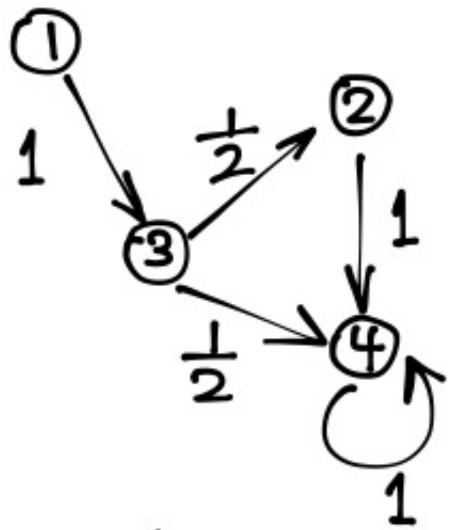
  So once the chain hits 2 it will never hit 1 again.

(b) Solution:
DOES NOT EXIST.
If $p_{ii} > 0$, period of i is 1. So state 1 should have period 1.

(c) Solution:
EXISTS.
Such Markov Chain can exist with:

- $S = \{1, 2, 3\}$,
- $P = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(d) Solution:
EXISTS.



Such Markov Chain can exist with:

- $S = \{1, 2, 3, 4\}$,
- $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

  $p_{12}^{(3)} = 0, 0 < p_{12}^{(2)} = \frac{1}{2} < 1$ and $p_{12} = 0$.

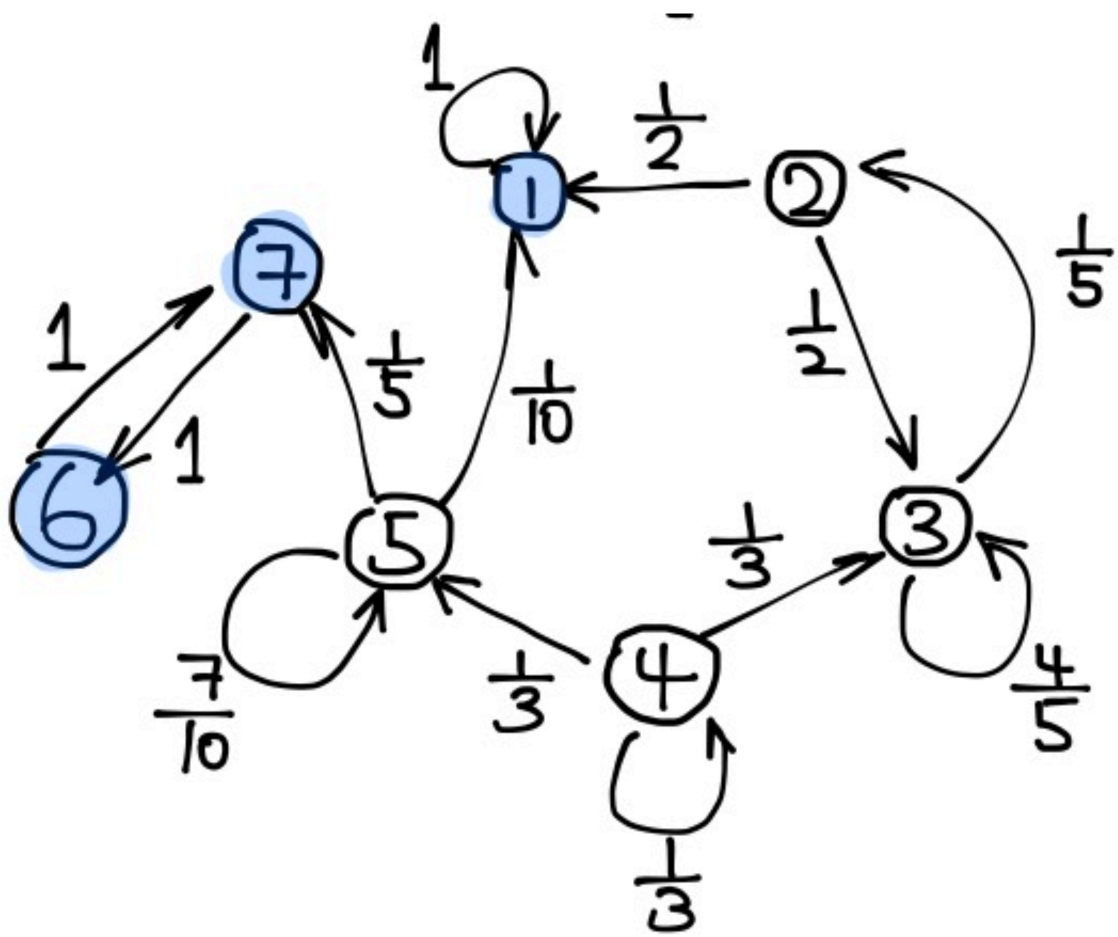
(e) Solution:
EXISTS.
Such Markov Chain can exist with:

- $S = \{1, 2, 3, 4\}$,
- $P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

  $f_{12} = p_{12} + p_{13} f_{32} = \frac{1}{2}$, as $f_{32} = 0$.
  $f_{13} = p_{13} + p_{12} f_{23} = \frac{1}{2} + \frac{1}{2} f_{23}$.
  $f_{23} = p_{23} + p_{22} f_{23} + p_{24} f_{43} = \frac{1}{4} + \frac{1}{4} f_{23}$ as $f_{43} = 0$.
  So $f_{23} = \frac{1}{3}$.
  Then $f_{13} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}$.

(f) Solution:
DOES NOT EXIST.
By the definition of transition probabilities, $p_{ij} \geq 0$ and $\forall i, \sum_j p_{ij} = 1$.
But if we have $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$, then $\lim_{n\to\infty} \sum_{j\in S} p_{ij}^{(n)} = \sum_{j\in S} \lim_{n\to\infty} p_{ij}^{(n)} = 0$, contradicting the definition of transition probabilities.
So such Markov Chain does not exist.

Problem 5



(a) Solution:

- $f_{11} = f_{66} = f_{77} = 1,$
- $f_{22}, f_{33}, f_{44}, f_{55} < 1,$ for the similar reason that, once they leave their state, reach state 1 (4, 5 could reach 7), they can never go back to their original state.
- So states 1, 6, 7 are recurrent, states 2, 3, 4, 5 are transient.

(b) Solution:

$$
\begin{aligned}
& f_{11} = p_{11} = 1 \\
& f_{21} = p_{21} + p_{23} f_{31} \\
& f_{31} = p_{32} f_{21} + p_{33} f_{31} \\
\therefore\, & f_{21} = \frac{1}{2} + \frac{1}{2} f_{31} \implies \frac{1}{2} f_{21} = \frac{1}{2} f_{31} \\
& f_{31} = \frac{1}{5} f_{21} + \frac{4}{5} f_{31} \implies \frac{1}{5} f_{31} = \frac{1}{5} f_{21} \\
\therefore\, & f_{21} = f_{31} = 1 \\
& f_{61} = f_{71} = 0 \\
& f_{51} = p_{51} + p_{57} f_{71} + p_{55} f_{51} \implies \frac{1}{10} = \frac{3}{10} f_{51} \implies f_{51} = \frac{1}{3} \\
& f_{41} = p_{45} f_{51} + p_{43} f_{31} + p_{44} f_{41} = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} + \frac{1}{3} f_{41} = f_{41} \\
\therefore\, & f_{41} = \frac{4}{9} \cdot \frac{3}{2} = \frac{2}{3}\,.
\end{aligned}
$$

To conclude, $f_{11} = 1 = f_{21} = f_{31} = 1, f_{41} = \frac{2}{3}, f_{51} = \frac{1}{3}, f_{61} = f_{71} = 0.$