SOME PROBLEMS FROM 2004–2012 EXAMS

Problem 1. Prove the Menelaus's theorem:

Take three lines l_1, l_2, l_3 and consider three points L, M, N at them: $L \in l_1$, $M \in l_2$, $N \in l_3$. Assume that A, B, C are points of intersections of these lines: $A = l_1 \cap l_2$, $B = l_2 \cap l_3$, and $C = l_3 \cap l_1$.

Points L, M, N belong to one line, if and only if

$$\frac{AL}{CL} \cdot \frac{BM}{AM} \cdot \frac{CN}{BN} = 1.$$

Problem 2. Prove Ceva's theorem: *Let the sides of a triangle ABC be divided at L, M, N in the respective ratios $\lambda\colon 1$, $\mu\colon 1$, $\nu\colon 1$. Then the three lines AL, BM, CN are passing through one point if and only if $\lambda\mu\nu = 1$.*

Problem 3. Consider a triangle ABC. Let D be a middle of the side AB, and let E be a middle of the median CD. In what proportion a line AE divides the side CB?

Hint: Put appropriate masses at the points A, B and C.

Problem 4. Consider a triangle ABC. Let D be the point on the side AB such that $AD : DB = 2$ and let E be the point on the segment CD such that $DE : EC = 2$. In what proportion the line AE divides the side CB? In what proportion the line BE divides the side CA?

Hint: Put appropriate masses at the points A, B and C.

Problem 5. Consider a tetrahedron $ABCD$. Let E be the point of intersection of the medians in the triangle ABC. Take a point F on the segment DE such that $DF : FE = 6$. In what proportion the plane passing through the points BCF divides the edge DA.

Hint: Put appropriate masses at points A, B, C and D.

Problem 6. Each vertex of a triangle has been connected to the two points on the opposite side that divide it to three equal parts. Consider the 6-gon formed by these three pairs of lines. Prove that the three diagonals joining opposite vertices of this 6-gon pass through one point.

Problem 7. Prove that in an arbitrary triangle the three points of intersection of the bisectors of its external angles with the opposite sides belong to one line.

Problem 8. Consider a regular triangle ABC. Find all points O for which the sum $4O_{AB} + O_{BC} + O_{CD}$ is the smallest possible. Here O_{AB}, O_{BC} and O_{CD} are the distances from point O to the sides AB, BC and CD respectively.

Problem 9. Take an angle between 2 rays l_1 and l_2 with vertex O and a point A inside the angle. Consider all triangles with vertex O such that two sides of them belong to l_1 and l_2 and the third side l passes through A. Find the location of line l for which the area of the triangle is minimal. Hint: consider the parallelogram with two sides in l_1 and l_2 and with center A and look how line l cuts this parallelogram.

Problem 10. Consider a regular triangle ABC. Find all points O for which the sum $O_{AB} + 2O_{BC} + 3O_{CA}$ is the smallest possible. Here O_{AB}, O_{BC} and O_{CA} are distances from point O to the sides AB, BC and CA respectively.

Problem 11. Consider angle $\alpha = 45^\circ$ between two rays l_1 and l_2 intersecting at point O. Take any point A inside the angle. Find points $B \in l_1$ and $C \in l_2$ such that polygonal path $ABCA$ has the smallest length. Find this smallest length assuming that the distance from A to O is a.

Problem 12. Let $A = (p, q)$ and $C = (-q, p)$ be a given pair of points in the plane. Assume that $q > p > 0$.

Find $x, y \in \mathbb{R}$ such that for points $B = (x, 0)$, $D = (0, y)$ the number $S = AB + BC - |CD - DA|$ is the smallest.

Find this number S.

Problem 13. Consider two circles S_1, S_2 with centers O_1, O_2 and radiuses R_1, R_2. Make inversion with respect to the circle S_1 and then make inversion with respect to the circle S_2. Describe all lines and circles which after two inversions will become straight lines. (Hint: to start with describe all lines and circles which become straight lines after one inversion with respect to the second circle S_2.)

Problem 14. Consider two non-concentric circles S_1 and S_2, one inside another. Assume that there exists a chain of circles $S_3, ..., S_{2005}$, such that each circle in the chain is tangent to the circles S_1 and S_2, and also to the next circle (i.e. S_3 is tangent to S_4, S_4 is tangent to S_5 and so on), and S_{2005} is tangent to S_3. Prove that for any other chain of circles $S'_3, \ldots, S'_{2005}$, such that each circle in the chain is tangent to the circles S_1 and S_2, and also to the next circle the last one S'_{2005} will be also tangent to S'_3. Hint: Using an inversion reduce the problem to a simpler form.

Problem 15. Consider two circles S_1, S_2 with centers O_1, O_2 and radiuses R_1, R_2. Make inversion with respect to the circle S_1 and then make inversion with respect to the circle S_2. Describe all lines and circles which become straight lines after these two inversions.

Problem 16. Take a circle S_0 and its diameter D. Take a chain of circles $S_1, S_2, S_3, \ldots$ such that circle S_1 is tangent to S_0 and is tangent to the diameter D at the center O; the circle S_2 is tangent to S_0, to D and to S_1; the circle S_3 is tangent to S_0, to D and to S_2 and so on. Let $A_1, A_2, \ldots$ be the sequence of points of tangency of the circles S_1 and S_2; the circles S_2 and S_3 and so on. Prove there exists a circle S which contains all the points $A_1, A_2, \ldots$

Problem 17. Consider triangle ABC such that $AB = 3$, $BC = 4$, $CA = 5$. Find the point O such that after an invertion centered at O the line passing through A, C becomes a line, and lines passing through A, B and through B, C become equal circles.

Problem 18. Take two intersecting circles S_1, S_2 on plane. Prove that there is a Moebius transformation which maps S_1, S_2 to two equal circles.

Problem 19. Let A be a circle of radius 2 centered at the origin $O = (0; 0)$ and let B and S_1 be circles of radius 1 centered at points $(1, 0)$ and $(1, 0)$. Consider the sequence of circles $S_2, S_3, \ldots, S_n, \ldots$ in the upper half plane $y > 0$ such that for $k > 1$ the circle S_k is tangent from inside to the circle A and is tangent from outside to the circles B and S_{k-1}. Let P_k be the point of tangency of circles S_k and B. Let L be the point $(-2, 0)$. Find the tangent of the angle $P_k L O$.

Hint: make an inversion about a circle centered at the point L.

Problem 20. Let $A < B < C < D$ be four points on the real line $\mathbb{R}$. Does there exist a Mobius transform f, such that the points $f(A)$, $f(B)$, $f(C)$, $f(D)$ belong to the real line $\mathbb{R}$ and $f(A) = -f(D)$, $f(B) = -f(C)$? Why?

Hint: consider circles in the plane whose diameters are on the real line. Use the theorem about classification of pairs of circles under Mobius transformations

Problem 21. Take a convex polyhedron in $\mathbf{R}_3$. Denote by f_0, f_1 and f_2 the number of its vertices, edges and faces, respectively. Prove:

1) $3f_0 \leq 2f_1$. Hint: at least 3 edges meet at each vertex of the polyhedron.

2) $2f_1/f_2 < 6$ – the average number of edges on faces of the polyhedron is strictly less that 6. Hint: use 1) and Euler formula $f_0 - f_1 + f_2 = 2$.

Problem 22. Consider points O_1, O_2, A and a segment PQ of length R on plane. Using this data and a compass and a straightedge construct two tangent lines to the ellipse $O_1X + XO_2 = R$ passing though the point A.