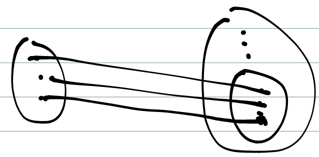
March 15th

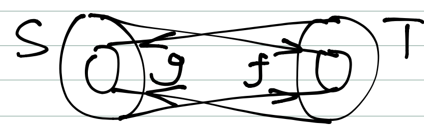
Recall that we say two sets S & T have the same cardinality if $\exists f: S \to T$ 1–1 & onto

Def: $|S| = |T|$ if there exists $f: S \to T$ 1–1

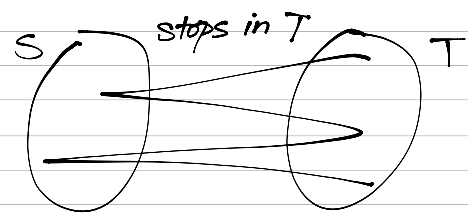


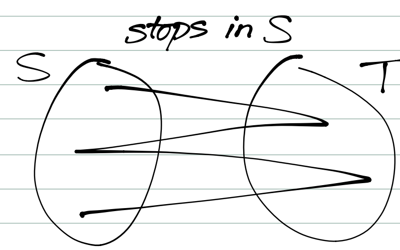
Theorem (Schroeder Berenstein)
if $|S| \le |T|$ and $|T| \le |S|$ then $|S| = |T|$

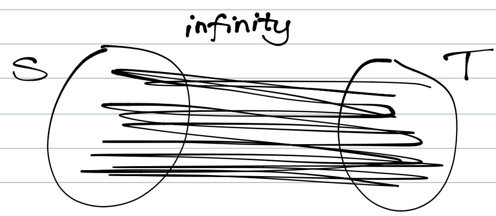
$f: S \to T$ 1–1
$g: T \to S$ 1–1



want $h: S \to T$ 1–1 & onto







$T = T_S \cup T_T \cup T_\infty$ similarly $S = S_S \cup S_T \cup S_\infty$

$\downarrow$
last ancestor in S

we proved
① $f: S_S \to T_S$ 1–1 & onto
② $g: S_T \to T_T$ 1–1 & onto
③ $f: S_\infty \to T_\infty$ 1–1 & onto

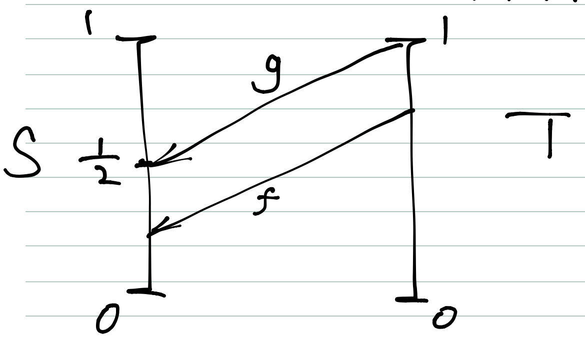
Define $h: S \longrightarrow T$ by the formula

$$h(s) = \begin{cases} f(s) & \text{if } s \in S_S \\ f(s) & \text{if } s \in S_\infty \\ g^{-1}(s) & \text{if } s \in S_T \end{cases}$$

1–1 & onto

Let $S=[0,1)$ $T=[0,1]$ $|S|\leq|T|$ $f(x)=x$ $f: S \xrightarrow[1-1]{} T$

$|T|\leq|S|$ $g: T \longrightarrow S$ $g(x)=\frac{x}{2}$



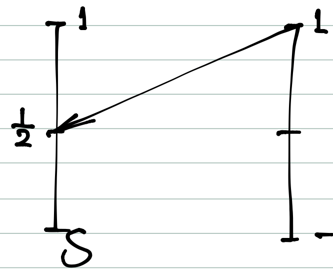
Let $x \in S$

$\frac{1}{2} < x < 1$ doesn't have an ancestor

$x \neq g(t)$ for any $t \in [0,1]$

but $x \in S_S$ $(\frac{1}{2}, 1) \subseteq S_S$

Let $s = \frac{1}{2}$

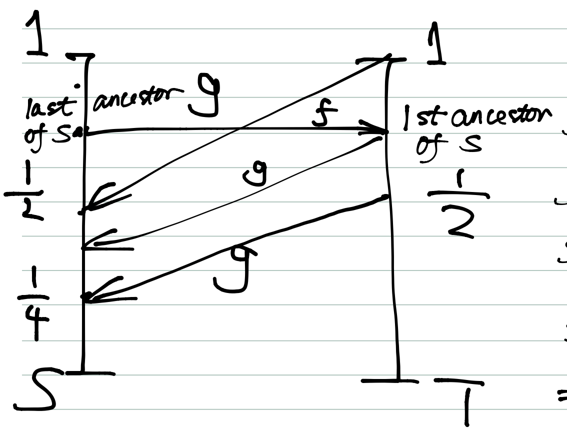


$\frac{1}{2} = g(1)$

$1 \notin f(S)$

$f(x) = x$

Let $\frac{1}{4} < s < \frac{1}{2}$



$g(1) = \frac{1}{2}$ $g(x) = \frac{x}{2}$

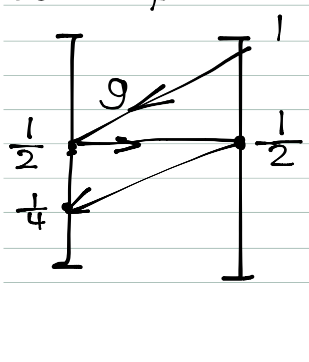
$g(\frac{1}{2}) = \frac{1}{4}$ $\frac{1}{4} < s < \frac{1}{2}$

$s = g(2s)$ $\frac{1}{2} < 2s < 1$

$f(x) = x$

$\Rightarrow (\frac{1}{4}, \frac{1}{2}) \subseteq S_S$

Let $s = \frac{1}{4}$



$g(\frac{1}{2}) = \frac{1}{4}$

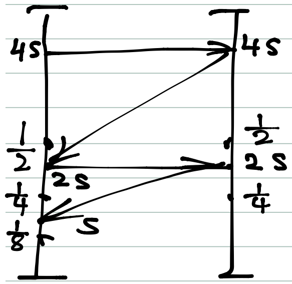
$\frac{1}{2}$ — 1st ancestor

$f(\frac{1}{2}) = \frac{1}{2}$

$\frac{1}{2} = g(1)$

$1 \in T - S$ of ancestor of $\frac{1}{4}$

Last ancestor $\Rightarrow \frac{1}{4} \in S_T$



if $s \in S$ $\quad \frac{1}{8} < s < \frac{1}{4}$

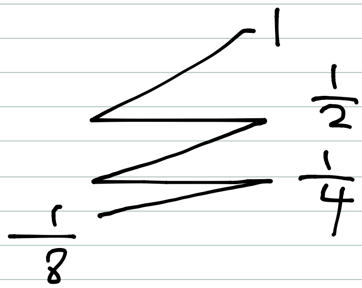
$s = g(2s)$

$\frac{1}{4} < 2s < \frac{1}{2}$

$2s = f(2s)$, $2s = g(4s)$

$\frac{1}{2} < 4s < 1$

last ancestor is in $S \Rightarrow (\frac{1}{8}, \frac{1}{4}) \subseteq S_S$

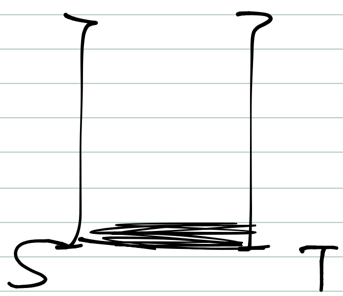


$\frac{1}{8} \in S_T$

$\Rightarrow$ by induction can prove $\frac{1}{2^n} \in S_T$ for any $n \geq 1$

$\frac{1}{2^{n+1}} < s < \frac{1}{2^n} \Rightarrow s \in S_S$

$s = 0 \Rightarrow f(0) = 0 \quad g(0) = 0 \Rightarrow S_\infty$



$S_\infty = \{0\}$

$S_T = \{\frac{1}{2^n} \mid n = 1, 2, 3, \cdots\}$

everything else is in S_S.

$S_S = \{x \in [0,1) \mid x \neq 0, x \neq \frac{1}{2^n}\}$

Now we can construct h

recall $h(s) = \begin{cases} f(s) & \text{if } s \in S_S \\ f(s) & \text{if } s \in S_\infty \\ g^{-1}(s) & \text{if } s \in S_T \end{cases}$

In this case $h(s) = \begin{cases} \overset{f(s)}{\overset{\shortparallel}{s}} & \text{if } s \neq \frac{1}{2^n} \\ \underset{2s}{\underset{\shortparallel}{g^{-1}(s)}} & \text{if } s = \frac{1}{2^n} \end{cases}$

if $s = \frac{1}{2^n}$

$2s = \frac{2}{2^n} = \frac{1}{2^{n-1}}$

$h(s) = \begin{cases} s & \text{if } s \neq \frac{1}{2^n} \\ \frac{1}{2^{n-1}} & \text{if } s = \frac{1}{2^n} \end{cases}$

$S=[0,1)$ if $s \neq \frac{1}{2^n}$

$s \to s$

$T=[0,1]$ $\frac{1}{2} \to 1$

$\frac{1}{4} \to \frac{1}{2}$

$\frac{1}{8} \to \frac{1}{4}$

h is onto

let $t \in T$ be any $t \in [0,1]$

① if $t \neq \frac{1}{2^n}$ for some n

$\Rightarrow t = h(t)$

② if $t = \frac{1}{2^n}$ then $t = h(\frac{1}{2^{n+1}})$

h is 1-1

let $s_1, s_2 \in S$ be different $s_1 \neq s_2$

Case ①. both s_1, s_2 are not of the form $\frac{1}{2^n}$

Then $h(s_1) = s_1 \Rightarrow h(s_1) \neq h(s_2)$

$h(s_2) = s_2$

Case ② $s_1 = \frac{1}{2^n}$ for some n

$s_2 \neq \frac{1}{2^m}$ for any m $h(s_1) = h(s_2)$

then $h(s_1) = \frac{1}{2^{n-1}}$ $h(s_2) = s_2$

Case ③ $s_1 = \frac{1}{2^n}$ $h(s_1) = \frac{1}{2^{n-1}}$

$\not\parallel$

$s_2 = \frac{1}{2^m}$ $h(s_2) = \frac{1}{2^{m-1}}$

$n \neq m$