Lecture 7

§5.5 Periodic Points

Example: Let $F(x)=x^2-1$

This function has 2 fixed points:

$x^2-1=x \Leftrightarrow x^2-x-1=0 \Leftrightarrow x=\frac{1\pm\sqrt{5}}{2}$

AND $F'(x)=2x$, so

$\left|F'\left(\frac{1\pm\sqrt{5}}{2}\right)\right| = |1\pm\sqrt{5}|>1$

So the fixed points are repelling

We find 2-cycles:

$$F^2(x)=x \Leftrightarrow (x^2-1)^2-1=x$$
$$\Leftrightarrow x^4-2x^2=x$$
$$\Leftrightarrow x(x+1)(x^2-x-1)=0$$
$$\Leftrightarrow x=0 \text{ or } x=-1 \text{ or } x=\frac{1\pm\sqrt{5}}{2} \to \text{fixed pts}$$

So $0,-1$ are 2-cycles for $F(x)$.

Since 2-cycles are fixed points for $F^2(x)$, we can define attracting/repelling 2-cycles as we did for fixed points.

so $(F^2)'(x) = (x^4-2x^2)'=4x^3-4x=4x(x^2-1)$

AND $(F^2)'(0)=0 =(F^2)'(-1)$

So $0,-1$ are attracting fixed points of $F^2(x)$,

so they are attracting cycles of $F(x)$.

HOW TO SIMPLIFY THE CALCULATION ABOVE?

The test $|(F^2)'(x)|<1$

using the chain rule:

$$[F(F(x))]'=F'(F(x))\cdot F'(x)$$

we apply to x_0 which is a 2-cycle:

$$\boxed{|(F^2)'(x_0)|=|F'(x_1)||F'(x_0)|}$$

To determine whether a periodic point is attracting or repelling. We need to compute the derivative of F^n at x_0.

Using the chain rule:

$$(F^2)'(x_0)=F'(x_1)F'(x_0)$$

$$(F^3)'(x_0)=[F(F^2(x_0))]'=F'(F^2(x_0))\cdot(F^2)'(x_0)=F'(x_2)F'(x_1)F'(x_0)$$

So we can prove by induction that:

$$(F^n)'(x_0)=F'(x_0)\cdot F'(x_1)\cdots F'(x_{n-1})$$

Remark: If we want to find the derivative of F^n, we only need to compute $F'(x)$ and apply it to all points of the orbit.

Corollary: Sps that $x_0, x_1, \cdots, x_{n-1}$ lie on an n-cycle of F. Then
$$(F^n)'(x_0) = (F^n)'(x_1) = (F^n)'(x_2) \cdots = (F^n)'(x_{n-1})$$

Types of cycles. Sps that $x_0, x_1, \cdots, x_{n-1}$ with $x_i = F^i(x_0)$ lie on an n-cycle of F, then

① The cycle is attracting if $|(F^n)'(x_i)| < 1$, that is $|F'(x_0)| \cdot |F'(x_1)| \cdots |F'(x_{n-1})| < 1$

② The cycle is repelling if $|(F^n)'(x_i)| > 1$, that is $|F'(x_0)| \cdot |F'(x_1)| \cdots |F'(x_{n-1})| > 1$

③ The cycle is neutral if $|(F^n)'(x_i)| = 1$: $|F'(x_0)| \cdot |F'(x_1)| \cdots |F'(x_{n-1})| = 1$

NOTE

A neutral FIXED/PERIODIC can be
- weakly attracting
- weakly repelling
- or neither

weakly attracting: if orbits starting near φ converge to φ (from both sides)
Weakly repelling: if orbits starting near φ will escape from φ. (both sides)
neither: one side repelling, one side attracting.

Example:
On the previous example, we can check that $(F^2)'(0) = (F^2)'(-1)$ and $|F'(0)F'(-1)| = 0 < 1$

Example: Let $F(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$ which has a 3-cycle:

$$\underline{0, 1, 2}, 0, 1, 2, \cdots$$

so $F'(x) = -3x + \frac{5}{2}$ and $F'(0)F'(1)F'(2) = \frac{5}{2}(-\frac{1}{2})(-\frac{7}{2}) = \frac{35}{8} > 1$

The cycle 0, 1, 2 is repelling.