Feb 27th

V ips $T: V \to V$ self-adjoint, i.e.

$$\langle T(v), w\rangle = \langle v, T(w)\rangle$$

Correction: standard inner product on $\mathbb{C}^n$

$$\langle v, w\rangle = \cancel{v^* w} = v^T \overline{w}$$

$$v^* = \overline{v}^T$$

Note: Complex inner products satisfy $\langle v, w\rangle = \overline{\langle w, v\rangle}$

Thm 1: Eigenvalues of T are real

Thm 2. Eigenvectors corresponding to distinct eigenvalues are orthogonal

T diagonalizable $\iff$ there is a basis of eigenvectors of T.

Spectral Thm: V ips, $T: V \to V$ self adjoint operator

Then there is an orthonormal basis of V consisting of eigenvectors of T. In particular, T is diagonalizable.

Proof: By induction on $n = \dim V$

Base case. $n=1$, i.e. V is one-dim

$T: V \to V$ is multiplication by a scalar z.

Take any $v \in V$, $v \neq 0$, $\|v\| = 1$

$T(v) = zv$ so v is an eigenvalue

and $\{v\}$ is an orthogonal basis of V

This proves base case

Inductive hyp: assume spectral thm holds for vector spaces of dim $n-1$

Inductive step: prove it holds for vector spaces of dim n

Let λ be an eigenvalue of T.

Let x_1 be a unit-eigenvalue corr. to λ.

$W = \text{span}\{x_1\}$. $W^\perp$ has dim $n-1$, since

$$\underset{n}{V} = \underset{1}{W} \oplus \underset{n-1}{W^\perp}$$

Claim: $v \in W^\perp$ then $T(v) \in W^\perp$

subproof: $\langle T(v), x_1\rangle = \langle v, T(x_1)\rangle = \langle v, \lambda x_1\rangle = \lambda\langle v, x_1\rangle = 0$

$W^\perp$ ips.

$S = T|_{W^\perp} : W^\perp \to W^\perp$ self-adjoint operator.

$\langle S(v), w\rangle = \langle T(v), w\rangle = \langle v, T(w)\rangle = \langle v, S(w)\rangle$.

By ind. hyp. there is an orthonormal basis $\{x_2, \cdots, x_n\}$ of $W^\perp$ consisting of eigenvectors of S.

$\{x_2, \cdots, x_n\}$ are eigenvectors of T

Claim: $\{x_1, \cdots, x_n\}$ is an orthonormal basis of evectors

- they're all eigenvectors ✓
- they're all unit length ✓
- the set is orthogonal since $\{x_2, \cdots, x_n\}$ is orth and $x_1 \in W$ while $x_2, \cdots, x_n \in W^\perp$. $\blacksquare$

<u>Given</u>: $T: V \to V$ Self-adj.

To find <u>orthogonal</u> basis of eigenvectors
1. Find eigenvalues of T
2. Find a basis for each eigenspace
3. Use GS-process to get an orthonormal basis of each eigenspace
4. Combine these basis into one set.

<u>Ex</u>: $T = \begin{bmatrix} 1 & i & 1 \\ -i & 1 & i \\ 1 & -i & 1 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$

T as self-adjoint. write inner product on $\mathbb{C}^3$.

1. $p(\lambda) = \begin{bmatrix} \lambda-1 & -i & -1 \\ i & \lambda-1 & -i \\ -1 & i & \lambda-1 \end{bmatrix} = (\lambda-2)^2(\lambda+1)$

$E_2 = \text{null} \begin{bmatrix} 1 & -i & -1 \\ i & 1 & -i \\ -1 & i & 1 \end{bmatrix}$

$= \text{span} \left\{ \underset{u_1}{\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

$v_1 = u_1$

$v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= \begin{bmatrix} 1/2 \\ i/2 \\ 1 \end{bmatrix}$

$\|v_1\| = \sqrt{[i \ 1 \ 0] \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$

$E_{-1} = \text{span} \left\{ \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix} \right\}$

Apply GS to $\left\{ \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix} \right\}$:

$\left\| \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix} \right\| = \sqrt{\langle v, v \rangle}$

$= \sqrt{v^T \bar{v}} = \sqrt{[-1 \ -i \ 1] \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}}$

$= \sqrt{1 - i^2 + 1} = \sqrt{3}$

$x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$ $x_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ i/2 \\ 1 \end{bmatrix}$ $x_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}$

$\{x_1, x_2, x_3\}$ is an orthonormal basis of $\mathbb{C}^3$ which consists of evectors of T.