Name: SOLUTIONS

MAT 334H
SUMMER 2013
TEST 2

Problem	1	2	3	4	5	6	Total
Points	10	10	10	10	10	10	60
Score							

- Solve the following problems, and write up your solutions neatly, in black or blue ink, in the space provided.
- Please make sure your name is entered at the top of this page.
- This test contains 8 pages. Please ensure they are all there.
- Please do not tear out any pages.
- You have 2 hours to complete this test.
- There are *no* aids allowed.
- There is some potentially useful information on the last page.

GOOD LUCK!

(1) Please answer the following questions in the space provided. (2 pts each)

(a) The function $f(z) = (1 - \cos z)\sin z$ has a zero of order $\underline{\quad 3 \quad}$ at $z_0 = 0$.

(b) If $f(z) = \sum_{k=\infty}^{\infty} \frac{k}{|k|!}(z-6)^k$ then $\text{Res}(f : 6) = \underline{\quad -1 \quad}$.

(c) If $f(z) = \frac{1}{(z-z_0)^4} + \frac{1}{2(z-z_0)^3} + \frac{1}{3(z-z_0)^2} + \frac{1}{4(z-z_0)} + \frac{1}{5} + \frac{1}{6}(z-z_0) + \cdots$, then z_0 is a pole of order $\underline{\quad 4 \quad}$.

(d) The function $f(z) = \frac{e^{\frac{1}{z}}(z^2-4)^3}{(z^3-8)}$ has a removable singularity at $z_0 = \underline{\quad 2 \quad}$.

(e) If $f(z) = e^{z-1}\frac{z^4-1}{(z^2-1)^2}$ then $z_0 = 1$ is: (circle one)

- a pole of order 1
- a zero of order 1
- a removable singularity
- an essential singularity

(a) $1 - \cos z = 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) = \frac{z^2}{2!} - \frac{z^4}{4!} + \cdots$

so $1-\cos z$ has zero of order 2.

$\sin z = z - \frac{z^3}{3!} + \cdots \Rightarrow \sin z$ has zero of order 1.

$\Rightarrow (1-\cos z)\cdot\sin z$ has zero of order $2+1=3$.

(b) $a_{-1} = \frac{-1}{1!} = -1$

(d) $f = \frac{e^{\frac{1}{z}}(z-2)^3(z+2)^3}{(z-2)(z^2+2z+4)} \Rightarrow \lim_{z\to 2} f(z) = 0.$

(e) $f = e^{z-1}\frac{(z^4-1)}{(z^2-1)^2} = e^{z-1}\frac{(z-1)(z+1)(z-i)(z+i)}{(z-1)^2(z+1)^2}$.

$\Rightarrow$ pole of order 1. at $z=1$.

(2) Let $f(z) = \dfrac{1 - e^{z-1}}{(z-1)^2}$.

(a) (5 pts) Find the Laurent series for f centred at $z_0 = 1$.

$$f(z) = \frac{1}{(z-1)^2} \cdot \left(1 - e^{z-1}\right)$$

$$1 - e^{z-1} = 1 - \left(\sum \frac{1}{n!}(z-1)^n \right) = 1 - \left(1 + (z-1) + \frac{1}{2}(z-1)^2 + \cdots \right)$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n!}(z-1)^n = -\left((z-1) + \frac{1}{2!}(z-1)^2 + \frac{1}{3!}(z-1)^3 + \cdots \right)$$

So $$f(z) = \frac{1}{(z-1)^2} \cdot \sum \frac{1}{n!}(z-1)^n = \frac{-1}{(z-1)^2}\left((z-1) + \frac{1}{2!}(z-1)^2 + \cdots \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}(z-1)^{n-2} = \frac{-1}{z-1} - \frac{1}{2!} - \frac{1}{3!}(z-1) - \cdots$$

(b) (2 pts) Determine the order of the pole of f at $z_0 = 1$.

From Laurent series above the order is 1.

(c) (3 pts) Determine $\dfrac{1}{2\pi i}\displaystyle\int_\gamma f(z)dz$, where γ is the circle of radius 1 centred at $z_0 = 1$, oriented positively.

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \mathrm{Res}(f:1) = \text{coeff of } \frac{1}{z-1}$$

$$= -1$$

(3) (10 pts) Find $\displaystyle\int_\gamma \frac{z^2+1}{z^3-3z^2+2z}\,dz$, where γ is the circle of radius $\frac{3}{2}$ centred at 0 oriented positively.

$Q(z) = z^3 - 3z^2 + 2z = z(z-1)(z-2)$ & $Q'(z) = 3z^2 - 6z + 2$

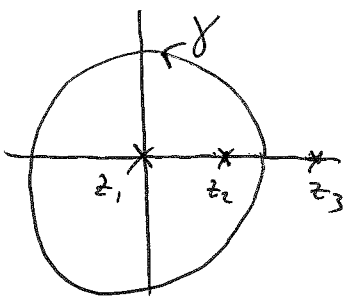
$P(z) = z^2 + 1 = (z-i)(z+i)$

So $f(z) = \dfrac{(z-i)(z+i)}{z(z-1)(z-2)}$ has poles of order 1 at $z_1 = 0$, $z_2 = 1$, $z_3 = 2$.

They have residues: $\mathrm{Res}(f : z_1) = \dfrac{P(0)}{Q'(0)} = \dfrac{1}{2}$

$\mathrm{Res}(f : z_2) = \dfrac{P(1)}{Q'(1)} = -2$

$\mathrm{Res}(f : z_3) = \dfrac{P(2)}{Q'(2)} = \dfrac{5}{2}$



By Residue Theorem:

$$\int_\gamma \frac{z^2+1}{z^3-3z+2z}\,dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \mathrm{Res}(f : z_k)$$

$$= 2\pi i\,(\mathrm{Res}(f : z_1) + \mathrm{Res}(f : z_2))$$

$$= 2\pi i\left(\frac{1}{2} - 2\right)$$

$$= 2\pi i\left(-\frac{3}{2}\right)$$

$$= -3\pi i$$

(4) (10 pts) Find $\int_\gamma \cot z \, dz$, where γ is the circle of radius 30 centred at 0, oriented positively.

Let $h(z) = \sin z$, then $h'(z) = \cos z$

So $\int_\gamma \cot z \, dz = \int_\gamma \frac{\cos z}{\sin z} dz = \int \frac{h'(z)}{h(z)} dz$

$= 2\pi i \left(\begin{matrix} \#\text{ zeros of} \\ \sin z \text{ inside } \gamma \end{matrix} - \begin{matrix} \#\text{ poles of} \\ \sin z \text{ inside } \gamma \end{matrix} \right)$

$= 2\pi i \left(\begin{matrix} \#\text{ zeros of} \\ \sin z \text{ inside } \gamma \end{matrix} - 0 \right)$

sin z has no poles!

$= 2\pi i \cdot 19$

$= 38\pi i$

The zeroes of $\sin z$ are $z_k = k\pi$, $k \in \mathbb{Z}$.

Since $h'(k\pi) = \cos k\pi = \pm 1$ they each have order (or multiplicity) 1.

We want zeroes inside γ: so we need $|z_k| < 30$.

Solve $|k| \cdot \pi < 30$ or $k < \frac{30}{\pi}$

$\Rightarrow |k| \leq 9.$ (Since $\pi > 3$)

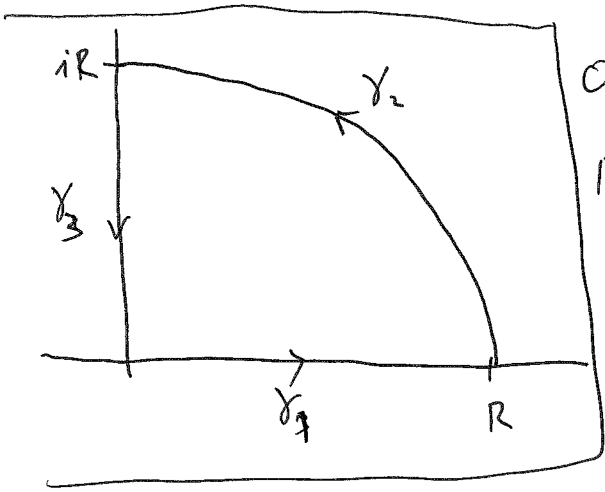
So zeros inside γ are: $0, \pm\pi, \pm 2\pi, \dots, \pm 9\pi$

and there are $1 + 2 \cdot 9 = 19$ of them.

(5) (10 pts) How many zeroes does $p(z) = z^4 + 2z^2 + 4z + 2$ have in the first quadrant? Justfiy your answer.

(Hint: Use the "Argument Principle" for an appropriate curve γ.)

Let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be the curve in the diagramme below with R large.



On γ_1, z is positive real & $p(z)$ is positive real, so $\arg(p(z)) = 0$ on γ_1.

On γ_2: $z = Re^{it}$ and

$$p(z) = R^4 e^{i4t}\left(1 + \frac{2}{R^2 e^{i2t}} + \frac{4}{R^3 e^{i3t}} + \frac{2}{R^4 e^{i4t}}\right)$$

$$\simeq R^4 e^{i4t} = z^4 \quad \text{for } R \text{ large.}$$

So $\arg(p(z)) \simeq \arg(z^4)$ on γ_2, and as t goes from 0 to $\frac{\pi}{2}$ $\arg p(z)$ goes from 0 to approximately $4 \cdot \frac{\pi}{2} = 2\pi$.

On γ_3: $z = iy$ $R \geq y \geq 0$; and $p(z) = i^4 y^4 + 2i^2 y^2 + 4iy + 2$
$= y^4 - 2y^2 + 2 + (4y)i$

So $p(z)$ is always in upper half plane $\operatorname{Im}(z) > 0$ for $R \geq y > 0$.

As $y \to 0$ $p(z) \to 2$. So $p(z)$ does not wind around 0 ~~as~~ for z on γ_3

So total change in argument is 2π

Argument principle $\Rightarrow$ $\frac{1}{2\pi} \cdot$ change in arg $= 1 = \#$ zeros of p

(6) Consider the real integral $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$.

(a) (4 pts) Find all four solutions to the equation $z^4 + 1 = 0$. (Hint: Polar coordinates.)

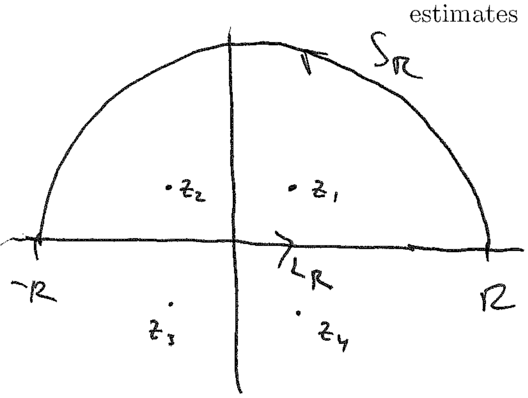
$z^4 = -1$, write $z = e^{i\theta} \Rightarrow e^{i4\theta} = e^{i\pi}$

$\Rightarrow 4\theta = \pi + 2k\pi$

$\theta = \frac{2k+1}{4} \cdot \pi$ for $k = 0, 1, 2, 3$

So solutions are: $z_1 = e^{i\pi/4}$, $z_2 = e^{i3\pi/4}$, $z_3 = e^{i5\pi/4}$, $z_4 = e^{i7\pi/4}$

(b) (6 pts) Compute the real integral $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$ using residues. (Hint: Use the estimates given on the last page.)



Let $\gamma = S_R \cup L_R$ as in figure, with R large.

Then $\int_\gamma \frac{z^2}{z^4+1} dz = 2\pi i \left(\text{Res}(f : z_1) + \text{Res}(f : z_2)\right)$

$= 2\pi i \left(\frac{(e^{i\pi/4})^2}{4e^{i3\pi/4}} + \frac{(e^{i3\pi/4})^2}{4e^{i\pi/4}} \right)$

$= 2\pi i \left(\frac{e^{i\pi/2}}{4e^{i3\pi/4}} + \frac{e^{i3\pi/2}}{4e^{i\pi/4}} \right) = 2\pi i \left(\frac{i}{4} e^{-i3\pi/4} - \frac{i}{4} e^{-i\pi/4} \right)$

$= \frac{-2\pi}{4} \left(e^{i5\pi/4} - e^{i7\pi/4} \right) = \frac{-2\pi}{4} \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right) \right)$

$= \frac{-\pi}{2} \left(-\frac{2}{\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}$.

(Problem 6 continued)

On the other hand: $\int_\gamma = \int_{S_R} f(z)dz + \int_{L_R} f(z)dz = \int_{S_R} f(z)dz + \int_{-R}^{R} \frac{x^2}{x^4+1}dx$

On S_R: $|f(z)| = \frac{|z^2|}{|z^4+1|} \leq \frac{R^2}{\frac{1}{2}R^4} = \frac{2}{R^2}$ by estimate below

So $\left|\int_{S_R} f(z)\,dz\right| \leq$ length $S_R \cdot \max|f(z)| \leq \pi \cdot R \cdot \frac{2}{R^2} = \frac{\pi}{R}$

So $\int_{S_R} f(z)\,dz \to 0$ as $R \to \infty$.

So letting $R \to \infty$ we get

$$\frac{\pi}{2} = \int_{S_R} f(z)dz + \int_{-R}^{R} \frac{x^2}{x^4+1}dx$$

$$\downarrow$$

$$\frac{\pi}{2} = 0 + \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx \qquad \text{so} \quad \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx = \frac{\pi}{2}.$$

Some useful estimates:

If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, then we get the following estimates for $|z| = R$ with R very large:

$$\frac{1}{2}|a_n|R^n \leq |p(z)| \leq 2|a_n|R^n$$

For a continuous function f and continuous curve γ, we get:

$$\left|\int_\gamma f(z)dz\right| \leq \text{length}(\gamma) \cdot \max_{z\in\gamma}|f(z)|$$