# Tutorial Problems - Sections 4 to 5 - MAT 327 - Summer 2014

## 4 Countability

1. Give an explicit bijection between $\mathbb{N}$ and the collection of all finite binary strings.
2. Prove, using countability, that there is transcendental number (i.e. a number that is not a solution to any polynomial with rational coefficients).
3. Prove that $\mathbb{R} \times \mathbb{R}$ can be put in bijection with $\mathbb{R}$.
4. Let $A$ be an infinite set. Prove that $A \times A$ can be put in bijection with $A$.
5. Prove that $\{[p, q) : p, q \in \mathbb{Q}\}$ forms a (countable) basis for a topology on $\mathbb{R}$. Show that this does not generate the Sorgenfrey Line.
6. Prove that the particular point topology on $\mathbb{R}$ is separable, but not second countable.
7. For two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, we say that $f \preceq g$ if there is an $N \in \mathbb{N}$ such that $f(n) \leq g(n)$ for all $N \leq n$. Show that given 1000 functions $f_1, \dots, f_{1000}$, there is always a function $g$ such that $f_i \preceq g$ for all $i = 1, \dots, 1000$.
8. Show that given *countably many* functions $f_i$ $(i \in \mathbb{N})$, there is always a function $g$ such that $f_i \preceq g$ for all $i \in \mathbb{N}$.
9. Use the previous result to create an interesting topological space on $\mathbb{N} \times \mathbb{N}$, or possibly $(\mathbb{N} \times \mathbb{N}) \cup \{0\}$. What properties does it have?

## 5 Convergence and Limit Points

1. Explain what types of sequences converge in discrete spaces.
2. Explain what types of sequences converge in $\mathbb{R}_{\text{co-finite}}$.
3. Explain what types of sequences converge in $\mathbb{R}_{\text{co-countable}}$.
4. Explain what types of sequences converge in $\mathbb{R}^2$ by making reference to the coordinates. Is there something you can say about general product spaces?
5. Give a clear explanation of how second countable spaces relate to first countable spaces. Help us to understand how these properties are the same and how they are different.

6. In your analysis course you defined Cauchy sequences, and said that a metric space was complete if and only if all Cauchy sequences converge. Is it possible to make sense of Cauchy Sequences in a general topological space? Why or why not?

7. A point $p \in X$, a topological space, is said to be a **cluster point** of a sequence $\langle x_n \rangle$ in $X$ if for all open sets $U$ containing $p$, there is an $x_n \in U$. How do cluster points differ from limit points? Give some examples that distinguish these notions.

8. Let $X$ be a first-countable space. Prove that each point $p \in X$ has a *nested* countable local base $B_1 \supseteq B_2 \supseteq \ldots$.