mgf

The moment generating function method (Thm 6.1)

Recall that the *moment generating function (mgf)* of a random variable X is
$m_X(t) = Ee^{Xt}$. $= \begin{cases} \sum_x e^{xt} f(x) & \text{if } X \text{ discrete} \\ \int e^{xt} f(x)\,dx & \cdots \text{cts} \end{cases}$

Mgf's can be used to identify distributions as follows:

If the mgf of a rv X is the same as that of another rv U,
we may conclude that X has the same distribution as U.

(Ie, if $m_X(t) = m_U(t)$, then $F_X(k) = F_U(k)$ and $f_X(k) = f_U(k)$ for all k.)

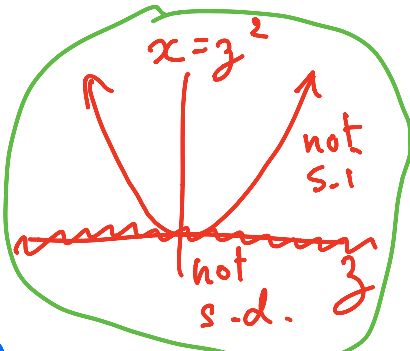
Let us now tackle the problem in Example 8. ($Z \sim N(0,1)$. Find the dsn of $X = Z^2$.)

$$m_X(t) = Ee^{Xt} = Ee^{Z^2 t} = \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2(1-2t)} dz$$
$$= c \underbrace{\int_{-\infty}^{\infty} \frac{1}{c\sqrt{2\pi}} e^{-\frac{1}{2c^2}z^2} dz}_{1}, \quad \text{where } c^2 = \frac{1}{1-2t}$$
$$= c. \qquad \text{(The integral must equal 1.)}$$

Thus $m_X(t) = (1-2t)^{-1/2}$.
But $(1-2t)^{-1/2}$ is the mgf of $U \sim \text{Gam}(1/2,2)$.
(Recall that if $W \sim Gam(a,b)$ then $m(t) = (1-bt)^{-a}$.)

It follows that $X \sim \text{Gam}(1/2,2)$.
Equivalently, $X \sim \chi^2(1)$. (Recall that if $R \sim \text{Gam}(k/2,2)$ then $R \sim \chi^2(k)$.)

Therefore the pdf of X is $f(x) = \dfrac{x^{\frac{1}{2}-1} e^{-x/2}}{2^{1/2}\Gamma(1/2)} = \dfrac{1}{\sqrt{2\pi x e^x}}$, $x > 0$.

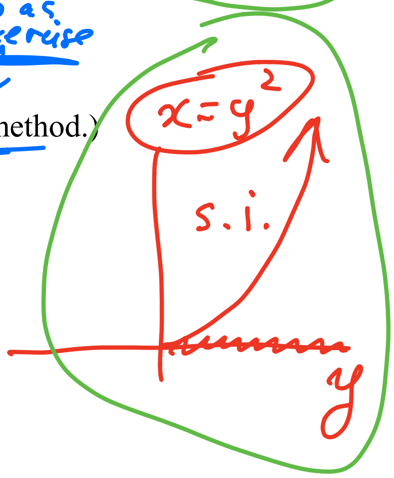


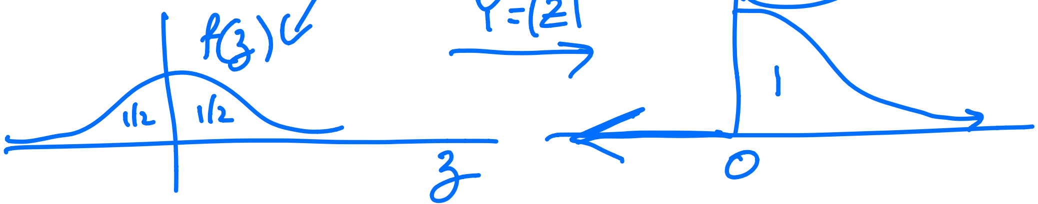
Another solution: Let $Y = |Z|$. Then $f(y) = \dfrac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$, $y > 0$.

Do as exercise

(This follows by symmetry about $z = 0$. It can also be proved using the cdf method.)
Now $x = y^2$ is a strictly increasing function, since y can't be negative.
So by the transformation method, $X = Y^2$ has pdf

$$f(x) = f(y)\left|\frac{dy}{dx}\right| = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x} \left|\frac{1}{2}x^{-\frac{1}{2}}\right| = \frac{1}{\sqrt{2\pi x e^x}}, \ x > 0, \text{ as before.}$$





Two useful results when applying the mgf technique

1. If $X = a + bY$, then
$$m_X(t) = e^{at} m_Y(bt).$$
(Prove this as an exercise.)

2. If $Y_1, \dots, Y_n$ are independent random variables and $X = Y_1 + \dots + Y_n$, then
$$m_X(t) = m_{Y_1}(t) \dots m_{Y_n}(t).$$
(This is Thm 6.2.)

Example 9 $Y \sim N(0,1)$. Find the dsn of $X = a + bY$. (This is an earlier exercise.)

$m_Y(t) = e^{\frac{1}{2}t^2}$. (This is proved in Tutorial 7.)

Therefore $m_X(t) = e^{at} m_Y(bt) = e^{at} e^{\frac{1}{2}(bt)^2} = e^{at + \frac{1}{2}b^2t^2}$, which is the mgf of $U \sim N(a, b^2)$.

It follows that $X \sim N(a, b^2)$.

Example 10 Suppose that $Y_1, \dots, Y_n$ are independent gamma rv's, such that the ith one has parameters a_i and b.
Find the distribution of $X = Y_1 + \dots + Y_n$.

$$\begin{aligned} m_X(t) &= m_{Y_1}(t) \dots m_{Y_n}(t) \\ &= (1-bt)^{-a_1} \dots (1-bt)^{-a_n} \\ &= (1-bt)^{-\dot{a}}, \quad \text{where } \dot{a} = a_1 + \dots + a_n. \end{aligned}$$

$\Downarrow$

Hence $X \sim \text{Gam}(\dot{a}, b)$.

Corollary: If $Y_1, \dots, Y_n \sim \text{iid } \chi^2(1)$, then $Y_1 + \dots + Y_n \sim \chi^2(n)$.
(NB: $\chi^2(r) = Gam(r/2, 2)$.)

Exercise Suppose that $Y_1, \dots, Y_n$ are independent normally distributed rv's such that the ith one has mean a_i and variance b_i^2.

Let $X = \sum_{i=1}^n k_i Y_i$. Show that $X \sim N\left(\sum_{i=1}^n k_i a_i, \sum_{i=1}^n k_i^2 b_i^2\right)$.

$$m_X(t) = Ee^{\left(\sum_{i=1}^n k_i Y_i\right)t} = E\prod_{i=1}^n e^{k_i Y_i t} = \prod_{i=1}^n Ee^{Y_i(k_i t)} = \prod_{i=1}^n m_{Y_i}(k_i t)$$

$$= \prod_{i=1}^n e^{a_i(k_i t) + \frac{1}{2}b_i^2(k_i t)^2} = e^{\left(\sum_{i=1}^n k_i a_i\right)t + \frac{1}{2}\left(\sum_{i=1}^n k_i^2 b_i^2\right)t^2}$$
(see Thm 6.3).

$X = 2Y + 1$, $X = Y^2$, ..., $X = \frac{\sin Y}{1-Y}$ etc.

Order statistics

Suppose that $Y_1, \ldots, Y_n$ are iid rv's.

Let: U_1 be the smallest of these (ie, $U_1 = \min(Y_1, \ldots, Y_n)$)

U_2 be the second smallest

....

U_n be the largest (ie, $U_n = \max(Y_1, \ldots, Y_n)$)

$= \begin{cases} Y_1 & \text{if } Y_1 \geq Y_i \, \forall i > 1 \\ Y_2 & \text{if } Y_2 \geq Y_i \, \forall i \neq 2 \\ \text{etc.} \end{cases}$

(Thus $U_1 \leq U_2 \leq \ldots \leq U_n$.)

We call U_k the *kth order statistic*. (Recall Problem 1 in Tutorial 6.)

Example 11 Suppose that $Y_1, Y_2 \sim$ iid $Expo(b)$.
Find the pdf of the second order statistic, $U_2 = \max(Y_1, Y_2)$.

cdf m.

$F_{U_2}(u) = P(U_2 < u) = P\{\max(Y_1, Y_2) < u\} = P(Y_1 < u, Y_2 < u)$ (and)

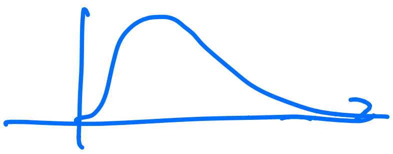
$= P(Y_1 < u)P(Y_2 < u)$ (by independence)

$= P(Y_1 < u)^2$

$= (1 - e^{-u/b})^2, \quad u > 0.$

So $f_{U_2}(u) = F'_{U_2}(u) = 2(1 - e^{-u/b})^1(-e^{-u/b})(-1/b)$

$= \boxed{2(1 - e^{-u/b})\frac{1}{b}e^{-u/b}}, \quad u > 0.$



Exercise: Show that $EU_2 = 3b/2$ (NB: $EU_2 > EY_i = b$, as one would expect.) (✓)

$$EU_2 = 2\int_0^\infty u\frac{1}{b}e^{-u/b}du - \int_0^\infty u\frac{1}{b/2}e^{-u/(b/2)}du \qquad = 2b - b/2 \qquad = 3b/2.$$

Thm:

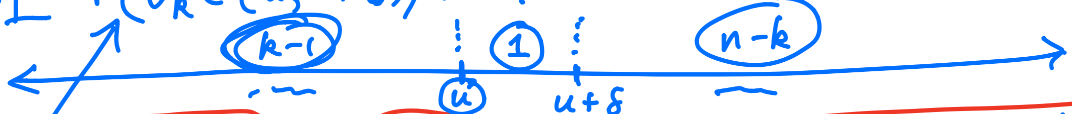
If $Y_1, \ldots, Y_n$ are continuous and iid, then the pdf of the *kth* order statistics U_k is

$$f_{U_k}(u) = \frac{n!}{(k-1)!(n-k)!}F(u)^{k-1}[1 - F(u)]^{n-k}f(u), \quad ?$$

where $f(y)$ and $F(y)$ are the pdf and cdf of Y_1, respectively. (See Thm 6.5.)

Note that this formula is in agreement with $f_{U_2}(u)$ in Example 11, where $n = k = 2$.

"Pf": $P(U_k \in (u, u+\delta)) = ?$



$\approx \frac{n!}{(k-1)!\,1!\,(n-k)!} F(u)^{k-1} (\cancel{\delta} f(u)) [1 - F(u)]^{n-k}$ by the multinomial dsn

$\approx (\cancel{\delta} f_{U_k}(u))$

QED.

Range restricted distributions

Example 12 Suppose that the number of accidents which occur each year at a certain intersection follows a Poisson distribution with mean λ.

Find the pdf of the number of accidents at this intersection last year if it is known that at least one accident occurred there during that year.

Let Y be the number of accidents at the intersection last year.

Then $\boxed{X = (Y \mid Y > 0)}$ has pdf

$$\begin{aligned}
f(x) &= P(X = x) \\
&= P(Y = x \mid Y > 0) \\
&= \frac{P(Y = x, Y > 0)}{P(Y > 0)} \quad \text{redundant if } x > 0 \\
&= \frac{P(Y = x)}{1 - P(Y = 0)} \quad \text{for } x > 0 \\
&= \frac{e^{-\lambda}\lambda^x / x!}{1 - e^{-\lambda}\frac{\lambda^0}{0!}} \quad x = 1, 2, 3, \ldots
\end{aligned}$$

$$\left(f(x) = \frac{P(Y=x)}{P(Y>0)}, \; x > 0 \right)$$

For example, if $\lambda = 3.2$ then $p_X(4) = \dfrac{e^{-3.2}3.2^4/4!}{1 - e^{-3.2}} = 0.186$, (✓)

which we note is slightly higher than $p_Y(4) = e^{-3.2}3.2^4/4! = 0.178$.

What is the expected number of accidents last year?

$$\begin{aligned}
E(Y \mid Y > 0) = EX &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x / x!}{1 - e^{-\lambda}} \\
&= \frac{1}{1 - e^{-\lambda}} \underbrace{\sum_{x=0}^{\infty} x \left(\frac{e^{-\lambda}\lambda^x}{x!} \right)}_{EY} \quad \text{(where the first term in the sum is zero)} \\
&= \frac{\lambda}{1 - e^{-\lambda}},
\end{aligned}$$

$(1 - e^{-\lambda} < 1)$

$E(Y \mid Y>0) > EY$ (✓)

which we note is higher that $EY = \lambda$.

For example, if $\lambda = 3.2$ then $EX = 3.336 > 3.2 = EY$.

Ex13 Find the pdf (pmf) of ~~(X | Y > 1)~~ $X = (Y \mid Y > 1)$.

$$f(x) = \frac{f(y = x)}{P(Y > 1)}, \quad x = 2, 3, \ldots$$

$$= \frac{e^{-\lambda}\lambda^{x}/x!}{1 - e^{-\lambda} - \lambda e^{-\lambda}}, \quad x = 2, 3, \ldots$$

Ex14 Find the pdf of

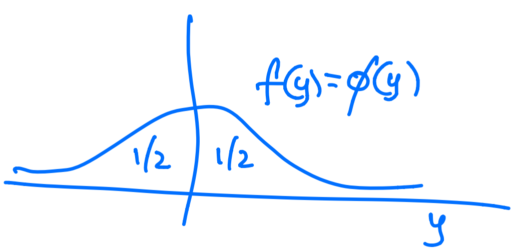
$$X = Y\,I(Y > 1) = \begin{cases} Y & \text{if } Y > 1 \\ 0 & \text{if } Y \le 1 \end{cases}$$

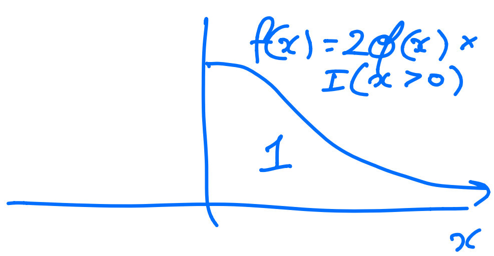
$$f(x = 0) = P(Y \le 1) = f(y = 0) + f(y = 1) = e^{-\lambda} + \lambda e^{-\lambda}$$

For $x > 1$: $f(x) = e^{-\lambda}\lambda^{x}/x!$ $\quad (x = 2, 3, 4, \ldots)$

Ex15 $Y \sim N(0, 1)$ & $X = (Y \mid Y > 0)$. Find the pdf of X.

$$f(x) = \frac{f(y = x)}{\underbrace{P(Y > 0)}_{1/2}} = 2\phi(x), \quad x > 0$$





Ex16 $Y \sim N(0, 1)$. Find the pdf of

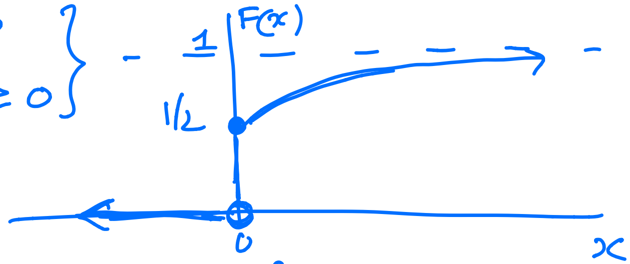
$$X = Y\,I(Y > 0) = \begin{cases} Y & \text{if } Y > 0 \\ 0 & \text{if } Y \le 0 \end{cases}.$$

Soln

$$P(X = 0) = P(Y \le 0) = 1/2.$$

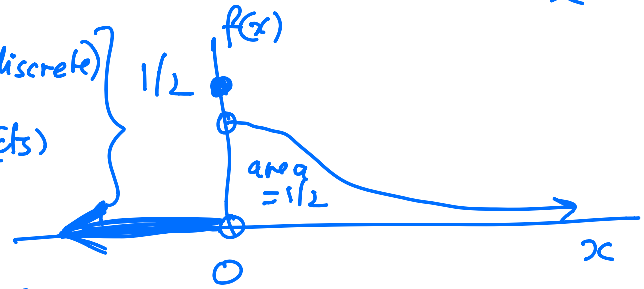
$$P(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ P(Y \le x) & \text{if } x > 0 \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \Phi(x), & x \geq 0 \end{cases}$$



X has a <u>mixed</u> dsn

$$f(x) = \begin{cases} 1/2, & x = 0 \text{ (discrete)} \\ \phi(x), & x > 0 \text{ (cts)} \\ 0, & x < 0 \end{cases}$$



Note: $\displaystyle\sum_{x \text{ discrete}} f(x) + \int_{x \text{ is cts}} f(x)\,dx$

$$= \frac{1}{2} + \int_0^\infty \phi(x)\,dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Ex. Find EX.

Soln: $\displaystyle EX = \sum_{x \text{ disc.}} x f(x) + \int_{x \text{ cts}} x f(x)\,dx$

$= ?$