MATH 315; HOMEWORK # 3

Due Feb. 4, 2015

1. (Exercise 11.2) (a) If $m \geq 3$, explain why $\phi(m)$ is always even.

(b) $\phi(m)$ is "usually" divisible by 4. Describe all of the m's for which $\phi(m)$ is not divisible by 4.

2. (Exercise 11.5 (a)) Find x that solves the simultaneous congruence: $x \equiv 3 \pmod 7$ and $x \equiv 5 \pmod 9$

3. (Exercise 11.11) Find at least five different numbers n with $\phi(n) = 160$. How many more can you find?

4. (Exercise 11.13 (a)) For each integer $2 \leq a \leq 10$, find the last four digits of a^{1000}. [Hint: We need to calculate a^{1000} mod 10000. Use Euler's theorem and Chinese remainder theorem. For example, $10000 = 2^4 \cdot 5^4$; $2^{1000} \equiv 0 \bmod 2^4$, and $2^{500} \equiv 1 \bmod 5^4$.]

5. (page 83, Exercise 12.2) (a) Show that there are infinitely many prime numbers that are congruent to 5 modulo 6. (Hint: use $A = 6p_1p_2 \cdots p_r + 5$.)

(b) Try to use the same idea (with $A = 5p_1p_2 \cdots p_r + 4$) to show that there are infinitely many primes congruent to 4 modulo 5. What goes wrong? In particular, what happens if you start with $\{19\}$ and try to make a longer list?

6. (page 84, Exercise 12.5, some parts) Recall that the number $n! = 1 \cdot 2 \cdots n$.

(1) Find the highest power of 2 dividing $5!, 10!$. If you find a pattern, find the highest power of 2 dividing 100!.

(2) For a prime p, find the highest power of p dividing $n!$. (Hint: use the Gauss notation $[x]$. It is defined as $[x] = n$ if $n \leq x < n+1$. So $[2.78] = 2$. The number of multiples of p among $1, ..., n$, are $[\frac{n}{p}]$.)

(3) Prove that if p^m divides $n!$, then $m < \frac{n}{p-1}$.