University of Toronto
**MAT237Y1Y TERM TEST 1**
**Thursday, Nov. 15, 2012**
Duration: 90 minutes

**No aids allowed**

**Instructions:** There are 11 pages including the cover page. Please answer all questions in the spaces provided (if you use back of a sheet please clearly specify that.) Document your arguments by briefly stating the results you use (in most cases you may find relevant definitions or results in an earlier question of this test.) The value for each question is indicated in parentheses beside the question number. The test is out of 80 and there are 10 bonus embedded in the test (total of 90 marks to be found.)

**NAME:** (last, first) Marking Scheme

**STUDENT NUMBER:** ____________________

**SIGNATURE:** ____________________

**CHECK YOUR TUTORIAL:**

| | | | | | | |
|---|---|---|---|---|---|---|
| ◯ TUT0101 Mon. 3-4 | ◯ TUT0201 Mon. 4-5 | ◯ TUT0301 Tue. 2-3 | ◯ TUT0401 Wed. 3-4 | ◯ TUT5101 Tue. 5-6 | ◯ TUT5201 Wed. 5-6 | ◯ TUT5301 Thu. 5-6 |

**MARKER'S REPORT:**

| Question | MARK |
|---|---|
| Q1 | /19 |
| Q2 | /24 |
| Q3 | /14 |
| Q4 | /16 |
| Q5 | /17 |
| TOTAL | /90 |

Martin Muñoz

1.

a) (2 marks) Complete the definition: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ if and only if ... There exists $\mathbb{C} \in \mathbb{R}^n$ and a function $E(\mathbb{h})$

S.t. $f(\mathbb{a}+\mathbb{h}) = f(\mathbb{a}) + \mathbb{C} \cdot \mathbb{h} + E(\mathbb{h})$ and $\lim_{\mathbb{h}\to\mathbb{0}} \frac{E(\mathbb{h})}{|\mathbb{h}|} = 0$.

1 .5 1

b) (4 marks) Use definition of differentiability to show that the function $f(x,y) = x^2 y$ is differentiable at $(2,0)$.

$$f(2+h, 0+k) = (2+h)^2(0+k) = (4+4h+h^2)k = 4k + 4hk + h^2k$$

3

$$= \underbrace{0}_{f(2,0)} + \underbrace{[0 \quad 4]\begin{bmatrix} h \\ k \end{bmatrix}}_{4k = \mathbb{C}\cdot\mathbb{h}} + \underbrace{hk(4+h)}_{E(\mathbb{h})}.$$

note $$\frac{E(\mathbb{h})}{|\mathbb{h}|} = \frac{hk(4+h)}{\sqrt{h^2+k^2}} < \frac{hk(4+h)}{\sqrt{h^2}} = \frac{hk(4+h)}{|h|} \longrightarrow 0$$ as $h \to 0$ & $k \to 0$

1

c) (4 marks) Estimate the value of $(\ln(1.02) + \sqrt{3.97})$ by using either the differential $df(\boldsymbol{a};\boldsymbol{h})$ or linear approximation for an appropriate function $f$.

let $f(x,y) = \ln x + \sqrt{y}$ and let $\mathbb{a} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and $\mathbb{h} = \begin{bmatrix} 0.02 \\ -0.03 \end{bmatrix}$ 1

1

$f(\mathbb{a}+\mathbb{h}) \simeq f(\mathbb{a}) + \underbrace{\nabla f(\mathbb{a}) \cdot \mathbb{h}}_{df(\mathbb{a};\mathbb{h})}$ so $f(1+0.02, 4-0.03) \simeq$

$$f(1,4) + \left[\frac{1}{1} \quad \frac{1}{2\sqrt{4}}\right]\begin{bmatrix} 0.02 \\ -0.03 \end{bmatrix} = (\ln 1 + \sqrt{4}) + \left[1 \quad \frac{1}{4}\right]\begin{bmatrix} 0.02 \\ -0.03 \end{bmatrix} =$$

$\frac{\partial f}{\partial x} = \frac{1}{x}$ $\quad \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ $\quad (0+2) + 0.02 - \frac{0.03}{4} = 2.0125$

1 1

2/11

d) (6 marks) Recall definition of $\partial_{\boldsymbol{u}} f(\boldsymbol{a})$. Use it and the definition from part (a) to prove $\partial_{\boldsymbol{u}} f(\boldsymbol{a}) = \nabla f(\boldsymbol{a}) \cdot \boldsymbol{u}$, where $\boldsymbol{u}$ is a unit vector.

$$\partial_u f(a) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t} \quad \text{but} \quad f(a+tu) = f(a) + \nabla f(a) \circ (tu) + E(tu)$$

2 2

$$\text{so} \quad \partial_u f(a) = \lim_{t \to 0} \frac{\nabla f(a) \circ (tu) + E(tu)}{t} =$$

1 $\lim_{t \to 0} \nabla f(a) \circ u + \frac{E(tu)}{|tu|} \cdot \frac{|tu|}{t}$ d.n.e but by Squeeze Thm

$\frac{E(tu)}{|tu|} \to 0$ as $t \to 0$

$= \nabla f(a) \circ u$

1

$$-\frac{E(tu)}{|tu|} < \frac{E(tu)}{|tu|} \frac{|tu|}{t} < \frac{E(tu)}{|tu|}$$

$\to 0 \qquad \to 0 \qquad 0$

e) (3 marks) Calculate $\dfrac{\partial f}{\partial \boldsymbol{u}}(2, -3)$ where $\boldsymbol{u} = (4/5, 3/5)$ and $f(x, y) = x^2 y$.

$$\nabla f(x,y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} \quad \text{so} \quad \nabla f(2,-3) = \begin{bmatrix} -12 \\ 4 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

1

$$\text{so} \quad \frac{\partial f}{\partial u}(2,-3) = \nabla f(2,-3) \circ u = \begin{bmatrix} -12 & 4 \end{bmatrix} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \frac{-48}{5} + \frac{12}{5} = \frac{-36}{5}$$

1 1

Louis-Philippe Thibault

2. Completeness Axiom

   a) (6 marks) each of the following statements is missing a necessary component and/or is mistakenly stated. Please correct them, to the version found in the textbook by either rewriting the correct statement or by adding the missing components:

   - completeness axiom for $\mathbb{R}$: Every subset $S \subset \mathbb{R}^n$ has a least upper bound.
     $S \neq \emptyset$ and bounded above

     1.5

   - monotone sequence theorem: Every monotone ~~increasing~~ bounded sequence in $\mathbb{R}$ is convergent.

     1.5

   - Nested interval theorem: Let $I_k = [a_k, b_k]$, $k = 1, 2, 3, \ldots$, be a nested sequence of intervals, i.e. $a_k \leq b_k$ and $I_k \subset I_{k-1}$ and $b_k - a_k \to 0$ as $k \to \infty$. Then there is a unique point $x_0$ that is in all $I_k$.

     1

   - Theorem 1.18 (BW I, for $\mathbb{R}$)): any bounded sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}$ has a convergent subsequence.

     1

   - Theorem 1.19 (BW II, for $\mathbb{R}^n$): Any subsequence of any sequence $\{\boldsymbol{x}_n\}$ must be convergent.

     Any bounded Sequence $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ has a Convergent Sub-Sequence.

     1

   b) (5 marks) What does it mean for the sequence $\{\boldsymbol{x}_n\}$ to be Cauchy? Prove that every Cauchy sequence $\{\boldsymbol{x}_n\}$ is bounded.

$\{\boldsymbol{x}_n\}$ is Cauchy if $\forall \epsilon > 0\ \exists N$ st. $\forall n, m > N$: $|\boldsymbol{x}_n - \boldsymbol{x}_m| < \epsilon$. 1

Let $\epsilon = 1$ and Choose $N$ St. $\forall n, m > N$ $|\boldsymbol{x}_n - \boldsymbol{x}_m| < 1$. In particular

$\forall n > N$ $|\boldsymbol{x}_n - \boldsymbol{x}_{N+1}| < 1$ 1 which means $\big||\boldsymbol{x}_n| - |\boldsymbol{x}_{N+1}|\big| < 1$

by triangle inequality 1

which means $-1 + |\boldsymbol{x}_{N+1}| < |\boldsymbol{x}_n| < 1 + |\boldsymbol{x}_{N+1}|$.

Let $M = \max\{|\boldsymbol{x}_1|, \ldots, |\boldsymbol{x}_N|, 1 + |\boldsymbol{x}_{N+1}|\}$ Then $\forall n$ $|\boldsymbol{x}_n| < M$ or $\boldsymbol{x}_n \in B(M, 0)$.

1 1

c) (5 marks) Prove that if a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ of a Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ converges to a point $x$ then $\{x_k\}_{k=1}^{\infty}$ also converges to the same point $x$

Assume $\{x_{k_j}\}_{j=1}^{\infty} \longrightarrow x$, That is $\forall \epsilon>0 \ \exists J$ s.t. $\forall j \quad j>J \Rightarrow |x_{k_j}-x|<\epsilon$

and assume $\{x_k\}$ is Cauchy, That is $\forall \epsilon>0 \ \exists K$ s.t. $\forall k,m>K \quad |x_k-x_m|<\epsilon$

Now given $\epsilon>0$ Choose $J$ s.t. $\forall j \quad j>J \Rightarrow |x_{k_j}-x|<\frac{\epsilon}{2}$ and

2 Choose $K$ s.t. $\forall k,m>K \quad |x_k-x_m|<\frac{\epsilon}{2}$. Now if

we select $j$ s.t. $k_j>K$ Then 1 $|x_k-x_{k_j}|<\frac{\epsilon}{2}$. But Then

$|x_k-x|=|x_k-x_{k_j}+x_{k_j}-x|\leq|x_k-x_{k_j}|+|x_{k_j}-x|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$. 2

so as long as $k>K \quad |x_k-x|<\epsilon$.

d) (8 marks) Use other parts of this question to prove that every Cauchy sequence in $\mathbb{R}^n$ is convergent. Explain how completeness axiom for $\mathbb{R}$ has been fundamentally involved in this process.

Let $\{x_k\}_{k=1}^{\infty}$ be Cauchy. By (b) $\{x_k\}$ is bdd. 1 By Thm 1.19 (a) $\{x_k\}$ has a convergent Sub-sequence. 1 By (c) $\{x_k\}$ must also converge to the same limit as the Sub-sequence, so it converges. 1

Completeness axiom $\overset{1}{\Rightarrow}$ monotone Sequence Theorem $\overset{1}{\Rightarrow}$ Nested interval Thm

$\Rightarrow$ 1.18 $\Rightarrow$ 1.19 $\Rightarrow$ $\left.\begin{matrix}(b)\\(a)\\c\end{matrix}\right\} \Rightarrow (d)$

1 1 1

5/11

Yannis

3. sequential characterization of continuity:

a) (3 marks) Define ($\epsilon$ definition in both cases):

1.5 - the function $f$ is continuous at $a$ if ... $\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

1.5 - $\{x_k\}_{k=1}^{\infty}$ converges to a point $a$ if ... $\forall \epsilon > 0 \ \exists K \ \forall k \ k > K \Rightarrow |x_k - a| < \epsilon$

b) (4 marks) Determine whether $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 y}{x^4 + y^4}$ exists; if so determine the limit and if not, explain why.

if the limit exists it must be unique. 1 → important

as $(x,y) \to (0,0)$ along the bath $y=0$, That is $(x,0) \to (0,0)$

$$\frac{x^3 y}{x^4 + y^4} = \frac{0}{x^4} \quad \text{so } \lim = 0$$ 1

The computation are related to the final result (i.e. ~~[illegible]~~ the existence or not of the limit)

However as $(x,y) \to (0,0)$ along the path $x=y$, Then $\dfrac{x^3 y}{x^4 + y^4} = \dfrac{x^4}{x^4 + x^4} = \dfrac{1}{2}$

$\lim = \frac{1}{2}$ $\neq 0$ so limit cannot exist.

1 1

6/11

c) (7 marks) Assume $\boldsymbol{a} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Show that if $f$ is not continuous at $\boldsymbol{a}$ then there exists a sequence $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ that converges to $\boldsymbol{a}$ but the sequence $\{f(\boldsymbol{x}_k)\}_{k=1}^{\infty}$ does not converge to $f(\boldsymbol{a})$.

if $f$ is NOT Cont at $a$ Then

$$\sim [\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \quad |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon] \quad 2$$

→ very important, if you don't mention that you are close to assuming what you need to prove.

which means $\exists \epsilon > 0 \ \forall \delta > 0 \ \exists x \quad |x-a| < \delta$ but $|f(x) - f(a)| \geq \epsilon$.

→ -1.

Note $\forall \delta \ \exists x$ ..allows Construction of such Seq:. Important part of construction

let $\delta = 1$ and Choose $x_1$ s.t. $|x_1 - a| < 1$ but $|f(x_1) - f(a)| \geq \epsilon$

⋮

2 3 $\delta = \frac{1}{n}$ choose $x_n$ st. $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \epsilon$

⋮

The Sequence $\{x_n\} \longrightarrow a$ b/c $\forall \delta \ \exists N$ (Say $N > \frac{1}{\delta}$)

1.5 1 s.t. $\forall n \quad n > N \Rightarrow n > \frac{1}{\delta} \Rightarrow \delta > \frac{1}{n}$, $|x_n - a| < \frac{1}{n} < \delta$.

But $f(x_n) \not\rightarrow f(a)$ b/c $\exists \epsilon > 0$ st. $\forall N \ \exists n$ $n > N$ but

say $n = N+1$

1.5 1 $|f(x_n) - f(a)| \geq \epsilon$

Fulgencio

4. Extreme Value Theorem

a) (3 marks) Give the following definition and complete the statement:

- $S$ is compact: if $S$ is Closed & bounded (1)

- Bolzano-Weierstrass theorem: A subset $S$ of $\mathbb{R}^n$ is compact if and only if

Whenever $\{x_k\}_{k=1}^{\infty} \subset S$ Then there exists a Subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ That Converges in $S$. (2)

b) (6 marks) Use Bolzano-Weierstrass characterization of compactness or definition of compactness to show only one of the following:

- the set $S = \{\frac{1}{k}\}_{k=1}^{\infty} \cup \{0\}$ is compact.
- the set $\{x \in \mathbb{R}^n : |x| = 1\}$ is compact. (Hint: you may use without proof that the norm function is continuous and if a function $f$ is continuous then $f^{-1}(\{c\})$ is also closed.)

$S = f^{-1}(\{1\})$ where $f(x) = |x|$. $\{1\}$ is Closed and $f$ is Cont so $f^{-1}_{\{1\}}$ is closed, so $S$ is Closed. Also $S$ is bdd b/c $S \subset B(2, 0)$. Proof (B) (2)

$S = \{\frac{1}{k}\}_{k=1}^{\infty} \cup \{0\}$ is Cpt:

(proof 1)

1. $S$ is bounded b/c every Converging Seq is bdd (2)

2. $S$ is Closed b/c Complement is open:

given $a \in S^c$ : Case 1 $a < 0$ let $\epsilon = \frac{|a|}{2}$

Case 2 $a > 1$ let $\epsilon = \frac{a-1}{2}$

Case 3 $\frac{1}{k+1} < a < \frac{1}{k}$ let $\epsilon = \min\{\frac{a - \frac{1}{k+1}}{2}, \frac{\frac{1}{k} - a}{2}\}$

note $B(\epsilon, a) \subset S^c$.

(4)

(proof 2) given any Sequence $\{X_j\} \subset S$

Case 1: $\{X_j\}$ Consists of infinitely many distinct elements of $S$, in This Case $\{X_j\}$ is a Subsequence of $\{\frac{1}{k}\}$, so it must Converge to $0 \in S$.

Case 2 $\{X_j\}_{j=1}^{\infty}$ Consists of only finitely many points of $S$; in This Case at least one of The $\frac{1}{k}$ is repeated infinitely many times so $\{Y_i\}_{i=1}^{\infty}$ where $Y_i = \frac{1}{k}$ $i = 1, 2 \ldots$ is a Constant Seq, so Converges in $S$

Now apply BW Theorem.

8/11

Fulgencio

c) (7 marks) State and prove the Extreme Value Theorem. Make sure you quote any theorem you are using in the process of this proof.

EVT

Suppose $S \subset \mathbb{R}^n$ is cpt and $f : S \to \mathbb{R}$ is cont. Then $f$ has an absolute min value and absolute max value on $S$; That is, There exists pts $a, b \in S$ st $f(a) \leq f(x) \leq f(b)$ for all $x \in S$.

1

1

Pf. Note That image of a Compact Set under a Cont. function must be compact, so $f(S) \subset \mathbb{R}$ is Compact, so it is bdd.

1

Then $lub(f(S))$ and $glb(f(S))$ exist. But $f(S)$ is also Closed,

1

so $lub f(S) \in f(S)$ so $\exists b \in S$ st $f(b) = lub(f(S))$

$glb f(S) \in f(S)$ $\exists a \in S$ " $f(a) = glb(f(S))$

2

now for any $x \in S$ $f(x) \in f(S)$ so $glb(f(S)) \leq f(x) \leq lub(f(S))$

i.e $f(a) \leq f(x) \leq f(b)$.

for all $x \in S$.

1

Ming Xiao

5. Intermediate Value Theorem

a) (4 marks) Define what it means for a set $S$ to be disconnected.

disconnection for S that is

S is disconnected if There is a pair of Sets $(S_1, S_2)$ s.t. (1) or

$S_1 \neq \emptyset \neq S_2$. (1)

$S = S_1 \cup S_2$ (1)

$\overline{S_1} \cap S_2 = \emptyset = S_1 \cap \overline{S_2}$ (1)

b) (5 marks) Is the set $S = \{(x, y) \in \mathbb{R}^2 : (x + 1)(x - y^2) = 0\}$ connected? Justify your answer.

$S = \{(x,y) \in \mathbb{R}^2 : (x+1)=0 \text{ or } (x-y^2)=0\}$

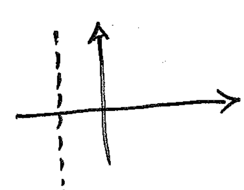
(1)

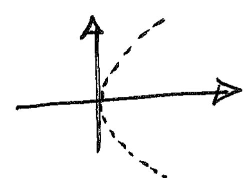
so S Consists of two pieces $S = \{(x,y) \in \mathbb{R}^2 : x+1=0\} \cup \{(x,y) : x-y^2=0\}$

(1) $= \{(x,y) \in \mathbb{R}^2 : x=-1\} \cup \{(x,y) : x=y^2\}$

let $S_1 = \{(x,y) \in \mathbb{R}^2 : x=-1\}$

and $S_2 = \{(x,y) \in \mathbb{R}^2 : x=y^2\}$ (3)





note : $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$

and $S_1 = V_1 \cap S$ where $V_1 = \{(x,y) : x < -\frac{1}{2}\}$ Since $\overline{V_1} \cap V_2 = \emptyset$

$S_2 = V_2 \cap S$ $V_2 = \{(x,y) : x > -\frac{1}{2}\}$ & $V_1 \cap \overline{V_2} = \emptyset$

Then $\overline{S_1} \cap S_2 = \emptyset = S_1 \cap \overline{S_2}$.

graph only & no argument (2)

argument (5)

10/11

c) (8 marks) State and prove the Intermediate Value Theorem for a function $f : S \longrightarrow \mathbb{R}$ and a set $S \subset \mathbb{R}^n$. Make sure to quote any theorem and property used in the course of this proof.

IVT

Suppose $f: S \to \mathbb{R}$ is Cont on $S$ and $V \subset S$ is Connected. If $\mathbf{a}, \mathbf{b} \in V$ and $f(\mathbf{a}) < t < f(\mathbf{b})$ or $f(\mathbf{b}) < t < f(\mathbf{a})$, There is a point $\mathbf{c} \in V$ s.t. $f(\mathbf{c}) = t$

1.5 1

Pf: 1 $f(V)$ is Connected b/c Cont. image of a Connected set is Connected

0.5 $f(V) \subset \mathbb{R}$ as $f: S \to \mathbb{R}$

1 $f(V)$ is an interval b/c Connected Sub-sets of $\mathbb{R}$ are intervals. Call it $I$

Since $f(\mathbf{a}), f(\mathbf{b}) \in I$ Then any point between $f(\mathbf{a})$ and $f(\mathbf{b})$ must also be in $I$, so $t \in I = f(V)$ so $t \in f(V)$ so $\exists\, \mathbf{c} \in V$ s.t. $f(\mathbf{c}) = t$.

1 1 1

11/11