PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 9
DUE FRIDAY, MAY 12, 4PM.

Warm-up problems. These are completely optional.

(1) When you go to your favorite restaurant, you order pasta 2/3 of the time and fish 1/3 of the time. When you order pasta, there is a 1/4 chance that it is out of stock, and when you order fish, there is a 1/2 chance that it is out of stock. What is the probability that the dish you order is out of stock?

(2) In bowling, a strike occurs when the bowler knocks down all the pins in one roll. Suppose that on each roll a bowler has probability p of rolling a strike. How high must p be so that the probability of a perfect game is at least 1 percent? First make a guess, then use a calculator to compute the answer.

Problems to be handed in. Solve four of the following five problems. One of the four must be Problem (2).

(1) Let X_1, X_2, X_3 be random variables such that $P(X_i = j) = 1/n$ for all $(i, j) \in [3] \times [n]$. Compute the probability that $X_1 + X_2 + X_3 \leq 6$, given that $X_1 + X_2 \geq 4$. You may assume that the random variables are *independent*, i.e.

9.16

$$P(X_1 = a_1, X_2 = a_2, X_3 = a_3) = P(X_1 = a_1)P(X_2 = a_2)P(X_3 = a_3).$$

(2) You hold a bag of ten coins, all superficially similar, but nine are fair, and one is foul (it shows heads with probability 9/10). You draw out a coin and begin flipping it.
   (a) The first five tosses are $HHHTH$. What is the probability that you are flipping one of the fair coins?
   (b) The next five tosses are $HHHHH$. Now what is the probability that you are flipping one of the fair coins?

(3) Suppose that a collection of $2n$ insects is randomly divided into n pairs. If the collection consists of n males and n females, what is the expected number of male-female pairs?

9.29

(4) Suppose that A, B, and n other people stand in a line in random order. Compute the expected number of people standing between A and B in two ways:
   (a) For each $k \in [n]$, compute the probability that there are exactly k people between A and B, and use the formula $E(X) = \sum_k kP(X = k)$.
   (b) Use linearity of expectation.

9.22

(5) Recall that in the finger game, players A and B show 1 or 2 fingers, and A then receives a payoff according to the following chart (a negative number indicates that A pays B).

	B shows 1	B shows 2
A shows 1	-2	+3
A shows 2	+3	-4

We considered a scenario where A shows 1 finger with probability x and B shows 1 finger with probability y, and showed that $x = 7/12$ gives an expected payoff of $1/12$ for A, and that this strategy is optimal. Here, *optimal* means that for any other choice of x, there exists a $y \in [0, 1]$ such that the expected payoff is lower than $1/12$.

(a) For what range of values $x \in [0, 1]$ can A guarantee a positive expected payoff, no matter how B plays?
(b) Prove that $y = 7/12$ is the optimal strategy for B.
(c) Assuming that both players play their optimal strategy, what proportion of the games do A and B actually win.