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Closure of a set S, denoted by $\overline{S}$ is the union of the set S together with boundary points of S: $\overline{S} = S \cup \partial S$. Interior and closure of a set are in a dynamic relationship, and most of the topological analysis takes place in this relationship. Of course, interior of a set S denoted by S^{int} is an open set, that is

$$x \in S^{int} \Longleftrightarrow \exists r > 0 \text{ such that } B(r, x) \subset S$$

This statement characterizes the interior of a set. Whereas the characteristic of a point in the closure of S is

$$x \in \overline{S} \Longleftrightarrow \forall r > 0 \; B(r, x) \cap S \neq \emptyset$$

(try to prove that this equivalence.)

See how the negation of this statement implies that the complement of the closure of a set S is the interior of the complement of S:

$$\exists r > 0 \text{ such that } B(r, \boldsymbol{a}) \cap S = \emptyset \text{ (which means } B(r, \boldsymbol{a}) \subset S^c$$

This implies (using proposition 1.4,) that the closure of a set is a closed set

An important characterization of the closure of a set is given in 1.14, which claims any point in the closure of S is the limit point of a sequence of points in S. Now combine this idea with the fact that $\overline{\overline{S}} = \overline{S}$ to conclude that if a sequence of points in $\overline{S}$ converges (to a point $\boldsymbol{a}$) then $\boldsymbol{a}$ must be in $\overline{S}$, so that there must be a sequence from S that converges to the point $\boldsymbol{a}$. What does this mean?

Some famous examples of closure of a set are:

- closure of any closed set is the set itself
- Let S be the set of rational numbers as a subset of $\mathbb{R}$. Then $\overline{S} = \mathbb{R}$, while $S^{int} = \emptyset$. Note also that in this case $\overline{S}^{int} = \mathbb{R}$, while $\overline{S^{int}} = \emptyset$.
- The previous example demonstrates that in general it is not true that for a set S, $S = \overline{S^{int}}$. But if for some set S we have $S = \overline{S^{int}}$ then such a set will be very special set which is very important for the theory of integration in chapter 5. A region of plane or of space for which this property holds is known as a regular region (see page 222.)