58

MAT337 HW4 Rui Qiu #999292509

§9.2

D. Proof: Our goal is to show for a random compact metric space X, $\exists Y \subseteq X$ s.t. Y is dense in X and Y is countable. (ie. X is separable).

Since for all $x \in X$ is contained in an open ball $B_{\frac{1}{n}}(x)$, $\forall n \in \mathbb{N}$, then $C_n = \{B_{\frac{1}{n}}(x) \mid x \in X\}$ is an open cover of X. (a union)

X is compact, so each C_n has a finite subcover

say $C'_n = \{B_{\frac{1}{n}}(x_k) \mid k = 1, \cdots, k_n\}$

~~So the index set with n elements, say $P = \{1, 2, \cdots, k_n\}$~~

~~Now we can use the~~

(2)

So the point sets $P_n = \{x_1, \cdots, x_{k_n}\}$

Then $Y = \bigcup_{n \in \mathbb{N}} P_n$

So far, for Y, it is a countable union with finite sets, hence Y is countable.

$\forall x \in X, \forall \varepsilon > 0, \exists m \in \mathbb{N}$ s.t. $m > \frac{1}{\varepsilon}$, then $\frac{1}{m} > \varepsilon$

Since $\{B_{\frac{1}{m}}(x_k) \mid k = 1, \cdots, k_m\}$ is a finite subcover of X,

$\exists n \in \{1, 2, \cdots, k_m\}$ s.t. $x \in B_{\frac{1}{m}}(x_n) \Rightarrow \rho(x, x_n) < \frac{1}{m} < \varepsilon \Rightarrow Y$ is dense in X

Therefore, every compact metric space has a compact dense subset

Hence, every compact metric space is separable.

§5.5

A. $g(x) = \sqrt{x}$.

Proof: $\varepsilon > 0$ is given, let $r = \varepsilon^2$

$\forall x, y \in [0, +\infty)$, $|x-y| < r$, $|g(x) - g(y)|^2 = |\sqrt{x} - \sqrt{y}|^2 = x - 2\sqrt{xy} + y$ ①

(10)

since $\sqrt{xy} \geq \sqrt{\min(x,y)}$ so ① $< x + y - 2\min(x,y) = |x-y| < r = \varepsilon^2$

Then $|g(x) - g(y)| < \varepsilon$

Thus $g(x)$ is uniformly continuous on its domain.

[redacted]

§5.6

D. Proof: Goal is to show polynomial

$p(x) = a_n x^n + \cdots + a_1 x + a_0$ with n is a odd number, $a_n \neq 0$

such that $\exists$ constant c, $p(c)=0$. [redacted]

Let's rewrite $p(x)$ first.

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$
$$= a_n\left(x^n + \cdots + \frac{a_1}{a_n}x + \frac{a_0}{a_1}\right)$$
$$= a_n x^n\left(1 + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_1 x^n}\right)$$

About the sign of a_n, we suppose $a_n > 0$ (WLOG)

For $a_n < 0$, $p(x) = -q(x)$. and $q(x)$ ~~is~~ has positive sign at first term.

8

$$\left.\begin{aligned}\lim_{x\to\infty} p(x) &= a_n \lim_{x\to\infty} x^n = \infty > 0 \\ \lim_{x\to-\infty} p(x) &= a_n \lim_{x\to\infty} x^n = -\infty < 0\end{aligned}\right\} \text{b/c } n \text{ is odd.}$$

As polynomials are continuous. and $\lim_{x\to-\infty} p(x) < 0 < \lim_{x\to\infty} p(x)$, by IVT, $\exists c \in \mathbb{R}$ s.t. $p(c) = 0$.

Done.

§5.7.

B. Proof: Know: f cont. on $[0,1]$

f is 1-1

10

wts: f monotone.

Since $[0,1]$ is compact and f continuous,

by EVT, $\exists\, a, b \in [0,1]$ s.t. $\forall x \in [0,1]$. $f(a) \le f(x) \le f(b)$

We claim that $a=0, b=1$ respectively (WLOG, $a=1, b=0$ would be similar).

We can use indirect proof, suppose $a \in (0,1)$, i.e. $a \neq 0$ or 1

let $\alpha' = \min\{f(0), f(1)\}$

Since f is 1-1, so $f(0) > f(a)$ & $f(1) > f(a)$

$$\Rightarrow \begin{cases} f(0) \ge \alpha' > f(a) \\ f(1) \ge \alpha' > f(a) \end{cases}$$

WLOG let $\alpha' = \min\{f(1), f(0)\} = f(0) \Rightarrow f(0) = \alpha'$ & $f(1) > f(0) > f(a)$

Since f continuous, by IVT, $\exists\, c\in(0,1)$ s.t. $f(c)=f(a)=\alpha$

$\Rightarrow f$ is not 1-1. (Contradiction)

So $a\notin(0,1)$

Similarly $b\notin(0,1)$

we have $a=0, b=1$ instead.

$\forall x, y\in(0,1)$, say $x<y$ then $f(x)\neq f(y)$

if $f(x)>f(y)$, f is continuous & $f(0)<f(y)<f(x)$

by IVT, $\exists\, p\in(0,x)$ s.t. $f(p)=f(y)$.

contradicts "1-1"!

So the only option is $f(x)<f(y)$.

So f monotone for $a=0, b=1$.

Similarly, for $a=1, b=0$, prove by the same pattern that

$\forall x,y\,(0,1)$, $x>y$, $f(x)<f(y)$.

§7.1

A. Solution: ~~$\|z\|=|x|+2\sqrt{y^2+z^2}$~~

10

① positive definiteness.

$\forall (x,y,z)\in\mathbb{R}^3$, $\|(x,y,z)\|=|x|+2\sqrt{y^2+z^2}\geq 0$ since two parts are nonnegative.

So we know this norm maps from $\mathbb{R}^3$ to $[0,+\infty)$.

$\|(x,y,z)\|=0 \iff |x|+2\sqrt{y^2+z^2}=0$

$\iff |x|=0,\ 2\sqrt{y^2+z^2}=0$

$\iff x=0,\ y^2+z^2=0$

$\iff x=0, y=0, z=0.$

Hence $(x,y,z)=(0,0,0)$

② homogeneous.

$\forall (x,y,z)\in\mathbb{R}^3$, $\alpha\in\mathbb{R}$, $\|\alpha(x,y,z)\|=\|(\alpha x,\alpha y,\alpha z)\|$

$=|\alpha x|+2\sqrt{\alpha^2y^2+\alpha^2z^2}$

$=|\alpha x|+2\alpha\sqrt{y^2+z^2}$

$=|\alpha|(|x|+2\sqrt{y^2+z^2})$

$=|\alpha|\,\|(x,y,z)\|$

③ Triangle inequality:

$\forall (x_1,y_1,z_1)$ & $(x_2,y_2,z_2)\in\mathbb{R}^3$, ~~aim to show~~: $\sqrt{(y_1+y_2)^2+(z_1+z_2)^2} \le \sqrt{y_1^2+z_1^2}+\sqrt{y_2^2+z_2^2}$

$$\|(x_1,y_1,z_1)+(x_2,y_2,z_2)\| = \|(x_1+x_2,\, y_1+y_2,\, z_1+z_2)\|$$
$$= |x_1+x_2| + 2\sqrt{(y_1+y_2)^2+(z_1+z_2)^2} \qquad ①$$

We need a tool: $\sqrt{(y_1+y_2)^2+(z_1+z_2)^2} \le \sqrt{y_1^2+z_1^2}+\sqrt{y_2^2+z_2^2}$

$$(y_1+y_2)^2+(z_1+z_2)^2 \le y_1^2+z_1^2+y_2^2+z_2^2+2\sqrt{(y_1^2+z_1^2)(y_2^2+z_2^2)}$$

$$y_1^2+y_2^2+z_1^2+z_2^2+2y_1y_2+2z_1z_2 \le y_1^2+y_2^2+z_1^2+z_2^2+2\sqrt{(y_1^2+z_1^2)(y_2^2+z_2^2)}$$

$$y_1y_2+z_1z_2 \le \sqrt{(y_1^2+z_1^2)(y_2^2+z_2^2)}$$

$$y_1^2y_2^2+z_1^2z_2^2+2y_1y_2z_1z_2 \le (y_1^2+z_1^2)(y_2^2+z_2^2)$$
$$< y_1^2y_2^2+y_1^2z_2^2+z_1^2y_2^2+z_1^2z_2^2$$

$$2y_1y_2z_1z_2 \le y_1^2z_2^2+z_1^2y_2^2$$

$$(y_1z_2-z_1y_2)^2 \ge 0$$

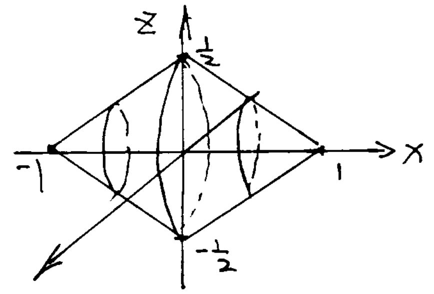
(done). (*)

Take (*) into ①

$$\|(x_1,y_1,z_1)+(x_2,y_2,z_2)\| = |x_1+x_2| + 2\sqrt{(y_1+y_2)^2+(z_1+z_2)^2}$$
$$\le |x_1|+|x_2| + 2\sqrt{(y_1^2+z_1^2)}+\sqrt{(y_2^2+z_2^2)}$$
$$= \|(x_1,y_1,z_1)\| + \|(x_2,y_2,z_2)\|$$

Hence we proved this is a norm on $\mathbb{R}^3$.

The sketch is below:



unit ball is $\{\|(x,y,z)\| = |x|+2\sqrt{y^2+z^2} \le 1\}$

$$\begin{cases} x=0,\ x^2+y^2\le 1 & \text{(circle)} \\ y=0,\ |x|+2|z|\le 1 \\ z=0,\ |x|+2|y|\le \end{cases}$$

§7.2

E. ~~Sd~~ Proof:

Suppose we have 2 norms: $\|\cdot\|$ and $\|\cdot\|'$ on V
with the same unit ball but they are not equal

i.e. $\exists x \in V$, s.t. $\|x\| \neq \|x\|'$

say $\|x\| < \|x\|'$ (WLOG)

let $\min\{\|x\|, \|x\|'\} = \|x\|$

Let's discuss m

① $m < 0$ (impossible, since $\|\ \|$ & $\|\ \|'$ are norms)

② $m = 0$.

$m = \|x\| = 0 \iff x = 0$

$\|x\|' = 0$ since $x = 0$

10

then $\|x\| = \|x\|'$ $\Rightarrow\Leftarrow$

③ $m > 0$

$\|\frac{x}{m}\| = \frac{1}{m}\|x\| = 1$

so $\|\frac{x}{m}\|' < 1$ since $\|\cdot\|$ and $\|\cdot\|'$ have the same unit ball.

but $\|\frac{x}{m}\|' = \frac{1}{m}\|x\|' > \frac{1}{m}\|x\| = 1$

$\Rightarrow\Leftarrow$ b/c they "should" have the same unit ball.

Hence $\|\cdot\|$ & $\|\cdot\|'$ must be equal.