5. Chain Rule

a) (2 marks) Suppose that F is a differentiable function on some open set $U \subset \mathbb{R}^3$, and suppose that the set

$$S = \{(x, y, z) \in U \mid F(x, y, z) = 0\}$$

is a smooth surface. For $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq 0$, state the geometric interpretation of $\nabla F(\mathbf{a})$.

$\nabla F(\vec{a})$ is normal to surface S at $\vec{a}$

or $\nabla F(\vec{a})$ is normal to tangent plane of S at $\vec{a}$.

b) (5 marks) Suppose that $u = F(x + e^y)$ where $F : \mathbb{R} \to \mathbb{R}$ is a function of class C^2. Show that

$$u_{yy} = e^y(u_x + u_{xy})$$

$u_x = F'(x+e^y) \cdot 1$

$u_{xy} = e^y F''(x+e^y)$

$u_y = F'(x+e^y) \cdot e^y$

$u_{yy} = e^y F'(x+e^y) + e^{2y} F''(x+e^y)$

$= e^y u_x + e^y u_{xy}$

$= e^y (u_x + u_{xy})$

c) (6 marks) Prove Chain Rule 1. That is, consider $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $\mathbf{g}(\mathbf{t})$ is differentiable at t=a, $f(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{b}$, and $\mathbf{b} = \mathbf{g}(a)$. Then, the composite function $\phi(t) = f(\mathbf{g}(t))$ is differentiable at $t = a$ and its derivative is given by

$$\phi'(a) = \nabla f(\mathbf{b}) \cdot \mathbf{g}'(a)$$

See Folland 2.26

6. Mean Value Theorem

a) (2 marks) Precisely state the Mean Value Theorem III for functions of n variables.

Let S be a region in $\mathbb{R}^n$ that contains the points $\vec{a}$ & $\vec{b}$ an the line segment L that contains them. Suppose f is a fcn defined on S that is continuous at each point of L and diff at each point of L except, possibly, the end points. Then $\exists c$ on L s.t.:

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$$

b) (5 marks) Suppose that a function is differentiable on

$$S = \{(x, y) \in \mathbb{R}^2 : \ 0 < x < 1 \ and \ 0 < y < 1\}$$

Show that this situation satisfies all of the hypotheses of the Mean Value Thereom III $\forall \mathbf{a}, \mathbf{b} \in S$.

If S is convex, then $\vec{a} + t(\vec{b} - \vec{a})$, $t \in [0,1]$ is in S and since f is diff on S, f is diff on this line segment and as diff $\Rightarrow$ cont cont on this line segment and so satisfies hypotheses of MVT.

S is convex as $0 < a_1 + t(b_1 - a_1) = (1-t)a_1 + tb_1 < 1 - t + t = 1$ as $t \in [0,1]$, $a, b \in S$.

Likewise $0 < (1-t)a_2 + tb_2 < 1 - t + t = 1$

So $\vec{a} + t(\vec{b} - \vec{a}) \in S$ so S convex.

c) (3 marks) State the definition of a subset of $\mathbb{R}^n$ being path (or arc) connected and prove that the set S defined in part b is path connected. (hint: given what you proved in part b, this should be very short)

$S \subset \mathbb{R}^n$ is path connected if $\forall \vec{a}, \vec{b} \in S$, $\exists$ a path connecting them in S. That is $\exists \vec{f} : [0,1] \to \mathbb{R}^n$ continuous s.t. $\vec{f}(0) = \vec{a}$, $\vec{f}(1) = \vec{b}$, $\vec{f}(t) \in S$ $\forall t \in [0,1]$.

$\vec{a} + t(\vec{b} - \vec{a})$ is thus a path so convex $\Rightarrow$ pathconnected

and so as b) showed convex, S is path connected.

d) (3 marks) Prove that if f is differentiable on an open convex set $S \subset \mathbb{R}^n$ such that $|\nabla f(\mathbf{x})| \le M$, $\forall \mathbf{x} \in S$, then $|f(\mathbf{b}) - f(\mathbf{a})| \le M|\mathbf{b} - \mathbf{a}|$, $\forall \mathbf{a}, \mathbf{b} \in S$.

Folland 2.40.

8. Taylor's Theorem

a) (2 marks) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^{k+1} on an open convex set S. For $\mathbf{a} \in S$ and $\mathbf{a} + \mathbf{h} \in S$, state the result of Taylor's Theorem in Several Variables with Lagrange's Remainder using Multi-index notation.

$$f(\vec{a}+\vec{h}) = \sum_{|\alpha| \le k} \frac{\partial^\alpha f(\vec{a})\, \vec{h}^\alpha}{\alpha!} + R_{\vec{a},k}(\vec{h})$$

b) (6 marks) State and prove Taylor's Theorem in One Variable with Lagrange's Remainder. You may use the following lemma that for a k+1 times differentiable fucntion g on [a,b], if g(a)=g(b) and $g^{(j)}(a) = 0$ for $1 \le j \le k$ then there is a point $c \in (a, b)$ such that $g^{(k+1)}(c) = 0$. Note that while this one variable theorem is a special case of the several variable theorem in part a) you should be proving the one variable situation directly.

Folland 2.63

c) (4 marks) Find the Taylor polynomial of order 4 for the function

$$f(x,y) = \frac{cos(xy)}{1+x^2}$$

based at (0,0).
Hint: You may use the following degree k expansions for cos(x) and $1/(1-x)$ respectively about 0: $\sum_{0\le j\le k/2} \frac{(-1)^j x^{2j}}{(2j)!}$ and $\sum_{0\le j\le k} x^j$.

Using Expansions with $z = xy$, $w = x^2$

$$f(x,y) = \left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \text{higher order}\right)\left(1 - w + w^2 - w^3 + w^4 + \text{high order}\right)$$

$$= \left(1 - \frac{(xy)^2}{2} + \frac{(xy)^4}{4!}\right)\left(1 - x^2 + x^4\right) + \text{higher order.}$$

$$= 1 - x^2 + x^4 - \frac{x^2y^2}{2} + \text{higher order}$$

so $P_{(0,0),4}(x,y) = 1 - x^2 + x^4 - \frac{x^2y^2}{2}$

d) (6 marks) Consider $f(x,y) = xy^3$. Compute $\partial^\alpha f(x,y)$ for all multi-indexes $|\alpha| \le 2$ and use these to write the degree 2 (i.e. k=2) Taylor Polynomial about the generic point (x,y). Finally, evaluate the Hessian matrix at the point (1,2).

$|\alpha| = 0 \Rightarrow \alpha = (0,0)$ so $\partial^{(0,0)} f(x,y) = f(x,y) = xy^3$

$|\alpha| = 1 \Rightarrow \alpha = (1,0)$ or $(0,1)$ so $\partial^{(0,1)} f(x,y) = \partial_y f(x,y) = 3xy^2$, $\partial^{(1,0)} f(x,y) = \partial_x f(x,y) = y^3$

$|\alpha| = 2 \Rightarrow \alpha = (1,1)$ or $(2,0)$ or $(0,2)$ so $\partial^{(1,1)} f(x,y) = \partial_y \partial_x f(x,y) = 3y^2$, $\partial^{(2,0)} f(x,y) = \partial_x^2 f(x,y) = 0$

$\partial^{(0,2)} f(x,y) = \partial_y^2 f(x,y) = 6xy$

$$\therefore P_{(x,y),2}(h,k) = \sum_{|\alpha|\le 2} \frac{\partial^\alpha f(x,y)}{\alpha!} = xy^3 + 3xy^2k + y^3h + \frac{6xyk^2}{2} + 3y^2hk$$

$$H(x,y) = \begin{pmatrix} \partial_x^2 f(x,y) & \partial_x \partial_y f(x,y) \\ \partial_y \partial_x f(x,y) & \partial_y^2 f(x,y) \end{pmatrix} = \begin{pmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{pmatrix}$$

$$\therefore H(1,2) = \begin{pmatrix} 0 & 12 \\ 12 & 12 \end{pmatrix}$$

7. Optimization

a) (2 marks) State the Extreme Value Theorem. Suppose $S \subset \mathbb{R}^n$ is compact & $f: S \to \mathbb{R}$ is continuous. Then f has an abs min value & abs max value on S.

b) (5 marks) Let f be a continuous function on an unbounded closed set $S \subset \mathbb{R}^n$. Prove that if $f(\mathbf{x}) \to -\infty$ as $|\mathbf{x}| \to \infty$ ($\mathbf{x} \in S$), then f has an absolute maximum on S.

Modifying 2.83: let $\vec{x}_0 \in S$, define $V = \{\vec{x} \in S \mid f(\vec{x}) \geq f(\vec{x}_0)\}$

- As $[f(\vec{x}_0), \infty)$ is closed, f cont, V is intersection of a closed set $\cap$ S which is closed via Thm 1.13.

- V is bounded as $f(\vec{x}) < f(\vec{x}_0)$ for large $|\vec{x}| > M$ and so such $\vec{x}$ are not in $V \Rightarrow \vec{x} \in V$ has $|\vec{x}| \leq M$

- Thus V is closed & bounded so compact.

By EVT, f has a max $\vec{a}$ on V. But then this is a max on S as $f(\vec{x}) < f(\vec{x}_0) \leq f(\vec{a})$ for $\vec{x} \in S \setminus V$.

c) (6 marks) Find all critical points and classify them for the function $f(x, y) = x(x - 1 + y^2)$.

$0 = \partial_x f(x,y) = 2x - 1 + y^2$

for a CP

$0 = \partial_y f(x,y) = 2yx$

$\Rightarrow y = 0$ or $x = 0$. If $x = 0 \Rightarrow y = \pm 1$

If $y = 0 \Rightarrow x = \frac{1}{2}$.

so CPs are $(\frac{1}{2}, 0), (0,1), (0,-1)$.

$\partial_x^2 f(x,y) = 2$

$\partial_y^2 f(x,y) = 2x$

$\partial_x \partial_y f(x,y) = 2y$

$\partial_y \partial_x f(x,y) = 2y$

so $H(x,y) = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$

so $D(0,1) = |H(0, \pm 1)| = -4 < 0 \Rightarrow$ saddle for $(0,1), (0,-1)$

$D(\frac{1}{2}, 0) = 2$ and $2 = \partial_x^2 f(\frac{1}{2}, 0) = 2 > 0$

min 2.82

$\Rightarrow$ local minimum.

d) (6 marks) Consider the line $L \subset \mathbb{R}^3$ of intersection between two planes defined by the equations $x + y = 1$ and $x - z = 0$ respectively. Using the method of Lagrange Multipliers with two constraints (and not some other method), compute the minimum distance from the origin to L. Hint: It is easier to minimize the square of the distance to the origin.

Let $G_1(x,y,z) = x+y-1$

$G_2(x,y,z) = x-z = 0$

$f(x,y,z) = x^2+y^2+z^2 = d^2$

Hence, $\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2$ at extrema.

so ① $2x = \lambda_1 + \lambda_2$

② $2y = \lambda_1$

③ $2z = -\lambda_2$

④ $x+y = 1$

⑤ $x - z = 0$

Solving, $x = z \Rightarrow \lambda_1 + \lambda_2 = -\lambda_2 \Rightarrow \lambda_2 = -\frac{1}{2}\lambda_1$

So $2x = \frac{1}{2}\lambda_1 = y = 1-x \Rightarrow 3x = 1$ so $x = \frac{1}{3} = z$

$\Rightarrow y = \frac{2}{3}$

$\therefore (x,y,z) = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$f\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right) = \frac{1}{9}(1+4+1) = \frac{6}{9} = \frac{2}{3}$