§2 - Basis

1 Motivation

We have already seen some topologies, but there are many more out there! The problem is that it is often difficult (or impossible!) to list out *all* of the open sets in a topology. Here we will use the notion of a **basis**, which will be a tool for describing many different topologies. Essentially, a basis is just a collection of sets that could form a topology if we added in all of the unions of sets in the basis. We will explain this in more detail below.

2 Definition and Examples

Definition. *Let X be a set. A subcollection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a* ***basis*** *(for some topology on X) if:*

- $\forall x \in X, \exists B \in \mathcal{B}$ *such that* $x \in B$. *(We say that "$\mathcal{B}$* ***covers*** *X".)*
- $\forall B_1, B_2 \in \mathcal{B}$ *if* $x \in B_1 \cap B_2$ *then there is a* $B \in \mathcal{B}$, *containing* x, *such that* $B \subseteq B_1 \cap B_2$.

While the second condition looks a bit strange, it is saying *in spirit* that a basis is closed under finite intersections. The examples of bases we give will often be closed under finite intersections.

Example 1. *The family $\mathcal{B} := \{ (a, b) \subseteq \mathbb{R} : a \leq b \}$ of open intervals in $\mathbb{R}$ is a basis (but is not a topology).*

Proof. [i] Any point $x \in \mathbb{R}$ is contained in the interval $(x - 100, x + 100) \in \mathcal{B}$.

[ii] We show the stronger property that the intersection of any two elements is itself an element of $\mathcal{B}$. You probably believe that, but let us prove it.

Let $A = (a_l, a_r)$ and $B = (b_l, b_r)$. If they are disjoint, then their intersection is $\emptyset$, so the "basis property" is vacuously true. If they are not disjoint, then $(a_l, a_r) \cap (b_l, b_r) = (\max\{a_l, b_l\}, \min\{a_r, b_r\})$ which is in $\mathcal{B}$.

[iii] You can see that it is not a topology as it is not closed under unions. For example $(2, \pi) \cup (7, 20)$ is not in $\mathcal{B}$. □

Example 2. *For any set X, the family $\mathcal{B} := \{ \{x\} : x \in X \}$, the family of singletons, is a basis on X.*

Example 3. *The family $\mathcal{B} := \{ [a, b) : a \leq b \}$ of half-open intervals, closed on the left, is a basis on $\mathbb{R}$. (The proof of this is identical to the proof of example 1.)*

Example 4. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Then*

$$\mathcal{B} := \mathcal{T} \times \mathcal{U} := \{ A \times B : A \in \mathcal{T}, B \in \mathcal{U} \}$$

*is a basis (but usually not a topology!) on $X \times Y$. (This is an important basis called **the usual basis for the product topology on X and Y**, that we will use later.)*

Proof. [i] Let $(x, y) \in X \times Y$. Then $(x, y) \in X \times Y$ which is in $\mathcal{B}$ because $X \in \mathcal{T}$ and $Y \in \mathcal{U}$.

[ii] It is pretty easy to see that for $A \times B, U \times V \in \mathcal{B}$ that

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V) \in \mathcal{B}$$

since both $\mathcal{T}$ and $\mathcal{U}$ are closed under intersections. $\square$

Rectangle Exercise: Give an example that shows that this is only a basis and not a full topology.

Example 5. *The family of open disks in the plane is a basis, and gives an example where the intersection of two basic open sets is not itself a basic open set, but the still contains many basic open sets. For example, the intersection of $B_2(0,0)$ and $B_2(3,0)$ is a sideways-eye thing.*

3 So, how do we get a topology from a basis?

It turns out that all you need to do to get a topology out of a basis is to throw in all unions of basic open sets!

Definition. *Let $\mathcal{B}$ be a basis on a set X. Define $\mathcal{T}_{\mathcal{B}} := \{ \bigcup \mathcal{C} : \mathcal{C} \subseteq \mathcal{B} \} \cup \{ \emptyset \}$, and this **the topology generated by $\mathcal{B}$**.*

For example, taking the basis $\mathcal{B}$ from example 3 (the half-open intervals basis) we see that $[0,1) \cup [100, 700)$ is open in $\mathcal{T}_{\mathcal{B}}$, so is $[6, 2\pi)$, and so is $\bigcup_{n \in \mathbb{N}} [\frac{1}{n}, 1) = (0,1)$.

Before we justify why we call this a topology, let us recall a tricky bit of notation. If $\mathcal{C}$ is a collection of subsets of X, then

$$\bigcup \mathcal{C} := \bigcup_{C \in \mathcal{C}} C.$$

For example, if $X = \mathbb{N}$ and $\mathcal{C} = \{ \{1,2,3\}, \{3,100\}, \{9,3\} \}$ then $\bigcup \mathcal{C} = \{1,2,3\} \cup \{3,100\} \cup \{9,3\} = \{1,2,3,9,100\}$. Another example, if $X = \mathbb{R}$ and $\mathcal{C} = \{ \{x, x-1\} : x > 0 \}$, then $\bigcup \mathcal{C} = (-1, \infty)$.

Proof that this is a topology. [i] By definition, $\emptyset \in \mathcal{T}_{\mathcal{B}}$, and since, by definition of a basis, $\mathcal{B}$ covers X we get $\bigcup \mathcal{B} = X$.

[ii] It is fairly clear that $\mathcal{T}_{\mathcal{B}}$ is closed under arbitrary unions, but there is a bit of notation. Let $\{ \bigcup \mathcal{A}_\alpha \in \mathcal{T}_{\mathcal{B}} : \alpha \in I \}$ be a collection of sets from $\mathcal{T}_{\mathcal{B}}$. Then

$$\bigcup_{\alpha \in I} (\bigcup \mathcal{A}_\alpha) = \bigcup (\bigcup_{\alpha \in I} \mathcal{A}_\alpha) \in \mathcal{T}_{\mathcal{B}}$$

since $\bigcup_{\alpha \in I} \mathcal{A}_\alpha \subseteq \mathcal{B}$.

Notational Nightmare Exercise: Go through this previous proof for the special case where you are trying to prove that the union of three elements of $\mathcal{T}_\mathcal{B}$ is again in $\mathcal{T}_\mathcal{B}$. Convince yourself that this is easy.

Intersection Property Exercise: Without looking at the proof below, convince yourself that there is actually something to show. That is, figure out why the proof won't just be "$\mathcal{B}$ is directed, so done!".

[iii] Let $\bigcup \mathcal{A}, \bigcup \mathcal{C} \in \mathcal{T}_\mathcal{B}$ and note that

$$(\bigcup \mathcal{A}) \cap (\bigcup \mathcal{C}) = \bigcup \{ A \cap C : A \in \mathcal{A}, C \in \mathcal{C} \}$$

This union will be in $\mathcal{T}_\mathcal{B}$ so long as each $A \cap C$ is. Let's show that!

Let $A \in \mathcal{A}$ and $C \in \mathcal{C}$. If they are disjoint, then certainly their intersection is in $\mathcal{T}_\mathcal{B}$. If not, then for each $x \in A \cap C$ there is a $B_x \in \mathcal{B}$ containing x such that $B_x \subseteq A \cap C$. This immediately gives

$$A \cap C \subseteq \bigcup_{x \in A \cap C} B_x \subseteq A \cap C$$

So

$$A \cap C = \bigcup_{x \in A \cap C} B_x \in \mathcal{T}_\mathcal{B}$$

$\square$

Note that the topology we generate contains $\mathcal{B}$ inside of it. That is, basic open sets remain open. This is because we are allowed to take the "union of just one set from $\mathcal{B}$" and this will be in $\mathcal{T}_\mathcal{B}$. This shows the (obvious) fact that any topology is a basis for itself. This also justifies why we call sets in the basis "basic *open* sets".

4 "What have you done for me lately?"

So now we know that given a basis, we can get a topology by taking unions of basic open sets. Let's get our hands dirty and look at the previous bases we described and look at their respective topologies.

Example 6. *The topology generated by the open intervals in $\mathbb{R}$ is precisely the usual topology on $\mathbb{R}$. Look at our original definition of the usual topology on $\mathbb{R}$ and you can see that it is exactly the union of open intervals.*

We will see later that the family of open intervals *with rational numbers as endpoints* is also a basis for this topology. This shows that $\mathbb{R}$ has a countable basis. (Who cares? Later in the course we will see that this property -frustratingly called being "**second**

countable"- is one way to say that a space is "small", and "small spaces are easier to understand than big spaces.")

Example 7. *For any set X, the family $\mathcal{B} := \{ \{x\} : x \in X \}$, the family of singletons generates the discrete topology on X.*

Discrete Basis Exercise: Present a different basis on X that also generates the discrete topology. Can you make something that looks *very* different from the one presented above, or is it basically (no pun intended) the same.

Example 8. *The family $\mathcal{B} := \{ [a, b) : a \leq b \}$ of half-open intervals, closed on the left, is a basis on $\mathbb{R}$, and generates a topology $\mathcal{S}$ on $\mathbb{R}$ called the **lower-limit topology**, and the space $(\mathbb{R}, \mathcal{S})$ is called the **Sorgenfrey Line**. This is a very weird topological space as we will see!*

Claim: The Sorgenfrey line is a weird space.

1. In the Sorgenfrey line, any interval $(0, b)$ is open. Here take $(0, b) = \bigcup_{n \in \mathbb{N}} [\frac{1}{n}, b)$ which is in $\mathcal{S}$. More generally, (but notationally ugly) every interval (a, b) is open in $\mathcal{S}$. This tells us that the Sorgenfrey Line refines the Euclidean Topology on $\mathbb{R}$.

2. In the Sorgenfrey line, no interval $(x, 0]$ is open. Any basic open set that contains 0 must also contain some positive numbers, so $(x, 0]$ is not the union of basic open sets.

3. The previous fact tells us the the Sorgenfrey line is different from the Discrete topology on $\mathbb{R}$, (because in the discrete topology, $(-1, 0]$ *is* open).

4. The Sorgenfrey line can be written as the disjoint union of non-empty open sets; namely $\mathbb{R} = (-\infty, 0) \cup [0, +\infty)$. (Later on we will see that this tells us that the Sorgenfrey line is **not connected**.)

Sorgenfrey Basis Exercise: Check that the basis we presented for the Sorgenfrey Line is uncountable. In analogy to the case with the reals, it seems plausible that

$$\{ [p, q) : p, q \in \mathbb{Q} \}$$

could be a basis for this topology, (which would show that the Sorgenfrey Line is second countable). Is it?

Rectangles do What?! Exercise The basis for the product topology on

$$(\mathbb{R}, \mathcal{T}_{\text{usual}}) \times (\mathbb{R}, \mathcal{T}_{\text{usual}})$$

generates a nice topology on the plane. Which topology does it generate?

5 When do two bases generate the same topology?

There are many bases that generate the same topology (as the previous exercise shows!). There turns out to be a simple critereon for checking that two different bases generate the same topology. (You may have guessed this already from the exercises.)

Proposition 9. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ both be bases on a set X. The following are equivalent:*

1. $\mathcal{T}_{\mathcal{B}_1} \subseteq \mathcal{T}_{\mathcal{B}_2}$;
2. *For every $B_1 \in \mathcal{B}_1$ and every $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.*

Proof. This proof amounts to just unwinding definitions. Spend an hour and prove it on your own. Draw a picture first, then use that to motivate the technically precise proof. □

Corollary 10. *A basis $\mathcal{B}$ generates $\mathcal{T}$ if and only if $\mathcal{B} \subseteq \mathcal{T}$ and for every open set $U \in \mathcal{T}$ and for every $x \in U$ there is a basic open $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

This is a very useful Corollary! It shows:

1. The family of open disks in the plane generates the same topology as the family of open rectangles in the the plane. Namely they both generate the usual topology of the plane!
2. The intervals with rational endpoints generate the usual topology on $\mathbb{R}$.

6 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Rectangle : Give an example that shows that the usual basis for the product topology is only a basis and not a full topology.

Not. Nightmare : Go through the proof that the topology generated by a basis is closed under unions for the special case where you are trying to prove that the union of three elements of $\mathcal{T}_{\mathcal{B}}$ is again in $\mathcal{T}_{\mathcal{B}}$. Convince yourself that this is easy.

$\bigcap$ Property : Without looking at the proof that the topology generated by a basis is closed under intersections, convince yourself that there is actually something to show. That is, figure out why the proof won't just be "$\mathcal{B}$ is directed, so done!".

Sorgenfrey Basis : Check that the basis we presented for the Sorgenfrey Line is uncountable. In analogy to the case with the reals, it seems plausible that $\{\, [p, q) : p, q \in \mathbb{Q} \,\}$ could be a basis for this topology. Is it?

Rectangles?! : The basis for the product topology on $(\mathbb{R}, \mathcal{T}_{\text{usual}}) \times (\mathbb{R}, \mathcal{T}_{\text{usual}})$ generates a nice topology on the plane. Which topology does it generate?