Assignment 1 - MAT 327 - Summer 2014

Due May 26th, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Let $(X, \mathcal{T})$ be a topological space, and let $f : X \longrightarrow Y$ be an injection.
Is $\{ f[A] : A \in \mathcal{T} \}$ a topology on the range of f?

[**C.2**] Let $(X, \mathcal{T})$ and (X, Γ) be topological spaces. Prove or disprove the following statements:

i. $(X, \mathcal{T} \cap \Gamma)$ is a topological space.

ii. $(X, \mathcal{T} \cup \Gamma)$ is a topological space.

[**C.3**] Let X be an infinite set. Prove that $\mathcal{T} := \{ A \subseteq X : A = \emptyset \text{ or } X \setminus A \text{ is finite} \}$ is a topology on X. This is called the "finite complement topology" or the "co-finite topology". If X is a finite space, what familiar topological space is X with the co-finite topology? (Support and prove your assertions.)

[**C.4**] Prove or disprove: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, then the image of any open set is open. (Here use the usual first-year calculus definition of continuous, real-valued function and open subset of $\mathbb{R}$.)

[**C.5**] Let $X = \{ 0, 1, 2, 3, 4 \}$. What is the smallest size of a basis that generates the discrete topology on X? (Note that this requires you to (1) prove that your lower bound is indeed a bound, and (2) give an example that shows that the lower bound is witnessed. What I mean is, if you think that the least size of a basis here has 47 elements you need to give me a basis with 47 elements and prove to me that 46 is not enough.) What can you say if $X = \{ 0, 1, 2, \ldots, n-1 \}$?

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] Let's go a step further than question C.1: Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, consider the family of sets in the range given by $\Gamma := \{ f[A] : A \text{ is open in } \mathbb{R}_{\text{usual}} \}$.

- Give an example where f is not an injection, but Γ is a topology on the range of f.
- Give an example where f is not an injection, and Γ is not a topology on the range of f.

[**A.2**] Let V be an open subset of $\mathbb{R}$, with the usual topology. Show that there is a countable family of open intervals $\{ I_n : n \in \mathbb{N} \}$, each with rational end-points such that $\bigcup_{n\in\mathbb{N}} I_n = V$. Can we instead do this by instead having each I_n be a closed interval? (If the answer is "Yes", then this shows that "every open subset of $\mathbb{R}$ is an F_σ set".)

For the next two questions you will need the following idea, which is related to the idea of a basis:

Definition 1. *Let X be a set. A subcollection $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a* ***subbasis*** *(for some topology on X) if the collection*

$$\mathcal{B} = \{ S_1 \cap S_2 \cap \ldots \cap S_N : N \in \mathbb{N}, S_1, S_2, \ldots, S_N \in \mathcal{S} \}$$

the family of all finite intersections of sets from $\mathcal{S}$, is a basis on X.

This tells us that given a subbasis, we can get a basis (by taking finite intersections), then generate a topology (by taking unions); this is called the **topology generated by** $\mathcal{S}$.

[**A.3**] Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{S}$ be a subbasis on X. Along the lines of the proposition we saw in lecture, state and prove a proposition that tells us when the topology generated by $\mathcal{S}$ is $\mathcal{T}$. Use this to prove that

$$\mathcal{S} := \{ (-\infty, b) : b \in \mathbb{R} \} \cup \{ (a, +\infty) : a \in \mathbb{R} \}$$

generates the usual topology on $\mathbb{R}$ (don't forget to check that this is a subbasis!). On your own, write down a "natural subbasis" for the Sorgenfrey line.

[**A.4**] *Let's go a bit further than C.5.* Let $X = \{ 0, 1, 2, 3, 4 \}$. What is the smallest size of a *subbasis* that generates the discrete topology on X? Write a sentence or two explaining if it is easy to generalize this to $X = \{ 0, 1, 2, \ldots, n-1 \}$.

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] Describe a topology $\mathcal{T}$ on $\mathbb{Z}$ such that:

- The set of all square numbers, $\mathbb{S}$ is open.
- For each $x \in \mathbb{Z}$ the set $\{x\}$ is not open.
- $\forall x, y \in \mathbb{Z}$ distinct, there is an open $U \ni x$ and an open $V \ni y$ such that $U \cap V = \emptyset$.

[**NI.2**] Write a computer program (in whatever language you like) or describe (in detail) an algorithm that counts the number of topologies on the set $\{0, 1, 2, 3, 4\}$. Is there any sense in which this can be done efficiently? (Please note that merely describing the brute force method and providing one or two cosmetic improvements to it will not earn many marks. I'm looking to see that you've thought about and grappled with the problem.)

[**NI.3**] Here's a game that two players could play: Player 1 chooses an uncountable subset $K_1 \subseteq \mathbb{R}$, then Player 2 chooses an uncountable $K_2 \subseteq K_1$. They continue alternating until they have an infinite chain $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq ...$, and we'll say that Player 2 wins iff $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$. Show that Player 2 has a strategy so that she can win, no matter what moves Player 1 makes. (There is a very tempting strategy that doesn't work, although it seems like it should work. At the very least find this strategy, explain why it is tempting, and show how it could fail.)