<u>Thm</u>: The initial value problem

$$y' = f(t,y) \quad y(t_0) = y_0$$

admits a unique solution $y(t)$, for t in some interval around t_0, provided f, $\frac{\partial f}{\partial y}$ are continuous near (t_0, y_0)

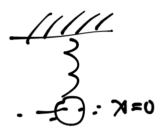
(Counter) example

Consider Newton's equations $m\frac{d^2x}{dt^2} = F(x)$ in the case of "potential forces".

$$F(x) = -\frac{dV}{dx} \text{ for some potential } V(x).$$

E.g. $V = mgx \rightsquigarrow$ free fall

$V = \frac{1}{2}\alpha x^2 \longrightarrow$ harmonic oscillator



Then one has conservation of energy

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E \qquad (*)$$

constant along solution curves. Check:

$$\frac{dE}{dt} = \frac{m}{2}\cdot 2\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{dV}{dx}\cdot\frac{dx}{dt}$$

$$= \frac{dx}{dt}\underbrace{\left(m\frac{d^2x}{dt^2} + \frac{dV}{dx}\right)}_{=0 \text{ by Newton}}$$

<u>Given</u> E, $(*)$ is a first order ODE.

Special case: $V(x) = mgx$, $E = 0$.

Then $x(t) = -\frac{1}{2}gt^2$ solves the eqn $(*)$

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - mgx = 0;\ x(0) = 0$$

However: $x(t) = 0$ is also a solution!!

Why doesn't this contradict existence & uniqueness?

Why is $x(t)=0$ "unphysical"?

- $x(t)$ solves $(*)$, but does not solve Newton's equation $m\frac{d^2x}{dt^2}=-mg$.
- Writing $(*)$ in standard form

$$\frac{dx}{dt}=\sqrt{-2gx}=f(t,x) \qquad (x\leq 0)$$

$$\frac{df}{dx}=-g\frac{1}{\sqrt{-2gx}} \text{ is not continuous.}$$

Remark:

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2=E-V(x)$$

$$\frac{dx}{dt}=\sqrt{\frac{2}{m}(E-V(x))}$$ is a separable equation, hence can be solved. (in principle)

Modeling:

Radioactive decay: Suppose some radioactive material given, initial amount is $Q(0)$. (e.g. in grams). Gradually the atoms "decay" by emitting particles and changing into other atoms. This happens with a certain probability.

Hence, for a sufficiently large sample, $\frac{dQ}{dt}=-rQ(t)$ r=rate of decay

Solution of the ODE is $Q(t)=Q(0)e^{-rt}$

The half-time T of the material is defined by $Q(T)=\frac{1}{2}Q(0)$

Thus $Q(0)e^{-rT}=\frac{1}{2}Q(0)$

$$e^{-rT}=\frac{1}{2}$$

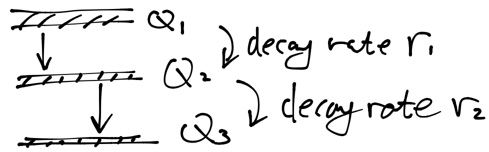
$$e^{rT}=2 \quad \Rightarrow rT=\ln 2 \Rightarrow T=\frac{\ln(2)}{r}$$

Example: ^{210}Po Polonium 210

decays into ^{206}Pb Lead 206

half-life: $T = 138$ days

Two-level systems



$$\frac{dQ_1}{dt} = -r_1 Q_1(t)$$

$$\frac{dQ_2}{dt} = -r_2 Q_2(t) + r_1 Q_1(t)$$

Solution of first ODE is $Q_1(t) = Q_1(0)e^{-r_1 t}$

Hence, 2nd equation becomes $\frac{dQ_2}{dt} = -r_2 Q_2(t) + r_1 Q_1(0)e^{-r_1 t}$

$$\frac{dQ_2}{dt} + r_2 Q_2(t) = r_1 Q_1(0)e^{-r_1 t}$$

This is a linear ODE.

Integrating factor: $\mu(t) = \exp(\int \overset{\text{const.}}{r_2}\, dt) = \exp(r_2 t)$

$$\underbrace{e^{r_2 t}\frac{dQ_2}{dt} + e^{r_2 t} r_2 Q_2(t)}_{\frac{d}{dt}(e^{r_2 t} Q_2)} = r_1 Q_1(0) e^{(r_2 - r_1)t}$$

$$e^{r_2 t} Q_2(t) - Q_2(0) = \frac{r_1 Q_1(0)}{r_2 - r_1}\left(e^{(r_2 - r_1)t} - 1\right)$$

$$Q_2(t) = e^{-r_2 t} Q_2(0) + \frac{r_1 Q_1(0)}{r_2 - r_1}\left(e^{-r_1 t} - e^{-r_2 t}\right)$$

(Here we're assuming $r_2 \neq r_1$. Homework, find solution $r_2 = r_1$)