PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 4
DUE FRIDAY, MARCH 24, 4PM.

Warm-up problems. These are completely optional.

(1) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective, but not surjective. Give an example of a function $g : \mathbb{N} \to \mathbb{N}$ which is surjective, but not injective.

(2) Let $a, b \in \mathbb{R}$ with $a \neq 0$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is a bijection. Compute a formula for f^{-1}.

Problems to be handed in. Solve four of the following five problems. One of the four must be Problem (1).

(1) Let $f : A \to B$ and let $g : B \to C$ be functions, and let $h = g \circ f$. Determine which of the following statements are true. Give proofs of the true statements and counterexamples for the false statements.
   (a) If h is injective, then f is injective.
   (b) If h is injective, then g is injective.
   (c) If h is surjective, then f is surjective. 4.34
   (d) If h is surjective, then g is surjective.

(2) Let $f : A \to B$ be any function. A function $g : B \to A$ is called a *left-inverse for f* if it satisfies
$$(g \circ f)(a) = a \text{ for all } a \in A.$$
It is called a *right-inverse for f* if it satisfies
$$(f \circ g)(b) = b \text{ for all } b \in B.$$
It is called *two-sided inverse for f* if it satisfies both these conditions.
   (a) Prove that if f has a two-sided inverse, then f is a bijection.
   (b) Give an example of a function f which has a left-inverse, but is not a bijection.
   (c) Give an example of a function f which has a right-inverse, but is not a bijection.

(3) Recall that $[n] = \{1, 2, \ldots, n\}$. Let A denote set of subsets of $[n]$ with an even number of elements, and let B denote the set of subsets of $[n]$ with an odd number of elements. Prove that $|A| = |B|$ by constructing an explicit bijection from A to B. 4.21

(4) Construct explicit bijections $f : (0, 1) \to [0, 1)$ and $g : (0, 1) \to [0, 1]$. 4.51

(5) Let L be the set of all sentences of the english language. Prove that L is countable. (For the purpose of this exercise, a sentence of the english language is any finite sequence of characters chosen from the set of characters visible on your computer's keyboard.)