Lecture 8 (continue §4.4)

The Cantor's intersection Thm

$A_1 \supset A_2 \supset \cdots$

$A_i \neq \emptyset$

A_i is compact $\forall i \in \mathbb{N} \Rightarrow \bigcap_{i \geq 1} A_i \neq \emptyset$

The Cantor Set

$0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

$S_0 = [0, 1]$

$S_1 = [0, 1/3] \cup [2/3, 1]$

$S_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

$S_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup \cdots \cup [\frac{26}{27}, 1]$

$\cdots$

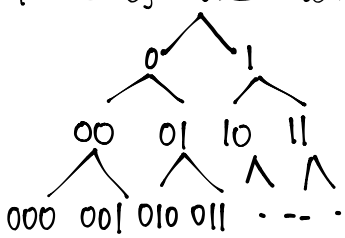
S_{i+1} from S_i by taking out the middle third from each of the intervals of S_i.

$S_i \neq \emptyset$

S_i are compact (closed & bounded)

$C = \bigcap S_i \neq \emptyset$

All of the endpoints of all of the intervals are in C.



Each path is a nested seq of interval

$\bigcap_{i \geq 1} I_i \neq \emptyset$

For each path $\exists$ a nonempty intersection

(1) C is compact

(2) $\overline{C} = C$

(3) $\text{int}\, C = \emptyset$. suppose $(a, b) \subset C \Rightarrow (a, b) \subset S_n \; \forall n$

$b - a \leq 3^{-n}, \forall n$ not possible

Def: A set whose closure has no interior is nowhere dense

Ex. $\boxed{\begin{array}{ll} \mathbb{Q} & \overline{\mathbb{Q}} = \mathbb{R} \\ \text{int}\, \mathbb{R} = \mathbb{R} & \end{array}}$

Note: $\text{int}\, \overline{\mathbb{Q}} = \mathbb{R}$

$\text{int}\, \mathbb{Q} = \emptyset$

Def: A point $x \in A$ is called isolated if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \{x\}$

$M = \{\frac{1}{n} : n \in \mathbb{N}\}$

Every point of M is an isolated pt.

$S = M \cup \{0\}$ is not b/c $\{0\}$ is not isolated b/c $\lim_{n \to \infty} \frac{1}{n} = 0$

Claim: C has no isolated pts (C is a perfect set)

Proof: Let $x \in C$.

<u>Case 1:</u> x is not the right end-point of any interval.

For each n if x is in one of the interval x_n

$$|x - x_n| \leq \frac{1}{3^n} \longrightarrow 0$$

$x_n \to x$

<u>Case 2</u>: If x is the right-end point $\Rightarrow$ let x_n be the left-end point.

Two sets A & B have the same cardinality if $\exists$ a bijection $f: A \to B$

$|A| = |B|$

$|A| \geq |B|$ if $\exists$ injective func. from $B \to A$

$|A| > |B|$ if $\exists$ an injective map from B to A, but no bijection map.

if $|S| = |\mathbb{N}| \Longrightarrow S$ is a countable

read as aleph-null written as $\aleph_0$

$|\mathbb{Q}| = |\mathbb{N}|$

$|\mathbb{R}| = C$ $\quad$ $|[0,1]| = C$

$C > \aleph_0$ $\quad$ (cardinality of continuum)

<u>Claim:</u> $|C| = |[0,1]|$

The Cantor – Bernstein–Schroeder thm

If $\exists$ injective functions:

$f: A \to B$

$g: B \to A$ $\Rightarrow |A| = |B|$

Consider the numbers in $[0,1]$

$x = (x_0 . x_1 x_2 x_3 \cdots)_{base3} = \sum 3^{-k} x_k$

$\sum_{k \geq 0} \frac{x_k}{3^k}$, $\quad x_i = \{0,1,2\}$

$\frac{1}{3} = (.1) = (0.022222\cdots)$

S_1 consists of all numbers of $[0,1]$ for which the first digit after a pt, is either 0 or 2.

S_2: it will consist of the numbers s.t. their dec. expansion will have the first two digits after "." to either 0 or 2

S_i: it will consist of all number of $[0,1]$ s.t the first i terms are all either 0 or 2

C: consists of all numbers of [0, 1] that have a ternary expansion using only 0's and 2's.

Goal is to construct an invective map (one-to-one map) from [0, 1] -> C.

Consider a binary expansion of number in [0, 1] y=$(0.y_1 y_2 y_3 \cdots)_{\text{base}2} \in [0,1]$

$\sum_{k \geq 0} \frac{y_k}{2^k}$ $\quad y_k = \{0,1\}$

Note that some points have two expansions

$\frac{1}{2} = 0.1 = 0.011111\cdots$

Pick the expansion that ends at 0's in this case.

$x = (0.x_1 x_2 x_3 \cdots x_i \cdots)_{\text{base}2}$

If $x_i = 0 \Rightarrow \tilde{x}_i = 0$

$\tilde{x}_i = 0$

$x_i = 1 \Rightarrow \tilde{x}_i = 2$

one-to-one map