$18/20$ Peter Cracks TUT0101

MAT224 Problem Set 2 Rui Qiu #999292509

#1.

Solution:

(a). $\begin{pmatrix} 1&3&2&0&-1 \\ 2&6&4&6&4 \\ 1&3&2&2&1 \end{pmatrix}\begin{pmatrix} x\\y\\z\\m\\n \end{pmatrix} = \begin{pmatrix} 0\\0\\0 \end{pmatrix}$

$\therefore \begin{pmatrix} x\\y\\z\\m\\n \end{pmatrix}$ is in $\ker T$.

$$\therefore \begin{pmatrix} 1&3&2&0&-1 \\ 2&6&4&6&4 \\ 1&3&2&2&1 \end{pmatrix} \to \begin{pmatrix} 1&3&2&0&-1 \\ 0&0&0&6&6 \\ 0&0&0&2&2 \end{pmatrix} \to \begin{pmatrix} 1&3&2&0&-1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \end{pmatrix} \to \begin{pmatrix} 1&3&2&1&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \end{pmatrix}$$

$$\therefore \left.\begin{matrix} x=-3y-2z-m \\ m=-n \end{matrix}\right\} \Rightarrow \begin{cases} x=-3y-2z-m \\ y=y \\ z=z \\ m=m \\ n=-m \end{cases}$$

$$\therefore \begin{pmatrix} x\\y\\z\\m\\n \end{pmatrix} = y\begin{pmatrix} -3\\1\\0\\0\\0 \end{pmatrix} + z\begin{pmatrix} -2\\0\\1\\0\\0 \end{pmatrix} + m\begin{pmatrix} -1\\0\\0\\1\\-1 \end{pmatrix}$$

Therefore the ~~basis $\{(-3,1,0,0,0),(-2,0,1,0,0),(-1,0,0,1,-1)\}$ is a basis for Ker~~ T a basis of $\ker T$ ~~is~~ $\{(-2,-2,-2,-2,-3)^T,(-1,-1,-2,-2,-2)^T,(0,-1,-1,-2,-1)^T\}$

(b). To find a basis for the image of T, note that the columns of T span its image.

So $\begin{pmatrix} 1&3&2&0&-1 \\ 2&6&4&6&4 \\ 1&3&2&2&1 \end{pmatrix}\begin{pmatrix} x\\y\\z\\m\\n \end{pmatrix} = \begin{pmatrix} x+3y+2z\quad -n \\ 2x+6y+4z+6m+4n \\ x+3y+2z+2m+n \end{pmatrix} = x\begin{pmatrix} 1\\2\\1 \end{pmatrix} + y\begin{pmatrix} 3\\6\\3 \end{pmatrix} + z\begin{pmatrix} 2\\4\\2 \end{pmatrix} + m\begin{pmatrix} 0\\6\\2 \end{pmatrix} + n\begin{pmatrix} -1\\4\\1 \end{pmatrix}$

we get a linear combination of T

It is ~~a~~ not a basis because it is linearly dependent.

Then reduce some tectors we get $\{(1,2,1),(0,6,2)\}$ as a basis for ~~image~~ $\operatorname{Im} T$

#2.

(a). Solution:

$T\begin{bmatrix} 1&0\\1&0 \end{bmatrix} = 1+1\cdot x+(0-1)x^2+1\cdot x^3 = \cancel{[1,1,-1,1]_\beta}\ 2x+1(x-x^2)+(-1)(x-x^3)+(-1)(x-1)$

$= [2,1,-1,-1]_\beta$

$T\begin{bmatrix} 0&1\\0&1 \end{bmatrix} = -1+(-1)x+1\cdot x^2+(-1)x^3 = -2x+(-1)(x-x^2)+1\cdot(x-x^3)+1(x-1) = [-2,-1,1,1]_\beta$

$T\begin{bmatrix} 1&0\\0&1 \end{bmatrix} = 1+0x+0x^2+(-1)x^3 = 0x+0(x-x^2)+1\cdot(x-x^3)+(-1)(x-1) = [0,0,1,-1]_\beta$

$T\begin{bmatrix} 0&0\\1&1 \end{bmatrix} = 0+(-1)x+(-1)x^2+0x^3 = -2x+1(x-x^2)+0(x-x^3)+0(x-1) = [-2,1,0,0]_\beta$

$$[T]_{\beta\alpha} = \begin{bmatrix} 2 & -2 & 0 & -2 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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(2). Solution

Say $\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x = y+m \\ z = m \\ m = 0 \end{matrix} \Rightarrow \operatorname{Ker} T = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$\cancel{\begin{bmatrix} x & y \\ z & m \end{bmatrix} = x\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + m\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}$

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$\notin M_{2\times 2}$

Then the basis of KerT could be $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$

(3). $\begin{pmatrix} 2 & -2 & 0 & -2 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ m \end{pmatrix} = \begin{pmatrix} 2x - 2y - 2m \\ x - y + m \\ -x + y + z \\ -x + y - z \end{pmatrix} = x\begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} + y\begin{pmatrix} -2 \\ -1 \\ 1 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + m\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Since $(-2,-1,1,1)$ and $(2,1,-1,-1)$ are linearly dependent.
So ~~the~~ a basis for $\operatorname{Im} T$ is $\{(-2,-1,1,1),(0,0,1,-1),(-2,1,0,0)\}$

(4). Solution:

Nullity of T $= \cancel{2}\ 1 = \dim \operatorname{Ker} T$
Rank of $T = 3 = $ dim of rowspace $=$ dim of column space
By Corollary 2.4.3, T is injective if and only if $\dim(\operatorname{Im} T) = \dim(M_{2\times 2})$
For this problem $\dim(\operatorname{Im} T) = 3 \neq 4 = \dim(M_{2\times 2})$
By Corollary 2.4.10, T is surjective if and only if ~~$\dim(M_{2\times 2}) \geq \dim(P_3(\mathbb{R}))$~~ it is injective when $\dim(M_{2\times 2}) = \dim(P_3(\mathbb{R}))$
For this problem $\dim(M_{2\times 2}) = 4 \geq 4 = \dim(P_3(\mathbb{R}))$, but it's not injective.
Therefore T is ~~surjective but not~~ neither injective nor surjective.

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Peter Crooks
TUT0101

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#3. §2.3 #12)

(1). Proof: Suppose $A, B \in M_{n\times n}(\mathbb{R})$, $c \in \mathbb{R}$

Then $cA+B \in M_{n\times n}(\mathbb{R})$.

So
$$
\begin{aligned}
Tr(cA+B) &= (ca_{11}+b_{11})+(ca_{22}+b_{22})+\cdots+(ca_{nn}+b_{nn}) \\
&= (ca_{11}+ca_{22}+\cdots+ca_{nn})+(b_{11}+b_{22}+\cdots+b_{nn}) \\
&= c(a_{11}+a_{22}+\cdots+a_{nn})+(b_{11}+b_{22}+\cdots+b_{nn}) \\
&= c\,Tr(A)+Tr(B)
\end{aligned}
$$

Therefore Tr is a linear transformation. ∎

(2). Solution:

Since $Tr: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $\dim \mathbb{R} = 1$

So $\dim(Im(Tr)) = 1$, or 0. as all subspaces of $\mathbb{R}$ have dimension 0 or 1.

But trace is nontrivial: the identity matrix has trace n.

So $\dim(Im(Tr)) \neq 0$, i.e. $\dim(Im(Tr)) = 1$.

Since $\dim(M_{n\times n}(\mathbb{R})) = n^2$ and Tr is a linear transformation

So
$$
\begin{aligned}
\dim(Ker(Tr)) &= \dim(M_{n\times n}(\mathbb{R})) - \dim(Im(Tr)) \\
&= n^2-1.
\end{aligned}
$$

(3). Solution:
$$
\left\{ \underbrace{\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & -1 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & -1 \end{bmatrix}}_{\text{PART I}}, \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_{\text{PART II}} \right\}
$$

Note: For PART I, set $a_{nn}=-1$, then each vector has a $a_{ii}=1$, $1\le i\le n-1$.
altogether $n-1$ vectors.

For PART II, each vector only has 1 non-zero entry, a_{ij} when $i\neq j$.
altogether ~~$\frac{(n-1)^2}{2}\times 2$~~ $= \frac{[1+(n-1)](n-1)}{2}\times 2 = n^2-n$ vectors.

And check that $(n-1)+(n^2-n) = n^2-n = \dim(Ker(Tr))$.

$\boxed{\text{\#4. See the last page}}$

Solution:

$$T(x)=Ax=\begin{bmatrix}1&0&2\\1&2&1\end{bmatrix}x=0$$

say $x=\begin{bmatrix}-4a&-4b&-4c\\1a&1b&1c\\2a&2b&2c\end{bmatrix}$

then $Ax=0$

So $\dim Ker(T)\neq 0 \Rightarrow Ker(T)\neq\{\vec{0}\}$

Then T is not injective, $Ker(T)=\left\{\begin{bmatrix}-4&0&0\\1&0&0\\2&0&0\end{bmatrix},\begin{bmatrix}0&-4&0\\0&1&0\\0&2&0\end{bmatrix},\begin{bmatrix}0&0&-4\\0&0&1\\0&0&2\end{bmatrix}\right\}$.

Since $\dim(\mathbb{Z}_3^3)=\dim(\mathbb{Z}_3^2)=3$ and T is not injective

By Proposition 2.4.10 that T is also not surjective.

say $x=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}$, $Ax=\begin{bmatrix}a_1+2a_3&b_1+2b_3&c_1+2c_3\\a_1+2a_2+a_3&b_1+2b_2+b_3&c_1+2c_2+c_3\end{bmatrix}$

Therefore $Im(T)=\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$.

#5.

(1). Proof: Let $v,w\in V$, $[v]_\alpha=[a_1,a_2,\dots,a_n]^T$, $[w]_\alpha=[b_1,b_2,\dots,b_n]^T$, $c\in F$

So $T(cv+w)=[cv+w]_\alpha=[ca_1,ca_2,\dots,ca_n]^T+[b_1,b_2,\dots,b_n]^T$

$=c[a_1,a_2,\dots,a_n]^T+[b_1,b_2,\dots,b_n]^T$

$=c[v]_\alpha+[w]_\alpha$

$=cT(v)+T(w)$

Therefore T is a linear transformation.

(2). Proof: ~~By Proposition 2.4.2 that~~

Since $\alpha=\{v_1,v_2,\dots,v_n\}$ is a basis for V and $T(v)=[v]_\alpha$

So $T(v)=0$ iff $v=\vec{0}$

i.e. the coefficients $a_i=a_1=a_2=\dots=a_n=0$

Therefore $Ker(T)=\{\vec{0}\}$

Thus T is injective.

As $\dim(V)=\dim(F^n)=n$ and T is injective (proved).

By Proposition (2.4.10) that T is also surjective.

Therefore T is bijective.

Peter Crooks
TUT0101

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#6.

Proof: Since $\{v_1, v_2, \cdots, v_n\}$ is a basis for V,
by Proposition (2.3.12), then $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ spans $Im(T)$
Since T is bijective (actually we use its surjectivity here only).
so $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ spans W. (spanning proved).
Since $\{v_1, v_2, \cdots, v_n\}$ is a basis for V.
then $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ iff $a_i = 0$, $i \in [1, n]$, $i \in \mathbb{Z}$.
Then
$$\begin{aligned} T(v) &= T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) \\ &= T(a_1v_1) + T(a_2v_2) + \cdots + T(a_nv_n) \\ &= a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) \\ &= 0 \end{aligned}$$
iff $a_i = 0$.

(Since T is bijective, we use its injectivity here only:
$T(v) = 0 \Longleftrightarrow a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$
(~~linear~~ linear independence proved).

Therefore $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis for W. ∎

#7.

(1) Proof: Say $C_1, C_2, \cdots, C_n \in F^m$ are columns of $[T]_{\beta\alpha}$.
($\Rightarrow$) Suppose T is surjective, and that $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in F^m$.

Suppose again we have $w = b_1w_1 + b_2w_2 + \cdots + b_mw_m \in W$
Since T is surjective, there must exist a $v \in V$ such that
$T(v) = w$
Let $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n$, then $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n \in V$

So $[T]_{\beta\alpha}[v]_\alpha = [w]_\beta$

Therefore
$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = [T]_{\beta\alpha}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = [C_1\, C_2 \cdots C_n]\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1C_1 + \cdots + a_nC_n$$

Hence, the columns of $[T]_{\beta\alpha}$ span F^m.

($\Leftarrow$) ~~On~~ In the other direction, assume the columns of $[T]_{\beta\alpha}$ span F^m
Let $b_1, \cdots, b_m$ be coefficients of w and $b_1, \cdots, b_m \in F$ such that $w = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$.
Since the columns span F^m, then there exist coefficients $a_1, \cdots, a_n \in F$ such that

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = a_1 C_1 + a_2 C_2 + \cdots + a_n C_n$$

Say $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in V$
~~As~~

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = [T]_{\beta\alpha} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

i.e. $[w]_\beta = [T]_{\beta\alpha} [v]_\alpha$

Therefore $w = T(v)$
Hence T is surjective.

(2) Proof.
($\Rightarrow$) Assume T is injective, with $a_1, \cdots, a_n \in F$ such that $a_1 C_1 + \cdots a_n C_n = 0$ (C is denoted as columns of $[T]_{\beta\alpha}$)
Let $v = a_1 v_1 + \cdots + a_n v_n \in V$
Then $[0]_\beta = 0 = [T]_{\beta\alpha} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = [T]_{\beta\alpha} [v]_\alpha$

i.e. $0 = T(v)$
Since T is injective, $v = 0$.
Since $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, which means $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ iff $a_1 = \cdots = a_n = 0$.
So $a_1 C_1 + a_2 C_2 + \cdots + a_n C_n = 0$ iff $a_1 = \cdots = a_n = 0$.
Hence the columns of $[T]_{\beta\alpha}$ are linearly independent in F^m.

($\Leftarrow$) ~~On~~ In the other direction, assume the columns of $[T]_{\beta\alpha}$ are linearly independent, and say $v \in Ker(T)$, ~~$v \in V$~~.
Let $a_1, \cdots, a_n$ be coefficients such that
$a_1 v_1 + \cdots + a_n v_n = 0$

(Sorry for changing the size of paper, cuz really ran out of the previous one.)

Peter Crooks
TUT0101

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Since $0 = T(v)$

then $0 = [0]_\beta = [T]_{\beta\alpha}[v]_\alpha = a_1C_1 + a_2C_2 + \cdots + a_nC_n$

Therefore $a_1 = a_2 = \cdots = a_n = 0$

so $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0v_1 + 0v_2 + \cdots + 0v_n = 0$

Hence T is injective.

#4.

Solution:

By row reduction we get $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

(a). Say $\mathbb{Z}_3^3 = \{(x_1, x_2, x_3)^T \mid x_1, x_2, x_3 \in \mathbb{Z}_3\}$

$\mathbb{Z}_3^2 = \{(x_4, x_5)^T \mid x_4, x_5 \in \mathbb{Z}_3\}$

since $2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \cancel{2}\,-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

so $[T]_{\beta\alpha} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}$, $\operatorname{Im} T = \{(x_4, x_5)^T, (0, 2x_5)^T\}$

Then $\dim(\operatorname{Im} T) = \dim(\mathbb{Z}_3^2) = 2$

Therefore T is surjective.

(b). Solve $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} x = 0$ that ~~well~~ we get $\operatorname{Ker} T = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

Then $\operatorname{Ker} T \neq 0$ so T is not injective.

$\operatorname{Ker} T = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$