§3 - Closed Sets and Closures

1 So Far ...

So far we have seen a couple examples of topological spaces ($\mathbb{R}$ with the usual topology, discrete topol- hey, why am *I* listing them? *You* should be the one trying to remember the 8 or so topologies we've discussed so far). We also discussed the idea of a basis of a topology, which above all was a useful way for describing topologies. (Did we say that a basis is almost closed under intersection or did we say that a basis is almost closed under unions?) We will meet some more topological spaces soon, but let's see what sort of phenomena we can investigate even with the few words (open, basis, topology, ...) that we currently know.

2 Motivation

We already have all of the tools to discuss the notion of convergence in a topological space. From our studies in analysis and calculus we learned what it means for the sequence $\{\frac{n+1}{n}\}_{n=1}^{\infty}$ to converge to 1. In general, we said that:

Definition. *A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to a real number p if $\forall \epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|p - x_n| < \epsilon$.*

In "English", a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to a real number p if for every $\epsilon > 0$ there is a tail of the sequence within ϵ of p.

Okay, this works fine, but we can actually translate this into the language of ϵ-balls $B_\epsilon(p)$, and it looks like this:

Definition. *A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to a real number p if $\forall \epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in B_\epsilon(p)$ for all $n \geq N$.*

Now here is where we can generalize this to topological spaces. So how do we extract the important aspects of this definition and state the definition of convergence in a topological space without making reference to distances or ϵ-balls? Think about this before we get to the official definition.

The second thing we will look at is what it means for "a point to be close to a set". For example, we want to know

- In $\mathbb{R}$ (with the usual topology), is 7 close to the set $(1, 7)$?
- In $\mathbb{R}$ (with the usual topology), is π close to $\mathbb{Q}$?
- In $\mathbb{R}$ (with the usual topology), is π close to $\mathbb{N}$?
- In $\mathbb{R}$ with the discrete topology, is 7 close to the set $(1, 7)$?

- In the Sorgenfrey Line, is 7 close to the set $(1, 7)$?

One way to answer these questions (once we have defined convergence) is to answer the related question: "Is there a sequence from that set that converges to that point?". This is mostly right, but sometimes, for technical reasons, we won't be able to talk about a *sequence* converging to a point. As a result, we instead use the language of "closed sets" and "closures", which won't reference sequences.

Okay, enough blabbing, let's get to business.

3 "It's Business. . . It's Business Time."

We start with the definition of what it means for a "point p to be close to a set A", in a topological space.

Definition. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. For a point $p \in X$ we define:*
$p \in \overline{A}$ iff whenever U is an open set that contains p, then $U \cap A \neq \emptyset$.
(Here $\overline{A}$ is called the ***closure of*** *A, in $\mathcal{T}$).*

Generally, when there is no confusion, we will just call $\overline{A}$ the closure of A, without referencing what topological space we are assuming A to be in.

Let's look at the closure of some sets in $\mathbb{R}$ (with the usual topology).

- If $A = (6, 8)$, then $\overline{(6, 8)} = [6, 8]$, as you would expect. To prove this, note first that if $x \in A$, then any open set U containing x has the property that $x \in U \cap A \neq \emptyset$.

  Let us show that $6 \in \overline{A}$ (and the proof that $8 \in \overline{A}$ will be similar). Take an open set U containing 6. Then, there is an $\epsilon > 0$ such that $B_\epsilon(6) \subseteq U$. We may also assume that $\epsilon < 2$. So then $6 + \frac{\epsilon}{2} \in (6, 8) \cap U \neq \emptyset$.

  Now if $p \notin [6, 8]$, then $(-\infty, 6) \cup (8, +\infty)$ is an open set containing p that is disjoint from $(6, 8)$.

- A similar (but easier) proof shows that $\overline{\{100\}} = \{100\}$. The same is true for any point in $\mathbb{R}$.

- It isn't so hard to see that $\overline{(6, 8) \cup (8, 90)} = [6, 90]$, using the same type of argument.

- Vacuously, $\overline{\emptyset} = \emptyset$, and (obviously) $\overline{\mathbb{R}} = \mathbb{R}$.

- Going through our first argument again also shows that $\overline{[6, 8]} = [6, 8]$.

Let's look at something a bit weirder, like the Sorgenfrey Line:

- Again we see that for general reasons $\overline{\{100\}} = \{100\}$, $\overline{\emptyset} = \emptyset$, and $\overline{\mathbb{R}} = \mathbb{R}$.

- What about $A = (6, 8)$? Again we see that $(6, 8) \subseteq \overline{(6, 8)}$.

  Well if $p < 6$ then $[p, 6)$ is an open set (in the Sorgenfrey Line) containing p, but disjoint from $(6, 8)$, so $p \notin \overline{(6, 8)}$. Also if $p \geq 8$, then $[8, p + 10000000)$ is an open set containing p, but disjoint from $(6, 8)$, so $p \notin \overline{(6, 8)}$.

  Finally, $6 \in \overline{(6, 8)}$ because any open set U containing 6, must contain an interval of the form $[6, b)$ for some $b > 6$. This interval will intersect $(6, 8)$.

  Thus $\overline{(6, 8)} = [6, 8)$.

Now that we have some intuition, let's extract some general phenomena:

Proposition. *Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$.*

i. $A \subseteq \overline{A}$.

ii. $\overline{\emptyset} = \emptyset$ *and* $\overline{X} = X$.

iii. *If $X \setminus A$ is open, then $\overline{A} = A$.*

iv. $\overline{\overline{A}} = \overline{A}$.

v. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

We will prove those in a second, but let us also make some guesses that *may or may not* be true:

Guesses. *Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$.*

- *For any point $p \in X$ we have $\overline{\{p\}} = \{p\}$.*
- $\overline{A \cap B} = \overline{A} \cap \overline{B}$
- *$\overline{A}$ is not an open set.*
- *If $\{A_\alpha : \alpha \in I\}$ is a collection of subsets of X, then $\overline{\bigcup_{\alpha \in I} A_\alpha} = \bigcup_{\alpha \in I} \overline{A_\alpha}$.*

Guess-ercise: Using the examples we have already worked through, as well as looking at some of the other topological spaces we have seen, try to find counterexamples to some (*all*?) of the above guesses.

Proof of Proposition. [i] If $x \in A$, then for any open set U containing x we get $x \in U \cap A \neq \emptyset$.

[ii] For the empty set there is nothing to prove, and $\overline{X} = X$ follows from [i] and that $\overline{A} \subseteq X$ for any $A \subseteq X$.

[iii] Suppose that $X \setminus A$ is open. By [i], to show that $\overline{A} = A$, it is enough to show that $\overline{A} \subseteq A$. This is equivalent to showing that if $x \in X \setminus A$ then x is not in $\overline{A}$. This is pretty straightforward because we know something about $X \setminus A$!

Let $x \in X \setminus A$, which is open, and $(X \setminus A) \cap A = \emptyset$. So then we have found an open set containing x which is disjoint from A. Thus $x \notin \overline{A}$.

[iv, v] I leave these two to you, as they are just unwinding definitions. They might even be on an assignment! □

In working through our examples and the proposition that followed it, we often used fact [i] which says $A \subseteq \overline{A}$. On Assignment 2 you will investigate "the boundary of a set A" which **for open sets** is defined by $\mathrm{bd}(A) := \overline{A} \setminus A$. In an intuitive sense, the boundary of a set A is the set of all points in $X \setminus A$ that are close to A.

Let us record a useful way to check that a point is in the closure of a set, if there is a basis floating around.

Proposition. *Let $\mathcal{B}$ be a basis for a topological space $(X, \mathcal{T})$, and let $A \subseteq X$. Then $x \in \overline{A}$ iff every basic open set B that contains x has $B \cap A \neq \emptyset$.*

Proof. Prove this yourself!

□

Foreshadowed Exercise: Go back to the questions from the motivation section and answer them, but replace the phrase "is close to" with "is in the closure of".

4 Dense Sets

One neat thing we can do now is investigate dense sets. Intuitively, a dense set is one that is close to everything!

Definition. *A set D in a topological space $(X, \mathcal{T})$ is **dense** if $\overline{D} = X$.*

Phrasing! Exercise Write down an equivalent defintion of dense that involves open sets. (Just unwind the definitions.)

Some examples:

1. $\mathbb{Q}$ is dense in $\mathbb{R}$, with the usual topology, because any basic open interval $B_\epsilon(x)$ around a real number x, contains a rational number.

2. $\mathbb{Q}$ is dense in the Sorgenfrey Line. To show this, let $x \in \mathbb{R}$, and let $[a, b)$ be a basic open interval containing x, with $a < b$. Since (a, b) contains a rational number, so does the larger set $[a, b)$.

3. For a topological space X, X is dense (in itself), because $\overline{X} = X$.

4. (Reason number 48 that the indiscrete topology is garbage.) Take any point in $p \in X$, with the indiscrete topology. We check that $\{ p \}$ is dense in X. Take any point x in X, and an open set U that contains x. Well, uh, there is only one possible choice for what U is. It has to be that $U = X$. And so $U \cap \{ p \} \neq \emptyset$, as it contains p. So $\overline{\{ p \}} = X$. Blech.

5. Every (yes, every!) infinite subset of $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$ is dense in $\mathbb{R}$. Let's prove this. Let $I \subseteq \mathbb{R}$ be an infinite set, and let $p \in \mathbb{R}$, with $p \in U$ an open set. Can U be disjoint from I? No way! $X \setminus U$ is finite, so $I \not\subseteq X \setminus U$.

Some non-examples:

- $\mathbb{N}$ is not dense in $\mathbb{R}$ with the usual topology. Notably, $\overline{\mathbb{N}} = \mathbb{N}$.
- No finite set is dense in $\mathbb{R}$ (with the usual topology). (Prove this!)

Small Sample Size Exercise: Our examples suggest that $\mathbb{Q}$ is dense in $\mathbb{R}$ *no matter what topology we put on* $\mathbb{R}$. Disprove this guess by finding a topology on $\mathbb{R}$ where $\mathbb{Q}$ is not dense.

Ok, so from these examples, what sort of properties can we extract:

Proposition. *Let $(X, \mathcal{T})$ be a topological space, with $A, D \subseteq X$.*

1. *If D is dense, and $D \subseteq A$, then A is dense.*
2. *If $D \cap A = \emptyset$, and A is a non-empty open set, then D is not dense.*
3. *Suppose Γ refines $\mathcal{T}$. If D is dense in (X, Γ), then it is dense in $(X, \mathcal{T})$.*

Proof. These are all straightforward unwinding of definitions. You should be able to prove these, in your head, on a busy subway car ... $\square$

As always, we should ask: "Why is this interesting?". Every time a new property is defined some justification needs to be made as to why this property is interesting. In this case, we will see soon that to define a continuous function (on a reasonable space) we only need to define it on a dense subset. This will tell us that to define a continuous function on $\mathbb{R}$ it is enough to define it on $\mathbb{Q}$. This is a bit of a big deal, because $\mathbb{Q}$ is countable and, in that sense, is much smaller that $\mathbb{R}$.

We should also check in on the questions we asked in the first section of notes: "When is a space large?". Well, one potential partial answer to this that $A \subseteq \mathbb{R}$ is large if it is dense in $\mathbb{R}$.

5 Case Closed

Back to business, we introduce one of the most fundamental properties in topology:

Definition. *A set C in a topological space X is* ***closed*** *if $X \setminus C$ is open.*

Yep, that's it. Let's look at some examples:

1. The interval $[0, 46]$ is a closed set in $\mathbb{R}$ (with the usual topology), because $\mathbb{R} \setminus [0, 46] = (-\infty, 0) \cup (46, +\infty)$ is open.

2. The previous example shows that $[0, 46]$ is closed in the Sorgenfrey Line and the discrete topology, because both of those topologies refine the usual topology. (Is the converse of this statement true? Does closed in the Sorgenfrey Line imply closed in the usual topology?)

3. As more evidence for the claim "The Sorgenfrey Line is weird", notice that every basic open set $[a, b)$ is also closed. This shows that the Sorgenfrey Line has a basis comprised of sets that are both open **and** closed. These spaces are highly disconnected and are called **Zero-Dimensional**.

4. The sets $\emptyset, X$ are closed in every topological space X. (In the Indiscrete topology these are the *only* closed sets.)

5. The closed sets are *very* easy to describe in the co-finite topology. The closed sets are precisely the co-co-finite sets ;) i.e. the finite sets (and the whole space of course).

Some non-examples:

1. $\{\frac{1}{n}\}_{n=1}^{\infty}$ is not closed in $\mathbb{R}_{\text{usual}}$ or the Sorgenfrey Line, because of that pesky 0. Take any basic open $B_\epsilon(0)$, for $\epsilon > 0$. This $B_\epsilon(0)$ must contain at least one $\frac{1}{N}$ (and in fact contains infinitely many). So $\mathbb{R} \setminus \{\frac{1}{n}\}_{n=1}^{\infty}$ is not open, and thus $\{\frac{1}{n}\}_{n=1}^{\infty}$ is not closed.

2. $(6, 8)$ is not closed in the Sorgenfrey Line. Why? Try to prove it, then look earlier in this section where we discuss the closure of $(6, 8)$ in the Sorgenfrey Line. That argument also shows that $\mathbb{R} \setminus (6, 8)$ is not open in the Sorgenfrey Line.

This last non-example suggest a very useful connection between **clos**ed sets and the **clos**ure of sets.

Proposition. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.*

1. *A is closed iff $A = \overline{A}$.*

2. *A is closed iff $\overline{A} \subseteq A$.*

3. *$\overline{A}$ is a closed set.*

Proof. [i] [$\Rightarrow$] Believe it or not, we have already shown this somewhere in these notes. Find it.

[$\Leftarrow$] Suppose that $A = \overline{A}$. Let $p \in X \setminus A = X \setminus \overline{A}$. So $x \notin \overline{A}$. Which means there is an open set U containing x, that is disjoint from A. That is, $U \subseteq X \setminus A$, which shows that $X \setminus A$ is open. i.e. A is closed.

[ii] Since $A \subseteq \overline{A}$ is always true, [i] gives the result.

[iii] This follows from [i] and the fact that $\overline{A} = \overline{\overline{A}}$. □

Thanks to our good friend Augustus De Morgan, we get the following (aesthetically) nice result:

Theorem. *Let $(X, \mathcal{T})$ be a topological space.*

1. *$\emptyset$ and X are closed sets.*

2. *The union of finitely many closed sets is closed.*

3. *The intersection of arbitrarily many closed sets is closed.*

Proof. [i] Immediate since $X \setminus X = \emptyset$ and $X \setminus \emptyset = X$.

[ii] Let $C_1, C_2, \ldots, C_N$ be closed sets. We need to show that $X \setminus \bigcup_{i=1}^{N} C_i$ is open. By DeMorgan's law, we have

$$X \setminus \bigcup_{i=1}^{N} C_i = \bigcap_{i=1}^{N} (X \setminus C_i)$$

which is open because it is a finite intersection of open sets.

[iii] Since unions of arbitrarily many open sets is open, an argument analogous to the previous one (using DeMorgan's *other* law) shows the result. □

This result tells us that we could (if we wanted) describe a topology by describing all of the closed sets, instead of all of the open sets. We won't ever do that in this class, but it does show up occasionally.

We are now able to present the following (amazing) proof that there are infinitely many primes.

6 There are Infinitely Many Primes

This is included *purely for interest*, because it is a ridiculuous proof. This is the proof adapted from "Proofs from the Book", by Aigner and Ziegler, 4th Ed..

Theorem (Fürstenbreg, 1955)**.** *There are infinitely many prime numbers.*

Proof. The many idea will be to define a topology on $\mathbb{Z}$ based on arithmetic progressions. We won't even mention the prime numbers until the end of the proof.

For $m, b \in \mathbb{Z}$ with $m > 0$ define $N(m, b) := \{mx + b : x \in \mathbb{Z}\}$, an arithmetic progression streching towards infinity in both directions. We will say that a set U is open if either:

1. $U = \emptyset$; *or*

2. For each $b \in U$ there is an $m > 0$ such that $N(m, b) \subseteq U$.

Claim: This is a topology.

$\emptyset, \mathbb{Z}$: We see that $\emptyset$ is open (as we explicitly included it), and $\mathbb{Z}$ is obviously open.

Unions: It is pretty easy to see that the union of a bunch of open sets is open.

Intersections: Let's show that the intersection of two open sets is open. Let U, V be open, and let $b \in U \cap V$. Since U is open, there is an $m_1 > 0$ such that $N(m_1, b) \subseteq U$. Since V is open, then is an $m_2 > 0$ such that $N(m_2, b) \subseteq V$. We can see that $N(m_1 \cdot m_2, b) \subseteq U \cap V$. (Now you can easily adapt this to show that the intersection of *finitely* many open sets are open, not just 2.)

[–End of claim–]

Now, we see that:

1. Each open set is infinite (because it contains an infinite arithmetic progression); and

2. Each $N(m, b)$ is open; and

3. Each $N(m, b)$ is closed.

[ii] Pick any $n \in N(m, b)$. It is of the form $n = my + b$ (for some $y \in \mathbb{Z}$), and notice that

$$N(m, n) = \{\, mx + n : x \in \mathbb{Z} \,\} = \{\, mx + my + b : x \in \mathbb{Z} \,\} = \{\, m(x + y) + b : x \in \mathbb{Z} \,\} = N(m, b)$$

[iii] Notice that $N(m, b) = \mathbb{Z} \setminus (N(m, 1) \cup N(m, 2) \cup \cdots \cup N(m, b - 1))$, and so $N(a, b)$ is the complement of an open set.

Now for the primes!

Any integer n that isn't ± 1 is divisible by some prime number p. Hence $n \in N(p, 0)$. This shows

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ a prime}} N(p, 0)$$

Now, what happens if there are only finitely many primes? Then

$$\bigcup_{p \text{ a prime}} N(p, 0)$$

is a closed set (as it is the *finite* union of closed sets.) So

$$\{-1, 1\} = \mathbb{Z} \setminus \bigcup_{p \text{ a prime}} N(p, 0)$$

is an open set. But this contradicts the fact that non-empty open sets are infinite! □

A 'Natural' exercise: This proof breaks down somewhere if you try to define the same topology on $\mathbb{N}$. Find that crux.

7 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Guess-ercise : Using the examples we have already worked through, as well as looking at some of the other topological spaces we have seen, try to find counterexamples to some (*all*?) of the guesses we made about closures.

Phrasing! : Write down an equivalent defintion of dense that involves open sets.

Foreshadow: Go back to the questions from the motivation section and answer them, but replace the phrase "is close to" with "is in the closure of".

Sample Size: Disprove the claim: "$\mathbb{Q}$ is dense in $\mathbb{R}$ *no matter what topology we put on* $\mathbb{R}$."

Natural : Find the place in Fürstenberg's proof that breaks down if you tried to define the topology on $\mathbb{N}$ instead of $\mathbb{Z}$.