1. a) (6 marks) Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$. We have seen that such a function may be considered continuous on all of $\mathbb{R}^n$ if it satisfies any of three equivalent properties: i) an $\epsilon - \delta$ property, ii) a property involving open sets, and iii) a property involving sequences. State precisely these three properties. (If you prefer, Folland gives i as the definition of continuity and proves that ii and iii are equivalent to i; regardless, state i,ii, and iii)

Property i) $\forall \vec{a} \in \mathbb{R}^n, \forall \varepsilon > 0 \ \exists \delta > 0$ such that $|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \varepsilon$ whenever $|\vec{x} - \vec{a}| < \delta$

Property ii) $\forall$ openset $U \subset \mathbb{R}^k$, $\vec{f}^{-1}(U) = \{\vec{x} \in \mathbb{R}^n : \vec{f}(\vec{x}) \in U\}$ is open in $\mathbb{R}^n$

Property iii) $\forall \vec{a} \in \mathbb{R}^n$ and $\forall$ sequences $\{\vec{x}_k\}$ in $\mathbb{R}^n$ that converge to $\vec{a}$, $\{\vec{f}(\vec{x}_k)\}$ converges to $\vec{f}(\vec{a})$

b) (8 marks) One way to prove the equivalence of these three properties described in part a) is to prove the following four implications between the properties: i implies ii, ii implies i, i implies iii, iii implies i. Choose any two of these four implications that you prefer, and present a rigorous proof of them.

i) $\Rightarrow$ iii) & iii) $\to$ i) is Theorem 1.15 of Folland.

i) $\to$ ii) is Theorem 1.13 of Folland. (First part)

ii) $\to$ i) is Exercise 8 which I will do here:

Let $\vec{a} \in \mathbb{R}^n$, $\varepsilon > 0$. $B(\varepsilon, \vec{f}(\vec{a}))$ is an openset so

$\vec{f}^{-1}(B(\varepsilon, \vec{f}(\vec{a})))$ is open by assumption.

Hence, $\exists \delta > 0$ s.t. $B(\delta, \vec{a}) \subset \vec{f}^{-1}(B(\varepsilon, \vec{f}(\vec{a})))$

ie $\exists \delta > 0$ s.t. $|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \varepsilon$ when $|\vec{x} - \vec{a}| < \delta$

$\Rightarrow$ i) $\blacksquare$

b) (extra space)

c) (4 marks) State whether the following function is or is not continuous on $\mathbb{R}^2$ and justify your answer:

$$f(x,y) = \frac{x(x^2-y^2)}{x^2+y^2}, \quad (x,y) \neq (0,0)$$

$$f(0,0) = 0$$

As $|x^2 - y^2| \leq x^2 + y^2$ via triangle inequality

$$|f(x,y)| \leq |x| \frac{\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = |x|.$$

As $|x| \to 0$ as $(x,y) \to (0,0)$ (Inequality 1.3)

$|f(x,y)| \to 0 \Rightarrow f(x,y) \to f(0,0) = 0$ $\therefore$ Continuous at $(0,0)$.

Away from zero, $f(x,y)$ is continuous as it is a composition of elementary continuous functions and so is continuous.

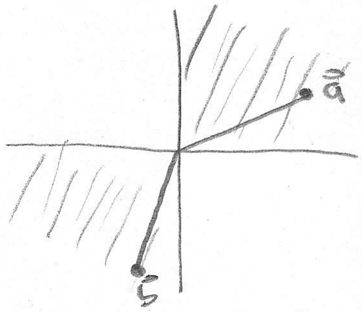
2. a) (2 marks) Define a disconnection on a set $S \subset \mathbb{R}^n$

A disconnection on S is a pair (S_1, S_2) of subsets of S such that
a) $S = S_1 \cup S_2$
b) $S_1 \cap \bar{S}_2 = \phi = \bar{S}_1 \cap S_2$
c) $S_1, S_2 \neq \phi$

b) (6 marks) Let

$$S = \overbrace{\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}}^{URQ} \cup \overbrace{\{(x,y) \in \mathbb{R}^2 : x \leq 0,\ y \leq 0\}}^{LLQ}$$

That is, S is the union of the top right and bottom left quadrants, including the axes. Prove that S is pathwise connected by explicitly constructing paths and showing they are in S.



For $\vec{a} \in$ upper right quadrant consider path from $\vec{a}$ to $\vec{0}$ via $f^{ao}(t) = (1-t)\vec{a}$

As $\vec{a}$ in upper right quadrant, $a_1 \geq 0$, $a_2 \geq 0$. Thus $f_1^{ao}(t) \geq 0$, $f_2^{ao}(t) \geq 0$ for $t \in [0,1]$.

$\therefore$ f^{ao} is in S. Clearly continuous & $f(0) = \vec{a}$, $f(1) = \vec{0}$

$\therefore$ f^{ao} is a path in S from $a \to 0$.

Likewise, for $f^{bo}(t) = (1-t)\vec{b}$, get a path from $\vec{b}$ to $\vec{0}$ in the lower left quadrant.

- Pick $\vec{a} \in URQ$, $\vec{b} \in LLQ$, get path from $\vec{a}$ to $\vec{b}$ via

$$f^{ab}(t) = \begin{cases} f^{ao}(2t), & 0 \leq t \leq 1/2 \\ f^{ob}(2t-1), & 1/2 \leq t \leq 1 \end{cases} \quad \text{where } f^{ob}(t) = f^{bo}(1-t)$$

- For $\vec{a} \in LLQ$, $b \in URQ$ use $f^{ab}(t) = \begin{cases} f^{bo}(2t) & 0 \leq t \leq 1/2 \\ f^{oa}(2t-1 & 1/2 \leq t \leq 1 \end{cases}$ with $f^{oa}(t) = f^{ao}(1-t)$

- For $a,b \in URQ$ $\quad f^{ab}(t) = \vec{a} + t(\vec{b} - \vec{a}) = (1-t)\vec{a} + t\vec{b}$, a path in URQ as $f_1(t) \geq 0$, $f_2(t) \geq 0$ as $a_1, b_1 \geq 0$, $a_2, b_2 \geq 0$. for $t \in [0,1]$

- For $a,b \in LLQ$ $\quad f^{ab}(t) = \vec{a} + t(\vec{b} - \vec{a}) = (1-t)\vec{a} + t\vec{b}$ a path is LLQ as $f_1(t) \leq 0$, $f_2(t) \leq 0$ as $a_1, b_1 \leq 0$, $a_2, b_2 \leq 0$ for $t \in [0,1]$

c) (6 marks) Prove that a continuous function maps connected sets to connected sets. That is, for $A \subset \mathbb{R}^n$ prove that if $\mathbf{f} : A \to \mathbb{R}^k$ is continuous $\forall \mathbf{x} \in A$ and A is connected, then $\mathbf{f}(A) = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in A\}$ is connected.

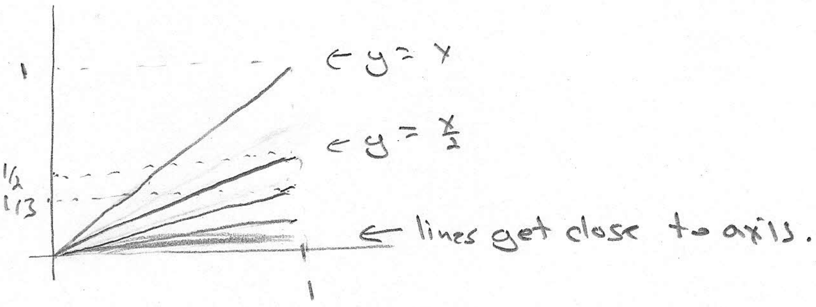
See Theorem 1.26 of folland.

3. Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 : \ y = x/n, \ 0 \leq x \leq 1 \ and \ n = 1, 2, 3, \dots\}$$

i.e., S is the collection of all the line segments y = x/n where $0 \leq x \leq 1$ and n is any positive integer.

a) (2 marks) Sketch S



b) (6 marks) Describe the interior S^{int} and the boundary ∂S of the set S.

Interior points have $B(r, a) \subset S$ for some r.
But $\forall a \in S$, $B(r,a) \cap S^c$ $\forall r$ as a point in S is on a line segment $y = \frac{x}{n}$ for $n \in \mathbb{Z}^+$ which intersects lines $y = \frac{x}{r}$ for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$

So $S^{int} = \phi$. Hence all $a \in S$ are boundary points but also for $a \in \{(x, 0) \mid 0 \leq x \leq 1\}$ Every ball about a intersects $y = \frac{x}{n}$ for some n.

Hence $\partial S = S \cup \{(x,0) \mid 0 \leq x \leq 1\}$

c) (6 marks) Justify whether S is or is not each of the following: open, closed, compact.

i) S is not open as $S \neq S^{int}$
ii) S is not closed as $\partial S \not\subset S$
iii) S is not compact as it is not closed even though it is bounded.

4. a) (2 marks) State the definition of convergence for a sequence $\{\mathbf{x_k}\}$ in $\mathbb{R}^n$

$\{\vec{x}_k\}$ converges to $\vec{L}$ if $\forall \varepsilon > 0$ $\exists K$ such that

$|\vec{x}_k - \vec{L}| < \varepsilon$ whenever $k > K$

b) (4 marks) Prove (with a rigorous ϵ argument) or provide a counterexample to the following statement: If a sequence $\{\mathbf{x_k}\}$ in $\mathbb{R}^n$ converges to $\mathbf{L}$, then every subsequence of $\{\mathbf{x_k}\}$ converges to $\mathbf{L}$.

Let $\varepsilon > 0$. As $\{\vec{x}_k\}$ converges to $\vec{L}$, $\exists K$ so that

$|\vec{x}_k - L| < \varepsilon$ whenever $k > K$. Let $\{\vec{x}_{k_j}\}$ be a subsequence.

Then as a subsequence is defined by a one to one increasing map $j \to k_j$, $\exists J$ so that $\forall j > J$, $k_j > K$

Thus $|\vec{x}_{k_j} - \vec{L}| < \varepsilon$ when $j > J$

$\Rightarrow x_{k_j}$ converges to $\vec{L}$

c) (4 marks) Prove (with a rigorous ϵ argument) or provide a counterexample to the following statement: For a sequence $\{\mathbf{x_k}\}$ in $\mathbb{R}^n$, if there exists a subsequence that converges to $\mathbf{L}$, then the sequence $\{\mathbf{x_k}\}$ converges to $\mathbf{L}$.

Consider the sequence $x_k = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases}$

This sequence does not converge as $|x_k - x_{k+1}| = 2$ $\forall k$ which is not less that all ε, hence not Cauchy $\Rightarrow$ doesn't converge

But the subsequence $\{x_{2k}\}$ of even terms converges to 1 as $|x_{2k} - 1| = 0 < \varepsilon$ $\forall k$

d) (4 marks) Consider the sequence defined by $x_k = 1 - k^{-2}$. Use the Monotone Sequence Theorem to prove that this sequence has a limit.

$\{x_k\}$ is increasing as $k+1 > k \Rightarrow (k+1)^2 > k^2 \Rightarrow \frac{1}{k^2} > \frac{1}{(k+1)^2}$

$\Rightarrow 1 - \frac{1}{(k+1)^2} > 1 - \frac{1}{k^2}$

$\{x_k\}$ is bounded above by 1 as k^{-2} is always positive

Hence M.S.T $\Rightarrow$ $\{x_k\}$ is convergent to a limit.

e) (4 marks) Consider the sequence defined by $x_k = 1 - k^{-2}$. Find, and justify, the limit of this sequence.

Postulate a limit of 1.

$|x_k - 1| = |1 - \frac{1}{k^2} - 1| = \frac{1}{k^2}$

But $\frac{1}{k^2} \to 0$ as k gets large.

That is, $\forall \varepsilon > 0$, $\exists K$ s.t $\frac{1}{k^2} < \varepsilon$ when $k > K$

$\Rightarrow |x_k - 1| < \varepsilon$ for $k > K$

$\Rightarrow x_k$ converges to 1.

5. For parts a) and b) following you MAY NOT use the Bolzano-Weierstrauss Theorem for subsets S of $\mathbb{R}^n$. Indeed, that theorem implies the equivalence of compactness (used in part a) and the sequential property (used in part b), and hence a) and b) are equivalent statements. Instead, I want you to "forget" about this theorem and provide two distinct proofs. First, you should prove the claim in part a) directly from the definition of compactness. Second, you should prove the claim in part b) directly using sequences..

a) (3 marks) Prove that if A and B are compact subsets of $\mathbb{R}^n$, then $A \cup B$ is compact. SEE ABOVE NOTE.

- Compact means closed & bounded.
- The union of closed sets is closed
- A, B bounded $\Rightarrow \exists\, C_1, C_2$ so $|x| < C_1$ for all $x \in A$
  $|x| < C_2$ for all $x \in B$

Using $C = \text{Max}\{C_1, C_2\}$, $x \in A \cup B$ has $x \in A$ or $x \in B \Rightarrow |x| < C$

so $A \cup B$ bounded $\Rightarrow A \cup B$ closed & bounded $\Rightarrow$ compact.

b) (5 marks) Suppose that every sequence of points in a subset A of $\mathbb{R}^n$ has a convergent subsequence whose limit lies in A. Likewise, suppose that every sequence of points in a subset B of $\mathbb{R}^n$ has a convergent subsequence whose limit lies in B. Now prove that every sequence of points in $A \cup B$ has a convergent subsequence whose limit lies in $A \cup B$. SEE ABOVE NOTE

Let $\{x_k\}$ lie in $A \cup B$. $\{x_k\}$ must have infinitly many points in either A or B or both.

WLOG, assume ∞ many in A.

Hence take subsequence $\{x_{k_j}\}$ of terms in A.

$\rightarrow$ This subsequence has a convergent sub sequence $\{x_{k_{j_\ell}}\}$ that converges in A, thus in $A \cup B$.

Hence $\{x_{k_{j_\ell}}\}$ is a convergent subsequence of $\{x_k\}$ in $A \cup B$. ∎

c) (5 marks) Consider a sequence $\{x_k\}$ with $x_k \in B(1+1/k, 0)$, $\forall k$. Prove that $\{x_k\}$ has a Cauchy subsequence.

As $\frac{1}{k}$ is decreasing, $B(1+\frac{1}{k},0) \subset B(2,0)$ $\forall k$.

$\Rightarrow \vec{x}_k \in B(2,0)$ $\forall k$ $\Rightarrow \{\vec{x}_k\}$ is a bounded subsequence. By Bolzano-Weirestrauss for $\mathbb{R}^n$, $\exists$ a convergent subsequence $\{\vec{x}_{k_j}\}$.

But all convergent sequences are Cauchy

$\Rightarrow \{\vec{x}_{k_j}\}$ is a Cauchy subsequence.