Name: SOLUTIONS

Student Number:

MAT 334H
FALL 2013 - LEC 5101
TERM TEST

Problem	1	2	3	4	5	Total
Points	10	10	10	10	10	50
Score						

- Solve the following problems, and write up your solutions neatly, in black or blue ink, in the space provided.
- Please make sure your name is entered at the top of this page.
- This test contains 7 pages. Please ensure they are all there.
- Please do not tear out any pages.
- You have 1 hour to complete this test.
- There are *no* aids allowed.
- There are some potentially useful formulae on the last page.

GOOD LUCK!

Name: ____________________

(1) Please answer the following questions in the space provided. You may use the bottom of the page, or page 7 for rough work. (2 pts each)

(a) $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{75} = \left(e^{i\pi/3}\right)^{75} = e^{i25\pi} = -1$

(b) $\lim_{z\to 2} \frac{z^3-8}{z-2} = \lim_{z\to 2} \frac{(z-2)(z^2+2z+4)}{(z-2)} = \lim_{z\to 2} z^2+2z+4 = 12$

(c) $\mathrm{Log}(-1-i) = \ln|-1-i| + i\,\mathrm{Arg}(-1-i) = \ln(\sqrt{2}) - i\frac{3\pi}{4}$

(where Log is the principal branch of logarithm.)

(d) The equation $z^5 = i$ has $\underline{\quad 5 \quad}$ solutions. (You don't need to find them all, just say how many.)

(e) Let $f(z) = 12z^3 + 2z$, and γ be the positively oriented circle arc joining $z_0 = e^{\frac{\pi}{4}i}$ to $z_1 = e^{\frac{3\pi}{4}i}$.

$$\int_\gamma f(z)dz = F(e^{3\pi i/4}) - F(e^{\pi i/4}) = 3(e^{3\pi i/4})^4 + (e^{3\pi i/4})^2 - 3(e^{\pi i/4})^4 - (e^{\pi i/4})^2$$

$$= 3e^{3\pi i} + e^{3\pi i/2} - 3e^{\pi i} - e^{\pi i/2}$$

$$= -3 - i - 3(-1) - i$$

$$= -2i$$

$F(z) = 3z^4 + z^2$ is antiderivative of $f(z)$.

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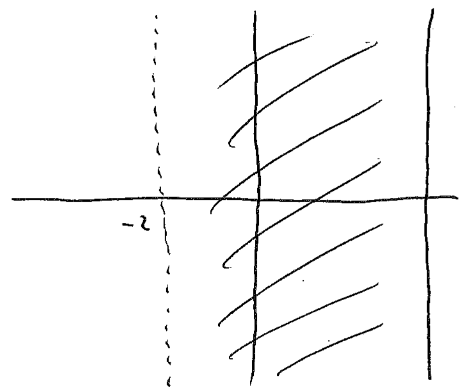
(2) Please sketch the following regions in the space provided. Use the convention that a dashed line (– – – –) is used to exclude points on the boundary of the set, and a solid line (——) is used to include points on the boundary of the set, and the interior of the set is shaded.

Use these sketches to determine if the given set is connected and simply-connected.

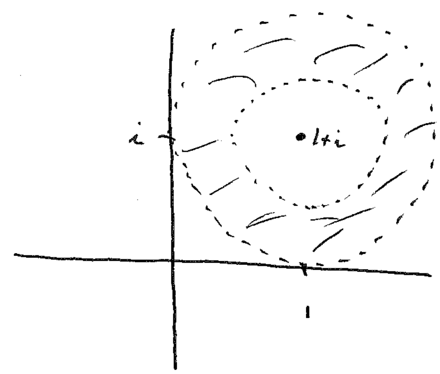
(a) $S = \{z \in \mathbb{C} \mid -2 < \mathrm{Re}(z) \leq 3\}$. Sketch:



Connected: Yes No

Simply Connected: Yes No

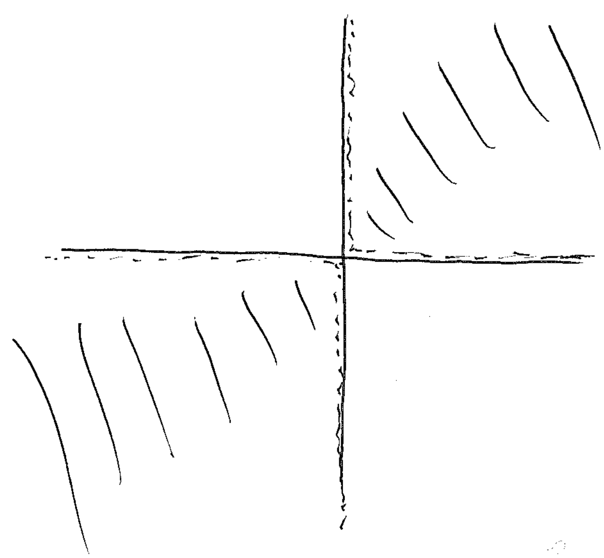
(b) $T = \{z \in \mathbb{C} \mid \frac{1}{2} < |z - 1 - i| < 1\}$. Sketch:



Connected: Yes No

Simply Connected: Yes No

(c) $U = \{z \in \mathbb{C} \mid \mathrm{Im}(z^2) > 0\}$. (Hint: Convert z into polar form, and describe the set U in terms of (r, θ) before sketching.) Sketch:



Connected: Yes No

Simply Connected: Yes No

Name:

(3) Consider the function $f(z) = \dfrac{z}{5+2z}$.

(a) Find the power series centred at $z_0 = 0$, for f.

$$f(z) = z\left(\frac{1}{5+2z}\right) = z\left(\frac{1/5}{1+\frac{2}{5}z}\right) = \frac{z}{5}\left(\frac{1}{1-\left(\frac{-2}{5}\right)z}\right)$$

$$= \frac{z}{5}\cdot\sum\left(\frac{-2}{5}z\right)^n$$

$$= \frac{z}{5}\sum\frac{(-1)^n\cdot 2^n}{5^n}\cdot z^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{5^{n+1}} z^{n+1}$$

(b) Find the radius of convergence for the series in (a).

$\frac{1}{1-w} = \sum w^n$ cges for $|w|<1$, we have $w = \frac{-2}{5}z$

$$\Rightarrow |w| = \left|\frac{-2}{5}z\right| = \frac{2}{5}\cdot|z| < 1$$

$$\Rightarrow |z| < \frac{5}{2}.$$

So $R = \frac{5}{2}$

(c) Find $\displaystyle\int_\gamma \frac{1}{z^{100}(5+2z)}dz$, where γ is the unit circle.

$$\int_\gamma \frac{1}{z^{100}(5+2z)}dz = \int_\gamma \frac{z}{z^{101}(5+2z)}dz = 2\pi i\cdot a_{100} \leftarrow$$

100th coeff of power series.

$$= 2\pi i\cdot\frac{(-1)^{100}\cdot 2^{100}}{5^{10\cancel{0}}}$$

$$= \left(\frac{2}{5}\right)^{101}\cdot \pi i$$

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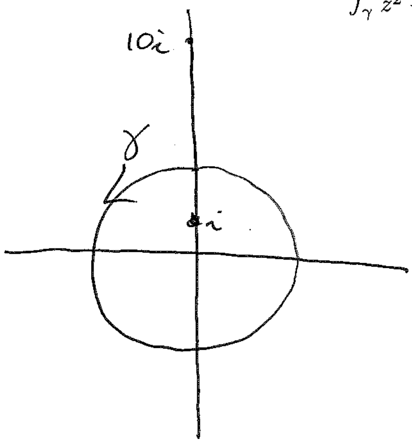
(4) Find $\int_\gamma \frac{\cos(\pi i z) e^{iz+1}}{z^2 - 11iz - 10} dz$, where γ is the circle of radius 4 centred at $z_0 = 0$.



$$z^2 - 11iz - 10 = (z - i)(z - 10i)$$

$z = i$ inside γ

$z = 10i$ outside γ.

Let $f(z) = \dfrac{\cos(\pi i z) e^{iz+1}}{z - 10i}$

Then f is analytic inside γ; so

$$\int_\gamma \frac{f(z)}{z - i} dz = \int_\gamma \frac{\cos(\pi i z) \cdot e^{iz+1}}{z^2 - 11iz - 10} dz = 2\pi i \cdot f(i)$$

by Cauchy Formula.

$$= 2\pi i \frac{\cos(-\pi) e^0}{i - 10i}$$

$$= \frac{-2\pi i}{-9i} = \frac{2\pi}{9}.$$

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(5) Consider the function $u(x, y) = x^2 - y^2 + x$.

(a) Is u a harmonic function? (Show your work. No credit is given for a simple yes/no answer.)

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}(2x+1) + \frac{\partial}{\partial y}(-2y)$$
$$= 2 - 2 = 0$$

So u is harmonic.

(b) Is u the real part of an analytic function? If yes, find a conjugate harmonic function v, so that $f = u + iv$ is analytic. If not, explain why.

Yes, since u is harmonic, it is the real part of an analytic function.
To find v, use C.-R. eq'ns:

$$\frac{\partial u}{\partial x} = 2x+1 = \frac{\partial v}{\partial y} \Rightarrow v = 2xy + y + g(x)$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} = -(2y + g'(x)) \Rightarrow g'(x) = 0$$
$$\Rightarrow g(x) = \text{const},$$

So $v(x,y) = 2xy + y + C$.

(In fact: $f = u + iv = z^2 + z + C$.)