STAT2001/6039 Mid-Semester Exam 2013 Solutions

Solution to Problem 1

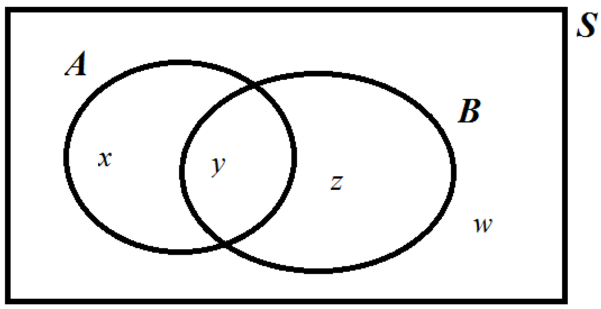
Draw a Venn diagram with x, y, z and w representing the probabilities $P(A-B)$, $P(AB)$, $P(B-A)$ and $P(\bar{A}\bar{B})$, respectively. Then

$$P(A\,|\,\bar{B}) = P(A\bar{B})/P(\bar{B}) = x/(x+w).$$

Thus $x/(x+w)$ = 3/7, $x + z$ = 1/2 and w/y = 4. Also, $x + y + z + w$ = 1.

Solving these four equations, we get x = 0.3, y = 0.1, z = 0.2 and w = 0.4.

Thus $P(B\,|\,\bar{A}) = P(B\bar{A})/P(\bar{A}) = z/(z+w) = 0.2/(0.2+0.4)$ = 1/3 = $\boxed{\mathbf{0.33333}}$.



Solution to Problem 2

On a single roll of the six dice, the probability of at least five sixes coming up is

$$\binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^1 + \binom{6}{6}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^0 = \frac{31}{6^6} = 0.00066444.$$

Now let Y be the number of rolls until the first roll with at least five sixes.

Then $Y \sim Geo(31/6^6)$ and we wish to find the smallest integer n such that

$$P(Y \le n) \ge 0.999999.$$

Now $P(Y \le n) = 1 - P(Y > n)$, where $P(Y > n)$ is the probability of no rolls with at least five sixes in the first n rolls. But that probability is simply $(1-31/6^6)^n$.

So, let us solve $1-(1-31/6^6)^n = 0.999999$, where n is allowed to be a real number.

This equation implies $(1-31/6^6)^n = 10^{-6}$, hence $n\log(1-31/6^6) = -6\log 10$, and hence $n = -6(\log 10)/\log(1-31/6^6)$ = 20785.88. From this we guess that the required answer is 20786. To check this, we calculate:

$P(Y \le 20785) = 1 - P(Y > 20785) = 1 - (1 - 31/6^6)^{20785}$ = 0.99999 89994 (too small)

$P(Y \le 20786) = 1 - P(Y > 20786) = 1 - (1 - 31/6^6)^{20786}$ = 0.99999 90001 (big enough).

We may conclude that the minimum number of rolls required is $\boxed{\textbf{20786}}$.

Alternative working: Let X be the number of rolls resulting in at least five sixes. Then $X \sim Bin(n, 31/6^6)$ and we wish to find the smallest n such that $P(X \ge 1) \ge 0.999999$.

But $P(X \ge 1) = 1 - (1 - 31/6^6)^n$, etc. We see that this logic leads to the same answer.

R Code for Problem 2 (not required)

```
-6*log(10)/log(1-31/6^6)  #  20785.88
options(digits=10); 1-(1-31/6^6)^c(20785,20786)   # 0.99999 89994 0.99999 90001
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Solution to Problem 3

Let 0 represent 1, 2, 3 or 4, and let A be the event that 66 comes up before 56.

Also, for example, let 5 denote the event that a 5 comes up on the first roll, and let 56 denote the event that 5 and 6 come up on the first and second rolls, respectively.

Then, conducting a first step analysis, we have:

$$P(A) = P(0)P(A \mid 0) + P(5)P(A \mid 5) + P(6)P(A \mid 6)$$

$$= \frac{4}{6}P(A) + \frac{1}{6}P(A \mid 5) + \frac{1}{6}P(A \mid 6)$$

$$P(A \mid 5) = P(50 \mid 5)P(A \mid 50) + P(55 \mid 5)P(A \mid 55) + P(56 \mid 5)P(A \mid 56)$$

$$= \frac{4}{6}P(A) + \frac{1}{6}P(A \mid 5) + \frac{1}{6}(0).$$

$$P(A \mid 6) = P(60 \mid 6)P(A \mid 60) + P(65 \mid 6)P(A \mid 65) + P(66 \mid 6)P(A \mid 66)$$

$$= \frac{4}{6}P(A) + \frac{1}{6}P(A \mid 5) + \frac{1}{6}(1).$$

Let $a = P(A)$, $b = P(A|5)$ and $c = P(A|6)$. Then the above three equations imply:

$a = (4/6)a + (1/6)b + (1/6)c$, $b = (4/6)a + (1/6)b$, $c = (4/6)a + (1/6)b + (1/6)$.

Solving, we get $a = 5/12$, $b = 1/3$ and $c = 1/2$.

So the answer is $a = P(A) = 5/12 = \boxed{\textbf{0.41667}}$.

Solution to Problem 4

$$E\sqrt{Y} = \sum_y \sqrt{y}\, f(y) = \sqrt{1}\left(\frac{1}{2}\right) + \sqrt{2}\left(\frac{1}{4}\right) + \sqrt{4}\left(\frac{1}{8}\right) + \ldots = 2^{-1} + 2^{-1.5} + 2^{-2} + \ldots = t^2 + t^3 + t^4 + \ldots$$

where $t=\frac{1}{\sqrt{2}}$. So $E\sqrt{Y}=t^2(1+t+t^2+...)=t^2\left(\frac{1}{1-t}\right)=\frac{1}{2-\sqrt{2}}$ = $\boxed{\mathbf{1.7071}}$.

Alternative working:

(1) $E\sqrt{Y}=(1+t+t^2+...)-1-t=\left(\frac{1}{1-t}\right)-1-t=\frac{1}{2-\sqrt{2}}$ = 1.7071.

(2) $E\sqrt{Y}=h^1+h^{1.5}+h^2+...=h(1+h+h^2+...)+h^{1.5}(1+h+h^2+...)$

where $h=\frac{1}{2}$. So $E\sqrt{Y}=\frac{h+h^{1.5}}{1-h}=\frac{0.5+0.5\sqrt{0.5}}{1-0.5}=1+\frac{1}{\sqrt{2}}$ = 1.7071.

Solution to Problem 5

Let A = "Room A is chosen", B = "Room B is chosen", etc., and N = "At least one of the selected persons is a man" (or equivalently "Not all are women"). Then we want

$$P(C\cup D\,|\,N)=1-P(A\cup B\,|\,N)=1-\frac{P(A\cup B)P(N\,|\,A\cup B)}{P(N)}.$$

Here: $P(A\cup B)$ = 2/4, $P(D)=1/4$, $P(A\cup B\cup C)=P(\bar{D})=3/4$

$$P(N\,|\,D)=1-\frac{\binom{8}{0}\binom{4}{3}}{\binom{12}{3}}=\frac{54}{55} = 0.98182$$

$$P(N\,|\,A\cup B)=P(N\,|\,\bar{D})=1-\frac{\binom{5}{0}\binom{9}{3}}{\binom{14}{3}}=\frac{10}{13} = 0.76923$$

$$P(N)=P(D)P(N\,|\,D)+P(\bar{D})P(N\,|\,\bar{D})=\frac{588}{715} = 0.82238.$$

So: $P(C\cup D\,|\,N)=1-\frac{(2/4)(10/13)}{588/715}=\frac{313}{588}$ = $\boxed{\mathbf{0.53231}}$.

R Code for Problem 5 (not required)

```
PNGivenD=1-choose(4,3)/choose(12,3); PNGivenNotD=1-choose(9,3)/choose(14,3)
c(PNGivenD, PNGivenNotD) # 0.9818182 0.7692308
PN = (1/4)* PNGivenD + (3/4)* PNGivenNotD; PN # 0.8223776
1-(2/4)* PNGivenNotD/PN # 0.5323129
```