Fundamental matrices.

$\vec{X}' = P(t)\vec{X}$

Recall: If $\vec{X}^{(1)}, \cdots, \vec{X}^{(n)}$ are fund. set of solutions. Then $\Psi(t) = (\vec{X}^{(1)}, \cdots, \vec{X}^{(n)}(t))$ is called fundamental matrix (Wronskain $= \det(\Psi(t))$).

$$\Psi(t) = \begin{pmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix}$$

Note: $\Psi'(t) = P(t)\Psi(t)$ (since $(\vec{X}^{(i)})' = P(t)\vec{X}^{(i)}$)

- If $\Psi(t)$ is a matrix of functions, with $\Psi'(t) = P(t)\Psi(t)$ $\det(\Psi(t)) \neq 0$, then Ψ is a fundamental matrix. (Can take this as the definition).

Basic properties:

- If Ψ fund. matrix, and C (constant) invertible $n \times n$ matrix, then $\tilde{\Psi} = \Psi C$ is a fund. matrix.

  Pf: $\tilde{\Psi}' = \Psi' C = P\Psi C = P\tilde{\Psi}$ $\det(\tilde{\Psi}) = \det(\Psi)\det(C) \neq 0$.

- Any two fund. matrices are related in this way:
  - Given t_0, there is a unique fund. matrix Φ with $\Phi(t_0) = I$ (Given Ψ, take $\Phi(t) = \Psi(t)\Psi(t_0)^{-1}$.) Normalized fund. matrix.

  The solution to $\vec{X}' = P(t)\vec{X}, \vec{X}(t_0) = \vec{\xi}$ is then $\vec{X}(t) = \Phi(t)\vec{\xi}$

  (Check: $\vec{X}' = \Phi'\vec{\xi} = P\Phi\vec{\xi} = P\vec{X}$, $\vec{X}(t_0) = \Phi(t_0)\vec{\xi} = \vec{\xi}$.)

Example: Find the normalized fund. matrix for $\vec{X}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A}\vec{X}$ $t_0 = 0$.

The eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are $r^{(1)} = i$ $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$r^{(2)} = -i$ $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\vec{X}^{(1)} = e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}, \vec{X}^{(2)} = e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$. (complex solutions).

$\Psi(t) = \begin{pmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{pmatrix}$ is a fund. matrix.

$\Phi(t) = \psi(t)\psi(0)^{-1}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\psi(0) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ $\quad \psi(0)^{-1} = \frac{1}{-2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & \frac{-1}{2i} \end{pmatrix}$

$$\Rightarrow \Phi(t) = \psi(t)\psi(0)^{-1} = \begin{pmatrix} \frac{e^{it}+e^{-it}}{2} & \frac{e^{it}-e^{-it}}{2i} \\ \frac{i(e^{it}-e^{-it})}{2} & \frac{e^{it}+e^{-it}}{2} \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

- More generally, consider $\vec{X}' = A\vec{X}$, $t_0 = 0$ with $n \times n$ matrix A.

  Normalized fund. matrix $\Phi(t)$

- If $n=1$, A is just a number $\Phi' = A\Phi$ $\Phi(0) = 1$. $\Phi(t) = e^{tA}$.
- This also holds for $n > 1$. using matrix exponentials.

$$e^C = \sum_{k=0}^{\infty} \frac{1}{k!} C^k = I + C + \frac{1}{2}C^2 + \frac{1}{3!}C^3 + \cdots$$

$$\Rightarrow \Phi(t) = e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

<u>check</u>: $\Phi'(t) = 0 + A + tA^2 + \frac{t^2}{2!}A^3 + \cdots = Ae^{tA} = A\Phi(t)$

<u>Example</u>: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$ $\quad A^3 = AA^2 = -A$ $\quad A^4 = AA^3 = I$ $\quad A^5 = A \cdots$

$A^{2m} = (-1)^m I$ $\quad A^{2m+1} = (-1)^m A$.

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k = \underbrace{\left(\sum_{m=0}^{\infty} \frac{1}{(2m)!} t^{2m}(-1)^m\right)}_{\cos(t)} I + \underbrace{\left(\sum_{m=0}^{\infty} \frac{t^{2m+1}(-1)^m}{(2m+1)!}\right)}_{\sin(t)} A$$

$$= \cos(t)\, I + \sin(t)\, A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

If A gets more complicated, can use this in reverse to calculate e^{tA}, (by calculating $\Phi(t) = \psi(t)\psi(t_0)^{-1}$). §7.7

Inhomogeneous equations: $\vec{X}' = P(t)\vec{X} + \vec{g}(t)$. $\quad \vec{X}(t_0) = \vec{\xi}$

Let $\Phi(t)$ be a normalized fund. matrix. Write $\vec{X}(t) = \Phi(t)\vec{u}(t)$

$$\vec{X}' = \Phi'\vec{u} + \Phi\vec{u}' = P\Phi\vec{u} + \Phi\vec{u}' = P\vec{X} + \Phi\vec{u}'$$

$P(t)\vec{x} + \vec{g} = P\vec{x} + \vec{g}$

$\Rightarrow$ Condition: $\Phi \vec{u}' = \vec{g} \Rightarrow \vec{u}' = \Phi^{-1}\vec{g}$

$\vec{u}'(t) = \Phi(t)^{-1}\vec{g}(t)$ integrate from t_0 to t.

$\vec{u}(t) - \vec{\xi} = \int_{t_0}^{t} \Phi(s)^{-1}\vec{g}(s)\,ds$ $\quad \vec{x}(t) = \Phi(t)\left(\vec{\xi} + \int_{t_0}^{t} \Phi(s)^{-1}\vec{g}(s)\,ds\right)$