PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 6
DUE FRIDAY, APRIL 21, 4PM.

Warm-up problems. These are optional, but it might be a good idea to do them at some point.

(1) Use the Euclidean algorithm to compute the gcd of 126 and 224. Express the gcd as an integer combination of 126 and 224.

(2) Determine whether the following equations have a solution in integers. If they do have solutions, find one explicitly.
 (a) $17x + 13y = 93$.
 (b) $60x + 42y = 104$.

Problems to be handed in. Solve three of the following four problems.

(1) Let $a, b \in \mathbb{Z}$. 6.17
 (a) Prove that $\gcd(a + b, a - b) = \gcd(2a, a - b) = \gcd(a + b, 2b)$.
 (b) Suppose that $\gcd(a, b) = 1$. What can you say about $\gcd(a^2, b^2)$? What about $\gcd(a, 2b)$? 6.18

(2) The royal treasury has 500 7-ounce weights, 500 11-ounce weights, and a balance scale. An envoy arrives with a bar of gold, claiming it weights 500 ounces. Can the treasury determine whether the envoy is lying? If so, how? What if the weights are 6-ounce and 9-ounce weights? 6.45

(3) Show that the gaps between primes can be arbitrarily large. Do this by constructing, for any positive integer n, a set of n consecutive integers that are not prime. (Hint: Determine a positive integer x such that x is divisible by 2, $x+1$ is divisible by 3, $x+2$ is divisible by 4, etc.) 6.35

(4) Let p be a prime number.
 (a) Prove that p divides $\binom{p}{k}$ for any $1 \leq k \leq p - 1$.
 (b) Prove that $n^p - n$ is divisible by p for every $n \in \mathbb{N}$. (Hint: Use the binomial theorem and part (a) in a proof by induction.)
 6.37