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In this textbook the language of sequences and the limit of sequences are used as practical parallel to the idea of limit. When I want to think of a variable approaching a point I think of it in terms of steps that the varible takes in order to reach its limit. Of course in this process there isn't a unique set of steps that the variable has to take. A sequence is the foot prints of a variable, either wandering about or focusing and approaching certain limit. As such these foot steps are points of certain subset of $\mathbb{R}^n$ (where the variable resides).

Like studying fossiles of the past geological era we use sequences as practical tools for analyzing the idea of limit and the way the environment imposes itselt on the limit. Sequences present us with insight into the topological nature of sets and functions involved in the process of limit. Also our minds work with our logical system in steps and as such sequences are natural models for studying limit and topology.

There is a distinction between a sequence and the range of a sequence. For the trivial sequence $(2, 2, 2, \dots)$ has all its terms equat to 2, that is for all i, $x_i = 2$. The range of the values of this sequence is $\{2\}$. Similarly the range of values of the sequence $(1, -1, 1, -1, \dots\}$ is $\{1. - 1\}$.

Another important idea is the *tail of a sequence.* This is all the terms of the sequence from some index onward, denoted by $\{x_k : k > K\}$. And to say that a sequence $\{x_k : k = 1, 2, \dots\}$ converges to a point a is to say that for any open ball around a some tail of the sequence is lying entirely inside the open ball. That is,

$$\forall \epsilon > 0, \exists K \text{ such that } \forall k > K \; x_k \in B(\epsilon, a)$$

The following is a list of important applications of the sequences in the textbook:

- Theorem 1.14 presents a characterization of a point in the closure of a set in terms of sequences.

- Theorem 4.15 gives an equivalent definition of continuity using sequences.

- We can always have trivial sequences like $(2, 2, 2, 2, \dots)$ which converg to a point (in this case 2 is the trivial limit). Exercises 6 and 7 of sectin 1.4 deal with some non trivial types of approach. An accumulation point is a point that is approached by a non constant sequence. These points are very intereting points of a set.

- in section 1.5 the idea of completeness of reals (which heavily depends of the idea of ordering of real number) is generalized to $\mathbb{R}^n$ which is not ordered. The sequence appraoch becomes very crucial in this generalization. Indeed the meaning completeness changes (from *any bounded non empty set has a least upper bound,*) to *any Cauchy sequence has a limit in the set.* To achieve this there are Theorems 1.16, 1.17, 1.18, 1.19 and 1.20 are gradually leading us to the new characterization of completeness. (See map 1.4 for more on this process.)

- Theorem 1.21 is a major theorem that gives a sequential characterization of compactness. According to this theorem a compact set is a microcosm of complete space. The proof of this theorem reveals some aspect of our mind which is captured by a composition of quantifiers: $\exists \epsilon, \forall \delta \exists x$ such that certain porpty holds true.