STA447/STA2006 Stochastic Processes

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Note

This lecture note is prepared for the course STA447/STA2006 Stochastic Processes. This lecture note may contain flaws. Please consult text book or reference books for confidence.

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* indicates graduate level. So you may skip those parts.

5.2 Martingale Convergence Theorem

Definition 43. Let X_t be a stochastic process and T be a stopping time. The stochastic process $X_{T\wedge n}$ is called a *stopped process*.

Exercise 37. Let T be a stopping time and X_n be a submartingale. Then the stopped process $X_{T\wedge n}$ is also a submartingale.

Fix $a < b$. Let $N_0 = -1$ and $N_{2k-1} = \inf\{m > N_{2k-2} : X_m \le a\}$ and $N_{2k} = \inf\{m > N_{2k-2} : X_m \ge b\}$. Then N_j's are stopping times and $\{N_{2k-1} < m \le N_{2k} = \{N_{2k-1} \le m-1\} \cap \{N_{2k} \le m-1\}^c \in \mathcal{F}_{m-1}$. Hence $H_m = 1(N_{2k-1} < m \le N_{2k}$ for some $k)$ is predictable. Define $U_n = \sup\{k : N_{2k} \le n\}$ is the number of upcrossings up to time n.

Theorem 65. Let X_n be a submartingale. $(b-a)\mathbb{E}U_n \le \mathbb{E}(X_n - a)^+ - \mathbb{E}(X_0 - a)^+$.

Proof. Define $Y_n = a + (X_n - a)^+$ so that it is a submartingale. It is easy to see that Y_n crosses $[a, b]$ whenever X_n does and that $(b-a)U_n \le (H \cdot Y)_n$. For $K_n = 1 - H_n$, $Y_n - Y_0 = (H \cdot Y)_n + (K \cdot Y)_n$ where $(K \cdot Y)_n$ is submartingale. Hence $\mathbb{E}(K \cdot Y)_n \ge \mathbb{E}(K \cdot Y)_0 = 0$ and $(b-a)\mathbb{E}U_n \le \mathbb{E}(H \cdot Y)_n \le \mathbb{E}(Y_n - Y_0)$, which proves the theorem. □

Theorem 66 (Martingale Convergence Theorem)**.** Let X_n be a submartingale. If $\sup_n \mathbb{E}X_n^+ < \infty$, then X_n converges a.s. to a limit X with $\mathbb{E}|X| < \infty$.

Proof. Using $(x-a)^+ \le x^+ + |a|$, $\mathbb{E}U_n \le (|a| + \mathbb{E}X_n^+)/(b-a)$. Since U_n is non-negative and increasing, there exists a limit U such that $U_n \nearrow U$ which is the number of upcrossing of $[a, b]$. From the assumption $\sup_n \mathbb{E}X_n^+ < \infty$, $\mathbb{E}U < \infty$ and $U < \infty$ a.s. Then for any rational numbers a and b,

$$P(\liminf X_n < a < b < \limsup X_n \text{ for some } a, b \in \mathbb{Q}) = 0.$$

Thus $\limsup X_n = \liminf X_n$ a.s. and there exists X such that $X = \lim X_n$. By Fatou's lemma, $\mathbb{E}X^+ \le \liminf \mathbb{E}X_n^+ < \infty$. Note that $\mathbb{E}X_n^- = \mathbb{E}X_n^+ - \mathbb{E}X_n \le \mathbb{E}X_n^+ - \mathbb{E}X_0$ and

$$\mathbb{E}X^- \le \liminf \mathbb{E}X_n^- \le \sup_n \mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty.$$

Hence $|X| < \infty$ a.s. and $\mathbb{E}|X| \le 2\sup_n \mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty$. □

Example 60. Let X_n be independent with $\mathbb{E}X_n = 0$. The sum of random variables $S_n = X_1 + \cdots + X_n$ is a martingale.

Suppose $\sum_n \mathbb{V}\mathrm{ar}(X_n) < \infty$. Then, $\mathbb{E}S_n^+ = \mathbb{E}S_n^+(1(S_n \leq 1) + 1(S_n > 1)) \leq 1 + \mathbb{E}S_n^2 1(S_n > 1) \leq 1 + \mathbb{V}\mathrm{ar}(S_n)$. Hence, $\sup_n \mathbb{E}S_n^+ \leq 1 + \sup_n \mathbb{V}\mathrm{ar}(S_n) \leq 1 + \sum_n \mathbb{V}\mathrm{ar}(X_n) < \infty$. Thus S_n converges a.s. by the martingale convergence theorem.

Theorem 67. Let X_t be a nonnegative supermartingale. Then X_t converges almost surely.

Proof. By assumption $X_t \geq 0$. Let $Y_t = -X_t$ be a submartingale. Then $\sup_t \mathbb{E}Y_t^+ = 0$. Hence Y_t converges to Y almost surely. Therefore X_t converges to $-Y$ almost surely. $\square$

Example 61. Let $N(t)$ be a nonhomogeneous Poisson process with rate process $\lambda(t)$. Then, for any $0 \leq s < t$, $N(t) - N(s) \sim \mathrm{Poisson}(\int_s^t \lambda(u)\ du)$ and $\mathbb{E}[N(t) \,|\, \mathcal{F}_s] = \mathbb{E}[N(s) + N(t) - N(s) \,|\, \mathcal{F}_s] = N(s) + \mathbb{E}[N(t) - N(s)] = N(s) + \int_s^t \lambda(u)\ du \geq N(s)$. Hence $N(t)$ is a submartingale. If $\int_0^\infty \lambda(u)\ du < \infty$, then $N(t)$ converges almost surely.

If $\int_0^\infty \lambda(u)\ du = \infty$, then $N(t)$ diverges almost surely. Let $\mu_t = \int_0^t \lambda(u)\ du$. Fix $M > 0$, then $P(N(t) > M) = P(\mathrm{Poisson}(\mu_t) > M) \approx P(N(0,1) > (M - \mu_t)/\sqrt{\mu_t}) \to 1$. Hence $N(t) \to \infty$ in probability as $t \to \infty$. It is easy to show the almost sure convergence (Excerise).

Example 62. Let X_n be a homogeneous Markov chain with the transition probability p and a stationary distiribution π. Hence $Y_n = \pi(X_n)$ is a martingale with nonnegative and bounded by 1. Therefore Y_n converges almost surely.

Question: what is the limit distribution of Y_n?