University of Toronto
FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, DECEMBER 2007

MAT 240H1F - ALGEBRA I

Instructor: F. Murnaghan
Duration - 3 hours

Total marks: 100

No calculators or other aids allowed.

Notation:

If m and n are positive integers, $M_{m\times n}(F)$ is the vector space of $m \times n$ matrices with entries in the field F.
$P(F)$ is the vector space of polynomials in one variable with coefficients in the field F.
If n is a nonnegative integer, $P_n(F)$ is the subspace of $P(F)$ consisting of polynomials of degree at most n.
If V and W are vector spaces over a field F, $\mathcal{L}(V, W)$ denotes the vector space of linear transformations from V to W. If $V = W$, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

[11] 1. In each case below, determine whether the subset W of the vector space V is a subspace of V. If W is a subspace of V, prove it. If not, demonstrate how one of the properties of subspace fails to hold.

a) Let $n \geq 3$ and let $V = \mathbb{R}^n$. Let

$$W = \{ \ x = (a_1, a_2, a_3, \dots, a_n) \in \mathbb{R}^n \mid \sqrt{3}a_1 - 4a_2 = a_1 a_n \ \}.$$

b) Let $V = \mathcal{L}(\mathbb{Q}^4)$. (Here, $\mathbb{Q}$ is the field of rational numbers.) Let

$$W = \{ \ T \in V = \mathcal{L}(\mathbb{Q}^4) \mid \{ (1,0,1,0), (0,1,0,-1) \} \subset N(T) \ \}.$$

[10] 2. In each case below, determine whether the function T is a linear transformation.

a) Let F be a field. Define $T : P(F) \to M_{2\times 2}(F)$ by

$$T(f(x)) = \begin{pmatrix} f(0) - f(1) & f(-1) \\ -f(1) & 0 \end{pmatrix}, \text{ for } f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in P(F).$$

b) Suppose that m and n are positive integers. Let V be a vector space of dimension n (over a field F) and let W be a vector space of dimension m (also over the field F). Let β be an ordered basis of V and let γ be an ordered basis of W. Suppose that $U \in \mathcal{L}(V, W)$ and $A \in M_{m\times n}(F)$. Define $T : V \to F^m$ by

$$T(x) = A[x]_\beta - [U(x)]_\gamma, \qquad x \in V.$$

[5] 3. Let A, $B \in M_{n\times n}(F)$, where $n \geq 2$ and F is a field. Suppose that A is similar to B and $A^3 = -A$. Prove that $B^3 = -B$.

[20] 4. Determine whether or not V and W are isomorphic vector spaces. (Justify your answers.)

a) Let $V = \mathcal{L}(P_2(\mathbb{C}), M_{2\times 2}(\mathbb{C}))$ and $W = \mathcal{L}(M_{2\times 3}(\mathbb{C}), \mathbb{C}^2)$.

b) Let $V = \{\, x = (\, a_1,\, a_2,\, a_3,\, a_4\,) \in \mathbb{R}^4 \mid a_1 + a_3 = 0 \,\}$ and $W = \{\, x = (\, a_1,\, a_2,\, a_3,\, a_4\,) \in \mathbb{R}^4 \mid a_1 = a_2 = a_3 \,\}$.

c) Let V_0 be an n-dimensional vector space over a field F, where $n \geq 2$. Let $\beta = \{\, x_1, \ldots, x_n \,\}$ be an ordered basis of V_0. Define

$$V = \{\, T \in \mathcal{L}(V_0) \mid [T]_\beta \text{ is a diagonal matrix. } \}$$
$$W = \{\, T \in \mathcal{L}(V_0) \mid T(x_1) = T(x_2) = \cdots = T(x_n) \,\}.$$

(Recall that $A = (A_{ij}) \in M_{n\times n}(F)$ is a diagonal matrix if $A_{ij} = 0$ whenever $i \neq j$.)

[18] 5. Let V be a vector space over a field F. Let $T \in \mathcal{L}(V)$. (For parts a) and b), do not assume that V is finite-dimensional.)

a) Prove that $T^2 = -T$ if and only if $T(x) = -x$ for all $x \in R(T)$.

b) Suppose that $T^2 = -T$. Prove that $N(T) \cap R(T) = \{\, \mathbf{0} \,\}$.

c) Assume that V is finite-dimensional. Prove that $T^2 = -T$ if and only if there exists an ordered basis of V such that

$$[T]_\beta = [T]_\beta^\beta = \begin{pmatrix} -I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where $r = \operatorname{rank}(T)$, I_r is the $r \times r$ identity matrix, and each 0 is a zero matrix of the appropriate size.

[8] 6. Let $A \in M_{n\times n}(F)$, where $n \geq 2$ and F is a field. Let $A^t \in M_{n\times n}(F)$ be the transpose of A. (Recall that A^t is obtained from A by interchanging the rows and columns of A: the jth column of A^t is equal to the jth row of A, $1 \leq j \leq n$.) Prove that it is possible to transform A into A^t using elementary row and column operations.

[28] 7. Let $V = P_2(\mathbb{C})$.
For parts a) and b), let $T \in \mathcal{L}(V)$ be defined by $T(f(x)) = f(ix) + f(2)\, x^2$, for $f(x) \in V$.

a) Find the characteristic polynomial and eigenvalues of T.

b) Prove that T is invertible and find $T^{-1}(a\,x^2 + b\,x + c)$ for all complex numbers a, b, and c.

For parts c) and d), let $\beta = \{\, x^2,\, x,\, 1 \,\}$ and suppose that $U \in \mathcal{L}(V)$ satisfies

$$[U]_\beta = [U]_\beta^\beta = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ i & i & i-1 \end{pmatrix}.$$

Let $\mathbf{1}_V \in \mathcal{L}(V)$ be the identity transformation (that is, $\mathbf{1}_V(f(x)) = f(x)$ for all $f(x) \in V$).

c) Compute $\operatorname{nullity}(U + \mathbf{1}_V)$ and $\operatorname{rank}(U + \mathbf{1}_V)$.

d) Find a basis of $N(U + \mathbf{1}_V)$.