$f(x,y,z) = x^2+4y^2+4z^2-2yz+3x+6y$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 8 \end{pmatrix}$$

$H - \lambda I$

$\det = (2-\lambda)(\lambda^2-16\lambda+60)$

$\lambda_1 = 2, \lambda_2 = 6, \lambda_3 = 10$ positive definite

Practice problems.

Minimize a function:

?

- ~~$\frac{\partial f}{\partial x} = 2x+3=0 \Rightarrow x = -\frac{3}{2}$~~

  ~~$\frac{\partial f}{\partial y} = 8y-2z+6=0 \Rightarrow y = \frac{2z-6}{8} = \frac{z-3}{4}$~~

  ~~$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ not positive definite~~

  ~~But we know $f(x,y) = x^2+4y^2+4z^2-2yz+3x+6y$ is convex~~

  ~~local min $\Rightarrow$ global min~~

not positive definite

- $f(x,y,z) = x^2+4y^2-4z^2-2yz+3x+6y$ (same idea)

- $f(x,y) = x^3+2y^2+4xy+4x+4y$ in $\{(x,y) \in E^2 : x \geq 0\}$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = 3x^2+4y+4 = 0 \\ \frac{\partial f}{\partial y} = 4y+4x+4 = 0 \end{array}\right\} \Rightarrow 3x^2-4x=0 \quad x(3x-4)=0$$

$x=0$ or $x=\frac{4}{3}$

$$\begin{bmatrix} 6x & 4 \\ 4 & 4 \end{bmatrix} > 0 \quad \text{so} \quad \cancel{\text{take}}\ x = \frac{4}{3}$$

thus $x=\frac{4}{3}$, $y = -1-\frac{4}{3} = -\frac{7}{3}$

Check eigenvalues whether positive definite

?

- $f(x,y) = x^3 + 2y^2 + 2xy + 4x + 4y$ in $\{(x,y) \in E^2 : x \geq 0\}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 + 2y + 4 = 0 \\ \frac{\partial f}{\partial y} &= 4y + 2x + 4 = 0 \end{aligned}\right\} \Rightarrow 3x^2 - x + 2 = 0$$

$$x^2 - \frac{1}{3}x + \frac{2}{3} = 0$$

$$2y = -x - 2 \qquad x^2 - \frac{1}{3}x + \frac{1}{36} - \frac{1}{36} + \frac{2}{3}$$

But when $x=0$, $\nabla f(x) = 0$, $y = -1$. $\quad (x - \frac{1}{6})^2 = \frac{1}{36} - \frac{2}{3} < 0$

Check $\nabla^2 f$, still not po. def. $\Leftarrow$ no answer.

- $f(x,y) = x^2 + 2y^2 + 2xy - 4x - 4y$ in $\{(x,y) \in E^2 : x + y \geq 1\}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 2y - 4 = 0 \\ \frac{\partial f}{\partial y} &= 4y + 2x - 4 = 0 \end{aligned}\right\} \quad \begin{aligned} x + y &= 2 \\ 2y + x &= 2 \end{aligned} \qquad y = 0,\ x = 2.$$

$$\nabla^2 f = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} > 0. \qquad \text{and } x + y = 2 + 0 \geq 1$$

so $x = 2, y = 0$ ✓

- ~~$f(x,y) = x^2 + 2y^2 + 2xy - 4x - 4y$ in $\{(x,y) \in E^2 : x + y \geq 1\}$~~

$$\left.\begin{aligned} \cancel{\frac{\partial f}{\partial x} = 2x + 2y - 4 = 0} \\ \cancel{\frac{\partial f}{\partial y} = 4y + 2x - 4 = 0} \end{aligned}\right\} \Rightarrow \begin{aligned} \cancel{x + y = 2} \\ \cancel{2y +} \end{aligned}$$

(Same problem)

Convexity:

- $f(x,y) = xy$ on $\Omega = \{(x,y) \in E^2 : x \geq 0, y \geq 0\}$

~~let $g_1(x) = x$~~ ~~$0 \leq \theta \leq 1$~~

~~it's convex on $x \geq 0$ since $g_1(\theta x + (1-\theta)x) \leq \theta g_1(x) + (1-\theta) g_1(x)$~~

~~$g_2(y) = y$~~ ~~$0 \leq \theta \leq 1$~~

~~also convex on $y \geq 0$ since $g_2(\theta y + (1-\theta)y) \leq \theta g_2(y) + (1-\theta) g_2(y)$~~

~~so $f(x,y) = g_1 \cdot g_2 = xy$ is convex. by proposition~~ ~~combination~~

Best way is to use Hessian matrix (Proposition 5).

Hessian ~~semi~~ positive semidefinite $\Leftrightarrow$ convex.

check eigenvalue $1, -1$. not positive definite

$$\nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ ~~✓ convex~~

- $\frac{1}{xy}$ $\quad \frac{\partial f}{\partial x} = \frac{1}{y} \cdot (-1) x^{-2} = -x^{-2} y^{-1}$ $\quad \frac{\partial^2 f}{\partial^2 x} = 2x^{-3} y^{-1}$

$\frac{\partial^2 f}{\partial x \partial y} = x^{-2} y^{-2}$

$\frac{\partial f}{\partial y} = -y^{-2} x^{-1}$ $\quad \frac{\partial^2 f}{\partial y \partial x} = x^{-2} y^{-2}$ $\quad \frac{\partial^2 f}{\partial^2 y} = 2y^{-3} x^{-1}$

so $$\nabla^2 f = \begin{bmatrix} \frac{2}{x^3 y} & \frac{1}{x^2 y^2} \\ \frac{1}{x^2 y^2} & \frac{2}{x y^3} \end{bmatrix} \geq 0$$ convex. ✓

on $\{x \geq 0, y \geq 0\}$

都应该通过算 eigenvalues 的正负 来判断是否 positive definite

- $f(x,y) = -\log(xy)$ on $x \geq 0, y \geq 0$.

$\frac{\partial f}{\partial x} = -\frac{1}{xy} \cdot y = -\frac{1}{x}$ $\quad \nabla^2 f = \begin{bmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix} \geq 0$ convex ✓

$\frac{\partial f}{\partial y} = -\frac{1}{xy} \cdot x = -\frac{1}{y}$

- $f(x,y) = -\log(x+y)$ on $x+y \geq 0$

$\frac{\partial f}{\partial x} = -\frac{1}{x+y}$ $\quad \nabla^2 f \begin{bmatrix} \frac{1}{(x+y)^2} & \frac{1}{(x+y)^2} \\ \frac{1}{(x+y)^2} & \frac{1}{(x+y)^2} \end{bmatrix} \geq 0$ convex ✓

$\frac{\partial f}{\partial y} = -\frac{1}{x+y}$ $\quad x+y \neq 0$

- $f(x,y)=(1-x^2)^2+y^2$ on $\Omega=E^2$

$$\frac{\partial f}{\partial x} = 2(1-x^2)\cdot(-2x)$$
$$= (2-2x^2)(-2x)$$
$$= 4x^3-4x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\nabla^2 f = \begin{bmatrix} 12x^2-4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(12x^2-4-\lambda)(2-\lambda)=0$$
$$\lambda_1=2,\ \lambda_2=12x^2-4$$

if ~~$x^2 < \frac{4}{12} = \frac{1}{3}$~~

~~Say $x = \frac{1}{100}$~~

then $\nabla^2 f$ not positive semidefinite

so. not convex.

Algorithms for minimizing a function of a single variable.

- $f(x) = x^4+e^x$ for $x>0$.

Newton's.

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$f'(x_k) = 4x^3+e^x$$
$$f''(x_k) = 12x^2+e^x$$

So $$x_{k+1} = x_k - \frac{4x^3+e^x}{12x^2+e^x}$$

- e.g. ...

Global Convergence Theorem

- def of a "closed point-to-set mapping":

A p-to-s mapping $A: X \to Y$ is said to be <u>closed</u> at $\vec{x}\in X$ if

① $\vec{x}_k \to \vec{x}$, $\vec{x}_k \in X$

② $\vec{y}_k \to \vec{y}$, $\vec{y}_k \in A(\vec{x}_k)$

①② ⟹ ③ $\vec{y} \in A(\vec{x})$

—— is said to be closed on X if it's closed at every pt $\vec{x}\in X$

$Ax = \{y \in E^n : y^T x \leq 1\}$

$x_k \to x$

$x_k = \begin{pmatrix} \frac{1}{k} \\ \frac{1}{k} \\ \vdots \end{pmatrix}$

① $\exists (x_k) \to X$

② $y_k \in A(x_k)$ $y_k \to y$

③ show $y \in A(x)$

$\frac{y_1}{k} + \frac{y_2}{k} + \cdots \frac{y_n}{k}$ $y^T = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$= y^T x \leq 1$

$\{y_1 + \cdots + y_n \leq k\} = A(x_k)$

pick $y_k \in A(x_k)$ & $y_k \to y$

$x = 0$

$\forall y^T x = 0 \leq 1$

$\vec{y_k} \in \{y_{k_1} + \cdots + y_{k_n} \leq k\}$

$\exists y_k = (\frac{1}{k}, \cdots \frac{1}{k})$ $y_k \in A(x_k)$.

• A on E^n : $A(x) = \{y \in E^n : y^T x \leq 1\}$

Show A is closed at $x = 0$.

① $\vec{x_k} \to x = 0$, $x_k \in X$

② $\vec{y_k} \to y$; $y_k \in A(x_k)$

$\Rightarrow$ ③ $y \in A(x) = 0$?

?

• $x_{n+1} = x_n - \nabla f^T(x_n)$ satisfies the hypotheses of GCT for all f?

• Steepest descent, condition number, etc.

• $Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, assume $a < c$

a. Let $d_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$d_0^T Q d^T = (1, 0) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (a \quad b) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a$

b. $r = \frac{A}{a}$. $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\lambda x = Ax$

$(\lambda - A)x = 0$

$\det \begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} - (a-\lambda)(c-\lambda) = 0$

large $\lambda = c$

small $\lambda = a$

~~$(a-\lambda)(c-\lambda) - b^2 = 0$~~

• f is quadratic on E^n:

$$f(x) = \frac{1}{2}x^TQx + b^Tx$$

$$g(s) = f(x+sd), \quad s \in \mathbb{R}$$

derive formula for s^* that minimizes g

$$g(s) = f(x+sd) = \frac{1}{2}(x+sd)^TQ(x+sd) + b^T(x+sd)$$

$$g(\cancel{x}) = \cancel{Qx+b}\ Qx+b \qquad \cancel{\text{differentiate w.r.t. } s}$$

$$\cancel{g(s) \quad Qs+b = \frac{1}{2}} \qquad \cancel{(d^TQd)s + b^Td = 0}$$

$$\cancel{s = \frac{-b^Td}{d^TQd}}$$

$$= \cancel{\frac{(g(s)-Qs)d}{d^TQd}}$$

$$= \cancel{\frac{d^Td}{d^TQd}} \ ?$$

• $f(x,y) = x^2 + xy + y^2 - 3x + y$.
method of steepest descent, $(x_0, y_0) = (5,5)$.

$$f(x,y) = \frac{1}{2}(x, y)\,Q\begin{pmatrix} x \\ y \end{pmatrix} - (x\ y)\begin{pmatrix} m \\ n \end{pmatrix} \quad b$$

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{1}{2}(ax + cy,\ bx + dy)\begin{pmatrix} x \\ y \end{pmatrix} - mx - ny$$

$$= \frac{1}{2}(ax^2 + cxy + bxy + dy^2) - mx - ny$$

$$= \left(\frac{1}{2}a\right)x^2 + \left(\frac{1}{2}c + \frac{1}{2}b\right)xy + \left(\frac{1}{2}d\right)y^2 - mx - ny$$

$$\Downarrow \qquad \Downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$a=2 \qquad c+b=2 \qquad d=2 \qquad m=-3 \qquad n=-1$

(symmetric so $b=c=1$)

then $f(x,y) = \frac{1}{2}(x,y)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \cancel{(x\ y)}(-3\ \ -1)\begin{pmatrix} x \\ y \end{pmatrix}$

$$\vec{g_0} = Q\vec{x_0} - \vec{b}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 10+5 & +3 \\ 5+10 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

$$\begin{array}{r} 18 \\ \times 18 \\ \hline 144 \\ 18 \\ \hline 324 \end{array} \qquad \begin{array}{r} 16 \\ \times 16 \\ \hline 96 \\ 16 \\ \hline \end{array}$$

$$x_1 = x_0 - \left(\frac{g_0^T g}{g^T Q g}\right) g_0$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \left(\frac{(18,16)\begin{pmatrix} 18 \\ 16 \end{pmatrix}}{(18\ \ 16)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 18 \\ 16 \end{pmatrix}}\right) \cdot \begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \left(\frac{324+256}{(36+16 \quad 18+32)\begin{pmatrix} 18 \\ 16 \end{pmatrix}}\right)\begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \left(\frac{580}{936+800}\right)\begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \frac{145}{434}\begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{103.249} \ -1.0138 \\ -0.3456 \end{pmatrix} \rightarrow \text{this is } (x_1, y_1)^T$$

Conjugate directions, conjugate gradient.

- $Q = \begin{pmatrix} 3 & 5 \\ 5 & 11 \end{pmatrix}$, find $d = (d_1, d_2)$ s.t. Q-orth. to $(-2, 3)$

$$(d_1 \ d_2)\begin{pmatrix} 3 & 5 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$-2(3d_1 + 5d_2) + 3(5d_1 + 11d_2) = 0$$

$$-6d_1 - 10d_2 + 15d_1 + 33d_2 = 0$$

$$\Rightarrow \quad 9d_1 + 23d_2 = 0$$

$$d_1 = -23$$

$$d_2 = 9$$

So $d = (-23, 9)$.

- $(g_1 + s d_0)^T Q d_0 = 0$

find $s = ?$ $\quad s \in \mathbb{R}$

$$g_1^T Q d_0 + s d_0^T Q d_0 = 0$$

$$s = \frac{-g_1^T Q d_0}{d_0^T Q d_0}$$