Note: This is a sample of answers(or ideas) to your first midterm exam. To some questions, there are several correct answers, but I will only write one.

Problem 1. [20 points]
Determine which of the following sequences converge:
(a)[10 points] $(a_n)_{n=2}^{\infty}$, where $a_n = (1-\frac{1}{2^2})(1-\frac{1}{3^2})...(1-\frac{1}{n^2})$.
(b)[10 points] $(b_n)_{n=1}^{\infty}$, where $b_n = x^{\frac{1}{n}}$.
Explain.

Answer 1. (a). $a_n = \frac{1}{2}\times\frac{3}{2}\times\frac{2}{3}\times\frac{4}{3}\times ... \times\frac{n-1}{n}\times\frac{n+1}{n} = \frac{n+1}{2n}$.
Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$$

So, $(a_n)_{n=2}^{\infty}$ converges.

(b). 1). For $x\in(0,1)$, $\forall n\in\mathbb{N}$, $\frac{b_{n+1}}{b_n} = x^{\frac{1}{n+1}-\frac{1}{n}} > 1$, and $\forall n\in\mathbb{N}$, $b_n < 1$.
Thus $(b_n)_{n=1}^{\infty}$ is an monotone increasing bounded above sequence.
By Monotone Convergence Theorem, $(b_n)_{n=1}^{\infty}$ converges.
2). For $x=1$, $b_n = 1, \forall n\in\mathbb{N}$.
Thus, $(b_n)_{n=1}^{\infty}$ converges.
3). For $x\in(1,+\infty)$, $\forall n\in\mathbb{N}$, $\frac{b_{n+1}}{b_n} = x^{\frac{1}{n+1}-\frac{1}{n}} < 1$, and $\forall n\in\mathbb{N}$, $b_n > 1$.
Thus $(b_n)_{n=1}^{\infty}$ is a monotone decreasing bounded below sequence.
By Monotone Convergence Theorem, $(b_n)_{n=1}^{\infty}$ converges.

Problem 2. [30 points]
The distance $d(x_0,S)$ between a real number x_0 and a non-empty set S of real numbers is defined by $d(x_0,S) = \inf_{x\in S}|x_0 - x|$. If S is bounded below and $x_0 = inf S$, prove that $d(x_0,S) = 0$.

Answer 2. 1). If $x_0\in S$, then $0\le d(x_0,S) = \inf_{x\in S}|x_0-x| \le |x_0-x_0| = 0$, i.e. $d(x_0,S)=0$.

2). If $x_0\notin S$, since $x_0 = \inf S$, thus $\forall\epsilon>0, \exists x_\epsilon\in S$, such that $x_0+\epsilon > x_\epsilon > x_0$. So, $0\le|x_0-x_\epsilon|<\epsilon$.
Since $\forall x\in S, |x_0-x|\ge 0$, i.e. 0 is a lower bound of $\{|x_0-x| : x\in S\}$.
By the definition of infimum, we know $\inf_{x\in S}|x_0-x| = 0$, i.e. $d(x_0,S)=0$.

Problem 3. [30 points]
(a)[10 points] State the rearrangement theorem for conditionally convergent series.
(b)[10 points] Consider an infinite series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. Prove that the series converges conditionally.

(c)[10 points] Show that if $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s$ then the sum of the rearranged series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12}$... converges to $\frac{s}{2}$.

Answer 3. (a). If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series, then for every real number L, there is a rearrangement that converges to L.

(b). 1). Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by using p-series test or integral test or any other suitable tests), thus $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent.
2). Let $b_n = \frac{1}{n}$, then b_n is a non-negative monotone decreasing sequence, with $\lim_{n\to\infty} b_n = 0$, thus by Leibniz Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n b_n$ converges. So, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = -\sum_{n=1}^{\infty}(-1)^n b_n$ converges.
By 1) and 2), we know the series converges conditionally.

(c). See your text book [Davidson&Donsig] Page 44, 3.3.1 Example. (Many of you proved $\lim_{n\to\infty} s_{3n} = \frac{s}{2}$, but no one proved $\lim_{n\to\infty} s_{3n-1} = \lim_{n\to\infty} s_{3n+1} = \lim_{n\to\infty} s_{3n} = \lim_{n\to\infty} s_n$)

Problem 4. [60 points] Let $S_0 = [0, 1]$. Construct S_{i+1} from S_i by removing an open middle interval from each interval in S_i. That is $S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$; $S_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc.
(a)[20 points] Is $C = \cap_{i\geq 1} S_i$ a closed set? Is it compact?
(b)[10 points] Prove that $C = \cap_{i\geq 1} S_i$ is not empty;
(c)[20 points] Consider $\sum_{i=1}^{\infty} \frac{y_i}{3^i}$, where $y_i \in \{0, 2\}$ for all natural number i. Prove that this series converges. Let $x = \sum_{i=1}^{\infty} \frac{y_i}{3^i}$. Is x an element in C?
(d)[10 points] What is the cardinality of C?

Answer 4. (a). Yes, C is closed.
By induction, we can prove that S_i are closed.(S_0 is closed. If S_n is closed, since S_{n+1} is derived from S_n by removing an open interval, say I, then $S_{n+1} = (\mathbb{R}/\ S_n^c)/\ I = \mathbb{R}/\ (S_n^c \cup I)$ is also closed, where S_n^c is the complement of S_n.)
Since any intersection of closed sets is closed, C is closed.
Yes, C is compact.
Since C is bounded($C \subset [-1, 1]$), thus C is a closed and bounded subset of $\mathbb{R}$. By Heine-Borel Theorem, C is compact.

(b). By induction, we can prove that $\forall i \in \mathbb{N}, 0 \in S_i$(leave this as a small exercise, hint: every time the interval removed is in the middle), thus $0 \in \cap_{i\geq 1} S_i = C$, i.e. C is non-empty.
(You can also use Cantor's Intersection Theorerm, which is also very nice.)

(c).

$$\sum_{i=1}^{\infty} \frac{y_i}{3^i} \leq \sum_{i=1}^{\infty} \frac{2}{3^i} = 1$$

Since every term of $\sum_{i=1}^{\infty} \frac{y_i}{3^i}$ is positive, by Monotone Convergence Theorem, this series converges.
Yes, x is an element in C.
Let $x_k = \sum_{i=1}^{k} \frac{y_i}{3^i}$, since $\sum_{i=1}^{\infty} \frac{y_i}{3^i}$ converges, we have $\lim\limits_{k\to\infty} x_k = x$. Notice $\forall i \in \mathbb{N}$, $\{x_k\}_{k=i}^{\infty} \in S_i$ (this follows from x_i are the left endpoint of interval in S_i and $\forall k > i, 0 \le x_k - x_i \le x - x_i \le \frac{1}{3^i}$, which is the length of each interval in S_i). Thus $x = \lim\limits_{k\to\infty} x_k$ is a limit point of S_i, and since S_i are closed, so $x \in S_i, \forall i \in \mathbb{N}$. Therefore, $x \in \cap_{i\ge 1} S_i = C$.

(d). $Card(C) = Card(\mathbb{R})$.
Let f be the Cantor function(defined in [Davidson&Donsig] Page 92, 5.7.8 Example), then f is a surjective function from C to $[0,1]$, thus $Card(C) \ge Card([0,1])$. Since $C \subset [0,1]$, we have $Card(C) \le Card([0,1])$(inclusion function is injective). Thus, $Card(C) = Card([0,1])$.
Let $g(x) = \frac{1}{\pi}\arctan y + \frac{1}{2}$, then g is an injective function from $\mathbb{R}$ to $[0,1]$, thus $Card([0,1]) \ge Card(\mathbb{R})$. $Card([0,1]) \le Card(\mathbb{R})$ comes from inclusion function($[0,1] \subset \mathbb{R}$). Thus, $Card([0,1]) = Card(\mathbb{R})$.
So, $Card(C) = Card([0,1]) = Card(\mathbb{R})$.

Problem 5. [40 points]
Determine whether the following series converge or diverge.
(a)[20 points] $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$
(b)[20 points] $\sum_{n=1}^{\infty} \mathrm{e}^{-n^2}$

Answer 5. (a).

$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$$

By using p-series test(or integral test, or any other suitable test), we know

$$\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$$

diverges. Thus, by comparison test, we know

$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$

diverges.

(b).

$$\limsup_{n\to\infty} \sqrt[n]{\mathrm{e}^{-n^2}} = \limsup_{n\to\infty} \mathrm{e}^{-n} = 0 < 1$$

By Root Test, we know $\sum_{n=1}^{\infty} \mathrm{e}^{-n^2}$ converges.

Problem 6. [40 points]
Prove that every conergent sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, ρ) is a Cauchy sequence.

Answer 6. Since $(x_n)_{n=1}^{\infty}$ is convergent, suppose $\lim\limits_{n\to\infty} x_n = x$, then $\forall \epsilon > 0, \exists N$, such that $\forall n > N$, we have $\rho(x, x_n) < \frac{\epsilon}{2}$.

Thus, for this N, $\forall m, k > N$, we have $\rho(x_m, x_k) \leq \rho(x_m, x) + \rho(x_k, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore, by definition, we know $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Problem 7. [30 points]
Let ρ be a discrete metric on a nonempty set X. Describe all of the open and closed subsets of X.

Answer 7. 1). For any point $x \in X$, the set $\{x\}$ is an open set.
This is because for this point, there exist $r = \frac{1}{2}$, such that $B_r(x)$ is contained in this point set.

2). Any subset $A \subset X$ is open.
Since $A = \cup\{x : x \in A\}$, is union of open sets, thus open.

3). Any subset $B \subset X$ is closed.
Since the complement of B is also a subset of X, and from 2) we know that the complement of B is open. Thus, B is closed.

Bonus 1. [50 points]
A sequence $(x_n)_{n=1}^{\infty}$ satisfies $0 < x_1 \leq x_2$ and $x_{n+2} = (x_{n+1}x_n)^{\frac{1}{2}}$ for $n = 1, 2, ...$ Prove that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and hence converges.

Solution 1. 1). Since $0 < x_1 \leq x_2$, thus $\frac{x_2}{x_1} \geq 1$.
Since $\forall n \in \mathbb{N}, x_{n+2} = (x_{n+1}x_n)^{\frac{1}{2}}$, thus, $\forall n \in \mathbb{N}$, $\frac{x_{n+2}}{x_{n+1}} = (\frac{x_{n+1}}{x_n})^{-\frac{1}{2}}$. So,

$$\frac{x_{n+2}}{x_{n+1}} = \left(\frac{x_{n+1}}{x_n}\right)^{-\frac{1}{2}} = \left(\left(\frac{x_n}{x_{n-1}}\right)^{-\frac{1}{2}}\right)^{-\frac{1}{2}} = \left(\left(\cdots\left(\frac{x_2}{x_1}\right)^{-\frac{1}{2}}\cdots\right)^{-\frac{1}{2}}\right)^{-\frac{1}{2}} = \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^n}$$

2). By induction, you can prove that $\forall n \in \mathbb{N}, x_n \leq x_2$.

3). Since

$$\lim_{n\to\infty}\left(\frac{x_2}{x_1}\right)^{(\frac{1}{2})^{n-1}} = \lim_{n\to\infty}\left(\frac{x_2}{x_1}\right)^{-(\frac{1}{2})^{n-1}} = 1$$

Thus, $\forall\epsilon \geq 0$, $\exists N_1$, such that $\forall n > N_1$,

$$\left(\frac{x_2}{x_1}\right)^{(\frac{1}{2})^{n-1}} -1 = \left|\left(\frac{x_2}{x_1}\right)^{(\frac{1}{2})^{n-1}} -1\right| < \frac{\epsilon}{|x_2|}$$

For this ϵ, $\exists N_2$, such that $\forall n > N_2$,

$$1 - \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}} = \left|1 - \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}}\right| < \frac{\epsilon}{|x_2|}$$

Let $N = N_1 + N_2$, then $\forall n > N$,

$$\max\{\left(\frac{x_2}{x_1}\right)^{(\frac{1}{2})^{n-1}} -1, 1 - \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}}\} < \frac{\epsilon}{|x_2|}$$

For this N, $\forall m > n > N$, we have

$$\begin{aligned}
|x_m - x_n| = |x_n| \cdot |\frac{x_m}{x_n} - 1| &= |x_n| \cdot |\frac{x_m}{x_{m-1}} \cdot \frac{x_{m-1}}{x_{m-2}} \cdot \cdots \cdot \frac{x_{n+1}}{x_n} - 1| \\
&= |x_n| \cdot \left|\left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{m-2}} \cdot \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{m-3}} \cdot \cdots \cdot \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}} -1\right| \\
&= |x_n| \cdot \left|\left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{m-2}+(-\frac{1}{2})^{m-3}+\ldots+(-\frac{1}{2})^{n-1}} -1\right| \\
&= |x_n| \cdot \left|\left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}\cdot(\frac{2}{3})\cdot[1-(-\frac{1}{2})^{m-n}]} -1\right| \\
&\leq |x_2| \cdot \max\{\left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}\cdot(\frac{2}{3})\cdot[1-(-\frac{1}{2})^{m-n}]} -1, 1 - \left(\frac{x_2}{x_1}\right)^{(-\frac{1}{2})^{n-1}\cdot(\frac{2}{3})\cdot[1-(-\frac{1}{2})^{m-n}]}\} \\
&\overset{*}{\leq} |x_2| \cdot \max\{\left(\frac{x_2}{x_1}\right)^{(\frac{1}{2})^{n-1}} -1, 1 - \left(\frac{x_2}{x_1}\right)^{-(\frac{1}{2})^{n-1}}\} \\
&\leq |x_2| \cdot \frac{\epsilon}{|x_2|} \\
&= \epsilon
\end{aligned}$$

$\left(\text{Note: } * \text{ is true because } -(\frac{1}{2})^{n-1} \leq (-\frac{1}{2})^{n-1} \cdot (\frac{2}{3}) \cdot [1 - (-\frac{1}{2})^{m-n}] \leq (\frac{1}{2})^{n-1} \right)$

That is, $\forall \epsilon > 0, \exists N, \forall m > n > N, |x_m - x_n| \leq \epsilon$.
Therefore, by definition of Cauchy sequence, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, hence it converges.

P.S. There might exits some typos in these answers, please tell me if you find any. Thanks!