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#2. Prove that $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. for $A, B \subseteq \mathbb{R}^n$

Recall: $\overline{A} = A \cup \partial A$
$\partial A = \{\vec{x} \in \mathbb{R}^n \mid \forall r>0, B(r,\vec{x}) \cap A \neq \emptyset \text{ and } B(r,\vec{x}) \cap A^c \neq \emptyset\}$

This means that $\overline{A} = \{x \in \mathbb{R}^n \mid \forall r>0, B(r,\vec{x}) \cap A \neq \emptyset\}$

Proof: Suppose that $\vec{x} \in \overline{A}$, need to show that $\vec{x} \in \overline{B}$.

Fix $r_0>0$ and consider $B(r_0,\vec{x})$
Since $\overline{A} = \{\vec{y} \in \mathbb{R}^n \mid \forall r>0, B(r,\vec{y}) \cap A \neq \emptyset\}$ we see that, $B(r_0,\vec{x}) \cap A \neq \emptyset$.
But $A \subseteq B \Rightarrow B(r_0,\vec{x}) \cap B \neq \emptyset$. Since $r_0>0$ was arbitrary it follows that
$(\forall r>0, B(r,\vec{x}) \cap B \neq \emptyset) \Rightarrow \vec{x} \in \overline{B}$.
Since $x \in \overline{A}$ was arbitrary, this shows that $\overline{A} \subseteq \overline{B}$. ■

#1. Quantifiers

$\forall, \exists, \ni_{\text{s.t.}}, \Rightarrow, \Leftrightarrow, \sim$ not/negation

$\sim[\forall x (\cdots)] \Leftrightarrow \exists x \sim(\cdots)$

Q 2.14
A function f is strictly decreasing if for every x and for every y, if $x<y$ then $f(x)>f(y)$.

(i) $\forall x \forall y, x<y \Rightarrow f(x)>f(y)$

(ii) $\sim(\forall x \forall y, x<y \Rightarrow f(x)>f(y))$

$\Leftrightarrow$ (iii) $\exists x \exists y, \sim(x<y \Rightarrow f(x)>f(y))$

$\Leftrightarrow$ (iv) $\exists x \exists y, x<y$ and $f(x) \leq f(y)$

#4. $f: \mathbb{R}^n \to \mathbb{R}^k$, $f^{-1}(U) = \{x \in \mathbb{R}^n \mid f(x) \in U\}$
From class, $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$

(a). Prove $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$

Proof: $f^{-1}(U \cap V) = \{x \in \mathbb{R}^n \mid f(x) \in U \cap V\} = \{x \in \mathbb{R}^n \mid f(x) \in U \text{ and } f(x) \in V\}$

If $f(x) \in U \Rightarrow x \in f^{-1}(U)$, Similarly $f(x) \in V \Rightarrow x \in f^{-1}(V)$.

So if $f(x) \in U \cap V \Rightarrow x \in f^{-1}(U) \cap f^{-1}(V) \Rightarrow f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$

(b). $U^c = \{x \in \mathbb{R}^n \mid x \notin U\}$
Prove that $f^{-1}(U^c) = (f^{-1}(U))^c$

Since $x \in f^{-1}(U) \iff f(x) \in U$, it follows that if $f(x) \notin U$ then $x \notin f^{-1}(U)$

$f^{-1}(U^c) = \{x \in \mathbb{R}^n \mid x \notin f^{-1}(U)\} = (f^{-1}(U))^c$

#3. Section 1.2

Q9 Let $a \in \mathbb{R}^n$, and $r \geq 0$. Prove $B(r, a) \subseteq B(r + |a|, 0)$

Pf: Let $x \in B(r, a) \Rightarrow |x - a| < r$
Now calculate $|x - 0| = |x - a + a - 0| \leq |x - a| + |a| \leq r + |a|$