Term 2 is coming!

§ 2.3 Two-phase method : can solve any canonical linear programming problem.

Eg. Maximize $z = 2x_1 + 3x_2$ s.t.

$$\begin{aligned} x_1 + x_2 &\le 3 \\ 2x_1 + x_2 &\ge 4 \\ 3x_1 - x_2 &= -6 \\ x_1 \ge 0,\ x_2 &\ge 0. \end{aligned}$$

Put in canonical form, so that each constraint has a non-negative right hand side :

$$\left.\begin{aligned} &\text{Maximize } z = 2x_1 + 3x_2 \text{ s.t.} \\ &x_1 + x_2 + x_3 = 3 \\ &2x_1 + x_2 - x_4 = 4 \\ &-3x_1 + x_2 = 6 \\ &x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0 \end{aligned}\right\} \circledast$$

Phase 1 will determine whether $\circledast$ has a solution and if so, will find one

Also : the solution should be basic.

(Now, only x_3 can serve as a basic variable).

We introduce an artificial variable (y_1 and y_2) into each constraint which lacks a basic variable and set up the phase 1 auxiliary problem.

Maximize $z = y_1 + y_2$ (or Maximize $z = -y_1 - y_2 \leftarrow -\sum$ all artificial variables)

$$\left.\begin{aligned} \text{s.t.}\quad &x_1 + x_2 + x_3 &&= 3 \\ &2x_1 + x_2 - x_4 + y_1 &&= 4 \\ &-3x_1 + x_2 + y_2 &&= 6 \\ &x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, y_1 \ge 0, y_2 \ge 0 \end{aligned}\right\} \oplus$$

This problem is bounded (that is, has an optimal solution) where either

① $z = 0$

② $z > 0$

If $z = 0$, then $y_1 = 0$, $y_2 = 0$, and $x_1, \dots, x_4$ are feasible for $\circledast$.

If $z > 0$, is optimal, then any $x_1, \dots, x_4, y_1, y_2$ that are feasible for $\oplus$ have $y_1 \ne 0$ or $y_2 \ne 0$, so $x_1, \dots, x_4$ are infeasible for $\circledast$ for any $x_1, \dots, x_4$.

A simplex solution of the auxiliary problem

Tableau 0

	x_1	x_2	x_3	x_4	y_1	y_2		x_2 with ratio x_2 col θ-ratio
x_3	1	1	1	0	0	0	3	$\frac{3}{1}$ ← x_3 ratio
y_1	2	1	0	-1	1	0	4	$\frac{4}{1}$
y_2	-3	1	0	0	0	1	6	$\frac{6}{1}$
	0	0	0	0	1	1	0	

The optimality criterion does not apply because some basic variables have non-negative coefficients in the objective row. we eliminate these according to routine by replacing the objective row with objective row $- y_1$ row $- y_2$.

Tableau 2

	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	1	1	1	0	0	0	3
y_1	1	0	-1	-1	1	0	1
y_2	-4	0	-1	0	0	1	3
	3	0	2	1	0	0	-4

Optimal tableau:
No feasible solution to $\oplus$ has $y_1=0, y_2=0$. $\otimes$ has no feasible solution.

Notes on "A Two-phase optimization"

Eg. Maximize $z = -2x_1 - 3x_2 - 2x_3$ s.t.

$$6x_1 - x_2 = 32$$

$$-2x_1 + 4x_2 + 3x_3 = 12$$

$$7x_1 - 5x_2 - 3x_3 \geq 20$$

$$3x_1 + 3x_2 + 3x_3 = 44,$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

After putting in the slack x_4, and setting up the auxiliary problem, one gets phase 1, tableau 0