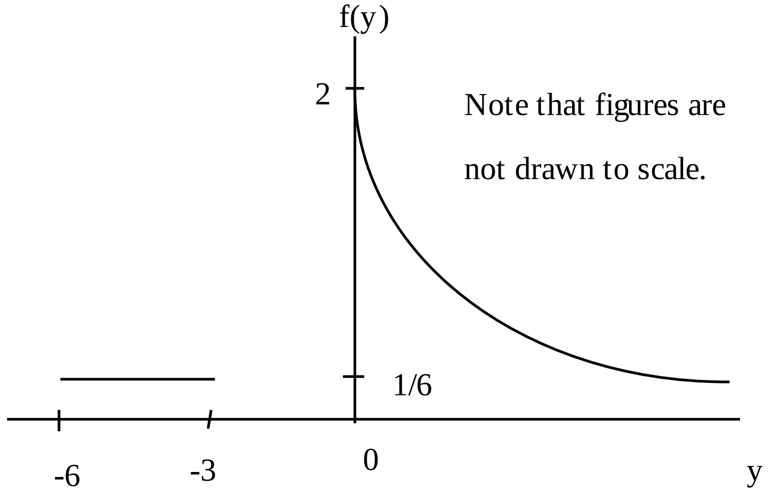
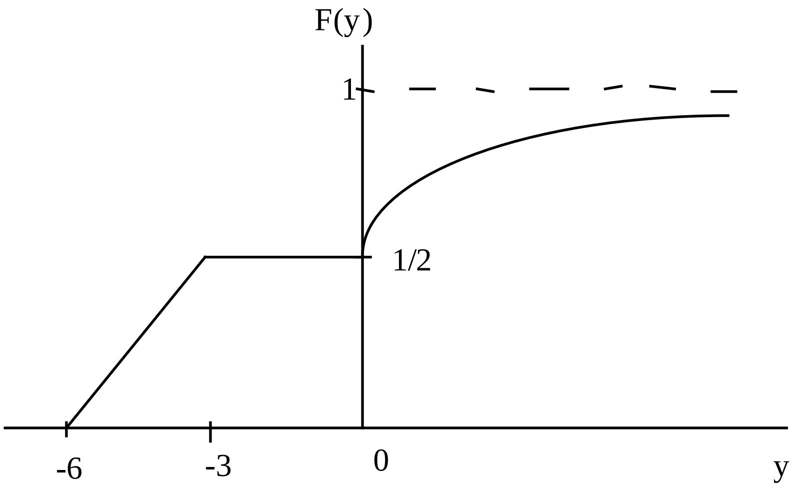
STAT2001/6039 Assignment 2 Solutions (2017)

Problem 1 **(a)** For -6 < y < -3, $F(y)=\int_{-6}^{y}\frac{1}{6}dt=1+\frac{y}{6}$. Thus $F(y)$ = 1/2 at y = -3, 0.

For y > 0, $F(y)=\frac{1}{2}+\int_{0}^{y}2e^{kt}dt=\frac{1}{2}+\frac{2}{k}(e^{ky}-1)$.

Now $F(y)\to 1$ as $y\to\infty$. So k = -4. Hence $F(y)=\begin{cases}0, & y\le -6\\ 1+y/6, & -6<y<-3\\ 1/2, & -3\le y\le 0\\ 1-e^{-4y}/2, & y>0\end{cases}$





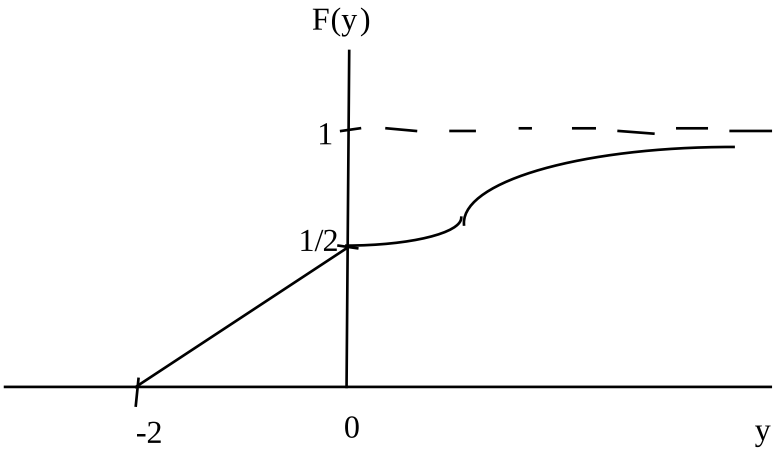
(b) $\mu = \int_{-6}^{-3} y \frac{1}{6} dy + \int_{0}^{\infty} y 2e^{-4y} dy$ = -2.125 (mean).

$EY^2 = \int_{-6}^{-3} y^2 \frac{1}{6} dy + \int_{0}^{\infty} y^2 2e^{-4y} dy$ = 10.5625. Hence $\sigma^2 = EY^2 - \mu^2$ = 6.047 (variance).

So σ = 2.459 (standard deviation).

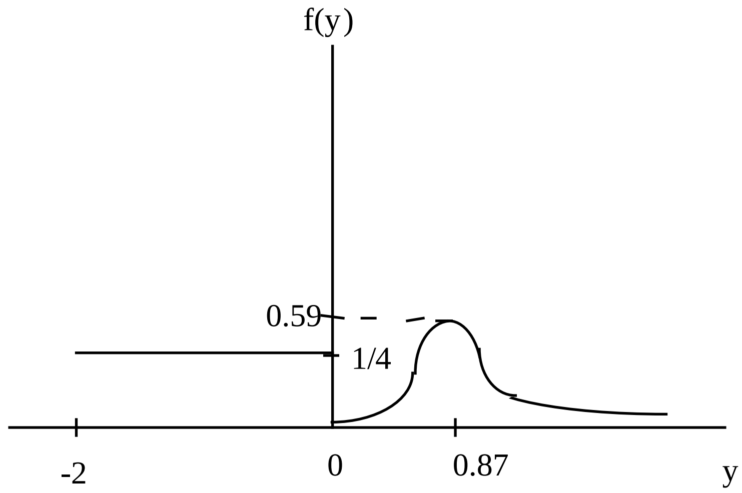
(c) Mode = 0. Median = any number from -3 to 0, inclusive.

Problem 2 **(a)**



$$f(y) = F'(y) = \begin{cases} a/2, & -2 < y < 0 \\ 3bcy^2 e^{-cy^3}, & y > 0 \end{cases}$$

By equating the derivative of $f(y)$ to zero, we find that $f(y)$ has a peak at (0.87, 0.59) when $a = b = 1/2$ and $c = 1$.

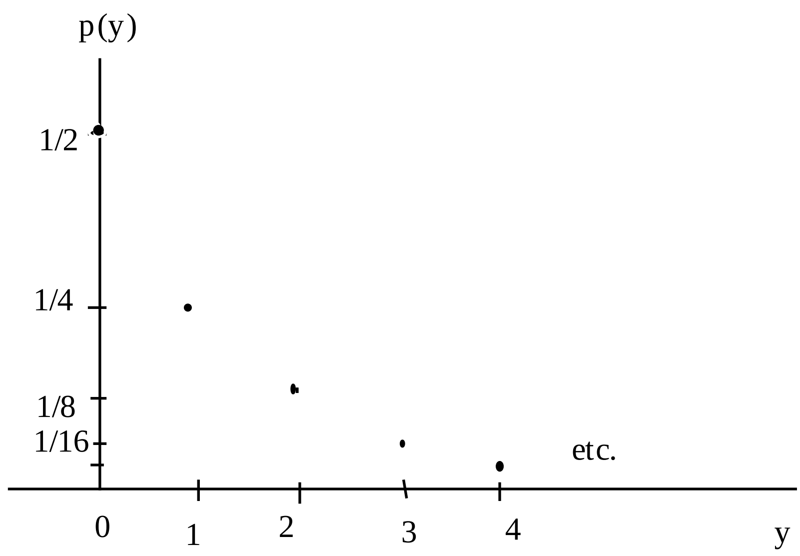


Range of possible values for *a*, *b* and *c*: $0 \le a \le 1$,
$b = 1 - a$,
$-\infty < c < \infty$ if $b = 0$, $c > 0$ if $b > 0$.

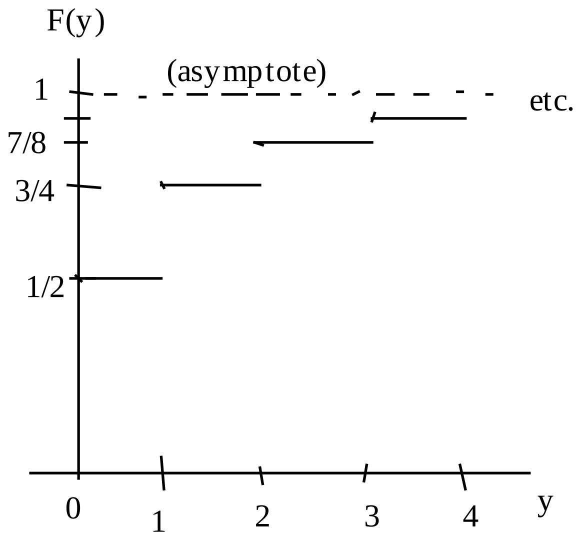
(b) The required probability is

$$\frac{P(-1<Y<1)}{P(Y<1)} = \frac{F(1)-F(-1)}{F(1)} = 1-\frac{F(-1)}{F(1)} = 1-\frac{1/4}{1-e^{-1}/2} = 0.694\,.$$

Problem 3 **(a)**



$$F(y) = \begin{cases} 0, & y < 0 \\ 1/2, & 0 \le y < 1 \\ 3/4, & 1 \le y < 2 \\ 7/8, & 2 \le y < 3 \\ 15/16, & 3 \le y < 4 \\ & \text{etc.} \end{cases}$$

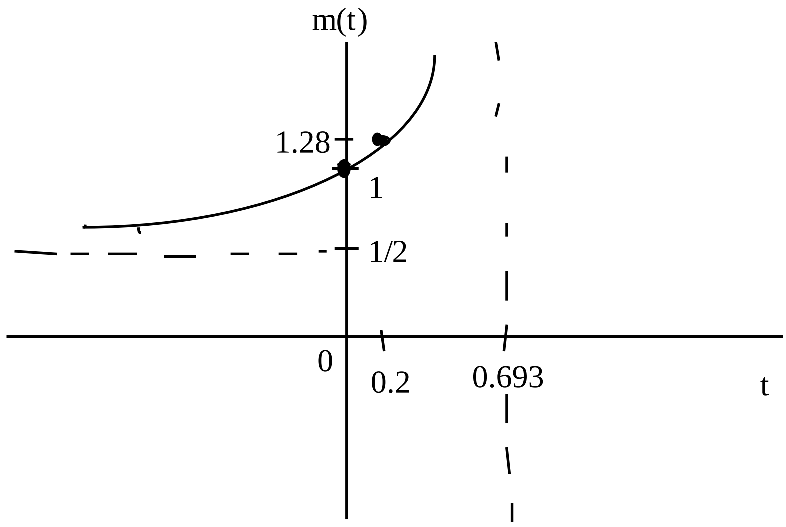


Range of possible values for b and k: $0 < k < 1$, $b = 1 - k$.

(Also possible are $b = 1$ and $k = 0$ if we define $0^0 = 1$.)

(b) $m(t) = Ee^{Yt} = \sum_{y=0}^{\infty} e^{yt} bk^y = b\sum_{y=0}^{\infty}(ke^t)^y = \dfrac{1-k}{1-ke^t}$, $t < -\log k$ $(0 < k < 1)$.

When $b = k = 1/2$: $m(0) = 1$, $m(0.2) = 1.284$, $m(t) \to 1/2$ as
$t \to -\infty$, $m(t) \to +\infty$ as $t \to \log 2 = 0.693$ from the left,
and $m(t)$ is undefined for $t \ge \log 2$ (for then, $|(1/2)e^t| \ge 1$).



$m'(t) = (1-k)ke^t(1-ke^t)^{-2}$. So $\mu = m'(0) = 1$.

Now we can also write $m'(t) = \mu e^t m(t)^2$.

Hence $m''(t) = \mu\{e^t 2m(t)m'(t) + e^t m(t)^2\}$.
Thus $\mu_2' = m''(0) = 2\mu^2 + \mu$.

Therefore the variance of Y is $\sigma^2 = \mu_2' - \mu^2 = (2\mu^2 + \mu) - \mu^2 = \mu^2 + \mu = 1^2 + 1 = 2$.

Alternatively, we observe that Y has the same variance as the geometric distribution with parameter 1/2.

(In fact, $Y = X - 1$ where $X \sim$ Geom(1/2).)

Thus $VarY = (1-1/2)/(1/2)^2 = 2$.

Problem 4 (a)
Approximately, $0.2 = P(Y > 91) = P\left(Z > \dfrac{91-\mu}{\sigma}\right)$, where $Z \sim \mathrm{N}(0,1)$.
But $P(Z > 0.84) = 0.2$ (from tables).
Hence $(91-\mu)/\sigma = 0.84$. (1)

Also, $0.25 = P(Y < 46) = P\left(Z < \dfrac{46-\mu}{\sigma}\right)$.

But $P(Z < -0.675) = 0.25$.

Hence $(46-\mu)/\sigma = -0.675$. (2)

By (1) and (2), μ = 66.05 and σ = 29.703.

Therefore the required proportion is

$$P(Y > 60) = P\left(Z > \frac{60-66.05}{29.703}\right) = P(Z > -0.20) = 0.5793.$$

(b) $$P(Y > 60 \mid Y < \mu) = \frac{P(60 < Y < \mu)}{P(Y < \mu)} = \frac{P(Y > 60) - P(Y > \mu)}{P(Y > \mu)}$$
$$= \frac{0.5793 - 0.5}{0.5} = 0.1586.$$

Problem 5 (a)

Let Y be the length of the segment to the LEFT of the saw point.

Then $Y \sim U(0,2)$.

So EY = 1, $VarY = (2-0)^2/12$ =1/3 and $EY^2 = VarY + (EY)^2 = 4/3$.

Let Z be the cost of the gold in dollars.

Now the cost of gold dots for a single die is $(1+2+3+4+5+6)\times 15 = 315$.

Also, 5400 = $30^2 \times 6$.

So both dice will have gold dots if $0.3 < Y < 1.7$;

otherwise only one die will have gold dots.

Thus Z = 630 with probability $(1.7-0.3)/2 = 0.7$,

and Z = 315 with probability 0.3.

Hence $EZ = 0.7(630) + 0.3(315) = 535.5$.

Next let X be the total cost of the two dice.

Then $X = 34 + 42(6)\{Y^2 + (2-Y)^2\} + 66\{Y^3 + (2-Y)^3\} + Z$

$= 1570 - 1800Y + 900Y^2 + Z$.

Hence $EX = 1570 - 1800(1) + 900(4/3) + 535.5 = 1505.5$.

So to make an expected profit of \$700, the two dice should be sold for \$2205.50.