APM 462 FINAL REVIEW

① In principle, first half material.
  (a) unconstrained minimization problems in E^n or in a subset Ω of E^n.
  (b). convex sets/ functions
  (c). algorithms for finding minima (steepest descent etc.)
    (see About midterm pdf for details)

② minimization problems with equality constraints.
 (a). first order conditions & Lagrange multipliers

FIRST ORDER NECESSARY CONDITION.
Let $\vec{x}^*$ be a regular point of the constraints $\vec{h}(\vec{x}) = \vec{0}$ and a local extremum point (a minimum or maximum) of f subject to those constraints. Then all $\vec{y} \in E^n$ satisfying

$$\vec{\nabla}\vec{h}(\vec{x}^*)\vec{y} = \vec{0}$$

must also satisfy

$$\vec{\nabla}f(\vec{x}^*)\vec{y} = 0$$

• $\vec{\nabla}f(\vec{x}^*)$ is orthogonal to the tangent plane. $\Rightarrow \vec{\nabla}f(\vec{x}^*)$ is a linear combination of the gradients of $\vec{h}$ at $\vec{x}^*$.

So: Thm: Let $\vec{x}^*$ be a local extremum point of f subject to the constraints $\vec{h}(\vec{x}) = \vec{0}$. Assume further that $\vec{x}^*$ is a regular point of these constraints. Then $\exists$ a $\vec{\lambda} \in E^m$ such that

$$\vec{\nabla}f(\vec{x}^*) + \vec{\lambda}^T\vec{\nabla}\vec{h}(\vec{x}^*) = \vec{0}$$

(b). regular points, ~~tag~~ tangent spaces.

Consider all differentiable curves on S passing through a point $\vec{x}^*$. The tangent plane at $\vec{x}^*$ is the collection of the derivatives at $\vec{x}^*$ of all these differentiable curves. The tangent plane is subspace of E^n.

Def of regular point: A point $\vec{x}^*$ satisfying the constraint $\vec{h}(\vec{x}^*)=\vec{0}$ is said to be a regular point of the constraint if the gradient vectors $\vec{\nabla}h_1(\vec{x}^*)$, $\vec{\nabla}h_2(\vec{x}^*)$, ---, $\vec{\nabla}h_m(\vec{x}^*)$ are linearly independent.

Thm: At a regular point $\vec{x}^*$ of the surface S defined by $\vec{h}(\vec{x})=\vec{0}$ the tangent plane is equal to

$$M=\{\vec{y}: \nabla\vec{h}(\vec{x}^*)\vec{y}=\vec{0}\}$$

(C). second-order conditions

SECOND-ORDER NECESSARY CONDITIONS

$f, \vec{h}\in C^2$

Sps $\vec{x}^*$ is a local minimum of f subject to $\vec{h}(\vec{x})=\vec{0}$ and that $\vec{x}^*$ is a regular point of these constraints. Then there is a $\vec{\lambda}\in E^m$ s.t.

$$\vec{\nabla}f(\vec{x}^*)+\vec{\lambda}^T\nabla\vec{h}(\vec{x}^*)=\vec{0}$$

If we denote by M the tangent plane $M=\{y: \nabla\vec{h}(\vec{x}^*)\vec{y}=\vec{0}\}$ then the matrix $\vec{L}(\vec{x}^*)=\vec{F}(\vec{x}^*)+\vec{\lambda}^TH(\vec{x}^*)$ is positive ~~defi~~ semidefinite on M, that is, $\vec{y}^T\vec{L}(\vec{x}^*)\vec{y}\geq\vec{0}$ $\forall \vec{y}\in M$

SECOND-ORDER SUFFICIENT CONDITIONS:

Sps $\exists$ $\vec{x}^*$ satisfying $\vec{h}(\vec{x}^*)=\vec{0}$ & a $\vec{\lambda}\in E^m$ s.t.

$$\vec{\nabla}f(\vec{x}^*)+\vec{\lambda}^T\vec{\nabla}\vec{h}(\vec{x}^*)=\vec{0}$$

Sps also that the matrix $\vec{L}(\vec{x}^*)=\vec{F}(\vec{x}^*)+\vec{\lambda}^T\vec{H}(\vec{x}^*)$ is positive definite on $M=\{\vec{y}: \nabla\vec{h}(\vec{x}^*)\vec{y}=\vec{0}\}$, that is, for $\vec{y}\in M$, $\vec{y}\neq\vec{0}$ there holds $\vec{y}^TL(\vec{x}^*)\vec{y}>\vec{0}$. Then $\vec{x}^*$ is a strict local minimum of f subject to $\vec{h}(\vec{x})=\vec{0}$.

③ minimization problems with both equality & inequality constraints.
Particularly KKT Thm.

"Mixed constraints"

Def. $\vec{x}^*$ be a point satisfying the constraints

$$(\circledast)\quad \begin{cases} \vec{h}(\vec{x}^*) = \vec{0} \\ \vec{g}(\vec{x}^*) \le \vec{0} \end{cases}$$

and let J be the set of indices j for which $g_j(\vec{x}^*) = 0$. Then $\vec{x}^*$ is said to be a regular point of the constraints above ($\circledast$) if the gradient vectors $\nabla h_i(\vec{x}^*)$, $\nabla g_j(\vec{x}^*)$, ~~1≤j≤m~~ $1 \le i \le m$, $j \in J$ are linearly independent.

Karush-Kuhn-Tucker conditions:
Let $\vec{x}^*$ be a relative minimum point for the problem

minimize $f(\vec{x})$
subject to $\vec{h}(\vec{x}) = \vec{0}$, $\vec{g}(\vec{x}) \le \vec{0}$.

and suppose $\vec{x}^*$ is a regular point for the constraints. Then there is a vector $\vec{\lambda} \in E^m$ and a vector $\vec{\mu} \in E^p$ with $\vec{\mu} \ge \vec{0}$ s.t.

$$\left.\begin{aligned} \nabla f(\vec{x}^*) + \vec{\lambda}^T \nabla \vec{h}(\vec{x}^*) + \vec{\mu}^T \nabla \vec{g}(\vec{x}^*) &= \vec{0} \\ \vec{\mu}^T \vec{g}(\vec{x}^*) &= \vec{0} \end{aligned}\right\} (\circledast\circledast)$$

SECOND-ORDER CONDITIONS
NECESSARY: Sps the function $f, \vec{g}, \vec{h} \in C^2$ & $\vec{x}^*$ is a regular point of the constraints ($\circledast$). If $\vec{x}^*$ is a regular ~~pt~~ minimum pt then $\exists$ a $\vec{\lambda} \in E^m$, $\vec{\mu} \in E^p$, $\vec{\mu} \ge \vec{0}$ s.t.
the KKT holds ($\circledast\circledast$)
& such that

$$L(\vec{x}^*) = F(\vec{x}^*) + \vec{\lambda}^T H(\vec{x}^*) + \vec{\mu}^T G(\vec{x}^*)$$

is positive semidefinite on the ~~tangent~~ subspace of the active constraints at $\vec{x}^*$.

SECOND-ORDER SUFFICIENT CONDITION:

Let $f, \vec{g}, \vec{h} \in C^2$.

a point $\vec{x}^*$ satisfying the "mixed constraints" (*)
is a strict minimum pt of problem is that
there exists $\vec{\lambda} \in E^m$, $\vec{\mu} \in E^p$ such that

$$\vec{\mu} \geq 0$$
$$\vec{\mu}^T \vec{g}(\vec{x}^*) = 0$$
$$\nabla f(\vec{x}^*) + \vec{\lambda}^T \nabla \vec{h}(\vec{x}^*) + \vec{\mu}^T \nabla \vec{g}(\vec{x}^*) = 0$$

and the Hessian matrix

$$L(\vec{x}^*) = F(\vec{x}^*) + \vec{\lambda}^T H(\vec{x}^*) + \vec{\mu}^T G(\vec{x}^*)$$

is positive ~~semi~~ definite on the subspace

$$M' = \{\vec{y} : \nabla \vec{h}(\vec{x}^*)\vec{y} = 0, \nabla g_j(\vec{x}^*)\vec{y} = 0 \ \forall j \in J\}$$
where $J = \{j : g_j(\vec{x}^*) = 0, \mu_j > 0\}$

(4) The $\underbrace{\text{Calculus of Variations}}$, § 4.1.1 on notes of Evans.

Basic problem of ↓
Find a curve $x^*(\cdot) : [0, T] \to \mathbb{R}^n$ that minimizes the functional

~~$I[x$~~
$$I[x(\cdot)] := \int_0^T L(x(t), x'(t))\, dt$$
among all functions $x(\cdot)$ satisfying $x(0) = x^0$ and $x(T) = x^1$

§4.1.1 Derivation of Euler-Lagrange equations

Note: We write $L = L(x, v)$ & regard x as "position", then var v as "velocity", the partial derivatives of L are

$$\frac{\partial L}{\partial x_i} = L_{x_i}, \quad \frac{\partial L}{\partial v_i} = L_{v_i} \quad (1 \leq i \leq n)$$

& we write $\nabla_x L = (L_{x_1}, \dots, L_{x_n})$,
$\nabla_v L = (L_{v_1}, \dots, L_{v_n})$

Find a curve $x^*(\cdot): [0,T] \to \mathbb{R}^n$ that minimizes $I[x(\cdot)] = \int_0^T L(x(t), x'(t))\,dt$
satisfying $x(0) = x^0$, $x(T) = x^1$

~~Let~~ Write $L = L(x, v)$.

$$\frac{\partial L}{\partial x_i} = L_{x_i} \quad \frac{\partial L}{\partial v_i} = L_{v_i}$$

$$\nabla_x L = (L_{x_1}, \dots, L_{x_n}) \quad \nabla_v L = (L_{v_1}, \dots, L_{v_n})$$

Thm (E-L equations): Let $x^*(\cdot)$ solves, then $x^*(\cdot)$ solves

$$\frac{d}{dt}[\nabla_v L(x^*(t), x^{*\prime}(t))] = \nabla_x L(x^*(t), x^{*\prime}(t))$$

ith component is

$$\frac{d}{dt}[L_{v_i}(x^*(t), x^{*\prime}(t))] = L_{x_i}(x^*(t), x^{*\prime}(t))$$

select ~~y~~ curve $y\,[0,T] \to \mathbb{R}^n$ that $y(0) = y(T) = 0$.
let $i(\tau) = I[x(\cdot) + \tau y(\cdot)]$
For $\tau \in \mathbb{R}$, $x(\cdot) = x^*(\cdot)$
$x(\cdot)$ is minimizer
$i(\tau) \geq I[x(\cdot)] = i(0)$
$\Rightarrow$ $i(\cdot)$ has min at $\tau = 0$, so $i'(0) = 0$.
must compute $i'(\tau)$.
Note that

$$i(\tau) = \int_0^T L(x(t) + \tau y(t), x'(t) + \tau y'(t))\,dt$$

$$i'(\tau) = \sum_{i=1}^n \int_0^T L_{x_i}(x(t), x'(t)) y_i(t) + L_{v_i}(x(t), x'(t)) y_i'(t)\,dt$$

$y(0) = y(T) = 0$

Integration by parts: $\int L_v(\cdot\sim) y'\,dt = L_v(\sim) y \Big|_0^T - \int \frac{d}{dt}(L_v(\cdot\sim)) y(t)\,dt$

Combine to find: $0 = i'(0) = \int_0^T [L_x(x^*(t), x^{*\prime}(t)) - \frac{d}{dt} L_v(x^*(t), x^{*\prime}(t))] y(t)\,dt$

~~impossible if $w(t) = 0$ for all $t \in [0,T]$~~

Thm. (E-L equations):
Let $\vec{x}^*(\cdot)$ solve the calculus of variations problem, then $\vec{x}^*(\cdot)$ solves the Euler-Lagrange differential equations:

$$\text{(E-L)} \qquad \frac{d}{dt}\left[\nabla_v L(\vec{x}^*(t), \vec{x}'^*(t))\right] = \nabla_x L(\vec{x}^*(t), \vec{x}'^*(t))$$

(n-second-order ODE, the ith component of system reads)

$$\frac{d}{dt}\left[L_{v_i}(\vec{x}^*(t), \vec{x}'^*(t))\right] = L_{x_i}(\vec{x}^*(t), \vec{x}'^*(t))$$

(5) system of ordinary differential equations

ODE has the form
$$\begin{cases} x'(t) = f(x(t)) & (t>0) \\ x(0) = x^0 \end{cases}$$
initial point $x^0 \in \mathbb{R}^n$ and the function $\mathbb{R}^n \to \mathbb{R}^n$.
The unknown is the curve $x: [0,\infty) \to \mathbb{R}^n$

controlled dynamics, say f depends also upon on some "control" parameters belonging to a set $A \subset \mathbb{R}^m$; so that $f: \mathbb{R}^n \times A \to \mathbb{R}^n$. Then if we select some $a \in A$ & consider the corresponding dynamics:
$$\begin{cases} x'(t) = f(x(t), a) & (t>0) \\ x(0) = x^0 \end{cases}$$

More generally, call a function $\alpha: [0,\infty) \to A$ is a control, corresponding to each control, we consider ODE
$$\begin{cases} x'(t) = f(x(t), \alpha(t)) & (t>0) \\ x(0) = x^0 \end{cases}$$
and the trajectory $x(\cdot)$ is the response of the system.

Def we define the reachable set for time t to be

$C(t)$ = set of initial ~~set~~ points x^0 for which there exists a control such that $x(t)=0$

and overall reachable set

C = set of initial points x^0 for which there exists a control such that $x(t)=0$ for some finite time t.

Note that $C = \bigcup_{t\geq 0} C(t)$

Let $\mathcal{M}^{n\times m}$ denote all $n\times m$ matrices.

$$\begin{cases} x'(t) = \mathcal{M}x(t) + \mathcal{N}\alpha(t) & (t>0) \\ x(0) = x^0 \end{cases}$$

Quick review for Linear ODE

$X(\cdot): \mathbb{R} \to \mathcal{M}^{n\times m}$ be the unique solution of ~~a~~ matrix ODE

$$\begin{cases} X'(t) = \mathcal{M}X(t) & (t\in\mathbb{R}) \\ X(0) = I \end{cases}$$

We call $X(\cdot)$ a fundamental solution & some times write $X(t) = e^{tM} := \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}$

- $X^{-1}(t) = X(-t)$

*Thm 2.1 (Solving ~~the~~ Linear systems of ODE)

① The unique solution of homogeneous system of ODE

$$\begin{cases} x'(t) = \mathcal{M}x(t) \\ x(0) = x^0 \end{cases}$$

is $x(t) = X(t)x^0 = e^{tM}x^0$

② The unique solution of nonhomogeneous system

$$\begin{cases} x'(t) = \mathcal{M}x(t) + f(t) \\ x(0) = x^0 \end{cases}$$

is $x(t) = X(t)x^0 + X(t)\int_0^t X^{-1}(s)f(s)\,dt$

$$\boxed{x(t) = e^{tM}x^0 + e^{tM}\int_0^t e^{-sM} N\alpha(s)\,ds}$$

$M=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $e^{tM}=\begin{pmatrix}1&t\\0&1\end{pmatrix}$ $M=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $e^{tM}=\begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}$ $M=\begin{pmatrix}a&0\\0&b\end{pmatrix}$

$e^{tM}=\begin{pmatrix}e^{at}&0\\0&e^{bt}\end{pmatrix}$

(6) CONTROLLABILITY OF LINEAR SYSTEMS

(a). In particular Thm 2.2, 2.3, 2.5.

Thm 2.2. (Structure of readable set)

(i) The reachable set C is symmetric & convex

(ii) Also, if $x^0 \in C(\bar{t})$, then $x^0 \in C(t)$ for all times $t \geq \bar{t}$.

def: (i). symmetric if $x \in S \Rightarrow -x \in S$

(ii). convex if $x, \hat{x} \in S$ & $0 \leq \lambda \leq 1$, imply $\lambda x + (1-\lambda)\hat{x} \in S$

Definition: the controllability matrix is

$$G = G(M,N) = \underbrace{[N, MN, M^2N, \cdots, M^{n-1}N]}_{n \times (mn) \text{ matrix}}$$

Thm 2.3 (CONTROLLABILITY MATRIX)

We have rank $G=n$ iff $0 \in C^\circ$.

Def: linear system (ODE) is controllable if $C = \mathbb{R}^n$.

Thm 2.5 (CRITERION FOR CONTROLLABILITY)

~~Let A be the cube $[-1,1]^n$ in $\mathbb{R}^n$~~,

Sps as well rank $G=n$, and $\mathrm{Re}\,\lambda \leq 0$ for each eigenvalue λ of the matrix M.

Then the system (ODE) is controllable.

(b). Bang-bang controls (Thm 2.8)

take A to be the cube $[-1,1]^m$ in $\mathbb{R}^m$

Def: A control $\alpha(\cdot) \in A$ is called bang-bang if each time $t \geq 0$ & each index $i=1,\cdots,m$ we have $|\alpha^i(t)|=1$

where $\alpha(t) = \begin{pmatrix}\alpha^1(t)\\ \vdots \\ \alpha^m(t)\end{pmatrix}$

Thm 2.8: Let $t>0$, suppose $x^0 \in C(t)$, for the system

$\dot{x}(t) = Mx(t) + N\alpha(t)$

Then $\exists$ a bang-bang control $\alpha(\cdot)$ which steers x^0 to 0 at time t.

⑦ LINEAR TIME-OPTIMAL CONTROL

Particularly, Pontryagin Maximum Principle:

Def: the control theory Hamiltonian is the function

$$H(x,p,a) = f(x,a)\cdot p + r(x,a) \quad (x,p \in \mathbb{R}^n, a \in A)$$

Thm: Assume $\alpha^*(\cdot)$ is optimal for (ODE), (P) & $x^*(\cdot)$ is the corresponding trajectory.

Then $\exists$ a function $p^*: [0,T] \to \mathbb{R}^n$ s.t.

(ODE) $\quad x'^*(t) = \nabla_p H(x^*(t), p^*(t), \alpha^*(t))$,

(ADJ) $\quad p'^*(t) = -\nabla_x H(x^*(t), p^*(t), \alpha^*(t))$,

&

(M) $\quad H(x^*(t), p^*(t), \alpha^*(t)) = \max_{a \in A} H(x^*(t), p^*(t), a) \quad (0 \le t \le T)$

In addition, the mapping

$$t \mapsto H(x^*(t), p^*(t), \alpha^*(t)) \text{ is a constant}$$

Finally, we have the terminal condition,

$$p^*(T) = \nabla g(x^*(T)).$$

$$h^T e^{-tM} \alpha = \max_{a \in A} \{h^T e^{-tM} N a\} \quad (\text{Ch}3.33\text{-}35)$$

§3.1 Existence of time-optimal controls

$$\begin{cases} x'(t) = Mx(t) + N\alpha(t) \\ x(0) = x^0. \end{cases}$$

optimal time problem

given starting pt $x^0 \in \mathbb{R}^n$, wts an optimal control $\alpha^*(\cdot)$ s.t.

$$P[\alpha^*(\cdot)] = \max_{\alpha(\cdot) \in \mathcal{A}} P[\alpha(\cdot)]$$

Then $\tau^* = -P[\alpha^*(\cdot)]$ is the minimum time to steer to origin.

§3.2 The maximum principle for linear time-optimal control.

def: $K(t, x^0)$ be the reachable set for time t,

$K(t,x^0) = \{x' \mid \exists\ \alpha(\cdot) \in A$ which steers from x^0 to x' at time $t\}$

~~Solve~~ Since $x(\cdot)$ ODE, have $x' \in K(t, x^0)$ iff

$$x' = X(t)x^0 + X(t)\int_0^t X^{-1}(s) N \alpha(s)\, ds = x(t)$$

for some control $\alpha(\cdot) \in A$

Thm: Potryagin Maximum principle. for LTO control.

$\exists\ \vec{h} \neq \vec{0}$ s.t.

$$(M) \qquad h^T X^{-1}(t) N \alpha^*(t) = \max_{a \in A} \{h^T X^{-1}(t) N a\}$$

for each time $t \in [0, \tau^*]$

Reachable set:

set of initial pts x^0 for which $\exists$ a control such that

$x(t) = 0$

Practice problem minimization with equality

minimize $f(x) = \frac{1}{2}(x^2+y^2) - z$

~~subject to $h(x) = \frac{1}{2}(y^2+z^2) - x = 0$~~ ~~$d^T x + e = 0$.~~

~~Q positive definite, b, d column vector.~~

subject to $h(x) = \frac{1}{2}(y^2+z^2) - x = 0$

$$\left.\begin{aligned} &x - \lambda = 0 \\ &y + \lambda y = 0 \\ &-1 + \lambda z = 0 \\ &h(x,y,z) = 0 \\ &(x,y,z) = \end{aligned}\right\}$$

$x = \lambda = -1$

$\cancel{y} \cdot \lambda = -1$ $\qquad \frac{1}{2}(y^2+1)+1 = 0$

$z = \frac{1}{-1} = -1$

Impossible !

So $y = 0$,

$\frac{1}{2}z^2 - x = 0 \quad x = \lambda, \quad z = \frac{1}{\lambda}$

$\frac{1}{2} \cdot \frac{1}{\lambda^2} - \lambda = 0$

$\lambda \cdot 2\lambda^2 = 1$

$2\lambda^3 = 1$

$\lambda^3 = \sqrt[3]{\frac{1}{2}}$

So $\vec{x} = (\cdot \text{——}), \lambda = 0.$

minimization with inequality constraints

$\cdot$ min $f(x) = x^2+2y^2+z^2$

subject to $g(x) = x+3y \geq 5$

$\Rightarrow$ ~~g(x)≥g~~ $g_0(x) = -x-3y+5 \leq 0$

~~$2x - \mu = 0$~~

~~$4y - 3\mu = 0$~~

~~$2z = 0$~~

$\mu \geq 0$

~~$2x - \mu_1 = 0$~~

$2x - \mu = 0 \quad \Rightarrow x = \frac{\mu}{2}$

$4y - 3\mu = 0 \quad \Rightarrow y = \frac{3}{4}\mu$

$2z = 0 \quad \Rightarrow z = 0$

$\mu(-x-3y+5) = 0$

$\left(-\frac{\mu}{2} - \frac{9}{4}\mu + 5\right)\mu = 0$

$\mu = 0$ or $-\frac{11}{4}\mu + 5 = 0$

$\mu = 5 \times \frac{4}{11} = \frac{20}{11}$

$\cdot$ min $f(x) = x^2+2y^2+z^2$

subject to $h(x) = x+2y+z = 5$, $g(x) = x+3y \geq 5$ $\Rightarrow$ $h(x) = x+2y+z-5 = 0$, $g(x) = -x-3y+5 \leq 0$

$2x + \lambda - \mu = 0$

$4y + 2\lambda - 3\mu = 0$

$2z + 1\lambda = 0$

$\lambda \geq 0, \ \mu \geq 0$

$x+2y+z-5 = 0$

$\mu(-x-3y+5) = 0$

$\Rightarrow$

$x = \frac{\mu - \lambda}{2}$

$y = \frac{3\mu - 2\lambda}{4}$

$z = -\frac{1}{2}\lambda$

$\frac{\mu-\lambda}{2} + \frac{3\mu-2\lambda}{2} - \frac{1}{2}\lambda = 5$

$\left(-\frac{\mu-\lambda}{2} - \frac{3(3\mu-2\lambda)}{4} + 5\right)\mu = 0$

Second order conditions for constrained minimization problems.

$\min \quad f(x,y,z) = \frac{1}{2}(x^2+y^2-z^2)$

Subject to $h(x) = x+2y+3z-4=0$.

$x+\lambda=0 \quad \Rightarrow x=-\lambda$
$y+2\lambda=0 \quad \Rightarrow y=-2\lambda$
$-z+3\lambda=0 \quad \Rightarrow z=3\lambda$
$\lambda \geq 0$
$x+2y+3z=4 \Rightarrow -\lambda-4\lambda+9\lambda = 4\lambda = 4 \Rightarrow \lambda=1$

So. $\vec{x}=(x,y,z)=(-1,-2,3)$, $\lambda=1$.

Check:

Whether $L(x) = F(x)+\lambda^T H(x)$ is positive ~~definite~~ semi definite

$= \nabla^2 f(x) + \lambda \nabla^2 h(x)$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Check this with a vector on ~~tangent space~~ plane.

Step ① Chose basis

So $\nabla h(x) \cdot \vec{V} = 0$

$1V_1 + 2V_2 + 3V_3 = 0.$

$\vec{V_1} = (2,-1,0)$

$\vec{V_2} = (3,0,-1)$

~~Check better~~

Step ② Write 2x2 matrix whose entry i,j is $V_i^T M V_j$

$\begin{pmatrix} 5 & 6 \\ 6 & 10 \end{pmatrix}$

positive definite ✓

So local min

$V_1^T M V_1 = (2,-1,0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}(2\ -1\ 0) = 5$

$V_1^T M V_2\ (2,-1,0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = (2\ -1\ 0)\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = 6$

$V_2^T M V_2 = (3\ 0\ -1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}(3, 0, -1) = 10$

Calculus of variations.

- Find a function $x(\cdot)$ that minimizes
$$I[x(\cdot)]=\frac{1}{2}\int_0^1 (x'(t)-t^2)^2 \text{ in the set } A=\{x(\cdot)\in C^2([0,1]): x(0)=0=x(1)=0\}$$

Sps: y^* min $I[x(\cdot)]$

$$\tau(s)=I[y^*(\cdot)+sz(\cdot)]\geq I[y^*(\cdot)]=i(0)$$

$i'(0)=0$

$$\tau(s)=\frac{1}{2}\int_0^1 (y'+sz'-t^2)^2dt=\frac{1}{2}\int_0^1 [sz'+(y'-t^2)]^2dt$$
$$=\frac{1}{2}\int_0^1 [s^2z'^2+2sz'(y'-t^2)+(y'-t^2)^2]dt$$

$$\frac{d}{ds}(\tau(s))=\frac{1}{2}\int_0^1 [2s(z')^2+2z'(y'-t^2)dt$$
$$=\int_0^1 s(z')^2+z'(y'-t^2)dt$$

So $\tau'(0)=\int_0^1 z'(y'-t^2)dt=0$

Also $y+sz$ satisfies constraints since optimality

$y(0)=y(1)=0$

$$y(0)+s(z(0))=y(1)+sz(1)=0$$
$$\Rightarrow z(0)=z(1)=0$$
$$\int_0^1 z'(y'-t^2)dt=0$$

Systems of ODE & matrix ~~ep~~ exponentials.

$M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (a). $e^{tM} = ?$

$$M^2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

& $M^{j+4k} = M^j \quad \forall k \,\&\, j \in \{0,1,2,3\}$

? Then $e^{tM} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$ ✓

$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow e^{tM} = \begin{pmatrix} \cos & \cdots \\ \cdots & \cdots \end{pmatrix}$

$M \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow e^{tM} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

b. Write down a formula for solution of $\frac{d}{dt} x(t) = Mx(t) + \begin{pmatrix} 0 \\ \alpha(t) \\ 0 \end{pmatrix}$

Sps $x(0) = x^0$

general solution is $x(t) = e^{tM} x^0 + e^{tM} \int_0^t e^{-sM} \begin{pmatrix} 0 \\ \alpha(s) \\ 0 \end{pmatrix} ds$

note: $e^{tM} = \sum \frac{t^k M^k}{k!}$ $\quad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= I\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) - M\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$

$= \cos t\, I - \sin t\, M$

$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$\boxed{e^{tM} = I + M + \frac{(tM)^2}{2!} + \frac{(tM)^3}{3!}}$

Controllability:

Consider equation

$$\frac{d}{dx}x(t) = Mx(t) + N\alpha(t), \quad \alpha(t) \in [-1,1] \text{ for all } t.$$

where $M = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$ $N = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or $N = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

For every $k = 0, 1, \cdots$, $M^k N$ has the form $\begin{pmatrix} \vdots \\ \vdots \\ 0 \end{pmatrix}$

So the controllability matrix G has the form $G = [N, MN, \cdots, M^4N] = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank $G = 4 \neq 5$ so not controllable

· Pontryagin Max Principle.

! Sps $\alpha^*(\cdot)$ is an optimal control, steering the system of equations from x^0 to the origin in the minimum possible time.

What can we deduce ~~from~~ about $\alpha^*(t)$ from PMP?

- $\frac{d}{dt}x(t) = Mx(t) + N\alpha(t), \quad \alpha(t) \in [-1,1]$ for all t.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Says that $\exists\, h = (h_1, h_2, h_3)^T$ s.t.

$$h^T e^{-tM} N\alpha(t) = \left(\tfrac{1}{2}h_1 t^2 + h_2 t + h_3\right)\alpha(t) = \max_{|a|\le 1}\left(h^T e^{-tM} N a\right)$$
$$= \max_{|a|\le 1}\left(\tfrac{1}{2}h_1 t^2 + h_2 t + h_3\right) a$$

$\Rightarrow \alpha(t) = \text{sign}\left(\tfrac{1}{2}h_1 t^2 + \cdots\right)$

So $\alpha(t) = \pm 1$ $\forall t$ that is not a root of poly $\tfrac{1}{2}h_1 t^2 + h_2 t + h_3$

change sign wc t is a root, (at most twice)

· Another e.g.

$$\frac{d}{dt} x_1(t) = x_2(t) + \alpha_1(t)$$

$$\frac{d}{dt} x_2(t) = -x_1(t) + \alpha_2(t)$$

with $\alpha_1^2(t) + \alpha_2^2(t) \le 1 \quad \forall t.$

a. what can be deduced?

the constraint is unusual,

rewrite $\frac{d}{dt} x(t) = M x(t) + N\alpha(t)$, $\quad \alpha_1^2(t) + \alpha_2^2(t) \le 1 \ \forall t$

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

it's easy to check the system is controllable.

PMP says exists a vector $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ s.t. $\forall t \in [0, \tau^*]$

$$h^T e^{-tM} N \alpha(t) = \max_{\{a : a_1^2 + a_2^2 \le 1\}} h^T e^{-tM} N a$$

N is identity,

i.e., $h^T e^{-tM} \alpha(t) = \max_{\{\ \}} h^T e^{-tM} a$

$$\Rightarrow (e^{tM} h)\, \alpha(t) = \max\, (e^{tM} h)\, a$$

max occurs when $a = \frac{e^{tM} h}{|e^{tM} h|}$, in fact equal to $e^{tM} \frac{h}{|h|}$

$\Rightarrow \forall t \in [0, \tau^*]$, $\alpha(t) = e^{tM} \tilde{h}$. where $\tilde{h} = \frac{h}{|h|}$

b. What's optical control? could figure out without PMP?

Note $x(t) = e^{tM}x^0 + e^{tM}\int_0^t e^{-sM}\alpha(s)\,ds$

substitute what we know about $\alpha(t)$ into

$$= e^{tM}\left[x^0 + \int_0^t e^{-sM}e^{sM}\tilde{h}\right]$$

$$= e^{tM}[x^0 + t\tilde{h}].$$

How to quickly make this 0?

$$\tilde{h} = \frac{-x^0}{|x^0|}$$

Then $x(t) = e^{tM}\left[\left(1 - \frac{t}{|x^0|}\right)x^0\right] = \left(1 - \frac{t}{|x^0|}\right)e^{tM}x^0$

$$\alpha(t) = -\frac{1}{|x^0|}e^{tM}x^0 = -\frac{x(t)}{|x(t)|}$$