Problem Set 1

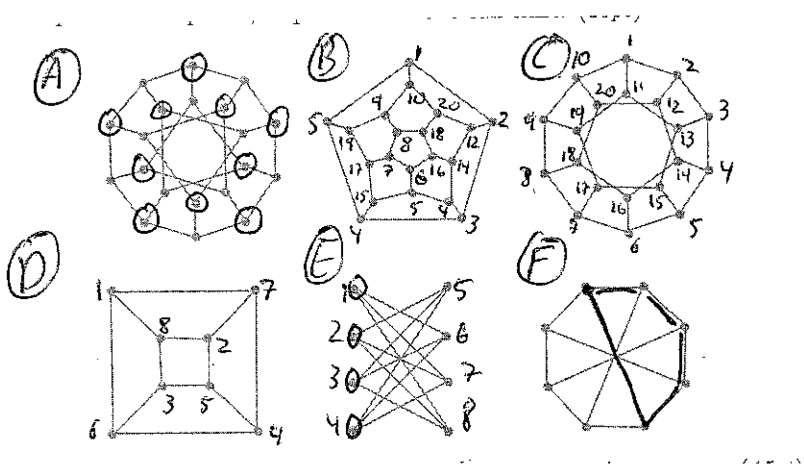
Solutions

1a) Prove: Two Graphs are isomorphic $\Longleftrightarrow$ their complements are isomorphic.

Answer: Consider an isomorphism $\phi : V(G) \to V(H)$ between graphs G and H. $V(G) = V(\bar{G})$ and $V(H) = V(\bar{H})$, so ϕ is also a bijection between $V(\bar{G})$ and $V(\bar{H})$.

$$G \cong H$$
$$\Updownarrow$$
$$\{u \text{ adj. to } v \text{ in } G \Longleftrightarrow \phi(u) \text{ adj. to } \phi(v) \text{ in } H\}$$
$$\Updownarrow$$
$$\{u \text{ \underline{not} adj. to } v \text{ in } G \Longleftrightarrow \phi(u) \text{ \underline{not} adj. to } \phi(v) \text{ in } H\}$$
$$\Updownarrow$$
$$\{u \text{ adj to } v \text{ in } \bar{G} \Longleftrightarrow \phi(u) \text{ adj to } \phi(v) \text{ in } \bar{H}\}$$
$$\Updownarrow$$
$$\bar{G} \cong \bar{H}$$

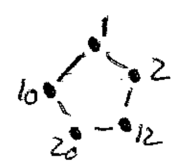
1 b) Which drawings represent isomorphic graphs, and which do not?



Answer: A, B, C have 20 vertices each.
D, E, F have 8 vertices each.

⇒ graphs in top row aren't isomorphic to graphs in bottom row.

A $\ncong$ B because A is bipartite (bipartition shown) but B contains the odd cycle

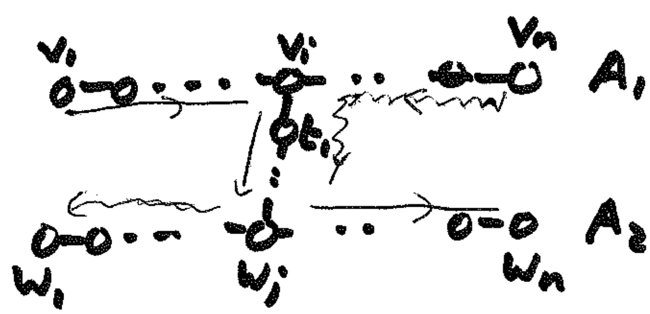


B $\cong$ C and D $\cong$ E as shown.

E $\ncong$ F because E is bipartite (as shown) while F contains an odd cycle (shown).

Q2) Prove that any two longest paths share a common vertex.

A) Let $A_1 = v_1 v_2 \dots v_n$ and
$A_2 = w_1 w_2 \dots w_n$
be two longest paths. Assume they do not share a common vertex. Then there exists (because G is connected) a path
$B = v_i t_1 t_2 \dots t_k w_j$ of length $k+1$
in G such that $t_1, \dots, t_k$ are not vertices in A_1 or in A_2 (k might be 0)



Let A_1^* be the longer among $v_1 \dots v_i$ and $v_n v_{n-1} \dots v_i$
and let A_2^* be the longer among $w_1 w_2 \dots w_j$ and $w_n w_{n-1} \dots w_j$

Then $\text{length}(A_1^* t_1 \dots t_k A_2^*) \geq \underbrace{\tfrac{n}{2}}_{\leq \text{length}(A_1^*)} + \overbrace{1}^{\leq \text{length}(B)} + \underbrace{\tfrac{n}{2}}_{\leq \text{length}(A_2^*)} > n$

<u>Contradiction</u>

Q3 Let S be a set of n points in a plane, s.t. the distance between any pair is at least 1. Show there cannot be more than 3n pairs of distance exactly 1.

A: Form a graph G on n vertices, with an edge between two vertices iff the distance between the corresponding points is 1.

Lemma: There is no vertex in G of valence ≥ 7

Proof: Assume there were such a vertex v. all points at distance 1 from the corresponding point would lie on the unit circle centred at v.

It's circumference is $2\pi < 7$. So there can't be ≥ 7 points on the circle of distance ≥ 1 from one another. $\square$

Now, by the handshake lemma:

$$|E| = \frac{1}{2} \sum_{v \in V(G)} \deg(G) \leq \frac{1}{2} \cdot 6n = 3n .$$

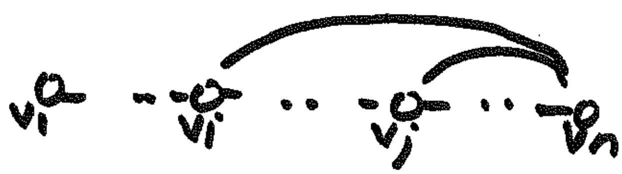
Q4) Prove a loopless graph whose vertices have valence ≥ 3 has a cycle of even length.

A) Assume G is simple, else we would have a cycle of length 2.

Let $A: v_1 \dots v_n$ be a maximal path in G.

$\deg(v_n) \geq 3 \Rightarrow v_n$ has 3 distinct neighbours v_{n-1}, u, w.

Because A is maximal, u and w must be vertices v_i, v_j in A. WLOG $i<j$.



Case 1: $v_i v_{i+1} \dots v_n$ is odd $\Rightarrow v_i v_{i+1} \dots v_n v_i$ is an even cycle.

Case 2: $v_j v_{j+1} \dots v_n$ is odd $\Rightarrow v_j v_{j+1} \dots v_n v_j$ is an even cycle.

Case 3: $v_i v_{i+1} \dots v_j$ is even $\Rightarrow v_n v_i v_{i+1} \dots v_j v_n$ is an even cycle.

13

(Q 5) G is bipartite $\Leftrightarrow$ every $H \subseteq G$ has an independent set containing at least half of $V(H)$

A) $\Rightarrow$: Let $[X, Y]$ be a bipartition of $V(G)$.
H is obtained by vertex, edge deletes, so it induces a bipartition $[X', Y']$ of H.
By the pidgeonhole principle, either X' or Y' is half of $V(H)$.

$\Leftarrow$: If G isn't bipartite, it contains an odd cycle C_{2n+1}. Take $H = C_{2n+1}$.

Every set which contains at least half the vertices in H must contain at least 2 consecutive vertices
$\Rightarrow$ H has no independent set containing at least half its vertices.

Q6)

a) Prove a simple graph on n vertices contains two vertices of equal valence.

A) Case 1: G contains no vertex of degree i for some $0 \leq i \leq n-1$
$\Rightarrow$ G contains two vertices of equal valence by the pidgeonhole principle.

Case 2: The degree sequence for G is $(0,1,2,\ldots,n-1)$ $\Rightarrow$ G contains a vertex u of degree 0 and v of degree $n-1$. This is a contradiction - all vertices in G are neighbours of v, but u has no neighbours.

b) Draw a loopless graph with four vertices all of different valence

A) 

Q 7):

A) Draw a bipartite graph G with bipartition:

(X) n vertices for n doctors, 1 per doctor

(Y) 3 vertices, A, B, C, for the questions.

connect a "doctor" to a vertex in Y if they got that question correct.

It is known that
$$\deg(A) = \frac{3n}{10}$$
$$\deg(B) = \frac{n}{4}$$
$$\deg(C) = \frac{n}{5}$$

So all three vertices in Y can connect to at most $\left(\frac{3}{10} + \frac{1}{4} + \frac{1}{5}\right)n = \frac{3}{4}n$ vertices in X (G has $\frac{3n}{4}$ edges). So at least $\frac{n}{4}$ vertices in X have valence 0, corresponding to the 25% of doctors who got all 3 questions wrong.

Alternative proof: Inclusion-exclusion.