Chapter 2 First Order Differential Equations

2.1 Linear Equations; Method of Integrating factors.

first order linear equation:

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$y' + p(t)y = g(t)$$

$\mu(t) = e^{\int p(t)dt}$

$$y = \frac{1}{\mu}\left[\int_{t_0}^{t} \mu(s)g(s)ds + C\right]$$

$\downarrow$

$y = \frac{1}{\mu(t)}\left[\int y(t)\mu(t)dt + c\right]$

$p(t) = a \Rightarrow \mu(t) = e^{at}$

$p(t) = \frac{2}{t} \Rightarrow \mu(t) = e^{2\ln|t|} = t^2$

$p(t) = \frac{t}{2} \Rightarrow \mu(t) = e^{\frac{t^2}{4}}$

Separable Equations:

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Homogeneous Equations

If RHS $\frac{dy}{dx} = f(x,y)$ can be expressed as a func. of $\frac{y}{x}$ only. then ....

like $\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}$ (*) homogeneous

Introduce variable v s.t. $v = \frac{y}{x}$, $y = xv(x)$

$\frac{dy}{dx} = y' = (xv)' = v + x\frac{dv}{dx}$

replace in (*) $v + x\frac{dv}{dx} = \frac{v-4}{1-v}$

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - \frac{v(1-v)}{1-v} = \frac{v^2-4}{1-v}$$

$\frac{1-v}{v^2-4}dv = \frac{1}{x}dx \Rightarrow \frac{1-v}{v^2-4} = \frac{A}{v+2} + \frac{B}{v-2}$ $\qquad A = -\frac{3}{4}$ $B = -\frac{1}{4}$

$$\int \frac{-\frac{3}{4}}{V+2}\,dV + \int \frac{-\frac{1}{4}}{V-2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{4}\ln|V-2| - \frac{3}{4}\ln|V+2| = \ln|x| - k$$

Exact Equations

$M(x,y) + N(x,y)\,y' = 0$ is an exact differential equation iff

$$M_y(x,y) = N_x(x,y)$$

at each point of R.

And $\psi_x(x,y) = M(x,y)$, $\psi_y(x,y) = N(x,y)$ iff

$$M_y(x,y) = N_x(x,y)$$

Integrating factors: convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor.

Assume μ is a function of x only, we have:

$$(\mu M)_y = \mu M_y, \quad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$$

Thus if $(\mu M)_y = (\mu N)_x$,

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$