Exerzitien VIII

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Nov. 17., in your tutorial.

Reading suggestion: Axler Chapter 5, omit last section.

Exercise 1. *Consider the matrix $T \in \mathbb{F}^{2\times 2}$ given by*

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1. *Using a careful row reduction, find a condition on a, b, c, d which holds if and only if T is invertible. Show derivation.*
2. *When the inverse exists, write an expression for T^{-1}. Note: the method above should yield an expression for the inverse matrix.*
3. *Suppose that we take $\mathbb{F} = \mathbb{R}$ above. What is the condition which holds if and only if T has an eigenvalue? Give one example of a real 2×2 matrix with no real eigenvalues.*

Exercise 2. *Find the general solution of the system of differential equations by following the steps below:*

$$\frac{dx_1}{dt} = 2x_1 + x_2$$
$$\frac{dx_2}{dt} = x_1 + 3x_2 + x_3$$
$$\frac{dx_3}{dt} = x_2 + 2x_3$$

1. *Write the system in matrix form $\frac{dX}{dt} = AX$, where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.*
2. *Make a change of variables $Y = PX$ (for an invertible matrix $P \in \mathbb{R}^{3\times 3}$) so that $\frac{dY}{dt} = P\frac{dX}{dt} = PAX$, i.e.*
$$\tfrac{dY}{dt} = (PAP^{-1})Y.$$
Find a matrix P so that PAP^{-1} is diagonal.
3. *Solve for $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in the diagonal system and then write the solution for X.*

Exercise 3. *A Jordan block of size k with eigenvalue λ is a $k \times k$ matrix of the form*

$$J_1(\lambda) = \begin{pmatrix} \lambda \end{pmatrix}, \quad J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad J_3(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \cdots$$

1. *Fix k. What is the dimension of* $\mathrm{null}(J_k(\lambda) - \lambda I)^n$ *for all n ?*
2. *Suppose that the square matrix A consists of i_k Jordan blocks of size k with the same eigenvalue λ arranged along the diagonal, where $(i_1, i_2, \cdots i_m)$ is a finite sequence of nonnegative integers. [For example, if the matrix A is*

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

   then it has $i_1 = 0$, $i_2 = 1$, and $i_3 = 1$.] Determine, in the general case, the dimensions of $\mathrm{null}(A - \lambda I)^n$ *for all n, in terms of the sequence $(i_1, \ldots, i_m)$.*

Exercise 4. *Let S, T be linear operators on V. Prove that ST and TS have the same eigenvalues.*