STA305/1004-Class20

March 16, 2016

Today's Class

- Factorial designs at two levels
- Cube plots
- Calculation of factorial effects

– inte-pretation of factorial designs

Difference between ANOVA and Factorial Designs

In ANOVA the objective is to compare the individual experimental conditions with each other. In a factorial experiment the objective is generally to compare combinations of experimental conditions.

Let's consider the food diary study above. What is the effect of keeping a food diary?

Expt condition	Keep food diary	Increase physical activity	Home visit	weight loss
1	No	No	No	y_1
2	No	No	Yes	y_2
3	No	Yes	No	y_3
4	No	Yes	Yes	y_4
5	Yes	No	No	y_5
6	Yes	No	Yes	y_6
7	Yes	Yes	No	y_7
8	Yes	Yes	Yes	y_8

We can estimate the effect of food diary by comparing the mean of all conditions where food diary is set to NO (conditions 1-4) and mean of all conditions where food diary set to YES (conditions 5-8). This is also called the **main effect** of food diary, the adjective *main* being a reminder that this average is taken over the levels of the other factors.

Factor 1 = Aspirin (Y/N)
Factor 2 = Beta-Carotene (Y/N)

4 treatments

Aspirin

Beta C	Y	N
Y	YY	YN
N	NY	NN

all possible 4 factor-level combinations

Difference between ANOVA and Factorial Designs

Expt condition	Keep food diary	Increase physical activity	Home visit	weight loss
1	No	No	No	y_1
2	No	No	Yes	y_2
3	No	Yes	No	y_3
4	No	Yes	Yes	y_4
5	Yes	No	No	y_5
6	Yes	No	Yes	y_6
7	Yes	Yes	No	y_7
8	Yes	Yes	Yes	y_8

The main effect of food diary is: $\mu_{Yes\text{-}Fd} - \mu_{No.Food\ Diary}$

$$\boxed{\frac{y_1+y_2+y_3+y_4}{4} - \frac{y_5+y_6+y_7+y_8}{4}}.$$

The main effect of physical activity is:

$$\underbrace{\hat{\mu}_{No\text{-}phys\ ACT}}\ \boxed{\frac{y_1+y_2+y_5+y_6}{4} - \frac{y_3+y_4+y_7+y_8}{4}}\ \hat{\mu}_{Yes\text{-}Pys\ activts}$$

The main effect of home visit is:

$$\frac{y_1+y_3+y_5+y_7}{4} - \frac{y_2+y_4+y_6+y_8}{4}.$$

Sps increase physical activity had 3 levels less, moderate, high then a one-way design (ANOVA)

$\mu_{less} = \mu_{mod} = \mu_{high}$

but levels of other factors would not be investgtd

Factorial designs at two levels

To perform a factorial design:

1. Select a fixed number of levels of each factor.
2. Run experiments in all possible combinations.

Factorial designs at two levels

qual. ≡ home visit (Y/N)
quat. ≡ weight (Y/N)

- We will discuss designs where there are just two levels for each factor.
- Factors can be quantitative or qualitative.
- Two levels of quantitative variable could be two different temperatures or concentrations.
- Two levels of a quantitative variable could be two different types of catalysts or presence/absence of some entity.

Pilot plant investigation - example of factorial design

A pilot plant invsetiagtion employed a 2^3 factorial design (Box, Hunter, and Hunter (2005)) with

	Factors	level 1	level 2	
2	Temperature	$160\mathrm{C}^{\circ}$(-1)	$180\mathrm{C}^{\circ}(+1)$	quant.
2	Concentration	20% (-1)	40% $(+1)$	
2	Catalyst	A (-1)	B$(+1)$	quali

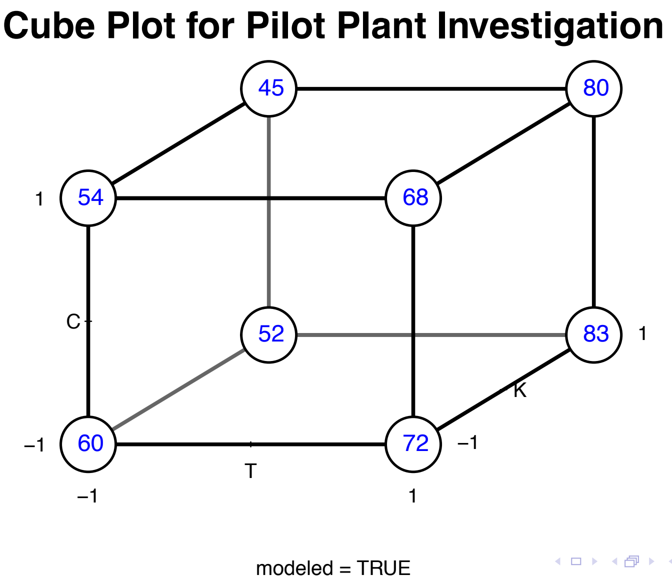
2^3

run	T	C	K	y
1	-1	-1	-1	60
2	1	-1	-1	72
3	-1	1	-1	54
4	1	1	-1	68
5	-1	-1	1	52
6	1	-1	1	83
7	-1	1	1	45
8	1	1	1	80

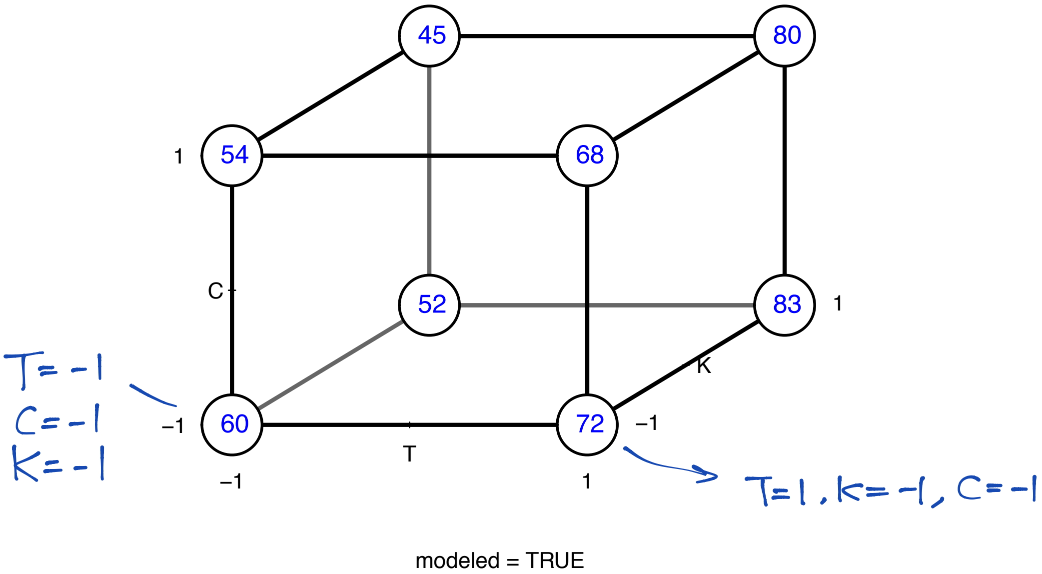
- Each data value recorded is for the response yield y averaged over two duplicate runs.

Cube plots

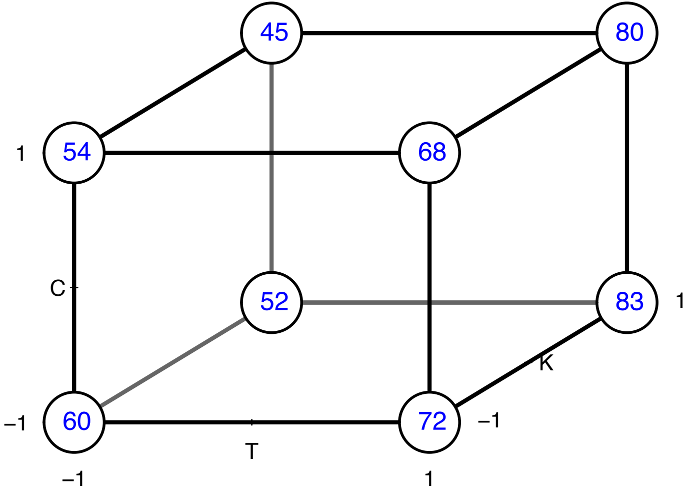
```
library("FrF2")
bhh54 <- lm(y~T*C*K,data=tab0502)
cubePlot(bhh54,"T","K","C",main="Cube Plot for Pilot Plant Investigation")
```



Cube plots



Cube plots



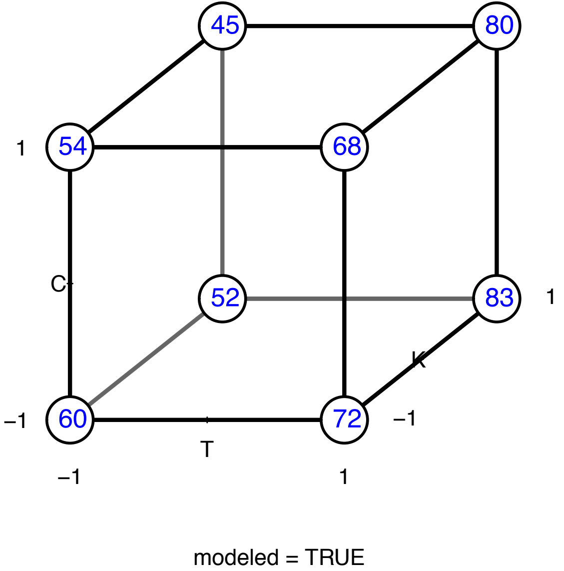
modeled = TRUE

$$T = \left[(72-60)+(68-54)+(80-45)+(83-52)\right]/4 = 23$$

$$T^{+} = \frac{68+83+72+80}{4}$$

$$T^{-} = \frac{-54-45-60-52}{C}$$

Cube plots



main effect of concentration

$$C = \frac{(54-60)+(45-52)+(80-83)+(68-72)}{4}$$

$$= -5$$

$$K = \frac{(83-72)+(52-60)+(80-68)+(45-54)}{4}$$

Cube plots

(treatment)

- 8 run design produces 12 comparisons
- Each edge of cube only one factor changed while other 2 held constant.
- Therefore experimenter that believes in only changing one factor at a time is satisfied.

one-at a time approach (HW#3, #4)
involves running an expt on
Factor 1 → apply results to
expt on factor 2
→ apply results to expt on
factor 3

Interaction effects - two factor interactions

When the catalyst K is A the temperature effect is:

$$\frac{68+72}{2} - \frac{60+54}{2} = 70 - 57 = 13.$$

When the catalyst K is B the temperature effect is:

$$\frac{83+80}{2} - \frac{52+45}{2} = 81.5 - 48.5 = 33.$$

The average difference between these two average differences is called the **interaction** between temperature and catalyst denoted by TK. This is the interaction between the two factors temperature and catalyst - the two factor interaction between temperature and catalyst.

$$TK = \frac{13+33}{2} = 23$$

K=A

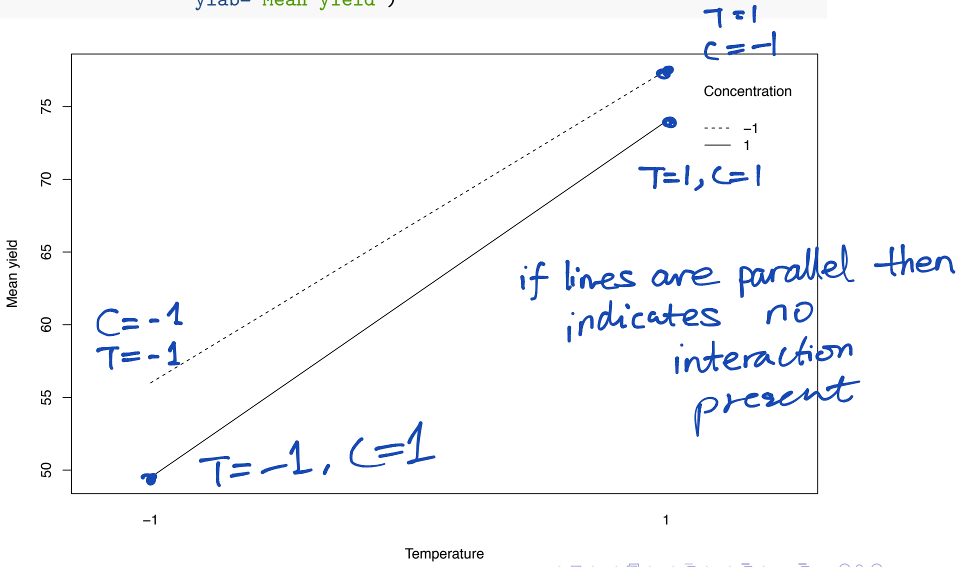
- Don't interpret main effect of temper

- Don't interpret main effect of catalyst w/out considering temp.

with considering catalyst.

Interaction plots - Concentration by temperature

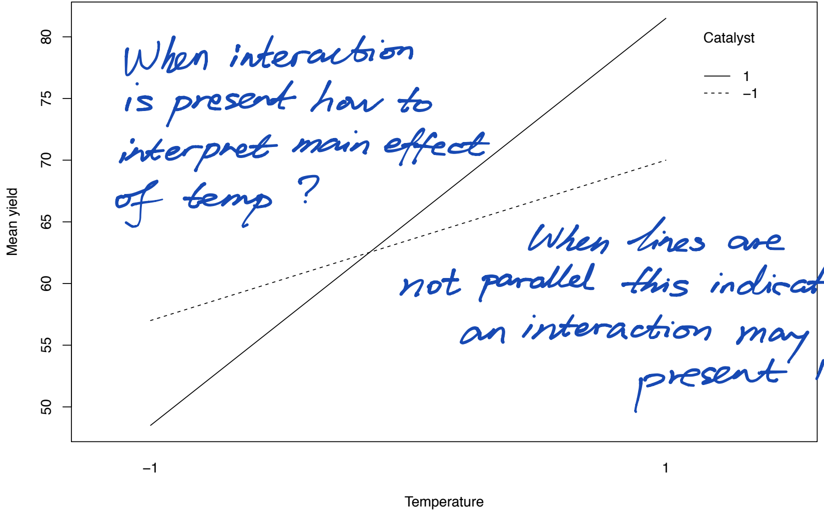
```r
interaction.plot(tab0502$T,tab0502$C,tab0502$y, type="l",
                 xlab="Temperature",trace.label="Concentration",
                 ylab="Mean yield")
```



Interaction plots - Temperature by catalyst

T=23

```
interaction.plot(tab0502$T,tab0502$K,tab0502$y, type="l",
                 xlab="Temperature",trace.label="Catalyst",
                 ylab="Mean yield")
```

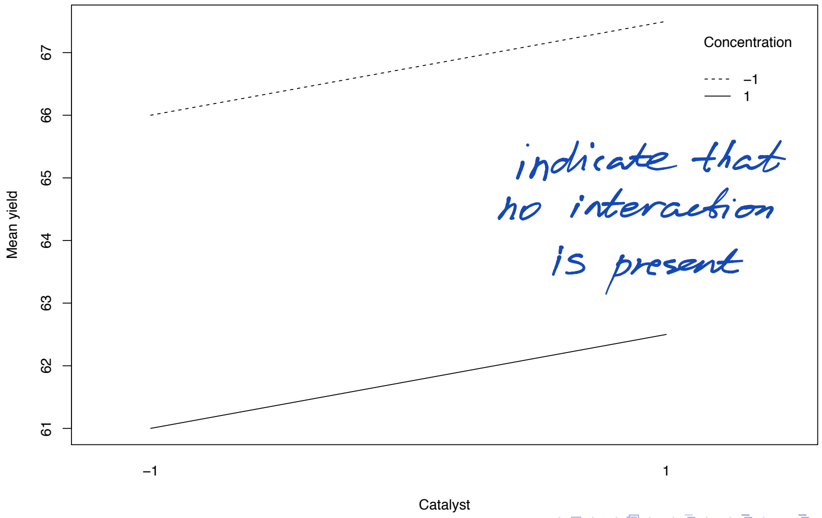


When interaction is present how to interpret main effect of temp ?

When lines are not parallel this indicates an interaction may be present !

Interaction plots - Concentration by catalyst

```
interaction.plot(tab0502$K,tab0502$C,tab0502$y, type="l",
                 xlab="Catalyst",trace.label="Concentration",
                 ylab="Mean yield")
```



indicate that
no interaction
is present

Three factor interactions

run	T	C	K	y
1	-1	-1	-1	60
2	1	-1	-1	72
3	-1	1	-1	54
4	1	1	-1	68
5	-1	-1	1	52
6	1	-1	1	83
7	-1	1	1	45
8	1	1	1	80

interaction is average diff of diff

The temperature by concentration interaction when the catalyst is B (at it's +1 level) is:

$$\text{Interaction TC} = \frac{(y_8 - y_7) - (y_6 - y_5)}{2} = \frac{(80 - 45) - (83 - 52)}{2} = 2.$$

The temperature by concentration interaction when the catalyst is A (at it's -1 level) is:

$$\text{Interaction TC} = \frac{(y_4 - y_3) - (y_2 - y_1)}{2} = \frac{(68 - 54) - (72 - 60)}{2} = 1.$$

$$\text{TCK} = \frac{2 - 1}{2} = \frac{1}{2}.$$

Three factor interaction

concentration × Temp
is the same as Temp × Concentration

- Interactions are symmetric in all factors.
- It could have been defined as half the difference between the temperature-by-catalyst interactions at each of the two concentrations.
- Mostly rely on statistical software such as R.

Replicate runs

- Each of the 8 responses in the table is the average of two (genuinely) replicated runs.
- Genuinely replicated run means that variation between runs made at same experimental conditions is a reflection of the total run-to-run variability.

run	T	C	K	y
1	-1	-1	-1	60
2	1	-1	-1	72
3	-1	1	-1	54
4	1	1	-1	68
5	-1	-1	1	52
6	1	-1	1	83
7	-1	1	1	45
8	1	1	1	80

average of two runs

Replicate runs

- Randomization of the run order for all 16 runs ensures the replication is genuine.
- `run1` is order of the first run and `run2` is order of the second run.

run1	run2	T	C	K	y1	y2	diff
6	13	-1	-1	-1	59	61	-2
2	4	1	-1	-1	74	70	4
1	16	-1	1	-1	50	58	-8
5	10	1	1	-1	69	67	2
8	12	-1	-1	1	50	54	-4
9	14	1	-1	1	81	85	-4
3	11	-1	1	1	46	44	2
7	15	1	1	1	79	81	-2

order of run
was randomized

Replicate runs

- Replication not always feasible or easy.
- For the pilot plant experiment a run involved: cleaning the reactor; inserting the appropriate catalyst charge; and running the apparatus at a given concentration for 3 hours, and sampling output every 15 minutes.
- A genuine run involved taking all of these steps all over again!

Replicate runs

- There are usually better ways to employ 16 independent runs than by fully replicating a 2^3 factorial.
- Other designs can study four or five factors with a 16 run two-level design.

Estimate of error variance of the effects from replicated runs

run1	run2	T	C	K	y1	y2	diff
6	13	-1	-1	-1	59	61	-2
2	4	1	-1	-1	74	70	4
1	16	-1	1	-1	50	58	-8
5	10	1	1	-1	69	67	2
8	12	-1	-1	1	50	54	-4
9	14	1	-1	1	81	85	-4
3	11	-1	1	1	46	44	2
7	15	1	1	1	79	81	-2

Ave

$$s_i^2 = \frac{(y_{i1} - y_{i2})^2}{2},$$

- y_{i1} is the first outcome from *ith* run.
- $\text{diff}_i = (y_{i1} - y_{i2})$.
- A pooled estimate of σ^2 is

$$s^2 = \frac{\sum_{i=1}^{8} s_i^2}{8} = \frac{64}{8} = 8.$$

- The variance of an effect is:

$$Var(\text{effect}) = \left(\frac{1}{8} + \frac{1}{8}\right) s^2 = 8/4 = 2$$

$\sqrt{Var(effect)}$

$= \sqrt{2} \cong 1.4$

$= s.e.(effect)$

Interpretation of results

- Which effects are real and which can be explained by chance?
- A rough rule of thumb: any effect that is 2-3 times their standard error are not easily explained by chance alone.

effect

$2 \times s.e(\text{effect})$

$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$\downarrow$

use t_{n-1} if n is small

Interpretation of results

- Assume that the observations are independent and normally distributed then

$$\text{effect}/se\,(\text{effect}) \sim t_8.$$

Two types of factorial effects

- main
- interaction

- A 95% confidence interval can be calculated as:

$$\text{effect} \pm \overset{2\text{-}3}{t_{8,.05/2}} \times se\,(\text{effect})\,.$$

where $t_{8,.05/2}$ is the 97.5th percentile of the t_8. This is obtained in R via the `qt()` function.

```
qt(p = 1-.025,df = 8)
```

any main effect or interaction effect

```
## [1] 2.306004
```

- In the pilot plant study

$$\text{effect} \pm 2.3 \times 1.4 = \text{effect} \pm 3.2.$$

any factorial effect

- The main effect of a factor should be individually interpreted only if there is no evidence that the factor interacts with other factors.
- Which effects should be considered jointly and which independently? ⊛

Do these intervals cover σ?

Effects	95% Confidence Interval
T	(19.8, 26.2)
C	(-8.2, -1.8)
K	(-1.7, 4.7)
TC	(-1.7, 4.7)
TK	(6.8, 13.2)
CK	(-3.2, 3.2)
TCK	(-2.7, 3.7)

main effect of T

$= \mu_{T-} - \mu_{T+}$

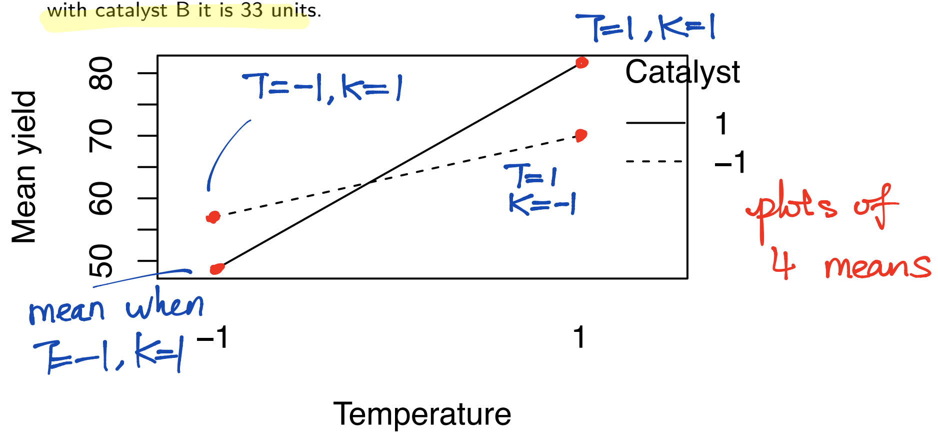
$\mu_{C-} - \mu_{C+}$

$\mu_{TC+} - \mu_{TC-}$

Consider Temp dependence on catalyst

Interpretation of results

- The effect of changing concentration over the ranges studied is to reduce yield by about 5 units. This is irrespective of the tested level of other variables.
- The effects of temperature and catalyst cannot be interpreted separately because of the large TK interaction. With catalyst A the temperature effect is 13 units and with catalyst B it is 33 units.



Linear model for factorial design

Let y_i be the yield from the i^{th} run,

$$x_{i1} = \begin{cases} +1 & \text{if } T = 180 \\ -1 & \text{if } T = 160 \end{cases}$$

$$x_{i2} = \begin{cases} +1 & \text{if } C = 40 \\ -1 & \text{if } C = 20 \end{cases}$$

$$x_{i3} = \begin{cases} +1 & \text{if } K = B \\ -1 & \text{if } K = A \end{cases}$$

A linear model for a 2^3 factorial design is:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i1} x_{i2} + \beta_5 x_{i1} x_{i3} + \beta_6 x_{i2} x_{i3} + \beta_7 x_{i1} x_{i2} x_{i3} + \epsilon_i .$$

The variables $x_{i1}x_{i2}$ is the interaction between temperature and concentration, $x_{i1}x_{i3}$ is the interaction between temperature and catalyst, etc.

Linear model for factorial design

The table of contrasts for a 2^3 design is the design matrix X from the linear model above.

-1×-1

Mean	T	K	C	T:K	T:C	K:C	T:K:C	yield average
1	-1	-1	-1	1	1	1	-1	60
1	1	-1	-1	-1	-1	1	1	72
1	-1	-1	1	1	-1	-1	1	54
1	1	-1	1	-1	1	-1	-1	68
1	-1	1	-1	-1	1	-1	1	52
1	1	1	-1	1	-1	-1	-1	83
1	-1	1	1	-1	-1	1	-1	45
1	1	1	1	1	1	1	1	80

1×-1

$-1 \times -1 = 1$

$$T = \frac{-60 + 72 - 54 + 68 - 52 + 83 - 45 + 80}{4}$$

$$T{:}K = (60 - 72 + 54 - 68 - 52 + 83 - 45 + 80)/4$$

- All factorial effects can be calculated from this table.
- Signs for interaction contrasts obtained by multiplying signs of their respective factors.
- Each column perfectly balanced with respect to other columns. → equal # of + & −
- Balanced (orthogonal) design ensures each estimated effect is unaffected by magnitude and signs of other effects.
- Table of signs obtained similarly for any 2^k factorial design.

END

Linear model for factorial design

What is the table of contrasts for a 2^4 factorial design?

Linear model for factorial design - calculating factorial effects from parameter estimates

The parameter estimates are obtained via the `lm()` function in R.

- Estimated least squares coefficients are one-half the factorial estimates.
- Therefore, the factorial estimates are twice the least squares coefficients.

$$\hat{\beta}_1 = 11.50 \Rightarrow T = 2 \times 11.50 = 23.26$$
$$\hat{\beta}_2 = 0.75 \Rightarrow K = 2 \times 0.75 = 1.5$$
$$\hat{\beta}_4 = 5.00 \Rightarrow TK = 2 \times 5.00 = 10.00$$

```
fact.mod <-lm(y~T*K*C,data=tab0502)
round(summary(fact.mod)$coefficients,2)
```

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    64.25        NaN     NaN      NaN
T              11.50        NaN     NaN      NaN
K               0.75        NaN     NaN      NaN
C              -2.50        NaN     NaN      NaN
T:K             5.00        NaN     NaN      NaN
T:C             0.75        NaN     NaN      NaN
K:C             0.00        NaN     NaN      NaN
T:K:C           0.25        NaN     NaN      NaN
```

Linear model for factorial design - significance testing

- When there are replicated runs we also obtain p-values and confidence intervals for the factorial effects from the regression model.
- For example, the p-value for β_1 corresponds to the factorial effect for temperature

$$H_0 : \beta_1 = 0 \quad \text{vs.} \quad H_1 : \beta_1 \neq 0.$$

If the null hypothesis is true then $\beta_1 = 0 \Rightarrow T = 0 \Rightarrow \mu_{T+} - \mu_{T-} = 0 \Rightarrow \mu_{T+} = \mu_{T-}$.

- μ_{T+} is the mean yield when the temperature is set at 180° and μ_{T-} is the mean yield when the temperature is set to 160°.

Linear model for factorial design - significance testing

To obtain 95% confidence intervals for the factorial effects we multiply the 95% confidence intervals for the regression parameters by 2. This is easily done in R using the function `confint.lm()`.

```
fact.mod <-lm(y~T*K*C,data=tab0503)
round(2*confint.lm(fact.mod),2)
```

```
             2.5 % 97.5 %
(Intercept) 125.24 131.76
T            19.74  26.26
K            -1.76   4.76
C            -8.26  -1.74
T:K           6.74  13.26
T:C          -1.76   4.76
K:C          -3.26   3.26
T:K:C        -2.76   3.76
```

Advantages of factorial designs over one-factor-at-a-time designs

- Suppose that one factor at a time was investigated. For example, temperature is investigated while holding concentration at 20% (-1) and catalyst at B (+1).
- In order for the effect to have more general relevance it would be necessary for the effect to be the same at all the other levels of concentration and catalyst.
- In other words there is no interaction between factors (e.g., temperature and catalyst).
- If the effect is the same then a factorial design is more efficient since the estimates of the effects require fewer observations to achieve the same precision.
- If the effect is different at other levels of concentration and catalyst then the factorial can detect and estimate interactions.