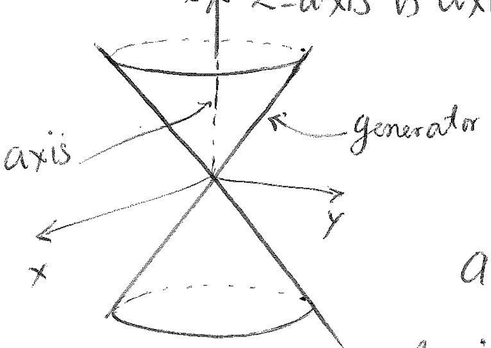
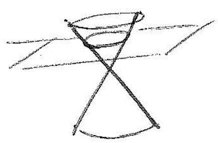
1. $F(x,y,z) = x^2+y^2-z^2=0$ represents a double cone

Z-axis is axis of revolution, and generator is $z=x$ or $z=y$.

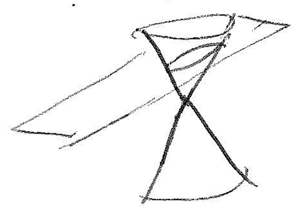


Circle, elipse, parabola and hyperbola are called conic sections because they are made of intersection of various planes with the cone:

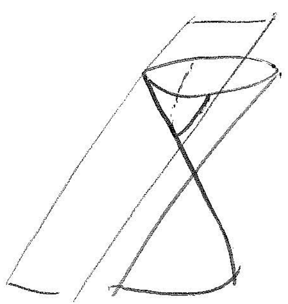
Circle: plane perp to the axis



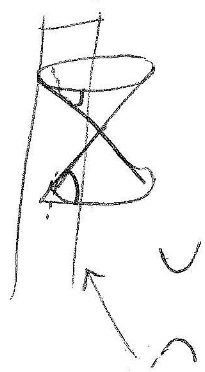
.) Slanted plane & meets only one cone → elipse



parabola: plane is parallel to the generator



hyperbola is when the plane is not parallel to the generator and meets the both cones:



a) $G(x,y,z) = z - 3 = 0$ Sln to $F = 0 = G$ is

$$\left\{\begin{array}{l} x^2+y^2-9=0 \\ z=3 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} x^2+y^2=9 \\ z=3 \end{array}\right. \qquad \mathbb{F} = \begin{bmatrix} F \\ G \end{bmatrix} \quad D\mathbb{F} = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 0 & 1 \end{bmatrix}$$

Conversion to (i) is possible as long as a 2x2 Sub-matrix of $D\mathbb{F}$ is detected with det $\neq 0$. for example as long as $y \neq 0$ Then y and z can be locally Solved in terms of x

$$\left\{\begin{array}{l} y = \sqrt{9-x^2} \\ z = 3 = f(x) \end{array}\right.$$

at pts $(0, \pm 3, 3)$ only y, z can be Solved and at The pts $(\pm 3, 0, 3)$ only x, z in terms of y.

$$D\mathbb{F} = \begin{bmatrix} \pm 6 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

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b) $y = 2$ (plane parallel to The axis of rotation)

$G(x,y,z) = y - 2 = 0$

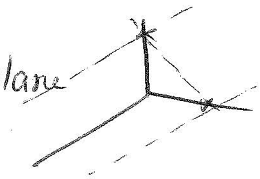
$$\mathbb{F}(x,y,z) = \mathbb{0} = \begin{bmatrix} F(x,y,z) \\ G(x,y,z) \end{bmatrix} = \begin{bmatrix} x^2+y^2-z^2 \\ y-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ so intersection

is $\left\{\begin{array}{l} x^2-z^2=4 \\ y=2 \end{array}\right.$ hyperbola, $D\mathbb{F} = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 1 & 0 \end{bmatrix}$ $z \neq 0$ b/c $z^2 = 4 + x^2 \geq 4$

so $\det \begin{bmatrix} 2y & -2z \\ 1 & 0 \end{bmatrix} \neq 0$

so we can always Solve ~~y~~ and z in terms of x: $\left\{\begin{array}{l} z = \pm\sqrt{4+x^2} \\ y = 2 \end{array}\right.$ (i)

c) $G(x,y,z) = z + y - 2 = 0$ is equation of a plane parallel to the x-axis and it is also parallel to the generator $z = y$ or $z = -y$



so the resulting curve is a parabola.

$F = 0 = G$ becomes

$x^2 + y^2 - z^2 = 0 \to$

$x^2 + y^2 - (2-y)^2 = x^2 + 4y - 4 = 0 \to y = 1 - \frac{x^2}{4}$

$$DF = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2y-4 \\ 0 & 1 & 1 \end{bmatrix}$$

of course $\det \begin{bmatrix} 2y & 2y-4 \\ 1 & 1 \end{bmatrix} = 4 \neq 0$

so one can solve y and z in terms of x:

$$\begin{cases} y = 1 - \frac{x^2}{4} \\ z = 2 - y = 2 - \left(1 - \frac{x^2}{4}\right) = 1 + \frac{x^2}{4} \end{cases} \quad (i)$$

d) $G(x,y,z) = 2z + y - 4 = 0$

is equation of a plane that is not perp to the axis & intersects the double cone in the upper part; $z = -\frac{y}{2}$ is shallower than $z = -y$ so the resulting curve is an elipse.

$F = 0 = G$ becomes:

$x^2 + (4-2z)^2 + z^2 = 0$

$x^2 + 5z^2 - 8z = 16$

$DF = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 1 & 2 \end{bmatrix}$, $\det \begin{bmatrix} 2y & -2z \\ 1 & 2 \end{bmatrix} \neq 0$ as long as

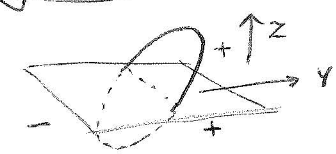
$$\begin{cases} 4y + 2z = 0 \\ 2z + y = 4 \end{cases} \quad y \neq \frac{-3}{4}$$

so solve y, z in terms of x.

$5z^2 - 8z - 16 + x^2 = 0$

$$\begin{cases} z = \dfrac{8 \pm \sqrt{64 + 20(16 - x^2)}}{10} \\ y = 4 - 2z = 4 - \dfrac{8 \pm \sqrt{\ }}{5} \end{cases}$$

$\pm$ refers to



If $y = -\frac{3}{4}$ Then $z = \frac{19}{8}$ Then $x \neq 0$ (from $x^2+y^2-z^2=0$)

Then $\begin{bmatrix} 2x & 2y \\ 0 & 1 \end{bmatrix}$ has non zero determinant and so x and y can be Solved in term of z: $\begin{cases} y = 4-2z \\ x^2+y^2-z^2=0 \end{cases}$

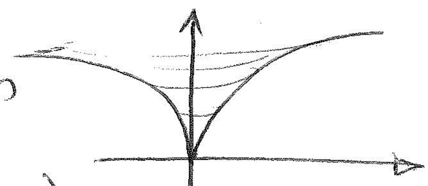
$$\begin{cases} y = 4-2z \\ x = \pm\sqrt{z^2-(4-2z)^2} \end{cases}$$

---

2. $f(u,v) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u\cos v \\ u\sin v \\ \sqrt{u} \end{bmatrix}$ note $x^2+y^2=u^2$ and $z=\sqrt{u}$

so $x^2+y^2=z^4$

So The location of $x^2+y^2=z^4$ is



so level curves are $z=c$ (are circles)

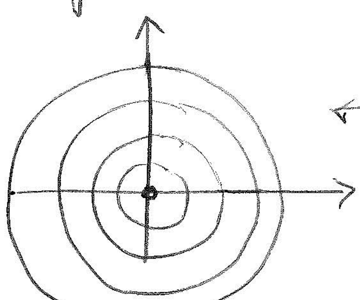
Bird's eye view:

$z=0 \Rightarrow u=0 \Rightarrow x^2+y^2=u^2=0$

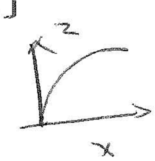
a point only

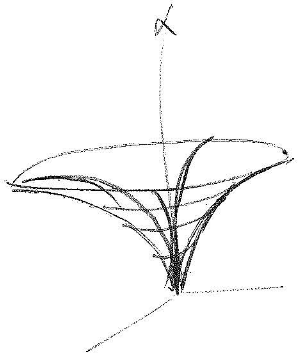
$z=1$ $x^2+y^2=1$ etc

$u=0, 1, 4, 9$ gives



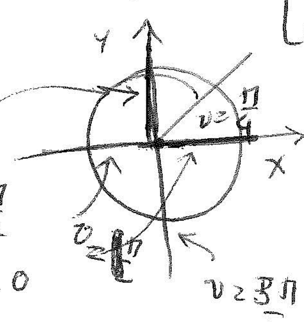
$v=0$ gives $\begin{bmatrix} u \\ 0 \\ \sqrt{u} \end{bmatrix}$ $z=\sqrt{x}$

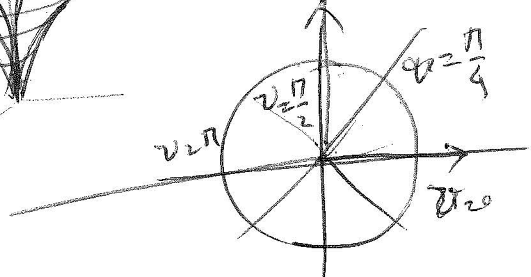




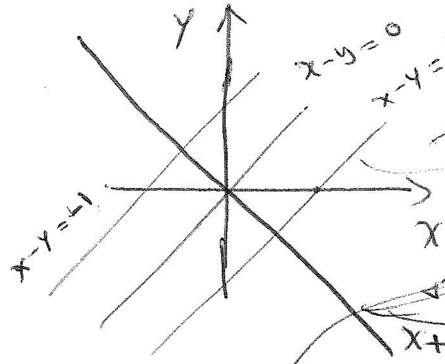
$\begin{bmatrix} 0 \\ u \\ \sqrt{u} \end{bmatrix}$ $v=\frac{\pi}{2}$

$v=0$



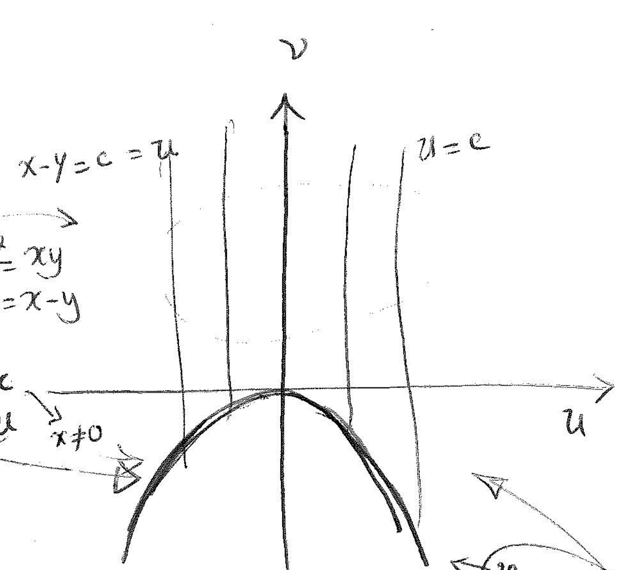


3. (a) $(u, v) = (x-y, xy)$



$\begin{cases} v = xy \\ u = x-y \end{cases}$

$\begin{cases} y = v/x \\ y = x+u \end{cases}$ $x \neq 0$



$x+y=0 \to x=-y$

$\Rightarrow v = xy = -x^2$

$u = x-y = -2y$ so $v = -x^2$

$= 2x$ $u = 2x$ so $v = -\left(\frac{u}{2}\right)^2$

to Solve for y:

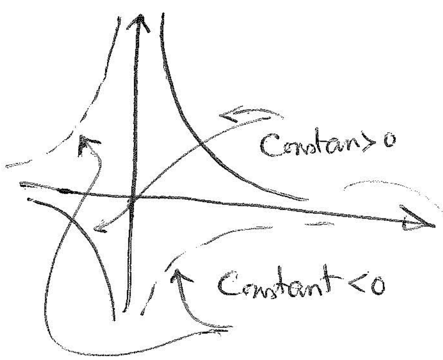
$\frac{v}{x} = x+u \Rightarrow v = x^2 - xu$ $x^2 - ux - v = 0$ We need $\Delta \geq 0$:

i.e $u^2 + 4v \geq 0$ That is $v \geq \frac{-u^2}{4}$ (*) eg $u=1$ $v \geq -\frac{1}{4}$

$u=0$ $v \geq 0$

$u=2$ $v \geq -1$

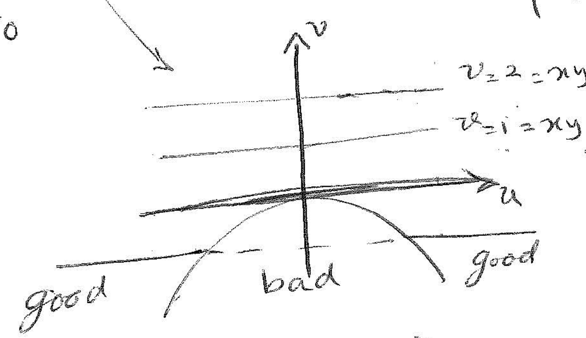
$xy = c$



Then $v = c$

and $u = x-y$ $\Rightarrow \begin{cases} v = xy \\ u = x-y \end{cases} \Rightarrow y = \frac{v}{x}$ $\neq 0$ $\to \frac{v}{x} = x-u$

$y = x-u$ $v = x^2 - xu$

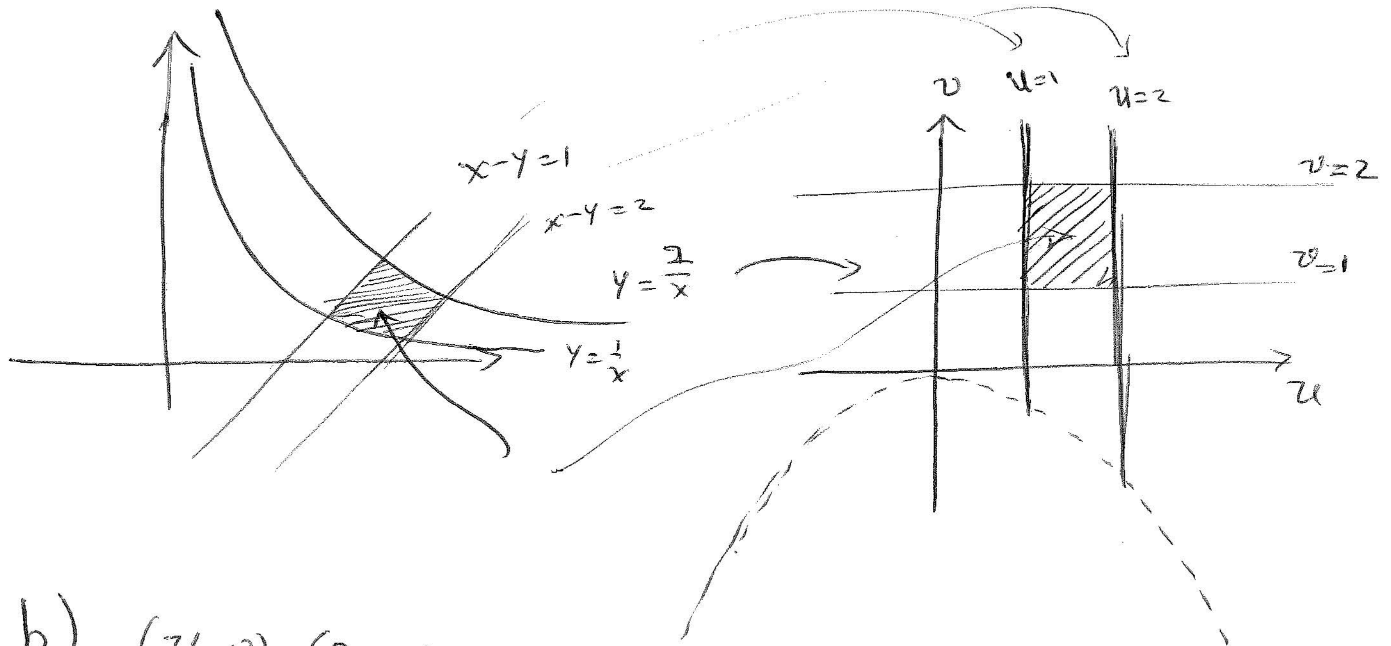


to have a sln we need $\Delta \geq 0$ so $u^2 + 4v \geq 0$

but if $v \geq \frac{-4}{u^2}$ we have a sln. if $v \geq 0$ Then There is no issues with The value of u so $-\infty < u < \infty$ allowed,

but if $v < 0$ we need u to Satisfy $u^2 > \frac{-4}{v}$ so

(6)



b) $(u,v)=(2,-2)$ implies

$\begin{cases} x-y=2 \\ xy=-2 \end{cases}$ Then $(*)$ we need $\frac{u^2}{4}+v\geq 1$ or $\frac{4}{4}+(-2)\geq 1$ ↯

so $(2,-2)$ does not belong to the range of f.

c) $\mathbf{x}=(x,y)$ $\quad Df=\begin{bmatrix} 1 & -1 \\ y & x \end{bmatrix}$ not invertible if both $x=0$ & $y=0$ or $y=-x$ else it is.

at $(u,v)=(1,2)=b$ we have $\begin{cases} x-y=1 \\ xy=2 \end{cases} \Rightarrow Df^{-1}(1,2)=[Df]^{-1}$

$y=\frac{2}{x} \rightarrow \frac{2}{x}=x-1 \rightarrow 2=x^2-x = x^2-x-2=0$

$y=x-1 \qquad x=\frac{1\pm\sqrt{9}}{2}=\frac{1\pm 3}{2} \begin{cases} 2 \rightarrow y=1 \\ -1 \rightarrow y=-2 \end{cases}$

function is two to one so we need to pick one of the options

so $x=2 \rightarrow y=1$

$x=-1 \rightarrow y=-2$ so $Df^{-1}(1,2)=[Df]^{-1}(2,1)=$ or $(-1,-2)$

$[Df(2,1)]^{-1}=\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}=\dfrac{\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}}{3}$

$[Df(-1,-2)]^{-1}=\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \rightarrow \dfrac{\begin{bmatrix} -1 & +1 \\ -2 & 1 \end{bmatrix}}{1}=\begin{bmatrix} -1 & +1 \\ -2 & 1 \end{bmatrix}$

but if we worked with the other branch we take the other sol'n.

4. Assume $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is C^1. Show that f cannot be a one to one function in two ways:

a) using implicit function theorem:

sln: This is an important technique: using a function like $f(x, y)$ to define a function $F(x, y)$ which satisfies the conditions of the IFT (3.1). At a point (a, b) we define $F(x, y) = f(x, y) - f(a, b)$ and note that since f is C^1 then so is F, and furthermore if either of the $f_x(a, b)$ or $f_y(a, b)$ is non-zero (which should be, as otherwise $\nabla f(a, b) = 0$ and therefore f would be constant,) then one of the $F_x(a, b)$ or $F_y(a, b)$ should be non-zero. Now the conditions of the IFT are satisfied and then there must be some $r_0 > 0$ such that $\forall x, \ |x - a| < r_0$ implies there is a unique y such that $F(x, y) = 0$. But this means $f(x, y) = f(a, b)$, which means that f is not 1-1 near (a, b). Indeed this function is not one to one in the neighborhood of any point.

b) using the Inverse Mapping Theorem

sln: Of course the inverse mapping theorem is to be applied to the functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. So to apply the IMT to this questions we must convert our problem into one that fits the description of the IMT. Assuming that $f_x(a, b) \neq 0$ we define a new function $\boldsymbol{g}(x, y) = (f(x, y), y)$ and notice that

$$D\boldsymbol{g}(a, b) = \begin{bmatrix} f_x(a, b) & 0 \\ f_y(a, b) & 1 \end{bmatrix}$$

which is invertible as $f_x(a, b) \neq 0$, so that by an application of IMT there are two neighborhoods U and V (of (a, b) and $(f(a, b), b)$ on which $\boldsymbol{g}$ is a one to one function, and on V the function $\boldsymbol{g}^{-1}$ is defined. In the neighborhood V consider two points with different values of y and same values for x (there must be a 'vertical' line in V); call these points P and Q. Because of the inverse function here must be two distinct points of U that correspond to these two points; call them (x_1, y_1) and (x_2, y_2), say $\boldsymbol{g}(x_1, y_1) = P$ and $\boldsymbol{g}(x_2, y_2) = Q$. Therefore $(f(x_1, y_1), y_1) = P$ and $(f(x_2, y_2), y_2) = Q$. Recall that the y coordinates of the points P and Q are different (that is $y_1 \neq y_2$) and the x coordinates are the same, (that is $f(x_1, y_1) = f(x_2, y_2)$.) This implies the function f is not one to one. (Of course $y_1 \neq y_2$ implies that the points (x_1, y_1) and (x_2, y_2) are distinct points.)

5. (in general when f is only integrable and not necessarily continuous part (b) of FTC does not apply)

$$\int_a^b 2x\, f(x^2)\, dx = \int_{u=a^2}^{b^2} f(u)\, du = \int_{x=a^2}^{b^2} f(x)\, dx \qquad (\text{when } 0<a<b)$$

let $u = x^2$ $\qquad (a^2 < b^2)$

but when $a < b \le 0$ $\quad \int_a^b 2x f(x^2)\, dx = \int_{a^2}^{b^2} f(u)\, du = -\int_{b^2}^{a^2} f(x) dx$

let $u = x^2$

$a < b \le 0 \longrightarrow a^2 > b^2$ $\qquad b^2 < a^2$

$$\int_{[a b]} 2x f(x^2) dx = -\int_{[b^2 a^2]} f(x) dx$$

or if $g(x) = x^2$

$$\int_{[a,b]} f(x^2)\, |g'(x)|\, dx = \int_{g([a\,b])} f(x)\, dx$$

$|g'(x)| = -2x$ as $x < 0$ $\qquad g([a\,b]) = [b^2, a^2]$

discussion on pg 177 suggests

$$\int_{[a b]} f(g(x))\, |g'(x)| dx = \int_{g([a b])} f(u)\, du$$

it is not known if g is inc or decreasing, so $|g'(x)|$ make a sign correction

$g([a b])$ we usually mean an interval $[c\ d]$, $c < d$