Ex. Let $X \sim \chi^2_{(\nu)}$

$X \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{1}{2}\right)$

$$m_X(t) = \left(\frac{1/2}{1/2 - t}\right)^{\nu/2} = \left(\frac{1}{1-2t}\right)^{\nu/2}$$

Let $X_1 \sim \chi^2_{(\nu_1)}, \ldots, X_n \sim \chi^2_{(\nu_n)}$ - indep.

$Y = X_1 + \ldots + X_n$

$$m_Y(t) = m_{X_1}(t) \cdot \ldots \cdot m_{X_n}(t) = \left(\frac{1}{1-2t}\right)^{\frac{\nu_1 + \ldots + \nu_n}{2}}$$

$\Rightarrow Y \sim \chi^2_{(\nu_1 + \ldots + \nu_n)}$

Note: $Z = N(0,1)$, $Z^2 \sim \chi^2_{(1)}$

$$m_{Z^2}(t) = E\left(e^{Z^2 t}\right) = \int_{-\infty}^{\infty} e^{z^2 t} \cdot \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{2}(1-2t)}}{\sqrt{2\pi}} \, dz = \left(\begin{array}{l} z\sqrt{1-2t} = u \\ dz = \frac{1}{\sqrt{1-2t}} \, du \end{array}\right)$$

$$= \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \, du}_{=1} \cdot \frac{1}{\sqrt{1-2t}} = \left(\frac{1}{1-2t}\right)^{1/2}$$

Ex. $X \sim N(0, \sigma^2)$

Find general formula for $E(X^k)$

$$m_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} = e^{\frac{t^2\sigma^2}{2}}$$

$$E(X) = m_X'(0) = t\sigma^2 e^{\frac{t^2\sigma^2}{2}} \Big|_{t=0} = 0$$

$$E(X^2) = m_X''(0) = \sigma^2 e^{\frac{t^2\sigma^2}{2}} + t^2\sigma^4 e^{\frac{t^2\sigma^2}{2}} \Big|_{t=0} = \sigma^2$$

$$E(X^3) = m_X'''(0) = t\sigma^4 e^{\frac{t^2\sigma^2}{2}} + 2t\sigma^4 e^{\frac{t^2\sigma^2}{2}} + t^3\sigma^6 e^{\frac{t^2\sigma^2}{2}} \Big|_{t=0} = 0$$

$$E(X^4) = m_X^{(iv)}(0) = \sigma^4 e^{\frac{t^2\sigma^2}{2}} + 2\sigma^4 e^{\frac{t^2\sigma^2}{2}} + \dots \Big|_{t=0} = 3\sigma^4$$

$E(X^k) = 0$, k is odd

$E(X^k) = \, ?$, k is even