Duration: **60 minutes (2:15pm – 3:15pm)**
Aids Allowed: **none**

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please fill out the identification section above.

MARKING GUIDE

# 1:	5	/ 5
# 2:	8	/ 9
# 3:	7	/ 9
# 4:	6	/ 8
TOTAL:	26	/31

This term test consists of 4 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, and* ***write your name on the back of the last page****.*

This test is double-sided.

Good Luck!

Question 1. [5 marks]

The Setun computer was developed in Moscow in the 1950s. It used a ternary (base 3) number system.

Part (a) [1 mark]

What is the decimal (base 10) representation of the ternary number 121? Show your work and place your final answer in the box.

$\cancel{(121)_{10} = (1\times 100 + 2\times 10 + 1\times 10)_{10} = (121)_{10}}$

$(121)_3 = (1\times 3^2 + 2\times 3^1 + 1\times 3^0)_{10} = (9+6+1)_{10} = (16)_{10}$

$\boxed{(16)_{10}\ \cancel{(121)_{10}}}$

Part (b) [1 mark]

What is the binary (base 2) representation of the ternary number 121? Show your work and place your final answer in the box.

$\cancel{(121)_{10} = (2^6 + 2^5 + 2^4 + 2^3 + 0\times 2^2 + 0\times 2^1 + 1\times 2^0)_{10}}$

$\cancel{= (1111001)_2}$

$(121)_3 = (2^4\times 1 + 2^3\times 0 + 2^2\times 0 + 2^1\times 0 + 2^0\times 0)_{10}$

$= (10000)_2$

$\boxed{\cancel{(1111001)_2}\ (10000)_2}$

Part (c) [1 mark]

Using only ternary numbers, determine the sum of the ternary numbers 10101 and 20102. Show your work and place your final answer in the box.

$$\begin{array}{r} (10101)_3 \\ +\ (20102)_3 \\ \hline (100210)_3 \end{array}$$

$\boxed{(100210)_3}$

Part (d) [2 marks]

Using only ternary numbers, determine the product of the ternary numbers 12 and 102. Show your work and place your final answer in the box.

$$\begin{array}{r} (12)_3 \\ \times\ (102)_3 \\ \hline 101 \\ 12 \\ \hline (2001)_3 \end{array}$$

$\boxed{(2001)_3}$

5/5

Question 2. [9 MARKS]

Recall that an integer n is even if and only if $\exists q \in \mathbb{Z}, n = 2q$. Also, an integer n is odd if and only if $\exists q \in \mathbb{Z}, n = 2q + 1$. Integers are either even or odd.

Let us define the predicates $E(n)$: "n is an even number" and $O(n)$: "n is an odd number".

Consider the following statement:

For every integer n, n^3 is even if and only if n is even.

Part (a) [1 MARK]

Translate the statement into symbolic notation. Quantify over the integers ($\mathbb{Z}$). Use the predicate $E(n)$.

1

~~($\forall n \in \mathbb{N}$)~~

$\forall n \in \mathbb{Z}$, ~~n^3 is even~~ $E(n^3) \Leftrightarrow E(n)$.

Part (b) [8 MARKS]

Write a detailed structured proof of the statement. Part marks will be given for having correct elements of the proof structure.

7/8

~~Proof: Assume $n \in \mathbb{Z}$,~~
~~Assume $E(n)$,~~
~~Then n is odd even~~
~~Let $n=2k$~~
~~Then $n^3=(2k)^3=8k^3=2(4k^3)$~~
~~Let $j=4k^3$~~
~~Then $n^3=2j$~~
~~Then $E(n^3)$~~
~~Then $E(n) \Rightarrow E(n^3)$.~~
~~Assume $E(n^3)$.~~
~~Let $n^3=2m$~~

Proof: Assume $n \in \mathbb{Z}$. # n is a typical integer
Assume $E(n)$. # one direction.
Let
Then $\exists k \in \mathbb{Z}$, $n=2k$. # definition of even numbers
Should use "Let" instead of "$\exists k$"
Then $n^3=(2k)^3=8k^3=2(4k^3)$. # algebra
Let $j=4k^3$.
Then $n^3=2(4k^3)=2j$.
Then $E(n^3)$. # definition of even numbers.
Then $E(n) \Rightarrow E(n^3)$. # implication
Assume $\neg E(n)$. # the other direction
Let
Then $\exists k' \in \mathbb{Z}$, $n=2k'+1$. # definition of odd numbers.
Then $n^3=(2k'+1)^3 = 8k'^3+12k'^2+6k'+1 = 2(4k'^3+6k'^2+3k')+1$. # algebra
Let $j' = 4k'^3+6k'^2+3k'$
Then $n^3=2j'+1$
Then n^3 is odd
Then $\neg E(n^3)$

→ (see the next page)

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each answer with the appropriate question and part number.

Then $\neg E(n) \Rightarrow \neg E(n^3)$. # implication
Then $E(n^3) \Rightarrow E(n)$. # contrapositive
Therefore $E(n) \Leftrightarrow E(n^3)$. # since $E(n) \Rightarrow E(n^3)$ and $E(n^3) \Rightarrow E(n)$
Then $\forall n \in \mathbb{Z},\ E(n^3) \Leftrightarrow E(n)$.

∎

$\neg E(n)$
$(2k+1)$
$(4k^2+4k+1)(2k+1)$
$8k^3+8k^2+2k+4k^2+4k+1$

Question 3. [9 MARKS]

Recall that an integer $p > 1$ is prime if and only if its only positive integer divisors are 1 and p.

Also, an integer n is odd if and only if $\exists q \in \mathbb{Z}, n = 2q+1$. An integer n is even if and only if $\exists q \in \mathbb{Z}, n = 2q$. Integers are either odd or even.

Let us define the predicates $P(n)$: "n is a prime number", $O(n)$: "n is an odd number" and $E(n)$: "n is an even number".

Consider the following statement:

All prime numbers greater than 2 are odd.

Part (a) [2 MARKS]

Translate the statement into symbolic notation. Quantify over the natural numbers ($\mathbb{N}$). Use the predicates $P(n)$, $O(n)$ and/or $E(n)$.

2

$\forall n \in \cancel{\mathbb{Z}}\,\mathbb{N}, n>2, \cancel{O(n)} \wedge P(n) \Rightarrow O(n)$. ✓

Part (b) [7 MARKS]

Write a detailed structured proof of the statement. Part marks will be given for having correct elements of the proof structure.

5

Proof: Assume $n \in \mathbb{N}$, # n is a typical integer. ✓

Assume $n > 2$,

Assume $\neg O(n)$, # negation of consequent

Then $E(n)$. # definition and since integers are either even or odd

Then $\exists k \in \cancel{\mathbb{Z}}\,\mathbb{N}, n = 2k$ # definition of even

Then $\neg P(n)$. # by definition of prime numbers

Then $E(n) \Rightarrow \neg(P(n))$.

Then $\neg O(n) \Rightarrow \neg P(n)$. # since $E(n) \Leftrightarrow \neg O(n)$

Then $P(n) \Rightarrow O(n)$. # contrapositive

Then $n>2, P(n) \Rightarrow O(n)$.

Then $\forall n \in \mathbb{N}, n>2, P(n) \Rightarrow O(n)$.

You have to show this!

prime: the only ~~int~~ integer ~~d~~ divisors of the number is 1 and itself.

Question 4. [8 MARKS]

Recall that for $x \in \mathbb{R}$, we can define $|x|$ by $|x| = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$

(This is the only **definition** of $|x|$ that you are allowed to use in your solution to this question.)

Consider the following statement:

For every real number x, if $|x - 3| < 3$ then $0 < x < 6$.

This statement is equivalent to the symbolic statement:

$\forall x \in \mathbb{R}, |x - 3| < 3 \Rightarrow 0 < x < 6.$

Now consider the following argument:

Assume $x \in \mathbb{R}$.
  Assume $|x - 3| < 3$.
    Then either $x - 3 \geqslant 0$ or $x - 3 < 0$.
    Case 1: Assume $x - 3 \geqslant 0$.
      Then $|x - 3| = x - 3$. # by the above definition
      Then $x - 3 < 3$. # since $|x - 3| < 3$
      Then $x < 6$. # add 3 to both sides
    Case 2: Assume $x - 3 < 0$.
      Then $|x - 3| = -(x - 3)$. # by the above definition
      Then $-(x - 3) < 3$. # since $|x - 3| < 3$
      Then $-x + 3 < 3$.
      Then $-x < 0$. # subtract 3 from both sides
      Then $0 < x$ # add x to both sides.
    Then we have proven both $0 < x$ and $x < 6$.
    Then $0 < x < 6$.
  Then $|x - 3| < 3 \Rightarrow 0 < x < 6$.
Then $\forall x \in \mathbb{R}, |x - 3| < 3 \Rightarrow 0 < x < 6$.

Part (a) [2 MARKS]

This argument is not a correct proof of the statement. Explain the flaw in the argument.

The argument ignores the ~~condition~~ restrains of x in two assumptions, which are $x-3 \geqslant 0 \Rightarrow x \geqslant 3$ and

1. $x-3<0 \Rightarrow x<3$.

Hence the conclusions of two cases should be $0<x<3$ and $3 \leq x<6$.

Why is this a flaw though?

The proof shows that, under the assumption that $|x-3|<3$, it follows that for $x-3 \geqslant 0$, $x<6$. It also ~~of~~ shows that under the same assumption, when $x-3<0$, $x>0$. Since either $x-3 \geqslant 0$ or $x-3<0$, we have shown that either $x<6$ or $x>0$. But we are required to show $0<x<6$. That is --- <u>and</u> ---. In other words, the disjunction of $0<x$ [illegible] $x<6$ has been proven, but not the conjunction.

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Part (b) [6 MARKS]

Give a correct proof of the statement $\forall x \in \mathbb{R}, |x-3| < 3 \Rightarrow 0 < x < 6$.

Proof: Assume $x \in \mathbb{R}$.

Assume $|x-3| < 3$.

Then either $x-3 \geq 0$ or $x-3 < 0$.

Case 1: Assume $x-3 \geq 0$.

Then $x \geq 3$. # algebra.

Then $|x-3| = x-3$. # by the above definition.

Then $x-3 < 3$. # since $|x-3| < 3$

Then $x < 6$. # algebra.

Then $3 \leq x < 6$.

Case 2. Assume $x-3 < 0$.

Then $x < 3$. #algebra

Then $-(x-3) < 3$. # since $|x-3| < 3$

Then $-x+3 < 3$.

Then $-x < 0$. #algebra

Then $0 < x$. #add x to both sides.

~~Then~~ Then $0 < x < 3$. −1

Then we have proven ~~both~~ either $0 < x < 3$ ~~and~~ or $3 \leq x < 6$.

Then $0 < x < 6$ why is this the case?

Then $|x-3| < 3 \Rightarrow 0 < x < 6$.

Then $\forall x \in \mathbb{R}, |x-3| < 3 \Rightarrow 0 < x < 6$. ■