PRACTICE QUESTIONS FOR MIDSEMESTER EXAM

(1) (a) State the fundamental theorem of arithmetic.
    (b) Use it to prove that $\sqrt{5}$ is irrational.

(2) If $f : A \to B$ is a function, and $S \subset B$ is a subset of the target, we define

$$f^{-1}(S) = \{x \in A \,|\, f(x) \in S\}.$$

  (a) Determine whether $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$. If true, provide a proof. If false, give a counterexample.
  (b) Determine whether $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$. If true, provide a proof. If false, give a counter example.

(3) Let $x \in \mathbb{Z}$. Prove that x^n has the same parity as x for any $n \in \mathbb{N}$. In other words, if x is even/odd, then x^n is even/odd.

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$.
  (a) Prove that $f(1) = 0$.
  (b) Prove that $f(u^n) = nu^{n-1}f(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

(5) (a) Give the definition of a countable set.
    (b) Prove that if A and B are countable sets, then $A \times B$ is countable.

(6) We roll a fair six-sided die four times. For each $k \in \{0, 1, 2, 3, 4\}$, determine the probability that we roll a 6 exactly k times.

(7) Determine (with proof) the number of nonnegative integer solutions to

$$x_1 + x_2 + \ldots + x_k = n.$$

You may express your answer in terms of binomial coefficients.

(8) (a) Let $a, b \in \mathbb{N}$. Define the greatest common divisor of a and b.
    (b) Suppose $\gcd(a, b) = 1$. What are the possible values of $\gcd(2a, 3b)$? You must justify your answer.