1.2 DEFINITION

Addition in $\mathbf{R}^n$ is defined as follows: for any $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbf{R}^n$, the sum $\mathbf{u} + \mathbf{v}$ is given by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

For any scalar a and any vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in $\mathbf{R}^n$, the **product** $a\mathbf{u}$ is defined by

$$a\mathbf{u} = (au_1, au_2, \ldots, au_n).$$

The operation that combines the scalar a and the vector $\mathbf{u}$ to yield $a\mathbf{u}$ is referred to as **multiplication of the vector u by the scalar** a, or simply as **scalar multiplication**. Also, the product $a\mathbf{u}$ is called a **scalar multiple** of $\mathbf{u}$.

The following theorem gives the basic properties of the two operations that we have defined.

1.3 THEOREM

The following properties are valid for any scalars a and b, and any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{R}^n$:

1. $\mathbf{u} + \mathbf{v} \in \mathbf{R}^n$. (Closure under addition)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. (Associative property of addition)
3. There is a vector $\mathbf{0}$ in $\mathbf{R}^n$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^n$. (Additive identity)
4. For each $\mathbf{u} \in \mathbf{R}^n$, there is a vector $-\mathbf{u}$ in $\mathbf{R}^n$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. (Additive inverses)
5. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. (Commutative property of addition)
6. $a\mathbf{u} \in \mathbf{R}^n$. (Absorption under scalar multiplication)
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$. (Associative property of scalar multiplication)
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$. (Distributive property, vector addition)
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$. (Distributive property, scalar addition)
10. $1 \cdot \mathbf{u} = \mathbf{u}$.

The proofs of these properties are easily carried out using the definitions of vector addition and scalar multiplication, as well as the properties of real numbers. As typical examples, properties 3, 4, and 8 will be proved here. The remaining proofs are left as an exercise.

Proof of Property 3 The vector $\mathbf{0} = (0, 0, \ldots, 0)$ is in $\mathbf{R}^n$, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$,

$$\mathbf{u} + \mathbf{0} = (u_1 + 0, u_2 + 0, \ldots, u_n + 0) = \mathbf{u}.$$

■ ■ ■

Since the points lie in the plane, their coordinates must satisfy the equation

$$ax + by + cz = d$$

of the plane. Substituting in order for $(0, 0, 0)$, $(1, 2, 3)$, and $(3, 5, 1)$, we obtain

$$\begin{aligned} 0 &= d \\ a + 2b + 3c &= d \\ 3a + 5b + c &= d. \end{aligned}$$

Using $d = 0$ and subtracting 3 times the second equation from the last one leads to

$$\begin{aligned} a + 2b + 3c &= 0 \\ -b - 8c &= 0. \end{aligned}$$

Solving for a and b in terms of c, we obtain the following solutions

$$\begin{aligned} a &= 13c \\ b &= -8c \\ c &\text{ is arbitrary.} \end{aligned}$$

With $c = 1$, we have

$$13x - 8y + z = 0$$

as the equation of the plane $\langle \mathcal{A} \rangle$. ■

In the remainder of our discussion, we shall need the following definition, which applies to real coordinate spaces in general.

1.18 DEFINITION

For any two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, the **inner product (dot product, or scalar product)** of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{k=1}^{n} u_k v_k.$$

The inner product defined in this way is a natural extension of the following definitions that are used in the calculus:

$$\begin{aligned} (x_1, y_1) \cdot (x_2, y_2) &= x_1 x_2 + y_1 y_2, \\ (x_1, y_1, z_1) \cdot (x_2, y_2, z_2) &= x_1 x_2 + y_1 y_2 + z_1 z_2. \end{aligned}$$

The distance formulas used in the calculus lead to formulas for the length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ in $\mathbf{R}^2$ or $\mathbf{R}^3$ as follows:

$$\|(x, y)\| = \sqrt{x^2 + y^2},$$
$$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}.$$

We extend these formulas for length to more general use in the next definition.

1.19 DEFINITION

For any $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbf{R}^n$, the **length** (or **norm**) of $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The following properties are direct consequences of the definitions involved, and are presented as a theorem for convenient reference.

1.20 THEOREM

For any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{R}^n$ and any a in $\mathbf{R}$:

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
(ii) $(a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v})$
(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
(iv) $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$, or $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
(v) $\|a\mathbf{u}\| = |a|\|\mathbf{u}\|$.

Our next theorem gives a geometric interpretation of $\mathbf{u} \cdot \mathbf{v}$ in $\mathbf{R}^2$ or $\mathbf{R}^3$. In the proof, we use the Law of Cosines from trigonometry: *If the sides and angles of an arbitrary triangle are labeled according to the pattern in Figure 1.9, then*

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

We state and prove the theorem for $\mathbf{R}^3$, but the same result holds in $\mathbf{R}^2$ with a similar proof.

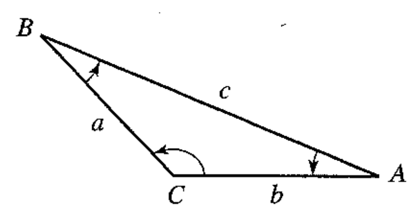

Figure 1.9

1.21 THEOREM

For any two nonzero vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbf{R}^3$,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta,$$

where θ is the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$ and $0° \le \theta \le 180°$.

Proof Suppose first that $\theta = 0°$ or $\theta = 180°$. Then $\mathbf{v} = c\mathbf{u}$, where the scalar c is positive if $\theta = 0°$ and negative if $\theta = 180°$. We have

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = \|\mathbf{u}\|\,(|c|\cdot\|\mathbf{u}\|)\cos\theta = |c|\cos\theta\,\|\mathbf{u}\|^2 = c\|\mathbf{u}\|^2$$

and

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot (c\,\mathbf{u}) = c\,(\mathbf{u} \cdot \mathbf{u}) = c\|\mathbf{u}\|^2.$$

Thus the theorem is true for $\theta = 0°$ or $\theta = 180°$.

Suppose now that $0° < \theta < 180°$. If $\mathbf{u} - \mathbf{v}$ is drawn from the head of $\mathbf{v}$ to the head of $\mathbf{u}$, the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ form a triangle with $\mathbf{u} - \mathbf{v}$ as the side opposite θ. (See Figure 1.10.)

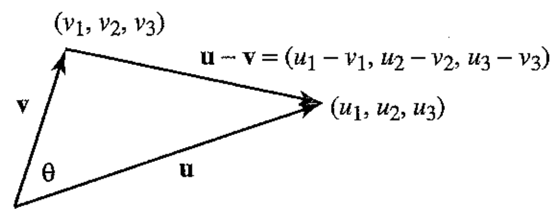

Figure 1.10

From the Law of Cosines, we have

$$\cos\theta = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Thus

$$\begin{aligned}
\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta &= \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2) \\
&= \tfrac{1}{2}\{u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 \\
&\qquad - [(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2]\} \\
&= \mathbf{u} \cdot \mathbf{v}.
\end{aligned}$$

■ ■ ■

1.22 COROLLARY

In $\mathbf{R}^2$ or $\mathbf{R}^3$, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (or orthogonal) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

18. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$. Prove that $\|\mathbf{u}\| = \|\mathbf{v}\|$ if and only if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.

19. Prove Theorem 1.20.

20. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$. Prove that $\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$.

21. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$. Prove that $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\ \|\mathbf{v}\|$.

22. Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbf{R}^n$. Prove that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

23. The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$, denoted by $d(\mathbf{u}, \mathbf{v})$, is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Find the distance between each of the following pairs of vectors.

(a) $\mathbf{u} = (3, 6), \mathbf{v} = (-1, 9)$ **(b)** $\mathbf{u} = (-1, 4, -2), \mathbf{v} = (5, 6, 5)$

(c) $\mathbf{u} = (0, 3, 4, 1), \mathbf{v} = (1, 2, 0, 0)$

(d) $\mathbf{u} = (-1, 1, 5, 3, 0), \mathbf{v} = (-2, 1, 4, 0, 1)$

24. Prove the following properties concerning the distance where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbf{R}^n$.

(a) $d(\mathbf{u}, \mathbf{v}) \ge 0$ **(b)** $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(d) $d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (The triangle inequality)

25. The **cross product** $\mathbf{u}\times\mathbf{v}$ of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is given by

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{e}_1 + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}\mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{e}_3,\end{aligned}$$

where $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$. The symbolic determinant below is frequently used as a memory aid, since "expansion" about the first row yields the value of $\mathbf{u}\times\mathbf{v}$.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Prove the following facts concerning the cross product.

(a) $\mathbf{u}\times\mathbf{v}$ is perpendicular to each of $\mathbf{u}$, $\mathbf{v}$. **(b)** $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

(c) $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$. **(d)** $(a\mathbf{u})\times\mathbf{v} = a(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(a\mathbf{v})$.

(e) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$. **(f)** $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.

(g) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$, where θ is the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$, and $0° \le \theta \le 180°$.

(h) $\|\mathbf{u}\times\mathbf{v}\|$ is the area of a parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.