APM462H1S, Winter 2014 , Assignment 4,

due: Monday March 31, at the beginning of the lecture.

Remark: If you need to repeat some arguments from the lecture and/or notes, it is fine to write "By arguing exactly as in the lecture notes of Evans, page ... , we find that ..." But you should only do this if the argument is really *exactly* the same. Also, whenever you refer to some source, it is best to give a reference that is precise and verifiable, so that a reader can in principle look it up and understand what you are talking about. For this reason, it is better to cite the notes of Evans, for example, than the lecture, since a reader will not know what you are talking about if you write "By arguing exactly as in the lecture, ...".

Exercise 1.
Assume that $x(\cdot)$ is a minimizer of the calculus of variations problem:

$$\text{minimize } I[x(\cdot\ )] = \int_0^1 \left[\frac{1}{2}x'(t)^2 + x'(t)x(t)\right]\ dt \qquad \text{subject to } x(0) = 1.$$

a. Show that if $y(\cdot)$ is any C^2 function such that $y(0) = 0$ (but $(y(1)$ need not equal zero), then

$$\int_0^1 [x'(t)\,y'(t) + x'(t)\,y(t) + x(t)\,y'(t)]\ dt = 0$$

solution. Let $y(\cdot)$ be any C^2 function such that $y(0) = 0$, and define

$$i(\tau) = I[x(\cdot) + \tau y(\cdot)]$$

Then $x(\cdot) + \tau y(\cdot)$ satisfies the constraint $x(1) + \tau y(1) = 1$ for every τ, so (because $x(\cdot)$ is a minimizer) it follows that

$$i(0) = I[x(\cdot)]\ \leq\ I[x(\cdot) + \tau y(\cdot)] = i(\tau)$$

for every τ. Thus $i(\cdot)$ has a global minimum at $\tau = 0$, so $i'(0) = 0$. And a straightforward computation, using the chain rule, shows that

$$0 = i'(0) = \int_0^1 [x'(t)\,y'(t) + x'(t)\,y(t) + x(t)\,y'(t)]\ dt.$$

b. Explain why $x''(t) = 0$ for all $t \in (0, 1)$.
solution. Let $y(\cdot)$ be any function such that $y(0) = 0$. Then

$$\begin{aligned}
0 &= \int_0^1 [x'(t)\,y'(t) + x'(t)\,y(t) + x(t)\,y'(t)]\ dt \\
&= \int_0^1 (x'(t) + x(t))\,y'(t)\,ndt\ +\ \int_0^1 x'(t)\,y(t)\,dt \\
&= -\int_0^1 (x'(t) + x(t))'\,y(t)\,dt\ + (x'(t) + x(t))y(t)\Big|_{t=0}^{t=1} + \int_0^1 x'(t)\,y(t)\,dt \\
&= -\int_0^1 x''(t)\,y(t)\,dt + [x'(1) + x(1)]y(1). \qquad (1)
\end{aligned}$$

In particular, if y is any function such that $y(0) = y(1) = 0$, then

$$-\int_0^1 x''(t)\, y(t)\, dt \;=\; 0.$$

Now, by arguing *exactly* as in the proof of Theorem 4.1 in the notes of Evans, we conclude that

$$x''(t) = 0 \qquad \text{for all } t \in [0,1].$$

c. Find an additional necessary condition that tells you something about the behaviour of $x(\cdot)$ at the right endpoint $t = 1$.

solution. Since we know that $x''(t) = 0$ for all t, we can look again at (1) and find that

$$[x'(1) + x(1)]y(1)$$

for every function $y(\cdot)$ such that $y(0) = 0$. It follows that

$$x'(1) + x(1) = 0,$$

d. Determine the minimizer $x(\cdot)$.

We know that the minimizer satisfies

$$x''(t) = 0, \qquad x(0) = 1, \qquad x(1) = -x'(1).$$

It follows from the first two of these facts that $x(\cdot)$ has the form

$$x(t) = 1 + ct \qquad \text{for some } c.$$

Then $x(1) = 1 + c$ and $x'(1) = +c$, so the condition $x'(1) + x(1) = 0$ implies that $c = -\frac{1}{2}$. Thus

$$x(t) = 1 - \frac{1}{2}t.$$

Exercise 2. Consider the problem:

$$\text{minimize} \quad I[x(\cdot)] = \frac{1}{2}\int_0^\pi x'(t)^2 \; dt \tag{2}$$

subject to the conditions $x(0) = x(\pi) = 0$ and the constraint

$$J[x(\cdot)] = \int_0^\pi x(t)^2 \; dt \;=\; 1. \tag{3}$$

Suppose that $x : [0,\pi] \to \mathbb{R}$ is a C^2 function that solves the above problem.

Let $y : [0,\pi] \to \mathbb{R}$ be any other C^2 function such that $y(0) = y(\pi) = 0$.

Define

$$\alpha(s) := \left(\int_0^\pi (x(t) + sy(t))^2 \; dt\right)^{1/2}$$

and

$$i(s) := I[\frac{x(\cdot) + sy(\cdot)}{\alpha(s)}].$$

a. Explain why $\alpha(0) = 1$ and $i'(0) = 0$.

solution. The fact that $x(\cdot)$ satisfies the constraints $J[x(\cdot)] = 0$ immediately implies that $\alpha(0) = 0$.

Also, we claim that for every s, the function $\frac{x(\cdot)+sy(\cdot)}{\alpha(s)}$ satisfies the constraints. It is easy to see that it equals zero at both endpoints. And

$$J[\frac{x(\cdot)+sy(\cdot)}{\alpha(s)}] = \int_0^1 \left[\frac{x(t)+sy(t)}{\alpha(s)}\right]^2 \, dt = \alpha(s)^{-2}\int_0^1 (x(t)+sy(t))^2 \, dt = 1$$

by the definition of $\alpha(s)$. This proves that $\frac{x(\cdot)+sy(\cdot)}{\alpha(s)}$ satisfies the constraints as claimed.

It follows, from the fact that $x(\cdot)$ is a minimizer, that

$$i(0) = I[x(\cdot)] \le I[\frac{x(\cdot)+sy(\cdot)}{\alpha(s)}] = i(s)$$

for every s. Thus $i(\cdot)$ has a minimum at $s=0$, so $i'(0)=0$.

b. Show that

$$i'(0) = \int_0^\pi x'(t)\,y'(t)\,dt - \lambda \int_0^\pi x(t)\,y(t)\,dt \tag{4}$$

for some constant λ, and find a formula for λ in terms of $x(t)$.

solution We will compute $i'(0)$. We have

$$i(s) = \alpha(s)^{-2} j(s) \qquad \text{for } j(s) = \frac{1}{2}\int_0^1 (x'(t)+sy'(t))^2 \, dt.$$

So by the product rule and chain rule,

$$i'(s) = -2\alpha(s)^{-3}\alpha'(s)j(s) + \alpha(s)^{-2}j'(s)$$

for every s, and in particular for $s=0$. Also, again by the chain rule,

$$\alpha'(s) = \frac{1}{2}\left(\int_0^\pi (x(t)+sy(t))^2 \, dt\right)^{-1/2} \frac{d}{ds}\int_0^\pi (x(t)+sy(t))^2 \, dt.$$

After a few calculations, and recalling that $\alpha(0)=0$, we deduce that

$$\alpha'(0) = \int_0^\pi x(t)\,y(t)\,dt.$$

Similarly, but easier,

$$j'(0) = \int_0^\pi x'(t)y'(t)\,dt.$$

Putting all this together, we obtain

$$\begin{aligned} 0 &= -2j(0)\int_0^\pi x(t)\,y(t)\,dt + \int_0^\pi x'(t)\,y'(t)\,dt \\ &= \int_0^\pi x'(t)\,y'(t)\,dt \; - \; \lambda\int_0^\pi (t)\,y(t)]\,dt \qquad \text{for } \lambda = 2j(0) = \int_0^\pi x'(t)^2\,dt. \end{aligned}$$

c. Show that if $x(\cdot)$ solves problem (2), (3), then

$$x''(t) + \lambda x(t) = 0 \qquad \text{for } 0 < t < \pi.$$

solution. Start from the conclusion of part **b** and integrate by parts to find that

$$\int_0^\pi (-x''(t) + \lambda x(t))y(t)\,dt = 0$$

for all $y(\cdot)$ such that $y(0) = y(\pi) = 0$. Then it follows by *exactly* the argument from Theorem 4.1 in the notes of Evans that

$$-x''(t) + \lambda x(t) = 0 \qquad \text{for all} t \in [0, \pi].$$

Exercise 3. Consider the linear system

$$\begin{cases} \dot{x}(t) &= Mx(t) + N\alpha(t) \qquad\qquad \text{for } t > 0 \\ x(0) &= x_0, \end{cases}$$

where $x(t) \in \mathbb{R}^3$ for all t, M is a 3×3 matrix, N is a 3×1 matrix (ie a column vector), and $\alpha(t) \in [-1, 1]$ for all t. Let G denote the controllability matrix

$$G = [N, MN, M^2N].$$

Find the rank of G in the following examples:

a.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

solution.
Here

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 6 & 12 \end{pmatrix}$$

This clearly has rank 2. because the first two rows are just multiples of each other, hence linearly dependent, and the third row is clearly independent of the other two.

b.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

solution Here

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 6 & 18 \end{pmatrix}$$

This has rank 3, since we can easily check that the determinant is 4.

c.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

solution Here

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has rank 2, since there are clearly two linearly indepdent rows.

d.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \quad N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which has rank 3.

Exercise 4.
a. Consider the single equation

$$x'(t) = mx(t) + n\alpha(t)$$

where $\alpha(t) \in [-1, 1]$ is a control parameter and m, n are positive numbers. Determine *exactly* the reachability set $\mathcal{C}$ for this equation. (Here $n = m = 1$, using notation from the lecture and the online notes of Evans, so the reachability set is a subset of the real line $\mathbb{R}$.)

solution.
We know from the note of Evans (page 17) that x^0 belongs to the time t reachbility set $\mathcal{C}(t)$ if and only if

$$x^0 = -\int_0^t X^{-1}(s)N\alpha(s)\,ds \text{ for some control } \alpha(\cdot).$$

In this case, this reduces to

$$x^0 = -\int_0^t e^{-ms}n\alpha(s)\,ds \text{ for some control } \alpha(\cdot).$$

Thus, x^0 belongs to the reachability set $\mathcal{C}$ if and only if

$$x^0 = -\int_0^t e^{-ms}n\alpha(s)\,ds \text{ for some control } \alpha(\cdot) \text{ and some } t > 0.$$

Note that in general, if $\alpha(t) \in [-1, 1]$ for all t, then

$$-\int_0^t e^{-ms}n\alpha(s)\,ds \;\leq\; \int_0^t e^{-ms}n\,ds = \frac{n}{m}(1 - e^{-mt}) < \frac{n}{m}$$

and

$$-\int_0^t e^{-ms}n\alpha(s)\,ds \;\geq\; -\int_0^t e^{-ms}n\,ds = -\frac{n}{m}(1 - e^{-mt}) > -\frac{n}{m}.$$

So the reachability set is contained in $(-\frac{n}{m}, \frac{n}{m})$. On the other hand, by choosing $\alpha = \pm 1$ (constant) and waiting a long enough time, we see that any x^0 such that $|x^0| < \frac{n}{m}$ belongs to the reachability set.

Thus (since we know that the reachability set is convex, hence an interval) we conclude that

$$\boxed{\mathcal{C} = (-\frac{n}{m}, \frac{n}{m}).}$$