Central Limit Theorem

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Central Limit Theorem

Theorem (CLT)

Suppose that $Y_1, Y_2, \ldots, Y_n \sim i.i.d.(\mu, \sigma^2)$, *where* $-\infty < \mu < +\infty$ *and* $0 < \sigma^2 < \infty$. *Then the statistic* $U_n = \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ *will have* $U_n \xrightarrow{d} N(0,1)$, *as* $n \to \infty$.

1. CLT usually needs two conditions: (1) relations between elements in the sequence; (2) differences in distributions.
2. CLT does not require specific distributions on the population.
3. It belongs to large sample theory.

Convergence in Distribution

A random sequence X_n **converges in distribution** to X is defined as follows.

Definition (Convergence in Distribution)

$X_n \xrightarrow{d} X$ means that $F_{X_n}(x) \longrightarrow F_X(x)$, $\forall x \in \mathbb{R}$, as $n \to \infty$.

1. There are many kinds of converges in different senses, including convergence almost surely, convergence in probability, convergence in distribution, convergence in L^p norm and etc.
2. **Slutsky's Theorem** If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} a$ with a being a constant, then (1) $Y_n X_n \xrightarrow{d} aX$; (2) $X_n + Y_n \xrightarrow{d} X + a$.

Example 1

As n is large, $\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \dot{\sim} N(0,1)$.

Example 1: 200 numbers are randomly chosen from between 0 and 1. Find the probability that the average of those numbers is greater than 0.53.

Analysis:

1. Let Y_i be the ith number, $i = 1, 2, \ldots, 200$. $Y_1, Y_2, \ldots, Y_n \sim U(0,1)$.
2. $P\left(\bar{Y} > 0.53\right) = P\left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} > \frac{0.53-1/2}{\sqrt{1/12}/\sqrt{200}}\right) \approx P(Z > 1.47) = 0.708.$

Example 2

Another way to consider application of CLT, as n is large, $\bar{Y} \dot{\sim} N(\mu, \sigma^2/n)$.

Example 2: still consider Example 1.

Analysis:

1. $\bar{Y} \sim N(1/2, (1/12)/200)$;
2. $P\left(\bar{Y} > 0.53\right) \approx P(U > 0.53) = P\left(Z > \frac{0.53 - 1/2}{(1/12)/200}\right) = 0.0708.$

Example 3

Another alternative way to consider application of CLT, as n is large, $\dot{Y} := \sum_{i=1}^{n} Y_i \dot{\sim} N(n\mu, n\sigma^2)$.

Example 3: A die is about to be rolled 50 times and each time you will win as many dollars as the number which comes up. Find the probability that you will win a total of at least $\$200$.

Analysis:

1. Let Y_i be the number of dollars you will win on the ith roll. $\mu = \mathbb{E}Y_i = 3.5$ and $\sigma^2 = VarY_i = 2.9167$.
2. $P\left(\dot{Y} \geq 200\right) \approx P\left(U > 200\right) = P\left(Z > \frac{200-50\times3.5}{\sqrt{50\times2.9167}}\right) = 0.0192.$

Example 4: Normal Approximation to Binomial

Suppose that $Y \sim Bin(n,p)$. Then $Y = \sum_{i=1}^{n} Y_i$, where $Y_i \sim Bern(p)$. By CLT, we have $Y \dot{\sim} N(np, np(1-p))$.

Example 4: A die is rolled $n = 120$ times. Find the probability that at least 27 sixes come up.

Analysis:

1. Let Y be the number of 6's. Then $Y \sim Bin(120, 1/6)$.
2. $Y \dot{\sim} N(120 \times \frac{1}{6}, 120 \times \frac{1}{6} \times (1 - \frac{1}{6}))$.
3. $P\left(Y \geq 27\right) \approx P(U \geq 27) = P\left(Z > \frac{27-20}{\sqrt{16.667}}\right) = 0.0436.$

Summary

1. The meaning of central limit theorem;
2. Application: Normal Approximation for some distributions.