Statistical Inference

Lecture 09b

ANU - RSFAS

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Beyond Point Estimation - Interval Estimation/Confidence Sets

$\hat{p} = 0.2$

$(0.1, 0.6)$

- Never be satisfied with a point estimate! We want to know something about the uncertainty!
- This leads to interval estimation/Confidence Sets.
- Construction methods for interval estimates:
  - parametric "exact" intervals
  - parametric asymptotic intervals
- Some general approaches:
  - Inverting a test statistic
  - Pivotal Quantities
  - Pivoting the CDF

Interval Estimation/Confidence Sets

Definition 5.1: Suppose that $\{f(x; \boldsymbol{\theta}); \theta \in \Omega\}$ define a family of distributions

- If $S_{\boldsymbol{X}}$ is a subset of Ω, depending on $\boldsymbol{X}$, such that

$$P(\boldsymbol{X} : S_{\boldsymbol{X}} \supset \boldsymbol{\theta}) = 1 - \alpha$$

then $S_{\boldsymbol{X}}$ is a **confidence set** for $\boldsymbol{\theta}$ with **confidence coeficient** $1 - \alpha$.

Interval Estimation

- There is a strong relationship between hypothesis testing and interval estimation. In general, every confidence set corresponds to a test and vice versa.

Eg. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2)$, where σ^2 is known. Consider testing:

$$H_0 : \mu = \mu_0 \quad vs \quad \mu \neq \mu_0$$

$$C = \left\{ \left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \right| \geq z_{\alpha/2} \right\}$$

- Now we know that under H_0 $P(\overset{C}{\cancel{R}}) = \alpha$. So the probability that H_0 is accepted is $\underline{1 - \alpha}$:

solve this for μ_0

$$P\left(-z_{\alpha/2} \leq \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_{\alpha/2} \right) = 1 - \alpha$$

Interval Estimation

use a generic μ

- Now fix α and determine the acceptance region. This is an interval estimator.

$$
\begin{aligned}
P\left(-z_{\alpha/2} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right) &= 1-\alpha \\
P\left(-z_{\alpha/2}\left(\sigma/\sqrt{n}\right) \le \bar{X}-\mu \le z_{\alpha/2}\left(\sigma/\sqrt{n}\right)\right) &= 1-\alpha \\
P\left(-\bar{X}-z_{\alpha/2}\left(\sigma/\sqrt{n}\right) \le -\mu \le -\bar{X}+z_{\alpha/2}\left(\sigma/\sqrt{n}\right)\right) &= 1-\alpha \\
P\left(\bar{X}+z_{\alpha/2}\left(\sigma/\sqrt{n}\right) \ge \mu \ge \bar{X}-z_{\alpha/2}\left(\sigma/\sqrt{n}\right)\right) &= 1-\alpha \\
P\left(\bar{X}-z_{\alpha/2}\left(\sigma/\sqrt{n}\right) \le \mu \le \bar{X}+z_{\alpha/2}\left(\sigma/\sqrt{n}\right)\right) &= 1-\alpha
\end{aligned}
$$

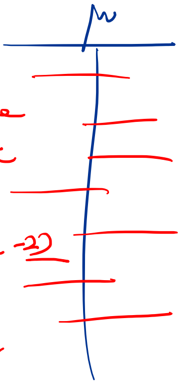
Interval Estimation

95%

- A $100(1-\alpha)\%$ confidence estimator for μ is: this one does not

$$[\bar{X} - z_{\alpha/2}\,(\sigma/\sqrt{n}),\ \bar{X} + z_{\alpha/2}\,(\sigma/\sqrt{n})]$$

(-3, -2)



- Remember, $\boldsymbol{X}$ is random not μ!!

once I put the specific value, an interval will contain or not the value (μ).

Over repeated sampling, 95% of the infinite intervals will contain μ.

(DATA YOU WILL NEVER SEE)

Hypothesis Testing & Confidence Sets/Intervals

Lemma 5.1: Suppose that $\overline{C}(\theta_0)$ is the acceptance region for a test of size α:

$$H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \quad \text{vs} \quad H_1 : \boldsymbol{\theta} \in \boldsymbol{\Omega} - \boldsymbol{\theta}_0.$$

Then a **conficence set** for $\boldsymbol{\theta}$ with **confidence coefficient** $(1 - \alpha)$, is given by

$$S_{\boldsymbol{X}} = \{\boldsymbol{\theta}_0 : \ \boldsymbol{X} \in \overline{C}(\boldsymbol{\theta}_0)\}$$

Proof:

θ is in S_x when $\theta = \theta_0$

by def.

$$P(\boldsymbol{X} : S_{\boldsymbol{X}} \supset \boldsymbol{\theta} | \boldsymbol{\theta} = \boldsymbol{\theta}_0) = P(\boldsymbol{X} \in \overline{C}(\boldsymbol{\theta}_0) | \boldsymbol{\theta} = \boldsymbol{\theta}_0) = 1 - \alpha$$

x is in acceptance reject

when $\theta = \theta_0$

Interval Estimation

Eg. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2)$, where σ^2 is unknown. Consider testing:

$$H_0 : \mu = \mu_0 \quad vs \quad \mu \leq \mu_0$$

- Based on a Likelihood Ratio Test we can find a rejection region of:

$$C = \left\{ \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \leq -t_{n-1,\alpha} \right\}$$

Interval Estimation

- This leads to an acceptance region of:

$$
\begin{aligned}
\overline{C} &= \left\{ \boldsymbol{x} : \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \geq -t_{n-1,\alpha} \right\} \\
&= \left\{ \bar{x} \geq \mu_0 - t_{n-1,\alpha}\, (s/\sqrt{n}) \right\}
\end{aligned}
$$

- This leads to a $(1-\alpha)$ upper bound confidence set for μ :

$$
\begin{aligned}
S_{\boldsymbol{X}} &= \left\{ \mu_0 : \bar{x} + t_{n-1,\alpha}\, (s/\sqrt{n}) \geq \mu_0 \right\} \\
&= (-\infty, \bar{x} + t_{n-1,\alpha}\, (s/\sqrt{n})]
\end{aligned}
$$

Interval Estimation

sample : {2.6, 1.2, 4.9}

$N(0, \sigma^2)$ ⇒ 99% CI for σ^2?

Eg.: Suppose that 2.6, 1.2 and 4.9 are a random sample from a normal distribution whose mean is zero and whose variance σ^2 is unknown. Derive and compute a central 99% confidence interval for σ^2.

3 Approaches with ($\chi_1^2, \chi_2^2, \chi_3^2$ respectively)

- Approach 1:

$\left(\frac{X_i - 0}{\sigma}\right)^2 = Z \sim N(0,1)$

$$\left(\frac{X_i-\mu}{\sigma}\right)^2 = \left(\frac{X_i}{\sigma}\right)^2 \sim Z^2 = \chi_1^2$$

$$\sum_{i=1}^{3}\left(\frac{X_i}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{3} X_i^2 \sim \chi_3^2$$

Sum of n χ_1^2 is χ_n^2.

- Let $Y = \sum_{i=1}^{3} X_i^2$.

$$P\left(\chi^2_{\alpha/2,3} \leq \frac{Y}{\sigma^2} \leq \chi^2_{1-\alpha/2,3}\right) = 1-\alpha$$

$$P\left(\frac{1}{\chi^2_{\alpha/2,3}} \geq \frac{\sigma^2}{Y} \geq \frac{1}{\chi^2_{1-\alpha/2,3}}\right) = 1-\alpha$$

$$P\left(\frac{Y}{\chi^2_{1-\alpha/2,3}} \leq \sigma^2 \leq \frac{Y}{\chi^2_{\alpha/2,3}}\right) = 1-\alpha$$

$$\left[\frac{Y}{\chi^2_{1-\alpha/2,3}} \quad , \quad \frac{Y}{\chi^2_{\alpha/2,3}}\right]$$

$$\left[\frac{32.21}{12.8381} \quad , \quad \frac{32.21}{0.0717212}\right]$$

$$[2.51 \quad , \quad 449]$$

- In R:

```
qchisq(0.01/2, 3)
```

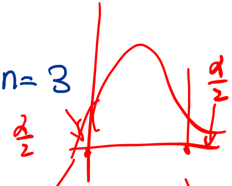
```
## [1] 0.07172177
```

```
qchisq(1-0.01/2, 3)
```

```
## [1] 12.83816
```

- Approach 2:

known $$\boxed{\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} = \chi^2_2}$$ $n = 3$



Similarly:

$$P\left(\chi^2_{\alpha/2,2} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2,2}\right) = 1-\alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{\alpha/2,2}}\right) = 1-\alpha$$

$$\left[\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,2}}, \quad \frac{(n-1)S^2}{\chi^2_{\alpha/2,2}}\right]$$

$$\left[\frac{(2)3.49}{10.5966}, \quad \frac{(2)3.49}{0.0100251}\right]$$

$$[0.659, \quad 696]$$

- Approach 3:

$\mu = 0$ known

Z

$$\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 = \left(\frac{\bar{X}}{\sigma/\sqrt{n}}\right)^2 = \frac{n\bar{X}^2}{\sigma^2} \approx Z^2 \sim \chi^2_1$$

$$P\left(\chi^2_{\alpha/2,1} \leq \frac{n\bar{X}^2}{\sigma^2} \leq \chi^2_{1-\alpha/2,1}\right) = 1-\alpha$$

$$P\left(\frac{n\bar{X}^2}{\chi^2_{1-\alpha/2,1}} \leq \sigma^2 \leq \frac{n\bar{X}^2}{\chi^2_{\alpha/2,1}}\right) = 1-\alpha$$

$$\left[\frac{n\bar{X}^2}{\chi^2_{1-\alpha/2,1}}, \quad \frac{n\bar{X}^2}{\chi^2_{\alpha/2,1}}\right]$$

$$\left[\frac{(3)2.9^2}{7.87944}, \quad \frac{(3)2.9^2}{0.0000393}\right]$$

$$[3.202 \quad , \quad 642468.3]$$

NB: since χ^2_3 has more information, its CI is the shortest among all possible three. (Prefer the 1st one)

Confidence Sets/Interval Estimation

- All three approaches, and everything we have considered thus far have a nice property. The distribution of the statistic does not contain parameters!

Definition 5.2: A random variable $g(\boldsymbol{X}, \boldsymbol{\theta})$ is a **pivotal quantity (or a pivot)** if the distribution of $g(\boldsymbol{X}, \boldsymbol{\theta})$ is independent of all parameters.

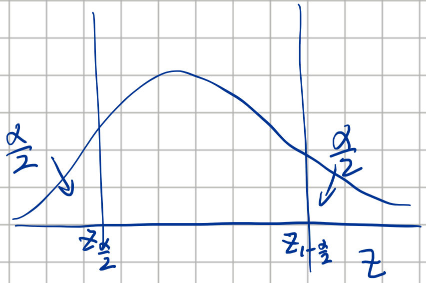
- If θ is a scaler, some definitions require that $g()$ be a monotonic function of θ. **See example 5.3 for a non-monotonic example.**

$X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$

Is $\bar{X}$ a pivot quantity? No!

$\bar{X} \sim N(\underset{\sim}{\theta}, \frac{1}{n})$

$\frac{\bar{X}-\theta}{1/\sqrt{n}} \sim N(0,1)$ now it is one ✓



$P\left(z_{\alpha/2} \le \frac{\bar{X}-\theta}{1/\sqrt{n}} \le z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$X \sim N(\theta, 1)$$

$$\frac{X-\theta}{1} \sim N(0,1)$$

$$P(Z_{\alpha/2} \le X-\theta \le Z_{1-\frac{\alpha}{2}}) = 1-\alpha$$

Confidence Sets/Interval Estimation

- Basic idea, is that the known distribution of a pivot quantity $g(\boldsymbol{X}, \theta)$ can be used to write a probability statement:

$$P(g_1 \leq g(\boldsymbol{X}, \theta) \leq g_2) = 1 - \alpha$$

Then this can be solved for θ (easier if g is a monotone function of θ):

$$P(\theta_1(\boldsymbol{X}) \leq \theta \leq \theta_2(\boldsymbol{X})) = 1 - \alpha$$

Interval Estimation - MLEs & Asymptotics

$$\hat{\theta} \dot{\sim} \text{normal}(\theta, I(\theta)^{-1})$$

$$\frac{\hat{\theta} - \theta}{1/\sqrt{I(\theta)}} \dot{\sim} \text{normal}(0, 1)$$

- We have a pivotal quantity. Based on the same approach as before we can construct an asymptotic $100(1-\alpha)\%$ confidence interval as:

$$\left[\hat{\theta} - z_{\alpha/2}\frac{1}{\sqrt{I(\hat{\theta})}} \; , \; \hat{\theta} + z_{\alpha/2}\frac{1}{\sqrt{I(\hat{\theta})}}\right]$$

Interval Estimation - MLEs & Asymptotics

- If we are interested in a function of θ, say $\tau(\theta)$, then we have:

asymptotically normal

$$\tau(\hat{\theta}) \dot{\sim} \text{normal}\left(\tau(\theta), \frac{[\tau'(\theta)]^2}{I(\theta)}\right)$$

$$\frac{\tau(\hat{\theta}) - \tau(\theta)}{\sqrt{\frac{[\tau'(\theta)]^2}{I(\theta)}}} \dot{\sim} \text{normal}(0,1)$$

- We can construct an asymptotic $100(1-\alpha)\%$ confidence interval as:

$$\left[\tau(\hat{\theta}) - z_{\alpha/2}\frac{\tau'(\hat{\theta})}{\sqrt{I(\hat{\theta})}}\ ,\ \tau(\hat{\theta}) + z_{\alpha/2}\frac{\tau'(\hat{\theta})}{\sqrt{I(\hat{\theta})}}\right]$$

Interval Estimation - MLEs & Asymptotics

Example: Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{exponential}(\theta)$:

$$f(x; \theta) = \theta \exp(-\theta x)$$

- Provide an equal tailed 95% CI for $\tau(\theta) = \theta^{-1}$.

$$\begin{aligned} \ell &= nlog(\theta) - \theta \sum x_i \\ \ell' &= \frac{n}{\theta} - \sum x_i \\ &\Rightarrow \frac{n}{\theta} - \sum x_i = 0 \\ &\hat{\theta} = \frac{1}{\bar{x}} \quad \Rightarrow \widehat{\left(\frac{1}{\theta}\right)} = \frac{1}{\hat{\theta}} = \bar{x} \end{aligned}$$

Interval Estimation - MLEs & Asymptotics

nothing random

$$
\begin{aligned}
\ell'' &= -\frac{n}{\theta^2} \\
\text{Fisher Information: } I(\theta) &= -E\left[-\frac{n}{\theta^2}\right] = \frac{n}{\theta^2} \\
CRLB(\theta^{-1}) &= \frac{\left[\frac{d}{d\theta}\frac{1}{\theta}\right]^2}{\frac{n}{\theta^2}} = \frac{\left[-\frac{1}{\theta^2}\right]^2}{\frac{n}{\theta^2}} = \frac{1}{n\theta^2} \\
CRLB(\hat{\theta}^{-1}) &= \frac{1}{n\hat{\theta}^2} = \frac{\bar{x}^2}{n}
\end{aligned}
$$

$CRLB(\theta) = I(\theta)^{-1}$

$= \frac{\theta^2}{n}$

- We end with the following interval for $\frac{1}{\theta}$:

$\tau(\hat{\theta})$

$$
\left[\bar{x} - z_{\alpha/2}\underbrace{\frac{\bar{x}}{\sqrt{n}}}_{CRLB},\ \bar{x} + z_{\alpha/2}\frac{\bar{x}}{\sqrt{n}}\right]
$$

L(ower) U(pper)

Interval Estimation - MLEs & Asymptotics

CLT $\bar{X} \to N(\theta, \sigma^2)$

$\tilde{\theta} = \bar{X}, \; s^2 = \hat{\sigma}^2$

- Note: $\tau(\hat{\theta}) = \bar{X}$, so why not use the following interval?

$$\left[\bar{x} - \overset{t}{z_{\alpha/2}} \frac{s}{\sqrt{n}} \, , \; \bar{x} + \overset{t}{z_{\alpha/2}} \frac{s}{\sqrt{n}}\right]$$

- If the data truly are exponentially distributed, then the previous interval will be more accurate.
- Of course, the this interval will be valid even in the case that the data are not truly exponentially distributed.

Interval Estimation - MLEs & Asymptotics

- Now suppose we are interested in a CI for θ:
- We constructed an interval $\tau = \frac{1}{\theta}$, so why not just take the the inverse? We can.

$$[u^{-1}, l^{-1}]$$

upper lower

- So we have for θ:

$$\left[\left\{\bar{x} + z_{\alpha/2}\frac{\bar{x}}{\sqrt{n}}\right\}^{-1}, \left\{\bar{x} - z_{\alpha/2}\frac{\bar{x}}{\sqrt{n}}\right\}^{-1}\right]$$

$y_1, \dots, y_n$

$\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ $\begin{pmatrix} x_{1n} \\ y_{2n} \end{pmatrix}$

$\log y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$

Prediction Interval

$\log \hat{y} = [\log(l), \log(u)]$

$(y) \Rightarrow [\exp(\log(l)), \exp(\log(u))]$

Interval Estimation - MLEs & Asymptotics

$\tau(\hat{\theta}) \pm z_{\frac{\alpha}{2}} \frac{[\tau'(\theta)]^2}{I(\theta)}$

- OK, but let's go back to the drawing-board and find the CI for θ from first principles:

$\hat{\theta} \pm z_{\alpha/2} I(\theta)^{-1}$

$$\left[\bar{x}^{-1} - z_{\alpha/2}\frac{1}{\sqrt{n}\bar{X}}\ ,\ \bar{x}^{-1} + z_{\alpha/2}\frac{1}{\sqrt{n}\bar{x}}\right]$$

- We see that the two approaches are not the same. This is because interval construction, as we have done it, is not functionally equivalent!!

Interval Estimation

$\theta \Rightarrow \hat{\theta}$

$(\frac{1}{\theta}) \Rightarrow (\hat{\frac{1}{\theta}}) = \frac{1}{\hat{\theta}}$

- Can we come up with an approach which does possess the equivariance property?
  - Yes, as long as the functional transformation in question is invertible.
  - Let's consider an asymptotic likelihood-based confidence interval procedure which is parameterization equivariant.
  - Specifically, this means that if we find a confidence region, C, for θ based on this new procedure and transform all of its values [which we sometimes denote as $\tau(C) = \{\tau(\theta) : \theta \in C\}$] then we will arrive at the same confidence region as if we had applied our new procedure to the parameter τ directly.

Asymptotic Maximimum LRT Interval Estimation

- Let's consider the following based on the maximum likelihood ratio test, where $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{exponential}(\theta)$; $f(x;\theta) = \theta \exp(-\theta x)$:

$$-2log\left(\frac{L(\theta;\boldsymbol{x})}{L(\hat{\theta};\boldsymbol{x})}\right) \dot{\sim} \chi^2_1$$

$$\begin{aligned}
-2[\ell(\theta;\boldsymbol{x}) - \ell(\hat{\theta};\boldsymbol{x})] &= 2[\ell(\hat{\theta};\boldsymbol{x}) - \ell(\theta;\boldsymbol{x})] \\
&= 2[nlog(\hat{\theta}) - \hat{\theta}\sum x_i - nlog(\theta) + \theta\sum x_i] \\
&= 2[nlog(\frac{1}{\bar{x}}) - \frac{1}{\bar{x}}\sum x_i - nlog(\theta) + \theta\sum x_i] \\
&= -2n\ log(\bar{x}) - 2n\frac{\bar{x}}{\bar{x}} - 2nlog(\theta) + 2\theta n\bar{x} \\
&= -2n\ log(\bar{x}\theta) + 2n(\theta\bar{x} - 1)
\end{aligned}$$

Asymptotic Maximimum LRT Interval Estimation

- We reject if:

$$-2n\ log(\bar{x}\theta) + 2n(\theta\bar{x} - 1) > \chi^2_{\alpha,1}$$

- We accept if:

$$-2n\ log(\bar{x}\theta) + 2n(\theta\bar{x} - 1) \leq \chi^2_{\alpha,1}$$

- So our confidence set is:

$$\begin{aligned} S_{\boldsymbol{x}} &= \left\{\theta \in \Theta : -2[\ell(\theta) - \ell(\hat{\theta})] \leq \chi^2_{\alpha,1}\right\} \\ &= \left\{-2n\ log(\bar{x}\theta) + 2n(\theta\bar{x} - 1) \leq \chi^2_{\alpha,1}\right\} \end{aligned}$$

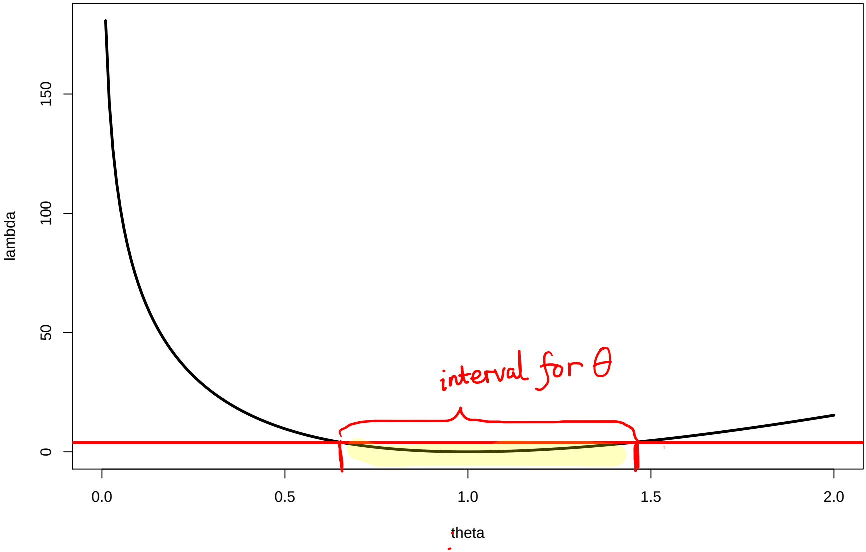
Asymptotic Maximimum LRT Interval Estimation

- We can't solve this analytically, but let's graph it:
- Suppose $\bar{X} = 1$, $n = 25$, and $\alpha = 0.05$:

```
x.bar <- 1
n <- 25

theta <- seq(0,2, by =0.01)
lambda <- -2*n*log(x.bar*theta) + 2*n*(theta*x.bar - 1)
plot(theta, lambda, lwd=3, type="l")
abline(h=qchisq(1-0.05, 1), lwd=3, col="red")
```

Asymptotic Maximimum LRT Interval Estimation



Asymptotic Maximimum LRT Interval Estimation

```
min( theta[lambda <= qchisq(1-0.05, 1)])
```

```
## [1] 0.66
```

```
max( theta[lambda <= qchisq(1-0.05, 1)])
```

```
## [1] 1.44
```

- So a 95% confidence interval for θ is:

$$[0.66\ ,\ 1.44]$$

Asymptotic Maximimum LRT Interval Estimation

- Now suppose we want the interval for $\tau = \frac{1}{\theta}$.
- Let's re-parameterize the log likelihood:

$\tau = \frac{1}{\theta}$

$\theta = \frac{1}{\tau}$

$$\begin{aligned} \ell(\tau) &= \ell(\theta = \tau^{-1}) = nlog(\tau^{-1}) - \tau^{-1} \sum x_i \\ &= -nlog(\tau) - n\frac{\bar{x}}{\tau} \end{aligned}$$

Asymptotic Maximimum LRT Interval Estimation

$$
\begin{aligned}
-2[\ell(\tau) - \ell(\hat{\tau})] &= 2[\ell(\hat{\tau}) - \ell(\tau)] \\
&= 2[-nlog(\hat{\tau}) - n\frac{\bar{x}}{\hat{\tau}} + nlog(\tau) + n\frac{\bar{x}}{\tau}] \\
&= 2[-nlog(\bar{x}) - n\frac{\bar{x}}{\bar{x}} + nlog(\tau) + n\frac{\bar{x}}{\tau}] \\
&= -2nlog(\bar{x}\tau^{-1}) + 2n(\tau^{-1}\bar{x} - 1)
\end{aligned}
$$

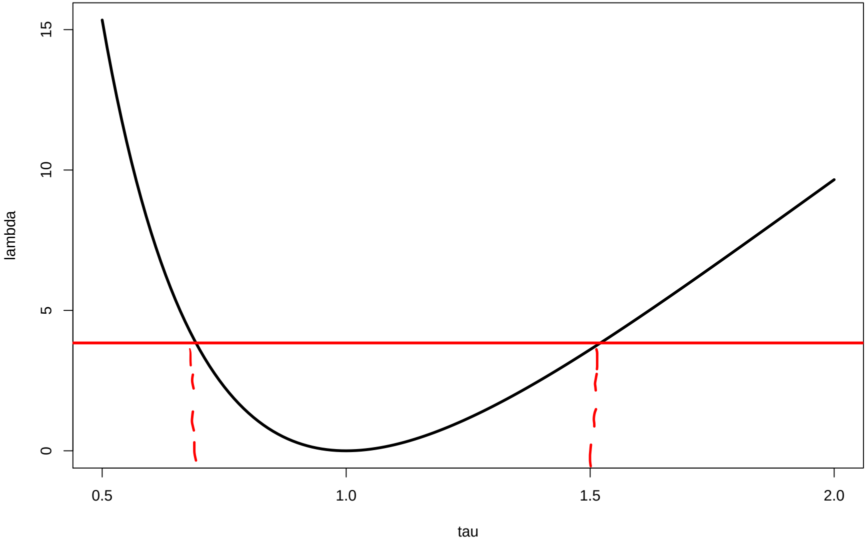
- All that was done through all the math was to replace θ with τ^{-1}!
- So our interval is:

$$[1/1.44\ ,\ 1/0.66] = [0.69\ ,\ 1.51]$$

- Let's see it in the plot
- Again, suppose $\bar{X} = 1$, $n = 25$, and $\alpha = 0.05$:

```
x.bar <- 1
n <- 25

tau <- seq(0.5, 2, by =0.01)
lambda <- -2*n*log(x.bar*(1/tau)) + 2*n*((1/tau)*x.bar - 1)
plot(tau, lambda, lwd=3, type="l")
abline(h=qchisq(1-0.05, 1), lwd=3, col="red")
```


15
10
5
0
lambda
0.5
1.0
1.5
2.0
tau

```r
min( tau[lambda <= qchisq(1-0.05, 1)])
```

```
## [1] 0.7
```

```r
max( tau[lambda <= qchisq(1-0.05, 1)])
```

```
## [1] 1.52
```

Maximum LRT Interval Estimation

$$-2\log\underbrace{\left(\frac{L(\theta)}{L(\hat{\theta})}\right)}_{\lambda = L(\theta)/L(\hat{\theta})} \dot{\sim} \chi^2_1$$

- Did we have to use the asymptotic result of the LRT for our interval. No, but it is more straightforward.

$$\hat{\theta} = \bar{X} \qquad X_1, \cdots, X_n \sim \exp(\theta)$$

$$\hat{\theta} \dot{\sim} N(\theta, I(\theta)^{-1})$$

Interval Estimation - CDF Method

- Pivoting the CDF (See pg 110)
  - A pivot g leads to a confidence set:

$$S_{\boldsymbol{X}} = \{\theta_0 : a \leq g(\boldsymbol{X}; \theta_0) \leq b\}$$

  - If for every $\boldsymbol{x}$ the pivot is a monotone function of θ then the confidence set $C(\boldsymbol{x})$ is guaranteed to be an interval.
  - Most pivots we have considered have this property.

Interval Estimation - CDF Method

Theorem:

- Let T be a statistic with a continuous cdf $F_T(t;\theta)$.
- Let $\alpha_1 + \alpha_2 = \alpha$ with $0 < \alpha < 1$.
- Suppose that for each $t \in T$, the functions $\theta_L(t)$ and $\theta_U(t)$ can be defined as:

1. If $F_T(t;\theta)$ is a decreasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by:

$$F_T(t;\theta_U(t)) = \alpha_1 \qquad F_T(t;\theta_L(t)) = 1 - \alpha_2$$

2. If $F_T(t;\theta)$ is an increasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by:

$$F_T(t;\theta_L(t)) = \alpha_1 \qquad F_T(t;\theta_U(t)) = 1 - \alpha_2$$

Then the interval $[\theta_L(t), \theta_U(t)]$ is a $1-\alpha$ confidence interval for θ.

- We can prove that $F_T(t;\theta)$ is monotone in θ. See C&B.

Interval Estimation - CDF Method

Example: Consider $X_1, \dots, X_n \overset{\text{iid}}{\sim} Unif(0, \theta)$.

$l'(\hat{\theta}) = 0$

- So we have the following CDF for X:

$$F_X(x; \theta) = \frac{x}{\theta} \mathbb{I}_{(0 \le x \le \theta)}$$

- We know the MLE for θ is $T = max(X_1, \dots, X_n)$

CDF at
$\hat{\theta} = max(x_1, \dots, x_n)$
$= x_{(n)}$

$$\begin{aligned} F_T(t; \theta) = Pr(T \le t) &= Pr\{max(X_1, \dots, X_n) \le t\} \\ &= Pr\{X_1 \le t, \dots, X_n \le t\} \\ &= Pr\{X_1 \le t\} \times \dots \times Pr\{X_n \le t\} \\ &= \{F_X(t; \theta)\}^n \\ &= \frac{t^n}{\theta^n} \mathbb{I}_{(0 \le t \le \theta)} \end{aligned}$$

Interval Estimation - CDF Method

- Note: $F_T(t;\theta)$ is a deceasing function for θ. Let $\alpha_1 = \alpha_2 = \alpha/2$. We have:

$$
\begin{aligned}
F_T(t;\theta_U(t)) &= \alpha/2 \\
\left(\frac{t}{\theta_U}\right)^n &= \alpha/2 \\
\theta_U &= t(\alpha/2)^{-(1/n)}
\end{aligned}
$$

$$
\begin{aligned}
F_T(t;\theta_L(t)) &= 1-\alpha/2 \\
\left(\frac{t}{\theta_L}\right)^n &= 1-\alpha/2 \\
\theta_L &= t(1-\alpha/2)^{-(1/n)}
\end{aligned}
$$

Interval Estimation - CDF Method

```
##
set.seed(1001)
n <- 15
X <- runif(n, 0, 10)
t <- max(X)
alpha <- 0.05

##
theta.u <- t*(alpha/2)^(-(1/n))
theta.l <- t*(1-alpha/2)^(-(1/n))

c(theta.l, theta.u)
```

θ

95% CI for θ

```
## [1]  9.873539 12.605028
```

Interval Estimation - CDF Method

- Interpretation: Over repeated sampling, we expect 95% of the intervals we create to contain the true value θ.
- Let's check: We set $\alpha = 0.05$, so 95% of the intervals should contain θ.

Interval Estimation - CDF Method

```r
set.seed(1001)
##
S <- 10000
coverage <- rep(0, S)
theta.true <- 10

##
n <- 15
alpha <- 0.05

##
for(s in 1:S){

##
X <- runif(n, 0, theta.true)
t <- max(X)

##
theta.u <- t*(alpha/2)^(-(1/n))
theta.l <- t*(1-alpha/2)^(-(1/n))

if(theta.l < theta.true && theta.u > theta.true){coverage[s] <- 1}
}

mean(coverage)
```

```
## [1] 0.9517
```