Lecture 9

Proposition: For $Q_c(x)=x^2+c$.

1. If $c>1/4$, then all orbits go to infinity
2. If $c=1/4$, it has one fixed point $x=1/2$, which is neutral.
3. If $c<1/4$, it has two fixed points p_- & p_+.

P_+ is repelling and
(a) if $-3/4<c<1/4$, then P_- is attracting
(b) if $c=-3/4$, then P_- is neutral.
(c) if $c<-3/4$, then P_- is repelling.

Remarks

① $Q_c(-x)=Q_c(x)$, Q_c is even, so the orbits of $-x_0$ and x_0 under Q_c are the same

② The point $-P_+$ is eventually fixed.

③ The orbit for $x_0>P_+$ under Q_c goes to $+\infty$ so the same happens to the orbit of $x<-P_+$ under Q_c.

$\Rightarrow$ The interesting dynamics will happen for $c<1/4$ and $-P_+<x_0<P_+$.

- observe that $P_-=\frac{1-\sqrt{1-4c}}{2}\in(-P_+,P_+)$
- one can derive that for $-3/4<c<1/4$, all orbits for $x_0\in(-P_+,P_+)$ under Q_c converge to P_-.

Exercise:

Prove that for $0\le c<1/4$, and $x_0\in(-P_+,P_+)$, the orbit of x_0 under Q_c tends to P_-.

(a). If $x_0>P_-$, then x_n is decreasing & $x_n>0$. So it converges. It can only converge to P_-.

(b). if $x_0<P_-$, then x_n is increasing & $x_n<P_-$. so it converges. It can only converge to P_-.

(c). if $-P_+<x_0<0$, then $0<x_1<P_+$

Remark: There are no cycles for $c>-3/4$, since we decreased all positive orbits

Q: what happens for $c<-3/4$?

By looking at the orbits, they seem to converge to a 2 cycle.
we find the 2 cycles.

$$Q_c^2(x)=x \iff (x^2+c)^2+c=x$$
$$\iff x^4+2x^2c-x+c^2+c=0$$

We know that $P_\pm$ are fixed pts, so they also solve $Q_c^2(x)=x$

so the polynomial above divide $(x-P_-)(x-P_+)=x^2-x+c$

Then $Q_c^2(x)=x \iff (x^2-x+c)(x^2+x+c+1)=0$

So 2-cycles are solutions of $x^2+x+c+1=0 \iff x=\frac{-1\pm\sqrt{1-4(c+1)}}{2}$

Define $q_-=\frac{-1-\sqrt{-3-4c}}{2}$, $q_+=\frac{-1+\sqrt{-3-4c}}{2}$

which are real numbers for $c\leq -3/4$.

So we have a new kind of bifurcation. We say that Q_c has a period-doubling bifurcation at $c=-3/4$.

Let us study 2-cycle

$$(Q_c^2)'(q_-)=Q_c'(q_-)Q_c'(q_+)=4q_-q_+=(-1)^2-(-3-4c)$$
$$=1+3+4c$$
$$=4(c+1)$$

$|Q_c'(q_-)Q_c'(q_+)|=4|c+1|<1$