Tutorial 10

STAT 3013/4027/8027

1. Consider a Poission regression model using the canonical link function (how do we determine the canonical link function?):

$$\begin{array}{rcl} Y_1, \ldots, Y_n & \overset{\text{indep.}}{\sim} & \text{Poisson}(\lambda_i) \\ \log(\lambda_i) & = & \beta_0 + \beta_1 x_i + \beta_2 x_i^2 \\ & \text{for } i = 1, \ldots, n. & \end{array}$$

- Data: A sample from a population of 52 female song sparrows was studied over the course of a summer and their reproductive activities were recorded. In particular, the age and number of new offspring were recorded for each sparrow (Arcese et al, 1992). Let $Y =$ fledged (number of offspring), and $X =$ age (age of mother).
- Based on the results from last week, test (through a frequetist approach):

$$H_0 : \beta_1 = 0 \text{ and } \beta_2 = 0 \quad vs. \quad H_1 : \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \ (H_0 \text{ is not true})$$

2. SI 7.1, 7.3, 7.10, 7.15.

Q7.1

(a). $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \text{Poisson}(\theta)$

$p(\theta)$ prior

$$p(\theta|\vec{x}) \propto p(\vec{x}|\theta)\, p(\theta)$$

$$\propto \left[\prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x}}{x!}\right] p(\theta)$$

$$\propto e^{-n\theta}\theta^{\Sigma x} p(\theta)$$

(b). $Y \sim \text{Poisson}(n\theta)$

$p(\theta)$ prior

$$p(\theta|y) \propto p(y|\theta)\,p(\theta)$$

$$\propto \left[e^{-n\theta}\frac{(n\theta)^{y}}{y!}\right] p(\theta)$$

$$\propto e^{-n\theta}\theta^{y} p(\theta)$$

(c) By Def 7.1, $p(\theta|\vec{x}) = p(\theta|\vec{y})$. $y = \Sigma x_i$ is a sufficient statistic.

Q7.3.

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(x+1)e^{-x\theta}, \quad (x>0)$$

is $\forall \theta > 0$, a p.d.f. A random sample of size n is taken from this pdf giving values $x_1, x_2, \cdots, x_n$.

it is sufficient to show $\int_0^\infty f(x;\theta)dx = 1$

$$\int_0^\infty \frac{\theta^2}{\theta+1}(x+1)e^{-x\theta}dx$$

$$= \left[\frac{\theta^2}{\theta+1}\right]\left\{\int_0^\infty x e^{-x\theta} dx + \int_0^\infty e^{-x\theta}dx\right\} = \left[\frac{\theta^2}{\theta+1}\right]\left\{\frac{\sqrt{2}}{\theta^2}\underbrace{\int\frac{\theta^2}{\sqrt{2}} x^{2-1}e^{-x\theta}dx}_{\text{gamma}} + \frac{1}{\theta}\underbrace{\int_0^\infty \theta e^{-x\theta}dx}_{\text{exponential}}\right\}$$

$$=\left[\frac{\theta^2}{\theta+1}\right]\left\{\frac{1}{\theta^2}+\frac{1}{\theta}\right\}$$

$$=1$$

a). $L(x;\theta)=\prod_{i=1}^{n}\left(\frac{\theta^2}{\theta+1}\right)(x_i+1)e^{-x_i\theta}$

$$=\boxed{\left(\frac{\theta^2}{\theta+1}\right)^n e^{-\theta\sum x_i}}\ \boxed{\left[\prod_{i=1}^{n}(x_i+1)\right]}$$

$$\propto\left(\frac{\theta^2}{\theta+1}\right)^n e^{-\theta\sum x_i}$$

b). $L(\theta)=$ $k(\theta|T)\,h(x)$

$\sum_{i=1}^{n} x_i$ is sufficient statistic

c). conjugate prior for θ.

let us replace "n" & "$\sum x_i$" by α_1 & α_2

$p(\theta)=\left(\frac{\theta^2}{\theta+1}\right)^{\alpha_1}e^{-\theta\alpha_2}$ is the prior dist.

Now the posterior is

$$p(\theta|\vec{x})\propto p(\vec{x}|\theta)\cdot p(\theta)$$

$$\propto\left(\frac{\theta^2}{\theta+1}\right)^n e^{-\theta\sum x_i}\cdot\left(\frac{\theta^2}{\theta+1}\right)^{\alpha_1}\cdot e^{-\theta\alpha_2}$$

$$\propto\left(\frac{\theta^2}{\theta+1}\right)^{(n+\alpha_1)}e^{-\theta(\sum x_i+\alpha_2)}$$

$p(\theta)$ is a conjugate prior.

Q.

7.10

mean

prior on θ of Poisson.

$X_1, \dots, X_n \sim \text{Poisson}(\theta)$

$$L(\theta) = \prod_{i=1}^{n} e^{-\theta} \frac{\theta^{x_i}}{x_i!}$$

$$\propto e^{-n\theta} \theta^{\sum x_i}$$

By looking at the likelihood, we see that we have a kernel of a Gamma dist'n.

$p(\theta) \sim \text{Gamma}(\alpha, \beta)$

We know $E(\theta) = \frac{\alpha}{\beta}$ & $V(\theta) = \frac{\alpha}{\beta^2}$

So have the mean & variance of the prior dist. unity.

$$E(\theta) = \frac{\alpha}{\beta} = 1 \Rightarrow \alpha = \beta$$

$$V(\theta) = \frac{\alpha}{\beta^2} = 1 \Rightarrow \beta = \alpha = 1$$

$$P(\theta) = \frac{\sqrt{1}}{1!} \theta^{1-1} e^{-\theta(1)}$$

$$= e^{-\theta}$$

$\Rightarrow p(\theta) \sim \text{Gamma}(1, 1)$

The posterior is $p(\theta | \vec{x}) \propto p(\vec{x} | \theta) p(\theta)$

$$\propto e^{-n\theta} \theta^{\sum x_i} e^{-\theta}$$

$$\propto e^{\theta(n+1)} \theta^{(\sum x_i + 1) - 1}$$

$$(\theta | \vec{x}) \sim \text{gamma}\left(\alpha = \sum_{i=1}^{n} x_i + 1, \beta = n+1\right)$$

(c).

we know that

$$2b(\theta|\vec{x}) \sim \chi^2_{2a}$$

$\therefore (1-\alpha)\%$ CI is

$$P[\chi^2_{\frac{\alpha}{2},2a} \le 2b(\theta|\vec{x}) \le \chi^2_{1-\frac{\alpha}{2},2a}] = 1-\alpha$$

$$\Rightarrow P[\frac{\chi^2_{\frac{\alpha}{2},2a}}{2b} \le (\theta|\vec{x}) \le \frac{\chi^2_{1-\frac{\alpha}{2},2a}}{2b}] = 1-\alpha$$

Q7.15. $X_1,\dots,X_n \overset{iid}{\sim} Exp(\theta)$

Test: $H_0: \theta=1$ vs $H_1: \theta\neq 1$

$P_r[H_0]=p$. $P_r[H_1]=1-p$

$$P(H_0|\vec{x}) = \frac{P(\vec{x}|H_0)\cdot P(H_0)}{P(\vec{x})}$$

$$P(H_1|\vec{x}) = \frac{P(\vec{x}|H_1)\cdot P(H_1)}{P(\vec{x})}$$

$$\theta_0^* = \frac{P(\vec{x}|H_0)\cdot P_r(H_0)}{\underbrace{P(\vec{x}|H_1)}_{\text{Bayes factor (BF)}}\cdot P_r(H_1)}$$

$$BF = \frac{\theta_0^n e^{-\theta_0\Sigma x_i}}{\int_{H_1}\theta^n e^{-\theta\Sigma x_i} p(\theta|H_1)d\theta} \Big\}\ \text{Let it be } A.$$

$$A = \int_{H_1}\theta^n e^{-\theta\Sigma x_i}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta\beta}d\theta =$$

See solutions.