PLEASE HAND IN

UNIVERSITY OF TORONTO

The Faculty of Arts and Science

FINAL EXAMINATIONS, DECEMBER 2010

MAT240H1F
Algebra 1

Duration – 3 hours

Instructor: M. Gualtieri

NO AIDS ALLOWED

NAME: ______________________________ **Student Number:** ____________________

INSTRUCTIONS:

I. **The exam consists of 8 pages**

ALL QUESTIONS MUST BE ANSWERED ON THESE QUESTION SHEETS

You may use the scratch paper but it is not to be handed in

II. **This is a closed-book exam with no materials allowed except the exam paper, the scratch paper and your writing utensil.**

III. **Good luck!**

Question	
1	
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7	
Total	

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Question 1 (20 points). True or false: (no justification required, grade=2(correct) + 0(incorrect))

i) Let $\mathbb{F}$ be a field, and let $a \in \mathbb{F}$. Then $a + a + a = 0$ implies that $a = 0$.

ii) Every n-dimensional vector space over the field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

iii) If $S : V \longrightarrow W$ and $T : W \longrightarrow V$ are linear maps, and $ST = \mathcal{I}_W$ (where $\mathcal{I}_W$ is the identity map on W), then it follows that $TS = \mathcal{I}_V$ (where $\mathcal{I}_V$ is the identity map on V).

iv) If $T : V \longrightarrow W$ is a linear map and $(v_1, \ldots, v_n)$ is a linearly independent list of vectors in V, then $(T(v_1), \ldots, T(v_n))$ is a linearly independent list of vectors in W.

v) If $T : V \longrightarrow W$ is a linear map and $(v_1, \ldots, v_n)$ is a list of vectors in V such that $(T(v_1), \ldots, T(v_n))$ is linearly independent in W, then $(v_1, \ldots, v_n)$ is linearly independent.

vi) A system of 438 homogeneous linear equations in 245 variables always has a solution.

vii) If a linear operator on $\mathbb{F}^n$ has n distinct eigenvalues, then we can find a basis of eigenvectors.

viii) If a linear operator on $\mathbb{F}^n$ has fewer than n distinct eigenvalues, then it is not diagonalizable.

ix) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

x) If c_1 and c_2 are distinct eigenvalues of the operator T, then $\mathrm{null}(T - c_1\mathcal{I}) \cap \mathrm{null}(T - c_2\mathcal{I}) = \{0\}$.

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Question 2 (16 points). Short answers, no justification required:

i) State the definition of an isomorphism from the vector space V to the vector space W.

ii) State the definition of the null space of a linear map $T : V \longrightarrow W$.

iii) State the definition of an eigenvector and eigenvalue for a linear operator $T : V \longrightarrow V$.

iv) State the definition of a generalized eigenvector for a linear operator $T : V \longrightarrow V$.

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Question 3 (16 points). Short answers, no justification required:

i) Give an example of linear operator on $\mathbb{R}^4$ which has no eigenvectors (Hint: give one on $\mathbb{R}^2$ to begin with).

ii) What is the explicit condition on $r, s, t \in \mathbb{Q}$ which implies and is implied by the linear independence of the vectors $((1, r, 1), (0, 1, s), (t, 0, 1))$ in $\mathbb{Q}^3$?

iii) Find all solutions to the inhomogeneous linear system, if any exist:

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 - 2x_4 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 + x_3 - x_4 &= 2. \end{aligned}$$

iv) What is the inverse of the following matrix in $\mathbb{Q}^{3\times 3}$: $\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$.

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Question 4 (16 points). Short answers, no justification required:
Consider the linear operator on $\mathcal{P}_2(\mathbb{R})$ (real polynomials of degree ≤ 2) defined by

$$T(f(x)) = f(x) + (x+1)f'(x),$$

where $f'(x)$ is the derivative of the polynomial $f(x)$.

i) Write the real 3×3 matrix $A \in \mathbb{R}^{3\times 3}$ of T in the standard basis for $\mathcal{P}_2(\mathbb{R})$.

ii) What are the eigenvalues of T?

iii) Find a list of polynomials which are a basis of eigenvectors for T.

iv) Write an invertible matrix $P \in \mathbb{R}^{3\times 3}$ such that PAP^{-1} is diagonal. Hint: consider the change of basis matrix.

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Question 5 (16 points). Let $T : \mathbb{Q}^4 \longrightarrow \mathbb{Q}^4$ be the linear operator given by

$$T(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1).$$

i) Compute T^k for $k = 1, 2, \ldots$.

ii) Find the minimal polynomial of T, i.e. the monic polynomial p of least degree such that $p(T) = 0$.

iii) Determine the eigenvalues of T, and find an eigenvector for each eigenvalue.

iv) Repeat the above question iii), viewing T as a map $\mathbb{C}^4 \longrightarrow \mathbb{C}^4$.

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Question 6 (16 points). Let $T : U \longrightarrow V$ be a linear map and let U be finite-dimensional. Prove that

$$\dim U = \dim \operatorname{null}(T) + \dim \operatorname{range}(T).$$

Hint: Begin by choosing a basis for $\operatorname{null}(T)$. You may use the fact that a basis for a subspace of U can always be extended to a basis of U.

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Question 7 (Bonus: 10 points). How many 2-dimensional *affine* subspaces are there in $(\mathbb{F}_3)^4$? An affine subspace is any subset of a vector space which can be obtained by translating a k-dimensional linear subspace by a fixed vector. In other words, it is a k-dimensional plane which does not necessarily pass through the origin.