Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Solutions for Problem Set 2

1. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ that has the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 & -1 \\ 2 & 6 & 4 & 6 & 4 \\ 1 & 3 & 2 & 2 & 1 \end{bmatrix}$$

relative to the bases $\{(1,1,1,1,1),(1,1,1,1,0),(1,1,0,0,0),(1,0,0,0,0),(0,0,0,1,0)\}$ of $\mathbb{R}^5$ and $\{(1,1,1),(0,1,0),(1,0,0)\}$ of $\mathbb{R}^3$.

(a) Find a basis for the kernel of T.

(b) Find a basis for the image of T.

Solution. (a) Let

$$\alpha = \{(1,1,1,1,1),(1,1,1,1,0),(1,1,0,0,0),(1,0,0,0,0),(0,0,0,1,0)\}$$

and

$$\beta = \{(1,1,1),(0,1,0),(1,0,0)\}$$

be the given bases for $\mathbb{R}^5$ and $\mathbb{R}^3$ respectively. For a vector $v \in \mathbb{R}^5$ if we set $x = [v]_\alpha$ then

$$v \in \ker(T) \Leftrightarrow T(v) = 0 \Leftrightarrow [T(v)]_\beta = 0 \Leftrightarrow [T]_\alpha^\beta [v]_\alpha = 0 \Leftrightarrow Ax = 0.$$

Therefore to find all $v \in \ker(T)$ we solve for all x such that $Ax = 0$. Row reducing A we get:

$$A \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 6 & 6 \\ 1 & 3 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{r_3 - r_1 \to r_3} \begin{bmatrix} 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$
$$\xrightarrow{\frac{r_2}{6} \to r_2} \begin{bmatrix} 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow{r_3 - 2r_2 \to r_3} \begin{bmatrix} 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns without leading 1s correspond to the parameters of the general solution. Thus if $x = (x_1, x_2, x_3, x_4, x_5)$ we can set

$$x_2 = r, \quad x_3 = s, \quad x_5 = t,$$

for parameters $r, s, t \in \mathbb{R}$. From the row reduced form of A, the other two variables are given by

$$\begin{aligned} x_1 &= -3x_2 - 2x_3 + x_5 = -3r - 2s + t, \\ x_4 &= -x_5 = -t. \end{aligned}$$

Then the general solution to $Ax = 0$ is given by

$$(-3r - 2s + t, r, s, -t, t) = r(-3,1,0,0,0) + s(-2,0,1,0,0) + t(1,0,0,-1,1),$$

for parameters $r, s, t \in \mathbb{R}$. This shows that the set

$$\{(-3,1,0,0,0),(-2,0,1,0,0),(1,0,0,-1,1)\}$$

spans the null space of A. We also have

$$(-3r-2s+t, r, s, -t, t) \Rightarrow r = s = t = 0,$$

therefore the set above is linearly independent, hence a basis for the null space.

Then a basis for $\ker(T)$ is given by $\{v_1, v_2, v_3\}$ with coordinates

$$\begin{aligned}
[v_1]_\alpha &= (-3,1,0,0,0)\\
[v_2]_\alpha &= (-2,0,1,0,0)\\
[v_3]_\alpha &= (1,0,0,-1,1).
\end{aligned}$$

Converting back to standard coordinates we get

$$\begin{aligned}
v_1 &= -3(1,1,1,1,1) + 1(1,1,1,1,0) = (-2,-2,-2,-2,-3)\\
v_2 &= -2(1,1,1,1,1) + 1(1,1,0,0,0) = (-1,-1,-2,-2,-2)\\
v_3 &= 1(1,1,1,1,1) - 1(1,0,0,0,0) + 1(0,0,0,1,0) = (0,1,1,2,1),
\end{aligned}$$

so that

$$\{(-2,-2,-2,-2,-3),(-1,-1,-2,-2,-2),(0,1,1,2,1)\}$$

is a basis for $\ker(T)$.

(b) For $v \in \mathbb{R}^5$ and $w \in \mathbb{R}^3$, setting

$$x = [v]_\alpha, \quad y = [w]_\beta,$$

we have

$$T(v) = w \Leftrightarrow [T]_\alpha^\beta [v]_\alpha = [w]_\beta \Leftrightarrow Ax = y.$$

Therefore $w \in \operatorname{im}(T)$ if and only if y is in the column space of A. In the row reduced form of A the first and fourth columns have leadings ones, so a basis for the column space of A is given by its the first and fourth columns:

$$\{(1,2,1),(0,6,2)\},$$

Thus if

$$[w_1]_\beta = (1,2,1), \quad [w_2]_\beta = (0,6,2),$$

then $\{w_1, w_2\}$ is a basis for $\operatorname{im}(T)$. We compute

$$\begin{aligned}
w_1 &= 1(1,1,1) + 2(0,1,0) + 1(1,0,0) = (2,3,1),\\
w_2 &= 6(0,1,0) + 2(1,0,0) = (2,6,0).
\end{aligned}$$

Thus

$$\{(2,3,1),(2,6,0)\}$$

is a basis for $\operatorname{im}(T)$.

2. Let $T\colon M_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = (a-b) + (a-d)x + (b-c)x^2 + (c-d)x^3.$$

Consider the bases $\alpha = \{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\}$ of $M_{2\times 2}(\mathbb{R})$, and $\beta = \{x, x-x^2, x-x^3, x-1\}$ of $P_3(\mathbb{R})$.

(a) Find $[T]_{\beta\alpha}$.

(b) Use $[T]_{\beta\alpha}$ to find a basis for the kernel of T.

(c) Use $[T]_{\beta\alpha}$ to find a basis for the image of T.

(d) State the nullity and rank of T. Is T injective? surjective?

Solution. (a) First let us denote the matrices in α by v_1, v_2, v_3, and v_4 in that order. Let us also denote the polynomials in β by w_1, w_2, w_3 and w_4 in the order in which they appear.

The columns of $[T]_{\beta\alpha}$ are from left to right: $[T(v_1)]_\beta$, $[T(v_2)]_\beta$, $[T(v_3)]_\beta$, and $[T(v_4)]_\beta$.

We have

$$\begin{aligned} T(v_1) &= 1 + x - x^2 + x^3, \\ T(v_2) &= -1 - x + x^2 - x^3. \\ T(v_3) &= 1 - x^3, \\ T(v_4) &= -x - x^2. \end{aligned}$$

To compute the coordinate vectors of the above polynomial, we first write the standard basis of $P_3(\mathbb{R})$ in terms of β:

$$\begin{aligned} 1 = x - (x-1) &= w_1 - w_4, & x &= w_1, \\ x^2 = x - (x - x^2) &= w_1 - w_2, & x^3 &= x - (x - x^3) = w_1 - w_3. \end{aligned}$$

Then we have

$$\begin{aligned} T(v_1) &= (w_1 - w_4) + w_1 - (w_1 - w_2) + (w_3 - w_1) = w_2 + w_3 - w_4, \\ T(v_2) &= -(w_1 - w_4) - w_1 + (w_1 - w_2) - (w_3 - w_1) = -w_2 - w_3 + w_4, \\ T(v_3) &= (w_1 - w_4) - (w_3 - w_1) = 2w_1 - w_3 - w_4, \\ T(v_4) &= -w_1 - (w_1 - w_2) = -2w_1 + w_2, \end{aligned}$$

and therefore

$$[T]_{\beta\alpha} = \begin{pmatrix} 0 & 0 & 2 & -2 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

(b) First we find the null space of $A = [T]_{\beta\alpha}$. Suppose $Ax = 0$ for $x = (x_1, x_2, x_3, x_4)$. From the first row of A we have $x_3 = x_4$. On the other hand the third and fourth columns give:

$$\left.\begin{aligned} x_1 - x_2 - x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \end{aligned}\right\} \Rightarrow -2x_3 = 0 \Rightarrow x_3 = 0.$$

Therefore $x_3 = x_4 = 0$. Then from the second row of A we get

$$x_1 - x_2 + x_4 = 0 \Rightarrow x_1 = x_2.$$

Thus we must have $(x_1, x_2, x_3, x_4) = (r, r, 0, 0)$ for some $r \in \mathbb{R}$. We also note that $Ax = 0$ for any x of this form, so $\{(1, 1, 0, 0)\}$ spans the null space of A.

Thus for any vector $v \in M_{2\times 2}(\mathbb{R})$ we have

$$\begin{aligned} T(v) = 0 &\Leftrightarrow [T(v)]_\beta = 0 \Leftrightarrow [T]_{\beta\alpha}[v]_\alpha = 0 \\ &\Leftrightarrow [v]_\alpha = (r, r, 0, 0) \text{ for some } r \in \mathbb{R} \\ &\Leftrightarrow v = rv_1 + rv_2 \text{ for some } r \in \mathbb{R} \\ &\Leftrightarrow v = r\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r & r \\ r & r \end{bmatrix} \text{ for some } r \in \mathbb{R} \end{aligned}$$

Therefore $\{v_1 + v_2\} = \{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\}$ is a basis for $\ker(T)$.

(c) First we extend the basis $\{v_1 + v_2\}$ for $\ker(T)$ to a basis for $M_{2\times 2}(\mathbb{R})$. For example we can take

$$\{v_1 + v_2, v_1, v_3, v_4\}.$$

Then the image under T of the extra vectors form a basis for the image of T:

$$\{T(v_1), T(v_3), T(v_4)\} = \{1 + x - x^2 + x^3, 1 - x^3, -x - x^2\}$$

(d) The nullity of T is $\dim\ker(T) = 1$ and its rank is $\dim\operatorname{im}(T) = 3$. The map T is not injective because $\dim\ker(T) > 0$. It is also not surjective because $\dim\operatorname{im}(T) < \dim P_3(\mathbb{R}) = 4$.

3. Textbook, Section 2.3, **12**.

Solution. (a) Let $A, B \in M_{n\times n}(\mathbb{R})$, and denote the ij^{th} entries of A and B by a_{ij} and b_{ij}. Let $c \in \mathbb{R}$ be a scalar. Then the ij^{th} entry of $cA + B$ is $ca_{ij} + b_{ij}$, so

$$\begin{aligned}\operatorname{Tr}(cA + B) &= (ca_{11} + b_{11}) + (ca_{22} + b_{22}) + ... + (ca_{nn} + b_{nn}) \\ &= c(a_{11} + a_{22} + ... + a_{nn}) + (b_{11} + b_{22} + ... + b_{nn}) \\ &= c\operatorname{Tr}(A) + \operatorname{Tr}(B).\end{aligned}$$

Therefore $\operatorname{Tr} : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.

(b) The map Tr is non-zero, since for example, $\operatorname{Tr}(I_n) = n$, where $I_n \in M_{n\times n}(\mathbb{R})$ is the identity matrix. Thus $\dim\operatorname{im}(T) > 0$. But also $\dim\operatorname{im}(T) \le \dim\mathbb{R} = 1$, so $\dim\operatorname{im}(T) = 1$ necessarily. Then by the dimension theorem we have

$$\dim\ker(T) = \dim M_{n\times n}(\mathbb{R}) - \dim\operatorname{im}(T) = n^2 - 1.$$

(c) Let E_{ij} be the matrix with ij^{th} entry equal to 1 and all others equal to zero. The n^2 vectors $\{E_{ij}\}_{1\le i,j\le n}$ form the standard basis for $M_{n\times n}(\mathbb{R})$. We have $\operatorname{Tr}(E_{ij}) = 1$ if $i = j$, and 0 otherwise. Thus the $n^2 - n$ vectors $\{E_{ij}\}_{i\neq j}$ belong to $\ker(\operatorname{Tr})$. We also have

$$\operatorname{Tr}(E_{ii} - E_{11}) = \operatorname{Tr}(E_{ii}) - \operatorname{Tr}(E_{11}) = 1 - 1 = 0.$$

Therefore the $n - 1$ vectors $\{E_{22} - E_{11}, E_{33} - E_{11}, ..., E_{nn} - E_{11}\}$ also belong to $\ker(\operatorname{Tr})$. Thus the set

$$\{E_{ij}\}_{i\neq j} \cup \{E_{ii} - E_{11}\}_{i>1}.$$

consists of $(n^2 - n) + (n - 1) = n^2 - 1$ vectors in $\ker(\operatorname{Tr})$. We show this set is linearly independent. Suppose a linear combination of these vectors is equal to zero:

$$\sum_{i\neq j} a_{ij}E_{ij} + \sum_{i>1} a_{ii}(E_{ii} - E_{11}) = 0,$$

with $a_{ij} \in \mathbb{R}$ for $1 \le i, j \le n$, $(i, j) \neq (1, 1)$. We show that all these a_{ij} must be zero. Since there is no a_{11} term above, it's safe to set $a_{11} = 0$. Then we can add $a_{11}(E_{11} - E_{11}) = 0$ to the above equation to write

$$0 = \sum_{i\neq j} a_{ij}E_{ij} + \sum_{i=1}^{n} a_{ii}(E_{ii} - E_{11}) = \sum_{i,j=1}^{n} a_{ij}E_{ij} - (a_{11} + a_{22} + ... + a_{nn})E_{11}.$$

If A is the matrix given by a_{ij} and $a = a_{11} + a_{22} + ... + a_{nn}$, the above equation gives

$$0 = A - aE_{11} \Leftrightarrow A = aE_{11}$$

But the matrix E_{11} has all but the top left entry equal to zero, so we have $a_{ij} = 0$ for all i, j.

Therefore

$$\{E_{ij}\}_{i \neq j} \cup \{E_{ii} - E_{11}\}_{i>1},$$

are $n^2 - 1$ linearly independent vectors in $\ker(\mathrm{Tr})$. As $\dim \ker(\mathrm{Tr}) = n^2 - 1$ by part (b), this set is also a basis.

4. Let $T\colon \mathbb{Z}_3^3 \to \mathbb{Z}_3^2$ be defined by $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

(a) Is T surjective? If not, find $Im(T)$.

(a) Is T injective? If not, find $Ker(T)$.

Solution. (a) The first two columns of A are non-zero and not multiples of each other, so they are linearly independent, therefore $\dim \mathrm{im}(T) = \dim \mathrm{col}(A) \geq 2$. Since $\mathrm{im}(T)$ is a subspace of $\mathbb{Z}_3^2$ we also have

$$\dim \mathrm{im}(T) \leq \dim \mathbb{Z}_3^2 = 2.$$

Therefore $\mathrm{im}(T) = 2 = \dim \mathbb{Z}_3^2$, so $\mathrm{im}(T) = \dim \mathbb{Z}_3^2$, and T is surjective.

(b) T can not be injective because $3 = \dim \mathbb{Z}_3^3 > \dim \mathbb{Z}_3^2 = 2$. In particular by the dimension theorem

$$\dim \ker(T) = \dim \mathbb{Z}_3^3 - \dim \mathrm{im}(T) = 3 - 2 = 1.$$

Therefore any non-zero vector in $\ker(T)$ forms a basis.

For $x = (a, b, c) \in \mathbb{Z}_3^3$, we have $T(x) = 0$ if and only if

$$\begin{aligned} a + 2c &= 0, \\ a + 2b + c &= 0. \end{aligned}$$

We look for a solution with $c = 1$. The first equation gives $a = -2 = 1$, and the second gives

$$-2 + 2b + 1 = 0 \Rightarrow 2b = 1 \Rightarrow b = 2.$$

We check that $(a, b, c) = (-2, 2, 1)$ is indeed a solution to the above system:

$$a + 2c = -2 + 2 \times 1 = 0, \quad a + 2b + c = -2 + 2 \times 2 + 1 = -2 + 1 + 1 = 0.$$

Therefore $\{(-2, 2, 1)\}$ is a basis for $\ker(T)$. In fact,

$$\ker(T) = \{0(-2, 2, 1), 1(-2, 2, 1), 2(-2, 2, 1)\} = \{(0, 0, 0), (-2, 2, 1), (-1, 1, 2)\}.$$

5. Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Let $T\colon V \to \mathbb{R}^n$ be defined by

$$T(v) = [v]_\alpha$$

for every $v \in V$.

(a) Show that T is a linear transformation.

(b) Show that T is bijective.

Solution. (a) Let $v, w \in V$. Since $\{v_1, v_2, ... v_n\}$ is a basis, we have

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n,$$

$$w = b_1v_1 + b_2v_2 + ... + b_nv_n,$$

for $a_i, b_i \in \mathbb{R}$. In other words

$$[v]_\alpha = (a_1, a_2, ..., a_n), \quad [w]_\alpha = (b_1, b_2, ..., b_n).$$

We also have

$$\begin{aligned} v + w &= a_1v_1 + a_2v_2 + ... + a_nv_n + b_1v_1 + b_2v_2 + ... + b_nv_n \\ &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + ... + (a_n + b_n)v_n, \end{aligned}$$

Therefore

$$\begin{aligned} T(v + w) &= [v + w]_\alpha = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n) \\ &= (a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n) = [v]_\alpha + [w]_\alpha \\ &= T(v) + T(w) \quad \checkmark \end{aligned}$$

For any scalar $c \in \mathbb{R}$,

$$cv = c(a_1v_1 + a_2v_2 + ... + a_nv_n) = ca_1v_1 + ca_2v_2 + ... + ca_nv_n,$$

so

$$T(cv) = [cv]_\alpha = (ca_1, ca_2, ..., ca_n) = c(a_1, a_2, ..., a_n) = c[v]_\alpha = cT(v). \quad \checkmark$$

Therefore T is a linear transformation.

(b) Let $v \in \ker(T)$. Then we have

$$T(v) = 0 \Rightarrow [v]_\alpha = (0, 0, ..., 0) \Rightarrow v = 0.v_1 + 0.v_2 + ... + 0.v_n = 0.$$

Therefore $\ker(T) = \{0\}$, so T is injective. By the dimension theorem

$$\dim \operatorname{im}(T) = \dim V - \dim \ker(T) = n - 0 = n = \dim \mathbb{R}^n,$$

so that $\operatorname{im}(T) = \dim \mathbb{R}^n$. Thus T is also surjective, and so bijective.

6. Let $T\colon V \to W$ be a bijective linear transformation. Prove that if $\{v_1, v_2, \dots, v_n\}$ is a basis for V, then $\{w_1, w_2, \dots, w_n\}$ is a basis for W.

Solution. First we show that the w_i span W. Let $w \in W$. The map T is bijective, in particular it is surjective, so $T(v) = w$ for some $v \in V$. Since $\{v_1, ..., v_n\}$ is a basis for V, there exist scalars $a_i \in F$ such that

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n.$$

Applying T to the above we get

$$w = T(v) = T(a_1v_2 + a_2v_2 + ... a_nv_n) = a_1T(v_1) + a_2T(v_2) + ... + a_nT(v_n) = a_1w_1 + a_2w_2 + ... + a_nw_n.$$

Therefore $w \in \operatorname{Span}\{w_1, w_2, ..., w_n\}$. Since w was arbitrary, this shows $W = \operatorname{Span}\{w_1, w_2, ..., w_n\}$.

Now we show the w_i are linearly independent. Suppose that for some $b_i \in F$

$$b_1w_1 + b_2w_2 + ... + b_nw_n = 0.$$

Since $w_i = T(v_i)$ we have

$$0 = b_1T(v_1) + b_2T(v_2) + ... + b_nT(v_n) = T(b_1v_1 + b_2v_2 + ... + b_nv_n).$$

Therefore $b_1v_1 + b_2v_2 + ... + b_nv_n \in \ker(T)$. But $\ker(T) = \{0\}$ because T is injective, so

$$b_1v_1 + b_2v_2 + ...b_nv_n = 0.$$

But we also know the set $\{v_1, v_2, ..., v_n\}$ is linearly independent, so the above equation implies

$$b_1 = b_2 = ... = b_n = 0.$$

This shows $\{w_1, w_2, ..., w_n\}$ is linearly independent. As we have already shown it spans W, it is a basis for it.

7. Let V and W be vector spaces over a field F. Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, and $\beta = \{w_1, w_2, \ldots, w_m\}$ a basis for W. Let $T\colon V \to W$ be a linear transformation.

(a) Prove that T is surjective if and only if the columns of $[T]_{\beta\alpha}$ span F^m.

(b) Prove that T is injective if and only if the columns of $[T]_{\beta\alpha}$ are linearly independent in F^m.

Solution. We let $C_1, C_2, ..., C_n \in F^n$ denote the columns of $[T]_{\beta\alpha}$.

(a) First the "only if" part:

Suppose T is surjective, and let $b = (b_1, b_2, ...b_m) \in F^m$. Then if $w = b_1w_1 + ...b_mw_m$ we have $[w]_\beta = b$. Since T is surjective, $T(v) = w$ for some $v \in V$. Let $(a_1, a_2, ..., a_n) = [v]_\alpha$. We have

$$T(v) = w \Rightarrow [T]_{\beta\alpha}[v]_\alpha = [w]_\beta = b.$$

Writing the matrix multiplication above in terms of linear combination of columns of $[T]_{\beta\alpha}$, we get

$$a_1C_1 + a_2C_2 + ... + a_nC_n = b.$$

Therefore b is in the column space of $[T]_{\beta\alpha}$. Since b was arbitrary, this shows the columns of $[T]_{\beta\alpha}$ span F^m.

Now the "if" part: Suppose the columns of $[T]_{\beta\alpha}$ span F^m. Let $w \in W$. Then there are scalars $a_1, ..., a_n$ such that $a_1C_1 + a_2C_2 + ... + a_nC_n = [w]_\beta$. Letting $v = a_1v_1 + ...a_nv_n$, we have

$$[T(v)]_\beta = [T]_{\beta\alpha}[v]_\alpha = a_1C_1 + a_2C_2 + ... + a_nC_n = [w]_\beta.$$

Then $T(v)$ and w have the same coordinates with respect to β, therefore they are equal, so $w = T(v) \in \mathrm{im}(T)$. Since w was arbitrary, this shows T is surjective.

The two paragraphs above together prove the if and only if statement.

(b) First we assume that T is injective, and suppose $a_1C_1 + a_2C_2 + ... + a_nC_n = 0$ for $a_i \in F$. Let $v = a_1v_1 + a_2v_2 + ... + a_nv_n$. Then $[v]_\alpha = (a_1, a_2, ..., a_n)$, and

$$[T(v)]_\beta = [T]_{\beta\alpha}[v]_\alpha = a_1C_1 + a_2C_2 + ... + a_nC_n = 0 \in F^m`.$$

Then $T(v) = 0$ because its coordinates with respect to β are zero. Since T is injective, $\ker(T) = \{0\}$, so $v = 0$, i.e.

$$a_1v_1 + a_2v_2 + ... + a_nv_n = 0.$$

But the v_i are linearly independent, so from the above we get $a_1 = a_2 = ... = a_n = 0$. This shows that $C_1, C_2, ..., C_n$ are linearly independent.

Conversely, we assume $C_1, C_2, ..., C_n$ are linearly independent, and suppose $T(v) = 0$. Then if $[v]_\alpha = (a_1, ..., a_n)$, and if $C_1, C_2, ..., C_n$ are the columns of $[T]_{\beta\alpha}$ we have

$$T(v) = 0 \Rightarrow [T(v)]_{\beta\alpha} = 0 \Rightarrow [T]_{\beta\alpha}[v]_\alpha = 0 \Rightarrow a_1C_1 + a_2C_2 + ... + a_nC_n = 0.$$

Since C_i are linearly independent, we have $a_1 = a_2 = ... = a_n = 0$. Thus $[v]_\alpha = 0$, and so $v = 0$. This shows that $\ker(T) = \{0\}$, i.e. T is injective.

Thus $C_1, C_2, ..., C_n$ are linearly independent if and only if T is injective.