SCHOOL OF FINANCE AND APPLIED STATISTICS

FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 12

Question 1

The annual effective interest rates in various years are assumed to be mutually independent. In each year the interest rate $\tilde{i}$ has a normal distribution with mean 0.08 and variance 0.0009. Find the mean and variance of $\tilde{S}(20)$.

Solution

Since interest rates are independent,

$$E\left[\tilde{S}(20)\right] = E\left[(1+\tilde{i})\right]^{20} = (1.08)^{20} = 4.6610$$

$$Var\left[\tilde{S}(20)\right] = E\left[\tilde{S}(20)^2\right] - (4.6610)^2 = E\left[(1+\tilde{i})^2\right]^{20} - (4.6610)^2$$

$$E\left[(1+\tilde{i})^2\right] = Var\left[1+\tilde{i}\right] + \left(E\left[1+\tilde{i}\right]\right)^2 = 0.0009 + 1.1664 = 1.1673$$

$$\Rightarrow E\left[\tilde{S}(20)^2\right] = (1.1673)^{20} = 22.0622$$

$$\Rightarrow Var\left[\tilde{S}(20)\right] = 0.337$$

Question 2

Smith invests $1,000 in a mutual fund on January 1, 1990. Smith estimates the growth for the fund in 1990 to be 20%, with probability 0.80, or –5% with probability 0.20.

(a) Find the expected accumulated value of Smith's fund on December 31,1990

(b) Find the variance (as calculated on January 1, 1990) of the accumulated value on December 31, 1990

(c) Smith estimates that the fund's growth rate in 1991 will be 30%, with probability 0.60, or –8% with probability 0.40, independent of the growth rate in 1990. Repeat parts (a) and (b) (as calculated on January 1, 1990) for the December 31, 1991 accumulated value.

(d) Based on the assumed growth distributions for 1990 and 1991, what is the probability that Smith will have annual average growth of at least 15% for the two-year period?

Solution

(a) $E[1000(1+\tilde{i}_1)] = 1000E[1+\tilde{i}_1] = 1000[(0.8)(1.2)+(0.2)(0.95)] = \$1,150$

where

$E[1+\tilde{i}_1] = 1+(0.8)(0.2)+(0.2)(-0.05) = 1.15$

(b) $Var\left[1,000(1+\tilde{i}_1)\right] = 1,000,000 \cdot Var\left[1+\tilde{i}_1\right]$

$E\left[(1+\tilde{i}_1)^2\right] = (1.2)^2(0.8)+(0.95)^2(0.20) = 1.3325$

$\Rightarrow Var\left[1+\tilde{i}_1\right] = 1.3325-(1.15)^2 = 0.01$

$\Rightarrow Var\left[1,000(1+\tilde{i}_1)\right] = 10,000$

(c)

$E[1+\tilde{i}_2] = 1+E[\tilde{i}_2] = 1+(0.3)(0.6)+(-0.08)(0.4) = 1.148$

$E\left[1000(1+\tilde{i}_1)(1+\tilde{i}_2)\right] = 1000E\left[1+\tilde{i}_1\right]E\left[1+\tilde{i}_2\right] = 1,000(1.15)(1.148) = 1,320.20$

$$Var\left[1000(1+\tilde{i}_1)(1+\tilde{i}_2)\right] = 1,000,000 \cdot Var\left[(1+\tilde{i}_1)(1+\tilde{i}_2)\right]$$
$$= 1,000,000\left[E\left[(1+\tilde{i}_1)^2(1+\tilde{i}_2)^2\right] - \left(E\left[(1+\tilde{i}_1)(1+\tilde{i}_2)\right]\right)^2\right]$$

$E\left[(1+\tilde{i}_1)^2(1+\tilde{i}_2)^2\right] = E\left[(1+\tilde{i}_1)^2\right]E\left[(1+\tilde{i}_2)^2\right]$

$E\left[(1+\tilde{i}_1)^2\right] = 1.3325$ (from part (b))

$E\left[(1+\tilde{i}_2)^2\right] = (1.3)^2(0.6)+(0.92)^2(0.40) = 1.35256$

$\Rightarrow E\left[(1+\tilde{i}_1)^2(1+\tilde{i}_2)^2\right] = (1.3325)(1.35256) = 1.802286$

$\Rightarrow Var\left[1000(1+\tilde{i}_1)(1+\tilde{i}_2)\right] = 1,000,000 \cdot (1.802286\text{-}1.32020^2)\text{=}59,358.16$

Alternatively, you may calculate the expected value and variance by setting up the following table and calculating $E[S(2)]$ and $Var[S(2)]$:

$S(2)$	$P(S(2))$
$1000\times1.2\times1.3 = 1,560$	$0.8\times0.6 = 0.48$
$1000\times1.2\times0.92 = 1,104$	$0.8\times0.4 = 0.32$
$1000\times0.95\times1.3 = 1,235$	$0.2\times0.6 = 0.12$
$1000\times0.95\times0.92 = 874$	$0.2\times0.4 = 0.08$

(d) Smith has average annual return of at least 15% for the two years if $(1+\tilde{i}_1)(1+\tilde{i}_2) \geq (1.15)^2 = 1.3225$.

The possible returns and corresponding average annual returns are:

$i_1 = 0.2, i_2 = 0.3$ $\quad$ $\left((1.2)(1.3)\right)^{1/2} - 1 = 24.9\%$

$i_1 = 0.2, i_2 = -0.08$ $\quad$ $\left((1.2)(0.92)\right)^{1/2} - 1 = 5.1\%$

$i_1 = -0.05, i_2 = 0.3$ $\quad$ $\left((0.95)(1.3)\right)^{1/2} - 1 = 11.1\%$

$i_1 = -0.05, i_2 = -0.08$ $\quad$ $\left((0.95)(0.92)\right)^{1/2} - 1 = -3.5\%$

The only way that an average annual return of at least 15% for the two years can happen is if $\tilde{i}_1 = 0.20$ and $\tilde{i}_2 = 0.30$, which has probability (0.80)(0.60)=0.48

Question 3

Suppose that $\tilde{i}_1 = \begin{cases} 0.08 & prob = 1/3 \\ 0.10 & prob = 1/3 \\ 0.12 & prob = 1/3 \end{cases}$

and in subsequent years the interest rate distribution is

$$\tilde{i}_t = \begin{cases} i_{t-1} - 0.02 & prob = 1/3 \\ i_{t-1} & prob = 1/3 \\ i_{t-1} + 0.02 & prob = 1/3 \end{cases}$$

Find the expected value and variance of $\tilde{S}(n)$ for $n = 2$.

Solution

For $n = 2$, $\tilde{S}(2)$ can arise as below with equal probablility for each of the 9 outcomes:

t=1	t=2	S(2)
0.08	0.06	1.1448
	0.08	1.1664
	0.1	1.1880
0.1	0.08	1.1880
	0.1	1.2100
	0.12	1.2320
0.12	0.1	1.2320
	0.12	1.2544
	0.14	1.2768

$\tilde{S}(2)$ has a 7-point distribution. That is, there are 7 possible accumulated values that can eventuate:

$$\Pr\left[(1.08)(1.06)\right] = \frac{1}{9}$$

$$\Pr\left[(1.08)^2\right]=\frac{1}{9}$$

$$\Pr\left[(1.08)(1.10)\right]=\frac{2}{9}$$

$$\Pr\left[(1.10)^2\right]=\frac{1}{9}$$

$$\Pr\left[(1.10)(1.12)\right]=\frac{2}{9}$$

$$\Pr\left[(1.12)^2\right]=\frac{1}{9}$$

$$\Pr\left[(1.12)(1.14)\right]=\frac{1}{9}$$

$$E\left[\tilde{S}(2)\right]=\frac{1.1448+1.1664+2(1.1880)+1.210+2(1.232)+1.2544+1.2768}{9}=1.210267$$

$$E\left[\tilde{S}(2)^2\right]=\frac{1.1448^2+1.1664^2+2(1.1880^2)+1.210^2+2(1.232^2)+1.2544^2+1.2768^2}{9}=1.46636$$

$$Var\left[\tilde{S}(2)\right]=1.46636-1.210267^2=0.0016$$

Question 4

Interest rates for the next 30 years are independently and identically distributed with the following distribution:

$$\tilde{i}=\begin{cases}-10\% & prob=0.05\\ 8\% & prob=0.75\\ 30\% & prob=0.20\end{cases}$$

Using the lognormal distribution, find the probability that the present value of $2,000 due in 30 years time will be less than $80.

Solution

This is equivalent to $\Pr\left[\tilde{S}(30)>\frac{2000}{80}=25\right]$

The distribution of $\tilde{\delta}$ is:

$$\tilde{\delta}=\begin{cases}\ln(0.90) & prob=0.05\\ \ln(1.08) & prob=0.75\\ \ln(1.30) & prob=0.2\end{cases}$$

$$E\left[\tilde{\delta}\right]=0.05\ln(0.9)+0.75\ln(1.08)+0.2\ln(1.3)=0.1049256$$

$$E\left[\tilde{\delta}^2\right]=0.05\left(\ln(0.9)\right)^2+0.75\left(\ln(1.08)\right)^2+0.2\left(\ln(1.3)\right)^2=0.0187643$$

$$\Rightarrow Var\left[\tilde{\delta}\right] = 0.0187643 - 0.1049256^2 = 0.0077549$$

So,

$$E\left[\ln\left[\tilde{S}(30)\right]\right] = 30E\left[\tilde{\delta}\right] = 3.147768$$

$$Var\left[\ln\left[\tilde{S}(30)\right]\right] = 30Var\left[\tilde{\delta}\right] = 0.232647$$

$$\Pr\left[\tilde{S}(30) > 25\right] = \Pr\left[\ln\left(\tilde{S}(30)\right) > \ln\left(25\right)\right]$$

$$= \Pr\left[\frac{\ln\left(\tilde{S}(30)\right) - E\left[\ln\left[\tilde{S}(30)\right]\right]}{\sqrt{Var\left[\ln\left[\tilde{S}(30)\right]\right]}} > \frac{\ln\left(25\right) - E\left[\ln\left[\tilde{S}(30)\right]\right]}{\sqrt{Var\left[\ln\left[\tilde{S}(30)\right]\right]}}\right]$$

$$= \Pr\left[\tilde{Z} > \frac{\ln\left(25\right) - 3.147768}{\sqrt{0.232647}}\right]$$

$$= \Pr\left[\tilde{Z} > 0.147\right]$$

where $\tilde{Z}$ is a standard normal variable with mean 0 and variance 1. Referring to tables for the standard normal distribution:

$$\Pr\left[\tilde{Z} > 0.147\right] = 0.442$$

<u>Question 5</u>

Interest rates are expected to follow a uniform distribution between 5% and 15% over each of the next 15 years. Interest rates for each year are not correlated with the previous year's rate.

Find the probability that $500 will accumulate to more than $2,500 in 15 years, assuming that the amount accumulated to over the 15 years follows a normal distribution.

<u>Solution</u>

$$E\left[500\tilde{S}(15)\right] = 500\left(E\left[1+\tilde{i}\ \right]\right)^{15} = 500\left(1+\frac{0.05+0.15}{2}\right)^{15}$$

$$= 500(1.1^{15}) = \$2,088.62$$

$$Var\left[500\tilde{S}(15)\right] = 500^2\left(Var(\tilde{S}(15))\right)$$

$$Var(\tilde{S}(15)) = \left(E\left[\left(1+\tilde{i}\ \right)^2\right]\right)^{15} - \left(E\left[1+\tilde{i}\ \right]\right)^{30}$$

$$E\left[\left(1+\tilde{i}\ \right)^2\right] = Var\left[1+\tilde{i}\ \right] + \left(E\left[1+\tilde{i}\ \right]\right)^2 = \frac{\left(1.15-1.05\right)^2}{12} + 1.1^2 = 1.2108\dot{3}$$

$$\therefore Var\left[500\tilde{S}(15)\right] = 250,000 \times (1.2108\dot{3}^{15} - 1.1^{30}) = 45,283.51$$

We now need to calculate $\Pr\left[500\tilde{S}(15) > 2500\right]$

$$= \Pr\left[\frac{500\tilde{S}(15) - E\left[500\tilde{S}(15)\right]}{\sqrt{Var\left[500\tilde{S}(15)\right]}} > \frac{2500 - E\left[500\tilde{S}(15)\right]}{\sqrt{Var\left[500\tilde{S}(15)\right]}}\right]$$

$$= \Pr\left[\tilde{Z} > \frac{2500 - 2088.62}{\sqrt{45,283.51}}\right]$$

$$= \Pr\left[\tilde{Z} > 1.933\right]$$

where $\tilde{Z}$ is a standard normal variable with mean 0 and variance 1. Referring to tables for the standard normal distribution:

$$\Pr\left[\tilde{Z} > 1.933\right] = 0.027$$

Past Exam Question – 2005 Final Exam Q5

Annual effective interest rates are expected to be independently and identically distributed according to the following distribution:

$$\tilde{i} = \begin{cases} -10\% & prob = 0.05 \\ 3\% & prob = 0.30 \\ 8\% & prob = 0.55 \\ 20\% & prob = 0.10 \end{cases}$$

Let $\tilde{S}(n)$ be the random variable denoting the accumulated value of \$1 for n years according to the interest rate distribution above.

a) Calculate the mean and variance of $\tilde{S}(5)$. (4 marks)

An investor has a liability of \$250,000 due in 15 years. She wishes to set aside an amount of money now in order to ensure that she will have enough to pay the liability when it falls due. Interest rates are expected to follow the distribution outlined above.

b) Assuming the accumulation factor of the amount invested to fund the liability due in 15 years is log-normally distributed, calculate what amount the investor will need to set aside in order to be 95% confident she will have enough to fund the liability. (8 marks)

Solution

a) $E\left[1+\tilde{i}\right] = 0.05\times 0.9 + 0.3\times 1.03 + 0.55\times 1.08 + 0.1\times 1.2 = 1.068$

$$E\left[\left(1+\tilde{i}\right)^2\right] = 0.05\times 0.9^2 + 0.3\times 1.03^2 + 0.55\times 1.08^2 + 0.1\times 1.2^2 = 1.14429$$

$$E\left[\tilde{S}(5)\right] = \left(E(1+\tilde{i}\,)\right)^5 = 1.068^5 = 1.3894927$$

$$E\left[\tilde{S}(5)^2\right] = \left(E\left[\left(1+\tilde{i}\,\right)^2\right]\right)^5 = 1.14429^5 = 1.9619165$$

$$Var\left[\tilde{S}(5)\right] = E\left[\tilde{S}(5)^2\right] - \left(E\left[\tilde{S}(5)\right]\right)^2 = 1.9619165 - 1.3894927^2 = 0.0312266$$

b) The distribution of $\tilde{\delta}$ is:

$$\tilde{\delta} = \begin{cases} \ln(0.90) & prob = 0.05 \\ \ln(1.03) & prob = 0.30 \\ \ln(1.08) & prob = 0.55 \\ \ln(1.20) & prob = 0.10 \end{cases}$$

$$E\left[\tilde{\delta}\right] = 0.05\times\ln(0.9) + 0.3\times\ln(1.03) + 0.55\times\ln(1.08) + 0.1\times\ln(1.2) = 0.0641603$$

$$E\left[\tilde{\delta}^2\right] = 0.05\times\left[\ln(0.9)\right]^2 + 0.3\times\left[\ln(1.03)\right]^2 + 0.55\times\left[\ln(1.08)\right]^2 + 0.1\times\left[\ln(1.2)\right]^2 = 0.007398925$$

$$\Rightarrow Var\left[\tilde{\delta}\right] = 0.007398925 - 0.0641603^2 = 0.003282381$$

So,

$$E\left[\ln\left[\tilde{S}(15)\right]\right] = 15\cdot E\left[\tilde{\delta}\right] = 0.9624045$$

$$Var\left[\ln\left[\tilde{S}(15)\right]\right] = 15\cdot Var\left[\tilde{\delta}\right] = 0.0492357$$

We are aiming to find X where:

$$\Pr\left[\tilde{S}(15) > \frac{250,000}{X}\right] = 0.95$$

$$\Pr\left[\ln\left(\tilde{S}(15)\right) > \ln\left(\frac{250,000}{X}\right)\right] = 0.95$$

$$\Pr\left[\frac{\ln\left(\tilde{S}(15)\right) - E\left[\ln\left[\tilde{S}(15)\right]\right]}{\sqrt{Var\left[\ln\left[\tilde{S}(15)\right]\right]}} > \frac{\ln\left(\frac{250,000}{X}\right) - E\left[\ln\left[\tilde{S}(15)\right]\right]}{\sqrt{Var\left[\ln\left[\tilde{S}(15)\right]\right]}}\right] = 0.95$$

$$\Pr\left[\tilde{Z} > \frac{\ln\left(\frac{250,000}{X}\right) - 0.9624045}{\sqrt{0.0492357}}\right] = 0.95$$

$$\therefore \frac{\ln\left(\frac{250,000}{X}\right) - 0.9624045}{\sqrt{0.0492357}} = -1.6449$$

$$\frac{250,000}{X} = \exp(0.9624045 - 1.6449\sqrt{0.0492357}) = 1.8203999$$

$$X = \$137,332.46$$

Therefore, the investor will need to set aside $137,332.46 now in order to be 95% confident she will have $250,000 in 15 years.