Assignment 1 - Solutions - MAT 327 - Summer 2014

Comprehension

[**C.1**] Let $(X, \mathcal{T})$ be a topological space, and let $f : X \longrightarrow Y$ be an injection.
Is $\mathcal{U} := \{ f[A] : A \in \mathcal{T} \}$ a topology on the range of f?

Answer for C.1. This is just straightforward application of set identities.

$\emptyset, \mathrm{ran}(f)$: Notice that $\emptyset = f[\emptyset] \in \mathcal{U}$, and $\mathrm{ran}(f) := f[X] \in \mathcal{U}$.

Intersection: Let $\{ f[A_1], f[A_2], \ldots, f[A_n] \}$ be a finite subcollection of $\mathcal{U}$. Notice that

$$f[A_1] \cap f[A_2] \cap \ldots \cap f[A_n] = f[A_1 \cap A_2 \cap \ldots \cap A_n]$$

We have equality because f is an injection, and since $(X, \mathcal{T})$ is a topological space, the finite intersection presented is an open set. Thus

$$f[A_1] \cap f[A_2] \cap \ldots \cap f[A_n] = f[A_1 \cap A_2 \cap \ldots \cap A_n] \in \mathcal{U}$$

If we want to be *very* careful we can prove (by induction) that set identity with functions and intersections.

Union: Let $\{ f[A_\alpha] : \alpha \in I \}$ be an arbitrary subcollection of $\mathcal{U}$. By a simple set identity (that does *not* depend on f being an injection),

$$\bigcup_{\alpha \in I} f[A_\alpha] = f[\bigcup_{\alpha \in I} A_\alpha]$$

Since $(X, \mathcal{T})$ is a topological space, the union presented is an open set. Thus

$$\bigcup_{\alpha \in I} f[A_\alpha] = f[\bigcup_{\alpha \in I} A_\alpha] \in \mathcal{U}$$

Again, if we want to be very careful we can prove (directly, not by induction!) that set identity with unions.

Thus we have shown that $\mathcal{U}$ is a topology on $\mathrm{ran}(f)$. $\square$

[**C.2**] Let $(X, \mathcal{T})$ and (X, Γ) be topological spaces. Prove or disprove the following statements:

i. $(X, \mathcal{T} \cap \Gamma)$ is a topological space.

ii. $(X, \mathcal{T} \cup \Gamma)$ is a topological space.

Answer for C.2.i. Yes, this is always a topology. Let us check the three conditions.

$\emptyset, X$: Notice that $\emptyset \in \mathcal{T} \cap \Gamma$ and $X \in \mathcal{U} \cap \Gamma$, since both $\mathcal{T}$ and Γ are topologies on X.

Intersection: Let $\{ A_1, A_2, \ldots, A_n \}$ be a finite subcollection of $\mathcal{T} \cap \Gamma$. Since $\mathcal{T}$ is a topology we have:

$$A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{T}$$

Since Γ is a topology we have:

$$A_1 \cap A_2 \cap \ldots \cap A_n \in \Gamma$$

Thus

$$A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{T} \cap \Gamma$$

Union: Let $\{ A_\alpha : \alpha \in I \}$ be an arbitrary subcollection of $\mathcal{U} \cap \Gamma$. Since $\mathcal{T}$ is a topology we have:

$$\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$$

Since Γ is a topology we have:

$$\bigcup_{\alpha \in I} A_\alpha \in \Gamma$$

Thus

$$\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T} \cap \Gamma$$

$\square$

Answer for C.2.ii. In general this need not be a topology. There are topologies on finite sets that show this, but let's give something a bit more ... real.

Let $X = \mathbb{R}$ and let $\mathcal{T}$ = Sorgenfrey topology. Let Γ = topology generated by the basis

$$\{ (a, b] : a \leq b \}$$

(which is basically the backwards Sorgenfrey topology). If we were being *very* careful we would check that this is indeed a basis on $\mathbb{R}$, but it is very similar to the proof that the usual basis for the Sorgenfrey line is a basis.

We readily see that $\mathcal{T} \cap \Gamma$ cannot be a topology on $\mathbb{R}$, because it fails to contain $\{ 7 \} = [7, 9) \cap (4, 7]$, sets which are open in their respective topolgies. $\square$

[**C.3**] Let X be an infinite set. Prove that $\mathcal{T} := \{ A \subseteq X : A = \emptyset \text{ or } X \setminus A \text{ is finite} \}$ is a topology on X. This is called the "finite complement topology" or the "co-finite topology".

Answer for C.3. This is a straightforward calculation. The fact that X is infinite is so that the $X_{\text{co-finite}}$ is not a discrete space. Even if X is finite, $\mathcal{T}$ will still be a topology.

$\emptyset, X$: By definition $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$, since $X \setminus X = \emptyset$, a finite set.

Intersection: Let $\{ A_1, A_2, \ldots, A_n \}$ be a finite subcollection of $\mathcal{T}$. (If any of the A_i is the empty set, we know the intersection will be the empty set, so assume no A_i is empty.) For each $i \leq n$ we know that $A_i = X \setminus B_i$, where B_i is a finite set. By DeMorgan's law we have:

$$A_1 \cap A_2 \cap \ldots \cap A_n = (X \setminus B_1) \cap (X \setminus B_2) \cap \ldots \cap (X \setminus B_n) = X \setminus (B_1 \cup B_2 \cup \ldots \cup B_n)$$

This is in $\mathcal{T}$ since the finite union of finite sets $B_1 \cup B_2 \cup \ldots \cup B_n$ is itself finite.

Union: Let $\{ A_\alpha : \alpha \in I \}$ be an arbitrary subcollection of $\mathcal{T}$. If each A_α is empty, then the union is the empty set, which is open by definition, so assume that A_β is a non-empty open set in this collection. So $A_\beta = X \setminus B_\beta$, where B_β is a finite set. Thus

$$\bigcup_{\alpha \in I} A_\alpha = A_\beta \cup (\bigcup_{\alpha \in I \setminus \{\beta\}} A_\alpha) = (X \setminus B_\beta) \cup (\bigcup_{\alpha \in I \setminus \{\beta\}} A_\alpha) \supseteq (X \setminus B_\beta)$$

or in a more palatable order,

$$X \setminus \bigcup_{\alpha \in I} A_\alpha \subseteq B_\beta$$

and since B_β is a finite set, we are finished.

$\square$

[**C.4**] Prove or disprove: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, then the image of any open set is open. (Here use the usual first-year calculus definition of continuous, real-valued function and open subset of $\mathbb{R}$.)

Answer for C.4. This is quite false. For example, take $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$, which we know is continuous from first-year calculus. We note that for $A = (-1, 1)$ (which is open in $\mathbb{R}$) we have $f[A] = [0, 1)$ which is not open in $\mathbb{R}$.

Another example could be any constant function. $\square$

[**C.5**] Let $X = \{ 0, 1, 2, 3, 4 \}$. What is the smallest size of a basis that generates the discrete topology on X? What can you say if $X = \{ 0, 1, 2, \ldots, n-1 \}$?

Solution for C.5. The smallest size of a basis that generates the discrete topology on $\{ 0, 1, 2, 3, 4 \}$ is 5. The basis $\mathcal{D} := \{ \{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \} \}$ shows you need *at most* 5 sets.

Claim: If $\mathcal{B}$ is a basis on X that generates the discrete topology, then $\mathcal{D} \subseteq \mathcal{B}$.

From a proposition in class, we see that for each $i \in X$, since $\mathcal{B}$ is a basis for the discrete topology, and $\{ i \}$ is an open set containing i, then there is a basic open $B \in \mathcal{B}$ so that $i \in B \subseteq \{ i \}$. The only possibility is that $\{ i \} = B \in \mathcal{B}$. So we have shown that $\mathcal{D} \subseteq \mathcal{B}$. $\square$

Application

[**A.1**] Let's go a step further than question C.1: Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, consider the family of sets in the range given by $\Gamma := \{\, f[A] : A \text{ is open in } \mathbb{R}_{\text{usual}} \,\}$.

- Give an example where f is not an injection, but Γ is a topology on the range of f.
- Give an example where f is not an injection, and Γ is not a topology on the range of f.

Answer for A.1.i. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the constant function $f(x) = 7$ for each $x \in \mathbb{R}$ (which is very much *not* an injection). It is clear that $\Gamma = \{\, \emptyset, \{\, 7 \,\} \,\}$ which is a discrete topology. $\square$

Answer for A.1.ii. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x$ if $x \neq 100$ and $f(100) = 0$. This is not an injection because both 0 and 100 get mapped to 0. We now observe that both $f[(-1, 1)] = (-1, 1) \in \Gamma$ and $f[(99, 101)] = \{\, 0 \,\} \cup (99, 100) \cup (100, 101) \in \Gamma$, but their intersection, the singleton $\{\, 0 \,\} \notin \Gamma$. So Γ is not a topology. $\square$

[**A.2**] Let V be an open subset of $\mathbb{R}$, with the usual topology. Show that there is a countable family of open intervals $\{\, I_n : n \in \mathbb{N} \,\}$, each with rational end-points such that $\bigcup_{n\in\mathbb{N}} I_n = V$. Can we instead do this by instead having each I_n be a closed interval? (If the answer is "Yes", then this shows that "every open subset of $\mathbb{R}$ is an F_σ set".)

Answer for A.2. First note that $\mathcal{B} := \{\, (p, q) : p, q \in \mathbb{Q} \,\}$ is a countable set (as there is an obvious injection into $\mathbb{Q} \times \mathbb{Q}$), and each subcollection of it is countable. So it is enough to find a subcollection $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{A} = V$.

For each $x \in V$, there is an open interval $(a, b) \subseteq V$ containing x. By the density of $\mathbb{Q}$ (see "Things you should know" section 5), we know that there are rational numbers p, q such that

$$a < p < x < q < b$$

and

$$x \in (p, q) \subseteq (a, b) \in \mathcal{B}$$

So let $A_x := (p, q) \in \mathcal{B}$, and let $\mathcal{A} := \{\, A_x : x \in V \,\}$. Clearly

$$V \subseteq \bigcup \mathcal{A} \subseteq V$$

so $V = \bigcup \mathcal{A}$.

For the "F_σ" part, we can do this by repeating the argument above, but using

$$\mathcal{B} := \{\, [p, q] : p, q \in \mathbb{Q} \,\}$$

$\square$

[**A.3**] Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{S}$ be a subbasis on X. Along the lines of the proposition we saw in lecture, state and prove a proposition that tells us when the topology generated by $\mathcal{S}$ is $\mathcal{T}$. Use this to prove that

$$\mathcal{S} := \{ (-\infty, b) : b \in \mathbb{R} \} \cup \{ (a, +\infty) : a \in \mathbb{R} \}$$

generates the usual topology on $\mathbb{R}$ (don't forget to check that this is a subbasis!). On your own, write down a "natural subbasis" for the Sorgenfrey line.

Proposition 1. *A subbasis $\mathcal{S}$ generates $\mathcal{T}$ if and only if $\mathcal{S} \subseteq \mathcal{T}$ and for every open set $U \in \mathcal{T}$ and for every $x \in U$ there is a finite subcollection $\{ S_1, S_2, \ldots, S_n \} \subseteq \mathcal{S}$ such that $x \in \bigcap_{i=1}^{n} S_i \subseteq U$.*

Proof. This follows immediately from the definition of a subbasis. $\square$

[The rest of this question is straightforward.]

[**A.4**] *Let's go a bit further than C.5.* Let $X = \{ 0, 1, 2, 3, 4 \}$. What is the smallest size of a *subbasis* that generates the discrete topology on X? Write a sentence or two explaining if it is easy to generalize this to $X = \{ 0, 1, 2, \ldots, n-1 \}$.

Solution for A.4. We can see that $\{ \{ 0, 1, 2 \}, \{ 0, 3, 4 \}, \{ 1, 3 \}, \{ 2, 4 \} \}$ is a subbasis with 4 elements that generates the discrete topology on $\{ 0, 1, 2, 3, 4 \}$.

Claim: If $\{ A, B, C \}$ is a collection of subsets of X, then it is not a subbasis for the discrete topology.

Note that there are only 4 possibly non-empty intersections we can make: $A \cap B, A \cap C, B \cap C$ and $A \cap B \cap C$. This shows that at least one of $\{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}$ or $\{ 4 \}$ is not represented.

In general, as we increase n, the number of intersections possible increases (exponentially), and we will not be able to extend our (simple) combinatorial argument. $\square$

New Ideas

These will only be sketches of solutions, and in the case of NI.3, the sketch of a solution that doesn't work.

[**NI.1**] Describe a topology $\mathcal{T}$ on $\mathbb{Z}$ such that:

- The set of all squares, $\mathbb{S}$ is open.
- For each $x \in \mathbb{Z}$ the set $\{x\}$ is not open.
- $\forall x, y \in \mathbb{Z}$ distinct, there is an open $U \ni x$ and an open $V \ni y$ such that $U \cap V = \emptyset$.

Note: The squares were a red herring; their algebraic structure had nothing to do with the problem. The only thing that matters here is that the set of squares is countable, infinite and co-infinite. The two other properties were included mostly to prevent you from taking the discrete topology, or a minor adjustment to the indiscrete topology.

Sketch 1 for NI.1. Adapt the proof of Furstenberg (that there are infinitely many primes), in two parts. First define a topology on $\mathbb{S} = \{ s_1, s_2, s_3, s_4, s_5, s_6, \dots \}$ by using the Furstenberg topology *on the indices*. That is, in the Furstenberg topology, sets of the form $N(m, b) := \{mx + b : x \in \mathbb{Z}\}$ are open sets. So now, define $S(m, b) := \{ s_{mx+b} : x \in \mathbb{Z} \}$. (This isn't *quite* right, because there is no s_{-10}, but it is easy to fix.) For example, here $S(10, 3) = \{ s_3, s_{13}, s_{23}, s_{33}, \dots \}$ is an open set.

Now do the same thing on $\mathbb{Z} \setminus \mathbb{S}$, to get a topology on $\mathbb{Z} \setminus \mathbb{S}$.

You now have a collection of open sets on Z. Generate a topology from them (by taking unions), and you will have the desired topology. It is straightforward to check that it has the desired properties. □

Sketch 2 for NI.1. You can find a sketch on Math Stack Exchange here:

http://math.stackexchange.com/a/401938

The idea is to fix a bijection $f : \mathbb{Q} \longrightarrow \mathbb{Z}$ which sends $(0, 1) \cap Q$ to the set of squares, and apply the result of C.1 to this function and $\mathbb{Q}$ with its induced topology. That is, a basis for the open sets in $\mathbb{Q}$ look like $(a, b) \cap \mathbb{Q}$, where $a, b \in \mathbb{Q}$. □

[**NI.2**] Write a computer program (in whatever language you like) or describe (in detail) an algorithm that counts the number of topologies on the set $\{ 0, 1, 2, 3, 4 \}$. Is there any sense in which this can be done efficiently?

A solution for NI.2. Here is a compact bit of Python code that runs in around 2 seconds for n=5, from Ivan Khatchatourian (a TA from last year). Here are his comments:

Turns out being able to treat numbers as bit strings is super useful.

Sets of elements from $X = \{ 1, 2, 3, 4, 5 \}$ are coded by bit strings which are in turn coded as numbers, all of which is automatic in Python. So 15 (in decimal) = 01111 (in binary) = $\{ 1, 2, 3, 4 \}$ (as a set [I usually imagine the smaller binary placeholders corresponding to earlier elements in the set). Another example is $19 = 10011 = \{ 1, 2, 5 \}$. I'm sure you see by now. So then we can do stuff like:

$$15\&19 = 01111\&10011 = \{ 1, 2, 3, 4 \} \cap \{ 1, 2, 5 \} = \{ 1, 2 \}$$

and

$$15|19 = 01111|10011 = \{ 1, 2, 3, 4 \} \cup \{ 1, 2, 5 \} = X$$

etc.

(Also Python has "set" objects which also work nicely. "-" is set difference, etc.)

With that in place, there's a natural enumeration of all the subsets of X (which I guess is more or less a lexicographic order). The function **S(i, T)** below counts the number of topologies you can make by starting with the collection T (of subsets of X) and adding sets enumerated above and including i.

Here's the code:

```
N=5;
def Num(i, T):
    check = 0;
    for j in range(i, 2**N):
        if len(set(j&u for u in T)-T)-1 == 0:
            check += Num(j+1, T|set(j|u for u in T))
    return 1 + check;
print Num(1, set([0, 2**N - 1]));
```

The clause on the if statement there is fulfilled when j is the only new thing added by taking the intersections of j with things in T (j is always added, since the whole space X, represented by $2^N - 1$, is always in T). If that's true, we add to T the unions of j with everything in T (this adds j itself, since the empty set, represented by 0, is always in T) and start again with j+1. And so on. As soon as we can't add anything without breaking stuff, we go back to the last thing we could add without breaking stuff and try adding the next thing. □

[**NI.3**] Here's a game that two players could play: Player 1 chooses an uncountable subset $K_1 \subseteq \mathbb{R}$, then Player 2 chooses an uncountable $K_2 \subseteq K_1$. They continue alternating until they have an infinite chain $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq ...$, and we'll say that Player 2 wins iff $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$. Show that Player 2 has a strategy so that she can win, no matter what moves Player 1 makes.

Sketch for NI.3. The main purpose of this exercise was for the student to find the (most) straightforward solution, then show that it doesn't work.

Here is that method.

Player 1 picks their uncountable set K_1. Write

$$\mathbb{R} = \bigcup_{z \in \mathbb{Z}} [z, z+1)$$

By the Uncountable Pigeonhole principle, one of these intervals must contain uncountably many elements from K_1. Without loss of generality, let $[0, 1)$ be that interval. Player 2 then chooses $K_2 := K_1 \cap [0, 1)$.

Player 1 picks an uncountable $K_3 \subseteq K_2$. Now one of $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$ must contain uncountably many elements from K_3. Let $K_4 := K_3 \cap$ "that interval".

Continue on in this way, always splitting the previous interval into halves and having Player 2 pick a half containing uncountably much of the previous set.

It is clear that $\bigcap_{n\in\mathbb{N}} K_n$ contains *at most* one element. Since $\bigcap_{n\in\mathbb{N}}(0, \frac{1}{2^n}) = \emptyset$, we might be lead to believe that the intersection is *always* empty. Unfortunately (for Player 2), it may be that the intersection is non-empty. For example, Player 1 can always leave an uncountable interval around $\frac{1}{3}$, so that Player 2 always chooses the "half" that contains $\frac{1}{3}$. Then player 2 will not win.

There is indeed a solution to this problem that shows Player 2 can force an empty intersection. And we will see it later in the course. $\square$