STA447 Homework 2

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Problem 1

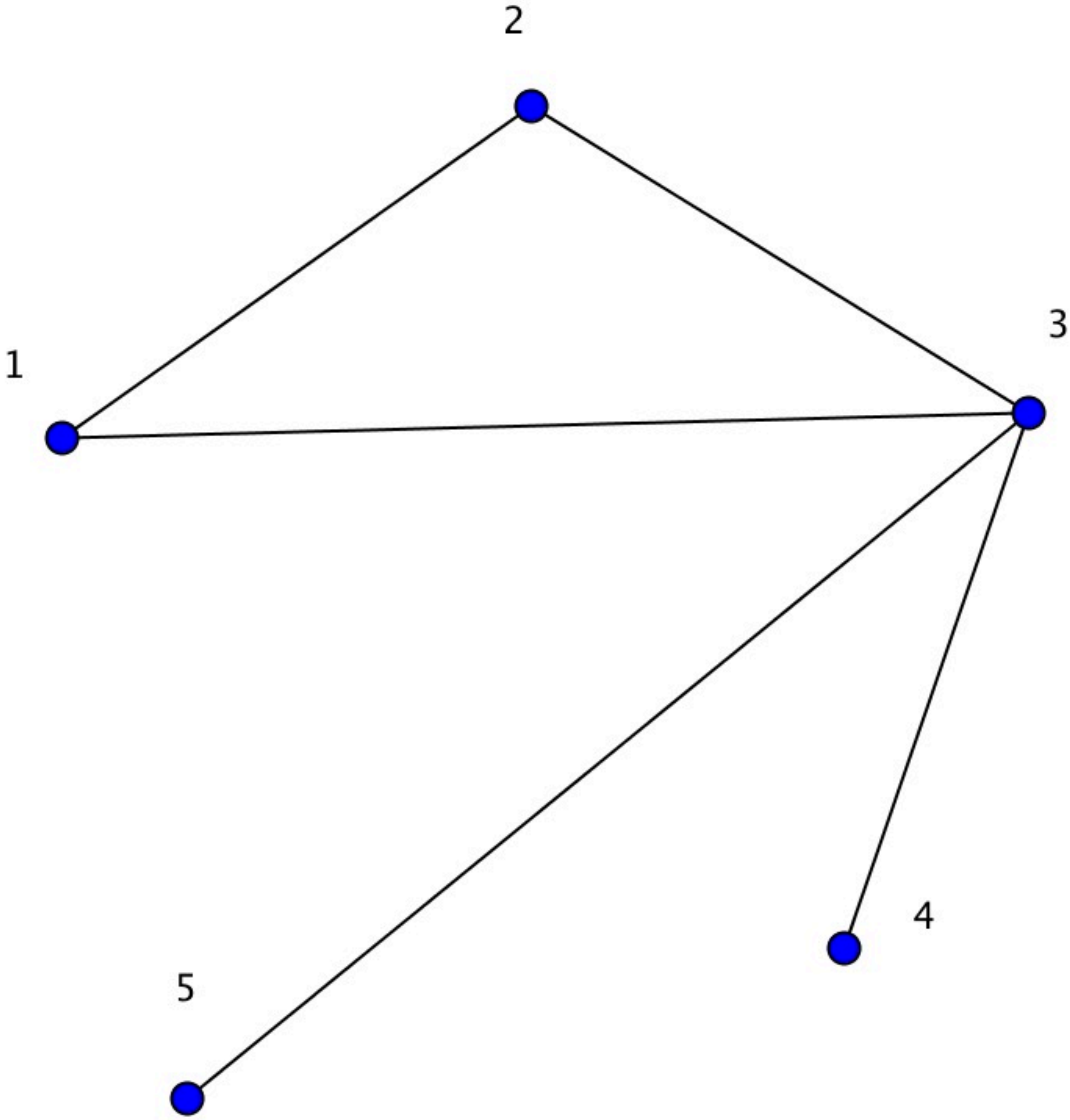
- (a)
    - Tried to open that link in Chrome but it showed that Chrome doesn't support Java applet since a certain version.
    - Then switched to Safari, and updated my Java version to the latest.
    - Successfully opened the link in Safari, but the section (which is supposed to be the applet is blank, and unable to click on).
    - Turned to Java control panel, tab security, white-list the link.
    - Also in "Preferences-Security-Java" settings, probability.ca is already white-listed as well.
    - Still cannot open the applet. ¯_(ツ)_/¯
- (b) **Metropolis algorithm:** In general, Metropolis algorithm is a Markov Chain Monte Carlo method to obtain a sequence of random samples from a probability distribution indirectly. The more iterations this algorithm runs, the more closely it approximates the exact distribution. In details, for each iteration, the algorithm picks a candidate for the next sample value depending on the current sample value. Then at certain probability, we take the candidate as next sample value. Otherwise, we take the use current value again as the next value. Such probability is determined by the comparison between density function values of the current and candidate (values) with respect to our targeted distribution.

Problem 2

- (a) See Problem part (a).
- (b) **Gambler's Ruin problem:** The basic setup of Gambler's Ruin is that suppose we have oen gambler with certain amount of starting stake a and winning probability p. He wins or loses 1 dollar each time. And we are interested in the probability of this gambler having stake c ($c > a$) before going broken. Then for each number a between $(0, c)$, we can set up a equation with unknowns. In total we can set up $c - 1$ equations but only with $c - 1$ unknowns, which are solvable. Then we can express the desirable probability with all known variables.

Problem 3

(a) Solution:



(b) Solution:
By theorem on notes: for a random walk on a connected non-bipartite graph, if $Z < \infty$, then $\lim_{n\to\infty} p_{uv}^{(n)} = \pi_v = \frac{d(v)}{Z}$ for all $u, v \in V$.

- For our graph, it is **connected** since there is always a path between any pair of two vertices.
- It is also **non-bipartite** because vertices 1,2,3 form an odd cycle, which is impossible to appear in a bipartite graph.
- Therefore, our theorem can be applied.

$$\lim_{n\to\infty} P[X_n = 3] = \frac{d(3)}{Z} = \frac{\sum_{v\in V} w(3,v)}{\sum_{u\in V} d(u)} = \frac{\sum_{v\in V} w(3,v)}{\sum_{u,v\in V} w(u,v)} = \frac{4}{10} = \frac{2}{5}$$

Problem 4

Solution:
Apply Gambler's Ruin formula:

$$\begin{aligned}
P_a(T_c < T_0) &= \begin{cases} \frac{((1-p)/p)^a - 1}{((1-p)/p)^c - 1} & , p \neq \frac{1}{2}, \\ \frac{1}{c} & , p = \frac{1}{2}. \end{cases} \\
P_5(T_c < T_0) &= \frac{(\frac{1-2/3}{2/3})^5 - 1}{(\frac{1-2/3}{2/3})^c - 1} = \frac{\frac{1}{32} - 1}{\frac{1}{2^c} - 1} \\
\therefore P_5(T_0 < \infty) &= \lim_{n\to\infty} P_5(T_0 < T_c) \\
&= \lim_{n\to\infty} \left[1 - \frac{1 - \frac{1}{32}}{1 - \frac{1}{2^c}} \right] \\
&= 1 - 1 + \frac{1}{32} \\
&= \frac{1}{32}
\end{aligned}$$

Note that since $c \to \infty, T_c = \inf\{n \geq 1 : X_n = c\} \to \infty$ as well.

Problem 5 (Special case of the *Gibbs Sampler*)

(a) Proof:
According to the transition probability given, we can divide elements in S into 4 different groups:

- element (x, y) which is equal to (z, w).
- elements (x, y) with the same x-coordinate as z, but $y \neq w$.
- elements (x, y) with the same y-coordinate as w, but $x \neq z$.
- other elements.

Also note that (z, w) is fixed, (x, y) is a variable.

$$\begin{aligned}
\sum_{(z,w)\in S} p_{(x,y),(z,w)} &= \frac{f(z,w)}{2C(z)} + \frac{f(z,w)}{2R(w)} \\
&+ \sum_{x=z,y\neq w} \frac{f(z,w)}{2C(z)} + \sum_{x\neq z,y=w} \frac{f(z,w)}{2R(w)} \\
&= \sum_{w\in\mathbb{Z}} \frac{f(z,w)}{2C(z)} + \sum_{z\in\mathbb{Z}} \frac{f(z,w)}{2R(w)} \\
&= \frac{C(z)}{2C(z)} + \frac{R(w)}{2R(w)} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

(b) Proof:
We want to show the chain is reversible w.r.t. a probability distribution π, this is equal to show that for any two elements $(x, y), (z, w) \in S$:

$$\pi_{(x,y)} p_{(x,y),(z,w)} = \pi_{(z,w)} p_{(z,w),(x,y)}$$

- If $x = z, y = w$, this is trivial since both sides are identical.
- If $x \neq z, y \neq w$, it's trivial too, since $p_{(x,y),(z,w)} = p_{(z,w),(x,y)} = 0$.
- If $x = z, y \neq w$.

$$
\begin{aligned}
LHS &= \pi_{(x,y)} p_{(x,y),(z,w)} \\
&= \frac{f(x,y)}{K} \frac{f(z,w)}{2C(x)} \\
RHS &= \pi_{(z,w)} p_{(z,w),(x,y)} \\
&= \frac{f(z,w)}{K} \frac{f(x,y)}{2C(x)} \\
\therefore LHS &= RHS.
\end{aligned}
$$

Similarly, if $x \neq z, y = w$.

$$
\begin{aligned}
LHS &= \pi_{(x,y)} p_{(x,y),(z,w)} \\
&= \frac{f(x,y)}{K} \frac{f(z,w)}{2R(y)} \\
RHS &= \pi_{(z,w)} p_{(z,w),(x,y)} \\
&= \frac{f(z,w)}{K} \frac{f(x,y)}{2R(y)} \\
\therefore LHS &= RHS.
\end{aligned}
$$

(c) Solution:
First we need some recaps about this state space $S = \mathbb{Z} \times \mathbb{Z}$. If we view S on a 2-dimensional x-y plot, then S contains all the points with integer coordinates. And we also know that the transition probabilityies between one point and all the other points on the same horizontal line and vertical line are **nonzero**.

Therefore, for any two points (elements), say $(x, y), (z, w)$ with $x \neq z, y \neq w.$, we always have $p_{(x,y),(z,w)} = 0$ but $p_{(x,y),(x,w)} > 0$ and $p_{(x,w),(z,w)} > 0$, so $p_{(x,y),(x,w)} p_{(x,w),(z,w)} > 0.$

Hence, the chain is **irreducible.**

The period of each state is 1, because $p_{(x,y),(x,y)} > 0$, so the chain is also **aperiodic.**

In addition, by proposition:

> If a Markov chain is reversible (proved in part (b)) with respect to $\{\pi_{(x,y)}\}$, then $\{\pi_{(x,y)}\}$ is **stationary**.

Then by **Markov chain convergence theorem**,

$$
\lim_{n\to\infty} p^{(n)}_{(z,w),(x,y)} = \pi_{(x,y)} = \frac{f(x,y)}{K} = \frac{f(x,y)}{\sum_{(x,y)\in S} f(x,y)}
$$

Problem 6

(a) Solution:
Suppose S is the state space. We want $\{X_n\}$ to be a martingale. Since $\{X_n\}$ is a Markov chain, and $E|X_n| < \infty$. Then a martingale "on average stays the same":

$$E(X_{n+1}|X_0, \dots, X_n) = X_n$$
$$E(X_1|X_0) = X_0$$
$$5 + (-1)P\left[Z_1 = -1\right] + C \cdot P\left[Z_1 = C\right] = 5 - \frac{3}{4} + \frac{C}{4} = 5$$
$$\therefore C = 3.$$

(b) Solution:
Because we proved $\{X_n\}$ is a martingale, then we have this "double-expectation formula":

$$E(X_{n+1}) = E\left[E\left(X_{n+1}|X_0, \dots, X_n\right)\right] = E(X_n),$$

Then $E(X_0) = E(X_1) = \cdots = E(X_9) = 5.$

(c) Solution:
Note that $T = \inf\{n \geq 1 : X_n = 0 \text{ or } Z_n > 0.\}$ is a random time.
T is **bounded**, i.e. $\exists M = 5 < \infty \text{ with } P(T \leq M) = 1.$ This is because once $Z_i = C = 3 > 0$ then it stops. Otherwise, we keep having $Z_i = -1$, but starting from $X_0 = 5$, we at most have 5 chances then $X_5 = 0$, then it stops anyway.

Therefore, by **optional stopping lemma**, $E(X_T) = E(X_0) = 5.$

Problem 7

(a) Solution:
$X_2 = i$ could be at 0, 2 or 4.

$$
\begin{aligned}
P(X_2 = 0) &= P(X_2 = 0|X_1 = 1, X_0 = 2) \\
&= P(X_1 = 1|X_0 = 2)P(X_2 = 0|X_1 = 1) \\
&= p_{21}p_{10} \\
&= (1 - \frac{2}{3}) \cdot (1 - \frac{2}{3}) \\
&= \frac{1}{9} \\
P(X_2 = 2) &= P(X_2 = 2|X_1 = 1, X_0 = 2) + P(X_2 = 2|X_1 = 3, X_0 = 2) \\
&= P(X_1 = 1|X_0 = 2)P(X_2 = 2|X_1 = 1) + P(X_1 = 3|X_0 = 2)P(X_2 = 2|X_1 = 3) \\
&= p_{21}p_{12} + p_{23}p_{32} \\
&= \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} \\
&= \frac{4}{9} \\
P(X_2 = 4) &= P(X_2 = 4|X_1 = 3, X_0 = 2) \\
&= P(X_1 = 3|X_0 = 2)P(X_2 = 4|X_1 = 3) \\
&= p_{23}p_{34} \\
&= \frac{2}{3} \cdot \frac{2}{3} \\
&= \frac{4}{9}
\end{aligned}
$$

Hence,

$$
P(X_2 = i) = \begin{cases} \frac{1}{9}, & i = 0, \\ \frac{4}{9}, & i = 2, 4, \\ 0, & \text{otherwise.} \end{cases}
$$

(b) Solution:

$$
\begin{aligned}
E(X_2) &= \sum_i iP(X_2 = i) \\
&= 0 + 2 \cdot P(X_2 = 2) + 4 \cdot P(X_2 = 4) \\
&= 2 \cdot \frac{4}{9} + 4 \cdot \frac{4}{9} \\
&= \frac{8}{3}.
\end{aligned}
$$

This is the same as the result in part (b).

(c) Solution:

By **Wald's Theorem**, here we have $a = X_0 = 2, m = E(Z_i) = 1 \cdot p + (-1) \cdot (1 - p) = \frac{1}{3}$, $T = 2$.

Therefore, $E(X_2) = a + mE(T) = 2 + \frac{1}{3} \cdot 2 = \frac{8}{3}$.

Homework 2 part 3

Problem 8

(a) Solution:
Since we have 2 parents, each individual could have 0, 1, or 2 children.

- $X_1 = Z_{0,1} + Z_{0,2}$, where $Z_{0,1} = 0, 1, 2, Z_{0,2} = 0, 1, 2,$ so $X_1 = 0, 1, 2, 3, 4$
  - $P(X_1 = 0) = \mu\{0\}^2 = \frac{1}{4}$
  - $P(X_1 = 1) = 2\mu\{0\}\mu\{1\} = \frac{1}{3}$
  - $P(X_1 = 2) = 2\mu\{0\}\mu\{2\} + \mu\{1\}^2 = \frac{5}{18}$
  - $P(X_1 = 3) = 2\mu\{1\}\mu\{2\} = \frac{1}{9}$
  - $P(X_1 = 4) = \mu\{2\}^2 = \frac{1}{36}$
  - $P(X_1 \geq 5) = 0$
  - $P(X_1 = 0) + P(X_1 = 1) + P(X_1 = 2) + P(X_1 = 3) + P(X_1 = 4) = 1$

(b) Solution:

- $X_2 = \sum_{i=1}^{X_1} Z_{1,i}$
- $X_1 = 0, 1, 2, 3, 4$
- If $X_1 = 0, X_2 = 0.$
- If $X_1 = 1, X_2 = Z_{1,1} = 0, 1, 2.$
- If $X_1 = 2, X_2 = Z_{1,1} + Z_{1,2} = 0, 1, 2, 3, 4.$
- If $X_1 = 3, X_2 = Z_{1,1} + Z_{1,2} + Z_{1,3} = 0, 1, 2, 3, 4, 5, 6.$
- If $X_1 = 4, X_2 = Z_{1,1} + Z_{1,2} + Z_{1,3} + Z_{1,4} = 0, 1, 2, 3, 4, 5, 6, 7, 8.$

So X_2 could be 0, 1, 2, 3, 4, 5, 6, 7, 8.
Then let $\mu_0 = \mu\{0\} = \frac{1}{2}$, $\mu_1 = \mu\{1\} = \frac{1}{3}$, $\mu_2 = \mu\{2\} = \frac{1}{6}$.

$$
\begin{aligned}
P(X_2 = 0) &= P(X_1 = 0) + P(X_1 = 1)\mu_0 + P(X_1 = 2)\mu_0^2 + P(X_1 = 3)\mu_0^3 + P(X_1 = 4)\mu_0^4 \\
&= \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{2} + \frac{5}{18} \cdot \frac{1}{4} + \frac{1}{9} \cdot \frac{1}{8} + \frac{1}{36} \cdot \frac{1}{16} \\
&= \frac{289}{576} \\
P(X_2 = 1) &= P(X_1 = 1)\mu_1 + P(X_1 = 2)2\mu_0\mu_1 + P(X_1 = 3)3\mu_0^2\mu_1 + P(X_1 = 4)4\mu_0^3\mu_1 \\
&= \frac{1}{3} \cdot \frac{1}{3} + \frac{5}{18} \cdot 2 \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{9} \cdot 3 \cdot \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{36} \cdot 4 \cdot \frac{1}{8} \cdot \frac{1}{3} \\
&= \frac{17}{72} \\
P(X_2 = 2) &= P(X_1 = 1)\mu_2 + P(X_1 = 2)\left[2\mu_0\mu_2 + \mu_1^2\right] \\
&+ P(X_1 = 3)\left[3\mu_0\mu_1^2 + 3\mu_0^2\mu_2\right] \\
&+ P(X_1 = 4)\left[6\mu_0^2\mu_1^2 + 4\mu_0^3\mu_2\right] \\
&= \frac{1}{3} \cdot \frac{1}{6} + \frac{5}{18} \cdot \left(2 \cdot \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{9}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{}{9} \cdot \left(3 \cdot \frac{}{2} \cdot \frac{}{9} + 3 \cdot \frac{}{4} \cdot \frac{}{6} \right) + \frac{}{36} \left(6 \cdot \frac{}{4} \cdot \frac{}{9} + 4 \cdot \frac{}{8} \cdot \frac{}{6} \right) \\
&= \frac{223}{1296} \\
P(X_2 = 3) &= P(X_1 = 2) \left[2\mu_1\mu_2 \right] \\
&+ P(X_1 = 3) \left[\mu_1^3 + 6\mu_0\mu_1\mu_2 \right] \\
&+ P(X_1 = 4) \left[4\mu_0\mu_1^3 + 12\mu_0^2\mu_1\mu_2 \right] \\
&= \frac{5}{18} \cdot \left(2 \cdot \frac{1}{3} \cdot \frac{1}{6} \right) + \frac{1}{9} \cdot \left(\frac{1}{27} + 6 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6} \right) + \frac{1}{36} \left(4 \cdot \frac{1}{2} \cdot \frac{1}{27} + 12 \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{6} \right) \\
&= \frac{13}{216} \\
P(X_2 = 4) &= P(X_1 = 2)\mu_2^2 \\
&+ P(X_1 = 3) \left[3\mu_1^2\mu_2 + 3\mu_0\mu_2^2 \right] \\
&+ P(X_1 = 4) \left[\mu_1^4 + 6\mu_0^2\mu_2^2 + 12\mu_0\mu_1^2\mu_2 \right] \\
&= \frac{5}{18} \cdot \frac{1}{36} + \frac{1}{9} \left(3 \cdot \frac{1}{9} \cdot \frac{1}{6} + 3 \cdot \frac{1}{2} \cdot \frac{1}{36} \right) + \frac{1}{36} \left(\frac{1}{81} + 6 \cdot \frac{1}{4} \cdot \frac{1}{36} + 12 \cdot \frac{1}{2} \cdot \frac{1}{9} \cdot \frac{1}{6} \right) \\
&= \frac{539}{23328} \\
P(X_2 = 5) &= P(X_1 = 3) \left[3\mu_1\mu_2^2 \right] \\
&+ P(X_1 = 4) \left[4\mu_1^3\mu_2 + 12\mu_0\mu_1\mu_2^2 \right] \\
&= \frac{1}{9} \left(3 \cdot \frac{1}{3} \cdot \frac{1}{36} \right) + \frac{1}{36} \left(4 \cdot \frac{1}{27} \cdot \frac{1}{6} + 12 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{36} \right) \\
&= \frac{21}{5832} \\
P(X_2 = 6) &= P(X_1 = 3) \left[\mu_2^3 \right] \\
&+ P(X_1 = 4) \left[4\mu_0\mu_2^3 + 6\mu_1^2\mu_2^2 \right] \\
&= \frac{1}{9} \cdot \frac{1}{216} + \frac{1}{36} \left(4 \cdot \frac{1}{2} \cdot \frac{1}{216} + 6 \cdot \frac{1}{9} \cdot \frac{1}{36} \right) \\
&= \frac{5}{3888} \\
P(X_2 = 7) &= P(X_1 = 4) \left[4\mu_1\mu_2^3 \right] \\
&= \frac{1}{36} \cdot 4 \cdot \frac{1}{3} \cdot \frac{1}{216} \\
&= \frac{1}{5832} \\
P(X_2 = 8) &= P(X_1 = 4)\mu_2^4 = \frac{1}{46656} \\
P((X_2 \geq 9) &= 0
\end{aligned}
$$

Checked that $\sum_i P(X_2 = i) = 1$.