Notes 9.1

tax < Income tax / capital gain tax $\Rightarrow$ Find net yield.

① $P = F \cdot r \cdot a_{\overline{n}|j} + C\, v_j^n$.

② $P = F \cdot r(1-t_I)\, a_{\overline{n}|j} + C\, v_j^n$.

③ $\boxed{P'} = Fr \cdot (1-t_I)\, a_{\overline{n}|j} + C\, v_j^n - t_c (C-P') \cdot v_j^n$.

$\boxed{P'} = Fr(1-t_I) \cdot a_{\overline{n}|j} + [C - t_c(C-P')]\, v_j^n$.

$P' = \boxed{}$

Ex. i.e. $P > F = C \Leftrightarrow j < r(1-t_I)$. ×

$F = 1000 = C$.

$n = 10 \times 2 = 20$

$r = \frac{6\%}{2} = 3\%$

$P = 800$.

$\boxed{t_I} = 40\%$

$\boxed{t_c} = 30\%$.

$\Rightarrow$ net yield.

Apply 3 :

$$800 = \cancel{P} = 1000 \cdot 3\% \cdot (1-40\%) \cdot a_{\overline{20|}j} + [1000 - 30\%(1000 - \overset{800}{\cancel{P}})] v_j^{20}$$

$$\boxed{800 = 18 \cdot a_{\overline{20|}j} + 940 \cdot v_j^{20}} \Rightarrow j ?$$

$P = 800$

$Fr = 18$

$n = 20$

$C = 940 = F$

$$\Rightarrow r = \frac{18}{940} = 0.019$$

$$P < F \iff \boxed{j > r = 0.019}$$

if $j = 2\% \Rightarrow P = 926.92 > 800$.

if $j = 3\% \Rightarrow P = 787.6 < 800$.

787.6	800	926.92
3%	j	2%

$$\Rightarrow j \cong 3\% + \frac{800 - 787.6}{926.92 - 787.6} \times (2\% - 3\%)$$

$$= 2.9\% \quad \Rightarrow i = (1+j)^2 - 1 = 5.9\%$$

Callable Bonds

at the option of the borrower.

Ex: $F = 1000 = C$.

* $\boxed{n = 24, 25, \ldots 30.}$ (year: 12, 13, .. 15)

$r = \frac{10\%}{2} = 5\%$

$\Rightarrow P < C$ (capital gain).

$j = 12\% / 2 = 6\%$

$\Rightarrow P =$

Sol: worst situation for an investor.

(best situation for a borrower).

at $12\%/2 = 6\%$.

$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n$

$= 1000 \cdot 0.05 \cdot a_{\overline{n}|0.06} + 1000 \cdot v_{0.06}^n$

$n = 24$, $P = 874.5$ ✓

$n = 25$, $P = 872.2$

⋮

$n = 29$, $P = 864.1$

$n = 30$, $P = 862.4$ → worst situation.

$P \downarrow$, $j \uparrow$

$P \uparrow$, $j \downarrow$

① If. $P = 874.5$ $(n=24)$.

if borrower redeemed the bond at $\boxed{n=25}$.

$874.5 = 1000 \cdot 0.05 \cdot a_{\overline{25}|j} + 1000 \cdot v_j^{25}$.

$\Rightarrow j = 11.96\% < 12\%$

If $P = 864.1$ $(n=29)$

redeemed at $n = 30$

$864.1 = 1000 \cdot 0.05 \cdot a_{\overline{30}|j} + 1000 \cdot v_j^{30}$

$\Rightarrow j = 11.97\% < 12\%$

$n=29$

$j = 12\%$ $P = \cancel{864.1}$

$\downarrow$ $\downarrow$

862.4

$j \geq 12\%$

The bond is redeemed at the option of the issuer.

① if $P < C$, Investor minimum yield.
$\Rightarrow$ latest redemption date.

② if $P > C$,
$\Rightarrow$ earliest redemption date.

More General Rule

$\boxed{\text{at the option of the issuer, An investor requires a minimum yield, then choose the lowest price for all possible redemption dates at this yield rate}}$

Ex: Callable bond (issuer)

$r = \frac{8\%}{2} = 4\%$

$t_2 = 25\%$

$i^{(2)} = 7\% \Rightarrow j = \frac{7\%}{2} = 3.5\%$

$F = 100 = C$

$n = (20, 21, \cdots 30)$; $(10 \to 15\text{yrs})$

$\Rightarrow P?$

Sol: $P < F = C \iff \underset{\underset{3.5\%}{\parallel}}{j} > \underset{\underset{3\%}{\parallel}}{\underline{r(1-t_2)}}$

$\Rightarrow P < F = C$, capital gain

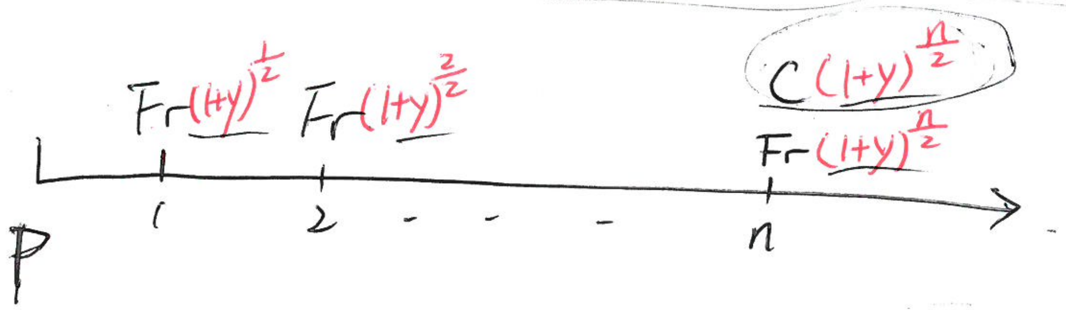
$\Rightarrow n = 30$

$P = Fr(1-t_2) \cdot a_{\overline{n}|j} + C \cdot v_j^n$

$$P = 1000\cdot 0.04\cdot(1-25\%)\cdot a_{\overline{30}|0.035} + 1000\cdot v_{0.035}^{30}$$

$$= \$90.80$$

Bonds with Inflation-Linked Payments.



price inflation rate $= y$ p.a.
effective interest rate $= i$ p.a.

$$P = \sum_{p=1}^{n} Fr\cdot(1+y)^{\frac{p}{2}}\cdot v_i^{p/2} + C\cdot(1+y)^{\frac{n}{2}}\cdot v_i^{n/2}.$$

$$= \sum_{p=1}^{n} Fr\cdot\left(\frac{1+y}{1+i}\right)^{p/2} + C\cdot\left(\frac{1+y}{1+i}\right)^{n/2}$$

Assume $\frac{1+y}{1+i} = \frac{1}{1+i'} \Rightarrow i' = \frac{i-y}{1+y}$.

real interest rate

$$\boxed{P = \sum_{p=1}^{n} Fr\cdot v_{i'}^{p/2} + C\cdot v_{i'}^{p/2}}$$

$$j' = (1+i')^{\frac{1}{2}} - 1$$

$\Updownarrow$

$a^{(2)}_{\overline{n/2}|i'}$

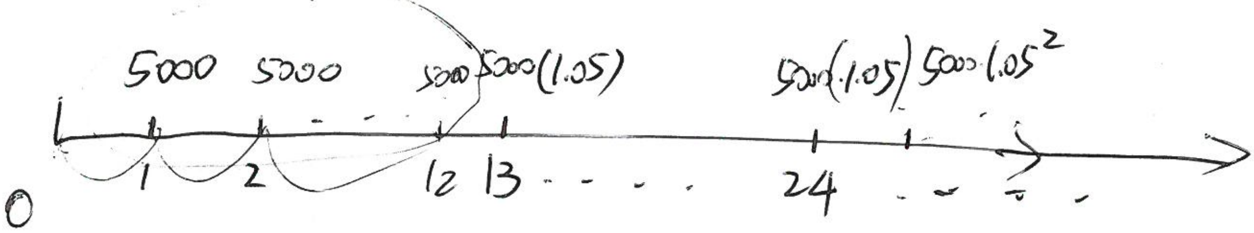
$$P = Fr\cdot a_{\overline{n}|j'} + C\cdot v_{j'}^{n}$$
$$P = 2Fr\cdot a^{(2)}_{\overline{n/2}|i'} + C\cdot v_{i'}^{n/2}$$

Shares & properties

Shares PV(Divs)

Properties PV(rent incomes)

Ex



$$P = 5000\left(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \cdots + v^{\frac{12}{12}}\right) + 5000(1.05)\left(v^{\frac{13}{12}} + v^{\frac{14}{12}} + \cdots v^{\frac{24}{12}}\right)$$

$$+ \cdots$$

$$= 5000\left(v^{\frac{1}{12}} + {}^{\frac{2}{12}} + \cdots + v^{\frac{12}{12}}\right) \cdot \left(1 + 1.05 \cdot v + 1.05^2 \cdot v^2 + \cdots\right)$$

$$= 5000 \quad 12 \cdot a^{(12)}_{\overline{1|}7\%} \cdot \left(1 + \frac{1.05}{1.07} + \left(\frac{1.05}{1.07}\right)^2 + \cdots\right)$$

$$= 5000 \cdot 12 \cdot a^{(12)}_{\overline{1|}7\%} \cdot \ddot{a}_{\overline{\infty|}i'}$$

$$= 3.095\ m$$

assume $\frac{1}{1+i'} = \frac{1.05}{1.07}$

$$\Downarrow$$

$$i' = \frac{1.07}{1.05} - 1$$

week 9

Forward contract.

Arbitrage.

Immediate profit — future profit.

no risk of loss *

Assume No arbitrage → For pricing

⇓

Law of one price → replicating payoffs.

Ex:

Securities	$t=0$ P_0	$t=1$ $P_1(u)$	$t=1$ $P_1(d)$
A	6	7	5
B	11	14	10.

at $t=0$, borrow 2 A, sell 2 A, buy 1 B

$6\times2-11=\$1$

at $t=1$, buy 2A, sell 1 B: up: $-7\times2+14=0$; down: $-5\times2+10=0$

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$$\frac{P_0^A}{P_0^B} = \frac{P_1^A}{P_1^B} = \frac{1}{2}$$

Forward Contract.

$t=0$ $\qquad$ $t=T$

agreement $\qquad$ $\boxed{K}$?

S

S_0.

S_T.

T

K. forward price.

δ risk-free force of interest

Ex: 1000 shares.

$S_0 = \$10.5$ per share.

$S_T = \$10.7$ per share

$T = \frac{1}{2}$ yr.

$K = \$11$ per share.

$t=0$

$S_0 \times 1000 = \$10{,}500$

$t=T$

$1000K = \$11{,}000$

$1000 S_T = \$10{,}700$

buyer : 300 loss at time T
seller : 300 profit at time T.

Find K?

① Securies with no income.

② - - with incomes.

① : ~~Portfolio~~ Portfolio A : * long forward contract to buy one unit S.

* $K \cdot e^{-\delta T}$

portfolio B : Buy one unit of asset S at S_0.