MAT 237, Quiz 3 Name ______________ **TUT0101**

Part A: (2 marks) Present the definition of differentiability for the function $f(x)$ at the point a. Make sure your definition is in terms of m and $E(h)$.

f is diff at a if exists a number m and a function E(h) s.t. (0.5)

f(a+h) can be written as $f(a+h) = f(a) + mh + E(h)$ and $\lim_{h \to 0} \frac{E(h)}{h} = 0$

(1) (0.5)

Part B: (5 marks) Use your definition in part (A) to show that the function $f(x) = 3x^2 - \sin x$ is differentiable at a given point a.

(1) $f(a+h) = 3(a+h)^2 - \sin(a+h) = 3(a^2+2ah+h^2) - \sin a \cos h + \cos a \sin h$

$= 3a^2 + 6ah + 3h^2 - \sin a \cos h + \cos a \sin h \underbrace{- \cos a\, h + \cos a\, h}_{\text{add \& Subtract}}$

$\underbrace{-\sin a \quad + \sin a}$

(2) $= \underbrace{3a^2 - \sin a}_{f(a)} + (\underbrace{6a - \cos a}_{m})h + \underbrace{\sin a - \sin a \cos h + \cos a \sin h + \cos a\, h}_{E(h)}$

(2) note $\frac{E(h)}{h} = \frac{\sin a(1-\cos h)}{h} + \cos a \frac{h - \sin h}{h} = 0 + 0$ b/c $\frac{1-\cos h}{h} \to 0$ & $\frac{h - \sin h}{h} \to 0$

Part C: (3 marks) Use the definition in part A to prove that that if f is differentiable at the point a then f is continuous at a.

if f is diff at a Then exists m and E(h) s.t.

(1.5) $f(a+h) - f(a) = mh + E(h)$, so as $h \to 0$ $mh \to 0$ and $E(h) \to 0$ (b/c $\frac{E(h)}{h} \to 0$)

Then $f(a+h) - f(a) \to 0$. so f is Cont at a.

(1.5)

Part A: (3 marks) Present the definition of a disconnection for a set S. Also present definition of I is an interval in $\mathbb{R}$.

a dis connection for S is a pair of Sets (S_1, S_2) s.t. $S_1 \neq \emptyset \neq S_2$,
and $S_1 \cup S_2 = S$ and $\overline{S_1} \cap S_2 = \emptyset = S_1 \cap \overline{S_2}$. (2)

I is an interval if $\forall a, b \in I$, $\forall c \in \mathbb{R}$ $a < c < b \Rightarrow c \in I$. (1)

Part B: (2 marks) (this seems to be obvious, but there is a very easy argument using transitivity of real numbers that still needs to be written down.) Use your definition in part (A) to show that the set $S = \{x : 1 \leq x \leq 2\}$ is an interval.

given $a, b \in S$, such That $1 \leq a < b \leq 2$, $\forall c \in \mathbb{R}$ if

$a < c < b$ Then $1 \leq a < c \Rightarrow 1 < c$
$c < b \leq 2 \Rightarrow c < 2$ Then $1 < c < 2$, so $c \in S$ (1)

(1)

Part C: (5 marks) Prove that the interval $[a, b]$ cannot have a disconnection (S_1, S_2).

assume for a Contradiction, That (S_1, S_2) is a disconnection for $[a b]$.
also assume w.l.o.g. That $a \in S_1$, $b \in S_2$. So $S_1 \neq \emptyset$ and bdd above
(2) by b b/c $b \notin S_1$ and $S_1 \subset [a b]$. by Completeness axiom let $c = \text{lub}\, S_1$.
Claim: $a < c < b$ but $c \notin [a b]$ (and This will be a Contradiction with The assumption That $[a b]$ is an interval.)
pf of Claim: $c = \text{lub} S_1 \Rightarrow c \in \overline{S_1} \Rightarrow c \notin S_2$ b/c $\overline{S_1} \cap S = \emptyset$.
(3) Since $c = \text{lub}\, S_1$ Then $(c, b] \subset S_2$ so $c \in \overline{S_2} \Rightarrow c \notin S_1$ so
$c \notin S_1 \cup S_2 = [a b]$ ⚡

Part A: (2 marks) Give the statement of Bolzano-Weierstrass, the sequential characterization of compactness.

A Subset $S \subset \mathbb{R}^n$ is compact iff every sequence of pts in S has a convergent subsequence whose limit lies in S.

Part B: (3 marks) Find a continuous function f and a converging sequence $\{x_k\}$ such that the sequence $\{f(x_k)\}$ has no converging subsequence.

Let $f(x) = \frac{1}{x}$ and let $x_k = \frac{1}{k}$. $\{x_k\} \to 0$ but $f(x_k) = k$

so $\{f(x_k)\} = \{1, 2, 3, \ldots\}$ which cannot have a convergent subseq.

Part C: (5 marks) Use the definition in part A to prove that that the image of a compact set S under a continuous functions f is compact.

Choose a sequence $\{y_k\} \subset f(S)$. $\forall k$ $\exists x_k \in S$ s.t. $f(x_k) = y_k$.

by BW (as in part A) exists a subsequence $\{x_{k_j}\}$ that converges

to a pt $a \in S$, Since f is cont. $f(x_{k_j}) \to f(a)$,

so $\{y_{k_j}\} \to f(a)$. $\therefore$ every seq in $f(S)$ has a

subsequence converging in $f(S)$.

now by (A) again, $f(S)$ is cpt.

MAT 237, Quiz 3 Name ____________ TUT0401

Part A: (2 marks) Present the statement of Mean Value Theorem for a function f on an interval $[a,b]$.

Suppose f is Cont. on $[a\,b]$ and differentiable on (a,b). There is a point $c \in (a,b)$ 1

s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ 1

Part B: (4 marks) Use part (A) to show that for any function f that satisfies the conditions of the Mean Value Theorem if $f'(x) = 0$ for all $x \in (a,b)$, then f must be constant on $[a,b]$.

given $x_1, x_2 \in [a\,b]$ Consider f on $[x_1, x_2] \subset [a\,b]$. f is Cont on $[x_1, x_2]$ 2

and differentiable on (x_1, x_2), Then $\exists c \in (x_1, x_2)$ s.t.

$0 = f'(c) = \dfrac{f(x_2)-f(x_1)}{x_2 - x_1}$ so $f(x_2) - f(x_1) = 0$ so $f(x_1) = f(x_2)$.

1 1 $\therefore$ f is Constant

Part C: (4 marks) Use Role's theorem to prove Mean Value Theorem.

Let $l(x) = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$ and $g(x) = f(x) - l(x)$

1

Note g is Cont on $[a\,b]$ and $g(x)$ is diff on $(a\,b)$, and $g(a) = 0 = g(b)$ 1

so by Role's Theorem $\exists\, c \in (a\,b)$ s.t. $g'(c) = 0$. But $g'(x) =$

1

$= f'(x) - \dfrac{f(b)-f(a)}{b-a}$ so $g'(c) = 0$ means $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

1

Part A: (2 marks) Present the statement of Extreme Value Theorem.

Assume $S \subset \mathbb{R}^n$ is Compact and $f: S \to \mathbb{R}$ is Continuous. Then f has an absolute min and an absolute max Value on S; That is, There exist points $a, b \in S$ s.t. $f(a) \leq f(x) \leq f(b)$ for all $x \in S$.

Part B: (3 marks) Use the Extreme Value Theorem to prove that if f is continuous, S is compact, and $f(x) > 0$ for all $x \in S$, then there is a point $a \in S$ such that $0 < f(a) \leq f(x)$ for all $x \in S$.

By EVT f has an absolute min value on S; That is, $\exists\, a \in S$ st.

of Course $0 < f(a)$ and $\forall x \in S$ $f(a) < f(x)$

Part C: (5 marks) Prove the Extreme Value Theorem (please quote any theorem that you need to use in your proof.)

Cont image of a Compact set is Compact, so $f(S)$ is Compact Subset of $\mathbb{R}$.

Compact Subsets of $\mathbb{R}$ are closed and bounded

$\sup f(S)$ and $\inf f(S)$ exist

$u =$ $w =$

and They belong to $f(S)$ b/c of Closedness

so $\exists\, a, b \in S$ st.

$f(a) = w$ and $f(b) = u$. so $\forall x \in S$ $w \leq f(x) \leq u$ or

$f(a) \leq f(x) \leq f(b)$

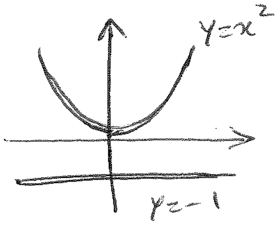
MAT 237, Quiz 3 Name ———————— TUT5201

Part A: (3 marks) What does it mean for a pair of sets (S_1, S_2) to be a disconnection for the set S? What does it mean for $S \subset \mathbb{R}^n$ to be connected?

- (S_1, S_2) is a disconnection for S if (i) $S_1 \neq \emptyset \neq S_2$ (ii) $S = S_1 \cup S_2$ (iii) $\bar{S}_1 \cap S_2 = \emptyset = S_1 \cap \bar{S}_2$ 2

- S is connected if it doesnot have a disconnection. 1

Part B: (3 marks) Is the set $S = \{(x, y) \in \mathbb{R}^2 : (x^2 - y)(y + 1) = 0\}$ connected? Explain your answer.



$S = \{(x,y) : x^2 - y = 0$ or $y + 1 = 0\} = \{(x,y) : y = x^2$ or $y = -1\}$ 1.5

S is disconnected b/c it has a disconnection (S_1, S_2)

$S_1 = \{(x,y) : y = x^2\}$ 1.5

$S_2 = \{(x,y) : y = -1\}$

Part C: (4 marks) State and prove the Intermediate Value Theorem (please quote any property/theorem that you need to use in your proof.)

IVT: Suppose $f : S \to \mathbb{R}$ is cont at every pt of S and $V \subset S$ is connected. If $a, b \in V$ and $f(a) < t < f(b)$ or $f(b) < t < f(a)$, there is a point $c \in V$ s.t. $f(c) = t$. 1

proof: f is cont., V is connected then $f(V)$ is connected subset of $\mathbb{R}$, hence 1

0.5 $f(V)$ is an interval. $\underset{\in f(V)}{f(a)} < t < \underset{\in f(V)}{f(b)}$ and $f(V)$ is an interval 1.5

Then $t \in f(V)$, so $\exists\, c \in V$ s.t. $f(c) = t$.

Part A: (2 marks) Present the definition of differentiability for a function f at a given point a. Make sure this is the new definition involving m and $E(h)$.

f is differentiable at a if exists a real number m, and a function $E(h)$ such that $f(a+h)$ can be written as $f(a+h) = f(a) + mh + E(h)$ and $\frac{E(h)}{h} \to 0$ as $h \to 0$.

1 1

Part B: (4 marks) Show that the function $f(x) = x|x|$ is differentiable at 0.

$$f(0+h) = h|h| = \underbrace{0}_{= f(0)} + \underbrace{0}_{m=0} h + E(h) \qquad E(h) = h|h| \quad \& \quad \frac{E(h)}{h} = |h| \to 0 \text{ as } h \to 0$$

1 1.5 1.5

so f is differentiable at 0.

part (C) (4 marks)

Part C: (4 marks) Prove that if both functions f and g are differentiable at a then the product $fg(x)$ is also differentiable at a.

1 f is diff at a, Then $\exists\, m_1$ and $E_1(h)$ s.t. $f(a+h) = f(a) + m_1 h + E_1(h)$

g " " " " " $\exists\, m_2$ " $E_2(h)$ " $g(a+h) = g(a) + m_2 h + E_2(h)$

now $fg(a+h) = f(a+h)g(a+h) = [f(a) + m_1 h + E_1(h)][g(a) + m_2 h + E_2(h)]$

1
$$= f(a)g(a) + f(a)m_2 h + m_1 g(a) h + f(a)E_2(h) + g(a)E_1(h) + m_1 m_2 h^2 + m_1 h E_2(h) + m_2 h E_1(h) + E_1(h)E_2(h)$$

1
$$= fg(a) + \underbrace{[f(a)m_2 + m_1 g(a)]}_{\text{derivative}} h + \underbrace{[f(a)E_2(h) + g(a)E_1(h) + m_1 m_2 h^2 + m_1 h E_2(h) + m_2 h E_1(h) + E_1(h)E_2(h)]}_{E(h)}$$

note
$$\frac{E(h)}{h} = f(a)\frac{E_2(h)}{h} + g(a)\frac{E_1(h)}{h} + m_1 m_2 h + m_1 E_2(h) + m_2 E_1(h) + \frac{E_1(h)E_2(h)}{h}$$

1