Name:________________________ **Student number:**________________________

Term Exam: Tuesday October 18

i) **Exam is 6 pages, duration 1 hour and 50 minutes**.

ii) **Answer on question sheet. Hand in only question sheet.**

iii) **No materials allowed except question sheet and rough paper.**

iv) **Write name and student number on every page**.

v) **Take more rough paper from the front if you need it. Raise your hand only to visit bathroom, if there is an error in the test, or to hand in your test early.**

Name:________________________ **Student number:**________________________

Question 1 (30 points)**.** Write T(rue) or F(alse).

i) It is not possible for two linear subspaces to have an empty intersection.

ii) If V is spanned by a list (v_1, v_2, v_3, v_4) and none of the v_i are zero, then $\dim V = 4$.

iii) If (v_1, v_2, v_3) is linearly dependent, then v_1 must be a linear combination of v_2 and v_3.

iv) If $v + v + v = 0$ for a vector v, then v must be the zero vector.

v) If U_1, U_2 are subspaces of a finite dimensional vector space V, then $\dim(U_1 + U_2) = \dim U_1 + \dim U_2$.

vi) $((1, 2, -3), (0, 1, -1), (0, 0, 2))$ is a basis for $\mathbb{R}^3$.

vii) Every vector space over the field $\mathbb{F}_2 = \{0, 1\}$ has a finite total number of vectors.

viii) If (v_1, v_2, v_3, v_4) is linearly independent, then the sum of subspaces $\mathrm{Span}(v_1, v_2) + \mathrm{Span}(v_3, v_4)$ is direct.

ix) If $\mathrm{Span}(v_1, v_2) + \mathrm{Span}(v_3, v_4)$ is direct, then (v_1, v_2, v_3, v_4) is linearly independent.

x) If $(v_1, \ldots, v_n)$ is a basis, then $(v_1 + w, \ldots, v_n + w)$ is a basis for any vector w.

Name:________________________ **Student number:**________________________

Question 2 (20 points)**.** Short answers, be precise:

i) What is the definition of linear dependence of a list $(v_1, \ldots, v_n)$ of vectors?

ii) Let X be a set, $\mathbb{F}$ be a field, and $\mathbb{F}^X$ be the set of functions from X to $\mathbb{F}$. Define the vector addition and scalar multiplication operations making $\mathbb{F}^X$ into a vector space over $\mathbb{F}$. (Only give the operations, don't verify axioms)

iii) Is the vector $(6, 7, 4, -8) \in \mathbb{R}^4$ contained in $\mathrm{Span}((2, -1, 0, 3), (3, 1, 1, 0), (-1, -2, -1, 2))$? (Hint: you may want to replace the list with a more convenient one)

iv) If U_1 and U_2 are five-dimensional subspaces of $\mathbb{R}^7$, what are the possible dimensions for $U_1 \cap U_2$?

Name:__________________________ **Student number:**__________________________

Question 3 (20 points)**.**

i) Let V be a vector space over $\mathbb{C}$, and let (u, v, w) be a linearly independent list of vectors in V. Prove that $(u + v, v + w, w + u)$ is linearly independent.

ii) What is the set of $(a, b, c) \in \mathbb{R}^3$ such that the list of vectors

$$((1, 1, a, b), (c, 0, -1, 1), (2, 1, 0, 1))$$

is linearly dependent in $\mathbb{R}^4$?

Name:________________________ **Student number:**________________________

Question 4 (40 points – four parts)**.** Let $X = \mathbb{F}_5$, and let V be the vector space of functions from X to $\mathbb{F}_5$. Recall that $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ is the field with five elements.

i) Give a basis for V and state the dimension of V (no justification required).

ii) Define the following subsets

$$V_e = \{f \in V \ : \ f(x) = f(-x), \quad \text{for all } x \in X\}$$
$$V_o = \{f \in V \ : \ f(x) = -f(-x), \quad \text{for all } x \in X\}$$

Prove that V_e and V_o are subspaces and determine their dimensions, justifying your answer.

Continued....

Name:__________________________ **Student number:**__________________________

iii) Find a nonzero even function $f \in V_e$ such that $f(0) + f(2) + f(4) = 0$ and $f(1) + f(3) = 0$. (No justification required)

iv) Prove that any function $f \in V$ can be written uniquely as a polynomial in one variable with coefficients in $\mathbb{F}_5$, of degree ≤ 4. State clearly any results you may need to use, including any relevant properties of the Lagrange interpolating polynomials $(f_0, \dots, f_n)$, defined for fixed distinct numbers $c_0, \dots, c_n$ by

$$f_i(x) = \prod_{\substack{0 \leq k \leq n \\ k \neq i}} \frac{x - c_k}{c_i - c_k}.$$