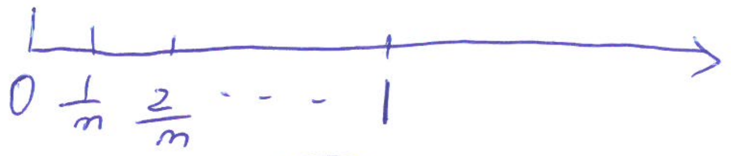
LN 4.1

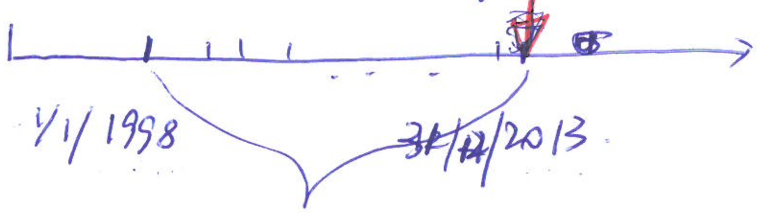
Ex.



$a^{(m)}_{\overline{n}|i}$, $s^{(m)}_{\overline{n}|i}$

?

$1000 per quarter.



$16 \text{ yrs} \times 4 = 64$ quarters.

$i = 10\%$ p.a. effective.

Sol:

(a). effective quarterly rate $j = (1+i)^{\frac{1}{4}} - 1$

$= 0.024114$.

Immediate annuity

$$1000\, s_{\overline{64}|j} = 1000 \cdot \frac{(1+j)^{64}-1}{j} = \$149,084$$

(b). $s^{(4)}_{\overline{16}|0.1} \times 1000 \times 4$.

$$= \frac{(1+0.1)^{16}-1}{i^{(4)}} \times 4000.$$

$$= \$149,084.$$

$i^{(4)} = m \cdot \left[(1+i)^{\frac{1}{4}} - 1\right]$

$= 0.096455$.

week 4.

- ① perpetuities
- ② continuous annuity
- ③ increasing / decreasing annuity.

①.

Immediate perpetuity.

$$a_{\overline{\infty}|} = \lim_{n\to\infty}\left(\frac{1-v^n}{i}\right) = \lim_{n\to\infty}\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right).$$

$$= \frac{1}{i}$$

<u>Pf</u>: (general reasoning).

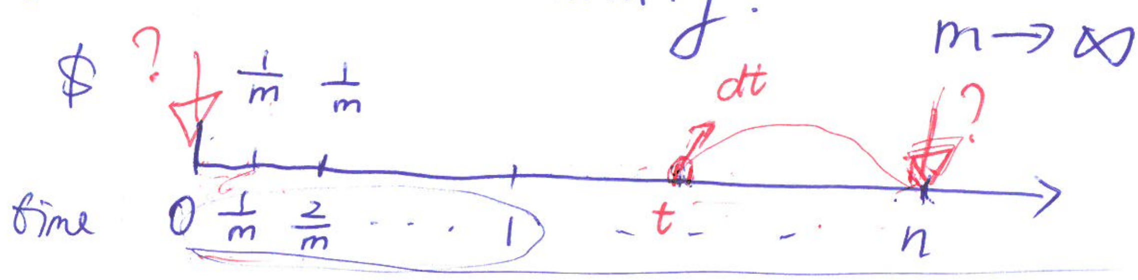
$$\boxed{X(1+i)-1 = \text{P.V. of } (CF_2, CF_3 \cdots CF_\infty) \text{ ~~at time 1~~}}$$

$= X$

$\Rightarrow X = \frac{1}{i}$

$$\Rightarrow \begin{cases} a_{\overline{\infty}|} = \frac{1}{i} \\ \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \\ a^{(m)}_{\overline{\infty}|} = \frac{1}{i^{(m)}} \\ \ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{d^{(m)}} \end{cases}$$

2. Continuous annuity.



1). effective interest rate i p.a — per year

$$\boxed{\bar{s}_{\overline{n}|i} = \int_0^n (1+i)^{n-t}\, dt}$$

$$= \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} \cdot s_{\overline{n}|i}$$

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

Pf: $\bar{s}_{\overline{n}|i} = \int_0^n (1+i)^{n-t}\, dt = \int_0^n \exp[\ln(1+i)^{n-t}]\, dt$

$$= -\frac{(1+i)^{n-t}}{\ln(1+i)}\Big|_0^n$$

$$= \int_0^n \exp[(n-t)\cdot \ln(1+i)]\, dt$$

$$\boxed{\int_0^n e^{at}\, dt = \frac{e^{at}}{a}}$$

$$= \frac{\exp(0)}{-\ln(1+i)} + \frac{\exp(n\ln(1+i))}{\ln(1+i)} \quad = \left. \frac{\exp[(n-t)\cdot\ln(1+i)]}{-\ln(1+i)} \right|_0^n$$ ④

$$= \frac{(1+i)^n - 1}{\ln(1+i)}$$

δ

$$= \frac{(1+i)^n - 1}{\delta}.$$

\#.

$$\Rightarrow. \quad \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \bar{s}_{\overline{n}|i} \cdot v^n$$

$\ln(1+i)$

$$= \boxed{\int_0^n \boxed{v^t} dt} = \left. \frac{v^t}{\ln v} \right|_0^n = \frac{1-v^n}{\delta}.$$

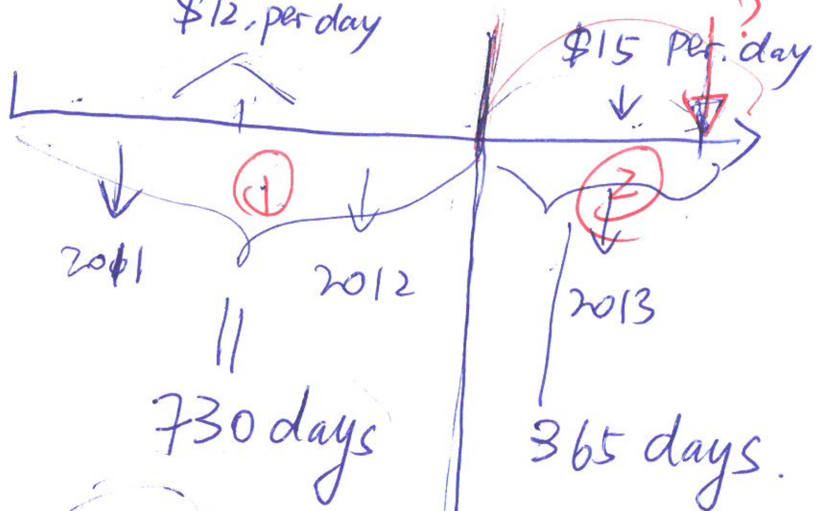
$$= \frac{i}{\delta} \cdot a_{\overline{n}|}.$$

2>. force of interest δ_r.

$$\bar{s}_{\overline{n}|\delta_r} = \int_0^n \boxed{\exp\left[\int_t^n \delta_r \, dr\right]} dt.$$

$$\bar{a}_{\overline{n}|\delta_r} = \int_0^n \boxed{\exp\left[-\int_0^t \delta_r \, dr\right]} dt.$$

Ex.



9% p.a. effective

12% p.a. effective.

1. exact
2. approximate

$S^{(m)}_{\overline{n}|i}$

Sol:

1. $A.V._{①} = \$12 \times S_{\overline{730}|j} \times (1+12\%) = 12 \times \frac{(1+j)^{730}-1}{j} \times 1.12$.

$$\boxed{j = (1+0.09)^{\frac{1}{365}} - 1}$$

$$A.V._{②} = \$15 \times S_{\overline{365}|k} = 15 \cdot \frac{(1+k)^{365}-1}{k}$$

$$\boxed{k = (1+12\%)^{\frac{1}{365}} - 1}$$

$\Rightarrow A.V. = A.V._{①} + A.V._{②} = \$16502.59.$

2 . Continuous annuity ① ②

$$\bar{S}_{\overline{n_1}|0.09} = \frac{(1+0.09)^{2}-1}{\ln(1+0.09)}$$

$$\bar{S}_{\overline{n_2}|0.12} = \frac{(1+0.12)^{1}-1}{\ln(1+0.12)}$$

Group 1: $\$12 \times 365 = \$4380$ p.a. 2011, 2012

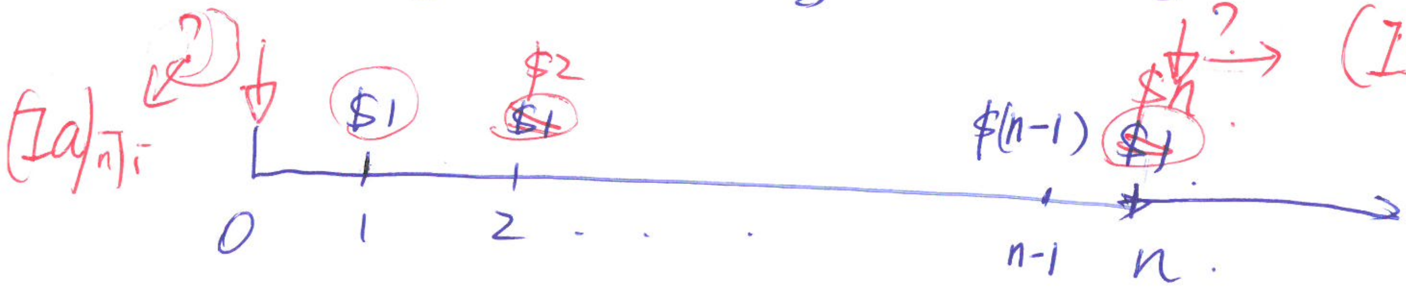
Group 2: $\$15 \times 365 = \$5475$ p.a. 2013

$$A.V. = 4380 \times \bar{S}_{\overline{2}|0.09} \times (1+0.12) + 5475 \times \bar{S}_{\overline{1}|0.12}$$

$$= 4380 \times \frac{(1+0.09)^2-1}{\ln(1+0.09)} \times (1+0.12) + 5475 \times \frac{(1+0.12)^1-1}{\ln(1+0.12)}$$

$$= \$\ 16504.75$$

3. Increasing / Decreasing Annuities



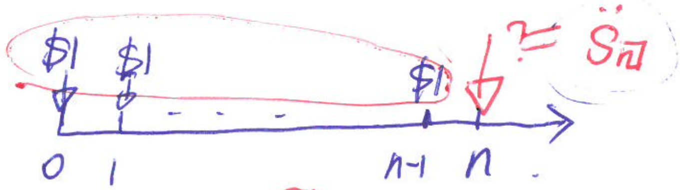
$$S(n) = 1\cdot(1+i)^{n-1} + 2\cdot(1+i)^{n-2} + \cdots + (n-1)(1+i) + n \quad ①$$

multiply $(1+i)$

$$(1+i)\cdot S(n) = (1+i)^n + 2\cdot(1+i)^{n-1} + \cdots + (n-1)\cdot(1+i)^2 + n(1+i) \quad ②$$

② − ①

$\Longrightarrow$



$$i\,S(n) = \boxed{(1+i)^n + (1+i)^{n-1} + \cdots + (1+i)} - n.$$

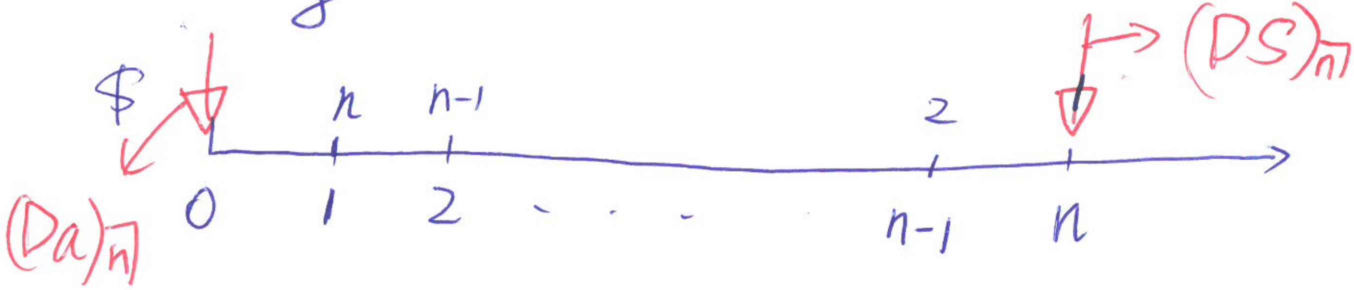
$$\Rightarrow S(n) = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

$\| \triangle$

$\boxed{(IS)_{\overline{n}|i}}$

$$\Rightarrow \begin{cases} (I\ddot{s})_{\overline{n}|i} = \dfrac{\ddot{s}_{\overline{n}|} - n}{d} \\ (Ia)_{\overline{n}|i} = (IS)_{\overline{n}|i}\cdot v^n = \dfrac{\ddot{a}_{\overline{n}|} - n\cdot v^n}{i} \\ (I\ddot{a})_{\overline{n}|i} = \dfrac{\ddot{a}_{\overline{n}|} - n v^n}{d} \end{cases}$$

Decreasing.



$$S(n) = n\cdot(1+i)^{n-1} + (n-1)\cdot(1+i)^{n-2} + \cdots + 2\cdot(1+i) + 1 \quad (1)$$

$$(1+i)\cdot S(n) = \quad (2)$$

(2) − (1)

$$\Rightarrow S(n) = \frac{n(1+i)^n - s_{\overline{n|}i}}{i} = (Ds)_{\overline{n|}i}$$

$$\Rightarrow \begin{cases} (D\ddot{s})_{\overline{n|}i} = \dfrac{n(1+i)^n - s_{\overline{n|}i}}{d} \\ (Da)_{\overline{n|}i} = (Ds)_{\overline{n|}i}\cdot v^n = \dfrac{n - a_{\overline{n|}i}}{i} \\ (D\ddot{a})_{\overline{n|}i} = \dfrac{n - a_{\overline{n|}i}}{d}. \end{cases}$$

Decompose ① $\to$ n level payments. $(A-B)$

$\to$ n $\uparrow$ payments $(B, 2B, 3B \cdots nB)$

$(A-B)+B \quad (A-B)+2B. \quad \cdots \quad (A-B)+n\cdot B.$

② $\to$ n level payments : A

$t=1,2,\cdots n$

$\to$ $(n-1)$ $\uparrow$ payments : $(B, 2B \cdots (n-1)B)$

$t=2,3,\cdots n.$

Decompose ①:

$$S(n) = A\cdot(1+i)^{n-1} + (A+B)\cdot(1+i)^{n-2} + \cdots + [A+(n-1)B]$$

\$1 \$1 \$1

0 1 2 n

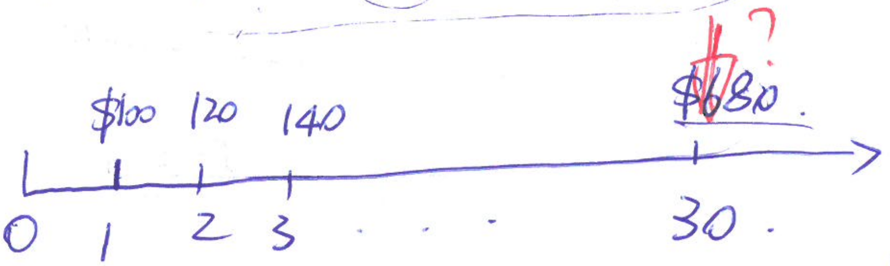
$$= (A-B)\left[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1\right] +$$

$$B\left[(1+i)^{n-1} + 2\cdot(1+i)^{n-2} + \cdots + (n-1)(1+i) + n\right]$$

$= (A-B)\cdot s_{\overline{n}|i} + B(Is)_{\overline{n}|i} \rightarrow$ (1)

$= A\cdot s_{\overline{n}|i} + B\cdot (Is)_{\overline{n-1}|i} \rightarrow$ (2)

Ex:



$i = 9\%$ effective p.a. $\quad A = \$100$, $B = \$20$

(1)

$$S(n) = (100-20)\cdot s_{\overline{30}|i} + 20\cdot \boxed{(Is)_{\overline{30}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}}$$

$$= 80\cdot \frac{(1+0.09)^{30}-1}{0.09} + 20\cdot \frac{\left(\frac{(1+0.09)^{30}-1}{\frac{0.09}{1.09}} - 30\right)}{0.09}$$

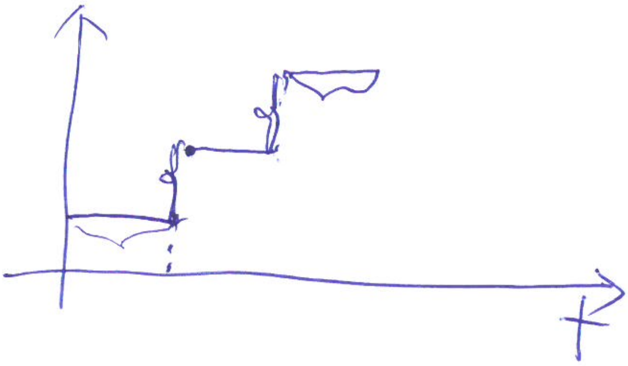
$$= 37254.65$$

(2)

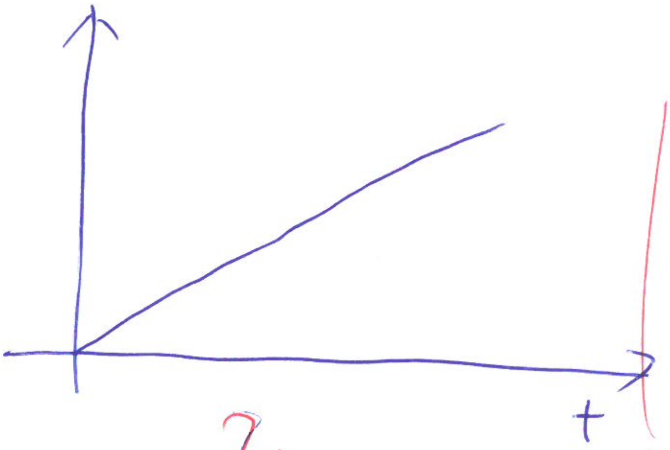
$(Is)_{\overline{n}|i}$ $\quad (Ia)_{\overline{n}|i}$ $\quad (I\ddot{a})_{\overline{\infty}|i}$

$\hookrightarrow n \rightarrow \infty$?

Continuous payments

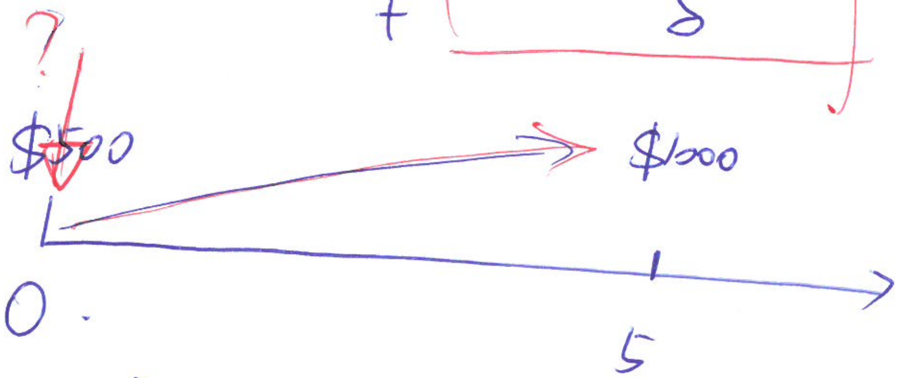


$$(I\bar{a})_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{\delta}$$



$$(\bar{I}\bar{a})_{\overline{n}|i} = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}$$

Ex:



$i = 4\%$ effective p.a.