Australian National University

RESEARCH SCHOOL OF FINANCE, ACTUARIAL STUDIES AND APPLIED STATISTICS

INTRODUCTORY MATHEMATICAL STATISTICS (STAT2001)
PRINCIPLES OF MATHEMATICAL STATISTICS (STAT6039)

Final Examination – June 2012

Study Period: 15 minutes
Time Allowed: 3 hours
Permitted Material: No restrictions

- ***Undergraduate students (those enrolled in STAT2001) should attempt only the first 5 problems. For these students the exam is out of 100 marks. Each of the first 5 problems is worth 20 marks.***

- ***Graduate students (those enrolled in STAT6039) should attempt all 6 problems. For these students the exam is out of 120 marks.***

- ***Draw a box around each solution and express each numerical solution as the simplest possible fraction (e.g. $\boxed{2/3}$) or to at least 4 significant digits (e.g. $\boxed{0.007204}$). Start your solution to each problem on a new page and do the problems in the order 1,2,3,....***

- ***To ensure full marks, show all the steps in working out your solutions. Marks may be deducted for not showing appropriate calculations or formulae, or for not clearly referencing the results in the text book or course material which you are using.***

Problem 1 ***(20 marks in total)***

A batch of nine widgets contains three which are defective and six which are nondefective. A testing procedure is 88% likely to be correct when applied to a defective widget and 71% likely to be correct when applied to a nondefective widget.

(a) A widget is selected randomly from the nine and tested.
The test indicates that the selected widget is defective.

Find the probability that this widget is actually defective. ***(5 marks)***

(b) Two widgets are randomly selected from the nine (without replacement) and tested independently. The tests indicate that both of the selected widgets are nondefective.

Find the probability that at least one of the two widgets is defective.
(15 marks)

Problem 2 ***(20 marks in total)***

Ben has recently bought a machine and insured it over the next seven-year period. The insurance company will pay him \$1000 each time the machine breaks down, with a cap of \$3000. (Thus, if the machine breaks down more than three times, Ben will receive a total of only \$3000.)

Suppose that the number of times the machine breaks down over the next seven-year period follows a Poisson distribution whose mean λ is known to be in the interval from 0 to 2. (Thus, the mean λ cannot be greater than 2.)

(a) How many dollars can we expect Ben to be paid
by the insurance company in total if $\lambda = 1.5$? ***(5 marks)***

(b) Find the maximum likelihood (ML) estimate of λ if the insurance company pays Ben nothing in total. Then derive a general expression for the ML estimate of λ as a function of how much Ben is paid in total. Next, derive the distribution of the corresponding ML estimator, and also a general expression for the expected value of that estimator. Finally,
calculate this expected value for the case $\lambda = 1$. ***(15 marks)***

Problem 3 ***(20 marks in total)***

Suppose that X and Y are two independent random variables with continuous distributions given by $X \sim U(0,2)$ and $Y \sim Beta(2,1)$.

(a) Find and sketch $f_W(w)$, the density of $W = 1/(1-Y)$.

Also calculate m, the mode of W, and $h = f_W(m)$.

Show the point (m,h) in your sketch of $f_W(w)$. ***(5 marks)***

(b) Find and sketch $f_R(r)$, the density of $R = (X-Y)^2$.

Also calculate $c = ER$ (the mean of R) and $k = f_R(c)$.

Show the point (c,k) in your sketch of $f_R(r)$. ***(15 marks)***

Problem 4 ***(20 marks in total)***

A factory produces bolts with a defective rate that changes randomly and independently from day to day but is constant throughout any given day.

Let p_i denote the defective rate on day i, and suppose that $p_1, p_2, \ldots \sim iid\ Beta(1,19)$.

For every bolt produced on day i, assume that there is a probability of p_i that it is defective, irrespective of which other bolts produced by the factory are defective.

(a) A random sample of 100 bolts is taken from the output of the factory on day 1. Find the probability that none of these 100 bolts are defective. ***(5 marks)***

(b) A random sample of 200 bolts is taken from the output of the factory on days 1,...,100, two bolts per day. Calculate, or approximate as best you can, q, the probability that at least 20 of these 200 bolts are defective.

(10 marks)

(c) Find an exact upper bound for q as defined in part (b). ***(5 marks)***

Problem 5 ***(20 marks in total)***

A random sample of 250 persons was taken from the population of Urbania (a country with millions of people), and the number of houses owned by each sampled person was determined.

The following table shows the results of this survey. (For example, 13 sampled persons were found to own exactly 2 houses each.)

Number of houses	*Frequency*
0	186
1	42
2	13
3	5
4	3
5	1

(a) Find an approximate 80% confidence interval for the average number of houses owned per person in Urbania. ***(10 marks)***

(b) We are interested in whether or not more than 50% of adults in Urbania own at least one house each.

Formulate and conduct an appropriate hypothesis test at the 5% level of significance. Use the fact that all houses in Urbania are owned by adults and the fact that exactly 105 of the 250 sampled persons are adults. Also report the p-value associated with the test. ***(10 marks)***

Problem 6 ***(to be done only by STAT6039 students)*** ***(20 marks)***

A standard six-sided die is rolled twice. Find the correlation between the number of sixes that come up and the total of the two numbers that come up.

END OF EXAMINATION