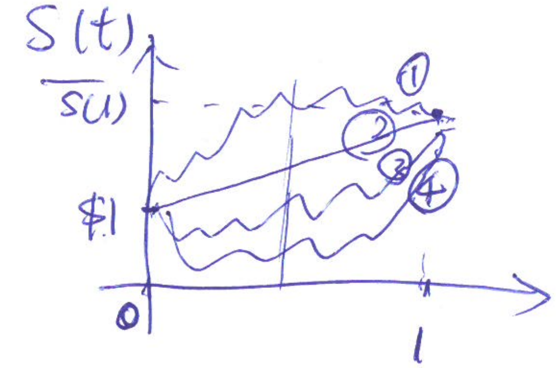
δ force of interest rate.

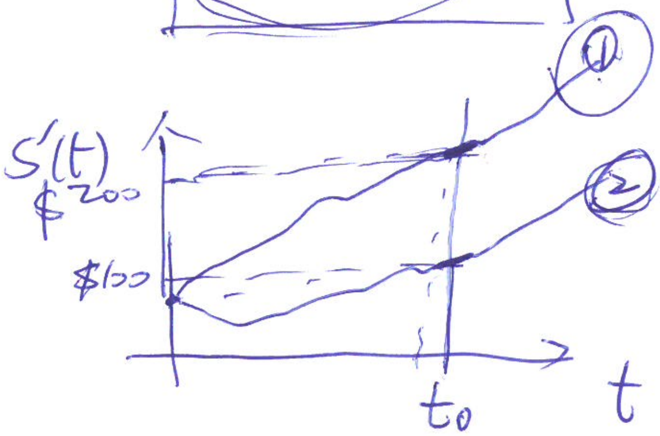


$\$1 \longrightarrow S(1)$

$t=0$ $\quad$ $t=1.$

$$\bar{i} = \frac{S(1)-1}{1}$$

$$\boxed{S'(t)\ ?}$$



$$\boxed{S_1'(t_0) = S_2'(t_0)}$$

\$600.

30 seconds.

$$\boxed{\frac{S'(t)}{S(t)}} = \text{force of interest} = \delta_t .$$

① Accumulated Value using $\delta_t = \frac{S'(t)}{S(t)} = \frac{d}{dt}\ln(S(t))$ ②

$$S(\cancel{n}\,t_2) = S(\cancel{0}\,t_1)\cdot \exp\left(\int_{\cancel{0}\,t_1}^{\cancel{n}\,t_2} \boxed{\delta_t}\, dt\right) \xrightarrow{\delta_t=\delta} \boxed{S_0 \cdot e^{\delta \cancel{n}\,(t_2-t_1)}}$$

$S(0)$ $\qquad\qquad\qquad\qquad$ $S(n)$

$\cancel{0}\ t_1$ $\qquad\qquad\qquad\qquad$ $\cancel{n}\ t_2$

Pf: $\int_{t=0}^{t=n} \delta_t \, dt = \int_0^n \frac{d}{d(t)} \ln[S(t)]\,dt$

$$= \ln[S(t)]\Big|_{t=0}^{t=n}$$

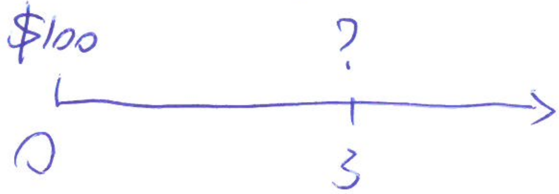
$$= \ln[S(n)] - \ln[S(0)]$$

$$\Rightarrow \quad S[n] = S_0 \cdot e^{\int_0^n \delta_t \, dt}.$$

② $S(\cancel{0}\,t_1) = S(\cancel{n}\,t_2)\cdot \exp\left\{-\int_{\cancel{0}\,t_1}^{\cancel{n}\,t_2} \delta_t \, dt\right\}$

$$\xlongequal{\delta_t=\delta} S(n)\cdot e^{-\delta\cdot \cancel{n}\,(t_2-t_1)}.$$

Ex : $\delta = 8\%$ p.a.



① . $\delta_t = \delta$

② $\delta_t = 0.08 + 0.005t$, $t \in [0, 3]$.

Sol:

①
$$S(0) = \$100$$
$$\delta_t = \delta = 0.08 .$$
$$\Rightarrow S(n) = S(0) \cdot e^{\delta \cdot 3} = 100 \cdot e^{0.24} = \$127.12 .$$

② $\Rightarrow S(n) = S(0) \cdot e^{\int_0^3 0.08 + 0.005t \, dt}$

$$= 100 \cdot e^{\left(0.08 \cdot 3 + \frac{0.005}{2}(9)\right)}$$

$$= \$130.02 .$$

$\begin{cases} i & i^{(m)} & v \\ d & d^{(m)} & \\ \delta & & \end{cases} \Longrightarrow \begin{cases} A.V. & S(t) \\ P.V. & S(0). \end{cases}$

$S(0)=1 \qquad S(t)$

$t=0 \qquad t=t$

$$\boxed{S(t) = \boxed{(i+1)^t} = v^{-t} = \boxed{\left(1+\frac{i^{(m)}}{m}\right)^{mt}} = \boxed{(1-d)^{-t}} = \left(1-\frac{d^{(m)}}{m}\right)^{-mt} = \boxed{e^{\delta t}}}$$

$(S(0)=\$1)$

$$S(0) = (1+i)^{-t} = v^t = \left(1+\frac{i^{(m)}}{m}\right)^{-mt} = (1-d)^t = \left(1-\frac{d^{(m)}}{m}\right)^{mt} = e^{-\delta t}.$$

$(S(t)=\$1)$

Ex: $i^{(12)} = 0.12$, then

$$i = \left(1+\frac{i^{(12)}}{12}\right)^{12} - 1 = 0.126825$$

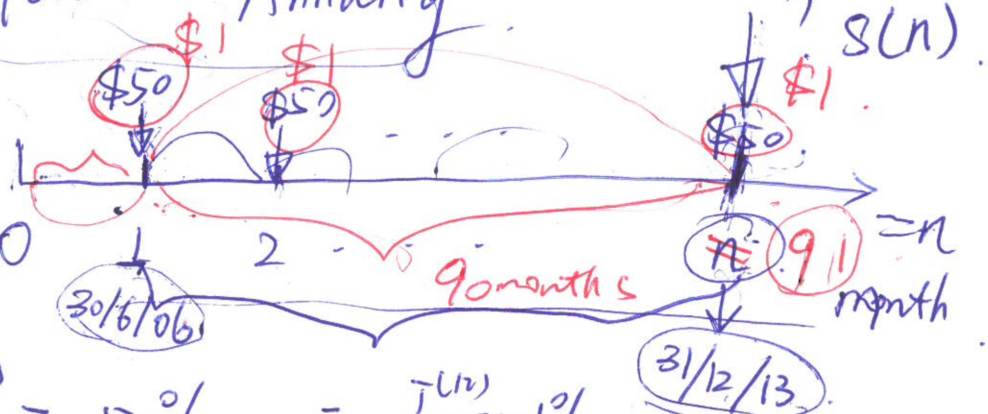
$$d = 1-\left(1+\frac{i^{(12)}}{12}\right)^{-12} = 0.112551$$

$$\delta = \ln\left(1+\frac{i^{(12)}}{12}\right)^{12} = 0.119404.$$

Annuities Certain

1. Immediate Annuity
2. Annuity Due
3. Deferred Annuity

(1)



$I^{(12)} = 12\%$. $\quad j = \frac{I^{(12)}}{12} = 1\%$

$n = 7.5$ yrs. $\times 12^{+1} = 90$ months $+1 = 91$.

$$S(n) = 50(1+j)^{\cancel{89}\,90} + 50(1+j)^{\cancel{88}\,89} + \cdots + 50.$$

$$= 50\left[(1+j)^{90} + (1+j)^{89} + \cdots + (1+j) + 1\right]$$

$$= 50\,\frac{(1+j)^{91}-1}{j} = \$7,365.60.$$

$$\boxed{1 + x + x^2 + x^3 + \cdots + x^k = \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1}-1}{x-1}}$$

A.V. of $\boxed{\text{Immediate Annuity}}$ = Annuity payable in arrears.

$$\boxed{s_{\overline{n}|i} = s_{\overline{n}|}} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1$$

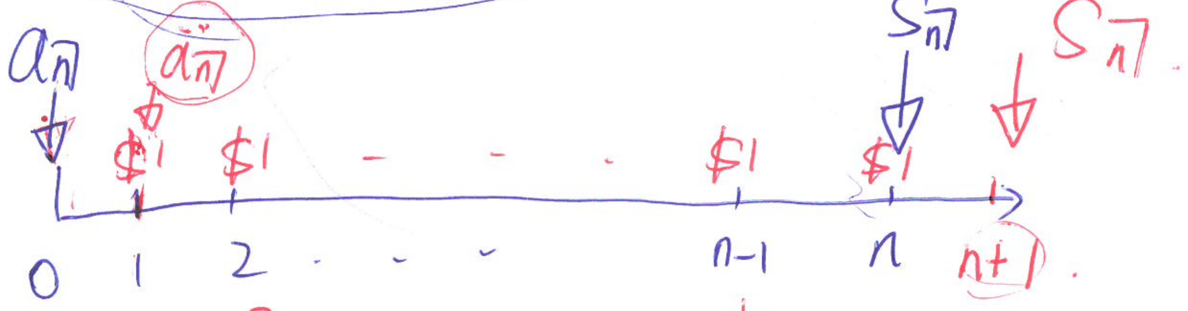
$$= \frac{(1+i)^n - 1}{(1+i) - 1} = \boxed{\frac{(1+i)^n - 1}{i}}$$

P.V. of immediate Annuity.

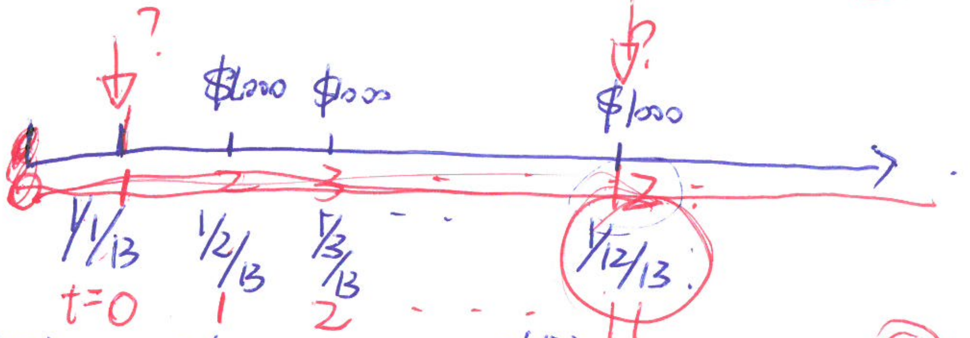
$$\boxed{a_{\overline{n}|i} = a_{\overline{n}|}} = s_{\overline{n}|} \cdot v^n = \boxed{\frac{1 - v^n}{i}} = \frac{1 - (1+i)^{-n}}{i}$$

$$\| $$

$$v + v^2 + \cdots v^n$$



Ex :

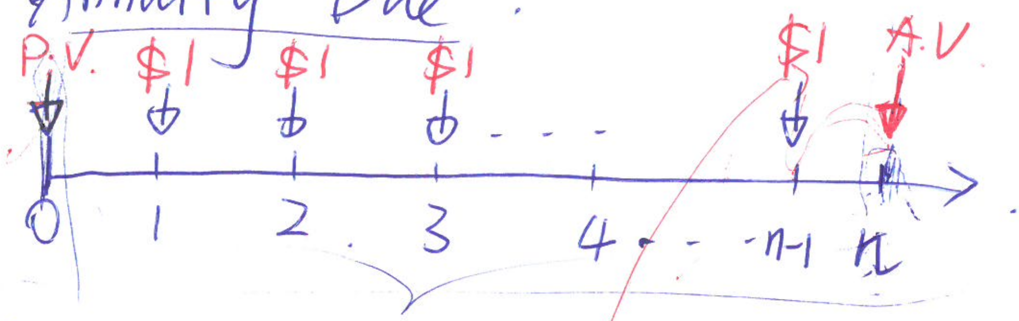


$i^{(12)} = 6\% \Rightarrow j = \frac{i^{(12)}}{12} = 0.005$, $n = 11$.

Sol : $\cancel{S(t)} = S(0) = a_{\overline{11}|0.005} \times 1000$

$$= 1000 \times \frac{1 - (1+0.005)^{-11}}{0.005} = \$10,677$$

② Annuity Due.



A.V. $\boxed{\ddot{s}_{\overline{n}|i}} = \ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \cdots + (1+i)^n$

$$= \boxed{\frac{(1+i)^n - 1}{d}}$$

P.V. $\boxed{\ddot{a}_{\overline{n}|i}} = \ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1}$

$$= \boxed{\frac{1 - v^n}{d}}$$

Ex Proof: $a_{\overline{n}|} = \frac{1 - v^n}{i}$ ~~with~~

from first principles.

Pf. $a_{\overline{n}|} = v + v^2 + \cdots + v^n$ ①

$\frac{1}{v} a_{\overline{n}|} = \cancel{v^2 + v^3 + \cdots + v^n}\ 1 + v + v^2 + \cdots + v^{n-1}$ ②

$\frac{1}{v} a_{\overline{n}|} = (1+i) \cdot a_{\overline{n}|}$

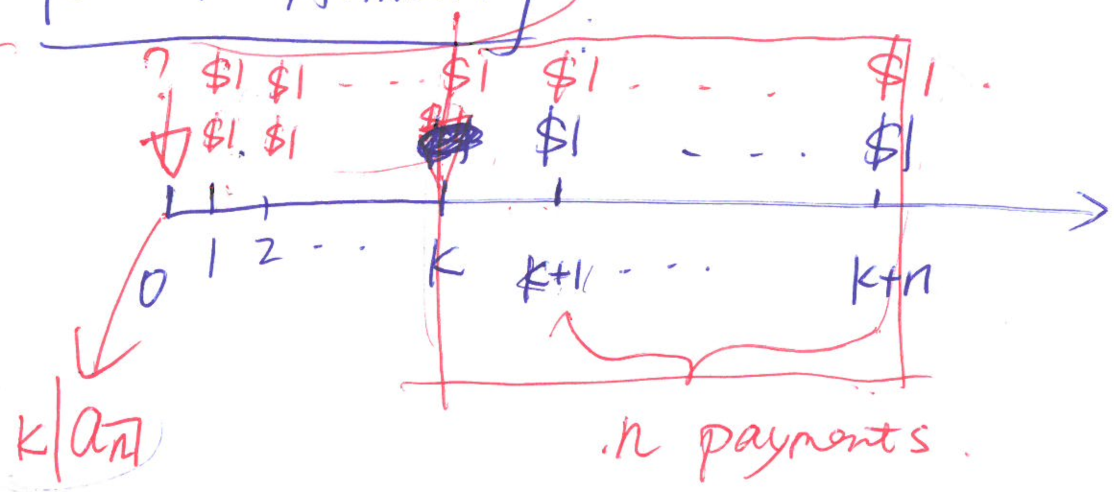
2 − 1. $i \cdot a_{\overline{n}|} = 1 - v^n$

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

#.

3

Deferred Annuity

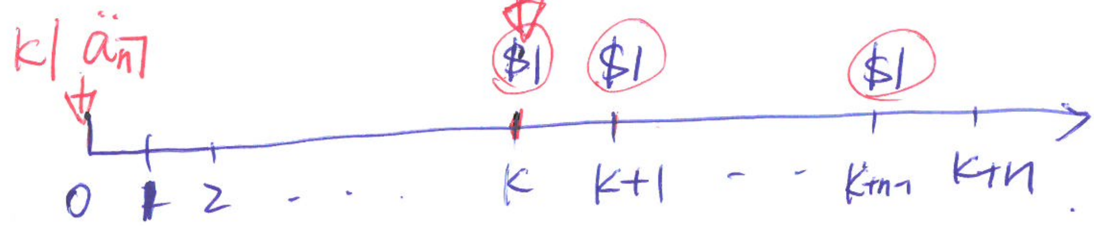


n-payment $\boxed{\text{Immediate annuity}}$ deferred for k payment periods.

$$k|a_{\overline{n}|} \overset{1}{=} \boxed{a_{\overline{n}|} \cdot v^k} = \frac{1-v^n}{i} \cdot v^k$$

$$\overset{2}{=} \boxed{a_{\overline{n+k}|} - a_{\overline{k}|}}$$

n-payment annuity-due deferred for k payment periods:



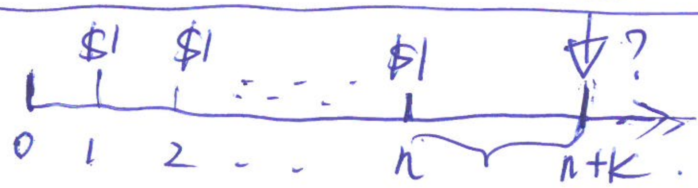
$$k|\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n}|} \cdot v^k = \ddot{a}_{\overline{n+k}|} - \ddot{a}_{\overline{k}|}$$

$\ddot{a}_{\overline{n}|}$ & $a_{\overline{n}|}$; $\ddot{s}_{\overline{n}|}$ & $s_{\overline{n}|}$.

$$\ddot{a}_{\overline{n}|} = (1+i) \cdot a_{\overline{n}|} \iff a_{\overline{n}|} = \ddot{a}_{\overline{n}|} \cdot v.$$

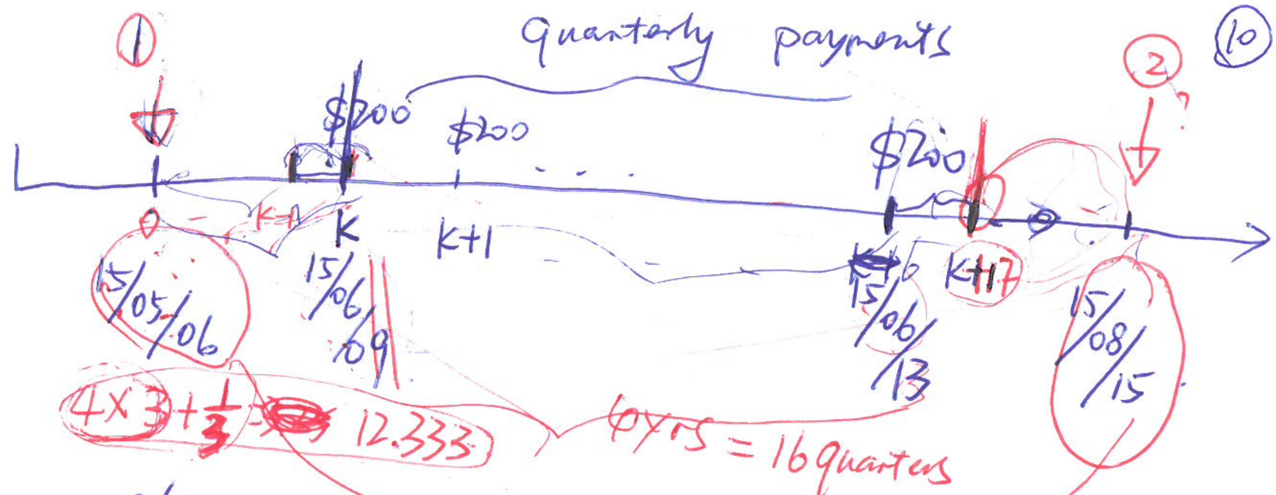
$$= \frac{i}{d} \cdot a_{\overline{n}|}$$

$$\ddot{s}_{\overline{n}|} = (1+i) \cdot s_{\overline{n}|} = \frac{i}{d} s_{\overline{n}|}.$$



$\Rightarrow s_{\overline{n}|} \cdot (1+i)^k$

Ex:



$I = 2.5\%$ per quarter.

9 yrs + 1 quarter

$= 9 \times 4 + 1 = 37$ quarters

① ${}_{12.333|}\ddot{a}_{\overline{17}|0.025} \times \$200$.

$$= 200 \cdot \underbrace{\left(\frac{1 - v_{0.025}^{17}}{d} \right)}_{= \ddot{a}_{\overline{17}|0.025}} \cdot \underbrace{1.025^{-12.333}}_{= v^k}$$

$= \$2073.$

② $2073 \times (1 + 2.5\%)^{37} = \$5168.71.$