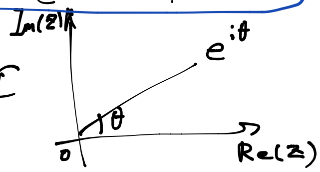
$\boxed{e^{i\theta}=\cos(\theta)+i\sin(\theta) \quad \text{Euler's formula}}$

$Re(e^{i\theta})=\cos(\theta)$

$Im(e^{i\theta})=\sin(\theta)$

$\mathbb{C}$



$z=\lambda+i\mu \qquad \bar{z}=\lambda-i\mu$

$Re(z)=\frac{z+\bar{z}}{2}$

$Im(z)=\frac{z-\bar{z}}{2i}$

$\boxed{\text{Thus } \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}}$

$\sin(3x)\cos(5x)$

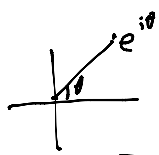
$=\left(\frac{e^{3ix}-e^{-3ix}}{2i}\right)\left(\frac{e^{5ix}+e^{-5ix}}{2}\right)$

$=\frac{1}{4i}\left(e^{8ix}-e^{-8ix}-(-e^{-2ix}+e^{2ix})\right)=\frac{1}{2}\left(\frac{e^{8ix}-e^{-8ix}}{2i}-\frac{e^{2ix}-e^{-2ix}}{2i}\right)=\frac{1}{2}\left(\sin(8x)-\sin(2x)\right)$

Eg. $\pi^{i}=\exp(i\ln(\pi))=\cos(\ln(\pi))+\sin(\ln(\pi))$

$a^{b}=\exp(b\ln(a))$

Eg. $i^{i}=\exp(i\ln(i))$



Note: $\exp(i\frac{\pi}{2})=i \Rightarrow \log(i)=i\frac{\pi}{2} \Rightarrow i^{i}=\exp(i\cdot i\frac{\pi}{2})=\exp(-\frac{\pi}{2})$

Back to $ay''+by'+cy=0 \quad (*)$

Char. eqn. $ar^2+br+c=0$ factor out $\sqrt{-1}=i$

Sps $b^2<4ac$

$r_1,r_2=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}=-\frac{b}{2a}\pm\frac{i}{2a}\sqrt{4ac-b^2}=\lambda\pm i\mu$

Two complex conjugate complex roots.

$\Rightarrow$ Solutions of our ODE are:

e^{r_1t}, e^{r_2t} as complex solutions.

$e^{r_1t}=e^{(\lambda+i\mu)t}=e^{\lambda t}e^{i\mu t}$

or

$y_1(t)=Re(e^{r_1t})=e^{\lambda t}\cos(\mu t)$

$y_2(t)=Im(e^{r_1t})=e^{\lambda t}\sin(\mu t)$

Example: $y'' + 4y = 0$

Char. eqn. $r^2 + 4 = 0$

roots: $r_1, r_2 = \pm\sqrt{-4} = \pm i2$

$\Rightarrow$ Two complex solution: e^{2it}, e^{-2it}

or two real solutions:

$y_1(t) = \cos(2t)$

$y_2(t) = \sin(2t)$

Example: $y'' + y' + y = 0$

char. eqn : $r^2 + r + 1 = 0$

roots: $r_1, r_2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$

$= \lambda \pm i\mu$

$\Rightarrow$ solutions:

$y_1(t) = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$

$y_2(t) = e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$

Summary: 2nd order homogeneous linear ODE with constant coefficient.

$ay'' + by' + cy = 0 \rightsquigarrow$ char. eqn $ar^2 + br + c = 0$

case	Roots	Solution
$b^2 > 4ac$	Real roots: $r_1, r_2 = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac}$	$\exp(r_1 t), \exp(r_2 t)$
$b^2 = 4ac$	Repeated (real) roots $r = -\frac{b}{2a}$	$\exp(rt)$, $t\exp(rt)$
$b^2 < 4ac$	Complex conj. roots $\lambda \pm i\mu$, $\lambda = -\frac{b}{2a}$, $\mu = \frac{1}{2a}\sqrt{4ac-b^2}$	$\exp(\lambda t)\cos(\mu t)$, $\exp(\lambda t)\sin(\mu t)$

Example: Free oszillator?

$mx'' + kx = 0$ ($k > 0$, spring constant)

char. eqn : $mr^2 + k = 0$

$\Rightarrow r_1, r_2 = \pm\frac{1}{2m}\sqrt{-4mk} = \pm i\sqrt{\frac{k}{m}}$

$\Rightarrow$ Two solutions : $x_1(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

$x_2(t) = \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

where $\omega_0 = \sqrt{\frac{k}{m}}$ characteristic frequency.

period $T = \frac{2\pi}{\omega_0}$

General solution:

$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

note: $x'(0) = B\omega_0$

Where $A = x(0)$

$B = \frac{1}{\omega_0} x'(0)$

Fact: $A\cos(\omega_0 t) + B\sin(\omega_0 t)$ can be written as $R\cos(\omega_0 t - \delta)$

where $R = \sqrt{A^2 + B^2}$, $\tan(\delta) = \frac{B}{A}$

$R\cos(\omega_0 t - \delta) = R(\cos(\omega_0 t)\cos(\delta) + \sin(\omega_0 t)\sin(\delta))$

$\Rightarrow$ want $R\cos(\delta) = A$

$R\sin(\delta) = B$