2

Sets and Functions

If there is one unifying foundation common to all branches of mathematics, it is the theory of sets. We have already seen the need for set notation in describing the context in which quantified statements are understood to apply. It is not our intention to develop set theory in a formal axiomatic way, but rather to discuss informally those aspects of set theory that are relevant to the study of analysis. In Section 5 we establish the basic notation for working with sets, and in the following two sections we apply this to the development of relations and functions. In Section 8 we compare the size of sets, giving special attention to infinite sets. In Section 9 (an optional section) we outline a set of axioms that can be used to develop formal set theory, and indicate some of the problems that are involved with the development.

Section 5 BASIC SET OPERATIONS

The idea of a set or collection of things is common in our everyday experience. We speak of a football *team*, a *flock* of geese, or a finance *committee*. Each of these refers to a collection of distinguishable objects that is thought of as a whole. In spite of the familiarity of the idea, it is essentially impossible to give a definition of "set" except in terms of synonyms that are also undefined. Thus we shall be content with the informal understanding that a set is a collection of objects characterized by some defining property that allows us to think of the objects as a whole. The objects in a set are called **elements** or **members** of the set.

It is customary to use capital letters to designate sets and the symbol $\in$ to denote membership in a set. Thus we write $a \in A$ to mean that object a is a member of set A, and $a \notin B$ to mean that object a is not a member of set B.

5.1 EXAMPLE If $A = \{1, 2, 3, 4\}$, then $2 \in A$ and $5 \notin A$.

To say that a set must be characterized by some defining property is to require that it be a clear question of fact whether a particular object does or does not belong to a particular set. We do not, however, demand that the answer to the question of membership be known. Another way to say this is to require that for any element a and any set A the sentence "$a \in A$" be a statement; that is, it must be true or false, and not both.

5.2 PRACTICE Which of the following satisfy the requirements of a set?

(a) all the current U. S. Senators from Massachusetts
(b) all the prime divisors of 1987
(c) all the tall people in Canada
(d) all the prime numbers between 8 and 10

To define a particular set, we have to indicate the property that characterizes its elements. For a finite set, this can be done by listing its members. For example, if set A consists of the elements 1, 2, 3, 4, we write $A = \{1, 2, 3, 4\}$, as in Example 5.1. If set B consists of just one member b, we write $B = \{b\}$. Thus we distinguish between the *element* b and the *set* $\{b\}$ containing b as its only element.

For an infinite set we cannot list all the members, so a defining rule must be given. It is customary to set off the rule within braces, as in

$$C = \{x : x \text{ is prime}\}.$$

This is read "C is the set of all x such that x is prime," or more simply, "C is the set of all prime numbers."

When considering two sets, call them A and B, it may happen that every element of A is also an element of B. In this case we say that A is a subset of B. This concept is of such fundamental importance that we distinguish it by our first formal definition:

5.3 DEFINITION Let A and B be sets. We say that A is a **subset** of B (or A is **contained** in B) if every element of A is an element of B, and we denote this by writing $A \subseteq B$.† (Occasionally, we may write $B \supseteq A$ instead of $A \subseteq B$.) If A is a subset of B

† When "if" is used in a definition, it is understood to have the force of "iff." That is, we are defining "$A \subseteq B$" to be the same as "every element of A is an element of B." Essentially, a definition is used to establish an abbreviation for a particular idea or concept. It would be more accurate to write "iff" between the concept and its abbreviation, but it is common practice to use simply "if."

and there exists an element in B that is not in A, then A is called a **proper subset** of B.

This definition tells us what we must do if we want to prove $A \subseteq B$. We must show that

$$\text{if } x \in A, \text{ then } x \in B$$

is a true statement. That is, we must show that each element of A satisfies the defining condition that characterizes set B.

5.4 DEFINITION Let A and B be sets. We say that A is **equal** to B, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

When this definition is combined with the definition of subset, we see that proving $A = B$ is equivalent to proving

$$x \in A \Rightarrow x \in B \quad \text{and} \quad x \in B \Rightarrow x \in A.$$

It is important to note that, in describing a set, the order in which the elements appear does not matter, nor does the number of times they are written. Thus the following sets are all equal:

$$\{1,2,3,4\} = \{2,4,1,3\} = \{1,2,3,2,4,2\}.$$

Since the repeated 2's in the last set are unnecessary, it is preferable to omit them.

Although we cannot give a formal definition of them now, it is convenient to name the following sets, which should already be familiar to the reader.

$\mathbb{N}$ will denote the set of positive integers (or natural numbers).

$\mathbb{Z}$ will denote the set of all integers.

$\mathbb{Q}$ will denote the set of all rational numbers.

$\mathbb{R}$ will denote the set of all real numbers.

In constructing examples of sets it is often helpful to indicate a larger set from which the elements are being chosen. We indicate this by a slight change in our set notation. For example, instead of having to write

$$\{x : x \in \mathbb{R} \text{ and } 0 < x < 1\},$$

we may abbreviate by writing

$$\{x \in \mathbb{R} : 0 < x < 1\}.$$

The latter notation is read "the set of all x in $\mathbb{R}$ such that $0 < x < 1$" or "the set of all real numbers x such that $0 < x < 1$."

There is also a standard notation that we shall use for interval subsets of the real numbers:

$$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}, \qquad (a,b) = \{x \in \mathbb{R} : a < x < b\}$$
$$[a,b) = \{x \in \mathbb{R} : a \le x < b\}, \qquad (a,b] = \{x \in \mathbb{R} : a < x \le b\}.$$

The set $[a,b]$ is called a **closed interval**, the set (a,b) is called an **open interval**, and the sets $[a,b)$ and $(a,b]$ are called **half-open** (or **half-closed**) **intervals**. We shall also have occasion to refer to the unbounded intervals:

$$[a,\infty) = \{x \in \mathbb{R} : x \ge a\}, \qquad (a,\infty) = \{x \in \mathbb{R} : x > a\}$$
$$(-\infty,b] = \{x \in \mathbb{R} : x \le b\}, \qquad (-\infty,b) = \{x \in \mathbb{R} : x < b\}$$

At present no special significance should be attached to the symbols "∞" and "$-\infty$" as in $[a,\infty)$ and $(-\infty,b]$. They simply indicate that the interval contains all real numbers greater than or equal to a, or less than or equal to b, as the case may be.

5.5 EXAMPLE Let

$$A = \{1,3\}$$
$$B = \{3,5\}$$
$$C = \{1,3,5\}$$
$$D = \{x \in \mathbb{R} : x^2 - 8x + 15 = 0\}.$$

Then the following statements (among others) are all true:

$$A \subseteq C \qquad 1 \notin D$$
$$A \not\subseteq B \qquad \{5\} \subseteq B$$
$$5 \in B \qquad B = D$$
$$5 \not\subseteq B \qquad B \ne C$$

Notice that the slash (/) through a connecting symbol has the meaning of "not." Thus $A \not\subseteq B$ is read "A is not a subset of B."

5.6 PRACTICE Let

$$A = \{1,2,3,4,5\}$$
$$B = \{x : x = 2k \text{ for some } k \in \mathbb{N}\}$$
$$C = \{x \in \mathbb{N} : x < 6\}.$$

Which of the following statements are true?

(a) $\{4,3,2\} \subseteq A$ (b) $3 \in B$
(c) $A \subseteq C$ (d) $\{2\} \in A$
(e) $C \subseteq B$ (f) $\{2,4,6,8\} \subseteq B$
(g) $C \subseteq A$ (h) $A = C$

In Practice 5.2(d) we found that the collection D of all prime numbers between 8 and 10 is a legitimate set. This is so because the statement "$x \in D$" is always false, since there are no prime numbers between 8 and 10. Thus D is an example of the **empty set**, a set with no members. It is not difficult to show (Exercise 5.18) that there is only one empty set, and we denote it by $\varnothing$. For our first theorem we shall prove that the empty set is a subset of every set. Notice the essential role that definitions play in the proof. At this point, we really have nothing else to use as building blocks.

5.7 THEOREM Let A be a set. Then $\varnothing \subseteq A$.

Proof: To prove that $\varnothing \subseteq A$, we must establish that the implication

$$\text{if } x \in \varnothing, \text{ then } x \in A$$

is true. Since $\varnothing$ has no members, the antecedent "$x \in \varnothing$" is false for all x. Thus, according to our definition of implies, the implication is always true. ♦

There are three basic ways to form new sets from old ones. These operations are called union, intersection, and complementation. Intuitively, union may be thought of as putting together, intersection is like cutting down, and complementation corresponds to throwing out. Their precise definitions are as follows:

5.8 DEFINITION Let A and B be sets. The **union** of A and B (denoted $A \cup B$), the **intersection** of A and B (denoted $A \cap B$), and the **complement** of B in A (denoted $A \backslash B$) are given by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}$$

If $A \cap B = \varnothing$, then A and B are said to be **disjoint**.

The three set operations given above correspond in a natural way to three of the basic logical connectives:

$$\begin{aligned} x \in A \cup B &\quad \text{iff} \quad (x \in A) \vee (x \in B) \\ x \in A \cap B &\quad \text{iff} \quad (x \in A) \wedge (x \in B) \\ x \in A \backslash B &\quad \text{iff} \quad (x \in A) \wedge {\sim}(x \in B). \end{aligned}$$

5.9 PRACTICE Which logical connective corresponds to the set relationship $A \subseteq B$?

The definition of complementation may seem to be unnecessarily complicated. Why didn't we just define "$\sim B$" to be $\{x: x \notin B\}$? The problem is that $\{x: x \notin B\}$ is too large. For example, suppose that $B = \{2, 4, 6, 8\}$. Then $\{x: x \notin B\}$ contains all of the following (and more!):

the integers 1, 3, 5, 7, 9, 11
the real numbers greater than 25
the function $f(x) = x^2 + 3$
the circle of radius 1 centered at the origin in the plane
the Empire State Building
my uncle Wilbur

It is quite reasonable that the integers 1, 3, 5, 7 should be included in "$\sim B$", and, depending on the context, we might want to include the real numbers greater than 25 as well. But it is quite unlikely that we would want to include any of the other items. Certainly, knowing that my uncle Wilbur is not a member of the set B would contribute little to any discussion of B.

As we have observed earlier, mathematical concepts and proofs always occur within the context of some mathematical system. It is customary for the elements of the system to be called the universal set. Then any set under consideration is a subset of this universal set. If the universal set in a particular discussion were the integers $\mathbb{Z}$, then the nonintegral real numbers greater than 25 would not be included in $\{x: x \notin B\}$. On the other hand, if the universal set were taken to be $\mathbb{R}$, then they would be included.

5.10 EXAMPLE Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ be subsets of the universal set $U = \{1, 2, 3, 4, 5, 6\}$. Then $A \cup B = \{1, 2, 3, 4, 6\}$, $A \cap B = \{2, 4\}$, $A \backslash B = \{1, 3\}$, and $U \backslash B = \{1, 3, 5\}$.

5.11 PRACTICE Fill in the blanks in the proof of the following theorem.

THEOREM: Let A and B be subsets of a universal set U. Then

$$A \cap (U \backslash B) = A \backslash B.$$

Proof: According to our definition of equality of sets, we must show that

$$[A \cap (U \backslash B)] \subseteq [A \backslash B] \text{ and } [A \backslash B] \subseteq [A \cap (U \backslash B)]$$

or, equivalently,

$$x \in A \cap (U \backslash B) \text{ iff } x \in A \backslash B.$$

Let us begin by showing that $x \in A \cap (U \backslash B)$ implies that $x \in A \backslash B$. If $x \in A \cap (U \backslash B)$, then $x \in A$ and $x \in$ ____________, by the definition of intersection. But $x \in U \backslash B$ means that $x \in U$ and ____________. Since $x \in A$ and $x \notin B$, we have $x \in$ ____________, as required. Thus $A \cap (U \backslash B) \subseteq A \backslash B$.

Conversely, we must show that __________ $\subseteq$ __________. If __________, then $x \in A$ and $x \notin B$. Since $A \subseteq U$, we have $x \in$ __________. Thus $x \in U$ and $x \notin B$, so __________. But then __________ and $x \in U \backslash B$, so $x \in A \cap (U \backslash B)$. Hence $A \backslash B \subseteq A \cap (U \backslash B)$. ♦

A helpful way to visualize the set operations of union, intersection, and complementation is by use of **Venn diagrams**, as in Figures 5.1 and 5.2. In Figure 5.1 the shaded area represents the union of A and B, and in Figure 5.2 the shaded area is the intersection of A and B. In each case the large rectangle represents the universal set U. While Venn diagrams (and other diagrams as well) are useful in seeing the relationship between sets, and may be helpful in getting ideas for developing a proof, they should not be viewed as proofs themselves. A diagram necessarily represents only one case, and it may not be obvious whether this is a general case that always applies or whether there may be other cases as well.

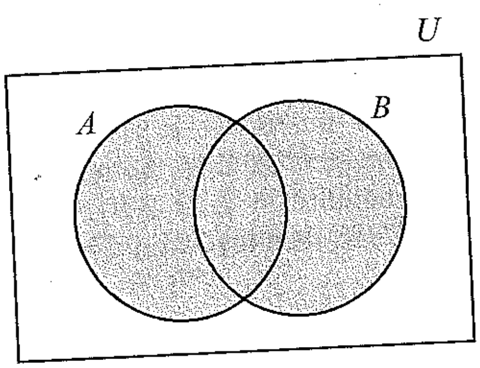


Figure 5.1 $A \cup B$

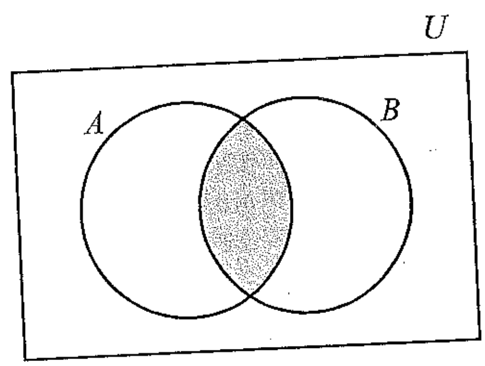


Figure 5.2 $A \cap B$

5.12 PRACTICE Use a Venn diagram to illustrate $A \backslash B$.

We close this section by stating some of the important properties of unions, intersections, and complements. Two of the proofs are sketched as practice problems and the others are left for the exercises.

5.13 THEOREM Let A, B, and C be subsets of a universal set U. Then the following statements are true.

(a) $A \cup (U \backslash A) = U$
(b) $A \cap (U \backslash A) = \varnothing$
(c) $U \backslash (U \backslash A) = A$
(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(f) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$
(g) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

.14 PRACTICE Complete the following proof of Theorem 5.13(d).

Proof: We begin by showing that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. If $x \in$ ____________, then either $x \in A$ or $x \in B \cap C$. If $x \in A$, then certainly $x \in A \cup B$ and $x \in A \cup C$. Thus $x \in$ ____________. On the other hand, if ____________, then $x \in B$ and $x \in C$. But this implies that $x \in A \cup B$ and ____________, so $x \in (A \cup B) \cap (A \cup C)$. Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Conversely, if $y \in (A \cup B) \cap (A \cup C)$, then ____________ and ____________. There are two cases to consider: when $y \in A$ and when $y \notin A$. If $y \in A$, then $y \in A \cup (B \cap C)$ and this part is done. On the other hand, if ____________, then since $y \in A \cup B$, we must have $y \in B$. Similarly, since $y \in A \cup C$ and $y \notin A$, we have ____________. Thus ____________, and this implies that $y \in A \cup (B \cap C)$. Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. ♦

Before going on to the proof of 5.13(f), let us make a couple of observations about the proof of 5.13(d). Notice how the argument divides naturally into parts, the second part being introduced by the word "conversely." This word is appropriate because the second half of the argument is indeed the converse of the first half. In the first part the point in $A \cup (B \cap C)$ was called x and in the second part the point in $(A \cup B) \cap (A \cup C)$ was called y. Why is this? The choice of a name is completely arbitrary, and in fact the same name could have been used in both parts. It is important to realize that the two parts are completely separate arguments; we start over from scratch in proving the converse and can use nothing that was derived about the point x in the first part. By using different names for the points in the two parts we emphasize this separateness. It is common practice, however, to use the same name (such as x) for the arbitrary point in both parts. When doing this we have to be careful not to confuse the properties of the points in the two parts.

We also notice that each half of the argument also has two parts or cases, the second case being introduced by the phrase "on the other hand." This type of division of the argument is necessary when dealing with unions. If $x \in S \cup T$, then $x \in S$ or $x \in T$. Each of the possibilities must be followed to its logical conclusion, and both "bridges" must lead to the same desired result (or to a contradiction, which would show that the alternative possibility could not occur).

Finally, when proving that one set, say S, is a subset of another set, say T, it is common to begin with the phrase "If $x \in S$, then..." It is also acceptable to begin with "Let $x \in S$" and then conclude that $x \in T$. The subtle difference between these phrases is that "Let $x \in S$" assumes that S is nonempty, so there is an x in S to choose. This might seem to be an unwarranted assumption, but really it is not. If S is the empty set, then of course $S \subseteq T$, so the only nontrivial case to prove is when S is nonempty.

5.15 PRACTICE Complete the following proof of Theorem 5.13(f).

Proof: We wish to prove that $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$. To this end, let $x \in A\backslash(B \cup C)$. Then ____________ and ____________. Since $x \notin B \cup C$, ____________ and $x \notin C$ (for if it were in either B or C, then it would be in their union). Thus $x \in A$ and $x \notin B$ and $x \notin C$. Hence $x \in A\backslash B$ and $x \in A\backslash C$, which implies that ____________. We conclude that $A\backslash(B \cup C) \subseteq (A\backslash B) \cap (A\backslash C)$.

Conversely, suppose that $x \in$ ____________. Then $x \in A\backslash B$ and $x \in A\backslash C$. But then $x \in$ ________ and $x \notin$ ________ and $x \notin$ ________. This implies that $x \notin (B \cup C)$, so $x \in$ ____________. Hence ____________ $\subseteq$ ____________ as desired. ♦

Up to this point we have talked about combinations of two or three sets. By repeated application of the appropriate definitions we can even consider unions and intersections of any finite collection of sets. But sometimes we want to deal with combinations of infinitely many sets, and for this we need a new notation and a more general definition.

5.16 DEFINITION If for each element j in a nonempty set J there corresponds a set A_j, then

$$\mathscr{A} = \{A_j : j \in J\}$$

is called an **indexed family** of sets with J as the index set. The union of all the sets in $\mathscr{A}$ is defined by

$$\bigcup\nolimits_{j \in J} A_j = \{x : x \in A_j \text{ for some } j \in J\}.$$

The intersection of all the sets in $\mathscr{A}$ is defined by

$$\bigcap\nolimits_{j \in J} A_j = \{x : x \in A_j \text{ for all } j \in J\}.$$

Other notations for $\bigcup_{j \in J} A_j$ include $\displaystyle\bigcup_{j \in J} A_j$ and $\bigcup \mathscr{A}$.

If $J = \{1, 2, \ldots, n\}$, we may write

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{j=1}^{n} A_j \quad \text{or} \quad \bigcup\nolimits_{j=1}^{n} A_j,$$

and if $J = \mathbb{N}$, the common notation is

$$\bigcup_{j=1}^{\infty} A_j \quad \text{or} \quad \bigcup\nolimits_{j=1}^{\infty} A_j.$$

Similar variations of the notation apply to intersections.

There are some situations where a family of sets has not been indexed but we still wish to take the union or intersection of all the sets. If $\mathscr{B}$ is a nonempty collection of sets, then we let

$$\bigcup_{B\in\mathscr{B}} B=\{x:\exists\, B\in\mathscr{B}\ni x\in B\}$$

and

$$\bigcap_{B\in\mathscr{B}} B=\{x:\forall\, B\in\mathscr{B},\ x\in B\}.$$

17 EXAMPLE For each $k\in\mathbb{N}$, let $A_k=[0,2-1/k]$. Then $\bigcup_{k=1}^{\infty}A_k=[0,2)$.

18 PRACTICE Let $S=\{x\in\mathbb{R}: x>0\}$. For each $x\in S$, let $A_x=(-1/x,\,1/x)$. Find $\bigcap_{x\in S}A_x$.

Review of Key Terms in Section 5

Subset	Empty set	Disjoint sets
Proper subset	Union	Indexed family
Equal sets	Intersection	
Interval	Complement	

VERS TO PRACTICE PROBLEMS

5.2 (a), (b), and (d) are sets. (c) is not a set unless "tall" and "in" are made precise.

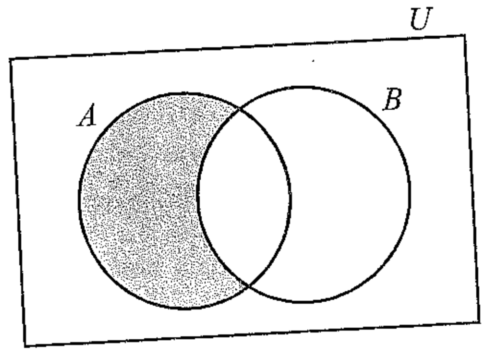
5.6 (a), (c), (f), (g), and (h) are true.

5.9 $x\in A\Rightarrow x\in B$.

5.11 If $x\in A\cap(U\backslash B)$, then $x\in A$ and $x\in U\backslash B$, by the definition of intersection. But $x\in U\backslash B$ means that $x\in U$ and $x\notin B$. Since $x\in A$ and $x\notin B$, we have $x\in A\backslash B$, as required. Thus $A\cap(U\backslash B)\subseteq A\backslash B$.

Conversely, we must show that $A\backslash B\subseteq A\cap(U\backslash B)$. If $x\in A\backslash B$, then $x\in A$ and $x\notin B$. Since $A\subseteq U$, we have $x\in U$. Thus $x\in U$ and $x\notin B$, so $x\in U\backslash B$. But then $x\in A$ and $x\in U\backslash B$, so $x\in A\cap(U\backslash B)$. Hence $A\backslash B\subseteq A\cap(U\backslash B)$. ♦

5.12



5.14 We begin by showing that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. If $x \in A \cup (B \cap C)$, then either $x \in A$ or $x \in B \cap C$. If $x \in A$, then certainly $x \in A \cup B$ and $x \in A \cup C$. Thus $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$, then $x \in B$ and $x \in C$. But this implies that $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Conversely, if $y \in (A \cup B) \cap (A \cup C)$, then $y \in A \cup B$ and $y \in A \cup C$. There are two cases to consider: when $y \in A$ and when $y \notin A$. If $y \in A$, then $y \in A \cup (B \cap C)$ and this part is done. On the other hand, if $y \notin A$, then since $y \in A \cup B$, we must have $y \in B$. Similarly, since $y \in A \cup C$ and $y \notin A$, we have $y \in C$. Thus $y \in B \cap C$, and this implies that $y \in A \cup (B \cap C)$. Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. ♦

5.15 We wish to prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. To this end, let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Since $x \notin B \cup C$, $x \notin B$ and $x \notin C$ (for if it were in either B or C, then it would be in their union). Thus $x \in A$ and $x \notin B$ and $x \notin C$. Hence $x \in A \setminus B$ and $x \in A \setminus C$, which implies that $x \in (A \setminus B) \cap (A \setminus C)$. We conclude that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.

Conversely, suppose that $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A \setminus B$ and $x \in A \setminus C$. But then $x \in A$ and $x \notin B$ and $x \notin C$. This implies that $x \notin (B \cup C)$, so $x \in A \setminus (B \cup C)$. Hence $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$, as desired. ♦

5.18 $\{0\}$

EXERCISES

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

5.1 Mark each statement True or False. Justify each answer.

(a) If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B.

(b) The symbol $\mathbb{N}$ is used to denote the set of all integers.
(c) A slash (/) through a symbol means "not."
(d) The empty set is a subset of every set.

5.2 Mark each statement True or False. Justify each answer.

(a) If $A \cap B = \varnothing$, then either $A = \varnothing$ or $B = \varnothing$.
(b) If $x \in A \cup B$, then $x \in A$ or $x \in B$.
(c) If $x \in A \backslash B$, then $x \in A$ or $x \notin B$.
(d) In proving $S \subseteq T$, one should avoid beginning with "Let $x \in S$," because this assumes that S is nonempty.

5.3 Let $A = \{2,4,6,8\}$, $B = \{3,4,5,6\}$, and $C = \{5,4\}$. Which of the following statements are true? ☆

(a) $\{6,8\} \subseteq A$
(b) $C \subseteq A \cap B$
(c) $(B \backslash C) \cap A = \{6\}$
(d) $(A \backslash B) \cap C \subseteq B$
(e) $\varnothing \in A$
(f) $C \subseteq B$
(g) $(A \cup B) \backslash C = \{2,3,6,8\}$
(h) $A \cap B \cap C = 4$

5.4 Let $A = \{2,4,6,8\}$, $B = \{1,2,3,4\}$, and $C = \{5,6,7\}$. Find the following sets.

(a) $A \cap B$
(b) $A \cup B$
(c) $A \backslash B$
(d) $B \cap C$
(e) $B \backslash C$
(f) $(B \cup C) \backslash A$
(g) $(A \cap C) \backslash B$
(h) $C \backslash (A \cup B)$

5.5 Use Venn diagrams with three overlapping circles to illustrate each identity.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ☆
(b) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$

5.6 Let A and B be subsets of a universal set U. Simplify each of the following expressions.

(a) $(A \cup B) \cup (U \backslash A)$
(b) $(A \cap B) \cap (U \backslash A)$
(c) $A \cap [B \cup (U \backslash A)]$
(d) $A \cup [B \cap (U \backslash A)]$
(e) $(A \cup B) \cap [A \cup (U \backslash B)]$
(f) $(A \cap B) \cup [A \cap (U \backslash B)]$

5.7 Let A and B be subsets of a universal set U. Define the symmetric difference $A \triangle B$ by

$$A \triangle B = (A \backslash B) \cup (B \backslash A).$$

(a) Draw a Venn diagram for $A \triangle B$.
(b) What is $A \triangle A$?
(c) What is $A \triangle \varnothing$?
(d) What is $A \triangle U$?

5.8 Let $S = \{\varnothing, \{\varnothing\}\}$. Determine whether each of the following is True or False. Explain your answers.

(a) $\varnothing \subseteq S$ (b) $\varnothing \in S$ (c) $\{\varnothing\} \subseteq S$ (d) $\{\varnothing\} \in S$

5.9 Fill in the blanks in the following proof of Theorem 5.13(a).

THEOREM: Let A be a subset of U. Then $A \cup (U \backslash A) = U$.

Proof: If $x \in A \cup (U \backslash A)$, then $x \in$ __________ or $x \in$ __________. Since both A and $U \backslash A$ are subsets of U, in either case we have __________. Thus __________ $\subseteq$ __________.

On the other hand, suppose that $x \in$ __________. Now either $x \in A$ or $x \notin A$. If $x \notin A$, then $x \in$ __________. In either case $x \in$ __________. Hence __________ $\subseteq$ __________. ♦

5.10 Fill in the blanks in the proof of the following theorem.

THEOREM: $A \subseteq B$ iff $A \cup B = B$.

Proof: Suppose that $A \subseteq B$. If $x \in A \cup B$, then $x \in A$ or __________. Since $A \subseteq B$, in either case we have $x \in B$. Thus __________ $\subseteq$ __________. On the other hand, if $x \in$ __________, then $x \in A \cup B$, so __________ $\subseteq$ __________. Hence $A \cup B = B$.

Conversely, suppose that $A \cup B = B$. If $x \in A$, then $x \in$ __________. But $A \cup B = B$, so $x \in$ __________. Thus __________ $\subseteq$ __________. ♦

5.11 Fill in the blanks in the proof of the following theorem.

THEOREM: $A \subseteq B$ iff $A \cap B = A$.

Proof: Suppose that $A \subseteq B$. If $x \in A \cap B$, then clearly $x \in A$. Thus $A \cap B \subseteq A$. On the other hand, ____.

Thus $A \subseteq A \cap B$, and we conclude that $A \cap B = A$.

Conversely, suppose that $A \cap B = A$. If $x \in A$, then ____.

Thus $A \subseteq B$. ♦

5.12 Suppose you are to prove that set A is a subset of set B. Write a reasonable beginning sentence for the proof, and indicate what you would have to show in order to finish the proof.

5.13 Suppose you are to prove that sets A and B are disjoint. Write a reasonable beginning sentence for the proof, and indicate what you would have to show in order to finish the proof.

5.14 Which statement(s) below would enable one to conclude that $x \in A \cup B$?

(a) $x \in A$ and $x \in B$.
(b) $x \in A$ or $x \in B$.
(c) If $x \in A$, then $x \in B$.
(d) If $x \notin A$, then $x \in B$.

5.15 Which statement(s) below would enable one to conclude that $x \in A \cap B$?

(a) $x \in A$ and $x \in B$.
(b) $x \in A$ or $x \in B$.
(c) $x \in A$ and $x \notin A \backslash B$.
(d) If $x \in A$, then $x \in B$.

5.16 Which statement(s) below would enable one to conclude that $x \in A \backslash B$?

(a) $x \in A$ and $x \notin B \backslash A$.
(b) $x \in A \cup B$ and $x \notin B$.
(c) $x \in A \cup B$ and $x \notin A \cap B$.
(d) $x \in A$ and $x \notin A \cap B$.

5.17 Which statement(s) below would enable one to conclude that $x \notin A \backslash B$?

(a) $x \notin A \cup B$.
(b) $x \in B \backslash A$.
(c) $x \in A \cap B$.
(d) $x \in A \cup B$ and $x \notin A$.
(e) $x \in A \cup B$ and $x \notin A \cap B$.

5.18 Prove that the empty set is unique. That is, suppose that A and B are empty sets and prove that $A = B$.

5.19 Prove: If $U = A \cup B$ and $A \cap B = \varnothing$, then $A = U \backslash B$. ☆

5.20 Prove: $A \cap B$ and $A \backslash B$ are disjoint and $A = (A \cap B) \cup (A \backslash B)$.

5.21 Prove or give a counterexample: $A \backslash (A \backslash B) = B \backslash (B \backslash A)$.

5.22 Prove or give a counterexample: $A \backslash (B \backslash A) = B \backslash (A \backslash B)$.

5.23 Let A and B be subsets of a universal set U. Prove the following.

(a) $A \backslash B = (U \backslash B) \backslash (U \backslash A)$
(b) $U \backslash (A \backslash B) = (U \backslash A) \cup B$
(c) $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$

5.24 Finish the proof of Theorem 5.13.

5.25 Find $\bigcup_{B \in \mathscr{B}} B$ and $\bigcap_{B \in \mathscr{B}} B$ for each collection $\mathscr{B}$.

(a) $\mathscr{B} = \left\{ \left[1, 1 + \frac{1}{n} \right] : n \in \mathbb{N} \right\}$ ☆
(b) $\mathscr{B} = \left\{ \left(1, 1 + \frac{1}{n} \right) : n \in \mathbb{N} \right\}$
(c) $\mathscr{B} = \{[2, x] : x \in \mathbb{R} \text{ and } x > 2\}$
(d) $\mathscr{B} = \{[0,3], (1,5), [2,4)\}$

***5.26** Let $\{A_j : j \in J\}$ be an indexed family of sets and let B be a set. Prove the following generalizations of Theorem 5.13.

(a) $B \cup \left[\bigcap_{j \in J} A_j \right] = \bigcap_{j \in J} (B \cup A_j)$
(b) $B \cap \left[\bigcup_{j \in J} A_j \right] = \bigcup_{j \in J} (B \cap A_j)$
(c) $B \backslash \left[\bigcup_{j \in J} A_j \right] = \bigcap_{j \in J} (B \backslash A_j)$
(d) $B \backslash \left[\bigcap_{j \in J} A_j \right] = \bigcup_{j \in J} (B \backslash A_j)$