CSC236 2015 Winter
Assignment 1: Solutions

(1) We prove that $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (1+mn) \le (1+m)^n$.

Proof. Let $m \in \mathbb{N}$.

Now by Simple Induction we prove $\forall n \in \mathbb{N}, (1+mn) \le (1+m)^n$.

<u>Base Case: 0.</u> $(1+m\cdot 0) = 1 \le 1 = (1+m)^0$.

<u>Inductive Step</u> Let $n \in \mathbb{N}$.

(IH) Assume $(1+mn) \le (1+m)^n$.

Then

$$\begin{aligned}(1+m)^{n+1} &= (1+m)^n \cdot (1+m) \ge (1+mn)(1+m)\,, \text{ by (IH)},\\ &= 1+mn+m+m^2n = (1+m(n+1)) + mn^2\\ &\ge (1+m(n+1)) \ \text{(since } mn^2 \ge 0\text{, since } m \ge 0\text{)}.\end{aligned}$$

(2) We prove that $r_n \le 236\left(\log_2\left(\log_2 n\right)\right)$ for all natural numbers $n \ge 4$.

Proof. By Complete Induction.

Let n be a natural number with $n \ge 4$.

<u>Base Cases $4 \le n \le 15$.</u>

Then $1 = \left\lfloor\sqrt{\lfloor\sqrt{4}\rfloor}\right\rfloor \le \left\lfloor\sqrt{\lfloor\sqrt{n}\rfloor}\right\rfloor \le \left\lfloor\sqrt{\lfloor\sqrt{15}\rfloor}\right\rfloor = 1$, so $\left\lfloor\sqrt{\lfloor\sqrt{n}\rfloor}\right\rfloor = 1$.

So

$$\begin{aligned}r_n &= 1 + r_{\lfloor\sqrt{n}\rfloor} = 1 + \left(1 + r_{\left\lfloor\sqrt{\lfloor\sqrt{n}\rfloor}\right\rfloor}\right) = 2 + r_1 = 3\\ &\le 236\cdot 1 = 236\log_2 2 = 236\log_2\left(\log_2 4\right) \le 236\log_2\left(\log_2 n\right)\end{aligned}$$

since $4 \le n$ and $\log_2 \circ \log_2$ is increasing.

<u>Inductive Step</u> Let $n \in \mathbb{N}$ with $16 \le n$.

(IH) Suppose $r_k \le 236\log_2\left(\log_2 k\right)$ for each $k \in \mathbb{N}$ such that $4 \le k < n$.

Since $n \ge 16$: $4 = \lfloor\sqrt{16}\rfloor \le \lfloor\sqrt{n}\rfloor \le \sqrt{n} < n$, so the (IH) applies for $k = \lfloor\sqrt{n}\rfloor$.

Then

$$\begin{aligned}r_n &= 1 + r_{\lfloor\sqrt{n}\rfloor}\\ &\le 1 + 236\log\left(\log_2\lfloor\sqrt{n}\rfloor\right), \text{ from (IH) as noted above,}\\ &\le 1 + 236\log_2\left(\log_2\sqrt{n}\right) \ \text{ since } \log_2\circ\log_2 \text{ is increasing and } \lfloor\sqrt{n}\rfloor \le \sqrt{n}\\ &= 1 + 236\log_2\left(\frac{1}{2}\log_2 n\right)\\ &= 1 + 236\left(-1 + \log_2\left(\log_2 n\right)\right)\\ &= (1-236) + 236\log_2\left(\log_2 n\right) \le 236\log_2\left(\log_2 n\right).\end{aligned}$$

(3) (a) Define b by:

$$\begin{aligned} b_0 &= 1, \\ b_h &= 2b_{h-1}\left(b_0 + \cdots + b_{h-1}\right) - b_{h-1}^2, \;\; h \geq 1. \end{aligned}$$

Claim: for all natural numbers h, b_h is the number of binary trees of height h.

Proof. By Complete Induction.

Base Case: 0.

There is exactly one empty tree, and $b_0 = 1$, so b_0 is the number of binary trees of height 0.

Inductive Step Let $h \in \mathbb{N}$ with $1 \leq h$.

(IH) Suppose b_i is the number of binary trees of height i, for each $i \in \mathbb{N}$ such that $0 \leq i < h$.

A binary tree of height $h \geq 1$ is determined by its left and right subtrees, which are binary trees of height less than h, with one of them having height exactly $h-1$.

A tree of height less than h has height $0, 1, \ldots,$ or $h-1$, and the number of trees of each of those heights is $b_0, b_1, \ldots,$ and b_{h-1} (by the (IH) for $i = 0, 1, \ldots, h-1 < h$).

So the number of trees of height less than h is $b_0 + \cdots + b_{h-1}$.

If the left subtree has height $h-1$ there are (by (IH)) b_{h-1} possibilities, multiplied by the $b_0 + \cdots + b_{h-1}$ possibilities for the right subtree. There are the same amount again if we switch left and right, doubling the total. That double-counts the case where the left and right subtrees both are of height $h-1$, so subtract off the number of those $(b_{h-1} \cdot b_{h-1})$.

(b) Claim: $b_{h+1} = a_{h+1}^2 - a_h^2$ for all natural numbers h.

Proof. By Complete Induction.

Base Case: 0.

$b_{0+1} = b_1 = 2b_0\left(b_0\right) - b_0^2 = 2 - 1 = 1 = \left(0^2+1\right)^2 - 0^2 = \left(a_0^2+1\right)^2 - a_0^2 = a_{0+1}^2 - a_0^2.$

Inductive Step Let $h \in \mathbb{N}$ with $1 \leq h$. Note that $h - 1 \in \mathbb{N}$, which we'll use a few times.

(IH) Suppose $b_{i+1} = a_{i+1}^2 - a_i^2$ for $i = 0, \ldots, h-1$.

Then

$$\begin{aligned} b_{h+1} &= 2b_h\left(b_0 + \cdots + b_h\right) - b_h^2 \\ &= 2b_h\left(b_0 + \left[b_1 + \cdots + b_h\right]\right) - b_h^2 \end{aligned}$$

where splitting out $[b_1 + \cdots + b_h]$ is valid since $1 \leq h$. From (IH) for $i = 0, \ldots, h-1$, we get

$= 2b_h\left(b_0 + \left[\left(a_1^2 - a_0^2\right) + \left(a_2^2 - a_1^2\right) + \cdots + \left(a_h^2 - a_{h-1}^2\right)\right]\right) - b_h^2$

$= 2b_h\left(b_0 + a_h^2 - a_0^2\right) - b_h^2$

$= 2b_h\left(1 + a_h^2\right) - b_h^2$

$= b_h\left(2 + 2a_h^2 - b_h\right)$

$= \left(a_h^2 - a_{h-1}^2\right)\left(2 + 2a_h^2 - \left(a_h^2 + a_{h-1}^2\right)\right)$ (from (IH) for $i = h-1$)

$= \left(a_h^2 - a_{h-1}^2\right)\left(2 + a_h^2 - a_{h-1}^2\right)$

$= \left(\left(a_{h+1} - 1\right) - \left(a_h - 1\right)\right)\left(2 + \left(a_{h+1} - 1\right) - \left(a_h - 1\right)\right)$ (from $a_n$for $n = h, h-1 \in \mathbb{N}$)

$= \left(a_{h+1} - a_h\right)\left(a_{h+1} + a_h\right)$

$= a_{h+1}^2 - a_h^2.$