PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 11
DUE FRIDAY, MAY 26, 4PM.

Warm-up problems. These are completely optional.

(1) The wheel with n vertices consists of a cycle with $n-1$ vertices and an additional vertex adjacent to all the vertices on that cycle. What is the chromatic number of the wheel with n vertices?

(2) Given two vertices $u, v \in V(G)$, prove that G contains a uv-path if and only if G contains a uv-trail.

Problems to be handed in. Solve four of the following five problems.

11.20 (1) Let G be a simple graph with n vertices.
   (a) Let x and y be nonadjacent vertices of degree at least $(n+k-2)/2$. Prove that x and y have at least k common neighbors.
   (b) Prove that if every vertex has degree at least $\lfloor n/2 \rfloor$, then G is connected. Show that this bound is the best possible whenever $n \geq 2$ by exhibiting a disconnected n-vertex graph where every vertex has at least $\lfloor n/2 \rfloor - 1$ neighbors.

11.4 (2) Let G be a connected graph with $m \geq 2$ vertices of odd degree. (Recall from the previous tutorial that m is even.) Prove that the minimum number of trails that together traverse each edge of G exactly once is $m/2$. (Hint: Transform G into a new graph G' by adding edges and/or vertices.)

(3) Let G be a graph with n vertices and no cycles of length three. Prove that G has at
11.25 most $n^2/4$ edges. (Hint: Consider the subgraph consisting of neighbors of a vertex of maximum degree and the edges among them.)

11.40 (4) Suppose that every vertex of a graph G has degree at most k. Prove that $\chi(G) \leq k+1$. Show that this bound is the best possible by exhibiting (for every k) a graph with maximum degree k and chromatic number $k+1$.

11.32 (5) Let T be a tree with m edges, and let G be a simple graph in which every vertex has degree at least m. Prove that G contains T as a subgraph. (Hint: Induction on m.)