July 17th

1. §3.4: For the transformation $u = \frac{x+y}{2\sqrt{2}}, v = \frac{y-x}{2\sqrt{2/3}}$,

a) If the "before" sketch is a grid in Euclidean space, draw the "after" sketch.

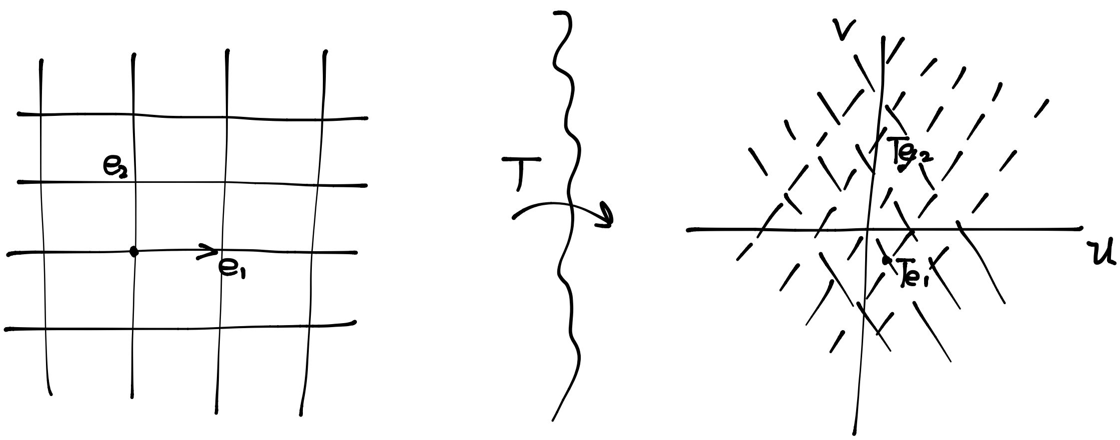
b) Determine the inverse of this transformation and see the effect of this transformation on the ellipse
$$x^2 - xy + y^2 = 2.$$

c) For the transformation $u = x\tan y, v = xy$, draw the effect of this transformation on the lines
$$x = 1, x = -1, x = 2, x = 0$$
and
$$y = 0, y = 1rad, y = -1rad,$$
as well as determining the Fréchet derivative and discuss possibility of finding the inverse, but no need to locally solve for the inverse.

(a) Since T is linear, Let's look at what T does to a basis.

$Te_1 = T(1,0) = (1/2\sqrt{2}\ ,\ -1/2\sqrt{2/3}) = 1/2\sqrt{2}\ (1, -\sqrt{3})$

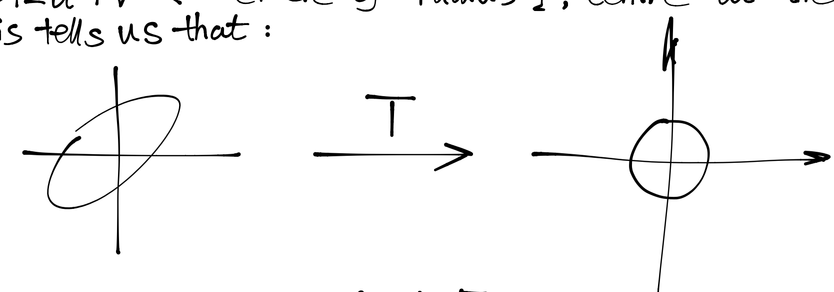
$Te_2 = T(0,1) = (1/2\sqrt{2},\ 1/2\sqrt{2/3}) = 1/2\sqrt{2}\ (1, \sqrt{3})$



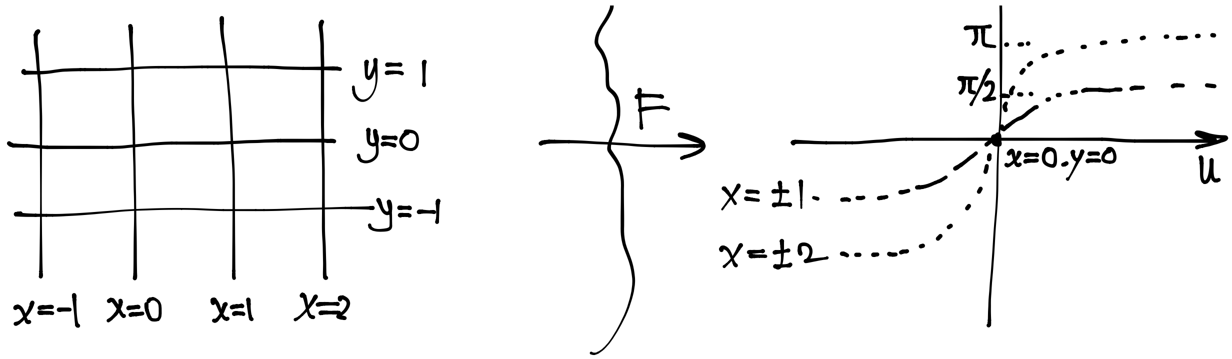
(b). Ellipse $x^2 - xy + y^2 = 2$ (*)

Solve the two eqns for x, y.
$$x = \sqrt{2}u - \sqrt{2/3}\ v,\quad y = \sqrt{2}u + \sqrt{2/3}\ v$$
Substitute these into (*)
$$2 = (\sqrt{2}u - \sqrt{2/3}\ v)^2 - (\sqrt{2}u - \sqrt{2/3}\ v)(\sqrt{2}u + \sqrt{2/3}\ v) + (\sqrt{2}u + \sqrt{2/3}\ v)^2 = 2u^2 + 2v^2$$
$\Rightarrow 1 = u^2 + v^2 \leftarrow$ circle of radius 1, centre at the origin.

This tells us that :



(c). $u = x\tan y \quad v = xy \quad (u,v) = F(x,y)$



For $x=1$, $u=\tan y$, $v=y \Rightarrow u=\tan v$
$x=-1$, $u=-\tan y$, $v=-y \Rightarrow u=\tan v$ as well
$x=2$, $u=2\tan y$, $v=2y \Rightarrow u=2\tan(v/2)$

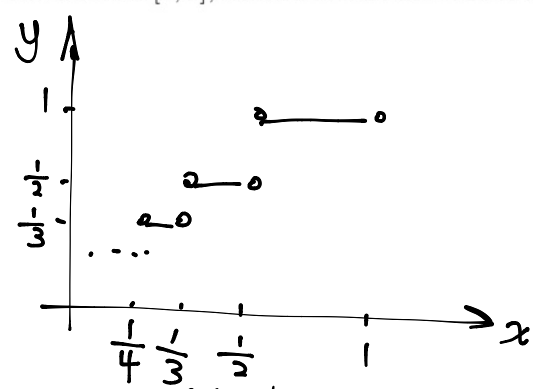
For $y=1$, $u=(\tan 1)x$, $v=x \Rightarrow u=(\tan 1)v$, $v=(1/\tan 1)u$
$y=-1$, $u=(\tan -1)x$, $v=-x \Rightarrow u=(\tan 1)v$

$$DF=\begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix}=\begin{pmatrix} \tan y & y \\ x\sec^2 y & x \end{pmatrix}$$

$\det DF = x\tan y - xy\sec^2 y$
This vanishes for some values of x, y (e.g. $x=0$, or $y=0$).
But there are many pts where $\det DF \neq 0$, e.g. $(x,y)=(1,\pi)$
$\det DF(1,\pi)=-\pi \neq 0$

By the IFT, F is locally invertible near $(x,y)=(1,\pi)$. But, this can't be a global inverse, since we saw that F is not injective (one-to-one)

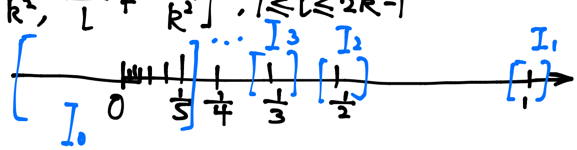
2. §4.1: Consider the ultimate step function, $f(x)$ defined on $[0,1]$ as follows: $f(0)=1$, and $f(x)=\frac{1}{n}$ for $\frac{1}{n+1}<x<\frac{1}{n}$ for all $n=1,2,3,\cdots$. Use Lemma 4.5 to prove that f is Riemann integrable on the interval $[0,1]$, and then calculate the Riemann integral $I_0^1 f$.



Let $S=\{0,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\}$
We're going to show that S has zero content ($\Rightarrow$ value of f on S doesn't affect the integral)

Fix some $k\in\mathbb{N}$ and consider the following intervals $I_0=[-\frac{1}{2k},\frac{1}{2k}]$,
$I_l=[\frac{1}{l}-\frac{1}{k^2},\frac{1}{l}+\frac{1}{k^2}]$, $1\leq l\leq 2k-1$

S :



Notice that if $\frac{1}{n} \leqslant \frac{1}{2k}$ then $\frac{1}{n} \in I_0$, otherwise $\frac{1}{n} > \frac{1}{2k} \Rightarrow \frac{1}{n} \in I_n$

$$\Rightarrow \bigcup_{l=0}^{2k-1} I_l \supseteq S$$

And $$\left|\bigcup_{l=0}^{2k-1} I_l\right| \leqslant |I_0| + \sum_{l=1}^{2k-1} |I_l| \leqslant \frac{1}{k} + \sum_{l=1}^{2k-1} \frac{2}{k^2} = \frac{1}{k} + \frac{2(2k-1)}{k^2} \xrightarrow{k\to\infty} 0$$

This shows S has zero content

(b) Since f is bdd on $[0,1]$ and the set of pts S where f is discontinuous has zero content, thm 4.13 $\Rightarrow$ f is Riemann intergrable

$$\int_0^1 f(x)dx = \sum \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{\pi^2}{6} - 1$$