Sta261H1 S

Midterm Exam

Feb. 13, 2013

Name (Print!): Zhou Zhou Student ID:________________________

- Do not turn the page until told to do so.
- Do not sit directly next to another student.
- This is a closed book examination. You should have your hand calculator and you may use one (double-sided) sheet of formulas. You should have no other written material with you during the exam.
- If a question asks you do some calculations, you must show your work to receive full credit. In particular, if you are basing your calculations on a formula, write down that formula before you substitute numbers into it.
- If a later part of a question depends on an earlier part, the later part will be graded conditionally on how you answered the earlier part, so that a mistake on the earlier part will not cost you points on the later part (unless perhaps the previous mistake was absolutely horrendous). If you can't work out the actual answer to an earlier part, put down your best guess and proceed.
- Write your answers in the spaces provided. If you do not have enough room to show all your work in the space provided, use the back of a nearby page; in such cases write something like "see also the back of page n" and be sure to mark clearly which problem the material on the back of any page refers to. If you pull the pages apart, sign all the pages. This exam has 8 pages, including a blank page at the end for scratch work.
- Use your time wisely, taking note of the points assigned to the various parts of the questions. There are a total of 100 points. It would not be wise to spend, say, 20 minutes on a part worth 8 points before you have tried to work the other parts of the exam.
- If you don't understand a question, or are having some other difficulty, see your instructor or TA.

1. **[34 points]** Suppose that 100 observations $X_1, X_2, \cdots, X_{100}$ are selected randomly from a population of size 1000 without replacement. The sample mean $\bar{X} = 5.0$ and the sample variance $s_X^2 = 25$.

   (a). **[8 points]** Which (if any) of the two numbers $\bar{X}$ and s^2 is a random variable? And why?

   They are both random variables. Because the values of $\bar{X}$ & S^2 will change from sample to sample.

   (b). **[9 points]** Find an estimate of the variance $\mathrm{Var}(\bar{X})$.

   Note $\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)$.

   Hence an estimate of it is

   $$\frac{S^2}{n}\left(\frac{N-n}{N-1}\right) = \frac{25}{100} \cdot \frac{1000-100}{1000-1}$$

   $$= 0.225.$$

(c). [**8 points**] Find a 95% confidence interval for the population mean μ_X.

A 95% CI for μ_X can be constracted as

$$\hat{\mu}_X \pm 1.96\sqrt{Var(\bar{X})}$$

$$= 5 \pm 1.96 \cdot \sqrt{0.225}$$

$$= [4.07, 5.93]$$

(d). [**9 points**] Now suppose quantities $Y_1, Y_2, \cdots, Y_{100}$ are observed from the same sample with sample mean $\bar{Y} = 3.0$ and sample variance $s_Y^2 = 36$. The sample correlation $\hat{\rho}_{XY} = 0.6$. Based on the information in (a), (b) and (c), construct a 95% confidence interval for the population ratio $r = \mu_Y/\mu_X$.

Note $E[R] \approx r$

& $Var(R) \approx \frac{1}{\mu_X^2} (r^2\sigma_X^2 + \sigma_Y^2 - 2r\sigma_{XY}) \cdot \frac{1}{n}(1 - \frac{n-1}{N-1})$.

An estimate of it

$$\hat{Var}(R) \approx \frac{1}{\bar{X}^2} \frac{1}{n}\left(\frac{N-n}{N-1}\right) \cdot (R^2 S_X^2 + S_Y^2 - 2R S_X S_Y \hat{\rho}_{XY})$$

$$= 0.0084.$$

$\therefore$ A 95% CI for R is

$$0.6 \pm 1.96\sqrt{0.0084} = [0.42, 0.78]$$

2. **[42 points]**

Consider the following distribution $f(x|\theta) = \frac{x}{\theta^2}\exp(-x^2/(2\theta^2))$, $x \geq 0$, $\theta > 0$. Suppose the following 5 i.i.d. observations are obtained from the distribution $f(x|\theta)$. $X_1 = 0.8$, $X_2 = 2$, $X_3 = 4.2$, $X_4 = 1.6$, $X_5 = 3.1$.

(a). **[8 points]** Find the method of moments estimate of θ. (Hint: For Gamma function $\Gamma(a) = \int_0^\infty x^{a-1}\exp(-x)\,dx$, we know that $\Gamma(a+1) = a\Gamma(a)$ for any $a > 0$, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$.)

Note $E[X] = \int_0^\infty \frac{x^2}{\theta^2} e^{-x^2/(2\theta^2)}\,dx$

$= \sqrt{2\pi}\,\theta/2.$

$\therefore \hat{\theta}_{ML} = \frac{2}{\sqrt{2\pi}} \frac{1}{n}\sum_{i=1}^n X_i = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{5}(0.8+2+4.2+1.6+3.1$

$= \cancel{0.934}.\ 1.87.$

(b). **[9 points]** Based on the method of moments estimate in (a) and the given data, find a 95% confidence interval for θ.

According to the CLT.

$\frac{\sqrt{2\pi}}{2}\sqrt{n}(\hat{\theta}_{ML} - \theta) \longrightarrow N(0, \mathrm{Var}(X))$

$EX^2 \cancel{\mathrm{Var}(x)} = \int_0^\infty \frac{x^3}{\theta^2}\exp(-x^2/(2\theta)^2)\,dx$

$= 2\theta^2. \Rightarrow \mathrm{Var}(X) = EX^2 - E(X)^2 = (2 - \frac{\pi}{2})\theta^2$

$\Rightarrow$ 95% CI for θ $= \hat{\theta}_{ML} \pm \frac{2 \times 1.96}{\sqrt{n}\cdot\sqrt{2\pi}} \cdot \sqrt{(2-\frac{\pi}{2})\hat{\theta}_{ML}^2}$

$\cancel{= [0.02, 3.71]}$

$= [1.01, 2.72]$

$\cancel{[1.243, 2.497]}$

(c). **[8 points]** Find the maximum likelihood estimate of θ^2.

Log Likelihood $\ell(\theta^2) = \log \prod_{i=1}^{n} x_i - \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} - n \log \theta^2$

$\ell'(\theta^2) = \frac{1}{\theta^4} \frac{\sum x_i^2}{2} - \frac{n}{\theta^2}$

$\Rightarrow \hat{\theta}^2_{MLE} = \frac{\sum_{i=1}^{n} x_i^2}{2n} = 3.443$

(d). **[9 points]** Based on the maximum likelihood estimate in (c) and the given data, find a 95% confidence interval for θ^2.

$\ell''(\theta^2) = -\frac{\sum x_i^2}{\theta^6} + \frac{n}{\theta^4}$

$\therefore -E\ell''(\theta^2) = \frac{2n\theta^2}{\theta^6} - \frac{n}{\theta^4} = \frac{n}{\theta^4}$

$\Rightarrow nI(\theta) = \frac{n}{\theta^4} \Rightarrow \underset{95\%}{CI} = \hat{\theta}^2_{MLE} \pm \frac{\hat{\theta}^2_{MLE}}{\sqrt{n}} \times 1.96$

$= [0.42, 6.46]$

(e). **[8 points]** Find a sufficient statistic for θ.

$\because \ell(\theta^2) = \log \prod_{i=1}^{n} x_i - \frac{\sum x_i^2}{2\theta^2} - n \log \theta^2$

$\Rightarrow$ Sufficient statistic is $\sum_{i=1}^{n} x_i^2$.

3. **[24 points]**

(a). **[8 points]** Let $X_1, X_2, \cdots, X_n$ be independent with uniform$[0, \theta]$ distribution. Prove that $\hat{\theta} = \max(X_1, X_2, \cdots, X_n)$ is a consistent estimator of θ.

For $\forall x > 0$.

$$P[|\hat{\theta} - \theta| > x] = P[\max(X_1 \cdots X_n] < \theta - x]$$
$$= P(X_1 < \theta - x, X_2 < \theta - x, \cdots X_n < \theta - x)$$
$$= P(X_1 < \theta - x) P(X_2 < \theta - x) \cdots P(X_n < \theta - x)$$
$$= \left(\frac{\theta - x}{\theta}\right)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$\Longrightarrow \hat{\theta}$ is consistent.

(b). **[8 points]** Suppose that $X_1, X_2, \cdots, X_n$ are independent and normally distributed with mean μ and variance 1. Let $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Is there any other unbiased estimator of μ which has smaller variance than $\hat{\mu}$? Prove your conjecture.

No, there isn't.

For this distribution

$$n I(\overset{\mu}{\theta}) = E[-l''(\theta\mu)] = n.$$

By the Cramer-Rao inequality, for any unbiased estimator of μ, $\hat{u}$),

$$Var(\hat{u}) \geq \frac{1}{n}.$$

On the other hand, $Var(\hat{u}) = \frac{1}{n}$. Hence $\hat{u}$ is efficient.

(c). [**8 points**] If $X_1, X_2, \cdots, X_n$ are independent with Geometric(θ) distribution. Find the maximum likelihood estimate of θ. (Reminder: A Geometric(θ) distribution has the following probability mass function : $P(X = x) = (1-\theta)^{x-1}\theta$, $x = 1, 2, \cdots$).

The log Likelihood

$$\ell(\theta) = \left(\sum_{i=1}^{n} x_i\right) \log(1-\theta) + \cancel{\log}\left(\frac{\theta}{1-\theta}\right) n$$

$$\ell'(\theta) = -\frac{\sum_{i=1}^{n} x_i}{1-\theta} + \frac{n}{\theta(1-\theta)}.$$

Let $\ell'(\theta) = 0$, we have

$$\hat{\theta}_{MLE} = \frac{1}{\frac{\sum_{i=1}^{n} x_i}{n}} = \frac{1}{X}.$$

Table entry for z is the area under the standard normal curve to the left of z.

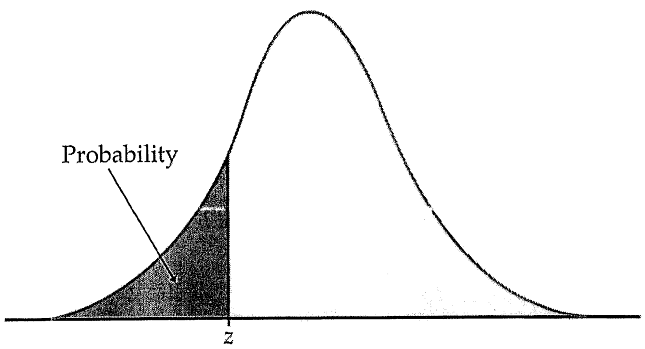

TABLE A
Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641