FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 5

Question 1

Calculate values for the following functions:

a) $\bar{a}_{\overline{7}|}$ at interest rate of 7.5% per annum effective.

b) $(I\bar{a})_{\overline{5}|}$ at interest rate of 10% per annum effective.

c) $\ddot{s}_{\overline{15}|}^{(4)}$ at a nominal rate of 10% convertible quarterly.

d) $s_{\overline{10}|}^{(1/2)}$ at interest rate of 1% per annum effective.

e) $(D\ddot{a})_{\overline{8}|}$ at interest rate of 6% per annum effective.

Solution

a) $\bar{a}_{\overline{7}|} = \dfrac{1-v^7}{\delta} = \dfrac{1-1.075^{-7}}{\ln(1.075)} = 5.4928$ where $i = 0.075$

b) $(I\bar{a})_{\overline{5}|} = \dfrac{\ddot{a}_{\overline{5}|} - 5v^5}{\delta} = 11.177$ where $i = 0.10$

c) 10% convertible quarterly is equivalent to an effective rate of 2.5% per quarter, or $i = 1.025^4 - 1 = 0.103813$ effective rate per annum.

$$\ddot{s}_{\overline{15}|}^{(4)} = \frac{(1+i)^{15}-1}{d^{(4)}} = \frac{(1+i)^{15}-1}{4(1-(1+i)^{-1/4})} = 34.848.$$

If we work in quarters, this is equivalent to $\dfrac{1}{4}\ddot{s}_{\overline{60}|0.025}$

d) $s_{\overline{10}|}^{(1/2)} = \dfrac{(1+i)^{10}-1}{i^{(1/2)}} = \dfrac{(1+i)^{10}-1}{0.5((1+i)^2-1)} = 10.410$ where $i = 0.01$

If we work in 2-year periods, this is equivalent to:

$2s_{\overline{5}|0.0201}$ where the effective 2-year interest rate is $1.01^2 - 1 = 0.0201$

e) $(D\ddot{a})_{\overline{8}|} = \dfrac{8 - a_{\overline{8}|}}{d} = 31.627$ where $i = 0.06$

Question 2

Find $\sum_{t=1}^{20}(t+5)v^t$ where $v=(1.05)^{-1}$

Solution

$$\sum_{t=1}^{20}(t+5)v^t = \sum_{t=1}^{20}tv^t + \sum_{t=1}^{20}5v^t = (Ia)_{\overline{20|}} + 5a_{\overline{20|}}$$

where $v=1.05^{-1} \Rightarrow i=0.05$

$$(Ia)_{\overline{20|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} = \frac{\left(\frac{1-v^n}{d}\right) - nv^n}{i} = \frac{\left(\frac{1-v^n}{\left(\frac{i}{1+i}\right)}\right) - nv^n}{i} = \frac{\left(\frac{1-1.05^{-20}}{\left(\frac{0.05}{1.05}\right)}\right) - 20(1.05)^{-20}}{0.05} \cong 110.95$$

$$5a_{\overline{20|}} = 5\left(\frac{1-v^n}{i}\right) = 5\left(\frac{1-(1.05)^{-20}}{0.05}\right) \cong 62.31$$

$$\therefore \sum_{t=1}^{20}(t+5)v^t \cong 110.95 + 62.31 = 173.26$$

Alternative method:

$$\sum_{t=1}^{20}(t+5)v^t = 6v + 7v^2 + 8v^3 + ... + 24v^{19} + 25v^{20} \qquad \ldots(1)$$

$$(1+i)\cdot\left(\sum_{t=1}^{20}(t+5)v^t\right) = 6 + 7v + 8v^2 + ... + 24v^{18} + 25v^{19} \qquad \ldots(2)$$

$$(2)\text{-}(1) = i\cdot\sum_{t=1}^{20}(t+5)v^t = 6 + (v + v^2 + ... + v^{19}) - 25v^{20}$$

$$= 6 + a_{\overline{19|}} - 25v^{20} = 6 + \left(\frac{1-v^{19}}{i}\right) - 25v^{20} = 6 + \left(\frac{1-(1.05)^{-19}}{0.05}\right) - 25(1.05)^{-20} = 8.66308$$

$$\Rightarrow \sum_{t=1}^{20}(t+5)v^t = \frac{8.66308}{i} \cong 173.26$$

Question 3
The first payment in a series of 30 annual payments is 1000 and each subsequent payment is 1% smaller than the previous one. What is the accumulated value of this series at the time of the final payment if (i) $i = 0.01$, (ii) $i = 0.05$, and (iii) $i = 0.10$?

Solution
Let AV be the accumulated value.

$$AV = 1000\left[(1+i)^{29} + (1+i)^{28}(0.99) + (1+i)^{27}(0.99)^2 + ... + (1+i)^1(0.99)^{28} + (0.99)^{29}\right]$$
$$AV = 1000(1+i)^{29}\left[1 + (1+i)^{-1}(0.99) + (1+i)^{-2}(0.99)^2 + ... + (1+i)^{-28}(0.99)^{28} + (1+i)^{-29}(0.99)^{29}\right]$$
$$AV = 1000(1+i)^{29}\left[1 + v_j + v_j^2 + ... + v_j^{28} + v_j^{29}\right]$$

where, $v_j = (1+i)^{-1}(0.99) = \dfrac{0.99}{1+i}$

$$AV = (1+i)^{29}1000\ddot{a}_{\overline{30|}j} = (1+i)^{29}1000\frac{1-v_j^{30}}{1-v_j}$$

(i) $\quad v_j = (1.01)^{-1}(0.99) \Rightarrow AV = (1.01)^{29}1000\dfrac{1-\left(\dfrac{0.99}{1.01}\right)^{30}}{1-\left(\dfrac{0.99}{1.01}\right)} = 30{,}407$

(ii) $\quad v_j = (1.05)^{-1}(0.99) \Rightarrow AV = (1.05)^{29}1000\dfrac{1-\left(\dfrac{0.99}{1.05}\right)^{30}}{1-\left(\dfrac{0.99}{1.05}\right)} = 59{,}704$

(iii) $\quad v_j = (1.10)^{-1}(0.99) \Rightarrow AV = (1.10)^{29}1000\dfrac{1-\left(\dfrac{0.99}{1.10}\right)^{30}}{1-\left(\dfrac{0.99}{1.10}\right)} = 151{,}906$

Alternatively, this problem could be solved by taking out $(0.99)^{29}$ instead of $(1+i)^{29}$ from the AV equation above.

This would give a formula of $AV = (0.99)^{29}1000 s_{\overline{30|}j}$ where $j = \left(\dfrac{1.01}{0.99}\right) - 1$

Question 4

A perpetuity provides payments every 6 months. The first payment is 1 and each subsequent payment is 3% greater than the last.

Find the present value of the perpetuity at the time of the first payment if the effective rate of interest is 8% per annum.

Solution

$i = 0.08$ is the annual effective rate of interest, but payments and increases are made every 6 months, so find the equivalent half-yearly effective rate of interest j.

$$j = (1.08)^{0.5} - 1 = 0.03923$$

$$PV = 1 + 1.03 \cdot v_j + (1.03 \cdot v_j)^2 + (1.03 \cdot v_j)^3 + \ldots$$

where $v_j = (1.08)^{-0.5} = (1.03923)^{-1}$

$$PV = 1 + \frac{1.03}{1.03923} + \left(\frac{1.03}{1.03923}\right)^2 + \left(\frac{1.03}{1.03923}\right)^3 + \ldots$$

This is a geometric series that can be reduced:

$$PV = \frac{1}{1 - \frac{1.03}{1.03923}} \cong 112.59$$

Alternatively, note that $PV = 1 + \frac{1.03}{1.03923} + \left(\frac{1.03}{1.03923}\right)^2 + \left(\frac{1.03}{1.03923}\right)^3 + \ldots$

can be written using annuity symbols as below:

$$PV = 1 + v_k + v_k^2 + v_k^3 + \ldots = \frac{1}{d_k}$$

where we introduce k so that $v_k = \frac{1.03}{1.03923} \Rightarrow d_k = 1 - v_k = 1 - \frac{1.03}{1.03923}$

Question 5

Smith has 100,000 with which she buys a perpetuity on January 1, 2013. Suppose that $i = 0.045$ and the perpetuity has annual payments beginning January 1, 2014. The first three payments are 2000 each, the next three payments are $2000(1+r)$ each,…, increasing forever by a factor of $1+r$ every three years. What is r?

Solution

$$100,000 = 2000\left(v_i + v_i^2 + v_i^3\right) + 2000(1+r)v_i^3\left(v_i + v_i^2 + v_i^3\right) + 2000(1+r)^2 v_i^6\left(v_i + v_i^2 + v_i^3\right) + \ldots$$

$$100,000 = 2000\left(v_i + v_i^2 + v_i^3\right)\left[1 + (1+r)v_i^3 + (1+r)^2 v_i^6 + \ldots\right]$$

$$100,000 = 2000\left(v_i + v_i^2 + v_i^3\right)\left[1 + \left(\frac{1+r}{(1+i)^3}\right) + \left(\frac{1+r}{(1+i)^3}\right)^2 + \ldots\right]$$

Now,

$$\ddot{a}_{\overline{\infty}|j} = 1 + \left(\frac{1+r}{(1+i)^3}\right) + \left(\frac{1+r}{(1+i)^3}\right)^2 + \ldots \qquad \text{where } v_j = \frac{1+r}{(1+i)^3}$$

$$\Rightarrow 100,000 = 2000\left(v_i + v_i^2 + v_i^3\right)\ddot{a}_{\overline{\infty}|j} = 2000\left(v_i + v_i^2 + v_i^3\right)\frac{1}{d_j}$$

$$\Rightarrow d_j = \frac{2000\left(v_i + v_i^2 + v_i^3\right)}{100,000} = 0.054979 = 1 - v_j = 1 - \frac{1+r}{(1.045)^3}$$

$$\Rightarrow r = \left(1 - 0.054979\right)(1.045)^3 - 1 \cong 0.0784$$

Question 6

Smith receives monthly family allowance payments of \$25 on the last day of each month, beginning January 31, 2011. The payments are increased *at the end of each calendar year* at 12% per annum effective to meet cost-of-living increases.

Payments are deposited in an account earning a nominal rate of interest convertible monthly of $i^{(12)}$=12%. The last payment is made on December 31, 2028 (18 years of payments).

Find the present value (on January 1, 2011) and the accumulated value (on December 31, 2028) of the payments.

Solution

$PV =$

$$25\left(v_j + v_j^2 + ... + v_j^{12}\right) + 25(1+r)v_j^{12}\left(v_j + v_j^2 + ... + v_j^{12}\right) + ... + 25(1+r)^{17} v_j^{12x17}\left(v_j + v_j^2 + ... + v_j^{12}\right)$$

where $j = \dfrac{0.12}{12} = 0.01$ is the effective monthly rate of interest, and $r = 0.12$

$$PV = 25\left(v_j + v_j^2 + ... + v_j^{12}\right)\left[1 + (1+r)v_j^{12} + ... + (1+r)^{17} v_j^{12x17}\right]$$

$$PV = 25a_{\overline{12|}j}\left[1 + (1+r)v_j^{12} + ... + (1+r)^{17} v_j^{12x17}\right]$$

$$PV = 25a_{\overline{12|}j}\left[1 + (1+r)(v_j^{12})^1 + ... + (1+r)^{17} (v_j^{12})^{17}\right]$$

$$PV = 25a_{\overline{12|}j}\left[1 + v_k + ... + v_k^{17}\right]$$

$$PV = 25a_{\overline{12|}j}\ddot{a}_{\overline{18|}k}$$

where $v_k = (1+r)(v_j^{12})$

$$PV = 25\left(\frac{1-v_j^{12}}{j}\right)\left(\frac{1-v_k^{18}}{1-v_k}\right) = 4812.27$$

The accumulated value can simply be found be increasing the present value with interest from January 1, 2011 to December 31, 2028; ie.

$$AV = 4812.27(1.01)^{18\times12} = 41,282.57$$

Past Exam Question – 2005 Final Exam Q1(c)(ii)

Prove the following equality $(I\ddot{a})_{\overline{n|}} = \ddot{a}_{\overline{n|}} + (Ia)_{\overline{n-1|}}$ (3 marks)

Solution

$$RHS = \ddot{a}_{\overline{n|}} + \frac{\ddot{a}_{\overline{n-1|}} - (n-1)v^{n-1}}{i} = \frac{i\ddot{a}_{\overline{n|}} + \ddot{a}_{\overline{n-1|}} + v^{n-1} - nv^{n-1}}{i}$$

$$= \frac{i\ddot{a}_{\overline{n|}} + \ddot{a}_{\overline{n|}} - nv^{n-1}}{i} = \frac{(1+i)\ddot{a}_{\overline{n|}} - nv^{n-1}}{i}$$

$$= \frac{\ddot{a}_{\overline{n|}} - n\dfrac{v^{n-1}}{(1+i)}}{i/(1+i)} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} = (I\ddot{a})_{\overline{n|}} = LHS$$