Assignment VI: week of Feb. 25th

This is the 6th assignment. You are encouraged to work on this by coming to the help sessions (Thursday 12-1, Friday 1-2 at MP202) and **grouping** *up with a few other students. But please* **first** *finish your reading assignment before attempting these, otherwise you won't learn well. Teaching assistants will be at hand to help.* **Submission deadline 5PM Wed. March 7th.**

Here, we investigate a few fundamental questions regarding the past and the future of the universe, using simple algebra. In questions 1-4 here, we assume that the universe contains only one form of energy, i.e., matter. Matter can be both luminous and dark, and it determines the geometry (curvature) of the universe.

1. Critical density. If the average density of matter (obtained by summing over masses of all galaxies and dividing it by the volume they occupy) is lower than the so-called critical density, mutual gravity is not able to slow down the space expansion and the universe is 'open' and 'infinite'. Conversely, it will be 'closed' and 'finite' if the density is larger than the critical density. At exactly the critical density, the universe is 'flat'. To our amazement, it appears that our measured matter density lies close to the critical density. Here, we try to understand what this critical density is.

   Imagine a galaxy with mass m at a distance r away from the center of a sphere, within which a total mass M reside. *(Think why we only care about mass inside the sphere, but not outside.)* The center can be anywhere in the universe. As viewed by an observer in the center, the galaxy appears to be receding according to the Hubble's law, $v = H_0 r$, where H_0 is the Hubble constant at the current age (all subscripts 0 denote 'now'). To heuristically derive the critical density ρ_{m0}, we associate a kinetic energy to the galaxy's hubble flow, symbolically, $\frac{1}{2}\, mv^2$, and balance this against its gravitational energy. If the density in the sphere is equal to the critical density, the total binding energy (kinetic plus gravitational) is zero, much like a space-ship that is moving away from the Earth at exactly the escape velocity. Show that this yields the following expression for ρ_{m0},

$$\rho_{m0} = \frac{3H_0^2}{8\pi G}. \tag{1}$$

   Evaluate this density using the currently measured Hubble constant (we adopt $H_0 = 70$ km/s/Mpc, where Mpc stands for mega-parcsec, and 1 pc $= 3.0857 \times 10^{16}$ m) and express its numerical value in units of number of hydrogen atoms per cubic meter.[1] Review Problem set II (question 1) if you find this puzzling.

2. Age in a 'flat' universe. Here, we relate age to the Hubble constant. Evolution of a flat and matter-dominated universe is simple and is described by equation (1). What does this mean? Both density and the Hubble constant in that equation are time-dependent. Let the (all-important) scale factor of the universe be $a(t)$ at time t, and a_0 be the current scale factor. A sphere that has a radius r_0 today has instead a radius $r(t)$ at time t, while containing the same number of galaxies (and therefore the same total mass). So we are able to relate the

[1] For comparison, the best man-made vacuum has a density of $\sim 10^9$ atoms per cubic meter.

density today with that at time t, ρ_{m0} and $\rho_m(t)$, as

$$\rho_m(t)\frac{4\pi r(t)^3}{3} = \rho_{m0}\frac{4\pi r_0^3}{3}. \tag{2}$$

The Hubble constant is related to the expansion of the universe as $H(t) = \dot{r}/r(t) = \dot{a}/a$, where $\dot{r}$ stands for dr/dt and so on. Equation (1) should then give us a differential equation for the evolution of the scale factor $a(t)$. Integrating this differential equation to solve for $a(t)$ and show that

$$H(t) = \frac{2}{3}t^{-1}, \tag{3}$$

where t is time since beginning of the universe (when $a = 0$). In other words, the age of a flat (and matter dominated) universe is related to the Hubble constant at that time as $t = 2/3H(t)^{-1}$. Evaluate the age using the measured H_0, expressed in unit of Gyrs (Giga-yrs) (*there is a help-sheet attached behind for this problem.*) Moreover, a universe that has critical density at one point in time will always have the critical density at all times – just like an marginally bound space-ship remains marginally bound forever.

3. Redshift and age. Recently, a galaxy 'candidate' at redshift $z = 10$ is reported (http://arxiv.org/abs/1211.3105). This galaxy, if confirmed, lies at a very large distance from us, and it is only detectable thanks to its light being amplified by gravitational lenses. Recall that cosmological redshift is caused by the expansion of the universe, with

$$1 + z = \frac{a_0}{a(t)}, \tag{4}$$

where $a(t)$ is the scale factor at the time the light was emitted. Can you calculate how young the universe was (in unit of Gyrs) when light left this galaxy, assuming a flat and matter-dominated universe?

4. Age in an 'empty' universe. When matter density is lower than the critical density, the universe is 'open'. Take the space-ship analogy, its energy $E = mv^2/2 - GMm/r$ is now positive and it is unbound. Show that this equation can be recast into one that is suitable for the universe,

$$H^2 = \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho_m - k\frac{c^2}{a^2}, \tag{5}$$

where energy E is now supplanted by a so-called curvature k as $E = -(kmc^2r_0^2)/(2a_0^2)$, with c being the speed of light. Curvature $k = -1$ if the universe is 'open' (unbound, $E > 0$), $+1$ if it is 'closed', and 0 if 'flat' (marginally bound, $E = 0$, as in the first problem). This equation is also called the 'Friedmann equation' and can be derived formally (as opposed to heuristically as we do here) from the Einstein's field equation. If the universe is empty (matter density $\rho_m = 0$), show that its age is $t = H(t)^{-1}$. Evaluate this again using H_0, in unit of Gyr.

5. Fate of a matter universe. The globular cluster (a compact cluster of stars) M4 in the Milky Way has been determined to have an age of 12.7 ± 0.7 Gyrs. Since the universe has to be older than its oldest stars, it necessarily means that our universe has a matter density that falls below the critical density. In 2010, the Wilkinson Microwave Anisotropy Probe (WMAP) project estimated the age of the universe to be 13.75 ± 0.11 Gyrs. This age falls in-between

that of a flat and an empty universe and we now know that matter density in our universe is currently $\sim 30\%$ of the critical value. If matter is the only form of energy in our universe, what does this imply for the fate of our universe? Specifically, draw a rough diagram showing how the scale factor $a(t)$ changes with time. To do so, it may be helpful to draw corresponding curves for a 'flat' and an 'empty' universe.

HELP SHEET for Question 2

Age in a 'flat' universe. Evolution of a flat (curvature-free) universe is simple and is described **at all times** by

$$\rho_m(t) = \frac{3H(t)^2}{8\pi G}. \tag{6}$$

Here, the Hubble constant is changing with time, and so is the matter density. The matter density is decreasing as

$$\rho_m(t) = \rho_{m0}\frac{a_0^3}{a(t)^3} \tag{7}$$

as the universe is expanding. This obtains from the fact that the total mass is conserved within any sphere.

Now you need a bit of calculus. In the following, I give a sketch on how to solve for $a(t)$ but when you follow it, make sure you keep all the proportionality constants.

The above equations combine to yield

$$\frac{da}{dt} \propto a^{-1/2}. \tag{8}$$

This differential equation is solved by moving dt to the right, and $a^{-1/2}$ to the left, and integrate both sides,

$$\int_{a(t=0)}^{a(t)} a^{1/2} da \propto \int_{t=0}^{t} dt \tag{9}$$

Now perform the definite integral to obtain

$$a^{3/2}(t) - a^{3/2}(t=0) \propto t - 0 \tag{10}$$

We are happy to take $a(t=0) = 0$ when $t = 0$, as it's the size of the universe when time begins (the big bang). So that gives us $a(t) \propto t^{2/3}$.

Use the definition for the Hubble constant and you should now be able to show that

$$H(t) = \frac{2}{3}t^{-1}, \tag{11}$$

Question 4 employs a similar logic to this problem.