University of Toronto
Department of Mathematics

MAT224H1S
Linear Algebra II

Midterm Examination
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Duration: 1 hour 50 minutes

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Tutorial Group: TUT0101 ? NOT SURE TUESDAY 12pm. BA (Peter Crooks) T/RA (?)

Sorry about this !!!

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	8 /10
2	6 /10
3	4 /10
4	10 /10
5	8 /10
6	10 /10
TOTAL	46 /60

[10] **1.** Let $T\colon \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ be the linear operator defined by

$(i,1)$

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Find the matrix of T with respect to the basis $\alpha = \{(1,0,0), (1,1,0), (1,1,1)\}$.

(b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution:

(a). $T(1,0,0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(1,1,0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(1,1,1) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore the matrix of T with respect to the basis α is

$$[T]_\alpha^\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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(b). $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row Reduction}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent

so $\{(1,0,0),(0,1,0)\}$ is a basis for $\mathrm{Im}(T)$. No

and for ~~Ker(T)~~,

suppose $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in $\mathrm{Ker}(T)$ such that

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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Hence $x=0, y=0$, z can be any number in $\mathbb{Z}_2$ (ie. 0 or 1)

Therefore $\{(0,0,1)\}$ is a basis for $\mathrm{Ker}(T)$.

Hence $\mathrm{Ker}([T]_\alpha) = \left(\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}_\alpha\right)$

$= \{(t,t,t)\}$

$= \{(1,1,1)t\}$

a basis for Ker T is $\{(1,1,1)\}$

Next, as the leading 1s in $\mathrm{rref}([T]_\alpha)$ occur in column 1 & 2, we conclude that the corresponding colums of $[T]_\alpha$ form a basis for its column space.

$\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_\alpha, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_\alpha\right\} = \{(1,1,1),(0,1,0)\}$ is a basis for $\mathrm{Im}(T)$.

[10] **2.** Let T: $\mathbb{R}^4 \to P_2(\mathbb{R})$ be the linear transformation that is represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

relative to the standard bases of $\mathbb{R}^4$ and $P_2(\mathbb{R})$. Find the matrix of T with respect to the bases $\alpha = \{(1,0,0,0), (0,0,1,0), (1,-1,0,0), (0,-1,1,1)\}$ and $\beta = \{x^2+1, x, 1\}$.

Solution. The standard basis of $\mathbb{R}^4$ is $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

The standard basis of $P_2\mathbb{R}$ is $\{1, x, x^2\}$

$T(1,0,0,0) = \begin{bmatrix} 1&1&0&1\\0&0&1&-1\\1&1&0&1 \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} = 1+x^2 = 1(x^2+1)+0\cdot x+0\cdot 1$

$T(0,1,0,0) = \begin{bmatrix} 1&1&0&1\\0&0&1&-1\\1&1&0&1 \end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} = 1+x^2 = 1(x^2+1)+0\cdot x+0\cdot 1$

$T(0,0,1,0) = \begin{bmatrix} 1&1&0&1\\0&0&1&-1\\1&1&0&1 \end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} = x = 0(x^2+1)+1\cdot x+0\cdot 1$

$T(0,0,0,1) = \begin{bmatrix} 1&1&0&1\\0&0&1&-1\\1&1&0&1 \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix} = 1-x+x^2 = 1(x^2+1)+(-1)x+0\cdot 1$

Therefore $[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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$T(1,0,0,0) = \begin{bmatrix} 1&1&0&1\\0&0&1&-1\\1&1&0&1 \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}_{std} = 1+x^2 = \begin{bmatrix}1\\0\\0\end{bmatrix}_\beta$

$T(0,0,1,0) =$ - - - - - $= x = \begin{bmatrix}0\\1\\0\end{bmatrix}_\beta$

$T(1,-1,0,0) = \quad = 0 = \begin{bmatrix}0\\0\\0\end{bmatrix}_\beta$

$T(0,-1,1,1) = \quad = 0 = \begin{bmatrix}0\\0\\0\end{bmatrix}_\beta$

$(1-i)(1+\frac{1}{2}i)$

$[T]_{\beta\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

[10] **3.** Let $W = \{p(x) \in P_2(\mathbb{R}) \mid p(0) = 0\}$. Show that W and $\mathbb{R}^2$ are isomorphic and find an isomorphism $T\colon W \to \mathbb{R}^2$.

Solution: ~~$T(p(x)) = (p(x), xp(x))$ is such an isomorphism that $T: W \to \mathbb{R}^2$.~~

Since $p(0)=0$, $p(x) = a + bx + cx^2$

hence $a=0$, W is the set of all polynomials in form of $p(x) = bx + cx^2$

Then $\dim(W) = \dim(\mathbb{R}^2) = 2$. 1

Then W and $\mathbb{R}^2$ are isomorphic.

Why $T(p(x)) = (p(x), x(p(x)))$ isomorphism? ✗

Since W and $\mathbb{R}^2$ are isomorphic,

$\mathrm{Ker}(T) = \{0\}$ ~~(as indicated in the question)~~

Then T is injective,

As $\dim(W) = \dim(\mathbb{R}^2)$ 1

So T is also surjective.

thus bijective.

Then it has an inverse.

Therefore T is an isomorphism.

A polynomial $p(x) = a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R})$ is in W iff $p(0)=0$ iff $a_0 = 0$.

That is,

$p(x) \in W \iff p(x) = a_1x + a_2x^2$

This shows that

$W = \{a_1x + a_2x^2 \in P_2(\mathbb{R}) \mid a_1, a_2 \in \mathbb{R}\}$
$= \mathrm{span}\{x, x^2\}$

Since $\{x, x^2\}$ is clearly lin. indpt.

$\dim W = 2 = \dim \mathbb{R}^2$

Hence W & $\mathbb{R}^2$ are isomorphic, as they have the same dimension.

An isomorphism $T: W \to \mathbb{R}^2$ is given by

$T(a_1x + a_2x^2) = (a_1, a_2)$

Indeed, T is linear:

$T((a_1x + a_2x^2) + \lambda(b_1x + b_2x^2))$
$= T((a_1 + \lambda b_1)x + (a_2 + \lambda b_2)x^2)$
$= (a_1 + \lambda b_1, a_2 + \lambda b_2)$
$= (a_1, a_2) + \lambda(b_1, b_2)$
$= T(a_1x + a_2x^2) + \lambda T(b_1x + b_2x^2)$

T is injective:

$T(a_1x + a_2x^2) = (0,0) \iff (a_1, a_2) = (0,0) \iff a_1x + a_2x^2 = 0$

hence T is also surjective b/c $\dim W = \dim \mathbb{R}^2$.

[10] **4.** Let T: $\mathbb{C}^3 \to \mathbb{C}^3$ be the linear transformation defined by

$$T(z_1, z_2, z_3) = ((1+i)z_1, -2iz_1 + (1+i)z_2 + 2iz_3, iz_1 + z_3),$$

where $\mathbb{C}^3$ is seen as a vector space over the field of complex numbers. Find the eigenvalues of T and bases for each of the corresponding eigenspaces.

Solution: Say we have a standard basis

$\{(1,0,0), (0,1,0), (0,0,1)\}$

Then $T(1,0,0) = (1+i, -2i, i)$

$T(0,1,0) = (0, 1+i, 0)$

$T(0,0,1) = (0, 2i, 1)$

Therefore the matrix of T is

$$A = \begin{bmatrix} 1+i & 0 & 0 \\ -2i & 1+i & 2i \\ i & 0 & 1 \end{bmatrix}.$$

Then $\det(A - \lambda I) = \det \begin{bmatrix} 1+i-\lambda & 0 & 0 \\ -2i & 1+i-\lambda & 2i \\ i & 0 & 1-\lambda \end{bmatrix}$

$= (1+i-\lambda)(1+i-\lambda)(1-\lambda)$

complex $= (1+i-\lambda)^2(1-\lambda)$

$T(x) = \lambda x$

So we have two eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1+i$

$E_{\lambda_1} = x$:

For $\lambda_1 = 1$, $T(x) = \lambda_1 x = x$

Say $x = (z_1, z_2, z_3)$ as mentioned in the problem.

Then $\begin{cases} z_1 = (1+i)z_1 & \Rightarrow z_1 = 0 \\ z_2 = -2iz_1 + (1+i)z_2 + 2iz_3 & \Rightarrow z_2 = -2z_3 \\ z_3 = iz_1 + z_3 & \Rightarrow z_3 = z_3 \end{cases}$

Hence a basis for eigenspace E_{λ_1} is $\{(0,-2,1)\}$

For $\lambda_2 = 1+i$, $T(x) = \lambda_2 x = (1+i)x$

Then $\begin{cases} (1+i)z_1 = (1+i)z_1 & \Rightarrow z_1 = z_1 \\ (1+i)z_2 = -2iz_1 + (1+i)z_2 + 2iz_3 & \Rightarrow z_2 = z_2 \\ (1+i)z_3 = iz_1 + z_3 & \Rightarrow z_3 = z_1 \end{cases}$

Hence a basis for eigenspace E_{λ_2} is $\{(1,0,1), (0,1,0)\}$.

[10] 5. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (ax_1 + bx_2, bx_1 + ax_2 + bx_3, bx_2 + ax_3).$$

Show that T is diagonalizable for all values of $a, b \in \mathbb{R}$.

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Prove: Here we need a standard basis for $\mathbb{R}^3$ again.

Say $\{(1,0,0),(0,1,0)(0,0,1)\}$

$T(1,0,0) = (a, b, 0)$

$T(0,1,0) = (b, a, b)$

$T(0,0,1) = (0, b, a)$

Then the matrix of T is $\begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}$

So $\det(T-\lambda I) = \det\begin{bmatrix} a-\lambda & b & 0 \\ b & a-\lambda & b \\ 0 & b & a-\lambda \end{bmatrix}$

$= (a-\lambda)[(a-\lambda)(a-\lambda) - b^2] - b(b)(a-\lambda)$

$= (a-\lambda)[(a-\lambda)^2 - b^2] - b^2(a-\lambda)$

$= (a-\lambda)[(a-\lambda)^2 - 2b^2]$

$= (a-\lambda)(a-\lambda+\sqrt{2}b)(a-\lambda-\sqrt{2}b)$

Hence the multiplicities are all 1. unless $b=0$

And $1+1+1 = 3 = \dim(\mathbb{R}^3)$

Therefore, T is diagonalizable for all values of $a, b \in \mathbb{R}$. ■

[10] **6.** Let $T\colon V \to W$ be an injective linear transformation. Prove that if $T(v_4)$ is dependent on $\{T(v_1), T(v_2), T(v_3)\}$, then v_4 is dependent on $\{v_1, v_2, v_3\}$.

Proof. If $T(V_4)$ is dependent on $\{T(V_1), T(V_2), T(V_3)\}$

i.e. $T(V_4) = a_1 T(V_1) + a_2 T(V_2) + a_3 T(V_3)$ for some $a_i \in \mathbb{R}$, $i = 1, 2, 3$.

Since T is a linear transformation.

So $T(V_4) = a_1 T(V_1) + a_2 T(V_2) + a_3 T(V_3)$

$= T(a_1 V_1) + T(a_2 V_2) + T(a_3 V_3)$

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$= T(a_1 V_1 + a_2 V_2 + a_3 V_3)$

Then since T is injective, which means it is one-to-one.

Then $V_4 = a_1 V_1 + a_2 V_2 + a_3 V_3$

Therefore V_4 is dependent on $\{V_1, V_2, V_3\}$.