CSC336 Tutorial 6 – Nonlinear equations

QUESTION 1 *Assume that five iterative methods applied to a non-linear problem exhibit the convergence behaviour indicated by the errors in the first four iterations below:*

method (a): $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$
method (b): $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$
method (c): $10^{-2}, 10^{-3}, 10^{-5}, 10^{-8}$
method (d): $10^{-2}, 10^{-3}, 10^{-5}, 10^{-9}$
method (e): $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$

Based on the above errors, estimate (approximately) the order (rate) of convergence and the asymptotic error constant, for each of the methods.

ANSWER: Recall the relation

$$\lim_{k->\infty} \frac{|e_{k+1}|}{|e_k|^p} = C$$

through which the order of convergence p of a sequence is defined, and note that it involves a limit, i.e., it refers to the asymptotic behaviour of the sequence.

Since we are given only few data, we make the assumption that the asymptotic behaviour has already been reached, so

$$\frac{|e_{k+1}|}{|e_k|^p} \approx C \tag{1}$$

for the iterations given. Thus, we can say that $\frac{|e_{k+1}|}{|e_k|^p} \approx \frac{|e_k|}{|e_{k-1}|^p} \Rightarrow \frac{|e_{k+1}|}{|e_k|} \approx \frac{|e_k|^p}{|e_{k-1}|^p}$, thus,

$$p \approx \frac{\log(|e_{k+1}|/|e_k|)}{\log(|e_k|/|e_{k-1}|)}. \tag{2}$$

Relation (2) means that a triple of consecutive elements of the sequence gives us an estimate of the order. Then, relation (1) gives us an estimate of C, for every pair of consecutive elements.

Applying (2) to both triples of method (a), we get $p = 1$ (linear conv.). Applying (1) to all three pairs of method (a), we get $C = 1/10$.
Applying (2) to both triples of method (b), we get $p = 1$ (linear conv.). Applying (1) to all three pairs of method (b), we get $C = 1/100$.
Applying (2) to the first triple of method (c), we get $p = 2$, and applying (2) to the second triple of method (c), we get $p = 3/2$. Usually, the higher-indexed elements

of the sequence exhibit the asymptotic behaviour of the method more closely, so we take the estimate $p = 3/2$ (superlinear but not quadratic conv.). Applying (1) to the three pairs of method (c), we get $C = 1, C = 0.31622..., C = 0.31622....$ We take the higher-indexed estimate for C, i.e. $C = 0.31622....$
Applying (2) to both triples of method (d), we get $p = 2$ (quadratic conv.). Applying (1) to all three pairs of method (d), we get $C = 10$.
Applying (2) to both triples of method (e), we get $p = 2$ (quadratic conv.). Applying (1) to all three pairs of method (e), we get $C = 1$.

Note:

When the errors fluctuate, an individual triple $(|e_{k-1}|, |e_k|, |e_{k+1}|)$ may lead to a negative order of convergence, and/or an individual pair of $(|e_k|, |e_{k+1}|)$ may lead to an erratic asymptotic error constant. In most cases, if there is a choice of triples or pairs that can be used, it is preferable to use triples or pairs with a high k, so that the method is studied when the asymptotic behaviour has been reached.
Alternatively, it may be advisable to use least squares approximations, which allow the incorporation of more data than unknowns. However, least squares approximations are a topic of another course.

QUESTION 2 *Consider the (nonlinear) equation $f(x) = 0$, with $f(x) = x^3 - 8$. Show that $f(x)$ has exactly one (real) root.*

ANSWER: For the above function f, we know (from standard algebra) that it has one (real) root at $x = 2$. (The other two roots are complex conjugate $-1 \pm \sqrt{-3}$.) However, we will apply techniques taught in this course to show that f has one real root.

Note that $f(0) = -8 < 0, f(3) = 27 - 8 = 19 > 0$ and $f(x)$ is continuous. Thus $f(x)$ must have at least one root in $(0, 3)$.

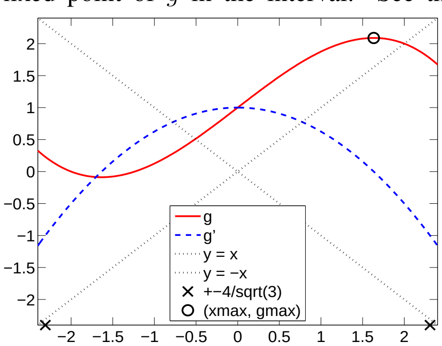
Also, $f'(x) = 3x^2 > 0$ in $(0, 3)$, i.e. $f(x)$ is monotonically increasing in $(0, 3)$. Thus it must have exactly one root in $(0, 3)$.

QUESTION 3 *Let $g(x) = -\frac{x^3}{8} + x + 1$. Find an interval where g is a contraction mapping, and where g has a unique fixed point.*
ANSWER: Taking into account that g is differentiable, consider $g'(x) = -\frac{3x^2}{8} + 1$.
We will make use of Theorem 4 in the notes. We will search for a (closed) interval I, where $|g'(x)| \le \lambda < 1$, and which (interval) g maps to itself.
First, find where $|g'(x)| < 1$:

$$-1 < -\frac{3x^2}{8} + 1 < 1 \Rightarrow \begin{cases} -\frac{3x^2}{8} < 0 \quad \text{true } \forall x \neq 0 \\ -2 < -\frac{3x^2}{8} \Rightarrow \frac{3x^2}{8} < 2 \Rightarrow 3x^2 < 16 \Rightarrow \\ \Rightarrow x^2 < \frac{16}{3} \Rightarrow x < \frac{4}{\sqrt{3}} \approx 2.309 \end{cases}$$

Thus, $|g'(x)| < 1$ $\forall x \in (0, \frac{4}{\sqrt{3}})$, and, therefore, g is a contraction mapping (systolic) in $I \equiv [0 + \delta, \frac{4}{\sqrt{3}} - \delta]$, for any small and positive δ. Note also that $g'(0) = 1$ and $g'(\frac{4}{\sqrt{3}}) = 1$, so we CANNOT use $[0, \frac{4}{\sqrt{3}}]$ with Theorem 4.
Next, prove that $g(x)$ maps I to itself: We study how g behaves in I. From $g'(x) = 0$, we have $x = \pm\frac{2\sqrt{2}}{\sqrt{3}} \approx \pm 1.63299$. In addition $g'(x) > 0$ in $(0, \frac{2\sqrt{2}}{\sqrt{3}})$ and $g'(x) < 0$ in $(\frac{2\sqrt{2}}{\sqrt{3}}, \frac{4}{\sqrt{3}})$. Thus, g is increasing in $(0, \frac{2\sqrt{2}}{\sqrt{3}})$ and decreasing in $(\frac{2\sqrt{2}}{\sqrt{3}}, \frac{4}{\sqrt{3}})$.
Therefore, in $(0, \frac{4}{\sqrt{3}})$, g attains its maximum at $x_{\max} = \frac{2\sqrt{2}}{\sqrt{3}}$, with $g(x_{\max}) = \frac{4\sqrt{2}}{3\sqrt{3}} + 1 \approx 2.08866 < 2.309 \approx \frac{4}{\sqrt{3}}$, which is also the maximum of g in I (subset of $(0, \frac{4}{\sqrt{3}})$).

For the minimum of g in I, consider that $g(\delta) > g(0) = 1$, and $g(\frac{4}{\sqrt{3}} - \delta) > g(\frac{4}{\sqrt{3}}) > 1$, so the minimum of g in I is found at $x_{\min} = \delta$, and $g(x_{\min}) > 1$.
Thus, when $x \in [0 + \delta, \frac{4}{\sqrt{3}} - \delta]$, we have that $g(x) \in (1, \frac{4\sqrt{2}}{3\sqrt{3}} + 1] \subset [0 + \delta, \frac{4}{\sqrt{3}} - \delta]$, as long as $\delta < \frac{4}{\sqrt{3}} - (\frac{4\sqrt{2}}{3\sqrt{3}} + 1) \approx 0.2207$. Thus g maps I to itself, for any δ such that $0 < \delta < 0.2207$.
So we have a systolic g that maps an interval to itself, therefore there exists a unique fixed point of g in the interval. See the plot, to visualize how g and g' behave.



Note: We could have worked with the relation $g(x) = x \Leftrightarrow x^3 - 8 = 0$ (as can be seen), but we wanted to make use of certain theorems relating to fixed points.

QUESTION 4 *Consider $f(x) = x^3 - 8$ and let $g(x) = -\frac{x^3}{8} + x + 1$. Is $g(x) = x$ equivalent to $f(x) = 0$? Consider the iteration $x_{k+1} = g(x_k)$. Find an interval where the iteration is convergent if started within. Is $x_{k+1} = g(x_k)$ convergent to the root of $f(x) = 0$, if started at $x_0 = 0$? How about $x_0 = 3$?*
ANSWER: Note that $f(x) = 0 \Leftrightarrow x^3 - 8 = 0 \Leftrightarrow -\frac{x^3}{8} + 1 = 0 \Leftrightarrow x = -\frac{x^3}{8} + x + 1 \Leftrightarrow x = g(x)$.
Thus $x = g(x)$ is equivalent to $f(x) = 0$. This means that the root of $f(x)$ is a fixed point of $g(x)$ and vice-versa.

To find an interval where the iteration is convergent, we go back to the previous question, in which we found an interval $I \equiv [0 + \delta, \frac{4}{\sqrt{3}} - \delta]$, with $0 < \delta < 0.2207$, where g is systolic, and which g maps to itself. According to Theorem 4b, convergence of the fixed-point iteration $x_{k+1} = g(x_k)$ to the unique fixed point of g in I (i.e. to the unique root of f in I) is guaranteed if $x_0 \in I$. (Essentially, convergence is guaranteed if $x_0 \in (0, \frac{4}{\sqrt{3}})$.)

What if we start with $x_0 = 0$? Note that $g'(0) = 1$, and 0 does not belong to I. Let's test one (or a few) iterations, starting at $x_0 = 0$.
Note that $x_1 = g(x_0) = 1 \in I$. Once we are in I, any subsequent approximations to

the root of $f(x)$ will converge to the root. Therefore $x_{k+1} = g(x_k)$ converges to the root of $f(x)$, if started at $x_0 = 0$.

Is $x_0 = 3$ an appropriate starting guess? Note that $|g'(3)| = |-2.3750| > 1$ and 3 does not belong to I. Let's test one (or a few) iterations, starting at $x_0 = 3$.
We have $x_1 = -\frac{x_0^3}{8} + 1 + x_0 = -\frac{27}{8} + 1 + 3 = \frac{-27+32}{8} = \frac{5}{8}$. Note that $x_1 = \frac{5}{8} \in I$. Once we are in I, any subsequent approximations to the root of $f(x)$ will converge to the root. Therefore $x_{k+1} = g(x_k)$ converges to the root of $f(x)$, if started at $x_0 = 3$.
(Actually, it can be shown that any $x_0 \in (0, 3.634)$ results in $x_1 \in I$ and the iteration is convergent.)

Moral: The theorems we have used give sufficient conditions for convergence. This does not mean that, when the conditions are not met, we do not have convergence. (That is, the conditions are not necessarily necessary.) We may have to study some cases separately.
For convergence of fixed point iteration, we usually first find an interval that contains the root and $|g'(x)| < 1$ within the interval. Then adjust the interval to match the conditions of some theorem. Once we have shown convergence within the interval,

we check whether starting outside the interval, after one (or few) iterations, get's us inside the interval. Thus, we may be able to enlarge the interval that matches the conditions of a theorem.

QUESTION 5 *Consider the (nonlinear) equation $f(x) = 0$, with $f(x) = x^3 - 8$. This function has one real root at $x = 2$. Apply 3 iterations of bisection in $[0, 3]$ to $f(x) = 0$. Find how many bisection iterations suffice to compute the root to tolerance 10^{-5}.*

ANSWER: Let $a = 0$ and $b = 3$. Then $f(a) = -8 < 0$ and $f(b) = 27 - 8 = 19 > 0$. Since f is continuous, we have found an initial range that contains the root. Proceed with the bisection iterations:

$m = \frac{a+b}{2} = \frac{3}{2}, \quad f(m) = \left(\frac{3}{2}\right)^3 - 8 = -\frac{37}{8} < 0;$ set $a = m = \frac{3}{2}$

$m = \frac{a+b}{2} = \frac{\frac{3}{2}+3}{2} = \frac{9}{4}, \quad f(m) = \left(\frac{9}{4}\right)^3 - 8 = \frac{217}{64} > 0;$ set $b = m = \frac{9}{4}$

$m = \frac{a+b}{2} = \frac{\frac{3}{2}+\frac{9}{4}}{2} = \frac{15}{8}, \quad f(m) = \left(\frac{15}{8}\right)^3 - 8 = -\frac{721}{512} < 0;$ set $a = m = \frac{15}{8}$

The computed approximations to the root, together with the initial range, are, in order, $0, 3, 3/2, 9/4, 15/8$.

How many iterations suffice to compute the root to tolerance 10^{-5}?

We assume that the iterates are indexed so that $x_{-1} = 0, x_0 = 3, x_1 = (0+3)/2 = \frac{3}{2}$, etc.

With 1 iteration, the error is no more than $\frac{b-a}{2} = \frac{3}{2}$

With 2 iterations, the error is no more than $\frac{3-3/2}{2} = \frac{3}{4} = 3 \cdot \frac{1}{2^2}$

With n iterations, the error is no more than $3 \cdot \frac{1}{2^n}$. That is, $|e_n| \le 3 \cdot \frac{1}{2^n}$. Want $|e_n| < 10^{-5}$.

It suffices to have $3 \cdot \frac{1}{2^n} < 10^{-5}$

$\Leftrightarrow \log(3) - n\log(2) < -5\log(10)$

$\Leftrightarrow \log(3) + 5\log(10) < n\log(2)$

$\Leftrightarrow \frac{\log(3)+5\log(10)}{\log(2)} < n$

$\Leftrightarrow n > 18.19...$

Thus $n = 19$ iterations suffice. If we actually run the bisection algorithm on this problem with $tol = 10^{-5}$, we see that it takes exactly 19 iterations.

QUESTION 6 *Consider the (nonlinear) equation $f(x) = 0$, with $f(x) = x^3 - 8$. This equation has one real root at $x = 2$.*

- *Apply 2 iterations of Newton's method to $f(x) = 0$ with starting guess $x_0 = 3$.*

$f'(x) = 3x^2 \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3-8}{3x_k^2}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3^3-8}{3\times 3^2} = 3 - \frac{27-8}{27} = \frac{62}{27} \approx 2.2962\ldots$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{62}{27} - \frac{(\frac{62}{27})^3-8}{3\times(\frac{62}{27})^2} \approx 2.036587\ldots$

- *Approximately, how many iterations will it take to reach tolerance 10^{-5}?*

For Newton's method, when the target root is simple, and $g = x - f/f'$ is at least twice differentiable near the root, we know that the rate of convergence is 2 and the number of digits of accuracy approximately doubles at each iteration (assuming the current approximation is close enough to the root).

Here, x_0 has error 1 ($|2-3| = 1$ fairly large)

But $x_1 \approx 2.29$ has error ≈ 0.29 (approximately 10^{-1})

And $x_2 \approx 2.0365$ has error ≈ 0.0365 (approximately 10^{-2})

Thus x_3 is expected to have error $\approx (0.0365)^2$ (approximately 10^{-4})

And x_4 is expected to have error $\approx (0.0365)^{2\times 2}$ (approximately 10^{-8})

We can simply say that 4 iterations are expected to suffice. However, it is better to be more precise and say that

x_n is expected to have error $\approx (0.0365)^{2^{n-2}}$

Want $(0.0365)^{2^{n-2}} < 10^{-5}$

$\Leftrightarrow 2^{n-2}\log(0.0365) < -5\log 10$

$\Leftrightarrow 2^{n-2} > -\frac{5\log 10}{\log(0.0365)} = \frac{5\log 10}{\log(\frac{1}{0.0365})}$

$\Leftrightarrow (n-2)\log 2 > \log(\frac{5\log 10}{\log(\frac{1}{0.0365})})$

$\Leftrightarrow n - 2 > \log(\frac{5\log 10}{\log(\frac{1}{0.0365})})/\log(2)$

$\Leftrightarrow n > \log(\frac{5\log 10}{\log(\frac{1}{0.0365})})/\log(2) + 2 \approx 3.8$

Thus 4 iterations are expected to suffice.

If we actually run Newton's with $x_0 = 3$ and $tol = 10^{-5}$ on this problem, we see that it takes 4 iterations and gives the root with error $\approx 2 \times 10^{-7}$.

- *Apply 1 Newton iteration to $f(x) = 0$ with starting guess $x_0 = 0$.*

$f'(x) = 3x^2 \quad , \quad f'(x_0) = 0.$

Thus Newton's is not applicable with $x_0 = 0$.

- *Find an interval for which Newton's is guaranteed to converge, if started within.*

Let $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3-8}{3x^2} = \frac{2}{3}x + \frac{8}{3x^2}$
Then $g'(x) = \frac{2}{3} - \frac{16}{3x^3}$
Find an interval I that contains the root and $|g'(x)| < 1 \; \forall x \in I$.
Notice that $|g'|$ becomes huge when $x \approx 0$.

Therefore, we seek an interval in the positive x-axis and away from 0.

Want $-1 < \frac{2}{3} - \frac{16}{3x^3} < 1 \quad \Leftrightarrow \quad -\frac{5}{3} < -\frac{16}{3x^3} < \frac{1}{3} \Leftrightarrow -5 < -\frac{16}{x^3} < 1$

Since we are looking for positive x's, we want

$$\begin{cases} x^3 > -16 & \text{true} \quad \forall\, x > 0 \\ -5x^3 < -16 & \Leftrightarrow \quad x^3 > \frac{16}{5} \quad \Leftrightarrow \quad x > (\frac{16}{5})^{\frac{1}{3}} \approx 1.474 \end{cases}$$

Thus, in the interval $I \equiv ((\frac{16}{5})^{\frac{1}{3}}, \infty)$, Newton's is guaranteed to converge.

Note: Newton's may converge for a larger interval and for certain starting values outside I.
e.g. If $x_0 = -1$, $x_1 = -\frac{2}{3} + \frac{8}{3} = 2$ (convergence in 1 iteration).
If $x_0 = 1$, $x_1 = \frac{2}{3} + \frac{8}{3} = \frac{10}{3} \in I$ (convergence guaranteed).
It turns out that, for this function, Newton's converges for just about any initial guess except $x_0 = 0$.

Let $s(x) \equiv \frac{8}{3x^2} - \frac{x}{3}$, and write $g(x) = \frac{2}{3}x + \frac{8}{3x^2} = x + \frac{8}{3x^2} - \frac{x}{3}$ as $g(x) = x + s(x)$, i.e., $s(x)$ is the "increment" at each Newton iteration.
Study the case $x_0 \in (0, (\frac{16}{5})^{\frac{1}{3}}]$:
Consider any $x_0 > 0$, and note that the function $s(x)$ is decreasing in $(0, \infty)$, since $s'(x) < 0$, and $s(x)$ has root at $x = 2$. Thus, in $(0, (\frac{16}{5})^{\frac{1}{3}}]$, we have $s(x) \geq s((\frac{16}{5})^{\frac{1}{3}}) \approx 0.7368$. Thus, if $x_k \in (0, (\frac{16}{5})^{\frac{1}{3}}]$, we have $x_{k+1} = g(x_k) = x_k + s(x_k) > x_k + 0.7368$. Thus, if $x_0 \in (0, (\frac{16}{5})^{\frac{1}{3}}]$, after a few (at most 2) iterations, x_k will fall into $((\frac{16}{5})^{\frac{1}{3}}, \infty)$, where convergence is guaranteed (as shown above). Note also that $2s((\frac{16}{5})^{\frac{1}{3}}) = (\frac{16}{5})^{\frac{1}{3}}$
Study the case $x_0 \in (-\infty, 0)$:
Consider any $x_0 < 0$, and note that the function $s(x)$ is decreasing in $(-\infty, -2 \cdot 2^{\frac{1}{3}})$ and increasing in $(-2 \cdot 2^{\frac{1}{3}}, 0)$. Thus, in $(-\infty, 0)$, we have $s(x) \geq s(-2 \cdot 2^{\frac{1}{3}}) \approx 2.5198$. Thus, if $x_k \in (-\infty, 0)$, we have $x_{k+1} = g(x_k) = x_k + s(x_k) > x_k + 2.5198$. Thus, if $x_0 \in (-\infty, 0)$, after several iterations, x_k will fall into $(0, \infty)$, where convergence is guaranteed (as shown above), assuming x_k will not fall on 0.
Note that, mathematically, $g(x)$ can never be 0, but if $x = 0$, $g(x)$ is undefined, and Newton's is not applicable. So, if $x_k \neq 0$, then $x_{k+1} = g(x_k) \neq 0$.

QUESTION 7 *Consider the (nonlinear) equation $f(x) = 0$, with $f(x) = x^3 - 8$. This equation has one real root at $x = 2$.*

- *Apply 2 iterations of secant to $f(x) = 0$ with starting guesses $x_{-1} = 0, \; x_0 = 3$.*

$f(x_{-1}) = -8, \quad f(x_0) = 19$
$x_1 = x_0 - f(x_0)\frac{x_0 - x_{-1}}{f(x_0) - f(x_{-1})} = 3 - 19\frac{3-0}{19+8} = \frac{8}{9}$
$f(x_1) = (\frac{8}{9})^3 - 8 = -\frac{5320}{729} \approx -7.297668\ldots$
$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = \frac{8}{9} + \frac{5320}{729} \times \frac{\frac{8}{9} - 3}{-\frac{5320}{729} - 19} = \frac{1488}{1009} \approx 1.474727\ldots$

- *Approximately, how many iterations will it take to reach tolerance 10^{-5}?*

For the secant method, the number of digits of accuracy approximately increases by a factor of 1.618 at each iteration (assuming the current approximation is close enough to the root).
For the above problem, it is hard to predict the number of iterations with the data we have so far, because the last computed approximation x_2 has error ≈ 0.53 (which is fairly large).
But we still base our calculations on it.

x_2 has error ≈ 0.53

x_3 is expected to have error $\approx (0.53)^{1.618}$
x_4 is expected to have error $\approx (0.53)^{1.618 \times 1.618} = (0.53)^{1.618^2}$
$\vdots$
x_n is expected to have error $\approx (0.53)^{1.618^{n-2}}$

Want $(0.53)^{1.618^{n-2}} < 10^{-5}$
$\Leftrightarrow 1.618^{n-2}\log(0.53) < -5\log 10$
$\Leftrightarrow 1.618^{n-2} > -\frac{5\log 10}{\log(0.53)} = \frac{5\log 10}{\log(\frac{1}{0.53})} \quad (\text{Note} : \log(0.53) < 0)$
$\Leftrightarrow (n-2)\log(1.618) > \log(\frac{5\log 10}{\log(\frac{1}{0.53})})$
$\Leftrightarrow n - 2 > \log(\frac{5\log 10}{\log(\frac{1}{0.53})}) / \log(1.618)$
$\Leftrightarrow n > \log(\frac{5\log 10}{\log(\frac{1}{0.53})}) / \log(1.618) + 2 \approx 8.022 \; (\approx 8)$
So we expect 9 iterations to suffice. (Possibly 8 iterations are enough.)

If we actually run secant with $x_{-1} = 0, \; x_0 = 3$ and $tol = 10^{-5}$ on this problem, we see that it takes 8 iterations and gives the root with error $\approx 9 \times 10^{-8}$.