FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 1

CASH FLOW MODELS

A cash flow model is a mathematical projection of payments arising from a financial contract or block of contracts.

Payments received are income and are positive cash flows. Payments made are expenses and are negative cash flows. The net cash flow at a certain time equals income minus expenses.

For example,

Time	Income ($)	Expense ($)	Net cash flow ($)
0	0	-500	-500
1	+150	-200	-50
2	+200	-100	+100
3	+250	0	+250

The purpose of this section is to give you a very brief introduction to the types of cash flows that are common in the actuarial field.

The timing of the cash flows may be known or unknown. The amount of the individual cash flows may also be known or unknown in advance. In addition, it may not be certain how many payments will actually be made.

Uncertainty in cash flow models:

- Timing known, amounts known, payments certain
  - eg. fixed coupon bonds (conventional bonds), zero coupon bonds, annuities certain, loans

- Timing known, amounts unknown, payments certain
  - eg. index linked bonds

- Timing known, amounts known, payments uncertain
  - eg. life annuities

- Timing known, amounts unknown, payments uncertain
  - eg. equities

- Timing unknown, amounts known, payments certain
  - eg. endowment assurance

- Timing unknown, amounts unknown, payments uncertain
  - eg. general insurance (motor vehicle policies).

When payments are uncertain, cash flows are projected allowing for the probability that a payment will be made. This is usually done by considering the *expected* payments for a group of policies.

INTEREST & ACCUMULATED VALUES

Most financial transactions involve a borrower and a lender. The borrower pays interest to the lender in return for use of the lender's capital (the sum of money offered).

For example, a bank acts as a borrower when a customer deposits money with the bank. The bank (borrower) pays interest to the customer (lender) in exchange for use of the customer's money.

A bank acts as a lender when a customer borrows money for a purchase such as a home or car. The customer (borrower) pays interest to the bank (lender) in exchange for use of the bank's money.

The simplest way to express an interest rate is by stating the **effective rate of interest**. This is equal to the amount of interest an investment will earn at the end of the time period as a proportion of the initial investment, or:

$$\textit{Effective rate of interest for a specified period} = \frac{\textit{amount of interest for the period}}{\textit{amount at the start of the period.}}$$

For example, if \$100 is invested and \$10 interest is paid at the end of the year, then the effective annual rate of interest is 10%.

The period of time need not be a year when quoting effective interest rates. For example, if \$100 is invested and \$1 is paid at the end of the first month, then the effective monthly interest rate is 1%.

We now define some notation. Let $S(t)$ represent the value of an investment at time t. The amount of interest earned on an investment from time t_1 to time t_2 is, therefore, $S(t_2) - S(t_1)$.

Using this notation, the effective annual rate of interest for a period from year u to $u+1$ is:

$$\boxed{i_{u+1} = \frac{S(u+1) - S(u)}{S(u)}}$$

Interest payments can be treated in one of two ways: simple or compound.

SIMPLE INTEREST

Interest payments are treated separately from the initial investment. Interest payments don't earn interest themselves. If the amount invested is $S(0)$, and the annual interest rate is i, then the accumulated amount after t years is:

$$\boxed{S(t) = S(0) + iS(0) + iS(0) + ... + iS(0) = S(0)\cdot(1+ti)}$$

COMPOUND INTEREST

For compound interest calculations, interest payments earn further interest themselves (ie. the interest earned is reinvested). If the amount invested is $S(0)$, and the annual interest rate is i, then the accumulated amount after t years is:

$$\boxed{S(t) = S(0)\cdot(1+i)^t}$$

Note that t doesn't have to be a whole number in these formulae.

The compound interest formula is derived below, where we assume that the amount invested at time 0 is X.

In the first year the amount of interest earned will be iX. Therefore, at the beginning of the second year, the new amount invested is $X + Xi = X(1+i)$. During the second year, this amount earns interest of $X(1+i)i$, which means that the total amount at the end of the second year is $X(1+i) + X(1+i)i = X(1+i)^2$. If we follow this logic through (using mathematical induction), then the accumulated amount at the end of the t^{th} year is equal to $X(1+i)^t$.

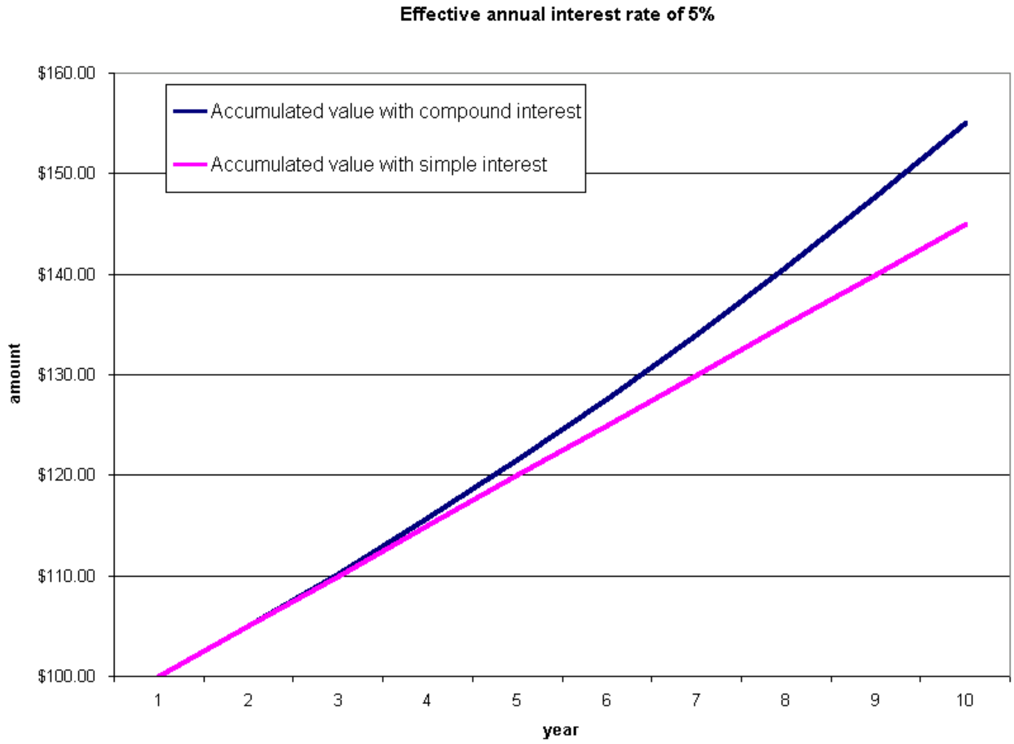
At various stages in the course we will also use the notation $A(t_1, t_2)$ (known as the accumulation factor) to represent the accumulation of 1 from time t_1 to time t_2, where $t_1 \le t_2$. Using this notation, under compound interest, $\dfrac{S(t)}{S(0)} = A(0,t) = (1+i)^t$

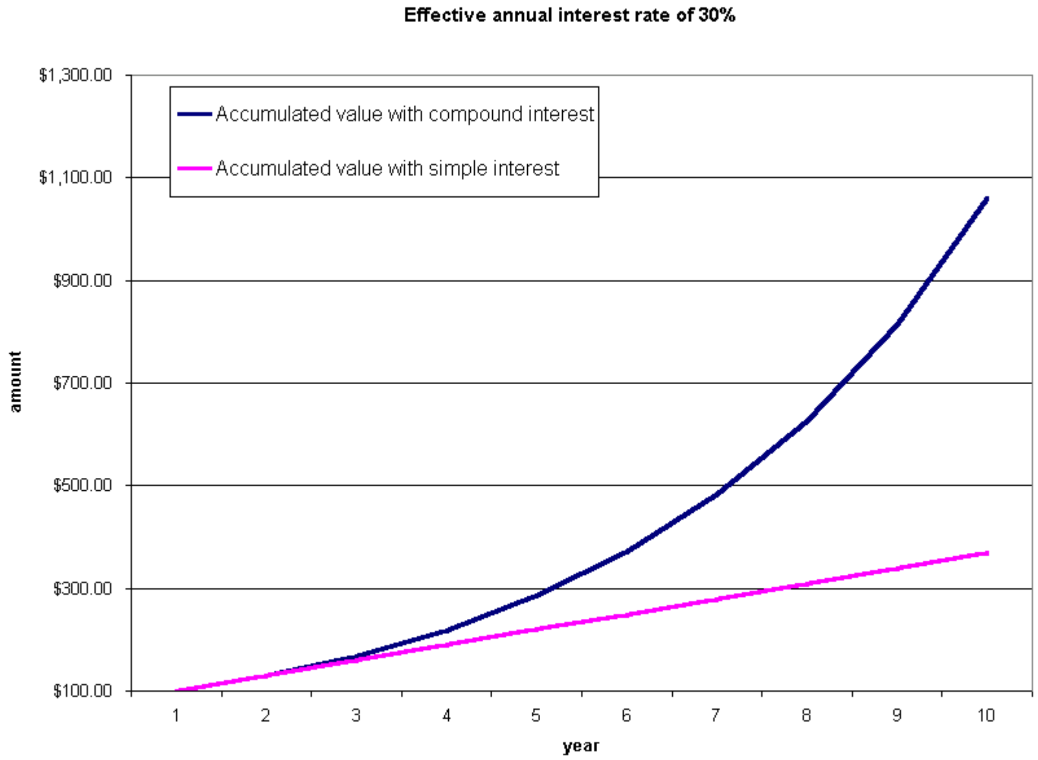
For example, if an amount of \$400 was invested and grew to \$500 in 3 years then $A(0,3) = 1.25$

EXAMPLE
Find the accumulated value of $100 in 4 years if:
a) the money is invested at a compound interest rate of 5% p.a.
b) the money is invested at a simple interest rate of 5% p.a.

Solution
Under compound interest: $X(1+i)^t = 100(1.05)^4 = 121.55$
Under simple interest: $X(1+ti) = 100(1+4(0.05)) = 120$

Since compound interest involves interest being paid on the interest already accumulated, a consequence is that the accumulated value under compound interest rises exponentially. Under simple interest the accumulated value rises as a straight line.





THE PRINCIPLE OF CONSISTENCY

Under compound interest, the proceeds of an investment should be the same whether an amount is invested for multiple periods or one single long period, so long as interest rates are the same.

In other words, when accumulating an investment under compound interest, accumulation factors can be multiplied together.

If the amount invested is $S(0)$, and the effective annual interest rate is i, then the accumulated amount after t_1 years is $S(t_1) = S(0)\cdot(1+i)^{t_1}$

If this amount is then reinvested at the same interest rate for $t_2 - t_1$ years then the accumulated amount at time t_2 is:

$$S(t_2) = S(t_1)\cdot(1+i)^{t_2-t_1} = S(0)\cdot(1+i)^{t_1}(1+i)^{t_2-t_1} = S(0)\cdot(1+i)^{t_2}$$

which is equivalent to an amount of $S(0)$ invested for t_2 years.

In other words, $(1+i)^{t_1}(1+i)^{t_2-t_1} = (1+i)^{t_2} \Rightarrow \frac{S(t_1)}{S(0)} \cdot \frac{S(t_2)}{S(t_1)} = \frac{S(t_2)}{S(0)}$, or if we define $A(t_1,t_2)$ as the accumulated amount at time t_2 of an investment of 1 at time t_1, where $0 \le t_1 \le t_2$, then:

$$A(0,t_1)A(t_1,t_2) = A(0,t_2)$$

Formally, if $0 \le t_1 \le t_2 \le ... \le t_n$, under the principle of consistency:

$$\boxed{A(0,t_n) = A(0,t_1)A(t_1,t_2)...A(t_{n-1},t_n)}$$

PRESENT VALUES

We have just looked at equations that can be used to answer the question: 'how much will a single investment accumulate to at a future date?'

In actuarial work financial analysis is required to be undertaken for payments which occur at a future date (eg. making life insurance payments to a policy-holder when they die). In these circumstances a more important question in financial calculations is 'how much money should be put aside today in order to have enough money to make the future payment?'.

The amount that should be put aside *now* to provide for payments in the future is **the present value** (PV) or **discounted value** of the payments.

The concept of present value factor, or discount factor, is introduced by considering the present value of an amount of 1.

If X equals the amount that must be invested at the start of the period to accumulate to 1 at the end of the period (ie. X is the present value of 1), and if an effective interest rate of i holds over the period, then:

$$X(1+i) = 1$$

This can be rearranged to give X:

$$X = \frac{1}{(1+i)} = (1+i)^{-1}$$

This is usually denoted by v, which is called the discount factor:

$v = (1+i)^{-1}$ = the present value at the beginning of a time period of an amount 1 due at the end of the period.

Recall that $A(0,t)$ is the accumulated value at time t of 1 invested at time 0.

Under compound interest,

$$A(0,t) = (1+i)^t = v^{-t}$$
$$\Rightarrow A^{-1}(0,t) = v^t$$

So, the accumulation to time t of an amount $KA^{-1}(0,t)$ is $\left(KA^{-1}(0,t)\right)A(0,t) = K$

Therefore, for an amount K due at time t, the present value is Kv^t.

From the example at the bottom of the Compound Interest section on page 3, this would mean that \$400 is the present value of \$500 due in 3 years time; ie.

K = 500 $A(0,3)$ = 1.25 $A^{-1}(0,3)$ = 0.8 Kv^t = 400

Under compound interest:

$Kv^t = K(1+i)^{-t}$ = the present value (at time 0) of an amount K due at time t.

Under simple interest:

$K(1+it)^{-1}$ = the present value (at time 0) of an amount K due at time t.

Since compound interest is predominantly used in practice, most of the calculations and examples for the remainder of the course will involve compound interest and not simple interest.

We don't have to restrict ourselves to finding the present value at time 0 – we can find the present value at any time t_1 of an amount K due at time t_2. Under compound interest:

$Kv^{t_2-t_1} = K(1+i)^{-(t_2-t_1)}$ = the present value at time t_1 of an amount K due at time t_2.

EXAMPLE

An investor must pay \$10000 in 8 years. How much should be invested in a deposit account now to provide for this payment if the effective annual compound interest rate earned in the deposit account is 8%?

Solution
If X is invested at time $t=0$, then $X(1+0.08)^8=\$10000$.

The present value of \$10000 in 8 years is:

$$\$10000(1+0.08)^{-8}=\$10000v^8=\$10000(0.54027)=\$5402.69$$

EXAMPLE
An investor must pay \$20000 in 10 years. How much should be invested in a deposit account in 6 years to provide for this payment if the effective annual compound interest rate earned in the deposit account is 6%?

Solution
If X is invested at time $t_1=6$, then $X(1+0.06)^{t_2-t_1}=X(1+0.06)^{10-6}=\$20000$.

The present value at time $t_1=6$ of \$20000 in 10 years is:

$$X=\$20000(1+0.06)^{-4}=\$20000v^4=\$20000\left(0.79209\right)=\$15842.$$

NOTE ON ROUNDING

Most of the questions in the course will involve calculating financial amounts. Many of these questions will involve intermediate steps which will result in the calculation of a figure which will be used later in the question. It is important that these figures are not rounded to an extent that will affect the overall result of the question. For example, in the example above, if $(1+0.06)^{-4}$ was taken to equal 0.79, the overall answer would be \$15800 and not \$15842 As a basic rule, intermediate calculations should be performed to **at least** 5 significant figures, with the exception of dollar values which should be calculated to the nearest cent.

Overall answers should be calculated to the detail specified in the question. If the question does not specify, you would generally report dollar values to the nearest cent and interest rates to one decimal place.

INVESTING WITH DIFFERENT INTEREST RATES

If we have different rates of interest each year, $i_1, i_2, i_3, ..., i_t$ then the accumulated value of an amount X after t years is:

$$X(1+i_1)(1+i_2)(1+i_3)...(1+i_t)$$

The present value at time $t=0$, of an amount K payable in t years is:

$$K(1+i_1)^{-1}(1+i_2)^{-1}(1+i_3)^{-1}...(1+i_t)^{-1} = \frac{K}{(1+i_1)(1+i_2)(1+i_3)...(1+i_t)}$$

EXAMPLE

Find the present value of \$300 due in 7 years, if the interest rate for the first 3 years is 6% p.a. and for the next 4 years is 8% p.a.

Solution

If X is invested at time $t=0$, then $X(1+0.06)^{3-0}(1+0.08)^{7-3} = \$300$.

The present value at time $t=0$ of \$300 in 7 years is:
$X = \$300(1+0.06)^{-3}(1+0.08)^{-4} = \$300 v_{0.06}^3 v_{0.08}^4 = \$300\left(0.83962\right)\left(0.73503\right) = \$185.14$.

CONVERTING BETWEEN EFFECTIVE RATES OF INTEREST

As mentioned earlier in these notes, effective rates of interest can apply to different time periods.

For example, if \$100 is invested and \$12 interest is paid at the end of the year, then the effective **annual** rate of interest is 12%. The accumulated amount at the end of the year will be $\$100(1.12) = \$112$

If \$100 is invested and \$1 is paid at the end of the first month, then the effective **monthly** rate of interest is 1%. An effective monthly rate of 1% will produce an accumulated amount at the end of the year of $\$100(1.01)^{12} = \$112.68$. Note that this is different to \$112 since interest is earned for the remainder of the year on the \$1 paid at the end of each month.

We may wish to find an effective monthly rate of interest that is **equivalent** to an effective annual rate of 12%. **Equivalent rates produce the same accumulated amounts over the same time period.**

The equivalent effective monthly rate j to an effective annual rate of $i = 12\%$ can be found by solving the equation:

$$\$100(1+j)^{12} = \$100(1+i)^{1} = \$112$$

To express an effective annual rate as an effective monthly rate, we take the annual growth factor and express it in terms of monthly growth factors. If we let the effective annual rate of interest be denoted i and the effective monthly rate be denoted j, then:

$$(1+i)^{1} = (1+j)^{12} \Rightarrow j = (1+i)^{\frac{1}{12}} - 1$$

In the example above, $j = (1.12)^{\frac{1}{12}} - 1 \cong 0.009489$.

Some of the common periods for which effective interest rates are quoted are:

	Compounding periods per year
Annual	1
Semiannual (or half-yearly)	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

EXAMPLE

What is the equivalent effective semiannual rate if the effective annual interest rate is 12%?

Solution

Let i be the effective annual rate, $i = 0.12$. Let j be the effective semiannual rate:

$$(1+i) = (1+j)^{2} \Rightarrow j = (1+i)^{1/2} - 1 = (1.12)^{1/2} - 1 \cong 0.0583$$

This is equal to 5.83%.

EXAMPLE

What is the equivalent effective weekly rate if the effective monthly interest rate is 1%?

Solution

Let i be the effective monthly interest rate, $i = 0.01$. Let j be the effective weekly rate. Since there are 52 weeks in a year and 12 months in a year, there are 52/12 weeks in a month.

$$(1+i) = (1+j)^{52/12} \Rightarrow j = (1+i)^{12/52} - 1 = (1.01)^{12/52} - 1 \cong 0.002299$$

This is equal to 0.2299%.