MAT 327 Midterm Test, University of Toronto July 3, 2013

Name (LAST, First): ____________________
Student Number: ____________________

- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and write "Solution continued on the back of this page".
- The test is from 4:10 pm - 6:00 pm. You have 110 minutes.
- The test is out of 100 marks. With bonus questions it is possible to earn a total of 109 marks.

FOR TA USE ONLY	
Question	Score
1	/30
2	/10
3	/10
4	/20
5	/30
BONUS	/9
TOTAL	/100

1. For each of the following questions (a)-(k), answer the question and provide **one or two sentences** of explanation (unless otherwise stated).

   (a) [2 marks] Which T_i property is equivalent to "points are closed"? (No explanation needed.)

   > **Solution:** T_1. (See §9 notes.)

   (b) [2 marks] Which T_i property implies that "each sequence converges to at most one point"? (No explanation needed.)

   > **Solution:** T_2. (See §5 notes.)

   (c) [2 marks] Let $\mathcal{T}_{\text{usual}}$ be the usual topology on $\mathbb{R}$. Is $\mathcal{T}_{\text{usual}} \times \mathcal{T}_{\text{usual}}$ a topology on $\mathbb{R}^2$?

   > **Solution:** No, because the union of two rectangles need not be a rectangle. For example, $(0,1)\times(0,1)\cup(1,2)\times(1,2) \neq A\times B$, with $A, B \in \mathcal{T}_{\text{usual}}$. However, it does form a *basis* for a topology. (See §2 notes, examples 4 and 5.)

   (d) [3 marks] Is every continuous function $f : \mathbb{R}_{\text{usual}} \longrightarrow \mathbb{R}_{\text{usual}}$ an open function?

   > **Solution:** No. Any constant function provides a counterexample. (See Assignment 1 solutions, C.5.)

   (e) [3 marks] Is $\mathbb{Z}_{\text{discrete}} \cong \mathbb{Q}_{\text{discrete}}$?

   > **Solution:** Yes. Any bijection between the two sets witnesses this fact. (See §7 notes, and the examples given in the homeomorphism section.)

   (f) [3 marks] Which of the following properties does ω_1 (with the order topology) have? First Countable, Second Countable, T_2, Separable. (No explanation needed.)

   > **Solution:** ω_1 is first countable and T_2, but not second countable or separable. (See §10 notes.)

(g) [3 marks] Does $\mathbb{R}_{\text{usual}}$ refine $\mathbb{R}_{\text{co-countable}}$?

Solution: No. For example, $\mathbb{R} \setminus \mathbb{Q}$ is open in $\mathbb{R}_{\text{co-countable}}$, but not in $\mathbb{R}_{\text{usual}}$. (Similar to material in §1 notes.)

(h) [3 marks] Without proof, write down a countable basis for the usual topology on $\mathbb{R}$.

Solution: $\{ (p, q) \subseteq \mathbb{R} : p, q \in \mathbb{Q} \}$ (See §4.6)

(i) [3 marks] Is S^{4327} (as a subspace of $\mathbb{R}^{4328}$) a T_3 space?

Solution: Yes. We know that T_3 is a hereditary property (Assignment 4, A.3) and a finitely productive property (§9.5 notes). We also know that $\mathbb{R}$ is a T_3 space (§9.2 notes). Together this gives us the answer.

(j) [3 marks] Which of the following two statements can be false in a topological space $(X, \mathcal{T})$, for $A \subseteq X$: (1) "$\overline{A \cap B} = \overline{A} \cap \overline{B}$" or (2) "$\overline{A \cup B} = \overline{A} \cup \overline{B}$". Provide a counterexample to the false statement.

Solution: In general, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$. For example, take $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}_{\text{usual}}$. Here $\overline{A \cap B} = \emptyset \neq \mathbb{R} = \overline{A} \cap \overline{B}$. (See §3.3.)

(k) [3 marks] Give an example of a continuous, surjective, open function

$$f : X \longrightarrow Y$$

that is not a homeomorphism. (Make sure to choose explicit topological spaces X and Y, and an explicit function.)

Solution: Let $X = \{ 1, 2 \}$ with the discrete topology, and let $Y = \{ 3 \}$ with the discrete topology, and let $f(1) = f(2) = 3$. This clearly has the required properties, but is not a homeomorphism because it is not a bijection (it is not 1-1). (Compare with Assignment 3, C.5.)

2. (a) [4 marks] State the definition of "$(X, \mathcal{T})$ is first countable space". Make sure to define all the terms you use.

> **Solution:** You can find the definition in §5.5.

(b) [6 marks] Suppose that $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are both first countable spaces. Prove that $X \times Y$ is a first countable space when given the product topology.

> **Solution:** For each $(x, y) \in X \times Y$, let $\mathcal{B}_x$ be a countable local basis for x, and let $\mathcal{C}_y$ be a countable local basis for y. Define $\mathcal{D} := \mathcal{B}_x \times \mathcal{C}_y$. We note that it is countable, since each factor is countable, so we only need to show that it is a local basis at (x, y). Since each $B \in \mathcal{B}_x$ is an open set in X containing x, and each $C \in \mathcal{C}_y$ is an open set in Y containing y, we see that any $B \times C \in \mathcal{D}$ is an open set in $X \times Y$ that contains (x, y).
>
> Now suppose that $A \subseteq X \times Y$ is an open set that contains (x, y). Then there is a basic open set $U \times V \subseteq A$ that contains (x, y), where U is open in X (and contains x) and V is open in Y (and contains y). Since $\mathcal{B}_x$ is a local basis, there is a $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$. Similarly there is a $C \in \mathcal{C}_y$ such that $y \in C \subseteq V$. Thus, $(x, y) \in B \times C \subseteq U \times V \subseteq A$, and $B \times C \in \mathcal{D}$.

3. This question will be about $\mathbb{R}_{\text{Sorgenfrey}}$, the Sorgenfrey Line. **Prove all assertions you make about the Sorgenfrey line** when answering these two questions.

   (a) [4 marks] Does the sequence $\{\,(0, -\frac{1}{n}) : n \in \mathbb{N}\,\}$ converge to $(0,0)$ in $\mathbb{R}_{\text{Sorgenfrey}} \times \mathbb{R}_{\text{Sorgenfrey}}$ with the product topology?

   Solution: It does not converge to $(0,0)$. In the product, $[0,1) \times [0,1)$ is an open set that contains $(0,0)$ but does not contain *any* element of the sequence. We can see this because the second coordinate of $[0,1) \times [0,1)$ is always non-negative, but every element of the sequence has negative second coordinate.

   (b) [6 marks] Prove or disprove: $\mathcal{B} := \{\,[p,q) : p,q \in \mathbb{Q}\,\}$ is a basis for $\mathbb{R}_{\text{Sorgenfrey}}$.

   Solution: This is not a basis for $\mathbb{R}_{\text{Sorgenfrey}}$. This was an exercise in §2.4. Here is the proof. Take $A = [\pi, \pi + 1)$ which is an open set in $\mathbb{R}_{\text{Sorgenfrey}}$. Since π is not a rational number, any element $[p,q) \in \mathcal{B}$ that contains π must have $p < \pi$. Thus $[p,q) \not\subseteq A$, and so $\mathcal{B}$ is not a basis for $\mathbb{R}_{\text{Sorgenfrey}}$.

   Also, we know the more general fact that the Sorgenfrey Line is not second countable (see Assignment 3, A.3 Solutions) so $\mathcal{B}$ could not possibly be a basis for the Sorgenfrey Line.

4. For this question, suppose that $\mathbb{R}$ has its usual topology, and ω_1 is given the order topology. You may reference, without proof, any theorems or propositions from class, or any assignment questions.

   (a) [5 marks] Give two (infinite) countable subspaces of $\mathbb{R}$ that are not homeomorphic.

   > **Solution:** Let $A := \mathbb{Z}$ and $B := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, which are both infinite, countable subsets of $\mathbb{R}$. Given their (respective) subspace topologies A is discrete (§7.2), but B is not, since $\langle \frac{1}{n} \rangle \to 0$. So they are not homeomorphic.

   (b) [5 marks] Give two uncountable subspaces of $\mathbb{R}$ that are not homeomorphic.

   > **Solution:** Let $A := (0,1)$ and $B := [0,1)$, which are both uncountable subsets of $\mathbb{R}$. Given their (respective) subspace topologies, A only contains cut-points and B contains one non-cut-point, namely the point 0. We saw that this was a topological invariant on Assignment 4, A.2, so A and B are not homeomorphic.

   (c) [5 marks] Give two (infinite) countable subspaces of ω_1 that are not homeomorphic.

   > **Solution:** From our §10 notes, we know that ω_1 contains an initial segment A that is homeomorphic to $\mathbb{N}$ (with the discrete topology), and we know that ω_1 has an initial segment B that is homeomorphic to $\omega + 1$. Here both spaces are infinite and countable, but A is discrete and B contains a non-trivial convergent sequence. So A and B are not homeomorphic.

(d) [5 marks] Give two uncountable subspaces of ω_1 that are not homeomorphic.

Solution: (This example was pointed out in the exercises in §10.) Let $A = \omega_1$ and let $B := \{ x \in \omega_1 : \{ x \} \text{ is open} \}$. Clearly A is an uncountable non-discrete space (see the previous question), and B is a discrete space (by construction), so they cannot be homeomorphic. It only remains to see that B is uncountable. Suppose it wasn't, then there would be an upper bound $b \in \omega_1$ for the set. Setting

$$S(b) := \min\{ x \in \omega_1 : x > b \}$$

(which exists since the set in question is nonempty, and ω_1 is well-ordered), we see that $\{ S(b) \}$ is an open set not in B, a contradiction.

Note: It is not enough to point out that B has an open point, because so does ω_1! We really need to use the invariant that *every* point in B is open.

5. Define a topological space $(X, \mathcal{T})$ to be **super second countable** if *every* basis $\mathcal{B}$ that generates $\mathcal{T}$ is countable.

   (a) [5 marks] Prove that every finite topological space is super second countable.

   > **Solution:** If $(X, \mathcal{T})$ is a finite topological space, then every basis $\mathcal{B}$ for $\mathcal{T}$ is, by definition, a subset of $\mathcal{P}(X)$, the power set of X, which is finite, since X is finite. Thus each such $\mathcal{B}$ is finite, hence countable.

   (b) [5 marks] Show that a super second countable space has only countably many open sets.

   > **Solution:** We prove the contrapositive. Suppose that $(X, \mathcal{T})$ is a topological space with $\mathcal{T}$ an uncountable collection. Observe that $\mathcal{T}$ is an (uncountable) basis for $\mathcal{T}$. So $(X, \mathcal{T})$ is not super second countable.

   (c) [5 marks] Show that a topological space with infinitely many mutually disjoint open sets has uncountably many open sets.

   > **Solution:** Let $\mathcal{D}$ be a collection of mutually disjoint open sets in some topological space $(X, \mathcal{T})$. If $\mathcal{D}$ is uncountable, there is nothing to prove, so assume that $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ is countable. (Moreover, assume that each D_n is non-empty.) Let
   >
   > $$\mathcal{U} := \{ \bigcup_{n \in A} D_n : A \in \mathcal{P}(\mathbb{N}) \}$$
   >
   > It is clear that each element of $\mathcal{U}$ is open, now we need that $\mathcal{U}$ is uncountable. We know that $\mathcal{P}(\mathbb{N})$ is uncountable (this is Cantor's Theorem from §4), so we just need to show that if $A \neq B$ (both in $\mathcal{P}(\mathbb{N})$) then $\bigcup_{n \in A} D_n \neq \bigcup_{n \in B} D_n$. Suppose $A \neq B$. Then there is a $m \in A \setminus B$ or there is an $m \in B \setminus A$. In the first case, we see that $D_m \subseteq \bigcup_{n \in A} D_n$ but $D_m \not\subseteq \bigcup_{n \in B} D_n$ (here we use $D_n \neq \emptyset$!), so the unions are different. The second case is analogous.

(d) [10 marks] Show that any infinite T_2 topological space contains infinitely many mutually disjoint open sets.

Solution: Let $(X, \mathcal{T})$ be an infinite T_2 space. If it is discrete, then the family $\{\{x\} : x \in X\}$ is the desired infinite collection of mutually disjoint, open sets.

Suppose it is not discrete. Let $p \in X$ be a point such that $\{p\}$ is not open. We will construct the collection of disjoint open sets recursively. Take any point $x_1 \neq p$ and find disjoint open sets A_1 and B_1 such that $p \in A_1$ and $x_1 \in B_1$.

Note that $A_1 \neq \{p\}$, since $\{p\}$ is not open. So find a point $x_2 \neq p$ such that $x_2 \in A_1$. Now find disjoint, open sets U_2, V_2 such that $p \in U_2$ and $x_2 \in V_2$. We then let $A_2 := A_1 \cap U_2$ and let $B_2 := A_1 \cap V_2$, which are both open sets. This has given us that B_2 and B_1 are non-empty, disjoint open sets.

We continue on in this way, finding $x_{n+1} \neq p$ with $x_{n+1} \in A_n$, then find disjoint open sets U_{n+1}, V_{n+1} so that $p \in U_{n+1}$ and $x_{n+1} \in V_{n+1}$. We then define $A_{n+1} := A_n \cap U_{n+1}$ and $B_{n+1} := A_n \cap V_{n+1}$. (We see that we can continue on in this way since A_n is never equal to $\{p\}$, and is always infinite.)

Now, by recursion, we get the collection $\{B_n : n \in \mathbb{N}\}$, which (by construction) is an infinite collection of mutually disjoint open sets.

(e) [5 marks] From the previous 4 facts, state and prove a new theorem that characterizes when a T_2 space is super second countable, based on its cardinality. (This theorem should contain the words "if and only if".)

Solution:

Theorem: A T_2 topological space $(X, \mathcal{T})$ is super second countable if and only if it is finite.

Proof: If it is finite, then by part (a) it is super second countable. If it is infinite, then by part (d) it has an infinite collection of mutually disjoint open sets. Thus by part (c) it contains uncountably many open sets. So by part (b) $(X, \mathcal{T})$ is not a super second countable space.

6. **These are BONUS questions**

   (a) [1 mark (bonus)] Spell the course instructor's (complete) first name and last name. $\boxed{\textbf{Micheal Pawliuk}}$

   (b) [1 mark (bonus)] Spell the TA Ivan's last name. $\boxed{\textbf{Khatchatourian}}$

   (c) [1 mark (bonus)] Spell the TA Ali's last name. $\boxed{\textbf{Mousavidehshikh}}$

   (d) [1 mark (bonus)] What was Hausdorff's first name? $\boxed{\textbf{Felix}\text{. See §5.}}$

   (e) [1 mark (bonus)] What was DeMorgan's first name? $\boxed{\textbf{Augustus}\text{. See §3.5.}}$

   (f) [1 mark (bonus)] Complete the classic "joke": "A topologist is a person who can't tell a $\boxed{\textbf{doughnut}}$ from a $\boxed{\textbf{coffee cup}}$." (See §6.)

   (g) [1 mark (bonus)] Name a famous Australian tennis player. $\boxed{\textbf{Rod Laver}\text{. See §8.}}$

   (h) [1 mark (bonus)] Name a type of "loose-fitting baggy form of trousers favoured by members of the University of Oxford, especially undergraduates, in England during the early 20th century from the 1920s to around the 1950s." $\boxed{\textbf{Oxford Bags}\text{. Assignment 3, A.3.}}$

   (i) [1 mark (bonus)] What word does the "T" stand for in the T_i property? $\boxed{\textbf{Trennungsaxiomen}\text{. See Assignment 4, A.3.}}$