Tutorial 5

February 26, 2015

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ f(\lfloor \frac{n}{2} \rfloor)^2 + 2f(\lfloor \frac{n}{2} \rfloor) & \text{if } n \geqslant 1 \end{cases}$$

prove if $m, n \in \mathbb{N}$ with $m \leqslant n$, then $f(m) \leqslant f(n)$.

Questions:
1. Complete Induction
2. $m = n, m < n$

Predicate:
For $n \in \mathbb{N}$, let $P(n)$ be for $m \in \mathbb{N}$ if $m \in \mathbb{N}$ if $m \leqslant n$ then $f(m) \leqslant f(n)$.

Base Case:
$n = 0$ this forms $m = 0$
$f(m) = f(0) = 2 = f(n) \therefore P(0)$ holds.

Inductive Step:
Let $m \in \mathbb{N}$ where $m \leqslant n$.
Then assume $P(k)$ holds for all $0 \leqslant k < n$ for $k \in \mathbb{N}$.
Consider if $n > 0$, then there are 2 cases to consider.

Case 1: $m = 0$
$f(m) = f(0) \leqslant f(\lfloor \frac{n}{2} \rfloor)$ by IH $P(\lfloor \frac{n}{2} \rfloor)$ because $$ \lfloor\frac{n}{2}\rfloor 0, \lfloor\frac{n}{2}\rfloor \in \mathbb{N} . \leqslant 2f(\lfloor\frac{n}{2}\rfloor) since f(\lfloor\frac{n}{2}\rfloor)\geqslant0 . \leqslant f(\lfloor\frac{n}{2}\rfloor)^2 + 2f(\lfloor\frac{n}{2}\rfloor) since f(\lfloor\frac{n}{2}\rfloor) \geqslant f(0) = f(n) P(n) $$ holds.

Case 2: $m > 0$
$f(m) = f(\lfloor \frac{m}{2} \rfloor)^2 + 2f(\lfloor \frac{m}{2})\rfloor$
$\leqslant f(\lfloor \frac{n}{2} \rfloor) + 2f(\lfloor \frac{n}{2} \rfloor)$
$= f(n)$
$\therefore P(n)$ holds.

because $f(\lfloor \frac{m}{2} \rfloor) \leqslant f(\lfloor \frac{n}{2} \rfloor)$ by I.H., $P(\lfloor \frac{n}{2} \rfloor)$ and since $m \leqslant n$, $\lfloor \frac{m}{2} \rfloor \leqslant \lfloor \frac{n}{2} \rfloor$.