UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2012 EXAMINATIONS

MAT335H1F
Chaos, Fractals and Dynamics
Examiner: D. Burbulla

Duration - 3 hours
Examination Aids: A Scientific Hand Calculator

Name: ______________________________ **Student Number:** ________________

INSTRUCTIONS: All six questions have equal weight. Attempt only five of them. Present your solutions in the exam booklets provided. Do not tear any pages from this exam. **Hand in this exam with your booklet(s).** The marks for each question are indicated in parentheses beside the question number. **MAXIMUM MARKS:** 100

1. [20 marks] Let A_i for $i = 0, 1, 2, 3$ be linear contractions with contraction factor $\beta = 1/3$ and fixed points

$$p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, p_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

respectively. Let $\mathcal{A}$ be the attractor generated by the iterated function system A_0, A_1, A_2, A_3.

   (a) [10 marks] Show that $\mathcal{A} = K \times K$, where K is the Cantor middle-thirds set.

   (b) [5 marks] What is the fractal dimension of $\mathcal{A}$?

   (c) [5 marks] Describe in your own words how the chaos game can be played to generate the fractal $\mathcal{A}$.

2. [20 marks] This question has four parts.

   (a) [4 marks] Define the Mandelbrot Set.

   (b) [6 marks] Define the Sierpinski triangle. What is its fractal dimension?

   (c) [5 marks] Define what it means, according to Devaney, for $F : X \longrightarrow X$ to be chaotic.

   (d) [5 marks] Prove that if $\mathbf{s} \in \Sigma$ then there is a sequence $\mathbf{t} \in \Sigma$ arbitrarily close to $\mathbf{s}$ for which $d[\sigma^n(\mathbf{s}), \sigma^n(\mathbf{t})] = 2$, for all sufficiently large n.

5. [20 marks] Determine the fate of the orbits of the following seeds z_0 under the following functions F. If the orbit is periodic, or eventually periodic, determine if the periodic cycle is attracting, repelling or neutral.

   (a) [4 marks] $z_0 = \dfrac{3}{10}$ and $F(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$

   (b) [4 marks] $z_0 = 1$ and $F(z) = \dfrac{iz}{2}$.

   (c) [4 marks] $z_0 = 0$ and $F(z) = z^2 + i$.

   (d) [4 marks] $z_0 = 0$ and $F(z) = z^2 + 2i - 1$.

   (e) [4 marks] $z_0 = 0$ and $F(z) = z^2 + \dfrac{i}{8} - 1$.

6. [20 marks] Let $Q_c : \mathbb{C} \longrightarrow \mathbb{C}$ by $Q_c(z) = z^2 + c$. Let K_c be the filled Julia set of Q_c; let J_c be the Julia set of Q_c.

   (a) [5 marks] Plot K_0 and J_0 in the complex plane.

   (b) [10 marks] Let $R = \{z \in \mathbb{C} \mid |z| > 1\}$; let $H : R \longrightarrow \mathbb{C} - [-2, 2]$ by

   $$H(z) = z + \frac{1}{z}.$$

   Show that H is a conjugacy between Q_0 on R and Q_{-2} on $\mathbb{C} - [-2, 2]$.

   (c) [5 marks] Plot K_{-2} and J_{-2} in the complex plane.