UNIT 7

INTERPRETATIONS AND MODELS:

SEMANTICS FOR PREDICATE LOGIC

7.1 Intension and Extension

When you know what a word means, what do you actually know?

Can you give a definition of it?

Do you know exactly what things it applies to?

For instance, ask yourself: What does ‘dog’ mean?

One way to give the meaning of ‘dog’ is the *intension* or sense of the word (often the dictionary meaning):

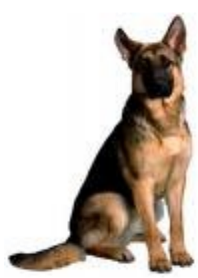
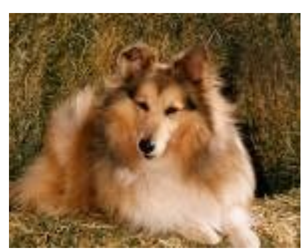
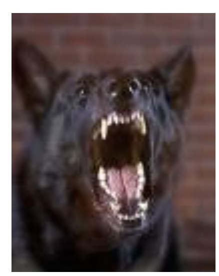
A domesticated carnivorous mammal of the genus, Canis, related to foxes and wolves.

It sounds like a good definition. But it probably isn’t the definition you would give if asked. Do you think it would help a new speaker of English learn the meaning of the word, ‘dog’? How did you learn the meaning of ‘dog’?

Most people learn the meaning of words like ‘dog’ and ‘cat’ through *ostensive* definitions: having examples pointed out to you.

The *extension* of the word ‘dog’ is all of those things that you might point at when giving an ostensive definition (all dogs!) That gives us another way of giving the meaning of a word: dog means…

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The *extension* of the word dogs includes all of those types of things!

Indeed, if we want to know whether a dictionary definition is a good one, we can check if it captures the meaning in the sense of the extension.

Chair: A piece of furniture used for sitting on.
Is this a good definition?

> No. It is too broad. It picks out couches, benches and stools. The extension of the word "chair" does not include such things.

Chair: A piece of furniture used for sitting on, normally for seating one person, with a back, legs and a seat.

Is this a good definition?

> No. Now it is too narrow. The extension of the word "chair" includes things that don't have legs – hanging chairs, some rocking chairs, etc.

The relation between the extension of a word and the 'intension' or sense of the word (what the dictionary definition is trying to capture) is more complicated than you might think, and philosophically fascinating! Philosophers such as Bertrand Russell, Hilary Putnam, Tyler Burge, Saul Kripke have written much about this part of philosophy of language, as well as its relation to the mind and intentionality.

For us, it is enough to recognize that extension is one central way of understanding meaning. It is an important part of a semantics for predicate logic.

7.2 Interpretations

By defining the extensions, an interpretation assigns meaning to predicates and singular terms.

Consider the symbolic sentences: $\forall x(Fx \rightarrow Gx)$ and $\exists x(Gx \wedge \sim Fx)$

The first one tells us that all things with property F also have property G. The second tells us that there is at least one thing that has property G but not property F.

But what are F and G? What do they mean?

Until we know what it means to have property F or property G, we cannot determine whether the sentences are true or false. We need to define or provide an interpretation of the predicates, F and G! We do this by giving the extension of the terms – stating which members of the universe of discourse are in the extension of the predicate. That is essentially what our abbreviation or symbolization schemes do.

> Universe of discourse: living things
>
> F^1: *a* is a dog
>
> G^1: *a* is a mammal

On this interpretation, the extension of F includes all dogs and the extension of G includes all mammals. Since all dogs *are* mammals, $\forall x(Fx \rightarrow Gx)$, and some mammals are *not* dogs, $\exists x(Gx \wedge \sim Fx)$, this interpretation makes both sentences true.

$\forall x(Fx \rightarrow Gx)$ and $\exists x(Gx \wedge {\sim}Fx)$

On these interpretations, also, both sentences are true…

Universe of discourse: positive integers	Universe of discourse: unrestricted
F^1: *a* is a multiple of 10	F^1: *a* is a reptile
G^1: *a* is even	G^1: *a* is cold-blooded

But on other interpretations, at least one of the sentences is false.

UD: positive integers	UD: unrestricted
F^1: *a* is a multiple of two	F^1: *a* is cold-blooded
G^1: *a* is even	G^1: *a* is a reptile

In the case of a one-place predicate, the interpretation directly defines the extension of the predicate. The ***extension*** of a predicate is the set of members of the universe that the predicate picks out (that satisfy the predicate).

However, or in the case of multi-place predicates, the extension will be a set of ordered pairs, triplets, etc. – those that satisfy the predicate.

UD: positive integers	UD: unrestricted
F^1: *a* is a multiple of four	F^1: *a* is a dog
G^1: *a* is even	G^1: *a* is a mammal
L^2: a is larger than b.	L^2: a chases b.
a: 2	a: Fido (a dog)

On the interpretation on the left, the extension of F and G are multiples of four and even numbers, respectively. But what is the extension of L? It would be a set of ordered pairs. Since L is the relation of "larger than", the extension would include all ordered pairs in which the first number is larger than the second number. For example: (2,1), (64, 7) and (534, 23). This would be an infinite set – the extension of L goes on forever.

In the interpretation on the right, the extension of L would be the set of ordered pairs in which the first member chases the second member. Thus, if Fido chases Smokey (a cat), the ordered pair, (Fido, Smokey), would be in the extension of L.

"a" is a name letter which picks out some individual in the universe of discourse. In the interpretation on the left, it is defined as the integer, 2. Thus, according to that interpretation Fa is false and Ga is true. In the interpretation on the right, it is defined as the individual, Fido – and according to that interpretation both Fa and Ga are true since Fido is a dog and a mammal.

We could also provide interpretations for three-place, four-place, or higher-order predicates. The extension of those predicates would be ordered triplets, ordered quadruplets, etc.

Consider the sentence: $\forall x(Fx \rightarrow \exists y(Gy \wedge L(xy)))$

And the two interpretations we just looked at.

UD: positive integers	UD: unrestricted
F^1: *a* is a multiple of four	F^1: *a* is a dog
G^1: *a* is even	G^1: *a* is a mammal
L^2: a is larger than b.	L^2: a chases b.

The sentence is true for an interpretation if everything that is in the extension of F is in the L relation to something in the extension of G.

Thus, the sentence is true on the interpretation on the left: every multiple of four is larger than at least one even number. In the extension of L are the ordered pairs, (4,2), (8,2), (12, 4), (16,6)… So, every individual in the extension of F {4, 8, 12, 16…} occurs in the first place of an ordered pair in the extension of L such that some individual in the extension G is in the second place.

According to the interpretation on the right, the sentence is true if every dog (everything in the extension of F) chases at least one mammal (is in the L relation to at least one thing in the extension of G.) It is unlikely that this sentence is true, if only because some dogs die before they are old enough to chase other mammals.

Determining the Extension of a Predicate from an Interpretation:

To determine the extension of a predicate, consider the interpretation and ask yourself, on that interpretation, which members of the universe is the predicate true of – which members of the universe satisfy the predicate?

Extension of a one-place predicate: the extension of a monadic predicate is the subset of members of the universe that satisfy the predicate. In other words, it is the set of those members of the universe that the predicate is true of.

Extension of a two-place predicate: the extension of a dyadic predicate is the set of ordered pairs of members of the universe that satisfy the predicate. In other words, it is the set of ordered pairs that the predicate is true of.

Extension of a three-place predicate: the extension of a triadic predicate is the set of ordered triplets of members of the universe that satisfy the predicate. In other words, it is the set of ordered triplets that the predicate is true of.

UD: natural numbers {1,2,3 …} G^1: *a* is odd H^1: *a* is negative. L^2: *a* is less than *b*.

The extension of G: the extension is the set of odd members of U. On this interpretation, these are the members of U (natural numbers) that satisfy the predicate, G, 'is odd'. {1,3,5,7…}

The extension of H: the extension is the set of negative natural numbers. But no members of U satisfy that predicate! On this interpretation, the extension of H is empty since no natural number is negative and thus, none can satisfy this predicate. { } or ∅

The extension of L: the extension is the set of ordered pairs of members of U such that the first of the ordered pair is less than the second of the ordered pair. So (1,2), (1,3), (2,3) and (5,100) will all be in the extension of L, but (2,1), (1,1) and (100,5) will not be. This set is an infinite set.

7.3 Truth-Value and Interpretations

Whether a sentence is true or false depends on how we interpret it – it may be true on some interpretations and false on others. Similarly, in sentential logic, many sentences were true on some truth-value assignments and false on others.

Consider this sentence: $\forall x \exists y L(xy)$

And these interpretations:

UD: people	UD: integers
L^2: *a* loans money to *b*.	L^2: *a* is less than *b*.

According to the first interpretation, the sentence says: Everybody loans money to somebody. That is false!

According to the second interpretation, the sentence says: Every integer is less than some integer. That's true!

In order to determine truth-value of a sentence, we need to know the meaning of the terms. We need an interpretation!

What is an Interpretation?

An interpretation defines:

- the universe (must not be empty)
- monadic, dyadic and higher-place predicates.
- the individual constants (zero place operation letters)
- truth-values of sentential atomic sentences (zero place predicates)
- monadic, dyadic or higher-place operations.

An abbreviation or translation scheme is an interpretation of the sentences we use the scheme to symbolize.

The translation schemes we have used provide interpretations by defining terms using ordinary English expressions. However, we don't have to define terms in this way. For instance, we can also use set notation to provide an interpretation (see 7.5).

Determining the Truth-Value of Sentences without Quantifiers:

i. Use the interpretation to determine the truth-value of each atomic formula.

ii. Proceed to determine the truth-value of the sentence on the basis of your knowledge of the connectives, just like we did in sentential logic (truth-tables).

What is the truth value of this sentence according to the following interpretations?

$$(Fa \vee Fb) \rightarrow (L(ca) \vee L(cb))$$

UD: unrestricted F^1: a is a building. L^2: a is taller than b.

a^0: Jackman Humanities Building b^0: The Statue of Liberty. c^0: The CN Tower

If either the Jackman Humanities Building (JHB) or The Statue of Liberty is a building then the CN tower is taller than JHB or The CN Tower is taller than The Statue of Liberty.

UD: integers F^1: a is prime. L^2: a is less than b.

a^0: 3 b^0: -4 c^0: -1

If either 3 or -4 is prime then either -1 is less than 3 or -1 is less than -4.

UD: integers F^1: *a* is prime. L^2: *a* is larger than *b*.

a^0: 4 b^0: 4 c^0: 4

If either 4 or 4 is prime then 4 is larger than 4 or 4 is larger than 4.

All of these interpretations make the sentence true. The interpretations on which the sentence is true have something in common: they make the consequent true by ensuring that (ca) or (cb) is within the extension of L or they make the antecedent false by ensuring that neither 'a' nor 'b' is within the extension of F.

But the same sentence will also be false on some interpretations: $(Fa \vee Fb) \rightarrow (L(ca) \vee L(cb))$

UD: unrestricted F^1: *a* is a building. L^2: *a* is shorter than *b*.

a^0: Jackman Humanities Building b^0: The Statue of Liberty. c^0: The CN Tower

UD: integers F^1: *a* is an integer. L^2: *a* is greater than *b*.

a^0: 2 b^0: 2 c^0: 2

The interpretations on which the sentence is false have something in common: they make the antecedent true by ensuring that either a or b is within the extension of F and they make the consequent false by ensuring that neither (ca) nor (cb) is within the extension of L.

Sentences with Quantifiers:

Determining the truth-value of sentences with quantifiers is not much more difficult. The interpretation defines the universe, predicates, individual constants (also operation letters.) Now we use both the logical connectives and the quantifiers to determine the truth-value of the sentence on that interpretation.

$\forall\alpha\ \phi$ A universally quantified sentence is true if and only if it is true for ***every*** single member of the universe, or for every single ordered pair or triplet (in the case of two and three place predicates).

$\exists\alpha\ \phi$ An existentially quantified sentence is true if and only if it is true for ***at least one*** member of the universe, or for at least on ordered pair or triplet (in the case of two and three place predicates).

i. Use the interpretation to determine the truth-value of any unquantified atomic formulas.

ii. Use the interpretation, the results of (i), and your knowledge of the connectives to determine the truth-value or partial truth-value of any quantified sentential components.

iii. Use the interpretation, the results of (i) and the results of (ii) to determine the truth-value of the sentence.

Consider the sentence: $\forall x(Fx \rightarrow Gx)$ on the following interpretations:

UD: unrestricted F^1: *a* is human. G^1: *a* is mortal.

The extension of F is a subset of U (everything) that includes all and only human beings.
The extension of G is a subset of U (everything) that includes all and only things that are mortal.
Everything in the extension of F is also in the extension of G: every human being is mortal.

UD: integers F^1: *a* is even G^1: *a* is divisible by 2.

The extension of F is a subset of U (integers) that includes all and only things that are even.
The extension of G is a subset of U (everything) that is all and only things that are divisible by two.
Everything in the extension of F is also in the extension of G: every even integer is divisible by two.

On both these interpretations, the sentence is true since everything in the extension of F is also in the extension of G.

$\forall x(Fx \rightarrow Gx)$

In contrast, the sentence is false on the following interpretation since not everything in the extension of F is also in the extension of G.

UD: unrestricted F^1: *a* can fly. G^1: *a* is a bird.

The extension of F is a subset of U (everything) that includes all and only things that can fly.
The extension of G is a subset of U (everything) that includes all and only things that are birds.
Not everything in the extension of F is in the extension of G: some things that can fly are NOT birds.

7.3 EG1 On the interpretations below, is the following sentence true or false?

$\sim Fa \wedge \forall x(Fx \rightarrow G(xa))$

a) UD: positive integers
a^0: 2
F^1: *a* is a multiple of 4.
G^2: *a* is divisible by *b*
True

b) UD: people
a^0: Bill Gates
F^1: *a* is Canadian
G^2: *a* is richer than *b*
True

c) UD: positive integers
a^0: 2
F^1: *a* is odd
G^2: *a* is smaller than *b*
False

7.3 EG2 On the interpretations below, is the following sentence true or false?

$\forall x(Fx \rightarrow \forall y(G(xy) \rightarrow H(yx))$

a) UD: People
G^2: *a* makes a promise to b
F^1: *a* is a politician
H^2: *a* believes *b*
False

b) UD: Positive Integers
G^2: *a* is a factor of *b*
F^1: *a* is even
H^2: *a* is less than or equal to *b*
False

c) UD: positive integers
G^2: *a* is less than *b*
F^1: *a* is purple
H^2: *a* is a multiple of *b*
True b/c no Purple integer

7.4 Interpretations and Properties of Sentences and Arguments

In sentential logic, we give meaning to symbolic sentences through truth-value analysis. If we know what makes it true, then we know what it means. We determine the truth-value of a complex sentence from the truth-values of its sentential components. The truth-table provides the truth-value of a sentence on every possible truth-value assignment. Thus, given any *particular* truth-value assignment, we can determine the truth-value of the sentence. Some sentences (tautologies) are true on any truth-value assignment. Other sentences (contradictions) are false on any truth-value assignment. But most were true on some truth-value assignments and false on others (contingent sentences).

Logical Truths, Logical Falsehoods and Contingent Sentences:

As we just saw, the truth-value of a symbolic sentence in predicate logic often cannot be determined without taking interpretations into consideration. Many sentences are true on some interpretations and false on others.

In sentential logic, a sentence that was true on some truth-value assignments and false on others was a contingent sentence (it can be true or false.) Likewise, the sentences we just looked at can be true or false, since they are true on some interpretations and false on others. They are also contingent sentences!

Logically True Sentences:

Some sentences are true on ***any*** interpretation. A ***logically true*** sentence is a sentence that is true on every interpretation. No matter how we interpret the predicates, name letters and operation letters, there is no interpretation on which such a sentence is false. In sentential logic, we called a sentence that was true on any truth-value assignment a tautology. Likewise, a sentence in predicate logic that is true on any interpretation *can* be called a tautology. However, the term 'tautology' more often refers to truth-functional tautologies only. 'Logical truth' is a wider term, and includes both truth-functional tautologies and the logical truths of predicate logic. And of course, when doing derivations all of these logical truths can be derived from the empty set, and thus are theorems in our system.

This sentence is true no matter how we interpret it: $(Fa \vee Fb) \leftrightarrow (\sim Fa \rightarrow Fb)$

UD: unrestricted. F^1: *a* is female. a: Alex b: Robin

Either Alex is female or Robin is female if and only if it's the case that if Alex isn't female, Robin is.

UD: integers. F^1: a is prime. a: 3 b: 4

Either 3 or 4 is prime if and only if it's the case that if 3 is not prime then 4 is prime.

Some quantified sentences are true on every possible interpretation:

$$\forall x \forall y F(xy) \rightarrow \forall x F(xx)$$

Try to think up an interpretation that makes this sentence false! If everything stands in the F relation to everything, then everything must stand in that relation to itself. Even though it is clearly true, we can't demonstrate it by giving interpretations – we would have to show that there is *no* interpretation on which it is false by considering *every possible* interpretation (an infinite set).

This is the situation for all logical truths… recall some of the theorems proved in Unit 6!

Consider the following interpretations for: $\exists x{\sim}Fx \leftrightarrow {\sim}\forall xFx$

F^1: *a* is purple.	Something is not purple if and only if not everything is purple.
F^1: *a* is a unicorn	Something isn't a unicorn if and only if not all things are unicorns.
F^1: *a* is made of matter	Something isn't made of matter if and only if not everything is made of matter.

On every possible interpretation, no matter what property F is, something exists that doesn't have property F if and only if not all things do have property F.

Consider the following interpretations for: $\forall x\forall y(B(xy) \rightarrow {\sim}B(yx)) \rightarrow {\sim}\exists xF(xx)$

B^2: *a* is taller than *b*.	If it is the case that if one thing is taller than another, the second is not taller than itself, then nothing is taller than itself. .
B^2: *a* loves *b*.	If it is the case that if one thing loves another, the second does not love the first, then nothing loves itself.

On every possible interpretation, no matter what relation B is, if B is an asymmetric relation, then B is also irreflexive. The sentence is logically true.

Logical falsehoods:

Some sentences are false on ***any*** interpretation. There is no interpretation on which they are true. In sentential logic, a sentence that was false on any truth-value assignment (always false) was a contradiction or logical falsehood. Likewise, a sentence that is false on any interpretation is a logical falsehood.

No matter how we interpret this sentence, it is false.

$Fa \leftrightarrow {\sim}Fa$

UD: unrestricted. F^1: *a* is a building. a: The Jackman Humanities Building

The Jackman Humanities Building is a building if and only if it is not a building.

UD: integers. F^1: a is prime. a: 3

3 is prime if and only if 3 is not prime.

Some quantified sentences are logical falsehoods: false on every possible interpretation.

$\forall x \exists y F(xy) \leftrightarrow \exists x \forall y \sim F(xy)$

UD: people F^2: *a* loves *b*.

UD: integers F^2: *a* is smaller than *b*.

Try to think up an interpretation that makes it true! If everything stands in the F relation to something then it can't be the case that something fails to stand in the F relation to everything. Likewise, if something fails to stand in the F relation to everything, it can't be the case that everything stands in the F relation to something.

Likewise, a sentence such as: $\forall x(Fx \to Gx) \wedge \exists x(Fx \wedge \sim Gx)$ will be false no matter what interpretation is given. After all, if everything with property F also has property G, then it can't be true that something has property F but not property G.

Even though the sentence is false on all interpretations, we can't demonstrate it by giving interpretations – we would have to show that there is *no* interpretation on which it is true by considering *every possible* interpretation (an infinite set).

We cannot demonstrate that a sentence is a logical falsehood; however, a single interpretation can show that a sentence is *not* logically false. Since a logical falsehood is false on every possible interpretation, a single interpretation on which a sentence is true will prove that the sentence is not logically false.

Contingent Sentences:

Since logically true sentences are true on any interpretation and logically false sentences are false on any interpretation, we can't prove that the sentences are logical truths or logical falsehoods by providing an interpretation. (We prove that a sentence is logically true or logically false with a derivation.)

BUT, a single interpretation that makes a sentence true can prove that the sentence is NOT logically false (since a logical falsehood is false for all interpretations). Likewise, a single interpretation that makes a sentence false can prove that the sentence is NOT logically true (since a logical truth is true for all interpretations.)

We can prove a sentence is NOT logically true by providing an interpretation that makes it false.
We can prove a sentence is NOT logically false by providing an interpretation that makes it true.

So, we can prove that a sentence is contingent by providing two interpretations: one that makes it true and one that makes it false.

S: $\forall x \exists y(Fx \to L(yx))$

UD: people	F^1: *a* is female	L^2: *a* is the parent of *b*.	*On this interpretation, S is True*
UD: people	F^1: *a* is female	L^2: *a* is the child of *b*.	*On this interpretation, S is False*
UD: positive integers	F^1: *a* is even.	L^2: *a* is smaller than *b*	*On this interpretation, S is True*
UD: positive integers	F^1: *a* is odd.	L^2: *a* is smaller than *b*.	*On this interpretation, S is False*

Some Concepts for Predicate Logic:

Logical Truth: A sentence P is logically true if and only if P is true on every interpretation.

Logical Falsehood: A sentence P is logically false if and only if P is false on every interpretation.

Contingent sentence: A sentence P is contingent if and only if P is neither a logical truth nor a logical falsehood. (It is true on some interpretations and false on some interpretations.)

Interpretations are to predicate logic what truth-value assignments are to sentential logic!

In determining whether a sentence of predicate logic is logically true, logically false or contingent, instead of considering every possible truth-value assignment, we need to consider every possible interpretation. But, there are an infinite number of possible interpretations. Thus, A finite set of interpretations ***cannot*** show that that a sentence is a logical truth or a logical falsehood.

However, we ***can*** use a single interpretation to prove that a sentence is not a logical truth. One need only provide an interpretation on which the sentence is false (a counterexample).

Likewise we can show that a sentence is not logically false by providing an interpretation on which the sentence is true (a counterexample).

We can also prove that a sentence in predicate logic is a contingent sentence. One need only provide two interpretations – one on which the sentence is true and one on which the sentence is false. In doing this we are proving that it is not a logical falsehood and not a logical truth.

Consistency and Equivalency

As we learned in Sentential Logic, a set of sentences is consistent if it is possible for all to be true (together). Thus, we can also show that a set of sentences is consistent by providing an interpretation on which they are all true.

A set of sentences is **consistent** if and only if there is at least one interpretation on which all the members of the set are true.

Two sentences are logically equivalent if and only if they have the same truth-value on every possible interpretation. Thus, we cannot prove that two sentences are logically equivalent, but we can show that two sentences are not equivalent by providing a single interpretation on which one is true and one is false.

Two sentences are **logically equivalent** if and only if there is no interpretation on which they have different truth-values. (They are logically equivalent if and only if they have the same truth-value on every interpretation.)

Validity and Tautological Implication

Since every valid argument can be transformed into a logical truth in which the antecedent is a conjunction of the premises and the consequent is the conclusion, valid arguments will be much like logical truths. No finite set of interpretations can show that an argument is valid, but a single interpretation can show that it is invalid.

To show an argument is invalid, provide an interpretation on which all the premises are true and the conclusion is false.

A single interpretation ***cannot*** show that a set of sentences tautologically implies another sentence. But a single interpretation can show that a set of sentences doesn't tautologically imply a further sentence. To show this, provide an interpretation on which all the sentences of in the original set are true, but the further sentence is false.

> **Validity:** An argument in predicate logic of is **valid** if and only if there is no interpretation on which every premise is true and the conclusion is false. An argument of is **invalid** if and only if the argument is not valid.
>
> **Tautological Implication**: A set of sentences **tautologically implies** a sentence ϕ if and only if there is no interpretation on which all the sentences in the set are true and ϕ is false.

We can demonstrate that the following argument is invalid by providing an interpretation on which the premises are true and the conclusion is false:

$\forall x \exists y L(xy)$. $\sim\exists x L(xx)$. $\forall x \forall y (L(xy) \rightarrow \sim L(yx))$. $\therefore\ \exists x \forall y L(yx)$.

UD: positive integers L^2: *a* is less than *b*

Premise 1: Every positive integer is such that it is less than some positive integer.
Premise 2: No positive integer is less than itself.
Premise 3: If a pos. integer is less than another pos. integer, then the second cannot be less than the first.
Conclusion: There is a positive integer that all positive integers are less than.

This interpretation makes the three premises true, but the conclusion is false: since no number is less than itself, there is no positive integer that all positive integers are less than.

Note that the universe of discourse in this interpretation is infinite – there is an infinite number of positive integers. This is important for making the first premise true and the conclusion false. Every positive integer is smaller than some other, but there is no largest positive integer. Yet, although the sequence is infinite, it has a fixed starting point: 1. So although there is no largest number, there is a smallest number, 1.

Often, when giving an interpretation to prove that an argument is invalid, it helps to use such a universe of discourse. Any argument that can be proven invalid with an infinite universe of discourse can be proven invalid with a finite universe of discourse, but the converse does not hold. Indeed, this argument cannot be shown to be invalid with a finite universe of discourse.

Can you see why a finite interpretation will fail to prove it is invalid?

Giving English Language Interpretations

It can sometimes be difficult to come up with an English language interpretation that will demonstrate that an argument is invalid, that a set of sentences is consistent, that two sentences are not equivalent or that a sentence is logically true or logically false.. It can be hard to find an English language translation that makes the sentences that are supposed to be true, true, and the sentences that are supposed to be false, false.

First: Define your Universe of Discourse (A restricted UD is simpler than an unrestricted one.)

- Pick a universe of discourse that you know well and that has well-defined properties and relations (it is easy to see which numbers are larger than which, but hard to know which people are richer than which).
- Think about properties in relation to your universe: there are some properties that apply to everything, some apply to some but not all things, some properties apply to nothing.
- Consider properties in relation to one another: if something has one property is cannot have a related property (if a number is even, it cannot be odd); if something has one property it must have a related property (if a number is divisible by four, then it is even), etc.
- Consider whether relations are symmetric, asymmetric, transitive, intransitive, reflexive, etc.

Second: Consider what you are trying to prove

- If you are proving that a sentence is not logically true, you need an interpretation on which it is false.
- If you are proving that a sentence is not logically false, you need an interpretation on which it is true.
- If you are proving that a set of sentences is consistent, you need an interpretation that makes them true.
- If you are proving that two sentences are not equivalent, you need an interpretation on which one is true and the other is false.
- If you are proving that an argument is invalid, then you need an interpretation on which the premises are true and the conclusion is false.

Third: Consider your sentences

- What is the main connective? When is such a sentence true or false?
- Start with the true existential sentences (or true existential sentential components) – they tell you which things must exist and have certain properties.
- Negative/false existential sentences/sentential components – they tell you which things cannot exist.
- True universal sentences (conditional form) – can be true because nothing satisfies the antecedent OR because everything that satisfies the antecedent also satisfies the consequent.
- False universal sentences (conditional form) – at least one thing exists that satisfies the antecedent but not the consequent.
- Think about quantified relations: Do all things stand in that relation to some things, or to nothing, or to all things? Does something stand in that relation to something, nothing, or all things?

HINT: Consider using positive integers for the universe. If it is possible to demonstrate a property of a sentence or argument by providing an interpretation, then it is possible to do so with an interpretation which has as its universe the set of positive integers.

It gives you easily defined properties such as: even, odd, prime (some are, some are not), larger than zero (all are), negative (none are). And it gives you easily understood relations such as: larger than (transitive, asymmetric, irreflexive); smaller or equal to (transitive, reflexive); equal to (transitive, symmetric, reflexive); exactly two more than (intransitive, asymmetric, irreflexive).

7.4 E1:

Show that the following sentences are contingent by providing, for each, an English language interpretation on which it is true and one on which it is false:

a) $Fa \leftrightarrow \sim Ga$

b) $(Fa \rightarrow \sim Ga) \wedge (Fb \rightarrow \sim Gb)$

c) $L(bc) \wedge L(cb) \rightarrow \sim L(bb)$

d) $\exists x(Gx \wedge \sim Fx)$

e) $\forall x(Fx \rightarrow \sim Gx)$

f) $\forall x\forall y\forall z(L(xy) \wedge L(yz) \rightarrow L(xz))$

g) $\forall x(Fx \rightarrow \exists y(Gy \wedge M(xy)))$

7.4 E2

Provide English language interpretations that show that the following sentences are not logical truths:

a) $\forall x\exists yF(xy) \rightarrow \exists y\forall xF(xy)$

b) $\exists x(Fx \rightarrow \exists y(Fy \wedge L(xy)) \rightarrow \forall x(Fx \rightarrow \exists y(Fy \wedge L(xy))$

c) $\sim\exists xGx \rightarrow \forall y(F(yy) \rightarrow Gy)$

d) $\exists x(Bx \wedge \forall y \sim L(xy)) \vee \sim\forall x\exists y(L(xy) \rightarrow \sim Bx)$

7.4 E3:

Provide English language interpretations that show that the following sentences are not logical falsehoods:

a) $\forall x\exists yF(xy) \rightarrow \forall x\exists y\sim F(xy)$

b) $\exists x(\sim Fx \wedge Gx) \wedge \forall x(Fx \rightarrow Gx)$

c) $\forall x(Gx \rightarrow \sim\exists yL(xy)) \rightarrow \forall x(\sim Gx \wedge \exists yL(xy))$

d) $\forall x(Bx \rightarrow \exists yC(xy)) \wedge \forall y\forall x(C(yx)\rightarrow\sim By)$

7.4 E4

Provide English language interpretations that show that the following arguments are invalid:

a) $\forall x(Fx \rightarrow \sim Gx)$. $\forall y(Hy \leftrightarrow \sim Gy)$. $\therefore \forall z(Hz \rightarrow Fz)$

b) $\forall x\exists y(Fx \rightarrow (Gy \wedge L(xy)))$. $\forall x\exists y(Gx \rightarrow (Fy \wedge L(xy)))$. $\exists x\forall y(Fx \wedge (Gy \rightarrow \sim L(yx)))$ $\therefore \exists xL(xx)$

c) $\exists x\forall y(Bx \wedge A(xy))$. $\forall x\exists y (Fx \rightarrow \sim A(xy))$. $\therefore \forall x(Fx \rightarrow \sim A(xx))$

7.4 E5

Provide English language interpretations that show that the set of sentences does not tautological imply the final sentence

a) $\{\ \forall x(Fx \rightarrow Gx),\quad \forall y(Hy \rightarrow {\sim}Fy)\ \}$ $\quad\quad \exists z(Gz \wedge {\sim}Hz)$

b) $\{\ \exists x\forall y(H(xy) \vee J(xy)),\quad \exists x\forall y{\sim}H(xy)\ \}$ $\quad\quad \exists x\forall yJ(xy)$

7.4 E6

Provide English language interpretations that show that the following sets of sentences are consistent:

a) $\forall x({\sim}Fx \rightarrow Gx)$. $\quad \forall y(Hy \leftrightarrow Gy)$. $\quad \exists z(Hz \wedge Fz)$ $\quad {\sim}\forall xFx$

b) $\exists x(Fx \wedge \forall y(Gy \rightarrow L(xy)))$. $\quad \forall y(Gy \vee Fy \rightarrow {\sim}L(yy))$. $\quad {\sim}\exists z(Gz \wedge \forall y(Fy \rightarrow L(zy))$

c) $\forall x(Fx \rightarrow Gx)$. $\quad \forall x(Gx \rightarrow Hx \vee Jx)$. $\quad {\sim}\exists x(Hx \wedge {\sim}Kx)$ $\quad \forall x(Fx \rightarrow ({\sim}Kx \wedge {\sim}Hx))$

d) $\exists x(Fx \wedge \forall y{\sim}H(xy))$. $\quad \exists x(Gx \wedge \exists yH(xy))$. $\quad \exists x(Fx \wedge Gx)$.

7.4 E7

Provide English language interpretations that show that the following pairs of sentences are not equivalent:

a) $\forall x(Fx \wedge Gx \rightarrow \exists yH(yx))$ $\quad\quad \forall x(Fx \wedge (Gx \rightarrow \exists yH(yx)))$

b) $\forall x(Fx \rightarrow {\sim}\forall y(Gy \rightarrow H(xy)))$ $\quad\quad {\sim}\forall y(Gy \rightarrow \forall x(Fx \rightarrow H(xy)))$

c) $\exists x(Fx \wedge \forall y(Gy \rightarrow \exists z(Hz \wedge B(xyz))))$ $\quad\quad \forall y(Gy \rightarrow \exists x(Fx \wedge \exists z(Hz \wedge B(xyz))))$

7.5 Abstract Finite Models

Many of these properties of sentences or sets of sentences can be demonstrated with very small, finite models.

A model is just an interpretation. However, so far we have mostly been using real world interpretations – we use a part of the world (or the world of mathematics) as an example or counterexample. Models are interpretations that are that created to demonstrate a particular feature or property.

A finite model is an interpretation with a finite universe of discourse (rather than an infinite universe of discourse) – and often a very small finite universe with only a few individuals in it.

Abstract Models

So far, in all of our interpretations, we defined the extensions of the terms by giving English words or expressions. When we provide an interpretation by giving the meanings in this way, we rely on the interpreter's understanding of the intension of the term in order to determine the extension – they have to know what 'multiple of two' or 'mammal' means in order to know what things are in the extension of the predicate.

But, we could also define the extension directly, using set notation.

UD: {Adam, Betty, Cameron, Diana}

F^1: {Adam, Betty}

G^1: {Betty, Cameron, Diana}

L^2: { (Adam, Betty), (Betty, Betty), (Cameron, Diana)}

In set notation, the members of the extension are listed in curly brackets. They are separated by commas. Ordered pairs and ordered triplets are put in parentheses.

By using set notation, we don't need to know anything about the individuals or the properties. The universe of discourse just provides us with a set of objects, and then the extension of each predicate is given with a subset.

UD: {0, 1, 2, 3}

F: {0, 1}

G: {1, 2, 3}

L: {(0,1), (1,1) , (2,3)}

When the extension is defined directly, the members of the universe of discourse could be anything. Indeed, we might as well just refer to them as 0 (the original member), 1 (the next member), 2 (the next member), etc.

According to this abstract interpretation: the UD has four members; the first and second members of the UD are in the extension of F; the second, third and fourth members of the UD are in the extension of G; and three ordered pairs are in the extension of L: (1^{st} member, 2^{nd} member), (2^{nd} member, 2^{nd} member) and (3^{rd} member, 4^{th} member). We can call this type of interpretation an abstract model. The members of the UD are not integers or numbers, they are just abstract things.

Instead of using a real world example with meaningful properties, we can leave the properties undefined, and simply state whether or not the individuals in the universe have those properties.

A finite model provides an interpretation by taking a non-empty finite class of individuals as the universe. A sub-class is any subset of the class (including the empty class and including the entire class).

The extension of a predicate is the subset of members of U that the predicate satisfies. We can give an interpretation by stating which members of U are in the extension of that predicate.

Thus, every monadic predicate is interpreted by giving the subclass of individuals that are within the extension of that predicate. A dyadic predicate is interpreted by giving the set of ordered pairs (each element of which is within the universe) within the extension of that predicate. The subclass can include all possible ordered pairs, given the members of the universe, just some of them, or none at all. Likewise, a triadic predicate is interpreted by giving the set of ordered triplets in the extension of the predicate.

Consider the sentence: $\forall x(Fx \rightarrow Gx)$.

We can give an interpretation on which this sentence is true.

Consider the simplest universe for an abstract, finite model: UD: {0}

> This universe of discourse has just one thing in it (we call this thing '0').
>
> The possible extensions of F and G: { } or {0}. These are all the possible subsets of the UD.
>
> UD: {0} F^1: {0} G^1: {0}
>
> In this interpretation the extension of F is {0}. Thus, 0 has property F.
>
> The extension of G is also {0}, so 0 also has property F.
>
> So, everything in the extension of F is also in the extension of G: '0' is in the extension of F and G.
>
> Hence, on this interpretation, the sentence: $\forall x(Fx \rightarrow Gx)$ is true.

We can also provide an abstract finite model that uses a larger universe of discourse: UD: {0, 1}

> This universe of discourse has two things in it ('0' and '1').
>
> The possible extensions of F and G are: { }, {0}, {1}, and {0,1} These are the subsets of the UD.
>
> UD: {0,1} F^1: {0,1} G^1: {1}
>
> In this interpretations, the extension of F is {0,1} and the extension of G is { 1}.
>
> 0 is in the extension of F, but not in the extension of G; so, it is not the case that everything in the extension of F is also in the extension of G.
>
> Hence, on this interpretation, the sentence $\forall x(Fx \rightarrow Gx)$ is false.

We can construct an abstract finite model to show that this sentence is not a logical truth:

$\forall x \exists y L(xy) \rightarrow \exists y \forall x L(xy)$

We might begin by trying a finite model with a UD of one member: UD: {0}

However, it quickly becomes clear that we cannot use the model to demonstrate that it is not logically true, since the only possible extensions of L are: {(0,0)} and { }.

> If (0,0) is in the extension of L, then the conditional is true: If every member of the UD (0) is in the L relation to some member of the UD (0), then some member (0) is in the L relation to all members (0).
>
> But, if (0,0) is not in the extension of L, the conditional is also true since the antecedent is false – it is not the case that every member of the UD is in the L relation to some member of the UD.

$\forall x \exists y L(xy) \rightarrow \exists y \forall x L(xy)$

We can demonstrate that the sentence is not a logical truth using a model with a universe of two members:

The model must show that the sentence is false.

> UD: {0,1} L^2: {(0,1), (1,0)}
>
> This shows that the universe has two members: '0' and '1'.
>
> The possible ordered pairs are: (0,0), (0,1), (1,0) and (1,1). Thus, the possible extensions of L, a two-place predicate include any subset of these four ordered pairs – including the empty set and all four ordered pairs.
>
> Here, the predicate, L, has two members, both ordered pairs: (0,1) and (1,0)
>
> Thus, every member of U stands in the L relation to some member of U – the antecedent is true.
>
> But, it is not the case that some member of U stands in the L relation to every member of U – the consequent is false.

How to construct a model:

- Decide what size of universe you want. You need at least one member and generally want to keep the number as small as possible. Most of the models we will be using will have no more than two or maybe three members.
- Define the universe. The first member will be '0', the second, '1', etc. (This is just a convention – we could instead use letters.) Use curly brackets to indicate the set of individuals in the universe. Example (a universe of two individuals): UD: {0,1}
- Define the predicates. State what subclass forms the extension of the predicate. One-place predicates will be sets of individuals, each of which is a member of the universe. Two-place predicates will be ordered pairs, each member of which is a member of the universe. Consider what extensions will make the premises true and the conclusion false. The extension of a predicate may be the empty set.
- Define the individual constants or name letters (one-place operation letters). State which member of the universe the individual constant or name letter refers to. Two name letters can be defined as picking out the same individual. Again, consider what will make the premises true and the conclusion false.

Advantages of Abstract Finite Models over Real World (Natural Language) interpretations:

- You don't have to think up predicates and relations that work together.
- You don't have to rely on empirical or mathematical facts to determine the truth-value of complex sentences When it is difficult to construct an abstract model directly, you can use a truth-functional expansions. (See 7.6)
- You aren't limiting the interpretation to a single natural language.

Disadvantage of Finite Models:

The sorts of things that you can show with a single interpretation cannot always be shown with a model that has a finite universe. For some sentences, we require an infinite universe to demonstrate that it is true or false; and some arguments can only be shown to be invalid using interpretations with an infinite universe. Finite models always have a finite universe (and we will normally work with universes of just one, two or three members!)

For example, no abstract finite model can show that the following set of sentences is consistent.

$\forall x \exists y L(yx))$. $\sim\exists x L(xx)$ $\forall x \forall y \forall z(L(xy) \wedge L(yx) \rightarrow L(xy))$

Can you see why? (Hint: begin with a universe of 1 member and try to make the sentences true in order, then add a member and try again, and another member…)

7.5 E1 On the interpretations below, is the following sentence true or false?

$\forall x(Fx \rightarrow \exists y G(xy))$

a)	UD: {0}	F^1: {0}	G^2: {(0,0)}	True
b)	UD: {0,1}	F^1: {0,1}	G^2: {(1,0), (1,1)}	False
c)	UD: {0,1}	F^1: {0,1}	G^2: {(0,0), (1,1)}	True
d)	UD: {0,1,2}	F^1: {1,2}	G^2: {(1,0), (1,2), (2,2)}	True

7.5 E2 Construct a finite model that shows that these sentences are not logical truths.

a) $\forall x(Fx \rightarrow \forall x Fx)$

b) $\exists x(\sim Fx \wedge Gx) \wedge \forall x(Fx \rightarrow Gx)$

c) $\forall x \exists y F(xy) \rightarrow \forall x \exists y \sim F(xy)$

7.5 E3 Construct a finite model that shows that these sentences are not logical falsehoods.

a) $\forall x(Fx \rightarrow \forall x Fx)$

b) $\exists x(\sim Fx \wedge Gx) \wedge \forall x(Fx \rightarrow Gx)$

c) $\forall x \exists y F(xy) \rightarrow \forall x \exists y \sim F(xy)$

7.6 Truth-Functional Expansions

One advantage of small finite, abstract models, is that they allow us to evaluate sentences and arguments truth-functionally, as we did in sentential logic, using truth-tables (or, more practically, shortened truth-tables.) But we can only do this if all of the logical operators are themselves truth-functional – and neither the existential nor the universal quantifier are.

In truth-functional analysis, we work with the truth-values of particular sentences. But, the quantifiers are general terms. We use them to talk in general about some things or all things of certain type, but they don't specify which particular members of the universe of discourse the predicates are true or false of. However, when we have a finite universe of discourse, we can interpret a quantified sentence relative to the universe. This will generate sentences about particulars, thereby making truth-functional analysis possible.

How can we understand the existential and universal quantifiers truth-functionally?

A 'truth-functional expansion' is a sentence without quantifiers that is equivalent to a quantified sentence for a specified, finite universe.

$\forall$: The universal quantifier says that a sentence is true for every member of the universe. It's true for the first member ***and*** the second ***and*** the third ***and*** the fourth etc.

$\forall xFx$ is true IFF every member of the UD has property F: $F0 \wedge F1 \wedge F2 \wedge F3...$

$\exists$: The existential quantifier says that a sentence is true for at least one member of the universe. It's true for the first member ***or*** the second ***or*** the third ***or*** the fourth etc.

$\exists xFx$ is true IFF at least one member of the UD has property F: $F0 \vee F1 \vee F2 \vee F3\ldots$

Consider the sentence: $\forall x(Fx \rightarrow Gx)$

It states that the conditional is true for all members of the universe of discourse.

In a UD with only one member, {0}, it is equivalent to: $F0 \rightarrow G0$.

In a UD with two members, {0,1}, it is equivalent to: $(F0 \rightarrow G0) \wedge (F1 \rightarrow G1)$

In a UD with three members, {0,1,2}, it is equivalent to: $(F0 \rightarrow G0) \wedge (F1 \rightarrow G1) \wedge (F2 \rightarrow G2)$

And so on for larger, finite universes of discourse.

The sentence is true for ALL members of the UD.

Consider the sentence: $\exists x(Fx \wedge Gx)$

It states that the conjunction is true for at least one member of the universe of discourse.

In a UD with only one member, {0}, it is equivalent to: $F0 \wedge G0$.

In a UD with two members, {0,1}, it is equivalent to: $(F0 \wedge G0) \vee (F1 \wedge G1)$

In a UD with three members, {0,1,2}, it is equivalent to: $(F0 \wedge G0) \vee (F1 \wedge G1) \vee (F2 \wedge G2)$

And so on for larger, finite universes of discourse.

The sentence is true for AT LEAST ONE member of the UD.

Now consider the sentence: $\forall x \exists y L(xy)$

We can expand the sentence in three steps… first consider $\forall$, then $\exists$, then put them together.

In a UD with only one member, {0}, $\forall x \exists y L(xy)$ is equivalent to: $\exists y L(0,y)$

In a UD with only one member, {0}, $\exists y L(0y)$ is equivalent to: $L(0,0)$

Thus, in a UD with only one member, {0}, $\forall x \exists y$ is equivalent to: $L(0,0)$

In a UD with two members, {0,1}, $\forall x \exists y L(xy)$ is equivalent to: $\exists y L(0,y) \wedge \exists y L(1,y)$

In a UD with two members, {0,1}, $\exists y L(0,y)$ is equivalent to: $L(0,0) \vee L(0,1)$

In a UD with two members, {0,1}, $\exists y L(1,y)$ is equivalent to: $L(1,0) \vee L(1,1)$

Thus, in a UD with two members, {0,1}, $\forall x \exists y L(1,y)$ is equivalent to: $(L(0,0) \vee L(0,1)) \wedge (L(1,0) \vee L(1,1))$

In a UD with three members, {0,1,2}, $\forall x \exists y L(xy)$ is equivalent to: $\exists y L(0,y) \wedge \exists y L(1,y) \wedge \exists y L(2,y)$

In a UD with three members, {0,1,2}, $\exists y L(0,y)$ is equivalent to: $L(0,0) \vee L(0,1) \vee L(0,2)$

In a UD with three members, {0,1,2}, $\exists y L(1,y)$ is equivalent to: $L(1,0) \vee L(1,1) \vee L(1,2)$

In a UD with three members, {0,1,2}, $\exists y L(2,y)$ is equivalent to: $L(2,0) \vee L(2,1) \vee L(2,2)$

Thus, in a UD with three members, {0,1,2}, $\forall x \exists y L(1,y)$ is equivalent to:

$$(L(0,0) \vee L(0,1) \vee L(0,2)) \wedge (L(1,0) \vee L(1,1) \vee L(1,2)) \wedge (L(2,0) \vee L(2,1) \vee L(2,2))$$

Each new member of the UD will increase the number of terms in the truth-functional expansion, exponentially. Even with this simple sentence, the number of terms in this sentence is equal to the square of the size of the UD. For just a slightly more complex sentence, $\forall x \exists y(Fx \rightarrow G(xy))$, the number of terms in the truth-functional equivalent sentence is the cube of the size of the UD.

It isn't practical to do a truth-functional expansion for universes of more than three individuals; however, in principle you can complete it for any finite UD. You can't complete the expansion for an infinite universe.

- Each expansion to a finite set of individuals is an interpretation of the sentence.
- When expanding, deal with one quantifier at a time, either starting with the innermost quantifier and moving outward, or starting with the outermost quantifier and moving inward.
- If the main quantifier is $\forall$ then the main connective of the expanded sentence is $\wedge$ in a universe of a size greater than 1.
- If the main quantifier is $\exists$ then the main connective of the expanded sentence is $\vee$ in a universe of a size greater than 1.
- For each quantifier, expand to every member of the universe (conjoined with $\wedge$ or disjoined with $\vee$).
- If the main connective is a truth-functional connective ($\sim$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$), provide expansions for each of the quantified sentences (leaving the truth-functional connectives where they are relative to the quantified sentences.)

Dealing with Name Letters, Operations and Identity:

Although we won't be doing it in this course, we can also provide truth-functional expansions for sentences that include name letters, operation letters and/or identity.

The interpretation must define the name letters and operation letters:

For example: Given the UD: {0,1}, "a=0" defines 'a' as the first member of the UD.
"a(0)=1; a(1)=0" defines the operation 'a' relative to each member of the UD.

Then, in giving the truth-functional expansions, you must make sure that any two individuals that are identical to one another have exactly the same properties.

7.6 E1 Provide a truth-functional expansion for each of the following sentences, using the specified universe of discourse.

a) $\exists x B(xx) \leftrightarrow \forall y(Gy \rightarrow Hy)$ UD: {0}

b) $\exists x B(xx) \leftrightarrow \forall y(Gy \rightarrow Hy)$ UD: {0,1}

c) $\exists x B(xx) \leftrightarrow \forall y(Gy \rightarrow Hy)$ UD: {0,1,2}

d) $\forall x(Gx \rightarrow \exists y(Hy \wedge K(yx)))$ UD: {0,1}

e) $\forall x \exists y(Gx \rightarrow (Hy \wedge K(yx)))$ UD: {0,1}

f) $\sim\exists x(Bx \wedge \forall y(D(xy) \leftrightarrow \sim D(yx)))$ UD: {0,1}

7.6 E2 Provide a truth-functional expansion for each of the following sentences, using the specified universe of discourse.

a) $\forall x Fx \leftrightarrow \exists y Gy$ UD: {0}

b) $\exists x(Bx \wedge \forall y(Cy \rightarrow D(yx)))$ UD: {0}

c) $\forall x(Bx \rightarrow \sim Cx)$ UD: {0,1}

d) $\exists x(\sim Dx \wedge Ex)$ UD: {0,1}

e) $\sim\forall x(Fx \rightarrow L(xx)$ UD: {0,1}

f) $\exists x \exists y L(xy)$ UD: {0,1}

g) $\forall x \exists y L(xy)$ UD: {0,1}

h) $\exists x \forall y(Fx \wedge L(xy))$ UD: {0,1}

i) $\exists x (Fx \wedge \forall y L(xy))$ UD: {0,1}

j) $\sim\forall x \exists y(Gx \rightarrow \sim F(yx))$ UD: {0,1}

k) $\sim\forall x(Gx \rightarrow \sim\exists y F(yx))$ UD: {0,1}

l) $\forall x(Gx \rightarrow \sim L(xx))$ UD: {0,1,2}

m) $\forall x Gx \rightarrow \exists y L(yy)$ UD: {0,1,2}

7.7 Truth-Functional Analysis of Quantified Sentences

Since we can, for very small, finite universes, provide truth-functional sentences that are equivalent to the quantified sentence, we can also use truth-functional methods of analysis, such as truth-tables.

Consider the argument:

$\forall x(Fx \rightarrow Gx)$

$\exists x(Fx \wedge Hx)$

$\therefore \forall x(Gx \rightarrow Hx)$

We want to know whether this argument is valid. Arguments are truth-functionally valid if for every truth-value assignment on which the premises are true, the conclusion is also true. In other words, it is valid if there is no truth-value assignment on which the conclusion is false and all the premises true. If there is such a truth-value assignment, the argument is invalid.

Before we evaluate it truth-functionally for a given UD, we need to provide a truth-functional expansion of the argument for that UD. We will use the simplest UD: {0}

$F0 \rightarrow G0$

$F0 \wedge H0$

$\therefore G0 \rightarrow H0$

For each of the predicates, the atomic sentence is true for 0 iff 0 is in the extension of that predicate.

Thus, if F0 is true, 0 is in the extension of F. If F0 is false, 0 is not in the extension of F.

Now, we can draw a truth-table for this interpretation of the sentence.

F0	G0	H0	F0	→	G0	F0	∧	H0	G0	→	H0
T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	F	T	F	F
T	F	T	T	F	F	T	T	T	F	T	T
T	F	F	T	F	F	T	F	F	F	T	F
F	T	T	F	T	T	F	F	T	T	T	T
F	T	F	F	T	T	F	F	F	T	F	F
F	F	T	F	T	F	F	F	T	F	T	T
F	F	F	F	T	F	F	F	F	F	T	F

There is only one truth-value assignment for which both the premises are true. The conclusion is also true on this TVA. Thus, this interpretation of the sentence does NOT show that the argument is invalid. But it also cannot show that the argument is valid since it is only one of the infinite number of interpretations for that argument. For an argument to be valid, it has to hold for every interpretation. Indeed, this argument is invalid!

When we give an truth-functional interpretation of the argument for a universe of discourse with only one member, there are only three atomic sentences. Thus, the truth-table would only have 8 lines. But the truth-table cannot prove that the argument is invalid. If the universe only has one member, whenever both premises are true (something is in the extension of F and H, and all things in the extension of F are also in that of G), the conclusion is true as well (all G's are H.).

We need at least two members in the universe of discourse to demonstrate the invalidity of the argument.

UD: {0,1}

But, a full truth-table would be very long! Since each member of the universe of discourse may or may not be in the extension of each of three predicates, we now we have 6 atomic sentences: F0, F1, G0, G1, H0, H1. So the truth-table would be 2^6 or 64 lines long! A larger UD, more predicates or multi-place predicates would quickly produce full truth-tables that would be hundreds or thousands of lines long. This is not practical!

7.7 EG1

Using a shortened truth-table, we can prove that this argument is invalid using a truth-functional expansion with a UD of two members:

$\forall x(Fx \rightarrow Gx)$
$\exists x(Fx \wedge Hx)$
$\therefore \forall x(Gx \rightarrow Hx)$

Here is the truth-functional expansion of the argument for the UD: {0,1}

$(F0 \rightarrow G0) \wedge (F1 \rightarrow G1)$
$(F0 \wedge H0) \vee (F1 \wedge H1)$
$\therefore (G0 \rightarrow H0) \wedge (G1 \rightarrow H1)$

To demonstrate the invalidity of the argument, we must find an interpretation that makes the premises true and the conclusion false. A single interpretation will prove that it is invalid.

F0	G0	H0	F1	G1	H1

(F0	→	G0)	∧	(F1	→	G1)
			T			

(F0	∧	H0)	∨	(F1	∧	H1)
			T			

(G0	→	H0)	∧	(G1	→	H1)
			F			

To prove that argument is invalid, we need to set the premises true and the conclusion false.

Now we can work from the desired truth-values of the expanded sentences to the truth-values of the atomic sentences (as we did in Unit 4).

There are four possible TVA's that make the premises true and the conclusion false.

F0	G0	H0	F1	G1	H1
T	T	F	T	T	T

F0	G0	H0	F1	G1	H1
T	T	T	T	T	F

F0	G0	H0	F1	G1	H1
F	T	F	T	T	T

F0	G0	H0	F1	G1	H1
T	T	T	F	T	F

Any one of them demonstrates the invalidity of the argument – and we can use it to create an abstract finite model. For instance, take the first. Since F0 and F1 are both true, 0 and 1 are both in the extension of F. Since G0 and G1 are both true, 0 and 1 are in the extension of G. And, since H1 is true, but H0 is false, 1 is in the extension of H:

UD: {0,1} F:{0,1} G:{0,1} H:{1}

This abstract model with a UD of two members demonstrates that the argument is invalid since, on this interpretation, the premises are true and the conclusion is false.

$\forall x(Fx \rightarrow Gx)$
$\exists x(Fx \wedge Hx)$
$\therefore \forall x(Gx \rightarrow Hx)$

Thus, there are an infinite number of interpretations that can prove that it is invalid. Indeed, for every universe of discourse with two members or more, it is possible to provide an interpretation that will demonstrate the invalidity of the argument.

7.7 EG2

Let's try another. We will create an abstract finite model to demonstrate the invalidity of the following argument. We will do so by providing a truth-functional expansion (to two individuals) for each sentence in the argument. Then, we will use our analysis of the expanded sentences to define a model with a universe of two individuals that shows that the argument is invalid.

$\exists x\ (Fx \wedge \forall y\ G(xy))$. $\sim\exists x\forall y\ (Hx \wedge G(xy))$. $\therefore\ \forall x(Hx \rightarrow \sim G(xx))$

First, we need to provide a truth-functional expansion for each sentence:

$(F0 \wedge (G(0,0) \wedge G(0,1))) \vee (F1 \wedge (G(1,0) \wedge G(1,1)))$
$\sim(((H0 \wedge G(0,0)) \wedge (H0 \wedge G(0,1))) \vee (((H1 \wedge G(1,0)) \wedge (H1 \wedge G(1,1))))$
$\therefore (H0 \rightarrow \sim G(0,0)) \wedge (H1 \rightarrow \sim G(1,1))$

Now, we set the premises true and the conclusion false, and we work from the truth-value of the premises and conclusion to the truth-values of the atomic sentences.

F0	F1	H0	H1	G(0,0)	G(0,1)	G(1,0)	G(1,1)

$$(F0 \wedge (G(0,0) \wedge G(0,1))) \vee (F1 \wedge (G(1,0) \wedge G(1,1)))$$

T

$$\sim (((H0 \wedge G(0,0)) \wedge (H0 \wedge G(0,1))) \vee (((H1 \wedge G(1,0)) \wedge (H1 \wedge G(1,1))))$$

T

$$(H0 \rightarrow \sim G(0,0)) \wedge (H1 \rightarrow \sim G(1,1))$$

F

UD: {0,1} F^1: {1} H^1: {0} G^2: {(0,0), (1,0), (1,1)}

The above interpretation shows that the argument is invalid.

$\exists x\, (Fx \wedge \forall y\, G(xy))$.	1 is in the extension of F and stands in the G relation to 0 & 1.
$\sim\exists x \forall y\, (Hx \wedge G(xy))$.	Nothing is in the extension of H and stands in the G relation to 0 & 1.
	0 is in the extension of H, but doesn't stand in the G relation to anything.
	1 stands in the G relation to 0 & 1, but is not in the extension of H.
$\therefore\ \forall x(Hx \rightarrow \sim G(xx))$	This is false: Not all things in the extension of H fail to stand in the G relation to itself. 0 is in the extension of H and stands in the G relation to itself.

7.7 E1:

Provide a truth-functional expansion for the following arguments, and use it to create a finite model that shows that the following arguments are invalid:

a) $\forall x(Fx \rightarrow (Gx \wedge Hx))$. $\exists x(\sim Fx \wedge Gx) \wedge \exists yFy$. $\therefore \forall x(\sim Hx \vee Fx)$

b) $\exists xFx$. $\exists xGx$. $\exists xHx$. $\forall x((Fx \wedge Gx) \vee (Fx \wedge Hx) \vee (Gx \wedge Hx) \rightarrow Jx)$. $\therefore \exists xJx$

c) $\exists x(Fx \wedge Kx)$. $\forall x(Jx \leftrightarrow Kx)$ $\sim\forall x(\sim Fx \wedge Kx)$. $\therefore \forall x(Jx \rightarrow Fx)$

d) $(\sim\exists yFy \rightarrow \exists yFy) \vee \sim Fa$. $\therefore \exists xFx$.

e) $Fa \wedge \exists yG(ya)$. $Fb \leftrightarrow \exists y\sim G(yb)$. $\therefore \exists yG(by)$

f) $\forall x\forall y(L(xy) \rightarrow H(xy))$. $\therefore \forall x\forall y(L(xy) \rightarrow (H(xy) \wedge H(yx))$

g) $\exists x\forall y(Gx \wedge L(xy))$. $\exists yL(yy) \rightarrow \forall x(Gx \rightarrow Hx)$ $\therefore \exists x(Hx \rightarrow \forall yL(yx))$

h) $\forall x\exists y\forall z(F(zy) \leftrightarrow Gz \wedge F(zx))$. $\therefore \exists x\forall y(F(yx) \leftrightarrow Gy)$

7.7 E2

Provide a truth-functional expansion for the following sentences and use it to create a finite model that show the property indicated:

a) $\exists x\exists y(Fx \wedge Gy) \wedge \sim\exists x(Fx \leftrightarrow Gx)$ Show this is not a logical falsehood.

b) $\forall x (Fx \wedge \exists yL(xy)) \wedge \sim\exists y\forall xL(yx))$ Show this is not a logical falsehood.

c) $\forall x(Fx \rightarrow \exists yL(xy)) \leftrightarrow \exists y\forall x(\sim Fx \vee L(xy))$ Show this is not a logical truth.

d) $\forall x(Fx \rightarrow \exists y(Gy \wedge A(xy))) \rightarrow \forall x(Gx \rightarrow \exists y(Fy \wedge A(yx)))$ Show this is not a logical truth.

e) $\forall xF(xa(x)) \rightarrow \exists xF(a(x)x)$ Show this is not a logical truth.

f) $\forall x(\sim Gx \rightarrow Fx)$. $\forall y(Hy \leftrightarrow Gy)$.
$\exists z(Hz \wedge Fz)$ $\sim\forall xFx$ Show that these are consistent.

g) $\exists x(Fx \wedge \forall y(Gy \rightarrow L(xy)))$. $\forall y(Gy \vee Fy \rightarrow \sim L(yy))$.
$\sim\exists z(Gz \wedge \forall y(Fy \rightarrow L(zy))$ Show that these are consistent.

7.7 E3:

Use an abstract finite model to demonstrate the invalidity of each of the following arguments:

i) provide a truth-functional expansion (to two individuals) for each sentence in this argument with respect to the model

ii) define a model with a universe of two individuals that shows that this argument is invalid.

a)	$\forall x(Fx \rightarrow \exists yG(xy))$.	$\exists x\forall y{\sim}G(yx)$.	$\therefore \exists x{\sim}Fx$
b)	$\forall x\exists y(F(xy) \rightarrow {\sim}F(yx))$.	$\exists x(Gx \wedge \exists yF(xy))$	$\therefore\ {\sim}\exists x(Gx \wedge F(xx))$
c)	${\sim}\forall x\exists y\ (Fx \rightarrow {\sim}G(xy))$.	$\forall x({\sim}Hx \vee \exists y{\sim}G(xy))$.	$\therefore\ \forall x(G(xx) \rightarrow {\sim}Hx)$
d)	$\exists x\forall y(Gx \wedge L(xy))$.	$\forall x(L(xx) \rightarrow \exists y{\sim}H(xy))$	$\therefore \exists x(Gx \wedge {\sim}\exists yH(yx))$
e)	$\exists x\forall y(Bx \wedge C(xy))$.	$\forall x(Ax \rightarrow \exists y{\sim}C(xy))$.	$\therefore\ \forall x(Ax \rightarrow {\sim}C(xx))$
f)	${\sim}\forall x\ (Gx \rightarrow \exists yB(xy))$.	$\exists x\forall y(Hx \rightarrow B(xy))$.	$\therefore\ {\sim}\exists x(Gx \wedge \exists y(Hy \wedge B(xy))$

7.8 Explanations for All Possible Interpretations

As we saw earlier, whether we are giving symbolization schemes or using set notation to define the extensions of terms, every interpretation defines the universe of discourse (which could be finite or infinite), the extension of each predicate within a non-empty universe, as well as any individuals (through name letters and operations). And each interpretation is just one of an *infinite* number of possible interpretations.

To show that an argument is valid requires that you consider *every possible* interpretation, checking that on each interpretation on which all the premises are true the conclusion is also true. Likewise, to show that a sentence *is* a logical truth or logical falsehood requires that you consider every possible interpretation. But there are an infinite number of interpretations, so we can't demonstrate these things by examining all the interpretations.

We can prove it using derivations. To prove that an argument is valid, use our derivation system to derive the conclusion from the premises. To show that a sentence is a logical truth, derive it from the empty set (derive a theorem). To show a sentence is a logical falsehood, derive a derive a contradiction from it (do an indirect derivation).

However, we can also provide an explanation that shows why, on every interpretation, a sentence will be true (in the case of logical truths) or false (in the case of logical falsehoods); and in the case of arguments, that whenever the premises are true then so is the conclusion. This informally accounts for the validity of an argument or the property of the sentence.

There is no ‘method’ for giving an explanation of this type. It is a matter of thinking it through based on your knowledge of the connectives and quantifiers. Try to be clear and make sure you have covered all of the infinite number of interpretations.

- Make sure you are choosing an arbitrary interpretation – any interpretation.
- Consider the main connective – what makes it true or false?
- Consider the quantifiers – a universal applies to every member of the universe, the existential to at least one member of the universe.
- Consider what is in the extension of each predicate (every member of U or some member of U).
- Consider all possibilities.

$\exists x Fx \lor \forall x(Fx \rightarrow \sim Gx)$ is a logical truth. Explain why:

This sentence is a disjunction. A disjunction is true if at least one of the disjuncts is true.

On any interpretation that the first disjunct is false, no member of the universe is in the extension of F. If nothing is in the extension of F, the second disjunct must be true since it says for each member of U, that if it is in the extension of F then it is not be in the extension of G. Since no member of the universe is in the extension of F, this is true for every member of U. Thus the sentence is true if the first disjunct is false. If the first disjunct is true on that interpretation, then the sentence is true. Since the first disjunct must be true or false on any interpretation, the sentence must be true on any interpretation and thus is a logical truth.

Hint: These types of explanations often involve reasoning about a sentential component that must be true or false. First make it true, then make it false. Show that the entire sentence is true/false either way.

Consolation: the more you do, the easier they will become.

We can also use this method to show why an argument is valid. You have to reason it out, in words. You will probably work on the assumption that all premises are true on some interpretation, and show that the conclusion must also be true on that interpretation. Be as clear as possible.

Explain why the following argument is valid:

$\forall x(Fx \rightarrow Gx)$. $\exists x Fx$. $\therefore \exists x Gx$

Consider any interpretation on which both $\forall x(Fx \rightarrow Gx)$. and $\exists x Fx$ are true. Since the second premise is true, some member of the universe is in the extension of F. Since the first sentence is true, every member of the universe is such that *if* it is within the extension of F *then* it is also in the extension of G. Thus, since some member of the universe is in the extension of F, that member must also be in the extension of G. Thus, some member of the universe is in the extension of G, $\exists x Gx$, on any interpretation on which both $\forall x(Fx \rightarrow Gx)$ and $\exists x Fx$ are true.

Thus the argument is valid.

7.8 E1

Explain why the following sentences are logically true:

a) $\forall xFx \vee \exists x{\sim}Fx$

b) $\forall x(Fx \rightarrow \exists yGy) \leftrightarrow (\exists xFx \rightarrow \exists yGy)$

c) $\exists x(Fx \rightarrow \forall yGy) \leftrightarrow (\forall xFx \rightarrow \forall yGy)$

e) $\exists x(Fx \rightarrow \forall y{\sim}Gy) \rightarrow (\forall xFx \rightarrow {\sim}\exists yGy)$

d) $\forall x\forall yF(xy) \rightarrow \forall xF(xx)$

Explain why the following sentences are logically false:

f) $\forall x{\sim}(Fx \vee Gx) \wedge \exists x({\sim}Fx \rightarrow Gx)$

g) $\forall x(Fx \rightarrow Gx) \wedge \forall y(Gy \rightarrow {\sim}Fy)$

h) $\forall x\exists yF(xy) \leftrightarrow \exists x\forall y{\sim}F(xy)$

i) $\exists x\forall y(Fx \wedge {\sim}G(xy)) \wedge \forall x(Fx \rightarrow \exists yG(xy))$

Explain why the following arguments are valid:

j) $\forall x(Fx \rightarrow Gx)$. $\exists x{\sim}Gx$. $\therefore {\sim}\forall xFx$.

k) $\forall x{\sim}(Fx \wedge Gx)$. $\therefore \exists x{\sim}Fx \vee \exists x{\sim}Gx$

l) $\exists x\forall yL(xy)$ $\therefore \forall y\exists xL(xy)$

m) $\exists x(Fx \wedge \forall yL(xy))$ $\therefore \forall x\exists y(Fy \wedge L(yx))$