Homework Assignment #6

MAT 335 – Chaos, Fractals, and Dynamics – Fall 2013

Partial Solution

Chapter 11.4. We can prove that the function of the first graph F_1 has a 3-cycle. Indeed, a 3-cycle satisfies

$$a \to b \to c \to a,$$

so given the graph of the function, the function is

$$F_1(x) = \begin{cases} x+1 & \text{if } 0 \leqslant x \leqslant 2 \\ 9-3x & \text{if } 2 \leqslant x \leqslant 3. \end{cases}$$

We can look for a 3-cycle of the form:

$$\underbrace{0 < a < b < 2}_{\text{increasing}} \qquad \text{and} \qquad \underbrace{2 < c < 3}_{\text{returns to the beginning}},$$

so that

$$\begin{aligned} b &= F_1(a) = a+1 \\ c &= F_1(b) = b+1 = a+2 \\ a &= F_1(c) = F_1(a+2) = 9 - 3(a+2). \end{aligned}$$

We solve this equation to obtain:

$$a = 9 - 3(a+2) \qquad \Rightarrow \qquad a = \frac{3}{4}.$$

We can check the orbit:

$$\frac{3}{4} \to \frac{7}{4} \to \frac{11}{4} \to \frac{3}{4}.$$

Since the exercise only asks us to match the graphs, we conclude that

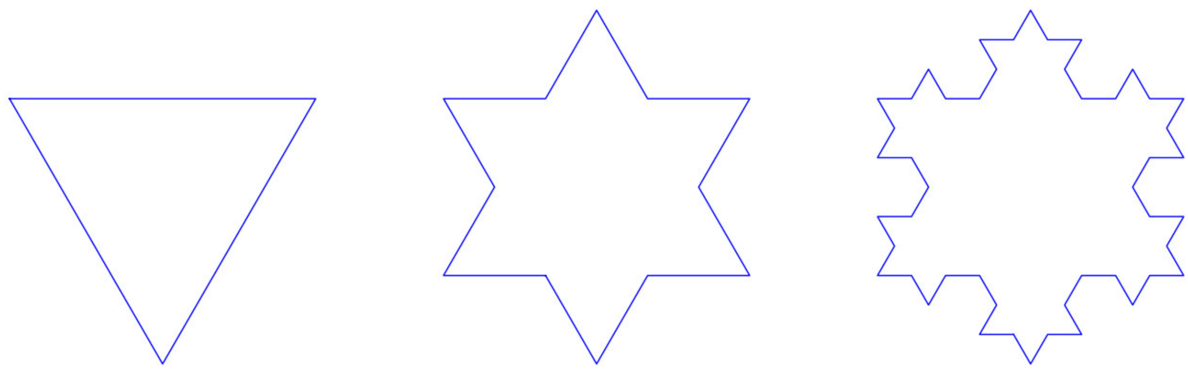
- Function F_1 of the first graph has cycles of all periods
- Function F_2 of the second graph only has cycles of periods 1,2, and 4.

Chapter 14.1.(a) This is an identical iterated system to the one that has the Sierpinski triangle as attractor. This one has $\beta = \frac{1}{3}$ instead of $\frac{1}{2}$, so the attractor will be a Sierpinski-like set where the triangles do not touch. Because it removes the middle-thirds, we get a hybrid of the Sierpinski Triangle with the Cantor set: Each of the edges of the triangle are rotated Cantor sets and each of the edges of the smaller triangles are rescaled rotated Cantor sets.

Chapter 14.1.(b) This iterated system looks similar to the one that resolves into the Cantor set, but with $\beta = \frac{1}{2}$. In fact, this set leaves the segment from $(0,0)$ to $(1,0)$ unaltered, so its attractor is the set $[0,1] \times \{0\}$.

Chapter 14.11. The topological dimension of this curve is 1, since the boundary of any small disk intersects the curve at a set of isolated points, which has dimension 0. This fractal has $k = 5$ (it transforms one line into 5 lines) and $M = 3$ (the lines have $\frac{1}{3}$ the original length), so the fractal dimension is $\dfrac{\ln 5}{\ln 3} \approx 1.46497$.

Chapter 14.15. First observe the figure below with the first three steps of the Koch curve:



Let us calculate the area of each iteration in search of a pattern:

0. The original set is an equilateral triangle with side length 1:

$$A_0 = \frac{\sqrt{3}}{4}.$$

1. We add 3 triangles with side length $\frac{1}{3}$ so each of the new triangles has area equal to the original one times $\frac{1}{9}$:

$$A_1 = \frac{\sqrt{3}}{4} + 3\frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4}\left(1 + 3\frac{1}{9}\right).$$

2. We add $3 \cdot 4$ triangles with side length $\frac{1}{3^2}$ so each of the new triangles has area equal to the original one times $\left(\frac{1}{9}\right)^2$:

$$A_2 = \frac{\sqrt{3}}{4}\left[1 + 3\frac{1}{9} + 3 \cdot 4\left(\frac{1}{9}\right)^2\right].$$

***k*.** At the step k, we add $3 \cdot 4^{k-1}$ triangles with side length $\frac{1}{3^k}$, so the area is

$$A_k = A_0 + \frac{\sqrt{3}}{4}\sum_{i=1}^{k} 3 \cdot 4^{i-1}\left(\frac{1}{9}\right)^i,$$

this implies that

$$A = A_0 + \frac{\sqrt{3}}{4}\frac{3}{4}\sum_{i=1}^{\infty}\left(\frac{4}{9}\right)^i = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{4}\frac{1}{1-\frac{4}{9}}\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{4}\frac{4}{9}\frac{9}{5}\right) = \frac{2\sqrt{3}}{5}.$$