UNIVERSITY OF TORONTO
Faculty of Arts and Science
Final Examinations, April 2012
MAT301H1S Groups and Symmetry

Instructor: Patrick Walls

8 questions :: 80 points total :: 3 hours :: No aids allowed

1. Let G be a group and let $a \in G$ such that $|a| = n$.
   (a) [5 points] Show that $|gag^{-1}| = n$ for all $g \in G$.
   (b) [5 points] Is the subset $G_n = \{g \in G : g^n = e\}$ a subgroup of G? Justify your answer. (Hint: Consider $G = S_3$.)

2. Let G be a group and let H and K be subgroups of G with $H \lhd G$. Define the subset of G
$$HK = \{hk : h \in H \text{ and } k \in K\}.$$
   (a) [5 points] Show that HK is a subgroup of G.
   (b) [5 points] If K is also normal in G, show that $HK \lhd G$.

3. Consider the following subgroup of S_4
$$H = \{\ (1)\ ,\ (1\ 3)\ ,\ (2\ 4)\ ,\ (1\ 2)(3\ 4)\ ,\ (1\ 3)(2\ 4)\ ,\ (1\ 4)(2\ 3)\ ,\ (1\ 2\ 3\ 4)\ ,\ (1\ 4\ 3\ 2)\ \}\ .$$
   (You don't need to prove that H is a subgroup of S_4.)
   (a) [5 points] Show that $K = \{\,(1), (1\ 3)(2\ 4)\,\}$ is a normal subgroup of H.
   (b) [5 points] Show that $H/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

4. Consider the following subgroup of $\mathrm{GL}(2,\mathbb{R})$
$$C = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}$$
   and define
$$\varphi : C \longrightarrow \mathbb{R}^\times : \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mapsto d^2\,.$$
   (You don't need to prove that C is a subgroup of $\mathrm{GL}(2,\mathbb{R})$.)
   (a) [5 points] Show that φ is a homomorphism.
   (b) [5 points] Find $\ker\varphi$ and $\operatorname{im}\varphi$.
   (c) [5 points] Show that $C/\ker\varphi$ has no elements of finite order (other than the identity).

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5. Let G be a finite abelian group of odd order and define

$$\varphi : G \longrightarrow G : g \mapsto g^2 .$$

(a) [5 points] Show that φ is a homomorphism and that $\ker \varphi = \{e\}$.

(b) [5 points] Show that φ is an automorphism of G.

6. [5 points] The group $\mathrm{Inn}(G)$ of inner automorphisms of a group G is the image of the homomorphism

$$c : G \longrightarrow \mathrm{Aut}(G) : a \mapsto c_a$$

where c_a is conjugation by $a \in G$, $c_a(g) = aga^{-1}$. Show that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

7. (a) [5 points] Classify abelian groups of order 175 (up to isomorphism).

   (b) [5 points] Show that every abelian group of order 175 has an element of order 35.

8. (a) [5 points] Show that the centralizer of $(1\ 2\ 3)$ in S_4 is $\langle (1\ 2\ 3) \rangle$.

   (b) [5 points] Show that not all 5-cycles in A_5 are conjugate (in A_5).