Lecture 8

Jan 29th, 2015

Let $\mathbb{E}$ be the set defined by:

Consider the set $\mathbb{E}$ "built" by

Let $\mathbb{E}$ be the smallest set such that:

- The string $x, y, z \in \mathbb{E}$
- If $e_1, e_2 \in \mathbb{E}$ then the strings

$$(e_1+e_2), (e_1 \times e_2), (e_1/e_2), (e_1-e_2) \in \mathbb{E}$$

in Python: '(' + 'e₁' + 'e₂' + ')' + ...

$x \in \mathbb{E}$

$y \in \mathbb{E}$

$z \in \mathbb{E}$

$x \in \mathbb{E}$ and $y \in \mathbb{E}$ so $(x+y) \in \mathbb{E}$

$x \in \mathbb{E}$ and $x \in \mathbb{E}$ so $(x+x) \in \mathbb{E}$

So are $(x-z)$, (y/y), $((x+y)\times y)$, $((x+x)\times(y/y))$, etc.

but $w \notin \mathbb{E}$

For $(x+y) \in \mathbb{E}$, we call x, y variables and call $+$ operator.

For $e \in \mathbb{E}$, let $v(e)$ be the number of occurences of variables in e; $O(e)$ be the number of occurences of operators in e.

Prove $\forall e \in \mathbb{E}$, $v(e) = O(e)+1$

By Structural Induction:

Base case: x, y, z each of those is a variable, there are no operations.

$$v(x) = 1 = 0+1 = O(x)+1,$$
$$v(y) = 1 = 0+1 = O(y)+1,$$
$$v(z) = 1 = 0+1 = O(z)+1,$$

Inductive step: Let $e_1, e_2 \in \mathbb{E}$.

Assume $v(e_1) = O(e_1)+1$, $v(e_2) = O(e_2)+1$ (IH)

Then $v(e_1+e_2) = v(e_1) + v(e_2)$

// the $+$ on the left stands for a string,

// while the one on the right means addition

$O((e_1+e_2)) = O(e_1)+1$ (for '$+$') $+ O(e_2)$. $(*)$

$v(e_1)+v(e_2) = (O(e_1)+1)+(O(e_2)+1)$ by IH

$= (O(e_1)+1+O(e_2))+1$

$= O(e_1+e_2)+1$ by $(*)$

$v((e_1 \times e_2)) = v(e_1)+v(e_2) = (O(e_1)+1)+(O(e_2)+1)$ by (IH)

Let $\odot$ be one of $+, \times, /, -$, then:

$$v((e_1 \odot e_2)) = \cdots$$

Let BT be defined by: Binary Tree

- empty $\in$ BT
- If $t_l, t_r \in BT$, then the tree t with a root node and left and right subtrees t_l, t_r is in BT.

Let's define for $t \in BT$, height(t).

- $\text{height}(\text{empty}) = 0$
- Let t be a tree with left and right subtrees $t_l \in BT$, $t_r \in BT$, then $h(t) = 1 + \max(h(t_l), h(t_r))$.

$$\text{number of leaves} \leq 2^{h(t)-1}$$

$$\text{leaves}(\text{empty}) = 0$$

For t with subtrees t_r, t_l : $\text{leaves}(t) = \text{leaves}(t_l) + \text{leaves}(t_r)$.