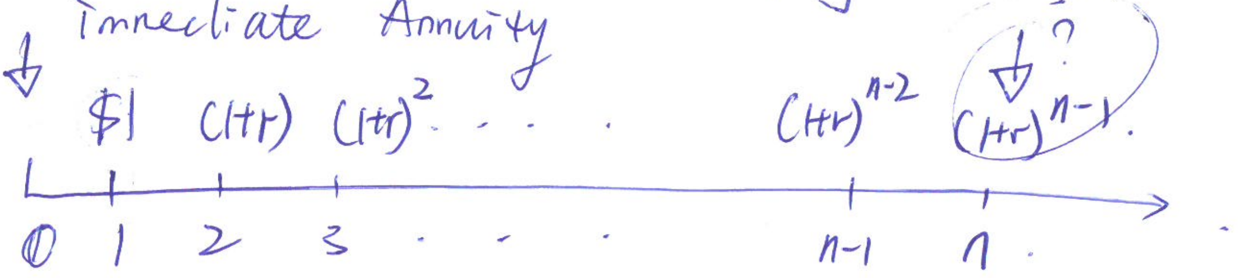
LN5.1

Annuities with indexation

Payments in Geometric Progression

① Immediate Annuity



$$S(n) = (1+i)^{n-1} + (1+r)\cdot(1+i)^{n-2} + \cdots + (1+r)^{n-2}\cdot(1+i) + (1+r)^{n-1}$$

$$= (1+i)^{n-1}\cdot\left[1 + \left(\frac{1+r}{1+i}\right) + \frac{(1+r)^2}{(1+i)^2} + \cdots + \frac{(1+r)^{n-1}}{(1+i)^{n-1}}\right]$$

$$\boxed{1+x+x^2+\cdots+x^k = \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1}-1}{x-1}}$$

$$= (1+i)^{n-1}\cdot\left[\frac{\left(\frac{1+r}{1+i}\right)^n - 1}{\left(\frac{1+r}{1+i}\right) - 1}\right]$$

$$= \frac{(1+i)^n\cdot\left[\frac{(1+r)^n}{(1+i)^n} - 1\right]}{r-i} = \left[\frac{(1+r)^n - (1+i)^n}{r-i}\right]$$

$$S(0) = S(n)\cdot v_i^n = \frac{(1+r)^n - (1+i)^n}{r-i}\cdot (1+i)^{-n}$$ ②

$$= \frac{\left(\frac{1+r}{1+r}\right)^n - 1}{r-i}$$

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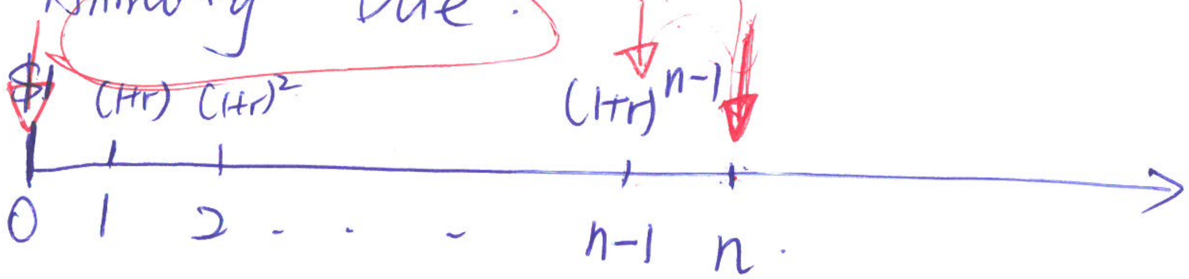
② . Annuity Due.



$$S(n) = (1+i)^n + (1+r)\cdot(1+i)^{n-1} + \cdots + (1+r)^{n-1}\cdot(1+i)$$

$$= (1+i)\cdot\left[\frac{(1+i)^n - (1+r)^n}{i-r}\right]$$

$$S(0) = S(n)\cdot v_i^n = (1+i)^{-n}\cdot\left[\qquad\right]$$

$$= \left[\frac{1 - \left[\frac{1+r}{1+i}\right]^n}{1 - \left[\frac{1+r}{1+i}\right]}\right] \quad \frac{1+r}{1+i} \to v_j$$

$$\ddot{a}_{\overline{n}|j} = \frac{1-(1+i)^{-n}}{d}$$

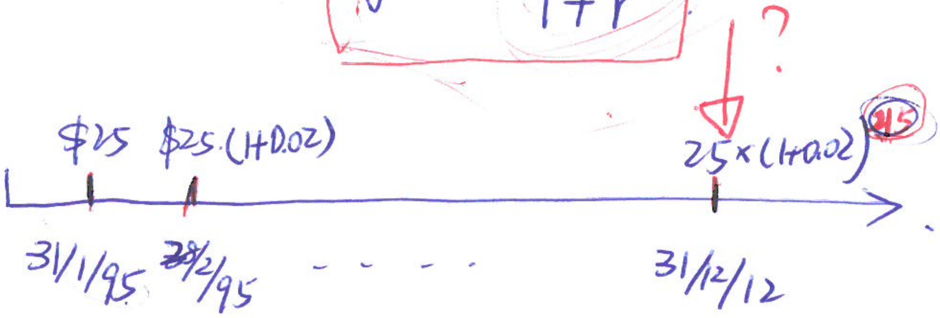
$$\frac{1+r}{1+i} = v_j \Rightarrow d = 1 - v$$

$$\Rightarrow j = \frac{i-r}{1+r}$$

$$\frac{1+r}{1+i} = v_j, \Rightarrow d = 1 - v_j = \frac{i-r}{1+i}$$ ③

$$\boxed{j = \frac{i-r}{1+r}}$$

Ex:



Sol. $[1995 \to 2012]$: 18 yrs × 12 = 216 months

M1

$$S(n) = 25 \cdot 1.01^{215} + 25 \cdot (1+0.02) \cdot 1.01^{214} + \cdots + 25 \cdot 1.02^{215}$$

$$= 25 \cdot 1.02^{215} \cdot \left[1 + \frac{1.01}{1.02} + \cdots + \left(\frac{1.01}{1.02}\right)^{214} + \left(\frac{1.01}{1.02}\right)^{215}\right]$$

$$= 25 \cdot 1.02^{215} \cdot \ddot{a}_{\overline{216}|j} \qquad v_j = \frac{1.01}{1.02}$$

$$= 25 \cdot 1.02^{215} \cdot \left(\frac{1 - v_j^{216}}{1 - v_j}\right)$$

$= \$$ [redacted]58,679.78

M2: $= 25 \cdot 1.01^{215} \cdot \left[1 + \left(\frac{1.02}{1.01}\right) + \cdots + \left(\frac{1.02}{1.01}\right)^{215}\right]$ $\rightarrow 1$

$= 25 \cdot 1.01^{25} \cdot s_{\overline{216}|j}$

$= 25 \cdot 1.01^{25} \cdot \dfrac{(1+j)^{216} - 1}{j}$

$1 + j = \dfrac{1.02}{1.01}$

$\Downarrow$

$j = \dfrac{1.02}{1.01} - 1$

$= \$158{,}679.78$

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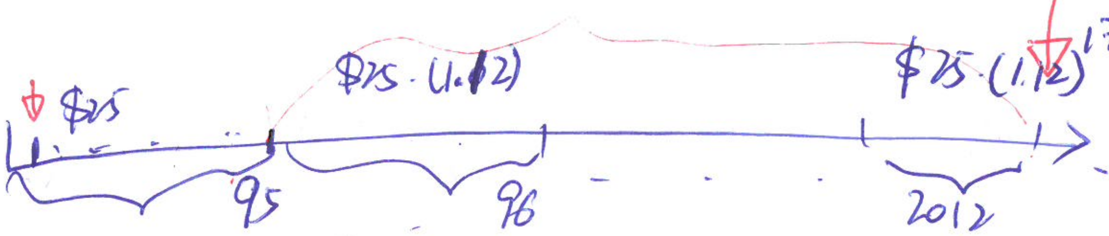
* payment period & index period don't coincide.

Ex: 

Sol: $S(n) = 25 \cdot \big(\underset{\substack{\| \\ 12\times17+11}}{1.01^{215}} + \cdots + \underset{\substack{\| \\ 12\times17}}{1.01^{204}}\big) +$ $\rightarrow$ 1995

$25 \cdot 1.12 \cdot \big(\underset{\substack{\| \\ 12\times16+11}}{1.01^{203}} + \cdots + \underset{\substack{\| \\ 12\times16}}{1.01^{192}}\big) + \cdots +$ $\rightarrow$ 1996

$25 \cdot 1.12^{17} \big(1.01^{11} + \cdots + 1\big)$ $\rightarrow$ 2012

$$= 25 \cdot (1.01^{11} + \cdots + 1)\left[1.01^{12\times 17} + 1.01^{12\times 16}\cdot 1.12 + \cdots + 1.12^{17}\right]$$

?

$$= 25 \cdot (1.01^{11} + \cdots + 1) \cdot 1.01^{12\times 17}\left[1 + \left(\frac{1.12}{1.01^{12}}\right) + \cdots + \left(\frac{1.12}{1.01^{12}}\right)^{17}\right]$$

<1

$$= 25\, s_{\overline{12}|0.01} \cdot 1.01^{12\times 17}\, \ddot{a}_{\overline{18}|w}$$

$$= \$41282.58$$

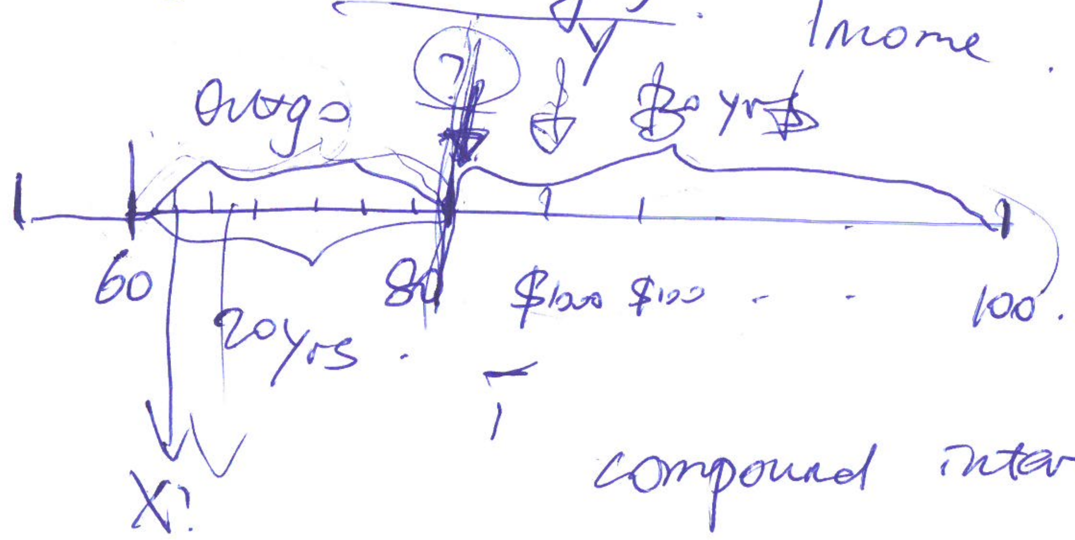
$$\boxed{\begin{gathered} v_w = \frac{1.12}{1.01^{12}} \\ \Downarrow \\ w = \frac{1.01^{12}}{1.12} - 1 \end{gathered}}$$

$$1.01^{12\times 17} \times 1.01^{-12} = 1.01^{12(17-1)} = 1.01^{12\times 16}$$

Equations of Value

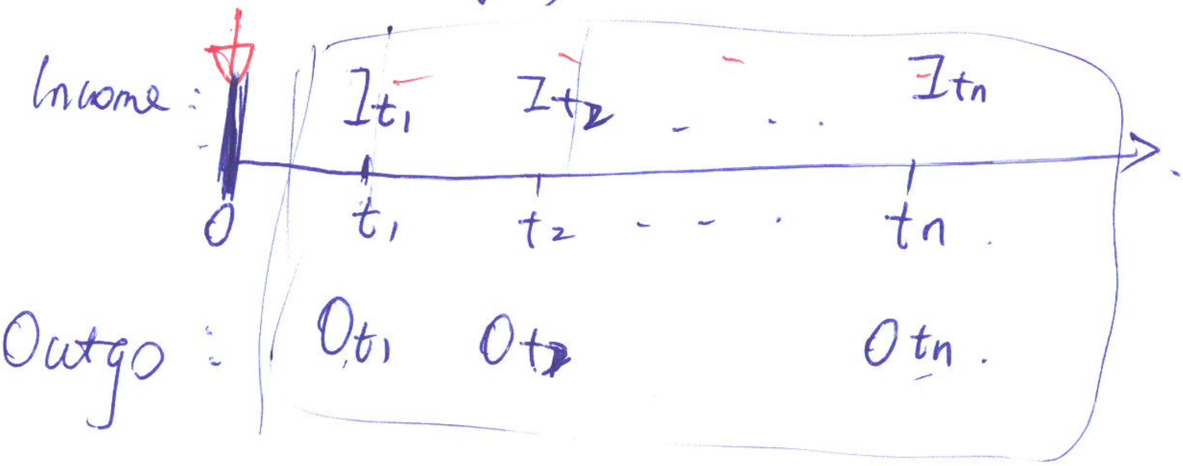
transaction $\nearrow$ Income, $\searrow$ Outgo.

PV (income) = PV (outgo).



compound interest.

①. Discrete payments.



PV (Income) = PV (outgo)

$$\sum_{k=1}^{n} I_{t_k} \underbrace{(1+i)^{-t_k}}_{= v^{t_k}} = \sum_{k=1}^{n} O_{t_k} \cdot \underbrace{(1+i)^{-t_k}}_{= v^{t_k}}.$$

$$\sum_{k=1}^{n}(I_{t_k}-O_{t_k})\cdot v^{t_k}=0$$

② Continuous payments.

$\rho_I(t)$ $\rho_O(t)$.

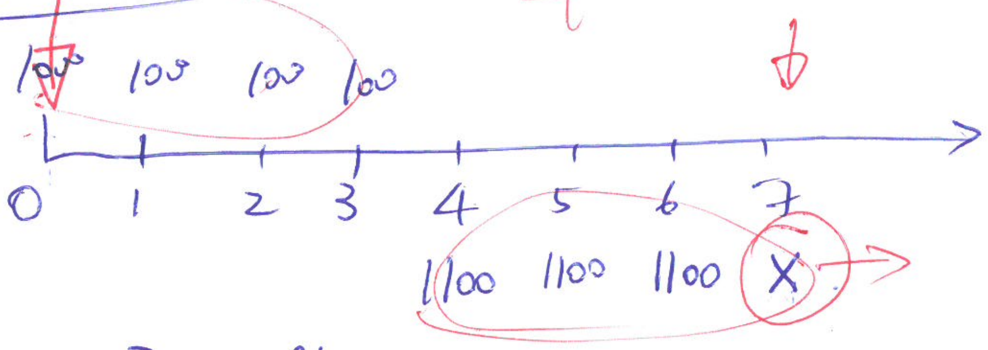
$$\int_0^n \rho_I(t)\cdot(1+i)^{-t}\,dt=\int_0^n \rho_O(t)\cdot(1+i)^{-t}\,dt.$$

③ Combination < Discrete / Cont.

$$\sum_{k=1}^{n} I_{t_k}\cdot(1+i)^{-t_k}+\int_0^n \rho_I(t)\cdot(1+i)^{-t}\,dt$$

$$=\sum_{k=1}^{n} O_{t_k}(1+i)^{-t_k}+\int_0^n \rho_O(t)\cdot(1+i)^{-t}\,dt.$$

Ex.:



Sol.:

$i=8\%$ p.a.

① $t=0$, 7, 3, 100

② $PV_I = 1000 \cdot \ddot{a}_{\overline{4}|0.08} = 1000\,(1+v+v^2+v^3)$

$PV_O = \boxed{1100\,(v^4+v^5+v^6)} + X \cdot v^7$

③ $PV_I = PV_O$

$$\Rightarrow X = \frac{1000\,(1+v+v^2+v^3) - 1100\,(v^4+v^5+v^6)}{v^7}$$

$$= 2273.79$$

Solving for unknowns.

- Payments
- n
- i

$\boxed{\text{Analytical Solution}}$

$\boxed{\text{Approximations}}$

- trial & error
- linear interpolation

$\underline{A} = \underline{B} \cdot a_{\overline{n}|i}$

$$\Rightarrow A = B \cdot \left(\frac{1-(1+i)^{-n}}{i}\right)$$

$$\Rightarrow \frac{1-iA}{B} = (1+i)^{-n}$$

$$\ln\left(1-\frac{iA}{B}\right) = -n\ln(1+i)$$

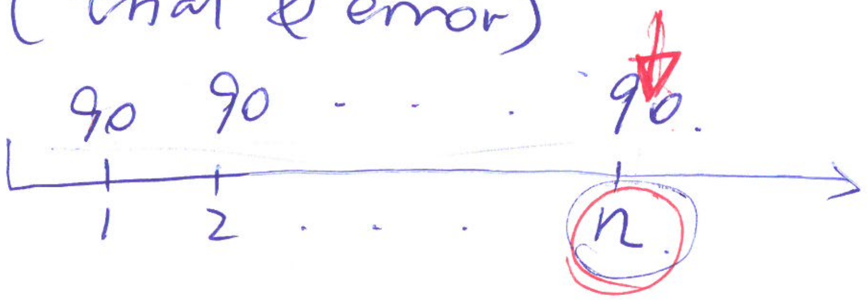
$$\Rightarrow n = \frac{-\ln\left(1-\frac{iA}{B}\right)}{\ln(1+i)}, \quad \ln(1+i) = \delta$$

$$\boxed{a(1+i)^{n2} + b(1+i)^{n} + c = 0}$$

$$\Rightarrow (1+i)^{n} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$i > 0$

Ex (trial & error)



$i = 1\%$ p.a.

$A.V(CF_1, CF_2 \cdots CF_n) > 100 \times n$

$\Downarrow$

$90 \cdot s_{\overline{n}|i} > 100 \cdot n$

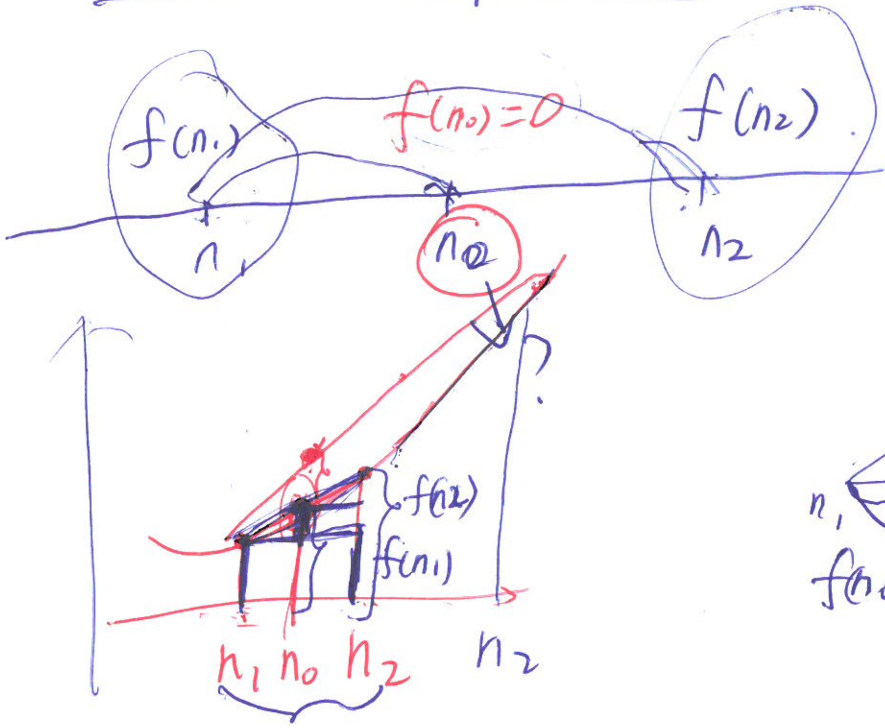
$$\Rightarrow \frac{90 \cdot (1.01^n - 1)}{0.01} > 100 \cdot n$$

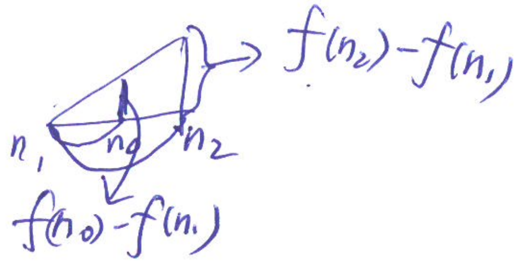
$$\Rightarrow 9000 \cdot (1.01^n - 1) - 100n > 0$$

$$\Rightarrow A(n) \triangleq \boxed{9000 \cdot 1.01^n - 100n - 9000} > 0$$

$$\begin{cases} n = 20 \Rightarrow A = -18.28 \\ n = 25 \Rightarrow A = 41.89 \Rightarrow n = 22 \\ n = 22 \Rightarrow A = 2.44 \\ n = 21 \Rightarrow A = \text{[illegible]} \end{cases}$$

Ex (linear interpolation)





$$\frac{f(n_0)-f(n_1)}{f(n_2)-f(n_1)} \simeq \frac{n_0-n_1}{n_2-n_1}$$

$$\Rightarrow \quad n_0 \simeq n_1 + \frac{f(n_0)-f(n_1)}{f(n_2)-f(n_1)} \cdot (n_2-n_1).$$

Ex: $2 \cdot s_{\overline{20}|i} + s_{\overline{8}|i} = 82.1938$

find i.

$$f(i) = 2 s_{\overline{20}|i} + s_{\overline{8}|i} = \frac{2\left[(1+i)^{20}-1\right]}{i} + \frac{(1+i)^{8}-1}{i}$$

① find n_1 & n_2.

$$\begin{cases} \bar{i}_1 = 15\% \Rightarrow f(\bar{i}_1) = \underline{218.6140} \\ \bar{i}_2 = 14\% \Rightarrow f(\bar{i}_2) = \underline{195.2826.} \end{cases}$$

Linear interpolation

$$\Rightarrow \bar{i}_0 \cong 15\% + \frac{182.1938 - 218.6160}{195.2826 - 218.6160}$$

$$\cong 0.134$$

$\bar{t}_1 = 0.132$

$\bar{i}_2 = 0.136 \quad \Rightarrow \quad \bar{i}_0 \cong 0.1338$