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A subset of $\mathbb{R}^n$ is said to be compact if it is closed and bounded. This is one of the many equivalent definitions of compactness. Indeed compactness is a property of a subset S of $\mathbb{R}^n$ which is very versatile; it has special significances in at least three different directions:

1. Topological/Geometric: a compact set S is closed and bounded, that is,

   a) $\partial S \subset S$

   b) $\exists M > 0$ such that $S \subseteq B(M, \mathbf{0})$.

   This characterization of compactness is a very convenient feature which is used in the proof of theorem 1.25, in the proof of corollary 1.23 (extreme value theorem), and in the proof of theorem 2.83 (section 2.9). And of course, all the discussions of integration in chpater 5 takes place on regions of the space which are bounded and their boundary belongs to them. Incidentally the extreme value theorem is used in theorems 4.24 (mean value theorem for integrals,) theorem 4.46 and 4,47.

2. Completeness: Theorem 1.21 (The Bolzano Weierstrass Theorem), which is the final version of theorems 1.17, 1.18, 1.19, claims that " any sequence in a compact set must have a convergent subsequence". That is the property of closed and bounded is equivalent to the property outlined in theorem 1.20. That is any compact set is complete on its own. This property makes a compact set into a complete set, and this is important because we can do Calculus on this set. This characterization of compactness is used in the proof of 1.22 (any continuous image of a compact set is compact.) This is a very important property of continuous functions. That is, under the continuous maps completeness of a set is preserved, so that the properties of Calculus are preserved.(Read the paragraph in the middle of page 31.)

   Also this characterisation of compactness is used in the proof of theorem 1.33, which claims any continuous function on a compact set is uniformly continuous. The property of uniform continuity is very useful in integration.

3. Logical: This characterization is outlined in theorem1.24 (the Heine-Borel Theorem, which we skip in this course.) This characterization of compactness asserts that for any arbitrary collection of closed set which collectively cover the set there must be a finite subcollection that cover the set. This justifies the term 'compact' used for such set. We don't need infinitely many closed sets to cover a compact set. The implication of this characterization is that if the property P is described by the compact set is a property that can be described by infinitely many statements/properties $P_1, P_2, \ldots$ each expressed in one closed set, then we need only a finite list of these properties which sufficiently express this property P. This idea is very important in studies in logic and applications of logic to models of mathematics.