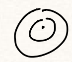
$\frac{dx}{dt} = (1-y)(2x-y) = 2x - y - 2xy + y^2$

$\frac{dy}{dt} = (2+x)(x-2y) = x^2 - 2xy - 4y + 2x$

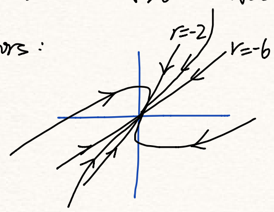
Criticle points: $\begin{pmatrix}-2\\1\end{pmatrix} \begin{pmatrix}2\\1\end{pmatrix} \begin{pmatrix}-2\\-4\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}$

Yacobian: $y = \begin{pmatrix} 2-2y & -1-2x+2y \\ 2x-2y+2 & -2x-4 \end{pmatrix}$

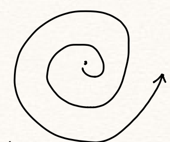
Lineared systems: ① $\begin{pmatrix}-2\\1\end{pmatrix}$ $A = \begin{pmatrix} 0 & 5 \\ -4 & 0 \end{pmatrix}$ Eigenvalues: $\pm\sqrt{-20} = \pm i\sqrt{20}$ center, clockwise 

② $\begin{pmatrix}2\\1\end{pmatrix}$ $A = \begin{pmatrix} 0 & -3 \\ 4 & -8 \end{pmatrix}$ Eigenvalues and eigenvectors:

$-2, \begin{pmatrix}3\\2\end{pmatrix}$ $-6, \begin{pmatrix}1\\2\end{pmatrix}$

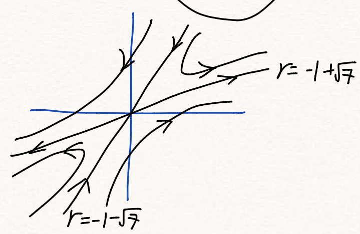


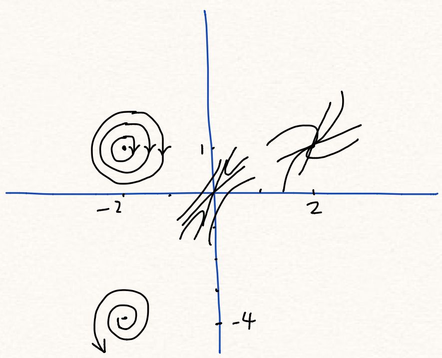
③ $\begin{pmatrix}-2\\-4\end{pmatrix}$ $A = \begin{pmatrix} 10 & -5 \\ 6 & 0 \end{pmatrix}$ Eigenvalues: $5 \pm i\sqrt{5}$ spiral, unstable, counter-clockwise



④ $\begin{pmatrix}0\\0\end{pmatrix}$ $A = \begin{pmatrix} 2 & -1 \\ 2 & -4 \end{pmatrix}$ Eigenvalues, eigenvectors: $-1+\sqrt{7}, \begin{pmatrix}1\\3-\sqrt{7}\end{pmatrix}, -1-\sqrt{7}, \begin{pmatrix}1\\3+\sqrt{7}\end{pmatrix}$

Saddle, unstable.





Population dynamics:

- Consider species of fish in a period. $\frac{dx}{dt} = rx$ r = birth rate.

  More realistic: $\frac{dx}{dt} = \underbrace{(r - ax)}_{\text{effective birth rate}} x$ (a = constant)

- Suppose more generally there are two species x, y.

  $\frac{dx}{dt} = x(r_2 - a_1 x - b_1 y)$ Nonlinear 2x2 system.

  $\frac{dy}{dt} = y(r_2 - a_2 x - b_2 y)$

Conciete example:

$\frac{dx}{dt} = x(4 - x - y) = f(x,y)$

$\frac{dy}{dt} = y(3 - y - \frac{1}{2}x) = g(x,y)$

Critical points:

$\begin{pmatrix}0\\0\end{pmatrix} \begin{pmatrix}0\\3\end{pmatrix} \begin{pmatrix}4\\0\end{pmatrix} \begin{pmatrix}2\\2\end{pmatrix}$

First three critical points mean extinction of one or both species. The last one means coexistence.

Are these stable or unstable?

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 4-2x-y & -x \\ -\frac{y}{2} & 3-2y-\frac{x}{2} \end{pmatrix}$$

① $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ eigenvalues $4, 3 \Rightarrow$ unstable node.

② $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & -3 \end{pmatrix}$ eigenvalues. $1, -3 \Rightarrow$ saddle (unstable)

③ $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ $A = \begin{pmatrix} -4 & -4 \\ 0 & 1 \end{pmatrix}$ eigenvalues $-4, 1, \Rightarrow$ saddle (unstable)

④ $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $A = \begin{pmatrix} -2 & -2 \\ -1 & -2 \end{pmatrix}$ eigenvalues $-2 \pm \sqrt{2} \Rightarrow$ stable node.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\frac{tr(A)}{2} \pm \sqrt{\left(\frac{tr(A)}{2}\right)^2 - det(A)}$

$\Rightarrow$ coexistence of the two species is stable.