FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 7

Question 1

A bank lends a company $5,000 at a fixed rate of interest of 10% pa. The loan is to be repaid by five level annual payments. Calculate the interest and capital payments at each repayment date.

Solution

First calculate the total amount of each payment X.

$$Xa_{\overline{5}|} = 5,000 \Rightarrow X = \frac{5,000}{3.7908} = \$1,318.98$$

The following table is the loan schedule.

Year	Loan outstanding at the start of the year (OB)	Interest due at the end of the year (I=10% of L)	Principal repaid at the end of the year (PR=1,318.98-I)	Loan outstanding at the end of the year (OB-PR)
1	5,000	500	818.98	4,181.02
2	4,181.02	418.10	900.88	3,280.14
3	3,280.14	328.01	990.97	2,289.17
4	2,289.17	228.92	1,090.06	1,199.11
5	1,199.11	119.91	1,199.07	0.04

The final loan amount is non-zero due to rounding error.

Question 2

A loan of $80,000 is repayable by eight annual payments, starting in one year's time, with interest payable at 4.5% per annum. Payments one to three are half as much as payments four to eight. Calculate the loan outstanding one year before the loan is completely repaid.

Solution

Let X equal the amount of the first installment, then:

$$80,000 = Xa_{\overline{3}|} + 2Xv^3 a_{\overline{5}|}$$

or equivalently:

$$80,000 = Xa_{\overline{8}|} + Xv^3 a_{\overline{5}|}$$

Therefore, $X = \dfrac{80,000}{6.5959 + (0.8763)(4.3900)} = 7,660.74$

The loan outstanding one year before the end of the term equals (prospectively) the present value one year before the end of the term of the final repayment, which equals:

$$2(7660.74)v = \$14,661.70$$

We can check this answer using the retrospective method:

$$80,000(1+i)^7 - Xs_{\overline{7}|} - Xs_{\overline{4}|} = \$14,661.70$$

Question 3

A loan of $120,000 is repayable by equal quarterly payments for 25 years. The effective rate of interest is 6% per annum. Find the interest portion of the first payment.

Solution

The amount of each installment is not needed because we are given the loan outstanding at the start. The interest portion of the first payment equals:

$$120,000(1.06^{1/4} - 1) = \$1,760.86$$

Question 4

A loan of $12,000 is repaid by 36 monthly payments starting one month after the loan. The first 12 payments are 395+X each, the next 12 payments are 395 each, and the final 12 payments are 395-X each. If $i^{(12)}$=0.12 find X.

Solution

$$12000 = (395+X)a_{\overline{12}|} + 395a_{\overline{12}|}v^{12} + (395-X)a_{\overline{12}|}v^{24}$$

where $j = \dfrac{0.12}{12} = 0.01$ is the effective monthly rate of interest.

$a_{\overline{12}|} = \dfrac{1-v^{12}}{j} = \dfrac{1-(1.01)^{-12}}{0.01} = 11.2551$ - this can also be obtained from the interest rate tables.

$$\Rightarrow 12000 = (395+X)11.2551 + 395(11.2551)v^{12} + (395-X)(11.255)v^{24}$$

$$\Rightarrow 12000 = X(2.39092) + 11892.46$$

$$\Rightarrow X \cong 45$$

Question 5
A 5-year loan made on July 1, 2009 is amortised with 60 level monthly payments starting August 1, 2009. If interest is at $i^{(12)}$=0.12, find the date on which the outstanding balance first falls below one-half of the original loan amount.

Solution
Let the monthly level payments be amount K and the loan amount L. Since the monthly effective interest rate $j = \frac{i^{(12)}}{12} = 0.01$, then:

$$OB_0 = L = Ka_{\overline{60}|0.01}$$

The outstanding balance at time t is (using the prospective method):

$$OB_t = Ka_{\overline{60-t}|}$$

We want to find t such that

$$OB_t < \frac{1}{2}OB_0 \Rightarrow Ka_{\overline{60-t}|} < \frac{1}{2}Ka_{\overline{60}|0.01} \Rightarrow a_{\overline{60-t}|} < \frac{1}{2}a_{\overline{60}|0.01} \Rightarrow a_{\overline{60-t}|} < 22.4775$$

$$\Rightarrow \frac{1-(1.01)^{-(60-t)}}{0.01} < 22.4775$$

$$\Rightarrow 0.77523 < (1.01)^{-(60-t)} \Rightarrow -\frac{\ln(0.77523)}{\ln(1.01)} > 60 - t \Rightarrow t > 60 + \frac{\ln(0.77523)}{\ln(1.01)}$$

$$\Rightarrow t > 34.4$$

$t = 35$ is June 1, 2012.

Past Exam Question – 2005 Final Exam Q3(a)
A \$20,000 loan is to be repaid by monthly installments payable in arrears over a period of 20 years. Monthly payments for the first 15 years of the loan are to be at the rate of \$*K* per month, while for the final 5 years payments are to be at the rate of \$2*K* per month. Interest is charged on the loan at the rate of 8% per annum effective.

i) Calculate the value of *K*. (3 marks)

ii) Calculate the total interest charged during the 17th year of the loan. (4 marks)

Solution

i)

$$20,000 = 12Ka^{(12)}_{\overline{15}|0.08} + 24Ka^{(12)}_{\overline{5}|0.08}v^{15}_{0.08}$$

$$= 12K\,\frac{i}{i^{(12)}}\left(a_{\overline{15}|0.08} + 2a_{\overline{5}|0.08}v^{15}_{0.08}\right)$$

$$= 12K\,\frac{0.08}{0.077208}\left(8.5595 + 2(3.9927)(0.31524)\right)$$

$$= 137.72854K \Rightarrow K = 145.21$$

ii) Total interest charged = Total Payment – Principal Repaid

$$= 24K - PR_{17} = 24K - (OB_{16} - OB_{17})$$

$$OB_{16} = 24Ka^{(12)}_{\overline{4}|0.08}$$

$$= 3,485.04\,\frac{0.08}{0.077208}(3.3121) = \$11,960.21$$

$$OB_{17} = 24Ka^{(12)}_{\overline{3}|0.08}$$

$$= 3,485.04\,\frac{0.08}{0.077208}(2.5771) = \$9,306.08$$

Therefore, total interest charged = $3,485.04 - (11,960.21 - 9,306.08) = \$830.91$

<u>Question 6</u>

This question refers to two projects relating to a small software company that has asked to set up a new computer system for a major client. :

Project A

Project A delegates all the development work to outside companies. The estimated cashflows for Project A are (where brackets indicate expenditure):

Beginning of year 1	contractors' fees	(\$150,000)
Beginning of year 2	contractors' fees	(\$250,000)
Beginning of year 3	contractors' fees	(\$250,000)
End of year 3	sales	\$1,000,000

Project B

Project B carries out all the development work in-house by purchasing the necessary equipment and using the company's own staff. The estimated cashflows for Project B are:

Beginning of year 1	new equipment	(\$325,000)
Throughout year 1	staff costs	(\$75,000)
Throughout year 2	staff costs	(\$90,000)
Throughout year 3	staff costs	(\$120,000)
End of year 3	sales	\$1,000,000

The staff costs can be assumed to be paid uniformly throughout the year.

(a) Calculate the net present value for Project A and Project B using a risk discount rate of 10% per annum. Using net present value as a criterion, which project is preferable?

(b) Find the internal rates of return for Project A and Project B, and hence determine which project is more favourable using this criterion.

Solution

(a)

For Project A, the net present value (in \$000) is:

$$NPV_A(0.10) = -150 - 250v - 250v^2 + 1000v^3 = 167.4$$

For Project B, the net present value (in \$000) is:

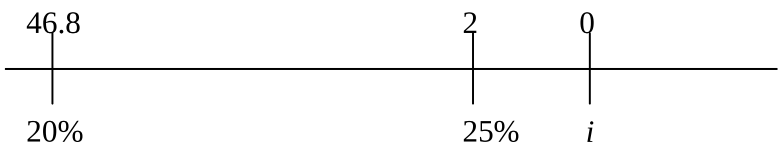
$$NPV_B(0.10) = -325 - 75\bar{a}_{\overline{1|}} - 90v\bar{a}_{\overline{1|}} - 120v^2\bar{a}_{\overline{1|}} + 1000v^3$$
$$= -325 - (75 + 90v + 120v^2)\frac{1-v}{\delta} + 1000v^3 = 182.1$$

So, using a risk discount rate of 10%, Project B appears more favourable.

(b)

For Project A, we need to find the interest rate i that satisfies the equation of value:
$$-150 - 250v - 250v^2 + 1000v^3 = 0$$

At 10%: $-150 - 250v - 250v^2 + 1000v^3 = 167.4$
At 20%: $-150 - 250v - 250v^2 + 1000v^3 = 46.8$
At 25%: $-150 - 250v - 250v^2 + 1000v^3 = 2.00$



We can approximate i by extrapolating (linearly) using these two values.

$$i \cong 20\% + \frac{0-46.8}{2-46.8}(25\% - 20\%) = 25.2\%$$

For Project B, we need to find the interest rate i that satisfies the equation of value:
$$-325 - (75 + 90v + 120v^2)\frac{1-v}{\delta} + 1000v^3 = 0$$

At 10%: $-325 - (75 + 90v + 120v^2)\dfrac{d}{\delta} + 1000v^3 = 182.1$

At 20%: $-325 - (75 + 90v + 120v^2)\dfrac{d}{\delta} + 1000v^3 = 40.41$

At 25%: $-325 - (75 + 90v + 120v^2)\dfrac{d}{\delta} + 1000v^3 = -13.59$

We can approximate i by interpolating (linearly) using these two values:

$$i \cong 25\% + \frac{0-(-13.59)}{40.41-(-13.59)}(20\%-25\%) = 23.7\%$$

Using IRR as our criterion, Project A has the higher yield and is more favourable.

Question 7

For each of the projects C and D outlined below, calculate:

(a) the internal rate of return.

(b) The range of interest rates at which money can be borrowed in order for the projects to be profitable.

(c) The accumulated profit at the end of 5 years, assuming that the projects are financed by a loan subject to interest at 6.25% and interest is earned at 5%.

Project C

Initial outlay	(\$100,000)
Proceeds (at the end of 5 years)	\$140,000

Project D

Initial outlay	(\$100,000)
Proceeds (at the end of each of the next 3 years)	\$38,850

Solution

(a) The internal rates of return i_C and i_D are given by:

$$100,000(1+i_C)^5 = 140,000 \Rightarrow i_C \cong 7\%$$

$$38,850 a_{\overline{3|}} = 100,000 \Rightarrow a_{\overline{3|}} = 2.574 \Rightarrow i_D \cong 8.1\%$$ (find by interpolation or trial and error).

(b) If the borrowing rate is less than 7%, both projects will be profitable.
If the borrowing rate is between 7% and 8.1%, only Project D will be profitable.
If the borrowing rate exceeds 8.1%, neither project will be profitable.

(c) The accumulated values of the profits at the end of 5 years, using a rate of interest of 6.25%, are:

$$140,000 - 100,000(1.0625^5) = \$4,592$$

$$\left(38,850 s_{\overline{3|}0.0625} - 100,000(1.0625^3)\right) \times 1.05^2 = \$4,454$$

Question 8
An investment requires an upfront outlay of $50,000 to produce income of $10,000 at the end of each of the first five years and $5,000 at the end of each of the next five years. If an investor is able to reinvest income at 6%p.a. effective, find the internal rate of return on the investment.

Solution
We need to solve the equation:

$$PV_I - PV_O = 0$$

As investment income is accumulated at a different rate to the IRR:

$$v_i^{10}\left(10,000 s_{\overline{5}|0.06} 1.06^5 + 5,000 s_{\overline{5}|0.06}\right) - 50,000 = 0$$

$$50,000 = 103,622.86 v_i^{10}$$

$$(1+i)^{-10} = 0.48252$$

$$i = 7.56\%$$

Question 9
A speculator borrows $50,000 at an effective interest rate of 8% per annum to finance a project that is expected to generate $7,500 at the end of each year for the next 15 years. Find the discounted payback period for this investment and the accumulated profit if the money invested after the loan is paid back is at a rate of 6%.

Solution
The accumulated profit at the end of year t will be:

$$-50,000(1+i)^t + 7,500 s_{\overline{t}|}$$

This will be positive when:

$$-50,000(1+i)^t + 7,500 s_{\overline{t}|} \geq 0$$

Dividing through by $7,500(1+i)^t$:

$$a_{\overline{t}|} \geq \frac{50,000}{7,500} = 6.6666$$

Looking at the interest rate tables (at 8%), we see that $a_{\overline{9}|} = 6.2469$ and $a_{\overline{10}|} = 6.7101$

So the discounted payback period is 10 years.

Alternatively,

$$-50,000(1+i)^t + 7,500 s_{\overline{t|}} \geq 0 \Rightarrow 43,750(1.08^t) \geq 93,750$$

$$t \geq \frac{\ln(2.14286)}{\ln(1.08)} = 9.9 \text{ years}$$

Thus, the discounted payback period is 10 years.

In order to calculate the total profit we must accumulate the amount after the discounted payback period, with any future income, at the reinvestment rate.

Amount after discounted payback period:

$$= -50,000(1.08)^{10} + 7,500 s_{\overline{10|}0.08} = \$702.97$$

Accumulated Profit:

$$= 702.97(1.06)^5 + 7,500 s_{\overline{5|}0.06} = \$43,218.93$$