STAT2032/6046 - Solutions to Mid-Semester Examination - First Semester 2014

1 Question 1 (13 marks)

(a) (i) [1 mark] Paul is saving to buy a house. Bank A offers a savings account which earns interest at a nominal rate of 10% pa payable half-yearly (that is, $i^{(2)} = 10\%$). What amount X must Paul invest at t=0 to accumulate $100,000 after 5 years (that is, at t=5)?

Solve for X such that $X(1 + \frac{i^{(2)}}{2})^{10} = 200000$. X= $\$61,391.33$

(a)(ii) [2 marks] Bank B also offers a savings account. Funds accumulate at a nominal rate of $d^{(2)}$ per annum payable half-yearly. What rate $d^{(2)}$ should Bank B offer, so that Paul prefers Bank B over Bank A?

Find the rate $d^{(2)}$ that is equivalent to $i^{(2)}$. Then the Bank B rate must be greater than this rate. That is
$d^{(2)} > 2(1 - (1 + \frac{i^{(2)}}{2})^{-1})$, so $d^{(2)} > 9.5238\%$.

(b) [5 marks] Simon borrows money from Bank C for 1 year at a nominal discount rate of 20% per annum payable half-yearly (that is, $d^{(2)} = 20\%$). In 1 year (that is, at t=1), Simon must repay $100,000 to Bank C. After 6 months (that is, at t=0.5), and just before the payment of any discount, Simon notices that Bank C is also offering loans at the nominal discount rate of $d^{(4)} = 20\%$ per annum payable quarterly

What action do you think Simon would like to take at t=0.5 (before the payment of any discount at this time)? Provide a written explanation for your answer and quantify any fee Bank C may charge at t=0.5.

The nominal discount rate is the same for a higher value of m. Simon can benefit from the increased number of periods of discounting and in total pay a lower discount over the period t=0.5 to t=1. Therefore, Simon would like to change the terms of his loan to the new discount rate $d^{(4)} = 20\%$.

More specifically, under the original loan, the outstanding balance at t=0.5 is \$90,000 before the payment of the discount of \$10000 at t=0.5. Suppose Simon borrows \$100,000 for 6 months under the new loan at t=0.5. Under the new loan, he will pay a discount of \$4750 at t=0.5, and a discount of \$5000 at t=0.75, and receive \$90,250 (at t=0.5). So Simon can pay off the outstanding balance (=\$90000) on the original loan at t=0.5 using the proceeds of the new loan \$90250, and be \$250 better off.

The bank should charge a fee of \$250 at t=0.5. This removes the \$250 benefit from switching loans.

(c) [3 marks] Prove that $\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \times \frac{d^{(m)}}{m}$.

$$\left(1 + \frac{i^{(m)}}{m}\right) = \left(1 - \frac{d^{(m)}}{m}\right)^{-1}$$

$$\left(1 + \frac{i^{(m)}}{m}\right) \cdot \left(1 - \frac{d^{(m)}}{m}\right) = 1$$

$$1 - \frac{d^{(m)}}{m} + \frac{i^{(m)}}{m} - \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m} = 1$$

Hence,

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \times \frac{d^{(m)}}{m}$$

(d) [2 marks] It is known that $1 + \frac{i^{(n)}}{n} = \frac{1+\frac{i^{(4)}}{4}}{1+\frac{i^{(5)}}{5}}$**. Find n.**

$\left(1 + \frac{i^{(5)}}{5}\right)\left(1 + \frac{i^{(n)}}{n}\right) = \left(1 + \frac{i^{(4)}}{4}\right)$.

Now $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. So n=20.

2 Question 2 (14 marks)

(a) [5 marks] At an annual effective interest rate of $i, i > 0$, both of the following annuities have a present value of X:

- a 20-year annuity in arrears with annual payments of 55;
- a 30-year annuity in arrears with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years.

Calculate X.

$$55a_{\overline{20}|} = 30a_{\overline{10}|} + 60v^{10}a_{\overline{10}|} + 90v^{20}a_{\overline{10}|}$$

Let $v^{10} = b$. We have the following quadratic equation:

$$90b^2 a_{\overline{10}|} + 5ba_{\overline{10}|} - 25a_{\overline{10}|}$$

$$5a_{\overline{10}|}(18b^2 + b - 5) = 0$$

Solve the quadratic equation, $v^{10} = b^2 = \frac{1}{0.5}$. Hence $i = 0.071773$. Therefore $X = \$574.72$

(b)(i) [5 marks] A 10-year annuity in arrears has the following sequence of annual payments:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5$$

Derive an algebraic expression for the present value of this annuity at an effective annual interest rate i. Simplify your answer, using actuarial notation where possible.

$$\begin{aligned}
PV &= v + v^2 + 2v^3 + 2v^4 + 3v^5 + 3v^6 + 4v^7 + 4v^8 + 5v^9 + 5v^{10} \\
&= v + 2v^3 + 3v^5 + 4v^7 + 5v^9 + v^2 + 2v^4 + 3v^6 + 4v^8 + 5v^{10} \\
&= v(1 + 2v^2 + 3v^4 + 4v^6 + 5v^8) + v^2(1 + 2v^2 + 3v^4 + 4v^6 + 5v^8) \\
&= (v + v^2)(1 + 2v^2 + 3v^4 + 4v^6 + 5v^8) \\
&= a_{\overline{2}|i} I\ddot{a}_{\overline{5}|j}
\end{aligned}$$

where $j = (1 + i)^2 - 1$ (that is, the 2-year effective interest rate)

(b)(ii) [2 marks] For the annuity in (b)(i), now suppose that payments are made continuously at the rate of

- 1 per annum in years 1 and 2;
- 2 per annum in years 3 and 4;
- 3 per annum in years 5 and 6;
- 4 per annum in years 7 and 8; and
- 5 per annum in years 9 and 10.

Will the present value of this continuous annuity be greater than, less than or equal to the present value of the annuity in (b)(i)? Provide a written explanation for your answer.

The present value will be greater because payments are received earlier (that is continuously throughout each year rather than at the end of each year). Hence, the effect of discounting is less, and the present value is greater.

(b)(iii) [2 marks] Find an algebraic expression for the present value of the annuity in part (b)(ii). Simplify your answer, using actuarial notation where possible.

$$\begin{aligned} PV &= \bar{a}_{\overline{2}|} + 2v^2\bar{a}_{\overline{2}|} + 3v^4\bar{a}_{\overline{2}|} + 4v^6\bar{a}_{\overline{2}|} + 5v^8\bar{a}_{\overline{2}|} \\ &= \bar{a}_{\overline{2}|}(1 + 2v^2 + 3v^4 + 4v^6 + 5v^8) \\ &= \bar{a}_{\overline{2}|i} I\ddot{a}_{\overline{5}|j} \end{aligned}$$

where $j = (1+i)^2 - 1$ (that is, the 2-year effective interest rate)

3 Question 3 (14 marks)

(a) [5 marks] A family wishes to provide an annuity of \$100 at the end of each month to their daughter now entering university. The annuity will be paid for only nine consecutive months each calendar year for four years. Find the present value of the annuity one month before the first payment at an annual effective interest rate of 12%.

Let $i = 0.12$, and so $v = (1.12)^{-1}$ and let $j = (1.12)^{1/12} - 1$

$$\begin{aligned}
PV &= 100(v^{1/12} + v^{2/12} + ... + v^{9/12}) + 100(v^{13/12} + v^{14/12} + ... + v^{21/12}) + \\
&\qquad 100(v^{25/12} + v^{26/12} + ... + v^{33/12}) + 100(v^{37/12} + v^{38/12} + ... + v^{45/12}) \\
&= 100(v^{1/12} + v^{2/12} + ... + v^{9/12})(1 + v + v^2 + v^3) \\
&= 100 a_{\overline{9}|j} \ddot{a}_{\overline{4}|i} \\
&= 100 \times 8.5874477 \times 3.4018313 \\
&= \$2921.30
\end{aligned}$$

Note: we can write $PV = 1200 a^{(12)}_{\overline{9/12}|i} \ddot{a}_{\overline{4}|i}$

(b) (i) [3 marks] Show that the present value of a continuous perpetuity $\bar{a}_{\overline{\infty}|}$, (that is, an annuity that pays continuously at a rate of \$1 per annum forever), is given by $\bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$.

$$\bar{a}_{\overline{\infty}|} = \lim_{n\to\infty} \bar{a}_{\overline{n}|} = \lim_{n\to\infty} \frac{1 - v^n}{\delta} = \frac{1}{\delta}.$$

(b) (ii) [1 mark] A perpetuity pays continuously at a rate of 100 per year and has a present value of 800. Calculate the annual effective interest rate used to calculate the present value.

$$800 = \frac{1}{\delta} = \frac{100}{\ln(1+i)}$$

$$i = 13.315\%.$$

(c) [5 marks] An investment fund is started with an initial deposit of 1 at time 0. New deposits are made continuously at the annual rate $(1+t)$ at time t over the next n years. The force of interest at time t is given by $\delta_t = (1+t)^{-1}$. Find the accumulated value in the fund at the end of n years.

Ignoring the initial deposit

$$\begin{aligned}
\text{Accum. value} &= \int_0^n (1+t)e^{\int_t^n \frac{1}{1+r}dr}dt \\
&= \int_0^n e^{[ln(1+r)]_t^n}dt \\
&= \int_0^n (1+t)\frac{1+n}{1+t}dt \\
&= \int_0^n (1+n)dt \\
&= n(n+1)
\end{aligned}$$

The accumulated value of the initial deposit is

$$e^{\int_0^n \frac{1}{1+r}dr} = e^{[ln(1+r)]_0^n} = n+1.$$

Hence the total accumulated value is $n(n+1)+(n+1)=(n+1)^2$.

4 Question 4 (11 marks)

(a) [5 marks] An annuity has 15 annual payments made at the end of each year. The first three payments are \$45,000 each, the next three payments are 6% smaller, and so on. That is, payments decrease by 6% every three years. Find the accumulated value of the annuity on the date of the last payment, if the effective annual interest rate is 6% per annum.

$$\begin{aligned}
AV &= 45000(1.06^{14} + 1.06^{13} + 1.06^{12}) + 45000(0.94)(1.06^{11} + 1.06^{10} + 1.06^{9}) \\
&\qquad + 45000(0.94)^2(1.06^8 + 1.06^7 + 1.06^6) + 45000(0.94)^3(1.06^5 + 1.06^4 + 1.06^3) \\
&\qquad + 45000(0.94)^4(1.06^2 + 1.06^1 + 1) \\
&= 45000(0.94)^4(1.06^2 + 1.06^1 + 1)\left[\left(\frac{1.06^3}{0.94}\right)^4 + \left(\frac{1.06^3}{0.94}\right)^3 + \left(\frac{1.06^3}{0.94}\right)^2 + \left(\frac{1.06^3}{0.94}\right) + 1\right] \\
&= 45000(0.94)^4 s_{\overline{3}|0.06} s_{\overline{5}|j} \text{ where } j = \frac{1.06^3}{0.94} - 1 = 0.267038 \\
&= 45000 \times (0.94)^4 \times 3.1836 \times 8.483774316 \\
&= \$948,924.22
\end{aligned}$$

(b) [3 marks] Now suppose for the term of the annuity in part (a), inflation occurs at the rate of 3% per annum. What is the annuity worth in 'real' dollars on the date of the final payment? (That is, express the accumulated value of the annuity in terms of purchasing power at time t=0)

The real interest rate $i' = \frac{0.06-0.03}{1.03} = 0.0291262$.
Hence $j' = \frac{1.029126621^3}{0.94} - 1 = 0.15951953$

$$AV = 45000 \times (0.94)^4 s_{\overline{3}|i'} s_{\overline{5}|j'} = 45000 \times (0.94)^4 \times 3.0882270 \times 6.87060 = \$745,466.34$$

(b) [3 marks] For the annuity in part (a), if payments decrease by 7% every three years (instead of 6% every three years), the accumulated value is \$933,544.35. If payments decrease by 8% each year, the accumulated value is \$918,449.03. Find the percentage decrease required every three years to obtain an accumulated value of \$920,000 (on the date of the last payment).

Use linear interpolation, and let k be the unknown percentage decrease.

$$
\begin{aligned}
k &\approx 0.07 + \frac{920000 - 933544.35}{918449.03 - 933544.35}(0.08 - 0.07) \\
&= 7.90\%
\end{aligned}
$$

5 Question 5 (8 marks)

(a) [2 marks] Amy borrows \$50,000 from Bank X and agrees to repay it back repay it back over six years, in quarterly instalments of principal and interest at the nominal interest rate of 8% per annum payable quarterly. Determine the size of Amys quarterly payment (assuming she makes payments at the end of every quarter).

The effective quarterly rate of interest is i=2%. Solve for X such that

$$50000 = Xa_{\overline{24|}0.02}$$

$$X = 2643.55$$

(b) [4 marks] Suppose that immediately after the sixth payment, Amy loses her job. Fortunately, it doesnt take long for Amy to find another job, and she misses only two quarterly payments. That is, she recommences payments 9 months after losing her job. If the term of the loan and interest rate remain unchanged, what is Amys revised quarterly repayment?

Amy's loan outstanding 3 months prior to recommencing payments is

$$2643.55a_{\overline{18|}}(1.02)^2 = \$41233.32446.$$

There are 16 quarterly payments left, hence solve for X', such that

$$41233.32446 = X'a_{\overline{16|}}$$

$$X' = \$3036.84$$

(c)[2 marks] What is the total amount of interest Amy pays on the loan?

The total amount of interest is

$$2643.55 \times 6 + 3036.84 \times 16 - 50000 = \$14,450.74$$