Exerzitien X

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Dec. 1., in your tutorial.

Complete readings: Axler, Chapter 1, 2, 3, 5 all but the last section, and Chapter 8, section 1 are the most important parts of the course, then Chapter 8 section 2, 3, 5, 6 but the classroom notes and documents may be more useful for these.

Exercise 1. *Let $T : U \to V$ and $S : V \to W$ be linear maps such that (S) and (T) are finite-dimensional.*

1. *Show that*
$$\dim \mathrm{null}(ST) = \dim \mathrm{null}(T) + \dim(\mathrm{null}(S) \cap \mathrm{range}(T)).$$
2. *Explain why this implies*
$$\dim \mathrm{null}(ST) \leq \dim \mathrm{null}(S) + \dim \mathrm{null}(T).$$
When does equality hold?
3. *If S, T are each injective, is ST necessarily injective? What if T is surjective and S injective?*
4. *Let D be the differentiation operator acting on the vector space of infinitely differentiable functions. Assuming that e^x spans $\mathrm{null}(D-1)$, prove that $(e^x, xe^x, \ldots, x^n e^x)$ spans $\mathrm{null}((D-1)^{n+1})$.*

Exercise 2. *Let L be the left shift operator on the vector space of sequences of numbers in a field. That is, if $s = (s_n)_{n=1}^{\infty}$ is a sequence then Ls is a new sequence defined by $(Ls)_n = s_{n+1}$.*

1. *Determine the eigenvalues and eigenvectors of L. Show that $(1) = (1, 1, 1, \ldots)$ spans the eigenspace for eigenvalue 1. Show that $(n) = (1, 2, 3, \ldots)$ is a generalized eigenvector for eigenvalue 1. Prove the following identity of sequences:*
$$(L-1)(n^2) = 2(n) + (1),$$
and conclude that (n^2) is in $\mathrm{null}(L-1)^3$.
2. *Use the above idea to prove that the sequence $t = (t_n)$, where $t_n = 1^3 + 2^3 + \cdots + n^3$, is a linear combination of the sequences $(1), (n), (n^2), (n^3), (n^4)$. Find the coefficients of the linear combination, and write the resulting formula for the sum of the first n cubes.*

Exercise 3. *(Bonus) Compute the determinant of the matrix and simplify the result as much as possible.*

$$V = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ 1 & c_2 & c_2^2 & \cdots & c_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$$

Advice: In addition to the conceptual material which includes definitions and theorems, I want to stress that there are some skills which you will need to know for the exam. Make sure you can solve systems of linear equations, homogeneous and inhomogeneous. Make sure you can find the null space and range of a matrix by row reduction. Make sure you can find inverses of matrices by row reduction. Make sure you can find eigenvalues and a basis of eigenvectors. Make sure you understand the change of basis formula very well, and that you remember which change of basis matrix goes where and what each of them means.