STA447/STA2006 Stochastic Processes

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Lecture on March 6, 2014

Note

This lecture note is prepared for the course STA447/STA2006 Stochastic Processes. This lecture note may contain flaws. Please consult text book or reference books for confidence.

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* indicates graduate level. So you may skip those parts.

4 Renewal Process

Example 45 (Machine repair)**.** A machine works for an amount of time τ_i before it fails, requiring an amount of time u_i to be repaired. Then the amount of time in ith cycle of failure and repair is $t_i = \tau_i + u_i$. If the machine works like a new machine, then the cycle length t_i are i.i.d. We may ask "How much proportion the machine is working?"

Example 46. The arrival time of customer in a coffee shop may not be exponentially distributed. Assume that there is only one server and the processing time of ith customer's order is s_i. We may ask "What it the average waiting time of each customer?"

Theorem 59. Let μ be the mean inter arrival time. If $P(\tau_i > 0) > 0$, then $N(t)/t \to 1/\mu$ almost surely as $t \to \infty$.

Proof. It is easy to see that $\tau_1 + \cdots + \tau_{N(t)} \le t < \tau_1 + \cdots + \tau_{N(t)} + \tau_{N(t)+1}$. Since $P(\tau_i > 0) > 0$, $N(t) \to \infty$ almost surely as $t \to \infty$. Hence $(N(t) + 1)/N(t) \to 1$ almost surely as $t \to \infty$ By the strong law of large numbers, $(\tau_1 + \cdots + \tau_{N(t)})/N(t) \to \mathbb{E}\tau_1 = \mu$ almost surely as $t \to \infty$. Then $(\tau_1 + \cdots + \tau_{N(t)+1})/N(t) \to \mu$ almost surely as $t \to \infty$. Since $N(t)/(\tau_1 + \cdots + \tau_{N(t)+1}) < N(t)/t \le N(t)/(\tau_1 + \cdots + \tau_{N(t)})$, we get $N(t)/t \to 1/\mu$ almost surely as $t \to \infty$. $\square$

Let R_i be the reward at the time of ith arrival which is assumed to be i.i.d. Let $R(t)$ be the accumulated rewards until time t, that is, $R(t) = \sum_{i=1}^{N(t)} R_i$.

Theorem 60. $R(t)/t \to \mathbb{E}R_i/\mathbb{E}\tau_i$ almost surely as $t \to \infty$.

Proof. By the strong law of large numbers,

$$\frac{R(t)}{t} = \frac{R(t)}{N(t)} \times \frac{N(t)}{t} \to \mathbb{E}R_1 \times \frac{1}{\mathbb{E}\tau_i} \quad \text{almost surely as} t \to \infty.$$

$\square$

Note. Heuristically, the average reward can be written as

$$\text{reward/time} = \frac{\text{expected reward/cycle}}{\text{expected time/cycle}}$$

Example 47. If the mean interarrival time of customer at a coffee shop is $\mu = 0.2$ and the mean purchase of each customer is $v = 5$ and assume the time of service is zero for convenience, then the mean purchase per unit time is $v/\mu = 5/0.2 = 25$.

Example 48 (Alternating renewal processes)**.** Let $S_1, S_2, \ldots$ be i.i.d. F with mean μ_F and $U_1, U_2, \ldots$ be i.i.d. G with mean μ_G. Consider a machine working for S_i unit time before needing a repair which takes U_i unit time. What is the limiting fraction of time in working status?

Let $N(t)$ be the renewal process with interarrival times $S_i + U_i$ and $R(t)$ be the accumulated reward until time t where S_i is the reward at ith renewal. Then the limiting reward is the limiting fraction of time in working status. Hence it is $R(t)/t \to \mathbb{E}R_i/\mathbb{E}\tau_i = \mathbb{E}S_i/\mathbb{E}(S_i + U_i) = \mu_F/(\mu_F + \mu_G)$.

Example 49 ($GI/G/1$ queue)**.** Suppose the interarrival time τ_i follows i.i.d. F with mean $1/\lambda$. The service time S_i of each customer follows i.i.d. G with mean $1/\mu$. Assume that $\lambda < \mu$.
Claim: If the queue starts with finite number of customers, say k, at time zero, then the queue will empty out almost surely.
Let $N(t)$ be the arrival time with $N(0) = k$. The claim can be written as $P(S_1 + \cdots + S_{N(t)} < t \text{ for some } t) = 1$. Note that $(N(t) - k)/t \to 1/\mu$, $(N(t) - k)/N(t) \to 1$ almost surely. Let $R(t)$ be the reward until time t defined by $R(t) = S_1 + \cdots + S_{N(t)}$. Then, $(R(t) - R(0))/t \to \mathbb{E}S_i/\mathbb{E}\tau_i = (1/\mu)/(1/\lambda) = \lambda/\mu < 1$ almost surely. Thus $R(t)/t = R(0)/t + (R(t) - R(0))/t \to 0 + \lambda/\mu < 1$ almost surely. Hence $P(R(t) < t \text{ for some } t) = 1$. Also we can conclude that the limiting fraction of time the server is busy is at least λ/μ.