Quiz #3. this Thursday. Material covered : 4.1–4.4 , 7.1–7.6 ( 7.2 & 7.3 )

## Linear first order systems

$$\vec{x}' = P(t)\vec{x} \quad \text{where } \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}$$

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n \end{aligned}$$

- If $\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}$ are solutions of $\vec{x}' = P(t)\vec{x}$ then $\vec{x}(t) = c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t)$ (*)

  is again a solution.

- Suppose we're looking at initial value Problem $\vec{x}' = P(t)\vec{x}$, $\vec{x}(t_0) = \vec{b}$ ( know: has unique sln. )

Then (*) satisfies $\vec{x}(t_0) = \vec{b}$ iff $c_1\vec{x}^{(1)}(t_0) + \cdots + c_n\vec{x}^{(n)}(t_0) = \vec{b}$

i.e. $\begin{cases} c_1x_1^{(1)}(t_0) + \cdots + c_nx_1^{(n)}(t_0) = b_1 \\ \vdots \\ c_1x_n^{(1)}(t_0) + \cdots + c_nx_n^{(n)}(t_0) = b_n \end{cases}$ is equation for $n$ unknowns $c_1, \cdots, c_n$

Coefficient matrix has $\vec{x}^{(1)}(t_0), \cdots, \vec{x}^{(n)}(t_0)$ as its columns.

Let $\Psi(t)$ be the $n \times n$-matrix with columns $\vec{x}^{(1)}(t), \cdots, \vec{x}^{(n)}(t)$

Then we get condition : $\Psi(t_0)\vec{c} = \vec{b}$ $\quad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

- By linear algebra, this has a unique sln provided $\det \Psi(t_0) \neq 0$.

Definition : The Wronskian of solutions $\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}$ is the function :

$$W[\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}](t) = \det(\Psi(t)) \quad \text{where } \Psi(t) = (\vec{x}^{(1)}(t), \cdots, \vec{x}^{(n)}(t)).$$

Fact : $W(t_0) \neq 0$ for some $t_0 \iff W(t) \neq 0$ for all $t$.
$\iff \vec{x}^{(1)}, \cdots, \vec{x}^{(n)}$ are linearly independent.

- In this case, $\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}$ are a fundemental set of solutions, and $\Psi(t)$ is called fundamental matrix.
- The general solution of $\vec{x}' = P(t)\vec{x}$ is then $\vec{x} = c_1\vec{x}^{(1)} + \cdots + c_n\vec{x}^{(n)}$.
- The general solution of inhomogeneous $\vec{x}' = P\vec{x} + \vec{g}$ is the general solution of $\vec{x}' = P\vec{x}$ plus

a particular solution of $\vec{x}' = P\vec{x} + \vec{g}$.

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Linear systems with constant coefficients.

$\vec{X}' = A\vec{X}$ where $A$ fixed $n\times n$ matric.

Trial solution: $\vec{x}(t) = e^{rt}\vec{\xi}$ (psy) $\vec{\xi} = \begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix}$

$\vec{X}' = A\vec{x} \Rightarrow re^{rt}\vec{\xi} = e^{rt}A\vec{\xi} \Rightarrow r\vec{\xi} = A\vec{\xi}$

$\Rightarrow \vec{X} = e^{rt}\vec{\xi}$ is a solution iff $\vec{\xi}$ is an eigenvector of $A$, with eigenvalue $r$.

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Example: $x' = 4x - 3y$.
$y' = -x + 2y$

eigenvectors, eigenvalues.

to solve $A\vec{\xi} = r\vec{\xi} \Rightarrow (A - rI)\vec{\xi} = 0$

but know $\vec{\xi} \neq \vec{0} \Rightarrow$

$r$ eigenvalue $\Leftrightarrow \underbrace{\det(A - rI)}_{\text{char. polynomial.}} = 0$

Sln: $A = \begin{pmatrix}4 & -3\\ -1 & 2\end{pmatrix}$ $\vec{X}' = A\vec{x}$ $\vec{X} = \begin{pmatrix}x\\ y\end{pmatrix}$

$\det(A - rI) = \begin{vmatrix}4-r & -3\\ -1 & 2-r\end{vmatrix}$

$= (4-r)(2-r) - (-3)(-1)$

$= r^2 - 6r + 5 = (r-1)(r-5)$ has roots $r^{(1)} = 5$ $r^{(2)} = 1$

- Eigenvalue: $r^{(1)} = 5$ $A - 5I = \begin{pmatrix}-1 & -3\\ -1 & -3\end{pmatrix}$ $\begin{pmatrix}-1 & -3\\ -1 & -3\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ solve $\begin{pmatrix}c_1\\ c_2\end{pmatrix}$

  Eigenvector: $\vec{\xi}^{(1)} = \begin{pmatrix}-3\\ 1\end{pmatrix}$ or $\begin{pmatrix}3\\ -1\end{pmatrix}\cdots$

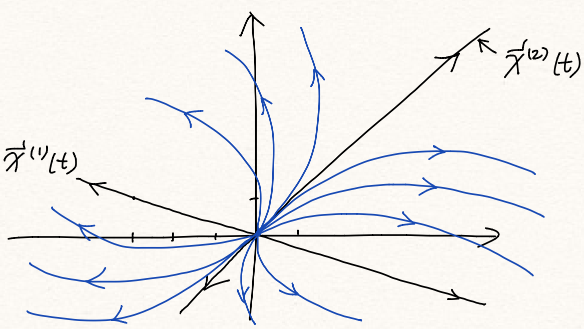
- Eigenvalue: $r^{(2)} = 1$ $A - I = \begin{pmatrix}3 & -1\\ -1 & 1\end{pmatrix}$ $\Rightarrow \vec{\xi}^{(2)} = \begin{pmatrix}1\\ 1\end{pmatrix}$ eigenvectors are non-zero vectors.

  $\vec{x}^{(1)} = e^{5t}\begin{pmatrix}-3\\ 1\end{pmatrix}$ $\vec{X}^{(2)} = e^{t}\begin{pmatrix}1\\ 1\end{pmatrix}$ linearly indep. $\Rightarrow$ fund. set of solution.

- Fundamental matrix $\psi(t) = \begin{pmatrix}-3e^{5t} & e^{t}\\ e^{5t} & e^{t}\end{pmatrix}$

- Wronskian: $W(t) = -4e^{6t}$

- Phase portrait



- General solution: $\vec{x}(t) = C_1 e^{5t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.