PS7 Sln

1. #19 $I = \iint_R \frac{x-2y}{3x-y}\,dA$

R is bounded between

$u = x - 2y = 0$

$u = 4$

$v = 3x - y = 1$

$v = 8$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\uparrow A$

$\mathbf{u} = A\mathbf{x}$ $\qquad \frac{\partial(u,v)}{\partial(x,y)} = A$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det(A^{-1})\right| = \left|\frac{1}{\det A}\right|$$

$$= \frac{1}{5}$$

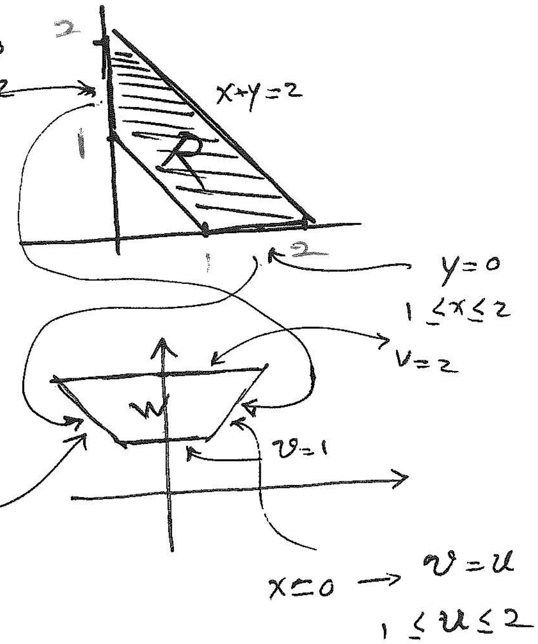
So $I = \iint_{W = AR} \frac{1}{5}\frac{u}{v}\,du\,dv = \frac{1}{5}\int_1^8 \int_0^4 \frac{u}{v}\,du\,dv$

$$= \frac{1}{5}\left[\int_1^8 \frac{dv}{v}\right]\left[\int_0^4 u\,du\right] =$$

$$\frac{1}{5}\left(1 - \frac{1}{64}\right)(8) = \left(8 - \frac{1}{8}\right)\frac{1}{5}$$

$$= \frac{63}{40}$$

#21 $I = \iint_R \cos \frac{y-x}{y+x} \, dA$



let $u = y - x$ Then $u + v = 2y$

$v = y + x$ $\quad v - u = 2x$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$I = \left| \frac{1}{\det A} \right| \int_{v=1}^{2} \int_{u=-v}^{v} \cos \frac{u}{v} \, du \, dv = \frac{1}{2} \int_1^2 \left[v \sin \frac{u}{v} \right]_{u=-v}^{u=v} dv$$

$$= \frac{1}{2} \int_1^2 v \left[\sin(1) - \sin(-1) \right] dv = \frac{1}{2} \int_1^2 v \, 2 \sin 1 \, dv$$

$$= \sin 1 \int_1^2 v \, dv = \sin 1 \left. \frac{v^2}{2} \right|_1^2 = \sin 1 \left(2 - \frac{1}{2} \right)$$

2.(a) $$\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} e^{-r^2}\,r\,dr\,d\theta = \left[\int_0^{2\pi} d\theta\right]\left[\int_{r=0}^{a} r e^{-r^2}\,dr\right]$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= 2\pi\ \frac{1}{2}\left[e^{-r^2}\right]_0^a = \pi\left(e^{-a^2}-1\right)$$ so $$\iint_{R^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\pi\left(e^{-a^2}-1\right)$$

$$= \pi$$

b) $$\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{y=-a}^{a}\int_{x=-a}^{a} e^{-x^2}\cdot e^{-y^2}\,dx\,dy = \left[\int_{-a}^{a} e^{-x^2}dx\right]\left[\int_{-a}^{a} e^{-y^2}dy\right]$$

So $$\iint_{\mathbb{R}^2} = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \pi$$ so

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

c) let $u = \dfrac{x}{\sqrt{2}\,\sigma}$ Then $e^{-u^2}du = e^{-\frac{x^2}{2\sigma^2}}\dfrac{dx}{\sqrt{2}\,\sigma}$ so

so $x = \sqrt{2}\cdot\sigma u$

$dx = \sqrt{2}\,\sigma\,du$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{\sqrt{2}\sigma}}dx = \int_{-\infty}^{\infty}\sqrt{2}\,\sigma\, e^{-u^2}du =$$

$$\sqrt{2}\,\sigma\cdot\sqrt{\pi} = \sqrt{2\pi}\,\sigma.$$

3 (a) of course the obvious way was

$$\int_0^{\frac{\pi}{2x}} x \sin xt \, dt = -\cos xt \Big|_0^{\frac{\pi}{2x}} = 1 - \cos\frac{\pi}{2} = 1$$

and $\frac{d}{dx} 1 = 0$ But this unfortunate problem was designed to be an opportunity to use Thm 4.47:

let $G(x_1, x_2, x_3) = \int_0^{x_1} x_2 \sin x_3 t \, dt$

$$\frac{dG}{dx} = x_2 \sin x_3 x_1 \cdot \frac{dx_1}{dx} + \int_0^{x_1} \sin x_3 t \, dt \underbrace{\frac{dx_2}{dx}}_{=1} + \int_0^{x_1} x_2 t \cos x_3 t \, dt \underbrace{\frac{dx_3}{dx}}_{=1}$$

$$= x \sin x \frac{\pi}{2x} \cdot \underbrace{\frac{dx_1}{dx}}_{=\frac{-\pi}{2x^2}} + \cancel{\int_0^{x_1} \sin x_3 t \, dt} + \underbrace{\int_0^{x_1} \underset{u}{x t} \cos xt \, dt}_{\text{by parts}}$$

$$t \sin xt \Big|_0^{x_1} - \int_0^{x_1} \sin xt \, dt$$

$$= -\frac{\pi}{2x} + t \sin xt \Big|_0^{\pi/2x} = \frac{-\pi}{2x} + \frac{\pi}{2x} = 0$$

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b) $\int_0^\infty e^{-ax} dx = \lim_{A \to \infty} -\frac{1}{a} e^{-ax} \Big|_0^A = \frac{1}{a}$

$\frac{d}{da} \int_0^\infty e^{-ax} dx = \frac{d}{da} \frac{1}{a}$ so $\int_0^\infty \frac{d}{da} e^{-ax} dx = \frac{-1}{a^2}$

$\int_0^\infty -x e^{-ax} dx = -\frac{1}{a^2}$

Similarly $\int_0^\infty x^2 e^{-ax} dx = \frac{2}{a^3}$ and $\int_0^\infty -x^3 e^{-ax} dx = \frac{-3!}{a^4}$