Jan 29-th

Problem Set II

Q6. Let $T: V \to W$ is a bijective transformation. Prove that if $\{v_1, \dots, v_n\}$ is a basis for V, then $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.

Solution: We claim that $\{T(v_1), \dots, T(v_n)\}$ is L.I.

Suppose that $a_1, \dots, a_n \in F$ and $a_1 T(v_1) + a_2 T(v_2) + \dots + a_n T(v_n) = 0$

We want to show that $a_1 = a_2 = \dots = a_n = 0$

$0 = T(a_1 v_1 + a_2 v_2 + \dots + a_n v_n)$

$0 = T(0)$ (injective)

...

Claim: $T(v_1), \dots, T(v_n)$ span W

Suppose that $w \in W$. We want to write a linear combination of $T(v_1), \dots, T(v_n)$.

Find $a_1, \dots, a_n \in F$ such that $w = a_1 T(v_1) + \dots + a_n T(v_n)$.

(T surjective means $\forall u \in W$, $\exists v \in V$ s.t. $T(v) = u$)

Since T is surjective, $\exists v \in V$ such that $T(v) = w$.

$v = a_1 v_1 + \dots + a_n v_n$ for some $a_1, \dots, a_n \in F$.

...

Problem Set I

Q6.

$T: \mathbb{R}^3 \to \mathbb{R}^2$

α basis for $\mathbb{R}^3$

β basis for $\mathbb{R}^2$

$$[T]_\alpha^\beta = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$[T(x,y,z)]_\beta = [T]_\alpha^\beta [(x,y,z)]_\alpha$$

PS II, Q7

V, W vector space over F.

$\alpha = \{v_1, \dots, v_n\}$ basis for V

$\beta = \{w_1, \dots, w_m\}$ basis for W

$T: V \to W$ lin. trans.

(a). Prove that T is surjective $\iff$ the column of $[T]_\alpha^\beta$ span F^m. Assume that T is surj.

Sps that $x \in F^m$. We want to write x as a linear combination of the columns on $[T]_\alpha^\beta$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Consider $w \in W$ defined by $w = x_1 w_1 + \dots + x_n w_n$ (Note: $[w]_\beta = x$)

Since T is surj. there exists $v \in V$ s.t. $T(v) = w$

$x = [w]_\beta = [T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$

$[T]_\alpha^\beta = [\underbrace{c_1, \dots, c_n}_{\text{columns}}]$

$$[v]_\alpha = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad b_1, \dots, b_n \in F$$

$$x = [T]_\alpha^\beta [v]_\alpha = [c_1, \cdots, c_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b_1 c_1 + \cdots + b_n c_n$$

Change of basis

$$[T]_{\alpha'}^{\beta'} = [I]_\beta^{\beta'} [T]_\alpha^\beta [I]_{\alpha'}^\alpha$$

$T: V \to W$ linear trans.
α, α' bases of V
β, β' bases of W.