FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 6

Question 1

The present value of two payments of \$100 each to be made at the end of n years and 2n years is \$100. If i=0.08, find n.

Solution

$100 = 100v^n + 100v^{2n} \Rightarrow 1 = v^n + v^{2n}$

$(v^n)^2 + v^n - 1 = 0$

$v^n = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -1}}{2 \times 1} = \dfrac{-1 + \sqrt{5}}{2} = 0.61803$ (ignore negative root)

$1.08^{-n} = 0.61083$

$n = \dfrac{-\ln(0.61803)}{\ln(1.08)} \cong 6.25$ years

Question 2

Smith buys 100 shares of stock ABC at the same time Brown buys 100 shares of stock XYZ. Both stocks are bought for 10 per share. Smith receives a dividend of 0.80 per share, payable at the end of each year, for 10 years, at which time (just after receiving the 10^{th} dividend) he sells his stock for 2 per share. Smith invests his dividends at annual rate 6%, and invests the proceeds of the sale of his stock at the same rate. Brown receives no dividends for the first 10 years, but starts receiving annual dividends of 0.40 per share at the end of 11 years. Brown also invests his dividends in an account earning 6%. If Brown sells his shares n years after purchase, what should be the sale price in order that his accumulated investment matches that of Smith, for each of n=15, 20 and 25?

Solution

The accumulated value for Smith at the time of sale of Smith's shares is:

$AV_{10} = 80s_{\overline{10}|0.06} + 200$

The accumulated value at $n = 15$ is: $AV_{15} = \left(80s_{\overline{10}|0.06} + 200\right)(1.06)^5 = 1678.755$

The accumulated value at $n = 20$ is: $AV_{20} = \left(80s_{\overline{10}|0.06} + 200\right)(1.06)^{10} = 2246.553$

The accumulated value at $n = 25$ is: $AV_{25} = \left(80s_{\overline{10}|0.06} + 200\right)(1.06)^{15} = 3006.395$

If P is the proceeds from the sale of Brown's shares, then the accumulated value for Brown at each time is:

$n = 15 \rightarrow AV_{15} = 40 s_{\overline{5}|0.06} + P = 225.4837 + P$

$n = 20 \rightarrow AV_{20} = 40 s_{\overline{10}|0.06} + P = 527.2318 + P$

$n = 25 \rightarrow AV_{25} = 40 s_{\overline{15}|0.06} + P = 931.0388 + P$

Equate the accumulated values for Smith and Brown and solve for P:
$n = 15 \rightarrow 225.4837 + P = 1678.755 \Rightarrow P \cong 1453$
$n = 20 \rightarrow 527.2318 + P = 2246.553 \Rightarrow P \cong 1719$
$n = 25 \rightarrow 931.0388 + P = 3006.395 \Rightarrow P \cong 2075$

The amount per share can be found by dividing by 100.

Question 3
An investor is to pay \$800 for a property. This property will return rent payments at the end of each year for 99 years. For the first 33 years the rental income is at a constant rate, increasing to double that rate for the next 33 years and triple the initial rate for the final 33 years. The property is expected to have a value of \$250,000 at the end of the 99 years.

If the investor expects a return of 8% p.a. on the investment, calculate the value of the rent payable in the first year.

Solution
We are aiming to solve for *X*, the rental payment in the first year, using the following equation of value:

$800 = X a_{\overline{33}|} + 2X a_{\overline{33}|} v^{33} + 3X a_{\overline{33}|} v^{66} + 250{,}000 v^{99}$

$= 11.513888X + 1.816637X + 0.214967X + 122.74059$

$677.25941 = 13.545492X$

$X = \$50.00$

Question 4
Jones invests \$1,000 at the end of each year for 15 years. If he knows that the present value of these payments is \$8,400, find the effective annual interest rate, using Linear Interpolation, with $i_1 = 0.08$ and $i_2 = 0.09$.

Solution
Need to solve the following equation:

$i_0 \cong i_1 + \dfrac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cdot (i_2 - i_1)$ with $i_1 = 0.08$, $i_2 = 0.09$, $f(i) = 1000 a_{\overline{15}|i}$, $f(i_0) = 8400$

$f(0.08) = 8559.48$

$f(0.09) = 8060.69$

$$i_0 \cong 0.08 + \frac{8400 - 8559.48}{8060.69 - 8559.48} \cdot (0.09 - 0.08)$$

$$i_0 \cong 0.0832$$

Test $1000a_{\overline{15}|0.0832} = 8394.74$

<u>Question 5</u>

On January 1, 2013, Smith deposits 500,000 in an account earning a monthly effective rate of 1%, with interest credited on the last day of each month. Withdrawals are made on the first day of each month starting February 1, 2013, with an initial withdrawal of 1000. Each subsequent withdrawal is 1% larger than the previous one, continuing in this pattern for as long as possible.

(a) When does the account finally become exhausted?

(b) What is the amount of the last regular withdrawal?

<u>Solution</u>

(a) $500{,}000 = 1000\left(v + v^2(1.01) + v^3(1.01)^2 + \ldots + v^n(1.01)^{n-1}\right)$

$$500{,}000 = v1000\left(1 + v(1.01) + v^2(1.01)^2 + \ldots + v^{n-1}(1.01)^{n-1}\right)$$

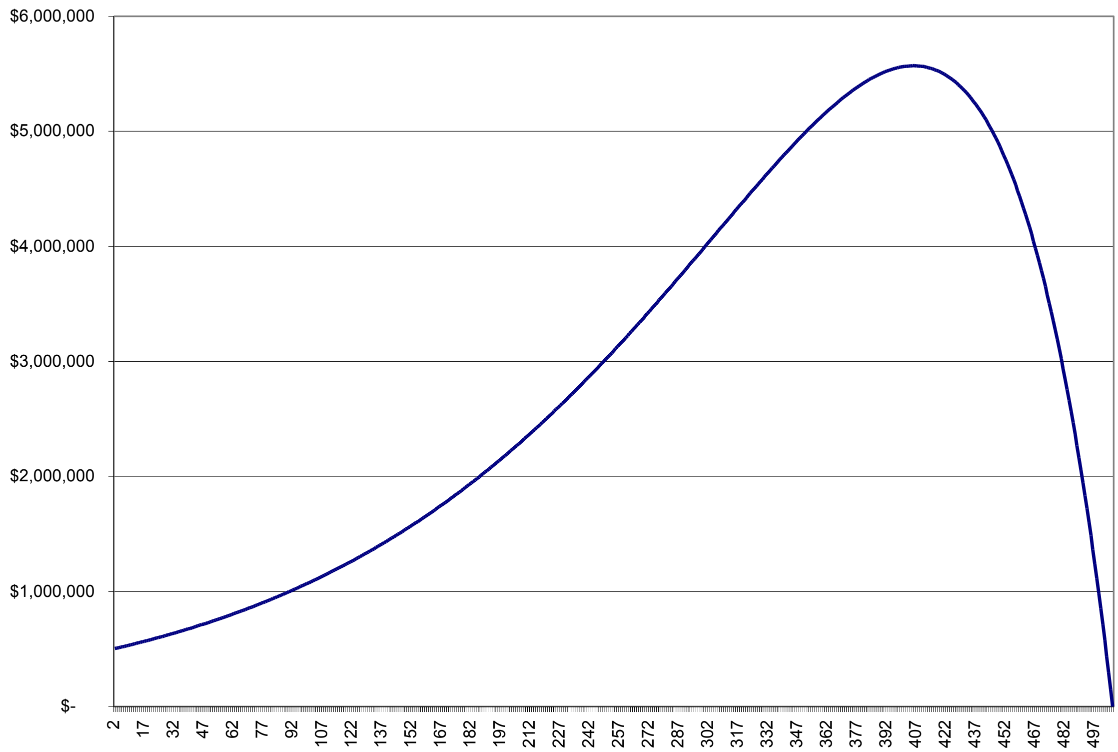
$$500{,}000 = v1000(1 + 1 + 1 + \ldots + 1) \qquad \text{since } v = 1.01^{-1}$$

$$500{,}000 = v1000n$$

$$\Rightarrow n = \frac{500000(1.01)}{1000} = 505 \text{ months}$$

(b) This is just $1000(1.01)^{504} = \$150{,}651.13$

As an extra note, the graph below shows how the balance changes over time.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