Today – §1.5 examples and classification
12 October – begin the simplex method (§2.1)

Recall: from linear algebra: If A is an $m \times n$ matrix, the row rank of A is the maximum number of linearly independent rows of A ($\leq m$). A's column rank is the maximum number of linearly independent columns ($\leq n$).

Theorem: row rank = column rank

Eg. The rank of the system

$$2x_1 + 3x_2 + x_3 = 5$$
$$x_1 + 2x_2 + x_4 = 3$$
$$3x_1 + 5x_2 + x_3 + x_4 = 8$$

is the rank of

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 1 \end{bmatrix}$$ which is only

2: 3^{rd} row $= 1^{st}$ row $+ 2^{nd}$ row

That is, rank $< m$ where $m = 3$.
So the system has no basic solutions (if $x_{i_1}, x_{i_2}, x_{i_3}$ were basic variables, the i_1^{st}, i_2^{nd}, i_3^{rd} columns would have to be linearly indep.

Geometrical fact: If S is the solution set of the canonical constraints.

$Ax = b$, $x \geq 0$ and S is non-empty, then S has an extreme point.

Remark: The system of the last example has at least one solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix} \begin{matrix} \\ \\ \geq 0 \\ \geq 0 \end{matrix}$

So the system

$$2x_1 + 3x_2 + x_3 = 5$$
$$x_1 + 2x_2 + x_4 = 3$$
$$3x_1 + 5x_2 + x_3 + x_4 = 8$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

has a solution set, including an extreme point (but no basic solution).

Theorem 1.8 and 1.9

Suppose A is $m \times n$ and has rank m. and let S denote the solution set of $\begin{matrix} Ax=b \\ x \geq 0 \end{matrix}$

Then x is an extreme point of S if and ~~if~~ only if x is a basic feasible solution of $\begin{matrix} Ax=b \\ x \geq 0 \end{matrix}$

Note Theorem 1.8 is "if"
1.9 is "only if"

Eg. To solve the problem

Maximize $z = x+y$

$2x+3y \leq 5$

$x+2y \leq 3$

$x \geq 0, y \geq 0$

In canonical form (u and v are slacks)

Maximize $z = x+y$ s.t.

$2x+3y+u=5$

$x+2y+v=3$

$x \geq 0, y \geq 0, u \geq 0, v \geq 0$

Coefficient matrix

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$ has rank 2

To find all basic solutions of equality constraints.

① u and v basic (x and y are non-basic; therefore $=0$)
Then $u=5$, $v=3$ and $\begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix}$ is basic.

② y and v basic ($x=0, u=0$ (non-basic))
$\left.\begin{matrix} 3y=5 \\ 2y+v=3 \end{matrix}\right\}$ $y=\frac{5}{3}, v=-\frac{1}{3}$ $\begin{bmatrix} 0 \\ \frac{5}{3} \\ 0 \\ -\frac{1}{3} \end{bmatrix}$ is basic

③ with x and v basic $(y=0, u=0)$:

$$\begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

④ with y and u basic $(x=0, v=0)$

$$\begin{bmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

⑤ with x and u basic:

$$\begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

⑥ with x and y basic

Solve $\left.\begin{aligned} 2x+3y&=5 \\ x+2y&=3 \end{aligned}\right\}$ $x=1, y=1$

so $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is basic

Discard the unfeasible solutions to get

$$\begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{extreme points}$$

Assuming the problem has an optimal solution, the extreme points theorem says an optimal solution is at one of the extreme points.

Test $z=x+y$ at each one: $0, \frac{5}{2}, \frac{3}{2}$ and 2 $\leftarrow$ corresponds to $\begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$

so $\begin{bmatrix} \vdots \end{bmatrix}$ is an optimal solution of the problem.