RESEARCH SCHOOL OF FINANCE, ACTUARIAL STUDIES AND APPLIED STATISTICS

INTRODUCTORY MATHEMATICAL STATISTICS (STAT2001)
PRINCIPLES OF MATHEMATICAL STATISTICS (STAT6039)

Final Examination – Semester 1, 2014

Study Period: 15 minutes

Time Allowed: 3 hours

Permitted Material: No restrictions

- *Undergraduate students (those enrolled in STAT2001) should attempt only the first six problems. For these students, the exam is out of 180 marks.*

- *Graduate students (those enrolled in STAT6039) should attempt all seven problems. For these students, the exam is out of 200 marks.*

- *Draw a box around each solution and express each numerical solution in decimal form to at least four significant digits (e.g.* $\boxed{0.007204}$*). Start your solution to each problem on a new page.*

- *To ensure full marks, show all the steps in working out your solutions. Marks may be deducted for not showing appropriate calculations or formulae, or for not clearly referencing the results in the text book or course material which you are using.*

Problem 1 ***(40 marks in total)***

Whenever a Komto car is manufactured, it has probability p of being Type B, and probability $1-p$ of being Type A, independently of all other Komto cars.

A Type A Komto car is completely reliable and never breaks down.

The number of times that a Type B Komto car breaks down in any one-year period follows a Poisson distribution with mean λ, independently of all other such cars.

A random sample of n = 5 Komto cars was observed over the last calendar year. Let $y=(y_1,...,y_n)$ be the vector of the numbers of times that these n cars broke down.

(a) Find the method of moments estimates of p and λ if y = (1, 0, 5, 1, 3). ***(10 marks)***

(b) Find the maximum likelihood estimates of p and λ if y = (1, 1, 4, 1, 3). ***(10 marks)***

(c) Suppose that p = 0.6 and λ = 0.2. Find the mean, variance and mode of the number of Type B cars in the sample, given that y = (1, 0, 2, 0, 0). ***(20 marks)***

Problem 2 ***(30 marks in total)***

Suppose that $Y \sim Bin(2,1/2)$ and $U \sim U(0,1)$, where $Y \perp U$.
Then define two new random variables $W = UY$ and $X = U + Y$.

(a) Calculate the value of the covariance between W and X. ***(10 marks)***

(b) Derive and sketch the probability density function of X. ***(10 marks)***

(c) Derive and sketch the cumulative distribution function of W. ***(10 marks)***

Note: For part (c), be sure to indicate precisely where on the real line the required function equals zero, where it equals one, and where it is undefined.

Problem 3 ***(20 marks in total)***

Suppose that $Y \sim Bin(n, p)$.

(a) Find a simple closed-form formula for $\lambda = E\left(\frac{1}{Y+1}\right)$

and evaluate this formula for the case n = 20 and p = 0.1. ***(10 marks)***

(b) Find a simple closed-form formula for $\eta = E(Y \mid 0 < Y < n)$

and evaluate this formula for the case n = 10 and p = 0.7. ***(10 marks)***

Problem 4 ***(40 marks in total)***

Consider n events, $A_1, ..., A_n$, such that $P(A_i) = p$ for all i = 1,...,n.

Then define X as the number of these n events which actually occur.

For each of the following assumptions, derive a simple closed-form formula for the variance of X and evaluate this formula for the case n = 30 and p = 0.02:

(a) Assume that the n events are independent. ***(10 marks)***

(b) Assume that the n events are disjoint. ***(10 marks)***

(c) Assume that $P(A_i A_j) = \dfrac{p \times I\left((i-j)^2 < 2\right)}{2 - I\left(i = j\right)} \;\; \forall\, i, j \in \{1, ..., n\}$. ***(20 marks)***

Note: In part (c), I denotes the standard indicator function, such that, for any event B, $I(B) = 1$ if B occurs, and $I(B) = 0$ if B does not occur.

Problem 5 ***(30 marks in total)***

Suppose that $U \sim U(0,1)$. Find and sketch the probability density function of:

(a) $Y = \dfrac{U+1}{U-1}$ ***(10 marks)***

(b) $X = \dfrac{U+1/2}{U-1/2}$. ***(20 marks)***

Problem 6 ***(20 marks in total)***

A standard six-sided die will be rolled a number of times which is to be determined. Find a minimum number of times the die should be rolled if the total of the numbers which will come up is required to be greater than 5000 with a probability that is:

(a) approximately 97.5% ***(10 marks)***

(b) definitely at least 97.5%. ***(10 marks)***

Problem 7 ***(To be done only by STAT6039 students)*** ***(20 marks in total)***

A simple linear regression model with independent and identically distributed normal errors is believed to fit the following data: $(x_1, y_1),...,(x_n, y_n)$ = (0,0), (0,1), (2,0), (2,3).

(a) Calculate point estimates of the unknown parameters in the model. ***(10 marks)***

(b) Predict the average of two future y-values with covariate values of 1 and 2, respectively, where these y-values are independent of one another and of the y-values in the sample. Also calculate a 95% prediction interval for that average. ***(10 marks)***

Note: The upper 0.025 quantiles of the t distributions with 1, 2, 3, 4 and ∞ degrees of freedom are 12.706, 4.303, 3.182, 2.776 and 1.960, respectively.

END OF EXAMINATION