Mat 337 Midterm 2 solutions

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March 21, 2014

Problem 1 (a) (10 points) If you had never seen this example before it would be hard to come up with it during a test:

$$f(x) = \begin{cases} \frac{1}{b} & x \in \mathbb{Q}, x = \frac{a}{b}, \gcd(a,b) = 1 \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

$0 = 0/1$, so $f(0) = 1$.[1]

Something like

$$\begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

does not work. It is not continuous at all irrational points, and thus doesn't satisfy the conditions that it has to satisfy..

(b) (20 points) The simplest way to do this question is to figure out a way to write $\max\{x, y\}$:

$$\max\{x, y\} = \frac{|x-y| + x + y}{2}.$$

Once we've figured this out it is straightforward to prove that $f(x, y) = \max\{x, y\}$ is continuous at every point in $\mathbb{R}^2$.

We can also do the problem without using the above formula. Let $(x_0, y_0) \in \mathbb{R}^2$ and let $\epsilon > 0$. Let $\delta = \epsilon$. $\|(x, y) - (x_0, y_0)\| < \epsilon$ means that $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \epsilon$, and this implies that

$$x_0 - \epsilon < x < x_0 + \epsilon$$

and

$$y_0 - \epsilon < y < y_0 + \epsilon.$$

Because $x < x_0 + \epsilon$ and $y < y_0 + \epsilon$, we get

$$f(x, y) = \max\{x, y\} < \max\{x_0 + \epsilon, y_0 + \epsilon\} = f(x_0, y_0) + \epsilon,$$

[1] The reason this function is continuous at irrationals is because if x is irrational, then although every neighborhood of x contains rational numbers, the smaller we make the neighborhood the larger the denominators of these rationals have to be, and thus we can make f arbitrarily small in a sufficiently small neighborhood of an irrational point. This is just an explanation, not a detailed argument.

and because $x > x_0 - \epsilon$ and $y > y_0 - \epsilon$, we get

$$f(x,y) = \max\{x,y\} > \max\{x_0 - \epsilon, y_0 - \epsilon\} = f(x_0,y_0) - \epsilon,$$

and therefore

$$|f(x,y) - f(x_0,y_0)| < \epsilon.$$

This shows that f is continuous at (x_0,y_0), and because (x_0,y_0) was an arbitrary point in $\mathbb{R}^2$, this shows that f is continuous on $\mathbb{R}^2$.

Problem 2 (a) (15 points) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Because $[0,1]$ is compact and f is continuous on $[0,1]$ (take that for granted; it is straightforward to prove), f is uniformly continuous on $[0,1]$. (A continuous function on a compact set is uniformly continuous: Theorem 5.5.9, p. 86, but you don't have to cite the theorem number when you use this fact.) Thus, if we can show that f is not Lipschitz then this will be a counterexample.

Suppose by contradiction that f were Lipschitz, with Lipschitz constant K. Then for all $x > y > 0$,

$$K \geq \frac{|f(x) - f(y)|}{|x-y|} = \frac{\sqrt{x} - \sqrt{y}}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} = \frac{1}{\sqrt{x} + \sqrt{y}} > \frac{1}{2\sqrt{x}},$$

i.e. $\sqrt{x} > \frac{1}{2K}$, i.e. $x > \frac{1}{4K^2}$. But for $0 < x \leq \frac{1}{4K^2}$ this is false, and it was claimed to be true for all $x > 0$, so this shows that f is not Lipschitz. Therefore, it is not true that any uniformly continuous function is Lipschitz.

(b) (15 points)

$$f(x,y) = \begin{pmatrix} 2 & 3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

f is linear, and it is a fact that a linear function is Lipschitz.

If you forgot that a linear function is Lipschitz, you can instead prove that f is Lipschitz directly.

(c) (15 points) If the absolute value of the derivative of a function f is bounded by K, then $|f(x) - f(y)| \leq K|x-y|$, i.e. f is Lipschitz. Now, $(\sin x)' = \cos x$ and $|\cos x| \leq 1$, so $\sin x$ is Lipschitz.

If you forgot the fact about the derivative of a function being bounded, you can also prove that $\sin x$ is Lipschitz directly, but it will be uglier.

Problem 3. (30 points) To prove that $f(C)$ is compact, let U_α be open sets in Y such that $f(C) \subseteq \bigcup U_\alpha$. Then

$$C = f^{-1}(f(C)) \subseteq \bigcup f^{-1}(U_\alpha).$$

(This is a general fact about taking the inverse images of any union of sets.) Because f is continuous, each of the sets $f^{-1}(U_\alpha)$ is open in X. Because C is compact and $C \subseteq \bigcup f^{-1}(U_\alpha)$ (this is an open cover of C) there are finitely many $U_{\alpha_1}, \ldots, U_{\alpha_n}$ such that $C \subseteq \bigcup_{k=1}^n f^{-1}(U_{\alpha_k})$. Then, applying f to both sides of this,

$$f(C) \subseteq f\left(\bigcup_{k=1}^n f^{-1}(U_{\alpha_k})\right) = \bigcup_{k=1}^n U_{\alpha_k}.$$

(The fact that f applied to the union is equal to the other union is not obvious, but is a general fact about sets and functions and does not involve the sets being open or f being continuous.) We had an arbitrary cover $f(C) \subseteq \bigcup U_\alpha$, and we have proved that $f(C) \subseteq \bigcup_{k=1}^n U_{\alpha_k}$, i.e. we proved that every opeen cover of $f(C)$ has a finite subcover. This shows that $f(C)$ is compact.

We can also prove this using sequential compactness: a metric space is compact if and only if every sequence has a convergent subsequence. Let $f(x_n)$ be a sequence in $f(C)$; we have to show that it has a convergent subsequence. Since x_n is a sequence in C and C is compact, x_n has a convergent subsequence $x_{a(n)} \to x \in C$. But then because f is continuous, we get $f(x_{a(n)}) \to f(x)$, and $x \in C$ so $f(x) \in f(C)$. This shows that $f(x_{a(n)}$ is a convergent subsequence of $f(x_n)$, and hence that $f(C)$ is compact.

Problem 4. (40 points) $f^{-1} : f(X) \to X$.

Let $y_n \to y \in f(X)$. We want to show that $f^{-1}(y_n) \to f^{-1}(y)$. Since f is one to one, there are unique $x_n \in X$ and $x \in X$ such that $y_n = f(x_n)$ and $y = f(x)$, i.e. $x_n = f^{-1}(y_n)$ and $x = f^{-1}(x)$, and we want to prove that $x_n \to x$. Because X is compact, the sequence x_n has at least one limit point. If we can show that every limit point of this sequence is equal to x, then it follows that the sequence converges to x.[2] Say there is a subsequence $x_{a(n)}$ that converges to A; our goal is to prove that $A = x$. But this would give us $f(x_{a(n)}) \to f(A)$, i.e. $y_{a(n)} \to f(A)$. Because the sequence y_n converges to y, any subsequence of y_n converges to y. This means that $f(A) = y$. But f is one to one, so applying f^{-1} to both sides gives us $A = f^{-1}(y) = x$. This shows that x is the only limit point of x_n, hence that $x_n \to x$, and this shows that f^{-1} is continuous.

Problem 5. (45 points) The difficulty of this question is fidning a way to use the intermediate value theorem. To be able to use it, we have to write the problem in terms of a function from $\mathbb{R}$ to $\mathbb{R}$, rather than a function from S^1 to $\mathbb{R}$. This is a clever idea, so don't feel bad for not thinking of it, but you have seen it now so add it to your memory.

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(\theta) = f(\cos\theta, \sin\theta) - f(-\cos\theta, -\sin\theta).$$

We have

$$\begin{aligned}
g(\theta+\pi) &= f(\cos\theta\cos\pi - \sin\theta\sin\pi, \sin\theta\cos\pi + \sin\pi\cos\theta) \\
&\quad - f(-\cos\theta\cos\pi + \sin\theta\sin\pi, -\sin\theta\cos\pi - \sin\pi\cos\theta) \\
&= f(-\cos\theta, -\sin\theta) - f(\cos\theta, \sin\theta) \\
&= -g(\theta).
\end{aligned}$$

If $g(0) = 0$, then $f(1,0) - f(-1,0) = 0$, i.e. $f(1,0) = f(-1,0)$, showing that f is not one-to-one. If $g(0) \neq 0$, then $g(0)$ is either positive or negative, and $g(\pi) = -g(0)$, so one of $g(0)$ and $g(\pi)$ is positive and the other is negative, and hence by the intermediate value theorem there is some θ_0, $0 < \theta_0 < \pi$, for which

[2] This depends on X being compact. The sequence $x_n = 0$ if n is even and $x_n = n$ is n is odd is a sequence in $\mathbb{R}$ that has exactly one limit point, 0, but which does not converge.

$g(\theta_0) = 0$, and then $f(\cos\theta_0, \sin\theta_0) = f(-\cos\theta_0, -\sin\theta_0)$, showing that f is not one-to-one.

Problem 6. (30 points) Because $I = [a, b]$ is compact and $f : I \to \mathbb{R}$ is continuous, there is some c, $a \leq c \leq b$, such that $|f(c)| \leq |f(x)|$ for all $x \in I$. If $|f(c)| = 0$ then $f(c) = 0$, which is what we want to prove. Suppose by contradiction that $|f(c)| > 0$. By hypothesis there is some $y \in I$ such that $|f(y)| \leq \frac{1}{2}|f(c)|$, and because $|f(c)| > 0$ we have $|f(y)| < |f(c)|$. But this contradicts the fact that $|f(c)| \leq |f(x)|$ for all $x \in I$. Therefore $f(c) = 0$.

Problem 7. (30 points) $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of X, so in particular it is an open cover of $f([0,1])$. f is continuous, so each $f^{-1}(U_\lambda)$ is open in $[0,1]$. And

$$[0,1] = f^{-1}(f([0,1])) \subseteq f^{-1}\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda),$$

so $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of $[0,1]$. Because $[0,1]$ is a compact metric space, by the Lebesgue number lemma there is some $\delta > 0$ such that: for each subset S of $[0,1]$ of diameter less than δ, there is some $\lambda \in \Lambda$ for which $S \subseteq f^{-1}(U_\lambda)$. Let $n > \frac{1}{\delta}$. Then, each of the sets $[\frac{k-1}{n}, \frac{k}{n}]$, with $1 \leq k \leq n$, has diameter less than δ. Thus, taking $s_k = \frac{k}{n}$, for each k, $1 \leq k \leq n$, there is some $\lambda \in \Lambda$ such that $[s_{k-1}, s_k] \subseteq f^{-1}(U_\lambda)$, and hence applying f to both sides of this we get $f([s_{k-1}, s_k]) \subseteq f(f^{-1}(U_\lambda)) = U_\lambda$. This proves the claim.

Bonus problem. (50 points) First, the statement that there is some $0 < \alpha < 1$ such that, for all $x, y \in I$, we have $|f(x) - f(y)| \leq \alpha|x - y|$ tells us that $f : I \to I$ is Lipschitz, and hence is continuous. Suppose that the sequence x_n converges; we haven't proved this yet. Say x_n converges to the limit l. Then x_{n+1} also converges to the limit l. And f is continuous, so $f(x_n) \to f(l)$. But $f(x_n) = x_{n+1}$, so $x_{n+1} \to f(l)$. But remember that $x_{n+1} \to l$, so we get $f(l) = l$. So, if we prove that the sequence x_n converges then we will have completed the solution.

The bonus question is proving a case of the contraction mapping theorem. I checked on Wikipedia and the proof there is readable: `http://en.wikipedia.org/wiki/Banach_fixed_point_theorem`