Tutorial 1

Some Basic proof techniques

1, Pigeon Hole Principle
2, Induction
3, Double-counting
4, Proof by contradiction
5, Direct proof (of course)

1) Pigeon Hole Principle

"If there are more pigeons than holes, and all pigeons are to be put in the holes, there must be 1 hole with more than 1 pigeon".

More formal statement:

If A, B are sets s.t. $|A| > |B|$, then all functions from A to B is not 1-1.

Equivalently: "Max is at least as big as average"

If a set A, $|A| = n$, is to be partitioned into $m (< n)$ partitions then the partition with the maximum number of element have at least $\lceil \frac{n}{m} \rceil$ elements.

Examples -

1. At any point in time, two people in NY have exactly the same number of hair.

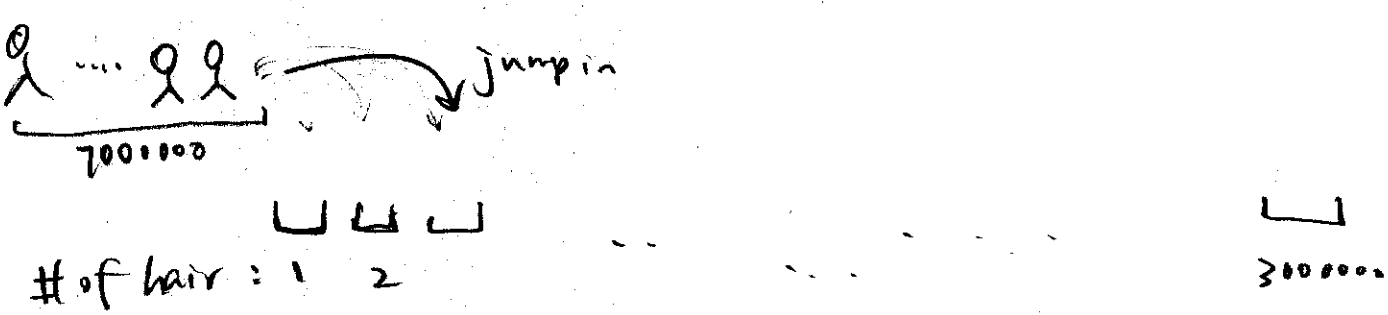
(i) Say each person has at most 1000 hair/sq inch on head and the biggest head has surface area less than that of a cube with side less than 20in.
so the total surface area of a head $\leq (20 \times 20 \times 6) in^2$
$< 3000\ in^2$

So hair on each person $< 3000 \times 1000$.

(ii) # of ppl in NY $> 7000\,000$

Let the hole be the # of hair on a person
pigeons be the ppl.

then at least 1 hole has more than 1 person in it



i.e. at least 2 ppl have the same # of hair.

2/ You are at a party with at least 1 other person
However big the party is, there will be
2 ppl who know the same number of other ppl.

proof Each person can know 0 to n-1 other ppl.
but if there is a person that does not
know anyone then there cannot be a person
who knows everyone.

pigeons: ppl
holes: # of ppl a person knows

Since either the "0" or the "n" hole is empty
then n pigeons are being put into n-1 hole
so use the pigeon hole principle to conclude.

3/ Any set of 19 distinct integers
chosen from the arithmetic progression

1, 4, 7, ..., 100
($a_{i+1} = 1+3i$)

contains a pair of distinct integers that
sum to 104.

proof: There are 34 integers that can be
partitioned into the following 18 subsets.

$\{1\}$,
$\underbrace{\{4, 100\}, \{7, 97\}, \ldots \{1+3i, 100-3i\}}_{16 \text{ pairs}}$
$\{52\}$

By the Pigeon hole Principle, at least 2 of the 19 distinct integers fall into the same partition, and that partition cannot be $\{1\}$, or $\{52\}$. The result follows.

Exercise

If n is odd, then for any permutation σ of $\{1, \ldots, n\}$, the product

$$P = (1-\sigma(1))(2-\sigma(2)) \cdots (n-\sigma(n))$$

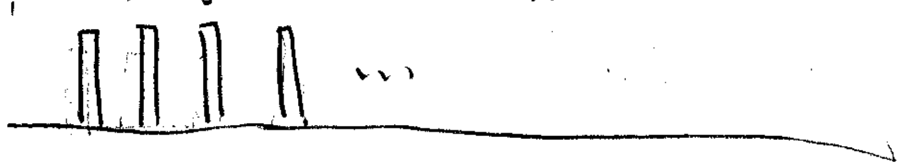
is even.

(2) Proof by Induction

Show that a statement S is true for all natural numbers $n \geq n_0$.

(Induction works on partially ordered sets in which all chains have a least elements too)

Like a domino, if you want the whole stack to fall, 2 things need to happen



(1) push the first tile:

$\Leftrightarrow$ base case: statement true for the smallest # n_0

(2).

"a falling tile can push the next tile"

$\Leftrightarrow$ Induction step:

If statement true for $n-1$ (or all numbers less than n)

then statement true for n.

Then all tiles fall! $\Leftrightarrow$ true for all $n \geq n_0$!

<u>Example</u> ATM machine with only \$2 coins and \$5 bills can handle all amounts $\geq$ \$4.

<u>proof</u> base case: $\$4 = \$2 \times 2$

Induction step: Suppose machine knows how to handle $\$n$ (≥ 4), then we need to show that the machine also knows how to handle $\$n+1$.

Case 1: $\$n$ contains at least 1 \$5.

i.e. $n = 2k + 5l$

where $l \geq 1$, $k \geq 0$

Then replace \$5 bill by 3 \$2.

i.e. $\$n+1 = (\$2)\cdot(k+3) + \$5\cdot(l-1)$

Case 2: \$ n is given by only \$2 coins

then replace 2 \$2 coins (there are at least 2 since $n \geq 4$) by a \$5 bill.

i.e. if \$ $n = \$2k$, $k \geq 2$

then \$ $n+1 = 2(k-2) + 5$

$\square$.