STAT2032/6046: Financial Mathematics

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Final Revision

Outline

Outline

Revision - basics

Interest: - compensation borrower of money pays the lender of money for its use.

The value of an investment changes over time due to interest - this is the **time value of money**.

Valuation of cashflows - must incorporate the effect of interest.

Revision - basics

For a single cashflow K at time t, and effective rate of interest i per period

	Compound Interest	**Simple Interest**
Accum. Val (at $t = n$)	$K(1+i)^{n-t}$	$K(1+i(n-t))$
Present Value (at t=0)	$K(1+i)^{-t} = Kv^t$	$\frac{K}{1+it}$

Accumulation - value cashflows at a future date in time, apply an accumulation factor.

Present value - value cashflows at t=0, apply discount factor to cashflows.

Revision - types of interest rates

1. **Effective interest rates** - interest is paid **once** in the specified period

   Example: Effective 6-month interest rate is 6% $\rightarrow$ \$100 invested now, will be worth \$106 in 6 months time

2. **Nominal interest rates**: Interest is paid **more** or **less** frequently than once per measurement period.

   Equivalence relationship between i and $i^{(m)}$

   $$\boxed{1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m}$$

   where i is the effective rate per annum

   $$\boxed{i > i^{(2)} > i^{(3)} > .... > \delta}$$

Revision - types of interest rates

3. **Discount rates**:
   (don't confuse with discount factor v)
   Interest - paid at the **end** of the period on the balance at the **beginning** of the period

   $$i = \frac{d}{1-d}$$

   Discount - paid at the **beginning** of the period on the balance at the **end** of the period

   $$d = \frac{i}{1+i}$$

Nominal discount rate:

$$\boxed{1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m}$$

$$\boxed{d < d^{(2)} < d^{(3)} < .... < \delta}$$

Revision - types of interest rates

4. **Force of interest**:
   For an effective annual rate of interest, the equivalent nominal rate of interest as the number of compounding periods m approaches infinity is called the **force of interest** δ. (In other words, the instantaneous rate of change of the investment value S(t)).

   Derive from first principles $\lim_{m\to\infty} d^{(m)} = \lim_{m\to\infty} i^{(m)} = \delta$

   $\delta_t = \frac{S'(t)}{S(t)} = \frac{\partial}{\partial t} \ln S(t)$.

Outline

Revision - Annuities

Distinguish between and value the cashflows of different types of annuities

- Annuity in arrears
- Annuity-due
- Deferred annuities
- Annuities payable more (or less frequently) than annually
- Perpetuities
- Continuous annuities
- Increasing annuities
- Decreasing annuities
- Indexed annuities
- Continuous varying annuities

Revision - Annuities

Formulas to value these annuities are in the formula sheet.

You should be able to derive all formulas from first principles as discussed in class.

Understand the relationship between the value of different types of annuities.

Revision - Equations of value

Equations of value set the value of all cash inflows equal to the value of all cash outflows at a common time point.

Analysing financial transactions involves constructing and solving equations of value.

Key step: Select a common reference time point t.

Example: If t=0, the equation of value is PV inflows=PV outflows.

Revision - Linear interpolation

$$x_0 \simeq x_1 + \frac{f(x_0) - f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1)$$

Approximation method when an analytic solution to the equation of value cannot be found.

Revision - Actuarial tables

Be familiar with what's available.

It may be more efficient to lookup pre-calculated values in the tables.

Revision - Inflation

$$(1 + i) = (1 + i_{real})(1 + r)$$

$$1 + i_{real} = \frac{1 + i}{1 + r}$$

$$i_{real} = \frac{i - r}{1 + r}$$

Real interest rate: the growth in investment value due to interest only, (that is, remove the effect of inflation).

Real cashflows (that is, with inflation effect removed, so all cashflows are in same units of purchasing power), should be valued at the real interest rate.

Outline

Revision - Loans

- Loan schedule - calculate all components (payment amount, interest, outstanding balance).
- Calculate repayment amount using amortisation method.
- Calculate outstanding balance - retrospective method and prospective method.
- Analyse loans with payments made m-thly.

Loan Schedule

t	Payment	Interest Due	Principal Repaid	Outstanding Balance
0				$L = OB_0$
1	K_1	$I_1 = i \cdot OB_0$	$PR_1 = K_1 - I_1$	$OB_1 = OB_0 - PR_1$
2	K_2	$I_2 = i \cdot OB_1$	$PR_2 = K_2 - I_2$	$OB_2 = OB_1 - PR_2$
...	...	...	...	...
t	K_t	$I_t = i \cdot OB_{t-1}$	$PR_t = K_t - I_t$	$OB_t = OB_{t-1} - PR_t$
...	...	...	...	...
n	K_n	$I_n = i \cdot OB_{n-1}$	$PR_n = K_n - I_n$	$OB_n = OB_{n-1} - PR_n$ $=0$

Amortisation method

How do we work out K_t , the repayment amount??

<u>Amortisation method</u>: set the present value of all amounts loaned out equal to the present value of all payments made to repay the loan. (that is, set up an equation of value and solve)

Outstanding Balance

- To determine outstanding balance at time t,

  Retrospective Method

$$OB_t = OB_0(1+i)^t - \sum_{a=1}^{t}(1+i)^{t-a}K_a$$

  Prospective Method

$$OB_t = vK_{t+1} + v^2K_{t+2} + ... + v^{n-t}K_n = \sum_{a=1}^{n-t} v^a K_{a+t}$$

Knowing OB_t at any time t is very important if you want to change the terms of your loan after t=0.

Outline

Capital Budgeting

Accumulated Profit

$AV(T) = \sum_t c_t(1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t}dt$

Net Present Value

$NPV(i) = \sum_t c_t(1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}dt$

Internal rate of return The risk discount rate that equates present value of income and outgo, that is, makes the NPV of the cashflows equal to zero.

Capital Budgeting

Link between NPV and IRR

Let $IRR = i_0$, and let i be the rate at which the company can borrow funds.

- Project is profitable if $i < i_0$
- Project is unprofitable if $i > i_0$
- Project breaks even if $i = i_0$

If the borrowing rate is less than the yield rate i_0, the project is profitable.

Capital Budgeting

Reinvestment rates
The borrowing rate is generally different from the rate at which proceeds can be invested at.

Discounted payback period
Length of time to break even from initial expenditure, allowing for the time value of money.

Outline

Measuring investment performance

Money weighted rate of return (MWRR)
Equivalent to internal rate of return.

Time weighted rate of return (TWRR)
Identifies the change in fund value due to interest credited and capital value change only

- Divide period into successive subintervals each time deposit/withdrawal is made. $(\mathbf{0 < t_1 < t_2 < ... < t_n = 1})$
- Calculate growth factor for each subinterval.
  $\mathbf{G_k = 1 + i_k = \frac{B_k}{A_k}}$
  $\mathbf{A_k}$ is the fund value at time t_{k-1} after all transactions are completed.
  $\mathbf{B_k}$: fund value at time t_k after interest is credited but just before the cash flow due at time t_k occurs.
- Multiply all growth factors together.

Outline

Fixed Interest Securities

The price of a bond is the present value of all future payments (coupons and redemption amount).

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

for $j = \frac{i^{(2)}}{2}$, assuming half-yearly coupon payments.

C=F $\rightarrow$ 'redeemable at par' (default assumption)

Fixed Interest Securities

- **Redemption yield**: internal rate of return
- **Nominal yield**: redemption yield expressed as a nominal rate of interest per annum
- **Running yield**: ratio of coupon rate per annum to the price of the bond per unit nominal. For example if a \$100 par value bond with coupons of \$9 per annum is selling for \$90, then the running yield on the bond is $\frac{0.09}{0.9} = 10\%$ per annum.

Fixed Interest Securities

Bond price between coupon dates

Let P_0 be the value of the bond just after the last coupon. Then

$$P_t = P_0(1+j)^t$$

where j is the effective yield over the coupon period.

Allowance for income tax

For tax payable at the same time as income is incurred

$$P = Fr(1-t_I) \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

Fixed Interest Securities

Allowance for capital gains tax

- If $P \geq C$, no capital gain, $P' = P = Fr(1 - t_I) \cdot a_{\overline{n}|j} + C \cdot v_j^n$
- If $P < C$, there is a capital gain,
  $P' = Fr(1 - t_I) \cdot a_{\overline{n}|j} + C \cdot v_j^n - t_C(C - P')v_j^n$

$t_C(C - P')$ is the additional liability for tax on a capital gain.

Fixed Interest Securities

Relationship between yield and coupon rate

Redemption yield is the same as the internal rate of return.
Suppose C=F (the bond is redeemable at par). Then the following relationships hold between the yield and the coupon rate:

- $P = F \Leftrightarrow j = r$ (bought at par)
- $P < F \Leftrightarrow j > r$ (bought at discount)
- $P > F \Leftrightarrow j < r$ (bought at premium)

Explain the inequalities above.

- Suppose $C \neq F$? Define modified coupon rate $g = \frac{Fr}{C}$, identify relationships between j and g.

Use the above relationships to identify bounds on the unknown yield.

Fixed Interest Securities

The effect of term to redemption on yield

For $\mathbf{P} > \mathbf{C}$ (bond is sold at a premium)

- as n increases, the yield $j \uparrow$
- as n decreases, the yield $j \downarrow$

How about when $\mathbf{P} < \mathbf{C}$?? (bond is sold at a discount)

- as n increases, the yield $j \downarrow$
- as n decreases, the yield $j \uparrow$

Fixed Interest Securities

The effect of term to redemption on yield

Suppose you can purchase either Bond A or Bond B. The two bonds are identical except Bond A matures in 10 years time, and Bond B matures in 8 years time.

- If $P > C$ the bond is sold at a premium, you are making a capital loss, select the bond with the latest redemption date, that is Bond A to maximise yield.
- If $P < C$ the bond is sold at a discount, you are receiving a capital gain, select the bond with the earliest redemption date, that is Bond B to maximise yield.

Fixed Interest Securities

Callable Bonds

A callable bond does not have a fixed redemption date.

Should assume the holder of the calling rights will call the bond to their advantage.

If the borrower has the calling rights, investors should assume the borrower exercises this right to the disadvantage of the investor, that is, the redemption time is chosen to maximise borrower yield, and minimise investor yield.

Fixed Interest Securities

Callable Bonds

If the borrower has the calling rights:

- For a bond bought at a discount, if an investor requires a particular minimum yield, the price paid should be based on the **latest** optional redemption date.
- For a bond bought at a premium, if an investor requires a particular minimum yield, the price paid should be based on the **earliest** optional redemption date.

In both cases, the minimum price is paid by the investor.

Fixed Interest Securities

Callable Bonds

If the borrower has the calling rights:

- For a bond bought at a discount, if an investor requires a particular minimum yield, the price paid should be based on the **latest** optional redemption date.
- For a bond bought at a premium, if an investor requires a particular minimum yield, the price paid should be based on the **earliest** optional redemption date.

In both cases, the minimum price is paid by the investor.

Bonds with inflation-linked payments

Valuation of shares and property

Outline

Arbitrage & Forward contracts

Arbitrage
Describes a **risk free trading profit**.

No-arbitrage principle - two securities or combinations of securities that give exactly the same payments must have the same price.

Replicating portfolio
Construct a portfolio of assets with exactly the same payments as the investment that we are interested in. By the no-arbitrage principle, the price of the investment must be the same as the price of the "replicating portfolio".

Arbitrage & Forward contracts

Forward contract - agreement between two counterparties to buy or sell a specific asset at a certain future time at a certain price.

Determining the forward price

$$\mathbf{K} = \mathbf{S_0 e}^{\delta \mathbf{T}} \text{ (securities with no income)}$$

$$\mathbf{K} = (\mathbf{S_0} - \mathbf{PV_I})\mathbf{e}^{\delta \mathbf{T}} \text{ (securities with income)}$$

Arbitrage & Forward contracts

Value of the forward contract

Suppose $0 < r < T$. Let V_L be the value of the forward contract at time r.

$$V_L = (K_r - K_0)e^{-\delta(T-r)}$$

where K_r is the forward price at time r for a forward contract also maturing at time T, on the same underlying asset.

For a security that pays no income,

$$V_L = S_r - S_0 e^{\delta r} = S_r - K_0 e^{-\delta(T-r)}$$

Outline

Spot rates and forward rates

Spot Rate: yield to maturity on a *zero coupon bond*. Applies on an investment that is available now and holds until some specified future time.

$\mathbf{s_t}$: t-period spot rate of interest - effective yield per period on a zero coupon bond with a term of t periods (that is, a measure of the average interest rate over the period from now until t periods time).

Spot rates and forward rates

Using spot rates, the bond price can be written:

$$P = \sum_{p=1}^{n} Fr \cdot v_{s_p}^{p} + C \cdot v_{s_n}^{n}$$

or, the price of a coupon-bond can be written as the sum of the prices of corresponding zero-coupon bonds:

$$P = Fr \cdot (P_1 + P_2 + ... + P_n) + C \cdot P_n$$

Spot rates and forward rates

Forward rates

Rates agreed today (t=0) for investment starting at a future time ($t > 0$).

Definition and notation:

$\mathbf{f_{t,T}}$: rate agreed at t=0, for investment from t to T.

Relationship between forward and spot rates

$$(1+s_t)^t(1+f_{t,T})^{T-t} = (1+s_T)^T$$

or, $\boxed{(1+f_{t,T})^{T-t} = \frac{(1+s_T)^T}{(1+s_t)^t}}$

Spot rates and forward rates

One-period forward rates:

$$\boxed{(1+f_{t,t+1}) = \frac{(1+s_{t+1})^{t+1}}{(1+s_t)^t}}$$

$$\boxed{(1+s_t)^t = (1+s_1)(1+f_{1,2})(1+f_{2,3})....(1+f_{t-1,t})}$$

For bond pricing

$$P = \frac{Fr}{(1+f_{0,1})} + \frac{Fr}{(1+f_{0,1})(1+f_{1,2})} + ... + \frac{Fr+C}{(1+f_{0,1})(1+f_{1,2})....(1+f_{n-1,n})}$$

Outline

Interest rate risk

Volatility

The effective duration (or volatility υ) of a series of cash flows is a measure of the rate of change of the present value of a series of cash flows as the interest rate changes.

Let $P(i) = \sum_{k=1}^{n} C_{t_k} v_i^{t_k}$ denote the present value of a series of future payments valued at an effective rate of interest i.

$$\upsilon = -\frac{P'(i)}{P(i)} = \frac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k+1}}{P(i)}$$

Interest rate risk

Duration

The mean term of the cashflows, weighted by the present value of the cashflows.

(Also known as the discounted mean term or Macaulay's duration)

$$DMT = \tau = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{P(i)} = \underline{(1+i)}\upsilon$$

Interest rate risk

Duration

What is the duration of a n-year zero coupon bond?

$$\tau = \frac{n \cdot C \cdot v_j^n}{C \cdot v_j^n} = n$$

Interest rate risk

<u>Duration</u> What is the duration of a bond with coupon rate r payable for n half-years and redemption amount C?

$$P = Fra_{\overline{n}|j} + Cv_j^n = \sum_{t=1}^{n} Frv_j^t + Cv_j^n$$

The duration of this bond is:

$$\tau = \frac{\sum_{t=1}^{n} t \cdot Fr \cdot v_j^t + n \cdot Cv_j^n}{P} = \frac{Fr(Ia)_{\overline{n}|j} + n \cdot Cv_j^n}{P}$$

Interest rate risk

Convexity

$$c(i) = \frac{\sum_{k=1}^{n} C_{t_k} t_k (t_k + 1) v_i^{t_k+2}}{P(i)} = \frac{P''(i)}{P(i)}$$

(captures the curvature of the yield curve)

Interest rate risk

<u>Approximation to $\mathbf{P(i_0 + \epsilon)}$</u>

$$P(i_0 + \epsilon) = P(i_0) + \epsilon P'(i_0) + \frac{\epsilon^2}{2} P''(i_0) + ...$$

We can rewrite this as:

$$\frac{P(i_0 + \epsilon) - P(i_0)}{P(i_0)} = \epsilon \frac{P'(i_0)}{P(i_0)} + \frac{\epsilon^2}{2} \frac{P''(i_0)}{P(i_0)} + ...$$

$\epsilon \frac{P'(i_0)}{P(i_0)} = -\epsilon \upsilon = -\epsilon \frac{\tau}{(1+i_0)}$

Interest rate risk

Immunisation

1. $S(i_0) = V_A(i_0) - V_L(i_0) = 0$
2. $\tau_A(i_0) = \tau_L(i_0)$ or $\upsilon_A(i_0) = \upsilon_L(i_0)$
3. $c_A(i_0) \geq c_L(i_0)$

Outline

Stochastic Interest Rate Models

$$\boxed{\text{Treat the interest rate as a random variable}}$$

$$\downarrow$$

Assign a probability distribution to explain the behaviour of future interest rates

Stochastic Interest Rate Models

Statistics Recap

For a random variable $\tilde{X}$, calculate

- $E[\tilde{X}]$
- $Var[\tilde{X}]$
- $E[g(\tilde{X})]$

Also note:

- $E[a\tilde{X} + b] = aE[\tilde{X}] + b$.
- $Var[a\tilde{X} + b] = a^2 Var(\tilde{X})$
- $SD(\tilde{X}) = \sqrt{Var(\tilde{X})}$
- Let $\tilde{X}$ and $\tilde{Y}$ be independent random variables. Then $Var[\tilde{X} + \tilde{Y}] = Var[\tilde{X}] + Var[\tilde{Y}]$

Also be familiar with the uniform and normal distributions

Stochastic Interest Rate Models

If interest rates are $\overset{\text{iid}}{\sim}$, with mean $=E[\tilde{i}]$, variance $=$ $Var[\tilde{i}]$, then the mean and variance of the accumulated value of 1 after n periods are

- $E[\tilde{S}(n)] = \big(E[(1+\tilde{i})]\big)^n$
- $E[\tilde{S}(n)^2] = \big(E[(1+\tilde{i})^2]\big)^n$
- $Var[\tilde{S}(n)] = \big(E[(1+\tilde{i})^2]\big)^n - \big(E[(1+\tilde{i})]\big)^{2n}$

Stochastic Interest Rate Models

Log-normal distribution

$\ln(\tilde{S}(n)) = \ln((1+\tilde{i}_1)) + \ln((1+\tilde{i}_2)) + ... + \ln((1+\tilde{i}_n)) = \tilde{\delta}_1 + \tilde{\delta}_2 + ... + \tilde{\delta}_n$

Apply Central Limit Theorem:

For large n, if the forces of interest $\tilde{\delta}_t$ are $\overset{\text{iid}}{\sim}$ with mean $E[\tilde{\delta}]$ and variance $Var[\tilde{\delta}]$, then $\ln[\tilde{S}(n)]$ is approx. normally distributed with:

- Mean: $E[\ln[\tilde{S}(n)]] = E[\tilde{\delta}_1 + \tilde{\delta}_2 + ... + \tilde{\delta}_n] = n \cdot E[\tilde{\delta}]$
- Variance: $Var[\ln[\tilde{S}(n)]] = Var[\tilde{\delta}_1 + \tilde{\delta}_2 + ... + \tilde{\delta}_n] = n \cdot Var[\tilde{\delta}]$

Stochastic Interest Rate Models

<u>Annuities</u>

$$E[\tilde{s}_{\overline{n}|}] = E[1+(1+\tilde{i}_n)+(1+\tilde{i}_n)(1+\tilde{i}_{n-1})+...+(1+\tilde{i}_n)(1+\tilde{i}_{n-1})...(1+\tilde{i}_2)]$$

Let $\tilde{i}_1, \tilde{i}_2, ..., \tilde{i}_n \overset{\text{iid}}{\sim}$ with mean $= E[\tilde{i}]$. Then

$$E[\tilde{s}_{\overline{n}|}] = 1 + E[1+\tilde{i}] + \left(E[1+\tilde{i}]\right)^2 + ... + \left(E[1+\tilde{i}]\right)^{n-1}$$

$$E[\tilde{s}_{\overline{n}|}] = s_{\overline{n}|j}$$

$$E[\tilde{\ddot{s}}_{\overline{n}|}] = \ddot{s}_{\overline{n}|j}$$

where $j = E[\tilde{i}]$

Course Overview

- 12 Weeks Lectures: Lecture Notes 1-11
- One Additional Lecture: the Review of Mathematics
- One Additional Workshop: 4 Case Studies using Excel
- 12 Tutorials + 5 Quizzes + Review Questions Problem Set + CT1 Past Exam Papers + Past Exam Papers (mid-term and final)
- One mid-term exam (with detailed examiners' report)
- One assignment (Excel is encouraged to be used.)

Final Exam - Reminder

- When: 14:00pm, Tuesday 13 June, 2017
- Where: Copland 24 G30, G31
- Duration: 15 minutes reading time; 3 hours writing time
- The mid-term is redeemable.
- Formula sheet and actuarial tables will be provided.
- Bring your own non-programmable calculator(s) (one or two), unmarked dictionaries also permitted. Personal copies of actuarial tables are not allowed.
- Explicit rounding rules will be provided as one of the instructions in your final exam paper.
- A Standard Normal Table will NOT be provided in your final exam. All probabilities will be provided along with the questions, if needed.
- **SHOW ALL WORKING!** (credit may be given for partial solutions)

Good Luck Everyone!