MATH6222 Week 9 Lecture Notes

Rui Qiu

2017-05-01

1 Monday's Lecture

1.1 equivalent relations

Proposition: If R is an equivalent relation often write $x \sim y$ to mean $(x, y) \in R$.

If "$\sim$" is an equivalent relation, then

1. If $x \sim y, [x] = [y]$.
2. If $x \not\sim y, [x] \cap [y] = \varnothing$
3. The distinct equivalence classes partition S, i.e. every element of S belongs in exactly one equivalence class.

Proof:

1. Suppose $z \in [x]$, i.e. $z \sim x$. We are given $x \sim y$. By symmetry, $z \sim y$, i.e. $z \in [y]$. This shows $[x] \subseteq [y]$, same argument in reverse shows $[y] \subseteq [x]$. Therefore, $[x] = [y]$.
2. Suppose $[x] \cap [y] \neq \varnothing$, i.e. $\exists\ z \in [x] \cap [y]$, i.e.$z \sim x$ and $z \sim y$. By symmetry, $x \sim z$. By transitivity $x \sim y$. Contradiction.
3. Given any $x \in S$. By reflexivity, $x \in [x]$. So every element of S is in at least one equivalence class. Suppose x was contained in two equivalence classes, say $[y]$ and $[z]$. Then $x \in [y] \cap [z] \implies y \sim z \implies [y] = [z]$.

1.2 Probability

Definition: a (finite) probability space is a finite set S together with a function $P : \{\text{subsets of } S\} \to [0, 1]$ satisfying:

1. $P(S) = 1$
2. If $A, B \in S$ and $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$.

We call a subset $A \subseteq S$ an "event". $P(A)$ is the probability of the event. If $A \cap B = \varnothing$, we say the events are mutually exclusive.

Example: Suppose S any finite set, define P by $P(A) = \frac{|A|}{|S|}$.

E.g. $A = \{(i,j) : 1 \leq i,j \leq 6\}$, $(|A| = 36)$.

This P defines the usual probability space for pairs of die.

Remark: Alternative definition. Finite set S together with a function $P : S \to [0,1]$ satisfying $\sum P(a) = 1, a \in S$. Define $P(A) = \sum_{a \in A} P(a)$.

1.3 Conditional probability

Pick a jar at random, pick a marble out of the jar. It's black.

What is the probability that I have picked for number 3?

...

Definition: Let A, B be two events in a probability space. The "probability of A given B"

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayesian Hypothesis Testing: Suppose $B_1 \ldots, B_r$ are mutually exclusive events which partition a finite probability space. Suppose wee know $P(B_i) = b_i$. Given another event $A \subseteq S$. Suppose we know $a_i = P(A|B_i)$ for each $i = 1, \ldots, r$.

Problem: Determine $P(B_i|A)$.

$B_i =$ We've picked for i, $b_i = \frac{1}{3}$.

$A =$ Picking a black marble

$P(A|B_1) = 0, P(A|B_2) = \frac{1}{2}, P(A|B_3 = 1$.

I asked $P(B_i|A)$,

so $P(B_1|A) = 0, P(B_2|A) = \frac{1}{3}, P(B_3|A) = \frac{2}{3}$.

2 Thursday's Lecture

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Suppose there are mutually exclusive events $B_1, \ldots, B_r$ which partition S. Then suppose there's another event A. We know $b_i = P(B_i)$.

Problem: Determine $P(B_i|A) = b_i^*$.

Bayes Formula:

$$b_i^* = P(B_i|A) := \frac{a_i b_i}{\sum_{j=1}^r a_j b_j}$$

Proof:

$$\begin{aligned} P(B_i|A) &= \frac{P(B_i \cap A)}{P(A)} \\ P(B_i \cap A) &= P(A \cap B_i) = P(A|B_i) \cdot P(B_i) = a_i b_i \\ A &= (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_r) \\ P(A) &= \sum_{j=1}^r P(A \cap B_j) = \sum_{j=1}^r a_j b_j \end{aligned}$$

Medical Testing: Suppose you have a test for condition X, which is 96% accurate. Question: If you test positive, what is the probability that you actually have condition X?

Suppose throughout the general population, the probability of having condition X is 1% (99% healthy).

A: testing positive +
B_1: you have condition $X, b_1 = P(B_1) = 0.01$
B_2 : you don't have condition $X, b_2 = P(B_2) = 0.99$
$a_1 = P(A|B_1) = 0.96$
$a_2 = P(A|B_2) = 0.04$

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)} = \frac{a_1 b_1}{a_1 b_1 + a_2 b_2} = \frac{.96 \cdot .01}{.96 \cdot .01 + .99 \cdot .04} = 19.59\%$$

Random variables and Expectation: a **random variable** on a probability space S is just a function $X : S \to R$.

The **expectation** of the random variable X is $E(X) := \sum_{a \in S} X(a)P(a)$

Simple Examples: $S = \{\text{people in our class}\}, X : S \to R$ (person $\to$ weight), $|S| = N$

$$E(X) = \sum_{\text{people in class}} \frac{1}{N}(\text{height of person}) = \text{average height in the class}$$

Given a random variable X on probability space S, we can define the probability $P(X = k) = \sum_{a \in S, \text{ such that } X(a)=k} P(a) = P(\{a \in S : X(a) = k\})$.

$$E(X) = \sum_k k \cdot P(X = k)$$

Suppose $S = \{\text{Sequences of length 10 flips} : HT \dots\}$, ($|S| = 2^{10}$).
$X = \{\text{any length 10 sequence of H T} \to \text{number of heads}\}$

$$E(X) = \sum_{k=0}^{10} k \cdot P(X = k) = \sum_{k=0}^{10} k \cdot \binom{n}{k} \frac{1}{2^{10}} = \frac{1}{2^{10}} \sum_{k=0}^{10} k \cdot \binom{n}{k}$$

3 Friday's Lecture

$S = \{\text{Length n sequences of } \{\text{H, T}\}\}$, $|S| = 2^n$.

Define X on S by taking the number of heads in any given sequences.

$$E(X) = \sum_{k=0}^{n} k \cdot P(X = k) = \sum_{k=0}^{n} k \binom{n}{k} 2^{-n} = \frac{n}{2}$$

Linearity of Expectation: Let $X_1, \dots, X_n$ be random variables on probability space S. Let $c_1, \dots, c_n \in \mathbb{R}$. Let $X = c_1X_1 + c_2X_2 + \dots + c_nX_n$.

Then $E(X) = c_1E(X_1) + c2E(X_2) + \dots + c_nE(X_n)$.

Proof:

$$\begin{aligned}
E(X) &= E(c_1X_1 + \dots + c_nX_n) \\
&= \sum_{a \in S} (c_1X_1(a) + c_2X_2(a) + \dots + c_nX_n(a)) \cdot P(a) \\
&= \sum_{a \in S} c_1X_1(a)P(a) + \dots + c_nX_nP(a) \\
&= \sum_{a \in S} c_1X_1(a)P(a) + \sum_{a \in S} c_2X_2(a)P(a) + \dots + \sum_{a \in S} c_nX_n(a)P(a) \\
&= c_1 \sum_{a \in S} X_1(a)P(a) + \dots + c_n \sum_{a \in S} X_n(a)P(a) \\
&= c_1E(X_1) + \dots + c_nE(X_n)
\end{aligned}$$

$$X_i = \begin{cases} 1, & \text{if the ith toss is heads} \\ 0, & \text{if the ith toss is tails} \end{cases}$$

Note: $i = 1, \dots, n$, $X = X_1 + \dots + X_n, E(X) = \sum_{i=1}^{n} E(X_i) = \frac{n}{2}$. As $E(X_1) = 1 \cdot P(X_1 = 1) + 0 \cdot P(X_1 = 0) = \frac{1}{2}$.

Problem: So n pairs of socks thrown into a laundry machine. Machine spits out a random subset of k socks. How many complete pairs of socks do we expect come out?

Outcomes $= \binom{2n}{k}, \{L_1, R_1, \ldots, L_n, R_n\}$

X is defined as the number of complete pairs in a particular subset.

$$X_i = \begin{cases} 1, & \text{if the } \{L_i, R_i\} \text{ comes out} \\ 0, & \text{otherwise} \end{cases}$$

We need to count subsets of size $k \subset \{L_1, R_1, L_2, R_2, \ldots, L_n, R_n\}$, which is $\binom{2n-2}{k-2}$.

So

$$E(X_1) = 1 \cdot P(X_1 = 1) + 0 \cdot P(X_1 = 0) = \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}$$

Then

$$E(X) = \sum_{i=1}^{n} E(X_i) = \frac{n\binom{2n-2}{k-2}}{\binom{2n}{k}}$$

Finger Game: A and B can hold up 1 or 2 fingers. If the total is odd, A wins; if the total is even, B wins. Whoever wins, the losing side has to pay the amount of money of the number of fingers.

- 1, 1, A pays B 2 dollars.
- 1, 2, B pays A 3 dollars.
- 2, 1, B pays A 3 dollars.
- 2, 2 A pays B 4 dollars.

Suppose A, B play randomly, each scenario has a probability of $\frac{1}{4}$.

$E(A) = \frac{1}{4}(-2) + \frac{1}{4}(3) + \frac{1}{4}3 + \frac{1}{4}(-4) = 0$. Seems fair.

A chooses

1. with probability x
2. with probability $1 - x$

B choose

1. with probability y

2. with probability $1 - y$

A wants to choose x to maximize expected pay-off.

Then the probabilities of the 4 scenarios above become $xy, (1-x)y, x(1-y), (1-x)(1-y)$.

$$\begin{aligned}
E(X) &= -2xy + 3(1-x)y + 3(1-y)x - 4(1-x)(1-y) \\
&= -12xy + 7x + 7y - 4 \\
&= (7 - 12x)y + (7x - 4) \\
(x = \frac{7}{12}) &= 7 \cdot \frac{7}{12} - 4 = \frac{1}{12} > 0
\end{aligned}$$

If $x = \frac{7}{12}$, we can make y irrelevant.

Suppose we look this from B's angle, then $= (7-12y)x + (7y-4), y = \frac{7}{12}$.

When A takes $x = \frac{7}{12}, 1 - x = \frac{5}{12}$. B holding 1 all the time! Then

$$E(A) = \frac{7}{12}(-2) + \frac{5}{12}(3) = \frac{1}{12}$$

If B holding 2 all the time!

$$E(A) = \frac{7}{12}(3) + \frac{5}{12}(-4) = \frac{1}{12}$$

No matter what B does here, A's expectation should always be $\frac{1}{12}$.

If $x < \frac{7}{12}$, B chooses $y = 0, (7x - 4) < \frac{1}{12}$.

If $x > \frac{7}{12}$, B chooses $y = 1, 7 - 12x + 7x - 4 = 3 - 5x < \frac{1}{12}$.