UNIVERSITY OF TORONTO
Faculty of Arts and Science
August 2012 Examinations
MAT301H1Y Groups and Symmetry

Instructor: Patrick Walls

8 questions :: 80 points total :: 3 hours :: No aids allowed

1. Let $G = \{x \in \mathbb{R} : x \neq 1/2\}$ and define $x * y = x + y - 2xy$.

   (*a*) [5 points] Show that $(G, *)$ is a group.

   (*b*) [5 points] Show that $\varphi : (G, *) \longrightarrow (\mathbb{R}^\times, \times) : x \mapsto 1 - 2x$ is an isomorphism.

2. The dihedral group of order 12 is

$$D_6 = \{e, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$$

   where $|r| = 6$, $|s| = 2$ and $sr = r^5 s$.

   (*a*) [4 points] Find the left and right cosets of $H = \{e, r^3, s, r^3s\}$ in D_6. Is H a normal subgroup of D_6?

   (*b*) [4 points] Find the left and right cosets of $K = \langle r^3 \rangle$ in D_6. Show that K is a normal subgroup of D_6.

   (*c*) [2 points] Prove or disprove: D_6/K is abelian.

3. Let $\varphi : G \longrightarrow \overline{G}$ be a homomorphism.

   (*a*) [5 points] Show that $\ker \varphi$ is a normal subgroup of G.

   (*b*) [5 points] Show that φ is injective if and only if $\ker \varphi = \{e\}$.

4. Let B be the subgroup of $\mathrm{GL}(2, \mathbb{R})$ defined by

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}$$

   and define

$$\varphi : B \longrightarrow B : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a/d & b/d \\ 0 & 1 \end{pmatrix}.$$

   (You don't need to prove that B is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.)

   (*a*) [4 points] Show that φ is a homomorphism.

   (*b*) [4 points] Find $\ker \varphi$ and $\operatorname{im} \varphi$.

   (*c*) [4 points] How many elements of $B/\ker \varphi$ have order 2?

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5. Let $\alpha = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8) \in S_8$ and $\beta = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8) \in S_8$.

   (*a*) [5 points] Find $|\alpha|$, $|\beta|$ and $|\alpha\beta^2|$.

   (*b*) [5 points] If H is a subgroup of S_8 containing α and β, show that $|H| \geq 840$.

6. Let G be a group and for each $a \in G$ define $c_a : G \longrightarrow G$ by $c_a(g) = aga^{-1}$.

   (*a*) [5 points] For each $a \in G$, show that c_a is an automorphism of G.

   (*b*) [3 points] Show that the map $c : G \longrightarrow \mathrm{Aut}(G) : a \mapsto c_a$ is a homomorphism.

   (*c*) [2 points] Find $\ker c$.

7. [8 points] Let G be a group and let $S = \{xyx^{-1}y^{-1} : x, y \in G\}$. If H is a subgroup of G, show that $S \subseteq H$ if and only if $H \lhd G$ and G/H is abelian.

8. (*a*) [5 points] Classify abelian groups of order 300 (up to isomorphism).

   (*b*) [5 points] Show that every abelian group of order 441 contains an element of order 21.