Adversarial search (game playing)

Chapter 5

Outline

- ♢ Games
- ♢ Perfect play
  - – minimax decisions
  - – α–β pruning
- ♢ Imperfect decisions in real time
- ♢ Stochastic games

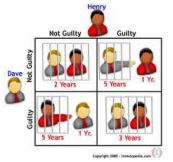
Adversarial search problems (games)

Arise in **competitive multi-agent** environments

In Game Theory, a multi-agent environment is called a game

In AI, a game is often a deterministic, turn-taking, two-player, zero-sum game of perfect information:

- deterministic
- two agents
- whose action alternate
- utility values are opposite e.g. (+1,-1)
- fully observable



We will write algorithms that play such games against an opponent.

Games Definition

A Game consists of:

♢ sets of players P, states S (board and player to play), and moves M

♢ an initial state $s_0 \in S$ which specifies how the game is set up

♢ $\textsc{Player}(s) \in P$: defines the player to move in state s

♢ $\textsc{Moves}(s) \in 2^M$: defines the set of legal moves in state s

♢ $\textsc{Result}(s,m) \in S$: defines the result of perfoming move m in state s

♢ $\textsc{Terminal}(s) \in \mathbb{B}$: the terminal test says whether the game is over

♢ $\textsc{Utility}(s,p) \in \mathbb{R}$: the utility function gives a numeric value to terminal states from the point of view of a given player, e.g. $\{+1, -1, 0\}$ for chess or $\{-192, \ldots, 192\}$ for backgammon

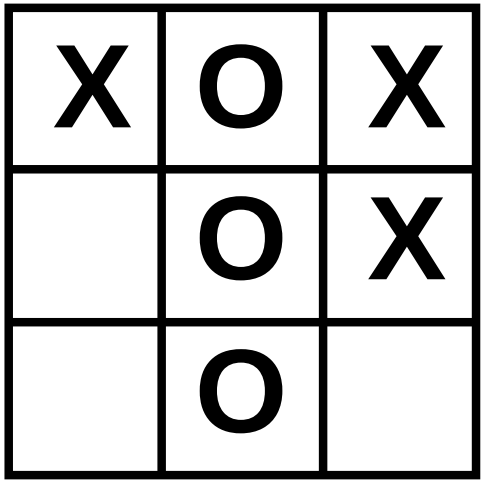
Strategy

♢ "Unpredictable" opponent
$\Rightarrow$ solution for a player is not a sequence of actions but a strategy

♢ A strategy for a player: a function mapping the player's states to (legal) moves.

♢ A winning strategy always lead the player to a win from s_0

Example: tic-tac-toe



states??: content of each cell {X,O,empty}, player to play

moves??: an empty cell

result??: content of chosen cell is X or O depending on the player playing; next player

terminal test??: are 3 O or 3 X aligned or is the board full?

utility function??: for a given player gives +1 if player has aligned 3 tokens, -1 if his opponent has, and 0 otherwise. There is a draw strategy.

Example: nim

states??: number of rows R, number of matches $n(r)$ in each row, player to play

moves??: a non-empty row $r \in R$ and a number of matches $0 < k \leq n(r)$ to remove

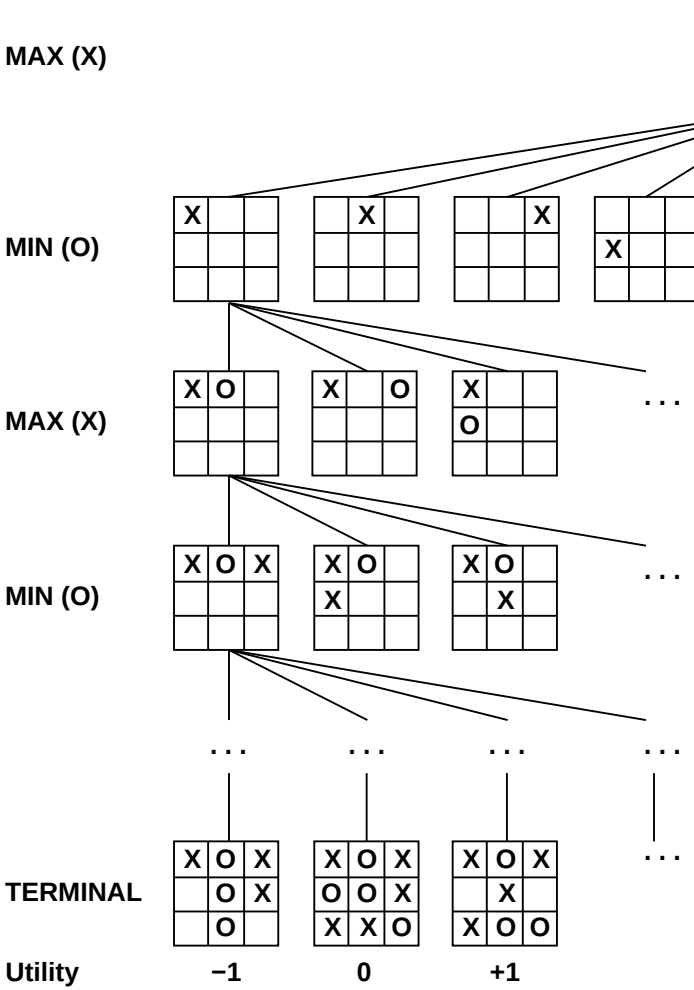
result??: $n(r) \leftarrow n(r) - k$, next player

terminal test??: $n(r) = 0$ for all $r \in R$

utility function??: for a given player, $+1$ if it is the player to play, -1 if his opponent is. There is a winning strategy for one of the players.

Game tree (2-player, deterministic, turns)

The game tree is defined by MOVES, RESULT and TERMINAL

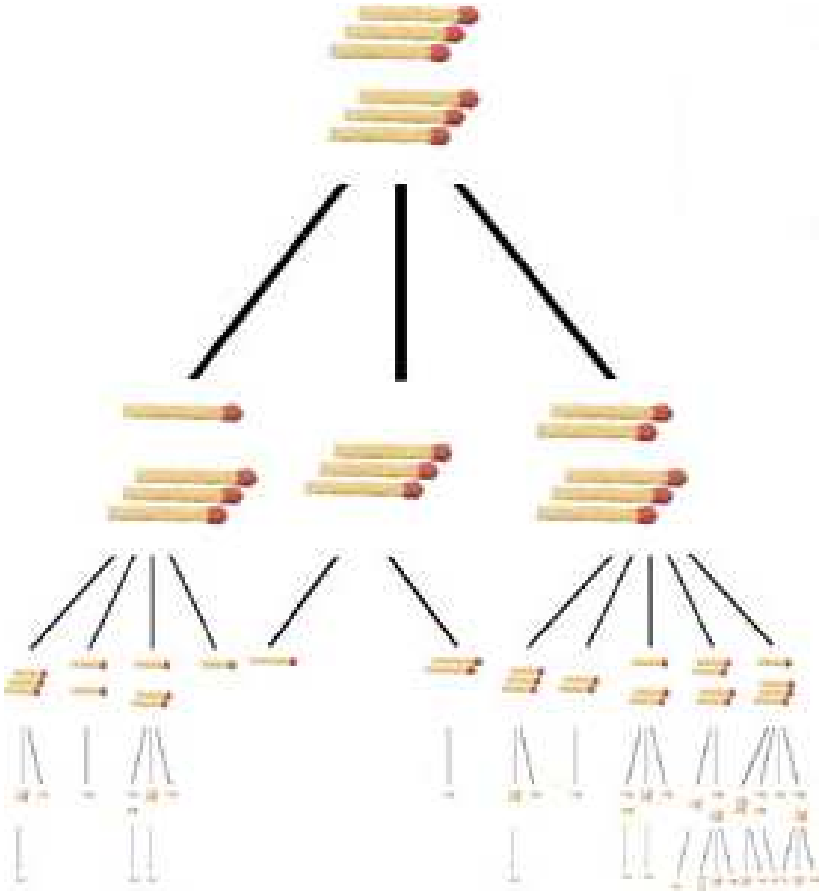


Max is the player using the algorithm to compute his strategy

Max is always at the root of the tree

Game tree (2-player, deterministic, turns)

The game tree is defined by MOVES, RESULT and TERMINAL



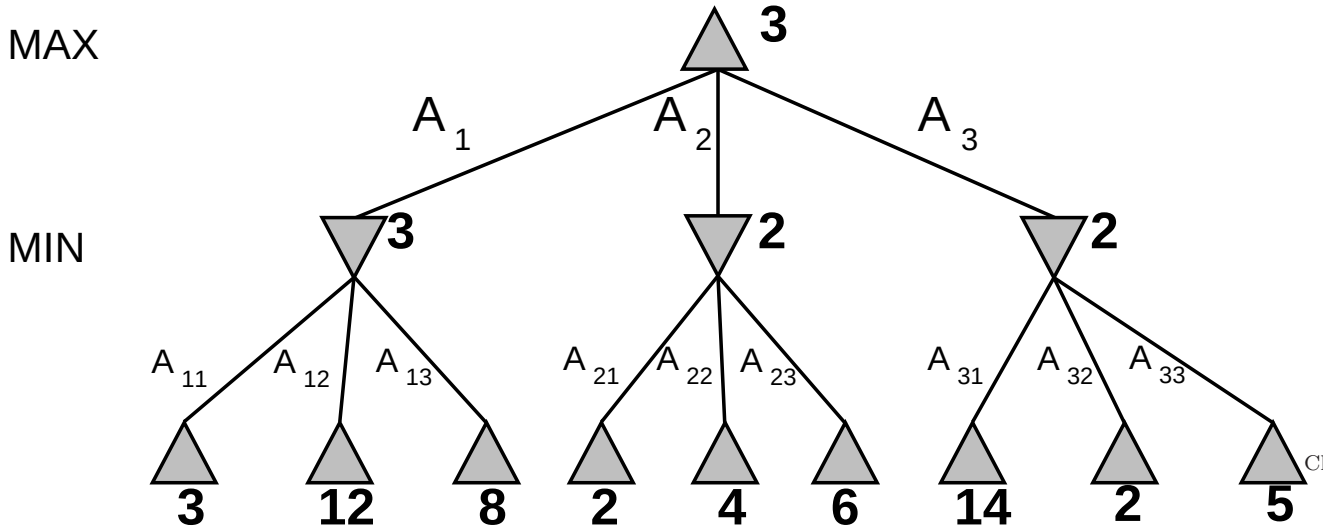
Minimax

Perfect play for deterministic, two-player, zero-sum, perfect-information games

Idea: choose move to position with highest minimax value
= best achievable utility against best possible opponent

$$\textsc{Minimax-Value}(s) = \begin{cases} \textsc{Utility}(s, \textsc{max}) & \text{if } \textsc{Terminal}(s) \\ \max_{m \in \textsc{Moves}(s)} \textsc{Minimax-Value}(\textsc{Result}(s,m)) & \text{if } \textsc{Player}(s) = \textsc{max} \\ \min_{m \in \textsc{Moves}(s)} \textsc{Minimax-Value}(\textsc{Result}(s,m)) & \text{if } \textsc{Player}(s) = \textsc{min} \end{cases}$$

E.g., 2-ply game:

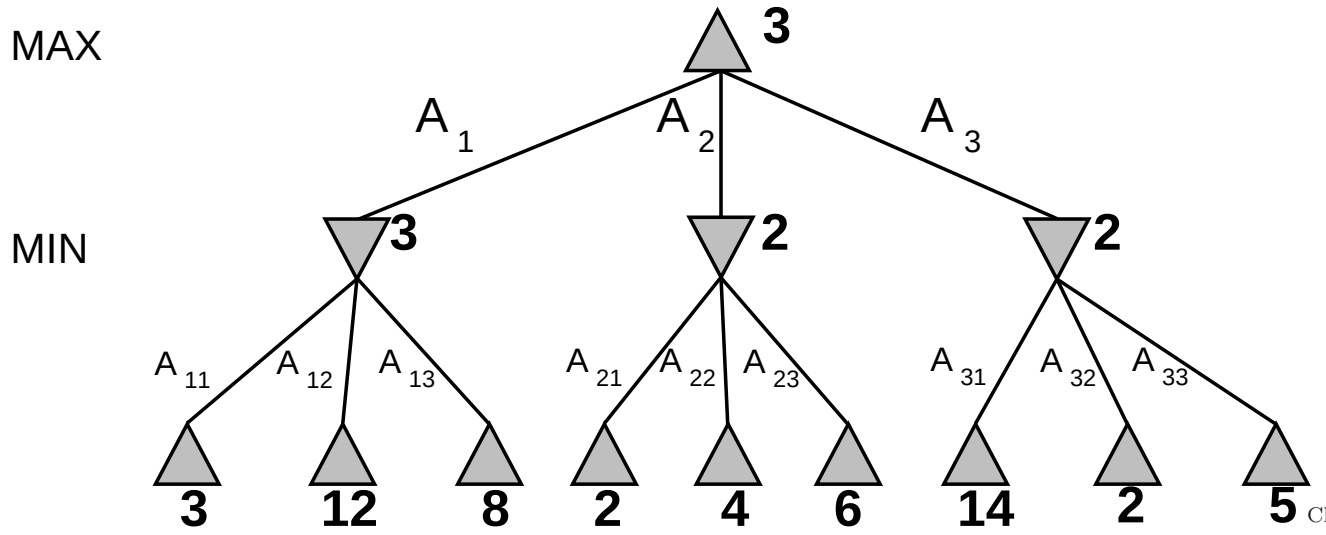


Minimax

Computing the Minimax value:

1. Apply utility function to each leaf of the game tree
2. Back-up values from the leaves through inner nodes up to the root:
   (a) MIN node: compute the $\min$ of its children values
   (b) MAX node: compute the $\max$ of its children values
3. At the root: choose the move leading to the child of highest value

Better method: use a depth-first like approach to save space



Minimax algorithm

```
function MINIMAX-DECISION(state) returns a move
    inputs: state, current state in game

    v ← MAX-VALUE(state)
    return the move m in MOVES(state) with value v

---

function MAX-VALUE(state) returns a utility value
    if TERMINAL(state) then return UTILITY(state,MAX)
    v ← −∞
    for m in MOVES(state) do
        v ← MAX(v, MIN-VALUE(RESULT(state,m)))
    return v

---

function MIN-VALUE(state) returns a utility value
    if TERMINAL(state) then return UTILITY(state,MAX)
    v ← +∞
    for m in MOVES(state) do
        v ← MIN(v, MAX-VALUE(RESULT(state,m)))
    return v
```

Properties of minimax

Complete?? Yes, if tree is finite

Optimal?? Yes, against an optimal opponent. Otherwise??

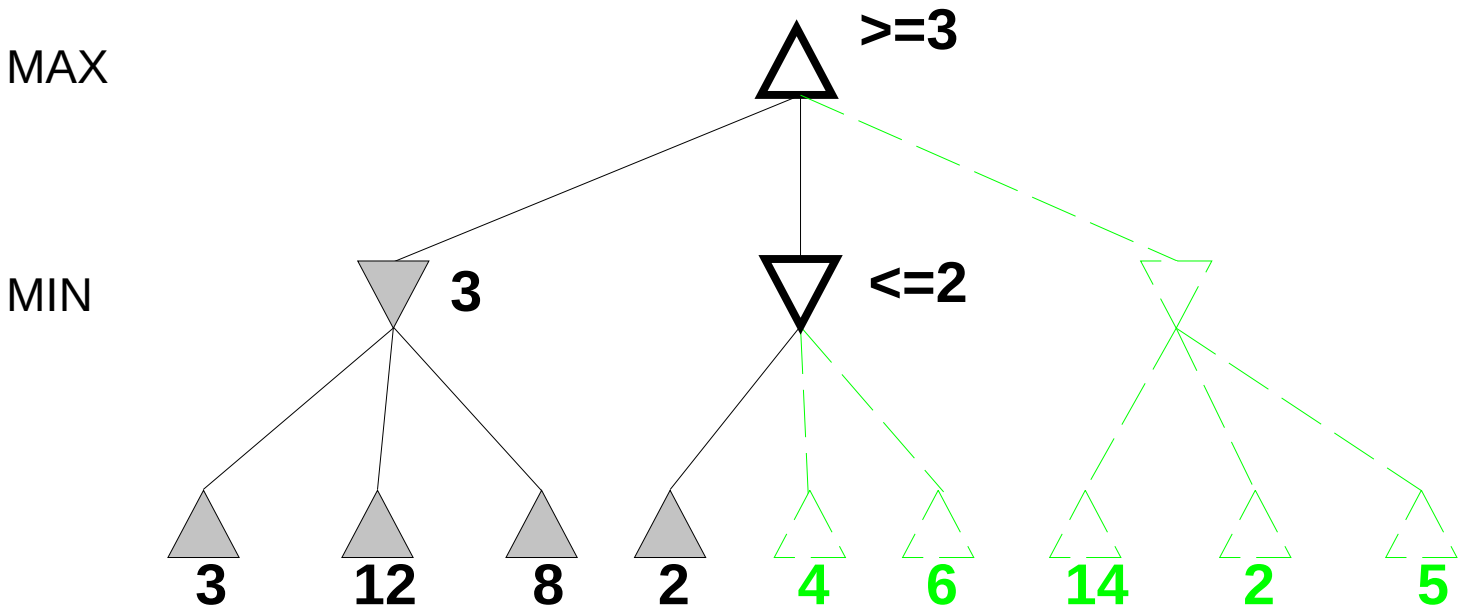
Time complexity?? $O(b^m)$

Space complexity?? $O(bm)$ (depth-first exploration)

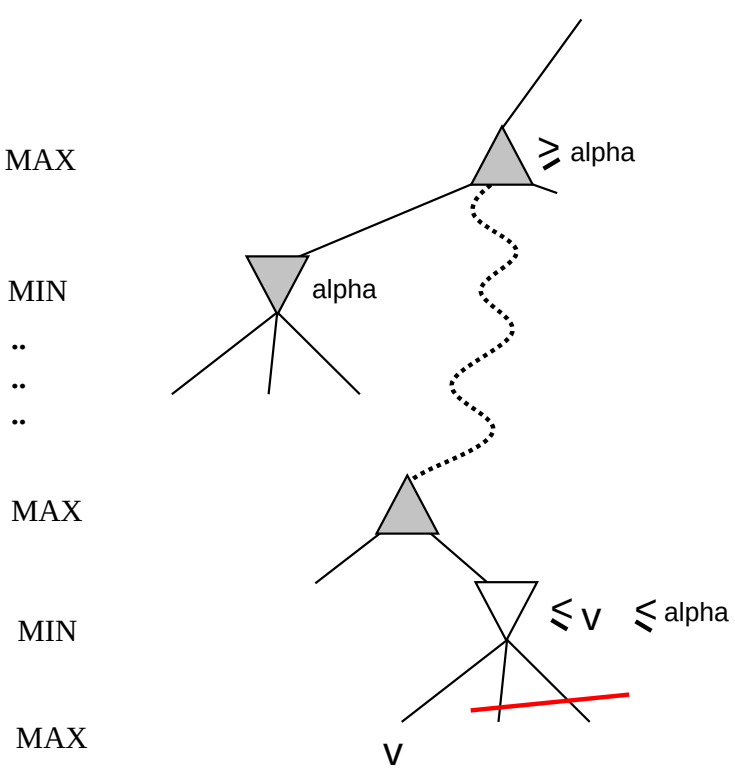
For chess, $b \approx 35$, $m \approx 100$ for "reasonable" games
$\Rightarrow$ exact solution completely infeasible

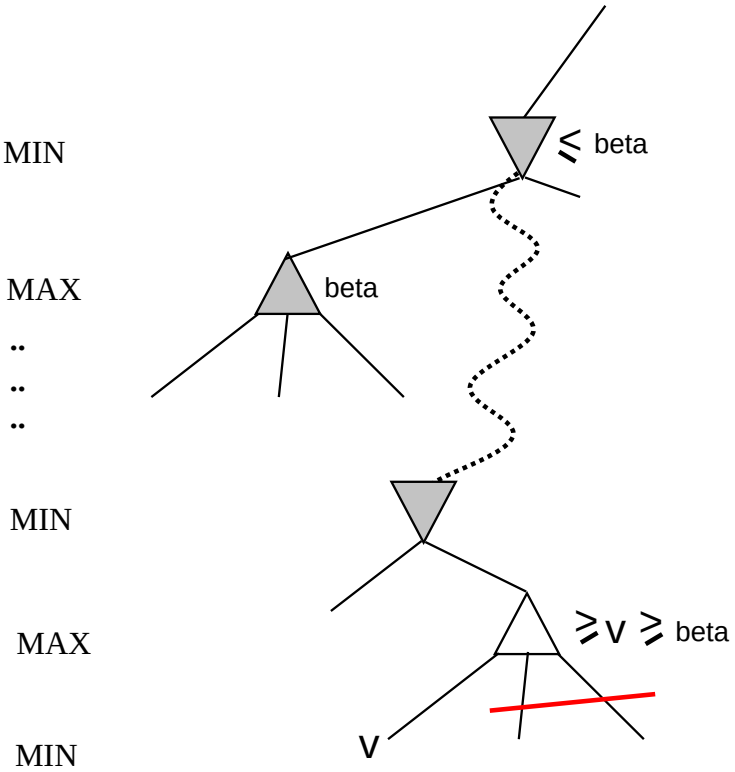
But do we need to explore every path?

Do we need to explore every path?



α–β pruning





α is the best value (to MAX, i.e. highest) found so far

If V is not better (greater) than α, MAX will avoid it $\Rightarrow$ prune that branch

Define β similarly for MIN

α–β Pruning

Idea:

- α is the best (largest) value found by MAX on path to current node
- β is the best (lowest) value found by MIN on path to current node
- Some values outside the interval $]\alpha, \beta[$ can be pruned

Algorithm:

- Node passes its current values for α and β to its children in turn
- Child passes back up its value to Node
- Node updates its current value v ($\max$ or $\min$ with child's value)
- Node checks whether $v \leq \alpha$ (MIN) or $v \geq \beta$ (MAX)
- If so, child's siblings can be pruned and v returned to Parent
- Otherwise β (MIN) or α (MAX) is updated

The α–β algorithm

function ALPHA-BETA-DECISION($state$) **returns** a move
  $v \leftarrow$ MAX-VALUE($state, -\infty, +\infty$)
  return the move m in MOVES($state$) with value v

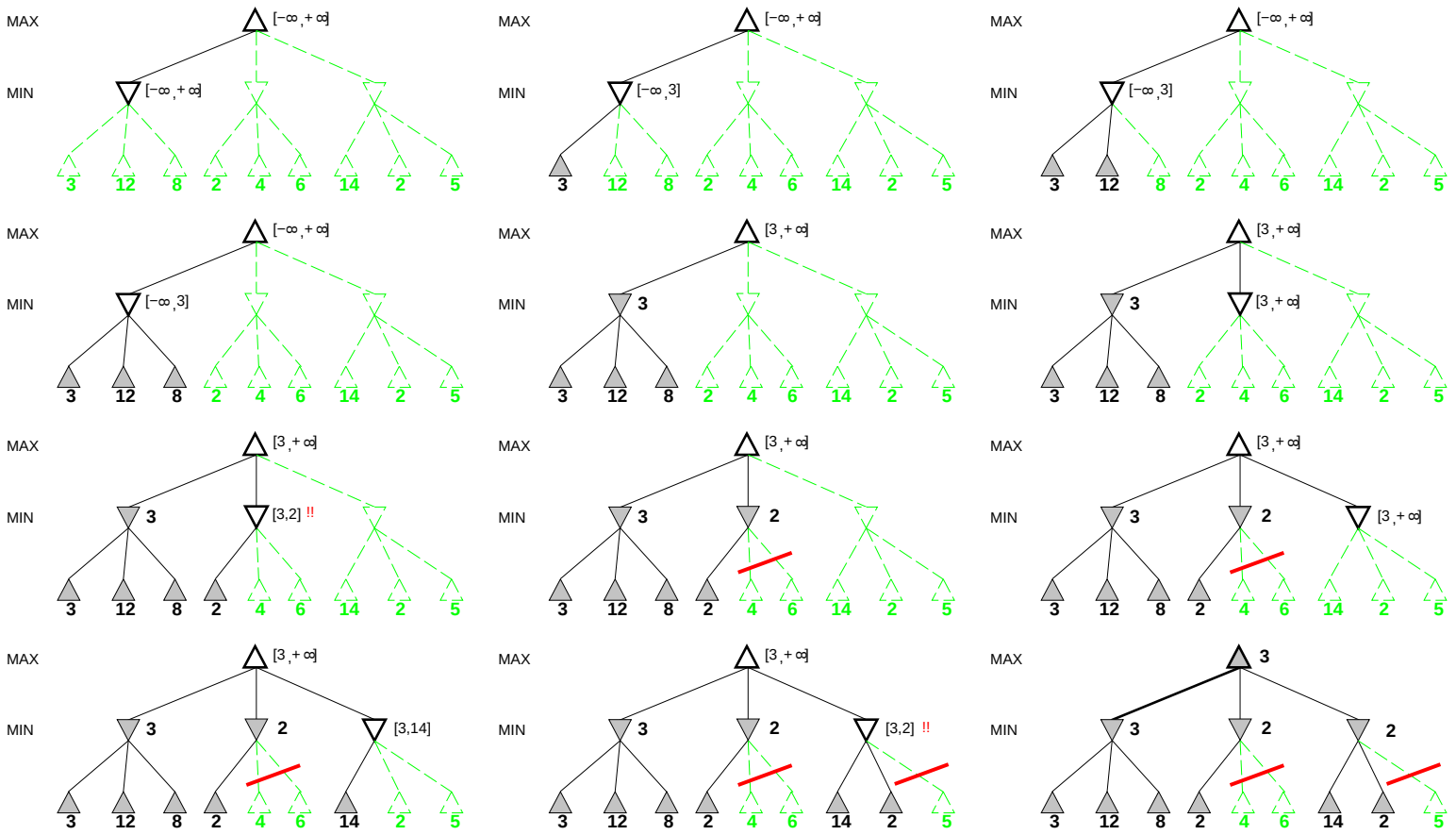
---

function MAX-VALUE($state, \alpha, \beta$) **returns** *a utility value*
  inputs: $state$, current state in game
    α, the value of the best choice for MAX so far
    β, the value of the best choice for MIN so far
  if TERMINAL($state$) **then return** UTILITY($state$,MAX)
  $v \leftarrow -\infty$
  for m in MOVES($state$) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(m,s), α, β))
    if $v \geq \beta$ **then return** v
    $\alpha \leftarrow$ MAX(α, v)
  return v

---

function MIN-VALUE($state, \alpha, \beta$) **returns** *a utility value*
  same as MAX-VALUE but with roles of α, β reversed

α–β pruning example



Remarks on and Properties of α–β

Alpha-Beta pruning is used with depth-first exploration of the game tree. (not complete generation!).

Pruning **does not** affect final result: value at the root of the tree is the same (but not the values of the inner nodes).

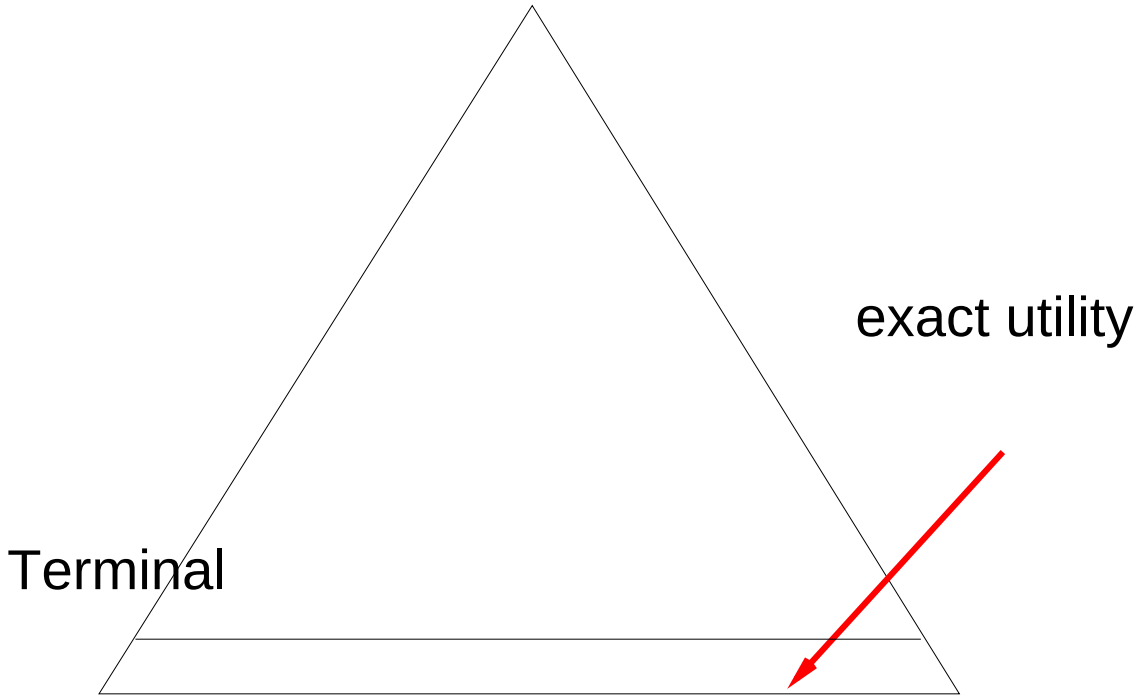
Good move ordering improves effectiveness of pruning. Perfect ordering (unachievable): increasing order for MAX and decreasing order for MIN.

With perfect ordering, the time complexity is asymptotically $O(b^{m/2}) \Rightarrow$ **doubles** solvable depth (random ordering yields $O(b^{3m/4})$).

Unfortunately, 35^{50} is still impossible!

Imperfect decisions in real-time

Approach: **limit search depth and estimate expected utility**



Imperfect decisions in real-time

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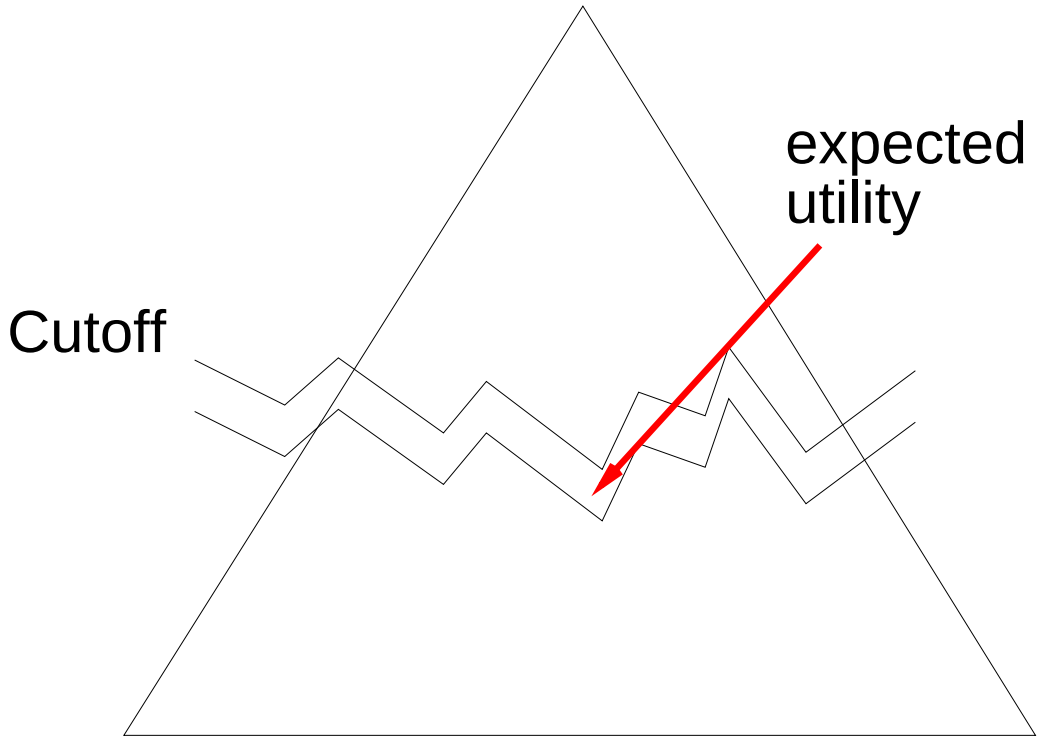

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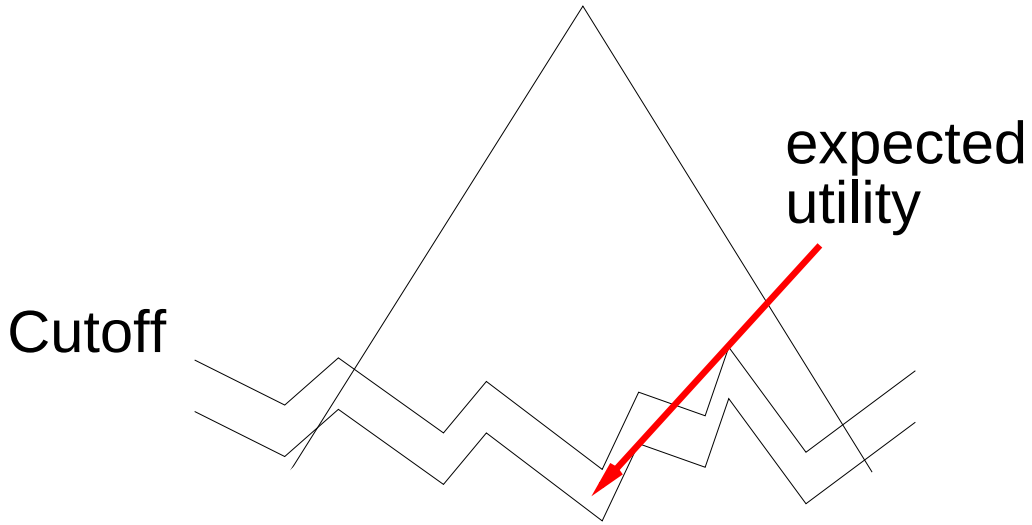

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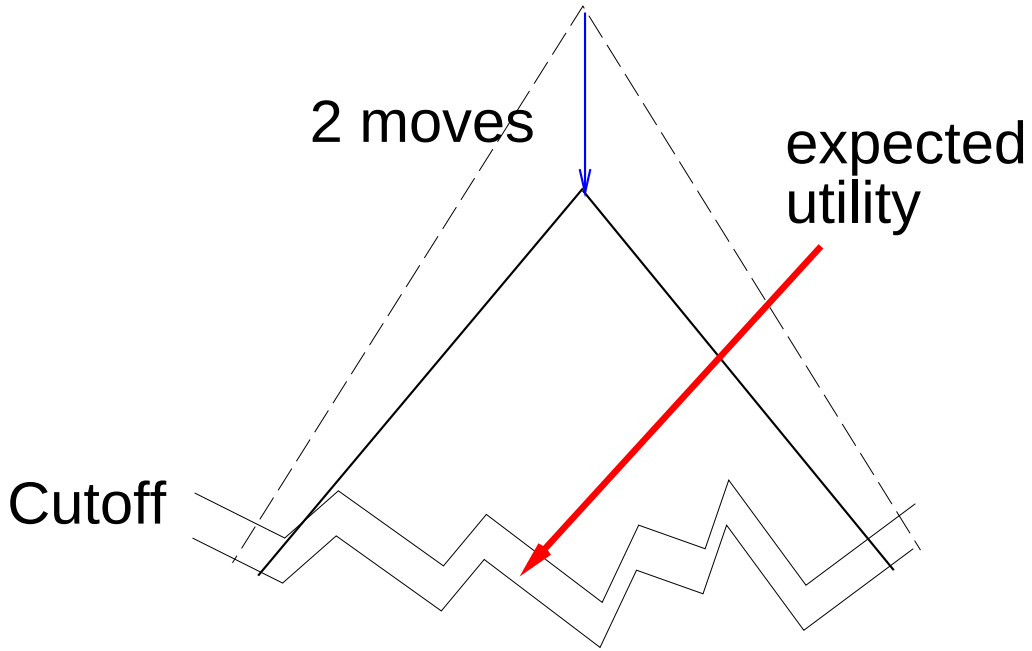

Imperfect decisions in real-time

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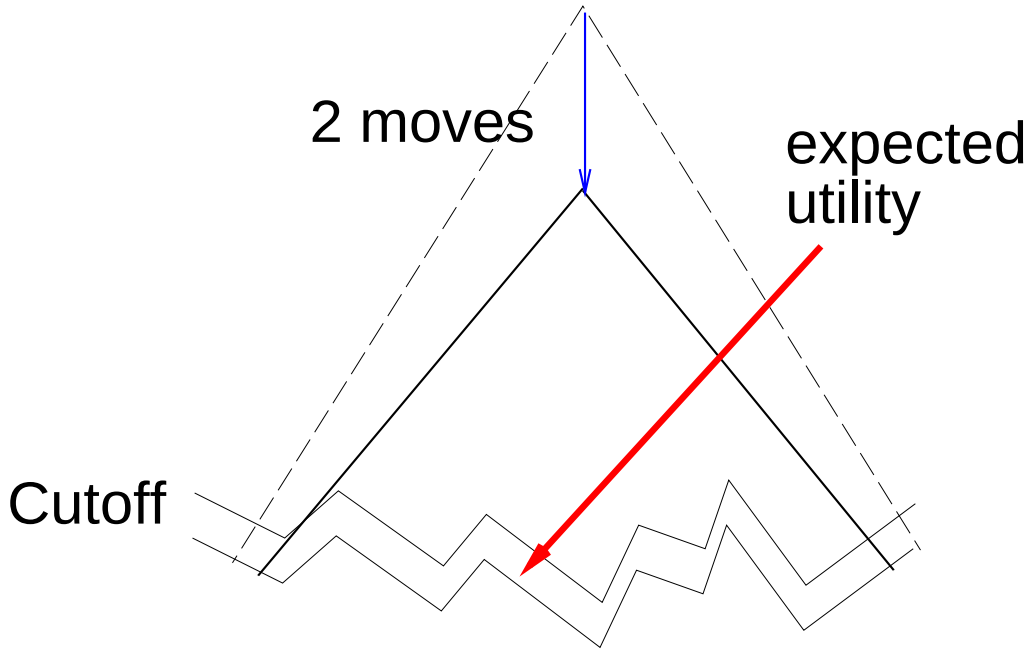


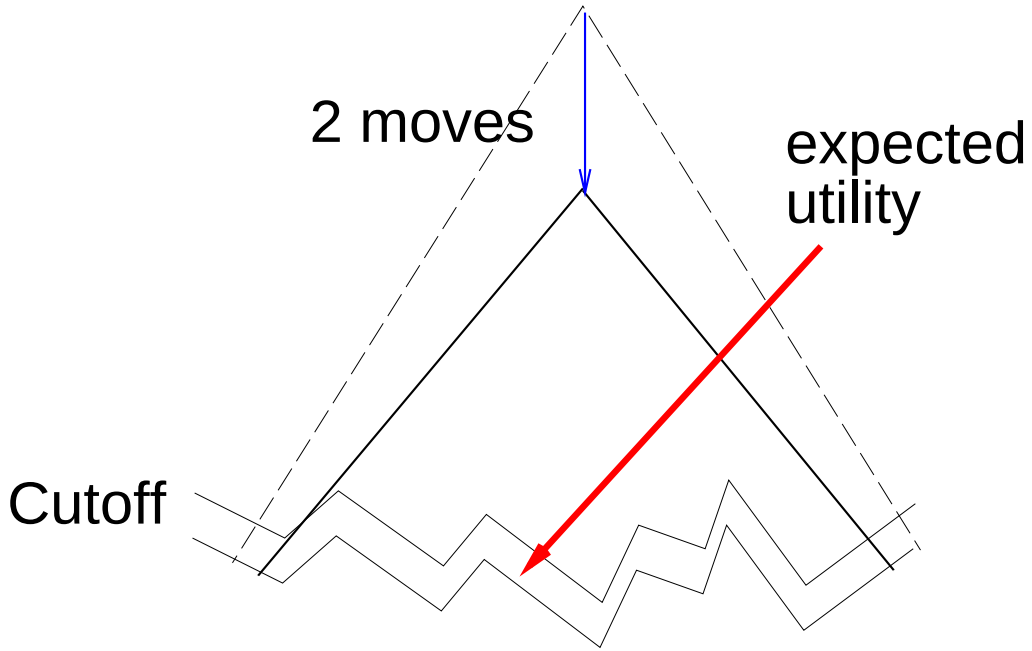
Suppose we have 100 seconds, explore 10^4 nodes/second

$\Rightarrow 10^6$ nodes per move $\approx 35^{8/2}$

$\Rightarrow \alpha$–β reaches depth 8 $\Rightarrow$ pretty good chess program

Imperfect decisions in real-time

Approach: **limit search depth and estimate expected utility**



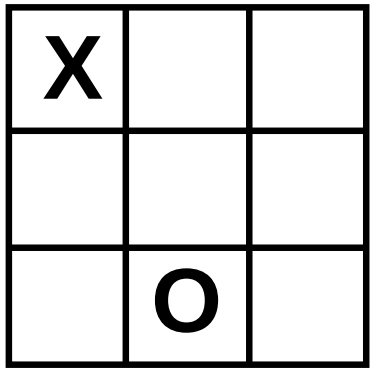
Changes to minimax: replace TERMINAL-TEST(s) with CUTOFF(s,d) and UTILITY(s,p) with EVAL(s,p) to estimate expected utility.

Changes to Minimax

- **Use CUTOFF test instead of TERMINAL test**
  - CUTOFF(s, d): true iff the state s encountered at depth d in the tree must be considered as a leaf (or s is terminal).
  - e.g., depth limit, estimated number of nodes expanded
  - perhaps add quiescence search
- **Use EVAL instead of UTILITY**
  - EVAL(s,p) i.e., evaluation function that estimates the expected utility of cutoff state s wrt player p, and correlates with chances of winning
  - should order the *terminal* states in the same way as UTILITY
  - should not take too long

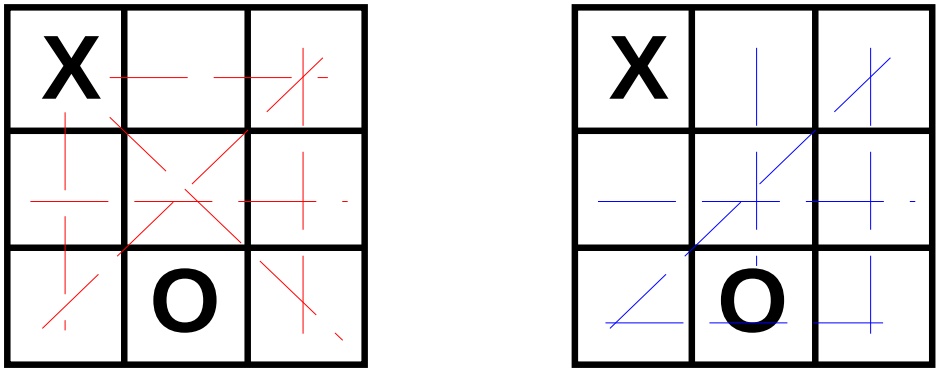
$$\text{D-MINIMAX-VALUE}(s, d) = \begin{cases} \text{EVAL}(s, \text{MAX}) & \text{if CUTOFF}(s, d) \\ \max_{m \in \text{MOVES}(s)} \text{D-MINIMAX-VALUE}(\text{RESULT}(s, m), d+1) & \text{if PLAYER}(s) = \text{MAX} \\ \min_{m \in \text{MOVES}(s)} \text{D-MINIMAX-VALUE}(\text{RESULT}(s, m), d+1) & \text{if PLAYER}(s) = \text{MIN} \end{cases}$$

Evaluation functions



What would be a good evaluation function for tic tac toe??

Evaluation functions

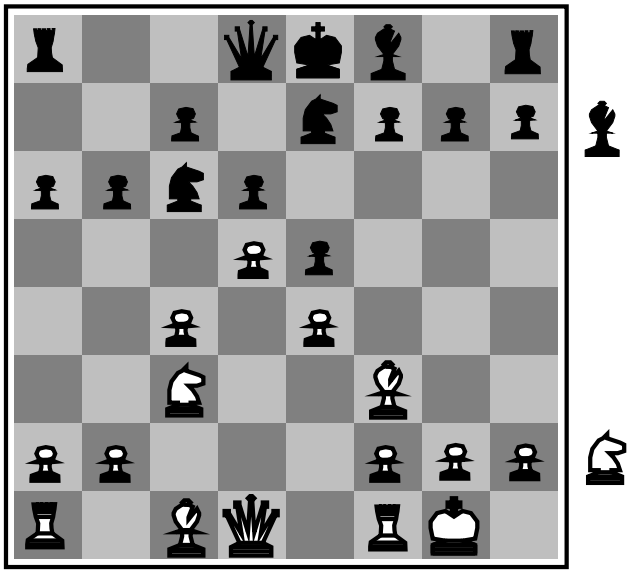


What would be a good evaluation function for tic tac toe??

EVAL(s,p) = winning-patterns(s,p) - winning-patterns(OPPONENT(s,p))

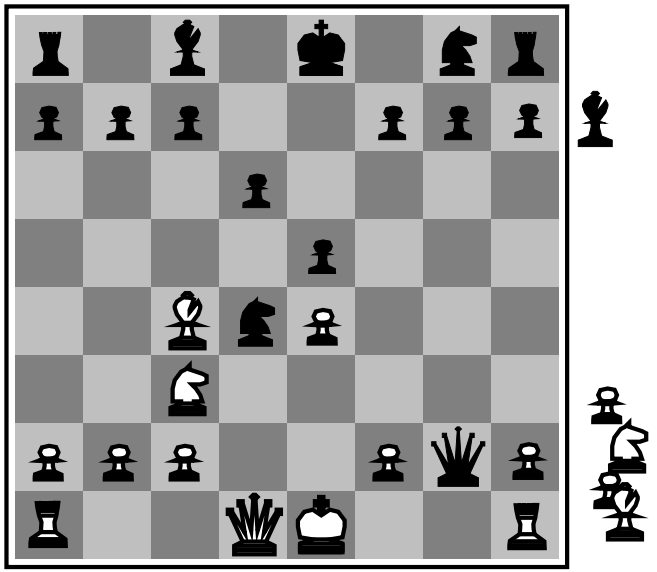
EVAL(s,X) = 6 - 5 = 1

Evaluation functions



Black to move

White slightly better



White to move

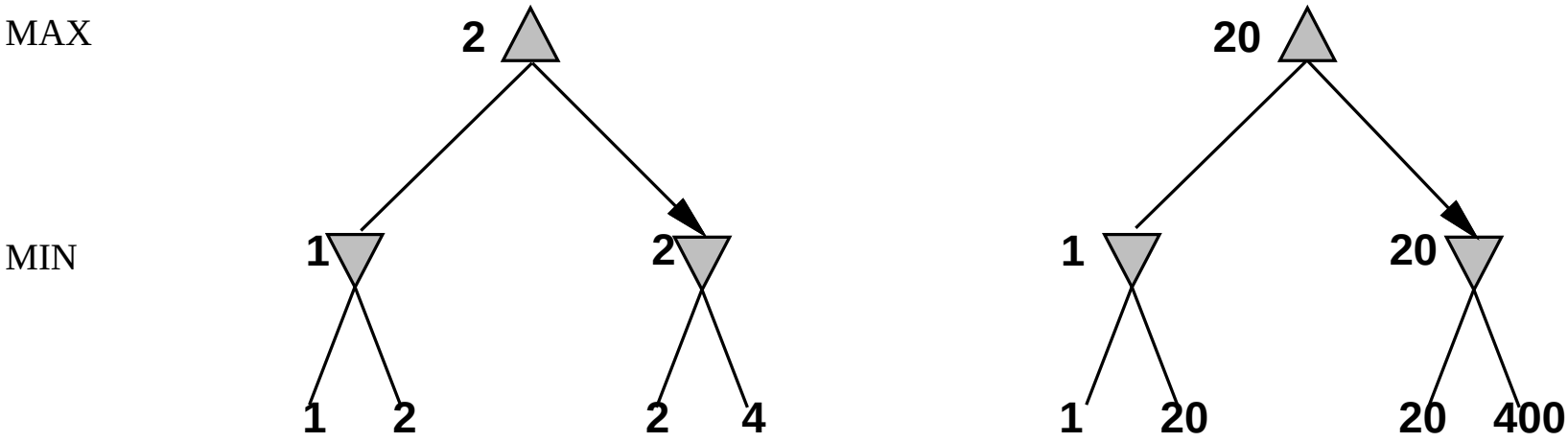
Black winning

For chess, typically linear weighted sum of features

$$Eval(s) = w_1 f_1(s) + w_2 f_2(s) + \ldots + w_n f_n(s)$$

e.g., $w_2 = 5$ with
$f_2(s) =$ (number of white castles) – (number of black castles), etc.

Observation: Exact values don't matter



Behaviour is preserved under any **monotonic** transformation of EVAL

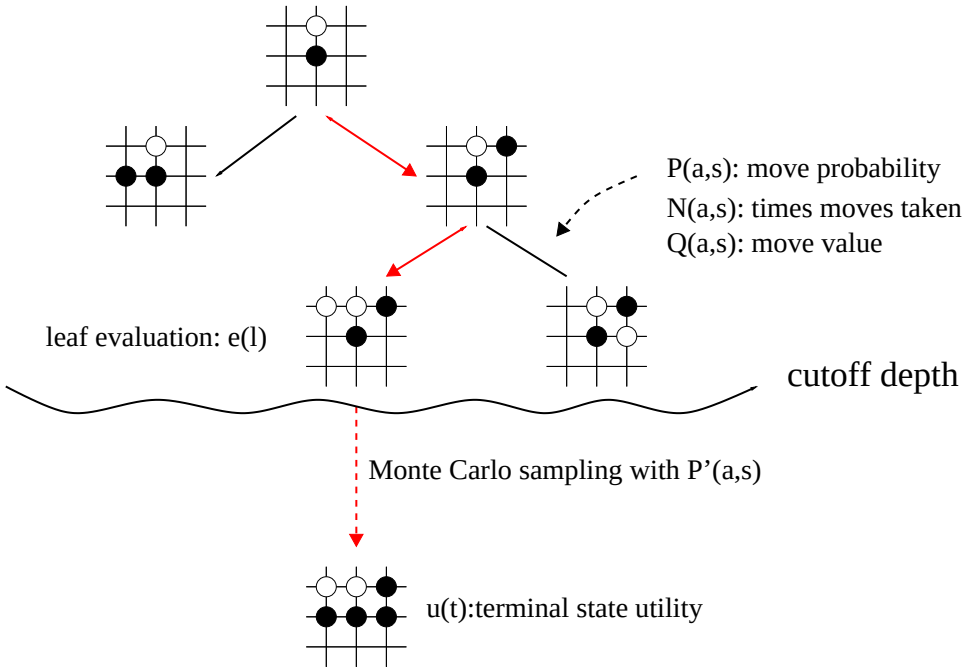
Only the order matters:

payoff in deterministic games acts as an ordinal utility function

Other techniques to tame complexity

- ♢ Symmetry prunning
- ♢ Singular extensions
- ♢ Monte carlo sampling
- ♢ Iterative deepening
- ♢ Pattern databases
- ♢ Deep learning
- ♢ Monte Carlo Tree Search (MCTS)

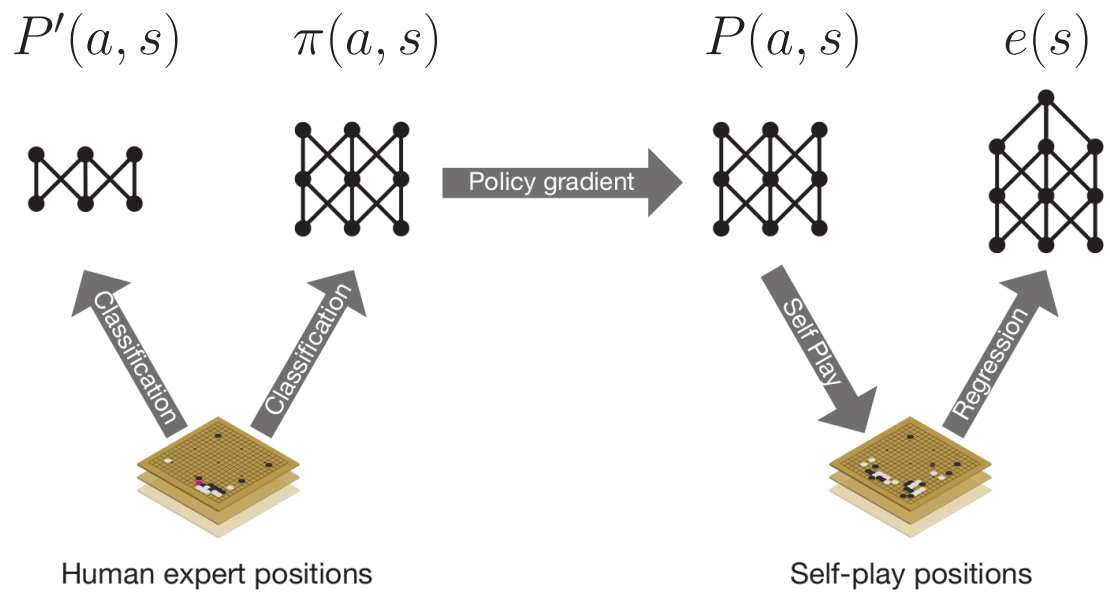
Alpha Go: MCTS + Deep Learning



Replace depth-first search by MCTS exploration of the game tree:

– In state s, select move a maximising $Q(a,s) + P(a,s)/(1 + N(a,s))$
– Leave states are evaluated using function $f(l) = \lambda e(l) + (1 - \lambda)u(t)$
– Terminal state t obtained by sampling from l with proba $P'(a,s)$
– After each simulation, $N(a,s)$ and $Q(a,s)$ are updated along path

Alpha Go: MCTS + Deep Learning



Use deep learning to go beyond simple linear combinations of input features:

– Learns policy $\pi(a,s)$ and $P'(a,s)$ from board features and expert moves
  $\pi(a,s)$: 13 layer convolutional net, 30M samples $s \rightarrow a$, 57% accuracy
  $P'(a,s)$: linear soft-max net, 24% accuracy, runs in 2 μs vs 3 ms
– Learns $P(a,s)$ by improving $\pi(a,s)$ using reinforcement learning
  uses policy gradient by playing against itself, beats 2nd Dan.
– Learns leaf evaluation function $e(l)$ from self-play data

Deterministic games in practice

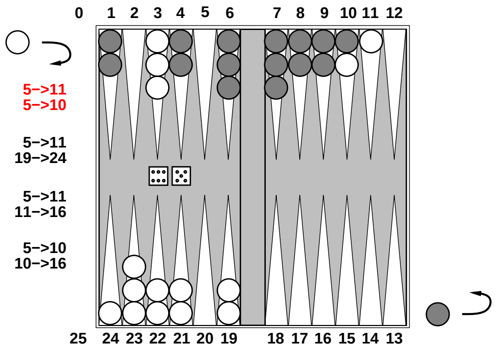
Checkers: Chinook ended 40-year-reign of world champion Marion Tinsley in 1994. Used an endgame database defining perfect play for all positions involving 8 or fewer pieces on the board (500 billion states). In 2007, checkers became the largest game to be completely solved (with 500×10^{20} states!).

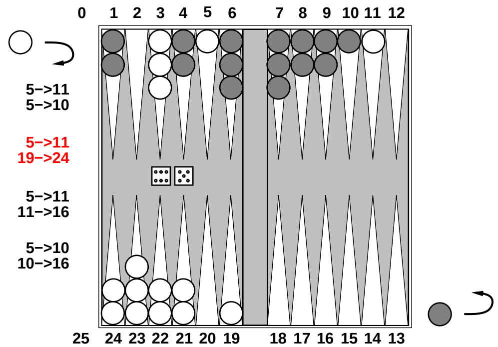
Chess: Deep Blue defeated human world champion Gary Kasparov in a six-game match in 1997. Deep Blue searches 200 million positions per sec, i.e. 300 billion per move (depth 14), uses very sophisticated evaluation (8000 features), and singular extensions to extend some search lines up to 40 ply.

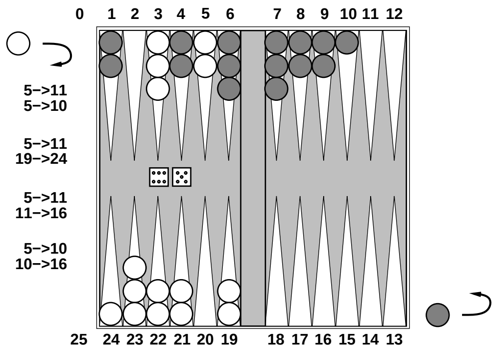
Go: branching factor $b > 300$ made this more challenging. Monte Carlo Tree Search (MCTS) is the method of choice. Zen defeated a 9 Dan in 2013. In 2016 AlphaGo made a surprising 4-1 win against Lee Sedol, using deep learning to learn a value function and policy to guide MCTS.

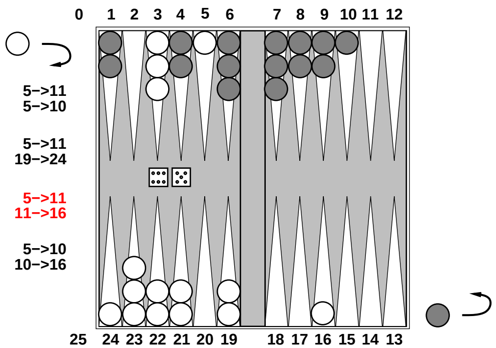
Poker: the next big thing! But it's a stochastic game!

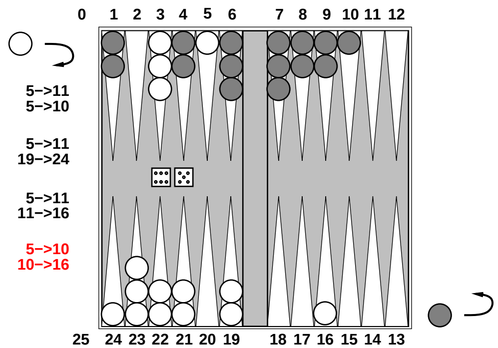
Stochastic games: backgammon









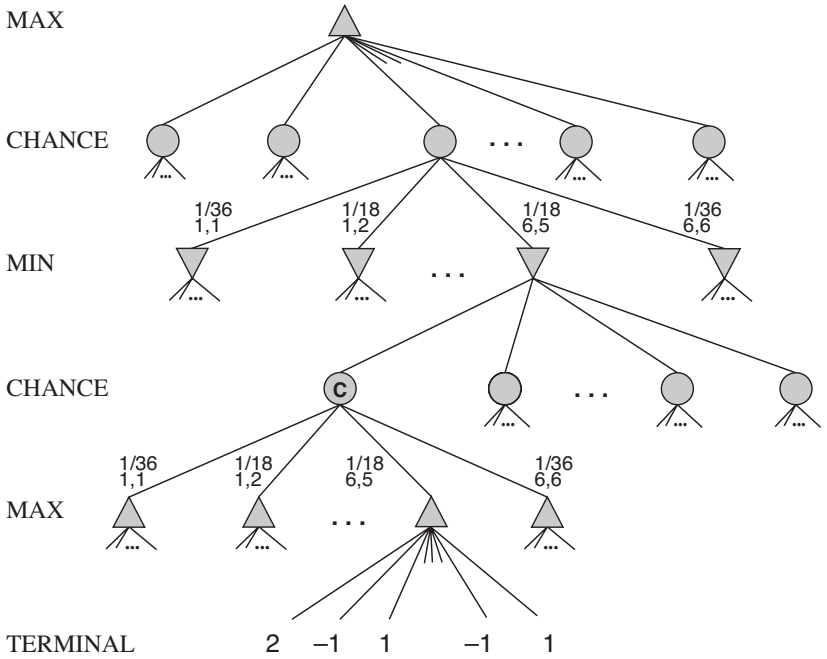


Stochastic games in general

Chance is introduced by dice, card-shuffling, coin flipping.

"Chance" can be seen as special kind of player whose move is the outcomes of a random event, which determines the space of legal moves down the tree.

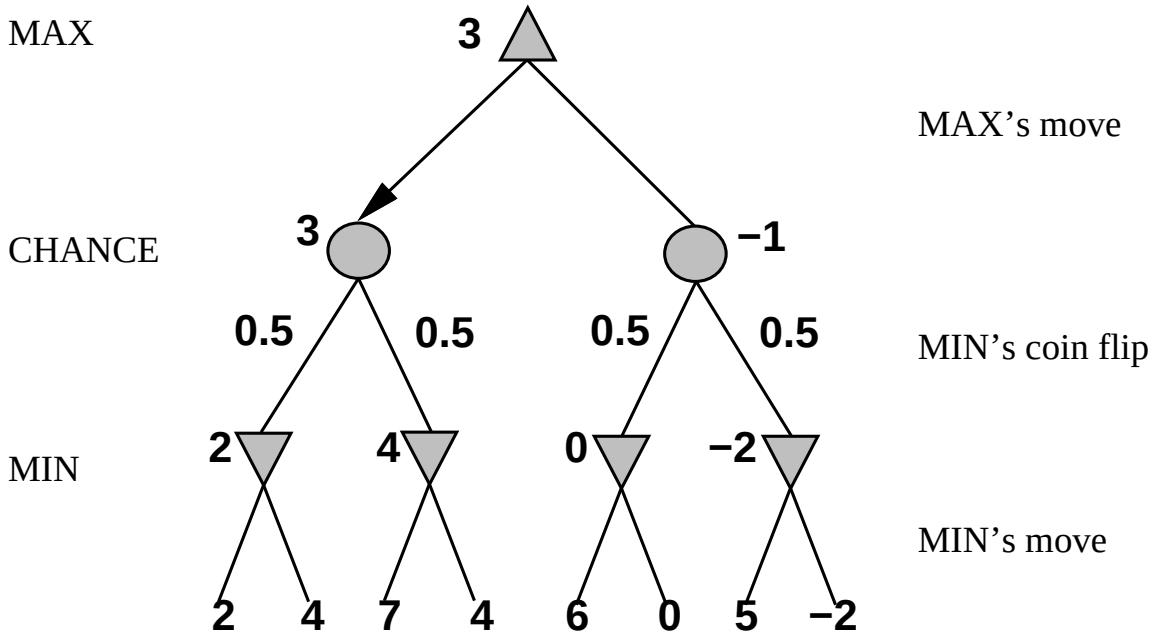
Chance is not adversarial: the value of chance positions is the expectation (average) over all possible outcomes of the value of the result.



Minimax in Stochastic Games

ExpectiMinimax gives perfect play. Like Minimax, except we must also handle chance nodes:

$$\textsc{ExpectiMinimax-Value}(s) = \begin{cases} \textsc{Utility}(s, \textsc{max}) & \text{if } \textsc{Terminal-Test}(s) \\ \max_{m \in \textsc{Moves}(s)} \textsc{ExpectiMinimax-Value}(\textsc{Result}(s,m)) & \text{if } \textsc{Player}(s) = \textsc{max} \\ \min_{m \in \textsc{Moves}(s)} \textsc{ExpectiMinimax-Value}(\textsc{Result}(s,m)) & \text{if } \textsc{Player}(s) = \textsc{min} \\ \sum_{o \in \textsc{Outcomes}(s)} \Pr(o) \textsc{ExpectiMinimax-Value}(\textsc{Result}(s,o)) & \text{if } \textsc{Player}(s) = \textsc{chance} \end{cases}$$



Stochastic games in practice

Time complexity: $O(b^m n^m)$ where n is the maximum number of outcomes of a chance event

Dice rolls increase the effective branching factor

$n = 21$ possible rolls with 2 dice

$b \approx 20$ legal moves in Backgammon (can be 4,000 with 1-1 roll)

$\approx 10^9$ nodes at depth 4 of the tree

α–β pruning is much less effective

lets us concentrate on the events that are likely to happen

but, as depth increases, probability of reaching a given node shrinks

$\Longrightarrow$ value of lookahead is diminished

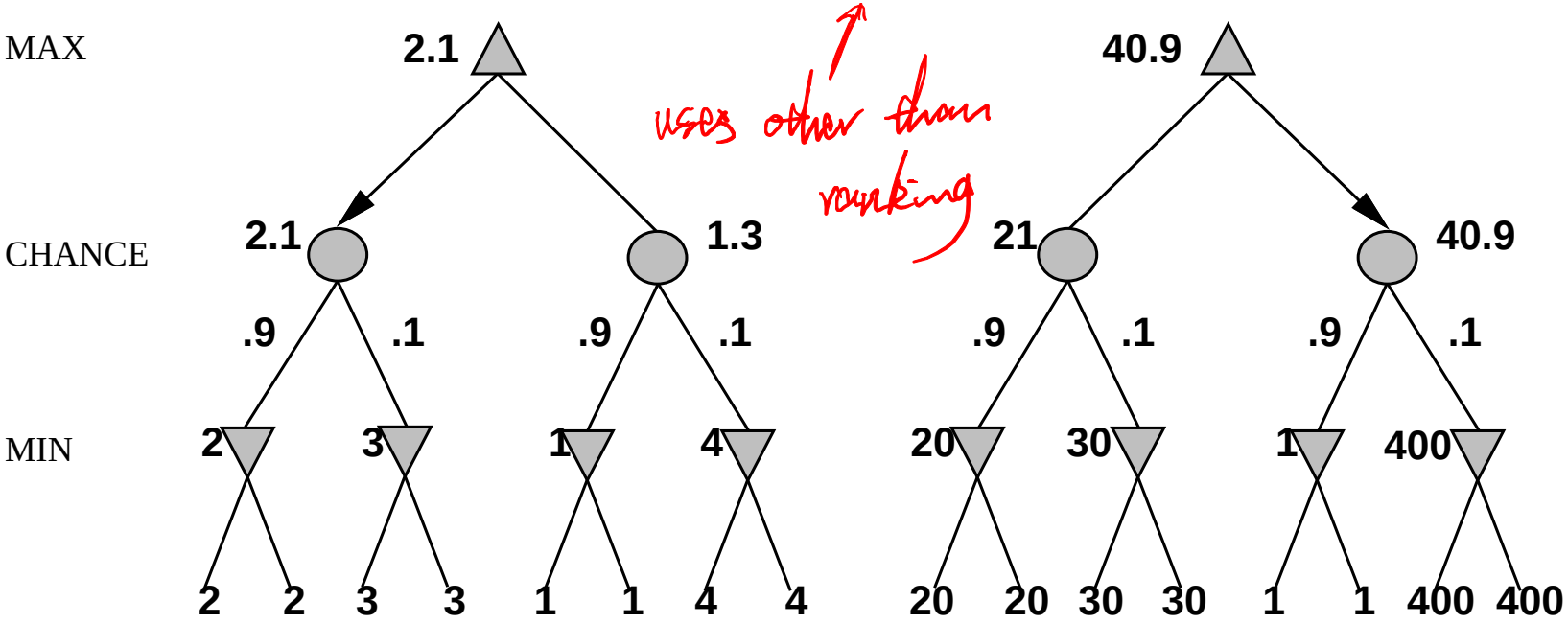
TDGammon uses depth-2 search + very good Eval

$\approx$ world-champion level

evaluation function learnt over millions of games

method combined reinforcement learning with neural nets

Evaluation function: exact values DO matter

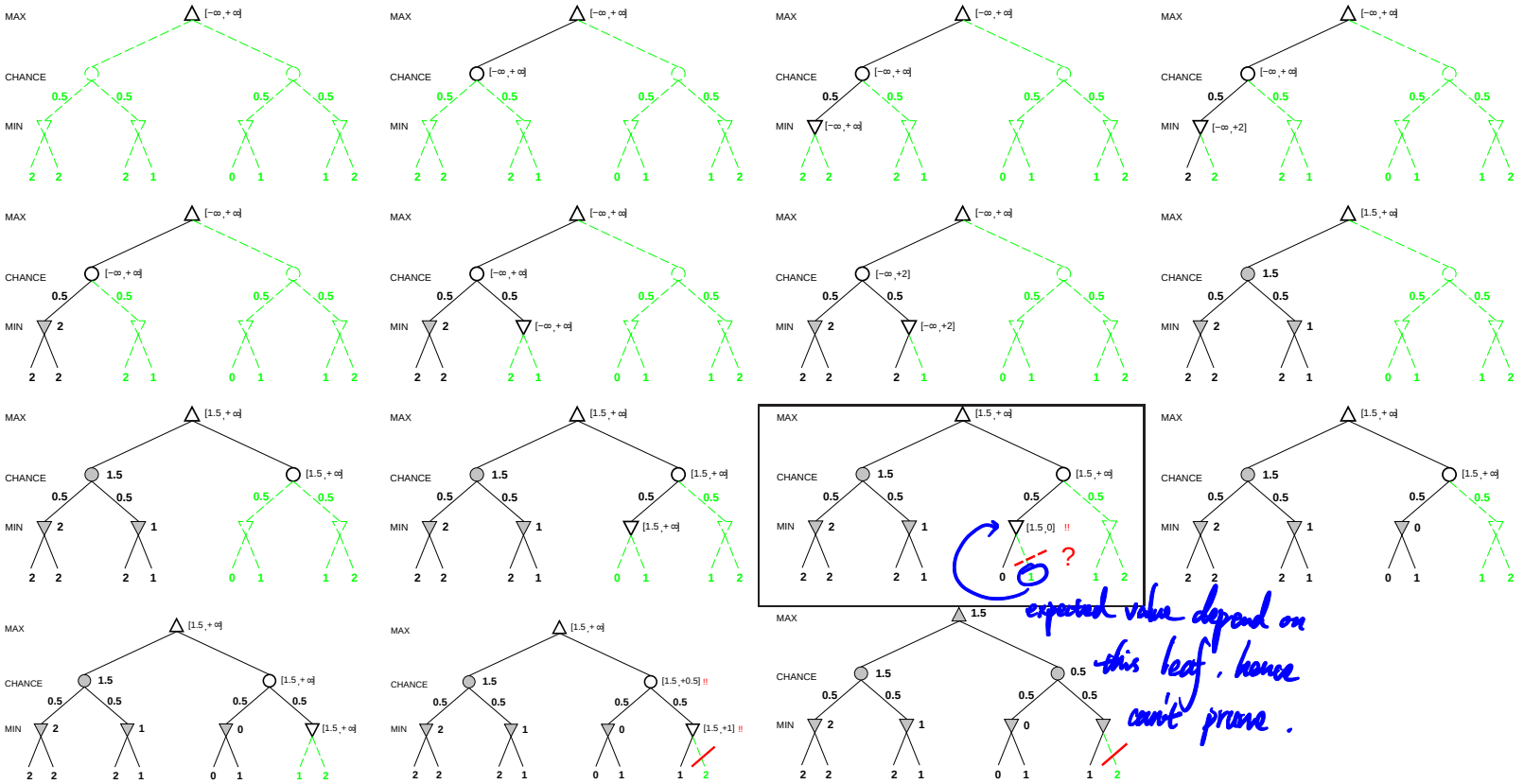


Behaviour is preserved only by positive linear transformation

Hence EVAL should be proportional to the expected payoff determined by UTILITY

Pruning in stochastic game trees

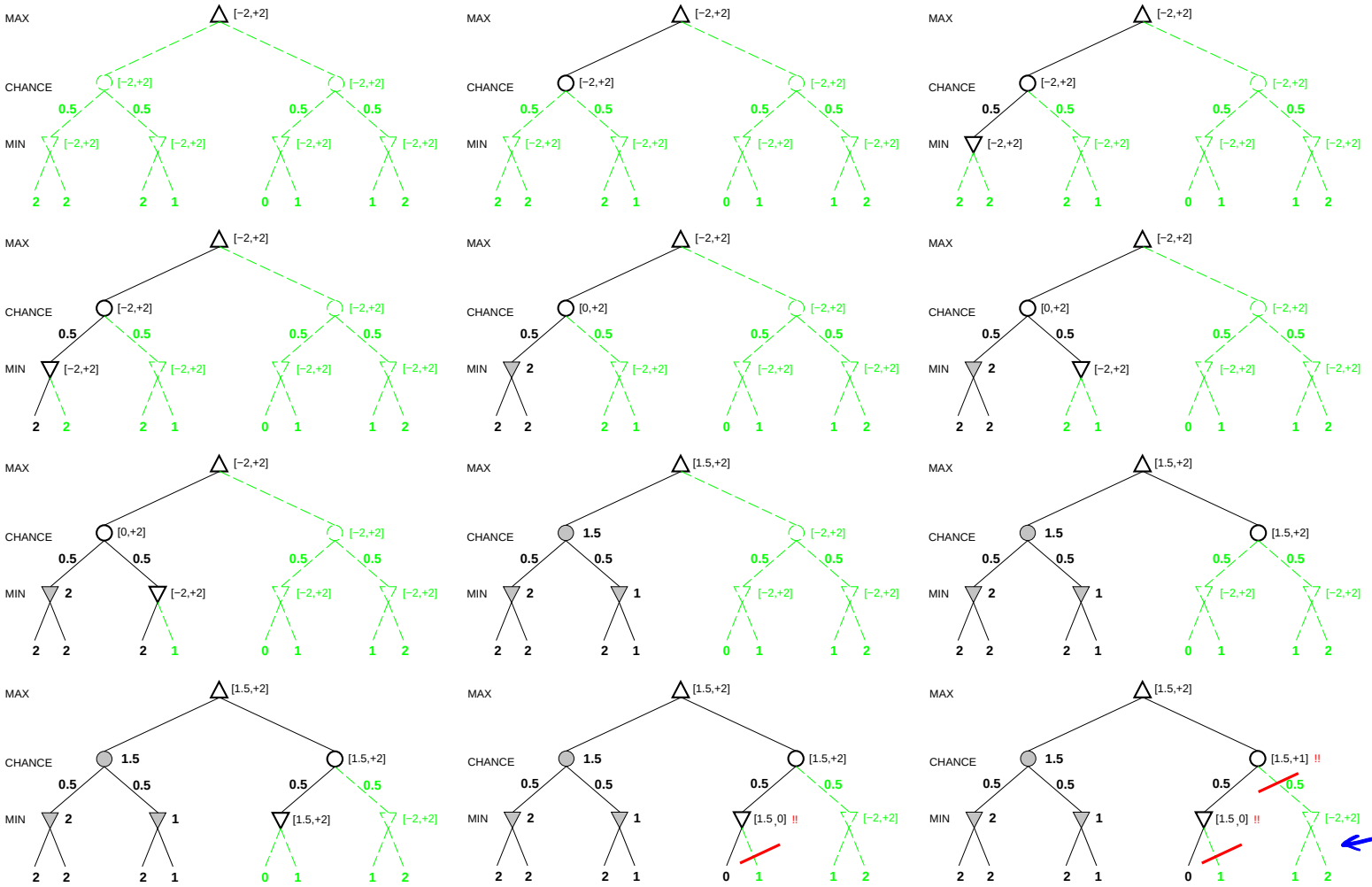
A version of α-β pruning is possible:



We cannot prune because the value of the chance node could still be high enough to be MAX's choice and we need to find out exactly how high it is.

Pruning contd.

More pruning occurs if we can bound the leaf values



We know that the right-hand MIN node will be worth at at most 2, therefore the chance node is worth at most 1 which is not high enough to be MAX's choice.

Summary

A **Game** is defined by an initial state, a successor function, a terminal test, and a utility function

The **minimax** algorithm select optimal actions for two-player zero-sum games of perfect information by a depth first exploration of the game-tree

Alpha-beta pruning does not compromise optimality but increases efficiency by eliminating provably irrelevant subtrees

It is not feasible to consider the whole game tree (even with alpha-beta), so we need to **cut the search off** at some point and apply an **evaluation function** that gives an estimate of the expected utility of a state

Game trees and minimax can be extended to stochastic games by introducing **chance nodes** whose value is the expectation of that of their successors

The value of alpha-beta pruning and lookahead is limited in stochastic games; the evaluation function needs to compensate