Exercise 1

$\theta \sim \text{Beta}(1,1)$ $\quad\quad$ $\theta|y \sim \text{Beta}(1+\Sigma y,\ 1+(n-\Sigma y))$

$(\Sigma y = 0)$ $\quad\quad$ $\sim \text{Beta}(1, n+1)$

$\therefore\ \Pr(\tilde{Y}=1 | y_1 \dots y_n) = E(\theta | y_1 \dots y_n) = \frac{1}{n+2}$.

$$\text{Mode}(\theta | y_1 \dots y_n) = \frac{1-1}{1+(n+1)-2} = 0$$

$\Pr(\tilde{Y}=1 | y_1 \dots y_n)$ is the better posterior summary for predicting the outcome of a future observation because we take into account the uncertainty in the value of θ (rather than relying on a plug in value such as $\hat{\theta} = y/n$ or $\text{Mode}(\theta | y_1 \dots y_n)$.

Exercise 2

$\theta | y \sim \text{Beta}(1+2, 1+8) = \text{Beta}(3, 9)$.

→ Use simulation or qbeta( ) function in R

→ 95% posterior interval for θ is

$[0.06, 0.52]$ (interval ~~lies~~ lies in range of plausible values for θ).

Frequentist

$$0.2 \pm 1.96 \sqrt{\frac{0.2(1-0.2)}{10}}$$

$$= \underbrace{(-0.048,\ 0.45)}_{\text{interval outside range } [0,1]}$$

interval outside range $[0,1]$.

Poisson model

$$p(\theta \mid y_1 \dots y_n) \propto p(\theta)\, p(y_1 \dots y_n \mid \theta)$$

$$\propto p(\theta) \prod_{i=1}^{n} \theta^{y_i} e^{-\theta}$$

$$= p(\theta)\, \theta^{\Sigma y_i} e^{-n\theta}$$

A <u>Conjugate</u> class of priors $p(\theta) \propto \theta^{c_1} e^{-c_2\theta}$.
has the form $\cancel{A}$.

The Gamma distribution has density of this form.

$$\theta \sim \text{Gamma}(a, b)$$

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.$$

and so $p(\theta \mid y) \propto \theta^{a + \Sigma y_i - 1} e^{-(n+b)\theta}$.

$$\theta \mid y \sim \text{Gamma}(a + \Sigma y_i, n+b)$$

b: # prior observations
a: sum of counts from b prior observations