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Remainders & left-overs

Recall: Let a, b, c be odd integers, then $ax^2 + bx + c = 0$ has no rational solution.

Suppose it had a rational solution $\frac{p}{q}$, then $a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0 \iff ap^2 + bpq + cq^2 = 0$.

Let $a = 2k+1, b = 2l+1, c = 2m+1$,

$$\begin{aligned}(2k+1)p^2 + (2l+1)pq + (2m+1)q^2 &= 0\\ 2kp^2 + p^2 + 2lpq + pq + 2mq^2 + q^2 &= 0\\ p^2 + pq + q^2 &= \text{some even number}\end{aligned}$$

Another approach that is by quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and it is equivalent to show that if a, b, c odd integers, then $\sqrt{b^2 - 4ac}$ is irrational.

Is $\sqrt{n}$ rational if and only if n is a perfect square.

Getting back to bounded and unbounded $f : \mathbb{R} \to \mathbb{R}$ is **bounded** if

$$\exists M, \forall x, |f(x)| < M$$

And this is equivalent to say

$$\exists M, \forall x, |f(x)| \leq M$$

Also, like this

$$\exists M > 0, \forall x \in \mathbb{R}, |f(x)| < M$$

or

$$\exists M > 0, \forall x \in \mathbb{R}, |f(x)| \leq M$$

Prove the first and the second statements are equivalent.

1st $\implies$ 2nd. We have a function f satisfying the 1st. Then $\exists M$ such that $|f(x)| < M$ for all $x \in \mathbb{R}$. But this implies $|f(x)| \leq M$ for all $x \in \mathbb{R}$. This means f satisfies the 2nd with the same M.

2nd $\implies$ 1st. Suppose f satisfies the 2nd. i.e. $\exists M$ such that $|f(x)| < M$ for all $x \in \mathbb{R}$. Let $N = M + 1$, then $|f(x)| < M + 1 = N$ for all $x \in \mathbb{R}$. Thus f satisfies the 1st condition as well.

Exercise: Try to prove condition 1 and 3 are equivalent.

An unbounded function satisfies

$$\begin{aligned} &\neg(\exists M\ \forall x\ , |f(x)| < M) \\ \iff &\forall M \exists x, \neg(|f(x)| < M) \\ \iff &\forall M \exists x, (|f(x)| \geq M) \end{aligned}$$

Let's prove that $f(x) = x^3$ is unbounded.

Given any real number M, consider $x = \sqrt[3]{M}$, then $f(x) = M \implies |f(x)| \geq M$. Done.

What does it mean to say

$$\lim_{x \to \infty} f(x) = \infty$$

. We can make f arbitrarily large by taking x arbitrarily large.

$$\forall x\ \forall k > 0 \text{ such that } f(x) < f(x + k)$$

$\forall M\ \exists N$ such that $(x > N) \implies (f(x) > M)$.