Lecture 13

Ex: $F_\lambda(x) = \lambda x - x^3$

fixed pts: $x=0$, $F_\lambda'(0) = \lambda$ $\begin{cases} \text{attract} & |\lambda|<1 \\ \text{repel} & |\lambda|>1 \end{cases}$

$x = \pm\sqrt{\lambda-1}$ $\quad F_\lambda'(\pm\sqrt{\lambda-1}) = 3-2\lambda$

attract: $1<\lambda<2$
repel: $\lambda>2$

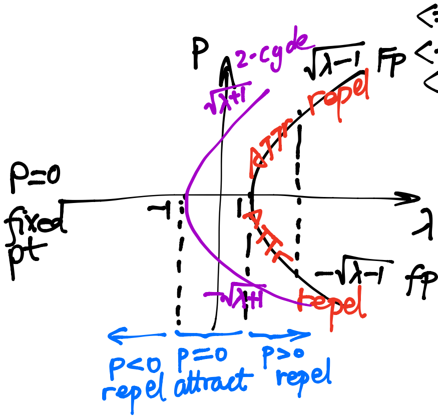
2-cycles: $F_\lambda(x)$ is odd $\longrightarrow F_\lambda(-x) = -F_\lambda(x)$

so if $F_\lambda(x_0) = -x_0$ Then $F_\lambda^2(x_0) = F_\lambda(F_\lambda(x_0)) = F_\lambda(-x_0) = -F_\lambda(x_0)$
$= -(-x_0)$
$= x_0$

we solve

$$F_\lambda(x) = -x \iff \lambda x - x^3 = -x$$
$$\iff (\lambda+1)x - x^3 = 0$$
$$\iff x(\lambda+1-x^2) = 0$$
$$\iff \underset{\text{Fixed pt}}{x=0} \text{ or } \underset{\text{2-cycle}}{x = \pm\sqrt{\lambda+1}}$$

is repelling always (check)



So F_λ has a period doubling bifurcation at $\lambda=-1$.

Recall: There're more 2-cycles for F_λ.
The one we found is the 2-cycle characteristic to odd functions. $F_\lambda(x) = -x$.

Let us find the other 2-cycle.

$$F_\lambda^2(x) = x \iff \lambda(\lambda x - x^3) - (\lambda x - x^3)^3 = x$$
$$\iff x(x^2-\lambda-1)$$
$$\iff x\underbrace{(x^2-\lambda-1)}_{\text{2-cycle}}(x^4-\lambda x^2+1)\underbrace{(x^2-\lambda+1)}_{\text{FPs}} = 0$$
$$\iff x=0 \text{ or } x = \pm\sqrt{\lambda+1} \text{ or } \pm\sqrt{\lambda-1} \text{ or}$$

$$x^2 = \frac{\lambda \pm \sqrt{\lambda^2-4}}{2}$$
$$x = \pm\sqrt{\frac{\lambda\pm\sqrt{\lambda^2-4}}{2}} \qquad \lambda \geq 2$$

these 4 pts form 2-cycles
check that $\lambda=2$, there are 2 Period-doubling bifurcations.

CHAPTER 7 Quadratic Family

recall that $Q_c(x) = x^2 + c$

① For $c > 1/4$, orbits $\to \infty$

② For $c = 1/4$, 1 fixed pt at $p = 1/2$, neutral

③ For $c < 1/4$, 2 fixed pts, P_-, P_+, P_+ is repelling,

(a). For $-3/4 < c < 1/4$, P_- attracting

(b). For $c = -3/4$, P_- neutral

(c). For $c < -3/4$, P_- repelling.

④ For $c < -3/4$, one 2-cycle, q_-, q_+

(a) For $-5/4 < c < -3/4$, attracting

(b) $c = -5/4$, neutral

(c). $c < -5/4$, repelling

§ 7.1 Case $c = -2$

- 2 fixed pts (repelling) $P_- = -1$, $P_+ = 2$

define: $I = [-P_+, P_+] = [-2, 2]$

Then $Q_{-2} : I \to I$

So for any seed $x_0 \in I$, its orbit for Q_{-2} will stay in I, but it won't converge.