29.11.11

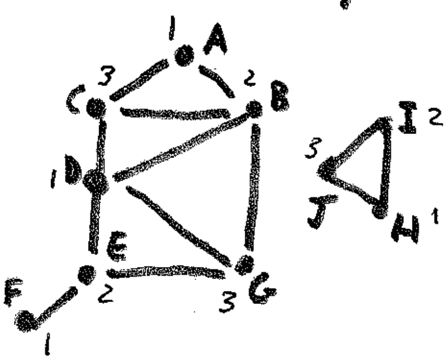
Lecture 12 handout

Bonus

(14.1)

The greedy colouring algorithm

- Order vertices of G $v_1, \ldots, v_n$.
- Colour vertices one by one in this order, assigning v_i the smallest positive integer not yet assigned to its neighbours.



- A **stable set** is a set of vertices no two of which are adjacent
- A **clique** is a set of mutually adjacent vertices.
- $w(G) = \alpha(\bar{G})$

size of biggest clique ($w(G)$) — size of biggest stable set ($\alpha(\bar{G})$).

Observation: $\chi(G) \geq w(G)$.

Observation: $\chi(G) \geq k+1$, k = max. deg. of G.

(14.4)

Definition: A perfect graph is a graph for which $\chi(G) = w(G)$, and $\chi(H) = w(H)$ for every induced subgraph H of G.

Perfect Graph Theorem (Lovász)

G is perfect $\Leftrightarrow$ $\bar{G}$ is perfect

Corollary:

König's Theorem (!!!)

Proof:

- Bipartite G is perfect
- matching M on G

$\Updownarrow$

$n-|M|$ colouring of $\bar{G}$

$\Updownarrow$ Thm

stable set with $n-|M|$ vertices on $\bar{G}$

$\Updownarrow$

vertex cover with $|M|$ elements on G.

Strong Perfect Graph Theorem (2006 !)

G is perfect iff it contains no odd cycle of length ≥ 5, or its complement, as an induced subgraph.

"Perfect graphs are P".

Dessert: Matroids.

Example: T_1, T_2 spanning trees of G.

$\forall_{e \in T_1} \exists_{f \in T_2}$ s.t. $(T_1 - \{e\}) \cup \{f\}$ is a spanning tree.

Example: $B_{1,2}$ bases of a vector space V.

$\forall_{v \in B_1} \exists_{w \in B_2}$ s.t. $(B_1 - \{v\}) \cup \{w\}$ is a basis.

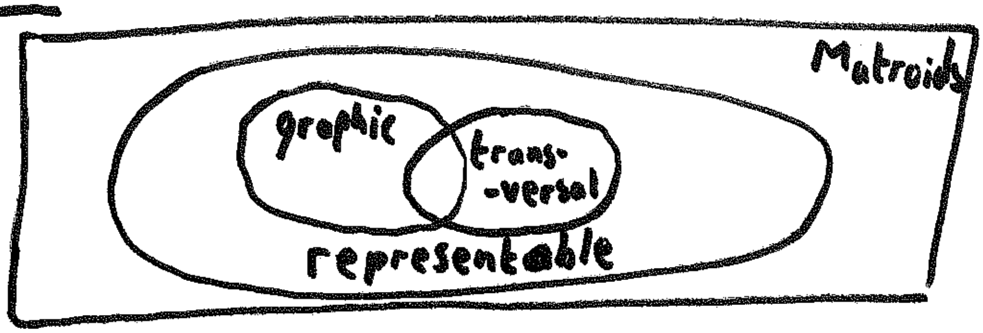
Greedy algorithms find bases and (minimal) spanning trees.
They're both matroids!

① **Definition:** (H. Whitney, T. Nakasawa, B van der Waerden...)

A matroid M is a pair $(E, \mathcal{B})$, E a non-empty finite set, $\mathcal{B}$ a non-empty collection of subsets of E (bases) s.t.

(i) no base properly contains another base.
(ii) Given bases $B_{1,2}$, $\forall_{e \in B_1} \exists_{f \in B_2}$ s.t. $(B_1 - \{e\}) \cup \{f\}$ is a base.

Examples: circuit matroid; vector matroid.



(2) <u>Definition</u>:

M is a pair $(E, \mathcal{I})$, $\mathcal{I}$ called <u>independent sets</u> such that:

- a subset of an independent set is independent
- I, J independent sets, $|J| > |I|$ then $\exists e \in J$, $e \notin I$ s.t. $I \cup \{e\} \in \mathcal{I}$.

- A maximal independent set is a <u>base</u>.
- A minimal dependent set is a <u>circuit</u>.

(3) <u>Definition</u>:

M is a pair (E, ρ), $\rho : \{\text{subsets of } E\} \to \mathbb{Z}_{\geq 0}$

- $\rho(A) \leq |A| \quad \forall A$ (Cardinality bound)
- If $A \subseteq B$ then $\rho(A) \leq \rho(B)$ (Increase)
- $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ (submodularity).

ρ is <u>rank</u>.

<u>Graphic matroid</u>: rank is size of spanning tree.

<u>Representable matroid</u>: M on E is representable over F if $\exists$ vector space V over F, $\phi : E \to V$ preserving (in)dependent sets.

Circuit matroid is representable over F_2

edge $e \mapsto$ row of e in incidence matrix.

This is what graph theory has to do with linear algebra!

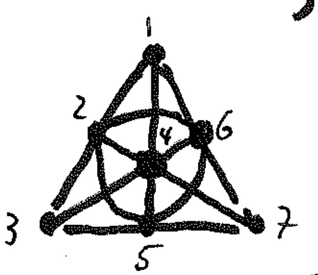
Example: Fano matroid $F = (E, \mathcal{B})$

$E := \{1,2,3,4,5,6,7\}$

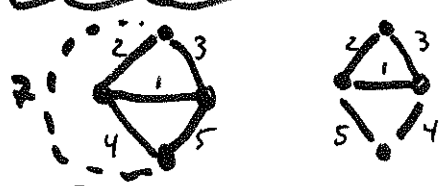
$\mathcal{B} = \{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,7\}, \{2,5,6\}, \{3,4,6\}, \{3,5,7\}$

Finite projective plane!!! $(\mathcal{P}, \mathcal{L})$

1) 2 distinct pts in $\mathcal{P}$ on one line.
2) 2 lines intersect in one point.
3) There are four points in $\mathcal{P}$, no three of which are collinear.



Not graphic! $C_1 = \{1,2,3\}$, $C_2 = \{1,4,5\}$, $C_3 = \{2,4,7\}$



$\{1,6,7\}$ cannot be a cycle.

Matroid dual: $M = (E, p)$; $M^* = (E, p^*)$

$$p^*(A) := |A| + p(E-A) - p(E).$$

Every matroid has a dual, and the dual is unique. M and M^* are both graphic iff they're represented as planar graphs.

Easy proofs: Intro. to Graph Theory, Wilson.

Thank you
for
taking this
course!