Jan 30th

<u>EX:</u> $T: M_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$\qquad\qquad \text{dim} = 4 \qquad\qquad \text{dim} = 3$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)x^2 + bx + c$$

$$\alpha = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\beta = \{ \underset{p_1}{1}, \underset{p_2}{1+x}, \underset{p_3}{1+x+x^2} \}$$

<u>Q1</u>: What's $[T]_\alpha^\beta = [T]_{\beta\alpha}$?

<u>Q2</u>: Use this to find a basis for Ker T

$$T\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = x^2 - x = p_3 - 2p_2 + p_1$$

$$T\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = x - 1 = p_2 - 2p_1$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right) = -x^2 + 1 = -p_3 + p_2 + p_1$$

$$T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2x^2 + x + 1 = 2p_3 - p_2$$

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix} = A$$

$$Q^2: A \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & -1 \\ 0 & 2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} x & y & z & w \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{matrix} w = 0 \\ y = z \\ x = z \end{matrix}$$

$$\text{Ker} A = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \Longrightarrow \text{Ker} T = \text{span}\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

Reverse engineer w/ get T

$$T\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = ?$$

$$\left[\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From yesterday: $\qquad [T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$

$$\left[T\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right)\right]_\beta = Ae_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = p_1 - 2p_2 + p_3 = 1 - 2(1+x) + 1 + x + x^2$$
$$= x^2 - x$$

Claim: $T: V \to W$, V has basis $\{v_1, \cdots, v_n\}$ T is completely determined by $T(v_1), \cdots, T(v_n)$

Proof: $v \in V$, $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n \Rightarrow T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$

§ 2.6 Inverses

Recall: A $n \times n$ matrix. What does it mean for A to be invertible?

$\exists$ a matrix B, s.t. $AB = BA = I$

DEF: Let $T: V \to W$ be a linear transf.
Then T is inventible of these exists a lin trans: $S: W \to V$ s.t. $T \cdot S = I_W$, $S \cdot T = I_V$

Claim: If $T: V \to W$, B is invertible then the transf is unique. We call it the "inverse of T" and denote T^{-1}!

Proof: Sps S, S' both satisfy:

$$T \cdot S = T \cdot S' = I_W \text{ and } S \cdot T = S' \cdot T = I_V$$

Want: $S(w) = S'(w)$ for all $w \in W$

$$S(w) = S(I_W(w))$$
$$S(w) = S(I_W(w))$$
$$= S(TS')(w)$$
$$= (ST)S'(w)$$
$$= I_V S'(w) = S'(w) \ \ldots$$

■

Claim 2: If T is bijective (i.e. inj and surj.)

Proof: (inj.) $= v \in \ker T \Rightarrow X(v) = 0$

$$\Rightarrow T^{-1}(T)(v) = T^{-1}(0)$$
$$\Rightarrow v = \{0\}$$

(surj.): $T: V \to W$

Want: given any $w \in W$, $\exists v \in V$ s.t. $T(v) = w$.

$w \in W$ $I_W(w) = w \Rightarrow \underbrace{TT^{-1}}_{I_W}(w) = w$

Let $v = T^{-1}w$ then $T(v) = w$. ■

Claim: $T: V \to W$ bijective, then T is invertible

Proof: We need to define $S: W \to V$

Given $w \in W$, need to define $(S(w))$

Since T is surj. there's a vector v. st $T(v) = w$

Sine ... injective, v must be unique.

We define $S(w) = v$. Remains to show S linear trans.