STAT6039 review examples

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0.1 multinomial distribution theorems

Suppose that Y_1, Y_2 and Y_3 are three rv's with means $2, -7$ and 5, variances $10, 6, 9$ and covariance $\sigma_{12} = -1, \sigma_{13} = 3, \sigma_{23} = 0$.

$$\begin{aligned}
E(3Y_1 - 2Y_2 + Y_3) &= 3\mu_1 - 2\mu_2 + \mu_3 \\
&= 3(2) - 2(-7) + 5 = 25 \\
Var(3Y_1 - 2Y_2 + Y_3) &= 3^2\sigma_1^2 + (-2)^2\sigma_2^2 + 1^2\sigma_3^2 + 2\{3(-2)\sigma_{12} + 3(1)\sigma_{13} + (-2)(1)\sigma_{23}\} \\
&= 9(10) + 4(6) + 1(9) + 2\{-6(-1) + 3(3) - 2(0)\} \\
&= 153 \\
Cov(3Y_1 - 2Y_2, Y_2 + 8Y_3) &= 3(1)\sigma_{12} + 3(8)\sigma_{13} + (-2)1\sigma_{22} + (-2)8\sigma_{23} \\
&= 3(-1) + 24(3) - 2(6) - 16(0) \\
&= 57
\end{aligned}$$

0.2 law of iterated expectation/variance/covariance

Twenty bolts just been randomly sampled from the production line. Count the number of defectives amongst them. From experience, the proportion of defectives is constant any given day, but varies from day to day in a uniform manner between 0.1 and 0.3.

(a) Let X be the number of defectives amongst the 20, Y be the proportion of defectives amongst all bolts produced in the factory today.

Then $(X|Y = y) \sim Bin(20, y)$, and $Y \sim U(0.1, 0.3)$. So $E(X|Y = y) = 20y, E(X|Y) = 20Y, EY = 0.2$.

Therefore $EX = EE(X|Y) = E(20Y) = 20EY = 4$.

(b) $Var(X|Y = y) = 20y(1 - y)$. Therefore $Var(X|Y) = 20Y(1 - Y) = 20(Y - Y^2)$. So

$$\begin{aligned}
Var(x) &= EVar(X|Y) + VarE(X|Y) \\
&= E\{20(Y - Y^2)\} + Var\{20Y\} \\
&= 20(EY - EY^2) + 400VarY
\end{aligned}$$

Now $VarY = \frac{(0.3-0.1)^2}{12} = \frac{1}{300}$, $EY^2 = VarY + (EY)^2 = \frac{13}{300}$. So...

0.3 MLE

A partly melted die is rolled repeatedly until the first 6 comes up. Then it is rolled again the same number of times. We are interested in p, the probability of 6 coming up on a single roll. Suppose that the first 6 comes up on the third roll, and the numbers which then come up are $6, 2, 6$. Find the MLE of p.

Let X=number of rolls until first 6, and Y=number of 6's on last half of rolls.

Then $X \sim Geo(p)$ (with $x = 3$) and $(Y|X = x) \sim Bin(x, p)$ (with $x = 2$).

So $p(x, y) = p(x)p(y|x) = (1-p)^{x-1}p\binom{x}{p}p^y(1-p)^{x-y}; x = 1, 2, \ldots; y = 0, \ldots, x$.

So $L(p) = (1-p)^{x-1+x-y}p^{1+y} = (1-p)^a p^b$, where $a = 2x - y - 1$ and $b = 1 + y$.

Then $l(p) = a\log(1-p) + b\log p, l'(p) = -\frac{a}{1-p} + \frac{b}{p} = 0 \implies p = \frac{b}{a+b}$.

$$\hat{p} = \frac{b}{a+b} = \frac{1+y}{(2x-y-1)+(y+1)} = \frac{1+y}{2x} = \frac{1+2}{2(3)} = \frac{1}{2}$$