Assignment 5 - MAT 327 - Summer 2014

Due July 7th, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Suppose that $\langle x_n \rangle$ is a (countable) increasing sequence in ω_1. Show that there is a point $p \in \omega_1$ such that $x_n \to p$. (**Hint**: You only know 2 ways to find points in ω_1.)

Proof. Consider the countable *set* $C = \{x_n : n \in \mathbb{N}\}$. If the sequence is eventually constant, then clearly it converges to that point. Otherwise, since it is increasing it must be strictly increasing.

Let $B := \{m : x_n \leq m, \forall n \in \mathbb{N}\}$. Since C is countable we know that B is non-empty (by a property of ω_1). So let $p := \min B$ (which exists since ω_1 is a well-order).

<u>Claim</u>: $\langle x_n \rangle$ converges to p.

Suppose not. Let (a, ∞) be a basic open set that contains p, but is disjoint from C. Note that $a < p$. So a must also be an upper bound for C. Thus a is a smaller element of B, which is a contradiction. $\square$

[**C.2**] Let $(L, \leq)$ be a linear order. Prove that $(L, \leq)$ is a well-order if and only if L does not contain any infinite decreasing chains. (A chain $C = \{\, x_n : n \in \mathbb{N} \,\}$ is decreasing provided that $n < m$ implies $x_m < x_n$.)

Proof. First [$\Rightarrow$], by proving the contrapositive. Suppose that L has an infinite decreasing chain C. Consider the collection

$$I := \{x \in C : \text{ there are infinitely many } y \in C \text{ with } y \leq x\}$$

If we show that I is non-empty, then clearly I will have no minimal element, so L will not be a well order. This is clear though, since C is a decreasing chain we have $C \subseteq I$.

[$\Leftarrow$] Again we prove it by contrapositive. Suppose that $(L, \leq)$ is not a well-order. So there is a non-empty $S \subseteq L$ such that S has no least element. If S was finite, then its minimum would exist (as non-empty finite subsets of a linear always have a minimal element), so S must be infinite. $\square$

[**C.3**] Suppose that ω_1 is in bijective correspondence with $\mathbb{R}$. Is $\mathbb{R}$ with its usual order a well-order? Why or why not?

Proof. Any such bijection has nothing to with the usual order on $\mathbb{R}$, (which is clearly not a well-order). Some students like to claim (for whatever bizarre reason) that $\mathbb{R}$ with it's usual order is a well order if such a bijection were to exist. I don't know why. $\square$

[**C.4**] By exhibiting an open cover that doesn't have a countable subcover, prove that ω_1 with the order topology is not a Lindelöf space. (Conclude that first countable does not imply Lindelöf.)

Proof. For $\alpha \in \omega_1$ let $U_\alpha := \{ x \in \omega_1 : x < \alpha \}$, which is a basic open set (which is also countable). Clearly $\{ U_\alpha : \alpha < \omega_1 \}$ is a cover of ω_1. If $A \subset \omega_1$ is countable, then $\bigcup_{\alpha \in A} U_\alpha$ is a countable union of countable sets, so is definitely not all of ω_1. $\square$

[**C.5**] Prove that every second countable space is a Lindelöf space.

Proof. Let $\mathcal{B}$ be a countable basis of a topological space X, and let $\mathcal{U} := \{ U_\alpha : \alpha \in I \}$ be a collection of open sets such that $X \subseteq \bigcup_{\alpha \in I} U_\alpha$.

For each point $x \in X$ there is a $U_{\alpha_x} \in \mathcal{U}$ such that $x \in U_{\alpha_x}$. Since $\mathcal{B}$ is a basis there is a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_{\alpha_x}$.

Now, clearly $X \subseteq \bigcup_{x \in X} B_x$ (and clearly there are only countably many such B_x), but, who cares? We wanted a subcollection of $\mathcal{U}$ that covers X. So for each B_x, choose one U'_{α_x} such that $B_x \subseteq U'_{\alpha_x}$. This (countable) collection of U'_{α_x} is exactly the countable collection that we wanted. $\square$

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] Suppose that $(L, \leq)$ is a linear order. Prove that L, with the order topology, has no non-trivial clopen subsets if and only if L is Dedekind complete and has no gaps.

Proof. The strategy here is to prove $\neg \Leftarrow \neg$ by showing that a gap gives a non-trivial clopen set, and any set that witnesses the failure of Dedekind completeness is a non-trivial clopen set. Then we show $\neg \Rightarrow \neg$ by showing that if L has a non-trivial clopen set and it is Dedekind complete, then it has a gap.//

[$\neg \Leftarrow \neg$] Suppose that L has a gap given by $x < y$. Clearly, $(-\infty, y) = (-\infty, x]$ is a non-trivial clopen set.

So now suppose instead that L is not Dedekind complete. That means that there is a non-empty set $S \subseteq L$ that is bounded above by M, but has no least upper bound. Let

$$A := \{b \in L : s < b, \forall s \in S\}$$

which is the set of all upper bounds for S. This is non-empty (since $M \in A$) and it is not all of L since S is non-empty. To show that A is open, note that if $b \in A$ then b is *not* a least upper bound for S, thus there is a $b' \in A$ such that $b' < b$. Clearly $b \in (b', \infty) \subseteq A$, so A is open. To show that $L \setminus A$ is open, note that if $x \in L \setminus A$ then it is not an upper bound for S. Thus there is an $s \in S \subseteq L \setminus A$ such that $x < s$. Thus $x \in (-\infty, s) \subseteq L \setminus A$, so $L \setminus A$ is open. Together this shows that A is a non-trivial clopen set.

[$\neg \Rightarrow \neg$] Let $C \subseteq L$ be a non-trivial clopen set and suppose that L is Dedekind complete. Since C is non-trivial, let $M \in L \setminus C$ and, without loss of generality, assume that there is a point $c \in C$ such that $c < M$. Now let $x = \sup\{a \in C : a < M\}$, which exists since L is Dedekind complete. Since C is closed, $x \in C$. Since C is open, there is an open set $(a, y) \subseteq C$ such that $x \in (a, y)$. Here we see that $(a, y) = (a, x]$, which clearly tells us that x, y form a gap. $\square$

[**A.2**] Prove that $\omega_1 + 1$ is a Lindelöf space. (Conclude that Lindelöf does not imply first countable or second countable.) If you are hungry, then prove that $\omega_1 + 1$ is actually *compact*, (that is, every open cover has a *finite* subcover).

Proof. Let $\mathcal{U}$ be an open cover of $\omega_1 + 1$. There is an open set $U_\Omega \in \mathcal{U}$ such that $\Omega \in U_\Omega$, where Ω is the largest element in $\omega_1 + 1$. Since U_Ω is open it contains a basic open set of the form $(\alpha, \Omega]$, where $\alpha \in \omega_1$. Now to show that $\mathcal{U}$ has a countable subcover it suffices to observe that $[0, \alpha]$ is a countable set, and for each $x \in [0, \alpha]$ we can pick an $U_x \in \mathcal{U}$ such that $x \in U_x$. Thus $\{U_x : x \in [0, \alpha] \cup \{\Omega\}\}$ is the desired countable subcover.

To show compactness, we start of the same way by choosing a U_Ω that contains Ω and covers $(\alpha_1, \Omega]$. Now we cover α_1 by a set $U_{\alpha_1} \in \mathcal{U}$. If U_{α_1} contains an interval of the form $(\alpha_2, \alpha_1]$ then repeat this process with α_2. If U_{α_1} does not contain such an interval, then $\{\alpha_1\}$ has an immediate predecessor, α_2. So repeat this process with α_2.

Either way we see that $\alpha_1 > \alpha_2 > \alpha_3 > \ldots$ and since $\omega_1 + 1$ is a well order, this descending chain must be finite. The corresponding U_{α_i} (together with U_Ω) will produce a finite subcover. $\square$

[**A.3**] Prove that every regular Lindelöf space is normal.

Proof. This proof is identical to the proof that regular second countable spaces are normal (in §9), except that we produce a countable cover of the closed sets using the Lindelöf property instead of second countability.

Let A, B be closed disjoint sets in X, a regular Lindelöf space. For each point $a \in A$ fix an open set U_a such that $a \in U_a$ and $\overline{U_a} \cap B = \emptyset$, which we can do since X is regular. Notice that $A \subseteq \bigcup_{a \in A} U_a$, and so $X = (X \setminus A) \cup \bigcup_{a \in A} U_a$. Since X is a Lindelöf space there is a countable set $\{\, U_n : n \in \mathbb{N} \,\}$ such that

$$A \subseteq \bigcup_{a \in A} U_a = \bigcup_{n \in \mathbb{N}} U_n$$

(Notice that $X \setminus A$ is disjoint from A, so once we have a countable cover of X, if that included the set $X \setminus A$ then we throw it away and still have a cover of A.) Similarly, find

$\{ V_n : n \in \mathbb{N} \}$ such that

$$B \subseteq \bigcup_{n \in \mathbb{N}} V_n$$

with the additional requirement that $\overline{V_n} \cap A = \emptyset$.

From here, repeat the proof in the notes line by line to ensure that $\bigcup_{n \in \mathbb{N}} U_n$ and $\bigcup_{n \in \mathbb{N}} V_n$ are disjoint. $\square$

[**A.4.**] Let $C, D \subseteq \omega_1$ be closed (in the order topology), unbounded (called **club**) subsets of ω_1. Prove that $C \cap D$ is a closed unbounded subset of ω_1. (Recall that $C \subseteq \omega_1$ is unbounded means that for all $\alpha \in \omega_1$, there is a $c \in C$ such that $\alpha < c$.) **Strategy**: Weave the two club sets together and use C.1.

Proof. There are a handful of proofs for this, but they all have the same basic idea. Note that $C \cap D$ is closed since the intersection of two closed sets is closed. We only need to show that $C \cap D$ is unbounded.

Let $\alpha \in \omega_1$, and let us construct (recursively) a $\beta \in C \cap D$ such that $\alpha < \beta$. Suppose that we already have chosen

$$c_1 < d_1 < c_2 < d_2 < \ldots < c_n < d_n$$

where $c_i \in C$ and $d_i \in D$ for all $1 \leq i \leq n$. Since C is unbounded, there is a $c_{n+1} \in C$ such that $d_n < c_{n+1}$. Since D is unbounded there is a d_{n+1} such that $c_{n+1} < d_{n+1}$.

By induction we get two "interweaved" sequences $\{ c_i : i \in \mathbb{N} \}$ and $\{ d_i : i \in \mathbb{N} \}$. Since the union of the two sequences is a countable set, there is a *least* $\beta \in \omega_1$ that is above $\{ c_i : i \in \mathbb{N} \} \cup \{ d_i : i \in \mathbb{N} \}$.

<u>Claim</u>: $\beta \in C \cap D$.

Since we chose β to be the least element above $\{ c_i : i \in \mathbb{N} \} \cup \{ d_i : i \in \mathbb{N} \}$ we can see that $\{ c_i : i \in \mathbb{N} \} \to \beta$ and $\{ d_i : i \in \mathbb{N} \} \to \beta$. Since C is closed, $\beta \in C$. Since D is closed, $\beta \in D$. Hence we have the claim. $\square$

New Ideas

[**NI.1**] Prove that every continuous function $f : \omega_1 \longrightarrow \mathbb{R}$ has a countable range (with both spaces given their order topologies). Conclude that every such function is eventually constant. (There are many different ways to prove this!)

Here is the proof of a TA from last year (Ivan):

Definition 1. *Let $(X, \mathcal{T})$ be an uncountable topological space, and $S \subseteq X$ an uncountable subset. A point $x \in X$ is called a* condensation point of S *if every open neighbourhood of x contains uncountably many points of S. That is, for ever U such that $x \in U \in \mathcal{T}$, $S \cap U$ is uncountable.*

Lemma 2. *Let $S \subseteq \mathbb{R}$ be uncountable. Then there exists $s \in S$ such that s is a condensation point of S.*

Proof. We work in the subspace topology on S. Second countability is hereditary, so fix a countable basis $\mathcal{B} = \{ B_n \mid n \in \mathbb{N} \}$ of this topology on S.
Assume for a contradiction that S contains no condensation points of S. This means that for every $s \in S$, there exists an open neighbourhood U containing s such that U is countable. Since $\mathcal{B}$ is a basis, there is an $n(s) \in \mathbb{N}$ such that $s \in B_{n(s)} \subseteq U$.
But then the collection $\{ B_{n(s)} \mid s \in S \} \subseteq \mathcal{B}$ is a countable collection of countable sets covering S, contradicting the fact that S is uncountable. □

Corollary 3. *Let $S \subseteq \mathbb{R}$ be uncountable. Then S contains infinitely many condensation points of S*

Proof. This is an easy induction using the lemma. Let $s \in S$ be a condensation point of S. Then $S \setminus \{s\}$ is uncountable, so apply the lemma again. Continue in this way. □

Proposition 4. *Let $f : \omega_1 \longrightarrow \mathbb{R}$ be continuous. Then the image of f is countable.*

Proof. Assume for a contradiction that the image of f, which we'll call S, is uncountable. Then by the corollary there exist two condensation points of S in S, which we'll denote s_1 and s_2. Using the regularity of S, find two open sets U_1 and U_2 (in the subspace topology on S) containing s_1 and s_2 respectively, such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Since the s_i are condensation points of S, the sets U_i (and in particular the sets $\overline{U_i}$) are uncountable.

Then, since f is continuous, $f^{-1}(\overline{U_1})$ and $f^{-1}(\overline{U_2})$ are two disjoint closed subsets of ω_1, which are uncountable and therefore unbounded. But this is impossible, since any two closed and unbounded subsets of ω_1 must intersect, by Problem **C.1** on Assignment 5. $\square$

I won't post "solutions" for these NI.2 and NI.3 since the point was to investigate the questions on your own, not read someone else's solution.