Tutorial Problems - Sections 11 to 12 - MAT 327 - Summer 2014

8 Axiom of Choice

1. Fill in the details that Zorn's Lemma implies every vector space has a basis.
2. Which direction of "In a first countable space, f is continuous is equivalent to $x_n \to x$ implies $f(x_n) \to f(x)$" uses the axiom of choice? Show us the proof and point out where the axiom of choice is used.
3. Where is the axiom of choice used in the proof that the countable union of countable sets is countable?
4. Prove that the "finite axiom of choice" is true. That is, every finite collection of non-empty sets has a choice function.

9 Metrizability

1. Let X be a set and let d be the discrete metric on X. Prove that (X, d) is complete.
2. Let A be a bounded subset of a metric space (X, d). Prove that $\overline{A}$ is bounded.
3. Let (X, d) be a metric space and let A be a dense subset of X such that every Cauchy sequence in A converges in X. Prove that (X, d) is complete.
4. Let d be the usual metric and let ρ be the square metric on $\mathbb{R}^2$. Prove that:
   (a) $\rho(a, b) \leq d(a, b)$ for all $a, b \in \mathbb{R}^2$;
   (b) $d(a, b) \leq \sqrt{2} \cdot \rho(a, b)$ for all $a, b \in \mathbb{R}^2$.
5. Fill in the details of the proposition in class that every metrizable space has a *bounded* metric that generates its topology.