March 27th

Recall: Given $T: V \to V$.

Find a canonical basis α s.t. $[T]_\alpha$ is the JCF of T.

Step 1: Compute the char. poly $p(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

Step 2: define $W_{\lambda_i} = \ker((T-\lambda_i I)^{r_i})$

↖ this is "generalized eigenspace"

Primary decomposition

$V = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_r}$

Thm: $\dim W_{\lambda_i} = r_i$

We proved that W_{λ_i} is T-invariant and $T|_{W_{\lambda_i}} \to W_{\lambda_i}$ has only one eigenvalue which is λ_i.

Compute canonical basis of W_{λ_i} s.t. $[T|_{W_{\lambda_i}}]_{\alpha_i}$ has JCF.

Step 3: put it all together. Define $\alpha = \alpha_1 \cup \cdots \cup \alpha_r$ a canonical basis of V. and $[T]_\alpha$ is the JCF of T.

Ex: $T: \mathbb{C}^4 \to \mathbb{C}^4$

$$T = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix} \qquad p(x) = (x-3)^2(x-2)^2$$

$\Rightarrow \mathbb{C}^4 = W_3 \oplus W_2$, where $W_3 = \ker((T-3I)^2)$

$W_2 = \ker((T-2I)^2)$

both two-dim spaces

• Find JCF of $T|_{W_3}$ and a canonical basis α_1 of W_3

To compute tableau of $T|_{W_3}$ $\dim\ker((T-3I)) = ?$ 1

$\dim\ker((T-3I)^2) = 2$

$$T - 3I = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 4 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Tableau of $(T-3I)|_{W_3}$ is $\boxed{}\boxed{}$

so JCF of $T|_{W_3}$ is $J_2(3)$

Now we want α_1 ~~$\alpha_1 = \{Nx, x\}$ where $x, Nx \in W_3$ and~~

Let $N = (T-3I)|_{W_3}$ nilpotent operators.

$\alpha_1 = \{N x, x\}$

Let $y = Nx \in \ker N \cap \operatorname{im} N$

$\ker N = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \right\}$

So take $y = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$ Solve for x $\left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 4 & 0 & 1 & -1 & 3 \end{array}\right]$

$\rightsquigarrow$ get solution x.

So $\alpha_1 = \{y, x\}$

Do same thing for $T|_{W_2}$

tableau : $\dim \ker(T-2I) = 2$ $\Leftarrow$ compute this to see that you get 2.
$\dim \ker((T-2I)^2) = 2$

So get tableau $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ $\Rightarrow$ JCF of $T|_{W_2}$ is $J_1(2) \oplus J_1(2)$

The JCF of T is

$$J_2(3) \oplus J_1(2) \oplus J_1(2) = \left[\begin{array}{cc|c|c} 3 & 1 & & \\ 0 & 3 & & \\ \hline & & 2 & \\ \hline & & & 2 \end{array}\right]$$

Last thing to do is find α_2 basis of W_2.
We're looking for a basis $\underbrace{\{z_1, z_2\}}_{\text{of } W_2}$ s.t. $Tz_i = 2z_i$

$$T - 2I = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \end{bmatrix}$$

$W_2 = \ker((T-2I)^2) = \ker(T-2I)$

$\ker(T-2I) = \operatorname{sp}\{e_2, e_4\}$ so $\{e_2, e_4\}$ is a basis of W_2 consisting of eigenvectors of T of evalue 2

<u>Upshot</u>: $\alpha = \{y, x, e_2, e_4\}$

<u>Claim:</u> If T is diagonalizable then its JCF is the diagonal matrix of eigenvalues

<u>Pf:</u> Sps λ is an eigenvalue of T.

$E_\lambda = W_\lambda$

E_λ: eigenspace

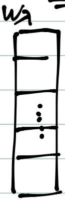
W_λ: generalized eigenspace.

If T is diagonalizable then $\dim E_\lambda$ equal to r_i, the multiplicity of λ in $p(x)$.

$\Rightarrow \dim E_\lambda = \dim W_\lambda$ and since $E_\lambda \subset W_\lambda$

$\Rightarrow E_\lambda = W_\lambda$

Now consider $T|_{W_\lambda} = T|_{E_\lambda}$ $(T-\lambda I)|_{E_\lambda}$ has ker equal to $E_\lambda \Rightarrow$



$\cdots$

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