MATH6222 week 12 lecture 15

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Exam: Friday, April 21st, 6:30-9:30

One more counting problem: How many ways to select n objects from k "types"? Equivalently, how many solutions in non-negative integers are there to $x_1 + x_2 + \cdots + x_k = n$?

e.g. apples, bananas, oranges, $x_1 + x_2 + x_3 = 5$.

$k = 1, \{c\}, x_1 = n$
$k = 2, \{c, v\}, x_1 + x_2 = n$
$k = 3, \{c, v, r\}, \ldots$

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

For k, $C(k) = \sum_{k=0}$???

Balls and Walls Bijection:

For $x_1 + x_2 + \cdots + x_k = n$, take n balls and $k-1$ walls, e.g. $x_1 + x_2 + x_3 + x_4 = 8$

$$\cdot \cdot | \cdot \cdot \cdot || \cdot \cdot \cdot$$

One-to-one between solutions $\implies$ {arrangements of n balls divided by $k-1$ walls}

Consider $n + k - 1$ "objects", choose $k-1$ of them to be walls. Also

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Number Theory

Question: What distances can we measure with a stick of length 15 meters and a one of length 6 meters?

We can get all multiples of 3!

can get $3 = 15 - 6 - 6$.

Given 2 fixed integers a, b, what integers can be represented in the form

$$ma + nb,\ \ m, n \in \mathbb{Z}.$$

Given fixed integers $a, b, c \in \mathbb{Z}$, when does $ax + by = c$ have a solution in integer?

$15x + 6y = c$ has integer solution $\iff$ c is a multiple of 3.

$12x + 5y = c$ for all c.

Def: Given $d, n \in \mathbb{Z}$, we write $d|n$ if $n = dk$ for some $k \in \mathbb{Z}$.

Remarks:

1. If $d|n$ and $d|m$, then $d|(n + m)$.

2. If $d|n$ and $m \in \mathbb{Z}$, then $d|mn$.

Proof1: $n = k_1 d, m = k2_d, n + m = d(k_1 + k_2)$, thus $d|(n + m)$.

Def: Greatest common divisor (GCD). Given $m, n \in \mathbb{Z}, \gcd(m, n)$ to be the largest integer d such that $d|m$ and $d|n$.

Missed today's lecture. The following notes are mainly textbook-based.

- walls and balls method for proving the number of ways to select n objects from k types is $\binom{n+k-1}{k-1}$. (See Theorem 5.23 on page 107.)
- The distances can be measured if one has only one rope of length of 15 and one of length 6 to work with, and concluded that the answer is all multiple of 3. (page 123-126, Theorem 6.12)

Theorem: With repetition allowed, there are $\binom{n+k-1}{k-1}$ ways to select n objects from k types. This also equals the number of nonnegative integer solutions to $x_1 + x_2 + \cdots + x_k = n$.

Proof: Selections are determined by how many objects are chosen of each type. Let x_i be the number chosen of type i. This establishes a one-to-one correspondence between the selections and the nonnegative integer solutions to $x_1 + \cdots + x_k = n$.

We model these solutions as arrangements of n dots and $k-1$ vertical separating bars. We represent selecting x_1 items of type 1 by recording x_1 dots and marking the end with a bar before continuing to the next type. Doing this for each type forms an arrangement of dots and bars.

We have n dots and $k-1$ bars.

Given an arrangement of n dots and $k-1$ bars, we can invert the process to obtain x_i; it equals the number of dots in the ith group. This establishes a one-to-one correspondence between solutions to $x_1 + \cdots + x_k = n$ and arrangements of n dots and $k-1$ bars. These arrangements are determined by choosing the locations for the bars in a list of length $n+k-1$, so there are $\binom{n+k-1}{k-1}$ of them. We have counted the solutions to the equation and hence also the selections of n objects from k types.

Divisibility

Theorem: The set of integer combinations of a and b is the set of multiples of $\gcd(a, b)$.

Proof: Let $d = \gcd(a, b)$. The set of integer combinations of a and b is $S = \{ra + sb : r, s \in \mathbb{Z}\}$. Let T denotes the set of multiples of d.

We first prove $S \subseteq T$. Since d divides both a and b, there are integers k and l such that $a = kd$ and $b = ld$. The distributive law now yields $ma + nb = mkd + nld = (mk + nl)d$, and thus d also divides $ma + nb$. Since this holds for every integer combination, we have $S \subseteq T$.

To prove $T \subseteq S$, we express each multiple of d as an integer combination of a and b. Since the integers $a \backslash d$ and $b \backslash d$ are relatively prime, by Lemma that if $a \backslash d$ and $b \backslash d$ are relatively prime, then there exists integer m and n

such that $m(a\backslash d)+n(b\backslash d)=1$. Thus $ma+nb=d$. For $k\in\mathbb{Z}$, we now have $(mk)a+(nk)b=kd$. Thus $T\subseteq S$.