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UNIVERSITY OF TORONTO

FACULTY OF ARTS AND SCIENCE

APRIL 2010 EXAMINATIONS

CSC 165H1S

DURATION — 3 HOURS

ONE 8.5"×11" HANDWRITTEN (BOTH SIDES) AID SHEET ALLOWED

STUDENT NUMBER: |_|_|_|_|_|_|_|_|_|

LAST NAME: ____________________

FIRST NAME: ____________________

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

# 1:	______/12
# 2:	______/10
# 3:	______/10
# 4:	______/10
# 5:	______/10
# 6:	______/10
# 7:	______/10
# 8:	______/10
# 9:	______/10
TOTAL:	______/92

This exam consists of 9 questions on 14 pages (including this one). *When you receive the signal to start, please make sure that your copy of the exam is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-14 of this exam.

Good Luck!

QUESTION 1. [12 MARKS]

Consider the three sentences below:

$$\begin{array}{ll} \text{S1:} & \forall x \in X, P(x) \Rightarrow Q(x) \\ \text{S2:} & \exists x \in X, P(x) \Rightarrow Q(x) \\ \text{S3:} & \forall x \in X, Q(x) \Rightarrow P(x) \end{array}$$

In each of the subquestions below "devise an example of set X and predicates P and Q" means you must suggest elements for X and meanings for predicates (boolean functions) P and Q that satisfy the condition in that subquestion.

PART (A) [2 MARKS]

Devise an example of set X and predicates P and Q that makes S1 true and S2 false.

PART (B) [2 MARKS]

Devise an example of set X and predicates P and Q that makes S1 false and S2 true.

PART (C) [2 MARKS]

Devise an example of set X and predicates P and Q that makes S1 true and S3 false.

PART (D) [2 MARKS]

Devise an example of set X and predicates P and Q that makes S1 false and S3 true.

PART (E) [2 MARKS]

Devise an example of set X and predicates P and Q that makes S2 true and S3 false.

PART (F) [2 MARKS]

Devise an example of set X and predicates P and Q that makes S2 false and S3 true.

QUESTION 2. [10 MARKS]

Consider the definition of $U(n)$ below:

$U(n) \Leftrightarrow n$ has remainder 2 when divided by 3. In other words $\exists i \in \mathbb{N}, n = 3i + 2$.

Use the given definition, and the proof structure from this course, to PROVE:

$$\forall n \in \mathbb{N}, U(n) \Rightarrow U(n^3)$$

HINT: You may find it helpful to recall the binomial expansion, for any natural numbers a and b,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

QUESTION 3. [10 MARKS]

In this question $\mathbb{N}^+$ denotes the positive (greater than 0) natural numbers, and $(h, i, j, k) \in \mathbb{N}^+$ denotes a quadruple of positive natural numbers. Use the proof structure from this course to prove the following statement:

$$\forall (h, i, j, k) \in \mathbb{N}^+, \frac{h}{i} < \frac{j}{k} \Rightarrow \frac{h}{i} < \frac{h+j}{i+k}.$$

QUESTION 4. [10 MARKS]

In this question $\mathbb{Z}$ denotes the integers and $\mathbb{R}$ denotes the real numbers. Consider the following definition of $\lfloor x \rfloor$:

$$y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y).$$

Use the given definition, and the proof structure from this course, to prove:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor.$$

QUESTION 5. [10 MARKS]

In this question $\mathbb{Z}$ denotes the integers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}^+$ denotes the real numbers that are greater than zero. Consider the following definition of $\lfloor x \rfloor$:

$$y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y).$$

Use the given definition and the proof structure from this course to DISPROVE:

$$\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < \delta \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < \varepsilon$$

QUESTION 6. [10 MARKS]

In this question $\mathbb{Z}$ denotes the integers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}^+$ denotes the real numbers that are greater than zero. Consider the following definition of $\lfloor x \rfloor$:

$$y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y).$$

Use the given definition and the proof structure from this course to PROVE:

$$\exists x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < \delta \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < \varepsilon$$

QUESTION 7. [10 MARKS]

Consider the definition of $\mathcal{O}(g)$ below:

$$\mathcal{O}(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} : \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)\}$$

Define $f(n) = n^3 - 3n + 1$ and $g(n) = 5n^2 + 7n + 9$. Use the proof structure from this course to PROVE $f \notin \mathcal{O}(g)$. You may NOT use the results and techniques of limits from calculus.

QUESTION 8. [10 MARKS]

Consider the definition of $\mathcal{O}(g)$ below:

$$\mathcal{O}(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} : \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)\}$$

Define $f(n) = 3n$ and $g(n) = \ln(n)$. Use the proof structure from this course to PROVE $f \notin \mathcal{O}(g)$. Unlike the previous question, you MAY use the techniques of limits from calculus, including l'Hôpital's rule. You may find it useful to note that the derivative of $\ln(x)$ is $1/x$.

QUESTION 9. [10 MARKS]

Suppose you have a floating-point system with base $\beta = 3$, exponents from the set $\{-2, \ldots, 2\}$, $t = 3$ digits in the significand, a single sign symbol $+$ or $-$, a radix point following the first digit, and the convention that the digit preceding the radix point is non-zero unless we are representing zero itself.

PART (A) [5 MARKS]

Which of the follwing can be represented exactly in the given system? In each case, explain why or why not.

(i) $3/4$

(ii) $4/9$

(iii) $4/27$

(iv) -25

(v) 27

PART (B) [5 MARKS]

Give an example of two numbers that each have representations (posssibly inexact) in the given system, but when one is subtracted from the other within the given system, yield a relative error of more than 100%. Explain.

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Total Marks = 92