-

Absolute value versus Norm

Recall that the convention of *absolute value* helped us define the concept of *magnitude* of a number, and the absolute value of the difference of two real numbers $|r - s|$ was considered as the *distance* between the two numbers r and s. These two ideas of *distance* and *magnitude* were foundations of a geometry on the real number. For example $|x - 2| < \epsilon$ was to be interpreted as the requirement that the variable x resides in the vicinity of 2, or that x be restricted to values near 2; more precisely $2 - \epsilon < x < 2 + \epsilon$. This is known as topology in $\mathbb{R}$ Now it is natural to assume that the Algebra of $\mathbb{R}^n$ should be able to induce a geometry on $\mathbb{R}^n$. Motivated by Pythagorean identity, the *dot product* is introduced at the bottom of page 4, to help introducing a notion of magnitude and then distance on $\mathbb{R}^n$. (try to understand the connection between Pythagorean identity and the definition of norm.) See exercise 2 and relate it to the well known cosine law of vector algebra. Also try to connect this exercise to the binomial expansion. And then see how the introduction of norm brings us to a generalized version of Pythagorean identity in exercise 3, (an identity that Pythagoras could not have imagined just because the concepts of $\mathbb{R}^n$ and norm were never part of the ancient language.) Try to see that when $n = 1$ then *norm* is the same as the *absolute value.* Therefore the textbook uses the same notation for both *absolute value* and for the *norm.* Also note that the scalar multiplication $k(x, y)$ helps defining *parallel* lines.

After defining the norm, then obviously the distance between two n-tuples is introduced (middle of page 6). The job of mathematics is not only to 'define concepts' (as in dot product and norm and distance,) but it is also, more importantly, to investigate the properties of these newly defined concepts, and the way they can be used as tools in further theories. To this end, the textbook presents two crucial propositions 1.1 (Cauchy's inequality) and 1.2 (triangle inequality.) These two inequalities determine the legitimate ways that the *norm* and the *dot product* can interact with one another (applications of Cauchy inequality appear in the future.) As a result of these inequalities the concept of *angle* between two n-tuples is established, and the idea of *orthogonality* is introduced.