FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 6

LOAN REPAYMENTS

A common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

Repayments can be decomposed into a principal and interest component.

Most loans operate like a repayment mortgage, where the initial principal is repaid during the term of the loan. This is done by making repayments that are greater than the amount of interest due. The remainder of the payment is used to repay part of the principal (or capital).

For some loans the repayments represent interest only, and at the end of the term the borrower will need to repay the capital.

AMORTISATION METHOD

The amortisation method of loan repayment corresponds to setting the present value of all amounts loaned out equal to the present value of all payments made to repay the loan. In other words, amortisation involves solving an equation of value for the transaction.

Each payment can be decomposed into a part that pays the interest that has accrued since the last payment, and a part that repays some of the outstanding principal.

Loan amount = $L = OB_0$
Interest rate per period = i

Amount paid at time t = K_t
Interest charged at the end of the t^{th} period = $I_t = OB_{t-1} \cdot i$
Principal repaid at the end of the t^{th} period = $PR_t = K_t - I_t$
Outstanding balance (principal) just after payment at time t = OB_t

Where $OB_t = OB_{t-1} + OB_{t-1} \cdot i - K_t$

$$\Rightarrow OB_t = OB_{t-1} + I_t - K_t = OB_{t-1} - \left(K_t - I_t\right) = OB_{t-1} - PR_t$$

Loan payments can be set out in a table called a "loan schedule".

t	Payment	Interest due	Principal Repaid	Outstanding balance
0				$L = OB_0$
1	K_1	$I_1 = OB_0 \cdot i$	$PR_1 = K_1 - I_1$	$OB_1 = OB_0 - PR_1$
2	K_2	$I_2 = OB_1 \cdot i$	$PR_2 = K_2 - I_2$	$OB_2 = OB_1 - PR_2$
…				
t	K_t	$I_t = OB_{t-1} \cdot i$	$PR_t = K_t - I_t$	$OB_t = OB_{t-1} - PR_t$
…				
n	K_n	$I_n = OB_{n-1} \cdot i$	$PR_n = K_n - I_n$	$OB_n = OB_{n-1} - PR_n$ =0

Note: if a particular payment K_t is not enough to pay the interest due I_t then the principal repaid is negative $PR_t = K_t - I_t < 0$. The shortfall in the interest payment due is $I_t - K_t$. The outstanding balance at the end of the year would then increase due to the shortfall: $OB_t = OB_{t-1} - PR_t > OB_{t-1}$

Consider the following simple example,

EXAMPLE
A bank lends an individual $1,000 for three years in return for three payments of $X at the end of each year. The bank charges an effective rate of 7% per annum.

The equation of value for the loan is:
$\$1{,}000 = Xa_{\overline{3}|0.07} \Rightarrow X = \$381.05$

Each payment of $381.05 includes the interest due and the capital.

Using the notation introduced above:

$OB_0 = \$1{,}000$ and $K_1 = K_2 = K_3 = \$381.05$

The interest charged at the end of the 1st period = $I_1 = i \cdot OB_0 = 0.07(1{,}000) = \$70$
The principal paid is $PR_1 = K_1 - I_1 = \$381.05 - \$70 = \$311.05$

At time 1 the outstanding balance is $OB_1 = OB_0 - PR_1 = \$1{,}000 - \$311.05 = \$688.95$
The interest due at the end of the 2nd period = $I_2 = i \cdot OB_1 = 0.07(688.95) = \$48.22$
This means principal is paid of $PR_2 = K_2 - I_2 = \$381.05 - \$48.22 = \$332.83$

At time 2 the balance is $OB_2 = OB_1 - PR_2 = \$688.95 - \$332.83 = \$356.12$
The interest due at the end of the 3rd period = $I_3 = i \cdot OB_2 = 0.07(356.12) = \$24.93$

The final amount of principal paid is $PR_3 = K_3 - I_3 = \$381.05 - \$24.93 = \$356.12$ which completely repays the remaining balance.

Putting this into a loan schedule:

t	Payment K_t	Interest due $OB_{t-1} \times i$	Principal Repaid $K_t - OB_{t-1} \times i$	Outstanding balance $OB_{t-1} - \left(K_t - OB_{t-1} \times i\right)$
0				1,000.00
1	381.05	70.00	311.05	688.95
2	381.05	48.22	332.83	356.12
3	381.05	24.93	356.12	0.00

EXAMPLE
A loan of \$1,000 charged at a rate of 1% per month is repaid by 6 monthly payments in arrears. The first three payments are amount X each and the last three are amount 2X each. Find X and draw up the repayments in a loan schedule.

Solution
Working in units of months,

$$\Rightarrow 1000 = Xa_{\overline{3}|0.01} + 2Xv^3 a_{\overline{3}|0.01}$$
$$\Rightarrow X = 115.61$$

t	Payment	Interest due	Principal Repaid	Outstanding balance
0				$L = OB_0 = 1000$
1	$K_1 = 115.61$	$I_1 = 10$	$PR_1 = 105.61$	$OB_1 = 894.39$
2	$K_2 = 115.61$	$I_2 = 8.94$	$PR_2 = 106.67$	$OB_2 = 787.72$
3	$K_3 = 115.61$	$I_3 = 7.88$	$PR_3 = 107.73$	$OB_3 = 679.99$
4	$K_4 = 231.21$	$I_4 = 6.80$	$PR_4 = 224.41$	$OB_4 = 455.58$
5	$K_5 = 231.21$	$I_5 = 4.56$	$PR_5 = 226.65$	$OB_5 = 228.93$
6	$K_6 = 231.21$	$I_6 = 2.29$	$PR_6 = 228.92$	$OB_6 = 0.01 \cong 0$

CALCULATING THE OUTSTANDING BALANCE

Rather than rolling forward the loan contract year by year to find the capital outstanding at each time, which is very time consuming, we can use two methods to quickly find the capital outstanding:

- retrospective method - calculate the accumulated value of the original loan and subtract the accumulated value of payments made to date.
- prospective method **-** calculate the present value of future payments that are required to pay off the loan.

RETROSPECTIVE METHOD

The retrospective method formulates the outstanding balance at time t as the amount of the original loan accumulated to time t, minus the accumulated value of all payments to time t, including the payment made at time t:

$$\boxed{OB_t = L(1+i)^t - \sum_{a=1}^{t}(1+i)^{t-a}K_a}$$

Proof

The outstanding balance of a loan at time t after the t^{th} payment is:

$OB_t = OB_{t-1}(1+i) - K_t$

We can prove the formula by using successive outstanding balance amounts:

$$OB_1 = OB_0(1+i) - K_1$$
$$OB_2 = OB_1(1+i) - K_2 = OB_0(1+i)^2 - K_1(1+i) - K_2$$
$$OB_3 = OB_2(1+i) - K_3 = OB_0(1+i)^3 - K_1(1+i)^2 - K_2(1+i) - K_3$$
$$\ldots$$
$$OB_t = OB_{t-1}(1+i) - K_t = OB_0(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - \ldots - K_{t-1}(1+i) - K_t$$
$$OB_t = OB_0(1+i)^t - \sum_{a=1}^{t}(1+i)^{t-a}K_a$$

EXAMPLE

For the previous example, the outstanding balance after the third payment is:

$$\begin{aligned} OB_3 &= 1000(1.01)^3 - \sum_{a=1}^{3}(1+i)^{3-a}K_a \\ &= 1000(1.01)^3 - (1.01)^2K_1 - (1.01)^1K_2 - K_3 \\ &= 1000(1.01)^3 - \left((1.01)^2 + (1.01)^1 + 1\right)115.61 \cong 679.99 \end{aligned}$$

PROSPECTIVE METHOD

The prospective method formulates the outstanding balance at time t as the present value, at time t, of all remaining payments from time t+1 onward, but not including the payment just made at time t.

$$\boxed{OB_t = vK_{t+1} + v^2 K_{t+2} + \ldots + v^{n-t} K_n = \sum_{a=1}^{n-t} v^a K_{a+t}}$$

EXAMPLE

For the previous example, the outstanding balance after the third payment is:

$$\begin{aligned} OB_3 &= v^1 K_4 + v^2 K_5 + v^3 K_6 \\ &= 231.21\left(1.01^{-1} + 1.01^{-2} + 1.01^{-3}\right) \cong 679.99 \end{aligned}$$

LOAN WITH LEVEL PAYMENTS

For a loan with level payments the amortisation schedule can be reduced to a simpler form.

Consider a loan where each repayment is of amount K. The loan amount is the present value of the repayments. Therefore, $L = OB_0 = Ka_{\overline{n}|}$ using the prospective method.

Loan Schedule

t	Payment	Interest due	Principal Repaid	Outstanding balance
0				$L = OB_0 = Ka_{\overline{n}\|}$
1	K	$I_1 = iKa_{\overline{n}\|} = K(1-v^n)$	$PR_1 = Kv^n$	$OB_1 = K\left(a_{\overline{n}\|} - v^n\right) = Ka_{\overline{n-1}\|}$
2	K	$I_2 = iKa_{\overline{n-1}\|} = K(1-v^{n-1})$	$PR_2 = Kv^{n-1}$	$OB_2 = K\left(a_{\overline{n-1}\|} - v^{n-1}\right) = Ka_{\overline{n-2}\|}$
…				
t	K	$I_t = iKa_{\overline{n-t+1}\|} = K(1-v^{n-t+1})$	$PR_t = Kv^{n-t+1}$	$OB_t = K\left(a_{\overline{n-t+1}\|} - v^{n-t+1}\right) = Ka_{\overline{n-t}\|}$
…				
n	K	$I_n = iKa_{\overline{1}\|} = K(1-v)$	$PR_n = Kv$	$OB_n = K\left(a_{\overline{1}\|} - v\right) = 0$

Total principal repaid is: $Ka_{\overline{n}|} = L$ (the amount of the loan)

Total interest paid is: $K(1-v^n) + K(1-v^{n-1}) + K(1-v^{n-2}) + \ldots + K(1-v) = K\left(n - a_{\overline{n}|}\right)$

LOAN WITH PAYMENTS MADE M-THLY

Most loans will be repaid in quarterly, monthly or weekly instalments. The methodology involved where payments are made more frequently than annually is the same.

If the rate of interest used is effective over the same time unit as the frequency of the repayment instalments, then the calculations proceed exactly as above, with the time unit redefined appropriately (as in the previous examples).

If the interest is expressed as an annual effective rate, with repayment instalments payable m-thly, we have the equation of value for the loan, given repayments of K_t at time $t = \frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \ldots, n$:

$$L = OB_0 = K_{1/m}v^{1/m} + K_{2/m}v^{2/m} + K_{3/m}v^{3/m} + \ldots + K_n v^n$$

If the loan is repaid by level instalments of amount K payable m-thly for n years (ie. mK payable annually), the loan equation simplifies to:

$$L = mKa_{\overline{n|}}^{(m)}$$

Recall that $a_{\overline{n|}}^{(m)}$ equals the present value $\frac{1}{m}$-th of a year before the first payment of a series of payments of $\frac{1}{m}$ each $\frac{1}{m}$-th of a year for n years (total of mn payments).

Alternatively, you may wish to change the annual effective rate to an effective rate for the period of time in question and perform the calculations in the same way as if they were made annually.

EXAMPLE

A loan of $900 is repayable by equal monthly payments for 3 years, with interest payable at 18.5% per annum effective. Calculate the amount of each monthly payment.

Solution

Let K equal the monthly payment, then $12K$ is paid annually:

$$900 = K12a_{\overline{3|}0.185}^{(12)} \Rightarrow K = \$32.13, \text{ where } i^{(12)} = 12((1.185)^{1/12} - 1)$$

Alternatively, if we work with months as the unit of time:

$$900 = Ka_{\overline{36|}j} \Rightarrow K = \$32.13 \text{ where } j = 1.185^{1/12} - 1 \text{ is the monthly effective rate of interest}$$

For loans with instalments payable m-thly, calculating the interest and principal components is done in the same way as for loans with annual instalments. ***However, care needs to be taken in calculating the interest due at each instalment date.***

If an annual effective interest rate of i is quoted for the loan, the effective rate of interest over a period $\frac{1}{m}$ is $\frac{i^{(m)}}{m} = \left((1+i)^{1/m} - 1\right)$.

The prospective and retrospective methods can still be applied to calculate the outstanding balance.

EXAMPLE
Calculate the outstanding balance immediately after the 12th payment and calculate the interest portion of the 13th payment of the loan given in the previous example.

Solution
The loan outstanding immediately after the 12th payment can be found by using the retrospective or prospective methods. Using the prospective method:

$$K12a^{(12)}_{\overline{2}|} = 32.13(12)a^{(12)}_{\overline{2}|} = \$649.25$$

The interest portion of the 13th payment is:

$$\$649.25 \cdot \left(\frac{i^{(12)}}{12}\right) = \$649.25 \cdot \left((1+i)^{1/12} - 1\right) = \$9.25$$

CAPITAL BUDGETING – COMPARING INVESTMENT PROJECTS

In this section, we are attempting to choose between various investment projects with different cash flow streams. There are a number of useful measures that help an investor select between potential investment projects. The criteria that we will consider are:

- Accumulated profit.
- Net present value.
- Internal rates of return.
- Discounted payback period.

CASH FLOWS

Comparison of investment projects involves comparing the cash flow payments for the projects. The net cash flow c_t at time t is:

c_t = cash inflow at time t - cash outflow at time t

If any payments may be regarded as continuous then $\rho(t)$, the net rate of cash flow per unit time t, is defined as:

$$\rho(t) = \rho_I(t) - \rho_O(t)$$

where $\rho_I(t)$ and $\rho_O(t)$ denote the rates of inflow and outflow at time t respectively.

By the time the project ends (at time T) the accumulated value (the balance in the account) will be:

$$\boxed{AV(T) = \sum_t c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t} dt}$$

This is one criterion that can be used to assess an investment project. This measure suffers from the disadvantage that it can only be used in situations where there is a definite fixed time horizon for the project.

Another problem is that the accumulated profits for two different projects cannot be compared directly if they have different time horizons, since the calculated values will relate to different dates, that is, the accumulated profit is dependent on the point of evaluation 't'.

EXAMPLE
Consider the following cash flows:

Immediate outflow of 100
Inflow of 200 after 3 months
Inflows of 1,000 per annum paid continuously

What is the accumulated value of the net cash flows at time t, where $t > 3$ months, using an annual effective interest rate of i?

Solution

$$AV(t) = -100(1+i)^t + 200(1+i)^{t-0.25} + \int_0^t 1000(1+i)^{t-s} ds$$

Recall from previous notes that: $\bar{s}_{\overline{n|}} = \int_0^n (1+i)^{n-t} dt$

$$\Rightarrow AV(t) = -100(1+i)^t + 200(1+i)^{t-0.25} + 1000\bar{s}_{\overline{t|}}$$

The problems mentioned above can be avoided by calculating the "net present value" instead.

NET PRESENT VALUE

The net present value at the rate of interest i of the net cash flows is usually denoted by NPV(i).

$$\boxed{NPV(i) = \sum_t c_t (1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}\,dt}$$

This is equivalent to the accumulated profit, except now we are looking at the value at the outset, rather than the value at the end of the project. A higher net present value indicates a more "profitable" project.

The rate of interest i used to calculate the net present value is often referred to as the **risk discount rate**. As the risk discount rate increases, the equivalent NPV decreases.

We will discuss how net present values are used in comparing projects after introducing the internal rate of return.

EXAMPLE

What is the present value at time 0, of the cash flows in the previous example, assuming the continuous inflow payment is paid indefinitely.

Solution

$$PV = -100 + 200(1+i)^{-0.25} + \int_0^\infty 1000(1+i)^{-s}\,ds$$

Recall from previous notes that: $\bar{a}_{\overline{n}|} = \int_0^n (1+i)^{-t}\,dt$

$$\Rightarrow PV = -100 + 200(1+i)^{-0.25} + 1000\bar{a}_{\overline{\infty}|}$$

$$= -100 + 200(1+i)^{-0.25} + \frac{1000}{\ln(1+i)}$$

YIELD RATES (INTERNAL RATE OF RETURN)

Previously we introduced problems where we had to solve for an unknown rate of interest. For each problem, an equation of value was developed, and an interest rate was found that solved the equation.

The yield rate or internal rate of return (IRR) is the effective rate of interest that equates the present value of income and outgo, ie makes the net present value of the cash flows equal to zero.

Methods of solving for unknown interest rates were covered in week 5 and include:

- Solving quadratic equations (when an analytical solution exists)
- Linear interpolation

EXAMPLE
\$100,000 is used to purchase an annuity-immediate
Institution A offers annual payments in arrears of \$17,000 for 10 years
Institution B offers annual payments in arrears of \$19,000 for 10 years

Solution
Obviously an investor would prefer Institution B since it offers higher payments.
The investment from Institution A has a yield that can be found by solving:
$17,000a_{\overline{10}|i} - 100,000 = 0 \Rightarrow i \cong 11.0\%$
The investment from Institution B has a yield that can be found by solving:
$19,000a_{\overline{10}|i} - 100,000 = 0 \Rightarrow i \cong 13.8\%$

EXAMPLE
Institution A offers annual payments of \$17,000 for 10 years in exchange for a purchase price of \$100,000
Institution B offers annual payments of \$19,000 for 12 years in exchange for a purchase price of \$130,000

Solution
In this example it is not immediately obvious which investment offers the higher yield.

The investment from Institution A has a yield that can be found by solving:
$17,000a_{\overline{10}|i} - 100,000 = 0 \Rightarrow i \cong 11.0\%$
The investment from Institution B has a yield that can be found by solving:
$19,000a_{\overline{12}|i} - 130,000 = 0 \Rightarrow i \cong 9.9\%$

Although Institution A offers the higher yield, this does not necessarily mean that the investment offered by Institution A is superior.

A more important criterion than internal rate of return for comparing different investment projects is to consider the rate of interest at which the investor may lend or borrow money. Rather than comparing yields, choosing between investment projects is better achieved by comparing net present values at specific interest preference rate (or hurdle rate).

LINK BETWEEN THE NPV AND IRR

Since NPV is the present value of the net cash flows associated with a project, if an investor lends or borrows money at an interest rate i_1, then the project will be profitable if: $NPV(i_1) > 0$.

If the internal rate of return on a project is i_0, then NPV(i_0)=0. In other words, a project turns from profitable to unprofitable when $i = i_0$. Assuming that our inflows are relatively in the future compared to our outflows, the NPV decreases as the risk discount rate increases, the NPV will fall below zero when $i > i_0$

Therefore, a project is profitable if $i_1 < i_0$. This makes sense intuitively: if the rate of interest at which an investor can lend or borrow funds is less than the yield on the investment, then the project will be profitable.

Many projects will need to provide a return to shareholders and so there will not be a specific fixed rate of interest that has to be exceeded. Instead a target, hurdle, or interest preference, rate of return may be set for assessing whether a project is likely to be sufficiently profitable. In this context, a project may be considered profitable if

$$NPV(i_1) > 0$$

where i_1 is the hurdle rate.

When comparing two investment projects A and B, project A is more profitable than project B, if:

$$NPV_A(i_1) > NPV_B(i_1)$$

Note: If the IRR for project A is greater than that for project B: $i_A > i_B$, this **DOES NOT** imply that $NPV_A(i_1) > NPV_B(i_1)$. This is because the NPV for each project depends on i_1.

EXAMPLE

Institution A offers annual payments of \$17,000 for 10 years in exchange for a purchase price of \$100,000

Institution B offers annual payments of \$19,000 for 12 years in exchange for a purchase price of \$130,000

Find the net present value of these two projects for an investor at:

(i) a risk discount rate of 9% per annum

(ii) a risk discount rate of 6% per annum

Solution

At a risk discount rate of 9% per annum, the net present values of both of these projects are:

$$NPV_A(0.09) = 17{,}000 a_{\overline{10}|0.09} - 100{,}000 = \$9{,}100$$

$$NPV_B(0.09) = 19{,}000 a_{\overline{12}|0.09} - 130{,}000 = \$6{,}054$$

At a risk discount rate of 6% per annum, the net present values of both of these projects are:

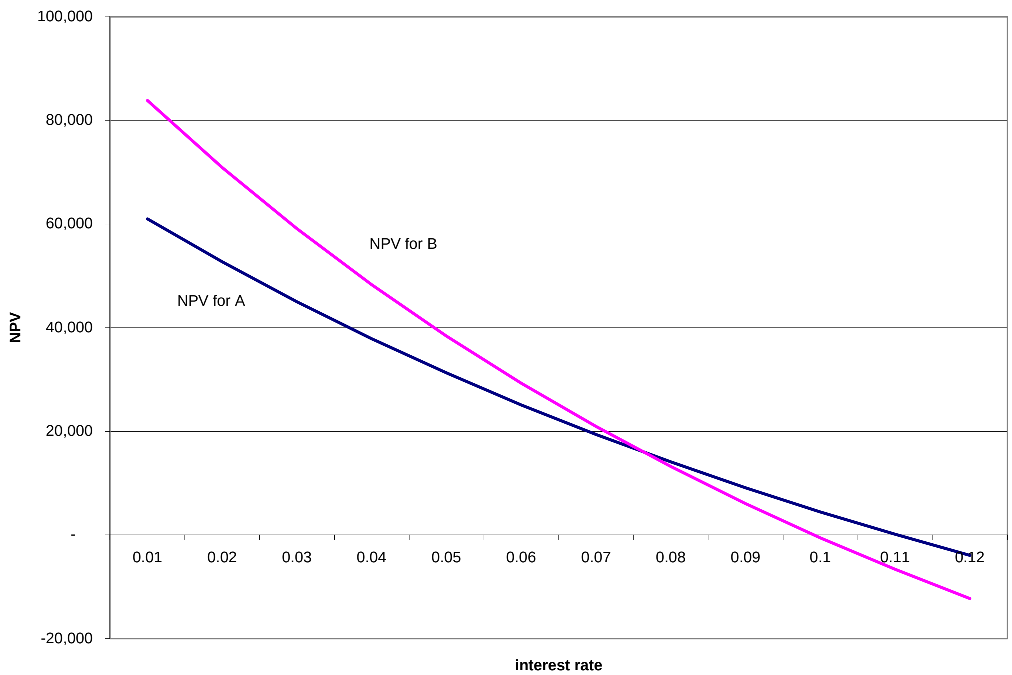
$$NPV_A(0.06) = 17{,}000 a_{\overline{10}|0.06} - 100{,}000 = \$25{,}121$$

$$NPV_B(0.06) = 19{,}000 a_{\overline{12}|0.06} - 130{,}000 = \$29{,}293$$

Both projects return a positive NPV. This means that both projects will be profitable to an investor if the investor borrows the purchase price and reinvests payments at the risk discount rates quoted.

Using net present values as a criterion, with a risk discount rate of 9%, Project A appears more favourable (higher NPV). However, with a risk discount rate of 6%, Project B appears more favourable. This is despite the fact that the yield on project A is 11% and the yield on project B is 9.9%.

The diagram below illustrates how the NPV changes for different interest rates. Note: the yields for the two projects are where the NPV=0.



In the above example there is one cross-over point. In more complicated examples there may be more than one cross-over point.

In summary, if an investor is trying to choose between different investment projects, when advising the investor of whether to invest in any of the projects, and if so, in order to choose the most profitable project:

1) find the yield i_0 (IRR) for each project by solving $NPV(i_0) = 0$
2) If the interest preference rate is i_1, then the projects with $i_1 < i_0$ are profitable and need to be compared. Those projects where $i_1 > i_0$ can be rejected.
3) For projects where $i_1 < i_0$, find $NPV(i_1)$. The project with the highest $NPV(i_1)$ will give the higher profit.

REINVESTMENT RATES

The calculations so far have assumed that the lender can reinvest payments received from the borrower at a reinvestment rate equal to the original investment rate.

In practice, a borrower may pay a different rate of interest i on the borrowings than the rate i_2 they would receive on investment of income.

EXAMPLE
Institution A offers annual payments at the end of each year of $17,000 for 10 years in exchange for a purchase price of $100,000.

What is the yield for a purchaser of the annuity if the rate of reinvestment of annual payments is 8%?

Solution
Before considering the impact of specific reinvestment rates, we consider the case where the yield is equal to the reinvestment rate.

From the borrower's point of view (Institution A), the yield on the investment is the solution to the equation:

$$NPV(i_0) = 0 \Rightarrow PV_I(i_0) - PV_O(i_0) = 0$$

$$100,000 - 17,000 a_{\overline{10}|i_0} = 0 \Rightarrow i_0 \cong 11\%$$

where $PV_I(i_0)$ and $PV_O(i_0)$ are the present values calculated at the interest rate i_0 of the income and outgo respectively.

For the investor purchasing the annuity, the yield depends on the rate of reinvestment applied to the payments received.

If the payments of $17,000 are reinvested by the investor at the same rate of interest of 11%, then the yield is also 11%. This is shown below.

If the payments of 17,000 are invested at 11% then the accumulated value of the payments at the date of the last payment is:

$$17{,}000 s_{\overline{10}|0.11}$$

The present value of these payments is:

$$17{,}000 s_{\overline{10}|0.11} v_{i_0}^{10}$$

So the yield in the case where payments are reinvested at 11% is the solution to the equation below:

$$PV_I(i_0) - PV_O(i_0) = 0$$

$$17{,}000 s_{\overline{10}|0.11} v_{i_0}^{10} - 100{,}000 = 0 .$$

If we solve this we find that the yield (the internal rate of return) is 11%.

However, if the payments of \$17,000 are reinvested at a rate of $i_2 = 8\%$, then the yield on the investment will differ from 11%.

If the payments of 17,000 are invested at 8% then the accumulated value of the payments at the date of the last payment is:

$$17{,}000 s_{\overline{10}|0.08} = \$246{,}272$$

In order to find the yield we need to solve the equation below:

$$17{,}000 s_{\overline{10}|0.08} v_{i_0}^{10} - 100{,}000 = 0$$

$$v_{i_0}^{10} = (1+i_0)^{-10} = \frac{100{,}000}{246{,}272} \Rightarrow i_0 \cong 9.43\%$$

In general, if the lender is not able to reinvest the repayments at the initial rate of investment, but reinvests them instead at rate i_2 per period, where $i_2 < i$, then the lender achieves an IRR less than i.

DISCOUNTED PAYBACK PERIOD

The discounted payback period is the number of years before the project starts making money (ie. positive balance).

If an investor can borrow and reinvest funds at the effective rate of interest i, then the time t_1 until the project is making money is the smallest value of t such that the accumulated value $AV(t)$ of the net cash flows is greater than or equal to zero, where:

$$AV(t) = \sum_{h \le t} c_h (1+i)^{t-h}$$

and where c_h is the net cash flow at time h (ie. income minus outgo).

In other words, the discounted payback period t_1 is the smallest value of t such that:

$$AV(t) = \sum_{h \le t} c_h (1+i)^{t-h} \ge 0$$

Here we are only assuming that the project consists of discrete cash flows. If continuous cash flows are also present, the discounted payback period t_1 is the smallest value of t such that:

$$AV(t) = \left(\sum_{h \le t} c_h (1+i)^{t-h} + \int_0^t \rho(s)(1+i)^{t-s} ds \right) \ge 0$$

We have assumed so far that the investor may borrow or lend money at the same rate of interest. In practice, however, an investor will probably have to pay a higher rate of interest (i_D, say) on borrowings than the rate (i_S, say) they receive on investments.

In this case accumulation of net cash flows must be calculated from first principles, where the rate of interest applied depends on whether the investor's account is in surplus (in which case use i_S), or in deficit (in which case use i_D).

We now consider the discounted payback period when it is assumed that the investor can borrow funds at the rate i_D and invest funds at the rate i_S. For simplicity, we will just work with discrete cash flows for this example.

In this case, the discounted payback period t_1 is the smallest value of t such that:

$$AV(t) = \sum_{h \le t} c_h (1+i_D)^{t-h} \ge 0$$

If the project is viable, the accumulated profit when the project ends at time T is:

$$P = AV(t_1)(1+i_S)^{T-t_1} + \sum_{t>t_1} c_t (1+i_S)^{T-t} = \sum_{t \le t_1} c_t (1+i_D)^{t_1-t} (1+i_S)^{T-t_1} + \sum_{t>t_1} c_t (1+i_S)^{T-t}$$

The profit consists of two components:

- the accumulated value of any cash flows to time t_1 (accumulated at the rate of interest i_D), accumulated a further $T - t_1$ years at the rate of interest i_S. This is because the balance is in deficit prior to time t_1, so interest is charged at i_D. Following time t_1 the balance is in surplus, so interest is received at the rate i_S
- any additional net cash flows after time t_1 accumulated to time T at the rate of interest i_S.

Other things being equal, a project with a shorter discounted payback period is preferable to a project with a longer discounted payback period because it will start producing profits earlier.

EXAMPLE
A project consists of a payment of $7000 at time 0 in return for an income stream of $1150 per annum (in arrears) for 10 years. Find the discounted payback period and accumulated profit for Project A using a borrowing rate of 6% p.a. and an investment rate of 3% p.a.

Solution
$$AV(t) = -7000(1.06)^t + 1150 s_{\overline{t|}0.06} \geq 0$$

Setting $AV(t) = 0$ and solving for t

$$-7000(1.06)^t + 1150 s_{\overline{t|}0.06} = 0$$

$$-7000(1.06)^t = -1150 s_{\overline{t|}0.06}$$

$$\frac{7000}{1150} = 6.086957 = s_{\overline{t|}0.06}(1.06)^{-t} = (1.06)^{-t}\frac{(1.06^t - 1)}{i} = \frac{1-(1.06)^{-t}}{0.06}$$

$$(1.06)^{-t} = 0.634626 \Rightarrow t = 7.80 \text{ years}$$

Since payments are made yearly, the first discrete time when the accumulated net cash flows exceeds 0 is 8 years.

So, the discounted payback period in this example is 8 years.

We now calculate the accumulated value after 8 years, and use this figure to calculate the accumulated profit after 10 years:

$$AV(8) = -7000(1.06)^8 + 1150 s_{\overline{8|}0.06} = \$225.15$$

Profit = $225.15 \times 1.03^2 + 1150 \times 1.03 + 1150 = \$2,573.36$

OTHER CONSIDERATIONS

At the simplest level, for projects involving similar amounts of money and with similar time horizons, the project that results in the highest accumulated profit will be the most favourable.

This is equivalent to selecting the project with the highest net present value. The internal rate of return provides a useful secondary criteria.

Where external borrowing is involved, the accumulated profit must be calculated directly by looking at the cash flows and taking into account the precise conditions of the loan. The discounted payback period can provide a useful secondary criteria.

In practice, it may not be straightforward to decide between investment projects purely on the basis of net present values, internal rates of return, or discounted payback periods.

Other considerations include:

- Cash flows
- Borrowing requirements
- Resources
- Risk
- Investment conditions
- Cost vs benefit
- Indirect benefits