map 1.3 Continuity & limit

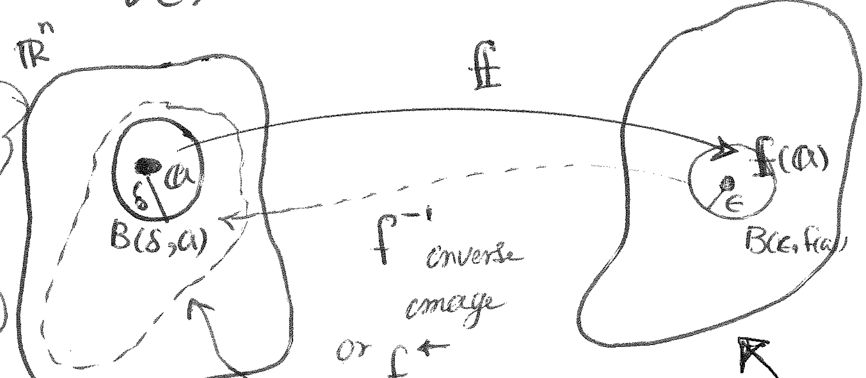
Thm 1.
f is Cont and U is open
Then $f^{-1}(U)$ is open:

proof:
$S = f^{-1}(U)$ is open
b/c $S \subset S^{int}$

proof:
$a \in S \Rightarrow f(a) \in U$
$\Rightarrow a \in U^{int} \Rightarrow$
$\Rightarrow \exists \epsilon > 0 : B(\epsilon, f(a)) \subset U$
$\Rightarrow f^{-1}(B(\epsilon, f(a))) \subset S$
$\Rightarrow \exists \delta$ s.t. $B(\delta, a) \subset f^{-1}(B(\epsilon, f(a))) \subset S$
f is Cont at a
so $\exists \delta > 0$ s.t. $B(\delta, a) \subset S$
so $a \in S^{int}$

$f : \mathbb{R}^n \to \mathbb{R}^k$ is Cont at $a \in \mathbb{R}^n$ if
$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R}^n \ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

Which is same as
$\forall \epsilon > 0 \ \exists \delta > 0 \ B(\delta, a) \subset f^{-1}(B(\epsilon, f(a)))$



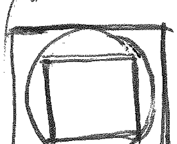
$f^{-1}(B(\epsilon, f(a)))$

Picture of Continuity in $\mathbb{R}^n \to \mathbb{R}^k$

$\begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} = f(x) \to L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$

iff $\forall i = 1 \dots n$
$f_i(x) \to L_i$
by inequality 1.3

So it is sufficient to study functions $f : \mathbb{R}^n \to \mathbb{R}$



1.3

$x \to a$ is same as $|x - a| \to 0$
Which is the same as
$\max\{|x_1 - a_1|, \dots, |x_n - a_n|\} \to 0$
or $|x_i - a_i| \to 0$ for all $i = 1 \dots n$

1.9 If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ are Cont

Then $f \circ g : \mathbb{R}^n \to \mathbb{R}^k$ is also Cont

$$f \circ g(x) = g(f(x))$$

1.10

$f_1(x, y) = x + y$

$f_2(x, y) = xy$

$g(x) = \frac{1}{x}$

$x \neq 0$

are all Cont.

$h : \mathbb{R}^n \to \mathbb{R}$ is Cont

$k : \mathbb{R}^n \to \mathbb{R}$ is Cont

Then $h + k : \mathbb{R}^n \to \mathbb{R}$ is Cont

proof:

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, bend left, "h"] \arrow[dr, bend right, "k"'] & \\ & \mathbb{R} \end{tikzcd}$$

so $\mathbb{R}^n \xrightarrow{F = \begin{bmatrix} h \\ k \end{bmatrix}} \mathbb{R}^2 \xrightarrow[\text{Cont}]{f_1} \mathbb{R}$

is Cont by 1.3

so by 1.9 $F \circ f_1$ is Cont

but $F \circ f_1(x) = f_1((h(x), k(x)))$

$= h(x) + k(x)$ is Cont