§5 - Convergence and Limit points

1 Motivation

So far we have seen how to describe when a point is close to a set; we used the notion of the closure of a set to describe closeness. This has the advantage that it makes sense in every topological space, and only references open sets and points.

Our previous experience in calculus and analysis is that "closeness is witnessed by convergent sequences". That is, we say that a sequence is close to a point if the sequence converges to that point. We know that we can make sense of convergence of sequences in $\mathbb{R}$ (thanks to first-year calculus) but can we make sense of it in a general topological space? We can! Although, we will need to be careful as a sequence is an "inherently countable" object, and not all topological spaces are small enough (or nice enough!) to be thought of as "inherently countable".

2 Convergent Sequences

Recall that a sequence $\langle x_n \rangle$ in a set X is really just a function from $\mathbb{N}$ to X; for each $n \in \mathbb{N}$ we have an $x_n \in X$.

Definition. *A sequence of $\langle x_n \rangle$ of points in a topological space $(X, \mathcal{T})$* ***converges to a point*** *$p \in X$ if for every open set U containing p, there is an $N \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq N$. In this case we write $\langle x_n \rangle \longrightarrow p$.*

Some examples:

- In $\mathbb{R}_{\text{usual}}$ and the Sorgenfrey line, $\left\langle \frac{1}{n} \right\rangle \longrightarrow 0$.
- In $\mathbb{R}_{\text{usual}}$ we have $\left\langle \frac{n}{n+1} \right\rangle \longrightarrow 1$, but in the Sorgenfrey line $\left\langle \frac{n}{n+1} \right\rangle$ does not converge to anything!
- If $\langle x_n \rangle$ converges to p in $\mathbb{R}_{\text{discrete}}$ then it is **eventually constant**, which means there is some $N \in \mathbb{N}$ such that $x_i = p$ for all $i \geq N$.
- (Reason 478 that the Indiscrete Topology is freaky). In the $\mathbb{R}_{\text{indiscrete}}$ every sequence converges to every point. Yes, that's right: If $\langle x_n \rangle$ is a sequence in $\mathbb{R}_{\text{indiscrete}}$ and $p \in \mathbb{R}$, then every open set containing p contains all of $\langle x_n \rangle$. So $\langle x_n \rangle \longrightarrow p$. **shudder**

3 Felix Hausdorff is your Friend

Really, the fact that everything converges to everything in an indiscrete space tells us that indiscrete spaces are very badly behaved. Our intuition tells us that if a sequence converges then it converges to only one point. Since this isn't always true, but we usually want it to be true, we usually restrict our attention to spaces where this *is* true.

Definition. *A topological space* $(X, \mathcal{T})$ *is a* ***Hausdorff Space*** *if for every two distinct* x *and* y *in* X *there are disjoint open sets* U *and* V *such that* $x \in U$ *and* $y \in V$.

This turns out to be exactly the condition we need so that each sequence converges to at most one point.

Theorem. *In a Hausdroff topological space each sequence converges to at most one point.*

Proof. Suppose that X is a Hausdorff space. Suppose that $\langle x_n \rangle \longrightarrow p$ and $q \neq p$. Then there are disjoint open sets U and V such that $p \in U$ and $q \in V$. Then there is a number $N \in \mathbb{N}$ such that $x_i \in U$ for $i \geq N$. Necessarily, if $i \geq N$ then $x_i \notin V$, so $\langle x_n \rangle$ does not converge to q. □

Converse Exercise: Does the converse hold? That is, if every sequence in a space converges to at most one point then is the space necessarily Hausdorff?

Some Examples:

- $\mathbb{R}_{\text{usual}}, \mathbb{R}_{\text{discrete}}$ and the Sorgenfrey line are Hausdorff spaces.
- The topological space you constructed in the New Ideas question 1 for assignment 1 was a Hausdorff space.

Some Non-Examples:

- $X_{\text{indiscrete}}$ is very much not a Hausdorff space as it only has one non-empty open set.
- $\mathbb{R}_{\text{co-finite}}$ and $\mathbb{R}_{\text{co-countable}}$ are not Hausdorff spaces as any two non-empty open sets intersect.

In the mathematical world, most spaces we encounter are Hausdorff spaces, and spaces that are not Hausdorff are quite strange and unintuitive. In fact, when Felix Hausdorff originally defined a topological space in the 1910s, he included the Hausdorff property in the definition; for early mathematicians, *every* topological space was a Hausdorff space!

As a result, we will see many theorems about topological spaces which have the additional assumption that the space is a Hausdorff space. This is an innocuous assumption, as many, many topological spaces are Hausdorff spaces. Later on in the course we will see weakenings of this property, and strengthenings of this property when we investigate so-called "separation properties".

4 Convergence and Closures

There are two main reasons why we care about sequences: (1) sequences tell us something about the closure of a set, and (2) sequences help us check that a function is continuous (as we will see later). So let's get to it!

Proposition. *Let $A \subseteq X$ a topological space, and let $\langle a_n \rangle$ be a sequence in A. If $\langle a_n \rangle \longrightarrow p$ then $p \in \overline{A}$.*

Proof. Suppose that $\langle a_n \rangle \longrightarrow p$, and let $p \in U$ an open set. Then there is an $N \in \mathbb{N}$ such that $a_i \in U$ for all $i \geq N$. So, in particular, $a_N \in U \cap A$, and thus $p \in \overline{A}$. □

Now, you might be tempted, *very* tempted, to claim that the converse of that proposition is true. Unfortunately, the converse is not true. Some spaces are just too "large" for this to be true.

Example. *Let $\langle x_n \rangle$ be any (non-zero) sequence in $R_{\text{co-countable}}$, then $0 \in \overline{\mathbb{R} \setminus \{0\}}$ but $\langle x_n \rangle$ does not converge to* 0.

Proof. Let's check that 0 is in the closure of $\mathbb{R} \setminus \{0\}$, then we'll check that x_n does not converge to 0.

[$0 \in \overline{\mathbb{R} \setminus \{0\}}$]

Let $0 \in U$ an open set. Then $U = \mathbb{R} \setminus C$, where C is a countable set. So it must be that $\mathbb{R} \setminus C$ is an uncountable set, and so must intersect $\mathbb{R} \setminus \{0\}$.

[$\langle x_n \rangle$ does not converge to 0]

Note that $C := \{x_n : n \in \mathbb{N}\}$ is a countable set (that does not contain 0), so $0 \in \mathbb{R} \setminus C$ is an open set which is disjoint from the sequence. So we have the desired result. □

That's too bad. It feels like the converse of the proposition should be true. Since it is so natural to have closure line up with sequences, we describe a property that gives us this.

5 First Countable Spaces

To describe how closures play with sequence convergence we need to describe a "local" property of a topological space.

Definition. *Let $(X, \mathcal{T})$ be a topological space and let $p \in X$. A* ***local basis at*** *p is a collection $\mathcal{B}_p$ of $\mathcal{T}$ such that:*

1. *If $B \in \mathcal{B}_p$, then $p \in B$.*
2. *If $U \in \mathcal{T}$ and $p \in U$, then there is a $B \in \mathcal{B}_p$ such that $B \subseteq U$.*

Some Examples:

1. In $\mathbb{R}_{\text{usual}}$, for any $x \in \mathbb{R}$ the family $\mathcal{B}_x := \{ (x - \frac{1}{n}, x + \frac{1}{n}) : n \in \mathbb{N} \}$ is a local basis at x.

2. In the Sorgenfrey Line, $\mathcal{B}_{10} := \{ [10, b) : b \in \mathbb{R}, b > 10 \}$ is a local basis at 10.

3. In $\mathbb{R}_{\text{discrete}}$, for any $x \in \mathbb{R}$, we have $\mathcal{B}_x = \{ x \}$ is a local basis at x.

We will care about when a space has lots of small local bases.

Definition. *We say that a topological space $(X, \mathcal{T})$ is **first countable** if every point in X has a countable local basis.*

Some Examples:

1. The following spaces are first countable: $\mathbb{R}_{\text{usual}}$, the Sorgenfrey Line, $\mathbb{R}_{\text{discrete}}$.

2. It takes some work to create spaces that are not first countable, and as a result so far in this course we only know about $\mathbb{R}_{\text{co-countable}}$ and $\mathbb{R}_{\text{co-finite}}$.

First Countable + Hausdorff Exercise: Notice that both examples of non-first countable spaces we gave are not Hausdorff spaces. Can you come up with a Hausdorff space that is not first countable?

Finally we can present the main reason we introduced this property:

Proposition. *Let $(X, \mathcal{T})$ be a first countable space, and let $A \subseteq X$ with $p \in X$. If $p \in \overline{A}$, then there is a sequence in A that converges to p.*

Proof. Let $\mathcal{B}_p := \{ B_n : n \in \mathbb{N} \}$ be a countable local basis at p. We will construct a sequence in A that converges to p.

Choose any $a_1 \in B_1 \cap A \neq \emptyset$.

In general, for $n \in \mathbb{N}$, notice that $\bigcap_{1 \leq i \leq n} B_i$ is an open set that contains p, so $(\bigcap_{1 \leq i \leq n} B_i) \cap A \neq \emptyset$. So choose any a_n in that intersection.

Claim: $\langle a_n \rangle$ is a sequence in A that converges to p.

It is clear from construction that this sequence is in A, so we only need to check convergence. So take an open set U containing p. Since $\mathcal{B}_p$ is a local basis, there is a $B_N \in \mathcal{B}_p$ such that $B_N \subseteq U$. Moreover, $(\bigcap_{1 \leq i \leq N} B_i) \subseteq U$, and in fact for all $n \geq N$ we see that

$$x_n \in \bigcap_{1 \leq i \leq n} B_i \subseteq \bigcap_{1 \leq i \leq N} B_i \subseteq U$$

So for all $n \geq N$ we have $x_n \in U$. $\square$

"It's basis elements all the way down" Exercise: From the previous proof, extract the fact that every first countable space has a countable local basis at every point of the form $B_1 \supseteq B_2 \supseteq B_3 \ldots$.

6 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Converse : If every sequence in a space converges to at most one point then is the space necessarily Hausdorff?

First Ctble + H : Give an example of a Hausdorff Space that is not first countable.

IBEALWD : Prove that every first countable space has a countable local basis at every point of the form $B_1 \supseteq B_2 \supseteq B_3 \ldots$