Lecture 6

Xiaoping Shi
Department of Statistics, University of Toronto
xpshi@utstat.toronto.edu

July 27, 2013

$$H_0 : \theta \in \omega_0 \quad \text{v.s.} \quad \theta \in \omega_1$$

where ω_0 is a subset of the set of all possible values of θ, ω_1 is disjoint from ω_0. Let $\Omega = \omega_0 \cup \omega_1$.

The generalized likelihood ratio is

$$\Lambda = \frac{\max_{\theta \in \omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

Theorem Under smoothness conditions on pdf, the null distribution of $-2 \log \Lambda$ tends to χ^2_d as the sample size tends to infinity, where d is dimΩ-dimω_0.

Example 1 Let $X_1, X_2, \cdots, X_n$ denote a random sample from $N(\mu, \sigma^2)$, where σ^2 is known. We wish to test

$$H_0 : \mu = \mu_0 \text{ v.s. } H_1 : \mu \neq \mu_0$$

where μ_0 is a prescribed number.

The role of θ is played by μ. $\omega_0 = \{\mu_0\}$; there are no free parameters under ω_0, so dim$\omega_0 = 0$. $\omega_1 = \{\mu | \mu \neq \mu_0\}$.
$\Omega = \{-\infty < \mu < \infty\}$; under Ω, μ is free, so dim$\Omega = 1$.
The generalized likelihood ratio is

$$\Lambda = \frac{\max_{\mu=\mu_0} L(\mu)}{\max_{-\infty<\mu<\infty} L(\mu)} = \frac{(1/\sqrt{2\pi})^n \exp[-\sum_{i=1}^n (X_i - \mu_0)^2/(2\sigma^2)]}{(1/\sqrt{2\pi})^n \exp[-\sum_{i=1}^n (X_i - \bar{X})^2/(2\sigma^2)]}$$

$$-2\log\Lambda = \frac{n}{\sigma^2}(\bar{X} - \mu_0)^2$$

By the Theorem, $-2\log\Lambda \sim \chi_1$, which is also shown by the fact that $\sqrt{n}(\bar{}-\mu_0)/\sigma \sim N(0,1)$.

The rejection region is $\frac{n}{\sigma^2}(\bar{X}-\mu_0)^2 \geq \chi_1^2(\alpha)$.