Practice Final 3

1. The Fibonacci sequence is the sequence of numbers $F(1), F(2), \dots$ defined by the following recurrence relations:
   $F(1) = 1, F(2) = 1,\ F(n) = F(n-1) + F(n-2)$ for all $n > 2$.
   For example, the first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \dots$

   (a) Prove by induction that for any $n \geq 1$ the consequtive Fibonacci numbers $F(n)$ and $F(n+1)$ are relatively prime.

   (b) Prove by induction that for any $n \geq 1$ the following identity holds

   $$F(2) + F(4) + \dots F(2n) = F(2n+1) - 1$$

2. (a) Find the remainder when $7^{3^{100}}$ is divided by 20.

   (b) Find $2^{p!} (\mod p)$ where p is an odd prime.

3. Prove that $q_1\sqrt{2} + q_2\sqrt{6}$ is irrational for any rational q_1, q_2 unless $q_1 = q_2 = 0$.

4. Suppose $(\phi(m), m) = 1$. Here m is a natural number and ϕ is the Euler function.

   Prove that $\sqrt{m}$ is irrational.

5. Let $p = 11, q = 5$ and $E = 11$. Let $N = 11 \cdot 5 = 55$. The receiver broadcasts the numbers $N = 15, E = 23$. The sender sends a secret message M to the receiver using RSA encryption. What is sent is the number $R = 2$.

   Decode the original message M.

6. (a) Find all complex roots of the equation

   $$z^6 + (1-i)z^3 - i = 0$$

(b) Express as $a + bi$ for some real a, b:

$$\frac{6^{100}}{(3+\sqrt{3}i)^{103}}$$

7. A complex number is called *algebraic* if it is a root of a polynomial with integer coefficients.

   Prove that the set of algebraic numbers is countable.

8. Suppose $0 < \alpha < \pi/2$ satisfies $\cos\alpha = \frac{2}{3}$. Prove that the angle α can not be trisected with a ruler and a compass.

9. Let S be that set of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

   Prove that $|S| > |\mathbb{R}|$.

10. For each of the following answer "true" or "false". Justify your answer.

    a) $\sqrt{\frac{\sqrt{5}}{\sqrt[3]{2}+\sqrt{11}}}$ is constructible.

    b) If x is not constructible then $\sqrt{x}$ is also not constructible.

    c) If x is constructible then $\sqrt[8]{x}$ is also constructible.