PROBLEM-SOLVING AND PROOFS
ASSIGNMENT 9 SOLUTIONS

(1) Let X_1, X_2, X_3 be random variables such that $P(X_i = j) = 1/n$ for all $(i, j) \in [3] \times [n]$. Compute the probability that $X_1 + X_2 + X_3 \le 6$, given that $X_1 + X_2 \ge 4$. You may assume that the random variables are *independent*, i.e.

$$P(X_1 = a_1, X_2 = a_2, X_3 = a_3) = P(X_1 = a_1)P(X_2 = a_2)P(X_3 = a_3).$$

Solution. We first assume that $n \ge 4$. For notational purposes, let $S = X_1 + X_2 + X_3$. By the definition of conditional probability,

$$P(\{S \le 6\} \mid \{X_1 + X_2 \ge 4\}) = \frac{P(\{S \le 6\} \cap \{X_1 + X_2 \ge 4\})}{P(\{X_1 + X_2 \ge 4\})}.$$

For the denominator, it is easier to compute the complement $P(\{X_1 + X_2 < 4\})$, because there are only two possibilities where $X_1 + X_2 \ge 4$ is false, namely when $X_1 + X_2 = 2$ and $X_1 + X_2 = 3$. Hence

$$P(\{X_1 + X_2 = 2\}) = P(X_1 = 1, X_2 = 1) = \frac{1}{n^2};$$
$$P(\{X_1 + X_2 = 3\}) = P(X_1 = 2, X_2 = 1) + P(X_1 = 1, X_2 = 2) = \frac{2}{n^2}.$$

Therefore,

$$\begin{aligned} P(\{X_1 + X_2 \ge 4\}) &= 1 - P(\{X_1 + X_2 < 4\}) \\ &= 1 - P(\{X_1 + X_2 = 2\}) - P(\{X_1 + X_2 = 3\}) = \frac{n^2 - 3}{n^2}. \end{aligned}$$

For the numerator, it is easier to split the event $\{S \le 6\} \cap \{X_1 + X_2 \ge 4\}$ into three mutually disjoint events:

$$\begin{aligned} P(\{S \le 6\} \cap \{X_1 + X_2 \ge 4\}) = P(&\{S \le 6\} \cap \{X_1 + X_2 = 4\}) \\ &+ P(\{S \le 6\} \cap \{X_1 + X_2 = 5\}) \\ &+ P(\{S \le 6\} \cap \{X_1 + X_2 \ge 6\}). \end{aligned}$$

When $X_1 + X_2 = 4$, the possibilities for X_1 and X_2 are $(1, 3), (2, 2), (3, 1)$. On the other hand, X_3 must be either be 1 or 2 for $S = X_1 + X_2 + X_3$ to be at most 6, Thus,

$$P(\{S \le 6\} \cap \{X_1 + X_2 = 4\}) = P(\{X_1 + X_2 = 4\} \cap \{X_3 = 1 \text{ or } 2\}) = \frac{3}{n^2} \cdot \frac{2}{n} = \frac{6}{n^3}.$$

When $X_1 + X_2 = 5$, the possibilities for X_1 and X_2 are $(1, 4), (2, 3), (3, 2), (4, 1)$. Also, X_3 must be equal 1 for $S \le 6$ to hold. Hence,

$$P(\{S \le 6\} \cap \{X_1 + X_2 = 5\}) = P(\{X_1 + X_2 = 5\} \cap \{X_3 = 1\}) = \frac{4}{n^2} \cdot \frac{1}{n} = \frac{4}{n^3}.$$

Finally, when $X_1 + X_2 \ge 6$, it is impossible for S to be at most 6, because we would need $X_3 \le 0$. So

$$P(\{S \le 6\} \cap \{X_1 + X_2 = 6\}) = 0.$$

Combining the computed probabilities for the numerator and denominator, we have

$$P(\{S \le 6\} \mid \{X_1 + X_2 \ge 4\}) = \frac{\frac{6}{n^3} + \frac{4}{n^3}}{\frac{n^2-3}{n^3}} = \frac{10}{n(n^2 - 3)}.$$

Note that for the previous calculations, we used the assumption that $n \ge 4$ for counting the possibilities for $X_1 + X_2 = k$ when $k = 2, 3, 4, 5$. When $n < 4$, some possibilities cannot occur (e.g. X_1 cannot equal 4). With modifications, $P = \frac{4}{9}$ when $n = 3$, or $P = 1$ when $n = 2$ (and undefined when $n = 1$ as $P(X_1 + X_2 \geq 4) = 0$).

(2) You hold a bag of ten coins, all superficially similar, but nine are fair, and one is foul (it shows heads with probability 9/10). You draw out a coin and begin flipping it.
   (a) The first five tosses are $HHHTH$. What is the probability that you are flipping one of the fair coins?
   (b) The next five tosses are $HHHHH$. Now what is the probability that you are flipping one of the fair coins?

Solution. Suppose A is the event "the coin is fair", and B is the event "the coin flips are $HHHTH$". The updated probability $P(A \mid B)$ of the coin being fair, by Bayes theorem, is

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

At the start, $P(A)$ is given as $\frac{9}{10}$. The probability $P(B \mid A)$ is the chance of obtaining $HHHTH$ from a fair coin, which is equal to $(\frac{1}{2})^5$. On the other hand, $P(B)$ is the sum of $P(B \mid A)P(A)$ (from a fair coin) and $P(B \mid A^c)P(A^c)$ (from the foul coin). The latter probabilities are $P(A^c) = \frac{1}{10}$ and $P(B \mid A^c) = (\frac{9}{10})^4 \frac{1}{10}$. Hence,

$$P(\text{coin is fair} \mid \text{flips are } HHHTH) = P(A \mid B) = \frac{(\frac{1}{2})^5 \cdot \frac{9}{10}}{(\frac{1}{2})^5 \cdot \frac{9}{10} + (\frac{9}{10})^4 \frac{1}{10} \cdot \frac{1}{10}} \approx 0.8108.$$

Now instead, we replace B with the event "the coin flips are $HHHTHHHHHH$". The differences are that the probability $P(B \mid A)$ is equal to $(\frac{1}{2})^{10}$, while the probability $P(B \mid A^c)$ is $(\frac{9}{10})^9 \frac{1}{10}$. Then,

$$P(\text{coin is fair} \mid \text{flips are } HHHTHHHHHH) = \frac{(\frac{1}{2})^{10} \cdot \frac{9}{10}}{(\frac{1}{2})^{10} \cdot \frac{9}{10} + (\frac{9}{10})^9 \frac{1}{10} \cdot \frac{1}{10}} \approx 0.1849.$$

A different method, where we compute $P(\text{coin is fair} \mid \text{flips are } HHHHH)$, gives the same answer of 0.1849. The modification required is changing the prior probability $P(A)$ of a fair coin to be 0.8108 (using the probability after the flips $HHHTH$) instead of 0.9.

(3) Suppose that a collection of $2n$ insects is randomly divided into n pairs. If the collection consists of n males and n females, what is the expected number of male-female pairs?

Solution. The names of the insects are not given, so with indignity, distinguish the female insects by a number instead, from 1 to n. Let X_i be the random variable which is 1 when the ith female insect is paired with any male insect, and 0 otherwise. Whenever $2n$ insects are paired, the ith female insect can pair with any of the $2n - 1$ other insects with equal probability, so there is a $\frac{n}{2n-1}$ chance of pairing with a male. The expectation $E(X_i)$ is then this probability, which is $\frac{n}{2n-1}$.

Due to the linearity of expectation, the expected value of the number of male-female pairs is just the sum of the expectations of male-female involving only female i (and a male):

$$E(\text{male-female pairs}) = \sum_{i=1}^{n} E(X_i) = \frac{n^2}{2n-1}.$$

(4) Suppose that A, B, and n other people stand in a line in random order. Compute the expected number of people standing between A and B in two ways:
   (a) For each $k \in [n]$, compute the probability that there are exactly k people between A and B, and use the formula $E(X) = \sum_k kP(X = k)$.

(b) Use linearity of expectation.

Solution.

Method 1. For a fixed $k \geq 1$, we want to count the number of arrangements with exactly k people between A and B, because each of the $(n+2)!$ arrangements of people in the line have equal probability. Our counting procedure involves first ordering the n faceless people first in $n!$ ways, then slotting A & B around the faceless people.

Assume that A is before B. Then B must be after the kth faceless person (not necessarily directly after), otherwise there would not be enough people before B for A to fit in while still maintaining the condition that k people are between A & B. So B can be directly after faceless person $k, k+1, \dots, n$, with $n-k+1$ possibilities. For each case, the position of A is fixed. The same can be said assuming that B is before A. Hence, there are $n! \cdot 2(n-k+1)$ arrangements where there are exactly k people between A and B. Therefore,

$$P(k \text{ people between A \& B}) = \frac{2(n-k+1)n!}{(n+2)!} = \frac{2(n-k+1)}{(n+1)(n+2)}.$$

So the expected value of people between A and B is

$$\begin{aligned} E(\text{People between A \& B}) &= \sum_{k=1}^{n} kP(k \text{ people between A \& B}) \\ &= \sum_{k=1}^{n} k\frac{2(n-k+1)}{(n+1)(n+2)} \\ &= \frac{2}{(n+1)(n+2)} \sum_{k=1}^{n} k(n+1-k). \end{aligned}$$

If you remember the 'Just For Fun' problem part (ii) from Tutorial 4, you possibly would have realised that $\sum_{k=1}^{n} k(n+1-k) = \binom{n+2}{3}$.[1] Otherwise, there are formulae for the sums $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$ which may help, namely[2]

$$\sum_{k=1}^{n} k = \frac{k(k+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Continuing on,

$$\begin{aligned} E(\text{People between A \& B}) &= \frac{2}{(n+1)(n+2)} \left[(n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 \right] \\ &= \frac{2}{(n+1)(n+2)} \left[(n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} \right] \\ &= \frac{n}{n+2} \left[(n+1) - \frac{(2n+1)}{3} \right] \\ &= \frac{n}{n+2} \left(\frac{n+2}{3} \right) = \frac{n}{3}. \end{aligned}$$

Therefore it is expected that $\frac{n}{3}$ faceless people are between A and B.

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[1] A combinatorial proof: suppose B has $n+2$ maths books in his bookshelf, and needs to select three books to give to A. Presumably, this is because A and B played some kind of "finger game" and B lost. In total, there are $\binom{n+2}{3}$ ways. However, B can first select the $(k+1)$th maths book as the middle book, choose a maths book before the middle book in k ways, then choose a maths book after the middle book in $(n+1-k)$ ways. Summing over all possible middle maths books (i.e. over k) gives the other side of the identity.

[2] Both formulae can be proved straightforwardly by induction.

Method 2. For each of the n faceless people, give a name to each person, with dignity, from Charlie$_1$ to Charlie$_n$. Consider the expectation of Charlie$_i$ (or C_i) being between A & B. For any configuration of $(n+2)$ people, we can first order A, B and C_i, and then place the $(n-1)$ other Charlie's in between. Out of the six reorderings ABC$_i$, AC$_i$B, BAC$_i$, BC$_i$A, C_iAB, C_iBA, two have Charlie$_i$ between A and B. This means that for every two configurations where Charlie$_i$ is between A & B, there are four other configurations where Charlie$_i$ is not between A & B.

Therefore, the expectation of Charlie$_i$ being between A & B is the probability of that situation occurring, which is $\frac{1}{3}$. By the linearity of expectation, the expected number of Charlie's between A & B is the sum of expectations of Charlie$_i$ between A & B (from $i = 1$ to n), which is $n \times \frac{1}{3} = \frac{n}{3}$.

(5) Recall that in the finger game, players A and B show 1 or 2 fingers, and A then receives a payoff according to the following chart (a negative number indicates that A pays B).

	B shows 1	B shows 2
A shows 1	-2	+3
A shows 2	+3	-4

We considered a scenario where A shows 1 finger with probability x and B shows 1 finger with probability y, and showed that $x = 7/12$ gives an expected payoff of $1/12$ for A, and that this strategy is optimal. Here, *optimal* means that for any other choice of x, there exists a $y \in [0,1]$ such that the expected payoff is lower than $1/12$.

(a) For what range of values $x \in [0,1]$ can A guarantee a positive expected payoff, no matter how B plays?
(b) Prove that $y = 7/12$ is the optimal strategy for B.
(c) Assuming that both players play their optimal strategy, what proportion of the games do A and B actually win.

Solution. Players A and B show a number of fingers independently, so the probabilities of each outcome occurring is encapsulated in the following table:

	B shows 1 finger	B shows 2 fingers
A shows 1 finger	xy	$x(1-y)$
A shows 2 fingers	$(1-x)y$	$(1-x)(1-y)$

The expected payoff of Player A is

$$\begin{aligned} E(\text{payoff for A}) &= xy(-2) + x(1-y)(3) + (1-x)y(3) + (1-x)(1-y)(-4) \\ &= -12xy + 7x + 7y - 4 \\ &= (7-12x)y + (7x-4). \end{aligned}$$

(a) For a fixed x, the expected payoff for Player A is a linear function on $y \in [0,1]$. The minimum (and maximum) of a linear function over $[0,1]$ must be achieved at the endpoints, at either 0 or 1. When $y = 0$, the payoff is $7x - 4$, which is positive iff $x > \frac{4}{7}$. When $y = 1$, the payoff is $3 - 5x$, which is positive iff $x < \frac{3}{5}$. When both inequalities are satisfied, i.e. $\frac{4}{7} < x < \frac{3}{5}$, A can guarantee a positive payoff.

(b) B's optimal strategy is to minimise the expected payoff for A. Write the payoff as $E(\text{payoff for A}) = (7-12y)x + (7y-4)$. When $y = \frac{7}{12}$, the expected payoff for A is $\frac{1}{12}$ no matter what x is. When $y < \frac{7}{12}$, A can choose to play the strategy $x = 1$, leading to a payoff of $3 - 5y > 3 - 5(\frac{7}{12}) = \frac{1}{12}$. When $y > \frac{7}{12}$, A can choose to play the strategy $x = 0$, leading to a payoff of $7y - 4 > 7(\frac{7}{12}) - 4 = \frac{1}{12}$. Thus, if B deviates from the optimal strategy $y = \frac{7}{12}$, then A has a strategy to increase the payoff above $\frac{1}{12}$ (and decrease the payoff for B).

(c) We say player A *wins* if A obtains a positive score from the game. A wins if A shows 1 finger and B shows 2 fingers (with probability $x(1-y) = \frac{7}{12} \cdot \frac{5}{12}$), or if A shows 2 fingers and B shows 1 finger (with probability $(1-x)y = \frac{5}{12} \cdot \frac{7}{12}$). The total chance of A winning is $\frac{35}{72}$. The total chance of B winning is $\frac{37}{72}$.