15.11.11

Lecture 10 handout

Planarity

PL topology preliminaries (bonus)

$A, B \subset \mathbb{R}^n$ • **Join** AB is subset $AB := \{\lambda a + \mu b \mid a \in A, b \in B\}$
$\lambda, \mu \in \mathbb{R}_{\geq 0}, \lambda + \mu = 1$

A  B

• **Cone** aB if each pt. expressed **uniquely** as $\lambda a + \mu b$.

cone

not cone

$P \subset \mathbb{R}^n$ is a **polyhedron** if each $a \in P$ has a cone nbd. in P.

 ← I'm a polyhedron

 ← I'm not a polyhedron

Triangle move:  (height might be 0)

Broken lines are **homeomorphic** if they are related by a finite sequence of triangle moves.

PL Jordan curve theorem: A closed self-avoiding broken line divides $\mathbb{R}^2$ into precisely two connected domains. (easy proof).

10.1

Embedding of a graph in M

- vertices $\mapsto$ distinct points in M
- edges $\mapsto$ disjoint (except at endpoints) broken lines between appropriate points in M.

A graph is <u>planar</u> if it can be embedded in $\mathbb{R}^2$.

planar

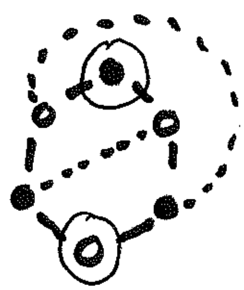
K_5

not planar

$K_{3,3}$

not planar

Proof: $K_{3,3}$:



2 of 3 edges in same domain.

K_5:

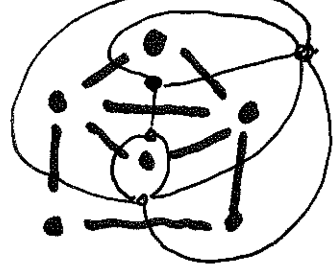
3 of 5 edges in one domain; 2 of them with no common vertex.

Kuratowski's Theorem: A graph is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.
(proof omitted).

(10.2) Duality

Graphs G and G^* are dual if there exists a bijective correspondence

$$\left\{ \begin{matrix} \text{cycles of} \\ \text{one graph} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{cut sets of} \\ \text{the other graph} \end{matrix} \right\}$$



for planar graphs, one vertex in each domain in $\mathbb{R}^2$; one edge per edge.

Whitney's Theorem: A graph is planar if and only if it has a dual graph.

Proof: ($\Rightarrow$) easy

($\Leftarrow$) Step 1: $K_{3,3}$ and K_5 have no duals.
Step 2: G has a dual $\Rightarrow$ $H \subset G$ has a dual.
Step 3: Triangle moves.

Proposition: The dual of any planar graph is connected.

Degree of a face $d(f)$ is number of edges on a bounding cycle.

Handshake lemma: $\sum_{f \in F} d(f) = 2|E|$.

Deletion-contraction duality:

$$(G \setminus e)^* = G^*/e^* \qquad (G/e)^* = G^* \setminus e^*$$

10.3

Theorem: For a planar graph G with s connected components, $\boxed{f = 1 + s - v + e}$

Corollary: A connected simple planar graph contains a vertex of degree at most 5.

Corollary: K_5 and $K_{3,3}$ are non-planar.

Next time: 11.1–11.2 Graph colouring