Lecture 24

ITINERARY: $x_0 \in \Lambda \subseteq I_0 \cup I_1$

$$S(x_0) = (s_0 s_1 s_2 \cdots) \text{ with } s_i = \begin{cases} 0 & \text{if } Q_c^i(x_0) \in I_0 \\ 1 & \text{if } Q_c^i(x_0) \in I_1 \end{cases}$$

$\Rightarrow Q_c^n(x_0) = I_{s_n}$

Thm: Let $c \leq -\frac{5+2\sqrt{5}}{4}$, then $S : \Lambda \to \Sigma$ is a homeomorphism

① S is 1-1
② S is onto
③ S is continuous
④ S^{-1} is continuous

A homeomorphism or topological isomorphism or bicontinuous function is a continuous function between topological spaces that has a continuous inverse function.

Proof: ① need to prove
$S(x) = S(y) \Rightarrow x = y$
Assume $S(x) = S(y)$ and by contradiction, assume also that $x \neq y$.

Recall that in 72 we proved that for $c \leq -\frac{5+2\sqrt{5}}{4}$

There is a number $\mu > 1$, such that
$\text{length}(Q_c(I)) \geq \mu \,\text{Length}(I)$ where I is an open interval.

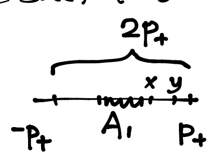
If $x < y$, then $\text{length}\, Q_c(x,y) \geq \mu \,\text{Length}(x,y)$

$|Q_c^n(y) - Q_c^n(x)| \geq \mu^n |y - x|$

$\Rightarrow \text{length}\, Q_c^n(x,y) \geq \mu^n \,\text{length}(x,y)$

we can choose n sufficiently large such that

$\mu^n |y - x| > 2p^+$



This implies that the distance between $Q_c^n(x)$ and $Q_c^n(y)$ is greater than $2P_+$, so one of them is not in $[-P_+, P_+] \supset \Lambda$ CONTRADICTION

② S is onto. Let $s \in \Sigma$, we will construct $x \in \Lambda$ s.t. $S(x) = s$

Define:
$$I_{s_0 s_1 \cdots s_n} = \{x \in I : x \in I_{s_0}, Q_c(x) \in I_{s_1}, \cdots, Q_c^n(x) \in I_{s_n}\}$$

Then $I_{s_0 s_1 \cdots s_n} = I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap Q_c^{-2}(I_{s_2}) \cap \cdots \cap Q_c^{-n}(I_{s_n})$ Note: $Q_c^{-n} = (Q_c^n)^{-1}$

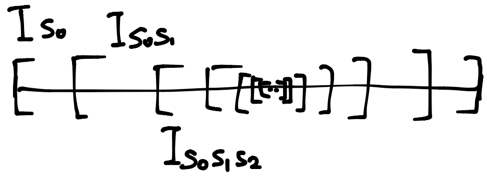
$$= I_{s_0} \cap Q_c^{-1}\left(I_{s_1} \cap Q_c^{-1}(I_{s_2}) \cap \cdots \cap Q_c^{-(n-1)}(I_{s_n})\right)$$

$$= I_{s_0} \cap Q_c^{-1}(I_{s_1 \cdots s_n})$$

We have the following properties.

$I_{s_0 s_1 \cdots s_n}$ is a closed interval

$I_{s_0 s_1 \cdots s_n} \subset I_{s_0 \cdots s_{n-1}}$



because the sets $I_{s_0 s_1 \cdots s_n}$ are closed and nested.

$\bigcap_{n\in\mathbb{N}} I_{s_0 \cdots s_n} \neq \emptyset$

So there is $x \in \bigcap_{n\in\mathbb{N}} I_{s_0 \cdots s_n} \in \Lambda$

and $S(x) = s$

③ S is continuous

$S : \Lambda \to \Sigma$ → sequence space

Λ ↓ real number space

we have to be careful that the spaces have different distances. We want to prove that for $x \in \Lambda$, S is continuous at x.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $y \in \Lambda$ and $|x-y| < \delta$, then $d[S(x), S(y)] < \varepsilon$