STA447/STA2006 Stochastic Processes

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Note

This lecture note is prepared for the course STA447/STA2006 Stochastic Processes. This lecture note may contain flaws. Please consult text book or reference books for confidence.

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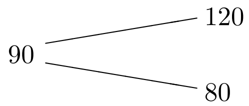
* indicates graduate level. So you may skip those parts.

7 Mathematical Finance

Example 68. Current stock price is 90. In the next unit time the stock price can be 120 or 80.



Suppose you are offered a European call option with strike price 100 and expiry 1, that is, you can buy the stock at price 100 in the next unit time regardless of the stock price. If the stock price goes up to 120, then you will buy the stock at price 100 then sell it to make 20. If the stock price goes down to 80, then you will ignore the option. Let X_0 be the initial price and X_1 be the price of the next unit time. Then your payoff can be $(X_1 - 100)^+$. Hence you won't lose any money. Then a natural question to ask is what is a reasonable price of this offer?

If you invested the stock, you may earn 30 or lose 10. In the other hand, if you pay c for the option, the you may earn $20 - c$ or lose $-c$. In sum,

	stock	option
up	30	$20 - c$
down	-10	$-c$

Suppose you buy x unit of stock and y unit of option. One investment strategy is to choose x and y to make the same outcome regardless of up or down. That is, $30x + (20 - c)y = -10x + (-c)y$ which solves $y = -2x$. Then the profit will be $(2c - 10)x = 2(c - 5)x$. If $c > 5$, then you can make money by buying stocks and selling twice many options. If $c < 5$, then still you can make money by reverting the scheme.

A scheme that makes money without any possibility of a loss is called an *arbitrage opportunity*. In the example, there is no arbitrage if $c = 5$. Hence 5 is a reasonable price for the option.

Theorem 74. Assume there are m different security with n possible outcomes. Let $a_{i,j}$ be the profit for the ith security when jth outcome occurs. Then only one of the following holds:
(a) There is an investment allocation x_i so that $\sum_{i=1}^m x_i a_{i,j} \ge 0$ for each j and strict inequality for some k.
(b) There is a probability vector $p_j > 0$ so that $\sum_{j=1}^n a_{i,j} p_j = 0$ for all i.

If there exists an allocation x satisfying (a) then there is no loss regardless of outcomes but gain for some outcomes. Hence it is an arbitrage opportunity. If there is no such allocation, there is a vector of probability p_j which makes the expected change in the price of the ith stock equal to 0 for all i. Such probability vector is called a *martingale measure*. The above theorem can rewritten as the follow.

Theorem 75. There is no arbitrage if and only if there is a strictly positive probability vector so that all the stock prices are martingales.

Proof. If (a) is true, then for any strictly positive probability $p_j > 0$, $\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{i,j} p_j > 0$, so (b) is false.

Suppose (a) is false. Let $y = A^\top x$ where $x = (x_1, \ldots, x_m)$ and $A = (a_{i,j})$, that is, $y_j = \sum_{i=1}^{m} x_i a_{i,j}$. Then $y_j \geq 0$ for all j implies $y_j = 0$ for all j. Otherwise there exists an allocation x satisfying (a). Hence there exists a strictly positive probability vector $p_j > 0$ such that $p^\top y = p^\top A^\top x = 0$ for all possible x. $\square$

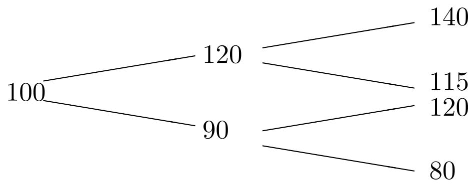
Example 69. In the previous example,

		$j=1$	$j=2$
stock	$i=1$	30	-10
option	$i=2$	$20-c$	$-c$

No arbitrage implies $30p_1 + (-10)p_2 = 0 = (20-c)p_1 + (-c)p_2$. By restricting $p_1 + p_2 = 1$ we get $p_1 = 1/4$ and $p_2 = 3/4$ by the first equality and the second equality implies $c = 20p_1 = 5$.

The first equation leads that the stock price is a martingale and the second equation is that the price of the option is the expected value under the martingale probabilities.

Example 70. Let X_0, X_1, X_2 be the stock prices at time 0, 1, 2 respectively. Assume $X_0 = 100$, $X_1 = 120$ or $X_1 = 90$; $X_2 = 140$ or $X_2 = 115$ if $X_1 = 120$; $X_2 = 120$ or $X_2 = 80$ if $X_1 = 90$.



The value of an European call option is the expected value under the probability which makes the stock price a martingale. Hence $P(X_1 = 120) = 1/3$ and $P(X_1 = 90) = 2/3$, $P(X_2 = 140 \,|\, X_1 = 120) = 1/5$, $P(X_2 = 120 \,|\, X_1 = 90) = 1/4$. If the value of the option is

$$\frac{1}{3}\frac{1}{5} \times 40 + \frac{1}{3}\frac{4}{5} \times 15 + \frac{2}{3}\frac{1}{4} \times 20 + \frac{2}{3}\frac{3}{4} \times 0 = 10.$$

Example 71 (Hedging strategy to replicate the option)**.** Consider four possible actions:
A_0 Keep a dollar in cache
A_1 Buy one share of stock at time 0 and sell it at time 1
A_2 Buy one share at time 1 if the stock is at 120, and sell it at time 2
A_3 Buy one share at time 1 if the stock is at 90, and sell it at time 2

These actions produce the following payoffs

X_1	X_2	A_0	A_1	A_2	A_3	option
120	140	1	20	20	0	40
120	115	1	20	-5	0	15
90	120	1	-10	0	30	20
90	80	1	-10	0	-10	0

It is possible to fine the coefficients z_i for the actions A_i to make the same payoff to the option, that is,

$$z_0 + 20z_1 + 20z_2 = 40, \quad z_0 + 20z_1 - 5z_2 = 15, \quad z_0 - 10z_1 + 30z_3 = 20, \quad z_0 - 10z_1 - 10z_3 = 0.$$

It solves $z_0 = 10, z_1 = 1/2, z_2 = 1, z_3 = 1/2$.

Note that z_0 is the option price. The vector $(z_1, z_2, z_3) = (1/2, 1, 1/2)$ is called a hedging strategy to replicate the option.

7.1 Binomial Model: One Period Case

Suppose there are only two possible outcomes for the stock: ups (U) and downs (D)

$$\begin{tikzcd}[column sep=large, row sep=small] & S_1(U) = S_0u \\ S_0 \arrow[ur, dash] \arrow[dr, dash] & \\ & S_1(D) = S_0d \end{tikzcd}$$

Also assume there is an interest rate r, that is, \$1 at time 0 is the same as \$$(1+r)$ at time 1. Then it is natural to assume

$$0 < d < 1 + r < u.$$

Consider two payoffs $V_1(U)$ or $V_1(D)$ at time 1 such as a call option $(S_1 - K)^+$ or a put option $(K - S_1)^+$.

To find the no arbitrage price when there are V_0 in cash and Δ_0 shares of the stock at time 0, that is,

$$V_0 + \Delta_0(\frac{1}{1+r}S_1(U) - S_0) = \frac{1}{1+r}V_1(U) \text{ and } V_0 + \Delta_0(\frac{1}{1+r}S_1(D) - S_0) = \frac{1}{1+r}V_1(D)$$

The risk neutral probability p^* must satisfy

$$\frac{1}{1+r}(p^*S_0u + (1-p^*)S_0d) = S_0$$

which solves $p^* = (1 + r - d)/(u - d) \in (0, 1)$. The expected value equates

$$V_0 = \frac{1}{1+r}(p^*V_1(U) + (1-p^*)V_1(D)).$$

Finally

$$\Delta_0\frac{1}{1+r}(S_1(U) - S_1(D)) = \frac{1}{1+r}(V_1(U) - V_1(D))$$

which implies

$$\Delta_0 = \frac{V_1(U) - V_1(D)}{S_1(U) - S_1(D)}$$

7.2 Binomial Model: Multiple Period Model

Let a_n be the history of ups (U) and downs (D) from time 1 to n, for example, $a_3 = UUD$ indicates S_1, S_2 have up price while S_3 has down price.

No arbitrage price problem becomes

$$\begin{aligned} V_n(a_n) + \Delta_n(a_n)(\frac{1}{1+r}S_{n+1}(a_nU) - S_n(a_n)) &= \frac{1}{1+r}V_{n+1}(a_nU) \\ V_n(a_n) + \Delta_n(a_n)(\frac{1}{1+r}S_{n+1}(a_nD) - S_n(a_n)) &= \frac{1}{1+r}V_{n+1}(a_nD) \end{aligned} \tag{1}$$

The risk neutral probability p^* equation

$$\frac{1}{1+r}(p_n^*(a_n)S_{n+1}(a_nU) + (1-p_n^*(a_n))S_{n+1}(a_nd)) = S_n(a_n) \tag{2}$$

which solves $p_n^*(a_n) = [(1+r)S_n(a_n) - S_{n+1}(a_nD)]/[S_{n+1}(a_nU) - S_{n+1}(a_nD)] \in (0, 1)$. The expected value equates

$$V_n(a_n) = \frac{1}{1+r}(p_n^*(a_n)V_{n+1}(a_nU) + (1-p_n^*(a_n))V_{n+1}(a_nD)).$$

Finally

$$\Delta_n(a_n)\frac{1}{1+r}(S_{n+1}(a_nU)-S_{n+1}(a_nD))=\frac{1}{1+r}(V_{n+1}(a_nU)-V_{n+1}(a_nD))$$

which implies

$$\Delta_n(a_n)=\frac{V_{n+1}(a_nU)-V_{n+1}(a_nD)}{S_{n+1}(a_nU)-S_{n+1}(a_nD)} \tag{3}$$

Suppose, in nth time, we have wealth W_n, stock Δ_n share, interest rate r per period. Then the wealth at time $n+1$ satisfies

$$W_{n+1}=\Delta_nS_{n+1}+(1+r)(W_n-\Delta_nS_n) \tag{4}$$

The equation (1) can be written as

$$V_{n+1}=\Delta_nS_{n+1}+(1+r)(V_n-\Delta_nS_n).$$

Hence we have the following theorem.

Theorem 76. If $W_0=V_0$ and we use the investment strategy in (3) then we have $W_n=V_n$.

Considering (2), we have the following theorem.

Theorem 77. In the binomial model, under the risk neutral probability measure, $M_n=S_n/(1+r)^n$ is a martingale with respect to $\mathcal{F}_n=\sigma(S_0,\dots,S_n)$. Similarly, for W_n and V_n in Theorem 76, $W_n/(1+r)^n$ and $V_n/(1+r)^n$ are martingales.

Theorem 78. Let V_P and V_C be the values of put and call option with the same strike K and expiration N. Then $V_P-V_C=K/(1+r)^N-S_0$.

Proof. Note that $S_N+(K-S_N)^+-(S_N-K)^+=K$. By taking expectation with respect to the risk neutral probability P^* after dividing by $(1+r)^N$, we get $\mathbb{E}^*[S_N/(1+r)^n+(K-S_N)^+/(1+r)^N-(S_N-K)^+/(1+r)^N]=K/(1+r)^N$ which is equivalent to $S_0+V_P-V_c=K/(1+r)^N$. Therefore the theorem follows. □

7.3 Black-Scholes Formula

In reality, the time domain is continuous which can be approximated by discrete time scenario. Further by assuming a binomial model, we get

$$S_{nh}=S_{(n-1)h}\exp(\mu h+\sigma\sqrt{h}X_n)$$

where $X_n\sim$ symmetric Bernoulli(1/2), that is, $P(X_n=1)=P(X_n=-1)=1/2$. This is a binomial model with $u=\exp(\mu h+\sigma\sqrt{h})$ and $d=\exp(\mu h-\sigma\sqrt{h})$. Inductively, we obtain

$$S_{nh}=S_0\exp(\mu nh+\sigma\sqrt{h}\sum_{m=1}^{n}X_m).$$

Let $t=nh$. Then send n to infinity, we get

$$S_t=S_0\exp(\mu t+\sigma\lim_{n\to\infty}t^{1/2}n^{-1/2}\sum_{m=1}^{n}X_m)\stackrel{d}{=}S_0\exp(\mu t+\sigma B_t). \tag{5}$$

The limit is a construction of a standard Brownian motion. Here μ is the exponential growth rate of the stock and σ is its volatility. If we also assume the interest rate is r per unit time, then the interest rate per h time is rh and the interest rate at time $t=nh$ is

$$\lim_{n\to\infty}\left(\frac{1}{1+rh}\right)^{t/h}=\lim_{n\to\infty}(1+rt/n)^{-n}=e^{-rt}.$$

Hence the discounted stock price is

$$e^{-rt}S_t = S_0 \cdot \exp((\mu - r)t + \sigma B_t).$$

Thus $e^{-rt}S_t$ is a martingale if and only if $\mu = r - \sigma^2/2$ by applying Exercise 5.3

Theorem 79. The risk neutral probability is achieved when $mu = r - \sigma^2/2$ and the value of a European option $g(S_T)$ is the expectation of $e^{-rT}g(S_T)$ with respect to the risk neutral probability.

Proof. We prove as limits of the discrete approximation. The risk neutral probabilities p_h^* are

$$p_h^* = (1 + rh - d)/(u - d)$$

Using $e^x = 1 + x + x^2/2 + O(|x|^3)$ for $x \approx 0$, $\exp(\mu h \pm \sigma h^{1/2}) = 1 + \mu h \pm \sigma h^{1/2} + \sigma^2 h/2 + O(h^{3/2})$

$$= \frac{1 + rh - (1 - \sigma h^{1/2} + (\mu + \sigma^2/2)h) + O(h^{3/2})}{2\sigma h^{1/2} + O(h^{3/2})} = \frac{1}{2} + \frac{r - \mu - \sigma^2/2}{2\sigma} h^{1/2} + O(h).$$

If $X_1^h, X_2^n, \ldots$ are i.i.d. with $P(X_1^h = 1) = p_h^*$ and $P(X_1^h = -1) = 1 - p_h^*$, then

$$\mathbb{E}X_i^h = 2p_h^* = [(r - \mu - \sigma^2/2)/\sigma]h^{1/2} + O(h), \quad \mathbb{V}\mathrm{ar}(X_i^h) = 1 - (\mathbb{E}X_i^h)^2 = 1 + O(h).$$

By the central limit theorem,

$$\sigma h^{1/2} \sum_{m=1}^{t/h} X_m^h = \sigma h^{1/2} \sum_{m=1}^{t/h} (X_m^h - \mathbb{E}X_m^h) + \sigma h^{1/2} \sum_{m=1}^{t/h} \mathbb{E}X_m^h \xrightarrow{d} \sigma B_t + (r - \mu - \sigma^2/2)t$$

Then under the risk neutral measure, say P^*,

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t).$$

Finally the value of the option $g(S_T)$ is the expectation under P^* discounted, that is, $e^{-rT}\mathbb{E}^* g(S_T)$. $\square$

Black-Scholes Partial Differential Equation

Let $V(t, s)$ be the value of the option $g(S_t)$ at time $t < T$ when the stock price is s. Then the value satisfies

$$V(t - h, s) = \frac{1}{1 + rh}[p^* V(t, su) + (1 - p^*)V(t, sd)]$$

which implies

$$\frac{V(t, s) - V(t - h, s)}{h} - rV(t - h, s) = p^* \frac{V(t, s) - V(t, su)}{h} + (1 - p^*)\frac{V(t, s) - V(t, sd)}{h}$$

By sending $h \to 0$,

$$\lim_{h \to 0} \Big[\frac{V(t, s) - V(t - h, s)}{h} - rV(t - h, s)\Big] = \frac{\partial V}{\partial t}(t, s) - rV(t, s).$$

Note that $V(t, s+\delta) - V(t, s) = (\partial V/\partial x)(t, s)\delta + (\partial^2 V/\partial x^2)(t, s)\delta^2/2 + O(\delta^3)$ implies $V(t, s) - V(t, s\exp(\mu h \pm \sigma h^{1/2})) = V(t, s) - V(t, s + s(\pm\sigma h^{1/2} + (\mu + \sigma^2/2)h + O(h^{3/2})) = -(\partial V/\partial x)(t, s)s(\pm\sigma h^{1/2} + (\mu + \sigma^2/2)h) - (\partial^2 V/\partial x^2)(t, s)s^2\sigma^2 h + O(h)$ and let $p^* = (1 + ch^{1/2})/2$ for simplicity. Then

$$\begin{aligned}
&p^*(V(t, s) - V(t, su))/h + (1 - p^*)(V(t, s) - V(t, sd))/h \\
&= \frac{1}{2h}\Big[(1 + ch^{1/2})(\frac{\partial V}{\partial x}(t, s)s(\mu h^{1/2} + (\mu + \frac{\sigma^2}{2})h) - \frac{\partial^2 V}{\partial x^2}(t, s)s^2\frac{\sigma^2}{2}h) \\
&\quad + (1 - ch^{1/2})(\frac{\partial V}{\partial x}(t, s)s(-\mu h^{1/2} + (\mu + \frac{\sigma^2}{2})h) - \frac{\partial^2 V}{\partial x^2}(t, s)s^2\frac{\sigma^2}{2}h) + O(h^{3/2})\Big] \\
&= \frac{1}{2h}\Big[2\frac{\partial V}{\partial x}(t, s)s(\mu + \frac{\sigma^2}{2})h - 2\frac{\partial^2 V}{\partial x^2}(t, s)s^2\frac{\sigma^2}{2}h + ch^{1/2}2\frac{\partial V}{\partial x}(t, s)s\mu h^{1/2} + O(h^{3/2})\Big] \\
&= -\frac{\partial V}{\partial x}(t, s)s(\mu + \frac{\sigma^2}{2} + r - \mu - \frac{\sigma^2}{2}) - \frac{\partial^2 V}{\partial x^2}(t, s)s^2\frac{\sigma^2}{2} + O(h^{3/2}).
\end{aligned}$$

Finally the Black-Scholes partial differential equation is derived as

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s) + rs\frac{\partial V}{\partial x}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial x^2}(t,s) = 0 \tag{6}$$

for $0 \leq t < T$ with the boundary condition $V(T,s) = g(s)$.

Theorem 80. The price of the European call option $(S_t - K)^+$ is

$$S_0\Phi(d_1) - e^{-rt}K\Phi(d_2) \tag{7}$$

where $d_1 = [\log(S_0/K) + (r+\sigma^2/2)t]/[\sigma\sqrt{t}]$ and $d_2 = d_1 - \sigma\sqrt{t}$.

Proof. Note that $\log(S_t/S_0) \sim N(\mu t, \sigma^2 t)$ with $\mu = r - \sigma^2/2$. Then

$$\mathbb{E}^*[e^{-rt}(S_t-K)^+] = e^{-rt}\int_{\log(K/S_0)}^{\infty}(S_0e^y - K)\frac{1}{(2\pi\sigma^2 t)^{1/2}}\exp(-\frac{(y-\mu t)^2}{2\sigma^2 t})\,dy$$

The change of variable $w = -(y-\mu t)/(\sigma t^{1/2})$ or $y = \mu t - w\sigma t^{1/2}$ gives

$$= e^{-rt}S_0e^{\mu t}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{d_2} e^{-w\sigma t^{1/2}}e^{-w^2/2}\,dw - e^{-rt}K\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{d_2} e^{-w^2/2}\,dw$$

where $d_2 = -(\log(K/S_0) - \mu t)/(\sigma t^{1/2}) = (\log(S_0/K) + (r-\sigma^2/2)t)/(\sigma t^{1/2})$

$$\begin{aligned} &= e^{-rt}S_0e^{\mu t+\sigma^2 t/2}\int_{-\infty}^{d_2}\frac{\exp(-(w+\sigma t^{1/2})^2/2)}{(2\pi)^{1/2}}\,dw - e^{-rt}K\Phi(d_2) \\ &= e^{-rt+\mu t+\sigma^2 t}S_0\Phi(d_2+\sigma t^{1/2}/2) - e^{-rt}K\Phi(d_2) \end{aligned}$$

Note $\mu = r - \sigma^2/2$ implies $e^{-rt}e^{\mu t+\sigma^2 t/2} = 1$. The for $d_1 = d_2 + \sigma t^{1/2}$,

$$= S_0\Phi(d_1) - e^{-rt}K\Phi(d_2).$$

Hence the theorem follows. ∎