CSC165H1 S - Exercise 7
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Question 1:

(a)

We want to prove:

$\forall a \in R, \forall b \in R, \quad (a \le b \Rightarrow (\exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow n^a \le c \cdot n^b))$

Proof:

Assume $a \in R, b \in R$ # domain assumption
    Assume $a \le b$ # antecedent
        Let $c_0 = 1$ and $B_0 = 1$
        Then $c_0 \in R^+$ and $B_0 \in N$
        Assume $n \in N$ # arbitrary natural number
            Assume $n \ge B_0$ # antecedent
                Then $n^a \le n^b$ # since $a \le b$ and function n^x monotone increasing
                Then $n^a \le n^b = c_0 \cdot n^b$ # since $c_0 = 1$
            Then $n \ge B_0 \Rightarrow n^a \le c_0 \cdot n^b$ # introduce $\Rightarrow$
        Then $\forall n \in N, \quad n \ge B_0 \Rightarrow n^a \le c_0 \cdot n^b$ # introduce $\forall$
        Then $\exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow n^a \le c \cdot n^b$ # introduce $\exists$
    Then $a \le b \Rightarrow \exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow n^a \le c \cdot n^b$
Then $\forall a \in R, \forall b \in R, \ a \le b \Rightarrow \exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow n^a \le c \cdot n^b$
Therefore $\forall a \in R, \forall b \in R, \ a \le b \Rightarrow n^a \in O(n^b)$ # by definition

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(b)

We want to prove

$\forall a \in R, \forall b \in R, \quad (1 < a \le b \Rightarrow (\exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow a^n \le c \cdot b^n))$

Proof:

Assume $a \in R, b \in R$ # domain assumption
    Assume $1 < a \le b$ # antecedent
        Let $c_0 = 1$ and $B_0 = 1$
        Then $c_0 \in R^+$ and $B_0 \in N$
        Assume $n \in N$ # arbitrary natural number
            Assume $n \ge B_0$ # antecedent
                Then $n \ge 1$ # since $B_0 = 1$
                Then $a^n \le b^n$ # since $1 < a \le b$, function x^n monotone increasing
                Then $a^n \le b^n = c_0 \cdot b^n$ # since $c_0 = 1$
            Then $n \ge B_0 \Rightarrow a^n \le c_0 \cdot b^n$ # introduce $\Rightarrow$
        Then $\forall n \in N, \quad n \ge B_0 \Rightarrow a^n \le c_0 \cdot b^n$ # introduce $\forall$
        Then $\exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow a^n \le c \cdot b^n$ # introduce $\exists$
    Then $1 < a \le b \Rightarrow \exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow a^n \le c \cdot b^n$
Then $\forall a \in R, \forall b \in R, \quad 1 < a \le b \Rightarrow \exists c \in R^+, \exists B \in N, \forall n \in N, n \ge B \Rightarrow a^n \le c \cdot b^n$
Therefore $\forall a \in R, \forall b \in R, 1 < a \le b \Rightarrow a^n \in O(b^n)$ # by definition

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(c)

Proof:

Assume the statement is true.

Then $\forall a \in R^+$, $\forall b \in R^+$, $a \neq 1 \wedge b \neq 1 \Rightarrow (\exists c_1 \in R^+, \exists c_2 \in R^+$, $\exists B \in N$, $\forall n \in N$,

n >= B $\Rightarrow$ $c_1 \log_b(n) \leq \log_a(n) \leq c_2 \log_b(n)$)

Let a = 2, b = 1/2. # arbitrary real numbers, $a \neq 1 \wedge b \neq 1$

Then $\log_a(n)$ is increasing in its domain and $\log_b(n)$ is decreasing in its domain. # by the graphs of log function

Let n = 2. # since n is an arbitrary natural number and n >= B, $B \in N$

Then $\log_b(n) = \log_{1/2}2 = -1$

Then $\log_a(n) = \log_2 2 = 1$

Then $c_1 \times 1 <= -1 <= c_2 \times 1$.

Contradiction. # since c_1 and c_2 are positive real numbers.

Then the assumption is not true.

Therefore the statement is disproved.

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Question 2:

Proof:

Prove the situation n = 0: $\sum_{i=0}^{0} t_i = 0$, which is obviously true.

Prove the situation of arbitrary natural number n:

Assume $n \in N$. # arbitrary natural number

Assume $\sum_{i=0}^{n} t_i$ = n(n+1)(n+2)/6. # antecedent

Then $t_0 = 0 \vee n > 0$. # natural numbers are non-negative

CASE 1 (assume n = 0):

Then $\sum_{i=0}^{n} t_i = 0 = t_0$

CASE 2 (assume n > 0):

Then $n \geq 1$ # n is an integer greater than 0

Then $\sum_{i=0}^{n+1} t_i = \sum_{i=0}^{n} t_i + t_{n+1}$

=n(n+1)(n+2)/6+(n+1)(n+2)/2

=(n+1)(n+2)(n/6)+(n+1)(n+2)(3/6)

=(n+1)(n+2)(n+3)/6

=(n+1)((n+1)+1)((n+1)+2)/6

Then $\sum_{i=0}^{n} t_i$ = n(n+1)(n+2)/6. # true in both possible cases

Then $\forall n \in N$, $\sum_{i=0}^{n} t_i$ = n(n+1)(n+2)/6. # introduce $\forall$

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