FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATION, April 2010

MAT 237 Y1Y, Advanced Calculus

PLEASE HAND IN

Examiners: I. Graham
R. Stanczak

Last Name:______________________ **Student #**______________

First Name:______________________

(a) TIME ALLOWED: 3 h

(b) NO AIDS ALLOWED.

(c) WRITE SOLUTIONS ON THE SPACE PROVIDED. USE THE REVERSE SIDE OF THE PAGE TO CONTINUE IF NECESSARY.

(d) DO NOT REMOVE ANY PAGES. THERE ARE 16 PAGES INCLUDING THIS ONE.

MARKER'S REPORT

Question	Mark
1	/12
2	/13
3	/14
4	/11
5	/11
6	/ 8
7	/12
8	/ 8
9	/11
TOTAL	/100

1. *[12 marks, 4 marks each part]* Let $f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$.

(a) Show that f is continuous at (0,0).

(b) Show, using the definition, that the directional derivative of f at (0, 0) in any direction $\mathbf{u}=(u_1,u_2)$ exists.

cont'd...

1. (c) Verify whether or not f is differentiable at $(0, 0)$.

cont'd...

2.(a) *[4 marks]* Find an equation for the tangent plane Π to the surface given by the equation $2xy^2 = 2z^2 - xyz$ at the point $P(2,-3,3)$ and verify that the plane Π and the y-axis intersect at the point $Q(0,-1,0)$.

(b) *[6 marks]* Evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$ if $\mathbf{F}(x,y,z) = \dfrac{z}{1+xz}\mathbf{i} + y\mathbf{j} + \dfrac{x}{1+xz}\mathbf{k}$ and C is the line segment from Q to P.

cont...

2. (c) *[3 marks]* Is the vector field **F** of part (b) conservative on the open ball $B(\mathbf{0},1)$? Justify your answer.

cont'd ...

3. (a) *[9 marks, 3 marks each part]* Consider the set $S = \{(x, y, z) \in R^3 : z^2 - x^2 - y^2 = 1\}$.

(i) Is S compact? Explain very shortly.

(ii) Is S connected? Why or why not?

(iii) The set S describes a smooth surface in R^3. An ant moves up along the path being the intersection of S and the plane $z = x + 1$ and parametrized by setting $y = t$. At what rate is his distance from the z-axis changing at the point $(2,2,3)$?

cont'd...

3. (b) *[5 marks]* Suppose that $w = f(u,v)$ is a differentiable function of $u = \dfrac{x}{y}$ and $v = \dfrac{z}{y}$. Then $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = m$, where m is a constant.

Find m.

cont'd...

4. (a) *[4 marks]* Suppose $S \subset \mathrm{R}^n$ is compact, $f\colon S \to \mathrm{R}$ is continuous, and $f(\boldsymbol{x}) > 1$ for every $\boldsymbol{x} \in S$. Show that there is a number $c > 1$ such that $f(\boldsymbol{x}) \geq c$ for every $\boldsymbol{x} \in S$.

(b) *[7 marks]* Let $f(x,y,z) = x^3 + y^2 + az^3 - 3az^2 - xy - y + 9$ where a is a non-zero constant. Find all points (if any) at which f has a local minimum.
(*continue your solution on the next page*)

cont'd...

4. (b) Continue your solution

cont'd...

5. (a) *[4 marks]* Knowing that the volume of the unit ball is $\frac{4}{3}\pi$ show, using the change of variables, that the volume of an elliptic ball $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ is $\frac{4}{3}\pi(abc)$.

cont'd...

5.(b) *[7 marks]* Find the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ that passes through the point $(1, 2, 3)$ such that the elliptic ball bounded by this ellipsoid has the smallest volume.

cont'd...

6.(a) *[4 marks]* A hole is bored through a sphere, the axis of the hole being a diameter of the sphere. The volume of the solid remaining is given by the iterated integral

$$V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta$$

Determine the radius of the hole and the radius of the sphere.

(b) *[4 marks]* Give an example of a function $f : R^2 \to R$ that is Riemann integrable on $[0,1]\times[0,1]$ but the iterated integral $\int_0^1 [\int_0^1 f(x,y)\,dy]\,dx$ does not exist.

cont'd ...

7. *[12 marks, 4 marks each part]* Let the transformation **G** from the uv-plane to the xy-plane be defined by $(x, y) = (u + v, u^2 - v)$. Let D be the region bounded by the u-axis, the v-axis, and the line $u + v = 2$.

(a) Near what points in the uv-plane is it possible to express u and v locally as functions of x and y?

(b) Find and sketch the image region **G**(D).

cont'd...

7. (c) Compute the integral $\iint\limits_{G(D)} \frac{dx\,dy}{\sqrt{1+4x+4y}}$.

cont'd...

8. *[8 marks]* Find the total mass of a spherical shell of radius a and negligible thickness having density at each point equal to the linear distance of the point from a single fixed point on the sphere.

cont'd...

9. (a) *[4 marks]* Prove or disprove the statement " If S is the level set of a C^1 function $f(x,y,z)$ and $\nabla f \neq \mathbf{0}$, then the flux of ∇f across S is never zero".

(b) *[7 marks]* Let $\mathbf{F}(x,y,z) = y^3\mathbf{i} - x^3\mathbf{j} + z^3\mathbf{k}$. Let S be the surface $x^2 + y^2 + z^4 = 5$, $z \geq 1$ oriented by the upward pointing normal. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

Remark: D*irect evaluation is difficult. S is a portion of a "deformed sphere" centered at the origin, so try to make use of either Stokes or Gauss theorem.*