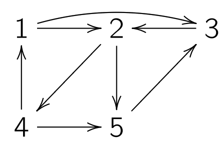
# Exerzitien VIII

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Nov. 24., in your tutorial.

Reading suggestion: Axler, Chapter 5 all but the last section and Chapter 8, section 1, 2, and 5 (Minimal polynomial)

**Exercise 1.** *A matrix $A \in \mathbb{R}^{n\times n}$ is called* stochastic *when its entries are nonnegative and the sum of the entries in each of its columns is equal to* 1*.*

1. *Prove that the product of two stochastic matrices is also stochastic.*
2. *Prove that* 1 *occurs as an eigenvalue for every stochastic matrix $A$. Hint: consider a vector in the image of $(A - \mathbf{I})$; does it satisfy any linear condition?*
3. *There are five bureaucrats at the University of Toronto – let's not name names, so we'll call them $1, 2, 3, 4, 5$. None of them want to help you, so they talk to you for 1 minute and then randomly send you to other bureaucrats according to the following directed graph:*



*This means, for example, that Bureaucrat 2 will send you to number 4 or number 5 with equal probability 1/2. During an eternity of going through red tape, following the directions of these bureaucrats, a certain fraction of your time will be spent with each of these bureaucrats (assume your travel between the offices is instantaneous).*

**Rank the bureaucrats in order from the most time-wasting to the least.**

*Hint: The expected fraction of time $x_i$ spent at bureaucrat $i$ will be a sum of the expected fractions $x_j$, weighted by the probability that $j$ sends you to $i$. For example $x_1 = \frac{1}{2}x_4$, because the only way to get to* 1 *is if you were at* 4 *and they happened to send you to* 1*, which only happens half the time (the other half, you would be sent to* 5*). Write the whole problem as an eigenvector problem*

$$Ax = x, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix}$$

*where $A$ is a stochastic $5 \times 5$ matrix. You can use* wolframalpha.com *to find the eigenvectors of $A$ (just type in "*`eigenvectors` *$\{\{a_{11}, \ldots, a_{15}\}, \{a_{21}, \ldots, a_{25}\}, \ldots, \{a_{51}, \ldots, a_{55}\}\}$").*

*Remark: This is how Google decides how to rank webpages in order of "importance": the bureaucrats are webpages and the arrows are hyperlinks between them. To learn more, follow* this link*.*

**Exercise 2.** *Find the minimal polynomial of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, and find the change of basis matrix $P$ such that $PAP^{-1}$ is in Jordan canonical form.*