UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2011 EXAMINATIONS

MAT335H1F
Chaos, Fractals and Dynamics
Examiner: D. Burbulla

Duration - 3 hours
Examination Aids: A Scientific Hand Calculator

Name: ______________________ **Student Number:** ______________

INSTRUCTIONS: All six questions have equal weight. Attempt only five of them. Present your solutions in the exam booklets provided. Do not tear any pages from this exam. Hand in this exam with your booklet(s). The marks for each question are indicated in parentheses beside the question number. **MAXIMUM MARKS:** 100

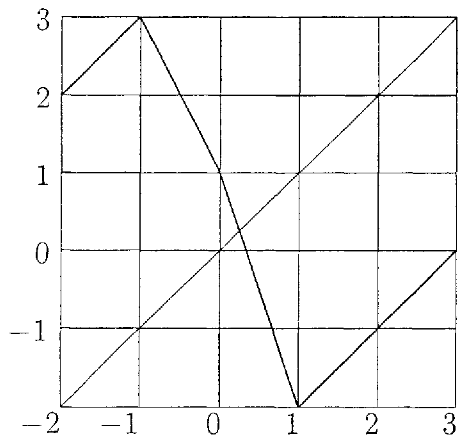
1. [20 marks] Let $V : [-2, 2] \longrightarrow [-2, 2]$ by $V(x) = 2|x| - 2$.
   (a) [5 marks] Plot the graphs of V and V^2.
   (b) [5 marks] Find all the fixed points and 2-cycles of V and determine if they are attracting or repelling.
   (c) [10 marks] Let $T : [0, 1] \longrightarrow [0, 1]$ by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} < x \leq 1 \end{cases};$$

   let $h : [0, 1] \longrightarrow [-2, 2]$ by $h(x) = -4x + 2$. Prove that h is a conjugacy between T and V.

2. [20 marks] This question has five parts.
   (a) [3 marks] Define: the subset D is dense in X.
   (b) [7 marks] Prove that the periodic points of σ are dense in Σ.
   (c) [6 marks] Define the Cantor middle-thirds set, K. What is its fractal dimension?
   (d) [2 marks] Show that K is not dense in $[0, 1]$.
   (e) [2 marks] Show that the complement of K is dense in $[0, 1]$.

3. [20 marks] The graph of $F : [-2,3] \longrightarrow [-2,3]$ is shown below, along with the line $y = x$.



(a) [5 marks] Show that -2 is on a 6-cycle for F.

(b) [5 marks] Explain why F has cycles with prime period p for any even number p.

(c) [10 marks] Prove that F has no cycles of any odd prime period $q > 1$.

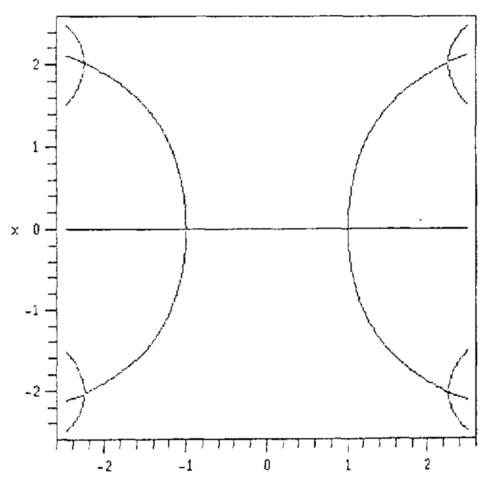
4. [20 marks] For $c \neq 0$, let $F_c : \mathbb{R} \longrightarrow \mathbb{R}$ by $F_c(x) = c \sin x$.

(a) [4 marks] Show that for $|c| < 1$, $x = 0$ is the only fixed point of F_c and its basin of attraction is $\mathbb{R} = (-\infty, \infty)$.

(b) [4 marks] Calculate the Schwarzian derivative of F_c and show it is negative.

(c) [4 marks] Give a graphical example of a fixed point of F_c for which the immediate basin of attraction does not extend to infinity.

(d) [8 marks] Below is the bifurcation diagram for F_c, for $|c| < 2.5$, $|x| < 2.5$.



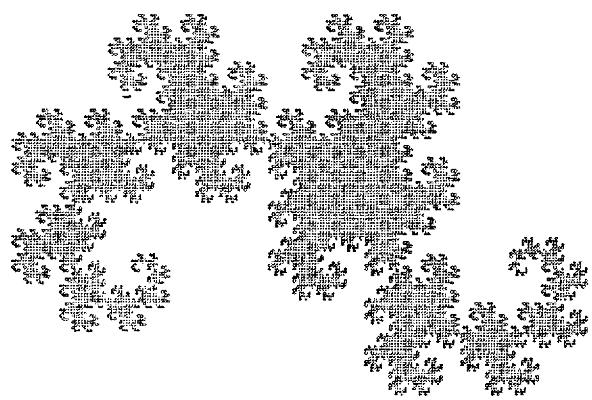
Classify each node in this diagram as a tangent (or saddle-node) bifurcation, a period-doubling bifurcation, or neither.

5. [20 marks] The following iterated function system

$$A_1\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$A_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

generates the following fractal, known as the dragon curve:



(a) [10 marks] Here is one way to generate the dragon curve:

Step 0: Draw the line segment I which joins the points $(0,0)$ and $(1,0)$.

Step 1: Replace I by the two line segments $A_1(I)$ and $A_2(I)$.

Step 2: Replace the two line segments of Step 1 by the four line segments

$$A_1 \circ A_1(I), A_1 \circ A_2(I), A_2 \circ A_1(I) \text{ and } A_2 \circ A_2(I).$$

Step k: Replace each line segment of Step $k-1$ by its images under A_1 and A_2.

Draw Steps 0 through 3 of this process.

(b) [5 marks] Calculate the fractal dimension of the dragon curve.

(c) [5 marks] Describe another algorithm that generates the dragon curve.

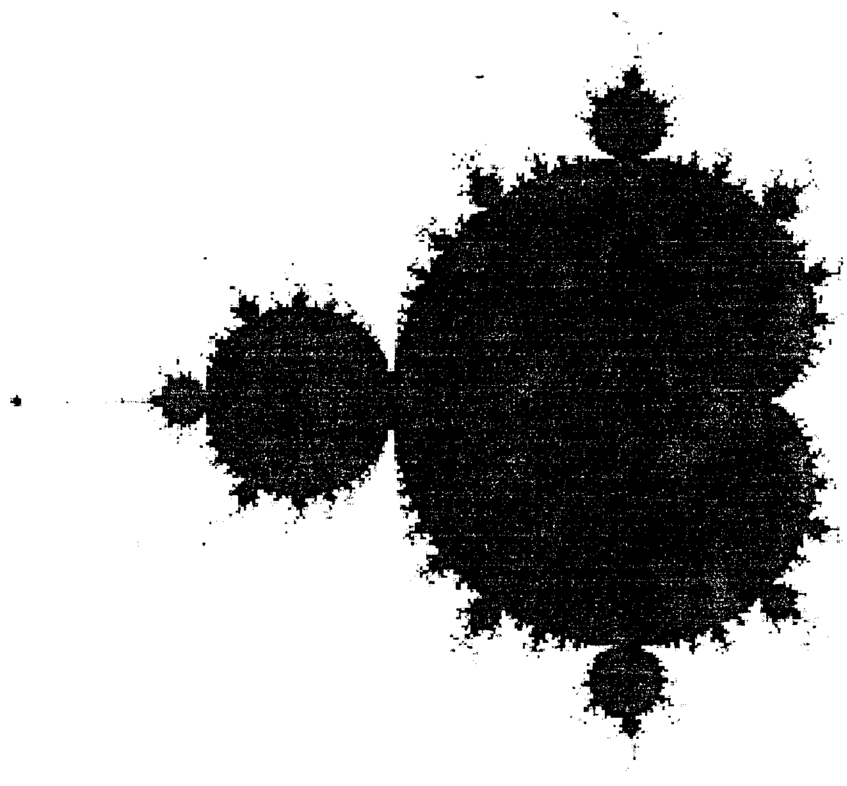
6. [20 marks] Let $Q_c : \mathbb{C} \longrightarrow \mathbb{C}$ by $Q_c(z) = z^2 + c$.

(a) [4 marks] Define the Mandelbrot set, $\mathcal{M}$.

(b) [4 marks] Show that the orbit of 0 under Q_{-2} is eventually fixed. Is this fixed point attracting or repelling? Is $-2 \in \mathcal{M}$?

(c) [6 marks] Show that the orbit of 0 under Q_i is eventually periodic. Is this cycle attracting or repelling? Is $i \in \mathcal{M}$?

(d) [2 marks] With respect to the following image of the Mandelbrot set, locate both -2 and i.



(e) [4 marks; 2 marks each] Let K_c be the filled Julia set of Q_c; let J_c be the Julia set of Q_c. Indicate whether the following statements are True or False, and give a brief justification for your choice.

I. K_{-2} is totally disconnected.

II. $K_{-2} = J_{-2}$