§16 - Tychonoff's Theorem

1 Motivation

In point-set topology, there are a variety of properties that are studied. When given a property of a topological space there are always 4 basic questions that need answering:

1. Is there any space with this property?
2. Is this property preserved under continuous maps?
3. Is this property preserved under products?
4. Is this property preserved under subspaces?

For example, the T_2 property is very nicely behaved in that it is preserved under open maps, products and subspaces. Separability is preserved under continuous surjections and countable products, but not subspaces.

With compactness we have already seen that it is preserved under continuous maps, *closed* subspaces and *finite* products. Tychonoff's Theorem asserts that an arbitrary (i.e. finite or infinite) product of compact spaces is again compact. In the proof that compactness is preserved under finite products, we use that there are only finitely many coordinates. We cannot easily extend this proof to infinitely many coordinates, so we have to do something clever.

Let us first look at some examples of infinite products of compact spaces:

- The simplest example is $\{0\}^{\mathbb{N}}$ which is just a single infinite sequence of zeros. Of course this is compact because it is finite.
- The first non-trivial example is $\{0,1\}^{\mathbb{N}}$. Showing that this is compact requires some thought, but can be seen by noting that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set, which is compact. Another strategy is to use König's Lemma (which you can find online).
- Think about how you can show that $[0,1]^{\mathbb{N}}$ is compact. It may help to observe that this space is a metric space.

Tychonoff's Theorem says something about the product of compact spaces *with the product topology*. It does not say anything about spaces with the box topology. For example, check that $\{0,1\}^{\mathbb{N}}$ and $[0,1]^{\mathbb{N}}$ (both with the box topology) are not compact.

2 Tools-Filters

In order to prove Tychonoff's Theorem we will use an equivalent notion of compactness.

Definition. *A collection $\mathcal{F}$ of subsets of some set X has the* ***finite intersection property*** *(FIP) if for every finite subcollection $\{F_1, F_2, \cdots, F_n\} \subseteq \mathcal{F}$ we have $\bigcap_{i=1}^{n} F_i \neq \emptyset$.*

Theorem. *A topological space X is compact iff for every collection $\mathcal{F}$ of closed subsets of X with the FIP, we have $\bigcap \mathcal{F} \neq \emptyset$.*

This has some hints that it will be helpful when looking at the product topology, because the basic open sets in the product topology are given by *finite intersections* of subbasic open sets and subbasic sets only give information about an individual coordinate. This will be how we exploit compactness of individual coordinates.

One of the main tools we use is that of a maximal collection of sets with the FIP. These maximal collections turn out to satisfy the properties of being maximal filters. On Assignment 6 you investigated maximal collections of sets with the FIP. There you also saw maximal filters, which are called ultrafilters. Of course, since we talked about *maximal* collections we needed to use Zorn's Lemma. Let us recall those notions.

Definition. *Let $\mathcal{C}$ be a collection of subsets of a topological space X, with the FIP. We say that $\mathcal{C}$ is* ***maximal*** *(with respect to the FIP) if whenever $\mathcal{A}$ is a collection of subsets with the FIP such that $\mathcal{C} \subseteq \mathcal{A}$, then $\mathcal{C} = \mathcal{A}$.*

Proposition. *For every collection of closed subsets $\mathcal{C}$ with the FIP, there is a maximal $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{A}$.*

Lemma (**Zorn's Lemma**). *Let $(P, \leq)$ be a (non-empty) partially ordered set such that every linearly ordered subset has an upper bound. Then P contains a maximal element.*

The partial order we will be using is the family $\mathcal{T}$ of all collections of subsets of a space X with the FIP that contain $\mathcal{C}$, ordered by $\subseteq$. So then $\mathcal{T} \neq \emptyset$ as $\mathcal{C} \in \mathcal{T}$.

Such a maximal extension will actually satisfy the stronger properties of being a filter.

Definition. *A collection of subsets $\mathcal{F}$ of a space X is called a* ***filter*** *if:*

1. *Finite intersections are contained in $\mathcal{F}$. That is:* $(\forall n \in \mathbb{N})(\forall F_1, \ldots, F_n \in \mathcal{F})[\bigcap_{i=1}^{n} F_i \in \mathcal{F}]$
2. *$\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$*
3. *$\mathcal{F}$ is closed upwards. That is if $B \subseteq A \subseteq X$ with $B \in \mathcal{F}$ then $A \in \mathcal{F}$.*

Remark. *Any filter satisfies the FIP by property (1) and (2).*

Definition. *Let $\mathcal{F}$ be a collection of subsets of a topological space X, that is a filter. We say that $\mathcal{F}$ is a **maximal filter** if it is maximal with respect to FIP.*

The following exercise allows us to just say "maximal" filters, without worrying about if the filter is maximal with respect to the FIP or filters.

Exercise. *Let $\mathcal{F}$ be a filter on a topological space X. TFAE:*

- *$\mathcal{F}$ is maximal with respect to the FIP.*
- *$\mathcal{F}$ is maximal with respect to filters.*

Example. *For $X = \mathbb{R}$, consider the following:*

- $\mathcal{F}_1 := \{(a,b) \subseteq \mathbb{R} : 100 \in (a,b)\}$;
- $\mathcal{F}_2 := \{A \subseteq \mathbb{R} : \exists (a,b) \ni 100, (a,b) \subseteq A\}$; *and*
- $\mathcal{F}_3 := \{A \subseteq \mathbb{R} : 100 \in A\}$.

Here $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$ and all have the FIP. However $\mathcal{F}_1$ is not a filter, $\mathcal{F}_2$ is a filter and $\mathcal{F}_3$ is a maximal filter.

Theorem. *Suppose that $\mathcal{F}$ is a collection of subsets of some set X that is maximal with respect to the FIP. Then $\mathcal{F}$ is a (maximal) filter.*

The most important part of a filter is property (1). The only other fact about maximal filters we will need is the following way to identify whether a set belongs to the maximal filter.

Theorem. *Suppose that $\mathcal{F}$ is a maximal filter on a set X and $A \subseteq X$. TFAE:*

1. $A \in \mathcal{F}$
2. $\forall F \in \mathcal{F}, A \cap F \neq \emptyset$

The thing to observe is that maximal filters have a strong form of the FIP (property (1) in the definition of a filter) and we have an alternate way of detecting when a set is in a maximal filter. It is also worth noting that Munkres proves Tychonoff's theorem without making mention of filters. Instead, the notion of a maximal collection with the FIP is used. This results in a proof with less notation, although on the other hand, filters are a useful notion in mathematics. Filters turn out to be a useful generalization of sequences.

3 Tychonoff's Theorem

We are now in a position to prove Tychonoff's Theorem:

Theorem (**Tychonoff's Theorem**). *Let $\{X_\alpha : \alpha \in I\}$ be a collection of compact spaces. Then $\prod_{\alpha \in I} X_\alpha$ is compact.*

Proof. Let $\mathcal{C}$ be a collection of closed subsets of $\prod X_\alpha$ with the FIP. We wish to show that $\bigcap \mathcal{C} \neq \emptyset$. Extend $\mathcal{C}$ to a maximal filter $\mathcal{F}$. Since $\mathcal{C} \subseteq \mathcal{F}$ we get $\bigcap \mathcal{F} \subseteq \bigcap \mathcal{C}$. In fact we get $\bigcap_{F \in \mathcal{F}} \overline{F} \subseteq \bigcap \mathcal{C}$, so it is enough to show that $\emptyset \neq \bigcap_{F \in \mathcal{F}} \overline{F}$. To do this we will define a function $f \in \prod_{\alpha \in I} X_\alpha$ that is also in $\bigcap_{F \in \mathcal{F}} \overline{F}$

We now transfer the filter down to the coordinates using the usual projection maps π_α. Let

$$\mathcal{F}_\alpha := \{\pi_\alpha(F) : F \in \mathcal{F}\}.$$

This collection has the FIP because

$$\emptyset \neq \pi_\alpha(F_1 \cap F_2) \subseteq \pi_\alpha(F_1) \cap \pi_\alpha(F_2).$$

Thus the collection of closed sets $\{\overline{\pi_\alpha(F)} : F \in \mathcal{F}\}$ has the FIP. Now since X_α is compact, we can choose

$$f(\alpha) \in \bigcap_{F \in \mathcal{F}} \overline{\pi_\alpha(F)}$$

for each $\alpha \in I$. Thus $f \in \prod_{\alpha \in I} X_\alpha$.

Claim: $f \in \bigcap_{F \in \mathcal{F}} \overline{F}$.

Here is where we do something clever. We show that every basic open set $B \subseteq \prod X_\alpha$ containing f has the property that $B \in \mathcal{F}$. This will tell us that for all basic open neighbourhoods B of f we have that $\forall F \in \mathcal{F}, B \cap F \neq \emptyset$, equivalently $\forall F \in \mathcal{F}, f \in \overline{F}$. Thus we have the claim and the theorem.

To show that every basic open set containing f is in $\mathcal{F}$ it is enough to show that every subbasic open set containing f is in $\mathcal{F}$, because $\mathcal{F}$ is closed under finite intersections (property (1) of filter), and every basic open set is a finite intersection of subbasic open sets.

Subclaim: All subbasic open sets containing f are in $\mathcal{F}$.

Let $S = \pi_\alpha^{-1}(U_\alpha)$ be a subbasic open set containing f, where U_α is open in X_α. Then $f(\alpha) \in U_\alpha$. By our choice of $f(\alpha)$, we have

$$f(\alpha) \in \bigcap_{F \in \mathcal{F}} \overline{\pi_\alpha(F)}.$$

Thus

$$f(\alpha) \in U_\alpha \cap \overline{\pi_\alpha(F)}$$

for each $F \in \mathcal{F}$. So

$$\emptyset \neq U_\alpha \cap \pi_\alpha(F)$$

and so

$$\emptyset \neq \pi_\alpha^{-1}(U_\alpha) \cap F = S \cap F.$$

Finally we conclude that $S \in \mathcal{F}$. $\square$

4 Axiom of Choice

Our use of Zorn's Lemma deserves some comment. It might seem to you that this proof doesn't need to use some theorem that we pulled from set theory. You might ask if we can accomplish this proof in a "purely topological way". As it turns out, set theorists have shown that you do need to use Zorn's Lemma, or something equivalent:

Theorem. *TFAE:*

1. *Zorn's Lemma*
2. *The axiom of choice*
3. *If $\{A_\alpha : \alpha \in I\}$ is a collection of non-empty sets, then $\emptyset \neq \prod_{\alpha \in I} A_\alpha$.*
4. *Every set can be well-ordered.*
5. *Tychonoff's Theorem*
6. *(The Compactness Theorem in Logic) Every finitely consistent model is consistent.*

It seems like (3) is obviously true, right? However, (1) seems pretty non-intuitive. Due to some highly non-intuitive consequences of Zorn's Lemma (like the Banach-Tarski Paradox) some mathematicians are skeptical about the use of the axiom of choice and its equivalences. As a result whenever there is a proof using the axiom of choice it is interesting to know whether or not you actually need to use the axiom of choice.

For example, consider the fact that "Every filter is contained in an ultrafilter". Our proof used Zorn's Lemma (to help streamline the proof), but it turns out that it is provable from a weaker principle than Zorn's Lemma called the boolean prime ideal theorem (BPIT).

In our proof of Tychonoff's theorem we used the axiom of choice somewhere. Can you find it? It turns out that we can remove this use of choice if we assume something about our spaces:

Exercise: Prove the following version of Tychonoff's Theorem without using the axiom of choice (or any of its equivalences), but assuming BPIT (thus also "Every filter is contained in an ultrafilter").

Theorem. *Let $\{X_\alpha : \alpha \in I\}$ be a collection of compact T_2 spaces. Then $\prod_{\alpha \in I} X_\alpha$ is compact and T_2.*

5 A Cool Application

Tychonoff's Theorem is counterintuitive because we think of compact spaces as being "small" or "tight", but taking (infinite) products usually makes a small space bigger. For example on assignment 7, A.1, we saw that an uncountable product of metrizable spaces need not be metrizable. However, Tychonoff's theorem says that a large product of small spaces can still be small (provided that "small" is "compact").

We have also seen that showing that a space is compact is rarely straightforward (for example think of our proof that $[0,1]$ is compact). However, now that we have proved Tychonoff's Theorem we can use it to show that a variety of spaces are actually compact. We will use the observation that a closed subset of a product of compact sets is compact.

Example: Let X be a (possibly enormous, possibly infinite) set. We may think of a transitive relation R on X formally, by thinking of R as being a collection of pairs of elements of X, that is $R \subseteq X^2$. (We say that xRy if and only if $(x,y) \in R$.) So, as in assignment 7, A.2, we can put a topology on $\mathcal{P}(X^2)$ by identifying this space with $\{0,1\}^{X^2}$, and taking the product topology on $\{0,1\}^{X^2}$. Then we will think of a transitive relation R on X as being a characteristic function $\chi_R : X^2 \longrightarrow \{0,1\}$.

Claim: The collection $\mathcal{R}$ of transitive relations on X is a compact space.

Since $\{0,1\}$ is a compact space, by Tychonoff's theorem we see that $\{0,1\}^{X^2}$ is a compact space, so it will be enough to show that $\mathcal{R}$ is a closed subset of $\{0,1\}^{X^2}$. This still seems challenging, so let's show that $\{0,1\}^{X^2} \setminus \mathcal{R}$ is open. Let $\chi_E \in \{0,1\}^{X^2} \setminus \mathcal{R}$ and let's show that there is an open set $U \subseteq \{0,1\}^{X^2} \setminus \mathcal{R}$ that contains χ_E.

Well what does it mean that $\chi_E \in \{0,1\}^{X^2} \setminus \mathcal{R}$? It means that E is *not* a transitive relation on X. Which means that there are $a,b,c \in X$ such that aEb and bEc but it isn't true that aEc. Translating this into the language of characteristic functions we wee that:

$$\chi_E(a,b) = 1, \chi_E(b,c) = 1, \chi_E(a,c) = 0.$$

Now we see that any function χ_R that agrees with χ_E on the points $(a, b), (b, c)$ and (a, c) cannot possibly be a transitive relation. Thus

$$U := \{\chi_R : \chi_R(a, b) = \chi_R(b, c) = 1, \chi_R(a, c) = 0\}$$

is an open set in $\{0, 1\}^{X^2}$ that contains χ_E and only contains non-transitive relations. As desired!

The important thing here was that if we know that E is *not* a transitive relation, then a witness to that fact is just a finite collection of pairs of elements of X. Since it was finite we were able to determine an open set in the product topology. If instead we needed *infinitely* many things to witness that a relation is not transitive, then this argument would not go through.