UNIVERSITY OF TORONTO
The Faculty of Arts and Science

MAT301H1F, Groups and Symmetry

Final Examination, December 2012

Instructor: Kasra Rafi
Duration: 3 hours

____________ First ____________ Last __________________ Student Number

Instructions: No aids allowed. Write solutions on the space provided. To receive full credit you must show all your work. If you run out of room for an answer, continue on the back of the page. This exam has 7 questions, for a total of 90 points.

Problem #	Grade
1	/10
2	/12
3	/28
4	/10
5	/10
6	/10
7	/10
Total	/90

1. (10 points) Prove the Lagrange's theorem:

   Theorem. Let G be a group and H be a finite subgroup. Then the order of H divides the order of G.

2. (12 points) Define the following terms and expressions:

   (a) Abelian group.

   (b) External direct product.

   (c) Internal direct product.

   (d) Isomorphism.

   (e) Normal subgroup.

   (f) Center of a group.

3. (28 points) Mark True of False. Justify your answer with a brief argument or a counter example.

   (a) $\square$ $6\mathbb{Z} \cap 8\mathbb{Z} = 48\mathbb{Z}$

   (b) $\square$ All subgroups of a cyclic group are normal.

   (c) $\square$ Every infinite group is cyclic.

(d) $\square$ If $H \lhd K$ and $K < G$ then $H \lhd G$.

(e) $\square$ If $f : G_1 \to G_2$ is a homomorphism and $H < G_1$ then $f(H) < G_2$.

(f) $\square$ If $H_1 \lhd G_1$ and $H_2 \lhd G_2$ then $(H_1 \oplus H_2) \lhd (G_1 \oplus G_2)$.

(g) $\square$ $\mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_9$.

4. (10 points) Let U be the subset of $\mathrm{GL}(2, \mathbb{R})$ defined by

$$U = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}.$$

Show that U is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.

5. (10 points) How many homomorphisms are there from $\mathbb{Z}$ to $\mathbb{Z}_6$? Write them down.

6. (10 points) Use the fundamental theorem of finitely generated abelian groups to write down all abelian groups of order 100, ($100 = 2^2 5^2$).

7. (10 points) Let G be a finite group and $H \leq G$ be a subgroup so that $|H| = |G|/2$. Prove that H is a normal subgroup of G.

 End of exam.