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A *set* is a well-defined collection of objects.

We call the objects of a set its elements or members.

If A is a set, we write $x \in A$ to mean that x is a member of A.

If $x \notin A$, x is not a member of A.

If A, B are sets, and every element of A is an element of B, then we say A is a subset of B. We write $A \subseteq B$.

$$\mathbb{R} \supseteq \mathbb{Q} \supseteq \mathbb{Z} \supseteq \mathbb{N}$$

Last we saw that $\sqrt{2} \notin \mathbb{Q}$, but $\sqrt{2} \in \mathbb{R}$. That implies $\mathbb{R} \neq \mathbb{Q}$.

We can describe subsets of given set A by writing $\{x \in A : x\text{satisfies some conditions}\}$

Let A be all cities in Australia, $\{x \in A : x\text{has a population of at least 3 million people} = \{Sydney, Melbourne\}$

Example: Find all real numbers satisfying $x^2 < x$.

$$\begin{aligned} x^2 &< x \\ x(x-1) &< 0 \\ \text{exactly one of } x, x-1 \text{is smaller than} 0 \\ 0 < x &< 1 \end{aligned}$$

$\{x \in \mathbb{R} : x^2 < x\} = \{x \in \mathbb{R} : 0 < x < 1\} = (0, 1)$

Coin Problem: We have a collection of piles of coins. Define a transformation that for each time, we take one coin from each pile to make a new pile.

Question: Describe all non-empty collections of coins unchanged under this transformation.

Let S be the set of collections of coins which are unchanged by transformation.

Let T be the set of all collections consisting of one pile of size 1, one of size 2, ..., one of size n for some natural number n. (Since we are referring to collections, order does not matter.)

Claim $S = T$. (must show $S \subseteq T$ and $T \subseteq S$.)

Proof:

First let's show $T \subseteq S$.

given a collection in T with piles of sizes of $1, 2, \ldots, n$ for some fixed integer n. When we transform, we get piles of sizes of $0, 1, \ldots, n-1$ plus one new pile of size n.

The result is a collection of piles of sizes $1, 2, \ldots, n$, exactly as we started with.

Thus this collection is unchanged by transformation and hence in S.

Next, we show $S \subseteq T$.

Let $a \in S$, i.e. a is a collection unchanged by transformation.

Let m be the number of piles in a.

Observe that when we transform a collection with m piles, we also create a new pile of size m. If a is unchanged by transformation, then a must have a pile of size m to begin with.

Now that we know a has a pile of size of m. We see that the transformation of a must have a pile of size $m-1$. But since a is unchanged under transformation, this implies a itself must have a pile of size $m-1$.

Continuing with this reasoning we see that a must have piles of size $m, m-1, m-2, \ldots, 1$. But a only has m piles, so these are all the piles of a. Thus a has the desired form and $a \in T$.