2017-03-02-lec05

Last time, we discussed the meaning of logical symbols like $\exists, \forall, \wedge, \vee, \neg$.

Using truth table, you can mechanically verify that $\neg(P \wedge Q)$ is logically equivalent to $(\neg P) \vee (\neg Q)$.

Another example: $P \implies Q$ logically equivalent to $\neg Q \implies \neg P$. This is called *contrapositive*.

(Proving the Contrapositive) Problem: Prove that if p is a prime and $p \neq 2$, then p is odd.

Contrapositive: If p is even, then either p is not prime, or $p = 2$.

Proof: If p is even, then $p = 2k$ for some $k \in \mathbb{N}$. Two cases:

- $k = 1$, then $p = 2$.
- $k > 1$, then $p = 2k$ is a nontrivial factorization of p (p can be broken down into smaller factor numbers). Therefore, p is not a prime.

$P \implies Q$ logically equivalent to $\neg(P \wedge \neg Q)$.

Proof by Contradiction: Assume both P and $\neg Q$, and derive a contradiction.

Example: If a, b, c odd integers, then $ax^2 + bx + c = 0$ has no solution in $\mathbb{Q}$.

Proof:

Suppose $ax^2 + bx + c = 0$ has a rational solution $\frac{p}{q}$.

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$$
$$ap^2 + bpq + cq^2 = 0$$

- p, q odd, p^2, q^2, pq odd. So ap^2, bpq, cq^2 all odd. Odd+Odd+Odd $\neq 0$.
- p odd, q even, ap^2 odd, bpq even, cq^2 even. Odd+Even+Even $\neq 0$.
- p even, q odd, ap^2 even, bpq even, cq^2 odd. Even+Even+Odd $\neq 0$.

We need to understand how negation interacts with quantifiers:

Let $P(x)$ be a well-defined mathematical statement for all $x \in S$.

$\neg(\forall x P(x))$ means the same as $\exists x(\neg P(x))$.

$\neg(\exists x P(x))$ means the same as $\forall x(\neg P(x))$.

Example: Every person likes ice cream.

This is not a negation! Every person does not like ice cream.

This is a negation! There is at least one person who does not like ice cream.

Recall: $f : \mathbb{R} \to \mathbb{R}$ bounded if $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, |f(x)| \leq M$.

What is the negation of this statement?

$\neg(\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, |f(x)| \leq M)$.

$(\forall M \in \mathbb{R}), \neg(\forall x \in \mathbb{R}), |f(x)| \leq M$.

$\forall M \in \mathbb{R}, \exists x \in \mathbb{R}, |f(x)| > M$.