Question 1. [5 MARKS]

The Setun computer was developed in Moscow in the 1950s. It used a ternary (base 3) number system.

Part (a) [1 MARK]

What is the decimal (base 10) representation of the ternary number 121? Show your work and place your final answer in the box.

$$\begin{aligned} & 1 \times 3^0 + 2 \times 3^1 + 1 \times 3^2 \\ = \; & 1 + 6 + 9 \text{ decimal} \\ = \; & 16 \text{ decimal} \end{aligned}$$

$\boxed{16}$

Part (b) [1 MARK]

What is the binary (base 2) representation of the ternary number 121? Show your work and place your final answer in the box.

$$\begin{aligned} 16 &= 1 \times 2^4 \\ &= 0 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 0 \times 2^3 + 1 \times 2^4 \end{aligned}$$

$\boxed{10000}$

Part (c) [1 MARK]

Using only ternary numbers, determine the sum of the ternary numbers 10101 and 20102. Show your work and place your final answer in the box.

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  10101
  20102
 ------
 100210
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$\boxed{100210}$

Part (d) [2 MARKS]

Using only ternary numbers, determine the product of the ternary numbers 12 and 102. Show your work and place your final answer in the box.

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  102
   12
 ----
  211
 1020
 ----
 2001
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$\boxed{2001}$

Question 2. [9 MARKS]

Recall that an integer n is even if and only if $\exists\, q \in \mathbb{Z}, n = 2q$. Also, an integer n is odd if and only if $\exists\, q \in \mathbb{Z}, n = 2q + 1$. Integers are either even or odd.

Let us define the predicates $E(n)$: "n is an even number" and $O(n)$: "n is an odd number".

Consider the following statement:

> For every integer n, n^3 is even if and only if n is even.

Part (a) [1 MARK]

Translate the statement into symbolic notation. Quantify over the integers ($\mathbb{Z}$). Use the predicate $E(n)$.

SAMPLE SOLUTION:

$$\forall\, n \in \mathbb{Z}, E(n^3) \iff E(n)$$

or

$$\forall\, n \in \mathbb{Z}, E(n^3) \Rightarrow E(n) \wedge E(n) \Rightarrow E(n^3)$$

Part (b) [8 MARKS]

Write a detailed structured proof of the statement. Part marks will be given for having correct elements of the proof structure.

SAMPLE SOLUTION: There are a few ways to prove this statement. Here is one proof.

Assume $n \in \mathbb{Z}$.
  Assume $E(n)$.
    Then $\exists\, q \in \mathbb{Z}, n = 2q$.
    Let $q_0 \in \mathbb{Z}$ be such that $n = 2q_0$.
    Then $n^3 = (2q_0)^3 = 8q_0^3 = 2(4q_0^3)$.
    Then $\exists\, r \in \mathbb{Z}, n^3 = 2r$.   # since $4q_0^3 \in \mathbb{Z}$
    Then $E(n^3)$.
  Then $E(n) \Rightarrow E(n^3)$.

  Assume $\neg E(n)$.
    Then $O(n)$.
    Then $\exists\, q \in \mathbb{Z}, n = 2q + 1$.
    Let $q_1 \in \mathbb{Z}$ be such that $n = 2q_1 + 1$.
    Then $n^3 = (2q_1 + 1)^3 = 8q_1^3 + 12q_1^2 + 6q_1 + 1 = 2(4q_1^3 + 6q_1^2 + 3q_1) + 1$.
    Then $\exists\, r \in \mathbb{Z}, n^3 = 2r + 1$.   # since $4q_1^3 + 6q_1^2 + 3q_1 \in \mathbb{Z}$
    Then $O(n^3)$.
    Then $\neg E(n^3)$.
  Then $\neg E(n) \Rightarrow \neg E(n^3)$.
  Then $E(n^3) \Rightarrow E(n)$.   # contrapositive
  Then $E(n^3) \Rightarrow E(n) \wedge E(n) \Rightarrow E(n^3)$.
  Then $E(n^3) \iff E(n)$.
Then $\forall\, n \in \mathbb{Z}, E(n^3) \iff E(n)$.

Question 3. [9 marks]

Recall that an integer $p > 1$ is prime if and only if its only positive integer divisors are 1 and p.

Also, an integer n is odd if and only if $\exists q \in \mathbb{Z}, n = 2q+1$. An integer n is even if and only if $\exists q \in \mathbb{Z}, n = 2q$. Integers are either odd or even.

Let us define the predicates $P(n)$: "n is a prime number", $O(n)$: "n is an odd number" and $E(n)$: "n is an even number".

Consider the following statement:

All prime numbers greater than 2 are odd.

Part (a) [2 marks]

Translate the statement into symbolic notation. Quantify over the natural numbers ($\mathbb{N}$). Use the predicates $P(n)$, $O(n)$ and/or $E(n)$.

Sample Solution:

$$\forall n \in \mathbb{N}, P(n) \wedge n > 2 \Rightarrow O(n)$$

Part (b) [7 marks]

Write a detailed structured proof of the statement. Part marks will be given for having correct elements of the proof structure.

Sample Solution:

A proof follows from the observation that even numbers that are more than 2 are divisible by 2 and hence cannot be prime. Here is a formal proof that uses this idea.

Assume $n \in \mathbb{N}$.
  Assume $P(n) \wedge n > 2$.
    Then $P(n)$.
    Then $n > 2$.
    Then the only integer divisors of n are $q_1 = 1$ and $q_2 = n$ with $q_2 > 2$.
    Assume $\neg O(n)$.
      Then n is even. # all natural numbers are either even or odd
      Then $\exists q \in \mathbb{N}, n = 2q$.
      Then 2 divides n.
      Then n has a divisor that is more than 1 and less than n.
      But n's only divisors are 1 and n.
      Then we have a contradiction and our assumption must be false.
    Then $O(n)$.
  Then $n > 2 \wedge P(n) \Rightarrow O(n)$.
Then $\forall n \in \mathbb{N}, n > 2 \wedge P(n) \Rightarrow O(n)$.

Alternatively, one could consider the contrapositive of $P(n) \wedge n > 2 \Rightarrow O(n)$, namely $\neg O(n) \Rightarrow \neg P(n) \vee \neg(n > 2)$, or the equivalent $E(n) \Rightarrow \neg P(n) \vee n \leqslant 2$. As seen in class, statments of the form $A \Rightarrow B \vee C$ are equivalent to $A \wedge \neg B \Rightarrow C$. And so, a proof can follow by considering the statement $E(n) \wedge P(n) \Rightarrow n \leqslant 2$. This is not to difficult to prove using some of the steps in the above proof by contradiction.

Question 4. [8 MARKS]

Recall that for $x \in \mathbb{R}$, we can define $|x|$ by $|x| = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$

(This is the only definition of $|x|$ that you are allowed to use in your solution to this question.)

Consider the following statement:

For every real number x, if $|x - 3| < 3$ then $0 < x < 6$.

This statement is equivalent to the symbolic statement:

$\forall x \in \mathbb{R}, |x - 3| < 3 \Rightarrow 0 < x < 6.$

Now consider the following argument:

Assume $x \in \mathbb{R}$.
    Assume $|x - 3| < 3$.
        Then either $x - 3 \geqslant 0$ or $x - 3 < 0$.
        Case 1: Assume $x - 3 \geqslant 0$.
            Then $|x - 3| = x - 3$. # by the above definition
            Then $x - 3 < 3$. # since $|x - 3| < 3$
            Then $x < 6$. # add 3 to both sides
        Case 2: Assume $x - 3 < 0$.
            Then $|x - 3| = -(x - 3)$. # by the above definition
            Then $-(x - 3) < 3$. # since $|x - 3| < 3$
            Then $-x + 3 < 3$.
            Then $-x < 0$. # subtract 3 from both sides
            Then $0 < x$ # add x to both sides.
        Then we have proven both $0 < x$ and $x < 6$.
        Then $0 < x < 6$.
    Then $|x - 3| < 3 \Rightarrow 0 < x < 6$.
Then $\forall x \in \mathbb{R}, |x - 3| < 3 \Rightarrow 0 < x < 6$.

Part (a) [2 MARKS]

This argument is not a correct proof of the statement. Explain the flaw in the argument.

SAMPLE SOLUTION: The proof shows that, under the assumption that $|x - 3| < 3$, it follows that for $x - 3 \geqslant 0$, $x < 6$. It also shows that under the same assumption, when $x - 3 < 0$, $x > 0$. Since either $x - 3 \geqslant 0$ or $x - 3 < 0$, we have shown that either $x < 6$ **or** $x > 0$. But we are required to show $0 < x < 6$. That is, we are required to show $0 < x$ **and** $x < 6$. In other words, the disjuction of $0 < x$, $x < 6$ has been proven, but not the conjuction.

Part (b) [6 marks]

Give a correct proof of the statement $\forall x \in \mathbb{R}, |x-3| < 3 \Rightarrow 0 < x < 6$.

Sample Solution: A correct proof follows from using the additional fact that $\forall\, z \in \mathbb{R}, 0 \leqslant |z|$. Students were allowed to use this as a known fact, thought it is not difficult to prove using a direct proof with two cases.

Assume $x \in \mathbb{R}$.
  Assume $|x-3| < 3$.
    Then either $x - 3 \geqslant 0$ or $x - 3 < 0$.
    Case 1: Assume $x - 3 \geqslant 0$.
      Then $|x-3| = x - 3$. # by the above definition
      Then $x - 3 < 3$. # since $|x-3| < 3$
      Then $x - 3 \geqslant 0$. # since $|x-3| \geqslant 0$
      Then $0 \leqslant x - 3 < 6$.
      Then $3 \leqslant x < 6$. # add 3 to all sides
    Then $x - 3 \geqslant 0 \Rightarrow 3 \leqslant x < 6$.
    Case 2: Assume $x - 3 < 0$.
      Then $|x-3| = -(x-3)$. # by the above definition
      Then $-(x-3) < 3$. # since $|x-3| < 3$
      Then $-(x-3) \geqslant 0$. # since $|x-3| \geqslant 0$
      Then $0 \leqslant -x + 3 < 3$.
      Then $-3 \leqslant -x < 0$. # subtract 3 from all sides
      Then $3 \geqslant x > 0$. # multiply by -1
      Then $0 < x \leqslant 3$.
    Then $x - 3 < 0 \Rightarrow 0 < x \leqslant 3$.

    Then $0 < x \leqslant 3 \wedge 3 \leqslant x < 6$.
    Then $0 < x < 6$.
  Then $|x-3| < 3 \Rightarrow 0 < x < 6$.
Then $\forall x \in \mathbb{R}, |x-3| < 3 \Rightarrow 0 < x < 6$.

Total Marks = 31