University of Toronto

MAT237Y1Y PROBLEM SET 5

DUE: End of tutorial, Thursday July 11th, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.

2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.

3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.

4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. (This exercise helps you read through the details of the statement and the proof of the Implicit Function Theorem 3.1)

   Consider the function
   $$F(x,y) = -x + y + \frac{1}{x - \frac{y}{3}}$$
   and the point $(a,b) = (1,0)$.

   a) Draw the largest possible neighborhood (connected open set) of the point (a,b) in $\mathbb{R}^2$ on which the function $F(x,y)$ is C^1. (This is the first part of the theorem)

   b) Next show that the other two conditions of the theorem also hold true.

   c) This fact about continuous functions is from MAT137 and is used is used in the proof of IFT, so let's revisit it: Prove that if $F(\mathbf{a}) > 0, \mathbf{a} \in \mathbb{R}^2$ and $F(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then there is a $\delta > 0$ such that for all $\mathbf{x}$, $|\mathbf{x} - \mathbf{a}| < \delta$ implies $F(\mathbf{x}) > 0$.

d) The first three lines of the proof of IFT describe how r_1 is determined. Determine the largest r_1 such that $\partial_y F$ remains positive for all (x, y) such that $|x - 1| < r_1$ and $|y - 0| < r_1$ (Indicate this region on your diagram in part (a).

e) Check that $F(1, r_1) > 0$ and $F(1, -r_1) < 0$. Find a $r_0 > 0$ such that $r_0 < r_1$ and for all x which satisfy $|x - 1| < r_0$ we have $F(x, r_1) > 0$ and $F(x, -r_1) < 0$.

f) For the implicitly defined function $y = f(x)$ in the conclusion of the theorem, find $f'(1)$.

2. Read the statement of the theorem 3.9 and make use of the matrix B to investigate the possibility of solving the following system of equations for the variables u and w in terms of variables x, y and v near the points $(u, v, w, x, y) = (1, 1, e, 0, -1)$ and $(u, v, w, x, y) = (0, 1, e^2, -1, -2)$. In case the solution is possible make sure to determine $\partial_x u$ and $\partial_x w$:

$$\begin{cases} \frac{xu}{w} + y \ln v = 0, \\ (x^2 + vy)e^{v-1} + x + \ln w = 0. \end{cases}$$

3. Show that the curve $x^2 - y^2 = 0$ is not smooth in any neighborhood of the point $(0, 0)$, as described on top of page 123. (Note: the definitions i, ii, or iii on page 120 are not sufficient conditions for the "smoothness" as described in the beginning of Section 3.2 as the tangent line varying continuously with the point of tangency.)

4. Consider the smooth surface S in $\mathbb{R}^3$ defined by the equation $x^2 + y^2 + z^2 = 1$ with the further restriction that $x > 0, y > 0$ and $z > 0$. Let U be the open set in $\mathbb{R}^2$ defined by the inequality $u^2 + v^2 < 1$ with the additional restriction that $u > 0$ and $v > 0$.

a) Draw U and S. (These are two separate drawings, one in $\mathbb{R}^3$ and the other in $\mathbb{R}^2$.)

b) Write S as the image of a map $\mathbf{G} : U \to \mathbb{R}^3$ in such a way that we have $x = G_1(u, v) = u$ and $y = G_2(u, v) = v$, where the component functions of $\mathbf{G}$ are $\mathbf{G} = (G_1, G_2, G_3)$..

c) Now write S as the image of a map $\mathbf{H} : U \to \mathbb{R}^3$ in such a way that $y = H_2(u, v) = u$ and $z = H_3(u, v) = v$. Now, you have just parameterized the surface S in two different ways.

d) Consider the function $h(x, y, z) = 7x^2 + 7y^2 + 10z^2 + 2xy + 4yz + 4xz$. In parts e) and f) we will find the maximum value of h constrained by S in two different ways. First consider, $h \circ G$ restricted to the boundary of U by writing it as a function of a single variable on each of the three sides of the boundary. You have now found the maximum of $h \circ G$ on the closure of U, but in this case you have also found its maximum on U: Why? Let's call the point at which the maximum of $h \circ G$ is attained (u_0, v_0).

e) Second, do the same thing as part e) for $h \circ H$. Let's call the point at which the maximum of $h \circ G$ is attained (u_1, v_1).

f) Finally, note that the maximum values obtained in parts d) and e) are the same, but that the points (u_0, v_0) and (u_1, v_1) are different. This is no mystery: check that $G(u_0, v_0) = H(u_1, v_1)$. This point is where h attains its maximum on S.

5. This question is to investigate the relationship between the three descriptions of a surface, and the statement of theorem 3.15 in the simplest case: a parameterized plane in $\mathbb{R}^3$.

   a) Parameterize the plane in $\mathbb{R}^3$ with direction vectors $\mathbf{u}$ and $\mathbf{v}$ and through the point $\mathbf{p}$ as in representation (iii) of a surface (use parameters s and t and use $\mathbf{f}(s,t)$).

   b) What is the relation of $\partial_s \mathbf{f}$ and $\partial_t \mathbf{f}$ to the vectors $\mathbf{u}$ and $\mathbf{v}$, and to $D\mathbf{f}$?

   c) Calculate $\partial_s \mathbf{f} \times \partial_t \mathbf{f}$. What is the relation of this vector to the plane?

   d) Write the equation of the plane as the locus of a linear function $F(x, y, z) = 0$. What is the relation of ∇F to $\partial_s \mathbf{f} \times \partial_t \mathbf{f}$?

   e) For vectors $\mathbf{u} = [1, 1, 1]^T$ and $\mathbf{v} = [0, 2, 2]^T$ is the plane a graph over the xy-plane? The xz-plane? The yz-plane? in each case use the components of $\partial_s \mathbf{f} \times \partial_t \mathbf{f}$ or ∇F to explain your reasoning. If it is a graph, write the equation describing it as a graph.

   Enjoy!