UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2013 EXAMINATIONS

CSC 336 H1F — Numerical Methods

Duration — 3 hours

No Aids Allowed

Answer ALL Questions

Do **NOT** turn this page over until you are **TOLD** to start.

Write your answers in the exam booklets provided.

Please fill-in **ALL** the information requested on the front cover of **EACH** exam booklet that you use.

The exam consists of 6 pages, including this one. **Make sure you have all 6 pages**.

The exam consists of 5 questions. **Answer all 5 questions**. The mark for each question is listed at the start of the question. Do the questions that you feel are easiest first.

To pass this course, you need a total mark of at least 50% and you must receive at least 35% on this the Final Exam.

The exam was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity**.

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.

1. [10 marks; 2 marks for each part]

For each of the five statements below, say whether the statement is <u>**true**</u> or <u>**false**</u> and briefly justify your answer.

(a) The choice of algorithm for solving a problem has no effect on the conditioning of the problem.

(b) If A is an $n \times n$ nonsingular real matrix, then $\text{cond}(A) = \text{cond}(A^{-1})$.

(c) If an $n \times n$ nonsingular real matrix A is badly conditioned (i.e., $\text{cond}(A)$ is very large), then the determinant of A must be close to zero.

(d) If x is a real vector with n-elements (i.e., $x \in \mathbb{R}^n$), then $0 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

(e) Let x_1, x_2, $x_3, \ldots$ be a sequence of real numbers that converges to a real number x^*. That is, $x_n \to x^*$ as $n \to \infty$. Suppose that

$$\lim_{n\to\infty} \frac{|x_{n+2} - x^*|}{|x_n - x^*|^2} = c$$

for some real constant c. Then the rate of convergence of x_n to x^* is 2.

2. [10 marks: 5 marks for each part]

Jack told Jill that an accurate approximation to e^x can be computed by summing the terms of the series

$$1 + x + x^2/2! + x^3/3! + \cdots$$

from left to right, provided that no overflows or underflows occur during the computation. Jill didn't believe that such a simple algorithm could work well for all real x (i.e., all $x \in \mathbb{R}$), even if no overflows or underflows occur during the computation. So, she wrote the MatLab function exp1 below to test whether or not Jack's assertion is correct.

```
function y = exp1(x)

   sum = 1;       % Initialize sum to be the first term in the series
                  %    1 + x + x^2/2! + x^3/3! + ...
   sumold = 0;    % Initialize sumold to be different from sum

   k = 1;         % Initialize k to be 1
   term = x;      % Initialize term = x^k/k! = x for k = 1.

   while sumold ~= sum,
      sumold = sum;          % Save the old value of sum in sumold
      sum = sum + term;      % Increment sum so that sum now holds
                             %    1 + x + x^2/2! + x^3/3! + ... + x^k/k!
      k = k+1;               % Increment k for the next iteration of the loop
      term = term * x / k;   % Increment term so that term = x^k/k!
   end

   y = sum;       % Return y = sum as the approximation to e^x
```

The table at the top of the next page lists values of x, $\mathrm{exp1}(x)$, e^x and the relative error in $\mathrm{exp1}(x)$ (i.e., $(\mathrm{exp1}(x) - e^x)/e^x$) for $x = -25, -20, -15, \ldots, 25$.

x	$\text{exp1}(x)$	e^x	$\dfrac{\text{exp1}(x) - e^x}{e^x}$
-25	-7.1298e-07	1.3888e-11	-51339
-20	5.6219e-09	2.0612e-09	1.7275
-15	3.0591e-07	3.0590e-07	2.3208e-05
-10	4.5400e-05	4.5400e-05	-3.0717e-09
-5	0.0067379	0.0067379	-2.1253e-13
0	1	1	0
5	148.41	148.41	-3.8301e-16
10	22026	22026	-3.3033e-16
15	3.2690e+06	3.2690e+06	2.8489e-16
20	4.8517e+08	4.8517e+08	-1.2285e-16
25	7.2005e+10	7.2005e+10	-2.1191e-16

You can see from the results in the table above that the MatLab function exp1 produces accurate results for all non-negative values of x tested (i.e., for $x = 0, 5, 10, 15, 20, 25$). In addition, if x is negative, but not too large in magnitude (e.g., $x = -5$), then exp1 produces a fairly accurate result also. However, if x is negative and large in magnitude (e.g., $x = -20$ or -25), then exp1 is very inaccurate, even though no overflows or underflows occurred in the computations reported in the table above.

(a) Explain the numerical results in the table above. In particular,

- explain why the MatLab function exp1 is accurate for all non-negative values of x and fairly accurate for x negative, but not too large in magnitude (e.g., $x = -5$), and
- explain why the MatLab function exp1 is very inaccurate for x negative and large in magnitude (e.g., $x = -20$ or -25).

You need to say more than there are rounding and truncation errors in the computation, since there are rounding and truncation errors in both the accurate and inaccurate approximations to e^x. You need to explain why the MatLab function exp1 is not seriously affected by the rounding and truncation errors when x is non-negative or negative but not too large in magnitude (e.g., $x = -5$). You also need to explain why the MatLab function exp1 is very badly affected by the rounding and truncation errors when x is negative and large in magnitude (e.g., $x = -20$ or -25).

(b) Make a small change to the MatLab function $\text{exp1}(x)$ so that it computes an accurate approximation to e^x for all $x = -25, -20, -15, \ldots, 25$. Call your new MatLab function $\text{exp2}(x)$ and write it out in full in your exam booklet. (Your MatLab code doesn't have to be syntactically correct; it just has to show that you have the right idea. Also, you don't have to include comments.)

Explain why you believe that your new MatLab function $\text{exp2}(x)$ computes an accurate approximation to e^x for all $x = -25, -20, -15, \ldots, 25$.

3. [10 marks: 5 marks for each part]

Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} -2 & 10 & 1 \\ 1 & -4 & 2 \\ 4 & -8 & 4 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$$

(a) Compute the LU factorization with partial pivoting of the matrix A. That is, compute a permutation matrix P, a unit-lower-triangular matrix L with all elements less than or equal to 1 in magnitude, and an upper triangular matrix U such that $PA = LU$.

Show all your calculations.

(b) Use the LU factorization of the matrix A from part (a) to solve the linear system $Ax = b$.

Show all your calculations.

4. [5 marks]

Dave needed to solve a system of n linear equations in n unknowns. He represented the system in matrix form as $Ax = b$ and used MatLab to compute an approximate solution, $\hat{x}$, to this system. To assess the accuracy of $\hat{x}$, Dave also used MatLab to compute

(a) $\|\hat{x}\|_\infty \approx 5$,

(b) $\|b\|_\infty \approx 6$,

(c) $\|r\|_\infty \approx 2 \times 10^{-15}$, where $r = b - A\hat{x}$, and

(d) $\text{cond}_\infty(A) \approx 3 \times 10^{10}$, where $\text{cond}_\infty(A) = \|A\|_\infty \times \|A^{-1}\|_\infty$.

Dave wanted to use the data above to bound the uncertainty in each component of the computed solution, $\hat{x}$. To be more specific, he wanted to find a real number ϵ so that he could guarantee that $x_i \in [\hat{x}_i - \epsilon, \hat{x}_i + \epsilon]$ for $i = 1, 2, \ldots, n$, where x_i is the i^{th} component of the exact solution x of $Ax = b$ and $\hat{x}_i$ is the i^{th} component of the computed solution $\hat{x}$ of $Ax = b$. Moreover, he wanted this ϵ to be close to as small as possible.

Describe how to compute such an ϵ.

Justify your answer and state any assumptions or approximations used in deriving your value for ϵ.

5. [10 marks: 5 marks for each part]

In class, we discussed finding the roots of the function $g(x) = x^2 - 4\sin(x)$. In this question, we'll consider finding the roots of the similar function

$$f(x) = e^{x^2} - 4\cos(x)$$

(a) How many real roots does $f(x)$ have? That is, how many different values $x_k \in \mathbb{R}$, $k = 1, 2, \ldots$, are there such that $f(x_k) = 0$?

For each root x_k, $k = 1, 2, \ldots$, determine an interval of length less than 1 that contains x_k. That is, for each x_k, $k = 1, 2, \ldots$, determine a_k and b_k such that $0 \le b_k - a_k < 1$ and $x_k \in [a_k, b_k]$.

Justify your answer.

(b) Give Newton's method for finding a root of $f(x) = 0$. That is, give the iteration associated with Newton's method for this particular function $f(x) = e^{x^2} - 4\cos(x)$.

What is a good starting guess for Newton's method for $f(x) = e^{x^2} - 4\cos(x)$ if you want to find a positive root of $f(x) = 0$?

Justify your answer.

You may find the following facts useful in solving this problem.

$$\pi/2 \approx 1.5708 \qquad e^{(\pm\pi/2)^2} \approx 11.792 \qquad 4\cos(\pm\pi/2) = 0$$

$$\pi/4 \approx 0.7854 \qquad e^{(\pm\pi/4)^2} \approx 1.8531 \qquad 4\cos(\pm\pi/4) \approx 2.8284$$

$$e^0 = 1 \qquad 4\cos(0) = 4$$

$$\frac{d\cos(x)}{dx} = -\sin(x) \qquad \frac{d e^{x^2}}{dx} = 2x\, e^{x^2}$$

Have a Happy Holiday

Total Marks = 45

Total Pages = 6