Statistical Inference

Lecture 03a

ANU - RSFAS

Last Updated: Mon Mar 6 16:57:57 2017

A Review of Distirbutions based on the Normal Distribution - Rice Chapter 6

Definition 1: If Z is a standard normal random variable, the $U = Z^2$ is a χ^2 distribution with 1 degree of freedom.

Definition 2: If $U_1, \ldots, U_n$ are independent and $X_1 \sim \chi^2_1$ then

$$\sum_{i=1}^{n} U_i \sim \chi^2_n$$

Proof of (1): Let's first consider the sums of independent gamma distributions.

Question: Suppose $X \sim \mathrm{gamma}(\alpha_1, \lambda)$ and $Y \sim \mathrm{gamma}(\alpha_2, \lambda)$, what is the distribution of $X + Y$?

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} \exp(-\lambda x)$$

- Let's get the moment generating function for X:

$$
\begin{aligned}
M_X(t) &= E[exp(xt)] = \int_0^\infty exp(xt)\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\lambda x)dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty exp(xt)x^{\alpha-1}\exp(-\lambda x)dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}\exp(-(\lambda - t)x)dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}\int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-(\lambda - t)x)dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha
\end{aligned}
$$

- Back to our question:

$$
\begin{aligned}
M_{X+Y}(t) &= M_X(t)M_Y(t) \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1}\left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2} \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1+\alpha_2}
\end{aligned}
$$

$$W = X + Y \sim \mathrm{gamma}(\alpha_1 + \alpha_2, \lambda)$$

- The MGF for a χ^2 distribution is:

$$M(t) = (1-2t)^{-n/2}$$

- If we take our MGF for a single gamma distribution and set $\alpha = n/2$ and $\lambda = 1/2$ we have:

$$\begin{aligned} M_X(t) &= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \\ &= \left(\frac{1/2}{1/2 - t}\right)^{n/2} \\ &= \left(\frac{1}{1-2t}\right)^{n/2} \end{aligned}$$

So a χ^2 distribution with n degrees of freedom.

- Now let's determine the sum of two χ^2 random variables. $U_1 \sim \chi^2_n$ and $U_2 \sim \chi^2_m$ then:

$$M_{U_1+U2}(t) = M_{U_1}(t)M_{U_2}(t) = (1-2t)^{-n/2}(1-2t)^{-m/2} = (1-2t)^{-(n+m)/2}$$

$$U_1 + U_2 \sim \chi^2_{n+m}$$

Definition 3 & Proposition A: If

- $Z \sim \mathrm{normal}(0, 1)$
- $U \sim \chi^2_n$
- Z and U are independent, then:

$$T = Z/\sqrt{U/n} \text{ is a t distrbution with } n \text{ degrees of freedom}$$

$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

- We have a transformation based on two independent random variables. We will use the standard transformation method.

$$t = z/\sqrt{u/n} \qquad v = u$$

- Now let's solve for the inverse of these solve for z and u in terms of t and v.

$$z = \frac{t\sqrt{v}}{\sqrt{n}} \qquad u = v$$

- Now let's get the determinant of the Jacobian:

$$|J| = \begin{vmatrix} \frac{\partial z}{\partial v} & \frac{\partial u}{\partial v} \\ \\ \frac{\partial z}{\partial t} & \frac{\partial u}{\partial t} \end{vmatrix} = \sqrt{v}/\sqrt{n}$$

$$f_{TV}(t,v) = f_{ZU}\left(\frac{t\sqrt{v}}{\sqrt{n}}, v\right)|J|$$

Note: the joint distribution of of Z and U is (remember they are independent):

$$f_{ZU}(z,u) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}z^2\right) \frac{1}{2^{n/2}\Gamma(n/2)} u^{n/2-1} exp(-u/2)$$

- So now we plug in for z and u.

$$\begin{aligned}
f_{ZU}(z,u) &= \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{t\sqrt{v}}{\sqrt{n}}\right)^2\right) \frac{1}{2^{n/2}\Gamma(n/2)} v^{n/2-1} exp(-v/2) \\
&= \frac{v^{n/2-1}}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} exp\left(-\frac{1}{2}\left(\frac{t\sqrt{v}}{\sqrt{n}}\right)^2\right) exp(-v/2) \\
&= \frac{v^{n/2-1}}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right)
\end{aligned}$$

$$
\begin{aligned}
f_{TV}(t,v) &= f_{ZU}\left(\frac{t\sqrt{v}}{\sqrt{n}}, v\right)|J| \\
&= \frac{v^{n/2-1}}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right) \\
&= \frac{v^{(n+1)/2-1}}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right)
\end{aligned}
$$

- Now we integrate out v to get t:

$$
\begin{aligned}
f_T(t) &= \int f_{TV}(t,v)dv \\
&= \int_0^\infty \frac{v^{(n+1)/2-1}}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right) dv \\
&= \frac{1}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} \int_0^\infty v^{(n+1)/2-1} exp\left(-\frac{v}{2}(1+t^2/n)\right) dv
\end{aligned}
$$

- So the integrand is a kernel of a gamma distribution with $a = (n+1)/2$ and $b = (1+t^2/n)/2$.

$$
\begin{aligned}
f_T(t) &= \frac{\Gamma\left((n+1)/2\right)}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} \frac{1}{[(1+t^2/n)/2]^{(n+1)/2}} \\
&= \frac{\Gamma\left((n+1)/2\right)}{\sqrt{\pi n}\Gamma(n/2)} \left[(1+t^2/n)\right]^{-(n+1)/2}
\end{aligned}
$$

Definition 4 & Proposition B: If

- $U \sim \chi^2_m$
- $V \sim \chi^2_n$
- U and V are independent then:

$$W = \frac{U/m}{V/n} \sim \mathrm{F}(m, n)$$

Proof: Through a similar approach we can show the result.

Sampling from the Normal Distribution

Theorem: If $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2)$, then

1. $\bar{X}$ and S^2 are independent
2. $\bar{X} \sim \text{normal}(\mu, \sigma^2/n)$ (already proved)
3. $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$

Proof: $\bar{X}$ and S^2 are independent.

- All we need to do is show that $\bar{X}$ and $Y_j = X_j - \bar{X}$ are independent for all j. **We will make the additional assumption that all the X_is are jointly normally distributed.** Then we just need to show that the $Cov(\bar{X}, X_j - \bar{X}) = 0 \Rightarrow$ independence (not generally the case for other distributions!)
- Rice makes a less restrictive assumption and uses moment generating functions.
- Now examine the functions $\bar{X}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$$
\begin{aligned}
Cov(\bar{X}, X_j - \bar{X}) &= Cov(\bar{X}, X_j) - Cov(\bar{X}, \bar{X}) \\
&= Cov(\bar{X}, X_j) - V(\bar{X}) \\
&= Cov(\bar{X}, X_j) - \sigma^2/n \\
&= Cov(\frac{1}{n}\left(X_1 + \cdots + X_j + \cdots + X_n\right), X_j) - \sigma^2/n \\
&= Cov\left(\frac{1}{n}X_1, X_j\right) + \cdots + Cov\left(\frac{1}{n}X_j, X_j\right) + \cdots - \sigma^2/n \\
&= 0 + \cdots + Cov\left(\frac{1}{n}X_j, X_j\right) + \cdots - \sigma^2/n \\
&= \frac{1}{n}Cov\left(X_j, X_j\right) - \sigma^2/n \\
&= \frac{1}{n}V\left(X_j\right) - \sigma^2/n \\
&= \sigma^2/n - \sigma^2/n = 0
\end{aligned}
$$

As S^2 is a function of $X_1 - \bar{X}, \ldots, X_n - \bar{X}$ then $\bar{X}$ and S^2 are independent (Corollary A).

Proof of 3 (Theorem B)

$$\sum (X_i - \mu)^2 = (n-1)S^2 + n(\bar{X} - \mu)^2$$

$$\sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2$$

$$\sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2$$

$$W = U + V$$

$$\left(\frac{X_i - \mu}{\sigma}\right)^2 = Z^2 \sim \chi_1^2$$

$$W = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

$$V = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 = Z^2 \sim \chi_1^2$$

Based on $\bar{X}$ and S^2 being independent then U and V are independent.

The MGF for a $\chi^2_p = (1-2t)^{-p/2}$.

$$
\begin{aligned}
W &= U + V \\
M_W(t) &= M_U(t)M_V(t) \\
(1-2t)^{-n/2} &= M_U(t)(1-2t)^{-1/2} \\
M_U(t) &= \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}} \\
M_U(t) &= (1-2t)^{-(n-1)/2}
\end{aligned}
$$

$$
U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}
$$

Corollalry B

- Consider the following statistic:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Proof: All we need to do is rewrite the statistic in the form of a t-distribution:

$$\begin{aligned}\frac{\bar{X} - \mu}{S/\sqrt{n}} &= \frac{\bar{X} - \mu}{S/\sqrt{n}} \left(\frac{\sigma}{\sigma}\right) \\ &= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{Z}{\sqrt{U/(n-1)}}\end{aligned}$$