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Problem Set I Sols

1) Theorem: $A \subseteq B$ iff $A \cup B = B$

Proof: Suppose that $A \subseteq B$, If $x \in A \cup B$ then $x \in A$ or $x \in B$. Since $A \subseteq B$, in ether case we have $x \in B$. Thus $A \cup B \subseteq B$.

(5)

On the other hand, if $x \in B$ then $x \in A \cup B$ so $B \subseteq A \cup B$. Hence $A \cup B = B$.

(1/2) per =

Conversely, Suppose $A \cup B = B$. If $x \in A$ then $x \in A \cup B$. But $A \cup B = B$ so $x \in B$. Thus $A \subseteq B$ ∎

2) 2.10 (1) a) As $4 \in [3,5]$, $\exists x$ in $[3,5]$ s.t. $x \geq 4$.
$\therefore$ True

(1) b) However $3 \in [3,5]$ but 3 is not ≥ 4.
$\therefore$ False

(1) c) As $1^2 \neq 3$, $\exists x \ni x^2 \neq 3$
$\therefore$ True

(1) d) As $\sqrt{3}$ has $\sqrt{3}^2 = 3$, "$\forall x, x^2 \neq 3$" is false

(10)

2.12) (1) Defining condition: $\exists k > 0$ such that $\forall x$, $f(x+k) = f(x)$

(1) Negation: $\forall k > 0$ $\exists x$ such that $f(x+k) \neq f(x)$

2.13) (1) Defining condition: $\forall x, y$ (or $\forall x$ & $\forall y$) such that $x \leq y$, $f(x) \leq f(y)$

(1) Negation: $\exists x, y$ such that $x \leq y$, $f(x) > f(y)$

2.15) (1) Defing condition: $\forall x, y \in A$ such that $f(x) = f(y)$, $x = y$

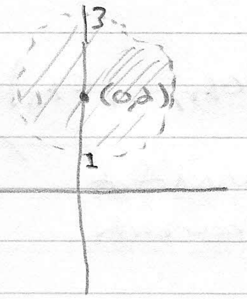
(1) Negation: $\exists x, y \in A$ such that $f(x) = f(y)$, $x \neq y$

Hilroy

(1.5) 3) a) $B(r,\vec{a}) = \{x \in \mathbb{R}^2 \mid |\vec{x}-\vec{a}| < r\} = \{x \in \mathbb{R}^2 \mid (x_1-a_1)^2 + (x_2-a_2)^2 < r^2\}$

Recall that $(x_1-a_1)^2 + (x_2-a_2)^2 = r^2$ is the equation of a circle of radius r centered at $\vec{a}$.

(2) Thus $B(1,(0,2))$ is



whose boundary is a circle about $(0,2)$ of radius 1

b) Our norm is now $|(x_1,x_2)| := |x_1| + |x_2|$

We consider $|(x_1,x_2) - (0,2)| < 1$

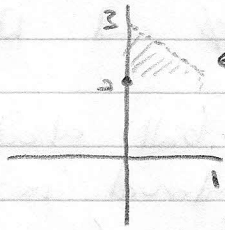
$$|x_1| + |x_2 - 2| < 1$$

Consider case 1: $x_1 \geq 0$, $x_2 - 2 \geq 0$ then $x_1 + x_2 - 2 < 1$

$\Rightarrow x_2 < 3 - x_1$

This gives us:

(7)



line of equation $x_2 = 3 - x_1$ for domain $x_1 \geq 0$, $x_2 - 2 \geq 0$

case 2: $x_1 \geq 0$, $x_2 - 2 < 0$ then $x_1 + -(x_2 - 2) < 1$

$\Rightarrow x_2 > 1 + x_1$

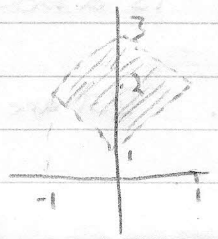
case 3: $x_1 < 0$, $x_2 - 2 \geq 0$ then $-x_1 + x_2 - 2 < 1$

$\Rightarrow x_2 = 3 + x_1$

case 4: $x_1 < 0$, $x_2 - 2 < 0$ then $-x_1 - (x_2 - 2) < 1$

$\Rightarrow x_2 > 1 - x_1$

Putting it together:



c) (2) i) $\vec{x} = \vec{0} \iff x_i = 0 \ \forall i \iff |x_i| = 0 \ \forall i$

Thus $\vec{x} = \vec{0} \Rightarrow \sum_{i=1}^{n} |x_i| = 0$

And if $\sum_{i=1}^{n} |x_i| = 0$, as each $|x_i|$ is positive, $|x_i| = 0 \ \forall i$
$\Rightarrow \vec{x} = \vec{0}$ ∎

(2) ii) $|c\vec{x}| = |(cx_1, \ldots, cx_n)| = |cx_1| + \ldots + |cx_n|$

via def of new norm

$= |c||x_1| + \ldots |c||x_n|$
$= |c||\vec{x}| \quad \forall c \in \mathbb{R}, \vec{x} \in \mathbb{R}^n$ ∎

(2) iii) $|\vec{x} + \vec{y}| = |(x_1+y_1, \ldots, x_n+y_n)|$

def. of norm → $= |x_1+y_1| + \ldots + |x_n+y_n|$

triangle inequality on abs. value → $\leq (|x_1| + |y_1|) + \ldots + (|x_n| + |y_n|)$

$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = |\vec{x}| + |\vec{y}|$ ∎

(15)

2 for ints
2 for ∂S
1 for $\bar{S}$

4) Recall that an interior point $\vec{x}$ means $\exists r > 0$ s.t. $B(r, \vec{x}) \subset S$ and a boundary point $\vec{x}$ means $\forall r > 0$ $B(r, \vec{x}) \cap S \neq \emptyset$ and $B(r, \vec{x}) \cap S^c \neq \emptyset$.

a) Consider a ball about each point in A. This is an open interval in the real line about a point $\frac{1}{n}, n \in \mathbb{Z}^+$: $(\frac{1}{n} - r, \frac{1}{n} + r) = I$

Clearly $\frac{1}{n} \in (\frac{1}{n} - r, \frac{1}{n} + r)$ so $I \cap A \neq \emptyset$

(5) As every open interval in $\mathbb{R}$ contains an irrational and A has no irrationals $I \cap A^c \neq \emptyset$. ∴ Every point in A is a boundary point (and thus not interior)

So $A^{int} = \emptyset$. But are there any other boundary points?

For $x \in A \cup \{0\}^c$, $\exists r > 0$ so $B(r,x) \cap A = \emptyset$
$\Rightarrow x \notin \partial A$

Hence, only other point to consider is 0.

Consider $B(r,0)$ $\quad \forall r > 0\ \exists N \in \mathbb{Z}^+$ s.t. $\frac{1}{n} < r$ for $n > N$.

$\Rightarrow B(r,0) \cap A \neq \emptyset$ and clearly $B(r,0) \cap A^c \neq \emptyset$
(take, say, $-\frac{1}{n} \notin A$)

$\therefore$ 0 is a boundary point.

Hence $A^{int} = \emptyset$, $\partial A = A \cup \{0\}$ and $\bar{A} := A \cup \partial A$
$= A \cup \{0\}$

b) A rational number is a $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

So $\frac{1}{n}$ for $n = \frac{p}{q}$ is just $\frac{q}{p}$. Thus $B = \mathbb{Q}^+$.

Let $x \in \mathbb{R}$, $x \geq 0$. Then the interval $(x-r, x+r) = I$

has both rational & irrational numbers in it.

(5)

so $I \cap \mathbb{Q}^+ \neq \emptyset$, $I \cap \mathbb{Q}^{+c} \neq \emptyset$. Thus all $x \in \mathbb{Q}^+$

are boundary points (and so not interior) and

additionally all $x \in \mathbb{R}$, $x \geq 0$ are boundary points. As in a), 0 is a boundary point as $\forall r > 0$ $B(r,0)$ intersects $\mathbb{Q}^+$ and $\mathbb{R}^- \subset \mathbb{Q}^{+c}$
and for $x < 0$ $\exists r > 0$ so $B(r,x) \cap \mathbb{Q}^+ = \emptyset$. Hence:

$\text{Int}\, \mathbb{Q}^+ = \emptyset$, $\partial \mathbb{Q}^+ = \{x \mid x \geq 0\}$, $\overline{\mathbb{Q}^+} = \partial \mathbb{Q}^+$.

c) $\frac{1}{n}$ for $n \in \mathbb{R}^+$ is just a positive real and

as all positive reals have positive multiplicative

(5) inverses, $C = \mathbb{R}^+ = (0, \infty)$, an open interval whose

only boundary point is the endpoint 0.

$C^{int} = C$, $\partial C = \{0\}$, $\bar{C} = \{x \in \mathbb{R} \mid x \geq 0\}$

(18) 5) Jacobi-Identity: $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}(b_2c_3 - b_3c_2) + \hat{j}(b_3c_1 - b_1c_3) + \hat{k}(b_1c_2 - b_2c_1)$$

So
$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}$$

$$= \hat{i}\left[\cancel{a_2b_1c_2} - \cancel{a_2b_2c_1} - \cancel{a_3b_1c_3} + \cancel{a_3b_3c_2}\right]$$
$$+ \hat{j}\left[-\cancel{a_1b_1c_2} + \cancel{a_1b_2c_1} + \cancel{a_3b_2c_3} - \cancel{a_3b_3c_2}\right]$$
$$+ \hat{k}\left[\cancel{a_1b_3c_1} - \cancel{a_1b_1c_3} - \cancel{a_2b_2c_3} + \cancel{a_2b_3c_2}\right]$$

(ignore / s for now)

We can repeat this computation for $\vec{b}\times(\vec{c}\times\vec{a})$ & $\vec{c}\times(\vec{a}\times\vec{b})$ or simply notice that this is just a permutation of symbols and so can write down the answer by doing the same permutation to the computation above:

So
$$\vec{b}\times(\vec{c}\times\vec{a}) = \hat{i}\left[\cancel{b_2c_1a_2} - \cancel{b_2c_2a_1} - \cancel{b_3c_2a_3} + \cancel{b_3c_3a_2}\right]$$
$$+ \hat{j}\left[-\cancel{b_1c_1a_2} + \cancel{b_1c_2a_1} + \cancel{b_3c_2a_3} - \cancel{b_3c_3a_2}\right]$$
$$+ \hat{k}\left[\cancel{b_1c_3a_1} - \cancel{b_1c_1a_3} - \cancel{b_2c_2a_3} + \cancel{b_2c_3a_2}\right]$$

via sending $a_i \to b_i$, $b_i \to c_i$, $c_i \to a_i$

And $\vec{c}\times(\vec{a}\times\vec{b}) =$
$$\hat{i}\left[\cancel{c_2a_1b_2} - \cancel{c_2a_2b_1} - \cancel{c_3a_2b_3} + \cancel{c_3a_3b_2}\right]$$
$$+ \hat{j}\left[-\cancel{c_1a_1b_2} + \cancel{c_1a_2b_1} + \cancel{c_3a_2b_3} - \cancel{c_3a_3b_2}\right]$$
$$+ \hat{k}\left[\cancel{c_1a_3b_1} - \cancel{c_1a_1b_3} - \cancel{c_2a_2b_3} + \cancel{c_2a_3b_2}\right]$$

Adding the three together I use like symbols / etc to cancel one another. All terms cancel, Hence:

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0} \quad \blacksquare$$

(12) 6) a) Clearly $f(x,y,z) = 1 + x + 2y$ is defined $\forall \vec{x} \in \mathbb{R}^3$.

One of our equivalent conditions for continuity is that $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $|f(\vec{x}) - f(\vec{a})| < \varepsilon$ when $|x_i - a_i| < \delta \;\; \forall i \in \{1,2,3\}$. Let $\varepsilon > 0$ and then

(3)

$$|f(\vec{x}) - f(\vec{a})| = |1 + x + 2y - (1 + a_1 + 2a_2)|$$
$$= |(x - a_1) + 2(y - a_2)|$$
$$\leq |(x - a_1)| + 2|y - a_2| \quad \text{via triangle inequality.}$$
$$< \frac{\varepsilon}{4} + \frac{2\varepsilon}{4} = \frac{3}{4}\varepsilon < \varepsilon \quad \text{via setting } \delta = \frac{\varepsilon}{4} \;\blacksquare$$

b) Since a composition of continuous functions is continuous, let us write $f(x,y) = \dfrac{(xy)^2}{\sin(x+y) + 2}$ as such a composition:

Recall $f_1(x,y) = x + y$ $\qquad f_4(x,y) = \frac{x}{y} \;\; (y \neq 0)$
$\qquad\;\; f_2(x,y) = xy$ $\qquad c(x,y) = 2$

(3) Then $\sin(x+y) = \sin(f_1(x,y))$
$\sin(x+y) + 2 = f_1(\sin(f_1(x,y)), c(x,y))$
$(xy)^2 = f_2(f_2(x,y), f_2(x,y))$

so

$$f = f_3\big(f_2(f_2(x,y), f_2(x,y)), f_1(\sin(f_1(x,y)), c(x,y))\big)$$

defined as denominator never zero

Via Theorem 1.9, f is continuous. $\blacksquare$

c) $f(x,y) = \dfrac{2x+y^2}{\sqrt{(2x)^2+y^2}}$, $\vec{x} \neq \vec{0}$

Let us approach (0,0) along a few convenient paths.

$f(0,y) = \dfrac{y^2}{\sqrt{y^2}} = |y| \to 0$ as $(x,y) \to (0,0)$

③

But $f(x,0) = \dfrac{2x}{\sqrt{(2x)^2}} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$

Thus the limit does not exist as it achieves different values along different paths.

d) $f(x,y) = \dfrac{x^2y - y^3}{x^2+y^2}$ $(x,y) \neq (0,0)$

Claim: The limit is zero if $|f(x,y)| \to 0$ as $|(x,y)| \to 0$

③ $|x^2y - y^3| = |y||x^2-y^2| \leq |y|(x^2+y^2)$ via triangle inequality

$\therefore |f(x,y)| \leq \dfrac{|y|\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = |y| \to 0$ as $|(x,y)| \to 0$

Hence the limit exists and is equal to 0.