ב״ה

18.10.11

Lecture 8 handout

(7.1)

Definition

A network $N(x,y)$ is a simple digraph D, a source x and a sink y, and a capacity function $c: E \to \mathbb{R}$

Intermediate vertices: $V(D) - \{x, y\} =: I$.

A flow is a function $f: E \to \mathbb{R}$, $0 \le f(e) \le c(e)$ for all $e \in E$.

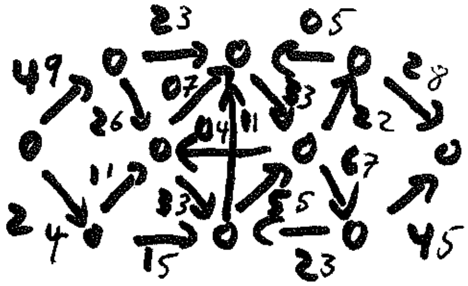
- conservation condition: $f^+(v) = f^-(v)$ for all $v \in I$.

For $X \subseteq V(D)$, $f^+(X) - f^-(X)$ is the net flow of X. $\left(f^{\pm}(X) := \sum_{v \in X} f^{\pm}(v)\right)$.

$\text{val}(f) := f^+(x) - f^-(x)$

Prop: $\text{val}(f) = f^+(X) - f^-(X)$ for any $X \subseteq V(D)$ with $x \in X$ and $y \in V(D) \setminus X$.

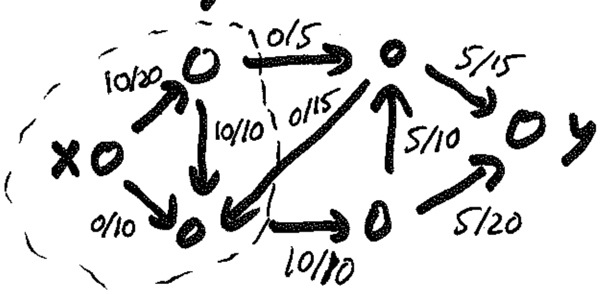
Problem: Find the maximal flow in $N(x,y)$



Cuts

A *cut* of $N(x,y)$ is a set $X \subseteq V(D)$ s.t. $x \in X$ and $y \in V(D) \setminus X$.

The *capacity* of X is the sum of capacities of edges from X to $V(D) \setminus X$.



$cap(X) = 15$
$val(X) = 10.$

Theorem: $val(f) \leq cap(X)$. Equality iff edges in $\partial^+(X)$ are f-saturated, and edges in $\partial^-(X)$ are f-zero. In this case, f is a max flow and K is a min cut.

Proof: $val(f) = f^+(x) - f^-(x) = \sum_{v \in X} \left(\sum_{w} f(v \to w) - \sum_{u} f(u \to v) \right)$

$$= \sum_{v \in X} \left(\sum_{w \in V \setminus X} f(v \to w) - \sum_{u \in V \setminus X} f(u \to v) \right) \underset{\substack{\uparrow \\ \text{equal if} \\ f\text{-zero}}}{\leq} \sum_{v \in X} \sum_{w \in V \setminus X} f(v \to w)$$

$$\overset{\text{equal if } f\text{-saturated}}{\leq} \sum_{v \in X} \sum_{w \in V \setminus X} c(v \to w) = cap(X).$$

Max flow min cut

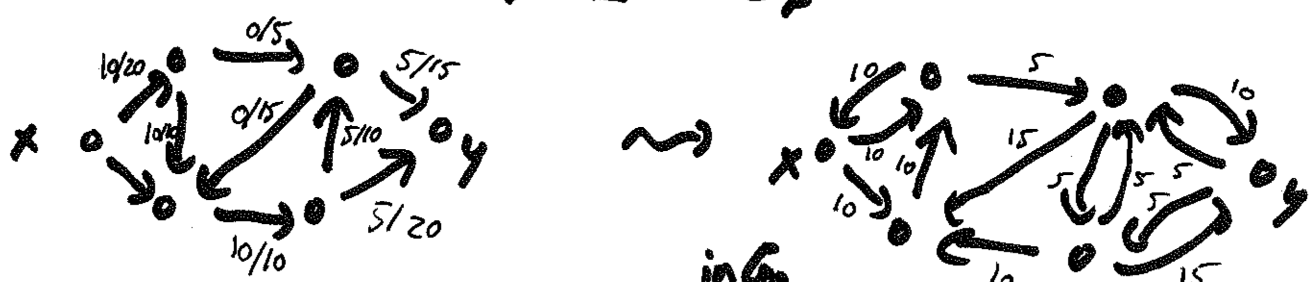
The value of the maximal flow equals the cost of the minimal cut.

PROOF • one-directional: 

Residual capacity: $c_f(u\to v) := \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v\in E \\ f(v\to u) & \text{if } v\to u\in E \\ 0 & \text{otherwise.}\end{cases}$

"how much edge operates below capacity".

Form a residual graph G_f:



If there is no path in G_f from x to y, set

$X = \{\text{vertices reachable from } x \text{ in } G_f\}$

Then X is a min cut, and f is a max flow from the proposition.

Otherwise, augment the flow!

Bonus topic

Inclusion-Exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A\cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example: Integers ≤ 100 divisible by none of $2, 3, 5, 7$.

Example: Surjections $f: X \to Y$.

$N(i)$: i not in $\mathrm{Im}(f)$. $= (n-1)^m$

$N(i_1, \dots, i_k) = (n-k)^m$

$$|\text{Surjections}| = |\text{functions } X \to Y| - \sum N(i) + \dots$$
$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \dots + (-1)^{n-1} \cdot n$$
$$= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m$$

Example: Cayley's formula $T(n) = n^{n-2}$

Next time: Midterm test.