Math776: Graph Theory (I)
Fall, 2013
Homework 4, solutions

1. [**page 54, #11**] Let G be a bipartite graph with bipartition $\{A, B\}$. Assume that $\delta(G) \geq 1$, and that $d(a) \geq d(b)$ for every edge ab with $a \in A$. Show that G contains a matching of A.

Solution by James Sweeney: Assume G has a minimal set S such that S does not satisfy the marriage condition. In other words $|N(S)| < |S|$. Remove one vertex of S, call it S'. Since S was minimal we are now guaranteed a matching in S'. Also the $|N(S')| = |N(S)|$ or else we would have had a matching in S. The edges from S' to $N(S') = \sum_{a_i \in S'} d(a_i) = \sum_{b_j \in N(S')} d(b_j)$. So for each $a_i \exists b_j$ such that $d(a_i) = d(b_j)$. When we add our one vertex back into S', it will be connected to one of the vertices in $N(S')$. This will disrupt the equality above and $\exists$ a b_j such that $d(a_i) < d(b_j) \rightarrow\leftarrow$ So there is no minimal set that violates the marriage condition. So all subsets have the marriage condition. So we have a matching on A. $\square$

2. [**page 55, #14**] Show that all stable matchings of a given graph cover the same vertices. (In particular, they have the same size.)

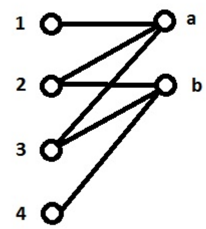
Solution by Wilson Harvey: Let M, M' be two stable matchings of G. For a contradiction, suppose $\exists\ v_0 \in M' \setminus M$. Then v_0 has a neighbor v_1 with $v_0v_1 \in M'$. Note that v_1 must be matched in M, otherwise we may add v_0v_1 to M to get a larger stable matching, a contradiction. Since v_1 is matched in M, then v_1 has a neighbor v_2 with $v_1v_2 \in M$. We have that $v_0v_1v_2$ is a path with edges alternately in M' and M. Continue in this manner to get a full path $P = v_0v_1 \cdots v_n$ (for some $n \in \mathbb{N}$) and consider v_{n-1}. We have the preferences $v_{n-2} <_{v_{n-1}} v_n$ in M, but $v_n <_{v_{n-1}} v_{n-2}$ in M', a contradiction.

Thus, such a v_0 cannot exist, so M and M' must cover the same vertices.

3. [**page 55, #15**] Show that the following 'obvious' algorithm need not produce a stable matching in a bipartite graph. Starting with any matching. If the current matching is not maximal, add an edge. If it is maximal but not stable, insert an edge that creates instability, deleting any current matching edges at its ends.

Solution by Melissa Bechard:

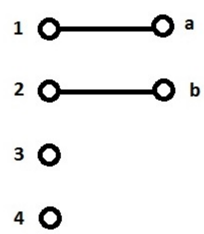
Consider the bipartite of G into classes A and B. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$ as seen below:



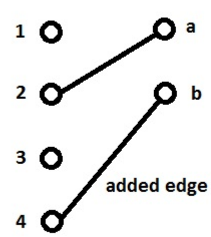
Suppose we have the following preferences:

- a: $3 > 2 > 1 > 4$
- b: $2 > 3 > 4 > 1$
- 1: $b > a$
- 2: $a > b$
- 3: $b > a$
- 4: $a > b$

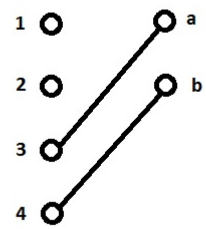
Suppose we begin with the following matching:



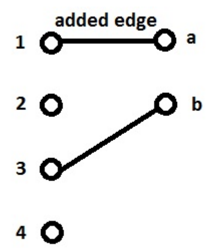
Notice, a prefers 2 over 1, and 2 prefers a, so we make this switch. When doing this we must delete the two edges we have started with and add in an edge.



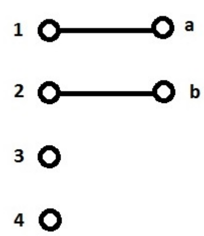
Now, a prefers 3 over 2, and 3 is unmatched so we add this edge in. Doing so requires we remove the topmost edge and leave the bottom most unchanged.



Notice, b prefers 3 over 4, and 3 prefers b over a, so we make this switch, which requires we remove both edges and place in a new edge.



Now, b prefers 2 over 3, and 2 is unmatched, so we add this edge. Doing so requires we remove the bottom-most edge and leave the other unchanged.

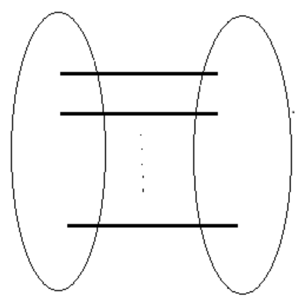


We have looped back to where we started. Thus, following this algorithm, we need not produce a stable matching.

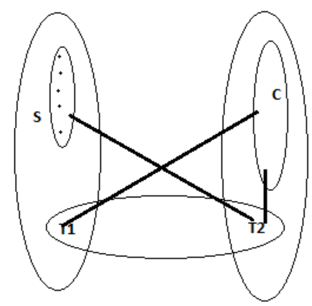
4. [**page 55, #20**] Derive the marriage theorem form Tutte's theorem.

Solution by Shuliang Bai:

Let G=(A, B) be a bipartite graph, if $|G|$ is odd, then add one vertex to B and add the edges on the verteices in B to make B a complete graph. Through this process, we obtain a graph H with $|H|$ is even.



G=(A, B')



H=(A, B)

1. Claim: G has a matching containing A if and only if H has a 1-factor.

   If H has a 1-factor M, Let M_A be the maximal subset of M, its edges incident on A, then M_A is a matching of G which contains A.

   If G has a matching M containing A, then all vertices in A have been matched to vertices in B, there are $|B| - |A|$ vertices in B are unmatched, since $|H|$ is even, then $|B| - |A|$ is even, and since those unmatched vertice is on a complete subgraph of B, then we can pick a matching M_1 among them, then $M \cup M_1$ is a 1-factor in H.

2. Claim : If G satisfies Hall's condition which says $|N(S)| \geq |S|$ for all $S \subseteq A$, then H satisfies Tutter's condition which says $q(H-T) \leq |T|$ for all $T \subseteq V(H)$.

   Let $T = T_1 \cup T_2$ be some vertices subset of H, $T \subseteq V(H)$, where $T_1 = T \cap A$ and $T_2 = T \cap B$. Since B is a complete subgraph of H, then the components of $H - T$ contains all of $B - T_2$, we call it C, and the isolated vertices in A. Let S be the set of these isolated vertices and $|S| = k$.

   Then $q(H - T) = k + 1$, if $|C|$ is odd; $q(H - T) = k$, if $|C|$ is even.

   If $|C|$ is even, $q(H - T) = k = |S| \leq N(S) \leq |T_2| \leq |T|$, where $|S| \leq N(S)$ is from Hall's condition, so H satisfies Tutter's condition.

   If $|C|$ is odd, if $T_1 \neq \varnothing$, $|T_1| \geq 1$, so $k \leq |T_2| = |T| - |T_1| \leq |T| - 1$, we have $k + 1 \leq |T|$, then H satisfies Tutter's condition. If $T_1 = \varnothing$, then $V(H) = |T_2| + k + |C|$. Since $|H|$ is even, then $|T_2| + k$ should be odd. Since $k \leq |T_2|$, then k should not be equal to $|T_2|$ in order to keep $|T_2| + k$ odd. So $k \leq |T_2| - 1$, then $q(H - T) = k + 1 \leq |T_2| = |T|$, so H satisfies Tutter's condition.

3. Claim : By Claim 2, for any $S \subseteq A$, $|N(S)| \geq |S|$, H satisfies Tutter's condition and therefore has a 1-factor, by Claim 1, so G has a matching containing A.

5. [page 83, #4] Let X and X' be minimal separators in G such that X meets at least two components of $G - X'$. Show that X' meets at least two components of $G - X$, and X meets all the components of $G - X'$.

Solution by Gregory Ferrin: Suppose that X' meets $G - X$ in only one component. Call this component C.

Then $X' \subseteq X \cup C$. So the components of $G - X'$ are components which come from $C - X'$ and a component which contains the rest of G. So, X meets only one component of $G - X'$. This a contradiction. Hence, X' meets at least two components of $G - X$.

Then, it follows from symmetry that X meets every component of $G - X'$.

6. [page 83, #10] Let e be an edge in a 3-connected graph $G \neq K_4$. Show that either $G \dot{-} e$ or G/e is again 3-connected.

Soution by Robert Wilcox: Let $e = xy$ be an edge in a 3-connected graph $G \neq K_4$. We want to show that either $G \dot{-} e$ or G/e is 3-connected. Suppose not, so neither of these graphs are 3-connected. Then each of these new graphs has a set of at most two vertices that disconnects it. First we look at G/e. If neither of the vertices are the compressed ends of e then these vertices would disconnect G, a contradiction. Let the other vertex in the separator be called z. Since $\{xy, z\}$ is a separator of G/e, $\{x, y, z\}$ will be a separator of G. And this set is a minimal separator in G so each of these connects to every component of $G - \{x, y, z\}$.

Now we look at $G \dot{-} e$. Neither x nor y can be in the separator or they would be part of a 2-separator of G, so let $\{u, v\}$ be a separator of $G \dot{-} e$. Now we consider where u and v live in $G - \{x, y, z\}$. If they are in the same component, then there is at least one component containing neither u nor v. Since x and y have edges to this connected component there must be a xy path that does not use e or go through u or v. Thus x and y are in the same component in $G - \{u, v\}$. This would mean that $\{u, v\}$ separates G, a contradiction, since removing e doesn't affect anything. Hence u and v are in different components of

$G - \{x, y, z\}$ and there are only these two components.

Let a, b, c be the ends of the edges to x, y, z respectively, in the component containing u. Since G is 3-connected there is an ab path that does not go through x or u. If such a path doesn't go through v then we can travel from x to a to this path to b then to y. This would mean that x and y are in the same component of $G - \{u, v\}$. This is a contradiction as argued above. Thus every such path goes through v. So there is some path from a to v that doesn't go through x, b, or u. This path must go through either y or z first to get to the other side. Going through y would place x and y in the same component of $G - \{u, v\}$, again a contradiction. Thus there is a path from a to z that doesn't go through x, v, or u. So a and z are in the same component of $G - \{u, v\}$, and thus x is as well since it is a neighbor of a.

Now we know that y and z are in different components of $G - \{u, v\}$ since x is in the same one as z. Since G is 3-connected there is a bc path that does not go through y or u. Every such path must go through v or else y and z would be in the same component of $G - \{u, v\}$. Thus there is a path from b to v that does not go through y, c or u. So this path must go through x or z first. But if it goes through x then x and y would be in the same component of $G - \{u, v\}$. And if it goes through z then y and z would be in the same component, also a contradiction. Therefore G must not be 3-connected and we have shown that there must be a contradiction so either $G \dot{-} e$ or G/e is 3-connected.