Venue

Student Number |__|__|__|__|__|__|__|__|

Research School of Finance, Actuarial Studies & Statistics

EXAMINATION

Semester 1 - Mid-Semester, 2017

STAT2032/6046 Financial Mathematics

Writing Time: 90 Minutes
Reading Time: 15 Minutes

Exam Conditions:
Central Examination
Students must return the examination paper at the end of the examination
This examination paper is not available to the ANU Library archives

Materials Permitted In The Exam Venue:
(No electronic aids are permitted e.g. laptops, phones)
- Calculator (non programmable),
- Unannotated paper-based dictionary (no approval required)

Materials to Be Supplied To Students:
- Script Book
- Scribble Paper
- Formula Sheet
- Interest Rate Tables

Instructions to Students:
- Please write your student number in the space provided in your examination paper and your Script Book.
- Attempt ALL questions. There are four questions, not equally weighted
- To ensure full marks show all the steps in working out your solution. Marks may be deducted for failure to show appropriate calculations or formulae.

Total Marks: 50

Question 1

Find the following amounts.

(a) The present value of \$750 due 7 years from now, at a nominal annual rate of interest of 6% compounded monthly. (3 marks)

Sol: effective interest rate per annum: $i = \left(1 + \frac{6\%}{12}\right)^{12} - 1 = 0.061678$

$$\text{PV} = 750 * 1.061678^{-7} = \$493.30$$

(b) The accumulated value of \$1000 at the end of 6 years, at a nominal annual rate of discount of 5% compounded semi-annually. (3 marks)

Sol: effective discount rate per annum: $d = 1 - \left(1 - \frac{5\%}{2}\right)^{2} = 0.049375$

$$AV = 1000 * (1 - 0.049375)^{-6} = \$1355.02$$

(c) The accumulated value of \$150 at the end of 9 years at a nominal annual rate of interest of 7% compounded semi-annually for the first 4 years and 5% compounded quarterly for the next 5 years. (4 marks)

Sol: effective interest rate per annum for the first 4 years: $i_1 = \left(1 + \frac{7\%}{2}\right)^{2} - 1 = 0.071225$

effective interest rate per annum for the next 5 years: $i_2 = \left(1 + \frac{5\%}{4}\right)^{4} - 1 = 0.050945$

$$AV = 150 * (1 + 0.071225)^4 (1 + 0.050945)^5 = \$253.23$$

(d) S(t) is the accumulated value at time t of an amount invested at time 0. You are given,

$S(t) = Kt^2 + Lt + M$ for $0 \le t \le 2$

$S(0) = 100$

$S(1) = 110$

$S(2) = 136$

Determine the force of interest at time $t = \frac{1}{2}$. (3 marks)

Sol: Firstly derive M=100, k=8, and L=2.

$S(t) = 8t^2 + 2t + 100$

$$\delta(t) = \frac{S'(t)}{S(t)} = \frac{16t + 2}{8t^2 + 2t + 100}$$

$$\delta\left(t = \frac{1}{2}\right) = \frac{10}{103} = 0.097087$$

(e) Given $a^{(12)}_{\overline{n|}i} = 10$, $a^{(12)}_{\overline{2n|}i} = 15$. Determine i. (4 marks)

Sol: $\frac{1 - v_i^n}{i^{(12)}} = 10$ (1)

$\frac{1 - v_i^{2n}}{i^{(12)}} = 15$ (2)

(1)/(2):

$$\frac{1 - v_i^n}{1 - v_i^{2n}} = \frac{10}{15}$$

$$2v_i^{2n} - 3v_i^n + 1 = 0$$

Using the formula to find the roots of the quadratic function, we have

$$v_i^n = 0.5 \ (1 \textit{ should not be considered due to the conditions})$$

Put $v_i^n = 0.5$ back in (1), we get $i^{(12)} = 0.05$. So $i = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 0.051162$

Question 2

A 3-year deferred continuous varying annuity is payable for 11 years. The rate of payment at time t is $t^2 - 1$ per annum and the force of interest at time t is $(1+t)^{-1}$. Calculate the value of the annuity at time t=0. (7 marks)

Sol: $PV = \exp\left\{-\int_0^3 \frac{1}{1+t} dt\right\} \int_3^{14} (t^2 - 1) \exp\{-\int_3^t \frac{1}{1+s} ds\} dt$

$$PV = \frac{1}{4} * \int_3^{14} \frac{4(t^2-1)}{t+1} dt = \int_3^{14} (t-1)\, dt = \frac{1}{2} t^2 - t|_{t=3}^{t=14} = 82.5$$

Question 3:

A 20-year annuity-certain provides payments annually of \$1000 at the end of the 1st year, \$900 at the end of the 2nd year, \$800 at the end of the 3rd year, and so on, until payments reduce to \$100. Payments then continue at \$100 per annum until the last payment of the annuity. At 4% effective interest rate per annum, find the accumulated value of the annuity at the end of twenty years. (8 marks)

Sol: By checking interest rate tables, we could get the basic annuity and interest related values.

$$\begin{aligned} AV &= \left[1000 * S_{\overline{10|}} - 100 * IS_{\overline{9|}}\right] * (1+i)^{10} + 100 * S_{\overline{10|}} \\ &= \left[1000 * 12.0061 - 100 * \frac{10.5828 * 1.04 - 9}{0,04}\right] * 1.48024 + 100 * 12.0061 \\ &= (12006.1 - 5015.28] * 1.48024 + 1200.61 \\ &= \$11548.70 \end{aligned}$$

Question 4:

A perpetuity with annual payments is to be payable at the end of 9 years from now. The first payment is \$250. Each annual payment thereafter is increased by \$50 until a payment of \$750 is reached. Subsequent payments remain level at \$750. This perpetuity is purchased by means of 10 annual premiums, with the first premium of P due immediately. Each premium after the first is 105% of the preceding one. The annual effective interest rates are 5% during the first 9 years and 3% thereafter. Find the Premium P. (9 marks)

Sol: Choose the reference time point as t=9 (it is ok to choose other reference time points.). By checking interest rate tables, we could get the basic annuity and interest related values.

$$10P * (1+0.05)^9 = 250\ddot{a}_{\overline{10}|0.03} + 50Ia_{\overline{9}|0.03} + \frac{750}{0.03} * (1+0.03)^{-9}$$
$$10P * 1.55133 = 250 * 8.5302 * 1.03 + 50 * \frac{7.7861 * 1.03 - 9 * 0.76642}{0.03} + \frac{750}{0.03} * v_{0.03}^{-9}$$

$$P = \frac{2196.5265 + 1869.838333 + 19160.5}{15.5133}$$
$$P = 1497.22$$

Question 5:
At time t=0 an investor purchased an annuity-certain which paid her $1,000 per annum annually in arrear for three years. The purchase price paid by the investor was $2,500.

The value of the retail price index at various times was as shown in the table below:

Time t (years)	t=0	t=1	t=2	t=3
Inflation Index	170.7	183.3	191.0	200.9

(i) Calculate, to the nearest 0.1%, the following effective rates of interest per annum achieved by the investor from her investment in the annuity. (Hint: Considering using linear interpolation approximation method when an analytical solution is hard to find.) (7 marks)
(a) the real rate of interest: and
(b) the money rate of interest

Sol: (a) Work in t=0 monetary values:
$$2500 = 1000 * (v\frac{170.7}{183.3} + v^2\frac{170.7}{191.0} + v^3\frac{170.7}{200.9})$$
Where $v = (1+i')^{-1}$, and $i' = the\ real\ rate\ of\ interest$
Try 4% RHS=2477.049
Try 3% RHS=2524.125
We approximate i' using linear interpolation method:
$$i' \cong 0.03 + \frac{2524.125 - 2500}{2524.125 - 2477.049} * (0.04 - 0.03)$$
$$i' = 0.0351 = 3.5\%$$
(b) $2500 = 1000\mathrm{a}_{\overline{3}|i}$
$$a_{\overline{3}|i} = 2.5$$
From tables, $a_{\overline{3}|0.09} = 2.5313$
$a_{\overline{3}|0.10} = 2.4869$
approximate i using linear interpolation method:
$$i \cong 0.09 + \frac{2.5313 - 2.5}{2.5313 - 2.4869} * (0.10 - 0.09)$$
$$i = 0.097 = 9.7\%$$

(ii) By considering the average rate of inflation over the three-year period, explain the relationship between your answers in (a) and (b) of (i). (2 marks)

Sol: We should find that $(1 + \mathrm{i}) \cong (1 + \mathrm{i'})(1 + \mathrm{r})$
Where r is the average rate of inflation over the three-year period
$$\frac{1+i}{1+i'} = \frac{1.097}{1.035} = 1.06$$
Which implies 6% p.a. inflation over the period.

The actual average inflation rate is

$$(1+r)^3 = \frac{200.9}{170.7}$$
$$r \cong 5.6\%$$

The inflation rate would not be expected to be exactly 6% p.a. since the Retail Price Index is not increasing by a constant amount each year.

END OF EXAMINATION