§12 - Metric Spaces and Metrizable Spaces

1 Motivation

Metric Spaces are beautiful spaces. In the world of topology, metric spaces are the supermodels and $\mathbb{R}$ is the Cindy Crawford. When we ask the question "How nice is this topological space?" we often mean "How similar is this topological space to a metric space?". On the first day of class we defined a topological space as a generalization of $\mathbb{R}$ with its distance function. We will expand upon that by looking at how metric spaces give us topological spaces.

We will also investigate what types of topological properties metric spaces have. A general theme will be that "metric spaces are very well-behaved"; metric spaces have all of the separation axioms we care about (T_2 and T_4) and has all the convergence properties we care about (first countability).

Since metric spaces are so nice, it is a very natural question to ask "When is a topological space given by a metric?". This question has motivated a lot of research in the 20th century, and we will look at some partial answers to this question.

Also, since this is a topology course we will be looking at metric spaces as topologists. That is, we will not be concerned with metric spaces, *per se*, but we will be interested in what sort of topological properties they have. As a result, there will be relatively few proofs of facts that are purely about metric spaces, and instead we will focus on proving facts of a topological nature.

2 Definitions

First and foremost we state the definition of a metric for a set. Even this is a generalization of some property of $\mathbb{R}$. (Isn't that a recurring theme? Identify some nice property that $\mathbb{R}$ has, then extract it, and see what sort of spaces have that property.)

Definition. *Let X be a set. A function $d : X \times X \longrightarrow \mathbb{R}$ is called a* ***metric*** *(or a distance function) provided that:*

1. $d(x, y) = 0$ *iff* $x = y$ $(\forall x, y \in X)$;
2. $d(x, y) \geq 0$ $(\forall x, y \in X)$;
3. $d(x, y) = d(y, x)$ $(\forall x, y \in X)$;
4. $d(x, z) \leq d(x, y) + d(y, z)$ $(\forall x, y, z \in X)$;

Notation: A pair (X, d), where d is a metric, is called a **metric space**. Property [1,2] together are called being "positive definite", property [3] is called being "symmetric" and property [4] is the "triangle inequality".

Some examples:

1. $\mathbb{R}$ has its **usual metric** defined by $d(x, y) = |x - y|$.
2. $\mathbb{R}^n$ has its **usual metric** defined by
$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}, \text{ where } \vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n).$$
Of course on $\mathbb{R}^1$ both "usual metrics" are the same.
3. Let X be any set. We can always give it the **discrete metric** defined by:
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$
We will prove that this is a metric in a moment. This metric is related to the discrete topology (answering a question from §1!).
4. $\mathbb{R}^2$ also has the **square metric** defined by
$$d(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$
This is called the square metric because the ϵ-Balls look like squares.
5. $\mathbb{R}^2$ also has the **taxicab metric** defined by
$$d(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$$
This is called the taxicab metric because this is the distance it takes a taxi to drive through a city. (It cannot drive through buildings! It can only drive down streets.)

Lemma. *The discrete metric d on a set X really is a metric.*

Proof. By definition d is positive definite, and it is immediate that it is symmetric. We only need to show that the trinagle inequality is true. So let $x, y, z \in X$. If $x = z$, then
$$d(x, z) = 0 \leq d(x, y) + d(y, z)$$
since d is positive definite. If $x \neq z$, then either $x \neq y$ or $y \neq z$ (or possibly both). In either case,
$$d(x, z) = 1 \leq d(x, y) + d(y, z).$$
□

Notation: For (X, d) a metric space, $p \in X$ and $\epsilon > 0$ we define
$$B_\epsilon(p) := \{\, x \in X : d(x, p) < \epsilon \,\}$$

3 Topologies from Metrics

As we saw in §1, we can extract a topology from ϵ-balls. There we only got the usual topology, but it turns out that we can get a topology from any metric space!

Definition. *Let (X, d) be a metric space. The* ***metric topology*** *(given by d, or generated by d) is the topology generated from the following basis*

$$\mathcal{B}_d := \{ B_\epsilon(p) : p \in X, \epsilon > 0 \}$$

Strictly speaking we should check that this really is a basis.

Proposition. *Given a metric space (X, d), the corresponding collection $\mathcal{B}_d$ is a basis for some topology on X.*

Proof. Let us check the definition! We first observe that $\mathcal{B}_d$ covers X since for each $p \in X$ we have $p \in B_1(p)$. It only remains to show that "$\mathcal{B}_p$ is directed" (that weird intersection property). The proof is easier to digest in a picture, so draw a picture!

Let $p \in B_{\epsilon_1}(p_1) \cap B_{\epsilon_2}(p_2)$. Define

$$\epsilon := \min\{ \epsilon_1 - d(p, p_1), \epsilon_2 - d(p, p_2) \}.$$

Notice that $\epsilon > 0$ since $0 < d(p, p_1) < \epsilon_1$ and $0 < d(p, p_2) < \epsilon_2$.

Claim: $B_\epsilon(p) \subseteq B_{\epsilon_1}(p_1) \cap B_{\epsilon_2}(p_2)$.

We will show that $B_\epsilon(p) \subseteq B_{\epsilon_1}(p_1)$, and the other direction is analogous. Let $z \in B_\epsilon(p)$. So we get

$$\begin{aligned} & d(z, p) < \epsilon \leq \epsilon_1 - d(p, p_1) \\ \Rightarrow\ & d(z, p) + d(p, p_1) < \epsilon_1 \\ \Rightarrow\ & d(z, p_1) < \epsilon_1 \end{aligned}$$

Where the last line is true by the triangle inequality. Thus $z \in B_{\epsilon_1}(p_1)$. □

Knowing this, we can figure out what topologies we get from the 5 metric spaces we have described. Recall that from §10 we know how to compare topologies if we know something about their bases.

Some Examples: On $\mathbb{R}^2$ the usual metric, square metric and taxicab metric all generate the usual topology. For a proof that the taxicab metric generates the usual topology, see:

`http://i.imgur.com/NhoAmVB.jpg`

In general, the usual metric on $\mathbb{R}^n$ generates the usual topology on $\mathbb{R}^n$. Finally, the discrete metric on a set X generates the discrete topology, which you can see because for any point $p \in X$ we have $B_{\frac{1}{2}}(p) = \{ p \}$.

Indiscrete Exercise: We saw that the discrete metric generates the discrete topology. Is there such a thing as the "indiscrete metric" which generates the indiscrete topology? What if we weaken the requirements on the distance function? Is this useful?

4 Metrizability

This leads us to two natural questions, but first a helpful word:

Definition. *A topological space $(X, \mathcal{T})$ is said to be metrizable if there is a metric d on X such that d generates $\mathcal{T}$.*

Note that many different metrics can generate the same topology, as we have seen with the square metric and the usual metric on $\mathbb{R}^2$. As a result we often talk about *metrizable* spaces rather than *metric* spaces. We should also mention a needed fact:

Proposition. *Metrizability is a topological invariant.*

Proof. Let $f : X \longrightarrow Y$ be a homeomorphism, and suppose that (X, d) is a metric space. Define $\rho : Y \times Y \longrightarrow \mathbb{R}$ by

$$\rho(y_1, y_2) := d(f^{-1}(y_1), f^{-1}(y_2)).$$

It is clear that $\rho(y_1, y_2) \geq 0$, and it is also clear that ρ is symmetric. Since f is injective ρ is positive definite, and since f is onto, the triangle inequality is true. (These last two facts are straightforward to check.) Thus ρ is a metric on Y.

All that remains to show is that ρ generates the topology on Y. This is left as an exercise for those interested. $\square$

Let us also (formally) remark that $\mathbb{R}^n_{\text{usual}}$ is metrizable, and discrete spaces are metrizable.

Now, on to the questions!

1. What topological properties do metrizable spaces have?
2. When is a topological space metrizable?

We are already in a position to answer question 1. The next proposition should be read as "metrizable spaces have nice separation properties".

Proposition. *Every metrizable space is T_2, T_3 and T_4.*

Proof. Our proof in §9 which shows that $\mathbb{R}^n$ is T_4 actually also shows that *any* metrizable space is T_4.

Let us show that it is T_2. Let (X, d) be a metric space, and let $x, y \in X$ be distinct points. Thus $d(x, y) > 0$, and let

$$\epsilon := \frac{d(x,y)}{2} > 0.$$

Notice that $x \in B_\epsilon(x)$ and $y \in B_\epsilon(y)$.

Claim: $B_\epsilon(x) \cap B_\epsilon(y) = \emptyset$.

Suppose that $z \in B_\epsilon(x) \cap B_\epsilon(y)$, so $d(x, z) < \epsilon$ and $d(y, z) < \epsilon$. Thus:

$$d(x,z) + d(y,z) < \epsilon + \epsilon = 2\epsilon = d(x,y) \leq d(x,z) + d(y,z)$$

where the last inequality is by the triangle inequality. Thus we have a contradiction. □

The next proposition says that "metrizable spaces have nice convergence properties".

Proposition. *Every metrizable space is first countable.*

Proof. Let (X, d) be a metric space, and let $p \in X$. Define

$$\mathcal{B}_p := \{\, B_\epsilon(p) : \epsilon \in \mathbb{Q}, \epsilon > 0 \,\}.$$

Observe that each of these open sets contains p, so we only need to show "basis" property.

Let $p \in U$, an open set in the metric topology. By definition, there is an $\epsilon > 0$ such that $p \in B_\epsilon(p) \subseteq U$. We also know that there is a rational $\delta > 0$ such that $0 < \delta < \epsilon$. Thus $p \in B_\delta(p) \subseteq B_\epsilon(p) \subseteq U$, and cleary $B_\delta(p) \in \mathcal{B}_p$. □

That proposition tells us way more than just metrizable spaces have a nice topological invariant. In §5, we remarked that first countability (and to a certain extent the T_2 property) was exactly what we needed so that convergence of a sequence in a topological space matched our intuition. Let us restate that important convergence proposition in the context of metrizable spaces:

Proposition. *Let $(\, X, \mathcal{T}\,)$ be a metrizable space, and let $A \subseteq X$ with $p \in X$. We have that $p \in \overline{A}$ if and only if there is a sequence in A that converges to p.*

Munkres referes to that proposition as "the sequence lemma".

What about the other countability properties, like second countability, ccc, and separability? Well we have no hope of getting that *all* metrizable spaces have these properties, because $\mathbb{R}_{\text{discrete}}$ is an example of a metrizable space that has none of those properties. However, all is not lost, metric spaces are still very nice!

Theorem. *A metrizable space is second countable iff it is ccc iff it is separable.*

Proof. This is question A.2 on Assignment 6. □

5 When is a Space Metrizable?

This is a very big question that motivated a lot of early research in the 20th century. Let's start by looking at some spaces that are not metrizable:

1. $X_{\text{indiscrete}}$ is not a metrizable space. It isn't even close to being a metric space. For example, it isn't T_2.
2. $\omega_1 + 1, \mathbb{R}_{\text{co-finite}}$ and $\mathbb{R}_{\text{co-countable}}$ are all not metrizable, because they are not first countable.
3. The Sorgenfrey line is not metrizable because it is separable, but not second countable, and we just saw that in metrizable spaces these are the same.

Later on in the course we will see a whole class of properties that are equivalent for metrizable spaces and this will help us to prove non-metrizability of some spaces (like ω_1!).

Two easy ways to get new metrizable spaces is to take subspaces or finite products.

Theorem. *The finite product of metrizable spaces is metrizable. (i.e. Metrizability is finitely productive.)*

Proof. There are many ways to prove this! Let $(X_1, d_1), \ldots, (X_N, d_N)$ be metric spaces. Here are a couple different ways that each work:

1. Let $d_1(\vec{x}, \vec{y}) := \sqrt{d(x_1, y_1) + \ldots + d(x_N, y_N)^2}$; or
2. Let $d_2(\vec{x}, \vec{y}) := \max\{|x_1 - y_1|, \ldots, |x_N - y_N|\}$; or
3. Let $d_3(\vec{x}, \vec{y}) := |x_1 - y_1| + \ldots + |x_N - y_N|$.

You can check that all three of these generate the product topology on $X_1 \times \ldots \times X_N$. □

Deja-Vu Exercise: Where have you seen these constructions before?

Theorem. *Any subspace of a metrizable space is metrizable. (i.e. Metrizability is a hereditary property.)*

Proof. Let (X, d) be a metric space and let $A \subseteq X$. Define $d_A : A \times A \longrightarrow \mathbb{R}$ by $d_A(a_1, a_2) = d(a_1, a_2)$, where $a_1, a_2 \in A$. This is clearly a metric on A, and it is easy to show that the subspace topology on A is the same as the metric topology generated by d_A. $\square$

Finally, we pay the proverbial piper by proving a fact whose proof we omitted in §7.4.

Theorem. *If X is a separable metrizable space, then it is hereditarily separable.*

Proof. Recall that this means that every subspace of X is separable. This can be proved directly, but let us show the power of your equivalence theorem above. Suppose the X is a separable metrizable space, with $A \subseteq X$ given the subspace topology. Then, by your equivalance theorem, X is also second countable. Second countability is clearly a hereditary property, so A is a metrizable second countable space. Again by your equivalance theorem, A is separable, as desired. $\square$

Now we have a large class of metrizable spaces that we know!

Some Example:

1. The sphere S^n is metrizable because it is a subspace of $\mathbb{R}^{n+1}$. However, think about our notion of distance on Earth. We think of distance in terms of walking along the surface of the Earth, not by drilling a hole straight through the Earth to our desired location. Our Earthling notion of distance on Earth can be formalized by defining the "sphere distance" between two points to be the shortest distance along a great circle that goes through those points. (A great circle is a circle whose radius is the same as the sphere.) You can check that this forms a metric, and that this metric generates the usual topology on the sphere.

2. The Torus $S^1 \times S^1$ is a metrizable space as it is a subspace of $\mathbb{R}^3$.

3. Let $\mathbb{M}_3(\mathbb{R})$ be defined to be the set of all 3×3 matrices with real coefficients. We may think of this as a subset of $\mathbb{R}^9$, so that it inherits the subspace topology. This makes $\mathbb{M}_3(\mathbb{R})$ a metrizable space! As a result it has all sorts of nice topological properties; it is a separable, first countable, Normal, metrizable space.

6 Two Useful Words: Bounded and Complete

Before leaving, we mention two words which are useful for studying metric spaces. You have already seen these notions in other classes, but let me remind you of them.

Definition. *Let (X, d) be a metric space. It is said to be **bounded** if*

$$diam(X) := \sup_{a,b \in X} d(a, b) = M < +\infty.$$

For example, $\text{diam}([0, 1]) = 1$ is a bounded metric space (where $[0, 1]$ is given its usual metric), but $\mathbb{R}$ with its usual metric is not.

Note that this definition is for metric spaces, *not* metrizable spaces. I'm not just being pedandic here, there is an important distinction to make! The following proposition illustrates the difference:

Proposition. $\mathbb{R}$ *(with its usual unbounded metric topology) is homeomorphic to* $(0, 1)$ *(with its usual bounded metric topology).*

This means that "boundedness" is not a topological invariant. In fact, every metrizable space has its topology generated by a bounded metric! This tells us that, in some sense, metrics are really local objects (sort of).

Proposition. *Let $(X, \mathcal{T})$ be a metrizable space. It is generated by a bounded metric.*

Proof. Let d be a (possibly unbounded) metric that generates $\mathcal{T}$. Verify that

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric that generates $\mathcal{T}$. $\square$

This notion of boundedness will be useful for us when we talk about infinite products of metrizable spaces.

Moving on let us mention the notion of a complete metric space, which has a particularly nice convergence property.

Definition. *A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges. A sequence $\langle x_n \rangle$ in X is a **Cauchy Sequence** if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n, m > N, |x_n - x_m| < \epsilon.$$

For example, $\mathbb{R}$ with its usual metric is a complete metric space, but $(0, 1)$ with its usual metric is not.

7 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Indiscrete : We saw that the discrete metric generates the discrete topology. Is there such a thing as the "indiscrete metric" which generates the indiscrete topology? What if we weaken the requirements on the distance function? Is this useful?

Deja-Vu : Where have you seen these constructions before?