UNIVERSITY OF TORONTO
Faculty of Arts and Sciences

MAT237Y1Y TERM TEST 2

Tuesday, July 23

Duration - 90 minutes

No Aids Allowed

Instructions: There are 5 questions and 12 pages including the cover page. There is a total of 85 marks which include 5 bonus marks. Please write your answers within the space provided, and clearly specify if you use back of a sheet to answer a question. The marking scheme allocates part marks to the details which are inherent in the process, so please use your judgement to include as much details as you know and you consider essential. No calculators, cell phones allowed.

NAME: (last, first) Solutions

STUDENT NUMBER:

SIGNATURE:

CHECK YOUR TUTORIAL:

◯ TUT5101 TA: Boris	◯ TUT5102 TA: James	◯ TUT5103 TA: Nan

MARKER'S REPORT:

Question	MARK
Q1 /19	
Q2 /12	
Q3 /10	
Q4 /19	
Q5 /25	
TOTAL /80	

1. a) (4 marks) What are the three representations of a smooth surface in $\mathbb{R}^3$?

i) graph : $z = f(x,y)$ or the other permutations of x, y, z, f is C'

ii) locus : $F(x,y,z) = 0$, F is C'

iii) parametric : $\vec{f}(u,v) = (\varphi(u,v), \psi(u,v), \phi(u,v))$, $\vec{f}$ is C'

b) (2 marks) Which condition on the parametric representation guarantees that it can be converted to the graph representation?

$\partial_u \vec{f} \times \partial_v \vec{f} \, (u_0, v_0) \neq \vec{0}$

or $\partial_u \vec{f}$ is linearly independent with $\partial_v \vec{f}$

c) (2 marks) What does it mean for $S \subset \mathbb{R}^3$ to be a smooth surface near a point **a**?

S is a smooth surface near $\vec{a}$, if there is a neighborhood N of $\vec{a}$, such that $N \cap S$ is the graph representation (i).

d) (4 marks) Is $\mathbf{f}(u,v) = (2u, 2v, \sqrt{u^2+v^2})$ the representation of a smooth surface near the origin $(0,0,0)$? How about near the point $(2,0,2)$? Explain why.

$\vec{f}_u = (2, 0, \frac{u}{\sqrt{u^2+v^2}})$, $\vec{f}_v = (0, 2, \frac{v}{\sqrt{u^2+v^2}})$

$(0,0,0)$ corresponds to $u=0, v=0$.

$\vec{f}_u$ and $\vec{f}_v$ is not continuous at $u=0, v=0$.

$\Rightarrow f$ is not the representation of a smooth surface near $(0,0,0)$

For $(2,0,2)$, it is not a point on the range of $\vec{f}(u,v)$.

e) (7 marks) Prove, using the IFT that under the regularity assumption (as in part b), the parametric representation (iii) of a surface can be locally converted to the graph representation (i).

Assume $\partial_u \vec{f}(u_0,v_0) \times \partial_v \vec{f}(u_0,v_0) \neq 0$, Then $\begin{vmatrix} i & j & k \\ \partial_u^x & \partial_u^y & \partial_u^z \\ \partial_v^x & \partial_v^y & \partial_v^z \end{vmatrix} \neq 0$.

and we have at least one of the

$\det\begin{bmatrix} \partial_u \varphi & \partial_u \psi \\ \partial_v \varphi & \partial_v \psi \end{bmatrix} \neq 0$ (or other permutation), where $(\varphi, \psi, \phi) = \vec{f}$

say $\det\begin{bmatrix} \partial_u \varphi & \partial_u \psi \\ \partial_v \varphi & \partial_v \psi \end{bmatrix} \neq 0$. Then define $F(x,y,u,v) = x - \varphi(u,v)$
$G(x,y,u,v) = y - \psi(u,v)$

and define $x_0 = \varphi(u_0,v_0)$
$y_0 = \psi(u_0,v_0)$ Then $F(x_0,y_0,u_0,v_0) = 0 = G(x_0,y_0,u_0,v_0)$

and $\left|\frac{\partial(\varphi,\psi)}{\partial(u,v)}(u_0,v_0)\right| \neq 0$, so we can apply IFT (3.9) to solve $F(x,y,u,v) = 0 = G(x,y,u,v)$ for u, v in terms of x, y near (x_0,y_0,u_0,v_0), so $(u,v) = w(x,y)$ and $z = \phi(u,v) = \phi(w(x,y))$, which is C^1 and is the graph representation of the surface.

2. Let the transformation $\mathbf{G}$ from the uv-plane to the xy-plane be defined by

$$(x, y) = (u + \frac{v}{2}, 2u^2 - v).$$

Let D be the region bounded by the u-axis, the v-axis, and the line $u + \frac{v}{2} = 2$.

a) (3 marks) Near what points in the uv-plane is it possible to express u and v locally as functions of x and y?

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 4u \\ \frac{1}{2} & -1 \end{pmatrix} = -1 - 2u \neq 0$$

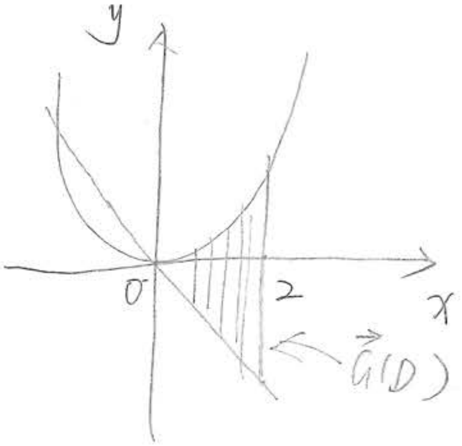
$\Rightarrow$ when $u \neq -\frac{1}{2}$, it is possible to express u and v locally as functions of x and y.

b) (5 marks) Find and sketch the image region $\mathbf{G}(D)$.

$$\left.\begin{array}{l} x = u + \frac{v}{2} \\ y = 2u^2 - v \end{array}\right\} \Rightarrow \text{ when } u = 0 \Rightarrow \begin{cases} x = \frac{v}{2} \\ y = -v \end{cases} \Rightarrow y = -2x$$

$$v = 0 \Rightarrow \begin{cases} x = u \\ y = 2u^2 \end{cases} \Rightarrow y = 2x^2$$

$$u + \frac{v}{2} = 2 \Rightarrow x = 2.$$

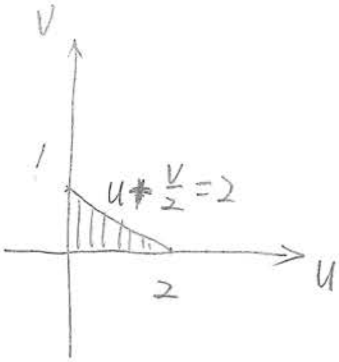


$\vec{G}(D)$ is the region bounded by $y = -2x$, $y = 2x^2$ and $x = 2$.

c) (4 marks) Compute the integral

$$\iint_{\mathbf{G}(D)} \frac{dxdy}{\sqrt{1+4x+2y}}.$$

$$\iint_{G(D)} \frac{dx\,dy}{\sqrt{1+4x+2y}} = \int_0^2 \int_0^{4-2u} \frac{1}{\sqrt{1+4x+2y}} \cdot |-1-2u| \, dv\, du$$



$$= \int_0^2 \int_0^{4-2u} \frac{|1+2u|}{\sqrt{1+4(u+\frac{v}{2})+2(2u^2-v)}} \, dv\, du$$

$$= \int_0^2 \int_0^{4-2u} \frac{|1+2u|}{\sqrt{1+4u+2v+4u^2-2v}} \, dv\, du$$

$$= \int_0^2 \int_0^{4-2u} \frac{|1+2u|}{\sqrt{(1+2u)^2}} \, dv\, du$$

$$= \int_0^2 \int_0^{4-2u} dv\, du$$

$$= \int_0^2 4-2u \, du$$

$$= 4u - u^2 \Big|_0^2$$

$$= 8-4 = 4$$

3. a) (3 marks) Give the statement of the implicit function theorem for a C^1 function $F(x, y)$ near a point (a, b).

Assume that $F(a,b) = 0$ and $\partial_y F(a,b) \neq 0$.

Then there is $r_0, r_1 > 0$ such that

a) for each x s.t. $|x-a| < r_0$, $\exists$ a unique y s.t. $|y-b| < r_1$ and $F(x,y) = 0$. [define this y by $f(x)$]

b) The function f is of class C^1 and

$$f'(x) = \frac{-\partial_x F(x, f(x))}{\partial_y F(x, f(x))}$$

b) (7 marks) Can the equation $(x^2 + y^2 + 2z^2)^{1/2} = 1 + \sin z$ be solved uniquely for z in terms of x and y near the point $(0, 1, 0)$? Explain why; and if the answer is yes, calculate the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$F(x,y,z) = (x^2+y^2+2z^2)^{1/2} - 1 - \sin z$$

$\Rightarrow \vec{a} = (0,1,0)$ satisfies $F(\vec{a}) = 0$ and

$$\partial_z F(\vec{a}) = \frac{2z}{\sqrt{x^2+y^2+2z^2}} - \cos z = -1 \neq 0$$

$\Rightarrow$ there is a unique solution $z = f(x,y)$.

$$\frac{\partial z}{\partial x} = -\frac{\partial_x F}{\partial_z F} = \frac{x}{\sqrt{x^2+y^2+2z^2}} = 0.$$

$$\frac{\partial z}{\partial y} = -\frac{\partial_y F}{\partial_z F} = \frac{y}{\sqrt{x^2+y^2+2z^2}} = \frac{1}{1} = 1.$$

4. a) (3 marks) What does it mean for a bounded function f to be Riemann integrable on the interval $[a, b]$? Present complete definition (include partition, the lower and upper sums and integrals etc.)

f is said to be Riemann integrable on $[a,b]$ if

least upper bound S_1 = greatest lower bound S_2

where $S_1 = \{ \underline{S}_P f : P$ is a partition of $[a,b] \}$

$S_2 = \{ \bar{S}_P f : P$ is a partition of $[a,b] \}$

$\underline{S}_P f = \sum_{i=1}^{n} m_i \Delta x_i$ where $P = \{a = x_0, x_1, \dots, x_n = b\}$

$\bar{S}_P f = \sum_{i=1}^{n} M_i \Delta x$ $\quad m_i = \min f(x)$ on $[x_{i-1}, x_i]$

$M_i = \max f(x)$ on $[x_{i-1}, x_i]$

b) (2 marks) State the ϵ characterization of integrability (Lemma 4.5).

For a bounded function f on $[a,b]$

f is integrable

$\Longleftrightarrow$

$\forall \varepsilon > 0, \exists P$ s.t. $\bar{S}_P f - \underline{S}_P f < \varepsilon$

c) (4 marks) Use part a to show that the constant function $f(x) = 1$ on the interval $[a, b]$ is integrable, and calculate the integral.

as $f(x) = 1$ is constant, $m_i = M_i = 1$ for any i, and any partition.

$$\Rightarrow \bar{S}_P f = \underline{S}_P f = \sum_{i=1}^{n} \Delta x_i = b-a$$

$$\Rightarrow \inf\{\bar{S}_P f\} = \sup\{\underline{S}_P f\} = b-a$$

$\Rightarrow f$ is integrable and $\int_a^b f(x)\,dx = b-a$.

d) (10 marks) Use part b to prove that if f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

f is continuous $\Rightarrow f$ is bounded.

$\Rightarrow \forall$ partion P, $\bar{S}_P f$, $\underline{S}_P f$ exist.

f is continuous, $[a,b]$ is compact

$\Rightarrow f$ is uniformly continuous

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$, s.t. $|f(x) - f(y)| < \varepsilon/(b-a)$

Let P be any partition of $[a,b]$ st. any subintervals $[x_{j-1}, x_j]$ of P have length less than δ.

$\Rightarrow \forall j, \forall x, y \in [x_{j-1}, x_j] \Rightarrow |x-y| < \delta. \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

$\Rightarrow |m_j - M_j| < \frac{\varepsilon}{b-a} \Rightarrow \bar{S}_P f - \underline{S}_P f = \sum_1^J (M_j - m_j)(x_j - x_{j-1})$

$< \frac{\varepsilon}{b-a} \sum_1^J (x_j - x_{j-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$

$\Rightarrow f$ is integrable

5. a) (4 marks) Give the statement of Fubini's theorem for double integrals.

Let $f(x,y)$ be integrable on the rectangle $R=[a,b]\times[c,d]$ and suppose for each $y\in[c,d]$, $f_y(x):=f(x,y)$ is integrable on $[a,b]$ & $g(y)=\int_a^b f_y(x)\,dx$ is integrable on $[c,d]$. then

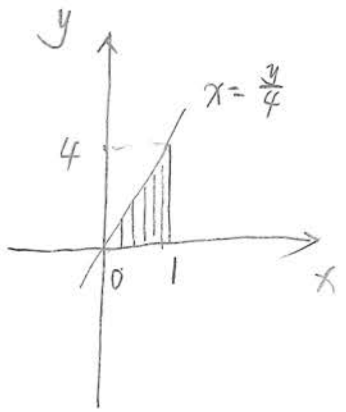
$$\iint_R f\,dA=\int_c^d\left[\int_a^b f\,dx\right]dy.$$

Similarly. if $f^x(y)$ is integrable on $[c,d]$ & $h(x)=\int_c^d f^x(y)\,dy$ is integrable on $[a,b]$. then

$$\iint_R f\,dA=\int_a^b\left[\int_c^d f\,dy\right]dx$$

b) (6 marks) Evaluate

$$\int_0^4 \int_{y/4}^1 y^2 e^{-x^4} dx dy.$$



$$= \int_0^1 \int_0^{4x} y^2 e^{-x^4} dy\, dx$$

$$= \int_0^1 \left[\frac{1}{3} y^3 e^{-x^4} \Big|_0^{4x} \right] dx$$

$$= \int_0^1 \frac{64}{3} x^3 e^{-x^4} dx$$

$$= \frac{64}{3} \cdot \frac{1}{4} (-e^{-x^4}) \Big|_0^1$$

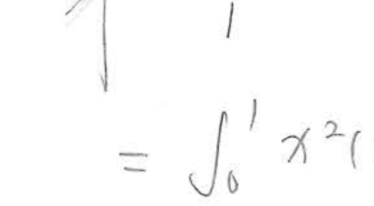
$$= \frac{16}{3} (1 - e^{-1})$$

c) (5 marks) Find the integral of

$$f(x, y) = x^2 + y^2,$$

on the domain $D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, x^2 \le y \le x\}$. Please integrate first in y-direction.

$$\begin{cases} y = x^2 \\ y = x \end{cases} \Rightarrow x = x^2 \Rightarrow x = 0 \text{ or } x = 1$$



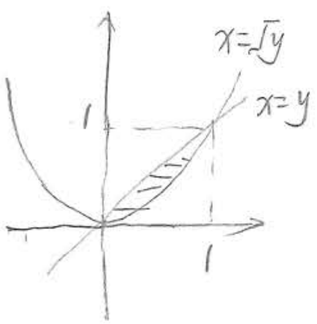
$\Rightarrow$ the intersection of the two curves are $(0, 0)$, $(1, 1)$.

$$\Rightarrow \iint_D f \, dA = \int_0^1 \int_{x^2}^{x} x^2 + y^2 \, dy \, dx$$

$$= \int_0^1 x^2(x - x^2) + \frac{1}{3}(x^3 - x^6) \, dx = \int_0^1 (x^3 - x^4 + \frac{1}{3} x^3 - \frac{1}{3} x^6) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^5}{5} + \frac{x^4}{12} - \frac{x^7}{21} \right] \Big|_0^1 = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{3}{35}$$

d) (5 marks) For the same integral of part c, please integrate first in x-direction.



$$\iint_D f\,dA = \int_0^1 \int_y^{\sqrt{y}} x^2+y^2\,dx\,dy$$

$$= \int_0^1 \frac{x^3}{3}\Big|_y^{\sqrt{y}} + y^2\left(x\Big|_y^{\sqrt{y}}\right) dy$$

$$= \int_0^1 \frac{1}{3}\left(y^{3/2}-y^3\right) + y^2\left(y^{1/2}-y\right) dy$$

$$= \int_0^1 \frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\,dy$$

$$= \frac{1}{3}\cdot\frac{2}{5}y^{5/2} - \frac{1}{3}\cdot\frac{1}{4}y^4 + \frac{2}{7}y^{7/2} - \frac{1}{4}y^4\Big|_0^1$$

$$= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4}$$

$$= \frac{3}{35}$$

e) (5 marks) Evaluate

$$\int_0^\pi e^x \sin x dx.$$

use integration by parts.

$$= -\int_0^\pi e^x (\cos x)' dx$$

$$= -\left[e^x \cos x \Big|_0^\pi - \int_0^\pi e^x \cos x \, dx \right]$$

$$= -\left[-e^\pi - 1 - \int_0^\pi e^x (\sin x)' dx \right]$$

$$= -\left[-e^\pi - 1 - \left(e^x \sin x \Big|_0^\pi - \int_0^\pi e^x \sin x \, dx \right) \right]$$

$$= e^\pi + 1 - \int_0^\pi e^x \sin x \, dx$$

$$\Rightarrow 2\int_0^\pi e^x \sin x \, dx = e^\pi + 1$$

$$\Rightarrow \int_0^\pi e^x \sin x \, dx = \frac{e^\pi + 1}{2}$$