FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 11

STOCHASTIC INTEREST RATE MODELS

So far we have taken a **deterministic** approach, where it was assumed that interest rates used in a financial transaction have been known in advance.

Although this is true in some practical situations, such as for loans with a fixed rate of interest, in other situations we will not know what future interest rates will be, for example, for variable interest rate loans. In these cases the rate of interest can be treated as a random variable and is said to be **stochastic**.

In practice, it is sensible to assume that interest rates are dependent. Historical experience suggests that it is more likely for rates to stay high or low for several successive periods than for it is for rates to bounce around randomly above and below some average rate. This seems even more plausible when we consider the fact that the level of interest rates is tied to economic conditions and government policy. There are numerous models that can be constructed to build in interest rate dependency, however, this is beyond the scope of the syllabus for this course.

For the remainder of the lectures we will only focus on discrete independent interest rate models where:

- we assume that interest accrues at specified (discrete) points in time, and
- interest rates at each period in time are independent of interest rates at other periods.

This section assumes a basic knowledge of statistics. The main results from statistics that we will be using are summarised below:

For a **discrete random variable** $\tilde{X}$, with probability mass function $p(x) = \Pr\left[\tilde{X} = x\right]$, the mean is: $E\left[\tilde{X}\right] = \sum_x x \cdot p(x)$

and the variance is: $Var\left[\tilde{X}\right] = E\left[\tilde{X}^2\right] - \left(E\left[\tilde{X}\right]\right)^2 = \sum_x x^2 \cdot p(x) - \left(\sum_x x \cdot p(x)\right)^2$

For a **continuous random variable** $\tilde{X}$, with probability density function $f(x)$, the probability $P\left[a < \tilde{X} < b\right] = \int_a^b f(x)dx$.

$\tilde{X}$ has mean: $E\left[\tilde{X}\right] = \int_{-\infty}^{\infty} x \cdot f(x)dx$

and variance: $Var[\tilde{X}] = E[\tilde{X}^2] - (E[\tilde{X}])^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - \left(\int_{-\infty}^{\infty} x \cdot f(x)dx\right)^2$

If a and b are constants then $Var[a\tilde{X} + b] = a^2 Var[\tilde{X}]$

The standard deviation of $\tilde{X}$ is $\sqrt{Var[\tilde{X}]}$.

For a function $h(\cdot)$: $E[h(\tilde{X})] = \int_{-\infty}^{\infty} h(x) \cdot f(x)dx$

If $\tilde{X}$ and $\tilde{Y}$ are independent random variables then $Var\left[\tilde{X} + \tilde{Y}\right] = Var\left[\tilde{X}\right] + Var\left[\tilde{Y}\right]$

We will also be using a number of continuous distributions:

Uniform distribution

$f(x) = \dfrac{1}{b-a}$ for $a < x < b$

$E[\tilde{X}] = \dfrac{a+b}{2}$

$Var[\tilde{X}] = \dfrac{(b-a)^2}{12}$

Normal distribution

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(\dfrac{-(x-\mu)^2}{2\sigma^2}\right)$ for $-\infty < x < \infty$

$E[\tilde{X}] = \mu$

$Var[\tilde{X}] = \sigma^2$

Recall that if $\tilde{X}$ is normally distributed with mean and variance as above, then

$$P[a < \tilde{X} < b] = P\left[\frac{a-\mu}{\sigma} < \frac{\tilde{X}-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] = P\left[\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right]$$

where Z has a standard normal distribution (ie. normal distribution with mean 0 and variance 1).

Statistical tables can be used with a standard normal variable to find $P[a < \tilde{X} < b]$

SINGLE CASHFLOWS

ACCUMULATED VALUES

The simplest example of the interest rate as a random variable is to consider the case where the exact interest rate for the coming period is unknown, but where it is known that the rate will be one of two levels with known probabilities:

$$\tilde{i} = \begin{cases} i_a & prob = a \\ i_b & prob = b \end{cases}$$

The expected value is $E[\tilde{i}] = \sum_i i \cdot p(i) = a \cdot i_a + b \cdot i_b$

To find the variance we can use $Var[\tilde{i}] = E[\tilde{i}^2] - (E[\tilde{i}])^2$

The second moment is $E[\tilde{i}^2] = a \cdot i_a^2 + b \cdot i_b^2$

So, the variance can be written: $Var[\tilde{i}] = (a \cdot i_a^2 + b \cdot i_b^2) - (a \cdot i_a + b \cdot i_b)^2$

EXAMPLE
Find the expected interest rate and standard deviation if

$$\tilde{i} = \begin{cases} 0.10 & prob = 0.5 \\ 0.15 & prob = 0.5 \end{cases}$$

Solution
$E[\tilde{i}] = 0.5 \cdot 0.1 + 0.5 \cdot 0.15 = 0.125$

The variance can be found by first finding the second moment,

$E[\tilde{i}^2] = 0.5 \cdot 0.1^2 + 0.5 \cdot 0.15^2 = 0.01625$, and then:

$Var[\tilde{i}] = E[\tilde{i}^2] - (E[\tilde{i}])^2 = 0.01625 - 0.125^2 = 0.000625$

The standard deviation is $\sqrt{0.000625} = 0.025$

The same method can be used to solve for more than two interest rates:

EXAMPLE
Find the expected interest rate and standard deviation if

$$\tilde{i} = \begin{cases} 0.06 & prob = 0.25 \\ 0.07 & prob = 0.15 \\ 0.08 & prob = 0.60 \end{cases}$$

Solution

$$E[\tilde{i}] = 0.25 \cdot 0.06 + 0.15 \cdot 0.07 + 0.60 \cdot 0.08 = 0.0735$$

$$E[\tilde{i}^2] = 0.25 \cdot 0.06^2 + 0.15 \cdot 0.07^2 + 0.60 \cdot 0.08^2 = 0.005475$$

$$\Rightarrow Var[\tilde{i}] = E[\tilde{i}^2] - \left(E[\tilde{i}]\right)^2 = 0.005475 - 0.0735^2 = 0.00007275$$

The standard deviation is $\sqrt{0.00007275} = 0.008529$

If we want to accumulate an investment of 1 to the end of the year, then:

$$E[\tilde{S}(1)] = E[1 + \tilde{i}] = 1 + E[\tilde{i}]$$

$$Var[\tilde{S}(1)] = Var[1 + \tilde{i}] = Var[\tilde{i}]$$

Suppose now that we wish to accumulate an amount over a two-year period instead of a one-year period. If $\tilde{i}_1$ is the random rate in the first year and $\tilde{i}_2$ is the random rate in the second year, then the random accumulated value at the end of two years of an initial investment of 1 is:

$$\tilde{S}(2) = (1 + \tilde{i}_1)(1 + \tilde{i}_2)$$

The expected value of the accumulation at the end of two years is:

$$E[\tilde{S}(2)] = E[(1 + \tilde{i}_1)(1 + \tilde{i}_2)]$$

If $\tilde{i}_1$ and $\tilde{i}_2$ are independent then

$$E[\tilde{S}(2)] = E[1 + \tilde{i}_1] \cdot E[1 + \tilde{i}_2]$$

Independence here means that the value of $\tilde{i}_1$ has no impact on the value of $\tilde{i}_2$.

Also under independence,

$$E[\tilde{S}(2)^2] = E[(1 + \tilde{i}_1)^2] \cdot E[(1 + \tilde{i}_2)^2]$$

If $\tilde{i}_1$ and $\tilde{i}_2$ are independent and are identically distributed, with mean $E[\tilde{i}]$ and variance $Var[\tilde{i}]$, then:

$E[\tilde{S}(2)] = (E[1+\tilde{i}])^2$, and

$E\left[\tilde{S}(2)^2\right] = \left(E\left[(1+\tilde{i})^2\right]\right)^2$, and

$Var[\tilde{S}(2)] = E[\tilde{S}(2)^2] - (E[\tilde{S}(2)])^2 = \left(E\left[(1+\tilde{i})^2\right]\right)^2 - (E[1+\tilde{i}])^4$

We can generalise this to find the mean and variance of the accumulated value of 1 after n periods, assuming independence between interest rates:

$$\tilde{S}(n) = (1+\tilde{i}_1)(1+\tilde{i}_2)\cdots(1+\tilde{i}_n)$$
$$E[\tilde{S}(n)] = E[1+\tilde{i}_1]\cdot E[1+\tilde{i}_2]\cdots E[1+\tilde{i}_n]$$
$$E[\tilde{S}(n)^2] = E\left[(1+\tilde{i}_1)^2\right]\cdot E\left[(1+\tilde{i}_2)^2\right]\cdots E\left[(1+\tilde{i}_n)^2\right]$$

If the interest rates are independent and identically distributed, with mean $E[\tilde{i}]$ and variance $Var[\tilde{i}]$, then the mean and variance of the accumulated value of 1 after n periods are:

$$E[\tilde{S}(n)] = (E[1+\tilde{i}])^n$$
$$E[\tilde{S}(n)^2] = \left(E\left[(1+\tilde{i})^2\right]\right)^n$$
$$Var[\tilde{S}(n)] = E[\tilde{S}(n)^2] - (E[\tilde{S}(n)])^2 = \left(E\left[(1+\tilde{i})^2\right]\right)^n - (E[1+\tilde{i}])^{2n}$$

EXAMPLE
Find the mean and variance of \$50 accumulated for 20 years if annual effective interest rates are independent and follow the distribution:

$$\tilde{i} = \begin{cases} 0.06 & prob = 0.25 \\ 0.07 & prob = 0.15 \\ 0.08 & prob = 0.60 \end{cases}$$

Solution
$$E[1+\tilde{i}] = 0.25\cdot 1.06 + 0.15\cdot 1.07 + 0.60\cdot 1.08 = 1.0735$$
$$E\left[(1+\tilde{i})^2\right] = 0.25\cdot 1.06^2 + 0.15\cdot 1.07^2 + 0.60\cdot 1.08^2 = 1.152475$$

$$E[\tilde{S}(20)] = (E[1+\tilde{i}])^{20} = (1.0735)^{20} = 4.1309$$
$$E[\tilde{S}(20)^2] = \left(E\left[(1+\tilde{i})^2\right]\right)^{20} = 1.152475^{20} = 17.0856$$

The mean of the accumulated value is $50\cdot E\left[\tilde{S}(20)\right] = \$206.54$

The variance is:

$$Var\left[50\cdot\tilde{S}(20)\right]=50^2\cdot Var\left[\tilde{S}(20)\right]=50^2\cdot\left(E\left[\tilde{S}(20)^2\right]-\left(E\left[\tilde{S}(20)\right]\right)^2\right)$$

$$=2500\left(1.152475^{20}-1.0735^{40}\right)=53.89$$

So far our examples have assumed a particular discrete distribution for the future interest rates. $\tilde{i}$ may have a continuous distribution, such as the uniform distribution.

EXAMPLE
Find the mean and variance of $50 accumulated for 20 years if annual effective interest rates are independent and have a uniform distribution on the interval $[0.08, 0.12]$.

Solution
Since $\tilde{i}_t$ is uniformly distributed on the interval $[0.08, 0.12]$ it follows that $1+\tilde{i}_t$ is uniformly distributed on the interval $[1.08, 1.12]$.

$$E\left[1+\tilde{i}\right]=\frac{a+b}{2}=\frac{1.08+1.12}{2}=1.10$$

Since $1+\tilde{i}_t$ are independent and identically distributed, we can use the result from above to find the mean and variance:

$$E\left[50\cdot\tilde{S}(20)\right]=50\cdot\left(E\left[1+\tilde{i}\right]\right)^{20}=50\cdot 1.10^{20}=\$336.38$$

We know the variance is calculated as follows:

$$Var\left[50\cdot\tilde{S}(n)\right]=2500\left[\left(E\left[\left(1+\tilde{i}\right)^2\right]\right)^{20}-\left(E\left[1+\tilde{i}\right]\right)^{40}\right]$$

To find the variance we need to find the second moment $E\left[\left(1+\tilde{i}\right)^2\right]$. Recall that the probability density function for a uniformly distributed variable $\tilde{X}$ is $f(x)=\dfrac{1}{b-a}$

$$E\left[\left(1+\tilde{i}\right)^2\right]=\int_a^b\left(1+\tilde{i}\right)^2 f(\tilde{i})\cdot d\tilde{i}=\int_{0.08}^{0.12}\frac{\left(1+\tilde{i}\right)^2}{0.12-0.08}\cdot d\tilde{i}=25\int_{0.08}^{0.12}\left(1+\tilde{i}\right)^2\cdot d\tilde{i}$$

$$=25\left[\frac{\left(1+\tilde{i}\right)^3}{3}\right]_{0.08}^{0.12}=25\left[\frac{(1.12)^3-(1.08)^3}{3}\right]=1.210133$$

Therefore, the variance is:

$$Var\left[50\cdot\tilde{S}(n)\right]=2500\left[\left(E\left[\left(1+\tilde{i}\right)^{2}\right]\right)^{20}-\left(E\left[1+\tilde{i}\right]\right)^{40}\right]=2500\left[1.210133^{20}-1.1^{40}\right]=\$249.00$$

Alternatively, we can use the fact that $Var\left[\tilde{i}\right]=\frac{(b-a)^{2}}{12}=\frac{(0.04)^{2}}{12}=0.000133333$

$$E\left[\left(1+\tilde{i}\right)^{2}\right]=Var\left[1+\tilde{i}\right]+\left(E\left[1+\tilde{i}\right]\right)^{2}=0.0001333+1.1^{2}=1.210133$$

THE LOG-NORMAL DISTRIBUTION

In this section we show that under a series of plausible assumptions, $\tilde{S}(n)$ has an approximate log-normal distribution. $\tilde{S}(n)$ is said to have a log-normal distribution if $\ln\left[\tilde{S}(n)\right]$ is normally distributed. The actual distribution of the $\tilde{i}_t$ for $t=1,2,\ldots,n$ does not affect this result, so long as $\tilde{i}_t$ are independent and identically distributed.

The main implication is that standard normal tables can be used to approximate the probability that $\tilde{S}(n)$ is greater than or less than some specified value.

If the annual rate of interest is a random variable, then so is the force of interest $\tilde{\delta}=\ln(1+\tilde{i})$.

Since $\tilde{S}(n)=\left(1+\tilde{i}_1\right)\left(1+\tilde{i}_2\right)\cdots\left(1+\tilde{i}_n\right)$

$$\Rightarrow \ln\left[\tilde{S}(n)\right]=\ln\left(1+\tilde{i}_1\right)+\ln\left(1+\tilde{i}_2\right)+\ldots+\ln\left(1+\tilde{i}_n\right)=\tilde{\delta}_1+\tilde{\delta}_2+\ldots+\tilde{\delta}_n$$

By the Central Limit Theorem, the sum of independent, identically distributed random variables is approximately normally distributed for large n.

Therefore, for large n, if the forces of interest $\tilde{\delta}_t$ are independent and identically distributed with mean $E\left[\tilde{\delta}\right]$ and variance $Var\left[\tilde{\delta}\right]$, then $\ln\left[\tilde{S}(n)\right]$ is approximately normally distributed with:

Mean: $E\left[\ln\left[\tilde{S}(n)\right]\right]=E\left[\tilde{\delta}_1+\tilde{\delta}_2+\ldots+\tilde{\delta}_n\right]=n\cdot E\left[\tilde{\delta}\right]$
Variance: $Var\left[\ln\left[\tilde{S}(n)\right]\right]=Var\left[\tilde{\delta}_1+\tilde{\delta}_2+\ldots+\tilde{\delta}_n\right]=n\cdot Var\left[\tilde{\delta}\right]$

We can use the fact that $\ln\left[\tilde{S}(n)\right]$ is approximately normal (for large n) to find the probability that the accumulated value is between specified value, that is, $\Pr\left[a<\tilde{S}(n)<b\right]$ for specified a and b.

EXAMPLE

Find the approximate probability $\Pr\left[\tilde{S}(10) > 3.247321\right]$, where $\tilde{i}_t$, $t = 1,2,...,10$ are independent and identically distributed with the distribution:

$$\tilde{i} = \begin{cases} 0.10 & prob = 0.5 \\ 0.15 & prob = 0.5 \end{cases}$$

Use the log-normal approximation for $\tilde{S}(10)$. In addition, calculate the exact value of $\Pr\left[\tilde{S}(10) > 3.247321\right]$ using binomial theory.

Solution

The distribution of $\tilde{\delta}$ is:

$$\tilde{\delta} = \begin{cases} \ln(1.10) & prob = 0.5 \\ \ln(1.15) & prob = 0.5 \end{cases}$$

$$E\left[\tilde{\delta}\right] = 0.5\left(\ln(1.10) + \ln(1.15)\right) = 0.117536$$
$$E\left[\tilde{\delta}^2\right] = 0.5\left(\left(\ln(1.10)\right)^2 + \left(\ln(1.15)\right)^2\right) = 0.014309$$
$$\Rightarrow Var\left[\tilde{\delta}\right] = 0.014309 - 0.117536^2 = 0.00049399$$

So,

$$E\left[\ln\left[\tilde{S}(10)\right]\right] = 10 \cdot E\left[\tilde{\delta}\right] = 1.17536$$
$$Var\left[\ln\left[\tilde{S}(10)\right]\right] = 10 \cdot Var\left[\tilde{\delta}\right] = 0.0049399$$

$$\begin{aligned}\Pr\left[\tilde{S}(10) > 3.247321\right] &= \Pr\left[\ln\left(\tilde{S}(10)\right) > \ln\left(3.247321\right)\right] \\ &= \Pr\left[\frac{\ln\left(\tilde{S}(10)\right) - E\left[\ln\left[\tilde{S}(10)\right]\right]}{\sqrt{Var\left[\ln\left[\tilde{S}(10)\right]\right]}} > \frac{\ln\left(3.247321\right) - E\left[\ln\left[\tilde{S}(10)\right]\right]}{\sqrt{Var\left[\ln\left[\tilde{S}(10)\right]\right]}}\right] \\ &= \Pr\left[\tilde{Z} > \frac{\ln\left(3.247321\right) - 1.17536}{\sqrt{0.0049399}}\right] \\ &= \Pr\left[\tilde{Z} > 0.035\right]\end{aligned}$$

where $\tilde{Z}$ is a standard normal variable with mean 0 and variance 1. Referring to tables for the standard normal distribution:

$$\Pr\left[\tilde{Z} > 0.035\right] = 0.486$$

To calculate the exact value of $\Pr\left[\tilde{S}(10) > 3.247321\right]$, we first need to test how many years of 15% return over the ten years are required to ensure $\tilde{S}(10) > 3.247321$

Test 5 years: $\tilde{S}(10) = 1.15^5 \times 1.10^5 = 3.239311$
Test 6 years: $\tilde{S}(10) = 1.15^6 \times 1.10^4 = 3.386552$

Thus we need at least 6 years of 15% to ensure $\tilde{S}(10) > 3.247321$

Therefore the probability is:

$$\Pr\left[\tilde{S}(10) > 3.247321\right] = \sum_{i=6}^{10} {}^{10}C_i \times 0.5^i \times 0.5^{10-i}$$
$$= 0.5^{10}\left(210 + 120 + 45 + 10 + 1\right) = 0.377$$

PRESENT VALUES

When finding the accumulated value or present value of a cashflow under stochastic assumptions we can use the distribution of $\tilde{i}$ or $1+\tilde{i}$.

EXAMPLE

Find the expected present value of \$100 paid in 5 years time if $\tilde{i}_t$, $t = 1,2,\ldots,5$ are independent and identically distributed with:

$$\tilde{i} = \begin{cases} 0.10 & prob = 0.5 \\ 0.15 & prob = 0.5 \end{cases}$$

Solution

$$\begin{aligned} EPV &= E[100(1+\tilde{i}_5)^{-1}(1+\tilde{i}_4)^{-1}(1+\tilde{i}_3)^{-1}(1+\tilde{i}_2)^{-1}(1+\tilde{i}_1)^{-1}] \\ &= 100E[(1+\tilde{i}_5)^{-1}]E[(1+\tilde{i}_4)^{-1}]E[(1+\tilde{i}_3)^{-1}]E[(1+\tilde{i}_2)^{-1}]E[(1+\tilde{i}_1)^{-1}] \\ &= 100\left(E\left[\frac{1}{1+i}\right]\right)^5 \\ &= 100\left(\frac{1}{1.1} \times 0.5 + \frac{1}{1.15} \times 0.5\right)^5 \\ &= \$55.63 \end{aligned}$$

Note, this is different from finding the present value now in order to have an **expected accumulation** 5 years from now of 100:

$$100 = PV \cdot E\left[(1+\tilde{i}_1)(1+\tilde{i}_2)(1+\tilde{i}_3)(1+\tilde{i}_4)(1+\tilde{i}_5)\right] = PV \cdot \left(E\left[1+\tilde{i}\,\right]\right)^5 = PV \cdot (1.125)^5$$

$$\Rightarrow PV = \frac{100}{1.125^5} = \$55.49$$

ANNUITIES

So far we have looked at estimating the accumulated and present values for single cashflows when interest rates are random variables. We now extend the theory to deal with annuities.

ACCUMULATED VALUES

For an annuity where the interest rate varies from year-to-year, the accumulated value at the date of the last payment (annuity-immediate) is:

$$1+(1+i_n)+(1+i_n)(1+i_{n-1})+(1+i_n)(1+i_{n-1})(1+i_{n-2})+\ldots+(1+i_n)(1+i_{n-1})(1+i_{n-2})\cdots(1+i_2)$$

The first payment is made at the end of the first year which is why the first interest rate applied to the payment stream is i_2.

If $i_1 = i_2 = \ldots = i_n$ this reduces to:

$$s_{\overline{n}|} = 1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}$$

If interest rates are random then the accumulated value for the annuity-immediate is also a random variable:

$$\tilde{s}_{\overline{n}|} = 1+(1+\tilde{i}_n)+(1+\tilde{i}_n)(1+\tilde{i}_{n-1})+(1+\tilde{i}_n)(1+\tilde{i}_{n-1})(1+\tilde{i}_{n-2})+\ldots+(1+\tilde{i}_n)(1+\tilde{i}_{n-1})(1+\tilde{i}_{n-2})\cdots(1+\tilde{i}_2)$$

If interest rates are independent and identically distributed with mean $E\left[\tilde{i}\,\right]$, then

$$E\left[\tilde{s}_{\overline{n}|}\right] = 1+E\left[1+\tilde{i}\,\right]+\left(E\left[1+\tilde{i}\,\right]\right)^2+\ldots+\left(E\left[1+\tilde{i}\,\right]\right)^{n-1} \Rightarrow$$

$$\boxed{E\left[\tilde{s}_{\overline{n}|}\right] = s_{\overline{n}|}}$$

where $s_{\overline{n}|}$ is evaluated at the interest rate $E\left[\tilde{i}\,\right]$.

Similarly it can be shown that if an annuity-due is expressed as a random variable $\tilde{\ddot{s}}_{\overline{n}|}$ then,

$$\boxed{E\left[\ddot{\tilde{s}}_{\overline{n|}}\right] = \ddot{s}_{\overline{n|}}}$$

The variance $\tilde{s}_{\overline{n|}}$ and $\ddot{\tilde{s}}_{\overline{n|}}$ can be complicated for large n. For those interested, the result for the variance is given below, though it is not proven here and it is not needed for the final exam:

$$Var\left[\tilde{s}_{\overline{n|}}\right] = Var\left[\ddot{\tilde{s}}_{\overline{n-1|}}\right] = \left(\frac{S^n - S}{S-1}\right)\left(\frac{S+R}{S-R}\right) - \left(\frac{R^n - R}{R-1}\right)\left(\frac{2S}{S-R}\right) - \left(\frac{R^n - R}{R-1}\right)^2$$

where $S = E\left[(1+\tilde{i}\,)^2\right]$ and $R = E\left[1+\tilde{i}\,\right]$

EXAMPLE
Assume that annual interest rates are independent and identically distributed with the distribution:

$$\tilde{i} = \begin{cases} 0.10 & prob = 0.5 \\ 0.15 & prob = 0.5 \end{cases}$$

Find the expected value of the accumulated amount of an immediate-annuity of \$20 per annum payable in arrears after 10 years.

Solution

$E\left[\tilde{i}\,\right]$=0.125

$$\Rightarrow E\left[20 \cdot \tilde{s}_{\overline{10|}}\right] = 20 \cdot s_{\overline{10|}0.125} = 20(17.97857) = 359.57$$