Peter Crooks
TUT0101

MAT224 PS 6 Rui Qiu #999292509

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Q1.

(a).

Proof:

Let $\vec{u} \in \ker(T)$. This means $T(\vec{u}) = 0$, and in particular for all $\vec{v} \in V$, we have

$$0 = \langle T \cdot \vec{u}, \vec{v} \rangle = \langle \vec{u}, T^* \vec{v} \rangle$$

In particular, we find that $\vec{u}$ is orthogonal to $T^* \vec{v}$ for all $\vec{v} \in V$, therefore $\vec{u} \in Im(T^*)^\perp$

so $\ker(T) \subseteq im(T^*)^\perp$

since then $\ker(T)^\perp = (im(T^*)^\perp)^\perp = im(T^*)$

(b). Let $\vec{u} \in \ker(T^*)$. This means $T^*(\vec{u}) = 0$ and in particular for all $\vec{v} \in V$ we have

$$0 = \langle T^* \cdot \vec{u}, \vec{v} \rangle = \langle \vec{u}, T \cdot \vec{v} \rangle$$

In particular, we find that $\vec{u}$ is orthogonal to $T \cdot \vec{v}$ for all $\vec{v} \in V$, therefore $\vec{u} \in Im(T)^\perp$

so $im(T)^\perp = \ker(T^*)$

then $\ker(T^*)^\perp = (im(T)^\perp)^\perp = im(T)$.

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Q2. Solution: say $\{1, x, x^2\}$ is a basis for $P_3(R)$

Let $T(p(x)) = p'(x)$, say $p(x) = a + bx + cx^2$ for $a, b, c \in R^1$.

so $p'(x) = b + 2cx$

Let $T(p(x)) = p'(x) = b + 2cx = 0$

Then $b = c = 0$, a is an arbitary real number.

Then $\ker(T) = \text{span}\{(1,0,0)\} = \text{span}\{1\}$

Since $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(P_2(R)) = 3$

$\dim(\text{im}(T)) = 2$

and $p'(x) = b + 2cx$

hence $\text{im}(T) = \text{span}\{(0,1,0), (1,0,0)\} = \text{span}\{1, x\}$

$T(1) = (0,0,0)$

$T(x) = (1,0,0)$

$T(x^2) = (0, 2, 0)$.

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we apply Gram-Schmidt to find an orthonormal basis from $\{1, x, x^2\}$. (Note that $\langle p(x), g(x) \rangle = \int_0^1 p(x) g(x)\, dx$

$V_1 = U_1 = 1$

$$V_2 = U_2 - \frac{\langle U_2, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - \frac{\frac{1}{2}}{1} \cdot 1 = x - \frac{1}{2}$$

$$V_3 = U_3 - \frac{\langle U_3, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2 - \frac{\langle U_3, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 = x^2 - \frac{\langle x^2, x - \frac{1}{2} \rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle}(x - \tfrac{1}{2}) - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$$

$$= x^2 - x + \frac{1}{6}$$

$\|V_1\| = 1$, $\|V_2\| = 2\sqrt{3}(x - \frac{1}{2})$, $\|V_3\| = 6\sqrt{5}(x^2 - x + \frac{1}{6})$

so orthonormal basis is $\alpha = \{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\}$.

$T(1) = 0$, $T(2\sqrt{3}(x - \frac{1}{2})) = 2\sqrt{3}$, $T(6\sqrt{5}(x^2 - x + \frac{1}{6})) = 2\sqrt{15}(x - \frac{1}{2})$

$$[T]_{\alpha\alpha} = \begin{bmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{bmatrix} \qquad [T^*]_{\alpha\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{bmatrix}$$

So $\ker T^* = \operatorname{span}\{(0,0,1)^T\}$

$= \operatorname{span}\{\cancel{12x-6}, 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\}$

$= \operatorname{span}\{x^2 - x + \frac{1}{6}\}$

$\operatorname{im} T^* = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 2\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2\sqrt{15} \end{pmatrix}\right\}$

$= \operatorname{span}\{12x-6, 60\sqrt{3}x^2 - 60\sqrt{3}x + 10\sqrt{3}\} = \operatorname{span}\{x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$

Thus, to conclude, we have.

$\ker T = \operatorname{span}\{1\}$

$\operatorname{im} T = \operatorname{span}\{1, 2x\}$

$\ker T^* = \operatorname{span}\{x^2 - x + \frac{1}{6}\}$

$\operatorname{im} T^* = \operatorname{span}\{x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$.

Peter Crooks
TUT 0101

MAT224 PS6 Rui Qiu #999292509

Q3.

Proof: Since $T: V \to V$, $\operatorname{im} T$ is a subspace of V.

By the fact that $V = W \oplus W^{\perp}$ (*)

then $\operatorname{im}(T) \oplus (\operatorname{im}(T))^{\perp} = V$

since $\operatorname{im}(T) \cap (\operatorname{im}(T))^{\perp} = \emptyset$, and $\operatorname{im}(T) + (\operatorname{im}(T))^{\perp} = V$

so $\dim(\operatorname{im}(T)) + \dim(\operatorname{im}(T))^{\perp} = \dim(V)$ ①

Since $\dim(\operatorname{im}(T)) + \dim(\ker(T)) = \dim(V)$ ② by rank-nullity thm

② − ① we get

$$\dim(\ker(T)) - \dim(\operatorname{im}(T))^{\perp} = 0$$

$$\Rightarrow \dim(\ker(T)) = \dim(\operatorname{im}(T))^{\perp}$$

according to Q2 & Q1

$$\dim(\ker(T)) = \dim(\operatorname{im}(T^*))^{\perp}$$

So $\dim(\operatorname{im}(T^*))^{\perp} = \dim(\operatorname{im}(T))^{\perp}$ ③

Use fact (*) again: $\operatorname{im}(T^*) \oplus (\operatorname{im}(T^*))^{\perp} = V$

$$\dim(\operatorname{im}(T^*))^{\perp} = \dim(V) - \dim(\operatorname{im}(T^*))$$

$$\dim(\operatorname{im}(T))^{\perp} = \dim(V) - \dim(\operatorname{im}(T))$$

By ③ then $\dim(V) - \dim(\operatorname{im}(T)) = \dim(V) - \dim(\operatorname{im}(T^*))$

Therefore $\dim(\operatorname{im}(T)) = \dim(\operatorname{im}(T^*))$ ∎

Q4.

(a). Solution:

$$N = \begin{pmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2+1 & -2+4-2 & -1+2-1 & -4+8-4 \\ 1-2+1 & -2+4-2 & -1+2-1 & -4+8-4 \\ -1+2-1 & 2-4+2 & 1-2+1 & 4-8+4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

Therefore N is nilpotent and the smallest k is 2 such that $N^k = 0$.

Peter Crooks

TUT0101

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(b).

Solution:

Since N is nilpotent

$$N\begin{pmatrix} x \\ y \\ z \\ m \end{pmatrix} = \begin{pmatrix} x-2y-z-4m \\ x-2y-z-4m \\ -x+2y+z+4m \\ 0 \end{pmatrix} = 0$$

then $\ker(N) = \{(4x+y+2z, z, y, x) : x, y, z \in \mathbb{R}\}$

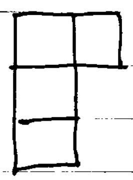
therefore a basis for $\ker(N)$ is $\{(4,0,0,1), (1,0,1,0), (2,1,0,0)\}$

so $\dim(\ker(N)) = 3$

And $\dim(\ker(N^2)) = 4$ (obviously since N^2 is a zero operator)

so there will be $3-0=3$ boxes in the first column and $4-3=1$ box in the second column.

The cycle tableau of a canonical basis will be



Hence the canonical form of N is

$$[N]_\beta^\beta = \cancel{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we want to get a canonical basis β.

Note that

$\ker(N) = \text{span}\{(4,0,0,1), (1,0,1,0), (2,1,0,0)\}$

$Im(N) = \text{span}\{(1,1,-1,0)\}$

We have 3 cycles with length of 2, 1, 1 respectively.

say $\alpha_1 = \{Nx_1, x_1\}$, $\alpha_2 = \{x_2\}$, $\alpha_3 = \{x_3\}$

Now we want to find the final vector of α_1, which is Nx_1

such that $Nx_1 \in Ker(N) \cap Im(N)$

$Nx_1 = (1, 1, -1, 0)$ is such a vector.

Then we want $x_1 \in R^4$ such that

$Nx_1 = (1, 1, -1, 0)$

$x_1 = (1, 0, 0, 0)$ satisfies.

For the second cycle $\alpha_2 = \{x_2\}$

we need to find an eigenvector of N that together with the vector $Nx_1 = (1, 1, -1, 0)$ gives a linearly independent set

such x_2 could be $(1, 0, 1, 0)$

For the third cycle $\alpha_3 = \{x_3\}$

similarly we find $x_3 = (4, 0, 0, 1)$.

Therefore $\beta = \{\{(1, 1, -1, 0), (1, 0, 0, 0)\} \cup \{(1, 0, 1, 0)\} \cup \{(4, 0, 0, 1)\}$.

the canonical basis

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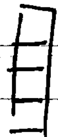
Q5.

Solution:

$k=1$. $\dim(\ker(N))=4$



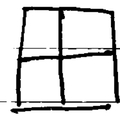
$$N=\cancel{\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}} \quad \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$k=2$, $\cancel{\dim(\ker(N))=1,\ \dim(\ker(N^2))=4}$

$$N=\cancel{\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}}$$

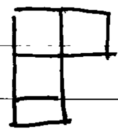
$\dim(\ker(N))=2$, $\dim(\ker(N^2))=4$

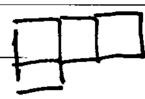
$$N=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$



or $\dim(\ker(N))=3$, $\dim(\ker(N^2))=4$

$$N=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$





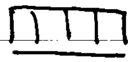
$k=3$, ① $\cancel{\dim(\ker(N))=1,\ \dim(\ker(N^2))=2,\ \dim(\ker(N^3))=4}$

~~or ②~~ $\cancel{\dim(\ker(N))=1,\ \dim(\ker(N^2))=3,\ \dim(\ker(N^3))=4}$

~~or ③~~ $\dim(\ker(N))=2$. $\dim(\ker(N^2))=3$, $\dim(\ker(N^3))=4$

$$N=\cancel{\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{②} \quad N=\cancel{\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}}_{①} \quad N=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}_{\cancel{③}}$$

$k=4.$ $\dim(\ker(N))=1,$ $\dim(\ker(N^2))=2,$
$\dim(\ker(N^3))=3,$ $\dim(\ker(N^4))=4.$



$$N=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Q6.

~~Proof~~ Solution:

$P_3(\mathbb{R}) = \text{span}\{1, x, x^2, x^3\}$

$\dim(P_3(\mathbb{R})) = 4$

$T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ with $T(p(x)) = p''(x) + p(x)$

name $\beta = \{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$

$T(1) = 0 + 1 = 1$

$T(x) = 0 + x = x$

$T(x^2) = 2 + x^2$

$T(x^3) = 6x + x^3$

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So $\det(T - \lambda I) = (\lambda - 1)^4 = 0$

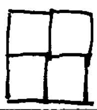
hence T has four equal value eigenvalues $\lambda = 1$.

To find Jordan canonical form of T and a canonical basis:

$$N = [T - I]_\beta = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad N^2 = 0$$

so $\dim(\ker(N)) = 2$, $\dim(\ker(N^2)) = 4$

so the tableau is



Hence JCF of N is $J_4(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

JCF of T is $J_4(1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Now we want to find a canonical basis α for T.

① first cycle $\alpha_1 = \{Nx_1, x_1\}$, $y_1 = Nx_1 \in Ker(N) \cap Im(N)$

since $Ker(N) = span\{(1,0,0,0), (0,1,0,0)\}$

$Im(N) = span\{(1,0,0,0), (0,1,0,0)\}$

so we set $y_1 = Nx_1 = \{(1,0,0,0)\}$

Now we need $Nx_1 = (1,0,0,0)$

so $x_1 = \cancel{(\frac{1}{2},0,0,0)}\ (0,0,\frac{1}{2},0)$

$\alpha_1 = \{(1,0,0,0), (0,0,\frac{1}{2},0)\}$

② second cycle, $\alpha_2 = \{Nx_2, x_2\}$, $y_2 = Nx_2 \in ker(N) \cap Im(N)$

set $y_2 = (0,1,0,0)$

need $Nx_2 = (0,1,0,0)$

so $x_2 = (0,0,0,\frac{1}{6})$

$\alpha_2 = \{(0,1,0,0), (0,0,0,\frac{1}{6})\}$

As $\alpha = \alpha_1 \cup \alpha_2$

We know that $\alpha = \{1, x, \frac{1}{2}x^2, \frac{1}{6}x^3\}$ and

$$[T]_{\alpha\alpha} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$