March 20th

Jordan Canonical form for nilpotent operator

$N: V \to V$ nilpotent

Then there exist $k_1 \geq k_2 \geq \cdots \geq k_r$ and a "canonical basis" α of V s.t.
$[N]_\alpha = J_{k_1} \oplus J_{k_2} \oplus \cdots \oplus J_{k_r}$

J_k = nilpotent Jordan matrix. e.g. $J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Moreover, $k_1 \geq k_2 \geq \cdots \geq k_r$ are uniquely determined by N.

whereas the basis α is not.

Today We consider $T: V \to V$ with one eigenvalue λ.

Given such T, the char. poly of T, $p(x) = (x-\lambda)^n$, $n = \dim V$

Cayley-Hamilton $\Rightarrow p(T) = (T - \lambda I)^n = 0$

i.e. $N = T - \lambda I$ is nilpotent.

So there is a canonical basis α, and $k_1 \geq \cdots \geq k_r$ s.t. $[T - \lambda I]_\alpha = J_{k_1} \oplus \cdots \oplus J_{k_r}$

$[T]_\alpha - [\lambda I]_\alpha = J_{k_1} \oplus \cdots \oplus J_{k_r}$
$[T]_\alpha - \lambda I_n = J_{k_1} \oplus \cdots \oplus J_{k_r}$

$[T]_\alpha = (J_{k_1} \oplus \cdots \oplus J_{k_r}) + \lambda I_n$

Def'n: The $n \times n$ Jordan matrix with eigenvalue λ is $J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$

E.g. $J_3(i) = \begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}$

$[T]_\alpha = J_{k_1} \oplus \cdots \oplus J_{k_r}$

This is the JCF of T.

Ex: $V = \text{sp}\{e^{2x}, xe^{2x}, x^2e^{2x}, x^3e^{2x}\}$

$\dim V = 4$ $\quad T: V \to V$, $T(f) = f' - f$
$\alpha = \{e^{2x}, \ldots, x^3e^{2x}\}$ basis of V.
$T(e^{2x}) = 2e^{2x} - e^{2x} = e^{2x}$

$T(xe^{2x}) = (e^{2x} + 2xe^{2x}) - xe^{2x} = e^{2x} + xe^{2x}$

$T(x^2e^{2x}) = (2xe^{2x} + 2x^2e^{2x}) - x^2e^{2x} = 2xe^{2x} + x^2e^{2x}$

$T(x^3e^{2x}) = 3x^2e^{2x} + 2x^3e^{2x} - x^3e^{2x} = 3x^2e^{2x} + x^3e^{2x}$

$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ So T has one eigenvalue $\lambda = 1$

To find JCF of T and a canonical basis ...

$N = [T - I]_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $N^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $N^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$N^4 = 0$ ...

$\left.\begin{array}{l} \dim \ker N = 1 \\ \dim \ker N^2 = 2 \\ \dim \ker N^3 = 3 \\ \dim \ker N^4 = 4 \end{array}\right\}$ tableau of this operator $\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }$

JCF of N is $J_4(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

JCF of T is $J_4(1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

We want to find a canonical basis α of T. It suffices to find one for N.

$\beta = \{N^3x, N^2x, Nx, x\}$. $y = N^3x$. $y \in \ker N \cap \operatorname{im} N^3$

$\operatorname{im} N^3 = sp\{e_1\} \Rightarrow \ker N \cap \operatorname{im} N^3 = sp\{e_1\}$, choose $y = \{e_1\}$

Solve. $N^3x = 6e_1 \Rightarrow x = e_4$

$\beta = \{6e_1, 6e_2, 3e_3, e_4\}$ canonical basis for N.

$T: V \to V$ $V = sp\{e^{2x}, \dots, x^3e^{2x}\}$

Canonical basis for T is $\{6e^{2x}, 6xe^{2x}, 3x^2e^{2x}, x^3e^{2x}\}$ with respect to this basis T has matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

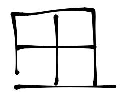
Ex: $T: \mathbb{C}^4 \to \mathbb{C}^4$

$$T = \begin{bmatrix} -2 & -8 & -3 & 8 \\ 0 & 3 & 2 & -5 \\ 0 & 0 & -2 & 0 \\ 0 & 5 & 2 & -7 \end{bmatrix} \qquad p(x) = (x+2)^4$$

$$\lambda = -2$$

$$N = T + 2I = \begin{bmatrix} 0 & -8 & -3 & 8 \\ 0 & 5 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & -5 \end{bmatrix}. \qquad N^2 = 0$$

$\dim\ker N = 2$
$\dim\ker N^2 = 4$



JCF of $T = J_2(-2) \oplus J_2(-2)$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

To find canonical basis: $\alpha_1 = \{Nx_1, x_1\}$
$\alpha_2 = \{Nx_2, x_2\}$

$y = Nx_1$, $y \in \ker N \cap \operatorname{im} N$

$$\operatorname{im} N = sp\left\{ \begin{bmatrix} -8 \\ 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$\ker N = sp\{e_1, e_2 + e_4\}$

$\operatorname{im} N \subset \ker N \Rightarrow \operatorname{im} N = N$ since they're both 2-dimensional

$$y = \begin{bmatrix} -8 \\ 5 \\ 0 \\ 5 \end{bmatrix}. \quad x_1 = e_2 \quad \text{so } \alpha_1 = \left\{ \begin{bmatrix} -8 \\ 5 \\ 0 \\ 5 \end{bmatrix}, e_2 \right\}, \quad \alpha_2 \cdots$$