TUTORIAL 7

(1) Let A and B be events in a finite probability space.
  (a) Prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  (b) Prove that if $P(A) > 1/2$ and $P(B) > 1/2$, then $P(A \cap B) > 0$.
  (c) What is the probability that, upon rolling two dice, you get doubles or a number divisible by 4.
  (d) You roll two dice, but you cannot see the result. Your friend sees them and notes that your total is divisible by 4. What is the probability that you have rolled doubles?

(2) In a famous game show on television, a prize is placed behind one of three doors, with probability $1/3$ for each. The contestant chooses a door. The host then opens one of the doors and says "As you can see, the prize is not behind this door. Do you want to stay with your original guess or switch to the remaining door?" When the contestant has chosen the wrong door, the host opens the other wrong door. When the contestant has chosen the right door, the host opens one of the two wrong doors, each with probability $1/2$.
  (a) Should the contestant should switch doors?
  (b) Can you construct a probability space corresponding to the problem, and explain your answer in terms of conditional probability?

(3) Suppose you have a biased coin, which shows heads with probability $2/3$ and tails with probability $1/3$. You flip the coin four times. Compute the probability of getting i heads, for each $i \in \{0, 1, 2, 3, 4\}$. Verify that the probabilities sum to one.

(4) Determine whether the following relations are equivalence relations.
  (a) $S = \mathbb{N} - \{1\}$; $(x, y) \in R$ if and only if $\gcd(x, y) > 1$.
  (b) $S = \mathbb{R}$; $(x, y) \in R$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$.

Just for fun.

(1) Let $f : [n] \to [n]$ be a permutation, and let f^k be the composition of f with itself k times. Prove that there is some $k \in \mathbb{N}$ such that $f^k = id$. (Hint: Consider the functional digraph of the permutation.) The minimum such k is called the order of the permutation.

(2) Let S_n denote the set of all nondecreasing lists of natural numbers summing to n, e.g. $S_4 = \{1111, 112, 13, 22\}$. Show that the coin game from the second week of class determines a function $f : S_n \to S_n$. Draw the functional digraph of the function when $n = 5$, $n = 6$. What can you say about the long-term behavior of the penny game in these two cases Determine all values of n such that f is injective. Determine all values of n such that f is surjective.