7.) $X \sim$ geometric(p) ; Sample of n (iid)

$$P(X=k) = p(1-p)^{k-1}$$

a.) MoM: From the table:

$$E(X) = \frac{1}{p} \quad \therefore \quad \frac{1}{p} = \bar{X}$$

$$\Rightarrow \tilde{p} = \frac{1}{\bar{X}}$$

b.) $L(p \mid \underset{\sim}{x}) = \prod_{i=1}^{n} p(1-p)^{k_i} = p^n (1-p)^{\sum k_i - n}$

$$\ell(p \mid \underset{\sim}{x}) = n \log(p) + (\sum k_i - n) \log(1-p)$$

$$\frac{\partial \ell}{\partial p} = \frac{n}{p} - \frac{\sum k_i - n}{(1-p)} = 0$$

Note: you should check the second derivative. $\ell''(p) < 0$

$$\Rightarrow \frac{n}{p} = \frac{\sum k_i - n}{(1-p)}$$

$$\Rightarrow \hat{p} = \frac{1}{\bar{X}}$$

c.) The asymptotic variance:

$$V(\hat{p}) \approx I(\hat{p})^{-1}$$

$$I(p) = -E(\ell''(p))$$

$\uparrow$

Fisher Information

$$\ell''(p) = -\frac{n}{p^2} - \frac{\Sigma k_i - n}{(1-p)^2}$$

$$\Rightarrow \quad I(p) = -E\left(-\frac{n}{p^2} - \frac{\Sigma k_i - n}{(1-p)^2}\right)$$

$$= \frac{n}{p^2} + \frac{1}{(1-p)^2}\left[E(\Sigma k_i) - n\right]$$

$$= \frac{n}{p^2} + \frac{1}{(1-p)^2}\left[n\left(\frac{1}{p}\right) - n\right]$$

$$= \frac{n}{p^2} + \frac{\frac{(n-np)}{p}}{(1-p)^2} = \frac{n}{p^2} + \frac{(n-np)}{(1-p)^2 p}$$

$$= \frac{n(1-p)^2 + (n-np)p}{p^2(1-p)^2}$$

$$= \frac{n(1-p)\left[(1-p)+p\right]}{p^2(1-p)^2}$$

$$= \frac{n}{p^2(1-p)}$$

$$\therefore \; V(\hat{p}) \approx \frac{\hat{p}^2(1-\hat{p})}{n}$$

d.) $P(p \mid \underset{\sim}{k}) \propto P(k \mid p)\, P(p)$

$$= p^{n} (1-p)^{\Sigma k_i - n} \; \underline{1}$$

Let's rewrite this a bit:

$$= p^{(n+1)-1} (1-p)^{(\Sigma k_i - n + 1) - 1}$$

This is a kernel for a beta distribution:

$[p \mid \underset{\sim}{k}] \sim \text{beta}(a, b)$; $a = (n+1)$

$b = (\Sigma k_i - n + 1)$

$$P(p \mid \underset{\sim}{k}) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1} (1-p)^{b-1}$$

19.) $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

a.) μ is known, σ is unknown.

$$L(\sigma^2 \mid \underset{\sim}{x}) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2\right)$$

$$\ell(\sigma^2 \mid \underset{\sim}{x}) = -n/2 \log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$= -n/2 \log(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2 - n/2 \log(2\pi)$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2[\sigma^2]^2}\Sigma(x_i-\mu)^2 = 0$$

$$\frac{1}{2[\sigma^2]^2}\Sigma(x_i-\mu)^2 = \frac{n}{2\sigma^2}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\Sigma(x_i-\mu)^2}{n}$$

$\therefore$ By invariance property of MLEs

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{\Sigma(x_i-\mu)^2}{n}}$$

Note: You should check the second derivative. $\ell''(\sigma^2) < 0$.

b.) σ is known, μ is unknown

$$L(\mu \mid \underset{\sim}{x}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right)$$

$$\ell(\mu \mid \underset{\sim}{x}) = -n/2 \log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum(x_i^2 - 2x_i\mu + \mu^2)\right]$$

$$= -n/2 \log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum x_i^2 - 2\mu\sum x_i + n\mu^2\right]$$

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{2\sigma^2}2\sum x_i - \frac{1}{2\sigma^2}2n\mu = 0$$

$$\sum x_i = n\mu \Rightarrow \hat{\mu} = \overline{X}$$

Note: You should check the second derivative. $\ell''(\mu) < 0$

c.) $V(\overline{X}) = \sigma^2/n$.

Let's determine the asymptotic variance:

$$\frac{\partial^2 \ell}{\partial \mu^2} = -\frac{2n}{2\sigma^2} = -\frac{n}{\sigma^2}$$

$$I(\hat{\mu}) = -E(\ell''(\mu)) = n/\sigma^2$$

$$V(\hat{\mu}) \approx I(\hat{\mu})^{-1} = \sigma^2/n \quad \text{(Same result)}$$

Note: The $\text{CRLB}(\hat{\mu}) = \dfrac{\left[\frac{\partial}{\partial\mu}\mu\right]^2}{I(\hat{\mu})} = \dfrac{1^2}{n/\sigma^2} = \sigma^2/n.$

Also: $E(\overline{X}) = \mu$.

$\therefore$ We have an unbiased estimator which achieves the CRLB. Thus their is no estimator with a smaller variance.

47.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x) = \theta x_0^{\theta} x^{-\theta-1}$

$$x \geq x_0 \,;\, \theta > 1$$

a.) $$E(X) = \int_{x_0}^{\infty} x \, \theta x_0^{\theta} x^{-\theta-1} \, dx$$

$$= \theta x_0^{\theta} \int_{x_0}^{\infty} x^{-\theta} \, dx$$

Note: $1-\theta < \theta$

$$= \theta x_0^{\theta} \left[\frac{x^{1-\theta}}{1-\theta} \right]_{x_0}^{\infty} = \theta x_0^{\theta} \left[\frac{\infty^{1-\theta}}{1-\theta} - \frac{x_0^{1-\theta}}{1-\theta} \right]$$

$$= \theta x_0^{\theta} \left[0 - \frac{x_0^{1-\theta}}{1-\theta} \right]$$

$$= \theta x_0^{\theta} x_0^{1-\theta} \left(-\frac{1}{1-\theta} \right)$$

$$= \frac{\theta x_0}{\theta - 1}$$

$\therefore$ MoM $\Rightarrow$ $$E(X) = \bar{X}$$

$$\frac{\theta x_0}{\theta - 1} = \bar{X}$$

$$\Rightarrow \tilde{\theta} = \frac{\bar{X}}{\bar{X} - x_0}$$

b.) $$L(\theta) = \prod_{i=1}^{n} \theta x_0^{\theta} x_i^{-\theta-1}$$

$$= \theta^n x_0^{n\theta} \prod_{i=1}^{n} x_i^{-\theta-1}$$

$$\ell(\theta) = n\log(\theta) + n\theta\log(x_0) - (\theta+1)\sum\log(x_i)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + n\log(x_0) - \sum\log(x_i) = 0$$

$$\frac{n}{\theta} = \sum\log(x_i) - n\log(x_0)$$

$$\Rightarrow \hat{\theta} = \frac{n}{\sum\log(x_i) - n\log(x_0)}$$

Note: You should check that the second derivative $\ell''(\theta) < 0$.

c.) $$V(\hat{\theta}) \approx I(\hat{\theta})^{-1} \Rightarrow -E(\ell''(\theta)) = I(\theta)$$

$$\therefore \frac{\partial \ell^2}{\partial \theta^2} = -\frac{n}{\theta^2} \Rightarrow -E(-n/\theta^2) = n/\theta^2$$

$$\therefore V(\hat{\theta}) \approx \frac{\hat{\theta}^2}{n}$$

d.) Let's use the factorization theorem to find a sufficient statistic;

$$f(x \mid \theta) = \theta^n x_0^{n\theta} \prod_{i=1}^{n} x_i^{-(\theta+1)} = \theta^n x_0^{n\theta} [t]^{-(\theta+1)}$$

$$t = \prod_{i=1}^{n} x_i \; ; \quad g(t \mid \theta) = \theta^n x_0^{n\theta} t^{-(\theta+1)}$$

$$h(x) = 1$$

$$\therefore f(x \mid \theta) = g(t \mid \theta)\, h(x)$$

50.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x \mid \theta) = \frac{x}{\theta^2} \exp\left(-x^2/2\theta^2\right) ; \; x \geq 0$

a.) MoM: $E(X) = \int_0^\infty x \, \frac{x}{\theta^2} \exp\left(-x^2/2\theta^2\right) dx$

$$= \int_0^\infty \frac{x^2}{\theta^2} \exp\left(-x^2/2\theta^2\right) dx$$

Let $y = x^2 \Rightarrow x = y^{1/2}$

$$\frac{dx}{dy} = \tfrac{1}{2} y^{-1/2} \Rightarrow dx = \tfrac{1}{2} y^{-1/2} dy$$

Let $\beta = 2\theta^2$

$$= \frac{1}{\theta^2} \int_0^\infty y \exp\left(-y/\beta\right) \left(\tfrac{1}{2}\right) y^{-1/2} \, dy$$

$$= \frac{1}{2\theta^2} \int_0^\infty y^{3/2 - 1} \exp\left(-y/\beta\right) dy$$

$$= \frac{1}{2\theta^2} \Gamma(3/2) \beta^{3/2} \underbrace{\int_0^\infty \frac{1}{\Gamma(3/2) \beta^{3/2}} y^{3/2-1} \exp\left(-y/\beta\right) dy}_{= 1}$$

$\uparrow$ gamma$(a = 3/2, \beta)$

$$= \frac{\Gamma(3/2) \left(2\theta^2\right)^{3/2}}{2\theta^2} = \theta \sqrt{2} \, \Gamma(3/2)$$

$$E(X) = \bar{X} \Rightarrow \tilde{\Theta} = \frac{\bar{X}}{\sqrt{2}\,\Gamma(3/2)}$$

b.) $$L(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} \exp\left(-x_i^2/2\theta^2\right)$$

$$= \frac{1}{\theta^{2n}} \left[\prod_{i=1}^{n} x_i\right] \exp\left(-\Sigma x_i^2/2\theta^2\right)$$

$$\ell(\theta) = -2n\log(\theta) + \Sigma\log(x_i) - \Sigma x_i^2/2\theta^2$$

$$\frac{\partial \ell}{\partial \theta} = \frac{-2n}{\theta} + \frac{2\Sigma x_i^2}{2\theta^3} = 0$$

Note: You should check the second derivative.

$$2n\theta^2 = \Sigma x_i^2$$

$$\hat{\Theta} = \sqrt{\frac{\Sigma x_i^2}{2n}}$$

c.) $V(\hat{\theta}) \approx \dfrac{\left[\frac{\partial}{\partial\theta}\theta\right]^2}{I(\theta)} = \dfrac{1}{I(\theta)}$

$I(\theta) = -E[\ell''(\theta)]$

$\Rightarrow \ell''(\theta) = \dfrac{2n}{\theta^2} - 3\dfrac{\Sigma x_i^2}{\theta^4}$

$-E(\ell''(\theta)) = -\dfrac{2n}{\theta^2} + \dfrac{3}{\theta^4} E(\Sigma x_i^2)$

$= -\dfrac{2n}{\theta^2} + \dfrac{3}{\theta^4} n E(x_i^2)$

$\Rightarrow E(x^2) = Var(x) + [E(x)]^2$

$\Rightarrow$ Raleigh is not in the table

So let's Calculate directly.

$E(x^2) = \int_0^\infty x^2 \dfrac{x}{\theta^2} \exp(-x^2/2\theta^2)\, dx$

$= \frac{1}{\theta^2}\int_0^\infty x^3 \exp(-x^2/2\theta^2)\, dx$

Let $y = x^2 \Rightarrow x = y^{1/2}$

$dx = 1/2\, y^{-1/2} dy$

$= \dfrac{1}{2\theta^2} \int y\, y^{1/2} \exp(-y/_{13})\, y^{-1/2} dy$

$$= \frac{1}{2\theta^2} \int_0^{\infty} y^{2-1} \exp(-y/\beta)\, dy$$

$$= \frac{1}{2\theta^2} \Gamma(2) \beta^2 \underbrace{\int_0^{\infty} \frac{1}{\Gamma(2)\beta^2} y^{2-1} \exp(-y/\beta)\, dy}_{=1}$$

$$= 2\theta^2$$

$$\therefore -E(\ell''(\theta)) = \frac{-2n}{\theta^2} + \frac{3n}{\theta^4} 2\theta^2$$

$$= \frac{4n}{\theta^2}$$

$$\therefore V(\hat{\theta}) \approx \frac{\theta^2}{4n}$$