Lecture 4

Series

A series is an infinite sum of complex numbers.

$$\sum_{n=k}^{\infty} z_n = z_k + z_{k+1} + \cdots$$

Ex: ① $1+z+z^2+z^3+\cdots = \sum_{n=0}^{\infty} z^n$

② $\sum_{n=1}^{\infty}\left(\frac{i^n}{n^2}\right) = \frac{i}{1}+\frac{i^2}{4}+\frac{i^3}{9}+\cdots$

Given a series $\sum_{n=0}^{\infty} z_n$ we define the kth partial to be:

$$S_k = z_0 + z_1 + \cdots + z_k$$

The series converges (cvgs) to L if $\lim_{k\to\infty} S_k = L$

If we write $z_n = x_n + iy_n$ & $\sum z_n \to L = x+iy$
then $\sum x_n \to x$
$\sum y_n \to y$

Moreover: If $\sum x_n \to x$ & $\sum y_n \to y$, then $\sum z_n \to L = x+iy$

Why does this help?
- We already know convergence tests for real series.

Ex: ① $\sum_{n=1}^{\infty} \frac{1+i}{n} = \sum_{n=1}^{\infty}\frac{1}{n} + i\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)$ as $z_n = \frac{1+i}{n} = \underbrace{\frac{1}{n}}_{x_n} + \underbrace{\frac{1}{n}}_{y_n} i$

since $\sum \frac{1}{n}$ diverges (harmonic), so the original series diverges.

② $\sum \frac{1+i}{n^2} = \sum \frac{1}{n^2} + i\left(\sum \frac{1}{n^2}\right)$ cvgs by p-test.

Triangle Inequality:

$$\text{If } z,w \in \mathbb{C} \text{ then } |z+w| \le |z| + |w|$$

$$\Rightarrow \left|\sum z_n\right| \le \sum |z_n|$$

So we get that if $\sum |z_n|$ converges then $\sum z_n$ converges.

Ex: Does $\sum_{n=0}^{\infty} \frac{(2+3i)^n}{n!}$ converge?

write as $\sum z_n$

Ratio Test: $\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} = \lim \left|\frac{(2+3i)}{(n+1)}\right| = 0 < 1$ so converges.

Exponential Functions

Def'n: For any $z \in \mathbb{C}$, $(z = x+iy)$, define $e^z = e^{x+iy} = e^x \cdot e^{iy}$
$= e^x(\cos y + i\sin y)$

This defines a function $f(z) = e^z$ whose domain is $\mathbb{C}$.

Properties:

① $e^{z+w} = e^z \cdot e^w$

② $|e^z| = |e^x \cdot e^{iy}| = |e^x| \cdot |e^{iy}| = |e^x| \cdot 1$ $\qquad |\cos y + i\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$

so $|e^z| = e^{\operatorname{Re} z} = e^x$

③ e^z is continuous (because $u = e^x\cos y$, $v = e^x \sin y$ are cts functions of x, y.)

④ If $w = e^z$, then $w = e^{z+2\pi i n} = e^{x+iy+2\pi i n}$

$= e^x e^{iy+2\pi i n}$

$= e^x \cdot e^{i(y+2\pi n)}$

$= e^x \cdot e^{iy}$

(NEW THING comparing with $\mathbb{R}$)

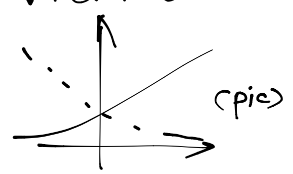
The function is not 1-1, there are infinitely many complex #s that can be mapped to the same number by the function.

i.e. $w = e^z$ has ∞ many solutions. $(w \neq 0)$

⑤ $e^z \neq 0$ for any $z \in \mathbb{C}$.

$e^z = e^x \cdot e^{iy}$ ← a point on the unit circle, also never 0.

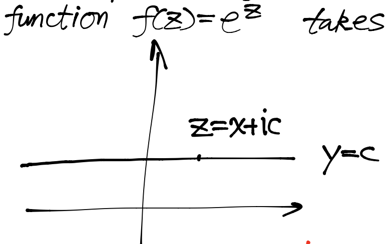
↓ never 0



(pic)

Visualize exponential function:

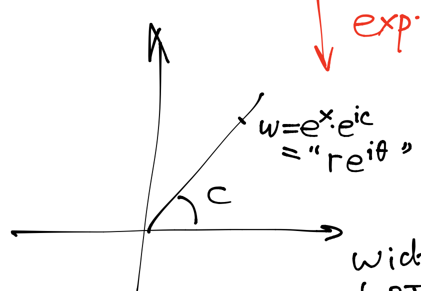
The function $f(z) = e^z$ takes a horizontal line to a ray extending from 0.

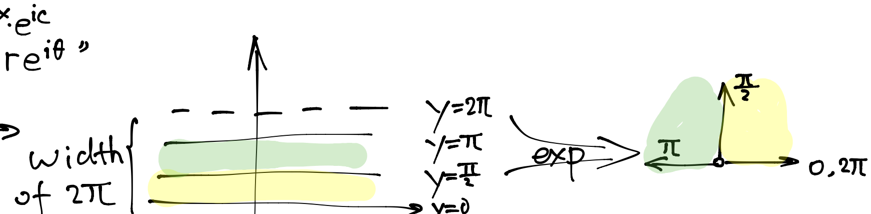


$e^z = e^{x+ic} = e^x \cdot e^{ic}$

output comes out naturally as a polar form

↓ exp· takes a horizontal line to a ray!





The set $\{z = x+iy \mid 0 \leq y < 2\pi\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} \setminus \{0\}$

The same is true for any strip $\{z=x+iy \mid c\leq y<c+2\pi\}$ (as long as the width is 2π)