UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2006

Math 240H1 Algebra I — Final Exam

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Solve all of the following 5 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Good Luck!

Problem 1. Prove the "replacement lemma": Let G be a set of g vectors in some vector space V and let L be some set of l linearly independent vectors in V (where g and l are both finite). Assume that $\operatorname{Span} L \subset \operatorname{Span} G$. Then $g \geq l$ and there is a subset R of G, consisting of $r := g - l$ vectors, so that $\operatorname{Span}(R \cup L) = \operatorname{Span} G$.

Problem 2.

1. Let $L : P_3(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation given by $L(p) = \begin{pmatrix} p(-2) \\ p(0) \\ p(2) \end{pmatrix}$. Find the matrix A representing L relative to the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and the standard basis of $\mathbb{R}^3$.

2. Let $w = a + bi$ be a complex number and let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = w \cdot z$. Considering $\mathbb{C}$ as a vector space over $\mathbb{R}$, find the matrix B representing T relative to the basis $\{1, i\}$ of $\mathbb{C}$.

Problem 3. Find all the solutions (if any exist) of the following two systems of linear equations:

$$\begin{pmatrix} -1 & 2 & -2 & -7 \\ -1 & 2 & 2 & 1 \\ -2 & 4 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 & 2 & -2 & -7 \\ -1 & 2 & 2 & 1 \\ -2 & 4 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -6 \end{pmatrix}.$$

Problem 4. Let $A \in M_{n \times n}(F)$ have the form

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

1. Compute $\det(A + tI)$, where I is the $n \times n$ identity matrix.

2. (3 point bonus). What does your result tell you about characteristic polynomials?

Problem 5. Let A be the matrix $A = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

1. Find a matrix P for which $P^{-1}AP$ is diagonal.

2. Compute A^7.

Good Luck!