

stat7017-hw1

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STAT7017 Homework 1

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Q1

5 **Proof:** This is proved by mathematical induction.

Basis

By definition, we know that

1. $|\mathbf{A}| = a_{11}$ when $p = 1$.
2. $|\mathbf{A}| = \sum_{j=1}^p a_{1j} |\mathbf{A}_{1j}| (-1)^{1+j}$ if $p > 1$ where \mathbf{A}_{1j} is the $(p-1) \times (p-1)$ matrix with the first row and j -th column deleted.

Induction hypothesis

Suppose we have $p = 1$, then our claim that the determinant is equation to the product of diagonal entry is automatically true.

Induction Step

Suppose again it is true that when $p = n$, i.e. $|\mathbf{A}_{n \times n}| = \prod_{j=1}^n a_{jj}$.

We want to show it is also true when $p = n + 1$, i.e. $|\mathbf{A}_{(n+1) \times (n+1)}| = \prod_{j=1}^{n+1} a_{jj}$.

$$\begin{aligned} |\mathbf{A}_{(n+1) \times (n+1)}| &= \sum_{j=1}^{n+1} a_{1j} |\mathbf{A}_{1j}| (-1)^{1+j} \\ &= a_{11} |\mathbf{A}_{-11}| (-1)^{1+1} + a_{12} |\mathbf{A}_{-12}| (-1)^{1+2} + \cdots + a_{1(n+1)} |\mathbf{A}_{-1(n+1)}| (-1)^{1+n+1} \\ &= a_{11} \prod_{j=2}^{n+1} a_{jj} + 0 + \cdots + 0 \\ &= \prod_{j=1}^{n+1} a_{jj} \end{aligned}$$

\therefore The determinant of diagonal matrix $\det(\mathbf{A})$ is the product of diagonal elements.

Q2

5

Proof: Suppose $\lambda_1, \dots, \lambda_p$ are eigenvalues of a $p \times p$ matrix \mathbf{A} . The characteristic polynomial of \mathbf{A} is

$$p(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| = \lambda^p + (-1) \cdot a_1 \lambda^{p-1} + \cdots + (-1)^{p-1} \cdot a_{p-1} \lambda + (-1)^p \cdot a_p$$

where a_1 is the trace of \mathbf{A} , a_p is $\det(\mathbf{A})$, a_i is the sum of i -rowed diagonal mirrors of \mathbf{A} .

Since $\lambda_1, \dots, \lambda_p$ are zeros of $p(\lambda)$,

$$p(\lambda) = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdots (\lambda - \lambda_p)$$

Now we focus on the constant term $(-1)^p \cdot a_p$. We can calculate it by inserting $\lambda = 0$ in both expressions:

- $|\mathbf{A} - 0 \cdot \mathbf{I}| = |\mathbf{A}| = a_p$
- $(-\lambda_1) \cdot (-\lambda_2) \cdots (-\lambda_p) = (-1)^p \prod_{i=1}^p \lambda_i = (-1)^p \cdot a_p$

Hence

$$\begin{aligned} (-1)^p \prod_{i=1}^p \lambda_i &= (-1)^p \cdot |\mathbf{A}| \\ |\mathbf{A}| &= \lambda_1 \cdot \lambda_2 \cdots \lambda_p \end{aligned}$$

Q3

Solution:

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Under the default setup, where $p = 100$, we generate $n = 500$ random normally distributed symmetric $p \times p$ matrices. The eigenvalues are plotted with grey histogram below, and the blue shaded curve indicates its density.

The density function of Wigner semicircle distribution can be written as:

$$f(x) = \frac{2}{\pi R^2} \sqrt{R^2 - x^2}$$

where the parameter R is the interval of possible x value with $x \in [-R, R]$. In our case, we take an upper bound of the maximal absolute values of eigenvalues as R . Specifically, $R = \lceil \max(\lambda_i) \rceil + 0.1$.

$$R = (4p)^{0.5}$$

The fitted Wigner semicircle distribution is plotted with a black curve.

```
library(ggplot2)
library(VGAM)

set.seed(7017)

wignerplot <- function(p, flag="N") {
  n <- p * 5

  matrices <- list()
  for (i in 1:n) {

    if (flag=="N") {
      A <- matrix(rnorm(p^2, 0, 1), p, p)
    } else if (flag=="T") {
      A <- matrix(rt(p^2, 10), p, p)
    } else if (flag=="inf") {
      # Pareto distribution with shape parameter < 2 has infinite variance
      A <- matrix(rpareto(p^2, shape=1.5), p, p)
    }

    A[lower.tri(A)] <- t(A)[lower.tri(A)]
    matrices[[i]] <- A
  }

  evs <- c()
  for (j in matrices) {
    evs <- c(evs, eigen(j)$values)
  }
}
```

```

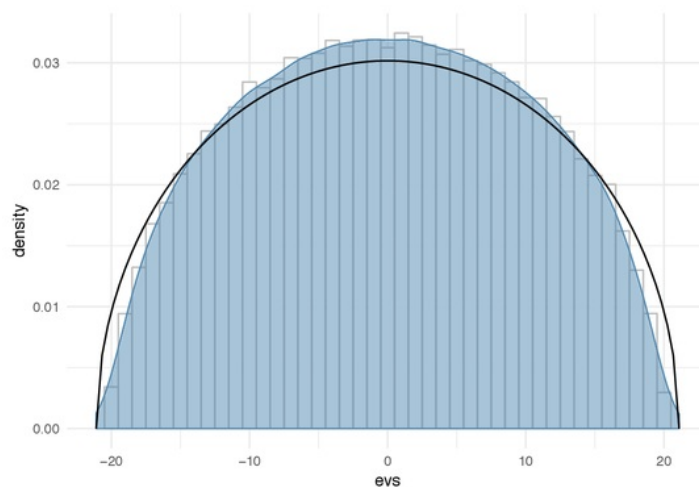
df <- data.frame(seq_along(evs), evs)

max(evs)
min(evs)
bound <- round(max(evs))+0.1

g1 <- ggplot(df, aes(evs)) +
  geom_histogram(aes(y=..density..), binwidth=1, colour="grey", fill="white") +
  stat_density(geom="area", alpha=0.5, fill='#5289B1', color="#5289B1") +
  stat_function(fun = function(x) 2/(pi*(bound)^2)*sqrt((bound)^2-x^2)) +
  xlim(-bound,bound) +
  theme_minimal()
return(g1)
}

wignerplot(100,flag="N")

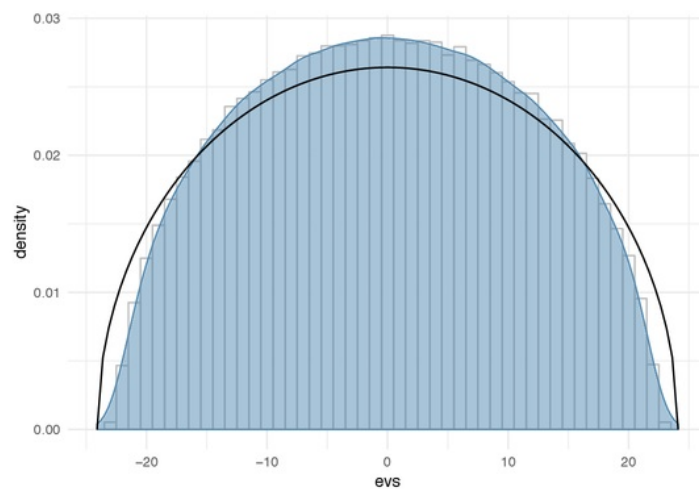
```



If we keep the n/p ratio constant (5 in this case), then the corresponding plots with $p = 125, 150, 200$ are plotted below:

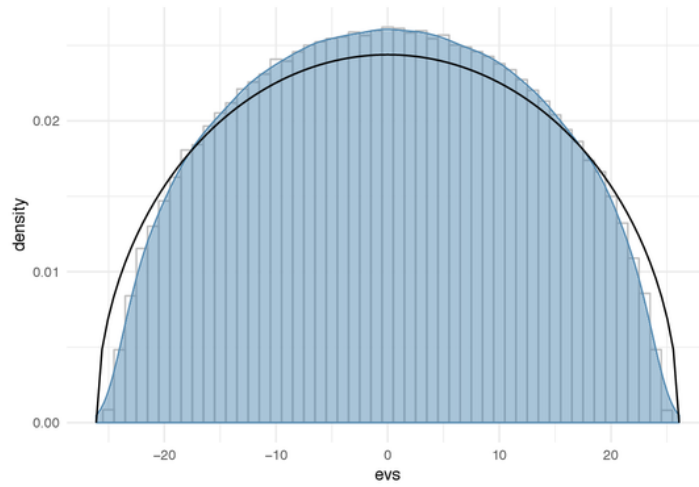
$p = 125$

```
wignerplot(125)
```



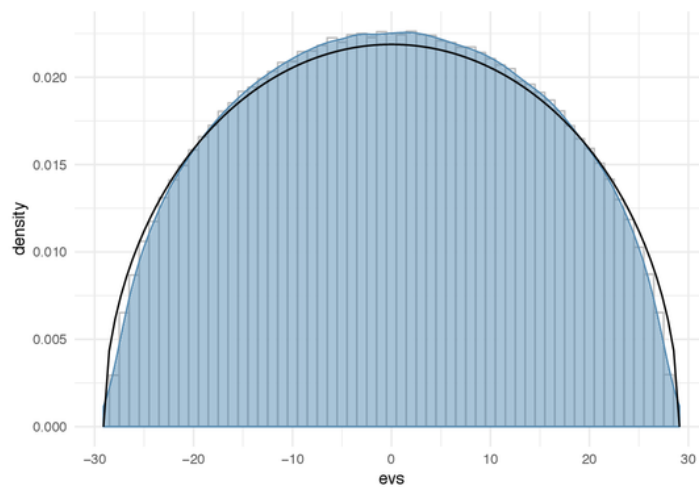
$p = 150$

```
wignerplot(150)
```



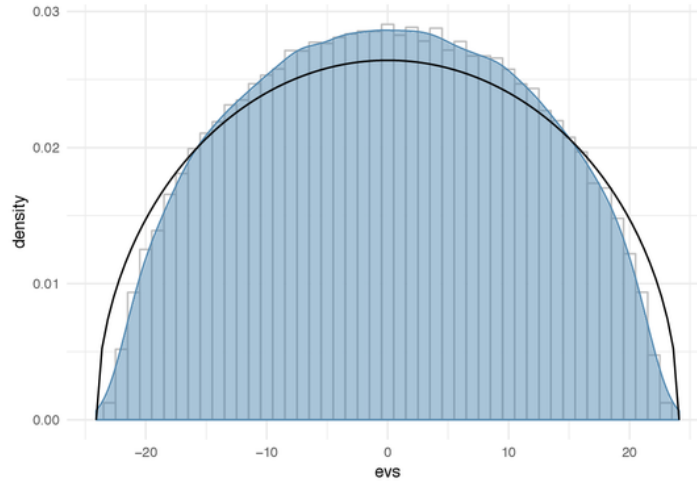
$p = 200$

```
wignerplot(200)
```

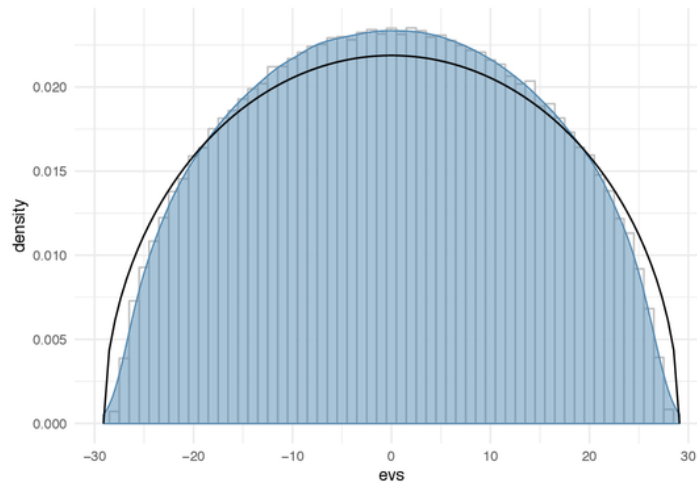


If we let the random matrices follow a Student's T distribution (let $\text{df} = 10$ though not specified) instead of a standard normal, with $p = 100, n = 500$, we have:

```
wignerplot(100,flag="T")
```



```
wignerplot(150,flag="T")
```



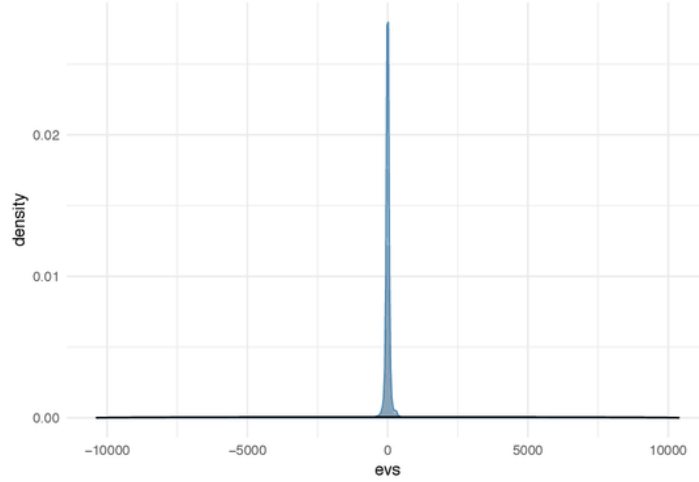
Some observations from Q3:

- Generally, we have shown the universality conjecture that the spectral density of large random symmetric matrices following a zero mean, finite variance converges to the density of the Wigner Semicircle distribution.
- There are some gaps between the observed density and the theoretical one. However, when n and p get larger (while holding the p/n ratio constant), the gaps seem to be more and more trivial.
- Also, since the scale of y-axes vary in the plots above, one fact could be masked that the “semicircle” are in fact flatter when n and p get larger.
- The last part shows that when sample size is large enough, the Student’s T distribution and standard normal distribution have similar effect in this scenario.

Q4

5 **Solution:** We need to find a distribution with finite mean but infinite variance. A Pareto distribution with shape parameter $\alpha \leq 2$ has infinite variance. When the distribution is settled, we repeat for $n = 500, p = 100$ setup. But this time, the semicircle law fails.

```
wignerplot(100,flag="inf")
```



References

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2. “Pareto distribution”, *Wikipedia*, 2018. [Online]. Available: https://en.wikipedia.org/wiki/Pareto_distribution. [Accessed: 04-Aug-2018].
3. “How can a distribution have infinite mean and variance?,” *Cross Validated*. [Online]. Available: <https://stats.stackexchange.com/questions/91512/how-can-a-distribution-have-infinite-mean-and-variance/91515>. [Accessed: 04-Aug-2018].
4. “How can a distribution have infinite mean and variance?,” *Cross Validated*. [Online]. Available: <https://stats.stackexchange.com/questions/91512/how-can-a-distribution-have-infinite-mean-and-variance/91515>. [Accessed: 04-Aug-2018].
5. A. Charpentier, “On Wigner’s law (and the semi-circle),” *Freakonometrics*, 16-Dec-2013. [Online]. Available: <https://freakonometrics.hypotheses.org/10964>. [Accessed: 04-Aug-2018].
6. F. Benaych-Georges and A. Knowles, “Lectures on the local semicircle law for Wigner matrices”, arXiv:1601.04055v3 [math.PR], 18 Oct. 2016.
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FINAL GRADE

GENERAL COMMENTS

Instructor

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