# Evolution of differentiability

**in $\mathbb{R}$**: $f:\mathbb{R}\to\mathbb{R}$ is differentiable at $a\in\mathbb{R}$ if

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists (say } = m \text{ for some } m)$$

Such $m$ is denoted by $f'(a)$

also tangent line to graph of $f$ at $a$.

This means: $f$ is differentiable at $a$ if $\exists m\in\mathbb{R}$ s.t.

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}-m=0 \quad \text{or} \quad \lim_{h\to 0}\frac{f(a+h)-f(a)-mh}{h}=0$$

$$\text{or} \quad \lim_{h\to 0}\frac{f(a+h)-(f(a)+mh)}{h}=0$$

$f(a)+mh$ is linear approximation of $f(a+h)$, and $mh$ is differential of $f$, at $a$, evaluated at $h$, denoted by $df_a(h)$

So the idea of differentiability is That the error of approximating $f(a+h)$ With its linear approximation is of order $h^2$, That is $E(h)=f(a+h)-(f(a)+mh)$ &

error of approximat.

$$\lim_{h\to 0}\frac{E(h)}{h}=0$$

This means $E(h)$ is $O(h^2)$

extending differentiability to $\mathbb{R}^n$

$f:\mathbb{R}^n\to\mathbb{R}$ is said to be differentiable at $a\in\mathbb{R}^n$

if $\exists\, \mathbb{c}\in\mathbb{R}^n$ s.t. $\displaystyle\lim_{\mathbb{h}\to\mathbb{0}}\frac{f(a+\mathbb{h})-f(a)-\mathbb{c}^T\mathbb{h}}{|\mathbb{h}|}=0$

numerator is in $\mathbb{R}$

Such a $\mathbb{c}$ will be denoted by $\nabla f(a)$ gradient of $f$

$$\lim_{\mathbb{h}\to\mathbb{0}}|f(a+\mathbb{h})-f(a)|=0$$

$f$ is Cont.

if $f$ is differentiable

$$\nabla f(a)=\begin{bmatrix}\partial_1 f(a)\\ \vdots \\ \partial_n f(a)\end{bmatrix}$$

$f(a)+\mathbb{c}^T\mathbb{h}$ is linear approximation of $f(a+\mathbb{h})$, and $\mathbb{c}^T\mathbb{h}$ or $\nabla f(a)\cdot\mathbb{h}$ is differential of $f$ at $a$ evaluated at $\mathbb{h}$ $df_a(\mathbb{h})$

or tangent hyperplane

it is possible That $\nabla f(a)$ does not exist but all $\partial_i f(a)$ exist.

But if all $\partial_i f$ exist and are Continuous Then

$f$ is said to be $C^1$: $f$ is $C^1 \Rightarrow f$ is differentiable $\Rightarrow \partial_i f$ exist.