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MAT224 Problem Set I. 20/20

1. Solution:

$+$	$[0]_3$	$[1]_3$	$[2]_3$
$[0]_3$	$[0]_3$	$[1]_3$	$[2]_3$
$[1]_3$	$[1]_3$	$[2]_3$	$[0]_3$
$[2]_3$	$[2]_3$	$[0]_3$	$[1]_3$

$\cdot$	$[0]_3$	$[1]_3$	$[2]_3$
$[0]_3$	$[0]_3$	$[0]_3$	$[0]_3$
$[1]_3$	$[0]_3$	$[1]_3$	$[2]_3$
$[2]_3$	$[0]_3$	$[2]_3$	$[1]_3$

$+$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[0]_5$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[1]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$	$[0]_5$
$[2]_5$	$[2]_5$	$[3]_5$	$[4]_5$	$[0]_5$	$[1]_5$
$[3]_5$	$[3]_5$	$[4]_5$	$[0]_5$	$[1]_5$	$[2]_5$
$[4]_5$	$[4]_5$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$

$\cdot$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[0]_5$	$[0]_5$	$[0]_5$	$[0]_5$	$[0]_5$	$[0]_5$
$[1]_5$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[2]_5$	$[0]_5$	$[2]_5$	$[4]_5$	$[1]_5$	$[3]_5$
$[3]_5$	$[0]_5$	$[3]_5$	$[1]_5$	$[4]_5$	$[2]_5$
$[4]_5$	$[0]_5$	$[4]_5$	$[3]_5$	$[2]_5$	$[1]_5$

2.(a). Solution: Suppose $(1,1,1,1) \in S$, $a_1, a_2 \in \mathbb{Z}_3$

Then $a_1(1,2,0,1) + a_2(2,0,1,2) = (1,1,1,1)$

So $\begin{cases} a_1 + 2a_2 = 1 \\ 2a_1 = 1 \Rightarrow a_1 = 2 \\ a_2 = 1 \\ a_1 + 2a_2 = 1 \end{cases}$

Take $a_1 = 2$ $a_2 = 1$ into $a_1 + 2a_2 = 2 + 2 = 1$ verified.

So $(1,1,1,1)$ is in S.

Similarly suppose $(1,0,1,1) \in S$, $b_1, b_2 \in \mathbb{Z}_3$

Then $b_1(1,2,0,1) + b_2(2,0,1,2) = (1,0,1,1)$

~~Take~~ b_1 So $\begin{cases} b_1 + 2b_2 = 1 \\ 2b_1 = 0 \\ b_2 = 1 \\ b_1 + 2b_2 = 1 \end{cases}$

Take $b_1 = 0$, $b_2 = 1$ into $b_1 + 2b_2 = 0 + 2 = 2 \neq 1$

So $(1,0,1,1)$ is not in S.

[redacted]

2. (b).

Solution:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore A basis of S is $B=\{(1,0,0,2,2),(0,1,2,0,0),(0,0,0,0,1)\}$

3. (a).

Solution:

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 2 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 3 & 4 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore the dimension of S is 3.

3 (b).

Solution: Let $V=P_n(\mathbb{Z}_3)$, consider $S=\{1, x, x^2, \cdots, x^n\}$

Clearly S spans V.

For the linearity of S.

$a_0+a_1x+a_2x^2+\cdots+a_nx^n=0$ in $P_n(\mathbb{Z}_3)$

Since $\mathbb{Z}_3=\{0,1,2\}$ spanning set

when $x=0$, $x^i=0$, S should be reduced to $\{1\}$ such that the only a_i make the final result 0 is $a_i=0$.

Similarly, when $x=1$, $S=\{1,1,1,1,\cdots,1\}$

if n is odd, then $1\times1+1\times2+1\times1+1\times2+\cdots+1\times1+1\times2=0$

if n is even, then $1\times1+1\times1+1\times1+1\times2+\cdots+1\times1+1\times2=0$

Hence S should also be reduced to $\{1\}$ in order to keep the linearity.

when $x=2$, $S=\{1,2,1,2,\cdots\}$, it's similar.

Therefore the basis is $B=\{1\}$.

And $\dim B = 1$

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(4). Solution:

$$\begin{bmatrix} 1 & i & -1+i & -1 \\ 2 & 1+2i & -2+3i & -2 \\ 1+i & i & -2+i & -1-i \end{bmatrix} \to \begin{bmatrix} 1 & i & -1+i & -1 \\ 0 & 1 & i & 0 \\ 0 & 1 & i & 0 \end{bmatrix} \to \begin{bmatrix} 1 & i & -1+i & -1 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the basis for row space of A is $\{(1, i, -1+i, -1), (0, 1, i, 0)\}$

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$$\to \begin{bmatrix} 1 & 0 & i & -1 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the basis for column space of A is $\{(1, 2, 1+i), (i, 1+2i, i)\}$

Name the pre-image $X = T^{-1}(U) = \{v \in V \mid T(v) \in U\}$

(5). Proof. Since $T(0) = 0$ (T is a linear transformation)

Then $0 \in X$

Suppose $v_1, v_2 \in X$, want to show $v_1 + v_2 \in X$

Since $T(v_1+v_2) = T(v_1) + T(v_2)$ by linearity

and $v_1, v_2 \in X$, X is the pre-image of U

so $T(v_1), T(v_2) \in U$

and since U is a subspace of W

so $T(v_1) + T(v_2) \in U$.

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Therefore X is closed under addition

Similarly suppose $v \in X$ and $a \in F$ which is the field of V.

want to show $av \in X$.

Since $T(av) = aT(v)$ by linearity

and $v \in X$, X is the pre-image of U

so $T(v) \in U$

and since U is a subspace of W

so $aT(v) = T(av) \in U$

Therefore X is closed under scalar multiplication.

Hence $T^{-1}(U)$ is a subspace of V.

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So $T(x,y,z) = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{3}{2} \\ \frac{-5}{2} & \frac{-3}{2} & \frac{-3}{2} \end{bmatrix}$

Say the Matrix is $M = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. then

$$M\begin{pmatrix}2\\3\\0\end{pmatrix} = \frac{3}{2}\begin{pmatrix}3\\-1\end{pmatrix} + \frac{-5}{2}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$M\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{5}{2}\begin{pmatrix}3\\-1\end{pmatrix} + \frac{-3}{2}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}6\\-1\end{pmatrix}$$

$$M\begin{pmatrix}2\\3\\1\end{pmatrix} = \frac{3}{2}\begin{pmatrix}3\\-1\end{pmatrix} + \frac{-3}{2}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

Therefore
$$\begin{cases} 2a+3b=2 \\ 2d+3e=1 \\ a+b+c=6 \\ d+e+f=-1 \\ 2a+3b+c=3 \\ 2d+3e+f=0 \end{cases} \Rightarrow \begin{cases} a=13 \\ b=-8 \\ c=1 \\ d=-1 \\ e=1 \\ f=-1 \end{cases}$$

Hence the matrix is $\begin{bmatrix} 13 & -8 & 1 \\ -1 & 1 & -1 \end{bmatrix}$.

(b). Solution:

According to the problem

$[T]_\beta^\alpha \cdot [(x,y,z)]_\alpha = [T(x,y,z)]_\beta$

$\because (x,y,z) = z(1,-1,1) + (y+z)(0,1,0) + (x-z)(1,0,0)$

$\therefore [(x,y,z)]_\alpha = \begin{pmatrix} z \\ y+z \\ x-z \end{pmatrix}$

$\because [T]_\beta^\alpha = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$\therefore [T(x,y,z)]_\beta = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} z \\ y+z \\ x-z \end{pmatrix} = \begin{pmatrix} x+3y+4z \\ x+2y+2z \end{pmatrix}$

$\therefore T(x,y,z) = (x+3y+4z)(3,2) + (x+2y+2z)\cdot(2,1)$
$= (5x+13y+16z,\ 3x+8y+10z)$

(7) Solution:

Let $\alpha = \{(1,2,0),(1,1,1),(1,1,0)\}$

$\beta = \{(1,1),(1,-1)\}$

$\alpha' = \{(2,3,0),(1,1,1),(2,3,1)\}$

$\beta' = \{(3,-1),(1,-1)\}$

say $(x,y,z) \in \mathbb{R}^3$

$$[T]_\beta^\alpha [(x,y,z)]_\alpha = [T(x,y,z)]_\beta$$

$\because (x,y,z) = (y-x)(1,2,0) + z(1,1,1) + (2x-y-z)(1,1,0)$

$$\therefore [(x,y,z)]_\alpha = \begin{pmatrix} y-x \\ z \\ 2x-y-z \end{pmatrix}$$

$$\because [T]_\beta^\alpha = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\therefore [T(x,y,z)]_\beta = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} y-x \\ z \\ 2x-y-z \end{pmatrix} = \begin{pmatrix} x-2z \\ y \end{pmatrix}$$

$$\therefore T(x,y,z) = (x-2z)(1,1) + y(1,-1)$$
$$= (x+y-2z,\ x-y-2z)$$

Then for $T(x,y,z)$:

$T(2,3,0) = (5,-1) = 2(3,-1) + 1(1,-1)$

$T(1,1,1) = (0,-2) = -1(3,-1) + 3(1,-1)$

$T(2,3,1) = (3,-3) = 0(3,-1) + 3(1,-1)$

Hence the matrix of $[T]_{\beta'}^{\alpha'} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 3 \end{bmatrix}$

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(8).

Proof: ($\Rightarrow$) Suppose that $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, then $\{v_1, v_2, \cdots, v_n\}$ is linearly independent and $V = \text{span}\{v_1, v_2, \cdots, v_n\}$.

1.1 For linearity, say $a_1, \cdots, a_n \in F$ are coefficients such that $a_1[v_1]_\beta + a_2[v_2]_\beta + \cdots + a_n[v_n]_\beta = 0$ in F^n; (**)

Note: in order to complete the rest of this proof, we need some other proofs.

Define $T: V \to F^n$ by $T(v) = [v]_\beta$

If $v, w \in V$, $\lambda \in F$; a_i and b_i are coefficient where $i = 1, 2, \cdots, n$.

and $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

$w = b_1v_1 + b_2v_2 + \cdots + b_nv_n$

Then $v + \lambda w = a_1v_1 + a_2v_2 + \cdots + a_nv_n + \lambda b_1v_1 + \lambda b_2v_2 + \cdots + \lambda b_nv_n$

$= (a_1 + \lambda b_1)v_1 + (a_2 + \lambda b_2)v_2 + \cdots + (a_n + \lambda b_n)v_n$

Therefore $T(v + \lambda w) = T(v) + \lambda T(w)$, hence it is a linear transformation. (*)

According to the result of (*), we can see that

$0 = a_1[v_1]_\beta + \cdots a_n[v_n]_\beta = [a_1v_1 + \cdots + a_nv_n]_\beta$

Since $\text{Ker}\,T = \{0\}$ by the property of linear transformation

then $a_1v_1 + \cdots + a_nv_n = 0$

Then because $\{v_1, \cdots, v_n\}$ is linearly independent, then we have:

$a_1 = a_2 = \cdots = a_n = 0$.

Therefore $\{[v_1]_\beta, [v_2]_\beta, \cdots, [v_n]_\beta\}$ is linearly independent in F^n.

1.2 For spanning, again since (*) we know T is surjective, which means $\forall w \in F^n, \exists v \in V$ such that $F(v) = w$.

As $V = \text{span}\{v_1, v_2, \cdots, v_n\}$, there exist $a_1, a_2, \cdots, a_n \in F$ such that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.

So $w = T(v) = a_1[v_1]_\beta + a_2[v_2]_\beta + \cdots + a_n[v_n]_\beta$.

Therefore $F^n = \text{span}\{[v_1]_\beta, [v_2]_\beta, \cdots, [v_n]_\beta\}$.

From 1.1 and 1.2, we proved that if $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, then $\{[v_1]_\beta, [v_2]_\beta, \cdots, [v_n]_\beta\}$ is a basis for F^n.

The other direction.

($\Leftarrow$) Then suppose conversely that $\{[V_1]_\beta, [V_2]_\beta, \cdots, [V_n]_\beta\}$ is a basis for F^n, ~~~~ which automatically means

2.1 $\{[V_1]_\beta, [V_2]_\beta, \cdots, [V_n]_\beta\}$ is linearly independent and $F^n = \text{span}\{[V_1]_\beta, [V_2]_\beta, \cdots, [V_n]_\beta\}$.

First, for linearity, assume $a_1, \cdots, a_n \in F$ are coefficients such that $a_1 v_1 + a_2 v_2 + \cdots a_n v_n = 0$.

Since T is linear (by $(*)$), both sides apply T:

$$T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = T(0)$$

$$a_1 [v_1]_\beta + a_2 [v_2]_\beta + \cdots + a_n [v_n]_\beta = 0$$

By hypothesis, $\{[V_1]_\beta, [V_2]_\beta, \cdots, [V_n]_\beta\}$ is linearly independent, so $a_1 = a_2 = \cdots = a_n = 0$.

Therefore $\{V_1, V_2, \cdots, V_n\}$ is linearly independent.

2.2 For spanning, a given $v \in V$, $\exists a_1, \cdots, a_n \in F$ such that

$$[V]_\beta = a_1 [V_1]_\beta + \cdots + a_n [V_n]_\beta$$

$$\text{i.e. } T(v) = T(a_1 v_1 + \cdots + a_n v_n)$$

Since $\text{Ker}\, T = \{0\} \Rightarrow$ injectivity $\Rightarrow$ one-to-one

$\Rightarrow v = a_1 v_1 + a_2 v_2 + \cdots a_n v_n$

Therefore $V = \text{span}\{V_1, V_2, \cdots, V_n\}$.

From 2.1 and 2.2, we proved that if $\{[V_1]_\beta, [V_2]_\beta, \cdots, [V_n]_\beta\}$ is a basis for F^n, then $\{V_1, V_2, \cdots, V_n\}$ is a basis for V.

Hence, conclude all ~~~~ 1.1, 1.2, 2.1, 2.2 of two ~~~~ directions: $\{V_1, V_2, \cdots, V_n\}$ is a basis for V if and only if $\{[V_1]_\beta, [V_2]_\beta, \cdots, [V_n]_\beta\}$ is a basis for F^n.

$\blacksquare$