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Inequality 1.3 is a very important inequality which can be used as a legitimate tool (fact) about norm of a n-tuple in relation to the absolute value of the components of the n-tuple. If we were to say the magnitude of a vector $\boldsymbol{x} = (x_1, x_2, \ldots x_n)$ is small, according to inequality 1.3 we could instead say that the maximum the magnitudes of the components is small. Similarly for $M < |\boldsymbol{x}|$ it is sufficient that $M < max\{|x_1|, |x_2|, \ldots |x_n|\}$. This inequality can simplify our calculations about limit of function of several variables, and later this can be important in working with vector valued functions. See for example the discussion in the middle of page 14, or the bottom of page 108. In general whenever the passage from vector to scalar is needed the inequality 1.3 appears to become a useful tool.

Let's understand what this inequality really is. Consider $n = 2$ and try to draw the following three regions of the plane in one coordinate system:

a) $|\boldsymbol{x}| \leq 1$

b) $max|x|, |y| \leq 1$

c) $\sqrt{2}max|x|, |y| \leq 1$

The norm as we study in this textbook is known as the Euclidean norm also known as tow-norm: $|\boldsymbol{x}| = (|x|^2 + |y|^2)^{\frac{1}{2}}$. See the role of 2 in the power of each component as well as the power $\frac{1}{2}$ of the entire expression. Try to replace this 2 with 1 and see what you get: $|\boldsymbol{x}| = |x| + |y|$ try to draw the region of the plane that is determined by $\boldsymbol{x} \leq 1$. (try to draw this region in the same diagram above.)

Prove that the abstract properties of a norm hold true of this new 1-norm. That is

a) $|\boldsymbol{x}| \geq 0$

b) if $|\boldsymbol{x}| = 0$ then $\boldsymbol{x} = \mathbf{0}$

c) for any real number k, we have $|k\boldsymbol{x}| = |k||\boldsymbol{x}|$.

d) for amy two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we have the triangle inequality holds: $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$.

This process can be repeated for any number $p > 0$. That is $|\boldsymbol{x}| = (|x|^p + |y|^p)^{\frac{1}{p}}$. Try using a graphing calculator to draw the same diagram with $p = 3$ and $p = \frac{1}{2}$. As you have guessed already as the value of p increases the shape of the region of the plane described by $|\boldsymbol{x}| \leq 1$ becomes more like a square. There is a limit to this process, and that is the infinity norm: $|\boldsymbol{x}| = max\{|x|, |y|\}$. Demonstrate that this is indeed defining a norm, that is it satisfies the properties. In a way inequality 1.3 suggests that while there are so many different norms that can be useful for different applications there seems to be a rather tight connection between them.