See the website to see a solved copy of Monday's test.

Today - finish § 3.1

- in § 3.2, discuss the Weak Duality Theorem (Thm 3.4)

Thm 3.2 and 3.3 (generalizd)

Given a dual pair of problems, each equality constraint in one problem is associated with an unrestricted variable in the other problem.

Proof: (in one instance):

Given the primal problem:

Maximize $z = x_1 + 2x_2$ s.t.

$$3x_1 - 4x_2 = 5$$
$$6x_1 + 7x_2 \le 8$$
$$x_1 \ge 0,\ x_2 \text{ unrestricted}$$

In primal standard form, this is equivalent to

Maximize $z = x_1 + 2x_2^+ - 2x_2^-$

$$3x_1 - 4x_2^+ + 4x_2^- \le 5$$
$$-3x_1 + 4x_2^+ - 4x_2^- \le -5$$
$$6x_1 + 7x_2^+ - 7x_2^- \le 8$$
$$x_1 \ge 0,\ x_2^+ \ge 0,\ x_2^- \ge 0$$

(where $x_2 = x_2^+ - x_2^-$)

Its dual is:

Maximize $z = 5w_1^+ - 5w_1^- + 8w_2$ s.t.

$$3w_1^+ - 3w_1^- + 6w_2 \ge 1$$
$$-4w_1^+ + 4w_1^- + 7w_2 \ge 2$$
$$4w_1^+ - 4w_1^- - 7w_2 \ge -2$$
$$w_1^+ \ge 0,\ w_1^- \ge 0,\ x_2 \ge 0$$

With $w_1 = w_1^+ - w_1^-$, the dual is equivalent to

Minimize $z = 5w_1 + 8w_2$ s.t.

$$3w_1 + 6w_2 \ge 1$$

$$-4w_1 + 7w_2 = 2$$

w_1 unrestricted, $w_2 \geq 0$.

Remark: Theorem 3.1, 3.2, and 3.3 can be used to find the dual of any maximization problem having $\leq$ constraints (plus equalities) and any minimization problem have $\geq$ constraints. (plus equalities)

Eg. The primal standard problem:

Maximize $z = x_1 + 2x_2$ s.t.

$$3x_1 + 4x_2 \leq 5$$
$$6x_1 + 7x_2 \leq 8$$
$$x_1 \geq 0, x_2 \geq 0$$

has dual

Maximize $z' = 5w_1 + 8w_2$ s.t.

$$3w_1 + 6w_2 \geq 1$$
$$4w_1 + 7w_2 \geq 2$$
$$w_1 \geq 0, w_2 \geq 0$$

In canonical form, the primal problem is

Maximize $z = x_1 + 2x_2 + 0 \cdot x_3 + 0 \cdot x_4$ s.t.

$$3x_1 + 4x_2 + x_3 \qquad = 5$$
$$6x_1 + 7x_2 \qquad + x_4 = 8$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

According to theorem 3.2, its dual is

Minimize $z' = 5w_1 + 8w_2$ s.t.

$$3w_1 + 6w_2 \geq 1$$
$$4w_1 + 7w_2 \geq 2$$
$$w_1 \qquad \geq 0$$
$$w_2 \geq 0$$

w_1 unrestricted, w_2 unrestricted

(Actually the same problem as the last version of the dual.)

Weak Duality Theorem

Given a dual pair of problem:

$$\text{Maximize } z = c^T x \text{ s.t.}$$
$$Ax \le b$$
$$x \ge 0 \in \mathbb{R}^n$$

and Maximize $z' = b^T w$ s.t.

$$A^T w \ge c$$
$$w \ge 0 \in \mathbb{R}^n$$

(A is $m \times n$)

If x_0 is feasible for the primal problem and w_0 is feasible for the dual problem, then $\underline{c^T x_0 \le b^T w_0}$

(comparison of objective values)

all components ≥ 0

Proof: We have: $Ax_0 \le b$, $x_0 \ge 0 \in \mathbb{R}^n$, $A^T w_0 \ge c$, $w_0 \ge 0 \in \mathbb{R}^m$

Then $w_0^T A x_0 \le w_0^T b$ (since $Ax_0 - b \ge 0 \in \mathbb{R}^m$, so $\underline{w_0^T}(\underline{b - Ax_0})$
≥ 0

Taking transposes $x_0^T A^T w_0 \le b^T w_0$

Since $A^T w_0 \ge c$ and $x_0 \ge 0$, we have $x_0^T A^T w_0 \ge x_0^T c = c^T x_0$

The 2 circled inequalities say $c^T x_0 \le x_0^T A^T w_0$ 1×1 matrix
$\le b^T w_0$, so $c^T x_0 \le b^T w_0$