# FINANCIAL MATHEMATICS
# (STAT 2032 / STAT 6046)

## TUTORIAL SOLUTIONS WEEK 2

**Question 1**
$2,500 is invested. Find the accumulated value of the investment 10 years after it is made for each of the following rates.
(a) 4% annual simple interest
(b) 4% effective annual compound interest
(c) 2% effective 6-month compound interest
(d) 1% effective 3-month compound interest

**Solution**
(a)
$i$ = 4% annual simple interest
$n$ = 10 years
$2500(1+ni) = 2500(1+(10)(0.04)) = 3500$

(b)
$i$ = 4% effective annual compound interest
$n$ = 10 years
$2500(1+i)^n = 2500(1+0.04)^{10} = 3700.61$

(c)
$i$ = 2% effective 6-month interest rate.
$n$ = 10 years, or 20 6-month periods.
$2500(1+i)^n = 2500(1+0.02)^{20} = 3714.87$

(d)
$i$ = 1% effective 3-month interest rate.
$n$ = 10 years, or 40 3-month periods.
$2500(1+i)^n = 2500(1+0.01)^{40} = 3722.16$

**NOTE:** An alternative, yet more labour intensive way of solving (c) and (d) could involve finding an equivalent effective annual interest rate and compounding this for 10 years.

For example, for part (c), 2% is the 6-month effective interest rate, so an equivalent effective annual interest rate is: $i = 1.02^2 - 1 = 0.0404$

The accumulated value is then:
$$2500(1+0.0404)^{10} = 3714.87$$

**Question 2**
If the effective 3-month compound interest rate is -3%, what is the accumulated value after 1 year of an initial investment of $1?

**Solution**
Negative rates are treated the same as positive rates.

$i$ = 3-month compound interest rate of –3%.
$n$ = 1 year, or four 3-month periods.

$$(1+i)^n = (1-0.03)^4 = 0.8853$$

Some questions for you to think about

- What is the present value now of $1 due in 1 year if the effective 3-month compound interest rate is -3%?

**Question 3**
It is known that the present value of $864 due in two years is $600. Find the accumulated value of $2000 invested at the same rate of compound interest for three years.

**Solution**
First we need to find the effective annual rate of interest $i$:

$$864v^2 = 600 \Rightarrow 864 = 600(1+i)^2 \Rightarrow i = 0.2$$

Therefore, the accumulated value of $2000 invested for three years is:

$$2000(1+i)^3 = 2000(1.2)^3 = 3456$$

**Question 4**
What is the present value of 1000 due in 10 years if the effective annual interest rate is 6% for each of the first 3 years, 7% for the next 4 years, and 9% for the final 3 years ?

**Solution**
We can solve this by discounting the 1000 back 3 years at 9%, then discounting this value back 4 years at 7%, and finally discounting this amount back to the present at 6%:

The discounted value at t=7 is $1{,}000\left(v_{0.09}^3\right)$
The discounted value at t=4 is $1{,}000\left(v_{0.09}^3\right)\left(v_{0.07}^4\right)$

The present value at t=0 is, therefore,

$$1{,}000\left(v_{0.09}^3\right)\left(v_{0.07}^4\right)\left(v_{0.06}^3\right) = 1000(1.09)^{-3}(1.07)^{-4}(1.06)^{-3} = 494.62$$

Some questions for you to think about

- If interest rates are quoted as effective per half-year, and everything else remains unchanged, will the present value be < or > or = to 494.62?
- If the term of the investment is 5 years, and effective semi-annual interest rates are 6% for the first 1.5 years, 7% for the next 2 years, and 9% for the final 1.5 years, then the present value will be < or > or = to 494.62?

**<u>Question 5</u>**

Smith needs to borrow $5,000 for one year.

Under one scenario (A) he is offered a loan at an effective annual rate of 5%. In other words, he has to pay interest of $5,000 \times 0.05 = 250$ at the end of the year.

Under another scenario (B), he is offered a loan of $10,000 at a lower effective annual rate of interest denoted by $i$. If he borrows the $10,000, he can invest the excess $5,000 for one year at 3%. In other words, at the end of the year he will have to pay interest of $(10,000 \times i)$, but he receives interest of $5000 \times 0.03 = 150$.

How low must the rate on the $10,000 loan (scenario B) be in order for Smith to prefer it to the $5,000 loan (scenario A)?

**<u>Solution</u>**

We are comparing two scenarios, so the first thing to do is write down the details of both.

**Scenario A**

Loan amount borrowed = $5,000
Duration of loan = 1 yr
Annual interest rate on borrowed loan = 5%

**Scenario B**

Loan amount borrowed= 10,000
Duration of loan= 1 yr
Annual interest rate on borrowed loan = $i$ (unknown)
Loan amount invested = 5,000
Duration of investment = 1 yr
Annual interest rate for investment = 3%

The question being asked is: *How low must be the rate on the $10,000 loan in order for Smith to prefer it to the $5,000 loan*? In order for Smith to prefer the $10,000 loan, the net interest paid by Smith should be lower than the net interest paid if he chose the $5,000 loan. We, therefore, need to find the net interest for each scenario.

**Scenario A**
Interest paid by Smith at the end of the year = 5,000(0.05) = 250

**Scenario B**
Interest paid by Smith at the end of the year = 10,000($i$)
Interest earned by Smith at the end of the year = 5,000(0.03) = 150
Net interest paid by Smith at the end of the year = 10,000($i$) - 150

Therefore, Smith will prefer the $10,000 loan if the net interest paid by Smith under Scenario B is less than the net interest paid by Smith under Scenario A, or

$$10{,}000(i) - 150 < 250$$

Solving for $i$,

$$i < 4\%$$

If $i$=4% then both scenarios are equally appealing. If $i$ is less than 4% Scenario B is preferred.

Some questions for you to think about

- Now suppose under Scenario B, Smith can borrow at i=4%. What is the range of the loan size Smith should borrow under Scenario B, to prefer it Scenario A? (note: we still assume any proceeds in excess of $5000 can be invested at 3%).

**Question 6**

Jones invests $100,000 in a 180-day short term investment at a bank, based on simple interest at an annual rate of 7.5%. After 120 days, interest rates have risen to 9% and Jones would like to redeem the certificate early and reinvest in a 60-day certificate at the higher rate.

In order for there to be no advantage in redeeming early and reinvesting at the higher rate, what early redemption penalty $P$ should the bank charge at the time of early redemption?

Hint: You can approach this question by equating the accumulated value of two scenarios and solving for $P$.

A) Jones invests $100,000 for 180 days at 7.5% simple interest per annum.
B) Jones invests $100,000 for 120 days at 7.5% simple interest per annum and then redeems the accumulated amount. Upon redemption he pays a penalty of $P$. He then reinvests the balance at 9% simple interest per annum for 60 days.

**Solution**
We want to find the level of early redemption penalty that the bank should charge so that there is no advantage in an investor redeeming early and reinvesting.

First, list the particulars of the investment. Under the first scenario (A), we assume that Jones does not redeem. Under the second scenario (B), we assume that Jones redeems and reinvests at a higher rate, and in doing so, is subject to an early redemption penalty.

**Scenario A**
Amount invested = 100,000
Duration = 180 days
Interest rate = 7.5% simple annual rate

**Scenario B**
Amount invested = 100,000
Duration = 180 days
Interest rate = 7.5% simple annual rate for first 120 days, then balance reinvested at 9% for the subsequent 60 days.
Redemption penalty = P (unknown)

We want to find P such that the two scenarios produce identical results. We can do this by calculating the accumulated amounts at the end of the 180 days:

Accumulated amount for Scenario A = $100{,}000\left(1+0.075\left(\frac{180}{365}\right)\right)=103{,}698.63$

To calculate the accumulated amount for Scenario B, first find the accumulated amount after 120 days:

$$= 100{,}000\left(1+0.075\left(\frac{120}{365}\right)\right) = 102{,}465.80$$

Before this amount is reinvested, the redemption penalty (P) is subtracted. The balance is then reinvested with simple interest of 9% for 60 days. The total accumulated amount for Scenario B is then:

$$= (102{,}465.80-P)\left(1+0.09\left(\frac{60}{365}\right)\right)=(102{,}465.80-P)(1.014795)$$

Equating these two amounts and solving for P, we get:

$$(102{,}465.80-P)(1.014795)=103{,}698.63$$
$$(102{,}465.80-P)=102{,}186.80$$
$$P = 278.93$$

Some questions for you to think about

- If interest rates only rise to 8%, how will P change and why??

**Question 7**

(a) Show that at an effective annual compound interest rate of $i$, the amount of interest earned in successive years on an investment of 1 grows by a factor of $(1+i)$ and these amounts are $i$, $(1+i)i$, $(1+i)^2 i$, …, $(1+i)^{n-1} i$ for the first, second, third, …, nth year, respectively.

(b) Using the fact that the amount of interest earned from time 0 to time n is $(1+i)^n - 1$, derive a formula for the sum $1+(1+i)+(1+i)^2+...+(1+i)^{n-1}$.

**Solution**

(a) The amount at time 0 is 1, so following the notation of week 1 lectures, let S(0) = 1.

Interest in the first year is $S(0)\cdot i = i$, so the accumulated value at the end of the first year is $S(1) = S(0) + i = 1 + i$.

Interest in the second year is $S(1)\cdot i = (1+i)\cdot i$, so the accumulated value at the end of the second year is $S(2) = S(1) + S(1)\cdot i = (1+i) + (1+i)\cdot i = (1+i)^2$.

Interest in the third year is $S(2)\cdot i = (1+i)^2 \cdot i$, so the accumulated value at the end of the third year is $S(3) = S(2) + S(2)\cdot i = (1+i)^2 + (1+i)^2 \cdot i = (1+i)^3$.

It follows that interest in the nth year is $S(n-1)\cdot i = (1+i)^{n-1}\cdot i$

So the sequence of interest payments form a geometric sequence, with common ratio (1+i)

(b)

The total interest earned from time 0 to time n is $(1+i)^n - 1$. This follows from the fact that the total interest component is simply the accumulated value $(1+i)^n$ minus the principal 1.

From part (a), we also know that the total interest earned from time 0 to time n is

$$i + (1+i)\cdot i + (1+i)^2 \cdot i + (1+i)^3 \cdot i + ... + (1+i)^{n-1} \cdot i$$

This can be written as:

$$i\left[1+(1+i)+(1+i)^2+(1+i)^3+...+(1+i)^{n-1}\right]$$

Therefore, $i\left[1+(1+i)+(1+i)^2+(1+i)^3+...+(1+i)^{n-1}\right] = (1+i)^n - 1$

and, $1+(1+i)+(1+i)^2+(1+i)^3+...+(1+i)^{n-1} = \dfrac{(1+i)^n - 1}{i}$

(which agrees with the result for the summation of a geometric series)

**<u>Past Exam Question – 2005 Final Exam Q1(d)</u>**

At a certain rate of compound interest, 1 will increase to 2 in *a* years, and 4 will increase to 20 in *b* years. If 6 will increase to 15 in *n* years, show that $n = b - a$. (3 marks)

**<u>Solution</u>**

$$1(1+i)^a = 2 \Leftrightarrow a = \frac{\log(2)}{\log(1+i)}$$

$$4(1+i)^b = 20 \Leftrightarrow b = \frac{\log(5)}{\log(1+i)}$$

$$6(1+i)^n = 15 \Leftrightarrow n = \frac{\log(15/6)}{\log(1+i)}$$

$$\therefore n = \frac{\log(5/2)}{\log(1+i)} = \frac{\log(5) - \log(2)}{\log(1+i)} = b - a$$

Note: we can also write $(1+i)^a(1+i)^n=(1+i)^b$. That is, we can invest in a single investment for b years, or equivalently, invest for a years, then reinvest the proceeds for n years, all at the same interest rate.