Assignment III: week of Jan. 23th

This is the 3nd of the 10 assignments. Attend the help sessions and work together with a couple other students. You **DO** *have to turn this one in, either in class, or slip it under my office door (MP1210).* **Submission Deadline: Wed. Jan 30th, 5PM**

[5 pts] 1. Satellites overhead. Here, we use a combination of Kepler's laws and basic calculus to investigate the orbit and orbital decay of satellites around Earth.

- The International Space Station (ISS) orbits the Earth in a circle at a height 380 kilometers above Earth's surface. Calculate how often it revolves around the Earth.
- The atmosphere at that height is rather tenuous, however, it exerts a notice-able drag on the satellite. At the said altitude, the latter experiences energy loss and a rate of orbital decay ~ 0.4 km/day. Constant boost needs to be applied to keep the station afloat. Assume the orbital decay goes as
$$\frac{dr}{dt} = -0.4\text{km/day}\left[\frac{380\text{km}}{r-R}\right]^6, \tag{1}$$
where r is the distance to the center of the Earth, R is Earth's radius, and the extra factor $(380\,\mathrm{km}/(\mathrm{r}-\mathrm{R}))^6$ is meant to crudely model how the decay rate increases at lower height (denser atmosphere). Solve this equation to find r as a function of t. Evaluate, in the absence of boosts, how soon (express in unit of days) will the station be crashing onto ground ($r = R$).
- The GPS satellites that we use for navigation orbits the Earth with a period of ~ 12 hours. Calculate their orbital heights. And assume that their orbits decay in the same rate and form as that in Equation (1), how long can they stay up in sky?

[5pts] 2. This problem is intended to introduce you to the important topic of dimensional analysis (courtesy Prof. B. Abraham).

At some point in its history after the big bang, the universe is so small that gravity and quantum mechanics have comparable effects on the properties of matter. No theory for quantum gravity exists at the present time, so once we reach this stage in our contemplation of the Universe, we simply cannot push our present knowledge further. In this question your goal is to work out (roughly) the time, length, and mass scales corresponding to the epoch at which our knowledge breaks down. These are called the Planck time, the Planck length and the Planck mass. We have no idea what the Universe looks like at times earlier than the Planck time.

To attack this problem, you should use the astrophysicist's best friend, dimensional analysis. Here is how dimensional analysis works. As you know, an excellent way to check the answers to calculations in science is to make sure the answers have the right units at the end of the calculation. If the units don't make sense (e.g. if you're trying to determine a velocity, but your answer has units of length), you known something has gone wrong. Dimensional analysis is a variation on this idea, wherein you run the process backwards. The goal is to estimate rough answers to tricky problems by simply coming up with the simplest expression in which the answer has the right units. The procedure is simple: (1) Make an intelligent guess about which physical constants and/or parameters are important to the problem. (2) Play around with these quantities until you come up with an algebraic expression that gives an answer in the right units. Amazingly, in most cases this simple procedure will give you results that are correct to an order of magnitude.

We have talked about the fundamental particles and the four basic forces. Now we introduce the concept of fundamental units. All physical quantities are associated with fundamental units of length L, time T, mass M, charge Q and temperature, θ. Of course, the specific values of the fundamental units depends on convention. For example, in the standard S.I. (Systeme Internationale) system of units, the fundamental unit of length is the meter, the funamental unit of time is the second, the fundamental unit of mass is the kilogram, the fundamental unit of charge is the coulomb, and the fundamental unit of temperature is the kelvin. In the c.g.s. system, the specific choice of fundamental units is different (eg. the fundamental unit of length is the cm), but the fundamental units are meant to represent the same measurable quantities (length, time, mass, charge, and temperature).

Other physical units are defined for convenience, but they are really combinations of these fundamental units. For example, in the S.I. system you are familiar with the Newton, Ohm, Hertz, Joule, etc etc. Each of these is a derived unit that can be reduced to powers of the fundamental units. For example, a Newton is simply 1 $kg/m/s^2$. We simply write "N" instead of $kg/m/s^2$ to save space. The specific choice of base units (m vs. cm, kg vs. g, second vs. year, etc) can vary; beings from another planet would no doubt have different choices. But the fundamental measurable quantities (length, mass, time, charge, and temperature) are the same everywhere.

So let us describe the dimensions of any quantity using a convenient notation, where a pair of square parentheses denotes a dimension. For example, the dimensions of velocity in S.I. units are m/s, which is dimensionally a length over a time. Similarly in the c.g.s. system the units would be cm/s, which is also a length over a time. We will describe this as: $[V] = [L][T]^{-1}$. Similarly, the dimensions of energy are $[E] = [M][V]^2 = [M][L]^2[T]^{-2}$. Note how these all reduce to powers of the fundamental units.

The specific question we are going to work out concerns the Planck time, Planck length, and Planck mass. To do this, we're going to guess that whatever complicated physics is needed to obtain these quantities, it is sure to depend on three fundamental constants of nature.These are: (1) Newton's gravitational constant, G. (2) Planck's constant, h, and (3) The speed of light, c. This is because G appears in most gravity calculations, h appears in most quantum mechanics calculations, and the speed of light is a central component of both relativity and quantum mechanics.

- Write the dimensions of the fundamental constants $[h]$, and $[c]$ in terms of powers of the mass M, length L, and time T. To get you started, first try it on $[G]$ and show that $[G] = [L]^3[M]^{-1}[T]^{-2}$.

- Find a set of 3 exponents α, β, γ such that the quantity $G^\alpha h^\beta c^\gamma$ has dimensions of time. In other words:

$$[T] = [G]^\alpha [h]^\beta [c]^\gamma. \tag{2}$$

  Hint: you should get 3 algebraic equations in the 3 unknowns α, β and γ that you can easily solve by hand.

- Now determine the numerical value for the so-called Planck time using $t_{\text{Planck}} = G^\alpha h^\beta c^\gamma$. Please express your results in the S.I. unit (below as well).

- If now we would like to have a quantity of dimension length

$$[L] = [G]^\alpha [h]^\beta [c]^\gamma, \tag{3}$$

  what should the exponents α, β, γ now be? Determine the numerical value for this length (called the Planck length).

- Use the same procedure again to get the Planck mass.

A list of physical constants in MKS (or SI) units, kg is kilogram, m is meter, K is Kelvin, s is second, N is Newton, J is Joule, C is coulomb.

elementary charge $e = 1.602 \times 10^{-19} C$

proton rest mass $m_p = 1.672 \times 10^{-27} kg$

neutron rest mass $m_n = 1.675 \times 10^{-27} kg$

hydrogen atom mass $m_H = 1.673 \times 10^{-27} kg$

electron rest mass $m_e = 9.109 \times 10^{-31} kg$

Planck constant $h = 6.625 \times 10^{-34} J \cdot s$

Speed of light in vacuum $c = 2.998 \times 10^8 m s^{-1}$

Gravitational constant $G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$

Fine structure constant $\alpha = 1/137.036$

Boltzmann constant $k = 1.380 \times 10^{-23} J K^{-1}$

Electron volt $eV = 1.602 \times 10^{-19} J$

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8} J s^{-1} m^{-2} K^{-4}$

Coulomb constant $k_e = \frac{1}{4\pi\epsilon_0^2} = \frac{c^2 \mu_0}{4\pi} = 8.988 \times 10^9 N m^2 C^{-2}$.