4.5

Using integration as an "operation" to define new functions!!

given any integrable function $f: \mathbb{R}^{n+m} \to \mathbb{R}$

We can use f and integration to define a new function $F: \mathbb{R}^m \to \mathbb{R}$

$$\boxed{F(\mathbf{x}) = \underbrace{\int\cdots\int}_{S\subset\mathbb{R}^n} f(\mathbf{x},\mathbf{y})\, d^n\mathbf{y}}$$

As usual, when we define a new function we should discuss the limit, continuity and differentiability in relation to the properties of the old function & the operation:

- **limit theorem:** $\lim\limits_{\mathbf{x}\to a} F(\mathbf{x}) \overset{?}{=} \int\cdots\int \lim\limits_{\mathbf{x}\to a} f(\mathbf{x},\mathbf{y})\, d^n\mathbf{y}$

  NO! See example 1 page 188

- **Continuity Theorem:** 4.46

  $\underset{\subset\mathbb{R}^n}{S}, \underset{\subset\mathbb{R}^m}{T}$ be compact, S is measurable, $f(\mathbf{x},\mathbf{y})$ cont. on $T\times S$. Then F is cont on T

pf: Cont + Cpt $\Rightarrow$ UC

$\therefore \exists\delta>0$ st. $\forall(\mathbf{x},\mathbf{y})\in T\times S$ & $\mathbf{x}'\in T$

$|\mathbf{x}-\mathbf{x}'|<\delta \Rightarrow |f(\mathbf{x},\mathbf{y})-f(\mathbf{x}',\mathbf{y})|<\frac{\epsilon}{|S|}$

so $|F(\mathbf{x})-F(\mathbf{x}')| \le \int\cdots\int |f(\mathbf{x},\mathbf{y})-f(\mathbf{x}',\mathbf{y})|\, d^n\mathbf{y} < \int\cdots\int_S \frac{\epsilon}{|S|}\, d^n\mathbf{y} = \epsilon$

- **differentiability Theorem:** (4.47)

  $S\subset\mathbb{R}^n$ is cpt, $T\subset\mathbb{R}^m$ open, f is cont on $T\times S$ & in C^1 on the 1st component for each $\mathbf{y}\in S$, Then F is C^1 on T &

$$\frac{\partial F}{\partial x_j}(\mathbf{x}) = \int\cdots\int_S \frac{\partial f}{\partial x_j}(\mathbf{x},\mathbf{y})\, d^n\mathbf{y} \quad \forall\, \mathbf{x}\in T$$

we use chain rule to extend these further

$$F(\mathbf{x}) = \int_a^{\varphi(x)} f(x,y)\,dy$$

$$= G(x_1,x_2) = \int_a^{x_1} f(x_2,y)\,dy$$

so $\frac{dG}{dx} = \frac{\partial G}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial G}{\partial x_2}\frac{dx_2}{dx}$

$= (\text{FTC}) + (4.47)$