UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2012 EXAMINATIONS

CSC 336 H1F — Numerical Methods

Duration — 3 hours

No Aids Allowed

Answer ALL Questions

Do **NOT** turn this page over until you are **TOLD** to start.

Please fill-in **ALL** the information requested on the front cover of **EACH** exam booklet that you use.

The exam consists of 6 pages, including this one. Make sure you have all 6.

The exam consists of 5 questions. **Answer all 5 questions**. The mark for each question is listed at the start of the question. Do the questions that you feel are easiest first.

The exam was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity.**

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.

1. [10 marks; 2 marks for each part]

For each of the five statements below, say whether the statement is **true** or **false** and briefly justify your answer.

(a) The choice of algorithm for solving a problem has no effect on the propagated data error.

(b) Let A be an $n \times n$ nonsingular matrix. Then $\|A^{-1}\| = \|A\|^{-1}$.

(c) If an $n \times n$ nonsingular matrix A has a very small determinant, then it is badly conditioned (i.e., $\text{cond}(A)$ is very large).

(d) For any $n \times n$ matrix A, there exist

- an $n \times n$ permutation matrix P,
- an $n \times n$ unit lower triangular matrix L, and
- an $n \times n$ upper triangular matrix U

such that $PA = LU$.

(e) Newton's method is an example of a fixed-point iteration scheme for finding a root of a nonlinear equation $f(x) = 0$.

2. [10 marks: 5 marks for each part]

Jim wrote the MatLab function

```
function [r1,r2] = roots(a,b,c)
   r1 = ( -b + sqrt(b^2 - 4*a*c) ) / (2*a) ;
   r2 = ( -b - sqrt(b^2 - 4*a*c) ) / (2*a) ;
```

to compute the two roots, r_1 and r_2, of the quadratic $ax^2 + bx + c$.

However, he later realized that his function was not calculating the roots very accurately if $r_1 \approx r_2$. For example, for

$$a = 1 + 10^{-9}, \qquad b = -2, \qquad c = 1 - 10^{-9},$$

the calculated roots are

$$r_1 = r_2 = 0.999999999000000 = 1 - 10^{-9}$$

but the true roots are

$$\hat{r}_1 = 1$$
$$\hat{r}_2 = \frac{1 - 10^{-9}}{1 + 10^{-9}} = 1 - 2 \cdot 10^{-9} + 2 \cdot 10^{-18} - 2 \cdot 10^{-27} + \cdots \approx 1 - 2 \cdot 10^{-9}$$

So, the errors in the roots are

$$r_1 - \hat{r}_1 = -10^{-9}$$
$$r_2 - \hat{r}_2 \approx +10^{-9}$$

Note that these errors are orders of magnitude larger than *machine epsilon* (which is about $2.22 \cdot 10^{-16}$ for this computation). In fact, the errors are about the order of magnitude of the square root of machine epsilon. To put it another way, Jim has only about half the number of correct digits in his computed roots compared to what would normally expected for a simple calculation of this sort.

(a) Explain why the computation is so inaccurate when $r_1 \approx r_2$.

(b) Is there any way to re-arrange the formulas

```
r1 = ( -b + sqrt(b^2 - 4*a*c) ) / (2*a) ;
r2 = ( -b - sqrt(b^2 - 4*a*c) ) / (2*a) ;
```

to obtain a more accurate method to compute the roots when $r_1 \approx r_2$ or is the inaccuracy illustrated above inherent in the problem?

Justify your answer.

3. [10 marks: 5 marks for each part]

Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

(a) Compute the LU factorization with partial pivoting of the matrix A. That is, compute a permutation matrix P, a unit-lower-triangular matrix L with all elements less than or equal to 1 in magnitude, and an upper triangular matrix U such that $PA = LU$.

Show all your calculations.

Note: you can take $P = I$ above if you don't need any pivoting.

(b) Use the LU factorization of the matrix A from part (a) to solve the linear system $Ax = b$.

Show all your calculations.

4. [10 marks: 5 marks for each part]

As we've discussed in class, the *elementary Gaussian elimination matrix* that arises at the k^{th} stage of Gaussian elimination for a system of n linear equations in n unknowns has the form

$$L_k = I - l_k e_k^T$$

where the top k elements of the n-vector l_k are zero and the bottom $n-k$ elements of l_k are the multipliers needed for the k^{th} stage of Gaussian elimination and e_k is the n-vector that has a one in position k and zeros everywhere else. We showed in class that L_k is always nonsingular and that

$$L_k^{-1} = I + l_k e_k^T$$

In this question, we'll generalize the result above.

An $n \times n$ real matrix A (i.e., $A \in \mathbb{R}^{n \times n}$) of the form

$$A = I - uv^T$$

where u and v are real n-vectors (i.e., u and $v \in \mathbb{R}^n$) is called an *elementary matrix*.

(a) What condition on u and v ensures that $A = I - uv^T$ is nonsingular?

(b) If $A = I - uv^T$ is nonsingular, show that there is a constant $\sigma \in \mathbb{R}$ such that

$$A^{-1} = I - \sigma uv^T$$

5. [10 marks: 5 marks for each part]

In class, we discussed finding the roots of the function $\hat{f}(x) = x^2 - 4\sin(x)$. In this question, we'll consider finding the roots of the similar function

$$f(x) = x^2 - 4\cos(x)$$

(a) How many real roots does $f(x)$ have? That is, how many different values $x_k \in \mathbb{R}$, $k = 1, 2, \ldots$, are there such that $f(x_k) = 0$?

For each root x_k, $k = 1, 2, \ldots$, determine an interval of length less than 1 that contains x_k. That is, for each x_k, $k = 1, 2, \ldots$, determine a_k and b_k such that $0 \le b_k - a_k < 1$ and $x_k \in [a_k, b_k]$.

Justify your answer.

(b) Give Newton's method for finding a root of $f(x) = 0$. That is, given the iteration associated with Newton's method for this particular function $f(x) = x^2 - 4\cos(x)$.

What is a good starting guess for Newton's method for $f(x) = x^2 - 4\cos(x)$ if you want to find a positive root of $f(x) = 0$?

Justify your answer.

You may find the following facts useful in solving this problem:

- $\pi \approx 3.1416$,
- $\cos(\pm\pi/4) = 1/\sqrt{2} \approx 0.7071$,
- $\cos(\pm\pi/2) = 0$,
- $\cos(\pm\pi) = -1$,
- $\cos(\pm 3\pi/2) = 0$,
- $\cos(\pm 2\pi) = 1$,
- $\dfrac{d\cos(x)}{dx} = -\sin(x)$.
- $\dfrac{d^2\cos(x)}{dx^2} = -\cos(x)$.

Have a Happy Holiday

Total Marks = 50

Total Pages = 6