Assignment 4 - MAT 327 - Summer 2014

Due June 16th, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Let $A := (-\infty, 0) \cup \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{2\} \cup (3, 4] \subseteq \mathbb{R}$, and give A the induced topology from $\mathbb{R}_{\text{usual}}$.

1. What is the interior (calculated in A) of $\{2\} \cup (3, 4]$?
2. Is $\{ \frac{1}{2n} : n \in \mathbb{N} \}$ a convergent sequence in the space $(A, \mathcal{T}_{\text{subspace}})$?
3. What are the non-trival clopen sets in this subspace?

[**C.2**] Using the list of topological invariants we gave at the end of the notes in §6, distinguish the following spaces: $\mathbb{R}_{\text{usual}}, \mathbb{R}_{\text{co-countable}}, \mathbb{R}_{\text{Sorgenfrey}}$ and $\mathbb{R}_{\text{discrete}}$. ("Distinguish" means show that no two spaces are homeomorphic.) You may wish to look at the table you made on Assignment 3, A.3.

[**C.3**] Prove that "X is separable" is a topological invariant.

[**C.4**] Show that any two (n-dimensional) epsilon balls $B_{\epsilon_1}(x_1)$ and $B_{\epsilon_2}(x_2)$ are homeomorphic, when thought of as subspaces of $\mathbb{R}^n_{\text{usual}}$. (You may wish to think of a different basis that generates the usual topology on $\mathbb{R}^n$). Does this imply that the 2 dimensional unit ball $B_1(0)$ is homeomorphic to $\mathbb{R}^2$?

[**C.5**] Prove the proposition at the end of §6.6:

Proposition. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$ be a bijection. TFAE:*

- *f is a homeomorphism*
- *f is continuous and an open function*
- *f is continuous and a closed function*

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] Let X be a topological space, with D a dense subset of X, and let Y be a Hausdorff space. Prove that, if $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are continuous functions that agree on D (i.e. $f(d) = g(d)$ for all $d \in D$) then f and g identical functions. (This says that the values of a continuous function are completely prescribed by its values on a dense set.)

The next question involves the notion of a cut-point, which intuitively is a point in a connected space, which once removed, leaves you with a disconnected space. We will investigate this more later in the course.

Definition. *Let $(X, \mathcal{T})$ be a topological space. We say that $p \in X$ is a "cut-point" of X if:*

i. X has no non-trivial clopen subsets.

ii. $X \setminus \{p\}$, as a subspace of X, has a non-trivial clopen subset.

For example, for $X = [0, 1]$, every point except 0 and 1 is a cut-point. In the circle S^1, no point is a cut-point.

[**A.2**] Show that "having a cut-point" is a topological invariant. Moreover, show that "cut-points get sent to cut-points", that is under a homeomorphism $f : X \longrightarrow Y$, $f(p)$ is a cut-point of Y iff p is a cut-point of X. Use this to distinguish the following spaces:

1. $(0, 1)$ as a subspace of $\mathbb{R}_{\text{usual}}$;
2. $[0, 1)$ as a subspace of $\mathbb{R}_{\text{usual}}$;
3. $[0, 1]$ as a subspace of $\mathbb{R}_{\text{usual}}$;
4. $\mathbb{R}^n$, for $n \geq 2$.

Conclude that $\mathbb{R}^1 \ncong \mathbb{R}^2$. (You may assume that $\mathbb{R}^n$ has no cut-points.) Does this argument also show that $\mathbb{R}^2 \ncong \mathbb{R}^3$?

[**A.3**] We saw that the Hausdorff property (sometimes called T_2, where the 'T' comes from *Trennungsaxiomen*) is a hereditary property. Later in the course we will study, in some depth, the related "separation axioms". Let's get a jump on that by considering the following separation property:

Definition. *A topological space $(X, \mathcal{T})$ is said to be T_3 provided that whenever C is a closed subset of X and $p \in X \setminus C$, there exist disjoint open sets U and V such that $C \subseteq U$ and $p \in V$.*

This says that in a T_3 space, **"we can separate any closed set from any point"** by using disjoint open sets.

Show that the T_3 property is hereditary. Why doesn't this argument also show that the T_4 property ("you can separate any two disjoint closed sets by disjoint open sets") is hereditary? Don't worry about coming up with an example of a T_4 space that is not hereditarily T_4, this is quite tricky.

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] Let's go further than A.1, with a fun problem. Suppose that $f : \mathbb{R}_{\text{usual}} \longrightarrow \mathbb{R}_{\text{usual}}$ is a continuous function that satisfies the following additive identity:

$$f(x+y) = f(x) + f(y), \forall x, y \in \mathbb{R}$$

Show that f is completely described by the value it takes on 1.

If you have taken a group theory course, conclude that

$$C^1_+(\mathbb{R}) := \{\, f : \mathbb{R} \longrightarrow \mathbb{R} | f \text{ is continuous and additive as above and } f \text{ is not the 0 function}\,\}$$

is a group (under what operation?) that is isomorphic to $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$.

What happens if we look at $C_+(\mathbb{R})$, the collection of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ (with *no* assumption about continuity)? How is this related to $\mathbb{R}$ as a vector space over $\mathbb{Q}$?

It is possible that you have seen this question before in another course. In that case, try to "push" this question further and say something new about it, or just submit a solution to a different question.

[**NI.2**] Let's investigate a method for distinguishing the various $\mathbb{R}^n$s using the notion of the **topological dimension of a space**. The definition is a bit strange because it is inductive, and doesn't look like any definition of "dimension" that we are used to.

Definition. *A topological space* $(X, \mathcal{T})$ *has* ***topological dimension 0*** *(or is Zero Dimensional) if it has a basis comprised of clopen sets.*

We have seen numerous examples of such spaces in this course (e.g. any discrete space).

Definition. *For* $n \in \mathbb{N}$*, we say that a topological space* $(X, \mathcal{T})$ *has* ***topological dimension less than or equal to*** n ***at a point*** $p \in X$ *provided that if* U *is any open set containing* p *then there is an open set* $V \ni p$ *such that* $V \subseteq U$ *and the dimension of the boundary of* V *has dimension less than or equal to* $n-1$*. Moreover, we say that* $(X, \mathcal{T})$ *has* ***topological dimension*** n ***at a point*** $p \in X$ *if the dimension of* X *at* p *is less than or equal to* n *and the dimension of* X *at* p *is not less than or equal to* $n-1$*.*

Definition. *For* $n \in \mathbb{N}$*, a topological space* $(X, \mathcal{T})$ *has* ***topological dimension less than or equal to*** n *if the dimension of* X *at each point* $p \in X$ *is less than or equal to* n*. We say that* $(X, \mathcal{T})$ *has* ***topological dimension*** n *if the dimension of* X *is less than or equal to* n *and the dimension of* X *is not less than or equal to* $n-1$*.*

Show that "having topological dimension n" is a topological invariant, then show that $\mathbb{R}^n$ has dimension n (or at the very least show that $\mathbb{R}^2$ and $\mathbb{R}^3$ have their expected dimensions). Conclude that you can distinguish $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$.

Which part of this problem is harder: showing that $\mathbb{R}^n$ has dimension n or that "topological dimension is a topological invariant"?

[**NI.3**] This is a question for people interested in "functional analysis", or "topological dynamics", which also makes use of A.1. In §6 we saw the definition of Homeo(X) the (auto)-homeomorphisms of X onto itself. Let's look at a slightly smaller space

$$\mathrm{Homeo}_*(X) := \{f \in \mathrm{Homeo}(X) : \forall a < b \in X, f(a) < f(b)\}, \text{ where } X \subseteq \mathbb{R}.$$

This is called "the orientation preserving homeomorphisms". Prove that

$$\mathrm{Homeo}_*([0,1]) \cong \mathrm{Homeo}_*((0,1)) \cong \mathrm{Homeo}_*(\mathbb{R}) \cong \mathrm{Homeo}_*(\mathbb{Q})$$

where each space (the "X"s) has its usual subspace topology inherited from $\mathbb{R}_{\text{usual}}$.
The topology on Homeo(X), where $X \subseteq \mathbb{R}$ is given by the basis

$$\mathcal{B} := \{\, V(\epsilon, F, g) : \epsilon > 0, F \text{ is a finite subset of } X, g \in \mathrm{Homeo}(X)\,\}$$

where

$$V(\epsilon, F, g) := \{\, f \in \mathrm{Homeo}_*(X) : d(g(a), f(a)) < \epsilon, \forall a \in F\,\}$$

Don't get distracted by the topology on Homeo$_*(X)$, here. The heart of this problem involves understanding the what the homeomorphisms on these spaces "look like".