Mat 337 Assignment 3 solutions

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1 Writing

1. Few people who do an undergraduate degree in mathematics become academic mathematicians. The value of being able to write a proof about Cauchy sequences is not the knowledge about Cauchy sequences, which few of you will care about once you complete your degrees. The value is being able to explain a complicated technical topic in a precise way with clear language. This is a third year course, and those for whom English is a second language should not be writing things like "the sequence is converges" when they want to say "the sequence converges"; supposing you only had basic English skills when you started university, you have now been through two and a half years in an English language university. I do not remove marks for this type of mistake, but it is a failing if you do not work to improve your ability to write clearly. It is a conceit to think that the ideas are more important than the presentation; if you are at the forefront of new research being done, then people will accept poor presentation because of how original your ideas are, but if you are doing a third year mathematics course, the presentation is at least as important as whether you think you really understand the ideas.

2. Do not start talking about things until you have quantified them. For example, it makes no sense to say there exists an s such that $B_s^\sigma(x) \subset B_r^\rho(x)$ unless I have already introduced r.

3. The following are saying two different things:

There is some $\delta > 0$ such that $B_\delta(x) \subset U$ for all $x \in U$.

For all $x \in U$ there is some $\delta > 0$ such that $B_\delta(x) \subset U$.

4. Knowing that a set is closed means that all the limit points of the set are contained in the set. Knowing this does not tell me anything directly about what sequences converge in a metric space.

5. The following does not make any sense: "Let $x \in U$. There is some $r > 0$ so $B_r^\rho(x) \subset U$. So $B_s^\sigma(x) \subset B_r^\rho(x)$." The reason it doesn't make sense is because the sentence has not stated what s is. Is this assertion applying to all s? (In this case it is false.) Is it saying that it applies to some s? (In this case, state why there is such an s.) It is nonsense unless you introduce s.

6. Read these solutions before asking any questions about marking. Most of the mistakes people make in writing a proof are not mistakes in addition or saying that something is $< \frac{1}{2}$ when it is actually < 1, but fundamentally wrong arguments or incomprehensible proofs. So merely because I did not circle an obvious mistake in your solution is not a reason why you should get more marks; if you lost marks and I did not circle a mistake, then there is some basic problem with your argument and it probably needs to be completely rewritten.

2 Solutions

These solutions might look long but they are done in complete detail. You need to be able to follow the reasoning here. There are no jumps that take deep understanding to follow. Following these solutions in detail would not be a bad way of studying and are certainly good practice learning how to write a precise argument.

p. 179, G. (a) (4 marks) Let U be a ρ-open set. That is, for all $x \in U$ there is some $r > 0$ such that $B_r^\rho(x) \subset U$. Moreover, for our definition of topologically equivalent we need a single s that does both of these; we don't get to choose a different s for each one. Because ρ and σ are topologically equivalent, there is some $s > 0$ such that $B_s^\sigma(x) \subset B_r^\rho(x)$, hence $B_s^\sigma(x) \subset U$. We have proved that for all $x \in U$ there is some $s > 0$ such that $B_s^\sigma(x) \subset U$, which shows that U is a σ-open set.

Let U be a σ-open set. That is, for all $x \in U$ there is some $r > 0$ such that $B_r^\sigma(x) \subset U$. Because ρ and σ are topologically equivalent, there is some $s > 0$ such that $B_s^\rho(x) \subset B_r^\sigma(x)$, hence $B_s^\rho(x) \subset U$. We have proved that for all $x \in U$ there is some $s > 0$ such that $B_s^\rho(x) \subset U$, which shows that U is a ρ-open set.

Let C be a ρ-closed set. Then $X \setminus C$ is a ρ-open set, and by the above therefore $X \setminus C$ is a σ-open set. Because $X \setminus C$ is σ-open, C is σ-closed.

Let C be a σ-closed set. Then $X \setminus C$ is a σ-open set, so $X \setminus C$ is a ρ-open set, and hence C is a ρ-closed set.

Therefore, ρ and σ have the same open and closed sets.

(b) (3 marks) Let $x_n \in X$ converge in ρ to some $x \in X$, and let $\epsilon > 0$. Because ρ and σ are topologically equivalent, there is some $s > 0$ such that $B_s^\rho(x) \subset B_\epsilon^\sigma(x)$. Because x_n converges in ρ to x, there is some N such that $n \geq N$ implies that $\rho(x, x_n) < s$. In other words, $n \geq N$ implies that $x_n \in B_s^\rho(x)$. But $B_s^\rho(x) \subset B_\epsilon^\sigma(x)$, so $n \geq N$ implies that $x_n \in B_\epsilon^\sigma(x)$, i.e. $\sigma(x_n, x) < \epsilon$, showing that x_n converges in σ to x.

Let $x_n \in X$ converge in σ to some $x \in X$, and let $\epsilon > 0$. Because ρ and σ are topologically equivalent, there is some $s > 0$ such that $B_s^\sigma(x) \subset B_\epsilon^\rho(x)$. Because x_n converges in σ to x, there is some N such that $n \geq N$ implies that $\sigma(x_n, x) < s$. That is, $n \geq N$ implies that $x_n \in B_s^\sigma(x)$. But $B_s^\sigma(x) \subset B_\epsilon^\rho(x)$, so $n \geq N$ implies that $x_n \in B_\epsilon^\rho(x)$, i.e. $\rho(x_n, x) < \epsilon$, showing that x_n converges in ρ to x.

Therefore ρ and σ have the same convergent sequences.

(c) (3 marks) I will give two solutions to this problem. The first solution is the method most students used. However to write it correctly takes some work. The second solution is shorter. **First solution.** Let $X = \{x \in \mathbb{R} : x > 0\}$, let

$\rho(x,y) = |x-y|$, and let $\sigma(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. First we prove that ρ and σ are topologically equivalent.[1] Let $x \in X$ and $r > 0$. We want to show that there is some s such that $B_s^\sigma(x) \subset B_r^\rho(x)$. Equivalently, we want to show that there is some s such that $\sigma(x,y) < s$ implies that $\rho(x,y) < r$.[2]

$$\sigma(x,y) < s \Leftrightarrow \left|\frac{1}{x} - \frac{1}{y}\right| < s \Leftrightarrow \left|\frac{y-x}{xy}\right| < s \Leftrightarrow \rho(x,y) < xy \cdot s.$$

If $1 - xs > 0$ then $\sigma(x,y) < s$ implies that $y < \frac{x}{1-xs}$. Therefore, if $s < \frac{1}{x}$ and $\sigma(x,y) < s$ then

$$\rho(x,y) < xy \cdot s < \frac{x^2}{1-xs} s.$$

We want to obtain $\rho(x,y) < r$, so we demand that

$$\frac{x^2}{1-xs} s \leq r,$$

i.e. that

$$s \leq \frac{r}{x(x+r)}.$$

Thus, if $s < \frac{1}{x}$ and $s \leq \frac{r}{x(x+r)}$ then

$$B_s^\sigma(x) \subset B_r^\rho(x).$$

Now let $x \in X$ and $r > 0$, and we want to show that there is some s such that $B_s^\rho(x) \subset B_r^\sigma(x)$. That is, we want to show that there is some s such that $\rho(x,y) < s$ implies that $\sigma(x,y) < r$, i.e. such that $|x-y| < s$ implies that $\left|\frac{x-y}{xy}\right| < r$. For this we need that $\frac{s}{xy} \leq r$. Now, if $|x-y| < r$ then $y > x-r$, and then $\frac{s}{xy} < \frac{s}{x(x-r)}$.[3] So we demand that s satisfy $\frac{s}{x(x-r)} \leq r$, i.e. $s \leq x(x-r)r$. Thus, if $s \leq x(x-r)r$, then

$$B_s^\rho(x) \subset B_r^\sigma(x).$$

We have established that σ and ρ are topologically equivalent.

We now give an example of a sequence that is Cauchy with respect to σ and not Cauchy with respect to ρ: the sequence n. First I will prove that this sequence is Cauchy with respect to σ. Let $\epsilon > 0$, and let $N > \frac{1}{\epsilon}$. If $n \geq m \geq N$ then

$$\sigma(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{|m-n|}{nm} < \frac{n}{nm} = \frac{1}{m} \leq \frac{1}{N} < \epsilon.$$

[1] In mathematics if you are asked to give an example of something you have to show that it actually is an example. It is worthless just to write down two metrics without showing that they are equivalent and stating a sequence that is Cauchy in one and not the other.

[2] s will depend on x and r, but may not depend on y.

[3] Suppose that $r < x$; if $r \geq x$, then apply this argument to $r_0 < x$, to be specific say $r_0 = \frac{x}{2}$, and of course $B_{r_0}^\sigma(x) \subset B_r^\sigma(x)$.

This shows that n is a Cauchy sequence with respect to σ. You can take it for granted that n is not a Cauchy sequence with respect to ρ. (That is something you should be able quickly to prove. It starts by taking a specific ϵ, then go from there.)

Therefore, ρ and σ are topologically equivalent metrics with different Cauchy sequences. The point of this exercise is that the topology induced by a metric does not say everything there is to be said about the metric; there are some things that can be talked about using metrics that cannot be talked about only using the language of open sets.

Second solution. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers. Let $\rho(x, y) = |x - y|$ and let σ be the discrete metric: $\sigma(x, y) = 0$ if $x = y$, and $= 1$ if $x \neq y$. We first prove that ρ and σ are topologically equivalent. Let $n \in \mathbb{N}$ and $r > 0$. We have to show that there is some s such that $B^\rho_s(\frac{1}{n}) \subset B^\sigma_r(\frac{1}{n})$ and $B^\sigma_s(\frac{1}{n}) \subset B^\rho_r(\frac{1}{n})$. There are two cases that I will deal with separately: either $0 < r \leq 1$ or $r > 1$. In the first case where $0 < r \leq 1$, $B^\sigma_r(\frac{1}{n}) = \{\frac{1}{n}\}$, and we want to show that there is some $s > 0$ such that $B^\rho_s(\frac{1}{n}) = \{\frac{1}{n}\}$. I assert that $s = \frac{1}{(n+1)^2}$ works. If $\frac{1}{m} \in B^\rho_s(\frac{1}{n})$, then $|\frac{1}{m} - \frac{1}{n}| < s$, which means that $|\frac{n-m}{mn}| < s$, which means that $|n-m| < mns = \frac{mn}{(n+1)^2}$. Check that this is only possible when $m = n$. That is, with $s = \frac{1}{(n+1)^2}$ we get that $B^\rho_s(\frac{1}{n}) = \{\frac{1}{n}\} = B^\sigma_r(\frac{1}{n})$. In the second case where $r \geq 1$, $B^\sigma_r(\frac{1}{n}) = X$. In this situation any $s > 0$, say $s = 1$ to be specific, gives us $B^\rho_s(\frac{1}{n}) \subset B^\sigma_r(\frac{1}{n})$. We have proved that for any $n \in \mathbb{N}$ and any $r > 0$ there is some $s > 0$ such that $B^\rho_s(\frac{1}{n}) \subset B^\sigma_r(\frac{1}{n})$.

For $n \in \mathbb{N}$ and $r > 0$, we still have to show that there is some s such that $B^\sigma_s(\frac{1}{n}) \subset B^\rho_r(\frac{1}{n})$. Take $s = 1$.[4] Then $B^\sigma_1(\frac{1}{n}) = \{\frac{1}{n}\}$, which is certainly contained in $B^\rho_r(\frac{1}{n})$. This completes the proof that ρ and σ are topologically equivalent.

Showing that ρ and σ are topologically equivalent is the part of the answer that takes most of the work. I will prove that the sequence $\frac{1}{n}$ is a Cauchy sequence with respect to ρ but not with respect to σ. Let $\epsilon > 0$ and let $N > \frac{1}{\epsilon}$. For $m \geq n \geq N$,

$$\rho(\frac{1}{m}, \frac{1}{n}) = |\frac{1}{m} - \frac{1}{n}| = |\frac{n-m}{mn}| < \frac{m}{mn} = \frac{1}{n} < \frac{1}{N} < \epsilon,$$

showing that $\frac{1}{n}$ is a Cauchy sequence with respect to ρ. The only part left of the solution is to show that $\frac{1}{n}$ is not a Cauchy sequence with respect to σ. It is not because the only Cauchy sequences for the discrete metric are sequences that are eventually constant, i.e., for which there is some constant c and some N such that $n \geq N$ implies that $x_n = c$. The sequence $\frac{1}{n}$ is not eventually constant, so it is not a Cauchy sequence with respect to the discrete metric, completing the solution.

p. 179, H. (a) (4 marks) Let $x \in X$ and $r > 0$. We want to prove that there is some $s > 0$ such that $B^\sigma_s(x) \subset B^\rho_r(x)$ and $B^\rho_s(x) \subset B^\sigma_r(x)$.[5] Let

[4] Sometimes people are cautious and use $\frac{1}{2}$ instead of 1, even when they don't have to. Since we are talking about open balls, we don't need to use $\frac{1}{2}$ here.

[5] It must be the same s for both of these inclusions according to the definition in the book.

$s = \min\{rc, \frac{r}{C}\}$. If $y \in B^\sigma_s(x)$, then $\sigma(x,y) < s$, so because σ and ρ are equivalent this means that $c\rho(x,y) < s$, which means that $\rho(x,y) < \frac{s}{c}$. But $s \leq rc$, so this means that $\rho(x,y) < r$, i.e. that $y \in B^\rho_r(x)$.

If $y \in B^\rho_s(x)$, then $\rho(x,y) < s$, so because σ and ρ are equivalent this means that $\frac{1}{C}\sigma(x,y) < s$, i.e. $\sigma(x,y) < Cs$. But $s \leq \frac{r}{C}$, so $\sigma(x,y) < r$, i.e. $y \in B^\sigma_r(x)$. We have proved that ρ and σ are topologically equivalent.

(b) (4 marks) Suppose that x_n is a Cauchy sequence with respect to ρ. We will show that x_n is a Cauchy sequence with respect to σ, and the other direction is the same. Let $\epsilon > 0$. Because x_n is a Cauchy sequence with respect to ρ, there is some N such that $n, m \geq N$ implies that

$$\rho(x_n, x_m) < \frac{\epsilon}{C}.$$

Because ρ and σ are equivalent, if $n, m \geq N$ then

$$\sigma(x_n, x_m) \leq C\rho(x_n, x_m) < C \cdot \frac{\epsilon}{C} = \epsilon,$$

showing that x_n is a Cauchy sequence with respect to σ.

(c) (2 marks) In **G (c)** we showed that ρ and σ were topologically equivalent and had different Cauchy sequences. If they were equivalent, then by **H (b)** they would have the same Cauchy sequences, which they do not. Hence ρ and σ are topologically equivalent metrics that are not equivalent.

p. 179, I. (5 marks for proving ρ is a metric) The only matrix with rank 0 is the 0 matrix. Thus, $\operatorname{rank}(A - B) = 0$ is equivalent to $A - B = 0$, which is equivalent to $A = B$. Therefore, $\rho(A, B) = 0$ implies that $A = B$. Furthermore,

$$\rho(A, C) = \operatorname{rank}(A - C) = \operatorname{rank}(A - B + B - C)$$

It is a fact that I will take as given that the rank of a sum of two matrices is $\leq$ the sum of their ranks. Using this, we get

$$\operatorname{rank}(A - B + B - C) \leq \operatorname{rank}(A - B) + \operatorname{rank}(B - C) = \rho(A, B) + \rho(B, C),$$

showing that ρ satisfies the triangle inequality. Finally, the rank of a matrix is equal to the rank of the negative of the matrix, so $\rho(A, B) = \operatorname{rank}(A - B) = \operatorname{rank}(B - A) = \rho(B, A)$. Therefore ρ is a metric on the set of $n \times n$ matrices.

(5 marks for proving topologically equivalent) We now prove that ρ is topologically equivalent to the discrete metric d. Let $A \in \mathscr{M}_n$ and let $r > 0$. Let $s = 1$:

$$B^\rho_1(A) = \{B \in \mathscr{M}_n : \operatorname{rank}(A - B) < 1\} = \{B \in \mathscr{M}_n : \operatorname{rank}(A - B) = 0\}$$

$$= \{A\} \subset B^d_r(A).$$

$$B^d_1(A) = \{B \in \mathscr{M}_n : d(A, B) < 1\} = \{B \in \mathscr{M}_n : d(A, B) = 0\} = \{A\} \subset B^\rho_r(A).$$

This shows that ρ and d are topologically equivalent.

p. 72, M. (a) (4 marks) Using the definition of C on p. 71, we have, because $a_{i,j}$ is either $-\frac{1}{2}$ or $\frac{1}{2}$,

$$\begin{aligned} C^2 &= \sum_{i=1}^{4}\sum_{j=1}^{4}|a_{i,j}|^2 \\ &= \sum_{i=1}^{4}\sum_{j=1}^{4}\frac{1}{4} \\ &= 4. \end{aligned}$$

Hence $C = 2$.

(b) (6 marks) Let a_1, a_2, a_3, a_4 be the columns of A. For $x \in \mathbb{R}^4$,

$$Ax = x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4.$$

Then,

$$\begin{aligned} \|Ax\|^2 = \langle Ax, Ax\rangle &= \langle x_1a_1 + \cdots + x_4a_4, x_1a_1 + \cdots + x_4a_4\rangle \\ &= \sum_{i=1}^{4}\sum_{j=1}^{4} x_ix_j\,\langle a_i, a_j\rangle \end{aligned}$$

If $i \neq j$ then $\langle a_i, a_j\rangle = 0$ and $\langle a_i, a_i\rangle = 1$; this is what it means to say that they are orthonormal. Hence

$$\sum_{i=1}^{4}\sum_{j=1}^{4} x_ix_j\,\langle a_i, a_j\rangle = \sum_{i=1}^{4} x_ix_i\,\langle a_i, a_i\rangle = \sum_{i=1}^{4} x_i^2 = \|x\|^2,$$

showing that

$$\|Ax\|^2 = \|x\|^2,$$

i.e.

$$\|Ax\| = \|x\|.$$

Using this, for $x, y \in \mathbb{R}^4$,

$$\|Ax - Ay\| = \|A(x-y)\| = \|x - y\|.$$

p. 76, A. (10 marks) (3 marks for showing that χ_A is continuous on interior of A) Let x_0 be a point in the interior of A, and let $\epsilon > 0$. Because x_0 is in the interior of A, there is some $r > 0$ such that $\|x - x_0\| < r$ implies $x \in A$ and hence $\chi_A(x) = 1$. If $\|x - x_0\| < r$, then $|\chi_A(x) - \chi_A(x_0)| = |1 - 1| = 0 < \epsilon$, showing that χ_A is continuous at x_0.

(2 marks for showing that χ_A is continuous on the interior of the complement of A) Let x_0 be a point in the interior of the complement of A and let $\epsilon > 0$. Because x_0 is in the interior of the complement of A, there is some $r > 0$ such that $\|x - x_0\| < r$ implies that $x \notin A$, hence $\chi_A(x) = 0$. If $\|x - x_0\| < r$, then $|\chi_A(x) - \chi_A(x_0)| = |0 - 0| = 0 < \epsilon$, showing that χ_A is continuous at x_0.

(5 marks for showing that χ_A is discontinuous on ∂A) Let x_0 be a point in ∂A. This means that for every $r > 0$, there is some $x \in A$ such that $\|x - x_0\| < r$, and there is some $x \notin A$ such that $\|x - x_0\| < r$. To prove that χ_A is discontinuous at x_0, we have to prove that there is some $\epsilon > 0$ such that for all $\delta > 0$ there is some x with $\|x - x_0\| < \delta$ and $\|\chi_A(x) - \chi_A(x_0)\| \geq \epsilon$. Let $\delta > 0$.

Either $x_0 \in A$ or $x_0 \notin A$. In the first case, there is some $x \notin A$ with $\|x - x_0\| < \delta$, and then

$$|\chi_A(x) - \chi_A(x_0)| = |0 - 1| = 1 \geq 1.$$

In the second case, there is some $x \in A$ with $\|x - x_0\| < \delta$, and then

$$|\chi_A(x) - \chi_A(x_0)| = |1 - 0| = 1 \geq 1.$$

Therefore, we have shown that for all $\delta > 0$ there is some x with $\|x - x_0\| < \delta$ and such that $|\chi_A(x) - \chi_A(x_0)| \geq 1$. This shows that χ_A is discontinuous at x_0.

p. 76, B. (10 marks) To prove that f has a removable singularity at $x = 0$ means to prove that $f(x)$ has a limit as $x \to 0$. If we want to prove that a particular function has a limit, we almost always have to actually figure out what the limit is. I claim that $f(x) \to 0$ as $x \to 0$. If I prove this, that will show that f has a removable singularity at $x = 0$. Using L'Hospital's rule,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x \log(x^2) = \lim_{x \to 0^+} 2\frac{\log x}{\frac{1}{x}} = \lim_{x \to 0^+} 2\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -2x = 0.$$

Again this,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x \log(x^2) = \lim_{x \to 0^+} (-x) \log((-x)^2) = -\lim_{x \to 0^+} x \log(x^2) = -0.$$

Since $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 0^-} f(x) = 0$, we have $\lim_{x \to 0} f(x) = 0$, showing that f has a removable singularity at $x = 0$.