16.4 GREEN'S THEOREM

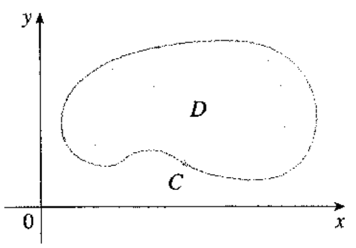


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

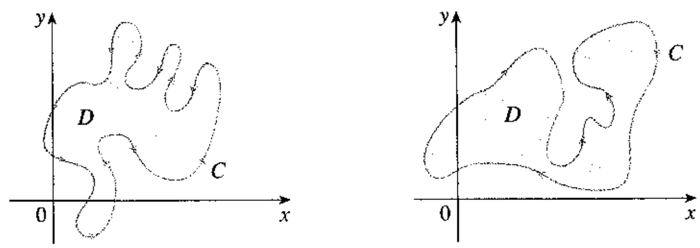


FIGURE 2

(a) Positive orientation (b) Negative orientation

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

NOTE The notation

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\int_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\boxed{1} \qquad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives (F', $\partial Q/\partial x$, and $\partial P/\partial y$) on the left side of the equation. And in both cases the right side involves the values of the original functions (F, Q, and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b.)

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both of type I and of type II (see Section 15.3). Let's call such regions **simple regions**.

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \qquad \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

and

$$\boxed{3} \qquad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \qquad \iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx$$

where the last step follows from the Fundamental Theorem of Calculus.

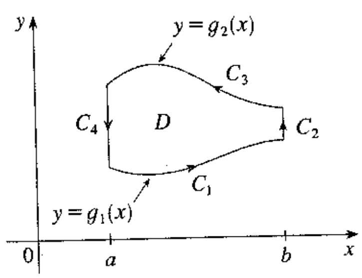


FIGURE 3

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1, C_2, C_3, and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leqslant x \leqslant b$. Thus

$$\int_{C_1} P(x, y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leqslant x \leqslant b$. Therefore

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\,dx = 0 = \int_{C_4} P(x, y)\,dx$$

Hence

$$\begin{aligned}\int_C P(x, y)\,dx &= \int_{C_1} P(x, y)\,dx + \int_{C_2} P(x, y)\,dx + \int_{C_3} P(x, y)\,dx + \int_{C_4} P(x, y)\,dx \\ &= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx\end{aligned}$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y)\, dx = -\iint_D \frac{\partial P}{\partial y}\, dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 28). Then, by adding Equations 2 and 3, we obtain Green's Theorem. □

EXAMPLE 1 Evaluate $\int_C x^4\, dx + xy\, dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

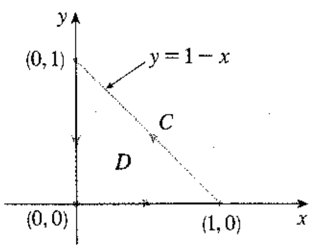


FIGURE 4

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\, dx + xy\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\, dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$ □

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leq 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi} \int_0^3 (7 - 3)\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\, dr = 36\pi$$ □

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\, dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\, dx + Q\, dy = 0$$

no matter what values P and Q assume in the region D.

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$$
$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

5
$$\boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx}$$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leqslant t \leqslant 2\pi$. Using the third formula in Equation 5, we have

$$\begin{aligned} A &= \tfrac{1}{2}\int_C x\,dy - y\,dx \\ &= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab \end{aligned}$$

□

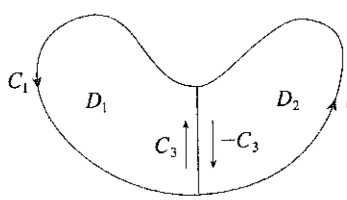


FIGURE 5

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 5, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

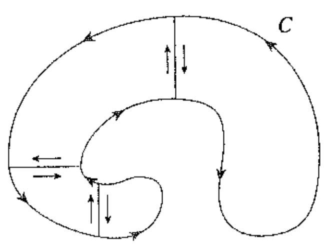


FIGURE 6

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 6).

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 7). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

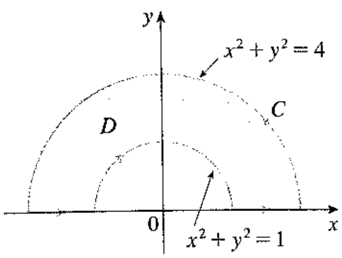


FIGURE 7

Therefore Green's Theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D y\,dA = \int_0^\pi \int_1^2 (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr = \left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3} \qquad \square$$

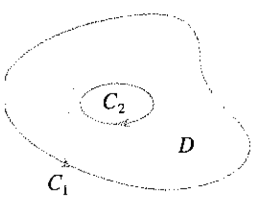


FIGURE 8

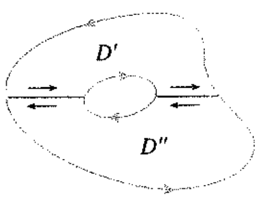


FIGURE 9

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 8 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. If we divide D into two regions D' and D'' by means of the lines shown in Figure 9 and then apply Green's Theorem to each of D' and D'', we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region D.

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

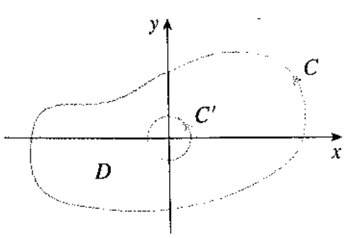


FIGURE 10

SOLUTION Since C is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle C' with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 10.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0$$

Therefore

$$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi \qquad \square$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that $\mathbf{F}$ is a conservative vector field.

16.4 EXERCISES

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C (x - y)\,dx + (x + y)\,dy$,
C is the circle with center the origin and radius 2

2. $\oint_C xy\,dx + x^2\,dy$,
C is the rectangle with vertices (0, 0), (3, 0), (3, 1), and (0, 1)

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C x\,dx + y\,dy$, C consists of the line segments from (0, 1) to (0, 0) and from (0, 0) to (1, 0) and the parabola $y = 1 - x^2$ from (1, 0) to (0, 1)

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy^2\,dx + 2x^2y\,dy$,
C is the triangle with vertices (0, 0), (2, 2), and (2, 4)

6. $\int_C \cos y\,dx + x^2\sin y\,dy$,
C is the rectangle with vertices (0, 0), (5, 0), (5, 2), and (0, 2)

7. $\int_C \left(y + e^{\sqrt{x}}\right)dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

8. $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$,
C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

9. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

10. $\int_C \sin y\,dx + x\cos y\,dy$, C is the ellipse $x^2 + xy + y^2 = 1$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y}\rangle$,
C consists of the arc of the curve $y = \sin x$ from (0, 0) to (π, and the line segment from (π, 0) to (0, 0)

12. $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$,
C is the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$

13. $\mathbf{F}(x, y) = \langle e^x + x^2 y, e^y - xy^2 \rangle$,
C is the circle $x^2 + y^2 = 25$ oriented clockwise

14. $\mathbf{F}(x, y) = \langle y - \ln(x^2 + y^2), 2 \tan^{-1}(y/x) \rangle$, C is the circle $(x - 2)^2 + (y - 3)^2 = 1$ oriented counterclockwise

CAS **15–16** Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15. $P(x, y) = y^2 e^x$, $Q(x, y) = x^2 e^y$,
C consists of the line segment from $(-1, 1)$ to $(1, 1)$ followed by the arc of the parabola $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$

16. $P(x, y) = 2x - x^3 y^5$, $Q(x, y) = x^3 y^8$,
C is the ellipse $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

18. A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

19. Use one of the formulas in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

20. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use (5) to find the area it encloses.

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A} \oint_C y^2\,dx$$

where A is the area of D.

23. Use Exercise 22 to find the centroid of a quarter-circular region of radius a.

24. Use Exercise 22 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3\,dx \qquad I_y = \frac{\rho}{3} \oint_C x^3\,dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 15.5.)

27. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

28. Complete the proof of the special case of Green's Theorem by proving Equation 3.

29. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\,dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

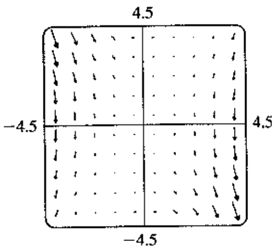
[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

16.5 CURL AND DIVERGENCE

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

11. II **13.** I **15.** IV **17.** III

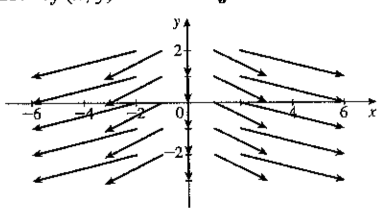
19. The line $y = 2x$



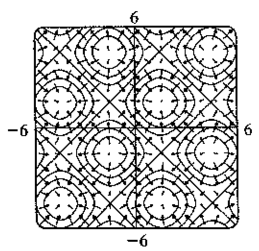
21. $\nabla f(x, y) = (xy + 1)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$

23. $\nabla f(x, y, z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$

25. $\nabla f(x, y) = 2x\,\mathbf{i} - \mathbf{j}$

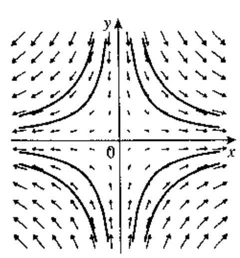


27.



29. III **31.** II **33.** (2.04, 1.03)

35. (a)



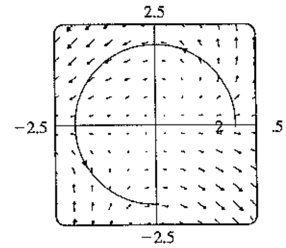
$y = C/x$

(b) $y = 1/x,\ x > 0$

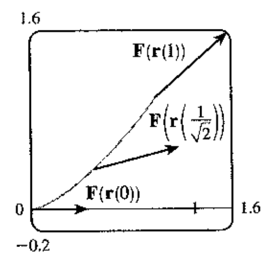
EXERCISES 16.2 ■ PAGE 1043

1. $\frac{1}{54}(145^{3/2} - 1)$ **3.** 1638.4 **5.** $\frac{243}{8}$ **7.** $\frac{17}{3}$ **9.** $\sqrt{5}\pi$
11. $\frac{1}{12}\sqrt{14}(e^6 - 1)$ **13.** $\frac{1}{5}$ **15.** $\frac{97}{3}$
17. (a) Positive (b) Negative
19. 45 **21.** $\frac{6}{5} - \cos 1 - \sin 1$ **23.** 1.9633 **25.** 15.0074

27. $3\pi + \frac{2}{3}$



29. (a) $\frac{11}{8} - 1/e$ (b)



31. $\frac{172{,}704}{5{,}632{,}705}\sqrt{2}(1 - e^{-14\pi})$ **33.** $2\pi k$, $(4/\pi, 0)$
35. (a) $\bar{x} = (1/m)\int_C x\rho(x, y, z)\,ds$,
$\bar{y} = (1/m)\int_C y\rho(x, y, z)\,ds$,
$\bar{z} = (1/m)\int_C z\rho(x, y, z)\,ds$, where $m = \int_C \rho(x, y, z)\,ds$
(b) $(0, 0, 3\pi)$
37. $I_x = k(\frac{1}{2}\pi - \frac{4}{3})$, $I_y = k(\frac{1}{2}\pi - \frac{2}{3})$
39. $2\pi^2$ **41.** 26 **43.** 1.67×10^4 ft-lb **45.** (b) Yes
47. $\approx$22 J

EXERCISES 16.3 ■ PAGE 1053

1. 40 **3.** $f(x, y) = x^2 - 3xy + 2y^2 - 8y + K$
5. $f(x, y) = e^x \sin y + K$ **7.** $f(x, y) = ye^x + x \sin y + K$
9. $f(x, y) = x \ln y + x^2y^3 + K$
11. (b) 16 **13.** (a) $f(x, y) = \frac{1}{2}x^2y^2$ (b) 2
15. (a) $f(x, y, z) = xyz + z^2$ (b) 77
17. (a) $f(x, y, z) = xy^2 \cos z$ (b) 0
19. 2 **21.** 30 **23.** No **25.** Conservative
29. (a) Yes (b) Yes (c) Yes
31. (a) Yes (b) Yes (c) No

EXERCISES 16.4 ■ PAGE 1060

1. 8π **3.** $\frac{2}{3}$ **5.** 12 **7.** $\frac{1}{3}$ **9.** -24π **11.** $\frac{4}{3} - 2\pi$
13. $\frac{625}{2}\pi$ **15.** $-8e + 48e^{-1}$ **17.** $-\frac{1}{12}$ **19.** 3π **21.** (c) $\frac{9}{2}$
23. $(4a/3\pi, 4a/3\pi)$ if the region is the portion of the disk $x^2 + y^2 = a^2$ in the first quadrant

EXERCISES 16.5 ■ PAGE 1068

1. (a) $-x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$ (b) yz
3. (a) $(x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$ (b) $z - 1/(2\sqrt{z})$
5. (a) $\mathbf{0}$ (b) $2/\sqrt{x^2 + y^2 + z^2}$
7. (a) $\langle 1/y, -1/x, 1/x \rangle$ (b) $1/x + 1/y + 1/z$
9. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$
11. (a) Zero (b) curl $\mathbf{F}$ points in the negative z-direction
13. $f(x, y, z) = xy^2z^3 + K$ **15.** $f(x, y, z) = x^2y + y^2z + K$
17. Not conservative **19.** No

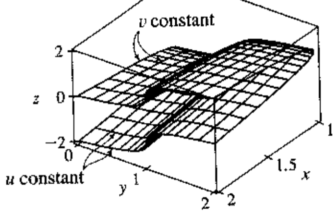
EXERCISES 16.6 ▫ PAGE 1078

1. P: no; Q: yes

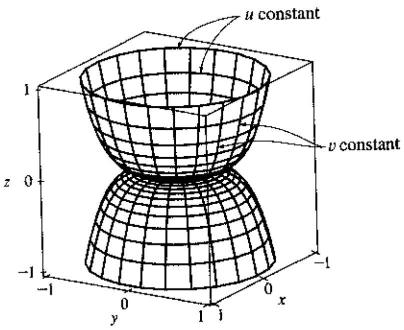
3. Plane through $(0, 3, 1)$ containing vectors $\langle 1, 0, 4\rangle$, $\langle 1, -1, 5\rangle$

5. Hyperbolic paraboloid

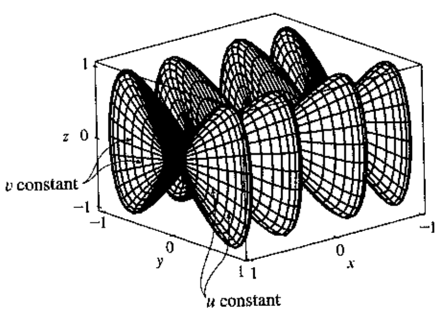
7.



9.



11.



13. IV **15.** II **17.** III

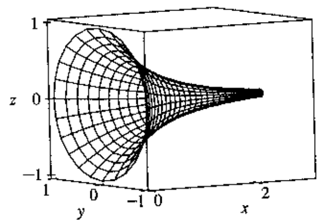
19. $x = 1 + u + v,\ y = 2 + u - v,\ z = -3 - u + v$

21. $x = x,\ z = z,\ y = \sqrt{1 - x^2 + z^2}$

23. $x = 2\sin\phi\cos\theta,\ y = 2\sin\phi\sin\theta,$
$z = 2\cos\phi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \theta \leqslant 2\pi$
$\left[\text{or } x = x,\ y = y,\ z = \sqrt{4 - x^2 - y^2},\ x^2 + y^2 \leqslant 2\right]$

25. $x = x,\ y = 4\cos\theta,\ z = 4\sin\theta,\ 0 \leqslant x \leqslant 5,\ 0 \leqslant \theta \leqslant 2\pi$

29. $x = x,\ y = e^{-x}\cos\theta,$
$z = e^{-x}\sin\theta,\ 0 \leqslant x \leqslant 3,$
$0 \leqslant \theta \leqslant 2\pi$



31. (a) Direction reverses (b) Number of coils doubles

33. $3x - y + 3z = 3$ **35.** $-x + 2z = 1$ **37.** $3\sqrt{14}$

39. $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1)$ **41.** $(2\pi/3)(2\sqrt{2} - 1)$

43. $(\pi/6)(17\sqrt{17} - 5\sqrt{5})$

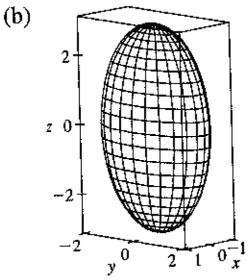
45. $\frac{1}{2}\sqrt{21} + \frac{17}{4}\left[\ln(2 + \sqrt{21}) - \ln\sqrt{17}\right]$ **47.** 4

49. 13.9783

51. (a) 24.2055 (b) 24.2476

53. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$

55. (b)



(c) $\int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv$

57. 4π **59.** $2a^2(\pi - 2)$

EXERCISES 16.7 ▫ PAGE 1091

1. 49.09 **3.** 900π **5.** $171\sqrt{14}$ **7.** $\sqrt{3}/24$

9. $5\sqrt{5}/48 + 1/240$ **11.** $364\sqrt{2}\pi/3$

13. $(\pi/60)(391\sqrt{17} + 1)$ **15.** 16π **17.** 12

19. $\frac{713}{180}$ **21.** $-\frac{1}{6}$ **23.** $-\frac{4}{3}\pi$ **25.** 0 **27.** 48

29. $2\pi + \frac{8}{3}$ **31.** 0.1642 **33.** 3.4895

35. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D [P(\partial h/\partial x) - Q + R(\partial h/\partial z)]\,dA$, where D = projection of S on xz-plane

37. $(0, 0, a/2)$

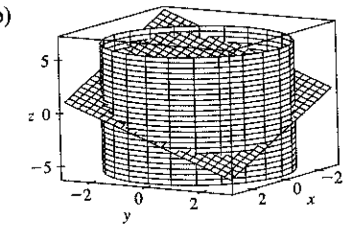
39. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\,dS$ (b) $4329\sqrt{2}\pi/5$

41. 0 kg/s **43.** $\frac{8}{3}\pi a^3\varepsilon_0$ **45.** 1248π

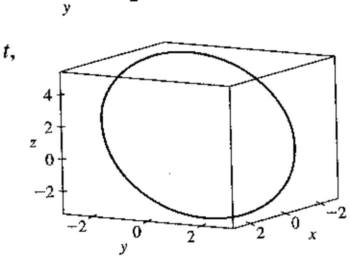
EXERCISES 16.8 ▫ PAGE 1097

3. 0 **5.** 0 **7.** -1 **9.** 80π

11. (a) $81\pi/2$ (b)



(c) $x = 3\cos t,\ y = 3\sin t,$
$z = 1 - 3(\cos t + \sin t),$
$0 \leqslant t \leqslant 2\pi$



17. 3

EXERCISES 16.9 ▫ PAGE 1103

5. 2 **7.** $9\pi/2$

9. 0 **11.** $32\pi/3$ **13.** 0

15. $341\sqrt{2}/60 + \frac{81}{20}\arcsin(\sqrt{3}/3)$ **17.** $13\pi/20$

19. Negative at P_1, positive at P_2

21. $\operatorname{div}\mathbf{F} > 0$ in quadrants I, II; $\operatorname{div}\mathbf{F} < 0$ in quadrants III, IV

CHAPTER 16 REVIEW ■ PAGE 1106

True-False Quiz

1. False **3.** True **5.** False **7.** True

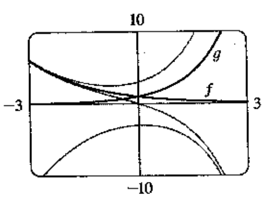
Exercises

1. (a) Negative (b) Positive **3.** $6\sqrt{10}$ **5.** $\frac{4}{15}$
7. $\frac{110}{3}$ **9.** $\frac{11}{12} - 4/e$ **11.** $f(x, y) = e^y + xe^{xy}$ **13.** 0
17. -8π **25.** $\frac{1}{6}(27 - 5\sqrt{5})$
27. $(\pi/60)(391\sqrt{17} + 1)$ **29.** $-64\pi/3$
33. $-\frac{1}{2}$ **37.** -4 **39.** 21

CHAPTER 17

EXERCISES 17.1 ■ PAGE 1117

1. $y = c_1e^{3x} + c_2e^{-2x}$ **3.** $y = c_1 \cos 4x + c_2 \sin 4x$
5. $y = c_1e^{2x/3} + c_2xe^{2x/3}$ **7.** $y = c_1 + c_2e^{x/2}$
9. $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$
11. $y = c_1e^{(\sqrt{3}-1)t/2} + c_2e^{-(\sqrt{3}+1)t/2}$
13. $P = e^{-t}\left[c_1 \cos\left(\frac{1}{10}t\right) + c_2 \sin\left(\frac{1}{10}t\right)\right]$
15.

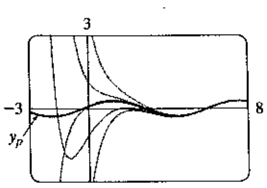


All solutions approach either 0 or $\pm\infty$ as $x \to \pm\infty$.

17. $y = 2e^{-3x/2} + e^{-x}$ **19.** $y = e^{x/2} - 2xe^{x/2}$
21. $y = 3 \cos 4x - \sin 4x$ **23.** $y = e^{-x}(2 \cos x + 3 \sin x)$
25. $y = 3 \cos\left(\frac{1}{2}x\right) - 4 \sin\left(\frac{1}{2}x\right)$ **27.** $y = \dfrac{e^{x+3}}{e^3 - 1} + \dfrac{e^{2x}}{1 - e^3}$
29. No solution
31. $y = e^{-2x}(2 \cos 3x - e^{\pi} \sin 3x)$
33. (b) $\lambda = n^2\pi^2/L^2$, n a positive integer; $y = C \sin(n\pi x/L)$

EXERCISES 17.2 ■ PAGE 1124

1. $y = c_1e^{-2x} + c_2e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$
3. $y = c_1 + c_2e^{2x} + \frac{1}{40} \cos 4x - \frac{1}{20} \sin 4x$
5. $y = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}$
7. $y = \frac{3}{2} \cos x + \frac{11}{2} \sin x + \frac{1}{2}e^x + x^3 - 6x$
9. $y = e^x\left(\frac{1}{2}x^2 - x + 2\right)$
11.

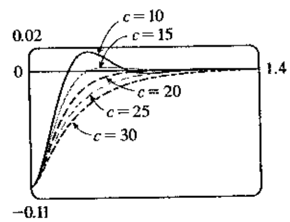


The solutions are all asymptotic to $y_p = \frac{1}{10} \cos x + \frac{3}{10} \sin x$ as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.

13. $y_p = Ae^{2x} + (Bx^2 + Cx + D) \cos x + (Ex^2 + Fx + G) \sin x$
15. $y_p = Ax + (Bx + C)e^{9x}$
17. $y_p = xe^{-x}[(Ax^2 + Bx + C) \cos 3x + (Dx^2 + Ex + F) \sin 3x]$
19. $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right) - \frac{1}{3} \cos x$
21. $y = c_1e^x + c_2xe^x + e^{2x}$
23. $y = c_1 \sin x + c_2 \cos x + \sin x \ln(\sec x + \tan x) - 1$
25. $y = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$
27. $y = e^x\left[c_1 + c_2x - \frac{1}{2}\ln(1 + x^2) + x \tan^{-1}x\right]$

EXERCISES 17.3 ■ PAGE 1132

1. $x = 0.35 \cos(2\sqrt{5}\, t)$ **3.** $x = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$ **5.** $\frac{49}{12}$ kg
7.



13. $Q(t) = (-e^{-10t}/250)(6 \cos 20t + 3 \sin 20t) + \frac{3}{125}$,
$I(t) = \frac{3}{5}e^{-10t} \sin 20t$
15. $Q(t) = e^{-10t}\left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t\right]$
$- \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$

EXERCISES 17.4 ■ PAGE 1137

1. $c_0 \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0e^x$ **3.** $c_0 \displaystyle\sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0e^{x^3/3}$
5. $c_0 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n + 1)!}x^{2n+1}$
7. $c_0 + c_1 \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = c_0 - c_1 \ln(1 - x)$ for $|x| < 1$
9. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$
11. $x + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^2 5^2 \cdot \cdots \cdot (3n - 1)^2}{(3n + 1)!}x^{3n+1}$

CHAPTER 17 REVIEW ■ PAGE 1138

True-False Quiz

1. True **3.** True

Exercises

1. $y = c_1e^{5x} + c_2e^{-3x}$ **3.** $y = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$
5. $y = e^{2x}(c_1 \cos x + c_2 \sin x + 1)$
7. $y = c_1e^x + c_2xe^x - \frac{1}{2} \cos x - \frac{1}{2}(x + 1) \sin x$
9. $y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$
11. $y = 5 - 2e^{-6(x-1)}$ **13.** $y = (e^{4x} - e^x)/3$
15. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n + 1)!}x^{2n+1}$
17. $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$
19. (c) $2\pi/k \approx 85$ min (d) $\approx$17,600 mi/h