University of Toronto
**MAT237Y1Y TERM TEST 2**
**Thursday, Feb. 14, 2013**
Duration: 100 minutes

**No aids allowed**

**Instructions:** There are 11 pages including the cover page. Please answer all questions in the spaces provided (if you use back of a sheet please clearly specify that.) Document your arguments by briefly stating the results you use (in most cases you may find relevant definitions or results in an earlier question of this test.) The value for each question is indicated in parentheses beside the question number. The test is out of 80 and there are 8 bonus marks embedded in the test (total of 88 marks to be found, and there are 15 marks to be added to your midterm mark.)

**NAME:** (last, first)

**STUDENT NUMBER:** MARKING SCHEME

**SIGNATURE:**

**CHECK YOUR TUTORIAL:**

| TUT0201 Mon. 4-5 | ○ TUT0301 Tue. 2-3 | ○ TUT0401 Wed. 3-4 | ○ TUT5101 Tue. 5-6 | ○ TUT5201 Wed. 5-6 | ○ TUT5301 Thu. 5-6 |
|---|---|---|---|---|---|

**MARKER'S REPORT:**

| Question | MARK |
|---|---|
| Q1 | /23 |
| Q2 | /20 |
| Q3 | /21 |
| Q4 | /25 |
| Q5 | MT |
| TOTAL | /80 |

adds up to 88 which includes 8 bonus marks

was added to the midterm

was not added

Nikita

1. Smooth curves

a) (8 marks) List three representations of a smooth curve in the plane.

0.5

i) graph: $y = f(x)$ or $x = f(y)$ 1 $f$ is $C^1$ 0.5

0.5

ii) locus or implicit: 2 $F(x,y)=0$ $F$ is $C^1$ 0.5

0.5

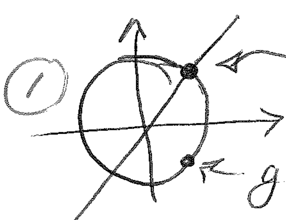
iii) parametric: $\begin{bmatrix} x \\ y \end{bmatrix} = f(t) = \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}$ $f$ is $C^1$ 0.5

2

b) (5 marks) Find a point at which the set $S = \{(x,y) : F(x,y) = (x-y)(x^2+y^2-1) = 0\}$ is a smooth curve, and find another point at which $S$ is **not** a smooth curve. Justify your answer with reference to definition of smooth curve.

Def of Smooth Curve: $S \subseteq \mathbb{R}^2$ is a Smooth Curve near $x_0 \in S$ if there exists a nbd $N$ of $x_0$ s.t. $N \cap S$ is the graph of a $C^1$ function.

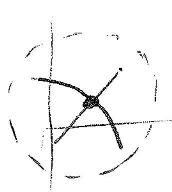
1.5




1

$S$ is union of $x-y=0$ and $x^2+y^2-1=0$

near the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ for any nbd $N$, $N \cap S$ looks like



1 fails the vertical line test.

at other pts such as $x_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$\frac{\partial F}{\partial y}(x_1) = -(x^2+y^2-1) + 2y(x-y)$

$= 0 + \sqrt{2}\left(\frac{1}{2}\right) \neq 0$

1.5

c) (10 marks) State a regularity condition which guarantees a curve represented parametrically can be written, locally, as the graph of a $C^1$ function. Then prove it.

Condition: $f'(t_0) \neq 0$ Then in a nbd of $t_0$ The parametric version can be converted to graph version.

1

proof

Let $f(t) = \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}$, and assume w.l.o.g $\varphi'(t_0) \neq 0$ 1

and define for any $x, t$ $F(x,t) = x - \varphi(t)$ 2

and set $x_0 = \varphi(t_0)$

Note $F(x_0, t_0) = 0$

and $\partial_t F(x_0, t_0) = -\varphi'(t_0) \neq 0$ 2

conditions of IFT

so by IFT exist $r_0, r_1$ s.t. $\forall x$ $|x - x_0| < r_0$
$r_0 > 0$, $r_1 > 0$

$\Rightarrow \exists! \, t$ s.t. $|t - t_0| < r_1$ & $F(x,t) = 0$

$t = \omega(x)$

$F(x, \omega(x)) = 0$

1

$F(x, \omega(x)) = 0$ implies

$x - \varphi(\omega(x)) = 0$

or $x = \varphi(\omega(x))$

now $f(t) = \begin{bmatrix} \varphi(\omega(x)) \\ \psi(\omega(x)) \end{bmatrix} = \begin{bmatrix} x \\ \underbrace{\psi(\omega(x))}_{h(x)} \end{bmatrix}$ 1

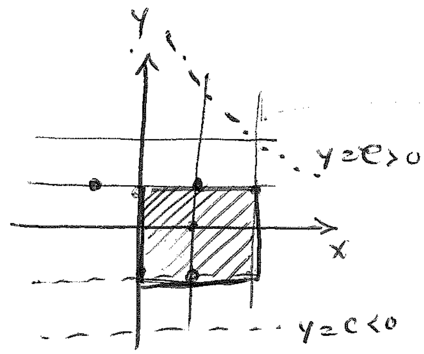
So $y = h(x)$ is The graph representation in The nbd $(x_0 - r_0, x_0 + r_0)$ of $x_0$.

1 1

Martin

2.

a) (10 marks) Consider the transformation $(u, v) = \boldsymbol{f}(x, y) = (x^2, \frac{y}{x})$ $x \neq 0$, and draw the image of this transformation on the vertical and horizontal lines in the domain, as well as the curves $xy = C$. what is the image of the transformation on the square with vertices on the points (0,1),(0,-1), (2,1), (2,-1) (excluding the y axis.) Is this function 1-1?




Vertical lines $x = c$ $\longrightarrow$ $(u,v) = (c^2, \frac{y}{c})$

$-\infty < y < \infty$

2 $c^2 > 0$ constant

so $u > 0$ $-\infty < v < +\infty$

another vertical line on the positive side of the $u$ axis

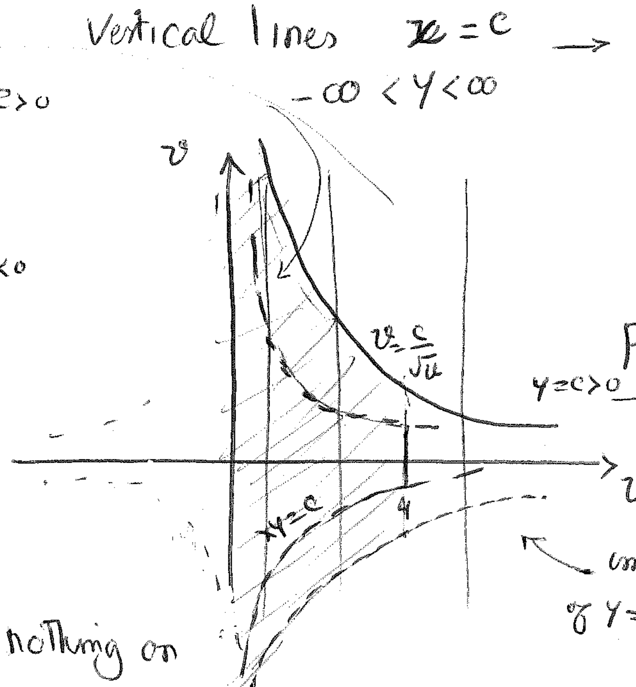



image of $y=c<0$

nothing on the $x<0$ side

horizontal lines

$y = C$ $-\infty < x < 0$

$0 < x < \infty$

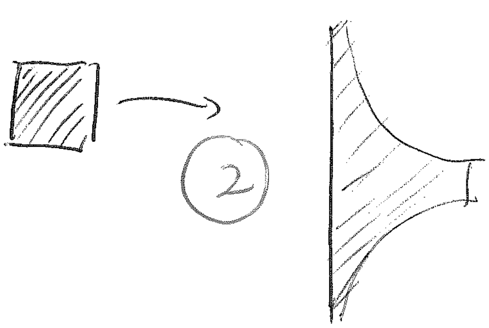
$(u,v) = (x^2, \frac{c}{x})$ 2

$u>0$ & $v = \frac{c}{\sqrt{u}}$

Curves $xy = C$

2 $uv = x^2 \frac{y}{x} = xy = c$

so $xy = c$ is mapped to $uv = c$




not 1-1 b/c

2 $(1,1) \longmapsto (1,1)$

$(-1,-1) \longmapsto (1,1)$

b) (8 marks) Appeal to the Inverse Mapping theorem to determine whether the transformation is invertible near the point $(2,2)$ in the domain (which maps to $(4,1)$). If so determine the Frechet derivative of the inverse transformation.

2

Frechet derivative of The transformation $D\mathbf{f}(x,y) = \begin{bmatrix} 2x & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$

at (2.2) ; $D\mathbf{f}(2,2) = \begin{bmatrix} 4 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

1

$\det(D\mathbf{f}(2,2)) \overset{=2}{\neq} 0$ so invertible by IMT

3

& $D\mathbf{f}^{-1}(4,1) = [D\mathbf{f}(2,2)]^{-1}$

$$= \frac{\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 4 \end{bmatrix}}{2} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & 2 \end{bmatrix}$$

2

Ming

3. Frechet derivative

a) (4 marks) Calculate the Frechet derivative of the function
$F(x, y, z) = (1 - x^2 - y^2 - (z-1)^2, z - x^2 - y^2)$ near a generic point $(x, y, z)$.

$$D\mathbb{F}_{(x,y,z)} = \begin{bmatrix} -2x & -2y & -2(z-1) \\ -2x & -2y & 1 \end{bmatrix}$$ 4

b) (8 marks) Determine a point $\boldsymbol{x}_0$ (that satisfies $\boldsymbol{F}(\boldsymbol{x}_0) = \boldsymbol{0}$) and near which the representation $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$ defines a smooth curve in space, and find a point near which this representation cannot be a smooth curve. Explain your answers.

$x_0$ smooth: need $D\mathbb{F}(x_0)$ to have rank 2 1 so eg $x=1, y=0, z=1$ 1

$F(1,0,1) = (0,0)$ & $D\mathbb{F}(1,0,1) = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} -2 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \end{array}$ 1 Since $\begin{array}{c} \begin{array}{cc} x & z \end{array} \\ \begin{bmatrix} -2 & 0 \\ -2 & 1 \end{bmatrix} \end{array}$

has non-zero det one can solve $x = f(y)$, $z = g(y)$. 1

$x_1$ non smooth: The best place to look for such a point is where $D\mathbb{F}(x_1)$ has rank $= 1 < 2$. 2.5

The only option is $x=0, y=0, z=0$ $\mathbb{F}(0,0,0) = (0,0)$ 0.5

and $D\mathbb{F}(0,0,0) = \begin{bmatrix} 0 & 0 & +2 \\ 0 & 0 & 1 \end{bmatrix}$ 1 no 2x2 sub matrix with det $\neq 0$ exists.

note: at $(0,0,0)$ The intersection of the two surfaces $\begin{cases} 1-x^2-y^2-(z-1)^2 = 0 \\ z - x^2 - y^2 = 0 \end{cases}$ is only a pt, and not a curve. 1

6/11

c) (9 marks) Present a condition on the Frechet derivative $\boldsymbol{DF}$ under which the representation $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$ defines a smooth curve. Proceed to prove that your condition guarantees that, locally, the solution to $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$ can be written as a graph. (Need to quote the system version of IFT, 3.9)

$\mathbb{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ to be able to convert $\mathbb{F}(x)=0$ representation to graph, near $x_0$ with $\mathbb{F}(x_0)=0$ we need $D\mathbb{F}(x_0)$ to have rank $= 2$ on a nbhd of $x_0$

0.5 1.5 0.5

Pf

If $D\mathbb{F}(x_0)$ has rank 2 then there is a 2×2 submatrix $B$ of $D\mathbb{F}(x_0)$ with $\det B \neq 0$. 2

assume the columns of $B$ w.l.o.g

This matrix $B$ correspond to variables $y$ and $z$ or $x_2$ and $x_3$, then by IFT there is a nbd of the pts $x_0$ $(x_0, y_0, z_0)$ on which $\begin{bmatrix} y \\ z \end{bmatrix} = \mathbb{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$ where $\mathbb{f}$ is $C^1$.

2.5

so $y = f_1(x)$ and $z = f_2(x)$ is the graph representation of the curve on space.

2

$f_1$, $f_2$ are $C^1$

7/11

yannis

4. Integration

a) (5 marks)

i) Write the definition of uniform continuity, i.e. complete the statement: $f\colon S \longrightarrow \mathbb{R}^k$ is uniformly continuous if... $\forall \epsilon>0\ \exists \delta\ \forall x,y \in S$ $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$. (2)

ii) State a condition guaranteeing that a function $f\colon [a,b] \longrightarrow \mathbb{R}$ is integrable which involves making the lower and upper Riemann sums near to one another. (Hint: it should begin, "For every $\epsilon > 0$, there is a partition $P$...")

$\forall \epsilon>0$ (1) $\exists P_k$ ~~st~~ partition $P$ of $[ab]$ (1) $S_P f - s_P f < \epsilon$. (1)

b) (12 marks) Use part a) to prove that any continuous function on an interval $[a, b]$ is integrable.

2

$[a\,b]$ is compact (0.5), by Thm 1.38 any cont $f$ on a compact set is u.c. (1.5)

so $f$ is u.c. To prove $f$ is integrable on $[ab]$:

4

given $\epsilon>0$ Choose $\delta$ (2) st. $\forall x,y$ $|x-y|<\delta \Rightarrow |f(x)-f(y)| < \frac{\epsilon}{b-a}$

Then Choose a partition $P$ st. any $\Delta x_i < \delta$. (2)

eg $P_k$ where $k > \frac{(b-a)}{\delta}$ & ..

Then $S_P f - s_P f = \sum_i (M_i - m_i)\Delta x_i$

applying uc of $f$ to $\frac{\epsilon}{(b-a)}$

4

Since $M_i = f(x_i^*)$ as $f$ is cont on $[x_{i-1}, x_i]$
$m_i = f(\bar{x}_i)$ (2) & if $x,y \in [x_{i-1}, x_i]$ Then $|x-y|<\delta$

and $|f(x)-f(y)| < \frac{\epsilon}{(b-a)}$ (1) Then

$M_i - m_i < \frac{\epsilon}{(b-a)}$ (1) so $S_P f - s_P f < \sum \frac{\epsilon}{(b-a)}\Delta x_i = \frac{\epsilon}{b-a}\sum \Delta x_i$
(1.5) $= \frac{\epsilon}{(b-a)}(b-a) = \epsilon$

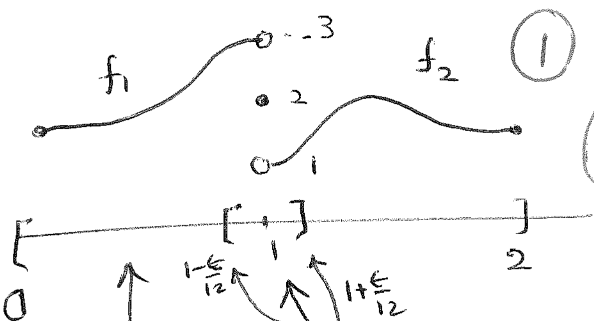
2

Now by the condition ii above (0.5) $f$ is integrable.

8/11

1 graph . 2+2 choice of 2 partitions for $f_1, f_2$
3 for treatment around 1

c) (8 marks) Consider the function $f$ defined piecewise on the interval $[0, 2]$ as follows: $f(x) = f_1(x)$ for $x \in [0, 1)$, $f(x) = f_2(x)$ for $x \in (1, 2]$ $f(1) = 2$, where $f_1$ and $f_2$ are continuous on the intervals $[0, 1]$ and $[1, 2]$ respectively, and $f_1(1) = 3$ and $f_2(1) = 1$. Show that $f$ is integrable on the interval $[0, 2]$. (First draw a hypothetical graph of the function $f$, then start with " Given $\epsilon > 0$ . . . ", and then use part (b) together with (ii) of part (a) applied to each one of $f_1$ and $f_2$ in order to find the necessary partitions.)




1

given $\epsilon > 0$ (Want to use Condition ii)

1 Since $f_1$ is Cont on $[0, 1-\frac{\epsilon}{12}]$ Then it is integrable $\exists\, \mathcal{P}_1$ of $[0, 1-\frac{\epsilon}{12}]$ st. 2

$S_{\mathcal{P}_1} f - s_{\mathcal{P}_1} f < \frac{\epsilon}{3}$

$S_{\mathcal{P}_1} f_1 - s_{\mathcal{P}_1} f_1 < \frac{\epsilon}{3}$

1 and Since $f_2$ is Cont on $[1+\frac{\epsilon}{12}, 2]$ Then it is integrable

So $\exists\, \mathcal{P}_2$ of $[1+\frac{\epsilon}{12}, 2]$ st. $S_{\mathcal{P}_2} f_2 - s_{\mathcal{P}_2} f_2 < \frac{\epsilon}{3}$

1

Choose $1+\frac{\epsilon}{12}$ and $1-\frac{\epsilon}{12}$

so $(M-m)\Delta x$
$= 2 \times (1+\frac{\epsilon}{12} - 1 + \frac{\epsilon}{12})$
$= \frac{\epsilon}{3}$

Now Construct $\mathcal{P}$ by adding all The pts of $\mathcal{P}_1$ and $\mathcal{P}_2$, That is 1 $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$

$\mathcal{P}$ is a partition of $[0\ 2]$ &

$S_{\mathcal{P}} f - s_{\mathcal{P}} f \le (S_{\mathcal{P}_1} f_1 - s_{\mathcal{P}_1} f_1) + (3-1)\Delta x + (S_{\mathcal{P}_2} f_2 - s_{\mathcal{P}_2} f_2) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$

1 $= \epsilon$

Now apply Condition ii to realize $f$ is integrable.

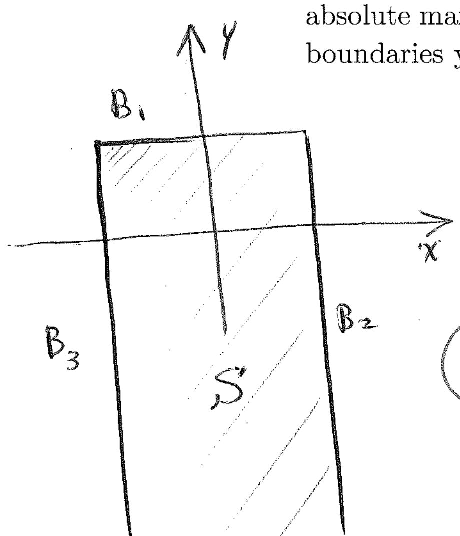
9/11

( 6 total ) Yiannis ||||

& small

5. (This question's mark is added directly to your midterm mark.)

a) (9 marks) Consider the function $f(x,y) = xye^{-y^2}$ defined on the set $S = \{(x,y) \in \mathbb{R}^2 : x \in [-1,1],\ y \leq 1\}$. Find and determine the nature of critical point(s), and determine the absolute max and min of the function $f$ on the set $S$. Justify your answers. (Note: since you are dealing with an unbounded region you need to quote a theorem (2.83 b) which guarantees the absolute max must exist on unbounded regions, and since there is interior and boundaries you need to use appropriate techniques of optimization.)




$|(x,y)| \to \infty$ as $y \to -\infty$, in This case $f(x,y) \to 0$. Also $f(-1,1) = \frac{-1}{e} < 0$ so

1.5 by Thm 2.83 There must exist an absolute min and $f(1,1) = \frac{1}{e} > 0$ so also There exist an absolute max.

inside $S$ : $\nabla f = \begin{bmatrix} ye^{-y^2} \\ -2xy^2e^{-y^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y=0$ so we have Critical pts along y-axis (1)

$xe^{-y^2}$

Note $f(x,y) > 0$ if $x > 0$ & $f(x,y) < 0$ if $x < 0$ (1) $f(0,y) = 0$ so The pts on The y-axis are not good Candidates for absolute max or min.

On The boundaries

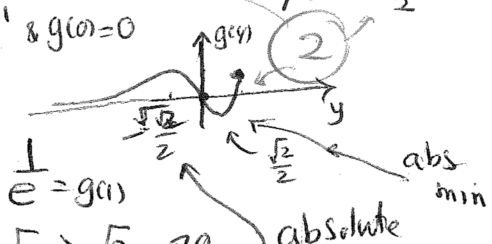
$B_1$ : $y = 1$ $-1 < x < 1$ $f(x,1) = xe^{-1}$ increases from $-e^{-1}$ to $e^{-1}$ 0.5

$B_2$ : $x = 1$ $-\infty < y < 1$ $f(1,y) = ye^{-y^2}$ $g(y)$ (1) $g'(y) = e^{-y^2} - 2y^2e^{-y^2} = 0 \Rightarrow 1 - 2y^2 = 0$, $y = \pm\frac{\sqrt{2}}{2}$

$g(1) = e^{-1}$ & $g(0) = 0$ (2)

$g(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}\frac{1}{\sqrt{e}} > \frac{1}{e} = g(1)$ (2) b/c $\sqrt{e} > \sqrt{2}$ so

abs max & $-\frac{\sqrt{2}}{2}\frac{1}{\sqrt{e}}$ abs min

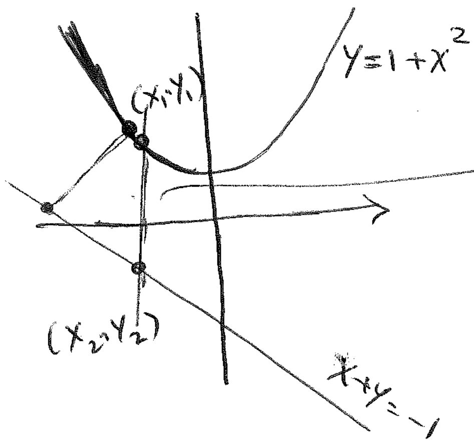



$B_3$ : $x = -1$ Similar to $B_2$ With opposit Signs.

$h(y) = -ye^{-y^2}$

$h'(y) = -e^{-y^2} + 2y^2e^{-y^2} = 0 \cdots$ $y = \pm\frac{\sqrt{2}}{2}$

b) (6 marks) Use the method of lagrange multipliers with two constraints (or other methods) to calculate the length of the shortest line connecting a point $(x_1, y_1)$ of the parabola $y = 1 + x^2$ to a point $(x_2, y_2)$ of the plane $x + y = -1$. (Draw a diagram and convince yourselves that your solution is actually minimum.)




Square of distance

$$f(x_1,y_1,x_2,y_2) = (x_1-x_2)^2+(y_1-y_2)^2$$

1.5

Constraints

$$F(x_1,x_2,y_1,y_2) = y_1-1-x_1^2 = 0$$

$$G(x_1,x_2,y_1,y_2) = x_2+y_2+1 = 0$$

need to solve the system $\begin{cases} \nabla f(x) = \lambda \nabla F(x) + \mu \nabla G(x), \\ F(x) = 0 = G(x) \end{cases}$

which is

2

$$\begin{cases} ① & 2(x_1-x_2) = -2x_1\lambda + 0 \\ ② & -2(x_1-x_2) = 0 + \mu \\ ③ & 2(y_1-y_2) = \lambda + 0 \\ ④ & -2(y_1-y_2) = 0+\mu \\ ⑤ & F(\cdot) = 0 = G(\ ) \end{cases}$$

$\Rightarrow x_1 - x_2 \neq 0$ $\qquad -\mu = -2x_1\lambda$

$\mu = -\lambda \neq 0$ $\qquad \lambda = -2x_1\lambda$

$y_1 - y_2 \neq 0$ b/c $y_1 > 0$ & $y_2 < 0$

so $\lambda \neq 0$ $\quad -2x_1 = 1$ $\quad \boxed{x_1 = -\frac{1}{2}}$

⑤ $y_1 = 1 + \frac{1}{4} = \frac{5}{4}$

2.5

next ① & ③ with $x_1 = -\frac{1}{2}$ give

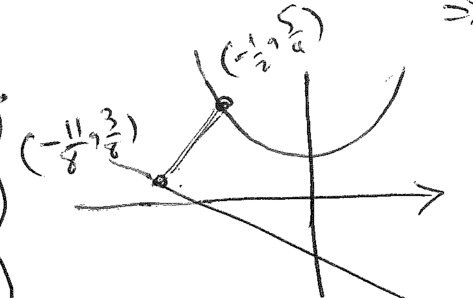
$$\begin{cases} (-1-2x_2) = \lambda \\ 2\left(\frac{5}{4} - y_2\right) = \lambda \end{cases} \longrightarrow \begin{cases} -1-2x_2 = \lambda \\ 2\left(\frac{5}{4}+1+x_2\right) = \lambda \end{cases} \longrightarrow \begin{cases} -1-2x_2 = \lambda \\ \frac{9}{2}+2x_2 = \lambda \end{cases}$$

⑤ $\Rightarrow y_2 = 1 + x_2$

$\Rightarrow \frac{11}{2} + 4x_2 = 0$

$\Rightarrow x_2 = -\frac{11}{8}$

$\Rightarrow y_2 = \frac{3}{8}$




11/11