UNIT 6

DERIVATIONS FOR PREDICATE LOGIC

6.1 Extending the Derivation System

Now that we've learnt how to symbolize sentences using predicates, operations and quantifiers, we need to extend our derivation system to handle these new features.

New Derivation Rules for the Quantifiers:

UI: Universal Instantiation

EG: Existential Generalization

EI: Existential Instantiation

QN: Quantifier Negation

UI and EI are both elimination rules – they act on sentences that have a quantifier as the main logical operator and they allow us to derive a sentence that does not contain that quantifier as the main logical operator.

EG is an introduction rule – it acts on sentences that may not have the existential quantifier (∃) as the main logical operator and it allows us to derive a sentence for which ∃ is the main logical operator.

New Type of Derivation:

UD: Universal Derivation

This derivation technique allows us to derive sentences for which the universal quantifier is the main logical operator. Like DD, CD and ID, it can be used for the main derivation or for subderivations.

The Rules and Derivation Types from Sentential Logic:

We will continue to use the three basic forms of derivation and subderivation:

DD: Direct Derivation

CD: Conditional Derivation

ID: Indirect Derivation

We will also continue to use the rules of inference that we have already learned:

The Basic Rules:

Modus Ponens (MP)

$$\frac{(\phi \to \psi) \quad \phi}{\psi}$$

Modus Tollens (MT)

$$\frac{(\phi \to \psi) \quad \sim\psi}{\sim\phi}$$

Double Negation (DN)

$$\frac{\phi}{\sim\sim \phi} \qquad \frac{\sim\sim\phi}{\phi}$$

Repetition (R)

$$\frac{\phi}{\phi}$$

Simplification (S)

$$\frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

Adjunction (ADJ)

$$\frac{\phi \quad \psi}{\phi \wedge \psi}$$

Addition (ADD)

$$\frac{\phi}{\phi \vee \psi} \qquad \frac{\psi}{\phi \vee \psi}$$

Modus Tollendo Ponens (MTP)

$$\frac{\phi \vee \psi \quad \sim \phi}{\psi} \qquad \frac{\phi \vee \psi \quad \sim \psi}{\phi}$$

Biconditional-Conditional (BC)

$$\frac{\phi \leftrightarrow \psi}{\phi \to \psi} \qquad \frac{\phi \leftrightarrow \psi}{\psi \to \phi}$$

Conditional-Biconditional (CB)

$$\frac{\phi \to \psi \quad \psi \to \phi}{\phi \leftrightarrow \psi}$$

Derived Rules:

Negation of Conditional (NC)

$$\frac{\sim(\phi \to \psi)}{\phi \wedge \sim\psi} \qquad \frac{\phi \wedge \sim\psi}{\sim(\phi \to \psi)}$$

Conditional as Disjunction (CDJ)

$$\frac{\phi \to \psi}{\sim\phi \vee \psi} \qquad \frac{\sim\phi \vee \psi}{\phi \to \psi} \qquad \frac{\sim\phi \to \psi}{\phi \vee \psi} \qquad \frac{\phi \vee \psi}{\sim\phi \to \psi}$$

Negation of Biconditional (NB)

$$\frac{\sim (\phi \leftrightarrow \psi)}{\phi \leftrightarrow \sim \psi} \qquad \frac{\phi \leftrightarrow \sim\psi}{\sim (\phi \leftrightarrow \psi)}$$

Separation of Cases (SC)

$$\frac{\phi \vee \psi \quad \phi \to \chi \quad \psi \to \chi}{\chi} \qquad \frac{\phi \to \chi \quad \sim\phi \to \chi}{\chi}$$

De Morgan's (DM)

$$\frac{\sim (\phi \vee \psi)}{\sim\phi \wedge \sim\psi} \qquad \frac{\sim\phi \wedge \sim\psi}{\sim (\phi \vee \psi)} \qquad \frac{\sim (\phi \wedge \psi)}{\sim\phi \vee \sim\psi} \qquad \frac{\sim\phi \vee \sim\psi}{\sim (\phi \wedge \psi)}$$

$$\frac{\sim (\sim\phi \vee \sim\psi)}{\phi \wedge \psi} \qquad \frac{\phi \wedge \psi}{\sim (\sim\phi \vee \sim\psi)} \qquad \frac{\sim (\sim\phi \wedge \sim\psi)}{\phi \vee \psi} \qquad \frac{\phi \vee \psi}{\sim (\sim\phi \wedge \sim\psi)}$$

And of course we can use any conditional or biconditional theorem as a rule (RT).

6.2 Universal Instantiation (UI)

Universal Instantiation is a derivation rule that can be used if the main logical operator is the universal quantifier ($\forall$). It is an elimination rule – allowing us to derive a sentence that does not contain "$\forall$" (or contains one fewer instances of "$\forall$".)

Consider the following argument:

All humans are mortal.	$\forall x(Fx \rightarrow Gx)$
Adam is human.	Fa
$\therefore$ Adam is mortal.	$\therefore$ Ga

In order to derive Ga from Fa, we need to be able to derive the conditional that is particular to Adam: $Fa \rightarrow Ga$ from $\forall x(Fx \rightarrow Gx)$. We can paraphrase the general universal sentence: for anything x, if x is human then x is mortal. Since is true about any x, it is true about Adam: If Adam is human then Adam is mortal. Adam is human. Therefore, the conclusion follows: Adam is mortal. This argument is about Adam, but we could draw the same sort of conclusion about Mary, Adam's father, Socrates or any other individual.

$Fa \rightarrow Ga$ is an ***instantiation*** or ***substitution instance*** of the general universal sentence: $\forall x (Fx \rightarrow Gx)$. This is the sentence that results if we substitute A for x and remove the universal quantifier – it is a substitution instance of the general universal.

UI: Universal Instantiation

This derivation rule allows us to derive a substitution instance from a universal generalization.

$$\frac{\forall\alpha\phi_\alpha}{\phi_\zeta}$$

Justification: line number of $\forall\alpha\phi_\alpha$ and UI.

This tells us that if we have a sentence that the main logical operator is the universal quantifier, $\forall\alpha\phi_\alpha$, (the entire sentence must be bound by that quantifier), we can derive any sentence ϕ_ζ that is a substitution instance of ϕ_α – substituting ζ (any singular term) for every instance of α (the variable governed by the universal quantifier) in ϕ_α (provided that ζ does not occur as a bound variable in ϕ_α.)

For instance, from $\forall x(Fx \rightarrow Gx)$ we can derive any of the following sentences:

$Fx \rightarrow Gx$	substituting x for x
$Fy \rightarrow Gy$	subsituting y for x
$Fa \rightarrow Ga$	substituting a for x.
$Fc(b) \rightarrow Gc(b)$	substituting c(b) for x

This makes sense, for if everything that is F is also G, then the sentence $F\alpha \rightarrow G\alpha$ is true no matter what singular term is substituted for α.

What would that argument look like as a derivation?

$\forall x(Fx \rightarrow Gx)$. Fa. $\therefore$ Ga

1	~~Show~~ Ga		
2		$Fa \rightarrow Ga$	Pr1 UI
3		Ga	2 Pr2 MP
4			3 DD

We've substituted a for x and removed the universal quantifier to get this substitution instance of the first premise. Although we *could* instantiate to any singular term, we want to instantiate to 'a' to match the second premise. That way we can use MP with this sentence and the second premise in order to derive Ga.

6.2 EG1 Let's try a derivation using UI

$\forall x(Fx \rightarrow Gx)$. $\forall y(Hy \rightarrow \sim Gy)$. Fb. $\therefore \sim$Hb.

1	~~Show~~ ~Hb	Show conc
2	$Fb \rightarrow Gb$	PR1, UI
3	Gb	PR3, 2 MP
4	$Hb \rightarrow \sim Gb$	PR2, UI
5	~~Gb	3 DN
6	~Hb	4,5 MT DD

6.2 EG2 Let's try one that uses more of our skills from sentential logic.

$\forall x((Fx \wedge \sim Gx) \rightarrow (Ax \vee Bx))$. $\forall y\sim(Gy \wedge Dy)$. $Fa \wedge Da$. $\therefore \forall x(Gx \leftrightarrow Bx) \rightarrow Aa$

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6.2 EG3 We can use UI twice on the same sentence!

$\forall x \forall y(Gx \wedge Hy \to L(xy))$. $\forall x(Bx \vee Hx)$. $Ga \wedge {\sim}Bb$. $\therefore L(ab)$

UI x2 UI

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6.2 EG4 When we use UI, we can instantiate to any constant… even one that we have already instantiated to in the same sentence. (This one is easiest as an indirect derivation!)

$\forall x \forall y(Fx \to (Cy \to B(xy)))$. $\forall x{\sim}B(xx)$. $\therefore {\sim}Fa \vee {\sim}Ca$

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6.3 Existential Generalization (EG)

Existential Generalization is a derivation rule that can be used to derive a sentence for which the main logical operator is the existential quantifier (∃). It is an introduction rule – allowing us to derive a sentence that contains "∃" from a sentence that doesn't (or that contains one fewer instance of "∃").

Adam is a person and Adam is mortal.	$Fa \wedge Ga$
∴ Some person is mortal.	∴ $\exists x(Fx \wedge Gx)$

In order to derive $\exists x(Fx \wedge Gx)$ we need to be able to derive the general existential sentence from the particular symbolic sentence: $Fa \wedge Ga$. We can paraphrase the general existential: there is some x such that x is human and x is mortal. If Adam is human and Adam is mortal then *something* is both human and mortal. Although this argument is about Adam, the same conclusion would follow if it were about Mary, Adam's father, Socrates or any other individual.

$Fa \wedge Ga$ is an instantiation or substitution instance of the general existential sentence: $\exists x (Fx \wedge Gx)$. This is the sentence that results if we substitute a for x and remove the existential quantifier – it is a substitution instance of the general existential sentence.

EG: Existential Generalization

This derivation rule allows us to derive the existential generalization from a substitution instance.

$$\frac{\phi_\zeta}{\exists\alpha\phi_\alpha}$$ Justification: line number of ϕ_ζ and EG.

This tells us that if we have a symbolic sentence ϕ_ζ that contains a singular term ζ, we can derive any sentence $\exists\alpha\phi_\alpha$ such that ϕ_ζ is a substitution instance of $\exists\alpha\phi_\alpha$ – replacing ζ (any singular term) with α (the variable governed by the existential quantifier). Note, we do not have to replace every instance of ζ with α.

For instance, from $Fa \wedge Ga$ we can derive any of the following sentences:

$\exists x(Fx \wedge Gx)$	substituting x for a
$\exists y(Fy \wedge Gy)$	substituting y for a
$\exists x(Fx \wedge Ga)$	substituting x for the first instance of a.
$\exists y(Fa \wedge Gy)$	substituting y for the second instance of a.
$\exists x(Fa \wedge Ga)$	substituting x for no instances of a.

Likewise we can derive $\exists x(Fx \wedge Gx)$ from any of the following sentences:

$Fb \wedge Gb$. $Fa(b) \wedge Ga(b)$. $Fx \wedge Gx$. $Fy \wedge Gy$.

This makes sense, for if a particular individual is F and G, then the sentence $\exists\alpha(F\alpha \wedge G\alpha)$ is true no matter what particular individual is both F and G.

Let’s try it out.

Show that the following argument is valid by deriving the conclusion from the premises.

$\forall xFx$. $\forall y(Fy \rightarrow Gy)$. $\therefore \exists xGx$

1	Show $\exists xGx$		First line of any derivation is show conclusion!
2	Fa	Pr1 UI	We’ve substituted a for x and removed the universal quantifier to get this substitution instance of the first premise. Although we use any singular term, if there isn’t a reason to instantiate to x, y, z or other variables, it makes the derivation clearer if we instantiate to singular terms such as a, b, c.
3	$Fa \rightarrow Ga$	Pr2 UI	We’ve substituted a for x and removed the universal quantifier to get this substitution instance of the second premise. Although we could instantiate to any singular term, we want to instantiate to a again because we want to be able to use MP with this sentence and line 2 in order to derive Ga. NOTE: Lines 2 and 3 give us sentences similar to that in sentential logic – atomic sentences and logical connectives. Now we can apply the rules from sentential logic.
4	Ga	2 3 MP	This is just a basic sentential derivation rule!
5	$\exists xGx$	4 EG	We’ve replaced a with x and introduced the existential quantifier to get this existential sentence. Although we could use any variable, we use x because that is the variable on the show line (it’s the sentence we want!) Now we just need to box, cancel and write DD.

Here is the proof:

1	~~Show~~ $\exists xGx$	
2	Fa	Pr1 UI
3	$Fa \rightarrow Ga$	Pr2 UI
4	Ga	2 3 MP
5	$\exists xGx$	4 EG, DD

The argument is valid since the conclusion can be derived from the premises. This makes sense: if everything is F and everything that is F is G, then something must be G! The reasoning here goes like this: take any arbitrary individual, a. Since everything is F, a must be F, so Fa. Everything that is F is G, thus if Fa then Ga. Thus Ga. Therefore, something is G.

GENERAL STRATEGY: UI is used to eliminate the universal quantifiers. Once they are eliminated, then the derivation can proceed using the derivation rules of sentential logic. Then, once the sentence is in the form that you want it, EG is used to turn it back into a quantified sentence.

6.3 EG1 Let's try another – although it is more complex, it uses the same general strategy.

$\sim Fc.\quad \forall x(Gx \rightarrow Fx).\quad \forall x \sim(Hx \wedge \sim Gx).\quad \forall y(Ly \vee Hy).\quad \therefore \exists z\,(Lz \vee Jz)$

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6.3 EG2 Sometimes you need to use the rules for sentential logic before you remove the quantifiers and/or after you introduce them.

$\forall x(Gx \rightarrow (Hx \wedge Jx)) \wedge \forall y(Jy \rightarrow Ly).\quad \forall xGx \therefore \exists xHx \wedge \exists yLy$

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6.3 EG3 Sometimes you need to use UI or EG several times for the same sentence.

Every use of UI is a separate step, likewise every use of EG is a separate step. We can, of course, use the short-cut of combining two steps on one line, but each step will still need its own justification.

Notice that this argument includes a two-place predicate.

$\forall x \forall y((Fx \wedge Gy) \rightarrow L(xy))$. $\sim L(ba)$. $\therefore \exists x \exists y(\sim Fy \vee \sim Gx)$

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2 Pr1 do UI twice

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4 do EG twice for conc.

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In some cases, when using EG, not all instances of the variable should be replaced.

$\forall x L(xx)$. $\therefore \exists x \exists y\, L(xy)$

1	~~Show~~ $\exists x \exists y\, L(xy)$		
2	$L(aa)$	PR1 UI	First we need an instantiated form of PR1
3	$\exists y L(ay)$	2 EG	If a is in relation L to itself, then a is L to something.
4	$\exists x \exists y L(xy)$	3 EG	So something is in relation L to something. (Since, everything is in relation L to itself!)

6.3 EG4 Let's try another:

$\forall x \forall y(F(xy) \rightarrow G(xx))$. $F(ca)$ $\therefore \exists x G(xc) \wedge \exists y G(yy)$

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Don't let operation letters confuse you!   You can use EG in lots of different ways on one or more place operations.

6.3 EG5   Try this one:

La(b(c)). ∴ ∃xLx ∧ ∃yLa(y) ∧ ∃zLa(b(z))

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CAREFUL: When using EG you must replace a singular term (an individual) with the variable.

Consider La(b(c)):   we can imagine an abbreviation scheme that would give us this symbolization.

c^0: Carol b^1: the brother of *a* c^1: the neice of *a*

c is a singular term – Carol.

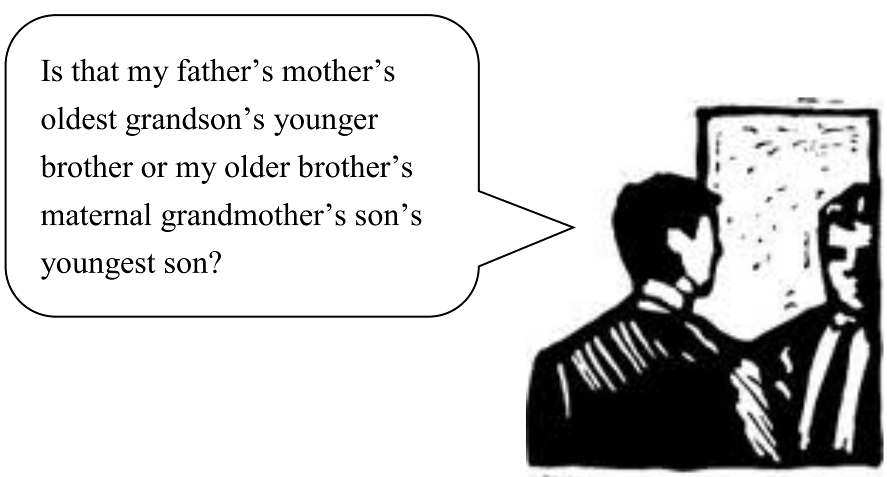
b(c) is a singular term – Carol's brother.

a(b(c)) is a singular term – Carol's brother's neice.

Any of those can be replaced with a variable when using EG since the rule is used to replace an individual's name or description – a singular term - with a variable.

However, the following are NOT legitimate existential generalizations: ∃xLa(x(c)) or ∃xLx(b(c)).

In La(b(c)), the expressions 'b' and 'c' are not singular term letters – so they cannot be replaced with a variable!



6.3 EG6 In this one, we need to show the instantiated form of the conclusion, so that we can use EG. But, it is difficult to do that directly – but we can easily do it with an indirect derivation. First show the instantiated sentence (which is a negated conjunction), then assume the unnegated form (the conjunction) for ID.

$\forall x(Fx \rightarrow \forall y(Gy \rightarrow \sim H(xy)))$. $\forall x(\sim Gx \vee H(xx))$.

$\therefore\ \exists x \sim(Fx \wedge Gx)$

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6.3 EG7 Sometimes we need to use CD inside a derivation.

$(Fa \rightarrow \exists y \sim Gy) \rightarrow \forall y(Hy \rightarrow \exists x L(xy))$. Ha. $\forall y(Fy \rightarrow By)$. $\forall y(Gy \rightarrow \sim By)$.

$\therefore\ \exists x L(xa)$

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6.3 E1 Show that the following syllogisms are valid by providing a derivation.

Syllogistic Logic began with Aristotle, and was further developed by Medieval logicians who used names like “Celeront” to help them remember which argument form was which. The ‘figure’ has to do with the position of the middle term (the term repeated in the premises.) These are all conditionally valid, moving from universals to existential statements. Thus, these conclusions are conditional and follow from the assumption that at least one such thing exists with the property of the universal.:

a) 1^{st} figure: Celeront	$\forall x(Cx \rightarrow \sim Jx)$.	$\forall y(Ky \rightarrow Cy)$.	$\therefore$ $Ka \rightarrow \exists z(Kz \wedge \sim Jz)$
b) 2^{nd} figure: Camestrop	$\forall x(Bx \rightarrow Fx)$.	$\forall y(Dy \rightarrow \sim Fy)$.	$\therefore Db \rightarrow \exists x(Dx \wedge \sim Bx)$
c) 3^{rd} figure: Darapti	$\forall x(Ax \rightarrow Fx)$.	$\forall x(Ax \rightarrow Ex)$.	$\therefore Aa \rightarrow \exists x(Ex \wedge Fx)$
d) 4^{th} figure: Fesapo	$\forall y(Fy \rightarrow \sim By)$.	$\forall x(Bx \rightarrow Dx)$	$\therefore Bc \rightarrow \exists y(Dy \wedge \sim Fy)$

6.3 E2 Show that the following arguments are valid by providing a derivation:

a) $\forall x(Fx \vee Gx)$. $\forall y(Hy \rightarrow \sim Fy)$. $\forall z(\sim Bz \rightarrow \sim Gz)$ $\therefore Ha \rightarrow Ba$

b) $\therefore \forall x(Ex \rightarrow (Fx \rightarrow Ax)) \rightarrow ((Ea \wedge Fa) \rightarrow \exists yAy)$

c) $\forall x(Ax \wedge (Bx \vee Cx))$. $\forall y(Ay \rightarrow (Cy \rightarrow Dy))$. $\sim(Ab \wedge Bb)$ $\therefore \exists x(Dx \wedge Cx)$

d) $\forall x(Ax \rightarrow Bx)$. $\sim Ba(c)$. $\forall z(\sim Az \rightarrow Cz)$. $\therefore$ $\exists yCy$

e) $\forall x(Fx \leftrightarrow \sim Gx \wedge Hx)$. $\forall y(Fy \wedge (\sim Cy \rightarrow \sim Hy))$. $\therefore$ $\exists x(\sim Gx \wedge Cx)$

f) $\forall x\forall y(Ax \wedge \sim By \rightarrow D(yx))$. $\sim(Ae \rightarrow Bd)$. $\forall x\forall y(D(yx) \rightarrow Gx \wedge Hy)$ $\therefore \exists x\exists y(Gy \wedge Hx)$

g) $\forall y((Fy \vee Gy) \rightarrow Hy)$. $\sim Hb$. $\forall x(Fx \vee \sim Bx)$. $\forall x\sim(Ax \leftrightarrow Bx)$ $\therefore \exists z(Az \wedge \sim Gz)$

h) $\therefore \forall x\forall yL(xy) \rightarrow \exists x\exists y(L(xy) \wedge L(yx))$

i) $\therefore \forall x\forall y\forall zA(xyz) \rightarrow \exists x\exists y(A(xxy) \wedge A(yxy) \wedge A(yyy))$

j) $\forall x\forall y(Ax \wedge By \rightarrow C(xy))$. Ab. $\forall x\forall y(C(xy) \leftrightarrow D(yx))$. $\therefore Be \rightarrow \exists zD(ez)$

k) $\forall x((Dx \wedge \sim Cx) \rightarrow Ex)$. $\forall y(Fy \leftrightarrow Ey)$. $\forall y\sim(Dy \wedge Fy)$. $\forall z(\sim Dz \vee \sim Cz)$ $\therefore \exists x(Dx \rightarrow Ax)$

l) $\forall x\forall y(Fx \rightarrow L(xy))$. $\forall z(Gz \rightarrow C(zz))$. $\forall y(\exists zC(zy) \rightarrow \sim\exists xL(xy))$ $\therefore \sim(Fa \wedge Gb)$

m) $Fa \wedge Gb$. $\forall x\forall y(Fx \wedge Gy \rightarrow L(xy))$. $\forall x\forall y(L(xy) \rightarrow (Hy \vee \sim L(xx)))$. $\therefore$ $\sim Hb \rightarrow \exists z\sim(L(zz) \vee Gz)$

6.4 Existential Instantiation (EI)

Existential Instantiation is a derivation rule that can be used if the main logical operator is the existential quantifier ($\exists$). It allows us to reason about 'some arbitrary individual' even when we don't have a particular individual in mind.

All girls are mortal.	$\forall x(Gx \rightarrow Hx)$
There are girls.	$\exists xGx$
$\therefore$ Some things are mortal.	$\therefore \exists xHx$

In this argument, we can reason as follows: The first premise tells us that all girls are mortal. The second premise tells us that at least one thing is a girl – but it doesn't tell us what member of the universe is a girl. So consider any arbitrary member of the universe that is a girl. If we call that girl, 'a', then Ga. Now, everything that is a girl is mortal. Thus, Ha. Since 'a' is mortal, something is H. $\exists xHx$.

In this argument, we named the girl, 'a'. But we could have used any arbitrary individual term – 'b', 'x', 'y' etc. What matters is that we picked an arbitrary individual term, since the argument doesn't tell us which member of the universe has the property F; it just tells us that *something* has that property. In order to ensure that we pick an arbitrary individual when instantiating from an existential statement, we need to instantiate to an individual term that does not occur in any previous line or premise.

Consider what could happen if we do NOT take an arbitrary individual.

Something is a girl.	$\exists xGx$
Something is a boy.	$\exists xFx$
$\therefore$ Something is a girl and a boy.	$\therefore \exists x(Gx \wedge Fx)$

This is clearly a fallacious argument – this conclusion CANNOT be legitimately drawn from the premises.

The POOR reasoning would go as follows: We know some member of the universe is a girl, so let's call that individual 'i'. Thus Gi. We know that some member of the universe is a boy, so let's call that individual 'i'. Thus, Fi. Now we can use adjunction to get $Gi \wedge Fi$. Since it is both a girl and a boy, something has properties G and F: $\exists x(Gx \wedge Fx)$.

The mistake is that we didn't pick an arbitrary individual when we predicated 'is a boy' of some member of the universe. We predicated 'is a boy' of 'i' – an individual which we have already said something about (Gi). 'i' is not an arbitrary member of the universe!

Existential Generalization must be restricted to an arbitrary member of the universe:

Make sure the individual term you are using does not occur in any previous line or premise!

EI: Existential Instantiation

This derivation rule allows us to derive a substitution instance from an existential generalization.

$$\frac{\exists\alpha\phi_\alpha}{\phi_\zeta}$$

Justification: line number of $\exists\alpha\phi_\alpha$ and EI.
Restriction: ζ cannot occur in any previous line or premise.

This tells us that if we have a symbolic sentence $\exists\alpha\phi_\alpha$ we can derive a sentence ϕ_ζ such that ϕ_ζ is a substitution instance of $\exists\alpha\phi_\alpha$ provided ζ is an arbitrary singular term – replacing every instance of α (the variable governed by the existential quantifier) with ζ (an arbitrary singular term). An arbitrary singular term is a variable letter (i-z) that does not occur unbound in any previous line or premise.

Because of the restriction, most of the time you want to use EI before UI – first pick out your arbitrary individual that you know something is true of and then also say about it anything that is true of everything.

$\exists x Fx.\ \ \forall x(Fx \rightarrow Gx)\ \ \therefore\ \exists y Gy$

1	show $\exists y Gy$	Goal: $G\alpha$ (It doesn't matter what term α is – as long as *something* is G. Then we can use EG to get the show line.)	
2	Fi	Pr1 EI	Since premise 1 says something is F, we can instantiate to an arbitrary individual which we will call 'i'.
3	$Fi \rightarrow Gi$	Pr2 UI	We can instantiate the second premise to any individual term, but we use 'i' so that the antecedent matches line 2.
4	Gi	2 3 MP	
5	$\exists y Gy$	4 EG	We can generalize the particular sentence on line 4, replacing 'i' with 'y' to match our goal sentence.
			Now we just need to box and cancel and write 'DD'.

If we used UI before we used EI, then line 2 would be $Fi \rightarrow Gi$. Then, if we tried to use EI, we would not be allowed to instantiate to Fi – since 'i' would no longer be an arbitrary individual. We would have to pick a new individual term, such as 'k'. This would give us Fk. But now we cannot use MP since the antecedent doesn't match. (At this point, we could begin again, or leave it use UI a second time, this time instantiating to 'k' rather than to 'i'. Then we could continue as above.)

EI first
UI to match

6.4 EG1 $\exists x Fx$. $\forall x \exists y(Fx \rightarrow (Gy \wedge L(xy)))$. $\forall x \forall y(L(xy) \rightarrow \sim H(yx))$. $\therefore \exists x \exists y(Gx \wedge \sim H(xy))$

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6.4 EG2 Sometimes, when the conclusion is an existential, it is best to first show the substitution instance with a conditional or indirect derivation.

$Bc(a)$. $\exists x Bx \rightarrow \forall y(Hy \rightarrow Gy)$. $\forall x(\sim Fx \vee Hx)$. $\therefore \exists x(Fx \rightarrow Gx)$

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A Few General Strategies:

Use EI first then UI (make the individual term match the instantiating term from EI).

Remember, you must always instantiate to a new term when using EI (one that doesn't appear on any previous line or premise.)

Remember you can use UI as many times as you like – with virtually no restrictions on the instantiating term.

If your conclusion is an existential sentence, your goal should be to derive an instantiated form of that sentence. Sometimes this can be done directly. Other times, you need to use a subderivation (CD or ID), following the strategies for sentential logic. Here are a few patterns you might try:

Conclusion: $\exists x(\phi x \rightarrow \psi x)$. After show conclusion:

Show $\phi x \rightarrow \psi x$. Assume for CD: ϕx. Derive: ψx.

Conclusion: $\exists x{\sim}\phi$. After show conclusion:

Show $\sim\phi$. Assume for ID: ϕx. Derive a contradiction.

Conclusion: $\exists x(\phi x \vee \psi x)$. After show conclusion:

Show $\sim\phi x \rightarrow \psi x$. Assume for CD: $\sim\phi x$. Derive: ψx. Use CDJ to get $\phi x \vee \psi x$.

Common Fallacies or Derivation Mistakes:

1	~~Show~~ $\exists y(Fy \wedge Gy)$		
2	$\exists xFx$	Premise 1	
3	$\exists xGx$	Premise 2	
4	Fi	2 EI	
5	Gi	3 EI	**MISTAKE!**
6	$Fi \wedge Gi$	4 5 ADJ	
7	$\exists y(Fy \wedge Gy)$	6 EG	

MISTAKE ON LINE 5: When using EI you cannot instantiate to a term that occurs on any previous line. Since 'i' occurs in line 4, one cannot instantiate to 'i' on line 5. When using EI one must instantiate to a new singular term. This makes sense: If something is F, and something is G, it does NOT follow that something is both F and G.

When using EI you must instantiate to an arbitrary singular term: use a new variable letter – one that does not occur unbound on any previous line or premise.

1	~~Show~~ $\exists yL(yy)$		
2	$\forall xFx$	Premise 1	
3	$\forall x(Fx \rightarrow \exists yL(xy))$	Premise 2	
4	Fy	2 UI	
5	$Fy \rightarrow \exists yL(yy)$	3 UI	**MISTAKE!**
6	$\exists yL(yy)$	4 5 MP DD	

MISTAKE ON LINE 5: When using UI on a sentence $\forall\alpha\phi_\alpha$ you cannot instantiate to a term that is bound in the sentence ϕ_α. Since 'y' is bound in $(Fx \rightarrow \exists yL(xy))$ one cannot instantiate to 'y'. This makes sense: If everything is F, and all F's stand in relation L to something, it does NOT follow that something stands in relation L to itself.

When using UI on $\forall\alpha\phi_\alpha$, you cannot instantiate to ζ if ζ occurs as a bound variable in ϕ_α.

ALSO: make sure (when using UI and EI) that you substitute the instantiating term for every instance of the variable.

$\forall x\forall y(Fx \wedge G(xy) \rightarrow H(yx))$

$\forall y(Fa \wedge G(xy) \rightarrow H(ya))$ UI **MISTAKE:** the 'x' in G(xy) must also be replaced with 'a'.

6.4 E1 Show that the following syllogisms are valid by providing a derivation.

a) 1^{st} figure: Ferio $\forall x(Mx \rightarrow \sim Ax)$. $\exists x(Bx \wedge Mx)$. $\therefore \exists x(Bx \wedge \sim Ax)$
b) 2^{nd} figure: Baroco $\forall x(Bx \rightarrow Ex)$. $\exists y(Ay \wedge \sim Ey)$. $\therefore \exists z(Az \wedge \sim Bz)$
c) 3^{rd} figure: Disamis $\exists x(Bx \wedge Fx)$. $\forall y(By \rightarrow Gy)$. $\therefore \exists x(Gx \wedge Fx)$
d) 4^{th} figure: Fresison $\forall y(Fy \rightarrow \sim Dy)$. $\exists x(Dx \wedge Bx)$ $\therefore \exists y(By \wedge \sim Fy)$

6.4 E2 Construct derivations that show that the following are valid arguments:

a) $\therefore \exists x(Fx \wedge Gx) \rightarrow \exists xFx \wedge \exists xGx$

b) $\exists x\exists yL(xy) \rightarrow \exists z\exists xL(zx)$

c) $\exists x(Fx \vee \sim Gx)$. $\forall y(Fy \rightarrow Ay)$. $\forall z(\sim Az \rightarrow Gz)$. $\therefore \exists xAx$

d) $\forall x(\sim Cx \vee Jx)$. $\forall y(By \rightarrow Cy)$. $\exists xBx$ $\therefore \exists z(Bz \wedge Jz)$

e) $\forall x(Bx \rightarrow Fx)$. $\forall y\sim(Dy \wedge Fy)$. $\therefore \exists xDx \rightarrow \exists x\sim(\sim Dx \vee Bx)$

f) $\exists x(Gx \wedge \forall y(By \rightarrow L(xy)))$. $\exists x(Bx \wedge \forall y(Gy \rightarrow L(xy)))$. $\therefore \exists x\exists y(L(xy) \wedge L(yx))$

6.5 UNIVERSAL DERIVATION (UD)

We also need a way to derive new universal sentences – a way to generalize to a universal sentence. A universal sentence $\forall\alpha\phi_\alpha$ claims that ϕ is true of *every* member of the universe! To prove that takes more than just a step in a derivation.

Consider the following argument:

> All politicians are corrupt.
> No corrupt people are poor.
> Therefore, no politicians are poor.

We need reasoning of the following type:

> Take any member of the universe – call that individual '*x*'. Suppose *x* is a politician. If *x* is a politician then *x* is corrupt – since all politicians are corrupt. If *x* is corrupt, then *x* is not poor – since no corrupt people are poor. We could run exactly the same argument with any other member of the universe. Therefore, no politicians are poor.

What matters in this type of reasoning is that we consider each and every member of the universe, and show that the argument works for that individual. Since the universe could be infinite, we can't run the argument for *every* individual – so we need to choose an arbitrary individual (something that can stand in for anything.) We can ensure that the variable is arbitrary if it does not occur free on any previous available line.

This is what the argument would look like:

All politicians are corrupt.	$\forall x(Ax \rightarrow Cx)$.
No corrupt people are poor.	$\forall x(Cx \rightarrow {\sim}Dx)$.
Therefore, no politicians are poor.	$\therefore\ \forall x(Ax \rightarrow {\sim}Dx)$

1	~~Show~~ $\forall x(Ax \rightarrow {\sim}Dx)$		
2	~~Show~~ $Ax \rightarrow {\sim}Dx$		We show it is true of some arbitrary individual, x. Here we instantiate it to 'x' by dropping the universal quantifier and substituting 'x' for every 'x' in the quantified sentence. 'x' is arbitrary because it does not occur unbound in any previous line of the derivation.
3	Ax	Ass CD	Assume the antecedent. Goal: the consequent, ~Dx.
4	$Ax \rightarrow Cx$	pr1 UI	We instantiate premise 1 to 'x' to match line 2 & 3.
5	Cx	3 4 MP	
6	$Cx \rightarrow {\sim}Dx$	pr2 UI	We instantiate premise 2 to 'x' to match lines 2-5
7	${\sim}Dx$	5 6 MP CD	The conditional derivation is complete.
		2 UD	Now we have shown it is true for one arbitrary individual. Thus, it must be true for every individual.

UD Universal Derivation

Show $\forall\alpha\phi_\alpha$ ψ_i ψ_{i+1} … ψ_{i+n}	On one of the lines from ψ_i to ψ_{i+n} will be ϕ_α (the instantiated form of $\forall\alpha\phi_\alpha$). To ensure arbitrariness, α (the instantiating term) cannot occur unbound in any previous line. The justification (when you box and cancel) will be the line number of ϕ_α and 'UD'.
Show $\forall\alpha\phi_\alpha$ ~~Show~~ ϕ_α χ_i χ_{i+1} … χ_{i+n}	In most cases it will take this form because usually ϕ_α cannot be reached with direct derivation techniques. Thus you will need a show line for ϕ_α (the instantiated form of $\forall\alpha\phi_\alpha$) and ϕ_α will be derived using CD or ID.

NOTE: what matters for arbitrariness is that the variable does not occur unbound on any previous line. For most of our derivations, variables do not occur unbound in premises since expressions with free variables are not symbolic sentences. However, they can be well-formed open formulas … and one *can* use open formulas as premises. For instance: If x is even then x is divisible by 2. $Ex \rightarrow Dx$. Open formulas function much like universals (think about the meaning of the open formula!) Thus, when setting up a UD, it is okay to use an unbound variable that occurs in a premise – that does not conflict with our need for arbitrariness.

6.5 EG1 Let's try some:

$\forall x(Fx \rightarrow Gx)$. $\forall y(Gy \rightarrow \sim Hy)$. $\therefore$ $\forall x(Fx \rightarrow \sim Hx)$

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When using UD, for your instantiating variable or term it is easiest to use the variable that occurs in the universal sentence – just copy the sentence dropping the universal quantifier.

Logically, the instantiating term does not need to match the variable in the universal sentence – provided the instantiating term does not occur free in any previous line. However, we will usually follow the convention of using the term that matches the variable in the universal sentence. This IS necessary in Logic2010!

Here is another version of the proof that derives Fx → ~Hx directly by using theorem 26 (Hypothetical Syllogism):

$\forall x(Fx \rightarrow Gx)$. $\forall y(Gy \rightarrow \sim Hy)$. $\therefore$ $\forall x(Fx \rightarrow \sim Hx)$

1	~~Show~~ $\forall x(Fx \rightarrow \sim Hx)$		Show conc (show conclusion)
2	$Fx \rightarrow Gx$	Pr1 UI	
3	$Gx \rightarrow \sim Hx$	Pr2 UI	
4	$Fx \rightarrow \sim Hx$	2 3 RT26	
5		4 UD	

6.5 EG2 Let's try another – a little more complex.

$\exists xFx \rightarrow \forall y(Jy \vee Hy)$. $\sim\exists xHx$. $\forall x(Jx \rightarrow Gx)$ $\therefore$ $\forall x(Fx \rightarrow Gx)$

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6.5 EG3 We can also use UD when we have two place predicates. Here's an easy one:

$\forall x \forall y(F(xy) \rightarrow F(yx))$. $\therefore$ $\forall x(F(ax) \rightarrow F(xa))$.

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6.5 EG4 Here's another:

$\forall x \forall y((Fx \wedge Fy) \rightarrow L(xy))$ $\therefore$ $\forall x(Fx \rightarrow L(xx))$

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6.5 EG5 $\forall x \exists y(Fx \rightarrow (Gy \wedge L(xy)))$. $\forall x \forall y(Gx \wedge {\sim}L(xy) \rightarrow {\sim}L(yx))$ $\therefore$ $\forall x(Fx \rightarrow \exists y L(yx))$

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6.5 EG6

Sometimes we need to derive a sentence that has two universal quantifiers at the beginning. The entire sentence is a universal, so you need a show line for the instantiated form of the sentence (drop the initial ∀.) That show line will also be a universal sentence, so you need another show line with the instantiated form of that sentence (again, drop ∀). Now you will have a sentence without any quantifiers! That is something you can work with!

$\forall x \forall y(L(xy) \rightarrow G(xy))$. $\forall z({\sim}Fz \vee \forall yL(zy))$. $\therefore$ $\forall x \forall y(Fx \rightarrow G(xy))$

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6.5 EG7 This one is a bit trickier. Be careful in parsing the second premise.

$\exists x(Hx \wedge \forall y(Gy \to K(xy)))$. $\forall x(Gx \to (\exists y(Hy \wedge K(yx)) \to \forall z(Hz \to L(xz))))$.
$\therefore\ \forall x \forall y((Gx \wedge Hy) \to L(xy))$

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6.5 EG8

This one looks complex, but most of the work is just removing and then putting back the quantifiers. The logic isn't very difficult at all! In this one, one must use UD to free the consequent of the second premise.

$\forall z\ \forall w(Gz \rightarrow \sim F(wz))$. $\forall x \forall y(F(xy) \rightarrow L(yx)) \rightarrow F(ab)$. $\therefore$ $\forall x \forall z(Gx \vee L(xz)) \rightarrow \exists x \exists y F(xy)$.

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DERIVATION TIP: When you are going to use UD, try to set up the show line as soon as possible. Then immediately put a show line for the instantiated form of the universal that you are trying to prove.

6.5 EG9

We can also prove theorems using our new rules. This theorem makes sense: it says that if something is not F then not all things are F. That's Quantifier Negation! (Theorem 203 to be precise!)

$\therefore\ \exists x{\sim}Fx \leftrightarrow {\sim}\forall xFx$

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6.5 E1 Show that the following syllogisms are valid by providing a derivation.

a) 1st figure: Celarent $\forall x(Mx \to \sim Ax)$. $\forall x(Bx \to Mx)$. $\therefore \forall x(Bx \to \sim Ax)$
b) 2nd figure: Camestres $\forall x(Bx \to Fx)$. $\forall y(Dy \to \sim Fy)$. $\therefore \forall z(Dz \to \sim Bz)$:
c) 4th figure: Camenes $\forall y(Fy \to By)$. $\forall x(Bx \to \sim Mx)$ $\therefore \forall y(My \to \sim Fy)$

6.5 E2 Construct derivations that show that the following are valid arguments:

d) $\therefore (\forall xFx \to \forall xGx) \to \forall x(Fx \to Gx)$

e) $\forall x(Fx \to Gx)$. $\forall x((Gx \vee Hx) \to (Ax \vee Bx))$. $\therefore \forall x((Fx \wedge \sim Bx) \to Ax)$

f) $\forall y(Fy \to \sim Ay)$. $\forall z(Bz \wedge Cz \to Az)$. $\therefore \forall x(Fx \to \sim Cx)$

g) $\forall x(Fx \to Gx) \to \forall y(\sim Ay \to By)$. $\exists xHx \to \forall y(\sim Gy \to \sim Fy)$. Ha. $\therefore \forall y(By \vee Ay)$

h) $\therefore \forall x\forall yL(xy) \to \forall x\forall yL(yx)$

i) $\forall x\forall y(B(xy) \to C(yx))$. $\exists xFx \to \forall yB(yy)$. $\therefore Fa \to \forall x\exists yC(xy)$

j) $\therefore \forall x\forall y(\sim(Fx \vee \sim Gy) \to B(xy)) \leftrightarrow \forall z(\sim Fz \to \forall w(Gw \to B(zw))$

k) $\forall x(\sim Fx \vee Gx) \wedge \exists y\forall x(Gx \to B(yx))$. $\forall x\forall y(B(xy) \leftrightarrow L(yx))$. $\therefore \forall x(Fx \to \exists yL(xy))$

Now you can do any derivation in predicate logic (at least those that don't involve identity!)

6.5 E3 Construct derivations that show that the philosophical arguments you symbolized in unit 5 are valid:

l) Everything that people desire is good.
Callicles (who is a person) pursues anything pleasurable.
Some pleasurable things are not good.
Therefore, Callicles pursues things that he does not desire.

(based on Plato, *Gorgias*)

$\forall x(Fx \to \forall y(D(xy) \to Gy))$ or $\forall y(\exists x(Fx \wedge D(xy)) \to Gy)$

$\forall x(Bx \to C(cx)) \wedge Fc$

$\exists x(Bx \wedge \sim Gx)$

$\therefore \exists x(C(cx) \wedge \sim D(cx))$

m) If Physicalism is true then all facts are within the scope of the physical sciences.
Mary knows all the facts about colour vision within the scope of the physical sciences.
Although Mary is a person, Mary has never seen any colour.
For a person to know certain facts about color vision it is necessary that the person sees something in color at some time or another.
Thus, some facts about colour vision are not within the scope of the physical sciences and physicalism is false.

(based on Jackson, *Epiphenomenal Qualia*.)

$P \rightarrow \forall x(Fx \rightarrow Ax)$

$\forall x(Fx \wedge Bx \wedge Ax \rightarrow K(ax))$

$Ha \wedge \sim\exists x\exists y(Dx \wedge Cy \wedge O(ayx))$

$\forall x(Hx \rightarrow \exists y(Fy \wedge By \wedge (K(xy) \rightarrow \exists z\exists w(Dz \wedge Cw \wedge O(xwz))))$

$\therefore \exists x(Fx \wedge Bx \wedge \sim Ax) \wedge \sim P$

c) An omniscient being (a being who knows everything) knows the causes of all evils.
An omnipotent being who knows the causes of an evil is able to prevent it from happening.
A perfectly good being who is able to prevent an evil thing from happening does not allow it to happen.
Things that are not allowed to happen (by some being) do not happen.
Evil things happen.
Therefore, there are no omniscient, omnipotent, perfectly good beings.

(based on The Problem of Evil, many sources from Epicurus on.)

$\forall x(\forall yK(xy) \rightarrow \forall z(Ez \rightarrow \forall w(C(wz) \rightarrow K(xw))))$

$\forall x(Fx \rightarrow \forall y(Ey \rightarrow \forall z((C(zy) \rightarrow K(xz)) \rightarrow B(xy)))))$

$\forall x(Gx \rightarrow \forall y(Ey \rightarrow (B(xy) \rightarrow \sim A(xy))))$

$\forall x(\exists y \sim A(yx) \rightarrow \sim Hx)$

$\exists x(Ex \wedge Hx)$

$\therefore \sim\exists x(Fx \wedge Gx \wedge \forall yK(xy))$

6.6 Quantifier Negation (QN)

easy part!

In the last example, we proved that: $\exists x{\sim}Fx \leftrightarrow {\sim}\forall xFx$

We could also prove that: $\forall x{\sim}Fx \leftrightarrow {\sim}\exists xFx$

We could use these theorems as rules. But, we would want to use it so often, it makes more sense to make it into a derived rule. We saw in symbolizing negations, we often want to change the negation of a universal (not all professors are boring) into a particular negation (there are some professors who are not boring) or the negation of a particular (it is not the case that some pigs fly) to a universal negation (no pigs can fly.)

Quantifier Negation is a derived rule. There are four forms:

${\sim}\forall\alpha\phi$	$\exists\alpha\,{\sim}\phi$	${\sim}\exists\alpha\,\phi$	$\forall\alpha\,{\sim}\phi$
———	———	———	———
$\exists\alpha\,{\sim}\phi$	${\sim}\forall\alpha\phi$	$\forall\alpha\,{\sim}\phi$	${\sim}\exists\alpha\,\phi$

$\forall \xrightarrow{QN} \exists$

$\exists \xrightarrow{QN} \forall$

All of them involve moving the negation from one side of the quantifier to the other and flipping the quantifier (from universal to existential or from existential to universal.) Each use of QN can negate or unnegate one (and only one) quantifier – either the negation sign or the quantifier must be the main logical operator!

The justification is the line number of the sentence and 'QN'.

6.6 EG1 Let's try it out

${\sim}\exists x{\sim}(Fx \rightarrow Gx)$. ${\sim}\exists y(Gy \wedge Hy)$. $\therefore \forall x\,{\sim}(Fx \wedge Hx)$

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6.6 EG2

Let's try another. Here we have to use QN twice – but in between we need to instantiate in order to make the negation sign the main logical operator.

$\sim\exists x\exists y(F(xy) \wedge F(yx))$. $\sim\forall xF(xd) \rightarrow \sim\forall yLy$ $\therefore F(de) \rightarrow \exists y{\sim}Ly$

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6.6 E1 Construct derivations (using QN) that show that the following are valid arguments:

a) $\therefore \sim\exists x(Fx \wedge \sim Gx) \rightarrow \forall x(Fx \rightarrow Gx)$

b) $\sim\forall x(Ax \rightarrow Mx)$. $\forall x(Bx \rightarrow Mx)$. $\therefore \exists x(Ax \wedge \sim Bx)$

c) $\forall x(Gx \rightarrow Hx)$. $(Ga \vee Gb)$. $\therefore \exists xHx$ (Try this one as an indirect derivation.)

d) $\exists x(\sim Fx \vee \sim Gx)$. $\sim\forall x(Fx \wedge Gx) \rightarrow \exists y{\sim}Ay$. $\forall x(\sim Gx \rightarrow Ax)$. $\therefore \sim\forall x{\sim}Gx$

e) $\sim\exists x\exists y(B(xy) \wedge C(yx))$. $\exists x{\sim}Fx \rightarrow \forall yB(yy)$. $\sim\forall xFx$. $\therefore \sim\exists x\forall yC(xy)$

f) $\therefore \forall x\exists y(B(xy) \rightarrow \exists w\forall zB(wz))$ (This one is a bit tricky.)

6.7 DERIVATION STRATEGIES & COMMON MISTAKES

In order to succeed in predicate derivations, make sure you are comfortable with the following:

Derivation rules: UI: Universal Instantiation
EG: Existential Generalization
EI: Existential Instantiation
QN: Quantifier Negation

and the rules from sentential derivations (Unit 3)

Derivation type: UD: Universal Derivation (as well as DD, CD, ID)

MAIN PREDICATE DERIVATION STRATEGY:

Deal with the arbitrary individuals FIRST:

When you are going to use UD, try to set up the show line as soon as possible. Then immediately put a show line for the instantiated form of the universal that you are trying to prove.

In general, use EI before UI. Make sure you are always using a new individual.

Before you use UI look at the arbitrary individuals … are there any you want to match? Remember you can use UI on the same sentence over and over.

Derivations:

Step One:

1. Show Conclusion.

Step Two:

Identify the main connective of the Show Line (this should be done for EVERY show line).
Make sure you parse the sentence correctly. Watch negations and brackets.

Main connective: $\rightarrow$ (Antecedent $\rightarrow$ Consequent)
Next line: Antecedent Ass CD
Next line: Show consequent note: it never hurts to make this a show line.

Main connective: $\leftrightarrow$ (Left side $\rightarrow$ Right side, Right side $\rightarrow$ Left side)
Next line: Left side Ass CD
Next line: Show Right side
Further down the page: Right side Ass CD
Next line: Show Left side
Further down the page: LS $\leftrightarrow$ RS Cite 2 canceled show lines, CB

Main connective: $\vee$

Ask: can I derive one of the two disjuncts separately? If so, do it!
Chances are that you *cannot!*

Option 1 (if you cannot derive them separately): ID

Next line: $\sim(\psi \vee \varphi)$	Ass ID
Next line: $\sim\psi \wedge \sim\varphi$	DM
Next line: $\sim\psi$	S
Next line: $\sim\varphi$	S

Now show a contradiction. You may want to use QN. Look for MT opportunities.

Option 2 (if you cannot derive them separately): CD

The correlated conditional for: $(\psi \vee \varphi)$ is $(\sim\psi \rightarrow \varphi)$

Next line: show correlated conditional
Next line: $\sim\psi$ Ass CD
Next line: show φ
Further down the page: $\psi \vee \varphi$ Cite canceled show line, CDJ

Main connective: $\wedge$

Ask: can I derive the two conjuncts separately?
If you can, then do so. If not …
Examine the premises, what can you do with them?

Consider an indirect proof (assume $\sim(\psi \wedge \varphi)$)

If you use DM - you get you $\sim\psi \vee \sim\varphi$. You could do it as separation of cases, showing that each disjunct leads to something. For instance, show $\sim\psi \rightarrow \alpha$ and show $\sim\varphi \rightarrow \alpha$. Use SC to derive α. Then, derive $\sim\alpha$ from your premises. (To make this work, you need to think about what α is … knowing that each disjunct contradicts a premise (or something following from the premises).

Main connective: $\forall$ … a universal.

Next line: Show substitution instance: just drop the initial quantifier.

If there is still a universal quantifier at the front, then repeat until there are no quantifiers left.

Now you have an unquantified show line OR a show line with an existential quantifier. Analyze this new show line and proceed on the basis of the main connective.

Main connective: $\exists$ … an existential.

Here you may want to just start working with the premises, deriving a substitution instance of this sentence directly. You want to be looking carefully at the premises – is a version of what you want the consequent of a conditional, or the negated antecedent, etc?

Other option: ID

Next line: $\sim\exists$ Ass ID

Next line: $\forall\sim$ QN

Now you have a universal to work with. Show a contradiction.

Main connective: ~ Negated sentence:

It is VERY important to look at the sentence that is being negated.
What is its main connective?

Option 1: ID

Next line: unnegated sentence Ass ID Show a contradiction.

Option 2: consider the QN correlate:

The QN correlate for $\sim\forall$ is $\exists\sim$. The QN correlate for $\sim\exists$ is $\forall\sim$.

Next line: show QN correlate

Now can you do this directly (in the case of a new show line with an existential quantifier) or with a universal derivation (in the case of a new show line with a negation). Watch the negations!

Derivation Rules for Predicate Logic:

Existential Generalization (EG)	**Universal Instantiation (UI)**	**Existential Instantiation (EI)**
ϕ_ζ	$\forall\alpha\phi_\alpha$	$\exists\alpha\phi_\alpha$
———	———	———
$\exists\alpha\phi_\alpha$	ϕ_ζ	ϕ_ζ
	Restriction: ζ does not occur as a bound variable in ϕ_α	Restriction: ζ does not occur in any previous line or premise.

Derived rule: Quantifier Negation (QN)

$\sim\forall\alpha\phi$	$\sim\exists\alpha\ \phi$	$\exists\alpha\ \sim\phi$	$\forall\alpha\ \sim\phi$
———	———	———	———
$\exists\alpha\ \sim\phi$	$\forall\sim\alpha\phi$	$\sim\forall\alpha\phi$	$\sim\exists\alpha\ \phi$

COMMON DERIVATION MISTAKES

DO NOT use UI or EI on part of a sentence.

You can only use EI and UI if the quantifier is the main logical operator.

That means that ∃ (for EI) and ∀ (for UI) must be at the front of the sentence and the whole sentence must be in the scope of the quantifier. It cannot even have a negation sign in front of it!

$$\forall x(Ox \rightarrow Mx) \rightarrow \exists y(Gy \vee Iy)$$

Can you use EI? NO!

- ∃ is not the main logical operator. This is a conditional sentence (→).
- ∃ isn't at the front of the sentence – only the consequent is in the scope of ∃.

Can you use UI? NO!

- ∀ is not the main logical operator. This is a conditional sentence (→).
- Although ∀ is at the front of the sentence, only the antecedent is in the scope of ∀.

What should you do?

- Use the rules from sentential logic (MP, MT, MTP, S, BC and DN) to isolate universal and existential sentences.
- Sometimes you will need to put a show line in before you use those rules. For instance here, if you show $\forall x(Ox \rightarrow Mx)$ then you can use it with MP to get $\exists y(Gy \vee Iy)$.

$$\sim\exists x\forall y(Bx \wedge (Gy \rightarrow L(xy)))$$

Can you use EI? NO!

- ∃ is not the main logical operator. This is a negated sentence (~).
- ∃ isn't at the front of the sentence. The whole sentence is not in the scope of ∃ (the negation sign lies outside its scope.)

Can you use UI? NO!

- ∀ is not the main logical operator. This is a negated sentence (~).
- ∀ isn't at the front of the sentence. The whole sentence is not in the scope of ∀ (the negation sign and ∃ lie outside its scope.)

What should you do?

- Use QN to turn it into a universal sentence.
- If you don't have QN, you could prove the universal: $\forall x\sim\forall y(Bx \wedge (Gy \rightarrow L(xy)))$ You may not want to do this, but it that difficult if you've proved the theorems justifying QN.
- Otherwise, you have to see how you can use it with your other sentences.

When using EI you must use a new variable letter – one that does not occur unbound on any previous line or premise.

Suppose this is part of your derivation…

	Show ∃yGy	
2	∃x(Fx → Gz)	Premise 1
3	∀y(Hy ∨ Fy)	Premise 2
4	∃x~Hx	Premise 3
5	Hi ∨ Fi	3 UI

EI

Now you want to use EI (acting on line 4)…

Can you use the term ‘z’ to get ~Hz?

- NO! ‘z’ occurs unbound on line 2 (Premise 1)

Can you use the term ‘i’ to get ~Hi?

- NO! ‘i’ occurs unbound on line 5.

MAKE SURE you use a new variable (a letter *i-z* that you haven’t used yet!)

ALWAYS USE an arbitrary term when setting up your UD.

When using UD, you generalize from an instantiation (the same sentence as the Universal that you are deriving, but using an arbitrary individual term.) Usually, this appears on a show line (show instantiation) immediately below the show line for the Universal.

Suppose this occurs in your derivation…

6	∃xGx	Premise 1	
7	Gx	2 EI	
8	Show ∀x(Fx → Gx)		
9	Show Fx → Gx	Show instantiation	**MISTAKE!!!**

You cannot use ‘x’ in show line 9 if you are later going to use it to prove line 8 (UD) because ‘x’ occurs unbound on a previous line (line 7).

How do you make sure that you are choosing an arbitrary variable?

- Set up your UD as soon as possible in your derivation .
- When using Logic 2010, you must use the SAME variable that is bound by ∀ in the Universal Sentence that you are deriving.
- When you put your show line for UD (show instantiation) double check that your variable doesn’t occur free on any previous lines (premises and boxed lines are okay.)

MAKE SURE you use an arbitrary term when setting up a UD!

When you use UI, you can use any term provided it does not occur as a bound variable in the scope of the ∀ you are instantiating.

That means that if universal sentence has other quantifiers (∃ or ∀) in its scope, you can't use the variables governed by those quantifiers.

$$\forall x(Hx \rightarrow \exists yL(xy))$$

Can you replace 'x' with 'y' when using UI? NO!

- The term 'y' is bound by ∃ which is in the scope of the universal you are using UI on.

What terms can you instantiate to?

- In this example, you can replace 'x' with any term except 'y'.

$$\forall x\forall y(Hx \wedge Gy \rightarrow \sim\exists yM(xyz))$$

Can you replace 'x' with 'y' when using UI? NO!

- The term 'y' is bound by ∀ which is in the scope of the universal you are using UI on.

Can you replace 'x' with 'z' when using UI? NO!

- The term 'z' is bound by ∃ which is in the scope of the universal you are using UI on.

What terms can you instantiate to?

- In this example, you can replace 'x' with any term except 'y' and 'z'.

DO NOT replace two different terms with a single variable when using EG

When you use EG, you can replace multiple instances of the same term, but you cannot replace two different terms with the same variable.

$$L(ab)$$

Can you use EG to get $\exists xL(xx)$? NO!

- That would be using the variable 'x' to replace two different terms: 'a' and 'b'.

$$L(xy)$$

Can you use EG to get $\exists xL(xx)$? NO!

- That would be using the variable 'x' to replace two different terms: 'x' and 'y'.
- It might seem like you are just replacing the 'y' with variable 'x', but the original x also gets captured by ∃.

6.8 THEOREMS FOR PREDICATE LOGIC

Try to derive as many of these as you can!

T201: $\forall x(Fx \rightarrow Gx) \rightarrow (\forall xFx \rightarrow \forall xGx)$

T202: $\forall x(Fx \rightarrow Gx) \rightarrow (\exists xFx \rightarrow \exists xGx)$

T203: $\sim\forall xFx \leftrightarrow \exists x\sim Fx$

T204: $\sim\exists xFx \leftrightarrow \forall x\sim Fx$

T205: $\forall xFx \leftrightarrow \sim\exists x\sim Fx$

T206: $\exists xFx \leftrightarrow \sim\forall x\sim Fx$

T207: $\exists x(Fx \vee Gx) \leftrightarrow \exists xFx \vee \exists xGx$

T208: $\forall x(Fx \wedge Gx) \leftrightarrow \forall xFx \wedge \forall xGx$

T209: $\exists x(Fx \wedge Gx) \rightarrow \exists xFx \wedge \exists xGx$

T210: $\forall xFx \vee \forall xGx \rightarrow \forall x(Fx \vee Gx)$

T211: $(\exists xFx \rightarrow \exists xGx) \rightarrow \exists x(Fx \rightarrow Gx)$

T212: $(\forall xFx \rightarrow \forall xGx) \rightarrow \exists x(Fx \rightarrow Gx)$

T213: $\forall x(Fx \leftrightarrow Gx) \rightarrow (\forall xFx \leftrightarrow \forall xGx)$

T214: $\forall x(Fx \leftrightarrow Gx) \rightarrow (\exists xFx \leftrightarrow \exists xGx)$

T215: $\forall x(P \wedge Fx) \leftrightarrow P \wedge \forall xFx$

T216: $\exists x(P \wedge Fx) \leftrightarrow P \wedge \exists xFx$

T217: $\forall x(P \vee Fx) \leftrightarrow P \vee \forall xFx$

T218: $\exists x(P \vee Fx) \leftrightarrow P \vee \exists xFx$

T219: $\forall x(P \rightarrow Fx) \leftrightarrow (P \rightarrow \forall xFx)$

T220: $\exists x(P \rightarrow Fx) \leftrightarrow (P \rightarrow \exists xFx)$

T221: $\forall x(Fx \rightarrow P) \leftrightarrow (\exists xFx \rightarrow P)$

T222: $\exists x(Fx \rightarrow P) \leftrightarrow (\forall xFx \rightarrow P)$

T223: $\forall x(Fx \leftrightarrow P) \rightarrow (\forall xFx \leftrightarrow P)$

T224: $\forall x(Fx \leftrightarrow P) \rightarrow (\exists xFx \leftrightarrow P)$

T225: $(\exists xFx \leftrightarrow P) \rightarrow \exists x(Fx \leftrightarrow P)$

T226: $(\forall xFx \leftrightarrow P) \rightarrow \exists x(Fx \leftrightarrow P)$

T227: $\forall xP \leftrightarrow P$

T228: $\exists xP \leftrightarrow P$

T229: $\exists x(\exists xFx \rightarrow Fx)$

T230: $\exists x(Fx \rightarrow \forall xFx)$

T231: $\forall xFx \leftrightarrow \forall yFy$

T232: $\exists xFx \leftrightarrow \exists yFy$

T233: $(Fx \rightarrow Gx) \wedge (Gx \rightarrow Hx) \rightarrow (Fx \rightarrow Hx)$

T234: $\forall x((Fx \rightarrow Gx) \wedge (Gx \rightarrow Hx) \rightarrow (Fx \rightarrow Hx))$

T235: $\forall x(Fx \rightarrow Gx) \wedge \forall x(Gx \rightarrow Hx) \rightarrow \forall x(Fx \rightarrow Hx)$

T236: $\forall x(Fx \leftrightarrow Gx) \wedge \forall x(Gx \leftrightarrow Hx) \rightarrow \forall x(Fx \leftrightarrow Hx)$

T237: $\forall x(Fx \rightarrow Gx) \wedge \forall x(Fx \rightarrow Hx) \leftrightarrow \forall x(Fx \rightarrow Gx \wedge Hx)$

T238: $\forall xFx \rightarrow \exists xFx$

T239: $\forall xFx \wedge \exists xGx \rightarrow \exists x(Fx \wedge Gx)$

T240: $\forall x(Fx \rightarrow Gx) \wedge \exists x(Fx \wedge Hx) \rightarrow \exists x(Gx \wedge Hx)$

T241: $\forall x(Fx \rightarrow Gx \vee Hx) \rightarrow \forall x(Fx \rightarrow Gx) \vee \exists x(Fx \wedge Hx)$

T242: $\sim\forall x(Fx \rightarrow Gx) \leftrightarrow \exists x(Fx \wedge \sim Gx)$

T243: $\sim\exists x(Fx \wedge Gx) \leftrightarrow \forall x(Fx \rightarrow \sim Gx)$

T244: $\sim \exists xFx \rightarrow \forall x(Fx \rightarrow Gx)$

T245: $\sim \exists xFx \leftrightarrow \forall x(Fx \rightarrow Gx) \wedge \forall x(Fx \rightarrow \sim Gx)$

T246: $\sim \exists xFx \wedge \sim\exists xGx \rightarrow \forall x(Fx \leftrightarrow Gx)$

T247: $\exists x(Fx \rightarrow Gx) \leftrightarrow \exists x\sim Fx \vee \exists xGx$

T248: $\exists xFx \wedge \exists x\sim Fx \leftrightarrow \forall x\exists y(Fx \leftrightarrow \sim Fy)$

T249: $\forall x\forall yF(xy)\rightarrow\exists x\exists yF(xy)$

T250: $\forall x\forall y\forall zF(xyz)\leftrightarrow\sim\exists x\exists y\exists z\sim F(xyz)$

T251: $\forall x\forall yF(xy)\leftrightarrow\forall y\forall xF(xy)$

T252: $\exists x\exists yF(xy)\leftrightarrow\exists y\exists xF(xy)$

T253: $\exists x\forall yF(xy)\rightarrow\forall y\exists xF(xy)$

T254: $\exists x\exists yF(xy)\leftrightarrow\exists x\exists y(F(xy) \vee F(yx))$

T255: $\forall x\forall yF(xy)\rightarrow\forall y\forall xF(yx)$

T256: $\forall x\exists y(F(xy) \wedge Gy)\rightarrow\exists x\exists y(F(xy) \wedge Gx)$

T257: $\forall xFA(x)\rightarrow\exists x(Fx \wedge FA(x))$

T258: $F(xA(x))\leftrightarrow\exists y(\forall z(F(zy)\rightarrow F(zA(x))) \wedge F(xy))$

T259: $\forall x\exists z(Fx\rightarrow\exists y(Gy\rightarrow Hz))\leftrightarrow(\exists xFx \wedge \forall xGx\rightarrow\exists xHx)$

T260: $\forall x(Fx\rightarrow\exists y(Gy \wedge (Hy \vee Hx)))\leftrightarrow\exists x(Gx \wedge Hx) \vee \sim\exists xFx \vee (\exists xGx \wedge \forall x(Fx\rightarrow Hx))$

T261: $\forall x(\exists yF(xy)\rightarrow\exists yG(xy))\leftrightarrow\forall x\forall y\exists z(F(xy)\rightarrow G(xz))$

T262: $\exists y(\exists xF(xy)\leftrightarrow Gy)\leftrightarrow\exists y\forall x\exists z((F(xy)\rightarrow Gy) \wedge (Gy\rightarrow F(zy)))$

T263: $\forall x\exists y(Fx\rightarrow Gy)\leftrightarrow\exists y\forall x(Fx\rightarrow Gy)$

T264: $\forall x\exists y(Fx \wedge Gy)\leftrightarrow\exists y\forall x(Fx \wedge Gy)$

T265: $\forall x\exists y(Fx \vee Gy)\leftrightarrow\exists y\forall x(Fx \vee Gy)$

T266: $\forall x\forall y\exists z(Fx \wedge Gy\rightarrow Hz)\leftrightarrow\forall y\exists z\forall x(Fx \wedge Gy\rightarrow Hz)$

T267: $\forall x(Fx\leftrightarrow\sim FA(x))\rightarrow\exists x(Fx \wedge \sim FA(x))$

T268: $\forall x(F(Ax) \vee \forall yF(xy))\rightarrow \exists x\forall yF(xy)$

T269: $\sim\exists y\forall x(F(xy)\leftrightarrow\sim F(xx))$

T270: $\forall z\exists y\forall x(F(xy)\leftrightarrow F(xz)\wedge \sim F(xx))\rightarrow\sim \exists z\forall xF(xz)$

T271: $\sim\exists y\forall x(F(xy)\leftrightarrow \sim\exists z(F(xz) \wedge F(zx)))$

T272: $\exists y\forall x(F(xy)\leftrightarrow F(xx))\rightarrow\sim\forall x\exists y\forall z(F(zy)\leftrightarrow\sim F(zx))$