FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 2

NOMINAL RATES OF INTEREST

For contracts where interest payments are credited or compounded more frequently than once a year, eg monthly, some financial institutions will think in terms of an effective time period shorter than a year, but to express the interest rate as an "annualised" rate. Interest rates expressed in this way are called **nominal** rates of interest.

A nominal rate can be associated with any interest compounding period: eg. six months, one month, one week, or one day. It can also be associated with periods greater than one year, for example, interest could be credited every two years.

There is a standard notation for nominal rates of interest:

We define $i^{(m)}$ as the nominal rate of interest per annum convertible m times per year. $i^{(m)}$ is payable in equal installments of $\frac{i^{(m)}}{m}$ at the ***end*** of each subinterval of length $\frac{1}{m}$ years (ie at times 1/m, 2/m,…,1).

A nominal rate of interest convertible m times per year is equivalent to an effective rate of interest of $\frac{i^{(m)}}{m}$ over a time period of $\frac{1}{m}$ years.

EXAMPLE
We will use the example of interest compounding each month to illustrate nominal rates.

Imagine that interest payments are made at the end of each month, and that the effective interest rate for the month is 1%. Using the notation above, since interest is compounded monthly, $m=12$.

Following the notation introduced above, $i^{(12)}$ is the annual nominal rate of interest convertible monthly. If $\frac{i^{(m)}}{m}$ is the effective rate of interest over a period of $\frac{1}{m}$ years, then $\frac{i^{(12)}}{12}=1\%$ is the effective rate of interest over a month. It follows that the annual nominal rate of interest $i^{(12)}=12\%$.

Therefore, an ***effective*** monthly interest rate of 1% is equivalent to a ***nominal*** interest rate of 12% per annum convertible (or payable) monthly.

Nominal interest rates are often given different names that mean the same thing. A nominal annual interest rate of 12% convertible monthly means the same as a:

> nominal annual interest rate of 12% compounded monthly, or
> nominal annual interest rate of 12% payable monthly.

'Monthly' may also be replaced by '12 times a year' in the above expressions.

An equivalent effective ***annual*** rate of interest, denoted i, can be found by compounding the effective periodic rate for m periods (m periods of length $\frac{1}{m}$ equals 1 year). For example, if $\frac{i^{(12)}}{12} = 1\%$ is the effective monthly interest rate, then \$1 accumulated for 12 months at this rate is:

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1+\frac{i^{(12)}}{12}\right)^{12} = (1+0.01)^{12} = (1+i)$$

In this example an equivalent effective annual rate of interest can be found by solving for i in the above equation : $i = (1+0.01)^{12} - 1 =$ 0.126825, or 12.6825%.

CONVERTING BETWEEN INTEREST RATES

Nominal and effective annual rates of interest are related by the following equation:

$$\boxed{1+i = \left(1+\frac{i^{(m)}}{m}\right)^{m}}$$

From this i and $i^{(m)}$ can be found be rearranging the equation:

$$i = \left(1+\frac{i^{(m)}}{m}\right)^{m} - 1$$

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

It follows from the equation above, that for an accumulation to time t, where t is measured in years:

$$\boxed{(1+i)^t = \left(1+\frac{i^{(m)}}{m}\right)^{mt}}$$

NOTE: We have defined nominal interest rates above as convertible m times per *year*. Strictly speaking, a nominal rate can be defined for any period. For example we may have a *monthly* nominal rate of interest that is convertible m times per *month*. In practice nominal rates are usually annual nominal rates, that is they are convertible m times per *year*. Throughout this course you can assume that when we refer to nominal rates we mean annual nominal rates, unless otherwise indicated.

The general rule to use with nominal rates is as follows:

(1) choose as the basic time unit the period corresponding to the frequency with which the nominal rate of interest is convertible (ie. if m =12, use months as the basic time period).

(2) use $\dfrac{i^{(m)}}{m}$ as the effective rate of interest per unit time.

For example, if we have a nominal rate of interest of 12% per annum convertible monthly, we can take one month as the unit of time and 1% as the effective rate of interest per month. We cannot use 12% as the effective annual interest rate, but we can convert 1% effective per month to an annual effective rate (equal to 12.6825% p.a. as per the previous example).

EXAMPLES

(a) What is the accumulated value of $100 in 24 months if it is invested at a nominal rate of 18% per annum convertible quarterly?

Using the formulae,
$m = 4$
$i^{(4)} = 18\%$
$t = 24$ months (2 years or 8 quarters)

$$S(t) = S(0)\left(1+\frac{i^{(m)}}{m}\right)^{mt} = S(0)\left(1+\frac{i^{(4)}}{4}\right)^{4x2} = 100\left(1+\frac{0.18}{4}\right)^{8} = \$142.21$$

Alternatively, we could have solved this quickly by using one quarter of a year as the unit of time, and $\frac{18\%}{4}$ is the effective rate of interest per quarter. In 24 months there are 8 quarters so, $S(t) = 100\left(1+\frac{0.18}{4}\right)^{8}$

(b) What is the accumulated value of \$50 in 6 months if it is invested at a nominal rate of 12% per annum convertible weekly

Using the formulae,
$m = 52$
$i^{(52)} = 12\%$
$t = 6$ months (0.5 years or 26 weeks)

$$S(t) = S(0)\left(1+\frac{i^{(m)}}{m}\right)^{mt} = 50\left(1+\frac{i^{(52)}}{52}\right)^{\frac{52}{2}} = 50\left(1+\frac{0.12}{52}\right)^{26} = \$53.09$$

To solve this quickly we can use one week as the unit of time, and $\frac{12\%}{52}$ is the effective rate of interest per week. In 6 months there are 26 weeks so,
$S(t) = 50\left(1+\frac{0.12}{52}\right)^{26}$

(c) What is the accumulated value of \$100 in 5 years if it is invested at an effective half-yearly rate of 8% **per half-year**?

We are dealing with an effective half-yearly rate in this example.
ie. $i_p = \frac{i^{(2)}}{2} = 8\%$ (***NOT*** $i^{(2)}$=8%)

We can use half-years as the unit of time, and 8% is the rate of interest per half-year. In 5 years there are 10 half-years so, $S(t) = 100(1+0.08)^{10}$= \$215.89

A consequence of the formulae is that as the interest conversion periods decrease in length (ie. as the number of compounding periods (m) per year increases), the equivalent annual effective interest rate increases.

The following table explores the relationship between the nominal interest rate and effective interest rates.

$$i^{(m)} = 0.12$$

m (effective period)	$\frac{1}{m}$-year effective interest rate $\frac{i^{(m)}}{m}$	Equivalent annual effective interest rate $i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$
1 (1 year)	$\frac{i^{(1)}}{1} = \frac{0.12}{1} = 0.12$	$(1.12)^1 - 1 = 0.12$
2 (6 months)	$\frac{i^{(2)}}{2} = \frac{0.12}{2} = 0.06$	$(1.06)^2 - 1 = 0.1236$
3 (4 months)	$\frac{i^{(3)}}{3} = \frac{0.12}{3} = 0.04$	$(1.04)^3 - 1 = 0.124864$
4 (3 months)	$\frac{i^{(4)}}{4} = \frac{0.12}{4} = 0.03$	$(1.03)^4 - 1 = 0.125509$
6 (2 months)	$\frac{i^{(6)}}{6} = \frac{0.12}{6} = 0.02$	$(1.02)^6 - 1 = 0.126162$
12 (1 month)	$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$	$(1.01)^{12} - 1 = 0.126825$
52 (1 week)	$\frac{i^{(52)}}{52} = \frac{0.12}{52} = 0.0023$	$(1.0023)^{52} - 1 = 0.127341$
365 (1 day)	$\frac{i^{(365)}}{365} = \frac{0.12}{365} = 0.00033$	$(1.00033)^{365} - 1 = 0.127475$
∞	$\lim_{m\to\infty}\left(1 + \frac{0.12}{m}\right)^m - 1 = e^{0.12} - 1 = 0.127497$	

As the number of compounding periods approaches infinity, the equivalent effective annual interest rate approaches a limit as seen in the last row of the table.

In this example, no matter how often compounding takes place the effective annual rate will not exceed 0.127497.

When $m = \infty$, $i^{(\infty)}$ can be regarded as the nominal annual rate of interest compounded continuously, and is a measure of the instantaneous rate of growth of the investment. This is also known as the force of interest and has the notation δ (ie. $i^{(\infty)} = \delta$).

If we have a constant force of interest, the accumulated value of \$1 in t years is equal to $e^{\delta t}$. If the force of interest changes as time (t) changes, then we use the notation δ_t. The force of interest is covered shortly.

In the previous example we found equivalent annual effective interest rates for a fixed nominal rate of $i^{(m)} = 0.12$. In the next example we find the equivalent nominal rates $i^{(m)}$ for a fixed annual effective interest rate of $i = 0.12$.

$i = 0.12$

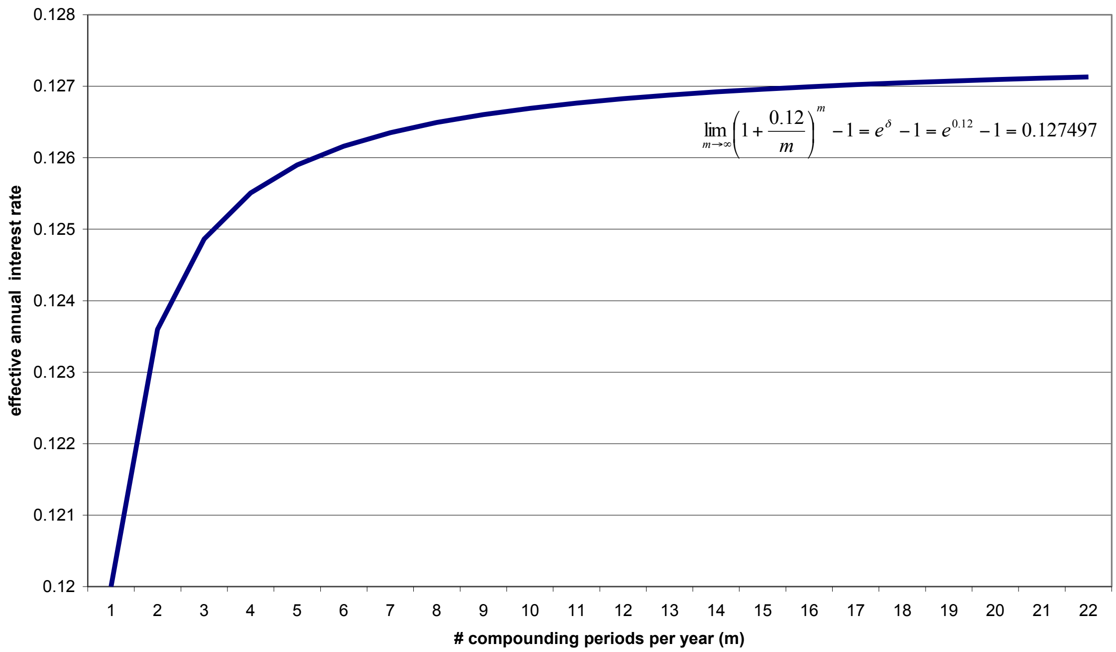
m (effective period)	$\frac{i^{(m)}}{m} = (1+i)^{1/m} - 1$	$i^{(m)} = m\left((1+i)^{1/m} - 1\right)$
1 (1 year)	$(1.12)^{1/1} - 1 = 0.12$	$1(0.12) = 0.12$
2 (6 months)	$(1.12)^{1/2} - 1 = 0.0583$	$2(0.0583) = 0.1166$
3 (4 months)	$(1.12)^{1/3} - 1 = 0.0385$	$3(0.0385) = 0.1155$
4 (3 months)	$(1.12)^{1/4} - 1 = 0.0287$	$4(0.0287) = 0.1149$
6 (2 months)	$(1.12)^{1/6} - 1 = 0.0191$	$6(0.0191) = 0.1144$
12 (1 month)	$(1.12)^{1/12} - 1 = 0.0095$	$12(0.0095) = 0.1139$
52 (1 week)	$(1.12)^{1/52} - 1 = 0.00218$	$52(0.00218) = 0.1135$
365 (1 day)	$(1.12)^{1/365} - 1 = 0.000311$	$365(0.000311) = 0.113346$
∞	$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} m\left((1+i)^{1/m} - 1\right) = \ln(1+i) = 0.113329$	

The last line gives the equivalent annual nominal interest rate when compounding is continuous. In other words, it gives the force of interest δ that is equivalent to an annual effective interest rate of 12%.

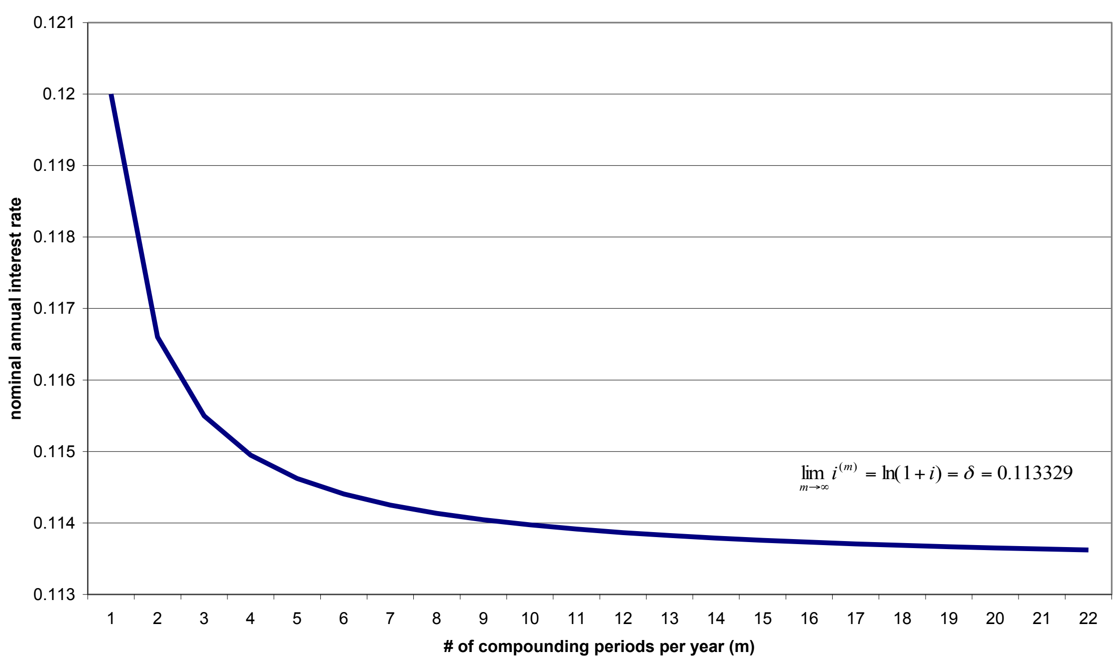
The main result is the following: For a particular annual effective interest rate i, the more frequent the compounding (ie. the higher the m), the lower the equivalent annual nominal interest rate. That is,

$$\boxed{i > i^{(2)} > i^{(3)} > \ldots > \delta}$$

Equivalent effective annual interest rate where nominal annual rate=12%



Equivalent nominal annual interest rate where effective annual rate=12%



PRESENT VALUES with nominal rates of interest

Present value calculations with nominal rates of interest can be extended to deal with more frequent transactions than yearly by using the formulae for nominal rates introduced above. The present value at time 0 of an amount K due at time t (in years), when nominal rates of interest of $i^{(m)}$ apply is:

$$K \cdot \left(1+\frac{i^{(m)}}{m}\right)^{-mt}$$

EXAMPLE

Find the present value at 1 January 2013 of payments of \$500 payable on 1 May 2013 and 1 July 2013, assuming an interest rate of 12% per annum convertible monthly.

Solution

$m = 12$

$i^{(m)} = 12\%$

$t =$ 4 months for first transaction; 6 months for second transaction (1/3 and 1/2 years respectively)

$$\$500 \cdot \left(1+\frac{0.12}{12}\right)^{\frac{-12}{3}} + 500 \cdot \left(1+\frac{0.12}{12}\right)^{\frac{-12}{2}} = \$500\left(1.01^{-4} + 1.01^{-6}\right) = \$951.52$$

It is easier to solve this equation by working directly with time units consistent with the interest conversion period.

12% per annum convertible monthly is equivalent to an effective monthly interest rate of 1%. So, working in terms of months, the present value is:

$$\$500(v^4 + v^6) = \$951.52$$

(where $v = (1.01)^{-1}$)

EFFECTIVE AND NOMINAL RATES OF DISCOUNT

To date we have only considered interest paid at the end of an interest compounding period. The interest rates considered so far can be referred to as ***interest payable in arrears***.

We now introduce ***interest payable in advance***. In this case the quoted rate is applied to obtain an amount of interest which is payable at the start of the period.

EXAMPLE
Smith borrows $1,000 for one year at a rate of 10% with interest payable in *advance*. The 10% is applied to the $1,000, resulting in an amount of $100 payable at the time the loan is made. Therefore, Smith receives $1,000-$100 = $900 at the start and repays $1,000 one year later. The $1,000 borrowed has been discounted at 10%. 10% in this instance is called the ***rate of discount*** and is given the notation $d = 0.10$.

The effective rate of discount d over a given time period is the amount of interest a single initial investment will require to be paid at the beginning of the time period, expressed as a proportion of the balance at the end of the time period.

The key distinction between effective rates of interest i and discount d can be expressed as follows:

i paid at the ***end*** of the period on the balance at the ***beginning*** of the period.
d paid at the ***beginning*** of the period on the balance at the ***end*** of the period.

By the definition of discount rate, for the example above:

$$d = \frac{\textit{amount of interest for the period}}{\textit{balance at the end of the period.}}$$

$$= \frac{100}{1000} = 10\%$$

$1,000 borrowed for one year at 10% discount means that Smith receives $900 at the start of the year and repays $1,000 one year later. The effective rate of interest is, therefore:

$$i = \frac{\textit{amount of interest for the period}}{\textit{balance at the start of the period.}}$$

$$= \frac{100}{900} \cong 11.11\%$$

Since d is the interest paid on an investment as a proportion of the accumulated investment at the end of the time period, it is equal to:

$$\boxed{d = \frac{i}{1+i}}$$

This can be rearranged to express i in terms of d:

$$\boxed{i = \frac{d}{1-d}}$$

For the example above, we can see that a rate of discount of 10% is equivalent to an interest rate of approximately 11.11%.

Using the notation defined in week 1 lectures, the effective annual rate of discount from time $t = 0$ to time $t = 1$ is:

$$d = \frac{S(1) - S(0)}{S(1)}$$

whereas, the effective rate of interest from time $t = 0$ to time $t = 1$ is:

$$i = \frac{S(1) - S(0)}{S(0)}$$

Since $v = (1+i)^{-1}$, it follows that:

$$d = \frac{i}{1+i} = 1 - \frac{1}{1+i} = 1 - v$$

and,

$$\boxed{v = 1 - d}$$

NOMINAL DISCOUNT RATES

We define $d^{(m)}$ to be the total amount of interest, payable in equal installments at the ***start*** of each subinterval (ie. at times 0, 1/m, 2/m,…,(m-1)/m).

$d^{(m)}$ implies a $\frac{1}{m}$ - year compound discount rate of $\frac{d^{(m)}}{m}$. Note: As with nominal interest rates, unless otherwise indicated, we will assume all nominal discount rates are annual, that is interest is convertible m times per *year*.

Nominal and effective annual rates of discount are related by the following equation:

$$\boxed{1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m}$$

d and $d^{(m)}$ can be found be rearranging the equation:

$$d = 1 - \left(1 - \frac{d^{(m)}}{m}\right)^{m}$$

$$d^{(m)} = m \cdot \left[1 - (1 - d)^{\frac{1}{m}}\right]$$

It follows from the equations above that the present value (at time 0) of 1 payable at time t is:

$$\boxed{v^{t} = (1 - d)^{t} = \left(1 - \frac{d^{(m)}}{m}\right)^{mt}}$$

Similarly, the accumulated value of 1 from time 0 to time t, is:

$$\boxed{(1 + i)^{t} = (1 - d)^{-t} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}}$$

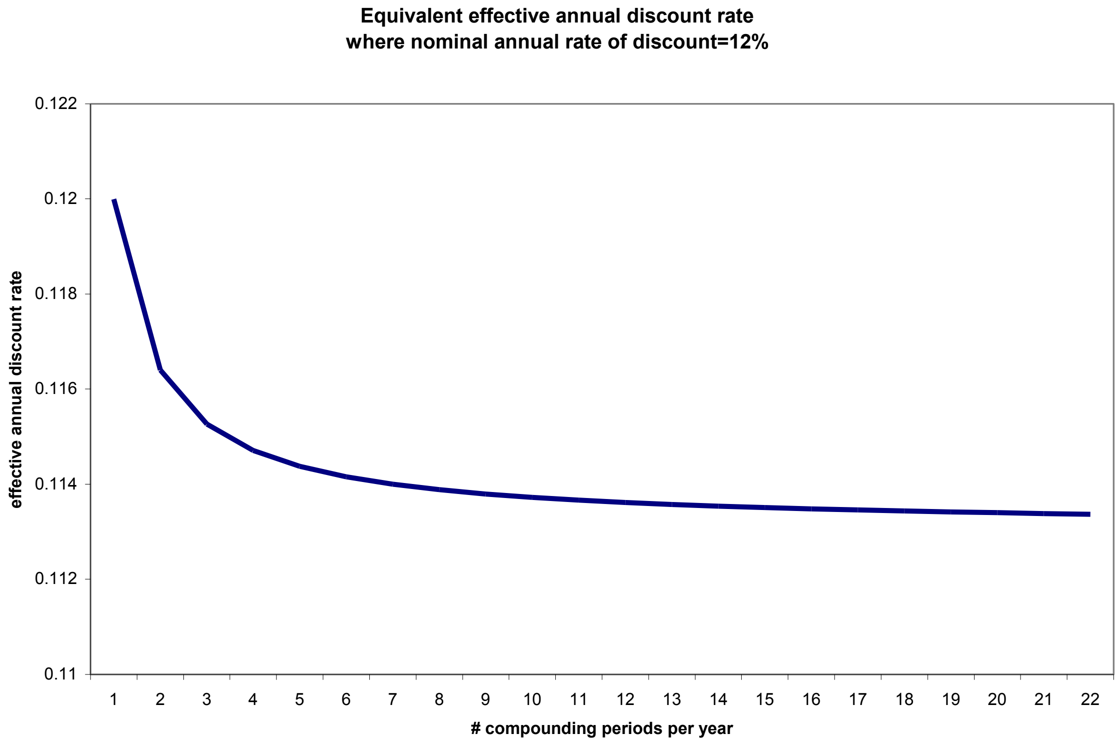
Present values in this context have been expressed in the form of ***compound discount***.

The calculation of present value over durations of less than one year is sometimes based on ***simple discount***:

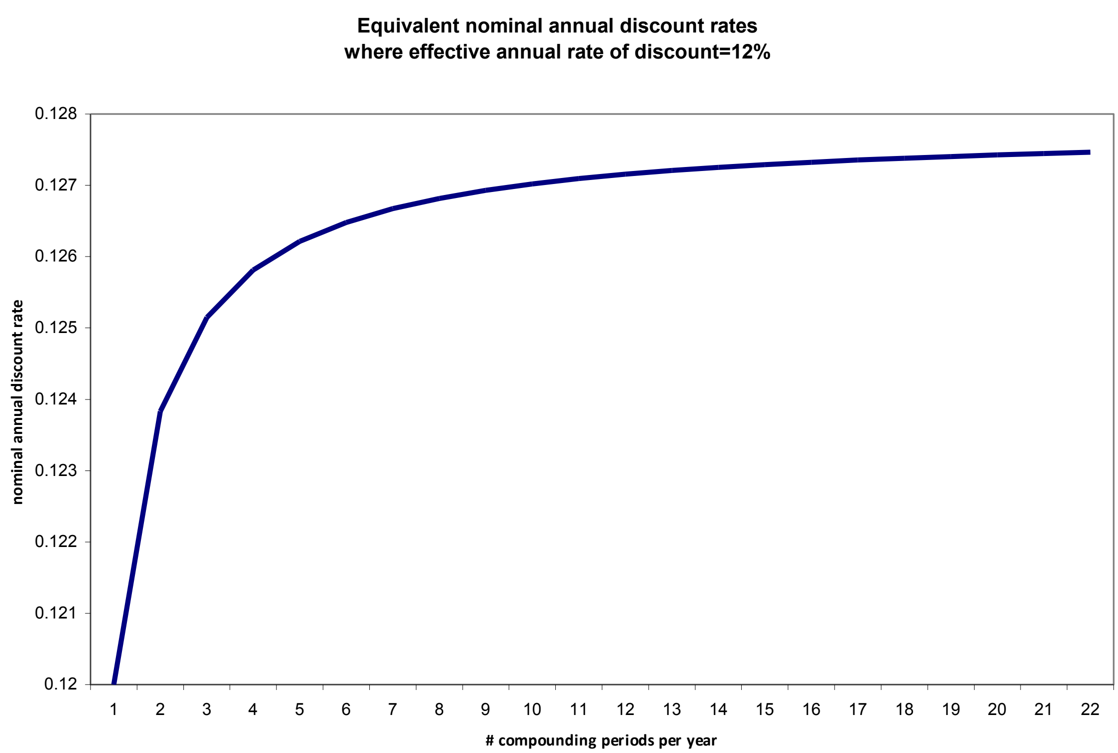
$$\boxed{(1 - d \cdot t)}$$

A consequence of the formulae relating nominal and effective rates of discount is that as the discount conversion periods decrease in length (ie. as the number of compounding periods (m) per year increases), the equivalent annual effective discount rate decreases.

In other words, for a fixed nominal rate of discount $d^{(m)}$, the effective annual discount rate d decreases as m increases. This is the opposite of what happens with interest rates. In a practical sense for a loan repayment, it can be considered that as the repayment frequency increases the effective discount rate decreases, due to the repayments being spread across time rather than up front.



For a fixed effective annual rate of discount, we can find the equivalent nominal rates of discount $d^{(m)}$ as m increases.



The main result is the following: For a particular annual effective discount rate d, the more frequent the compounding (ie. the higher the m), the higher the equivalent annual nominal discount rate. That is,

$$\boxed{d < d^{(2)} < d^{(3)} < \ldots < \delta}$$

This contrasts with interest rates, where we showed that: $i > i^{(2)} > i^{(3)} > \ldots > \delta$

EXAMPLE
Find the present value of $500 due in 10 years time if the rate of discount is 12% p.a. convertible once every two years. Express the discount rate as an effective annual interest rate.

Solution
$m = 0.5$
$d^{(m)} = 12\%$
$t = 10$ years

$$\$500\left(1-\frac{d^{(m)}}{m}\right)^{mt} = \$500\left(1-\frac{0.12}{0.5}\right)^{0.5\times 10} = \$500\times 0.76^5 = \$126.78$$

To calculate the effective annual interest rate, we can use the result above:

$$126.78 = 500\left(\frac{1}{1+i}\right)^{10} \Rightarrow i = \left(\frac{126.78}{500}\right)^{-\frac{1}{10}} - 1 = 14.71\%$$

Alternatively, the original discount rate can be converted directly to an annual effective interest rate:

$$(1+i)^{10} = \left(1-\frac{0.12}{0.5}\right)^{-5} \Rightarrow i = 14.71\%$$

FORCE OF INTEREST

We have already informally introduced the force of interest when we found equivalent nominal interest rates to a particular effective annual interest rate as m increased.

For an effective annual rate of interest, the equivalent nominal rate of interest as the number of compounding periods m approaches infinity is called the ***force of interest***:

$$\lim_{m\to\infty} i^{(m)} = \delta$$

We will now formally derive the force of interest:

The amount of investment growth (interest) from time t to time $t+\dfrac{1}{m}$ is given by:

$$S\left(t+\frac{1}{m}\right)-S(t)$$

so, the $\dfrac{1}{m}$- year effective compound rate $\dfrac{i^{(m)}}{m}$ for the period, equals

$$\frac{growth}{initial.amount} = \frac{S\left(t+\frac{1}{m}\right)-S(t)}{S(t)}.$$

The nominal rate is, therefore, given by:

$$i^{(m)} = \frac{S\left(t+\frac{1}{m}\right)-S(t)}{\frac{1}{m}\cdot S(t)}$$

Taking the limits of both sides as $m\to\infty$:

$$i^{(\infty)} = \lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty}\frac{S\left(t+\frac{1}{m}\right)-S(t)}{\frac{1}{m}\cdot S(t)}$$

where $i^{(\infty)}$ is the annual nominal rate of interest compounding continuously.

If we replace $\dfrac{1}{m}$ with h, then this becomes:

$$i^{(\infty)} = \frac{1}{S(t)}\cdot\lim_{h\to 0}\frac{S(t+h)-S(t)}{h} = \frac{1}{S(t)}\frac{d}{dt}S(t) = \frac{S'(t)}{S(t)} = \frac{d}{dt}\ln(S(t))$$

which is written as δ_t and is known as ***the force of interest at time t*** or the ***instantaneous rate of growth at time t***.

If δ_t is constant (ie. doesn't change as t changes) it is written δ.

Using a similar approach to the one above it can also be shown that

$$\lim_{m\to\infty} d^{(m)} = \delta.$$

There are a number of useful mathematical results that follow from $i^{(\infty)} = \frac{S'(t)}{S(t)} = \frac{d}{dt}\ln(S(t))$. These are given below:

If accumulation is based on simple interest then:

$$S(t) = S(0)(1+i\cdot t)$$

$$\delta_t = \frac{S'(t)}{S(t)} = \frac{S(0)\cdot i}{S(0)(1+i\cdot t)} = \frac{i}{(1+i\cdot t)}$$

If accumulation is based on compound interest then:

$$S(t) = S(0)(1+i)^t$$

$$\begin{aligned}
\delta_t &= \frac{d}{dt}\ln(S(t)) \\
&= \frac{d}{dt}\ln\left(S(0)(1+i)^t\right) \\
&= \frac{d}{dt}\ln(S(0)) + \frac{d}{dt}\left(\ln(1+i)^t\right) \\
&= 0 + \frac{d}{dt}\left(t\cdot\ln(1+i)\right) \\
&= \ln(1+i)
\end{aligned}$$

Therefore, under compound interest at an annual effective rate i, the equivalent force of interest is:

$$\boxed{\delta_t = \ln(1+i).}$$

So the force of interest is constant as long as the effective annual rate of interest is constant.

Rearranging,

$$\boxed{i = e^{\delta_t} - 1}$$

We can derive a general expression for investment growth when a force of interest is operating on the investment as follows:

From earlier,

$$\delta_t = \frac{S'(t)}{S(t)} = \frac{d}{dt}\ln(S(t))$$

Integrating from $t = 0$ to $t = n$:

$$\int_0^n \delta_t \, dt = \int_0^n \frac{d}{dt}\ln(S(t))dt = \ln(S(n)) - \ln(S(0)) = \ln\left(\frac{S(n)}{S(0)}\right)$$

Exponentiating and rearranging, we get a formula for the accumulation at time n under a force of interest of δ_t:

$$\boxed{S(n) = S(0)\cdot\exp\left(\int_0^n \delta_t \, dt\right)}$$

Similarly, the present value of an amount $S(n)$ at time n is:

$$\boxed{S(0) = S(n)\cdot\exp\left(-\int_0^n \delta_t \, dt\right)}$$

The formulae can be generalised. For two time periods t_1 and t_2, where $t_1 \le t_2$, the accumulation of an amount $S(t_1)$ from time t_1 to t_2 is:

$$\boxed{S(t_2) = S(t_1)\exp\left(\int_{t_1}^{t_2} \delta_t \, dt\right)}, \text{ or, } \boxed{A(t_1,t_2) = \exp\left(\int_{t_1}^{t_2} \delta_t \, dt\right)}$$

It follows from this formula that an investment of

$S(t_1) = \dfrac{S(t_2)}{A(t_1,t_2)} = \dfrac{S(t_2)}{\exp\left(\int_{t_1}^{t_2} \delta_t \, dt\right)} = S(t_2)\exp\left(-\int_{t_1}^{t_2} \delta_t \, dt\right)$ will accumulate to $S(t_2)$.

In other words, the present value at time t_1 of an amount $S(t_2)$ due at time t_2 is:

$$\boxed{S(t_1) = S(t_2) \cdot \exp\left(-\int_{t_1}^{t_2} \delta_t dt \right)}$$

The case when $\delta_t = \delta$ (ie. the force of interest is constant over time) is important and will come up often during the course.

When $\delta_t = \delta$ the accumulation from time 0 to time n simplifies to:

$$\boxed{S(n) = S(0) \cdot \exp\left(\int_0^n \delta dt \right) = S(0) \cdot e^{\delta n}}$$

When $\delta_t = \delta$ the present value at time 0 of an amount $S(n)$ payable at time n is:

$$\boxed{S(0) = S(n) \cdot e^{-\delta n}}$$

If δ_t is not constant, we can substitute an expression for δ_t into the equations above to find the accumulated or present values.

EXAMPLE

The current force of interest is 8% p.a. Find the accumulated value of $100 after 3 years if:

(i) the force of interest remains at this rate over the 3 years;

(ii) the force of interest increases constantly at a linear rate over the 3 years at a rate of 0.5% p.a., i.e. $\delta_t = 0.08 + 0.005t$

Solution

(i) $S(n) = S(0) \times e^{\delta n} = 100e^{0.24} = \$127.12$

(ii) $S(n) = S(0) \cdot \exp\left(\int_0^n (0.08 + 0.005t) dt \right) = S(0) \cdot \exp\left(0.08n + \frac{0.005}{2} n^2 \right)$

$$S(3) = \$100 \cdot \exp\left(0.08(3) + \frac{0.005}{2}(9) \right) = \$100 \cdot \exp(0.2625) = \$130.02$$