Lecture 11 handout

11.1 Colouring of planar maps

- A face colouring is a map $c : F(G) \to C$ such that neighbouring faces get different colours.
- An edge colouring is $c : E(G) \to C$ s.t. neighbouring edges get different colours.
- A vertex colouring is $c : V(G) \to C$ s.t. neighbouring vertices get different colours.

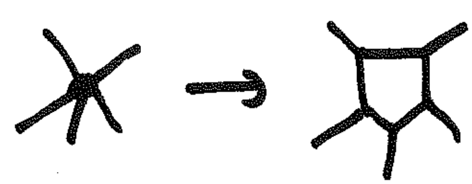
Four-colour theorem (1)

Any bridgeless planar graph is 4-face colourable

Four-colour theorem (2) (dual graph).

Any loopless planar graph is 4-colourable.

Reduction to cubic graphs:

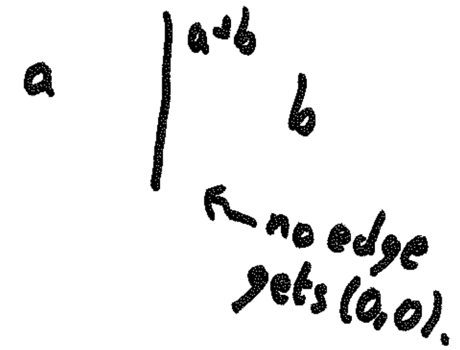


Four colour theorem (3) (Tait)

A bridgeless cubic planar graph is 3-edge colourable.

Proof of equivalence:

$C := \{(0,0), (0,1), (1,0), (1,1)\}$.



Conversely:

$G[E_i] :=$ subgraph spanned by edges coloured i.

It is a spanning subgraph.

$G_{ij} := G[E_i \cup E_j]$ is a spanning 2-regular subgraph

$\Rightarrow$ 2-face colourable.

Intersect G_{12} and $G_{2,3}$.

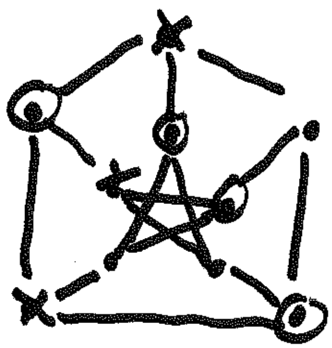
The chromatic number $\chi(G)$ of G is the minimal number of colours with which it can be coloured.

Examples:

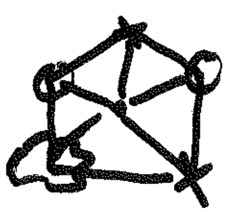
$$\chi(K_n) = n$$

$$\chi(\text{null graph}) = 1$$

$$\chi\left(\begin{matrix}\text{bipartite}\\ \text{graph}\end{matrix}\right) = 2$$



$$\chi(G) = 3$$



$$\chi(G) = 4$$

11.2 Five colour theorem

A loopless planar graph is 5-colourable.

Proof: (1)

G has vertex v of degree ≤ 5. If its neighbours don't exhaust 5 colours we are finished. Else, denote its neighbours v_1, v_2, v_3, v_4, v_5 with colours $1,2,3,4,5$ correspondingly, cyclically ordered.

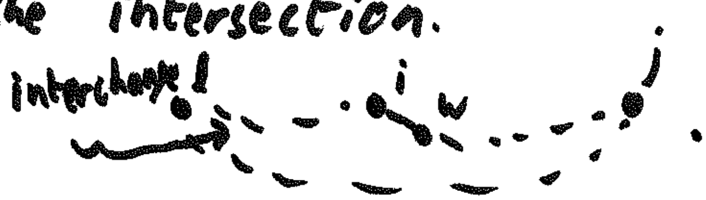
If v_1, v_3 are disconnected in $G_{1,3}$ (minus v), change colours of connected component of v_1. Else, there is a path between them in $G_{1,3}$

Same argument for v_2, v_4, and we get a contradiction by the Jordan curve theorem.

Bonus

<u>Brooks Theorem</u>: If G is simple, connected, not complete, largest vertex degree is $p \geq 3$, then $\chi(G) \leq p$.

<u>Proof</u>: As in 5-colour theorem. Delete v, cyclically order its neighbours $v_1, \ldots, v_p$. v_i, v_j connected in H_{ij} $(H := G - \{v\})$.
Colours of all neighbours of v must be different (or recolour v), so C_{ij} = connected component of v_i, v_j is a path. C_{ij}, C_{ik} can intersect only at v_i, or recolour the intersection.
To complete



Applications of colouring:

- Scheduling
- Storage
- Timetabling

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bonus

Chromatic polynomial

$C(G,k)$: number of ways to k-colour G.

$C(G,k) = k(k-1)^2$.

$C(K_n,k) = k(k-1)\cdots(k-n+1)$

Deletion-contraction

$P(G,k) = C(G,k)$ for all $k \geq 0$

$P(G,x) = P(G \setminus e,x) - P(G/e,x)$.

Proof: $P(G,x)$: # colourings in which incident edges to e have different colours

$P(G/e,x)$: # colourings of $P(G\setminus e,x)$ in which incident edges to e have the same colour.

Procedure: add edges or identify until we obtain a complete graph

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Next time: Review + Colouring bonus material.