Today – finish §2.1 ("An unbounded Problem")

– §2.2 ("A Degenerate Optimalization")

Remark: If, in any tableau, you get a column where objective row coefficient is < 0 (≠ 0) and where other coefficient are all ≤ 0, then the tableau represents an unbounded problem.

Eg. (based on "An Unbounded Problem") ← the eg2. pdf

An Unbounded Problem

Tableau 1:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	2	2	−3	−2	1	0	6
x_6	−6	−1	9	−1	0	1	7
	−9	−10	8	9	0	0	0

Tableau 2:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	1	1	$-\frac{3}{2}$	−1	$\frac{1}{2}$	0	3
x_6	−5	0	$\frac{15}{2}$	−2	$\frac{1}{2}$	1	10
	1	0	−7	−1	5	0	30

Tableau 3:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	1	0	$-\frac{7}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	5
x_3	$-\frac{2}{3}$	0	1	$-\frac{4}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{3}$
	$-\frac{11}{3}$	0	0	$-\frac{43}{15}$	$\frac{82}{15}$	$\frac{14}{15}$	$\frac{118}{3}$

From tableau 3, one would enter x_1.

x_1	θ-ratios
x_2	$\frac{5}{0}$
x_3	$-\frac{7}{6}$

no valid ratio

We will construct a half-line in the feasible region, where z can take abitrarily large value.

Tableau 3 represents the problem:

$$\text{Maximize } z = \frac{11}{3}x_1 + \frac{43}{15}x_4 - \frac{82}{15}x_5 - \frac{14}{15}x_6 + \frac{118}{3}$$

$$\text{s.t. } 0x_1 + x_2 - \frac{7}{5}x_4 + \frac{3}{5}x_5 + \frac{1}{5}x_6 = 5$$

$$-\frac{2}{3}x_1 + x_3 - \frac{4}{15}x_4 + \frac{1}{15}x_5 + \frac{2}{15}x_6 = \frac{4}{3}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$$

Tableau 3 also represents the basic feasible solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ \frac{4}{3} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we increase to $M \geq 0$, say, we can stay in the feasible region by making appropriate increases in the basic variables x_3 and x_4, to get the non-basic solution (if $M > 0$)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} M \\ 5 \\ \frac{4}{3} + \frac{2}{3}M \\ 0 \quad 0 \\ 0 \quad 0 \\ 0 \quad 0 \end{bmatrix}$$

(feasible for any M)

where $z = \frac{11}{3}M + \frac{118}{3}$

Eg. (still based on "An Unbounded Problem")
The x_4-column of tableau 2 also shows the problem is unbounded

Tableau 2 represents the basic feasible solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

, and for any $M \geq 0$, $\begin{bmatrix} 0 \\ 3+M \\ 0 \\ M \\ 0 \\ 10+2M \end{bmatrix}$ is also feasible (and non-basic if $M > 0$) where $z = M + 30$.

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§ 2.2 Definition A basic solution of a system of equations is degenerate provided at least one basic variable is 0.

Notes on "A Degenerate Optimal Solution"

A Degenerate Optimal Solution

Tableau 1:

	x_1	x_2	x_3	x_4	x_5	
x_3	1	5	1	0	0	19
x_4	1	-1	0	1	0	1
x_5	-1	2	0	0	1	2
	-3	-7	0	0	0	0

Tableau 2:

	x_1	x_2	x_3	x_4	x_5	
x_3	$\frac{7}{2}$	0	1	0	$-\frac{5}{2}$	14
x_4	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	2
x_2	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1
	$-\frac{13}{2}$	0	0	0	$\frac{7}{2}$	7

Tableau 3:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{2}{7}$	0	$-\frac{5}{7}$	4
x_4	0	0	$-\frac{1}{7}$	1	$\frac{6}{7}$	0
x_2	0	1	$\frac{1}{7}$	0	$\frac{1}{7}$	3
	0	0	$\frac{13}{7}$	0	$-\frac{8}{7}$	33

Tableau 4 is optimal:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{1}{6}$	$\frac{5}{6}$	0	4
x_5	0	0	$-\frac{1}{6}$	$\frac{7}{6}$	1	0
x_2	0	1	$\frac{1}{6}$	$-\frac{1}{6}$	0	3
	0	0	$\frac{5}{3}$	$\frac{4}{3}$	0	33

Tableau 1 indicates the problem being solved is Maximize $z = 3x_1 + 7x_2$

$$x_1 + 5x_2 \leq 19$$
$$x_1 - x_2 \leq 1$$
$$x_1 \geq 0,\ x_2 \geq 0)\ -x_1 + 2x_2 \leq 2$$

From tableau ①, x_2 will enter, and x_2-column θ-ratios

$$\begin{array}{c|l} x_3 & \frac{19}{5} \\ x_4 & \cancel{*} \\ x_5 & \frac{2}{2} \longrightarrow x_5 \text{ will exit} \end{array}$$

This leads to Tableau ②, from which x_1 will enter x_1-column θ-ratios.

$$\begin{array}{c|l} x_3 & \frac{14}{7/2} = 4 \\ x_4 & \frac{2}{1/2} = 4 \\ x_2 & \cancel{\frac{*}{1/2}} \end{array}$$

a tie. But either x_3 or x_4 (The other tied variable(s) will drop to 0.)

We arbitrarily exited x_3, to get tableau ③.

From Tableau ③, x_5 will enter.

x_5-column θ-ratios

$$\begin{array}{c|l} x_1 & \cancel{\frac{22}{-5/7}} \\ x_4 & \frac{0}{6/7} = 0 \quad \text{, smallest, so } x_4 \text{ exits} \\ x_2 & \frac{3}{1/7} \end{array}$$

This leads to tableau ④ which verifies

$$\begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ is optimal (Tableau ③ did not)