UNIVERSITY OF TORONTO
Faculty of Arts and Science
APRIL EXAMINATIONS 2014
CSC336H1 LEC 5101 S (20141)
Duration - 2 hours
Aids allowed: pocket calculators

Last (family) name :

First (given) name :

Student id :

Question	Marks
1	/ 15
2	/ 30
3	/ 15
4	/ 25
5	/ 15
Total	/ 100

As noted in the syllabus of the course, you must achieve at least 33% in the final exam in order to pass the course.

Please write legibly. Unreadable answers are worthless.

You can use both sides of all seven (7) sheets for your answers, except the front (cover) page.
You must return all 7 sheets.

1. [*15 points*] Consider the following data:

x	$1/2$	1	2	4
f	-1	0	1	2

Obtain the least squares fit to this data (i.e. to $f(x)$) by a linear polynomial. More specifically, formulate the problem into a (non-square) linear system, indicate the size of the (non-square) matrix arising, then solve the system by the normal equations method.

2. Consider computing the root(s) of the function $f(x) \equiv x + \ln x$, defined in $(0, \infty)$.

(a) [*2 points*] Using appropriate graphs, locate approximately the root of $f(x)$. (Choose functions easy to graph, so that you can draw the graphs by hand.) Indicate an interval of length no more than 1, where the root lies. Give a (very) rough approximation to the root (numerical value). (You don't need a calculator to give a rough approximation to the root.)

(b) [*3 points*] Let $e = \exp(1) \approx 2.7183$. Note that $e^{-1} = \exp(-1) \approx 0.3679$. Using mathematical arguments, show that there exists exactly one positive root of $f(x)$, and indicate an interval $[a, b]$ of length not greater than $1 - e^{-1}$, where the positive root lies. Indicate a and b.

(c) [*5 points*] Using $x^{(0)} = e^{-1}$ as initial guess, apply (by hand) one Newton iteration to compute an approximation $x^{(1)}$ to the root. Indicating how $x^{(1)}$ is computed, simplify as much as you can, and write the result in terms of e. You don't have to give a numerical value.

(d) [*10 points*] Show that Newton's iteration applied to $f(x)$ converges to the root for any initial guess in $[e^{-1}, 1 - \delta]$, for any small positive δ.

(e) [*4 points*] Show that Newton's iteration applied to $f(x)$ converges to the root, if the initial guess is $x^{(0)} = 1$.

(f) [*6 points*] The equation $x + \ln x = 0$ can be written in several equivalent forms:

(i) $x = -\ln x$, (ii) $x = e^{-x}$, (iii) $x = \dfrac{x + e^{-x}}{2}$.

Each of the above forms gives rise to a fixed-point iteration scheme. Which of the three schemes is guaranteed to converge, if started close enough to the root? Explain. Which of the schemes guaranteed to converge will converge faster? Explain.

Note: You only need a rough approximation to the root to answer these equations.

You do NOT need to specify the interval which the starting guess belongs to.

At certain points, it may be useful to consider the following approximate values:

$e = \exp(1) \approx 2.7183$, $\dfrac{1}{e} = \exp(-1) \approx 0.3679$, $\dfrac{2}{1+e} \approx 0.5379$, $e^{-1/2} = \exp(-\dfrac{1}{2}) \approx 0.6$.

3. Consider the system of two nonlinear equations with respect to the two unknowns x and y

$$x + y + \log(xy) = 0$$
$$x^2 + 4y^2 = 9.$$

(a) [*8 points*] Formulate the Jacobian matrix for the above nonlinear system. The entries of the Jacobian should be given in terms of the variables x and y.

(b) [*7 points*] Assume one wants to apply Newton's method to the above system with starting guess $[x^{(0)}, y^{(0)}]^T$. Find conditions on $x^{(0)}$ and $y^{(0)}$, so that Newton's method is applicable and explain. (Simplify the condition(s) as much as you can.)

4. Consider the function $f(x) = \log_2 x$, and the data $(\frac{1}{2}, -1)$, $(1, 0)$ and $(2, 1)$ arising from f.

(a) *[10 points]* Using the basis functions of your choice (monomial, Lagrange or NDD), construct the quadratic polynomial $p_2(x)$ interpolating $f(x)$ at the above data. Bring it to monomial form, if it is not already in that form.

(b) *[5 points]* Give the error formula for this interpolation problem (i.e. for *this* f and *these* data points). The formula should involve an unknown point ξ. Any other functions involved in the formula should be given explicitly in terms of x.

(c) *[10 points]* Assume we evaluate p_2 at some $x \in [\frac{1}{4}, 2]$. Indicate the interval where ξ belongs to and explain. Describe how you can find an upper bound (as sharp as you can) for the error $|f(x) - p_2(x)|$ when $x \in [\frac{1}{4}, 2]$. You do not have to give a numerical value to the bound. You may specify it as the maximum of a fixed number of quantities simplified as much as possible.

Excerpt from notes:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j),$$

where ξ is an unknown point in $\mathrm{ospr}\{x_0, x_1, \cdots, x_n, x\}$, that depends on x.

Other: If you do not have a calculator, you are welcome to use the approximate values $\frac{7+\sqrt{7}}{6} \approx 1.6$, $\frac{7-\sqrt{7}}{6} \approx 0.7$. Also recall that $(\log(x))' = \frac{1}{x}$, and that $\log_2(x) = \frac{\log(x)}{\log(2)} \approx \frac{\log(x)}{0.6931}$.

5. Assume we interpolate a function $f \in C^4$ in some interval $[a, b]$, by linear splines and by cubic splines, in both cases using equidistant knots.

(a) *[5 points]* Using n subintervals of $[a, b]$, the maximum error of the linear spline interpolant over all knots turns out to be 1.024. How many subintervals should be used, to that the maximum error of the linear spline interpolant over all knots is approximately 10^{-3}? Explain. The answer is to be given in terms of n.

(b) *[10 points]* Using n subintervals of $[a, b]$, the maximum error of the cubic spline interpolant over all knots turns out to be 1.024. How many subintervals should be used, to that the maximum error of the cubic spline interpolant over all knots is approximately 10^{-3}? Explain. The answer is to be given in terms of n.

Excerpt from notes:

$$|f(x) - L(x)| \le \frac{1}{8} \max_{x \in [a,b]} |f''(x)| \max_{i=1,\cdots,n} (x_i - x_{i-1})^2$$

$$|f(x) - C(x)| \le \frac{5}{384} \max_{x \in [a,b]} |f^{(4)}(x)| \max_{i=1,\cdots,n} (x_i - x_{i-1})^4$$

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