MAT 237, Quiz 3

NAME: **STUDENT ID NUMBER :**

Check your tutorial:

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Part A: (2 marks) Precisely state the definition of Directional Derivatives.

The directional derivative of f at $\vec{a}$ in the $\vec{u}$ direction is $\partial_{\vec{u}} f(\vec{a}) := \frac{d}{dt} f(\vec{a}+t\vec{u})\Big|_{t=0}$ $\left(= \lim_{t\to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} \right)$

either fine.

Part B: (3 marks) Find $\partial_x w$ and $\partial_y w$ in terms of the partial derivatives $\partial_1 f$, $\partial_2 f$, and $\partial_3 f$ for $w = f(2x-y^2, xsin(3y), x^4)$.

$$\partial_x w = \partial_1 f \frac{\partial(2x-y^2)}{\partial x} + \partial_2 f \frac{\partial(x\sin 3y)}{\partial x} + \partial_3 f \frac{\partial(x^4)}{\partial x}$$
$$= \partial_1 f \cdot 2 + \partial_2 f \cdot \sin 3y + \partial_3 f \cdot 4x^3$$

likewise,

$$\partial_y w = -2y\,\partial_1 f + 3x\cos(3y)\,\partial_2 f + 0$$

with $\partial_1 f, \partial_2 f, \partial_3 f$ evaluated at $(2x-y^2, x\sin 3y, x^4)$

Part C: (5 marks) Prove the Mean Value Theorem III. That is, for $S \subset \mathbb{R}^n$ such that **a**, **b**, and the line segment L that joins them are all in S. Let f be continuous on L and differentiable on L except perhaps the endpoints **a** and **b**. Then prove that there exists a point **c** on L such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Let $\vec{h} = \vec{b} - \vec{a}$, then $L = \{\vec{a} + t\vec{h} \mid 0 \le t \le 1\}$. Let $\phi = f(\vec{a}+t\vec{h})$, $0 \le t \le 1$

Since f is cont on L, ϕ is cont on $[0,1]$, and so differentiable on $(0,1)$ via chain rule. Hence,

$$\phi'(t) = \nabla f(\vec{a}+t\vec{h}) \cdot \frac{d}{dt}(\vec{a}+t\vec{h}) = \nabla f(\vec{a}+t\vec{h}) \cdot \vec{h} = \nabla f(\vec{a}+t\vec{h}) \cdot (\vec{b}-\vec{a})$$

By MVT for one variable, $\exists u \in (0,1)$ s.t. $\phi(1) - \phi(0) = \phi'(u)(1-0) = \phi'(u)$.

Letting $\vec{c} = \vec{a} + u\vec{h}$, $f(\vec{b}) - f(\vec{a}) = \phi(1) - \phi(0) = \phi'(u) = \nabla f(\vec{a}+u\vec{h}) \cdot (\vec{b}-\vec{a})$

$$= \nabla f(\vec{c}) \cdot (\vec{b}-\vec{a}) \quad \blacksquare$$