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8.3 Some Common Unbiased Point Estimators

Some formal methods for deriving point estimators for target parameters are presented in Chapter 9. In this section, we focus on some estimators that merit consideration on the basis of intuition. For example, it seems natural to use the sample mean

$\overline{Y}$ to estimate the population mean μ and to use the sample proportion $\hat{p} = Y/n$ to estimate a binomial parameter p. If an inference is to be based on independent random samples of n_1 and n_2 observations selected from two different populations, how would we estimate the difference between means $(\mu_1 - \mu_2)$ or the difference in two binomial parameters, $(p_1 - p_2)$? Again, our intuition suggests using the point estimators $(\overline{Y}_1 - \overline{Y}_2)$, the difference in the sample means, to estimate $(\mu_1 - \mu_2)$ and using $(\hat{p}_1 - \hat{p}_2)$, the difference in the sample proportions, to estimate $(p_1 - p_2)$.

Because the four estimators $\overline{Y}$, $\hat{p}$, $(\overline{Y}_1 - \overline{Y}_2)$, and $(\hat{p}_1 - \hat{p}_2)$ are functions of the random variables observed in samples, we can find their expected values and variances by using the expectation theorems of Sections 5.6–5.8. The standard deviation of each of the estimators is simply the square root of the respective variance. Such an effort would show that, when random sampling has been employed, all four point estimators are unbiased and that they possess the standard deviations shown in Table 8.1. To facilitate communication, we use the notation $\sigma^2_{\hat{\theta}}$ to denote the variance of the sampling distribution of the estimator $\hat{\theta}$. The standard deviation of the sampling distribution of the estimator $\hat{\theta}$, $\sigma_{\hat{\theta}} = \sqrt{\sigma^2_{\hat{\theta}}}$, is usually called the *standard error* of the estimator $\hat{\theta}$.

In Chapter 5, we did much of the derivation required for Table 8.1. In particular, we found the means and variances of $\overline{Y}$ and $\hat{p}$ in Examples 5.27 and 5.28, respectively. If the random samples are independent, these results and Theorem 5.12 imply that

$$E(\overline{Y}_1 - \overline{Y}_2) = E(\overline{Y}_1) - E(\overline{Y}_2) = \mu_1 - \mu_2,$$

$$V(\overline{Y}_1 - \overline{Y}_2) = V(\overline{Y}_1) + V(\overline{Y}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

The expected value and standard error of $(\hat{p}_1 - \hat{p}_2)$, shown in Table 8.1, can be acquired similarly.

Table 8.1 Expected values and standard errors of some common point estimators

Target Parameter θ	Sample Size(s)	Point Estimator $\hat{\theta}$	$E(\hat{\theta})$	Standard Error $\sigma_{\hat{\theta}}$
μ	n	$\overline{Y}$	μ	$\frac{\sigma}{\sqrt{n}}$
p	n	$\hat{p} = \frac{Y}{n}$	p	$\sqrt{\frac{pq}{n}}$
$\mu_1 - \mu_2$	n_1 and n_2	$\overline{Y}_1 - \overline{Y}_2$	$\mu_1 - \mu_2$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ *†
$p_1 - p_2$	n_1 and n_2	$\hat{p}_1 - \hat{p}_2$	$p_1 - p_2$	$\sqrt{\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}}$ †

*σ_1^2 and σ_2^2 are the variances of populations 1 and 2, respectively.

†The two samples are assumed to be independent.

Although unbiasedness is often a desirable property for a point estimator, not all estimators are unbiased. In Chapter 1, we defined the sample variance

$$S^2 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n-1}.$$

It probably seemed more natural to divide by n than by $n-1$ in the preceding expression and to calculate

$$S'^2 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n}.$$

Example 8.1 establishes that S'^2 and S^2 are, respectively, biased and unbiased estimators of the population variance σ^2. We initially identified S^2 as the *sample variance* because it is an unbiased estimator.

EXAMPLE **8.1** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$. Show that

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

is a biased estimator for σ^2 and that

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

is an unbiased estimator for σ^2.

Solution It can be shown (see Exercise 1.9) that

$$\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2 = \sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2.$$

Hence,

$$E\left[\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = E\left(\sum_{i=1}^{n} Y_i^2\right) - nE(\overline{Y}^2) = \sum_{i=1}^{n} E(Y_i^2) - nE(\overline{Y}^2).$$

Notice that $E(Y_i^2)$ is the same for $i = 1, 2, \ldots, n$. We use this and the fact that the variance of a random variable is given by $V(Y) = E(Y^2) - [E(Y)]^2$ to conclude that $E(Y_i^2) = V(Y_i) + [E(Y_i)]^2 = \sigma^2 + \mu^2$, $E(\overline{Y}^2) = V(\overline{Y}) + [E(\overline{Y})]^2 = \sigma^2/n + \mu^2$, and that

$$\begin{aligned} E\left[\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] &= \sum_{i=1}^{n}(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) \\ &= n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) \\ &= n\sigma^2 - \sigma^2 = (n-1)\sigma^2. \end{aligned}$$

It follows that

$$E(S'^2) = \frac{1}{n} E\left[\sum_{i=1}^{n} (Y_i - \overline{Y})^2\right] = \frac{1}{n}(n-1)\sigma^2 = \left(\frac{n-1}{n}\right)\sigma^2$$

and that S'^2 is biased because $E(S'^2) \neq \sigma^2$. However,

$$E(S^2) = \frac{1}{n-1} E\left[\sum_{i=1}^{n} (Y_i - \overline{Y})^2\right] = \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2,$$

so we see that S^2 is an unbiased estimator for σ^2. ■