Mean Value Theorem

on $\mathbb{R}$

if $f:[ab]\longrightarrow\mathbb{R}$ is Cont on $[a\,b]$ and diff on $(a\,b)$ Then $\exists c\in(a\,b)$ st.

$$\frac{f(b)-f(a)}{b-a}=f'(c)\quad\text{or}$$

$$\boxed{\exists c\in(a\,b)\text{ st. }[f(b)-f(a)]=f'(c)(b-a)}$$

if $|f'(x)|<M$ for all $x\in(a\,b)$ Then $|f(b)-f(a)|<M|b-a|$

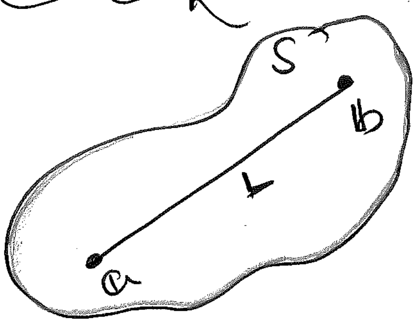
$\Rightarrow$ bounded f

$\Rightarrow$ to estimate $f(a)$ ...

$\Rightarrow$ Continuity of f

To $\mathbb{R}^n$

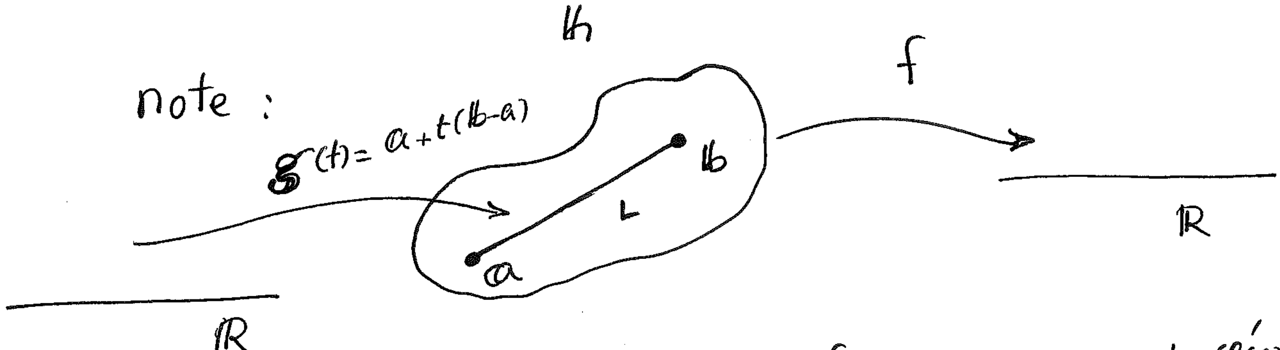
if f is Cont on L & diff on L (perhaps not at a,b) Then $\exists c\in L$ st. $f(b)-f(a)=\nabla f(c)\cdot(b-a)$



We study MVT on a line Segment L which Connects a to b.

$$L=\{a+t\underbrace{(b-a)}_{h}:t\in[0,1]\}$$

note:



$\varphi(t)=f(g(t))$ $\qquad \varphi(0)=f(a)$, $\varphi(1)=f(b)$

$\exists c\in(0\,1)$ s.t. $\varphi'(c)(1-0)=\varphi(1)-\varphi(0)$

$$\nabla f(\underbrace{a+c(b-a)}_{c})\cdot\underbrace{(b-a)}_{g'(t)}=f(b)-f(a)$$

$\boxed{\text{Corollary 2.40}}$

let f be differentiable on an Open, Convex set S, and $|\nabla f(x)|<M$ $\forall x\in S$, Then $|f(b)-f(a)|<M|b-a|$ $\forall a,b\in S$

f has to be differentiable on L

differentiability at the boundary points is not defined

f is diff on S means in some small nbd of a we need f to be defined & and limit to exist

need This Condition So That for any two points $a,b\in S$, The line L falls inside S. and f is defined & cont on L & differentiable too

Mean Value Theorem (cont'd)

Corollary 2.41

if f is diff on an open Convex S and $\nabla f(x) = 0 \ \forall x \in S$, Then $\forall a, b \in S \ |f(b) - f(a)| = 0$

$\therefore f(b) - f(a) = 0 \ \therefore f(b) = f(a)$

i.e f is Constant on S

useful for Fundamental Theorem of Calculus type argument (integration)

$\Rightarrow$

Thm 2.42

assumption of Convex can be replaced with Connected

of Course Convex $\Rightarrow \not\Leftarrow$ arc Connected $\Rightarrow \not\Leftarrow$ Connected

if f is diff on open Connected Set S, and $\nabla f(x) = 0$ for all $x \in S$, Then f is Constant on S

proof:

pick $a \in S$, and define two subsets of S:

$S_1 = \{x \in S : f(x) = f(a)\}$ & S_2 = The rest of The points of S $= \{x \in S : f(x) \neq f(a)\}$

$f(a) = v \in \mathbb{R}$

$S_1 \cup S_2 = S$

$S_1 \cap S_2 = \emptyset$

- $S_1 = f^{-1}(\{v\})$ so S_1 is closed b/c f is cont. & $\{v\}$ is a closed subset of $\mathbb{R}$.

- S_2 is open b/c $S_2 = S \cap f^{-1}(\{v\}^c)$.

  S open, f cont, $\{v\}^c$ open

- S_1 is open also: pick $b \in S_1$ and let $r > 0$ be st $B(r, b) \subset S$ b/c S is open.

B is open, Convex & $\nabla f(x) = 0$ so by 2.41 f is Constant on B, so $f(x) = f(b) = f(a)$ for all $x \in B$, so $B \subset S_1$. so S_1 is open

$\bar{S}_1 \cap S_2 = \emptyset$

S_1 is open, S_2 open

$S_1 \cap S_2 = \emptyset$ so $S_2 \subseteq S_1^c$ ← Closed

so $\bar{S}_2 \subset S_1^c$ so

$\bar{S}_2 \cap S_1 = \emptyset$

Now if f is not Constant on S, Then $S_2 \neq \emptyset$, and Since $\bar{S}_1 \cap S_2 = \emptyset$, $S_1 \cap \bar{S}_2 = \emptyset$ & $S_1 \cup S_2 = \emptyset$ Then (S_1, S_2) is a disconnection of S, so S is NOT Connected

Contradiction