

Assignment 1 - Solutions - MAT 327 - Summer 2013

Comprehension

[C.1] Let (X, \mathcal{T}) be a topological space, and let $f : X \rightarrow Y$ be an injection.

Is $\mathcal{U} := \{f[A] : A \in \mathcal{T}\}$ a topology on the range of f ?

Answer for C.1. This is just straightforward application of set identities.

$\emptyset, \text{ran}(f)$: Notice that $\emptyset = f[\emptyset] \in \mathcal{U}$, and $\text{ran}(f) := f[X] \in \mathcal{U}$.

Intersection: Let $\{f[A_1], f[A_2], \dots, f[A_n]\}$ be a finite subcollection of \mathcal{U} . Notice that

$$f[A_1] \cap f[A_2] \cap \dots \cap f[A_n] = f[A_1 \cap A_2 \cap \dots \cap A_n]$$

We have equality because f is an injection, and since (X, \mathcal{T}) is a topological space, the finite intersection presented is an open set. Thus

$$f[A_1] \cap f[A_2] \cap \dots \cap f[A_n] = f[A_1 \cap A_2 \cap \dots \cap A_n] \in \mathcal{U}$$

If we want to be *very* careful we can prove (by induction) that set identity with functions and intersections.

Union: Let $\{f[A_\alpha] : \alpha \in I\}$ be an arbitrary subcollection of \mathcal{U} . By a simple set identity (that does *not* depend on f being an injection),

$$\bigcup_{\alpha \in I} f[A_\alpha] = f\left[\bigcup_{\alpha \in I} A_\alpha\right]$$

Since (X, \mathcal{T}) is a topological space, the union presented is an open set. Thus

$$\bigcup_{\alpha \in I} f[A_\alpha] = f\left[\bigcup_{\alpha \in I} A_\alpha\right] \in \mathcal{U}$$

Again, if we want to be very careful we can prove (directly, not by induction!) that set identity with unions.

Thus we have shown that \mathcal{U} is a topology on $\text{ran}(f)$. □

[C.2] Let (X, \mathcal{T}) and (X, Γ) be topological spaces. Prove or disprove the following statements:

- i. $(X, \mathcal{T} \cap \Gamma)$ is a topological space.
- ii. $(X, \mathcal{T} \cup \Gamma)$ is a topological space.

Answer for C.2.i. Yes, this is always a topology. Let us check the three conditions.

\emptyset, X : Notice that $\emptyset \in \mathcal{T} \cap \Gamma$ and $X \in \mathcal{U} \cap \Gamma$, since both \mathcal{T} and Γ are topologies on X .

Intersection: Let $\{A_1, A_2, \dots, A_n\}$ be a finite subcollection of $\mathcal{T} \cap \Gamma$. Since \mathcal{T} is a topology we have:

$$A_1 \cap A_2 \cap \dots \cap A_n \in \mathcal{T}$$

Since Γ is a topology we have:

$$A_1 \cap A_2 \cap \dots \cap A_n \in \Gamma$$

Thus

$$A_1 \cap A_2 \cap \dots \cap A_n \in \mathcal{T} \cap \Gamma$$

Union: Let $\{A_\alpha : \alpha \in I\}$ be an arbitrary subcollection of $\mathcal{U} \cap \Gamma$. Since \mathcal{T} is a topology we have:

$$\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$$

Since Γ is a topology we have:

$$\bigcup_{\alpha \in I} A_\alpha \in \Gamma$$

Thus

$$\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T} \cap \Gamma$$

□

Answer for C.2.ii. In general this need not be a topology. There are topologies on finite sets that show this, but let's give something a bit more ... real.

Let $X = \mathbb{R}$ and let \mathcal{T} = Sorgenfrey topology. Let Γ = topology generated by the basis

$$\{(a, b] : a \leq b\}$$

(which is basically the backwards Sorgenfrey topology). If we were being *very* careful we would check that this is indeed a basis on \mathbb{R} , but it is very similar to the proof that the usual basis for the Sorgenfrey line is a basis.

We readily see that $\mathcal{T} \cap \Gamma$ cannot be a topology on \mathbb{R} , because it fails to contain $\{7\} = [7, 9) \cap (4, 7]$, sets which are open in their respective topologies. □

[C.3] Let X be an infinite set. Prove that $\mathcal{T} := \{A \subseteq X : A = \emptyset \text{ or } X \setminus A \text{ is finite}\}$ is a topology on X . This is called the “finite complement topology” or the “co-finite topology”.

Answer for C.3. This is a straightforward calculation. The fact that X is infinite is so that the $X_{\text{co-finite}}$ is not a discrete space (as we will see in C.4). Even if X is finite, \mathcal{T} will still be a topology.

\emptyset, X : By definition $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$, since $X \setminus X = \emptyset$, a finite set.

Intersection: Let $\{A_1, A_2, \dots, A_n\}$ be a finite subcollection of \mathcal{T} . (If any of the A_i is the empty set, we know the intersection will be the empty set, so assume no A_i is empty.) For each $i \leq n$ we know that $A_i = X \setminus B_i$, where B_i is a finite set. By DeMorgan's law we have:

$$A_1 \cap A_2 \cap \dots \cap A_n = (X \setminus B_1) \cap (X \setminus B_2) \cap \dots \cap (X \setminus B_n) = X \setminus (B_1 \cup B_2 \cup \dots \cup B_n)$$

This is in \mathcal{T} since the finite union of finite sets $B_1 \cup B_2 \cup \dots \cup B_n$ is itself finite.

Union: Let $\{A_\alpha : \alpha \in I\}$ be an arbitrary subcollection of \mathcal{T} . If each A_α is empty, then the union is the empty set, which is open by definition, so assume that A_β is a non-empty open set in this collection. So $A_\beta = X \setminus B_\beta$, where B_β is a finite set. Thus

$$\bigcup_{\alpha \in I} A_\alpha = A_\beta \cup \left(\bigcup_{\alpha \in I \setminus \{\beta\}} A_\alpha \right) = (X \setminus B_\beta) \cup \left(\bigcup_{\alpha \in I \setminus \{\beta\}} A_\alpha \right) \supseteq (X \setminus B_\beta)$$

or in a more palatable order,

$$X \setminus \bigcup_{\alpha \in I} A_\alpha \subseteq B_\beta$$

and since B_β is a finite set, we are finished.

□

[C.4] Let X be a finite space. What familiar topological space is X with the co-finite topology? (Support and prove your assertions.)

Answer for C.4. This is the discrete topology on X . Let $A \subseteq X$. Since X is finite, so is $X \setminus A$. Thus $A \in X_{\text{co-finite}}$. □

[C.5] Prove or disprove: If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then the image of any open set is open. (Here use the usual first-year calculus definition of continuous, real-valued function and open subset of \mathbb{R} .)

Answer for C.5. This is quite false. For example, take $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, which we know is continuous from first-year calculus. We note that for $A = (-1, 1)$ (which is open in \mathbb{R}) we have $f[A] = [0, 1)$ which is not open in \mathbb{R} .

Another example could be any constant function. □

Application

[A.1] Let's go a step further than question C.1: Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, consider the family of sets in the range given by $\Gamma := \{ f[A] : A \text{ is open in } \mathbb{R}_{\text{usual}} \}$.

- Give an example where f is not an injection, but Γ is a topology on the range of f .
- Give an example where f is not an injection, and Γ is not a topology on the range of f .

Answer for A.1.i. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the constant function $f(x) = 7$ for each $x \in \mathbb{R}$ (which is very much *not* an injection). It is clear that $\Gamma = \{ \emptyset, \{7\} \}$ which is a discrete topology. \square

Answer for A.1.ii. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x$ if $x \neq 100$ and $f(100) = 0$. This is not an injection because both 0 and 100 get mapped to 0. We now observe that both $f[(-1, 1)] = (-1, 1) \in \Gamma$ and $f[(99, 101)] = \{0\} \cup (99, 100) \cup (100, 101) \in \Gamma$, but their intersection, the singleton $\{0\} \notin \Gamma$. So Γ is not a topology. \square

[A.2] Let $\{ \mathcal{T}_\alpha : \alpha \in I \}$ be a family of topologies on a set X . Prove that there is a (unique) topology \mathcal{T} such that:

- $\forall \alpha \in I, \mathcal{T}_\alpha \subseteq \mathcal{T}$; and
- If Γ is a topology on X such that " $\forall \alpha \in I, \mathcal{T}_\alpha \subseteq \Gamma$ " then $\mathcal{T} \subseteq \Gamma$.

Answer for A.2. We know by question C.2.ii, that we cannot hope to just take $\bigcup_{\alpha \in I} \mathcal{T}_\alpha$. We will consider the family \mathfrak{T} all possible topologies that refine *all* of the \mathcal{T}_α , then we will take the intersection of all elements in \mathfrak{T} (to ensure that we end up with the *smallest* possible topology that refines all the \mathcal{T}_α .)

First we note that $\mathcal{T}_{\text{discrete}} \in \mathfrak{T}$ since the discrete topology refines every topology on X . In particular this tells us that $\mathfrak{T} \neq \emptyset$.

Let

$$\mathcal{T}_\infty := \bigcap \mathfrak{T} := \bigcap \{ \mathcal{T} : \mathcal{T} \in \mathfrak{T} \}$$

Claim 1: \mathcal{T}_∞ is a topology on X .

This is *analogous* to C.2.i, (but it does not *follow* from C.2.i, even by induction).

Claim 2: $\forall \alpha \in I, \mathcal{T}_\alpha \subseteq \mathcal{T}_\infty$.

Fix \mathcal{T}_α . We note that since each $\mathcal{T} \in \mathfrak{T}$ refines \mathcal{T}_α , then we have, by definition of intersection, that

$$\mathcal{T}_\alpha \in \bigcap \{ \mathcal{T} : \mathcal{T} \in \mathfrak{T} \} = \mathcal{T}_\infty$$

Claim 3: If Γ is a topology on X such that " $\forall \alpha \in I, \mathcal{T}_\alpha \subseteq \Gamma$ " then $\mathcal{T}_\infty \subseteq \Gamma$.

Let Γ be such a topology. Then we must have included Γ in \mathfrak{T} . So we get

$$\mathcal{T}_\infty = \bigcap \{ \mathcal{T} : \mathcal{T} \in \mathfrak{T} \} \subseteq \Gamma$$

□

[A.3] Let V be an open subset of \mathbb{R} , with the usual topology. Show that there is a countable family of open intervals $\{I_n : n \in \mathbb{N}\}$, each with rational end-points such that $\bigcup_{n \in \mathbb{N}} I_n = V$. Can we instead do this by instead having each I_n be a closed interval? (If the answer is “Yes”, then this shows that “every open subset of \mathbb{R} is an F_σ set”.)

Answer for A.3. First note that $\mathcal{B} := \{(p, q) : p, q \in \mathbb{Q}\}$ is a countable set (as there is an obvious injection into $\mathbb{Q} \times \mathbb{Q}$), and each subcollection of it is countable. So it is enough to find a subcollection $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{A} = V$.

For each $x \in V$, there is an open interval $(a, b) \subseteq V$ containing x . By the density of \mathbb{Q} (see “Things you should know” section 5), we know that there are rational numbers p, q such that

$$a < p < x < q < b$$

and

$$x \in (p, q) \subseteq (a, b) \in \mathcal{B}$$

So let $A_x := (p, q) \in \mathcal{B}$, and let $\mathcal{A} := \{A_x : x \in V\}$. Clearly

$$V \subseteq \bigcup \mathcal{A} \subseteq V$$

so $V = \bigcup \mathcal{A}$.

For the “ F_σ ” part, we can do this by repeating the argument above, but using

$$\mathcal{B} := \{[p, q] : p, q \in \mathbb{Q}\}$$

□

New Ideas

These will only be sketches of solutions, and in the case of NI.3, the sketch of a solution that doesn't work.

[NI.1] Describe a topology \mathcal{T} on \mathbb{Z} such that:

- The set of all primes, \mathbb{P} is open.
- For each $x \in \mathbb{Z}$ the set $\{x\}$ is not open.

- $\forall x, y \in \mathbb{Z}$ distinct, there is an open $U \ni x$ and an open $V \ni y$ such that $U \cap V = \emptyset$.

Note: The primes were a red herring; their algebraic structure had nothing to do with the problem. The only thing that matters here is that the primes are countable, infinite and co-infinite. The two other properties were included mostly to prevent you from taking the discrete topology, or a minor adjustment to the indiscrete topology.

Sketch 1 for NI.1. Adapt the proof of Furstenberg (that there are infinitely many primes), in two parts. First define a topology on $\mathbb{P} = \{p_1, p_2, p_3, p_4, p_5, p_6, \dots\}$ by using the Furstenberg topology *on the indices*. That is, in the Furstenberg topology, sets of the form $N(m, b) := \{mx + b : x \in \mathbb{Z}\}$ are open sets. So now, define $P(m, b) := \{p_{mx+b} : x \in \mathbb{Z}\}$. (This isn't *quite* right, because there is no p_{-10} , but it is easy to fix.) For example, here $M(10, 3) = \{p_3, p_{13}, p_{23}, p_{33}, \dots\}$ is an open set.

Now do the same thing on $\mathbb{Z} \setminus \mathbb{P}$, to get a topology on $\mathbb{Z} \setminus \mathbb{P}$.

You now have a collection of open sets on \mathbb{Z} . Generate a topology from them (by taking unions), and you will have the desired topology. It is straightforward to check that it has the desired properties. \square

Sketch 2 for NI.1. You can find a sketch on Math Stack Exchange here:

<http://math.stackexchange.com/a/401938>

The idea is to fix a bijection $f : \mathbb{Q} \rightarrow \mathbb{Z}$ which sends $(0, 1) \cap \mathbb{Q}$ to the set of primes, and apply the result of C.1 to this function and \mathbb{Q} with its induced topology. That is, a basis for the open sets in \mathbb{Q} look like $(a, b) \cap \mathbb{Q}$, where $a, b \in \mathbb{Q}$.

In general, I would rather you asked *me* for hints/discussion, before you ask the internet for help. Doing research online is one thing, asking Math Stack Exchange for a solution is something else (although in this case I do believe that the student was merely looking for guidance). I have given you a lot of freedom for the New Ideas questions, please use it wisely. \square

[NI.2] Write a computer program (in whatever language you like) or describe (in detail) an algorithm that counts the number of topologies on the set $\{0, 1, 2, 3, 4\}$. Is there any sense in which this can be done efficiently?

A solution for NI.2. This question produced many interesting observations and strategies. It was very enjoyable to read through the different attacks people made. A word about the marking for this question: the New Ideas questions are intended to engage you and the intent is for you to intimately engage the problem in return. I was looking for evidence that you had engaged the problem, and become familiar with it. As a result, any solution that was merely the (obvious) brute force method, written in non-specific pseudo-code,

was given relatively few marks. (Imagine if a company contracted you to come up with a hiring procedure for them, and your procedure was: “Hire everyone, then one-by-one fire people who don’t do good work.” You would not get paid for that contract.) If you wrote up the brute force method, specifically explaining each step, and you included a couple of ways to (non-trivially) improve the efficiency, then your mark for this question should be high.

Here is a compact bit of Python code that runs in around 2 seconds for $n=5$, from Ivan Khatchatourian. Here are his comments:

Turns out being able to treat numbers as bit strings is super useful.

Sets of elements from $X = \{1, 2, 3, 4, 5\}$ are coded by bit strings which are in turn coded as numbers, all of which is automatic in Python. So 15 (in decimal) $= 01111$ (in binary) $= \{1, 2, 3, 4\}$ (as a set [I usually imagine the smaller binary placeholders corresponding to earlier elements in the set]). Another example is $19 = 10011 = \{1, 2, 5\}$. I’m sure you see by now. So then we can do stuff like:

$$15 \& 19 = 01111 \& 10011 = \{1, 2, 3, 4\} \cap \{1, 2, 5\} = \{1, 2\}$$

and

$$15 | 19 = 01111 | 10011 = \{1, 2, 3, 4\} \cup \{1, 2, 5\} = X$$

etc.

(Also Python has “set” objects which also work nicely. “-” is set difference, etc.)

With that in place, there’s a natural enumeration of all the subsets of X (which I guess is more or less a lexicographic order). The function **S(i, T)** below counts the number of topologies you can make by starting with the collection T (of subsets of X) and adding sets enumerated above and including i .

Here’s the code:

```
N=5;
def Num(i, T):
    check = 0;
    for j in range(i, 2**N):
        if len(set(j&u for u in T)-T)-1 == 0:
            check += Num(j+1, T|set(j|u for u in T))
    return 1 + check;
print Num(1, set([0, 2**N - 1]));
```

The clause on the if statement there is fulfilled when j is the only new thing added by taking the intersections of j with things in T (j is always added, since the whole space X ,

represented by $2^N - 1$, is always in T). If that's true, we add to T the unions of j with everything in T (this adds j itself, since the empty set, represented by 0, is always in T) and start again with $j+1$. And so on. As soon as we can't add anything without breaking stuff, we go back to the last thing we could add without breaking stuff and try adding the next thing. \square

[NI.3] Here's a game that two players could play: Player 1 chooses an uncountable subset $K_1 \subseteq \mathbb{R}$, then Player 2 chooses an uncountable $K_2 \subseteq K_1$. They continue alternating until they have an infinite chain $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$, and we'll say that Player 2 wins iff $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$. Show that Player 2 has a strategy so that she can win, no matter what moves Player 1 makes.

Sketch for NI.3. The main purpose of this exercise was for the student to find the (most) straightforward solution, then show that it doesn't work.

Here is that method.

Player 1 picks their uncountable set K_1 . Write

$$\mathbb{R} = \bigcup_{z \in \mathbb{Z}} [z, z+1)$$

By the Uncountable Pigeonhole principle, one of these intervals must contain uncountably many elements from K_1 . Without loss of generality, let $[0, 1)$ be that interval. Player 2 then chooses $K_2 := K_1 \cap [0, 1)$.

Player 1 picks an uncountable $K_3 \subseteq K_2$. Now one of $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$ must contain uncountably many elements from K_3 . Let $K_4 := K_3 \cap$ "that interval".

Continue on in this way, always splitting the previous interval into halves and having Player 2 pick a half containing uncountably much of the previous set.

It is clear that $\bigcap_{n \in \mathbb{N}} K_n$ contains *at most* one element. Since $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{2^n}) = \emptyset$, we might be lead to believe that the intersection is *always* empty. Unfortunately (for Player 2), it may be that the intersection is non-empty. For example, Player 1 can always leave an uncountable interval around $\frac{1}{3}$, so that Player 2 always chooses the "half" that contains $\frac{1}{3}$. Then player 2 will not win.

There is indeed a solution to this problem that shows Player 2 can force an empty intersection. And we will see it later in the course. \square