Feb 13th

Today: V a v.s $/F$ $F=\mathbb{R}$ or $\mathbb{C}$

def: : a (hermetian) inner product on V is a map $V\times V \to F$, denoted $\langle v,w\rangle$

satisfies: ① $\langle av_1+bv_2, w\rangle = a\langle v_1,w\rangle + b\langle v_2,w\rangle$

② $\langle v,w\rangle = \overline{\langle w,v\rangle}$ ($\overline{\quad}$ = complex conjugate)

③ $\langle v,v\rangle \geq 0$ with $\langle v,v\rangle = 0 \iff v=0$

NOTES: ① if $F=\mathbb{R}$, $\bar{a}=a$

so axiom 2 $\iff \langle v,w\rangle = \langle w,v\rangle$

② linearly in the 2nd argument: $\langle v, aw_1+bw_2\rangle$

$$= \overline{\langle aw_1+bw_2, v\rangle} = \overline{a\langle w_1,v\rangle} + \overline{b\langle w_2,v\rangle}$$
$$= \bar{a}\cdot\overline{\langle w_1,v\rangle} + \bar{b}\cdot\overline{\langle w_2,v\rangle}$$
$$= \bar{a}\cdot\langle v,w_1\rangle + \bar{b}\langle v,w_2\rangle$$

[This property is called "anti-linearity in the 2nd argument] [$F=\mathbb{R}$, this is just linearity]

③ $\langle v,v\rangle = \overline{\langle v,v\rangle} \Rightarrow \langle v,v\rangle \in \mathbb{R} \Rightarrow$ axiom 3 is sensible

Example: ① $V=F^n$ the space of column vectors

$$v=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad w=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$\langle v,w\rangle = \sum_{i=1}^{n} a_i \bar{b_i}$ "standard inner product"

$\Rightarrow$ addition axiom: $\langle v_1+v_2, w\rangle$

$$v_1=\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad v_2=\begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix} \quad w=\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\langle v_1+v_2, w\rangle = \sum_{i=1}^{n}(a_i+a_i')\bar{b_i} = \sum_{i=1}^{n} a_i\bar{b_i} + a_i'\bar{b_i} = \sum_{i=1}^{n} a_i\bar{b_i} + \sum_{i=1}^{n} a_i'\bar{b_i}$$
$$= \langle v_1,w\rangle + \langle v_2,w\rangle$$

$\langle v,w\rangle$ v.s. $\langle w,v\rangle$

③ $\langle v,v\rangle \geq 0$ ($\langle v,v\rangle = 0 \iff v=0$) say $\langle v,v\rangle = \sum_{j=1}^{n} a_j\cdot\bar{a_j}$

$a_j = x+iy$

$\bar{a_j} = x-iy$ Let $x^2+y^2 = \|a\|^2$

$a_j\cdot\bar{a_j} = x^2+y^2$ $\langle v,v\rangle = \sum_{j=1}^{n} a_j\cdot\bar{a_j} = \sum_{j=1}^{n}\|a_j\|^2 \geq 0$

$\langle v,v\rangle = 0 \iff$ all $\|a_j\|^2=0 \iff$ all $a_j=0 \iff v=0$

real case : $\langle v,v\rangle = \sum_{i=1}^{n} a_i^2$

ex: take the square $P_2(\mathbb{R})$ = polynomial in $\mathbb{R}$ of degree ≤ 2
$= \{a_0 + a_1x + a_2x^2\}$

product : $\langle p,q\rangle = \int_0^1 p(t)q(t)\,dt \in \mathbb{R}$

① $\langle ap_1+bp_2, q\rangle = \int_0^1 (ap_1+bp_2)q\,dt$
$= a\int_0^1 p_1 q\,dt + b\int_0^1 p_2 q\,dt = a\langle p_1,q\rangle + b\langle p_2,q\rangle$

② $\langle p,q\rangle = \langle q,p\rangle$
③ $\langle p,p\rangle \ge 0$ $\quad \langle p,p\rangle = \int_0^1 p^2\,dt \ge 0$
since $p^2 \ge 0$

if $\int_0^1 p^2\,dt = 0$

since p^2 is a continuous function, $p^2 \ge 0$
$\Rightarrow p^2 = 0$ on $[0,1]$

$\Rightarrow p = 0$ on $[0,1]$

$\Rightarrow p$ is the zero polynomial

General properties
$v \in V$, then $w \in V$ is orthogonal to v.
if $\langle v,w\rangle = 0$

if $W \subseteq V$ is a subspace, $W^\perp = \{v \in V \mid \langle v,w\rangle = 0 \text{ for all } w \in W\}$
orthogonal compliment to W.

$V = W \oplus W^\perp$
FACT: if $v \in V$, then $w \mapsto \langle w,v\rangle \in F$ is a linear functional on V.
linear map $V \to F$

$\hookrightarrow$ This gives a linear isomorphism $V \longrightarrow V^*$

$V^* = \{\text{linear maps } V \to F\}$

$V \to V^*$ is given by $v \mapsto (w \to \langle w,v\rangle)$.

Consequence : $V = W \oplus W^\perp$
Pf: $v \in V$, then $v \rightsquigarrow$ linear map $W \mapsto F$
(by the same recipe, $v \mapsto (w \to \langle w,v\rangle)$)

But: W is an inner product space!
since $\langle \ , \ \rangle$ restricts to W.
$\Rightarrow$ any linear map $W \to \mathbb{C}$ is represented by $w_1 \in W$
$\therefore \langle v - w_1, \ \rangle$ is trivial on W
since $\langle v, \ \rangle$ and $\langle w, \ \rangle$ are the same map
$\Rightarrow v - w_1 \in W^{\perp} \Rightarrow V = W + W^{\perp}$