§15 - Compact Spaces

1 Motivation

Compactness is great. Really great. Like super-awesome, every-day-is-Christmas-in-Disneyland, I-just-found-a-twenty-in-my-old-jeans great.

Topology (and mathematics in general) is often concerned with objects that are fundamentally infinite. Calculus is fundamentally about infinite limits and making sense of them. $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ are all infinite. We love these things, but the problem is that humans aren't too good with handling infinite things. This is where compactness steps in. Compact objects are ones that might be infinite but "behave like finite spaces". This means that we can perform many of our clumsy finite operations (like takings maxs and mins) on compact sets. This is a bit vague, but this is the main reason that compactness is used in many branches of mathematics.

We will look at some examples, notably in $\mathbb{R}^n$ and prove the Heine-Borel theorem which says that compact sets in $\mathbb{R}^n$ are precisely the closed and bounded sets. Our proof is a "creeping along" proof, which might be different from the one you've seen in analysis, but it highlights the connection between completeness and compactness.

We will then look at some techniques that use compactness, and what compactness means in a broader mathematical sense. Finally, we will look at some related notions to compactness.

2 The Definition

The definition of compactness has to do with open covers and subcovers.

Definition. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{U}$ be a collection of open sets. We say that $\mathcal{U}$* ***is a cover of*** *X provided that $X = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} U$. If $\mathcal{U}$ is an open cover of X and $\mathcal{A}$ is a subcollection of $\mathcal{U}$ such that $X = \bigcup \mathcal{A}$, then we say that $\mathcal{A}$* ***is a subcover of*** *$\mathcal{U}$.*

For example, $\mathcal{U} = \{(-x, x) : x \in \mathbb{R}, x > 0\}$ is an open cover of $\mathbb{R}$ since $\bigcup_{x>0}(-x, x) = \mathbb{R}$. Also, $\mathcal{A} = \{(-n, n) : n \in \mathbb{N}\}$ is a (countable) subcover of $\mathcal{U}$.

Subcover Exercise: Show that the cover $\mathcal{U}$ above has no *finite* subcover.

Definition. *A topological space $(X, \mathcal{T})$ is said to be* ***compact*** *provided that every open cover $\mathcal{U}$ of X has a finite subcover.*

Invariant Exercise: Prove that compactness is a topological invariant.

This property might seem a bit strange, but it is *extremely* useful. We have already seen a similar sounding property on Assignment 5: The Lindelöf property (ever open cover has a *countable* subcover). Let's formally record the relation between compact spaces and Lindelöf spaces:

Proposition. *If X is a compact space, then it is a Lindelöf space.*

Let us give some examples and some non-examples of compact spaces.

Some Non-Examples:

- $\mathbb{R}_{\text{usual}}$ is not compact, as our previous exercise shows.
- $(0, 1)$ is not compact (since it is homeomorphic to $\mathbb{R}$).
- $\mathbb{R}_{\text{Sorgenfrey}}$ is not compact, for similar reasons to $\mathbb{R}_{\text{usual}}$.
- $\mathbb{N}_{\text{usual}}, \mathbb{Z}_{\text{usual}}$ and $\mathbb{Q}_{\text{usual}}$ are not compact for similar reasons.
- $B_\epsilon(x) \subseteq \mathbb{R}^n$ is not compact, again for similar reasons.
- ω_1 is not compact (as you saw on Assignment 5, C.4).
- $\mathbb{R}^{\mathbb{N}}$ is not compact.

Some Examples:

- Any finite topological space is compact.
- $[0, 1] \subseteq \mathbb{R}_{\text{usual}}$ is compact. (We will see this in a moment.)
- *Closed* ϵ balls in $\mathbb{R}^n$ are compact. (This will be a result of the Heine-Borel Theorem.)
- $\omega + 1$ is a compact space. (We will see this in a moment.)
- $\omega_1 + 1$ is a compact space. (See assignment 5, A.2.)
- $[0, 1]^{\mathbb{N}}$ is compact. (This will be a result of Tychonoff's Theorem, which we will see later.)

Lind/Compact Exercise: Use the previous list and Assignment 5, C.5 to come up with a Lindelöf space that is not compact.

Proposition. *The space $\omega + 1 = \mathbb{N} \cup \{\omega\}$ with the order topology, is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $\omega + 1$. There is a $U_\omega \in \mathcal{U}$ such that $\omega \in U_\omega$. Now U_ω contains a basic open set of the form $(N, \omega] \subseteq U_\omega$. Now for $m \leq N$, find a $U_m \in \mathcal{U}$ such that $m \in U_m$.

Clearly, $\{U_x : x \in \{1, 2, \ldots, N, \omega\}\}$ is a finite subcover of $\mathcal{U}$, as desired. $\square$

This seemed very easy (and it was!) but it actually shows something interesting.

Proposition. *Let $(X, \mathcal{T})$ be a topological space, and let $\langle x_n \rangle \to x$ all in X. Then*

$$\{x_n : n \in \mathbb{N}\} \cup \{x\}$$

is a compact subspace of X.

Convergent Sequence Exercise: Prove the previous proposition, that a convergent sequence, together with its limit point is a compact space.

3 The Heine-Borel Theorem

We will prove the Heine-Borel theorem in this section (whose statment I will postpone till later), but first lets give another example of a compact space: $[0, 1]$. The proof you saw in Analysis probably used Cauchy Sequences, but since we have already talked about linear orders we can give the much shorter (and more beautiful!) "creeping along proof".

Theorem. *The closed unit interval $[0, 1]$ is compact.*

Proof. Of course we mean that $[0, 1]$ will have the usual topology. Let $\mathcal{U}$ be an open cover of $[0, 1]$. We define the following set

$$B := \{x \in [0, 1] : [0, x] \text{ can be covered by finitely many elements of } \mathcal{U}\}$$

and our goal is to show that $1 \in B$, in which case we have proved the theorem.

$\underline{0 \in B}$: Since $\mathcal{U}$ is a cover, there is a $U \in \mathcal{U}$ such that $[0, 0] = \{0\} \subseteq U$.

$\underline{B \text{ is bounded above by } 1}$: Since $B \subseteq [0, 1]$ we see that 1 is an upper bound of B.

Now define

$$b := \sup B$$

which exists, since B is non-empty, bounded above and $\mathbb{R}$ contains all of its suprema. (This known as "$\mathbb{R}$ is a complete linear order", which is unrelated to Cauchy sequences.)

$\underline{b \in [0,1]}$: This is because $[0,1]$ is closed.

<u>If $s < 1$, and $s \in B$, then there is a $t \in B$ such that $s < t$</u>: (This is where the proof gets its name.) Let $s < 1$ with $s \in B$. Then there is a finite collection $\{U_1, U_2, \ldots, U_N\} \subseteq \mathcal{U}$ such that $[0,s] \subseteq U_1 \cup \ldots \cup U_N$.

Without loss of generality, say that $s \in U_N$. Notice that since $s < 1$, and U_N is open, there is an $\epsilon > 0$ such that $(s-\epsilon, s+\epsilon) \subseteq U_N$. Taking any t such that $s < t < s+\epsilon$ we see that $[0,t] \subseteq U_1 \cup \ldots \cup U_N$, so $t \in B$.

$\underline{b \in B}$: Since $\mathcal{U}$ is a cover, take some $U \in \mathcal{U}$ such that $b \in U$. There is an $\epsilon > 0$ such that $(b-\epsilon, b] \subseteq U$. For any s such that $b-\epsilon < s < b$ we see that since $b = \sup B$ we have $s \in B$. That is, $[0,s]$ can be coverered by finitely many elements of $\mathcal{U}$, call them $U_1 \cup \ldots \cup U_N$. Thus we see that $\{U_1, U_2, \ldots, U_N, U\}$ is a finite collection of elements of $\mathcal{U}$ that covers $[0,b]$.

$\underline{b = 1}$: If not, then the two previously given facts lead to a contradiction. $\square$

This proof was quite simple really. In a way it was very naive and greedy; we kept just creeping along, adding in sets as needed.

Did we mess up? Exercise: Go through the previous proof, but replace $[0,1]$ with spaces like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q} \cap [0,1], [0,1), (0,1]$ or $\mathbb{R}$. Make sure that this proof technique does not prove that those sets are compact. In each case, Figure out where the proof breaks down.

The creeping along proof is also able to prove a stronger result:

Theorem. *Let $(X, \leq)$ be a linear order, and let $(X, \mathcal{T})$ be its order topology. X is compact if and only if every non-empty set in X has a least upper bound (supremum) and a greatest lower bound (infimum).*

Note that this second property is slightly different from Dedekind Completeness which was defined on Assignment 5. This theorem tells us that $\omega_1 + 1$ is compact.

Now that we have seen this key example, let us state the Heine-Borel Theorem which we will prove in little pieces (and we have already done most of the hard work).

Theorem (Heine-Borel)**.** *A subspace of $\mathbb{R}_{usual}$ is compact if and only if it is closed and bounded (in the usual metric).*

To complete the proof we will need the following useful lemma:

Lemma. *Let $(X, \mathcal{T})$ be a compact space. If $C \subseteq X$ is closed, then C (with the subspace topology) is compact.*

Proof. Let C be a closed subset of X, a compact space. Let $\mathcal{U}$ be an open cover of C (where each element of $\mathcal{U}$ is open in C.) For each $U \in \mathcal{U}$ there is a corresponding V_U, open in X such that $V_U \cap C = U$. Since C is closed, $X \setminus C$ is an open set in X. Now it is easy to see that $\{V_U : U \in \mathcal{U}\} \cup \{X \setminus C\}$ is an open cover of X. Thus there is a finite subcover of X, and it is again easy to see that it must be of the form $\{V_{U_1}, \ldots, V_{U_N}, X \setminus C\}$. Thus we see that $\{U_1, \ldots, U_N\}$ is an open cover of C that is a (finite) subcover of $\mathcal{U}$. $\square$

Is it easy? Exercise: In the proof that closed subsets of compact spaces are compact, complete all of the details that I said "were easy".

Proposition. *If $C \subseteq \mathbb{R}_{usual}$ is a closed and bounded set, then it is compact.*

Proof. Such a C is contained in an interval $[-N, N]$ for some $N \in \mathbb{N}$, since it is bounded. Here we see that since $[-N, N] \cong [0, 1]$, we have that $[-N, N]$ is compact. Since C is closed (in $\mathbb{R}$), and $[-N, N]$ is closed in $\mathbb{R}$, we have that C is closed in $[-N, N]$. Thus by the previous Lemma, C is compact. $\square$

That was the [$\Leftarrow$] direction of the Heine-Borel Theorem. Now we need to prove the [$\Rightarrow$] part of the Heine-Borel Theorem.

Proposition. *Let K be a compact subspace of $\mathbb{R}$, then K is bounded.*

Proof. We know that $K \subseteq \mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$, so $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of K. Since K is compact, there is a finite set $F \subseteq \mathbb{N}$ such that $K \subseteq \bigcup_{n \in F} (-n, n)$. Letting $N = \max F$ (which exists since F is finite and non-empty), we see that

$$K \subseteq \bigcup_{n \in F} (-n, n) = (-N, N).$$

Hence K is bounded. $\square$

The last piece of our Heine-Borel puzzle only uses the fact that $\mathbb{R}$ is a Hausdorff space.

Proposition. *Let K be a compact subspace of X, a Hausdorff space, then K is closed.*

Proof. We will show that $X \setminus K$ is open. Suppose that $p \in X \setminus K$. For each point $k \in K$, by the Hausdorff property, find disjoint open U_k, V_k such that $p \in U_k$ and $k \in V_k$. Notice that $\{V_k : k \in K\}$ is an open cover of X. (Yes, yes, if we're being perfectly technical, $\{V_k \cap K : k \in K\}$ is an open cover of K according to our definition of open cover.)

Since K is compact, there is a finite set $F \subseteq K$ such that $\{V_k : k \in F\}$ is an open cover of K. Thus $\cap_{k \in F} U_k$ is an open set (hooray for finite intersections!) that contains p, and is disjoint from K. So $X \setminus K$ is open. $\square$

With that we have proved the Heine-Borel Theorem.

Order Exercise: This proof of the Heine-Borel Theorem was presented in a slightly longer than usual way in order to help your understanding. The down side of this is that the proof seems to be a bit "all over the place". Rewrite a proof of the Heine-Borel Theorem that can be read in a more linear fasion. Since this is for your own benefit, feel free to include as many or as few details as you like.

WARNING: The Heine-Borel theorem *only* says something about $\mathbb{R}$ (and later we will see that it says something about $\mathbb{R}^n$). It does not say anything about compactness in other metric spaces.

Example: $(\mathbb{R}^{\mathbb{N}}, \rho_{\text{uniform}})$ is a metric space that has a closed and bounded set that is not compact. We will see that the collection $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ is a closed set (this is clear) in the closed unit ball centred at the constant 0 function (also clear), but $\mathcal{E}$ is not compact (we will see this later). Here $e_n : \mathbb{N} \longrightarrow \mathbb{R}$ is the function that is 1 only at n, and is 0 everywhere else.

4 Compactness and Separation

Compact spaces have very nice separation properties. Part of the idea is that compactness lets us turn infinite operations (like taking infinite intersections) into finite operations (like taking finite intersections). Since we know that the *finite* intersection of open sets is still open we will be able to construct open sets that separate closed sets from a point.

Let us quickly mention a useful corollary of the previous proposition.

Corollary. *Let $(X, \mathcal{T})$ be a compact Hausdorff space. A subspace C of X is compact if and only if C is closed.*

Let us also notice a quick corollary of the proof of the previous proposition:

Corollary. *Every compact Hausdorff space is regular.*

Really? Exercise. Convince yourself that the previous corollary really does follow in two or three lines from the previous proofs we have done.

That corollary reminds us of an assignment question: Assignment 5, A.3 "Every regular Lindelöf space is normal". From this we get the following satisfying fact:

Corollary. *Every Hausdorff compact space is normal.*

We will actually see a "more direct" proof of this fact in the next section (although morally speaking it will be the same proof).

5 What does Compactness "mean"?

One way of looking at the definition of compactness is that a space is compact if "whenever you think you need infinitely much information to describe the space, really you only needed finitely much information." This idea is shown in the following theorem.

Theorem. *Let K be a compact subspace of $\mathbb{R}$, then K is bounded.*

Go through the proof of this in the previous section. See? We *thought* we needed infinitely many intervals to "describe" K, but really it could be described using only finitely many intervals!

Metric Generalization Exercise: Prove that every compact subset of a metric space is bounded. Does it make sense to say "every compact subset of a metrizable space is bounded?"

Compactness also allows us to do things to infinite sets that we can only usually do to finite sets, such as taking a supremum and guaranteeing that it exists. The following example illustrates this:

Theorem. *Every real-valued continuous function $f : X \longrightarrow \mathbb{R}$ from a compact space X is bounded.*

Proof. Notice that $\{ f^{-1}(-n, n) : n \in \mathbb{N} \}$ is an open cover of X. Thus since X is compact, there is a finite set $F \subseteq \mathbb{N}$ such that $\{ f^{-1}(-n, n) : n \in F \}$ is an open cover of X. It is easy to see that $X = \bigcup_{n \in F} f^{-1}(-n, n) = f^{-1}(-\max F, \max F)$. Thus we see that $f(X) \subseteq (-\max F, \max F)$, so it is bounded. □

Next we see that compactness allows us to (sometimes) turn infinite intersections of open sets into finite intersections. We have already seen this when we proved that closed subsets of compact Hausdorff spaces are compact. Let's see one more example of this:

Theorem. *If $(X, \mathcal{T})$ is a compact Hausdorff space then it is a normal space.*

Proof. Let $C, D \subseteq X$ be disjoint closed sets. We notice that they are both compact. For each $d \in D$, use regularity to find U_d, V_d disjoint open sets such that $C \subseteq U_d$ and $d \in V_d$. Clearly $\{ V_d : d \in D \}$ is an open cover of D, so there is a finite set $F \subseteq D$ such that $\{ V_d : d \in F \}$ is a cover of D.

Notice that $C \subseteq \bigcap_{d \in F} U_d$, an open set, and $D \subseteq \bigcup_{d \in F} V_d$, also open, and both sets are disjoint. □

We can also think of compactness in the contrapositive: A space X is compact if "Whenever $\mathcal{U}$ is an open collection where all finite subcollections are not covers, then the whole collection is not a cover". The idea is that compactness allows us to say something about the whole collection, provided that we know something about all of its finite subcollections.

Now compactness means "when you describe the space with infinitely much information you really only need finitely much information"; so what what does Lindelöf mean? It means "when you describe the space with infinitely much information you really only need *counatbly* much information". In general the Lindelöf condition won't be as useful for us because the most common way to use compactness is to turn an "infinite max" into a "finite max" (which always exist). You can't play the same game with Lindelöf spaces; turning an "infinite max"into a "countable max" is not as useful as a countable set might not have a max!

All is not lost though. Lindelöf spaces become more interesting when you are working with properties that are preserved under countable operations (but possibly not under arbitrarily many operations). This is all vague, but is a useful high-level perspective of topology. If you pursue your study of topology you will see these sorts of ideas coming through.

6 Finite Products of Compact spaces

Let us remark that finite products of compact spaces are compact. The main idea here is to cover a compact set by a "tube of open sets" (whatever that means).

Proposition. *If X, Y are compact spaces then $X \times Y$ is compact. (i.e. compactness is finitely productive.)*

Proof. First we prove something about "tubes". Let $\{U\}$ be an open cover of $X \times Y$, and let $x \in X$. Notice that $\{x\} \times X \cong Y$, a compact space, so $\{x\} \times X$ is compact. Since $\{U\}$ is an open cover of $X \times Y$ it is also an open cover of $\{x\} \times X$, so let $\mathcal{U}_x$ be a finite subcover of $\mathcal{U}$. Notice that

$$T_x := \bigcup \mathcal{U}_x$$

is an open "tube" in $X \times Y$ that contains $\{x\} \times Y$.

Now our tube might be a bit lumpy, but it concentrates around $\{x\} \times Y$. What I mean is that

$$S_x := \bigcap_{A \in \mathcal{U}_x} \pi_1(A)$$

is a (non-empty) open set in X that contains x, since it is a finite intersection of open sets that contain x.

Observe that $\{S_x : x \in X\}$ is an open cover of X, so there is a finite set $F \subseteq X$ such that $\{S_x : x \in F\}$ is a finite open cover of X. Thus we see that

$$\{T_x : x \in F\}$$

is an finite open cover of $X \times Y$.

Our only problem is that this set of tubes is not a *subcover* of $\mathcal{U}$. That's fixable though, since each tubes is made up of *finitely many* elements of $\mathcal{U}$. So let our finite subcover just be the collection of the finitely many elements of $\mathcal{U}$ from each of the finitely many tubes in $\{T_x : x \in F\}$. $\square$

7 Various Versions of Compactness

Since compactness is such a useful property, mathematicians have found many equivalent conditions to compactness. We will state a couple that are always true, then we will give some that are only true in metrizable spaces.

Theorem. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent:*

1. *X is compact;*
2. *Every collection of closed subsets of X with the finite intersection property has non-empty intersection.*

Proof. Let us show the [$\Rightarrow$] direction, first. Let $\mathcal{C}$ be a collection of closed sets with the finite intersection property. This means that for any finite subcollection $\mathcal{D} \subseteq \mathcal{C}$, $\{X \setminus D : D \in \mathcal{D}\}$ is *not* a cover of X (but it is a collection of open subsets of X). Thus by compactness, $\{X \setminus C : C \in \mathcal{C}\}$ is an open collection that is not a cover of X. Equivalently, $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

[$\Leftarrow$] Suppose that X is not compact. Let $\mathcal{U}$ be an open cover without a finite subcover. This means that $\bigcap_{U \in \mathcal{U}} X \setminus U = \emptyset$, but $\bigcap_{U \in \mathcal{F}} X \setminus U \neq \emptyset$ for each finite subset $\mathcal{F} \subseteq \mathcal{U}$. So $\{X \setminus U : U \in \mathcal{U}\}$ is a collection of closed sets with the finite intersection property whose intersection is empty. $\square$

Now we look at some notions that are all the same in metrizable spaces.

Theorem. *Let (X, d) be a metric space. The following are equivalent:*

1. *X is compact;*
2. *X is sequentinally compact. (i.e Every sequence in X has a convergent subsequence);*
3. *X is countably compact. (i.e. Every countable open cover has a finite subcover);*
4. *Every infinite subset of X has a limit point.*

[More details to come, but this theorem won't be tested on.]