Lecture Week 3

MOM: normal distribution $X \sim N(\mu, \sigma^2)$

$f(x)$ - pdf $\qquad \hat{\mu}_{mom}$, $\hat{\sigma}^2_{mom}$?

① $E(X) = \frac{1}{n}\sum X_i$ first moment

$\Rightarrow \hat{\mu} = \frac{1}{n}\sum X_i$

② $E(X^2) = \frac{1}{n}\sum X_i^2$ second moment.

$\Rightarrow \sigma^2 + \mu^2 = \frac{1}{n}\sum X_i^2$

plug in $\hat{\mu} = \frac{1}{n}\sum X_i$

you can get $\hat{\sigma}^2 = \frac{1}{n}\sum X_i^2 - \left(\frac{1}{n}\sum X_i\right)^2$

MLE : exponential distribution

$X \sim \exp(\lambda) \qquad f(x) = \lambda e^{-\lambda x}$

$$L(\lambda / x_1 \cdots x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$l(\lambda / x_1 \cdots x_n) = n \underbrace{\log}_{\ln} \lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\Rightarrow \quad l'(\lambda / x_1 \cdots x_n) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \qquad l'' < 0 ?$$

setting $l' = 0$

$$\Rightarrow \quad \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\lambda} = \left(\frac{\sum_{i=1}^{n} x_i}{n} \right)^{-1}$$

same with MOM

Asymptotically $(n \to \infty)$

$$\mathrm{Var}(\hat{\lambda}) = \frac{1}{I(\hat{\lambda})}$$

$\hat{\lambda}$ is random

λ is fixed

Note : find expect second derivative first, then plug in $\hat{\lambda}$

$$I(\lambda) = -E(l''(\lambda)) = \frac{n}{\lambda^2}$$

$$l''(\lambda) = -\frac{n}{\lambda^2}$$

check < 0

Therefore, $\mathrm{Var}(\hat{\lambda}) = \dfrac{1}{I(\hat{\lambda})} = \dfrac{1}{\frac{n}{\hat{\lambda}^2}} = \dfrac{\hat{\lambda}^2}{n} = \dfrac{n}{(\sum_i x_i)^2}$

Poisson MLE

$$f(X,\theta) = \frac{\theta^{x_i} \exp(-\theta)}{x_i!}$$

$$L(\theta / x_1 \cdots x_n) = \prod_{i=1}^{n} \frac{\theta^{x_i} \exp(-\theta)}{x_i!}$$

$$\propto \theta^{\sum_{i=1}^{n} x_i} \cdot \exp(-n\theta)$$

$$l(\theta / x_1 \cdots x_n) = \sum_{i=1}^{n} x_i \cdot \ln\theta - n\theta$$

$$l'(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - n = 0$$

$$\Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{54}{11}$$

Note: find $I(\theta)$ first, then plug in $\hat{\theta}$

Compute variance

$$l''(\theta) = -\frac{\sum_{i=1}^{n} x_i}{\theta^2} \Rightarrow E(-l''(\theta)) = \frac{\sum_{i=1}^{n} E(x_i)}{\theta^2}$$

$$= \frac{n}{\theta}$$

$$Var(\hat{\theta}) = \frac{1}{I(\theta)} \approx \frac{\hat{\theta}}{n} = \frac{54}{121}$$

Non-parametric Approach

$$d(t) = \sum_{i=1}^{n} t_i \quad \text{where} \quad t_i = \begin{cases} 1 & i\text{th dead by } t \\ 0 & i\text{th alive by } t \end{cases}$$

$$t_i \sim \text{Bern}(\underbrace{F(t)}_{P(T<t)})$$

$$d(t) = \sum_{i=1}^{n} t_i \sim Bin(N, F(t))$$

$$\hat{F}(t) = \frac{d(t)}{N}$$

mean of binomial

$$E(\hat{F}(t)) = \frac{1}{N} E(d(t)) = \frac{1}{N} \cdot N \cdot F(t)$$

$$= F(t) \quad (\text{unbiased})$$

$$Var(\hat{F}(t)) = \frac{1}{N^2} Var(d(t)) = \frac{1}{N^2} \cdot \underbrace{N F(t)(1-F(t))}_{\text{variance of binomial } npq}$$

$$= \frac{F(t)(1-F(t))}{N}$$

Exercise solution. Week 3.

a) $l(\theta)$ e.g. $\theta = \binom{\alpha}{\beta}$ $\frac{\partial l}{\partial \alpha} = 0$ $\frac{\partial L}{\partial \beta} = 0$

In this case, β is known. $\beta = 3$

$$L(\alpha) = \prod_{i=1}^{n} \frac{\alpha 3^{\alpha}}{t_i^{\alpha+1}}$$

$$l(\alpha) = \sum_{i=1}^{n} \log \alpha + \sum_{i=1}^{n} \alpha \log 3 - \sum_{i=1}^{n} (\alpha+1) \log t_i$$

$$= n \log \alpha + n\alpha \log 3 - (\alpha+1) \sum_{i=1}^{t} \log t_i$$

$$l'(\alpha) = \frac{n}{\alpha} + n \log 3 - \sum_{i=1}^{n} \log t_i = 0$$

$$\Rightarrow \frac{n}{\alpha} = \sum_{i=1}^{n} \log t_i - n \log 3$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log t_i - n \log 3} = 1.023$$

b) $l''(\alpha) = -\frac{n}{\alpha^2}$ $\quad E(-l''(\alpha)) = \frac{n}{\alpha^2}$

$$V(\hat{\alpha}) = \frac{\hat{\alpha}^2}{n} = \frac{1.023^2}{11} = 0.095$$

c) $\hat{S}(10) = 1 - \underline{\underline{\hat{F}(10)}} \rightarrow \frac{\text{no. of } t_i \leq 10}{n}$

$$\hat{F}(10) = \frac{7}{11} \quad \Rightarrow \quad \hat{S}(10) = \frac{4}{11}$$

$$SD(\hat{S}(t)) = \sqrt{V(\hat{S}(t))} = \sqrt{V(\hat{F}(t))}$$

$$= \sqrt{\frac{\hat{F}(t)(1-\hat{F}(t))}{N}}$$

$$= \sqrt{\frac{\frac{4}{11} \times \frac{7}{11}}{11}} = 0.145$$