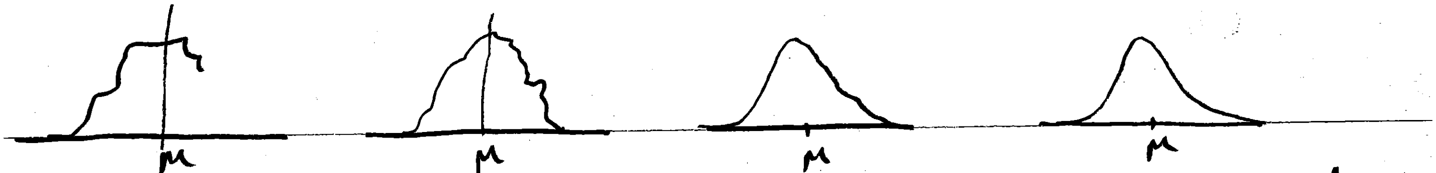
Lecture 6

Central limit theorems.

Recall that central limit theorems (CLTs) describe how the sum of random variables fluctuates around some quantity (eg. the mean).

The classic CLT case is to consider a sequence $X_1, X_2, \cdots$ of I.I.D. random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$ then the (Lindeberg-Lévy) CLT says if $S_n := \sum_{k=1}^{n} X_k$ then

$$\sqrt{n}(S_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$



This lecture we will look at some equivalent statements in our random matrix setting. In particular of linear spectral statistics of the form

$$T_n = \frac{1}{p} \sum_{k=1}^{p} \varphi(\lambda_k) = \int \varphi(x)\, dF^{A_n}(x) =: F^{A_n}(\varphi).$$

of some sample matrix A_n, eg.

$$A_n = \begin{cases} \mathbf{S}_n, & \text{sample covar matrix} \\ \mathbb{F}_n, & \text{Fisher matrix} \end{cases}$$

Some examples that we will see later in the course are:

Example 1: The generalised variance is

$$T_n = \frac{1}{p}\log|S_n| = \frac{1}{p}\sum_{k=1}^{p}\log(\lambda_k)$$

$$\varphi(x) = \log(x).$$

Example 2: Later in the course, we shall look at testing equality of sample covariance matrices. To test the hypothesis $H_0 : \Sigma = I_p$ we shall look at the log-likelihood ratio statistic

$$LRT_1 = \operatorname{tr} S_n - \log|S_n| - p = \sum_{k=1}^{p}(\lambda_k - \log(\lambda_k) - 1)$$

i.e. $\varphi(x) = x - \log(x) - 1$.

Example 3:

We shall also look at the two-sample test of the hypothesis $H_0 = \Sigma_1 = \Sigma_2$ that two populations have a common covariance matrix

$$LRT_2 = -\log|I_p + \alpha_n F_n| = -\sum_{k=1}^{p}(1 + \alpha_n \log(\lambda_k))$$

where α_n is some constant.

$$\varphi(x) = -\log(1 - \alpha_n x)$$

CLT for Linear Spectral Statistics of S_n.

We shall consider simple case

"independent vectors without cross-correlation"

$$S_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^*$$

In other words, the data matrix $X = (x_1, x_2, \dots, x_n) = (x_{ij})$ of size $p \times n$ has IID entries with $\mathbb{E}[x_{ij}] = 0$ $\mathbb{E}|x_{ij}|^2 = 1$.

$$S_n = \frac{1}{n} X X^*$$

The LSD of S_n is the Marchenko-Pastur law F_y where $y = \lim p/n$. This means, $F^{S_n}(\varphi) \longrightarrow F_y(\varphi)$ for any continuous function φ.

Making an analogy to the classic CLT we would like to understand how $F^{S_n}(\varphi)$ fluctuates around $F_y(\varphi)$, as $n \to \infty$ ($p \to \infty$)

From RMT, we know that $F^{S_n}(\varphi)$ fluctuates around its mean in such a way that $p[F^{S_n}(\varphi) - \mathbb{E}(F^{S_n}(\varphi))] \sim$ Normal.

We can decompose

$$p\left[F^{S_n}(\varphi) - F_y(\varphi)\right] = p\underbrace{\left[F^{S_n}(\varphi) - \mathbb{E}F^{S_n}(\varphi)\right]}_{\sim\text{Normal.}} + p\underbrace{\left[\mathbb{E}[F^{S_n}(\varphi)] - F_y(\varphi)\right]}_{\text{Bias.}}$$

The "bias" term is often a function of $y_n - y = p/n - y$.

y_n is called the dimension-to-sample ratio and the difference to y can be of any order. For example, if

$$y_n - y \approx p^{-\alpha}, \quad \alpha > 0.$$

then the bias term behaves like $p^{1-\alpha}$ and the value depends on α. if α small then $p^{1-\alpha}$ can blow-up and if α large then $p^{1-\alpha}$ converges to zero or constant as $p \to \infty$.

We need more restrictions on $y_n - y$.

We also need to accurately estimate $\mathbb{E}F^{S_n}(\varphi)$. One way is to estimate $\mathbb{E}F^{S_n}(\varphi) \approx F_{y_n}(\varphi)$. "finite horizon proxy"

We saw last week that the ST $\underline{s}$ of $\underline{F}_y := (1-y)\delta_0 + yF_y$ satisfies the equation that we found for the Generalised MP $(H=\delta_1)$:

$$z = -\frac{1}{\underline{s}} + \frac{y}{1+\underline{s}}, \quad z \in \mathbb{C}.$$

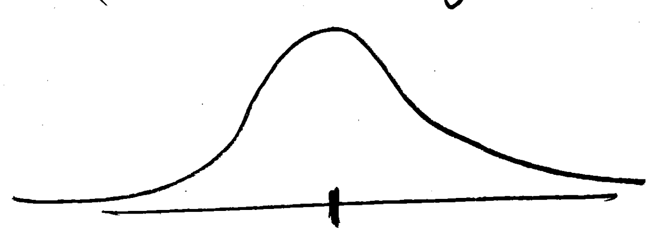
Let $\beta = E|x_{ij}|^4 - 1 - \kappa \qquad h = \sqrt{y}$

set $\kappa = 2$ if entries of X are real and $\kappa = 1$ if complex values.

If entries are Gaussian, $\beta = 0$.

The following theorem quantifies the fluctuations of

$$p\left(F^{S_n}(\varphi) - F_{y_n}(\varphi)\right).$$



[Bai & Silverstein; 2004]

Theorem: Assume $p \times n$ data matrix $\mathbb{X} = (\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n)$ has IID entries $\mathbb{E} x_{ij} = 0$, $\mathbb{E}|x_{ij}|^2 = 1$. $\mathbb{E}|x_{ij}|^4 = \beta + 1 + \kappa < \infty$

Also,
$$p \to \infty, \quad n \to \infty, \quad p/n \to y > 0.$$

Let $f_1, f_2, \ldots f_k$ be analytic functions on a open region containing support of F_y.

The random vector $(X_n(f_1), X_n(f_2), \ldots, X_n(f_k))$

where
$$X_n(f) := p\left(F^{S_n}(f) - F_{y_n}(f)\right)$$

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converges weakly to a Gaussian vector
$$(X_{f_1}, \ldots X_{f_k})$$

with mean $\mathbb{E} X_f = (\kappa - 1) I_1(f) - \beta I_2(f)$
$$\mathrm{Cov}(X_f, X_g) = \kappa J_1(f,g) + \beta J_2(f,g).$$

where
$$I_1(f) = -\frac{1}{2\pi i} \oint \frac{y(\underline{s}/(1+\underline{s}))^3(z) f(z)}{[1 - y(\underline{s}/(1+\underline{s}))^2]^2} dz.$$
$$I_2(f) = -\frac{1}{2\pi i} \oint \frac{y(\underline{s}/(1+\underline{s}))^3(z) f(z)}{1 - y(\underline{s}/(1+\underline{s}))^2} dz.$$

and

$$J_1(f,g) = -\frac{1}{4\pi^2} \oint\oint \frac{f(z_1)g(z_2)}{(\underline{s}(z_1) - \underline{s}(z_2))^2} \underline{s}'(z_1)\underline{s}'(z_2)\,dz_1\,dz_2 .$$

$$J_2(f,g) = -\frac{y}{4\pi^2} \oint f(z_1) \frac{\partial}{\partial z_1}\left(\frac{\underline{s}}{1+\underline{s}}(z_1)\right) dz_1 \times \oint g(z_2)\frac{\partial}{\partial z_2}\left(\frac{\underline{s}}{1+\underline{s}}(z_2)\right) dz_2$$

where the integrals are over contours enclosing the support of F_y.

Remarks:

- The asymptotic mean $E[X_f]$ is non-null and depends on fourth moment.

- This theorem is difficult to use in practice because the limiting parameters are integrals on contours that are not given explicitly.
- This theorem, from 2004, was a big breakthrough as it gave explicit formulas for the limiting mean and covariance.

A more explicit version of this theorem can be obtained:

<u>Proposition</u>: We have

$$I_1(f) = \lim_{r \downarrow 1} I_1(f,r)$$

$$I_2(f) = \frac{1}{2\pi i} \oint_{|\xi|=1} f(|1+h\xi|^2) \frac{1}{\xi^3} d\xi$$

$$J_1(f,g) = \lim_{r \downarrow 1} J_1(f,g,r).$$

$$J_2(f,g) = -\frac{1}{4\pi^2} \oint_{|\xi_1|=1} \frac{f(|1+h\xi_1|^2)}{\xi_1^2} d\xi_1 \oint_{|\xi_2|=1} \frac{g(|1+h\xi_2|^2)}{\xi_2^2} d\xi_2.$$

with $I_1(f,r) = \frac{1}{2\pi i} \oint_{|\xi|=1} f(|1+h\xi|^2) \left[\frac{\xi}{\xi^2 - r^{-2}} - \frac{1}{\xi}\right] d\xi.$

$$I_2(f,g,r) = -\frac{1}{4\pi^2} \oint_{|\xi_1|=1} \oint_{|\xi_2|=1} \frac{f(|1+h\xi_1|^2) g(|1+h\xi_2|^2)}{(\xi_1 - r\xi_2)^2} d\xi_1 d\xi_2$$

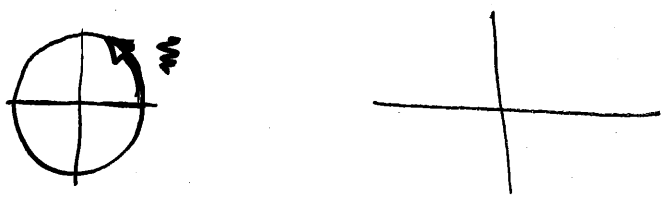
<u>Proof</u>: We are just going to look at the simplest case of $I_2(f)$.

The idea is to perform change of variable

$$z = 1 + hr\xi + hr^{-1}\bar{\xi} + h^2$$

with $r > 1$ but close to 1, and $|\xi| = 1$ $\qquad h = \sqrt{y}$

As ξ runs anticlockwise, z runs on contour $\mathcal{C}$ encloses support $[a,b] = [(1 \mp h)^2]$.



Since $z = -\frac{1}{\underline{s}} + \frac{y}{1+\underline{s}}$, $z \in \mathbb{C}^+$. We have

$$\underline{s} = -\frac{1}{1+hr\xi} \quad \text{and} \quad dz = h(r - r^{-1}\xi^{-2})\,d\xi$$

Applying this to $I_2(f)$ in Thm:

$$I_2(f) = \lim_{r \downarrow 1} \frac{1}{2\pi i} \oint_{|\xi|=1} f(z) \frac{1}{\xi^3} \frac{r\xi^2 - r^{-1}}{r(r^2\xi^2 - 1)} \, d\xi$$

$$= \frac{1}{2\pi i} \oint_{|\xi|=1} f(|1+h\xi|^2) \frac{1}{\xi^3} \, d\xi.$$

as
$$\begin{aligned} |1+h\xi|^2 &= (1+h\xi)\overline{(1+h\xi)} \\ &= (1+h\xi)(1+h\bar{\xi}) \\ &= 1 + h\xi + h\bar{\xi} + h^2|\xi| \\ &= 1 + h\xi + h\bar{\xi} + h^2. \end{aligned}$$

An example application of CLT

Proposition: Consider two linear spectral statistics

$$\sum_{i=1}^{p} \log(\lambda_i) \quad , \quad \sum_{i=1}^{p} \lambda_i .$$

where (λ_i) are eigenvalues of sample covariance S_n.
Then, under assumptions of Theorem, the vector

$$\begin{pmatrix} \sum_{i=1}^{p} \log(\lambda_i) - pF_{y_n}(\log x) \\ \sum_{i=1}^{p} \lambda_i - pF_{y_n}(x) \end{pmatrix} \xrightarrow{d} N(\mu_1, Q_1)$$

$$\mu_1 = \begin{pmatrix} \frac{\kappa-1}{2}\log(1-y) - \frac{1}{2}\beta y \\ 0 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} -\kappa\log(1-y) + \beta y & (\beta+\kappa)y \\ (\beta+\kappa)y & (\beta+\kappa)y \end{pmatrix}$$

$$F_{y_n}(x) = 1 \qquad F_{y_n}(\log x) = \frac{y_n - 1}{y_n}\log(1) - y_n - 1.$$

<u>Proof:</u> In the Theorem, take $k=2$ with

$$f(x) = \log(x) \quad g(x) = x \ , \quad x > 0.$$

and we are going to consider the vector (X_f, X_g).

$$\mathbb{E}[X_f] = (k-1)I_1(f) + \beta I_2(f) \quad \mathbb{E}[X_g] = (k-1)I_1(g) + \beta I_2(g)$$

etc.

We shall use the proposition to calculate

$$I_1(f,r) = \frac{1}{2\pi i}\oint_{|\xi|=1} f(|1+h\xi|^2)\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right] d\xi.$$

$$= \frac{1}{2\pi i}\int_{|\xi|=1} \log(|1+h\xi|^2)\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right] d\xi.$$

recall $|1+h\xi|^2 = (1+h\xi)(1+h\bar{\xi})$ $\qquad |\xi| = 1 \Rightarrow \bar{\xi} = e^{-i\theta} = \frac{1}{\xi}$

$$= (1+h\xi)(1+h/\xi)$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=1}\left[\log(1+h\xi) + \log(1+h/\xi)\right]\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right] d\xi$$

$$= \frac{1}{2\pi i}\left[\oint_{|\xi|=1}\log(1+h\xi)\frac{\xi}{\xi^2-r^{-2}}d\xi - \oint_{|\xi|=1}\log(1+h\xi)\frac{1}{\xi}d\xi\right.$$

$$\left. + \oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{\xi}{\xi^2-r^{-2}}d\xi - \oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{1}{\xi}d\xi\right]$$

For the first integral, the poles are $\pm \frac{1}{r}$

$$\frac{1}{2\pi i}\oint_{|\xi|=1} \log(1+h\xi)\frac{\xi}{\xi^2-r^{-2}}\,d\xi = \left.\frac{\log(1+h\xi)\,\xi}{\xi-r^{-1}}\right|_{\xi=-r^{-1}}$$

$$+\left.\frac{\log(1+h\xi)\,\xi}{\xi+r^{-1}}\right|_{\xi=r^{-1}}$$

$$=\frac{1}{2}\log\left(1-\frac{h^2}{r^2}\right).$$

For second integral, singularity at $\xi=0$.

$$\frac{1}{2\pi i}\oint_{|\xi|=1}\log(1+h\xi)\frac{1}{\xi}\,d\xi = \log(1+h\xi)\Big|_{\xi=0} = 0.$$

For third integral, we perform a change of variable $z=\frac{1}{\xi}$ so $d\xi = -z^{-2}\,dz$

$$\frac{1}{2\pi i}\oint_{|\xi|=1}\log(1+h\xi^{-1})\frac{\xi}{\xi^2-r^{-2}}\,d\xi = -\frac{1}{2\pi i}\oint_{|z|=1}\log(1+hz)\frac{z^{-1}}{z^{-2}-r^{-2}}\cdot\frac{-1}{z^2}\,dz$$

$$=\frac{1}{2\pi i}\oint_{|z|=1}\frac{\log(1+hz)\,r^2}{z(z+r)(z-r)}\,dz = \left.\frac{\log(1+hz)\,r^2}{(z+r)(z-r)}\right|_{z=0} = 0.$$

Fourth integral: $\qquad z = \xi^{-1} \quad d\xi = -z^{-2}dz.$

$$\frac{1}{2\pi i}\oint_{|\xi|=1} \log(1+h\xi^{-1})\frac{1}{\xi}\,d\xi = -\frac{1}{2\pi i}\oint_{|z|=1} \log(1+hz)\frac{-z}{z^2}\,dz$$

$$= \log(1+hz)\Big|_{z=0} = 0.$$

Collecting all terms gives $I_1(f,r) = \frac{1}{2}\log(1-h^2/r^2)$

$$I_1(g,r) = \frac{1}{2\pi i}\oint_{|\xi|=1} g(|1+h\xi|^2)\cdot\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right]d\xi.$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=1} |1+h\xi|^2\left[\frac{\xi}{\xi^2-r^{-2}} - \frac{1}{\xi}\right]d\xi$$

and $|1+h\xi|^2 = (1+h\xi)(1+h\bar{\xi}) = 1+h\xi^{-1}+h\xi+h^2$

$$= \frac{\xi+h+h\xi^2+h^2\xi}{\xi}$$

So $$= \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\xi+h+h\xi^2+h^2\xi}{\xi} \times \frac{\xi}{\xi^2-r^{-2}}\,d\xi$$

$$+ \frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\xi+h+h\xi^2+h^2\xi}{\xi}\,\frac{1}{\xi}\,d\xi$$

The first integral

$$\frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\xi+h+h\xi^2+h^2\xi}{(\xi-r)(\xi+r)}\,d\xi = \left.\frac{\xi+h+h\xi^2+h^2\xi}{\xi-r}\right|_{\xi=-r^{-1}}$$

$$+\left.\frac{\xi+h+h\xi^2+h^2\xi}{\xi+r}\right|_{\xi=r^{-1}} = 1+h^2$$

and second integral

$$\frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\xi+h+h\xi^2+h^2\xi}{\xi^2}\,d\xi = \left.\frac{\partial}{\partial\xi}\left(\xi+h+h\xi^2+h^2\xi\right)\right|_{\xi=0} = 1+h^2$$

2nd-order pole at $\xi=0$.

hence $I_1(g,r)=0$.

$$I_2(f)=\frac{1}{2\pi i}\oint_{|\xi|=1}\log(|1+h\xi|^2)\frac{1}{\xi^3}\,d\xi$$

$$=\frac{1}{2\pi i}\left[\oint_{|\xi|=1}\frac{\log(1+h\xi)}{\xi^3}\,d\xi+\oint_{|\xi|=1}\frac{\log(1+h\xi^{-1})}{\xi^3}\,d\xi\right].$$

First integral:

$$\frac{1}{2\pi i}\oint_{|\xi|=1}\frac{\log(1+h\xi)}{\xi^3}\,d\xi = \frac{1}{2}\frac{\partial^2}{\partial\xi^2}\log(1+h\xi)\Big|_{\xi=0} = -\frac{1}{2}h^2$$

3rd order pole

Second integral: $z = \xi^{-1}$ $d\xi = -z^{-2} dz$.

$$\frac{1}{2\pi i}\oint_{|\xi|=1} \frac{\log(1+h\xi^{-1})}{\xi^3} d\xi = -\frac{1}{2\pi i}\oint_{|z|=1} \frac{\log(1+hz)}{z^{-3}} \cdot \frac{-1}{z^2} dz$$

$$= \log(1+hz)\Big|_{z=0} = 0.$$

Now for the covariance terms.

$$J_1(f,g,r) = -\frac{1}{4\pi^2}\oint_{|\xi_1|=1}\oint_{|\xi_2|=1} \frac{\log(|1+h\xi_1|^2)\,|1+h\xi_2|^2}{(\xi_1 - r\xi_2)^2} d\xi_1 d\xi_2$$

$$= \frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{\log(|1+h\xi_1|^2)}{(\xi_1 - r\xi_2)^2} d\xi_1 \cdot \frac{1}{2\pi i}\oint_{|\xi_2|=1} |1+h\xi_2|^2 d\xi_2.$$

First integral,

$$\frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{\log(|1+h\xi_1|^2)}{(\xi_1 - r\xi_2)^2} d\xi_1 = \frac{1}{2\pi i}\left[\oint_{|\xi_1|=1} \frac{\log(1+h\xi)}{(\xi_1 - r\xi_2)^2} d\xi_1 + \oint_{|\xi_1|=1} \frac{\log(1+h\xi^{-1})}{(\xi_1 - r\xi_2)^2} d\xi_1\right]$$

$$= \frac{1}{2\pi i}\left[A + B\right].$$

Notice for A, for $|\xi_2| = 1$ fixed, $|r\xi_2| > 1$ so $r\xi_2$ not a pole.

$$A = 0.$$

$$B = \frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{\log(1+h\xi_1^{-1})}{(\xi_1 - r\xi_2)^2}\, d\xi_1 \qquad z = \frac{1}{\xi_1} \quad d\xi_1 = -z^{-2} dz$$

$$= \frac{-1}{2\pi i}\oint_{|z|=1} \frac{\log(1+hz)}{(z^{-1} - r\xi_2)^2}\,\frac{-1}{z^2}\, dz = \frac{1}{2\pi i}\cdot\frac{1}{(r\xi_2)^2}\oint_{|z|=1} \frac{\log(1+hz)}{\left(z - \frac{1}{r\xi_2}\right)^2}\, dz$$

2nd order at $z = \frac{1}{r\xi_2}$

$$= \frac{1}{(r\xi_2)^2}\frac{\partial}{\partial z}\left(\log(1+hz)\right)\Big|_{z=\frac{1}{r\xi_2}} = \frac{h}{r\xi_2(r\xi_2+h)}$$

Now, $$J_1(f,g,r) = \frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{\log(|1+h\xi_1|^2)}{(\xi_1 - r\xi_2)^2}\, d\xi_1 \cdot \frac{1}{2\pi i}\oint_{|\xi_2|=1} |1+h\xi_2|^2\, d\xi_2.$$

$$= \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1} \frac{(1+h\xi_2)(1+h\bar{\xi}_2)}{\xi_2(\xi_2 + hr^{-1})}\, d\xi_2.$$

$$= \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1} \frac{\xi_2 + h\xi_2^2 + h + h^2\xi_2}{\xi_2^2(\xi_2 + hr^{-1})}\, d\xi_2.$$

$$= \frac{h}{2\pi i r^2}\left[\underbrace{\oint_{|\xi_2|=1} \frac{1+h^2}{\xi_2(\xi_2+hr^{-1})}\, d\xi_2}_{0} + \underbrace{\oint_{|\xi_2|=1} \frac{h}{(\xi_2+hr^{-1})}\, d\xi_2}_{2\pi i h} + \underbrace{\oint_{|\xi_2|=1} \frac{h}{\xi_2^2(\xi_2+hr^{-1})}\, d\xi_2}_{0}\right]$$

$$= \frac{h^2}{r^2}$$

$$J_1(f,f,r) = \frac{1}{2\pi i}\int_{|\xi_2|=1} f(|1+h\xi_2|^2)\cdot \underbrace{\frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{f(|1+h\xi_2|^2)}{(\xi_1 - r\xi_2)^2}d\xi_1}_{\text{DONE BEFORE.}} d\xi_2 .$$

$$= \frac{1}{2\pi i}\oint_{|\xi_2|=1} f(|1+h\xi_2|^2)\,\frac{h}{r\xi_2(r\xi_2+h)}\,d\xi_2 .$$

$$= \underbrace{\frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{\log(1+h\xi_2)}{\xi_2(\frac{h}{r}+\xi_2)}d\xi_2}_{A} + \underbrace{\frac{h}{2\pi i r^2}\oint_{|\xi_2|=1}\frac{\log(1+h\xi_2^{-1})}{\xi_2(\frac{h}{r}+\xi_2)}d\xi_2}_{B} .$$

$$A = \frac{h}{r^2}\left[\left.\frac{\log(1+h\xi_2)}{\frac{h}{r}+\xi_2}\right|_{\xi_2=0} + \left.\frac{\log(1+h\xi_2)}{\xi_2}\right|_{\xi_2=-\frac{h}{r}} \right]$$

$$= -\frac{1}{r^2}\log\left(1-\frac{h^2}{r}\right).$$

$$B = \frac{-h}{2\pi i r^2}\oint_{|z|=1}\frac{\log(1+hz)}{z^{-1}(\frac{h}{r}+z^{-1})}\cdot\frac{-1}{z^2}\,dz = \frac{1}{2\pi i r}\oint_{|z|=1}\frac{\log(1+hz)}{z+rh^{-1}}\,dz = 0$$

NOT a pole since $|\frac{r}{h}|>1$.

hence, $J_1(f,f,r) = -\frac{1}{r}\log\left(1-\frac{h^2}{r}\right)$

$$J_1(g,g,r) = \frac{1}{2\pi i}\oint_{|\xi_2|=1} |1+h\xi_2|^2 \cdot \frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{|1+h\xi_1|^2}{(\xi_1 - r\xi_2)^2}\, d\xi_1 d\xi_2 .$$

and
$$\frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{|1+h\xi_1|^2}{(\xi_1-r\xi_2)^2} d\xi_1 = \frac{1}{2\pi i}\oint_{|\xi_1|=1} \frac{\xi_1 + h\xi_1^2 + h + h^2\xi_1}{\xi_1(\xi_1 - r\xi_2)^2} d\xi_1$$

$$= \frac{1}{2\pi i}\Bigg[\underbrace{\oint_{|\xi_1|=1} \frac{1+h^2}{(\xi_1-r\xi_2)^2} d\xi_1}_{0} + \underbrace{\oint_{|\xi_1|=1} \frac{h\xi_1}{(\xi_1-r\xi_2)^2} d\xi_1}_{0} + \underbrace{\oint_{|\xi_1|=1} \frac{h}{\xi_1(\xi_1-r\xi_2)^2} d\xi_1}_{2\pi i \left.\frac{h}{(\xi_1-r\xi_2)^2}\right|_{\xi_1=0} = \frac{2\pi i h}{r^2\xi_2^2}} \Bigg]$$

$$= \frac{h}{r^2\xi_2^2}.$$

Therefore,

$$J_1(g,g,r) = \frac{h}{2\pi i r^2}\oint_{|\xi_2|=1} \frac{\xi_2 + h\xi_2 + h + h^2\xi_2}{\xi_2^3} d\xi_2 .$$

$$= \frac{h}{2\pi i r^2}\left[\oint_{|\xi_2|=1} \frac{1+h^2}{\xi_2^2} d\xi_2 + \oint_{|\xi_2|=1} \frac{h}{\xi_2} d\xi_2 + \oint_{|\xi_2|=1} \frac{h}{\xi_2^3} d\xi_2 \right]$$

$$= \frac{h^2}{r^2}.$$

Now we have to calculate all the J_2 terms:

$$J_2(f,g),\ J_2(f,f),\ J_2(g,g).$$

$$J_2(F,G) = -\frac{1}{4\pi^2} \oint_{|\xi_1|=1} \frac{F(|1+h\xi_1|^2)}{\xi_1^2} d\xi_1 \oint_{|\xi_2|=1} \frac{G(|1+h\xi_2|^2)}{\xi_2^2} d\xi_2 .$$

First notice that.

$$\frac{1}{2\pi i} \oint_{|\xi_1|=1} \frac{\log(|1+h\xi_1|^2)}{\xi_1^2} d\xi_1 = \frac{1}{2\pi i} \oint_{|\xi_1|=1} \frac{\log(1+h\xi_1)+\log(1+h\xi_1^{-1})}{\xi_1^2} d\xi_1$$

$$= \frac{1}{2\pi i}\left[2\pi i \underbrace{\left[\frac{\partial}{\partial \xi_1} \log(1+h\xi_1)\right]\Big|_{\xi_1=0}}_{h} - \underbrace{\oint_{|z|=1} \frac{\log(1+hz)}{z^{-2}} \frac{-1}{z^2} dz}_{0} \right]$$

$$= h.$$

And we have.

$$\frac{1}{2\pi i} \oint_{|\xi_2|=1} \frac{g(|1+h\xi_2|^2)}{\xi_2^2} d\xi_2 = \frac{1}{2\pi i} \oint_{|\xi_2|=1} \frac{\xi_2 + h\xi_2^2 + h + h^2\xi_2}{\xi_2^3} d\xi_2 = h$$

$$J_2(f,g) = \frac{1}{2\pi i} \oint_{|\xi_1|=1} \frac{f(|1+h\xi_2|^2)}{\xi_1^2} d\xi_1 \cdot \frac{1}{2\pi i} \oint_{|\xi_2|=1} \frac{g(|1+h\xi_2|^2)}{\xi_2^2} d\xi_2 = h^2$$

$$J_2(f,f) = \frac{1}{2\pi i} \oint_{|\xi_1|=1} \frac{f(|1+h\xi_2|^2)}{\xi_1^2} \cdot \frac{1}{2\pi i} \oint_{|\xi_2|=1} \frac{f(|1+h\xi_2|^2)}{\xi_2^2} d\xi_2 = h^2$$

$$J_2(g,g) = \frac{1}{2\pi i} \oint_{|\xi_1|=1} \frac{g(|1+h\xi_1|^2)}{\xi_1^2} d\xi_1 \cdot \frac{1}{2\pi i} \oint_{|\xi_2|=1} \frac{g(|1+h\xi_2|^2)}{\xi_2^2} d\xi_2 = h^2 \quad \blacksquare$$