SOME QUESTIONS AND PROBLEMS RELATED TO LECTURES 1-3

Affine Geometry

Theoretical questions.

Around Ceva's theorem

1. Ceva's theorem and its proof that uses calculation of areas.

2. The center of masses, its properties including calculation of its coordinates.

3. Proof of Ceva's theorem by means of the center of masses.

4. Assume that three lines passing through vertices of a triangle satisfy the Ceva's condition. On each line consider the segment between the vertex and the point of intersection with the opposite side. In what proportion the point of intersection of the three lines divides each such segment? Find an answer a) by means of areas, b) by means the center of masses.

5. For a point P and a triangle ABC denote by S_1, S_2, S_3 the oriented area of the triangle PBC, PCA and PAB correspondingly. Put the mass S_1 at the point A, put the mass S_2 at the point B, and put the mass S_3 at the point C. Prove that the point P is the center of masses. Prove that $S_1 + S_2 + S_3 = S$ where S is the oriented area of the triangle ABC.

An additional section: Linear Algebra and Ceva's Theorem

6. Consider a parallelogram or a triangle on a plane. Choose one of its vertices and consider each side passing through it as a vector. Prove that the oriented area of the parallelogram or of the triangle is a bilinear skew symmetric function in the couple of vectors which correspond to the couple of sides passing through the chosen vertex. Prove the similar statement about the oriented volume of a parallelepiped and about the oriented volume of a tetrahedron in the space. How one can compute the oriented area of a parallelogram or of a triangle and the oriented volume of a parallelepiped or a tetrahedron using determinants?

7. Consider a triangle ABC on a plane L and fix a point $O \notin L$ on the space. Prove that a vector $m_1\vec{OA} + m_2\vec{OB} + m_3\vec{OC}$ belong to the plane L if and only if $m_1 + m_2 + m_3 = 1$. Let m_1, m_2 and m_3 be a triple of numbers such that $m_1 + m_2 + m_3 = 1$. Put the mass m_1 at the point A, put the mass m_2 at the point B, and put mass m_3 at the point C. Let P be the center of masses. Prove that $\vec{OP} = m_1\vec{OA} + m_2\vec{OB} + m_3\vec{OC}$. Prove the statement 4 using the Cramer's formula for solutions of a linear system.

Affine maps

8. Prove the Menelaus's theorem using a parallel projection of a triangle on a line.

9. Prove that an automorphism of the field of real numbers is the identity map. Hint: 1) assume first that the automorphism is continuous and prove the statement

under this extra assumption, 2) note that the inequality $a \geq b$ is equivalent to the solvability of the equation $x^2 = a - b$. Using this fact prove that the automorphism is continuous.

10. Assume that the segments x, y and 1 are given. Construct the segment $x+y$ and the segment xy using parallel lines only.

11. Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be one to one map, such that an image of each straight line is a straight line. Prove that $A(x) = a + Bx$, where $a \in \mathbb{R}^2$ and $B\colon \mathbb{R}^2 \to \mathbb{R}^2$ is one to one linear map. Hint: use 9–10.

Convex geometry

12. Take a convex closed body in $\mathbb{R}^n$. Take a point in its complement. Prove that there exist a hyper-plane L such that the body and the point belong to a different half-spaces with the common boundary L.

13. Prove that any closed convex set in $\mathbb{R}^n$ is an intersection of closed half-spaces.

14. Prove that any bounded convex closed body in $\mathbb{R}^n$ is the convex hull of the set of its extreme points.

15* (an additional problem). Prove Minkovsky theorem: any convex body in $\mathbb{R}^n$ which is symmetric with respect to the origin and which volume is bigger than 2^n contains a non zero integral point. Hint. Prove lemma: any set in $\mathbb{R}^n$ which volume is bigger than 1 contains a couple of points a, b such that $a - b$ is a non zero integral point.

Combinatorial analysis of simple convex polyhedra

16. Simple polyhedra in $\mathbb{R}^n$ (as an intersection of generic half-spaces), its f-vector and its h-vector. Index of a linear function at a vertex of a simple polyhedron. Prove that the number of vertices which index is k is equal to the number h_k. Prove Dehn-Zommerville theorem ($h_k = h_{n-k}$). Prove the inequality $h_k > 0$ for $0 \leq k \leq n$.

17. Prove that the f-vector (f_0, f_1, f_2, f_3) of a simple polyhedron in $\mathbb{R}^3$ has the following properties: $f_0 - f_1 + f_2 = 2$, $f_3 = 1$, $3f_0 = 2f_1$, $f_2 \geq 4$, $f_3 = 1$. For each such f-vector construct an example of corresponding polyhedron. Prove the Euler formula for any (not necessary simple) convex polyhedron in $\mathbb{R}^3$.

Some problems

1. Prove that three heights of triangle pass through one point.

2. Prove that three medians of triangle pass through one point.

3. Prove that three bisectors of triangle pass through one point.

4. Consider a triangle and take an inscribed circle. Join each vertex with the point of tangency at the opposite side. Prove that three lines you constructed pass through one point.

5–8. In what proportion the point of intersection divides three segments in problems 1–4?

9. Consider a tetrahedron. Join each vertex with the point of intersection of the medians of the opposite face. Prove that all 4 segments you constructed pass through one point. Prove that this point divides each segment in the proportion $1\colon 3$.

10. Take a 4-gon, $ABCD$. Let K, L, M, N be the middle point at the segment AB, BC, CD, CA correspondingly. Let Q be the point of intersection of the lines LN and KM. Let P and Q be the middle point at the diagonal AC and BD correspondingly. Prove that Q is the middle point of the segment PQ.

11. Take a 6-gon $A_1A_2A_3A_4A_5A_6$. Let B_1, B_2, B_3, B_4, B_5, B_6 be the middle of the side A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_6A_1 correspondingly. Let O_1 be the point of intersection of the medians of the triangle $B_1B_3B_5$ and let O_2 be the point of intersection of the medians of the triangle $B_2B_4B_6$. Prove that $O_1 = O_2$.

12. Consider a triangle ABC. Take points A_1, B_1, C_1 such that $A_1 \in BC$ and $BA_1\colon A_1C = 2\colon 1$, $B_1 \in CA$ and $CB_1\colon B_1A = 2\colon 1$, $C_1 \in AB$ and $AC_1\colon C_1B = 2\colon 1$. Consider the triangle which sides belong to the lines AA_1, BB_1, CC_1. Prove that its area is equal to $\dfrac{1}{7}$ of the area of the triangle ABC.

13. Take a circle and points A, B, C and P on it. Take triangle ABC. Consider orthogonal projection C_1, A_1, B_1 of the point P on the line AB, BC and CD correspondingly. Prove that points A_1, B_1, C_1 belong to one line.

14. Consider 3 circles. For every couple of circles consider their two common external tangent lines and take its point of intersection. Prove that these 3 points of intersection belong to one line.

15. Prove that for any triangle three points of intersections of bisectors of its external angles with opposite sides belong to one line.