

Formula Sheet

Effective rate of interest: $i_{u+1} = \frac{S(u+1) - S(u)}{S(u)}$

Payment of 1	Compound interest	Simple interest
Accumulated value after t years	$(1+i)^t$	$(1+ti)$
Present value at time 0	$v^t = (1+i)^{-t}$	$(1+it)^{-1}$

i paid at the **end** of the period on the balance at the **beginning** of the period.

d paid at the **beginning** of the period on the balance at the **end** of the period.

$$d = \frac{i}{1+i}$$

Present value with simple discount: $(1-d \cdot t)$

$$\text{Real interest rate: } 1+i_{\text{real}} = \frac{1+i}{1+r}$$

The accumulated value of 1 from time 0 to time t under compound interest:

$$S(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = (1+i)^t = v^{-t} = (1-d)^{-t} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt} = e^{\delta t}$$

The present value at time 0 of 1 payable at time t under compound interest is:

$$S(0) = \left(1 + \frac{i^{(m)}}{m}\right)^{-mt} = (1+i)^{-t} = v^t = (1-d)^t = \left(1 - \frac{d^{(m)}}{m}\right)^{mt} = e^{-\delta t}$$

Force of interest

For constant force of interest, (under compound interest) $\delta = \ln(1+i)$

	Constant force of interest ($\delta_t = \delta$)	Variable force of interest
Accumulation at time t_2 of an amount $S(t_1)$ invested at t_1	$S(t_2) = S(t_1) \cdot e^{\delta(t_2-t_1)}$	$S(t_2) = S(t_1) \cdot \exp\left(\int_{t_1}^{t_2} \delta_t dt\right)$
Present value at time t_1 of an amount $S(t_2)$ due at time t_2	$S(t_1) = S(t_2) \cdot e^{-\delta(t_2-t_1)}$	$S(t_1) = S(t_2) \cdot \exp\left(-\int_{t_1}^{t_2} \delta_t dt\right)$

Annuities

n payments of 1	Payments made in arrears (at end of each period)	Payments made in advance (at start of each period)
Accumulated value	$s_{\overline{n} } = \sum_{t=0}^{n-1} (1+i)^t = \frac{(1+i)^n - 1}{i}$	$\ddot{s}_{\overline{n} } = \sum_{t=1}^n (1+i)^t = \frac{(1+i)^n - 1}{d}$
Present value	$a_{\overline{n} } = \sum_{t=1}^n v^t = \frac{1-v^n}{i}$	$\ddot{a}_{\overline{n} } = \sum_{t=0}^{n-1} v^t = \frac{1-v^n}{d}$

Payments of $\frac{1}{m}$ made each $\frac{1}{m}$th of a year for n years	Payments made in arrears	Payments made in advance
Accumulated value	$s_{\overline{n} }^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}}$	$\ddot{s}_{\overline{n} }^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}}$
Present value	$a_{\overline{n} }^{(m)} = \frac{1-v^n}{i^{(m)}}$	$\ddot{a}_{\overline{n} }^{(m)} = \frac{1-v^n}{d^{(m)}}$
Payments of 1 for perpetuity	Payments made in arrears	Payments made in advance
Present value of 1 per period	$a_{\overline{\infty} } = \frac{1}{i}$	$\ddot{a}_{\overline{\infty} } = \frac{1}{d}$
Present value of $\frac{1}{m}$ per period of length $\frac{1}{m}$	$a_{\overline{\infty} }^{(m)} = \frac{1}{i^{(m)}}$	$\ddot{a}_{\overline{\infty} }^{(m)} = \frac{1}{d^{(m)}}$
Continuous annuity of 1 per period for n periods	Fixed rate of interest	Variable rate of interest
Accumulated value	$\overline{s}_{\overline{n} } = \int_0^n (1+i)^{n-t} dt = \frac{(1+i)^n - 1}{\delta}$	$\overline{s}_{\overline{n} \delta_r} = \int_0^n \exp\left(\int_t^n \delta_r dr\right) dt$
Present value	$\overline{a}_{\overline{n} } = \int_0^n v^t dt = \frac{1-v^n}{\delta}$	$\overline{a}_{\overline{n} \delta_r} = \int_0^n \exp\left(-\int_0^t \delta_r dr\right) dt$
Arithmetically increasing annuity of n payments (first payment amount = 1, subsequent payments increase by 1 per period)	Payments made in arrears	Payments made in advance
Accumulated value	$(Is)_{\overline{n} } = \frac{\ddot{s}_{\overline{n} } - n}{i}$	$(I\ddot{s})_{\overline{n} } = \frac{\ddot{s}_{\overline{n} } - n}{d}$
Present value	$(Ia)_{\overline{n} } = \frac{\ddot{a}_{\overline{n} } - nv^n}{i}$	$(I\ddot{a})_{\overline{n} } = \frac{\ddot{a}_{\overline{n} } - nv^n}{d}$
Arithmetically decreasing annuity of n payments (first payment amount= n, subsequent payments decrease by 1 per period)	Payments made in arrears	Payments made in advance
Accumulated value	$(Ds)_{\overline{n} } = \frac{n \cdot (1+i)^n - s_{\overline{n} }}{i}$	$(D\ddot{s})_{\overline{n} } = \frac{n \cdot (1+i)^n - s_{\overline{n} }}{d}$
Present value	$(Da)_{\overline{n} } = \frac{n - a_{\overline{n} }}{i}$	$(D\ddot{a})_{\overline{n} } = \frac{n - a_{\overline{n} }}{d}$

Increasing annuity	with discrete increases and continuous payments	with continuous increases and continuous payments
Present value	$(\bar{Ia})_{\overline{n} } = \int_0^n \lceil t \rceil v^t dt = \frac{\ddot{a}_{\overline{n} } - nv^n}{\delta}$	$(\bar{Ia})_{\overline{n} } = \int_0^n tv^t dt = \frac{\bar{a}_{\overline{n} } - nv^n}{\delta}$

Geometric series summation formula: $1 + x + x^2 + x^3 + \dots + x^k = \frac{1 - x^{k+1}}{1 - x} = \frac{x^{k+1} - 1}{x - 1}$

Present value at time 0 of a series of n payments, each of amount 1, commencing at time $k + 1$: ${}_k|a_{\overline{n}|} = v^k \cdot a_{\overline{n}|} = a_{\overline{n+k}|} - a_{\overline{k}|}$

General formula for increasing annuity

n -payment annuity with first payment A and subsequent payment B larger (or smaller) than the previous one. Payments made in arrears. Accumulated value at time n is

$$S(n) = (A - B)s_{\overline{n}|i} + B(Is)_{\overline{n}|i}$$

Solving Equations of Value

Quadratic form: $a(1+i)^n + b(1+i)^n + c = 0$ solution: $(1+i)^n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Linear interpolation: Given $i_1, i_2, f(i_1)$ and $f(i_2)$: $\frac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cong \frac{i_0 - i_1}{i_2 - i_1}$

$$\Rightarrow i_0 \cong i_1 + \frac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cdot (i_2 - i_1)$$

Loan calculations

Loan amount = $L = OB_0$

Interest rate per period = i

Amount paid at time $t = K_t$

Interest charged at the end of the t^{th} period = $I_t = OB_{t-1} \cdot i$

Principal repaid at the end of the t^{th} period = $PR_t = K_t - I_t$

Outstanding balance (principal) just after payment at time $t = OB_t$

Where $OB_t = OB_{t-1} + OB_{t-1} \cdot i - K_t$

$OB_t = OB_{t-1} + I_t - K_t = OB_{t-1} - (K_t - I_t) = OB_{t-1} - PR_t$

Outstanding balance:

Retrospective method: $OB_t = L(1+i)^t - \sum_{a=1}^t (1+i)^{t-a} K_a$

Prospective method: $OB_t = vK_{t+1} + v^2K_{t+2} + \dots + v^{n-t}K_n = \sum_{a=1}^{n-t} v^a K_{a+t}$