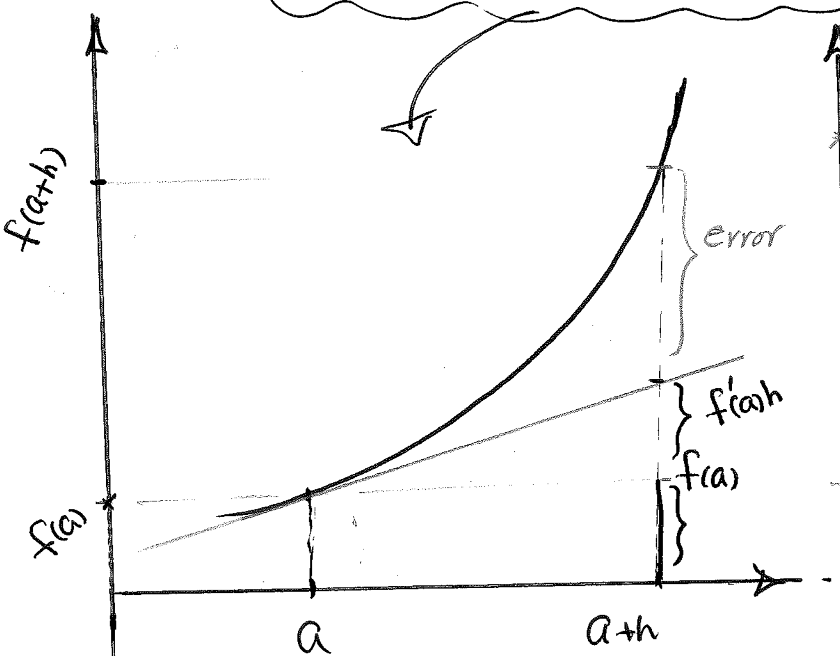
MVT in $\mathbb{R}$ {from MVT to Taylor polynomials & Taylor Theorem / Lagrange remainder}

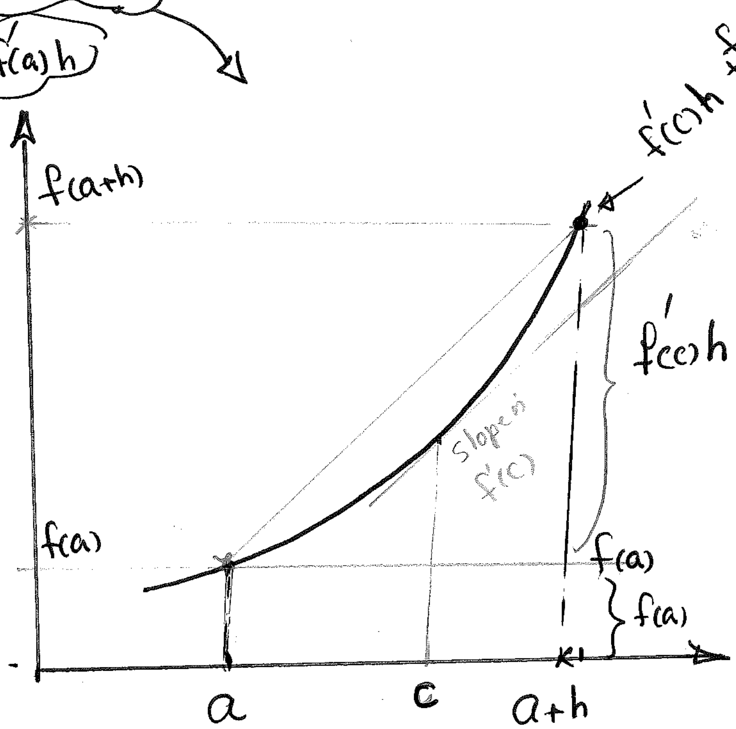
$\boxed{\text{MVT}}$ $\Rightarrow$ $\exists\, c \in (a,b)$ st. $f(b)-f(a) = f'(c)(b-a)$

or $f(b) = f(a) + f'(c)(b-a)$ or

$\exists\, c \in (a, a+h)$ st. $f(a+h) = f(a) + f'(c)h$

whereas $f(a+h) \approx f(a) + f'(a)h$





differential or linear approximation

$f(a+h) \approx f(a) + f'(a)h$

error of approximation $= f(a+h) - (f(a) + f'(a)h)$

MVT:

$f(a+h) = f(a) + f'(c)h$

Lagrange remainder of degree 0 $R_{a,0}(h)$

Taylor polynomial of degree 1 at a is

$P_{a,1}(h) = f(a) + f'(a)h$

Taylor Theorem for k=1: $f(a+h) = f(a) + f'(a)h + R_{a,1}(h)$

$= f(a) + f'(a)h + \frac{f''(c)}{2}h^2$

if There were a higher order MVT it would've looked like This: $f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2$

or $\frac{f(b)-f(a)}{b-a} - f'(a) = f''(c)(b-a)$ or even $\frac{\frac{f(b)-f(a)}{b-a} - f'(a)}{b-a} = f''(c)$

Taylor polynomials
Lagrange Remainder
Taylor Theorem

on $\mathbb{R}$

$$P_{a,2}(h) = f(a) + \frac{f'(a)}{1}h + \frac{f''(a)}{2!}h^2$$ is quadratic approximation of a function $f(x)$ at a

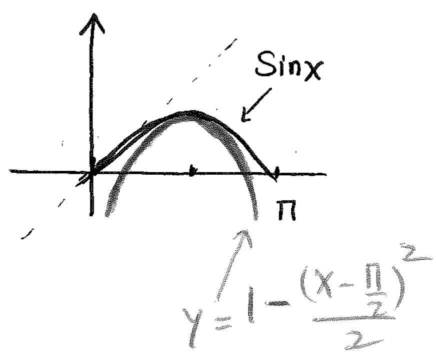
also denoted by $P_{a,2}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

eg $f(x) = \sin x$

$a = \frac{\pi}{2}$

$$P_{a,2}(x) = 1 - \frac{(x-\pi/2)^2}{2}$$

graphic



$P_{a,2}(x)$ matches
1. position
2. Slope
3. Concavity

of $f(x)$ at $\frac{\pi}{2}$

if an object were to move along $f(x)$ vs The path $P_{a,2}(x)$ That at $\underline{a}$ The object's 1. position, 2. Velocity & momentum, 3. accelaration would be The same (on both paths)

physics

analytically

$P_{a,2}(a) = f(a)$

$P'_{a,2}(a) = f'(a)$

$P''_{a,2}(a) = f''(a)$

Question what would The interpretation of $P_{a,3}(h)$ or $P_{a,k}(h)$ be?

Taylor's Theorem

If f is $k+1$ time differentiable on an interval $I \subset \mathbb{R}$ and $a \in I$, and $a+h \in I$

Then $\exists\, c \in [0\ 1]$ st. $f(a+h) = P_{a,k}(h) + \frac{f^{(k+1)}(a+c)}{(k+1)!}h^{k+1}$