SCHOOL OF FINANCE AND APPLIED STATISTICS

FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 10

Question 1
A ten-year zero-coupon bond is issued on 1 February 2000 at a price of \$79 per \$100 nominal. On 1 February 2002 an investor entered into a forward contract to buy \$1000 nominal of the bond in 5 years' time. The price of the bond was \$83 per \$100 nominal on 1 February 2002 and \$92 per \$100 nominal on 1 February 2007.

Calculate the profit or loss made by the investor on 1 February 2007 if the risk-free force of interest was 3% pa.

Solution
The investor has entered into a contract on 1 February 2002 to purchase \$1000 nominal on 1 February 2007. The forward price (the price agreed at 1 February 2002) can be found by using the equation derived during lectures.

The forward price of the contract per \$100 nominal is $K = S_0 e^{\delta T}$, where $S_0 = \$83$ per \$100 nominal, $T = 5$ years, and $\delta = 0.03$.

$\Rightarrow K = 83e^{(5)0.03} = \$96.43$

Therefore on 1 February 2007, the investor pays \$96.43 for \$100 nominal of bond that has a price of \$92 per \$100 nominal. Since the investor had agreed to purchase \$1000 nominal, the investor (the long-party in the contract) therefore makes a loss of:

$10(96.43 - 92) = \$44.30$

Question 2

(i) A fixed interest stock is redeemable at 106% (ie. 1.06 per unit nominal) in 15 years' time and pays coupons of 9% per annum payable half-yearly in arrears. What price should an investor pay per $100 nominal to obtain a gross redemption yield of 9% per annum?

(ii) Instead of purchasing the stock, the investor decides to agree a forward contract to buy the security in six years' time immediately after the coupon payment then due. Calculate the forward price based on a risk-free rate of return of 6% pa effective and no arbitrage. The current price of the stock is that calculated in part (i).

Solution

(i) Each coupon payment is of amount $\frac{0.09}{2}(100) = \$4.5$ per $100 nominal. The price of the stock per $100 nominal is given by:

$$P = 4.5a_{\overline{30|}} + 106v_j^{30} = \$103.24$$

where $j = (1.09)^{1/2} - 1$

(ii) The forward price is calculated from the equation $K = (S_0 - PV_I)e^{\delta T}$ where S_0 is the price of the security at time 0 and PV_I is the present value of the fixed income payments due during the term of the forward contract which will not be received by the purchaser of the contract.

$S_0 = \$103.24$
$T = 6$ years
$PV_I = 4.5a_{\overline{12|}}$ (there are 12 coupon payments prior to maturity of the contract)
$\delta = \ln(1.06)$
$j = (1.06)^{0.5} - 1$

Therefore the forward price (per $100 nominal) is:
$$K = (S_0 - PV_I)e^{\delta T} = (103.24 - 4.5a_{\overline{12|}j})(1.06)^6 = \$82.74$$

Question 3

On 30 June 2006 an investor wishes to enter a forward contract to buy 10,000 shares of Company ABC at 30 June 2016. The current share price is \$2.50 and the annual dividend payable at 31 December 2006 is expected to be \$0.08 per share.

If the risk free force of interest is 5% p.a. and dividends are expected to remain constant, calculate the value of the long forward contract at 30 June 2012 if the share price at that date is \$2.90 and dividends have remained constant as expected.

Solution

We know from lecture notes the value of a long forward contract is.

$V_L = \left(K_r - K_0\right)e^{-\delta(T-r)}$

In this case:

$\delta = 0.05$

$T = 10$

$r = 6$

The forward price can be valued using the following formula:

$K = (S_0 - PV_I)e^{\delta T}$

with the annuity function for dividend payments needing to be adjusted to allow for payments mid-way through the year.

$$K_0 = (2.5 - 0.08e^{0.05\times0.5}a_{\overline{10}|e^{0.05}-1})e^{0.05\times10} = 3.083957$$

$$K_6 = (2.9 - 0.08e^{0.05\times0.5}a_{\overline{4}|e^{0.05}-1})e^{0.05\times4} = 3.187860$$

Thus the value of the long forward contract is:

$V_L = \left(3.187860 - 3.083957\right)e^{-0.05\times4} = 0.0850686$

and the overall value is $10,000 \times 0.0850686 = \$850.69$

Question 4

If the n year spot rates can be approximated by the function $0.09 - 0.03e^{-0.1n}$,

Calculate the following quantities:

a) the one-year forward rate at time 10 years.
b) the price of \$100 nominal of a 10-year zero coupon bond redeemable at par.
c) the 5 year spot rate in 20 years' time.
d) the price of \$100 nominal of a 10-year zero coupon bond redeemable at par purchased in 5 years' time

Solution

a) We want to find $f_{10,11}$.

The one-year forward rate can be written:

$$(1+f_{10,11})=\frac{(1+s_{11})^{11}}{(1+s_{10})^{10}}$$

So using the formula given, we can find the 10-year and 11-year spot rates:

$$s_{10}=0.09-0.03e^{-0.1(10)}=0.07896$$
$$s_{11}=0.09-0.03e^{-0.1(11)}=0.08001$$

$$\Rightarrow (1+f_{10,11})=\frac{(1+s_{11})^{11}}{(1+s_{10})^{10}}=1.0906$$

$$\Rightarrow f_{10,11}=9.06\%$$

b) We need to find the spot rate for $n=10$

$$0.09-0.03e^{-0.1(10)}\cong 0.07896$$

Therefore, the price is $100\cdot v_{s_{10}}^{10}=100(1.07896)^{-10}\cong \$46.8$

c) The 5 year spot rate in 20 years' time may also be referred to as the 5 year forward rate at time 20 years, or the 5 year spot rate 20 years forward, $f_{20,25}$.

$$(1+f_{20,25})^5=\frac{(1+s_{25})^{25}}{(1+s_{20})^{20}}=\frac{(1+0.09-0.03e^{-0.1(25)})^{25}}{(1+0.09-0.03e^{-0.1(20)})^{20}}\cong 1.5667$$

$$f_{20,25}=1.5667^{1/5}-1=0.09395$$

d) We want to find $f_{5,15}$, which is the 10-year spot rate that holds in 5 years time.

$$(1+f_{5,15})^{10}=\frac{(1+s_{15})^{15}}{(1+s_5)^5}=\frac{(1+0.09-0.03e^{-0.1(15)})^{15}}{(1+0.09-0.03e^{-0.1(5)})^5}\cong\frac{3.321}{1.414}=2.348$$

This is the implied accumulation factor between time 5 and time 15 years. So the price of the bond purchased in 5 years time will be:

$$100\left(1+f_{5,15}\right)^{-10}=100/2.348=\$42.6$$

Question 5
Zero coupon bonds redeemable at par are available with the following prices for $100 nominal:

Term	Price
4 years	$79
5 years	$74
6 years	$69
7 years	$64

Find the one-year forward rate of interest starting in 5 years' time implied by these prices.

Solution
We want to find $f_{5,6}$.

$$(1+f_{5,6})=\frac{(1+s_6)^6}{(1+s_5)^5}$$

We can find the spot rates s_5 and s_6 from the prices of the 5 year and 6 year zero-coupon bonds.

$$74=100(1+s_5)^{-5}\Rightarrow(1+s_5)^5=\frac{100}{74}$$

$$69=100(1+s_6)^{-6}\Rightarrow(1+s_6)^6=\frac{100}{69}$$

$$\Rightarrow(1+f_{5,6})=\frac{(1+s_6)^6}{(1+s_5)^5}=\frac{74}{69}=1.0725$$

Therefore, the one-year forward rate of interest starting in 5 years' time is 7.25%.

Question 6
Find the price of a 3-year $100 bond, redeemable at par, with annual coupons of 6% per annum, if the 3-year spot rate is 9% per annum and the following annual forward rates of interest apply:

$s_1=10\%$ per annum
$f_{2,3}=7\%$ per annum

where $f_{t,T}$ is the annual rate of interest agreed at time 0 for an investment made from time t until time T.

Solution

We want the price, $P = \frac{6}{(1+s_1)} + \frac{6}{(1+s_2)^2} + \frac{106}{(1+s_3)^3}$,

so we need to find the spot rates of interest s_1, s_2 and s_3.

We are told that $s_1 = 10\%$ and $s_3 = 9\%$

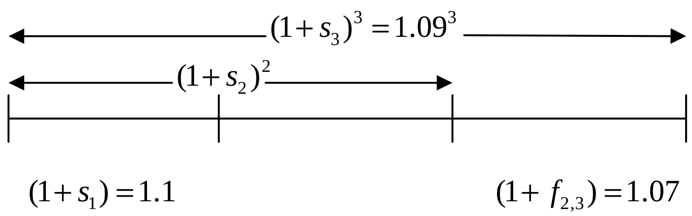
$$\left(1+f_{2,3}\right)^1 = 1.07 = \frac{(1+s_3)^3}{(1+s_2)^2} = \frac{1.09^3}{(1+s_2)^2} \Rightarrow (1+s_2)^2 = \frac{1.09^3}{1.07} = 1.210307 \Rightarrow \mathrm{s}_2 = 10.01\%$$

So, $P = \frac{6}{(1+s_1)} + \frac{6}{(1+s_2)^2} + \frac{106}{(1+s_3)^3} = 92.26$

Alternatively, we can write the bond equation in terms of the known quantities, and solve in one step:

$$P = \frac{6}{(1+s_1)} + \frac{6}{\left(\frac{(1+s_3)^3}{(1+f_{2,3})}\right)} + \frac{106}{(1+s_3)^3} = 92.26$$

A simple diagram such as the one below may help you when answering a question like this:



Past Exam Question – 2005 Final Exam Q2(c)
For discrete time periods (t = 0,1,2,3…..), the forward effective rate per annum can be calculated as follows:

$$f_{t,t+1} = \exp(0.07 + 0.001t^2) - 1$$

Calculate the following (on an effective per annum basis):

i) s_3 (2 marks)

ii) $f_{6,9}$ (2 marks)

iii) $f_{t,t+2}$ (2 marks)

Solution
i)

$$s_3 = \left[(1+f_{0,1})(1+f_{1,2})(1+f_{2,3})\right]^{1/3} - 1$$
$$= \left(e^{0.07+0.071+0.074}\right)^{1/3} - 1 = 7.43\%$$

ii)

$$f_{6,9} = \left[(1+f_{6,7})(1+f_{7,8})(1+f_{8,9})\right]^{1/3} - 1$$
$$= \left(e^{0.106+0.119+0.134}\right)^{1/3} - 1 = 12.71\%$$

iii)

$$f_{t,t+2} = \left[(1+f_{t,t+1})(1+f_{t+1,t+2})\right]^{1/2} - 1$$
$$= \left(e^{0.07+0.001t^2+0.07+0.001(t+1)^2}\right)^{1/2} - 1$$
$$= \left(e^{0.141+0.002t+0.002t^2}\right)^{1/2} - 1$$
$$= \left(e^{0.0705+0.001t+0.001t^2}\right) - 1$$