Multivariate Analysis of Variance (MANOVA)

<u>Model:</u> $\underset{\sim}{X}_{ij} = \underset{\sim}{\mu}_i + \underset{\sim}{\varepsilon}_{ij}$ $\quad i = 1, \cdots, k$ (treatments)
$\quad j = 1, \cdots, n$

$\underset{\sim}{\varepsilon}_{ij} \sim N_p(\underset{\sim}{0}, C)$

$\Rightarrow \underset{\sim}{x}_{ij} \sim N_p(\underset{\sim}{\mu}, C)$

Test $H_0: \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2$

$$S_T = \sum_{i=1}^{k} \sum_{j=1}^{n} (\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}})^T$$

$$= \sum_{i=1}^{k} n_i (\bar{\underset{\sim}{x}}_i - \bar{\underset{\sim}{x}})(\bar{\underset{\sim}{x}}_i - \bar{\underset{\sim}{x}})^T + \sum_{i=1}^{k} \sum_{j=1}^{n} (\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}}_i)(\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}}_i)^T$$

$$= S_B + S_W$$

<u>Idea:</u> If $\underset{\sim}{\mu}_1, \cdots, \underset{\sim}{\mu}_k$ are not all equal (i.e. H_0 is false) then S_B should be much bigger than S_W.

$\rightarrow$ matrix

<u>Sol'n:</u> Look at eigenvalues of $S_W^{-1} S_B$
i.e. Test statistics should be based on these eigenvalues.

<u>Why?</u> Model $\underset{\sim}{X}_{ij} = \underset{\sim}{\mu}_i + \underset{\sim}{\varepsilon}_{ij}$

$$\underset{\sim}{y}_{ij} = \underset{\text{invertible}}{A} \underset{\sim}{x}_{ij} + \underset{\sim}{b} = A\underset{\sim}{\mu}_i + \underset{\sim}{b} + A\underset{\sim}{\varepsilon}_{ij} = \underset{\sim}{\mu}_i^* + \underset{\sim}{\varepsilon}_{ij}^*$$

$$\underset{\sim}{\mu}_1 = \cdots = \underset{\sim}{\mu}_k \iff \underset{\sim}{\mu}_1^* = \cdots = \underset{\sim}{\mu}_k^*$$

Look at S_B & S_W based on $\{\underset{\sim}{x}_{ij}\}$ & $\{\underset{\sim}{y}_{ij}\}$

$$S_B^x = \sum_{i=1}^{k} n (\bar{\underset{\sim}{x}}_i - \bar{\underset{\sim}{x}})(\bar{\underset{\sim}{x}}_i - \bar{\underset{\sim}{x}})^T$$

$$S_B^y = A S_B^x A^T$$

$$S_W^y = A S_W^x A^T$$

Therefore, $(S_W^y)^{-1} S_B^y = (A S_W^x A^T)^{-1} (A S_B^x A^T) = (A^T)^{-1} (S_W^x)^{-1} A^{-1} A S_B^x A^T$
$= (A^T)^{-1} (S_W^x)^{-1} S_B^x A^T$ vs. $(S_W^x)^{-1} S_B^x$

Ideally, conclusions based on $(S_W^y)^{-1} S_B^y$ should be the same as those based on $(S_W^x)^{-1} S_B^x$

non-trivial

Need to build function ϕ such that
$\phi((S_W^y)^{-1} S_B^y) = \phi((S_W^x)^{-1} S_B^x)$ for any invertible matrix A

Look at eigenvalues $\underbrace{(A^T)^{-1} (S_W^x)^{-1} S_B^x A^T}_{\overset{||}{(S_W^y)^{-1} S_B^y}} \underset{\sim}{v} = \lambda \underset{\sim}{v}$

$$(S_W^x)^{-1} S_B^x \underbrace{A^T \underset{\sim}{v}}_{\underset{\sim}{v}^*} = \lambda \underbrace{A^T \underset{\sim}{v}}_{\underset{\sim}{v}^*} \longrightarrow (S_W^x)^{-1} S_B^x \underset{\sim}{v}^* = \lambda \underset{\sim}{v}^*$$

Thus the eigenvalues of $(S_W^y)^{-1} S_B^y$ are the same as those of $(S_W^x)^{-1} S_B^x$ for any A.

Therefore to test $H_0: \underset{\sim}{\mu}_1 = \cdots = \underset{\sim}{\mu}_k$, look at test statistics based on eigenvalues $\lambda_1, \cdots, \lambda_p$ of $S_W^{-1} S_B$. $(\lambda_1 \geq \cdots \geq \lambda_p \geq 0)$

Need to be very careful!

Test statistics

① Wilk's lambda (Likelihood ratio statistics)

$$\Lambda = \prod_{i=1}^{p} (1+\lambda_i)^{-1} \qquad (0<\Lambda<1)$$

– reject Λ for $\Lambda \le k_\alpha$

② Pillai's Trace

$$T_p = \sum_{i=1}^{p} \frac{\lambda_i}{1+\lambda_i} \qquad (\text{reject } H_0 \text{ for } T_p \ge k_\alpha)$$

③ Lawley-Hotelling Trace ?

$$T_{LH} = \sum_{i=1}^{p} \lambda_i \quad (\text{reject } H_0 \text{ for } T_{LH} \ge k_\alpha)$$

④ Roy's maximal root

$$R = \max(\lambda_1, \cdots, \lambda_p) = \lambda_i \quad (\text{reject for } R \ge k_\alpha)$$

What are the null dist'ns of these statistics?

- in general, difficult to derive exact dist'n.
- approximations $\begin{cases} F \text{ or } \chi^2 \text{ dist'n} \\ \text{Normal approximations} \end{cases}$
- in certain cases, Λ can be transformed to an exact F dist'n.

$n = n_1 + \cdots + n_k$

p	k	Transformation	degree of freedom
any	2	$\left(\frac{1-\Lambda}{\Lambda}\right)\left(\frac{n-p-1}{p}\right)$	$p,\ n-p-1$
any	3	$\left(\frac{1-\Lambda^{\frac{1}{2}}}{\Lambda^{\frac{1}{2}}}\right)\left(\frac{n-p-2}{p}\right)$	$2p,\ 2(n-p-2)$
1	any	$\left(\frac{1-\Lambda}{\Lambda}\right)\left(\frac{n-k}{k-1}\right)$	$k-1,\ n-k$
2	any	$\left(\frac{1-\Lambda^2}{\Lambda^2}\right)\left(\frac{n-k-1}{k-1}\right)$	$2(k-1),\ 2(n-k-1)$

Example: Group: male & female painted turtles $(k=2)$

$p=3$ variables $\begin{cases} \text{length} \\ \text{width} \\ \text{height} \end{cases}$ $\quad n_1 = n_2 = 24$

$$S_W^{-1} S_B = \begin{pmatrix} -1.61 & -1.02 & -0.81 \\ -0.81 & -0.51 & -0.40 \\ 7.39 & 4.66 & 3.70 \end{pmatrix}$$

– $S_W^{-1} S_B$ has only 1 non-zero eigenvalue $\lambda_1 = 1.574$, $\lambda_2 = \lambda_3 = 0$

$R = 1.574 \quad T_{LH} = \lambda_1 + \lambda_2 + \lambda_3 = 1.574$

$$T_p = \frac{\lambda_1}{1+\lambda_1} = 0.611 \qquad \Lambda = \frac{1}{1+\lambda_1} = 0.389$$

Conversion to F-statistic

$$F = \left(\frac{1-\Lambda}{\Lambda}\right)\left(\frac{48-3-1}{3}\right) = 23.07$$

Hostelling's T^2 statistic $(k=2)$

$$\text{p-value} = P[F_{(3,44)} \ge 23.07] = 4.06 \times 10^{-9}$$