MAT335 - Chaos, Fractals, and Dynamics - Fall 2013

Term Test 1 - October 21, 2013

Time allotted: 50 minutes. Aids permitted: None.

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Instructions

- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work. **Mark clearly any rough work** (not to be marked).

GOOD LUCK!

1	2	3	TOTAL
18 /20	? /10	18/20	46 /50

1. Consider the function

$$F(x) = \begin{cases} 1 & \text{if } x \leqslant \frac{1}{2} \\ 2x & \text{if } \frac{1}{2} < x \leqslant \frac{3}{2} \\ 6(2-x) & \text{if } x > \frac{3}{2} \end{cases}$$

(a) Find the fixed points of F and determine whether they are attracting, repelling, or neutral.

Solution: For $x \leq \frac{1}{2}$, $1 = x$, but $x > \frac{1}{2}$. then no fixed point.

For $\frac{1}{2} < x \leq \frac{3}{2}$, $F(x) = 2x = x$,

$x = 0$, not in $(\frac{1}{2}, \frac{3}{2}]$

still no fixed point.

For $x > \frac{3}{2}$, $6(2-x) = x$

$12 - 6x = x$

$12 = 7x$

$x = \frac{12}{7}$

As $F'(x) = -6$

$|F'(\frac{12}{7})| > 1$

So the fixed point $\frac{12}{7}$ is repelling. 6

Continues on page 3...

(b) Show that the orbit of $x_0 = 1$ is periodic. What is its prime period? Is it attracting, repelling, or neutral?

Solution: $x_0 = 1$, $F(x_0) = 2$, $F^2(x_0) = 0$, $F^3(x_0) = 1$, ...

~~So the~~

As $x_0 = F^3(x_0)$. So its prime period is 3.

$$|F^3(x)'| = |F'(x_0)F'(x_1)F'(x_2)|$$
$$= |2 \cdot (-6) \cdot 0|$$
$$= 0$$

Hence the 3-cycle is neutral.

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(c) Show that any point $x_0 < \frac{1}{2}$ such that $x_0 \neq 0$, is eventually periodic.

Solution: Note that for $x_0 < \frac{1}{2}$, $F(x_0) = x_1 = 1$,
$F^2(x_0) = x_2 = 2$,
$F^3(x_0) = x_3 = 0$
$F^4(x_0) = x_4 = 1$,
...

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Then it turns to a 3-cycle just shown in part (b).
Hence x_0 is eventually periodic.

(d) Show that any point $x_0 > 3$ is eventually periodic.

Solution: For $x_0 > 3$, $F(x_0) = x_1 = 6(2 - x_0) < -6 \leq \frac{1}{2}$
$F^2(x_0) = x_2 = 1$
$F^3(x_0) = x_3 = 2$
$F^4(x_0) = x_4 = 0$
$F^5(x_0) = x_5 = 1$

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Again, we have the same 3-cycle.
Hence x_0 ~~≥~~ is eventually periodic.

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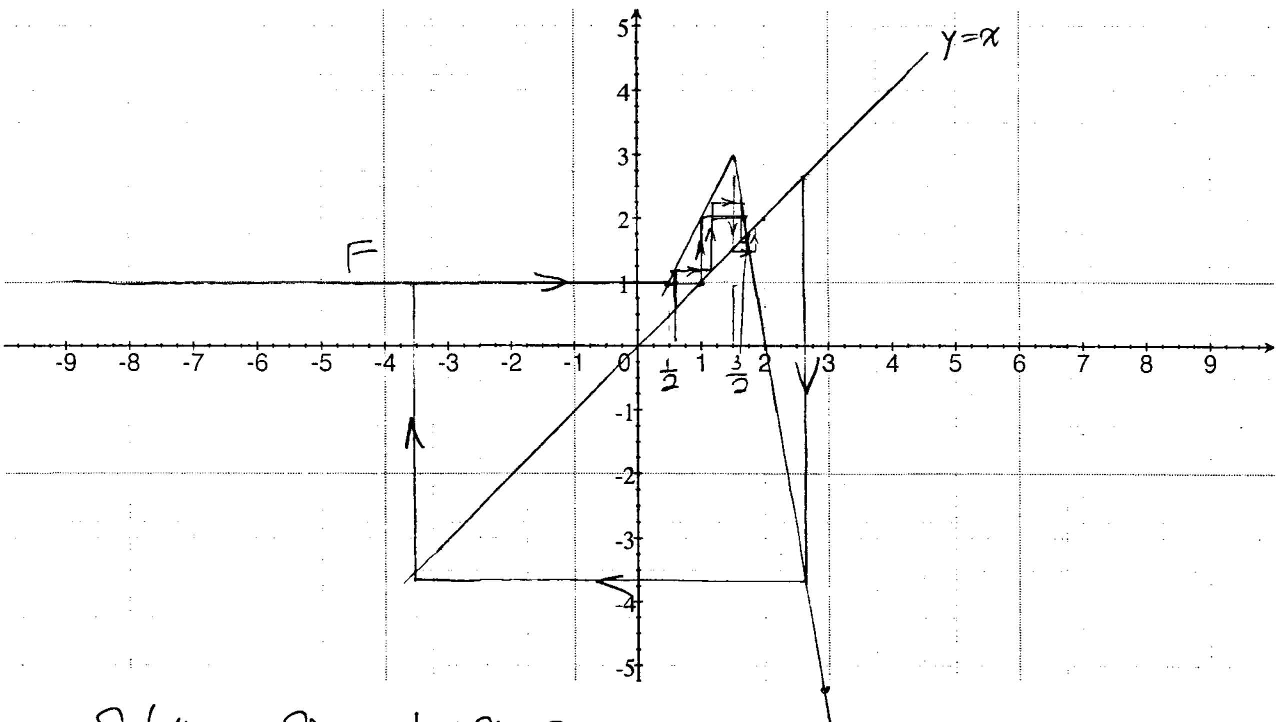
(e) Plot the graphs $y = F(x)$ and $y = x$. What happens to the orbit of x_0 under F if $\frac{1}{2} < x_0 < 3$?



Solution: ~~Since $\frac{1}{2} < x_0 < 3$~~

~~$F(x_0)$~~ ① $\frac{1}{2} < x_0 \leq \frac{3}{2}$

② $\frac{3}{2} < x_0 < 3$

① ~~$F(x_0) \in (1, 3]$~~ ~~a. $1 < F(x_0) \leq \frac{3}{2}$~~ ~~b. $\frac{3}{2} < F(x_0)$~~

The orbit of x_0 under $\frac{1}{2} < x_0 < 3$

is ~~attracted to a fixed~~ point $x = \frac{12}{7}$.

Because 3-cycles is the only attracting cycle, it seems that the orbit of a typical $x_0 \in (\frac{1}{2}, 3)$ under F is like the one sketched on the graph, which merges with the periodic orbit $(0,1,2)$ marked in red. So it's eventually periodic.

In fact, there are all periods.

Continues on page 5...

2. Let $F(x)$ be an odd function: $F(-x) = -F(x)$ for all x.

Show that if $F(x_0) = -x_0$, then x_0 lies on a 2-cycle of $F(x)$.

Solution: Since $F(x_0) = -x_0$,

$$x_0 = -F(x_0)$$

$$F(x_0) = -x_0$$

$$F^2(x_0) = F(-x_0) = -F(x_0) = x_0$$

$$F^3(x_0) = F(x_0) = -x_0$$

...

Hence we have the orbit as

$x_0, -x_0, x_0, -x_0, \ldots$

Therefore, x_0 lies on a 2-cycle of $F(x)$.

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Continues on page 6...

3. Consider the family of functions $F_\lambda(x) = \lambda x \cos x$ for $\lambda \neq 0$.

(a) Show that there is one **unique fixed point** for F_λ when $-1 < \lambda < 1$. Is it attracting, repelling, or neutral?

Solution: $F_\lambda(x) = \lambda x \cos x = x$

$\lambda x \cos x - x = 0$

$x(\lambda \cos x - 1) = 0$

$x = 0$ or $\cos x = \frac{1}{\lambda}$

i.e $x = 0$ or $x = \arccos \frac{1}{\lambda}$

~~but~~ notice that $-1 < \lambda < 1, \lambda \neq 0$

$\frac{1}{\lambda} > 1$ or $\frac{1}{\lambda} < -1$

but $\cos x \in [-1, 1]$ for any x.

hence $x = 0$ is the unique fixed point

$F_\lambda'(x) = \lambda \cos x + \lambda x(-\sin x)$

$= \lambda(\cos x - x \sin x)$

$|F_\lambda'(0)| = |\lambda(\cos 0 - 0 \sin 0)|$

$= |\lambda(1 - 0)|$

$= |\lambda| < 1$ since $-1 < \lambda < 1$.

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Hence $x = 0$ is attracting.

(b) When $\lambda < -1$, is the fixed point from (a) attracting, repelling, or neutral?

Solution: $|F_\lambda'(0)| = |\lambda| > 1$ since $\lambda < -1$

Then $x = 0$ is repelling.

1

Continues on page 7...

(c) Find the two periodic points q_1 and q_2 of prime period 2 that have the smallest absolute value. Are they attracting, repelling, or neutral?

(**Hint 1.** $F_\lambda(x)$ is an odd function)

(**Hint 2.** You can use arccos in your answer and remember that $\arccos : [-1, 1] \to [0, \pi]$)

Solution:

$$F_\lambda^2(x) = \lambda(\lambda x\cos x)\cos(\lambda x\cos x)) = x$$
$$= \lambda^2 x\cos x\cos(\lambda x\cos x) = x$$
$$= \quad \lambda x\cos(-x)$$
$$F(x) = \lambda x\cos x = -x_0$$
$$= F(x_0)$$
$$\lambda x\cos x = -x$$

By Question 2, and the fact that $F_\lambda(x)$ is odd.

$F_\lambda^2(x) = \lambda(\lambda x\cos x)\cos(\lambda x\cos x)) = x$

we want $\lambda x\cos x = -x$

such that $F_\lambda^2(x) = \lambda(-x)\cos(-x) = x$ ... 2-cycle

So $\lambda x\cos x = -x$ needs to be solved:

$$\lambda x\cos x + x = 0$$
$$x(\lambda\cos x + 1) = 0$$
$$x = 0 \text{ or } \lambda\cos x + 1 = 0$$

fixed point $\qquad \cos x = \frac{-1}{\lambda}$ $\qquad$ 5

$$x = \arccos\frac{-1}{\lambda}$$

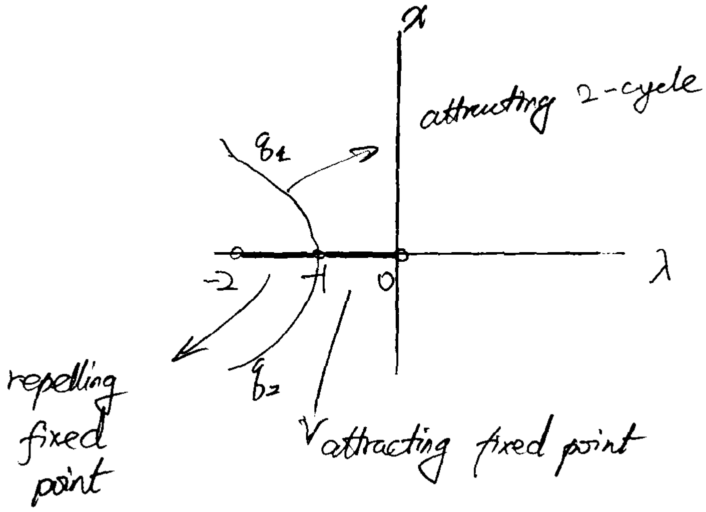
So $q_1 = \arccos\frac{-1}{\lambda}$, $q_2 = -q_1 = -\arccos\frac{-1}{\lambda}$

They are attracting. since

$$|F'(q_1)F'(q_2)| = |F'(q_1)F'(q_1)|$$
$$= |\lambda(\tfrac{-1}{\lambda} - \arccos\tfrac{-1}{\lambda}\cdot\sin\arccos\tfrac{-1}{\lambda}|$$
$$< 1$$

Continues on page 8...

(d) Based on your results in the previous parts, sketch the bifurcation diagram for $F_\lambda(x)$ for $-2 < \lambda < 0$. Label the nodes and indicate if each node is a saddle-node bifurcation, a period-doubling bifurcation, or neither.



A period-doubling bifurcation.

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