Lecture 23

Chapter 9

- $\Lambda = \{x_0 \in [-P_+, P_+],\ Q_c^n(x_0) \in [-P_+, P_+]$ for all $n\}$
  $S(x_0) = (s_0 s_1 s_2 \cdots)$ where $s_i = \begin{cases} 0 & \text{if } Q_c^i(x_0) \in I_0 \\ 1 & \text{if } Q_c^i(x_0) \in I_1 \end{cases}$
  itinerary

- $\Sigma = \{(s_0 s_1 s_2 \cdots) s_i \in \{0, 1\}\}$ sequence space
- $d[s,t] = \sum_{i=0}^{n} \frac{|s_i - t_i|}{2^i}$
- $s_i = t_i$ for $i = 0, 1, \cdots, n \Rightarrow d[s,t] \le \frac{1}{2^n}$
- $d[s,t] < \frac{1}{2^n} \Rightarrow s_i = t_i$ for $i = 0, 1, \cdots, n$

- $\sigma : \Sigma \to \Sigma \qquad \sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$

Theorem: The shift map $\sigma : \Sigma \to \Sigma$ is continuous in the metric d
This means that: for all $\Sigma > 0$, there exists $\delta > 0$ s.t.
$d[s,t] < \delta \Rightarrow d[\sigma(s), \sigma(t)] < \Sigma$
Proof: consider an arbitrary $\Sigma > 0$, chose $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \Sigma$
Then take $\delta > 0$ s.t. $\delta < \frac{1}{2^{n+1}}$
Let $s, t \in \Sigma$ with $d[s,t] < \delta < \frac{1}{2^{n+1}}$

By the proximity thm $s_i = t_i$ for $i = 0, \cdots, n+1$
And $\sigma(s) = (s_1 s_2 s_3 \cdots s_{n+1} \cdots)$, $\sigma(t) = (t_1 t_2 t_3 \cdots t_{n+1} \cdots)$

SAME

so $\sigma(s)$ and $\sigma(t)$ have the same $n+1$ components $(0, \cdots, n)$
By the proximity thm, $d[\sigma(s), \sigma(t)] \le \frac{1}{2^n} < \Sigma$

we conclude that σ is continuous.

$\blacksquare$

§ 9.3 Conjugacy

Thm: if $x \in \Lambda$ then $(S \circ Q_c)(x) = (\sigma \circ S)(x)$

$$\begin{tikzcd} \Lambda \arrow[r, "Q_c"] \arrow[d, "S"] & \Lambda \arrow[d, "S"] \\ \Sigma \arrow[r, "\sigma"] & \Sigma \end{tikzcd}$$

Proof: Let $x \in \Lambda$ and write $S(x) = (s_0 s_1 s_2 \cdots)$

This means that $Q_c^n(x) \in I_{s_n}$

$S(Q_c(x)) = (s_1 s_2 s_3 \cdots)$ SAME

$Q_c(x) \in I_{s_1}$, $Q_c^2(x) \in I_{s_2}$, $\cdots$

and $\sigma(S(x)) = \sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$

So $(S \circ Q_c)(x) = (\sigma \circ S)(x)$

We can repeat this reasoning

$$\begin{tikzcd} \Lambda \arrow[r, "Q_c"] \arrow[d, "S"'] & \Lambda \arrow[r, "Q_c"] \arrow[d, "S"'] & \Lambda \arrow[d, "S"] \\ \Sigma \arrow[r, "\sigma"] & \Sigma \arrow[r, "\sigma"] & \Sigma \end{tikzcd} \Rightarrow (S \circ Q_c^2)(x) = (\sigma^2 \circ S)(x)$$

So if x is a periodic point for Q_c with period 2, then $Q_c^2(x) = x$

$\Rightarrow S(x) = \sigma^2(S(x))$

In general, $\boxed{(S \circ Q_c^n)(x) = (\sigma^n \circ S)(x)}$

- S connects orbits of x under Q_c to orbits of $S(x)$ under σ.
- orbit of x_0 under Q_c is HARD to calculate.
  $x_0, Q_c(x_0), Q_c^2(x_0), \cdots, Q_c^n(x_0)$
- orbit of $S(x_0)$ under σ is EASY to calculate
  $S(x_0), \sigma(S(x_0)), \sigma^2(S(x)), \cdots, \sigma^n(S(x))$

Q: How does S work?

Theorem: let $c < -\frac{5+2\sqrt{5}}{4}$. The map $S: \Lambda \to \Sigma$ is a homeomorphism

1. S is 1-1
2. S is onto
3. S is continuous
4. S^{-1} is continuous