Lecture 17

Properties of Cantor Set K.

③ $a \in K$ iff there is a ternary expansion of a
$a = 0.s_1 s_2 s_3 \cdots$ with $s_i \in \{0, 2\}$

④ K is uncountable

Ternary Expansion

Let $a \in [0,1]$, we can write in base 3 as $a = 0.s_1 s_2 \cdots$

which means $a = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$

Where $s_i \in \{0, 1, 2\}$

This is called a ternary expansion of a.

Example:

① $0.0220220220\cdots$ corresponds to

$$\left(\frac{0}{3^1} + \frac{2}{3^2} + \frac{2}{3^3}\right) + \left(\frac{0}{3^4} + \cdots\right)$$

3-digits pattern

$$= 2\left(\frac{1}{3^2} + \frac{1}{3^5} + \frac{1}{3^8} + \cdots\right) + \left(\frac{2}{3^3} + \frac{2}{3^6} + \frac{2}{3^9} + \cdots\right)$$

$$= \frac{2}{3^2}\left(1 + \frac{1}{3^3} + \frac{1}{3^6} + \cdots\right) + \frac{2}{3^3}\left(1 + \frac{1}{3^3} + \frac{1}{3^6} + \cdots\right)$$

$$= \left(\frac{2}{3^2} + \frac{2}{3^3}\right) \sum_{i=0}^{\infty} \frac{1}{3^{3i}} = \frac{8}{27} \frac{1}{1 - \frac{1}{27}} = \frac{8}{27} \cdot \frac{27}{26} = \frac{4}{13}$$

② $0.11111\cdots$

$$\frac{1}{3^1} + \frac{1}{3^2} + \cdots = \frac{1}{3} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{1}{2}$$

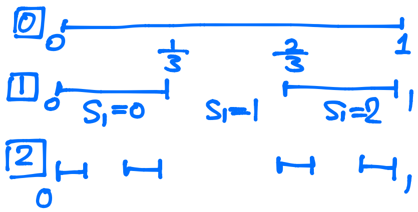
③ 0.1 corresponds to $\frac{1}{3}$

$0.02222\cdots$ corresponds to $\frac{2}{9} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{2}{9} \frac{1}{1 - \frac{1}{3}} = \frac{1}{3}$

not unique! has different expressions in base 3, so

so $\frac{4}{13}$ and $\frac{1}{3}$ are in K, $\frac{1}{2} \notin K$

Remark: The ternary digits of $a \in [0,1]$ tell at each step with 'third' the third number is in:



① First iteration implies that $s_1 \neq 1$

② The second iteration implies that $s_2 \neq 1$

So the nth iteration $\Rightarrow s_n \neq 1$. (Basically proved property ③)

Proof of 4:
Suppose that K is countable.
then there is a bijection $\phi: N \to K$
Let $k_n = \phi(n)$, So $K = \{k_1, k_2, k_3, \cdots\}$

Then
$$\left.\begin{aligned} k_1 &= 0.s_1^1 s_2^1 s_3^1 \cdots \\ k_2 &= 0.s_1^2 s_2^2 s_3^2 \cdots \\ k_3 &= 0.s_1^3 s_2^3 s_3^3 \cdots \end{aligned}\right\} \ \& \ s_j^i \in \{0,2\}$$
$$\vdots$$
$$k_n$$

Define a number $k = 0.s_1 s_2 s_3 \cdots$ by
$$\left.\begin{aligned} s_j &= 0 \text{ if } s_j^j = 2 \\ s_j &= 2 \text{ if } s_j^j = 0 \end{aligned}\right\}$$

The ternary expansion of k has only 0s and 2s. So $k \in K$.
Then there is $m \in N$ such that $k_m = k$
Thus $k_m = 0.s_1^m s_2^m \cdots s_m^m \cdots$ $\quad s_j^m \in [0,2]$
$k = 0.s_1 s_2 \cdots s_m \cdots$ $\quad s_j \in \{0,2\}$

But those 2 are not equal!

$s_m \neq s_m^m \Rightarrow k \neq k_m \Rightarrow\Leftarrow$

Exercise: Read page 79 in textbook. Direct proof that K is uncountable $\psi: K \to [0,1]$

Use base 2, cos the base 3 in Cantor set is just like base 2
Only need to change $1 \leftrightarrow 2$.