MATH6222 week 5 lecture 12

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Midterm on April 21st.

4 Questions:

1. Is $\mathbb{N}$ smallest infinite set?
2. Other sizes besides $\mathbb{N}$ and $\mathbb{R}$?
3. What is a real number?
4. Is our definition of $\leq$ sensible?

Face: Every real number has a unique decimal expansion if we disallow expansions that ends in an infinite sequence of 9's.

$$12.999\dots$$

$$13.000\dots$$

Let's call such an expansion "good".

Theorem: $|\mathbb{N}| < |\mathbb{R}|$.

Proof: Suppose $\exists$ bijection $f : \mathbb{N} \to \mathbb{R}$.

$$1 \to \dots\dots.a_{11}a_{12}a_{13}\dots$$

$$2 \to \dots\dots.a_{21}a_{22}a_{23}\dots$$

$$3 \to \dots\dots.a_{31}a_{32}a_{33}\dots$$

Let's produce a real number **not** in the image of f.
Let's define $b_1, b_2, b_3, \dots$ by rule:

$$b_i := \begin{cases} 4, & \text{if } a_{ii} \neq 4; \\ 5, & \text{if } a_{ii} = 4. \end{cases}$$

Claim that the real number $\ldots\ldots .b_1b_2b_3\ldots$ is not in image of f.
Same diagonal argument as before.

Is $\mathbb{R}^2 \to \mathbb{R}$?
$(0,1) \times (0,1) \to (0,1)$
$.x_1x_2x_3x_4\ldots$
$.y_1y_2y_3y_4\ldots$
$\to .x_1y_1x_2y_2x_3y_3x_4y_4\ldots$
Check injective and surjective.

Injectivity:
$(.x_1x_2x_3\ldots, .y_1y_2y_3\ldots) \to .x_1y_1x_2y_2x_3y_3\ldots x_i$
$(.x_1'x_2'x_3'\ldots, .y_1'y_2'y_3'\ldots) \to .x_1'y_1'x_2'y_2'x_3'y_3'\ldots x_i'$
For some i, either $x_i \neq x_i'$ or $y_i \neq y_i'$

Surjectivity:
Think about $.1919191919\ldots$, not surjective.
Question: Does $|A| \leq |B|$ and $|B| \leq |A| \implies |A| = |B|$?

This is to say, $\exists$ injection $A \to B$, $\exists$ injection $B \to A$, but *exists* bijection $A \to B$?

This is true. Pretty obvious for finite sets, but for infinite sets? (It is true as well)

Corollary: $|(0,1) \times (0,1)| = |(0,1)|$.

Theorem: $|\mathbb{N}| < |S|$ where S is the set of $0, 1$ sequences.

I claim $S \to 2^{\mathbb{N}} := \{\text{subsets of } \mathbb{N}\}$

$$01100010\cdots \to \{2,3,7,\ldots\}$$

$$10110010\cdots \to \{1,3,4,7,\ldots\}$$

Theorem: For any set S, we claim $|S| < |2^S|$.

Corollary: $|\mathbb{N}| < |2^{\mathbb{N}}| < |2^{2^{\mathbb{N}}}| < \ldots$

Proof:

Injection: $S \to 2^S$.
$x \to \{x\}(\subseteq S)$

Bijection:
Suppose $\exists$ bijection $f : S \to 2^S$
We want to produce a subset $T \subseteq S$ such that T is not in the image of f.
Define T as follows:

$$x \in T \iff x \notin f(x)$$

It follow that $T \neq f(x)$ for any $x \in S$. Done.

We have 2 sets from $A \to B$
$f : A \to B$ and $g : B \to A$