Correction to last time:

Variation of parameters:

$$\vec{x}(t) = \Psi(t)\Psi(t_0)^{-1}\vec{x}(t_0) + \Psi(t)\int_{t_0}^{t} \Psi(s)^{-1}\vec{g}(s)\,ds$$

Nonlinear autonomous 2x2 system.

$$* \left\{ x' = F(x,y) \quad y' = G(x,y) \right.$$

Can write $\vec{x}' = \vec{f}(\vec{x})$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \vec{f} = \begin{pmatrix} F \\ G \end{pmatrix}$$

Special case: linear systems

$$x' = ax + by \qquad \vec{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x}$$
$$y' = cx + dy$$

---

x_0, y_0 is a critical point for $(*)$ if $F(x_0, y_0) = G(x_0, y_0) = 0$. Then $x(t) = x_0$, $y(t) = y_0$ is then a solution of $(*)$.

We're interested in "nearby" solutions to critical points (x_0, y_0). Introduce

$u = x - x_0$, $v = y - y_0$

(Then the critical point cor. to $u=0, v=0$)

$(*)$ becomes
$$u' = F(x_0+u, y_0+v)$$
$$v' = G(x_0+u, y_0+v)$$

By Taylor theorem for function of two variables, $F(x_0+u, y_0+v) = F(x_0, y_0)$

$$= \frac{\partial F}{\partial x}(x_0, y_0)u + \frac{\partial F}{\partial y}(x_0, y_0)v$$
$$+ R(u,v)$$

(for F twice differentiable)

where $R(u,v) \leq C(u^2+v^2)$ for u,v sufficiently small.

Similar for $G(x_0+u, y_0+v)$.

Have $F(x_0, y_0) = G(x_0, y_0) = 0$ by assumption.

For $u, v \to 0$, remainder term becomes very small

$\rightsquigarrow$ linearized system

$$\begin{cases} u' = \frac{\partial F}{\partial x}(x_0, y_0)u + \frac{\partial F}{\partial y}(x_0, y_0)v \\ v' = \frac{\partial G}{\partial x}(x_0, y_0)u + \frac{\partial G}{\partial y}(x_0, y_0)v \end{cases}$$

Summary: If (x_0, y_0) is a critical point, put $u = x - x_0$, $v = y - y_0$, and approximate the DE by the linear system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \qquad A = J(x_0, y_0)$$

where

$$J = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$$

is the Jacobian of F and G

Expect that linearized system gives approximate behaviour near (x_0, y_0)

Example: $x' = x - y = F(x, y)$
$y' = 1 - x^2 = G(x, y)$

- Critical points:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

- Jacobian $J = \begin{pmatrix} 1 & -1 \\ -2x & 0 \end{pmatrix}$

- linearized system

a) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

eigenvalues: $\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \begin{cases} 2 \\ -1 \end{cases}$

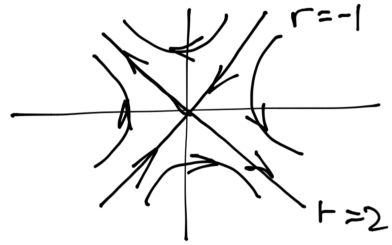
$\Rightarrow$ saddle

Note: A eigenvalue:

$\frac{tr(A)}{2} \pm \sqrt{\cdots\cdots - det(A)}$

$r=2$ has eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$r=-1$ has eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

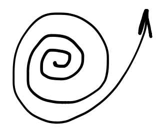


(b) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & -1 \\ +2 & 0 \end{pmatrix}$

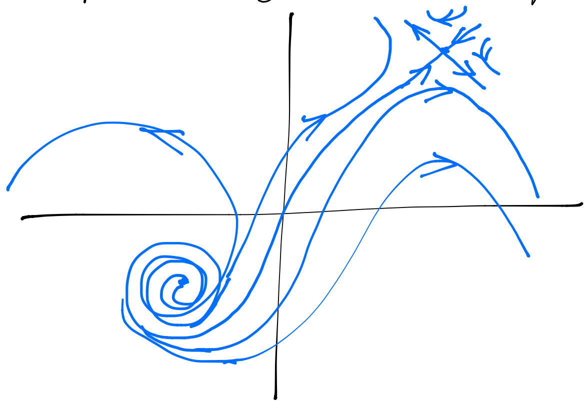
eigenvalues : $\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2} = -\frac{1}{2} \pm \sqrt{\frac{-7}{4}}$ $\Longrightarrow$ unstable spiral

$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

conterclockwise



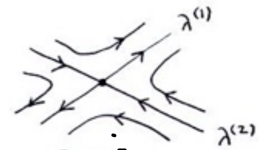
Phase portrait of nonlinear eqn (*)



Phase portraits for linear 2×2 systems $\vec{x}' = A\vec{x}$ (A real)

I) Distinct real eigenvalues

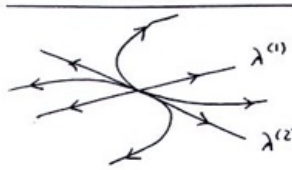
Ia) $\lambda^{(1)} > 0 > \lambda^{(2)}$



Saddle

Ib) $\lambda^{(1)} > \lambda^{(2)} > 0$

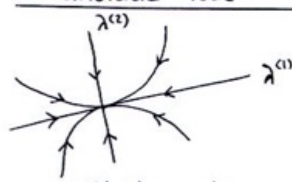
Note that $\lambda^{(2)}$ dominates for $t \to -\infty$ while $\lambda^{(1)}$ dominates for $t \to \infty$.



Unstable node

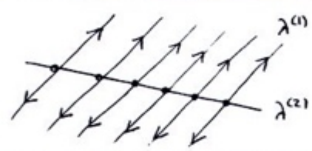
Ic) $0 > \lambda^{(1)} > \lambda^{(2)}$

Here $\lambda^{(2)}$ dominates for $t \to -\infty$ while $\lambda^{(1)}$ dominates for $t \to \infty$



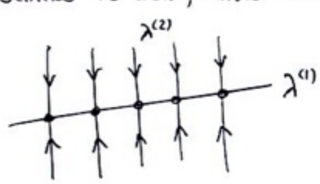
Stable node

Id) $\lambda^{(1)} > \lambda^{(2)} = 0$



Ie) $\lambda^{(1)} = 0 > \lambda^{(2)}$

Similar to Id), arrow reversed:



II) Repeated real eigenvalue λ

IIa) $\lambda > 0$

IIa1) A has two indep. eigenvectors with eigenvalue λ

unstable proper node

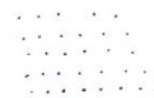
IIa2) A has only one eigenvector (up to factor)

or

unstable improper node

IIb) $\lambda = 0$

IIb1) A has two independent eigenv. with eigenvalue 0: Happens only if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Every sol. is constant



IIb2) A has only one eigenvector (up to factor)

or

IIc) $\lambda < 0$ (Like IIa, arrows reversed)

IIc1) A has two independent eigenvectors

stable proper node

IIc2) A has only one eigenvector (up to factor)

or

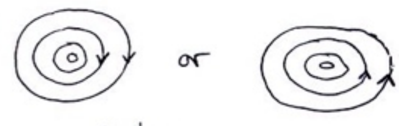
stable improper node

III) Complex eigenvalues $\lambda = a + ib$, $\bar{\lambda} = a - ib$ $(b \neq 0)$

IIIa) $a > 0$

unstable spiral

IIIb) $a = 0$ (purely imaginary eigenvalues)



Center

IIIc) $a < 0$

stable spiral

Remark: A simple way of deciding between the two possibilities in cases IIa2, IIb2, IIc2 and IIIa, IIIb, IIIc is to compute $\vec{x}'$ at a suitable point $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

For example, in IIIa)–IIIc) one can take $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$\vec{x}' = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

showing that one has counterclockwise motion for $a_{21} > 0$, clockwise motion for $a_{21} < 0$.