(1) Suppose the following interest rates apply:

In year 1 there is a 40% chance that the interest rate is 10% and a 60% chance that the interest rate is 15%;

In year 2 there is a 20% chance that the interest rate will be 2% higher than that in year 1, and an 80% chance it will be 1% higher than that in year 1.

What is the expected accumulation of $50 over the next 2 years?

[5 marks]

<u>Solution</u>

First need to calculate $E\left[\tilde{S}(2)\right]$. Note that interest rates are NOT independent → the possible outcomes of $\tilde{S}(2)$ are:

Year 1	Year 2	$\tilde{S}(2)$	Prob$[\tilde{S}(2)]$
	11% (prob = 0.8)	(1.1)(1.11) = 1.221	0.32
10% (prob = 0.4)	12% (prob = 0.2)	(1.1)(1.12) = 1.232	0.08
	16% (prob = 0.8)	(1.15)(1.16) = 1.334	0.48
15% (prob = 0.6)	17% (prob = 0.2)	(1.15)(1.17) = 1.3455	0.12

Therefore get E($50\tilde{S}(2)$) = 50(1.29106) = $64.55

(2) The table below gives information about an investment fund (all figures in \$m):

	2005	2006	2007
Value of fund at 1 January	620		
Value of fund at 30 April	700	710	850
Net cashflow received on 1 May	-50	+150	X
Value of fund at 31 December	680	825	400

(a) What is the Money weighted (annual) rate of return for this fund between 1 Jan 2005 and 31 December 2006? (to the nearest 0.5%) (3 marks)

(b) If the time weighted rate of return between 1 January 05 and 31 December 2007 is 5.5% per annum effective, what is the value of X? (2 marks)

(a) Need to solve the following for i:

$$620(1+i)^2 - 50(1+i)^{5/3} + 150(1+i)^{2/3} = 825$$

Try i = 10% → get \$851.43
Try i = 8% → get \$824.22
Interpolate…. i = 8.06% → so answer is 8.0%

(b)Need to solve the following for X:

$$1.055^3 = \frac{700}{620} \text{ x } \frac{710}{700\text{-}50} \text{ x } \frac{850}{710+150} \text{ x } \frac{400}{850+X} \rightarrow X = -435$$

(3) The annual effective forward rates are given as:

$$f_{0,1} = 8\%$$
$$f_{1,2} = 7\%$$
$$f_{2,3} = 6\%$$
$$f_{3,4} = 5\%$$

What is the gross redemption yield at the issue date for a 4-year bond, redeemable at par, with a 5% coupon payable annually in arrears? [5 marks]

V (1) = 1.08^{-1} = 0.925926
V (2) = 1.08^{-1} 1.07^{-1} = 0.86635
V (3) = 1.08^{-1} 1.07^{-1} 1.06^{-1} = 0.81637
V (4) = 1.08^{-1} 1.07^{-1} 1.06^{-1} 1.05^{-1}= 0.77749

P = 5 x (0.925926 + 0.86635 + 0.81637 + 0.77749) + 100 x 0.77749 = 94.67

Now need to find *i* that solves $94.67 = 5a_{\overline{4}|} + 100v^4$

Try *i* = 6%, *i* = 7% …. Interpolate… get *i* = 6.56%

(4) As at 1 June 2008 the price of an Asset is currently \$120. It pays six-monthly coupons of \$5, with the next coupon payment due in exactly 1 month's time. If a 12-month forward contract is entered into on 1 June 2008, what is the forward price for this asset if:

- the risk-free rate of interest is 5% convertible half-yearly; and
- there is no arbitrage. [3 marks]

Risk free interest rate $i = 1.025^2 - 1 = 5.0625\%$

PV of coupons = $5v_{5.0625\%}^{1/12} + 5v_{5.0625\%}^{7/12} = 9.8375$

Forward price = $(120 - 9.8375)(1.050625) = \$115.74$

(5) An investor borrows $120,000, at 7% effective per annum, in order to purchase an annuity of $28,000 per annum payable half-yearly in arrears (i.e. the annuity pays $14,000 at the end of every six months) for 25 years.

What is the discounted payback period of this investment?

[4 marks]

Need to find *t* that solves:

$$120,000 = 28,000 a_{\overline{t}|}^{(2)}$$

$$120,000 = 28,000 \frac{i}{i^{(2)}} a_{\overline{t}|}$$

$$120,000 = 28,000 \text{x} 1.017204 a_{\overline{t}|}$$

$$\frac{1 - (\frac{1}{1.07})^t}{0.07} = 4.21323$$

→ t = 5.165 years

So DPP is 5.5 years…. why?

(6) You have the following option to pay off (amortise) a mortgage of $175,000 over 15 years:

	Payments
Payments in years 1-5	Monthly of amount $Y per month
Payments in years 6-10	Quarterly of amount $2Y per quarter
Payments in years 11-15	Annually of amount $3Y per year

The annual effective interest rate is 10% and all payments are made in arrears.

(i) What is the value of $Y? [3 marks]

(ii) How much interest is paid over the last 13 years of the loan? [4 marks]

(i) Need to solve the following for Y:

$$175,000 = 12Ya_{\overline{5}|}^{(12)} + v^5 4(2Y)a_{\overline{5}|}^{(4)} + v^{10} 3Ya_{\overline{5}|}$$

$$175,000 = Ya_{\overline{5}|}[(12\frac{i}{i^{(12)}} + 8v^5\frac{i}{i^{(4)}} + 3v^{10})$$

$$175,000 = Y(3.7908)[12\frac{0.10}{0.09569} + 8(0.62092)\frac{0.10}{0.096455} + 3(0.38554)]$$

175,000 = Y (71.4454)

Y = 2,449.42

(ii) Loan balance at end of second year can be determined prospectively or retrospectively. Using the retrospective method:

Loan has grown to 175,000 x 1.1^2 = 211,750

Value at t = 2, of payments made in the first 2 years is:

$$12Ys_{\overline{2}|}^{(12)} = 12Y\frac{i}{i^{(12)}}s_{\overline{2}|} = 12(2449.42)\frac{0.10}{0.09569}(2.1) =$$

64,505.57

So value of loan at t = 2 is 211,750 – 64,505.57 = 147,244.43

Total payments made over the last 13 years of the loan is equal to

3(12)Y + 5(4)(2Y) + 5(3Y) = 91Y = 222,897.22

So total interest paid over the final 13 years is

222,897.22 – 147,244.43 = 75,652.79

(7) You purchase a bond that is callable at the discretion of the issuer anytime between (and including) 6.5 and 17.5 years from time of issue.

The bond has a face value of $1,000, is redeemable at par, and pays half yearly coupons of 10% per annum.

You incur 30% in income tax which is paid as soon as it is due. You do not pay capital gains tax.

(a) What price do you pay to guarantee a net yield to redemption of 6.0% per annum? [5 marks]

(b) If income tax was payable 3 years after it was incurred, would you expect to pay a lower price or higher price in part (a) above? Why? [2 marks]

(c) An investor buys the bond off you for $1,100 in exactly 5 years time, just after the coupon due at that time. Assuming the investor pays no tax at all, what return is this investor guaranteed to make? Give your answer to the nearest 0.1%. [5 marks]

[a] You earn coupons at 10(1 - 0.3) = 7% per annum convertible half yearly or 7.13% effective per annum. This exceeds the return you require of 6% per annum so you will pay a premium for the bond. This means you will want to price the bond at the earliest redemption date to guarantee your required return.

So $P = 1000(\frac{0.1}{2})(0.7)a_{\overline{13}|j\%} + 1000v_{j\%}^{13}$ where $j = 1.06^{0.5} - 1$

So $P = 373.27 + 684.72 = 1{,}057.99$

{check – Price at t = 17.5 would be 1,117.57}

[b] If income tax is paid later, then that is deferring a liability i.e. deferring a cost further into the future than what it currently is. Therefore the PV of that cost would be lower. <u>This would mean that the price is higher</u>. {in other words, having tax paid later means that the bond is worth more now}

[c] The investor is also buying the bond at a premium. So their best return is possible when the bond is held for as long as possible. So their guaranteed return is when the bond is redeemed as early as possible. That is, need to solve the following for *j%*:

$$1{,}100 = 1{,}000(\frac{0.1}{2})a_{\overline{3}|j\%} + 1{,}000v_{j\%}^{3}$$

Try j = 2%: $1{,}000(\frac{0.1}{2})a_{\overline{3}|j\%} + 1{,}000v_{j\%}^{3} = 1086.52$

Try j = 1%: $1{,}000(\frac{0.1}{2})a_{\overline{3}|j\%} + 1{,}000v_{j\%}^{3} = 1117.64$

Interpolate to get j = 1.56% → so i = 3.1%.

(8) In Fund A money accumulates at a force of interest $\delta(t)$ at time t, measured in years, which is given by:

$$\delta(t) = \begin{cases} 0.01t + 0.1 & 0 \le t < 10 \\ 0.02t & 10 \le t < 20 \end{cases}$$

In Fund B money accumulates at an effective rate of interest $i\%$ per annum.

a) You invest \$100 in Fund A at time 0. What is the accumulated value after 10 years? [2 marks]

b) Find i if \$1 is invested at time 0 in each fund for 20 years, and the accumulated value of Fund A and B are equal at the end of 20 years. [4 marks]

Solution

a)

$$\$100\exp\left(\int_0^{10}(0.01t+0.1)dt\right)=\$100\exp\left(0.005t^2+0.1t\Big|_0^{10}\right)$$
$$=\$100\exp(1.5)=\$448.17$$

b) Accumulated amount of $1 in Fund A:

$$\exp\left(\int_0^{10}(0.01t+0.1)dt\right)\exp\left(\int_{10}^{20}0.02t\,dt\right)=\exp(1.5)\exp\left(0.01t^2\Big|_{10}^{20}\right)$$
$$=\exp(1.5)\exp(3)=90.02$$

Accumulated amount of $1 in Fund B: $(1+i)^{20}$

$(1+i)^{20}=90.02\Rightarrow i\cong 25.2\%$

(9) You deposit $10,000 into an account. In return, you receive either:

i) $16,000 in five years time, or
ii) $18,400 in seven years time, or
iii) a series of six yearly payments of $3,000 each, the first such payment being made after four years.

a) Find the yields for the three alternatives using linear interpolation or otherwise. Based on yield, what is the most attractive alternative?

[5 marks]

b) If the $10,000 that is initially invested is borrowed at an effective annual interest rate of 8.5%, what is the most attractive alternative?

[4 marks]

Solution

a) i) $10{,}000 = 16{,}000(1+i)^{-5} \Rightarrow i = 9.856\%$

ii) $10{,}000 = 18{,}400(1+i)^{-7} \Rightarrow i = 9.102\%$

iii) $10{,}000 = 3{,}000\ddot{a}_{\overline{6}|}v^4 = 3{,}000a_{\overline{6}|}v^3$

$$10{,}000 - 3{,}000\left(\frac{v^3 - v^9}{i}\right) = 10{,}000 - 3{,}000\left(\frac{(1+i)^{-3} - (1+i)^{-9}}{i}\right) = 0$$

Try 9%	RHS = -391.86
Try 9.5%	RHS = -99.131
Try 10%	RHS = 183.4845

Use interpolation to find the yield:

$$= 0.095 + \frac{0 + 99.131}{183.4845 + 99.131}(0.10 - 0.095) = 0.09675$$

$i \cong 9.675\%$

Based on yield, (i) is the most attractive investment.

b) Find the NPV of all three investments, discounting at 8.5%.

(i) $NPV(0.085) = -10{,}000 + 16{,}000(1+i)^{-5} = 640.73$

(ii) $NPV(0.085) = -10{,}000 + 18{,}400(1+i)^{-7} = 394.64$

(iii) $NPV(0.085) = -10{,}000 + 3{,}000\left(\frac{v^3 - v^9}{i}\right) = 695.12$

Therefore (iii) is the most attractive alternative.

(10) A financial institution has agreed to pay \$10,000 at the end of each year for 6 years. In exchange for this liability, the institution receives \$50,756.92 at time 0 which it invests in various assets as follows:

- \$7,500 of the \$50,756.92 is invested in a 7-year zero coupon bond;
- And the remaining funds are invested in a 1-year and 3-year zero coupon bond.

The investments yield an effective rate of interest of 5% per annum.

Find the amount invested in the 1 and 3-year zero coupon bonds so that the first two conditions for immunisation are satisfied.

[7 marks]

<u>Solution:</u>

Condition 1: $PV_L = PV_A$

$PV_L = 10,000 a_{\overline{6}|} =$ 50,756.92

Now let:

- X_1 = the redemption payment (face value) of the 1 year bond;
- X_2 = the redemption payment (face value) of the 3 year bond;
- X_3 = the redemption payment (face value) of the 7 year bond;

Then $PV_A = X_1 v + X_2 v^3 + X_3 v^7$

$X_3 v^7 = 7,500 \Rightarrow X_3 =$ 10,553.25

$\Rightarrow X_1 v + X_2 v^3 =$ 50,756.92-7,500 $\cong$ 43,256.92

(1)

Condition 2: $PV_L' = PV_A'$

$$PV_L' = -10000 \sum_{t=1}^{6} t v^{t+1} = -10000 v (Ia)_{\overline{6}|} = -10000 v(17.0437) = \text{-162,321}$$

$$PV_A' = -X_1 v^2 - 3X_2 v^4 - 7X_3 v^8$$

$$\Rightarrow -X_1 v^2 - 3X_2 v^4 = -162,321 + 7X_3 v^8 \cong \text{-112,321}$$

(2)

Solve (1) and (2) for X_1 and X_2

$X_1 =$ 6,212.80 → Amount invested in 1-yr bond = $X_1 v =$ 5,917

$X_2 =$ 43,226 → Amount invested in 3-yr bond = $X_2 v^3 = 37,340$

Check: 7,500 + 5,917 + 37,340 = 50,757.