8.4) $$\left.\begin{aligned}[A(X-c)]^2 &= [A(X)-A(c)]^2\\ &= [A(X)-c]^2\end{aligned}\right\}\ \text{ii and iii pg 7 combined}$$
$$= [A(X)]^2 - 2c\,A(X) + c^2 \quad \boxtimes$$

8.5) Let $\mathcal{T} = \{k \text{ such that } p_k > 0\}$ $\quad X(\omega_k) = c\ , k \in \mathcal{T}$

"$\Rightarrow$"

$$A(X) = \sum_k p_k X(\omega_k) = \sum_{k\in\mathcal{T}} p_k c = c\sum_{k\in\mathcal{T}} p_k = c$$

(Assuming $X(\omega_k)$ is real valued $\forall k$, ie $X(\omega_k) < \infty$)

$$A(X^2) = \sum_k p_k X(\omega_k)^2 = \sum_{k\in\mathcal{T}} p_k c = c^2\sum_{k\in\mathcal{T}} p_k = c^2$$

Therefore $[A(X)]^2 = A(X^2)$

"$\Leftarrow$"

$[A(x)]^2 = A(x^2) \quad \Rightarrow$

$$0 = A\left[(X - A(X))^2\right] = \sum_k p_k\left(X(\omega_k) - \sum_k p_k X(\omega_k)\right)^2$$

if $p_k > 0$ then $X(\omega_k) - \sum_k p_k X(\omega_k) = 0$

that is, $X(\omega_k) = \underbrace{\sum_k p_k X(\omega_k)}_{\text{constant}} = c \quad , \quad k \in \mathcal{T} \quad \boxtimes$

16.4) $|X_1 + X_2| \leq |X_1| + |X_2|$ (triangle inequality)

$\Rightarrow \quad |X_1| + |X_2| - |X_1 + X_2| \geq 0$

$\Rightarrow \quad E(|X_1| + |X_2| - |X_1 + X_2|) \geq 0$ Axiom 1 pg 15

$\Rightarrow \quad E(|X_1|) + E(|X_2|) \geq E(|X_1 + X_2|)$ Axiom 3 pg 15

$\square$.

16.5) $|X_n - X| \leq Y_n$

$\Rightarrow \quad -Y_n \leq X_n - X \leq Y_n$

$\Rightarrow \quad -E(Y_n) \leq E(X_n) - E(X) \leq E(Y_n)$ (positive linear operator)

$\Rightarrow \quad |E(X_n) - E(X)| \leq E(Y_n) \quad \forall n$ (*)

Since $E(Y_n) \to 0$ as n increases, we have $\forall n$, $\exists \varepsilon > 0$ such that $|E(Y_n) - 0| < \varepsilon$

Therefore from (*), $\forall n$ $\exists \varepsilon > 0$ such that

$|E(X_n) - E(X)| \leq E(Y_n) \leq |E(Y_n)| < \varepsilon$

which means $E(X_n) \to E(X)$ with n increasing

20.1) $\boxed{\text{Let us show that } f(t) \geq 0 \text{ !}}$

Assume $f(t) < 0$, we can take $H(t) > 0$, $t > 0$

and $H(0) = 0$ $\quad (H(t) \geq 0)$

Then $E[H(\tau)] = \int_0^\infty \underbrace{H(t)}_{\geq 0} \underbrace{f(t)}_{< 0} dt < 0$ (conflict with axiom 1) pg 15

$\boxed{\text{Let us show } p \geq 0}$

Take $H(t) = \begin{cases} 1 & \tau = 0 \\ 0 & \text{otherwise} \end{cases}$ (Note $H(t) \geq 0$)

Thus $E[H(\tau)] = p \geq 0$ (Axiom 1)

$$\boxed{p + \int_0^\infty f(t).dt = 1}$$

$$E(1) = p.1 + \int_0^\infty f(t) dt) = 1 \quad \text{(Axiom 4)}$$

21.3) $$E(X) = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X(w)\,dw$$

Axiom 1: $X \geq 0 \Rightarrow$

$$0 = \int_{-D}^{D} 0\,dw \leq \int_{-D}^{D} X(w)\,dw$$

$$\Rightarrow E(X) = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X(w)\,dw \geq 0$$

Axiom 2: $$E(cX) = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} c\,X(w)\,dw$$

$$= c \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X(w)\,dw = c\,E(X)$$

Axiom 3: $$E(X_1 + X_2) = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X_1(w) + X_2(w)\,dw$$

$$= \lim_{D\to\infty} \frac{1}{2D} \left[\int_{-D}^{D} X_1(w)\,dw + \int_{-D}^{D} X_2(w)\,dw \right]$$

$$= \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X_1(w)\,dw + \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X_2(w)\,dw = E(X_1) + E(X_2)$$

Axiom 4: $$E(1) = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} 1\,dw = \lim_{D\to\infty} \frac{1}{2D}\, w\Big|_{-D}^{D}$$

$$= \lim_{D\to\infty} \frac{1}{2D} [D - (-D)] = \lim_{D\to\infty} \frac{2D}{2D} = \lim_{D\to\infty} 1 = 1$$

Axiom 5: Using $X_n(w) = \begin{cases} 1 & |w| \le n \\ 0 & \text{otherwise} \end{cases}$

We have $X_1 \le X_2 \le X_3 \le \ldots$

and $\lim_{n\to\infty} X_n(w) = X = 1$ (increases monotomcally)

$\boxed{\text{But when } D > n \text{ we have}}$

$$\lim_{n\to\infty} E(X_n(w)) = \lim_{n\to\infty} \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} X_n(w)\, dw$$

$$= \lim_{n\to\infty} \lim_{D\to\infty} \frac{1}{2D} \, w \Big|_{-n}^{n}$$

$$= \lim_{n\to\infty} \lim_{D\to\infty} \frac{n}{D}$$

$$= \lim_{n\to\infty} 0 = 0 \ne E(X) = 1 \ldots$$

24.1) The Loss function can be written as:

$$g(t) = \begin{cases} a(t-T) & \text{if } T < t \\ b(T-t) & \text{if } T > t \end{cases}$$

Or $$g(t) = a(t-T)\,\mathbb{1}(T<t) + b(T-t)\,\mathbb{1}(T>t)$$

The expected Loss become:

$$\begin{aligned} E[g(t)] &= a(t-T)\,P(T<t) + b(T-t)\,P(T>t) \\ &= a(t-T)\,P(T<t) - b(t-T)\,P(T>t) \\ &= (t-T)\left[a\,P(T<t) - b\,P(T>t)\right] \end{aligned}$$

Taking the derivative with respect to t and set to 0 we have

$$a\,P(T<t) - b\,P(T>t) = 0$$

$$\Rightarrow \quad a\,P(T<t) = b\,P(T>t) \qquad (P(T>t) = 1 - P(T<t))$$

Or

$$a\,P(T<t) = b - b\,P(T<t)$$

$$\Rightarrow \quad P(T<t) = \frac{b}{a+b}$$

32.7) The relation $\hat{X} - E(X) = a^T(Y - E(Y))$

become $\hat{Y}_{t+s} - \mu = \beta^s(Y_t - \mu)$

By Theorem 2.8.2 $\hat{Y}_{t+s}$ is an LLS estimate

of Y_{t+s} if and only if a statisfies the

linear relations $V_{Y_t Y_t} a = V_{Y_t Y_{t+s}}$ (we have to find $\underline{a}$)

$$V_{Y_t Y_t} = \begin{bmatrix} Cov(Y_1, Y_1) & \cdots & Cov(Y_1, Y_t) \\ Cov(Y_1, Y_t) & \cdots & Cov(Y_t Y_t) \end{bmatrix} = \alpha \begin{bmatrix} 1 & \cdots & \beta^{|1-t|} \\ \vdots & & \vdots \\ \beta^{|t-1|} & & 1 \end{bmatrix}$$

$$V_{Y_t Y_{t+s}} = \begin{bmatrix} Cov(Y_{t+s}, Y_1) \\ \vdots \\ Cov(Y_{t+s}, Y_t) \end{bmatrix} = \alpha \begin{bmatrix} \beta^{|t+s-1|} \\ \vdots \\ \beta^{|t+s-t|} \end{bmatrix}$$

Set $a = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta^s \end{bmatrix}$ and we have $V_{Y_t Y_t} a = V_{Y_t Y_{t+s}}$

Thus $\hat{Y}_{s+t} = \mu + \beta^s(Y_t - \mu)$ (scalar form) is the L.L.S predictor

37.7) Let $X_1 = X - E(X)$ and $X_2 = Y - E(Y)$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$XX^T = \begin{bmatrix} X_1^2 & X_1 X_2 \\ X_1 X_2 & X_2^2 \end{bmatrix} \quad \text{symmetric}$$

Then $E(XX^T) = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$ symmetric

$$c^T E(XX^T) c = E(c^T X X^T c) = E(c_1^2 X_1^2 + 2c_1c_2 X_1 X_2 + c_2^2 X_2)$$

$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$= E(c_1 X_1 + c_2 X_2)^2$$

$$= E\left[(c^T X)^2\right] \geq 0$$

Therefore $E(XX^T)$ is positive definite matrix

By Theorem 2.9.1 $|E(X_1 X_2)|^2 \leq E(X_1^2)E(X_2^2)$ that is,

$[Cov(X,Y)]^2 \leq Var(X) Var(Y)$, with equality if there is a non-trivial relation $c_1 X_1 + c_2 X_2 \overset{m.s}{=} 0$, i.e,

$$c_1(X - E(X)) + c_2(Y - E(Y)) \overset{m.s}{=} 0 \quad \text{or}$$

$$c_1 X + c_2 Y \overset{ms}{=} c_1 E(X) + c_2(Y) = c_0$$