University of Toronto

MAT237Y1Y PROBLEM SET 4

DUE: End of tutorial, Thursday June 20th, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.
2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.
3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.
4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. Find the Taylor polynomial of order 3 of

$$f(x,y) = (x-1)^2 + sin(\pi y) + xlny$$

based at $(x,y) = (2,1)$.
Hint: Find an appropriate change of variables to center the Taylor polynomial at the origin, and use that the Taylor expansion of $ln(1+t) = t - t^2/2 + t^3/3 - t^4/4 + \ldots$

2. Consider the function $f(x,y) = \frac{1}{1-x-y}$

   a) Let's compute in two different ways the 2nd order Taylor polynomial, $P_{(0,0),2}(x,y)$ of f based at the origin. First we will use the expansion for $\frac{1}{1-t}$ given in Prop 2.65.

   b) Now let's compute the same 2nd order Taylor polynomial directly from the definition as in 2.68 where we are computing partial derivatives.

c) Show that $|\partial^\alpha f(x,y)| \leq 6 \times 2^4$ for every multi-index $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| = 3$ and every point $(x,y) \in \mathbb{R}^2$ such that $|x| \leq 1/4$, $|y| \leq 1/4$

d) Estimate the error, $|R_{(0,0),2}(x.y)|$ in the Taylor approximation of $f(x,y)$ on the square $|x| \leq 1/4$, $|y| \leq 1/4$ using the definition of the error in equation 2.72 and following Corollary 2.75.

3.

a) Starting at the bottom of page 75 a derivation is given for a general situation with two constraint equations and two dependent variables using Cramer's Rule. Derive a generalization of this to a situation with n differentiable constraints and n dependent variables. That is, begin with $F_1(x_1, \ldots, x_m, u_1, \ldots, u_n) = 0, \ldots, F_n(x_1, \ldots, x_m, u_1, \ldots, u_n) = 0$ and assume that we can solve each u_i in terms of $x_1, \ldots, x_m$. Now derive a computation of $\frac{\partial u_i}{\partial x_j}$. (Note that there would be a minor notational change of some ∂s to ds if there was only one independent variable x, you may assume for convenience there is more than 1.)

a) Read Example 3 from 2.5. Now suppose x,y,z,w are initially equal to (1,0,1,1) and constrained by

$$F_1(x,y,z,w) = 2x^2y + zw - 1 = 0$$

$$F_2(x,y,z,w) = sin(\pi xz) + y = 0$$

$$F_3(x,y,z,w) = z + w - 2 = 0$$

If x is changed to .97, approximate how much y,z, and w change? You may use your formula derived in part a, or by other means, to solve this.

4. In class I emphasized how the proofs of the one variable and multivariable Taylor's theorem modeled the proofs of first the one variable MVT I and then the extension from one variable MVT I to multivariable MVT III. Hence the multivariable Taylor's theorem can be thought of as a generalization of the original MVT I in two ways, first generalizing from one derivative to higher order derivatives, and then generalizing from one variable to multiple variables. What I want you to do here is the reverse. That is, start with the multivariable Taylor's Theorem 2.68, and show how first MVT III and then MVT I are both special cases of this theorem. (Note that this should be quite short, you should be doing things like, say, interpreting what α might mean in the special case or figure out what k is).

5. Do Exercises 10 and 11 from 2.6. These prove a generalized product rule for partial derivatives known as Leibniz Rule and a "Multi-Binomial" theorem respectively. As brief answers are provided by Folland, make sure to work out all details carefully justifying each step. Please show the inductive argument for the one variable higher order product rule (but you can just quote the one variable binomial theorem).

6. Find all critical points for the function $f(x, y) = (y^2 - y)x$. Write out the Hessian Matrix for each critical point. Compute the eigenvalues for the Hessian Matrix at each critical point and use this to write out the 2nd order Taylor polynomial about each critical point. Finally, classify each critical point using 2.82.

   Enjoy!