MAT 246S **Solutions to Practice Term Test 2** **Winter 2013**

(1) Find the formula for the sum $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots + (2n) \cdot (2n-1) - (2n) \cdot (2n+1)$ and prove it by mathematical induction.

Solution

Observe that $(2n)(2n-1) - (2n)(2n+1) = (2n) \cdot (-2) = -4n$ Thus we need to find $-4 \cdot 1 - \ldots - 4n = -4(1 + \ldots n) = -4\frac{n(n+1)}{2} = -2n(n+1)$.

We prove this by induction.

When $n = 1$ we have $1 \cdot 2 - 2 \cdot 3 = 2 - 6 = -4 = -2 \cdot (1) \cdot (2) = -4$.

Induction step. Suppose $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots - (2n) \cdot (2n+1) = -2n(n+1)$ then $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots - (2n) \cdot (2n+1) + (2n+1) \cdot (2n+2) - (2n+2) \cdot (2n+3) = -2n(n+1) + (2n+1) \cdot (2n+2) - (2n+2) \cdot (2n+3) = -2n(n+1) - 2(2n+2) = -2(n+1)(n+2)$.

(2) Find the remainder when 6^{100} is divided by 14.

Solution

First we observe that $6 \equiv -1 \pmod 7$. Hence $6^{100} \equiv (-1)^{100} = 1 \pmod 7$. Thus $6^{100} \equiv 1 \pmod 7 \equiv 8 \pmod 7$. This means that 7 divides $6^{100} - 8$. But $6^{100} - 8$ is even and 2 also divides $6^{100} - 8$. Since $(2,7) = 1$ this means that 14 divides $6^{100} - 8$, i.e. $6^{100} \equiv 8 \pmod{14}$.

Answer: 8.

(3) Find the integer a, $0 \le a < 37$ such that $(34!)a \equiv 1 \pmod{37}$.

Solution

Since 37 is prime, by Wilson's theorem, $36! \equiv -1 \pmod{37}$.

We rewrite $34! \cdot 35 \cdot 36 \equiv = -1 \pmod{37}$. Since $36 \equiv -1 \pmod{37}$ this gives $34! \cdot 35 \equiv 1 \pmod{37}$.

Answer: $a = 35$.

(4) Let $n = pq$ where p, q are distinct odd primes. Find the remainder when $\phi(n)!$ is divided by n.

Solution

Since p and q are distinct odd, without loss of generality $2 < p < q$. We have $\phi(n) = (p-1)(q-1)$. Since $q > p > 2$ we have $\phi(n) = (p-1)(q-1) > (p-1)$ and hence $\phi(n) \geq p$. Similarly, $\phi(n) = (p-1)(q-1) > (q-1)$ and hence $\phi(n) \geq q$. Therefore both p and q occur as factors in the product $\phi(n)! = 1 \cdot 2 \ldots \cdot p \cdot \ldots \cdot q \cdot \ldots \cdot \phi(n)$. Hence $n = pq$ divides $\phi(n)!$ i.e.

Answer: $\phi(n)! \equiv 0 (\mod n)$.

(5) Find all integer solutions of the equation

$$34x + 50y = 22$$

Solution

First we divide the equation by 2 and get an equivalent equation $17x + 25y = 11$. Note that $gcd(17, 25) = 1$.

Next we use the Euclidean algorithm to find a solution of the equation

$$17x + 25y = 1$$

We have $25 = 1 \cdot 17 + 8, 17 = 2 \cdot 8 + 1$. Hence $8 = 25 \cdot 1 - 17 \cdot 1$ and $1 = 17 \cdot 1 - 2 \cdot 8$. Plugging in the former equation into the latter we get $1 = 17 \cdot 1 - 2(25 \cdot 1 - 17 \cdot 1) = 17 \cdot 3 - 25 \cdot 2$. Hence $x_0 = 3, y_0 = -2$ is a solution of $17x + 25y = 1$. Multiplying this equation by 11 we see that $\tilde{x}_0 = 3 \cdot 11 = 33, \tilde{y}_0 = (-2) \cdot 11 = -22$ is a solution of $17x + 25y = 11$.

Recall that if $\tilde{x}_0, \tilde{y}_0$ solves $ax + by = c$ with $(a, b) = 1$ then $x = x_0 + kb, y = y_0 - ka$ with $k \in \mathbb{Z}$ is the general integer solution of $ax + by = c$.

In our case this gives

Answer: $x = 33 + 25k, y = -22 - 17k$ with $k \in \mathbb{Z}$ is the general integer solution of $17x + 25y = 11$.