Lecture 12

Period-Doubling Bifurcation

F_λ has a period-doubling bifurcation in I (open interval) at λ_0 if $\exists\ \varepsilon > 0$ s.t.

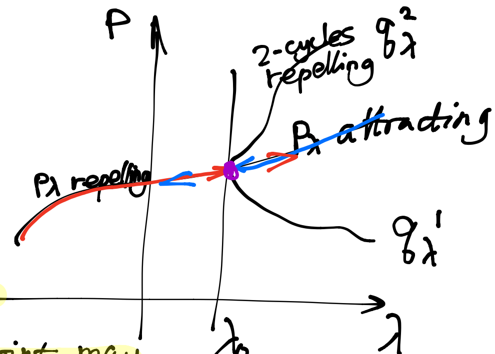
i.) Unique fixed pt. P_λ for F_λ in I for all $\lambda < (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$

ii). λ in half of $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ including λ_0, F_λ has no 2-cycles in I and P_λ attracting (resp. repelling)

iii). λ in other half excluding λ_0, F_λ has unique 2-cycle $q_\lambda^1, q_\lambda^2 \in I$ attracting (resp. repelling)

P_λ is repelling (resp. attracting)

(iv) $q_\lambda^i \xrightarrow{\lambda \to \lambda_0} P_{\lambda_0}$



Remarks

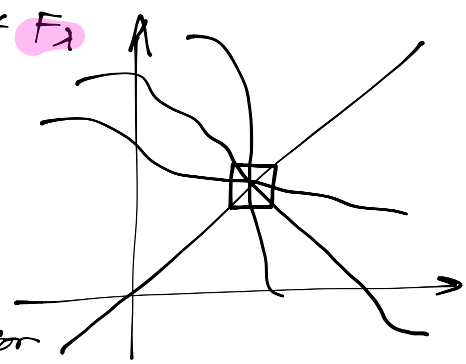
① There're 2 Typical cases for a period-dbling bfct'n.

- As the parameter changes, a fixed point may change from repelling to attracting, and at the same time give birth to a repelling 2-cycle
- As the parameter changes, a fixed point may change from attracting to repelling, and at the same time give birth to an attracting 2-cycle.

② Cycles can also undergo a period-doubling biferc**a**tion: in this case an n-cycle will give birth to a 2n-cycle.

③ A PDB occurs when the graph of F_λ is perpendicular to the diagonal $y=x$. That is, when $F'_{\lambda_0}(P_{\lambda_0}) = -1$ which implies that $(F^2_{\lambda_0})'(P_{\lambda_0}) = 1$, which means that

$F^2_{\lambda_0}$ is tangent to the diagonal.



Example:

Let $Q_c(x) = x^2 + c$

recall:

$P_- = \dfrac{1 - \sqrt{1-4c}}{2}$ is an attracting fixed pt for $-3/4 < c < 1/4$.

$q_- = \dfrac{-1 - \sqrt{-3-4c}}{2}$ $\quad q_+ = \dfrac{-1 + \sqrt{-3-4c}}{2}$ is an attracting 2-cycle for $-5/4 < c < -3/4$

Find I!

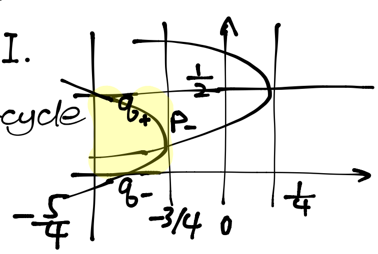
At $c_0 = -3/4$, $\exists$ a period-doubling biferc**a**tion. Take $\varepsilon = 1/2$ (in I)

(i). For $\lambda \in (-5/4, -1/4)$, Q_c has a unique fixed pt P_- which is attrac-ting in I.

(ii). For $c \in [-3/4, 1/4)$, Q_c has no 2-cycles in I. and P_- is attracting.

(iii). For $c \in (-5/4, -3/4)$, Q_c has a unique 2-cycle q_-, q_+ which is attracting, P_- is repelling.

(iv). $q_\pm \xrightarrow{c \to -3/4} -\frac{1}{2} = P_-$



Exercise: Show that $Q_c^2(x)$ has a PDB at $c_0=-5/4$.

Example: Consider the family $F_\lambda(x)=\lambda x-x^3$

Find FP.

$F_\lambda(x)=x \iff \lambda x-x^3=x \iff x(\lambda-1-x^2)=0$

$\iff x=0$ or $x=\pm\sqrt{\lambda-1}$ and $F_\lambda'(x)=\lambda-3x^2$

so $F_\lambda'(0)=\lambda$ and $F_\lambda'(\pm\sqrt{\lambda-1})=\lambda-3(\lambda-1)=3-2\lambda$

- $x=0$ is a fixed pt.: attracting for $-1<\lambda<1$.
- $x=\pm\sqrt{\lambda-1}$ are fixed pts: attracting for $-1<3-2\lambda<1 \iff 1<\lambda<2$

- Find the 2-cycles: First observe that F_λ is odd:

$$F_\lambda(-x)=-F_\lambda(x)$$

so to find 2-cycles, we need to solve:

$$F_\lambda(x)=(-x)$$