Tutorial Problems - Sections 1 to 3 - MAT 327 - Summer 2013

1 Topological Spaces

1. Let X be an infinite set with $p \in X$ and let $\mathcal{T}_p := \{ U \in \mathcal{P}(X) : U = \emptyset \text{ or } p \in U \}$. Prove that $\mathcal{T}_p$ is a topology on X. (Called the *particular point* topology on X, at p.)

2. Is there an opposite version of the particular point topology? State a precise definition of such a topology and prove that it is a topology. How does this new topology relate to the particular point topology? What is a good name for this topology?

3. Using how we defined $(\mathbb{R}, \mathcal{T}_{\text{usual}})$ in terms of a distance function, describe the topological space $(X, \mathcal{T}_{\text{discrete}})$ by making reference to the discrete metric on X. (Look at your previous course notes for the definition of the discrete metric.)

4. Let $\{ \mathcal{T}_\alpha : \alpha \in \Lambda \}$ be a collection of topologies on a set X. Prove that there is a unique topology $\mathcal{T}$ on X such that: (1) for each $\alpha \in \Lambda$, $\mathcal{T}$ is finer than $\mathcal{T}_\alpha$, and (2) if $\mathcal{T}'$ is a topology on X that is finer than $\mathcal{T}_\alpha$ for each $\alpha \in \Lambda$, then $\mathcal{T}$ is coarser than $\mathcal{T}'$.

5. Let $\{ \mathcal{T}_\alpha : \alpha \in \Lambda \}$ be a collection of topologies on a set X. Prove that there is a unique topology $\mathcal{T}$ on X such that: (1) for each $\alpha \in \Lambda$, $\mathcal{T}$ is coarser than $\mathcal{T}_\alpha$, and (2) if $\mathcal{T}'$ is a topology on X that is coarser than $\mathcal{T}_\alpha$ for each $\alpha \in \Lambda$, then $\mathcal{T}$ is finer than $\mathcal{T}'$.

2 Basis for a Topology

1. Let $\mathcal{T}$ be the usual topology on $\mathbb{R}$. Prove that $\mathcal{B} = \{ (a, b) : a < b \text{ and } a, b \in \mathbb{Q} \}$ is a countable basis for $\mathcal{T}$.

2. Let $\mathcal{B}$ be the collection of all intervals of the form $[a, b)$, where $a < b$ and a and b are rational. Prove that $\mathcal{B}$ is a (countable) basis for a topology $\mathcal{T}$ on $\mathbb{R}$. Is $\mathcal{T}$ the Sorgenfrey topology on $\mathbb{R}$?

3. Let X be the set of all functions that map $[0, 1]$ into $[0, 1]$. For each subset A of $[0, 1]$, let $B_A = \{ f \in X : f(x) = 0, \forall x \in A \}$. Prove that $\mathcal{B} = \{ B_A : A \subseteq [0, 1] \}$ is a basis for a topology on X.

4. In our proof that a basis generates a topology we showed that the topology was closed under arbitrary unions. Write down the proof that the union of three sets in the topology is again in the topology. Make sure to simplify the notation appropriately.

5. In our proof that a basis generates a topology we showed that the topology was closed under finite intersections. Explain this proof in words and pictures. Also explain why this property is not just immediate from the fact that a basis is directed.

6. Prove proposition 9 in Section 2.

3 Closed Sets, Closures, and Interiors of Sets

1. Let $\mathcal{T}$ be the usual topology on $\mathbb{R}$ and let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b)$ is neither open nor closed.

2. Let $\mathcal{T}$ be the finite complement topology on $\mathbb{R}$, and let $A = [0, 1]$. Find $\overline{A}$ and $\text{int}(A)$ and prove your answers.

3. Let $\mathcal{T} = \{ U \in \mathcal{P}(\mathbb{R}) : 0 \notin U \text{ or } U = \mathbb{R} \}$.

   (a) Justify that $\mathcal{T}$ is a topology on $\mathbb{R}$.

   (b) Describe the closed subsets of $\mathbb{R}$.

   (c) Find $\overline{\{1\}}$.

4. Let X be a set, and let $\text{cl} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be a function such that the following conditions hold:

   (a) For each $A \in \mathcal{P}(X)$, $A \subseteq \text{cl}(A)$.

   (b) For each $A \in \mathcal{P}(X)$, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

   (c) $\text{cl}(\emptyset) = \emptyset$.

   (d) If $A, B \in \mathcal{P}(X)$, then $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

   Let $\mathcal{T} = \{ U \in \mathcal{P}(X) : \text{there is a subset } C \text{ of } X \text{ such that } \text{cl}(C) = C \text{ and } U = X \setminus C \}$. Prove that $\mathcal{T}$ is a topology on X. Properties (a)-(d) are called the **Kuratowski Closure Properties** in honour of K. Kuratowski (1896-1980).

5. Let X be a set and let $D \subseteq X$. Define a function $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ by $f(A) = A \cup D$ for each $A \in \mathcal{P}(X)$. Prove that f (almost) satisfies the Kuratowski Closure Properties. (What should you change to make this true?)

6. Let f be as in the previous question.

   (a) Describe the members of $\mathcal{T}$, where $\mathcal{T}$ is the topology defined in the previous exercise.

   (b) What is the topology $\mathcal{T}$ when $D = \emptyset$?

   (c) What is the topology $\mathcal{T}$ when $D = X$?

7. Let $(X, \mathcal{T})$ be a topological space, let $C \subseteq X$ be closed, and let $U \subseteq X$ be open. Prove that $C \setminus U$ is closed and $U \setminus C$ is open.

8. Let U be an open subset of a topological space $(X, \mathcal{T})$. Prove that

$$\overline{X \setminus \overline{X \setminus \overline{U}}} = \overline{U}$$

9. Give an example of a countable, dense subset of $\mathbb{R}$ that contains only irrational numbers.

10. Give 4 examples of zero-dimensional spaces, three of which we have seen in class and one which you have come up with on your own.

11. Let $\mathcal{B}$ be a basis for a topological space $(X, \mathcal{T})$, and let $A \subseteq X$. Then $x \in \overline{A}$ iff every *basic* open set B that contains x has $B \cap A \neq \emptyset$.

12. Prove that no finite set in $\mathbb{R}$ is dense in $\mathbb{R}_{\text{usual}}$.

13.

    Proposition 1. *Let $(X, \mathcal{T})$ be a topological space, with $A, D \subseteq X$.*

    (a) If D is dense, and $D \subseteq A$, then A is dense.
    (b) If $D \cap A = \emptyset$, and A is a non-empty open set, then D is not dense.
    (c) Suppose Γ refines $\mathcal{T}$. If D is dense in (X, Γ), then it is dense in $(X, \mathcal{T})$.

14. Fürstenbreg's proof of infinitely many primes (secretly) uses a basis to describe a topological space. Identify that basis, prove that it is a basis and then decide whether or not the space is zero-dimensional.