Prove or disprove each of the following statements. Write detailed proof structures and justify your work.

1. For all real numbers r, s, if r and s are both positive, then $\sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.

FIRST, *write the statement symbolically:*

$$\forall r \in \mathbb{R}^+, \forall s \in \mathbb{R}^+, r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$$

SECOND, *try a direct proof:*
Assume $r \in \mathbb{R}^+$ *and* $s \in \mathbb{R}^+$
    Assume $r > 0$ *and* $s > 0$
        Then, $\sqrt{r} + \sqrt{s} = \ldots$ NO OBVIOUS WAY TO CONTINUE.

NEXT, *try an indirect proof:*
Assume $r \in \mathbb{R}^+$ *and* $s \in \mathbb{R}^+$.
    Assume $\sqrt{r} + \sqrt{s} = \sqrt{r+s}$.
        Then, $(\sqrt{r} + \sqrt{s})^2 = (\sqrt{r+s})^2$.   # *square both sides*
        Then, $(\sqrt{r})^2 + 2\sqrt{r}\sqrt{s} + (\sqrt{s})^2 = r + s$.   # *expand both sides*
        Then, $2\sqrt{rs} = 0$.   # *subtract* $r + s$ *from both sides*
        Then, $rs = 0$.   # *divide by 2 and square both sides*
        Then, $r = 0 \vee s = 0$.
        # NOW, DO A SUB-PROOF BY CASES.
        Assume $r = 0$.
            Then, $r \not> 0$.
            Then, $r \not> 0 \vee s \not> 0$.
            Then, $\neg(r > 0 \wedge s > 0)$.
        Assume $s = 0$.
            Then, $s \not> 0$.
            Then, $r \not> 0 \vee s \not> 0$.
            Then, $\neg(r > 0 \wedge s > 0)$.
        In either case, $\neg(r > 0 \wedge s > 0)$.
    Then, $r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.
Then, $\forall r \in \mathbb{R}^+, \forall s \in \mathbb{R}^+, r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.

2. For all real numbers x and y, $x^4 + x^3y - xy^3 - y^4 = 0$ exactly when $x = \pm y$.

   FIRST, *write the statement symbolically (be careful to handle that "±" correctly):*

   $$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow (x = y \lor x = -y)$$

   SECOND, *start the proof structure for the universal quantifiers:*

   Assume $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$.
     # TO PROVE AN EQUIVALENCE, WE PROVE THE IMPLICATION IN EACH DIRECTION.
     First assume $x^4 + x^3y - xy^3 - y^4 = 0$.
       Then, $x^3(x+y) - y^3(x+y) = 0$. # *factor out the expression*
       Then, $(x^3 - y^3)(x+y) = 0$. # *factor out the expression*
       Then, $x^3 - y^3 = 0 \lor x + y = 0$. # $ab = 0 \Leftrightarrow a = 0 \lor b = 0$
       # NOW, DO A SUB-PROOF BY CASES.
       Assume $x^3 - y^3 = 0$
         Then, $x^3 = y^3$ # *add* y^3 *to both sides*
         Then, $x = y$ # *take cube roots on both sides*
         Then, $x = y \lor x = -y$ # *introduce* $\lor$
       Assume $x + y = 0$
         Then, $x = -y$ # *subtract* y *from both sides*
         Then, $x = y \lor x = -y$ # *introduce* $\lor$
       In either case, $x = y \lor x = -y$.
     Then, $x^4 + x^3y - xy^3 - y^4 = 0 \Rightarrow x = \pm y$.
     Next assume $x = \pm y$.
       Then, $x = y \lor x = -y$. # *expand "±"*
       # NOW, DO A SUB-PROOF BY CASES.
       Assume $x = y$.
         Then, $x^3 = y^3$. # *cube both sides*
         Then, $x^3 - y^3 = 0$. # *subtract* y^3 *from both sides*
         Then, $(x^3 - y^3)(x+y) = 0$. # *multiply both sides by* $(x+y)$
         Then, $x^4 + x^3y - xy^3 - y^4 = 0$. # *expand*
       Assume $x = -y$.
         Then, $x + y = 0$. # *add* y *to both sides*
         Then, $(x^3 - y^3)(x+y) = 0$. # *multiply both sides by* $(x^3 - y^3)$
         Then, $x^4 + x^3y - xy^3 - y^4 = 0$. # *expand*
       In both cases, $x^4 + x^3y - xy^3 - y^4 = 0$.
     Then, $x = \pm y \Rightarrow x^4 + x^3y - xy^3 - y^4 = 0$.
     Then, $x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow x = \pm y$. # *introduce* $\Leftrightarrow$
   Then, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow (x = y \lor x = -y)$.

   Notice how the detailed proof structure makes it easy to keep track of assumptions, and cases and sub-cases, and to know exactly when we are done.