March 19th

Question 2 $P_2(\mathbb{R})$
Inner product $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$

$T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(p(x)) = p'(x)$
Find a basis for each of the four fundamental subspaces of T.
Four fundamental subspaces: $\ker(T)$, $\operatorname{Im}(T)$, $\ker(T^*)$, $\operatorname{Im}(T^*)$

From Q1. $\ker(T^*) = \operatorname{Im}(T)^\perp$ (1)
$\operatorname{Im}(T^*) = \ker(T)^\perp$ (2)

Find $\ker(T)$ and $\operatorname{Im}(T)$ first Then, use (1) and (2) to find $\ker(T^*)$ and $\operatorname{Im}(T^*)$

Suggested question

Let $N: \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $N = \begin{pmatrix} 6 & 2 & 1 & -1 \\ -7 & -1 & -1 & 2 \\ -9 & -7 & -2 & -1 \\ 13 & 3 & 2 & -3 \end{pmatrix}$

$N^2 = 0$
2 is the smallest k s.t. $N^k = 0$

$\ker(N^2) = \mathbb{R}^4$

Consider $e_1, e_2 \in \mathbb{R}^4$

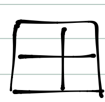
$Ne_1 = \begin{pmatrix} 6 \\ -7 \\ -9 \\ 13 \end{pmatrix} \neq 0$ $\quad N^2 e_1 = 0$

$Ne_2 = \begin{pmatrix} 2 \\ -1 \\ -7 \\ 3 \end{pmatrix} \neq 0$ $\quad N^2 e_2 = 0$

$\alpha_1 = (Ne_1, e_1)$ is a cycle of length 2
$\alpha_2 = (Ne_2, e_2)$ is a cycle of length 2
Since e_1 & e_2 are linearly indpt. α_1 & α_2 are <u>non-overlapping</u>

<u>cycles</u> $\Rightarrow \alpha_1 \cup \alpha_2 = \{Ne_1, e_1, Ne_2, e_2\}$ is linearly indpt

$\Rightarrow \alpha_1 \cup \alpha_2$ is a basis of $\mathbb{R}^4$

$\rightsquigarrow$ Cycle Tableau  $\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Cannonical form of N

Canonical basis is $\alpha_1 \cup \alpha_2$.

OVERALL PHILOSOPHY

F field

$N \in M_{n\times n}(F)$ a nilpotent matrix

Exercise: Find the canonical form of N, as well as a canonical basis

Lemma: Sps $x \in F^n$, $x \neq 0$

Sps also that $N^k x = 0$, but $N^{k-1}x \neq 0$, $k \geq 1$

Then $\{N^{k-1}x, N^{k-2}x, \cdots, Nx, x\}$ is linearly independent

We call this the cycle for N & x

It has length k.

The idea is to construct a basis of F^n built out of non-overlapping cycles.

$\alpha_1 = (N^{k_1-1}x_1, N^{k_1-2}x_1, \cdots Nx_1, x_1)$

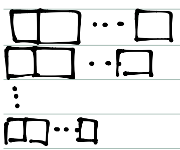
$\vdots$ $\qquad k_1 \geq k_2 \geq \cdots \geq k_m$

$\vdots$

$\alpha_m = (N^{k_m-1}x_m, N^{k_m-2}x_m, \cdots, Nx_m, x_m)$

Assume that $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m$ is a basis of F^n

Having found $\alpha_1, \cdots, \alpha_m$, we construct a cycle tableau



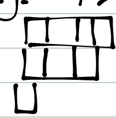
k_1 boxes

k_2 boxes

$\vdots$

k_m boxes

e.g. $4 \geq 3 \geq 1$



Canonical form

$$\begin{pmatrix} \left.\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}\right\} k_1 & & \\ & \left.\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}\right\} k_2 & \\ & & \left.\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}\right\} k_m \end{pmatrix}$$

HINT

Question 5 Create all cycle tableaux with 4 boxes.

Use a theorem from the book.

$$\dim(\ker(N^j)) - \dim(\ker(N^{j-1})) = \#\text{ of boxes in column of the cycle tableau}$$