FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 3

Question 1
Find the present value of 1000 due at the end of 10 years if
(a) $i^{(2)} = 0.09$, (b) $i^{(6)} = 0.09$, and (c) $i^{(12)} = 0.09$.

Solution

$$S(0) = S(t)(1+i)^{-t} = S(t)\left(1+\frac{i^{(m)}}{m}\right)^{-mt}$$

It is easiest to work in units of time relating to the compounding periods for each example. For example, for $i^{(2)}$, $m = 2$ and the effective half-yearly interest rate is $0.09/2 = 0.045$.

(a) Number of periods given unit of time is half-year $(m = 2)$ = 10 x 2 = 20

$1000 \cdot v_{0.045}^{20} = 1000(1.045)^{-20} = 414.64$

(b) Number of periods given $(m = 6)$ = 10 x 6 = 60. Effective interest rate corresponding to units of time of 1/6-years is 0.09/6=0.015.

$1000 \cdot v_{0.015}^{60} = 1000(1.015)^{-60} = 409.30$

(c) Number of periods given $(m = 12)$ = 10 x 12 = 120. Effective interest rate corresponding to units of time of 1 month is 0.09/12=0.0075.

$1000 \cdot v_{0.0075}^{120} = 1000(1.0075)^{-120} = 407.94$

Question 2
Mountain Bank pays interest at a nominal rate convertible half-yearly of $i^{(2)} = 0.15$. River Bank pays interest compounded daily. What minimum nominal annual rate convertible daily must River Bank pay in order to be as attractive as Mountain Bank ?

Solution
River Bank (RB) will be as attractive as Mountain Bank (MB) if an investment in RB accumulates to an amount equal to or greater than the same investment accumulated in MB. This is equivalent to finding $i^{(365)}$ so that the effective annual rate of interest for RB is greater than or equal to that of MB.

Accumulated value of an investment of X in MB

$$= X\left(1+\frac{i^{(2)}}{2}\right)^2 = X(1.075)^2 = X(1.155625)$$

Accumulated value of an investment of X in RB = $X\left(1+\frac{i^{(365)}}{365}\right)^{365}$

We want to solve for $i^{(365)}$ such that:

$$X\left(1+\frac{i^{(365)}}{365}\right)^{365} \geq X(1.155625)$$

Solving for $i^{(365)}$:

$$i^{(365)} \geq 365\cdot\left((1.155625)^{1/365}-1\right)$$

$$i^{(365)} \geq 0.144670$$

<u>Question 3</u>

Bank A has an effective annual rate of 18%. Bank B has a nominal annual rate of 17% convertible m times per year. What is the smallest whole number of times per year (m) that Bank B must compound its interest in order that the rate at Bank B be at least as attractive as that at Bank A on an effective annual basis? Repeat the exercise with a nominal rate of 16% per annum at Bank B.

<u>Solution</u>

Bank A has an effective rate of interest of $i = 18\%$.
Bank B has a nominal annual rate of interest of $i^{(m)} = 17\%$.
The number of compounding periods m is unknown.

From lectures we know that effective and nominal rates of interest are related by:

$$1+i = \left(1+\frac{i^{(m)}}{m}\right)^m$$

Therefore, we want to find the smallest m so that,

$$f(m) = \left(1+\frac{0.17}{m}\right)^m \geq 1.18$$

If $m=1$, $f(1)=1.17$
If $m=2$, $f(2)=1.1772$
If $m=3$, $f(3)=1.1798$
If $m=4$, $f(4)=1.1811$

Therefore, the smallest number of compounding periods per annum is 4 in order for Bank B to be at least as attractive as Bank A.

We now repeat the exercise with a nominal rate of 16% at Bank B, or $i^{(m)} = 16\%$
Therefore, we want to find the smallest m so that,

$$f(m) = \left(1 + \frac{0.16}{m}\right)^m \geq 1.18$$

If $m = 1$, $f(1) = 1.16$
If $m = 12$, $f(12) = 1.1723$
If $m = 52$, $f(52) = 1.1732$
It appears that no matter how many compounding periods per annum, we may not be able to achieve an accumulation of 1.18.

How can we check this?

We can check this by finding the limit as m approaches infinity of $f(m)$:

$$\lim_{m\to\infty} f(m) = \lim_{m\to\infty}\left(1 + \frac{0.16}{m}\right)^m = e^{0.16} = 1.1735$$

Therefore, no matter how many times per annum compounding takes place, a nominal rate of 16% cannot accumulate to an effective rate of greater than 17.35%.

nb: an exact (non-integer) solution for m requires numerical methods, not required for this course.

Question 4
Nominal interest can be defined even if m is not an integer. The algebraic definition $1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$ is still valid. Suppose a bank advertises a nominal rate of 10% per annum convertible every 45 days on short-term deposits. Find m and the equivalent effective annual rate of interest.

Solution
m is the number of compounding periods per year. Since we are dealing with a term of 45 days, the number of 45-day terms in a year is $m = \frac{365}{45} = 8.1111$.

If the nominal rate is 10% convertible every 45 days then $i^{(m)} = i^{(365/45)} = 10\%$.

The equivalent effective annual rate of interest can be found by $i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$:

$$i = \left(1 + \frac{0.10}{365/45}\right)^{365/45} - 1 = 0.104495.$$

Question 5 (a)

The nominal rate of interest $i^{(m)}$ can be defined for values of $m<1$. Algebraically the definition follows the relationship in the equation $1+i=\left(1+\dfrac{i^{(m)}}{m}\right)^m$

If $i=0.10$, find the equivalent $i^{(0.5)}$, $i^{(0.25)}$, $i^{(0.1)}$, and $i^{(0.01)}$. Rank the values in increasing size, and compare with the relationship $i^{(m)}<i$ for $m>1$.

Solution

$i^{(m)}=m\left[(1+i)^{1/m}-1\right]$

(i) compounding every 2 years

$i^{(0.5)}=0.5\left[(1+0.10)^{1/0.5}-1\right]=0.105$

(ii) compounding every 4 years

$i^{(0.25)}=0.25\left[(1+0.10)^{1/0.25}-1\right]=0.116025$

(iii) compounding every 10 years

$i^{(0.1)}=0.1\left[(1+0.10)^{1/0.1}-1\right]=0.159374$

(iv) compounding every 100 years

$i^{(0.01)}=0.01\left[(1+0.10)^{1/0.01}-1\right]=137.796$

Recall that when $m>1$ the equivalent effective annual rate of interest is greater than nominal rates: $i>i^{(2)}>i^{(3)}>...>\delta$.

When $m<1$ the equivalent effective annual rate of interest is less than nominal rates:
$i<i^{(0.5)}<i^{(0.25)}<i^{(0.1)}<i^{(0.01)}$

Question 5 (b)

Find the equivalent effective annual rate i if (i) $i^{(0.5)}=0.10$, (ii) $i^{(0.25)}=0.10$, (iii) $i^{(0.1)}=0.10$, and (iv) $i^{(0.01)}=0.10$.

Solution

$$i=\left(1+\frac{i^{(m)}}{m}\right)^m-1$$

(i) $i=\left(1+\dfrac{0.10}{0.5}\right)^{0.5}-1=0.0954$

(ii) $i=\left(1+\dfrac{0.10}{0.25}\right)^{0.25}-1=0.0878$

(iii) $i = \left(1+\frac{0.10}{0.1}\right)^{0.1} - 1 = 0.0718$

(iv) $i = \left(1+\frac{0.10}{0.01}\right)^{0.01} - 1 = 0.0243$

Question 6

If the effective rate of interest is 10% per annum, calculate (a) d and (b) $d^{(12)}$.

Solution

(a) $d = \frac{i}{1+i} = \frac{0.1}{1.1} = 0.090909$

(b) $d^{(12)} = 12\left(1-(1-d)^{1/12}\right) = 12\left(1-(1+i)^{1/12}\right) = 12\left(1-(1.1)^{1/12}\right) = 0.094933$

Question 7

Find the accumulated value of $100 at the end of two years if:

(a) the nominal annual rate of interest is 6% convertible quarterly.

(b) the nominal annual rate of discount is 4% convertible monthly.

(c) the nominal annual rate of discount is 6% convertible once every four years.

Solution

(a) $m = 4, t = 2$

$$S(t) = S(0)\left(1+\frac{i^{(m)}}{m}\right)^{tm} \Rightarrow S(2) = 100\left(1+\frac{0.06}{4}\right)^{8} = 112.6493$$

(b) $m = 12, t = 2$

$$S(t) = S(0)\left(1-\frac{d^{(m)}}{m}\right)^{-tm} \Rightarrow S(2) = 100\left(1-\frac{0.04}{12}\right)^{-24} = 108.3432$$

(c) $m = \frac{1}{4}, t = 2$

$$S(t) = S(0)\left(1-\frac{d^{(m)}}{m}\right)^{-tm} \Rightarrow S(2) = 100\left(1-\frac{0.06}{(1/4)}\right)^{-\frac{2}{4}} = 114.7079$$

nb: note the negative exponent when accumulating with discount rates (it's the reverse of finding the present value with discount rates). DON'T CONFUSE DISCOUNT RATES WITH DISCOUNT FACTORS!!)

Question 8

An investment of $1,000 accumulates to $1,360.86 at the end 'of 5 years. If the force of interest is δ during the first year and 1.5δ in each subsequent year, find the equivalent effective annual interest rate in the second year.

Solution

The accumulated value of \$1,000 after 1 year at a force of interest of δ is $1000e^{\delta}$.
The accumulated value of this amount after an additional 4 years at a force of interest of 1.5δ is:

$$1000e^{\delta}e^{1.5\delta}e^{1.5\delta}e^{1.5\delta}e^{1.5\delta}=1000e^{\delta}e^{4(1.5\delta)}=1000e^{7\delta}$$

Therefore,

$$1000e^{7\delta}=1360.86$$

Solving for δ:

$$\delta=\frac{1}{7}\ln\left(\frac{1360.86}{1000}\right)=0.044017$$

The effective annual interest rate in the first year is:

$$e^{\delta}=(1+i)\Rightarrow i=e^{\delta}-1=4.5\%$$

The effective annual interest rate in the second year is:

$$e^{1.5\delta}=(1+i)\Rightarrow i=e^{1.5\delta}-1=6.8253\%$$

Question 9

Smith forecasts that interest rates will rise over a 5-year period according to a force of interest function given by $\delta_t=0.08+\dfrac{0.025t}{t+1}$ for $0\le t\le 5$.

(a) According to this scheme, what is the average annual compound effective rate for the 5-year period?
(b) What is the present value at t=2 of \$1,000 due at t=4?

Hint: $\int\left(\frac{t}{t+1}\right)dt=t-\ln(t+1)$ (Why? Write $\frac{t}{t+1}=1-\frac{1}{t+1}$)

Solution

(a) From lectures we know that the accumulation at time n of an amount 1 is given by:

$$S(n)=\exp\left(\int_0^n\delta_t dt\right)$$

Under compound interest at an annual effective rate of i this is also equal to $(1+i)^n$.

Therefore, for a 5-year period:

$$(1+i)^5=\exp\left(\int_0^5\delta_t dt\right)=\exp\left(\int_0^5\left(0.08+\frac{0.025t}{t+1}\right)dt\right)$$

Evaluate the integral. Note that:

$$\int\left(\frac{t}{t+1}\right)dt=t-\ln(t+1)$$

Therefore,

$$\int_0^5 \left(0.08+\frac{0.025t}{t+1}\right)dt = \left(0.08t+0.025(t)-0.025\ln(t+1)\right)\Big]_0^5$$
$$= 0.08(5)+0.025(5)-0.025\ln(6)$$
$$= 0.480206$$

$$\Rightarrow \exp\left(\int_0^5 \left(0.08+\frac{0.025t}{t+1}\right)dt\right) = 1.616407$$
$$\Rightarrow i = 1.616407^{1/5} - 1 = 0.1008$$

(b) The present value at time 0 of an amount $S(n)$ due at time n is:

$$S(0) = S(n)\cdot\exp\left(-\int_0^n \delta_t dt\right)$$

The present value at time 2 of $1,000 due at time 4 is :

$$S(2) = 1000\cdot\exp\left(-\int_2^4 \left(0.08+\frac{0.025t}{t+1}\right)dt\right)$$
$$= 1000\cdot\exp\left[-0.08(4)-0.025\left(4-\ln(5)\right)+0.08(2)+0.025\left(2-\ln(3)\right)\right] = 821$$

Question 10

The present value of K payable after 2 years is $960. If the force of interest is cut in half the present value becomes $1,200. Find K.

What is the present value if the **equivalent** effective annual discount rate is cut in half?

Solution

The present value of K payable after 2 years is:

$$960 = Ke^{-2\delta}$$

The present value of K payable after 2 years if the force of interest is cut in half is:

$$1200 = Ke^{-2\frac{\delta}{2}} = Ke^{-\delta}$$

$$\Rightarrow e^{-\delta} = \frac{1200}{K} \Rightarrow e^{-2\delta} = \left(\frac{1200}{K}\right)^2$$

$$960 = Ke^{-2\delta} = K\left(\frac{1200}{K}\right)^2 \Rightarrow K = \frac{1200^2}{960} = 1500$$

In terms of the effective discount rate, we can write: $e^{-\delta} = (1-d)$.

When K=1500,

$$e^{-\delta} = (1-d) = \frac{1200}{1500} = 0.8 \Rightarrow d = 0.2$$

Therefore, if we halve the discount rate $(d = 0.1)$, the PV of 1500 payable after 2 years is:

$$PV = K\left(1 - \frac{d}{2}\right)^2 = 1500(1-0.1)^2 = 1215$$

nb: note the difference in relative impact on the PV between halving the force of interest and halving the discount rate.

<u>Past Exam Question – 2005 Final Exam Q1</u>

(a) Find the accumulated value of $100 at the end of two years if the nominal annual rate of interest is 6% per annum convertible weekly. (1 mark)

(c) An investment is made at a compound force of interest of 8% p.a. for a period of 5 years. Find the equivalent rate of ***simple interest*** per annum. (2 marks)

<u>Solution</u>

a) Accumulated Value = $100\left(1 + \frac{0.06}{52}\right)^{2(52)} = \$112.74$

c)

$$(1+it) = e^{\delta t}$$

$$(1+5i) = e^{0.4}$$

$$i = \frac{e^{0.4} - 1}{5} = 9.84\%$$