SCHOOL OF FINANCE AND APPLIED STATISTICS

FINANCIAL MATHEMATICS
(STAT 2032 / STAT 6046)

TUTORIAL SOLUTIONS WEEK 9

Question 1

You invest $75,000 on 30 April 2005 on a $100,000 nominal bond redeemable at par on 31 December 2020. Coupons are paid annually at 5% p.a., with the next coupon due on 30 June 2005 (the last coupon payment is on 30 June 2020). Find the net effective redemption yield if tax on income and capital gains is 25%.

Solution

We need to set up an equation of value to solve this problem. Note that there are 16 coupons which are paid annually (not half-yearly) and the bond is not purchased on a coupon date.

$$75,000 = 100,000 \times 0.05 \times (1-0.25)\ddot{a}_{\overline{16}|i} v_i^{\frac{2}{12}} + \left(100,000 - 0.25(100,000 - 75,000)\right) v_i^{15\frac{8}{12}}$$

$$75,000 = 3,750 v_i^{\frac{2}{12}} \left(\frac{1-v^{16}}{d}\right) + 93,750 v_i^{15\frac{8}{12}}$$

Test 7% p.a. RHS = $69,960.35
Test 6% p.a. RHS = $77,410.84

Using linear interpolation:

$$i \cong 7\% + \frac{75,000 - 69,960.35}{77,410.84 - 69,960.35}(6\% - 7\%) = 6.3\%$$

Question 2

Calculate the net effective yield on the bond in Question 1, if it is redeemable at par on 31 December 2015 and the last coupon payment is on 30 June 2015. Why has the net yield increased/decreased?

Solution

Using the same approach as above:

$$75,000 = 3,750 v_i^{\frac{2}{12}} \left(\frac{1-v^{11}}{d}\right) + 93,750 v_i^{10\frac{8}{12}}$$

Test 7% p.a. RHS = \$75,306.92
Test 8% p.a. RHS = \$69,796.90

Using linear interpolation:

$$i \cong 7\% + \frac{75,000 - 75,306.92}{69,796.90 - 75,306.92}(8\% - 7\%) = 7.1\%$$

The net yield has increased. This is because the bond is making a capital gain and thus a higher yield is achieved the earlier that capital gain is realised.

<u>Question 3</u>

A pension fund is about to purchase a fixed-interest stock, which has an optional redemption date. The fund's actuary wishes to calculate an appropriate price for the stock using a specified interest rate, but based on conservative assumptions. The following statements have been made:

I If the calculated price is less than the total redemption payment for the entire range of possible redemption dates, the price will not depend on the assumed redemption date.

II If the price is always greater than the total redemption payment, the earliest possible redemption date should be assumed.

Which of these assertions are necessarily true?

<u>Solution</u>

I is false. The price will depend on the assumed redemption date, unless the effective valuation rate and the effective coupon rate happen to be equal (or, equivalently if $j = r$), in which case the price is equal to the face value (assuming that the redemption amount equals the face value, or $C = F$).

II is true. If the purchase price is greater than the redemption amount, then the effective coupon rate must be higher than the yield obtained (or, equivalently $j < r$), and the minimum yield for the investor (and maximum yield for the bond issuer) occurs at the earliest possible redemption date.

Question 4

A 10% p.a. coupon payable half yearly bond with face amount 100 is callable on any coupon date from 15.5 years after issue up to the maturity date which is 20 years from issue.

(a) Find the price of the bond to yield a minimum nominal annual rate convertible half-yearly of (i) 12%, (ii) 10%, and (iii) 8%.

(b) Find the minimum annual nominal yield (convertible half-yearly) to maturity if the bond is purchased for (i) 80, (ii) 100, and (iii) 120.

Solution

In earlier lectures we showed that when a bond can be redeemed at optional redemption dates, the minimum yield occurs:

- at the latest date if the bond is bought at a discount.
- at the earliest date if the bond is bought at a premium.

(a) Since the redemption amount isn't provided, we assume that redemption is at par.

(i) A nominal yield of 12% convertible half-yearly corresponds with $j = 6\%$. Since $r = 5\%$, and $j > r$, it follows that $P < F$ and the bond is bought at a discount. Therefore, the minimum price paid is based on the latest date of 20 years (40 coupon periods) for the valuation.

$$5a_{\overline{40}|0.06} + 100v_{0.06}^{40} = 84.95$$

(ii) A nominal yield of 10% corresponds with $j = 5\%$. Since $j = r$, it follows that $P = F = 100$

(iii) A nominal yield of 8% corresponds with $j = 4\%$. Since $j < r$, it follows that $P > F$ and the bond is bought at a premium. Therefore, choose the earliest date of 15.5 years (31 coupon periods) for the valuation.

$$5a_{\overline{31}|0.04} + 100v_{0.04}^{31} = 117.59$$

(b)

(i) A price of 80 means that the bond is bought at a discount. Therefore, the minimum yield occurs at the latest redemption date of 20 years.

$$5a_{\overline{40}|j} + 100v_j^{40} = 80 \Rightarrow i^{(2)} = 0.127923$$

(I used the Goal Seek tool in Excel to find the yield. In an exam you would need to use interpolation or an alternative method).

(ii) A price of 100 means that $i^{(2)} = 0.10$. (Recall that $P = F \Leftrightarrow j = r$)

(iii) A price of 120 means that the bond is bought at a premium. Therefore, the minimum yield occurs at the earliest redemption date of 15.5 years:

$$5a_{\overline{31}|j} + 100v_j^{31} = 120 \Rightarrow i^{(2)} = 0.077597$$

Question 5

A bond of nominal amount $200,000 is redeemable at par in 4 equal annual installments of $50,000, the first being in 11 years' time.

Coupons of 7.5% per annum are payable half yearly in arrears. The coupons are payable on the nominal amount outstanding at the time of coupon payment. For example, the coupons payable during year 12 after the first redemption payment of $50,000 are based on the nominal amount outstanding at that time of $150,000.

All investors require a net effective yield of 9% per annum.

(a)

Calculate the price P_1 for the whole bond for an investor.

Solution

The value of the capital repayments is:

$$PV_{cap} = 50,000 v_{0.09}^{10} a_{\overline{4}|0.09} = \$68,424$$

The value of the interest payments is:

$$PV_{\text{int}} = \left(\frac{0.075}{2}\right)\left(200,000 a_{\overline{22}|j} + 150,000 v^{22} a_{\overline{2}|j} + 100,000 v^{24} a_{\overline{2}|j} + 50,000 v^{26} a_{\overline{2}|j}\right)$$

where $j = (1.09)^{1/2} - 1$

$$\Rightarrow PV_{\text{int}} = \$112,060$$

The price for the loan is:

$$P_1 = PV_{\text{int}} + PV_{cap} = \$112,060 + \$68,424 = \$180,484$$

(b)

Calculate the revised price P_2 for an investor if the final installment is redeemable at a price of 1.2 per unit nominal.

Solution

The price P_2 can be found by adding on the value of the premium (ie. the additional amount) payable at redemption:

$$\begin{aligned} P_2 &= P_1 + 0.20(50,000)v^{14} \\ &= 180,484 + 0.20(50,000)v^{14} = \$183,476 \end{aligned}$$

(c)
Calculate the revised price P_3 for an investor if coupon payments are made at a rate of 9% for the first 3 years (and 7.5% thereafter).

Solution
The price P_3 can be found by adding on the value of the extra coupon payments. The original coupon payments were $\frac{0.075}{2}$ multiplied by the nominal amount per coupon payment. The new coupon payments for the first three years are $\frac{0.09}{2}$, so the extra coupon payments for the first six coupon payments are $\frac{0.09-0.075}{2}=\frac{0.015}{2}$

The present value of the extra coupon payments is:

$$\left(\frac{0.015}{2}\right)(200,000)a_{\overline{6}|j}$$

where $j=(1.09)^{1/2}-1$

The new price is, therefore:

$$P_3=P_1+\left(\frac{0.015}{2}\right)(200,000)a_{\overline{6}|j}=180,484+7,761=\$188,245$$

Question 6
On June 15, 1990 a corporation issues an 8% bond with a face value of $1,000,000. The bond can be redeemed, at the option of the corporation, on any coupon date in 2001 or 2002 at par, on any coupon date in 2003 through 2005 for amount $1,200,000, or on any coupon date in 2006 through June 15, 2008 at redemption amount $1,300,000. Coupons are payable half-yearly.

(a) Find the price to yield a minimum nominal annual rate convertible half-yearly of (i) 10% and (ii) 6.5%

(b) Find the minimum nominal annual yield convertible half-yearly if the bond is bought for (i) $800,000, (ii) $1,000,000, or (iii) $1,200,000.

Solution
In earlier lectures we showed that in the case where the redemption amount is not equal to the face value it is useful to define a modified coupon rate g where $g=\frac{Fr}{C}$. In this situation the following results hold:

- If $P = C \iff j = g$
- If $P > C \iff j < g$
- If $P < C \iff j > g$

In the lecture notes we also showed that when a bond can be redeemed at optional redemption dates, the minimum yield occurs:

- at the latest date if the bond is bought at a discount.
- at the earliest date if the bond is bought at a premium.

In this question:

$g = 0.04$ for 2001 and 2002 (redeemed at par)

$g = \frac{Fr}{C} = \frac{1,000,000(0.04)}{1,200,000} = 0.0333$ for 2003-2005

$g = \frac{Fr}{C} = \frac{1,000,000(0.04)}{1,300,000} = 0.0308$ for 2006-2008

Coupons are assumed to be paid half-yearly on June 15 and on December 15.

The equation for the bond is:

$$P = 1,000,000\left(\frac{0.08}{2}\right)a_{\overline{n}|j} + Cv_j^n = 40,000a_{\overline{n}|j} + Cv_j^n$$

For redemption during 2001 and 2002: $P = 40,000a_{\overline{n}|j} + 1,000,000v_j^n$

For redemption during 2003, 2004 and 2005: $P = 40,000a_{\overline{n}|j} + 1,200,000v_j^n$

For redemption during 2006, 2007 and 2008: $P = 40,000a_{\overline{n}|j} + 1,300,000v_j^n$

(a) (i) minimum yield of $i^{(2)} = 10\% \Rightarrow j = 0.05$. In this case $j > g$ for all redemption dates. $j > g \Leftrightarrow P < C$, so the bond is bought at a discount. The minimum yield occurs at the latest date.

For redemption during 2001 and 2002, we should price at December 15, 2002

$$n = 25 \Rightarrow P = 40,000a_{\overline{25}|0.05} + 1,000,000v_{0.05}^{25} = 859,056$$

For redemption between 2003 and 2005, we should price at December 15, 2005

$$n = 31 \Rightarrow P = 40,000a_{\overline{31}|0.05} + 1,200,000v_{0.05}^{31} = 888,144$$

For redemption between 2006 and 2008, we should price at June 15, 2008

$$n = 36 \Rightarrow P = 40,000a_{\overline{36|}0.05} + 1,300,000v_{0.05}^{36} = 886,334$$

The minimum price paid (overall) should be \$859,056. If redemption occurs at any date during 2001 through to June 15, 2008, this price will guarantee a nominal yield of at least 10% convertible half-yearly.

(ii) minimum yield of $i^{(2)} = 6.5\% \Rightarrow j = 0.0325$.

In this case $j < g$ for 2001-2002 and 2003-2005. For redemption during 2001-2002 $j < g \Leftrightarrow P > C$, so the bond is bought at a premium to achieve this yield, and we should price at June 15, 2001 (earliest date):

$$n = 22 \Rightarrow P = 40,000a_{\overline{22|}0.0325} + 1,000,000v_{0.0325}^{22} = 1,116,588$$

For 2003-2005, we should price at June 15, 2003 (earliest date)

$$n = 26 \Rightarrow P = 40,000a_{\overline{26|}0.0325} + 1,200,000v_{0.0325}^{26} = 1,217,373$$

$j > g$ for 2006-2008. For redemption during 2006-2008 $j > g \Leftrightarrow P < C$, so the bond is bought at a discount to achieve this yield, and we should price at June 15, 2008 (latest date):

$$n = 36 \Rightarrow P = 40,000a_{\overline{36|}0.0325} + 1,300,000v_{0.0325}^{36} = 1,252,660$$

The minimum price paid (overall) should be \$1,116,588. If redemption occurs at any date during 2001 through to June 15, 2008, this price will guarantee a nominal yield of at least 6.5% convertible half-yearly.

(b)
Note that in these examples the Goal Seek tool in Excel has been used to find the yield. In an exam you would need to use interpolation or an alternative method.

(i) The bond is bought for 800,000 which is at a discount to any redemption. The minimum yield occurs at the latest date in each interval.

For 2001-2002, we should find the yield at December 15, 2002 (latest date)

$$j > 0.04,\ n = 25 \Rightarrow 800,000 = 40,000a_{\overline{25|}j} + 1,000,000v_j^{25} \Rightarrow i^{(2)} = 0.1098$$

For 2003-2005, we should find the yield at December 15, 2005 (latest date)

$$j > 0.0333,\ n = 31 \Rightarrow 800,000 = 40,000a_{\overline{31|}j} + 1,200,000v_j^{31} \Rightarrow i^{(2)} = 0.1126$$

For 2006-2008, we should find the yield at June 15, 2008 (latest date)

$$j > 0.0308,\ n = 36 \Rightarrow 800,000 = 40,000a_{\overline{36|}j} + 1,300,000v_j^{36} \Rightarrow i^{(2)} = 0.1115$$

Therefore, the minimum annual yield convertible half-yearly if the bond is bought for $800,000 is 10.98% p.a. and occurs when the bond is redeemed at December 15, 2002.

(ii) The bond is bought for 1,000,000. This is at par for 2001-2002, and at discount for the other redemption intervals.

For 2001-2002, $P = C \Leftrightarrow j = 0.04 \Rightarrow i^{(2)} = 0.08$

For 2003-2005, we should find the yield at December 15, 2005 (latest date)

$$j > 0.0333,\ n = 31 \Rightarrow 1,000,000 = 40,000 a_{\overline{31}|j} + 1,200,000 v_j^{31} \Rightarrow i^{(2)} = 0.0864$$

For 2006-2008, we should find the yield at June 15, 2008 (latest date)

$$j > 0.0308,\ n = 36 \Rightarrow 1,000,000 = 40,000 a_{\overline{36}|j} + 1,300,000 v_j^{36} \Rightarrow i^{(2)} = 0.0872$$

Therefore, the minimum annual yield convertible half-yearly if the bond is bought for $1,000,000 is 8% p.a. and occurs when the bond is redeemed at any coupon date in 2001 or 2002.

(iii) The bond is bought at 1,200,000. This is at a premium for 2001-2002, at par for 2003-2005 and at a discount for 2006-2008.

For 2001-2002, the minimum yield corresponds with redemption at June 15, 2001 (earliest date)

$$j < 0.04,\ n = 22 \Rightarrow 1,200,000 = 40,000 a_{\overline{22}|j} + 1,000,000 v_j^{22} \Rightarrow i^{(2)} = 0.0555$$

For 2003-2005, $P = C \Leftrightarrow j = 0.0333 \Rightarrow i^{(2)} = 0.0667$

For 2006-2008, we should find the yield at June 15, 2008 (latest date)

$$j > 0.0308,\ n = 36 \Rightarrow 1,200,000 = 40,000 a_{\overline{36}|j} + 1,300,000 v_j^{36} \Rightarrow i^{(2)} = 0.0691$$

Therefore, the minimum annual yield convertible half-yearly if the bond is bought for $1,200,000 is 5.55% p.a. and occurs when the bond is redeemed at June 15, 2001.

Question 7

An inflation linked bond is purchased on 30 April 1994 for $900. It has a face value of $1,000 and pays half yearly coupons indexed to inflation of 5% p.a. It is redeemable at an indexed value to par at 30 April 2018. Calculate the gross redemption yield assuming a constant rate of inflation of 3% p.a.

Solution
Using the formula form the lecture notes, we can set up an equation to solve for the real rate of interest using linear interpolation:

$P = Fra_{\overline{n|}i'} + Cv_{i'}^{n}$ where *i'* is the effective half-yearly real interest rate

$900 = 25a_{\overline{48|}i'} + 1000 \cdot v_{i'}^{48}$

Test $\begin{aligned} i' &= 0.03 \Rightarrow LHS = 873.67 \\ i' &= 0.025 \Rightarrow LHS = 1000 \end{aligned}$

Therefore, using linear interpolation:

$$i' \cong 3\% + \frac{900 - 873.67}{1000 - 873.67}(2.5\% - 3\%) = 2.90\%$$

The effective yearly real interest rate is $1.029^2 - 1 = 5.88\%$

From the lecture notes we know that

$i' = \dfrac{i - r}{1 + r} \Rightarrow 0.0588 = \dfrac{i - 0.03}{1.03}$

$i = 9.1\%$

Thus the gross redemption yield is 9.1% p.a.

Past Exam Question – 2005 Final Exam Q4
You are currently running a small business from a property you own. You decide to mortgage the property in order to finance further expansion of the business. The terms of the mortgage are as follows:

- You receive a lump sum payment of $*X* on 15 July 2004.
- You must make payments of $80,000 every 6 months, the first payment being on 15 January 2005.
- The institution holding the mortgage may demand a closing payment of $1,000,000 on any regular payment date from 15 January 2015 – 15 January 2020 (inclusive). This is paid in addition to the regular $80,000 on that date. All regular payments cease from the date the mortgage is closed.

The maximum interest rate at which you wish to take out the mortgage is 9.0% p.a. convertible half-yearly.

a) Using the techniques of valuing a callable bond, calculate *X*, the initial lump sum payment which you will receive. (5 marks)

b) The mortgage holder decides to close out the mortgage on 15 January 2017. Assuming you received the lump-sum payment as per your answer to a), and using linear interpolation, calculate the annual interest rate, convertible half-yearly, that is effective on the mortgage (to one decimal place). (5 marks)

<u>Solution</u>

a) The first thing to do is work out which date the mortgage will be closed to ensure a maximum interest rate of 9.0% convertible half-yearly (4.5% per half year).

Coupon Rate $r = \dfrac{80,000}{1,000,000} = 8\%$ per half-year

Interest Rate $j = \dfrac{i^{(2)}}{2} = \dfrac{0.09}{2} = 4.5\%$ per half-year

$\therefore r > j$ and "bond" is bought at a premium (ie. $X > \$1,000,000$)

The borrower wishes to set a maximum interest rate (or maximum yield), so X should be based on latest closing date (ie. 15 January 2020). This gives $n = 31$ as the number of half-years till the closing date.

$$X = 80,000 a_{\overline{31}|0.045} + 1,000,000 v_{0.045}^{31}$$
$$= \$1,579,053.68$$

Alternatively, you may calculate the price at 15 January 2015 and 15 January 2020 and find the highest value of X.

b) If the mortgage is closed on 15 January 2017, $n = 25$ is the number of half-years. We need to find j to solve the equation:

$$1,579,053.68 = 80,000 a_{\overline{25}|j} + 1,000,000 v_j^{25}$$

$$\text{Test} \quad \begin{aligned} j = 4.5\% &\Rightarrow LHS = 1,518,987.31 \\ j = 4.0\% &\Rightarrow LHS = 1,624,883.20 \end{aligned}$$

$$\frac{1,579,053.68 - 1,518,987.31}{1,624,883.20 - 1,518,987.31} \cong \frac{j - 4.5\%}{4.0\% - 4.5\%}$$
$$j = 4.2164\%$$
$$i^{(2)} = 2j = 8.4\%$$

As expected, when the mortgage is closed out earlier than 15 January 2020, the interest rate on the mortgage is less than 9.0% convertible half-yearly.

Past Exam Question – 2005 Final Exam Q6(a)
You inherit an empty block of land from a distant relative and decide to build a house on it with a view to living in it in the future. The house you wish to build costs $300,000 and is expected to be completed in two years from now. You are required to pay the $300,000 on the completion of the house.

In the meantime you wish to set aside an amount of money to ensure you will have enough to pay for the house in two years time. You are considering investing in inflation-linked bonds with the following characteristics:

- Term = 3 years
- Indexed Coupon rate = 8% p.a.
- Coupon Frequency = Annual in arrears
- Redemption is at the indexed amount of the initial par value

Assuming an interest rate of 6% p.a. and an expected future inflation rate of 3% p.a.:

a) Show that the initial face value of the indexed linked bond that will need to be purchased to ensure that the present value of the inflation linked bond is equal to the present value of the house payment is $233,361.25. (4 marks)

Solution
a) $PV_L = 300,000(1.06^{-2}) = 266,998.93$

$$PV_A = 0.08F\left(\frac{1.03}{1.06} + \left(\frac{1.03}{1.06}\right)^2 + \left(\frac{1.03}{1.06}\right)^3\right) + F\left(\frac{1.03}{1.06}\right)^3$$

$$= 0.2266696F + 0.9174747F = 1.144144F$$

$$266,998.93 = 1.144144F \Rightarrow F = \$233,361.25$$