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Suggested problems

Section 2.1 #4 Linearily

Prove that if $(a_1, \dots, a_n)$ is a fixed vector in $\mathbb{R}^n$, then $T: \mathbb{R}^n \to \mathbb{R}$ defined by $T(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ $\quad (x = (x_1, \dots, x_n))$ is a linear transformation.

V, W vectors spaces over a field F Use this instead to test linearity.

$T: V \to W$ function

T is a linear transformation $\iff T(cv_1 + v_2) = cT(v_1) + T(v_2)$ $\forall c \in F, \forall v_1, v_2 \in V$

$x, y \in \mathbb{R}^n$, $c \in \mathbb{R}$

$$\begin{aligned} T(cx+y) &= T(c(x_1, \dots, x_n) + (y_1, \dots, y_n)) = T(cx_1+y_1, cx_2+y_2, \dots, cx_n+y_n) \\ &= a_1(cx_1+y_1) + a_2(cx_2+y_2) + \cdots + a_n(cx_n+y_n) \\ &= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + (a_1y_1 + a_2y_2 + \cdots + a_ny_n) \\ &= cT(x) + T(y) \end{aligned}$$

$\blacksquare$

§5.1 #8

Prove that the set of real numbers of the form $a + b\sqrt{2}$, where a and b are natural numbers, is a field with the usual operations of addition and multiplication of real numbers.

i.e.

Prove $A = \{a + b\sqrt{2} \in \mathbb{R}, a, b \in \mathbb{Q}\}$ is a field.

(i). Commutativity of addition : $a + b = b + a$

(ii). Associativity of addition : $(a+b)+c = a+(b+c)$

(iii) Existence of an additive identity

$0 = \underset{\mathbb{Q}}{0} + \underset{\mathbb{Q}}{0}\sqrt{2}$

Therefore, $0 \in A$

Since $x + 0 = x$ $\forall x \in \mathbb{R}$, $a + 0 = a$ $\forall a \in A$

(iv). Existence of additive inverses:

$x \in A$

$x = a + b\sqrt{2}$, $a, b \in \mathbb{Q}$

Note that $(-a) + (-b)\sqrt{2} \in A$

$\cdots + \cdots = 0$

(v). Commutativity of multiplication: $ab = ba$

(vi). Associativity of multiplication $(ab)c = a(bc)$

(vii) Distributivity $(a+b)c = ac + bc$ & $a(b+c) = ab + ac$

(viii). Existence of multiplicative identity :

$1 = 1 + 0\sqrt{2} \in A$

(ix). Existence of multiplicative inverse :

$x \in A$, $x \neq 0$

$x = a + b\sqrt{2}$, $a, b \in \mathbb{Q}$

What is x^{-1} in $\mathbb{R}$?

(in $\mathbb{R}$) $x^{-1} = \frac{1}{a+\sqrt{2}b}$

$$\overset{\text{justify}}{=} \frac{a-b\sqrt{2}}{(a+\sqrt{2}b)(a-\sqrt{2}b)} = \frac{a}{a^2-2b^2} + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2}$$

mult by $\frac{a-b\sqrt{2}}{a-b\sqrt{2}}$

We need to verify that $a-b\sqrt{2} \neq 0$

Assume that $a-b\sqrt{2} = 0$

2 Cases: (i) $b=0$

$\Rightarrow a=0$

$\Rightarrow x = a+b\sqrt{2} = 0$, a contradiction

ii). $b \neq 0$

$a-b\sqrt{2} = 0 \Rightarrow \sqrt{2} = \frac{a}{b} \in \mathbb{Q}$

contradiction. since $\sqrt{2}$ is irrational.

Therefore $a-\sqrt{2}b \neq 0$