Summary

Methods of 1st order ODE's

Name	Equation	Solution Method
Linear	$y'+p(t)y=g(t)$	integrating factors
Separable	$y'=\alpha(t)\beta(y)$	separation of variables $\frac{1}{\beta(y)}dy=\alpha(t)\,dt$
Exact	$M(t,y)dt+N(t,y)dy=0$ with $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$	find $\psi(t,y)$ with $\frac{\partial\psi}{\partial t}=M$, $\frac{\partial\psi}{\partial y}=N$; $\psi(t,y)=C$ is solution
Homogeneous	$y'=F(\frac{y}{t})$	substitute $v=\frac{y}{t}$, get separable equation $\frac{dv}{dt}=\dots$

Examples — Which method applies?

(a) $\cancel{\frac{dy}{dt}=\frac{ty+t}{y^2-ty^2}}$ (a) $\cancel{\frac{dy}{dt}=\frac{t(y+1)}{y^2(1-t)}}$

(b) $\frac{dy}{dt}=\frac{t-y+2ty}{t}$ (b) $\frac{dy}{dt}=1-\frac{1}{t}y+2y=1+(2-\frac{1}{t})y$ linear

(c) $\frac{dy}{dt}=\tan(y)$ (c) separable

(d) $\frac{dy}{dt}=\frac{3t^2+2y}{y-2t-3}$ (d) $(y-2t-3)dy+(-3t^2-2y)dt=0$ $\frac{d}{dt}(\quad)=-2$ $\frac{d}{dy}(\quad)=-2$ Exact

(e) $e^{-y}\frac{dt}{dy}=1+e^{-t}-e^{-y}-e^{t-y}$ (e) $e^{-y}\frac{dy}{dt}=(1+e^{t})(1-e^{-y})$ separable

(f) $ty^2y'=t^3+y^3$ (f) $y'=(\frac{t}{y})^2+(\frac{y}{t})=(\frac{y}{t})^{-2}+(\frac{y}{t})$

Existence and uniqueness

Consider the initial value problem

$$y'=f(t,y) \qquad y(t_0)=y_0 \qquad (*)$$

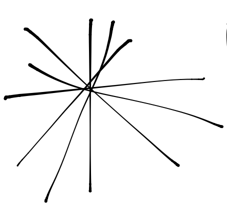
(given t_0, y_0)

Thm: Suppose $f, \frac{\partial f}{\partial y}$ are continuous near (t_0,y_0). Then IVP $(*)$ admits a unique solution $y=y(t)$ for t in some open interval around t_0.

I.e. Solution exists and is unique.

Example: $ty'=y$, $y(0)=0$.

has solutions $y=mt$, for any $m\in\mathbb{R}$.



But this doesn't contradict theorem: $y'=(\frac{y}{t})$

here f not continuous near (t_0,y_0)

Example: $y' = \underbrace{2ty^2}_{f(t,y)}$, $y(0) = 1$

Theorem applies, and guarantees unique solution $y(t)$, for t in some interval around 0.

Let's solve the equation:

$$\frac{1}{y^2}\,dy = 2t\,dt \quad \int$$

$$-\frac{1}{y} = t^2 + C$$

$C = -1$ (initial conditions)

So, $-\frac{1}{y} = t^2 - 1 \Rightarrow y = \frac{1}{1-t^2}$ defined for $-1 < t < 1$

Thus: Even if f, $\frac{\partial f}{\partial y}$ are continuous everywhere, the solution may go to infinity in finite time.