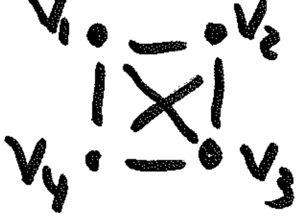
Problem Set 2 Solutions

Question 1 Solution

a) Sum of outdegrees must be 6 by the handshake lemma, and there can at most be one source and sink.
Asign outdegrees (d_1, d_2, d_3, d_4) to



in all possible ways. (actually for 4 vertices there is only one way).
Because any permutation of $\{v_1, v_2, v_3, v_4\}$ induces an automorphism of G, this is everything:

$(3,2,1,0)$:

$(3,1,1,1)$:

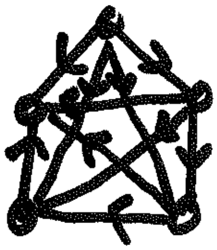
(two orientations of $v_2 v_3 v_4 v_2$ are isomorphic)

$(2,2,2,0)$:

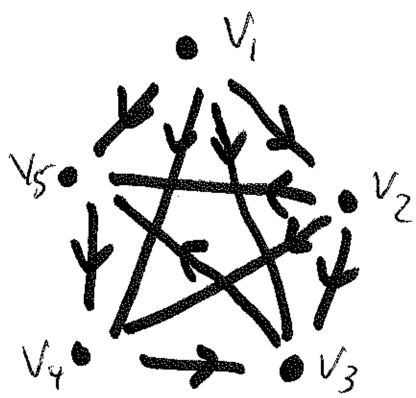
$(2,2,1,1)$:

(b) Same strategy as part (a)

(4, 3, 2, 1, 0):

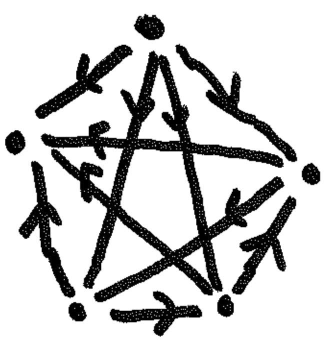


(4, 3, 1, 1, 1):



(2 orientations of the triangle $V_3 V_4 V_5 V_3$ are isomorphic).

(4, 2, 2, 2, 0)



(2 orientations of the triangle $V_2 V_3 V_4 V_2$ are isomorphic).

(4, 2, 2, 1, 1):

unique way up to isomorphism to glue in the (2, 2, 1, 1) tournament ~~graph~~ from part (a) to complete graph on V_2, V_3, V_4, V_5

(3, 3, 3, 1, 0)

(orientations of the triangle $V_1 V_2 V_3 V_1$ are isomorphic

(b) (continued)

$(3,3,2,2,0)$

(unique up to iso. way to glue in $(2,2,1,1)$ tournament to tournament on v_1, v_2, v_3, v_4)

$(2,2,2,2,2)$

$(3,3,2,1,1)$:

Edge $v_1 \to v_3$:

distinguished by whether or not $v_1 v_2 v_3 v_1$ is a directed cycle

Edge $v_3 \to v_1$:

(orientations of triangle v_3, v_4, v_5 isomorphic)

$(3,2,2,2,1)$

$v_5 v_1 v_2 v_5$ directed triangle

(orientations of $v_2 v_3 v_4 v_2$ are isomorphic)

$v_5 v_4 v_2 v_3 v_5$ directed cycle

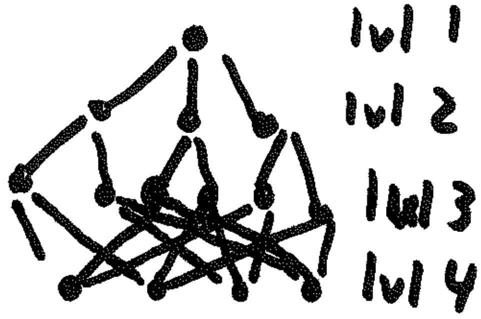
neither

Question 2 Solution

a) Vertex $v_0 \in V(G)$ has k neighbours, $v_1, \dots, v_k$. None of these can be neighbours or we would have a 3-cycle. So each has $k-1$ neighbours outside the set $v_0, v_1, \dots, v_k$. None of these coincide or we'd have a 4-cycle.

So $|V(G)| \geq 1 + k + k(k-1) = k^2 + 1$.

b) Build G from a rooted tree. The first 3 levels must be distinct so as not to create cycles of length ≤ 5. For the same reason, neighbours on level 4 of neighbours of a level 2 vertex must be distinct. So there are at least 4 vertices on level 4.



$1 + 3 + 6 + 4 = 14$

14 vertices

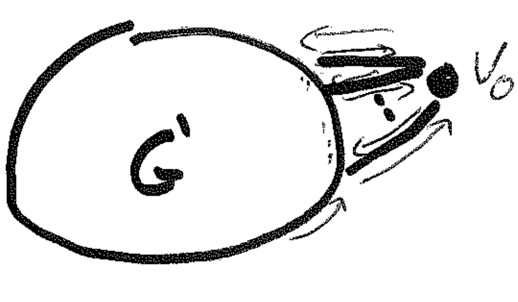
Question 3 solution:

Induction.

Base case: The trivial graph with one vertex satisfies the condition trivially.

Induction hypothesis: There exists a closed walk on a connected graph G on n vertices which transverses each of its edges precisely once in each direction.

Induction: Let G be a connected graph on $n+1$ vertices. Let v_0 be a leaf of a spanning tree of G. $G' := G - \{v_0\}$ is connected, so has such a closed walk by hypothesis. For each edge incident to v_0, walk to the edge, to v_0, and back. This creates the desired walk on G.



Question 4 solution:

Set $V(T) = \{v_1, \ldots v_n\}$

$$W(T) = \sum_{i<j} d_{ij}$$

There is a unique path between any v_i and any v_j in T, and each edge e on the path is counted as 1 towards d_{ij}.

So each edge e contributes 1 to $W(T)$ for each pair of vertices $v_i \in T_1(e)$ and $v_j \in T_2(e)$. So it contributes $n_1(e) \cdot n_2(e)$ towards $W(T)$.

Summing over all edges gives

$$W(T) = \sum_{e \in E} n_1(e) \cdot n_2(e).$$

Question 5 solution

Count $|E|$ in two ways:

- Handshake lemma: $|E| = \frac{1}{2}\sum_{i=1}^{\infty} i\, n_i$
- T is a tree: $|E| = |V| - 1 = \sum_{i=1}^{\infty} n_i - 1$

So

$$\frac{1}{2}\sum_{i=1}^{\infty} i\, n_i = \sum_{i=1}^{\infty} n_i - 1$$

$$\Rightarrow \sum_{i=1}^{\infty} (i-2) n_i + 2 = 0$$

$$\Rightarrow -n_1 + 0 + \sum_{i=3}^{\infty} (i-2) n_i + 2 = 0$$

$$\Rightarrow n_1 = \sum_{i=3}^{\infty} (i-2) n_i + 2$$

n_1 is the number of leaves in T.

Question 6 solution:

Lemma: For G loopless, connected

"If T is a spanning tree of G then T is unique"

$$\Updownarrow$$

"G is a tree"

Proof: $(\Rightarrow)$ If $E(G)-E(T)$ is non-empty, pick $e \in E(G)-E(T)$. Then $T+\{e\}$ contains a cycle of length ≥ 2. Delete a different edge e' from this loop. $T+\{e\}-\{e'\}$ is another spanning tree for G.

$(\Leftarrow)$ Deleting an edge from a tree disconnects it. (spanning trees are obtained via edge deletion).

Q.E.D.

Corollary: "If T is a spanning tree of G then T is unique"

$$\Updownarrow$$

"G becomes a tree after deletion of loops (or is disconnected)"