March 26th

Problem Set 7

Q2: W subspace of an inner product space V
$T: V \to V$ lin. transformation

Prove that W is T-invariant, then $W^\perp$ is T^*-invariant

Recall W is T-invariant $\iff$ if $w \in W$ then $T(w) \in W$

Example: $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(x_1, x_2, x_3) = (x_3, x_1, x_2)$

- $W_1 = \text{span}\{e_1\}$ a subspace of $\mathbb{R}^3$

Is W_1 a T-invariant subspace? In other words, is it true that if $w_1 \in W_1$ then $T(w_1) \in W_1$?

Note that $T(e_1) = e_2 \notin W_1$. Therefore W_1 is not T-invariant.

- $W_2 = \text{span}\{e_1 + e_2 + e_3\}$

Is W_2 a T-invariant subspace?

$W_2 = \{(c, c, c) : c \in \mathbb{R}\}$

$T(c, c, c) = (c, c, c) \in W_2, \forall c \in \mathbb{R}$

Therefore, W_2 is T-invariant.

Recall that: $W^\perp = \{x \in V : \langle x, y \rangle = 0 \ \forall y \in W\}$
so, if $x \in V$ then $x \in W^\perp \iff \langle x, y \rangle = 0, y \in W$

$S = T^*: V \to V$
$\langle S(w), y \rangle = \langle w, S^*(y) \rangle$
$\langle T^*(w), y \rangle = \langle w, T^*(y) \rangle$

Solution: We want to show that $\forall w \in W^\perp, T^*(w) \in W^\perp$. Suppose that $w \in W^\perp$. We must show that $\langle T^*(w), y \rangle = 0 \ \forall y \in W$. If $y \in W$, then $\langle T^*(w), y \rangle = \langle w, (T^*)^*(y) \rangle = \langle w, T(y) \rangle = 0$ (since $w \in W^\perp$ and $T(y) \in W$)
W since $y \in W$

So, $T^*(w) \in W^\perp \ \forall w \in W^\perp \Rightarrow W^\perp$ is T^* invariant

Q4. $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$

$T(A) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} A - A \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ Find a basis α for $M_{2\times2}(\mathbb{R})$ such that $[T]_\alpha^\alpha$ is in Jordan canonical form and find $[T]_\alpha^\alpha$.

$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & a+b-d \\ -c & c \end{pmatrix}$

Note that $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff c=0$ and $a+b-d=0$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a+b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$\ker(T) = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right\}$

$\|$

$E_0(T)$

There are two other eigenvalues, namely 1 and -1

$E_{-1}(T) = \operatorname{span}\left\{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right\}$

$E_1(T) = \operatorname{span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$

Have 4 lin. ind. eigenvectors since $M_{2\times2}(\mathbb{R})$ is 4-dim, these vectors form a basis α

$$[T]_{\alpha\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Hint</u>: Use the standard basis β of $M_{2\times2}(\mathbb{R})$

Find $[T]_\beta^\beta$. Find its JCF as well as a Jordan basis γ of $\mathbb{R}^4$. Convert this to a basis of $M_{2\times2}(\mathbb{R})$. call it α.