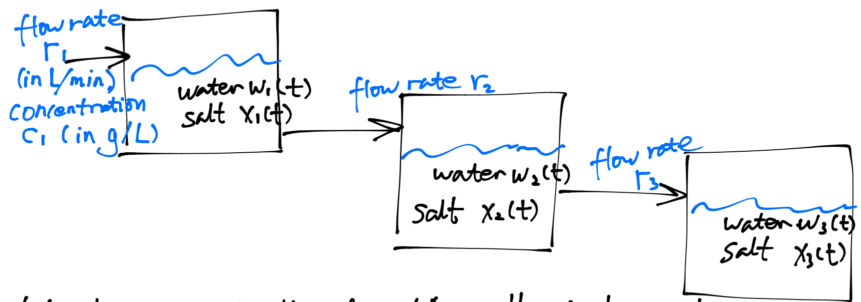
Midterm #1

Oct 10. MP203 18:10–19:00

covers Chapters 1, 2.1–2.6 (no tools allowed)

Mixing problems

Mixing tanks in series



How does concentration of salt in the tank evolve in time? Differential equation:

$$\frac{dx_1}{dt} = \underbrace{r_1 \cdot c_1}_{(g/min)} - r_2 \cdot \frac{x_1}{w_1}$$

$w_1(t) = w_1(0) + t(r_1 - r_2)$

$w_2(t) = w_2(0) + t(r_2 - r_1)$

$w_3(t) = w_3(0) + t\, r_3$

$$\frac{dx_2}{dt} = r_2 \cdot \frac{x_1(t)}{w_1(t)} - r_3 \cdot \frac{x_2(t)}{w_2(t)}$$

$$\frac{dx_3}{dt} = r_3 \cdot \frac{x_2(t)}{w_2(t)}$$

IF. A NEW CONNECTION FROM THE FIRST TANT TO THE THIRD?

Population dynamics

$y(t)$ population as function of time.

current world population 7.1×10^9 people.

in '98 : 5.9×10^9 people

Rough model: $\frac{dy}{dt} = r \cdot y$

Solution $y(t) = y(0) \cdot \exp(rt)$

(In reality, world population has grown even more than exponentially

Realistically, at some point the population has to saturate or decrease.

Model: r is not quite constant, but will have to decrease with large y.

Simplest model: $r(y) = r - ay$ where a is constant.

$\frac{dy}{dt} = r(y)\cdot y = (r-ay)y = r(1-\frac{y}{k})y \quad (k=\frac{r}{a})$ "Verhulst Equation"

Two special solutions of $\frac{dy}{dt} = r(1-\frac{y}{k})y$

$y(t)=0$
$y(t)=k$ constant "equalibrium solution"

It's a separable equation, hence solved by separation of variables.

$$\frac{1}{(1-\frac{y}{k})y}\,dy = r\cdot dt$$ integrate

partial fraction

$$\left(\frac{\frac{1}{k}}{1-\frac{y}{k}} + \frac{1}{y}\right) dy = r\,dt$$

integrate

$$\left(\frac{1}{k-y} + \frac{1}{y}\right) dy = r\,dt$$

$$-\ln|k-y| - \ln|y| = rt + c$$

$$\ln\left|\frac{y}{k-y}\right| = rt+c$$

$$e^{rt+c} = e^{rt}\cdot e^{c} = \left|\frac{y}{k-y}\right|$$

$$\frac{y}{k-y} = A\cdot e^{rt} \qquad A = \pm e^{c}$$

$$(1+A\cdot e^{rt})y = k\cdot Ae^{rt}$$

$$\boxed{y = \frac{k\cdot Ae^{rt}}{1+A\cdot e^{rt}}}$$ A depends on y_0

$$y = \frac{KA}{e^{-rt}+A}$$

Note: for $t\to\infty$, $y(t)\to K$. if $A>0$.