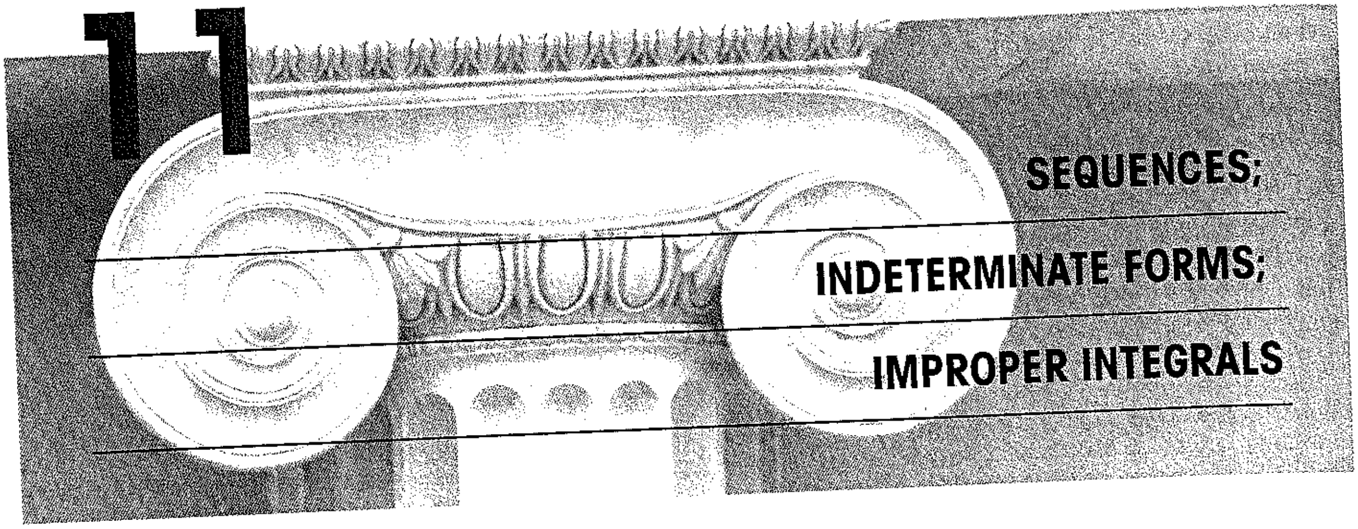
CHAPTER 11



SEQUENCES; INDETERMINATE FORMS; IMPROPER INTEGRALS

11.1 THE LEAST UPPER BOUND AXIOM

So far our approach to the real number system has been somewhat primitive. We have simply taken the point of view that there is a one-to-one correspondence between the set of points on a line and the set of real numbers, and that this enables us to measure all distances, take all roots of nonnegative numbers, and, in short, fill in all the gaps left by the set of rational numbers. This point of view is basically correct and has served us well, but it is not sufficiently sharp to put our theorems on a sound basis, nor is it sufficiently sharp for the work that lies ahead.

We begin with a nonempty set S of real numbers. As indicated in Section 1.2, a number M is an *upper bound* for S if

$$x \leq M \qquad \text{for all} \qquad x \in S.$$

It follows that if M is an upper bound for S, then every number in $[M, \infty)$ is also an upper bound for S. Of course, not all sets of real numbers have upper bounds. Those that do are said to be *bounded above*.

It is clear that every set that has a largest element has an upper bound: if b is the largest element of S, then $x \leq b$ for all $x \in S$. This makes b an upper bound for S. The converse is false: the sets

$$S_1 = (-\infty, 0) \qquad \text{and} \qquad S_2 = \{1/2, 2/3, 3/4, \cdots, n/(n+1), \cdots\}$$

both have upper bounds (for instance, 2 is an upper bound for each set), but neither has a largest element.

Let's return to the first set, S_1. While $(-\infty, 0)$ does not have a largest element, the set of its upper bounds, namely $[0, \infty)$, does have a smallest element, namely 0. We call 0 the *least upper bound* of $(-\infty, 0)$.

Now let's reexamine S_2. While the set of quotients

$$\frac{n}{n+1} = 1 - \frac{1}{n+1}, \; n = 1, 2, 3, \cdots,$$

does not have a greatest element, the set of its upper bounds, $[1, \infty)$, does have a least element, 1. The number 1 is the *least upper bound* of that set of quotients.

In general, if S is a nonempty set of numbers which is bounded above, then the *least upper bound* of S is the least number which is an upper bound for S.

We now state explicitly one of the key *assumptions* that we make about the real number system. This assumption, called the *least upper bound axiom*, provides the sharpness and clarity that we require.

AXIOM 11.1.1 THE LEAST UPPER BOUND AXIOM

Every nonempty set of real numbers that has an upper bound has a *least* upper bound.

Some find this axiom obvious; some find it unintelligible. For those of you who find it obvious, note that the axiom is not satisfied by the rational number system; namely, it is not true that every nonempty set of rational numbers that has a rational upper bound has a least rational upper bound. (For a detailed illustration of this, we refer you to Exercise 33.) Those who find the axiom unintelligible will come to understand it by working with it.

We indicate the least upper bound of a set S by writing lub S. As you will see from the examples below, the least upper bound idea has wide applicability.

(1) $\text{lub}\,(-\infty, 0) = 0, \qquad \text{lub}(-\infty, 0] = 0.$

(2) $\text{lub}\,(-4, -1) = -1, \qquad \text{lub}(-4, -1] = -1.$

(3) $\text{lub}\,\{1/2, 2/3, 3/4, \cdots, n/(n+1), \cdots\} = 1.$

(4) $\text{lub}\,\{-1/2, -1/8, -1/27, \cdots, -1/n^3, \cdots\} = 0.$

(5) $\text{lub}\,\{x : x^2 < 3\} = \text{lub}\{x : -\sqrt{3} < x < \sqrt{3}\} = \sqrt{3}.$

(6) For each decimal fraction

$$b = 0.b_1b_2b_3, \cdots,$$

we have

$$b = \text{lub}\,\{0.b_1, 0.b_1b_2, 0.b_1b_2b_3, \cdots\}.$$

(7) If S consists of the lengths of all polygonal paths inscribed in a semicircle of radius 1, then lub $S = \pi$ (half the circumference of the unit circle).

The least upper bound of a set has a special property that deserves particular attention. The idea is this: the fact that M is the least upper bound of set S does not guarantee that M is in S (indeed, it need not be, as illustrated in the preceding examples), but it guarantees that we can approximate M as closely as we wish by elements of S.

THEOREM 11.1.2

If M is the least upper bound of the set S and ϵ is a positive number, then there is at least one number s in S such that

$$M - \epsilon < s \leq M.$$

PROOF Let $\epsilon > 0$. Since M is an upper bound for S, the condition $s \leq M$ is satisfied by all numbers s in S. All we have to show therefore is that there is some number s in S such that

$$M - \epsilon < s.$$

Suppose on the contrary that there is no such number in S. We then have

$$x \leq M - \epsilon \qquad \text{for all} \qquad x \in S.$$

This makes $M - \epsilon$ an upper bound for S. But this cannot be, for then $M - \epsilon$ is an upper bound for S that is *less* than M, which contradicts the assumption that M is the *least* upper bound. ❑

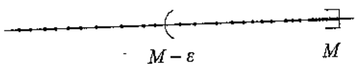


Figure 11.1.1

The theorem we just proved is illustrated in Figure 11.1.1. Take S as the set of points marked in the figure. If $M = \text{lub}\, S$, then S has at least one element in every half-open interval of the form $(M - \epsilon, M]$.

Example 1

(a) Let $S = \{1/2, 2/3, 3/4, \cdots, n/(n+1), \cdots\}$ and take $\epsilon = 0.0001$. Since 1 is the least upper bound of S, there must be a number $s \in S$ such that

$$1 - 0.0001 < s < 1.$$

There is: take, for example, $s = \dfrac{99,999}{100,000}$.

(b) Let $S = \{0, 1, 2, 3\}$ and take $\epsilon = 0.00001$. It is clear that 3 is the least upper bound of S. Therefore, there must be a number $s \in S$ such that

$$3 - 0.00001 < s \leq 3.$$

There is: $s = 3$. ❑

We come now to lower bounds. Recall that a number m is a *lower bound* for a nonempty set S if

$$m \leq x \qquad \text{for all} \qquad x \in S.$$

Sets that have lower bounds are said to be *bounded below*. Not all sets have lower bounds, but those that do have *greatest lower bounds*. We don't have to assume this. We can prove it by using the least upper bound axiom.

> **THEOREM 11.1.3**
>
> Every nonempty set of real numbers that has a lower bound has a *greatest* lower bound.

PROOF Suppose that S is nonempty and that it has a lower bound k. Then

$$k \leq s \qquad \text{for all} \qquad s \in S.$$

It follows that $-s \leq -k$ for all $s \in S$; that is,

$$\{-s : s \in S\} \qquad \text{has an upper bound } -k.$$

From the least upper bound axiom we conclude that $\{-s : s \in S\}$ has a least upper bound; call it m. Since $-s \leq m$ for all $s \in S$, we can see that

$$-m \leq s \qquad \text{for all} \qquad s \in S,$$

and thus $-m$ is a lower bound for S. We now assert that $-m$ is the greatest lower bound of the set S. To see this, note that, if there existed a number m_1 satisfying

$$-m < m_1 \leq s \qquad \text{for all} \qquad s \in S,$$

then we would have

$$-s \leq -m_1 < m \qquad \text{for all} \qquad s \in S,$$

and thus m would not be the *least* upper bound of $\{-s : s \in S\}$.† ❑

The greatest lower bound, although not necessarily in the set, can be approximated as closely as we wish by members of the set. In short, we have the following theorem, the proof of which is left as an exercise.

THEOREM 11.1.4

If m is the greatest lower bound of the set S and ϵ is a positive number, then there is at least one number s in S such that

$$m \leq s < m + \epsilon.$$

The theorem is illustrated in Figure 11.1.2. If $m = \text{glb}\, S$ (that is, if m is the greatest lower bound of the set S), then S has at least one element in every half-open interval of the form $[m, m + \epsilon)$.

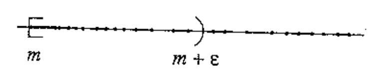


Figure 11.1.2

Remark Given that a function f is defined and continuous on $[a, b]$, what do we know about f? Certainly the following:

(1) We know that if f takes on two values, then it takes on every value in between (the intermediate-value theorem). Thus f maps intervals onto intervals.
(2) We know that f takes on both a maximum value and minimum value (the extreme-value theorem).

We "know" this, but actually we have proven none of it. With the least upper bound axiom in hand, we can prove both theorems. (Appendix B.) ❑

†We proved Theorem 11.1.3 by assuming the least upper bound axiom. We could have proceeded the other way. We could have set Theorem 11.1.3 as an axiom and then proved the least upper bound axiom as a theorem.

EXERCISES 11.1

Exercises 1–20. Find the least upper bound (if it exists) and the greatest lower bound (if it exists).

1. $(0, 2)$.
2. $[0, 2]$.
3. $(0, \infty)$.
4. $(-\infty, 1)$.
5. $\{x : x^2 < 4\}$.
6. $\{x : |x - 1| < 2\}$.
7. $\{x : x^3 \geq 8\}$.
8. $\{x : x^4 \leq 16\}$.
9. $\{2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \cdots\}$.
10. $\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \cdots\}$.
11. $\{0.9, 0.99, 0.999, \cdots\}$
12. $\{-2, 2, -2.1, 2.1, -2.11, 2.11, \cdots\}$.
13. $\{x : \ln x < 1\}$.
14. $\{x : \ln x > 0\}$.
15. $\{x : x^2 + x - 1 < 0\}$.
16. $\{x : x^2 + x + 2 \geq 0\}$.
17. $\{x : x^2 > 4\}$.
18. $\{x : |x - 1| > 2\}$.
19. $\{x : \sin x \geq -1\}$.
20. $\{x : e^x < 1\}$.

Exercises 21–24. Find a number s that satisfies the assertion made in Theorem 11.1.4 for S and ϵ as given below.

21. $S = \left\{\frac{1}{11}, \left(\frac{1}{11}\right)^2, \left(\frac{1}{11}\right)^3, \cdots, \left(\frac{1}{11}\right)^n, \cdots\right\}, \quad \epsilon = 0.001.$

22. $S = \{1, 2, 3, 4\}, \quad \epsilon = 0.0001.$

23. $S = \left\{\frac{1}{10}, \frac{1}{1000}, \frac{1}{100{,}000}, \cdots, \left(\frac{1}{10}\right)^{2n-1}, \cdots\right\},$
$\epsilon = \left(\frac{1}{10}\right)^k (k \geq 1).$

24. $S = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \left(\frac{1}{2}\right)^n, \cdots\right\}, \quad \epsilon = \left(\frac{1}{4}\right)^k (k \geq 1).$

25. Prove Theorem 11.1.4 by imitating the proof of Theorem 11.1.2.

26. Let $S = \{a_1, a_2, a_3, \cdots, a_n\}$ be a finite nonempty set of real numbers.

(a) Show that S is bounded.
(b) Show that lub S and glb S are elements of S.

27. Suppose that b is an upper bound for a set S of real numbers. Prove that if $b \in S$, then $b =$ lub S.

28. Let S be a bounded set of real numbers and suppose that lub $S =$ glb S. What can you conclude about S?

29. Suppose that S is a nonempty bounded set of real numbers and T is a nonempty subset of S.

(a) Show that T is bounded.
(b) Show that glb $S \leq$ glb $T \leq$ lub $T \leq$ lub S.

30. Show by example

(a) that the least upper bound of a set of rational numbers need not be rational.
(b) that the least upper bound of a set of irrational numbers need not be irrational.

31. Let c be a positive number. Prove that the set $S = \{c, 2c, 3c, \cdots, nc, \cdots\}$ is not bounded above.

32. (a) Show that the least upper bound of a set of negative numbers cannot be positive.
(b) Show that the greatest lower bound of a set of positive numbers cannot be negative.

33. The set S of rational numbers x with $x^2 < 2$ has rational upper bounds but no least rational upper bound. The argument goes like this. Suppose that S has a least rational upper bound and call it x_0. Then either

$$x_0^2 = 2, \quad \text{or} \quad x_0^2 > 2, \quad \text{or} \quad x_0^2 < 2.$$

(a) Show that $x_0^2 = 2$ is impossible by showing that if $x_0^2 = 2$, then x_0 is not rational.
(b) Show that $x_0^2 > 2$ is impossible by showing that if $x_0^2 > 2$, then there is a positive integer n for which $(x_0 - \frac{1}{n})^2 > 2$, which makes $x_0 - \frac{1}{n}$ a rational upper bound for S that is less than the least rational upper bound x_0.
(c) Show that $x_0^2 < 2$ is impossible by showing that if $x_0^2 < 2$, then there is a positive integer n for which $(x_0 + \frac{1}{n})^2 < 2$. This places $x_0 + \frac{1}{n}$ in S and shows that x_0 cannot be an upper bound for S.

34. Recall that a *prime number* is an integer $p > 1$ that has no positive integer divisors other than 1 and p. A famous theorem of Euclid states that there are an infinite number of primes (and therefore the set of primes is unbounded above). Prove that there are an infinite number of primes. HINT: Following the way of Euclid, assume that there are only a finite number of primes $p_1, p_2, \cdots, p_n$ and examine the number $x = (p_1 p_2 \cdots p_n) + 1$.

35. Let $S = \left\{2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \cdots, \left(\frac{n+1}{n}\right)^n, \cdots\right\}$.

(a) Use a graphing utility or CAS to calculate $\left(\frac{n+1}{n}\right)^n$ for $n = 5, 10, 100, 1000, 10,000$.
(b) Does S have a least upper bound? If so, what is it? Does S have a greatest lower bound? If so, what is it?

36. Let $S = \{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ with $a_1 = 4$ and for further subscripts $a_{n+1} = 3 - 3/a_n$.

(a) Calculate the numbers $a_2, a_3, a_4, \cdots, a_{10}$.
(b) Use a graphing utility or CAS to calculate $a_{20}, a_{30}, \cdots, a_{50}$.
(c) Does S have a least upper bound? If so, what is it? Does S have a greatest lower bound? If so, what is it?

37. Let $S = \{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\}$. Thus $S = \{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ with $a_1 = \sqrt{2}$ and for further subscripts $a_{n+1} = \sqrt{2a_n}$.

(a) Use a graphing utility or CAS to calculate the numbers $a_1, a_2, a_3, \cdots, a_{10}$.
(b) Show by induction that $a_n < 2$ for all n.
(c) What is the least upper bound of S?
(d) In the definition of S, replace 2 by an arbitrary positive number c. What is the least upper bound in this case?

38. Let $S = \{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots\}$. Thus $a_1 = \sqrt{2}$ and for further subscripts $a_{n+1} = \sqrt{2 + a_n}$.

(a) Use a graphing utility or CAS to calculate the numbers $a_1, a_2, a_3, \cdots, a_{10}$.
(b) Show by induction that $a_n < 2$ for all n.
(c) What is the least upper bound of S?
(d) In the definition of S, replace 2 by an arbitrary positive number c. What is the least upper bound in this case?

■ 11.2 SEQUENCES OF REAL NUMBERS

To this point we have considered sequences only in a peripheral manner. Here we focus on them.

What is a sequence of real numbers?

DEFINITION 11.2.1 SEQUENCE OF REAL NUMBERS

A *sequence of real numbers* is a real-valued function defined on the set of positive integers.