Exercise 4 - Prior predictive distribution $p(y)$ for Poisson model.

(assume $n=1$).

$y \mid \theta \sim \underbrace{Pois(\theta)}_{\text{\sout{Lik} sampling model}}$.

$\underbrace{\theta \sim Gamma(a, b)}_{\text{Prior}} \rightarrow \theta \mid y \sim \underbrace{Gamma(a+y, b+1)}_{\text{Posterior}}$

$$p(y) = \frac{p(y \mid \theta)\, p(\theta)}{p(\theta \mid y)} \quad \text{(Bayes' rule)}.$$

$$= \frac{\dfrac{e^{-\theta}\theta^{y}}{y!} \times \dfrac{b^{a}}{\Gamma(a)} \theta^{a-1} e^{-b\theta}}{\dfrac{(1+b)^{a+y}}{\Gamma(a+y)} \theta^{a+y-1} e^{-(b+1)\theta}}$$

$$= \frac{b^{a}\, \Gamma(a+y)}{\Gamma(a)\, y!\, (1+b)^{a+y}}$$

$$= \binom{a+y-1}{y} \left(\frac{b}{b+1}\right)^{a} \left(\frac{1}{b+1}\right)^{y}.$$

$y \sim$??