The employee makes less than 55,000. → SENTENCE
It may refer to unquantified objects (for example "the employee"). Once the objects are specified (subs are made for the variables), a sentence is either true or false.

Every employee makes less than 55,000 ⇒ STATEMENT
Doesn't refer to any unquantified variables, and it is either true ~~or~~ or false.

implication: IF P THEN Q
P: ANTECEDENT (ASSUMPTION)
Q: CONSEQUENT (CONCLUSION)

$\bar{L}$: complement of L.

VACUOUS TRUTH: Whenever the antecedent is false and the consequent is either true or false, the implication is as a whole TRUE. Another way of thinking of this is that the set where antecedent is true is empty (vacuous), and hence a subset of every set.

We use $\wedge$ to combine 2 sentences into a new sentence that claims that both of the original sentences are true.
We use $\vee$ ... at least one of ... is true.

In logic, a PREDICATE is a boolean function.

TRUTH TABLE

P Q R	$Q\Rightarrow R$	$P\Rightarrow(Q\Rightarrow R)$	$P\wedge Q$	$(P\wedge Q)\Rightarrow R$	$P\Rightarrow(Q\Rightarrow R)\Leftrightarrow(P\wedge Q)\Rightarrow R$
T T T	T	T	T	T	T
T T F	F	F	T	F	T
T F F	T	T	F	T	T
T F T	T	T	F	T	T
F T T	T	T	F	T	T
F F T	T	T	F	T	T
F T F	F	T	F	T	T
F F F	T	T	F	T	T

IF D is the set of domains, and P(D) is the set of all predicates in domain D, then
$\forall D\in D, \forall P\in P(D), \forall Q\in P(D), \forall x\in D, (P(x)\Rightarrow Q(x)) \Leftrightarrow (\neg P(x) \vee Q(x))$
TAUTOLOGY (statement is satisfiable)

summary of manipulation rules:

identity laws $P\wedge(Q\vee\neg Q)\Leftrightarrow P$
$P\vee(Q\wedge\neg Q)\Leftrightarrow P$
idempotency laws $P\wedge P\Leftrightarrow P$ $\quad P\vee P\Leftrightarrow P$
commutative laws $P\wedge Q\Leftrightarrow Q\wedge P$ $\quad P\vee Q\Leftrightarrow Q\vee P$
$(P\Leftrightarrow Q)\Leftrightarrow(Q\Leftrightarrow P)$
associative laws $(P\wedge Q)\wedge R\Leftrightarrow P\wedge(Q\wedge R)$
$(P\vee Q)\vee R\Leftrightarrow P\vee(Q\vee R)$
distributive laws $P\wedge(Q\vee R)\Leftrightarrow(P\wedge Q)\vee(P\wedge R)$
$P\vee(Q\wedge R)\Leftrightarrow(P\vee Q)\wedge(P\vee R)$
contrapositive $P\Rightarrow Q\Leftrightarrow\neg Q\Rightarrow\neg P$
implication $P\Rightarrow Q\Leftrightarrow\neg P\vee Q$
equivalence $(P\Leftrightarrow Q)\Leftrightarrow(P\Rightarrow Q)\wedge(Q\Rightarrow P)$
double negation $\neg(\neg P)\Leftrightarrow P$
DeMorgan's laws $\neg(P\wedge Q)\Leftrightarrow\neg P\vee\neg Q$
$\neg(P\vee Q)\Leftrightarrow\neg P\wedge\neg Q$
implication negation $\neg(P\Rightarrow Q)\Leftrightarrow P\wedge\neg Q$
equivalence negation $\neg(P\Leftrightarrow Q)\Leftrightarrow\neg(P\Rightarrow Q)\vee\neg(Q\Rightarrow P)$
quantifier negation $\neg(\forall x\in D, P(x))\Leftrightarrow\exists x\in D, \neg P(x)$
$\neg(\exists x\in D, P(x))\Leftrightarrow\forall x\in D, \neg P(x)$
quantifier distributive laws $\forall x\in D, P(x)\wedge Q(x)\Leftrightarrow(\forall x\in D, P(x))\wedge(\forall x\in D, Q(x))$
$\exists x\in D, P(x)\vee Q(x)\Leftrightarrow(\exists x\in D, P(x))\vee(\exists x\in D, Q(x))$.

Absorption: $P\wedge(Q\wedge\neg Q)\Leftrightarrow Q\wedge\neg Q$
$P\vee(Q\vee\neg Q)\Leftrightarrow Q\vee\neg Q$

A proof is an argument that convinces sb. who is logical, careful & precise.
LEMMA: a small result needed to prove sth. we really care about.
THEOREM: the main result that we care about (at the moment).
COROLLARY: easy consequence of another result.
CONJECTURE: sth. suspected to be true, but not yet proven.
AXIOM: sth. we assent to be true, without justification. "self-evident".

summary of inference rules
INTRODUCTION RULES:

[¬I] negation introduction
Assume A
⋮
contradiction
——
¬A

introduction rules
[⇒I] implication introduction
(direct) Assume A ⋮ B —— A⇒B
(indirect) Assume ¬B ⋮ ¬A —— A⇒B

[∀I] universal intro
Assume a∈D
⋮
P(a)
——
∀x∈D, P(x)

[∧I] conjunction introduction
A
B
——
A∧B

[⇔I] equivalence/bi-implication introduction
A⇒B
B⇒A
——
A⇔B

[∃I] existential introduction
P(a)
a∈D
——
∃x∈D, P(x)

[∨I] disjunction introduction
A
——
B∨A

A∨¬A

A∨B

elimination rules

[¬E] negation elimination
¬¬A
——
A

A
¬A
——
contradiction

[⇒E] impli. e. (Modus Ponens)
A⇒B
A
——
B

(Modus Tollens)
A⇒B
¬B
——
¬A

[∃E] ∃x∈D, P(x)
——
Let a∈D. such that P(a)
⋮

[∀E] ∀x∈D. P(x)
a∈D
——
P(a)

[∧E] conjunction elimination
A∧B
——
A
B

[⇔E] equiv. ...
A⇔B
——
A⇒B
B⇒A

[∨E] disj e.
A∨B
¬A
——
B

A∨B
¬B
——
A

A LOOP INVARIANT is a relationship between the variables that is always true at the start and at the end of a loop iteration.

- METHOD CALL: 1 step + steps to evaluate each argument + steps to execute the method.
- RETURN STATEMENT: 1 step + steps to evaluate return value
- IF STATEMENT: 1 step + steps to evaluate condition
- ASSIGNMENT STATEMENT: 1 step + steps to evaluate each side.
- Arithmetic, comparison, boolean operators: 1 step + steps to evaluate each operand.
- ARRAY ACCESS: 1 step + steps to evaluate index
- member access: 2 steps
- constant, variable evaluation: 1 step

$\lceil\log_2(x+1)\rceil = \lfloor\log_2 x\rfloor+1$ forall $x\in\mathbb{R}^+$

Big "O", Let $\mathbb{F}=\{\mathbb{N}\to\mathbb{R}^{\geq 0}\}$
$\forall f,g\in\mathbb{F}: f\in O(g)\Leftrightarrow\exists c\in\mathbb{R}^+, \exists B\in\mathbb{N}, n\geq B\Rightarrow f(n)\leq c\cdot g(n)$
$\forall f,g\in\mathbb{F}: f\in\Omega(g)\Leftrightarrow\exists c\in\mathbb{R}^+, \exists B\in\mathbb{N}, n\geq B\Rightarrow f(n)\geq c\cdot g(n)$
$\forall f,g\in\mathbb{F}, f\in\theta(g)\Leftrightarrow\exists c_1,c_2\in\mathbb{R}^+, \exists B\in\mathbb{N}, \forall n\in\mathbb{N}, n\geq B\Rightarrow c_1\cdot g(n)\leq f(n)\leq c_2\cdot g(n)$
$\forall f,g\in\mathbb{F}, \forall z\in\mathbb{R}^+, f=zg\Rightarrow f\in\theta(g)$
$\forall m,n,r\in\mathbb{N}, r=m\,\%\,n\Leftrightarrow(0\leq r<n)\wedge(\exists q\in\mathbb{N}, m=q*n+r)$
$W_p(n)\in\Omega(f(n))\Leftrightarrow\exists c\in\mathbb{R}^+, \exists B\in\mathbb{N}, \forall n\in\mathbb{N}, \exists x\in I, size(x)=n\wedge n\geq B\Rightarrow t_p(x)\geq c*f(n)$

Proof outline: By definition of "O", have to show for $5n^4-3n^2+1\in O(6n^5-4n^3+2n)$
$\exists c\in\mathbb{R}^+, \exists B\in\mathbb{N}, \forall n\in\mathbb{N}, n\geq B\Rightarrow 5n^4-3n^2+1\leq c(6n^5-4n^3+2n)$.
Let $c'=\ldots$, Then $c'\in\mathbb{R}^+$.
Let $B'=\ldots$. Then $B'\in\mathbb{N}$.
Assume $n\in\mathbb{N}$ and $n\geq B'$.
... show that $5n^4-3n^2+1\leq c'(6n^5-4n^3+2n)$.
Then $\forall n\in\mathbb{N}, n\geq B'\Rightarrow 5n^4-3n^2+1\leq c'(6n^5-4n^3+2n)$.
Then $\exists c\in\mathbb{R}^+, \exists B\in\mathbb{N}, \forall n\in\mathbb{N}, n\geq B\Rightarrow 5n^4-3n^2+1\leq c'(6n^5-4n^3+2n)$

some theorems:
general rules:
- $f\in O(f)$
- $(f\in O(g))\wedge(g\in O(h))\Rightarrow f\in O(h)$
- $g\in\Omega(f)\Leftrightarrow f\in O(g)$
- $g\in\theta(f)\Leftrightarrow g\in O(f)\wedge g\in\Omega(f)$

Find a tight bound on the $W_p(t)$:

```
# Precondition : L is a list that contains n>0 real #s.
1. max=0
2. for i=0,1,...,n-1:
3.     for j=i,i+1,...,n-1:
4.         sum=0
5.         for k=i,i+1,...,j:
6.             sum=sum+L[k]
7.         if sum>max:
8.             max=sum
```

Proof. Assume $n \in \mathbb{N}$ and $n \geq 3$ and L is a list of n real numbers.
Then the list takes $1 < n < n^3$ steps.
Also, the loop over i iterates exactly n times, and for each iteration ...
The loop over j iterates at most n times, and for each iteration ...
The loop over k iterates at most n times and each iter- takes 1 step, for a total of at most n steps.
The other statements in the loop body for j take at most 3 steps
So the loop over k takes at most $n+3 \leq 2n$ steps.
... So the ... j ... $2n^2$ steps.
... So ... i ... $2n^3$ steps.
Then ~~entire~~ entire algorithm therefore takes at most $n^3+2n^3=3n^3$ steps.
Then, $\forall n \in \mathbb{N}, n \geq 3 \Rightarrow \forall L \in \{\text{all lists of real numbers}\}, len(L)=n \Rightarrow t(L) \leq 3n^3$.
Hence, $T(n) \in O(n^3)$.

Proof. Assume $n \in \mathbb{N}$, and $n \geq 1$.
Let $L=\{1,2,\ldots,n\}$.
Then for each value of $i \in \{0,\ldots,\lfloor n/3 \rfloor\}$ ..
For each value of $j \in \{\lfloor 2n/3 \rfloor, \ldots, n-1\}$
The loop for k iterates over every value in $\{i,\ldots,j\}$ and executes 1 step at each iteration.
So the loop for k takes at least $n/3$ steps (since there are at least $\lfloor 2n/3 \rfloor - \lfloor n/3 \rfloor + 1 \geq n/3$ values for k)
... so the ... j ... at least $n^2/9$ steps (since ... $n - \lfloor 2n/3 \rfloor \geq n/3$ values for j).
... so ... i ... $n^3/27$ steps ($\lfloor n/3 \rfloor + 1 > n/3$ values for i)
Then $\forall n \in \mathbb{N}, n \geq 1 \Rightarrow \exists L \in \{\text{all lists of real \#s}\}. len(L)=n \wedge t(L) \geq n^3/27$.
$n = len(L)$
Hence, $T(n) \in \Omega(n^3)$.

```
# Precondition : L is a list that contains n>0 #s and
1. step=1.
2. index=0
3. while index<len(L):
4.     print L[index]
5.     index=index+step
6.     step=step+1
```

Proof: Assume ...
Then line 1 & 2 take $2 \leq 2(n^{\frac{1}{2}})$ steps
Then the while loop iterates k times & each iteration takes 3 steps and 1 step of condition check.
Also it takes 1 step to exit.
Then $(k-1)k/2 \leq len(L) < k(k+1)/2$
Then $(k-1)k/2 \leq n < k(k+1)/2$
$(k-1)^2/2 \leq n < (k+1)^2/2$
Then $k-1 \leq \sqrt{2n} < k+1$
Then $k \leq 1+\sqrt{2n} \leq \sqrt{n} + \sqrt{4n} = 3\sqrt{n}$
Then $4k+1 \leq 4(3\sqrt{n})+1 \leq 12\sqrt{n}+\sqrt{n} = 13\sqrt{n}$
Then ... $2\sqrt{n} + 13\sqrt{n} \leq 15\sqrt{n}$ steps
Then ..
Hence $T(n) \in O(\sqrt{n})$

Hence $T(n) \in \Omega(\sqrt{n})$

COUNTABILITY:

def: Suppose that $f: A \to B$ (i.e. f is a function that ...).
- f is ONE-TO-ONE if $\forall a_1 \in A, \forall a_2 \in A, f(a_1)=f(a_2) \Rightarrow a_1=a_2$;
- f is ONTO if $\forall b \in B\ \exists a \in A.\ f(a)=b$ (every element in B can be 'reached' from at least one element of A.

Set A is COUNTABLE if
1. there is a function $f: A \to \mathbb{N}$ that is one-to-one, or
2. there is a function $f: \mathbb{N} \to A$ that is onto.

Claim $\mathbb{Q}^+$ is countable:
sublist 0 : 1/1
1 : 1/2, 2/1,
2 : 1/3, 2/2, 3/1
...

Then $f(n): \mathbb{N} \to \mathbb{Q}^+$. Also $f(n)$ is onto: every fraction p/q with $p>0$, $q>0$ is eventually listed (in position p of sub-list number $p+q-2$).
$\mathbb{Q}^+$ is countable.

CLAIM: $\mathbb{R}$ is uncountable:
Proof: Assume $\mathbb{R}$ is countable.
Then $\exists f: \mathbb{N} \to \mathbb{R}$ that is onto. # def of "countable"
$f(0) = i_0 . d_{0,0} d_{0,1} d_{0,2} \ldots d_{0,n} \ldots$
$f(1) = i_1 . d_{1,0} d_{1,1} d_{1,2} \ldots d_{1,n} \ldots$
$f(2) = i_2 . d_{2,0} d_{2,1} d_{2,2} \ldots d_{2,n} \ldots$
$f(n) = i_n . d_{n,0} d_{n,1} d_{n,2} \ldots d_{n,n} \ldots$
where $i_0, \ldots i_n \ldots \in \mathbb{Z}$ are int. parts of real # $f(0), f(1), \ldots$,
and $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, d_{i,j} \in \{0,\ldots,9\}$
Now let $r = 0.d_0 d_1 d_2 \ldots d_n \ldots$, where $\forall i \in \mathbb{N}$,
$$d_i = \begin{cases} 1 & \text{if } d_{i,i}=0, \\ 0 & \text{otherwise} \end{cases}$$
Then $r \in \mathbb{R}$ # r is an infinite decimal that does not end with repeating 9's.
Then $\exists k \in \mathbb{N}, f(k)=r$, # f is onto.
Since $f(k)=i_k . d_{k,0} d_{k,1} \ldots d_{k,n}$ and $r = 0.d_0 d_1 d_2 \ldots d_n \ldots$ this implies $i_k=0$ & $\forall n \in \mathbb{N}$,
$$d_{k,n} = d_n = \begin{cases} 1 & \text{if } d_{n,n}=0 \\ 0 & \text{otherwise} \end{cases}$$
i.e. $d_{k,k}=0 \Leftrightarrow d_{k,k}=1$, a contradiction
...

PSI

Prove set $S_1 = \{(a,b) : a \in \mathbb{N}, b \in \mathbb{N}\}$ is countable.
Assume $(a_1,b_1) \in S_1$, $(a_2,b_2) \in S_1$.
Assume $f_1((a_1,b_1)) = f_1((a_2,b_2))$
Then $2^{a_1}3^{b_1} = 2^{a_2}3^{b_2}$.
Then $a_1=a_2$ & $b_1=b_2$ # by the Fundamental Theorem of Arithmetic.
Then $f_1'(a_1,b_1) = f_1'((a_2,b_2)) \Rightarrow (a_1,b_1)=(a_2,b_2)$
$\leftarrow$ Then f_1' is one-to-one,

disProve set $S_2 = \mathcal{P}(\mathbb{N})$ countable.
power set.
Assume S_2 is countable.
Then $\exists f: \mathbb{N} \to S_2$ that is onto.
Let $f_0: \mathbb{N} \to S_2$ be onto.
Then $\forall D \in S_2, \exists n \in \mathbb{N}, D = f_0(n)$
D is a set of natural #s,
We can think about the value of $f_0(n)$, $\forall n \in \mathbb{N}$.
Construct a special element of S_2.
Let $D = \{m \in \mathbb{N} : m \notin f_0(m)\}$
$f_0(m)$ is a set of natural #s (since $f_0: \mathbb{N} \to S_2$)
D is the set of natural numbers that are not in $f_0(m)$.
Then $D \in S_2$

Assume $n \in \mathbb{N}$
Then either $n \in f_0(n)$ or $n \notin f_0(n)$
Case 1: Assume $n \in f_0(n)$
Then $n \notin D$ # since $n \in f_0(n)$.
Then $D \neq f_0(n)$. # since $n \in f_0(n)$ & $n \notin D$
Case 2: Assume $n \notin f_0(n)$
Then $n \in D$. # since $n \notin f_0(n)$
Then $D \neq f_0(n)$.
Then $D \neq f_0(n)$ in either cases.
Then $\forall n \in \mathbb{N}, D \neq f_0(n)$
Then $\neg \exists n \in \mathbb{N}, D = f_0(n)$.
Then this contradicts f_0 being onto.
Then ...