Lecture 2

For $n \in \mathbb{N}$, let $A_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$

an element is contained in a set

E.g. $A_2 = \{i \in \mathbb{N} : 1 \leq i \leq 2\} = \{1, 2\}$

$A_0 = \{i \in \mathbb{N} : 1 \leq i \leq 0\} = \{\ \} = \emptyset$

$A_1 = \{i \in \mathbb{N} : 1 \leq i \leq 1\} = \{1\}$

* $\emptyset$ is both an element and a set

Notations

a set is a subset

Set of subsets of A_n

E.g. subsets of A_2 are : $\{\ \}, \{1\}, \{2\}, \{1, 2\}$

$\{\ \} \in P(A_2)$	$\{\ \} \subset A_2$	
$\{1\} \in P(A_2)$	$\{1\} \subset A_2$	$1 \in A_2$
$\{2\} \in P(A_2)$	$\{2\} \subset A_2$	$2 \in A_2$
$\{1,2\} \in P(A_2)$	$\{1,2\} \subset A_2$	

Set of subsets of A_2 is called "Power set" of A_2

[Notation $P(A_2)$], it's a set containing these sets: $\{\{\ \}, \{1\}, \{2\}, \{1, 2\}\}$

$P(A_0) = P(\{\ \}) = \{\{\ \}\}$

$|P(A_0)| = 1 = 2^0$

How many subsets does A_2 have ? 4 $\quad |P(A_2)| = 4 = 2^2$

subsets of A_3, i.e. subsets of $\{1, 2, 3\}$: $\quad |P(A_3)| = 8 = 2^3$

subsets with 3, and the ones without

$\{\ \}, \{1\}, \{2\}, \{1, 2\}$

$\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

For $n \in \mathbb{N}$, Let $Q(n)$ be the # of subsets of $\{1, \cdots, n\}$ is 2^n

i.e. $|P(A_n)| = 2^n$

Claim : Q is true for all natural #s

$\forall n \in \mathbb{N}, Q(n)$ (Boolean)

Proof : By induction.

Base case : Prove $Q(0)$

Prove : $|P(A_0)| = 2^0 = 1$

$|P(A_0)| = |P(\{\ \})| = |\{\{\ \}\}| = 1 = 2^0$

Inductive Step : Prove $\forall n \in \mathbb{N}, (Q(n) \to Q(n+1))$:

Let $n \in \mathbb{N}$

Assume $Q(n)$, i.e. $\boxed{|P(\{1, 2, \cdots, n\})| = 2^n} \longrightarrow$ inductive hypothesis

Now prove $Q(n+1)$, i.e. $|P(\{1, \cdots, n+1\})| = 2^{n+1}$

$\{1, \cdots, n+1\} = \{1, \cdots, n, n+1\}$

let the subsets of $\{1, \cdots, n\}$ be $S_1, \cdots, S_{2^n}$

(there are 2^n by Inductive hypothesis)

The subsets of $\{1, \cdots, n, n+1\}$ are the subsets without $n+1$ and the ones containing $n+1$

i.e. the subsets $S_1, \cdots, S_{2^n}$ of $\{1, \cdots, n\}$ and $S_1 \cup \{n+1\}, S_2 \cup \{n+1\}, \cdots, S_{2^n} \cup \{n+1\}$

That's $2^n + 2^n$ subsets $= 2^{n+1}$ subsets