Lecture 11
Feb 12th, 2015

$T(1)=1$
$T(n)=1+\max(T(\lceil \frac{n}{2} \rceil), T(\lfloor \frac{n}{2} \rfloor)), n \geq 2$
T is non-decreasing on the positive natural numbers

$$\forall n \in \mathbb{N}, 1 \leq n \Rightarrow (T(n) \leq T(n+1))$$
$$\forall m,n \in \mathbb{N}, 1 \leq m \leq n \Rightarrow T(m) \leq T(n)$$

Claim: $P(1): T(1) \leq T(1)$
$P(2): T(1) \leq T(2)$
$T(2) \leq T(2)$
$P(3): T(1) \leq T(3)$
$T(2) \leq T(3)$
$T(3) \leq T(3)$
$P(4): T(1) \leq T(4)$
$T(2) \leq T(4)$
$T(3) \leq T(4)$
$T(4) \leq T(4)$
Prove $T(3) \leq T(4)$

$T(3) = 1+\max(T(2), T(1))$
$T(4) = 1+\max(T(6), T(5))$

For $n \in \mathbb{N}$, let $P(n)$ be: $\forall m \in \mathbb{N}, 1 \leq m \leq n \Rightarrow T(m) \leq T(n)$

e.g. $P(3): \forall m \in \mathbb{N}, 1 \leq m < 3 \Rightarrow T(m) \leq T(3)$

so $T(1) \leq T(3), T(2) \leq T(3), T(3) \leq T(3)$

e.g. $T(123) \leq T(236)$
$T(123) = 1+\max(T(62), T(61))$
$T(236) = 1+\max(T(119), T(118))$
$P(118): T(62) \leq T(118)$
$P(119): T(62) \leq T(119)$

Proof of $\forall n \in \mathbb{N}, n \geq 1 \Rightarrow P(n)$
Base case $P(1)$. proof: $1 \leq m \leq 1 \Rightarrow T(m) \leq T(1) \Rightarrow T(1) \leq T(1)$
Inductive Step: Let $n \in \mathbb{N}, n \geq 1$
IH: Assume $P(1), P(2), P(3), \cdots P(n-1)$ # dangerous
Assume $P(k)$ for such $k \in \mathbb{N}$ with $1 \leq k < n$ that $\forall m \in \mathbb{N}, 1 \leq m \leq k \Rightarrow T(m) \leq T(k)$
Prove: $\forall l \in \mathbb{N}, 1 \leq l < n \Rightarrow T(l) \leq T(n)$
Let $l \in \mathbb{N}, 1 \leq l \leq n$
case: $l \leq 2$
$T(l) = 1+\max(T(\lceil \frac{l}{2} \rceil), T(\lfloor \frac{l}{2} \rfloor))$
$T(n) = 1+\max(T(\lceil \frac{n}{2} \rceil), T(\lfloor \frac{n}{2} \rfloor))$, since $n \geq 2$
Use $P(\lceil \frac{n}{2} \rceil), P(\lfloor \frac{n}{2} \rfloor)$
$k = \lceil \frac{n}{2} \rceil, k = \lfloor \frac{n}{2} \rfloor$ $\quad k \in \mathbb{N}, 1 \leq k < n$ because $n \geq 2$ $\Rightarrow T(\lceil \frac{l}{2} \rceil) \leq T(\lceil \frac{n}{2} \rceil)$
$T(\lfloor \frac{l}{2} \rfloor) \leq T(\lfloor \frac{n}{2} \rfloor)$