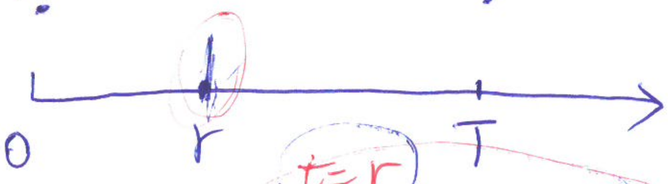
Notes 10.1.

$t=r \overset{?}{\Longrightarrow} V_L$, V_S

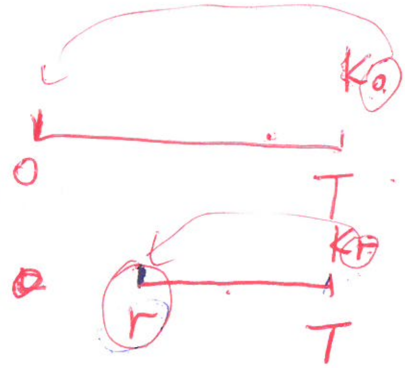
$r \in (0, T)$



$t=r$

Portfolio A: $V_L + K_0 \cdot e^{-\delta(T-r)}$

Portfolio B: $K_r \cdot e^{-\delta(T-r)}$



$t=T$

Portfolio A $\quad K_0 - K_0 + S_T = S_T$

Portfolio B. $\quad K_r - K_r + S_T = S_T$

law of one price.

$$\Rightarrow \quad V_L + K_0 \cdot e^{-\delta(T-r)} = K_r \cdot e^{-\delta(T-r)}$$

$$\Rightarrow \quad \boxed{V_L = (K_r - K_0) \cdot e^{-\delta(T-r)}}$$

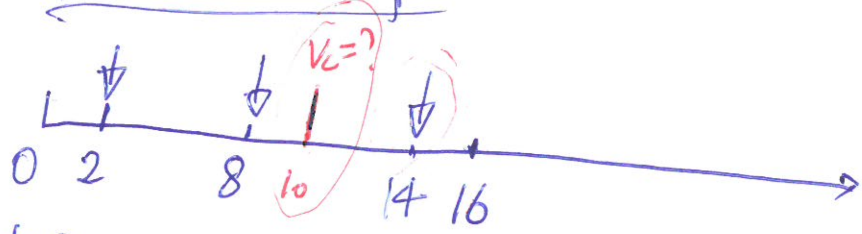
For example $\boxed{\text{No income}}$

$$V_L = (S_r \cdot e^{\delta(T-r)} - S_0 \cdot e^{\delta T}) \cdot e^{-\delta(T-r)}$$

$$\boxed{V_L = S_r - S_0 \cdot e^{\delta r}}$$

$$\boxed{V_S = -V_L}.$$

Ex:



$K_0 = \$19693.97$, $r = 10$. $T = 16$ months

$$K_r = (S_r - PV_I) \cdot e^{\delta(T-r)}.$$

$$= \left(\frac{45.60}{100} \times 50{,}000 - 2{,}500 \cdot e^{-0.07 \cdot \frac{4}{12}} \right) \cdot e^{0.07 \cdot \left(\frac{16-10}{12}\right)}$$

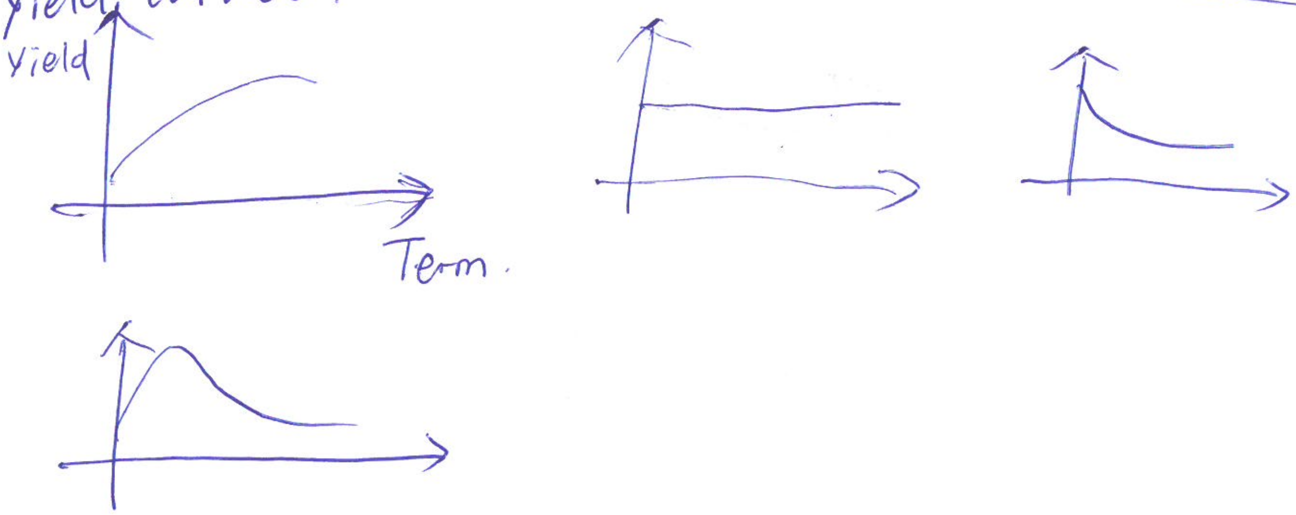
$$= \$21{,}082.79$$

$$V_L = (K_r - K_0) \cdot e^{-\delta(T-r)}.$$

$$= (21082.79 - 19693.97) \cdot e^{-0.07 \cdot \left(\frac{16-10}{12}\right)}.$$

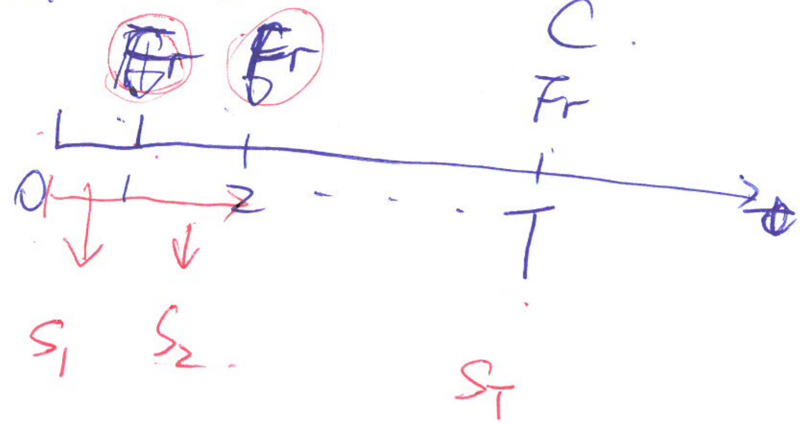
$$= \$1341.05$$

Yield curves.

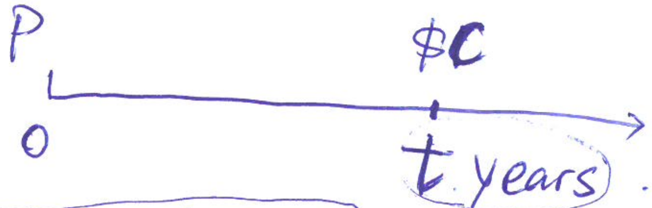


① spot rates

② forward rates.

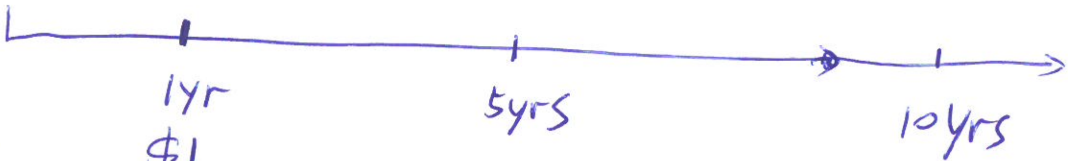


Spot Rates   zero-coupon bond.



$$\boxed{P \cdot (1+S_t)^t = C} \iff \boxed{P = (1+S_t)^{-t} \cdot C}$$

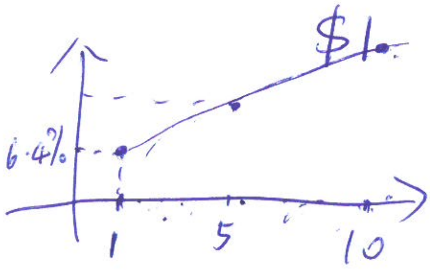
Ex:



① \$0.94

② \$0.70

③ \$0.47.

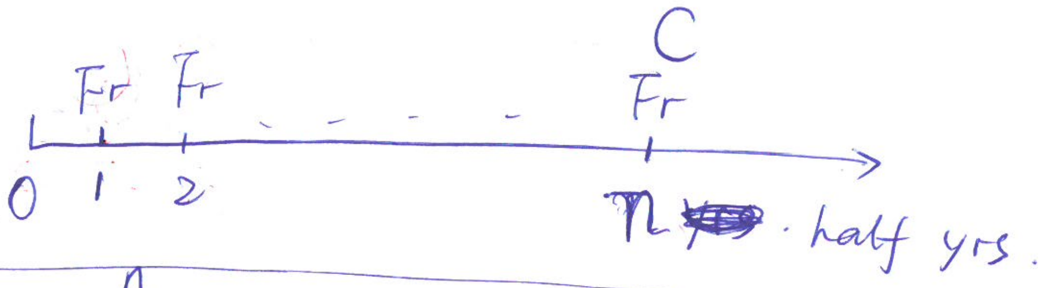


\$1

Sol: ①: $0.94 = (1+S_1)^{-1} \cdot \$1 \Rightarrow S_1 = 6.4\%$.

②: $0.70 = (1+S_5)^{-5} \cdot \$1 \Rightarrow S_5 = 7.4\%$

③: $0.47 = (1+S_{10})^{-10} \cdot \$1 \Rightarrow S_{10} = 7.8\%$.



$$\boxed{P = \sum_{p=1}^{n} Fr \cdot v_j^p + C \cdot v_j^n}$$

j: half-year.

$$P = \sum_{p=1}^{n} Fr \cdot v_{s_p}^p + C \cdot v_{s_n}^n$$

s_p: half-year spot rates.

$$P = Fr\left(v_{s_1} + v_{s_2}^2 + \cdots + v_{s_n}^n\right) + C \cdot v_{s_n}^n$$

$$\boxed{P = \frac{Fr}{1+s_1} + \frac{Fr}{(1+s_2)^2} + \cdots + \frac{Fr + C}{(1+s_n)^n}}$$

$$\boxed{P_p = \frac{1}{(1+s_p)^p}}$$

$(p = 1, 2 \cdots n)$: price of a unit zero-coupon bond maturing in p half years.

$$\boxed{P = Fr\left(P_1 + P_2 + \cdots + P_n\right) + C \cdot P_n}$$

Ex. $\boxed{P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n} \Rightarrow j \Rightarrow i = (1+j)^2 - 1$.

$n = 4 \times 2 = 8$ half yrs

$$P = Fr \cdot (v_{S_1} + v_{S_2}^2 + \cdots + v_{S_n}^n) + C \cdot v_{S_n}^n$$

$$= 100 \cdot \frac{10\%}{2} \cdot \left(\frac{1}{(1+\frac{7.5\%}{2})} + \frac{1}{(1+\frac{7.75\%}{2})^2} + \cdots + \frac{1}{(1+\frac{9\%}{2})^8} \right)$$

$$+ 100 \cdot \frac{1}{(1+\frac{9\%}{2})^8}$$

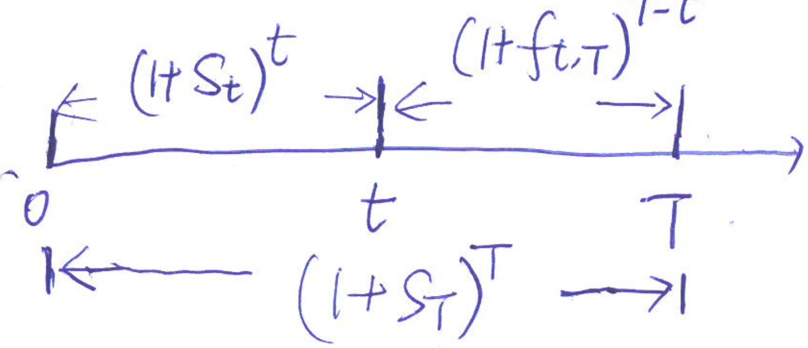
$$= \$103.72$$

$$103.72 = 100 \cdot \frac{10\%}{2} \cdot a_{\overline{8}|j} + 100 \cdot v_j^8$$

$\Rightarrow$ linear interpolation $\quad j =$

$\Rightarrow \quad i = (1+j)^2 - 1 = 9.07\%$.

Forward Rates.

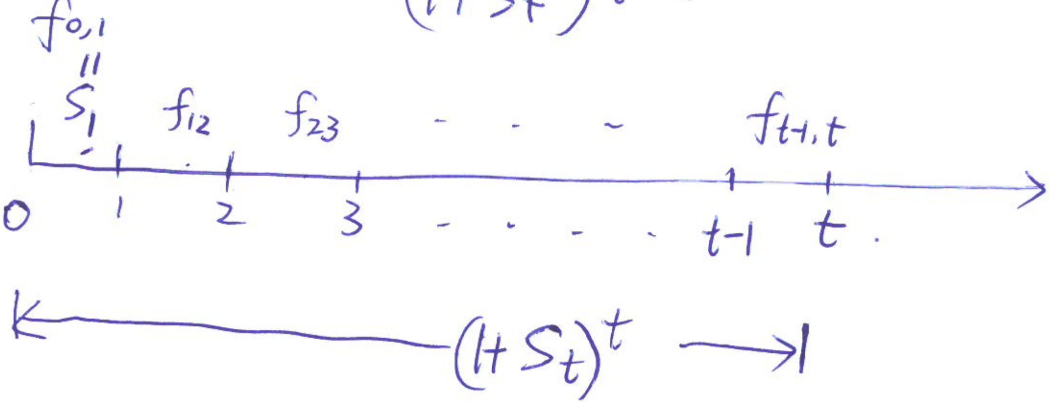


$$(1+S_T)^T = (1+S_t)^t \cdot (1+f_{t,T})^{T-t}.$$

if $T = t+1$

$$(1+S_{t+1})^{t+1} = (1+S_t)^t \cdot (1+f_{t,t+1})$$

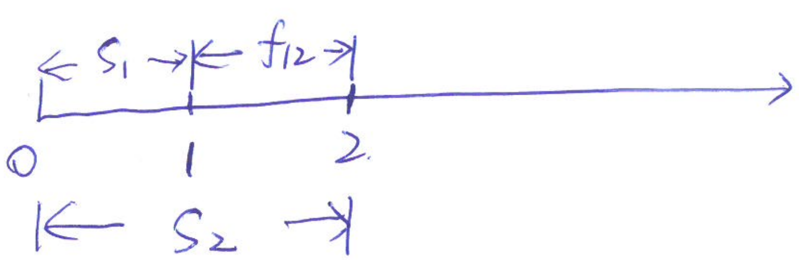
$$\Rightarrow \quad 1+f_{t,t+1} = \frac{(1+S_{t+1})^{t+1}}{(1+S_t)^t}.$$



$$(1+S_t)^t = (1+S_1)\cdot(1+f_{12})(1+f_{23})\cdots(1+f_{t-1,t}).$$

$$\boxed{P = \frac{Fr}{(1+f_{0,1})} + \frac{Fr}{(1+f_{0,1})(1+f_{12})} + \cdots + \frac{Fr+C}{(1+f_{0,1})(1+f_{12})\cdots(1+f_{n-1,n})}}$$

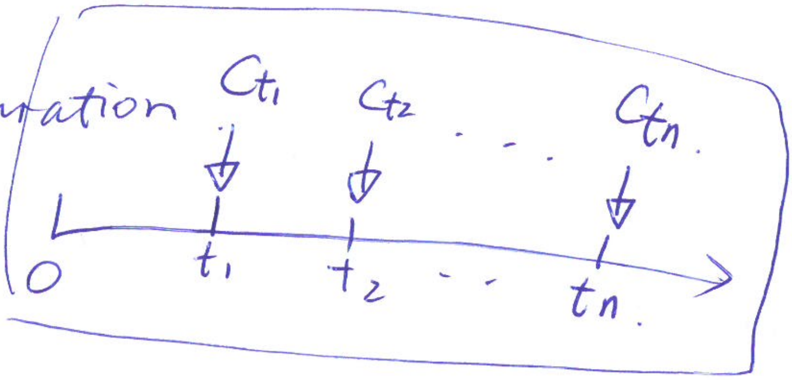
Ex: $S_1 = 6\%$, $S_2 = 8.08\%$ $\Big\} \Rightarrow f_{1,2}$:



$$(1+S_2)^2 = (1+S_1)\cdot(1+f_{12}) \Rightarrow f_{12} = \frac{(1+S_2)^2}{1+S_1} - 1$$

① Volatility / Effective Duration
② Duration
③ Covexity.

①&② Volatility & Duration



$$PV=\sum_{k=1}^{n} C_{t_k}\cdot v_i^{t_k}$$

$$\boxed{PV=\sum_{k=1}^{n} C_{t_k}\cdot(1+i)^{-t_k}}$$

$$v=-\frac{dPV}{di}\cdot\frac{1}{PV}$$ = Effective Duration OR Volatility

$$v=-\frac{\sum_{k=1}^{n} C_{t_k}\cdot(-t_k)\cdot(1+i)^{-t_k-1}}{\sum_{k=1}^{n} C_{t_k}\cdot(1+i)^{-t_k}}$$

$$v=\frac{\sum_{k=1}^{n} C_{t_k}\cdot t_k\cdot(1+i)^{-(t_k+1)}}{\boxed{\sum_{k=1}^{n} C_{t_k}\cdot(1+i)^{-t_k}}=PV}$$

8

$$DMT = \tau = \frac{\sum_{k=1}^{n} C_{t_k} \cdot t_k \cdot (1+i)^{-t_k}}{\sum_{k=1}^{n} C_{t_k} \cdot (1+i)^{-t_k}} : \boxed{\text{Duration}}$$

$\downarrow$ discounted mean term.

Ex 1.

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

$$P = \sum_{t=1}^{n} Fr \cdot v_j^t + C \cdot v_j^n$$

$$\Rightarrow \tau = \frac{\sum_{t=1}^{n} t \cdot Fr \cdot v_j^t + n \cdot C \cdot v_j^n}{\sum_{t=1}^{n} Fr\, v_j^t + C \cdot v_j^n = P}$$

Ex 2. $\boxed{n}$-year zero-coupon bond.

$$\tau = \frac{n \cdot C \cdot v_i^n}{C \cdot v_i^n} = \boxed{n}.$$