University of Toronto
Faculty of Arts and Science

MAT224H1S
Linear Algebra II

Final Examination
April 2012

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Duration: 3 hours

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

[10] **1.** Let $V = P_2(\mathbb{R})$ together with inner product

$$< p, q > = \int_0^1 p(x)q(x)\, dx.$$

(a) Find the matrix of the orthogonal projection onto the subspace $W = Span\{1, x\}$.

(b) What is the (minimum) distance of $1 + x + x^2$ to W?

EXTRA PAGE FOR QUESTION 1 - please do not remove.

[10] **2.** Let $V = P_2(\mathbb{C})$ together with inner product

$$< a_0 + a_1 x + a_2 x^2, b_0 + b_1 x + b_2 x^2 >= a_0 \bar{b}_0 + a_1 \bar{b}_1 + a_2 \bar{b}_2.$$

Show that T: $V \to V$ defined by $T(a_0 + a_1 x + a_2 x^2) = -ia_2 - a_1 x + ia_0 x^2$ is self-adjoint and find the spectral decomposition of T.

EXTRA PAGE FOR QUESTION 2 - please do not remove.

[10] **3.** Consider $P_1(\mathbb{R})$, the vector space of real linear polynomials, with inner product

$$< p(x), q(x) >= \int_0^1 p(x)q(x)\,dx.$$

Let T: $P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be defined by $T(p(x)) = p'(x) + 3p(x)$. Find $T^*(a + bx)$.

EXTRA PAGE FOR QUESTION 3 - please do not remove.

[10] **4.** Let $N\colon \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$N = \begin{bmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Show that N is nilpotent and find the smallest k such that $N^k = 0$.

(b) Find a canonical basis for $\mathbb{R}^4$ and the canonical form of N.

EXTRA PAGE FOR QUESTION 4 - please do not remove.

[10] **5.** Let T: $M_{2\times 2}\mathbb{R} \to M_{2\times 2}(\mathbb{R})$ be the linear operator defined by

$$T(A) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} A - A \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a basis α for $M_{2\times 2}(\mathbb{R})$ such that $[T]_{\alpha\alpha}$ is in Jordan canonical form and determine $[T]_{\alpha\alpha}$.

EXTRA PAGE FOR QUESTION 5 - please do not remove.

[10] **6.** Suppose V is an inner product space and $T\colon V \to V$ is a linear operator that satisfies $T^2 = T$.

(a) Show that $v - T(v) \in ker(T)$.

(b) Prove that $V = ker(T) \oplus im(T)$.