STAT2001 Tutorial 6 Solutions

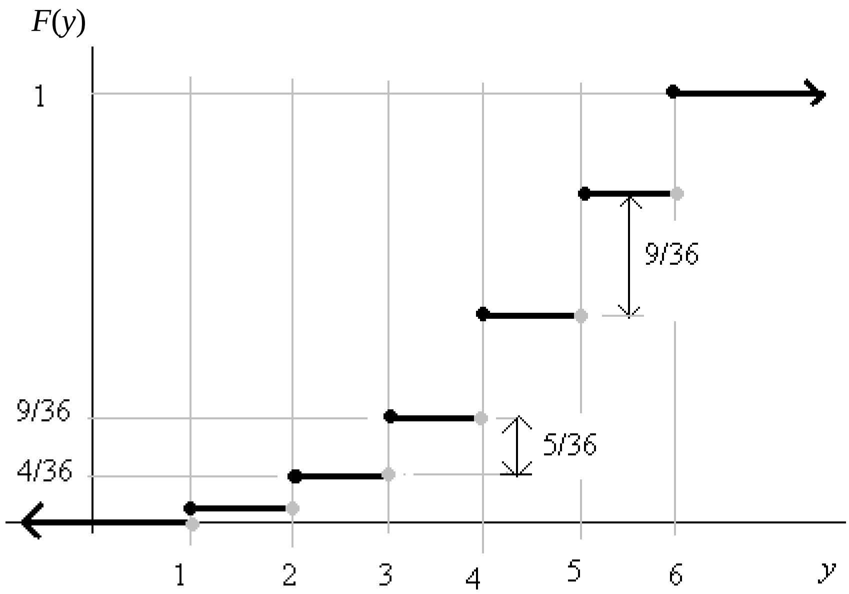
Problem 1

(a) $F(y) = P(Y \le y) = (y/6)^n$, $y = 1,2,\ldots,6$.

More precisely, $F(y) = \begin{cases} 0, & y < 1 \\ (k/6)^n, & k \le y < k+1 \\ 1, & y \ge 6 \end{cases}$ for each $k = 1,2,3,4,5$

or $$F(y) = \begin{cases} 0, & y < 1 \\ (1/6)^n, & 1 \le y < 2 \\ (2/6)^n & 2 \le y < 3 \\ (3/6)^n & 3 \le y < 4 \\ (4/6)^n & 4 \le y < 5 \\ (5/6)^n & 5 \le y < 6 \\ 1 & y \ge 6 \end{cases}$$

If $n = 2$, $F(y) = \begin{cases} 0, & y < 1 \\ 1/36, & 1 \le y < 2 \\ 4/36, & 2 \le y < 3 \\ 9/36, & 3 \le y < 4 \\ 16/36, & 4 \le y < 5 \\ 25/36, & 5 \le y < 6 \\ 1, & y \ge 6 \end{cases}$

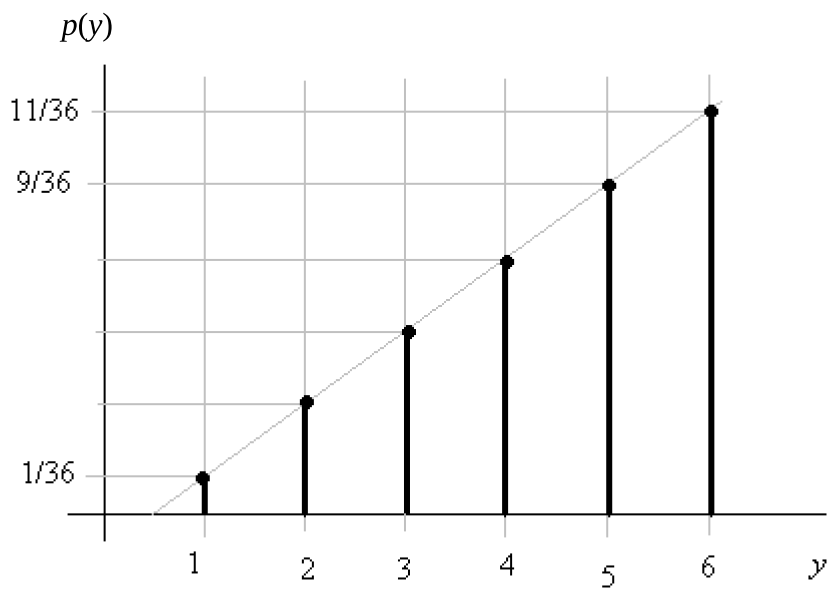


(b) $(3/6)^2 - (2/6)^2 = 9/36 - 4/36 = 5/36$.

Check: *P*(Largest number = 3) = *P*({13,23,33,32,31}) = 5/36.

(c) $p(y) = F(y) - F(y-1) = \left(\frac{y}{6}\right)^n - \left(\frac{y-1}{6}\right)^n$, y = 1,...,6.

If n = 2 then $p(y) = \begin{cases} 1/36, & y=1 \\ 3/36, & y=2 \\ 5/36, & y=3 \\ 7/36, & y=4 \\ 9/36, & y=5 \\ 11/36, & y=6 \end{cases}$



What is *Y*'s median? 5, since: $P(Y \le 5) = 25/36 \ge 1/2$ and

$P(Y \ge 5) = 1 - P(Y \le 4) = 1 - 16/36 = 20/36 \ge 1/2$

.

What is *Y*'s mode? 6, since $p(y)$ is a maximum at y = 6.

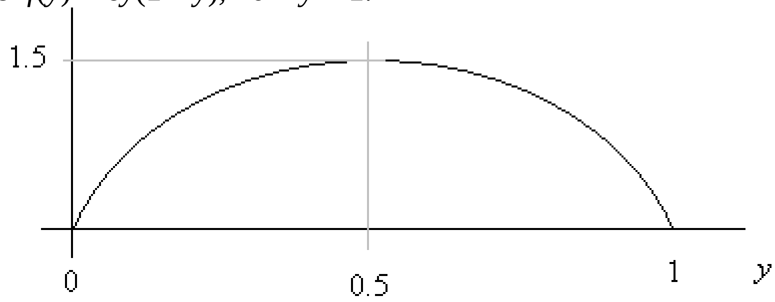
What is *Y*'s mean?

$$EY = \sum_{y=1}^{y} yp(y) = 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right) = \frac{161}{36} = 4.472.$$

Problem 2

(a) $1 = k\int_0^1 (y - y^2)dy = k\left[\left.\frac{y^2}{2} - \frac{y^3}{3}\right|_0^1\right] = k\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{k}{6}$. Therefore $k = 6$.

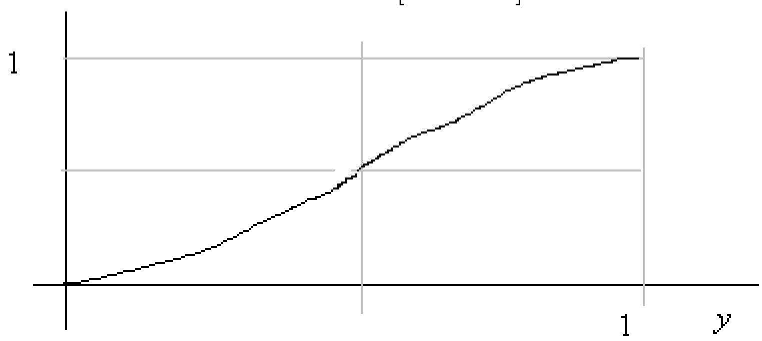
So $f(y) = 6y(1-y)$, $0 < y < 1$.



We can also write $f(y) = \dfrac{y^{2-1}(1-y)^{2-1}}{B(2,2)}$, which means that $Y \sim$ Beta(2,2).

What is Y's median? 0.5.
What is Y's mode? 0.5 also.
What is Y's mean? 2/(2 + 2) = 0.5 (all obvious by symmetry).

(b) $F(y) = 6\int_0^y (t - t^2)dt = 6\left[\left.\frac{t^2}{2} - \frac{t^3}{3}\right|_0^y\right] - 6\left(\frac{y^2}{2} - \frac{y^3}{3}\right) = 3y^2 - 2y^3$, $0 < y < 1$.



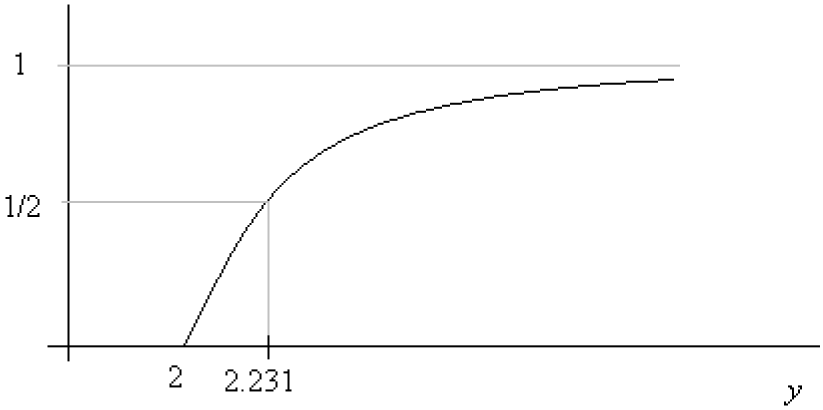
(c) $P(0.4 < Y < 0.5) = F(.5) - F(.4)$.
Now $F(.5) = 0.5$ (by symmetry), and $F(.4) = 3(.4)^2 - 2(.4)^3 = 0.352$.
Therefore $P(0.4 < Y < 0.5) = .5 - .352 = .148$.

(d) $P(.4 < Y \mid Y < .5) = \dfrac{P(.4 < Y, Y < .5)}{P(Y < .5)} = \dfrac{P(.4 < Y < .5)}{.5} = \dfrac{.148}{.5} = 0.296$.

Problem 3

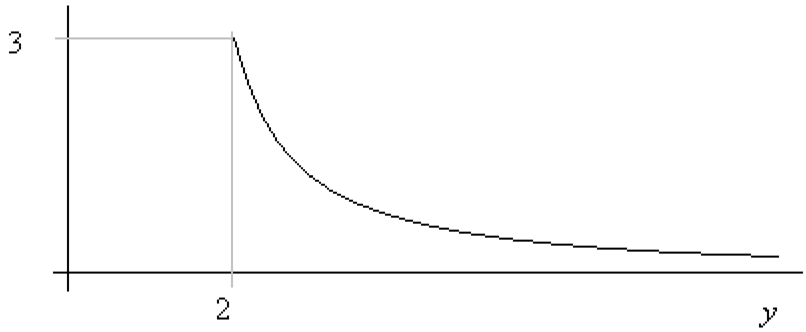
(a) By the properties of cdf's, $F(\infty)=1 \Rightarrow c-ke^{-\infty}=1 \Rightarrow c=1$.
Then also, $F(2) = 0 \Rightarrow 1-ke^{-3(2)}=0 \Rightarrow k=e^{6}$.

So $F(y)=1-e^{6}e^{-3y}=1-e^{-3(y-2)}, \quad y>2$.



What is Y's median? Solve $1-e^{-3(y-2)}$ = 1/2 to get 2.231.

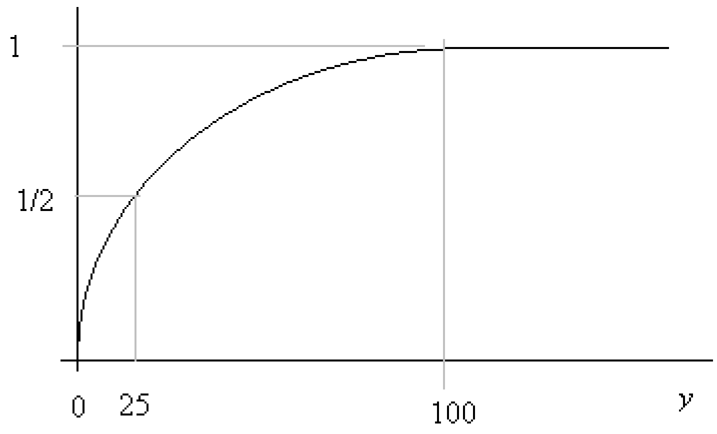
(b) $f(y)=F'(y)=-e^{-3(y-2)}(-3)=3e^{-3(y-2)}, y>2$.



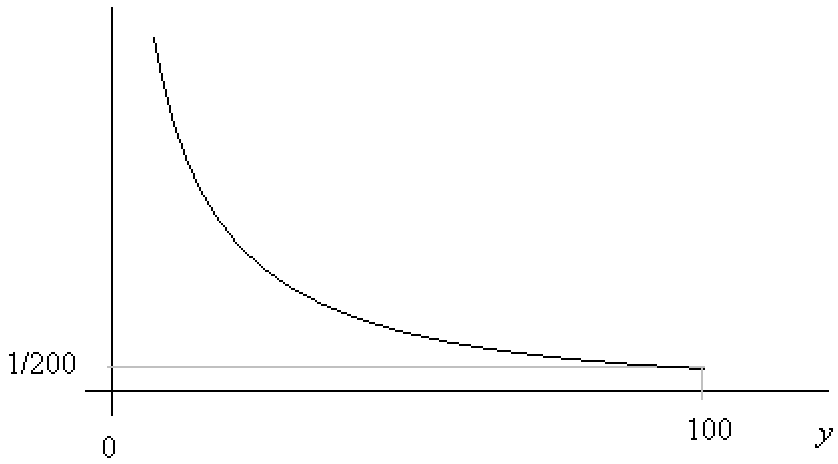
Note that Y has an exponential dsn shifted to the right by 2 units (Y ~ *Expo*(1/3) + 2, or $Y-2 \sim Expo(1/3)$). What is Y's mode? 2. What is Y's mean? 1/3 + 2 = 2.333.

Problem 4

(a) Observe that $Y=X^2$, where X = width of square, and $X \sim U(0,10)$.
Thus: $P(Y < 4) = P(X < 2) = 2/10$, $P(Y < 9) = P(X < 3) = 3/10$, etc.
We see that $P(Y \le y)=P(Y<y)=P(X<\sqrt{y})=\sqrt{y}/10$.
(Note that $P(Y \le q)=P(Y<q)$ for all q, since Y is continuous.)
Hence Y has cdf $F(y)=\dfrac{1}{10}y^{1/2}, 0 \le y \le 100$.
(Also, $F(y)=0$ for $y<0$ and $F(y)=1$ for $y > 100$.)



(b) $f(y) = F'(y) = \frac{1}{20} y^{-1/2}, 0 < y < 100$.



(c) *Mode*(*Y*) = 0.

This means the square is most likely to have a small area. For example, the area will more likely be between 5 and 10 than between 10 and 15.

$F(y) = \frac{1}{2} \Rightarrow \frac{1}{10} y^{1/2} = \frac{1}{2} \Rightarrow y = 25$. So *Median*(*Y*) = 25.   Thus an area of less that 25 square metres is equally likely as an area of more than 25 square metres.

This makes sense, since the median length of the sticks is 5, which corresponds to an area of 25.

What's the mean of *Y*? $EY = \int_0^{100} y \frac{1}{20} y^{-1/2} dy = ... = 33.3333$,

or more simply, $EY = EX^2 = VX + (EX)^2 = \frac{(10-0)^2}{12} + 5^2 = \frac{100}{3}$,

using basic facts regarding the uniform distribution .