Worth: 3% **Due:** By 12 noon on Tuesday 6 March.

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1. - *In Symbolic Notation:* $\neg\left(\exists x \in \mathbb{R}, x^2 - 6x + 10 = 0\right)$ or $\forall x \in \mathbb{R}, x^2 - 6x + 10 \neq 0$.
   - Proof structure — a proof by contradiction.
   - **Proof:**

     Assume $\exists x \in \mathbb{R}, x^2 - 6x + 10 = 0$.
       Let $x_0 \in \mathbb{R}$ be such that $x_0^2 - 6x_0 + 10 = 0$.
       Then $(x_0^2 - 6x_0 + 9) + 1 = 0$.
       Then $(x_0 - 3)^2 + 1 = 0$. # fact 1
       But $x_0 - 3 \in \mathbb{R}$. # reals are closed under subtraction
       And $(x_0 - 3)^2 \geqslant 0$. # square of real number is nonnegative
       And $1 > 0$. # square of real number is nonnegative
       Then $(x_0 - 3)^2 + 1 > 0$. # fact 2
       But fact 2 contradicts fact 1.
     Then $\neg\left(\exists x \in \mathbb{R}, x^2 - 6x + 10 = 0\right)$.

2. (a)
   - *In Symbolic Notation:* $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.
   - Proof structure — a direct proof, using proof by cases.
   - Note — we need to be careful with the value 0 since for example $x < 0$ and $y \geqslant 0$ means $xy \leqslant 0$ not $xy < 0$, and these cases are not consistent with the cases in the definition.
   - **Proof:**

     Assume $x \in \mathbb{R}$.
       Then $x < 0$ or $x = 0$ or $x > 0$.
       Case 1: Assume $x < 0$.
         Then $|x| = -x$. # by definition of $|\cdot|$.
         Assume $y \in \mathbb{R}$.
           Then $y < 0$ or $y = 0$ or $y > 0$.
           Case 1a: Assume $y < 0$.
             Then $|y| = -y$. # by definition of $|\cdot|$.
             Then $\begin{aligned} |x||y| &= (-x)(-y) \\ &= xy \\ &= |xy| \quad \text{\# since } xy \geqslant 0 \text{ when } x < 0 \text{ and } y < 0. \end{aligned}$
             Then $|x||y| = |xy|$.
           Then $y < 0 \Rightarrow |x||y| = |xy|$.
           Case 1b: Assume $y = 0$.
             Then $|y| = 0$. # by definition of $|\cdot|$.
             Then $\begin{aligned} |x||y| &= (-x)(0) \\ &= 0 \end{aligned}$
             And $\begin{aligned} |xy| &= |x0| \\ &= 0 \end{aligned}$
             Then $|x||y| = |xy|$.
           Then $y = 0 \Rightarrow |x||y| = |xy|$.
           Case 1c: Assume $y > 0$.
             Then $|y| = y$. # by definition of $|\cdot|$.
             Then $\begin{aligned} |x||y| &= (-x)(y) \\ &= -xy \end{aligned}$
             And $|xy| = -xy$ # since $xy < 0$ when $x < 0$ and $y > 0$.
             Then $|x||y| = |xy|$.
           Then $y > 0 \Rightarrow |x||y| = |xy|$.
         Then $\forall y \in \mathbb{R}, |x||y| = |xy|$.

Then $x < 0, \forall y \in \mathbb{R}, |x||y| = |xy|$.

Case 2: Assume $x = 0$.
Then $|x| = 0$. # by definition of $|\cdot|$.
Assume $y \in \mathbb{R}$
Then $|x||y| = 0|y| = 0$.
And $|xy| = |0y| = 0$
Then $|x||y| = |xy|$.
Then $\forall y \in \mathbb{R}, |x||y| = |xy|$.
Then $x = 0 \Rightarrow |x||y| = |xy|$.

Case 3: Assume $x > 0$.
Then $|x| = x$. # by definition of $|\cdot|$.
Assume $y \in \mathbb{R}$
Then $y < 0$ or $y \geqslant 0$.
Case 3a: Assume $y < 0$.
Then $|y| = -y$. # by definition of $|\cdot|$.
Then $|x||y| = x(-y) = -xy$.
And $|xy| = -xy$ # since $xy < 0$ when $x > 0$ and $y < 0$.
Then $|x||y| = |xy|$.
Then $y < 0 \Rightarrow |x||y| = |xy|$.
Case 3b: Assume $y \geqslant 0$.
Then $|y| = y$. # by definition of $|\cdot|$.
Then $|x||y| = (x)(y) = xy = |xy|$ # since $xy \geqslant 0$ when $x > 0$ and $y \geqslant 0$.
Then $|x||y| = |xy|$.
Then $y \geqslant 0 \Rightarrow |x||y| = |xy|$.

Then $\forall y \in \mathbb{R}, |x||y| = |xy|$.
Then $x > 0, \forall y \in \mathbb{R}, |x||y| = |xy|$.
Then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

(b)

- *In Symbolic Notation:* $\forall x_1 \in \mathbb{R}, \forall x_2 \in \mathbb{R}, \forall y_1 \in \mathbb{R}, \forall y_2 \in \mathbb{R}, |x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1 y_1| > |x_2 y_2|$.
- Proof structure — a direct proof, using proof by cases.
- Note — if $|a| > |b|0$, we can conclude that $|a| > 0$ but $|b| \geqslant 0$.
- Note — the proof uses the given result that for real $t > 0$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow tx > ty$.
- **Proof:**

Assume $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, y_1 \in \mathbb{R}, y_2 \in \mathbb{R}$.
Assume $|x_1| > |x_2| \wedge |y_1| > |y_2|$
Then $|x_1| > |x_2|$.
Then $|y_1| > |y_2|$.
Then $|y_1| > 0$. # since $|y_2| \geqslant 0$.
Then $|y_1||x_1| > |y_1||x_2|$. # by given result
Then $|x_1||y_1| > |y_1||x_2|$. # commutivity of multiplication
Then $|x_1 y_1| > |y_1||x_2|$. # by part (a)

Also $|x_2| = 0$ or $|x_2| > 0$.
Case 1: Assume $|x_2| = 0$.

$$\text{Then } |y_1||x_2| = |y_1|0 = 0.$$

$$\text{Then } |x_2y_2| = |y_2x_2| = |y_2||x_2| = |y_2|0 = 0.$$

Then $|y_1||x_2| = |x_2y_2|$.
Then $|y_1||x_2| \geqslant |x_2y_2|$.

Case 2: Assume $|x_2| > 0$.

$$\begin{aligned}\text{Then } |y_1||x_2| &= |x_2||y_1| \\ &> |x_2||y_2| && \text{\# by given result} \\ &= |x_2y_2| && \text{\# bypart(a)}\end{aligned}$$

Then $|y_1||x_2| > |x_2y_2|$.
Then $|y_1||x_2| \geqslant |x_2y_2|$.

Then, in either case, $|y_1||x_2| \geqslant |x_2y_2|$.
Then $|x_1y_1| > |x_2y_2|$.
Then $|x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1y_1| > |x_2y_2|$
Then $\forall x_1 \in \mathbb{R}, \forall x_2 \in \mathbb{R}, \forall y_1 \in \mathbb{R}, \forall y_2 \in \mathbb{R}, |x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1y_1| > |x_2y_2|$

3. (a)

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Working in base 2:

carry:  11111

           1011              110110
       + 110110           x    1011
       --------          ---------
        1000001              110110
                            1101100
                                  0
                          110110000
                          ---------
                         1001010010
```

(b)

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Working in base 4:

carry:
           3130                3130      note: 3x0=0, 3x3 = 21, 3x1=3.
       +    103           x     103
       --------          ---------
           3233               22110
                                  0
                             313000
                          ---------
```

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1001110
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(c) If $a = (342)_8$ and $b = (173)_8$, find $a - b$ without coverting to base 10.

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See: http://www.lyricsfreak.com/t/tom+lehrer/new+math_20138395.html

Or do the work ..

Working in base 8:

borrow:     11        note: 12 - 3 = 7
            342             13 - 7 = 4
       -    173
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            147
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