Exerzitien II

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Sept. 29, in your tutorial.

Reading suggestion: **Sums and direct sums**, Review all of Chapter 1 in Axler, and read first section of Chapter 2.

Exercise 1.

1. *Let U_1, U_2 be subspaces of the vector space V. Prove that their intersection $U_1 \cap U_2$ is also a subspace of V.*
2. *Let $U_1, U_2, \ldots$ be an infinite collection of subspaces, where each subspace U_n is labeled by a natural number $n \in \mathbb{N}$. Prove that the intersection of all the U_n is a subspace of V. In other words, show that the following is a subspace of V:*

$$\bigcap_{n\in\mathbb{N}} U_n = \{x \in V \ : \ x \in U_n \text{ for all } n \in \mathbb{N}\}$$

3. *Is the union $U_1 \cup U_2$ of subspaces also a subspace? If yes, prove it, if no, give a counterexample.*

Exercise 2. *Let $V = \mathbb{R}^{\mathbb{R}}$ be the real vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Define the "even" and "odd" functions as follows:*

$$V_e := \{f \in V \mid f(-x) = f(x) \ \forall x \in \mathbb{R}\},$$
$$V_o := \{f \in V \mid f(-x) = -f(x) \ \forall x \in \mathbb{R}\}.$$

1. *Prove that V_e and V_o are subspaces.*
2. *Show that the sum $V_e + V_o$ is all of V.*
3. *Prove that the sum is direct, i.e. $V = V_e \oplus V_o$.*
4. *Give the decomposition of the function $f(x) = e^x$ according to the above direct sum. That is, write f as a sum $f = f_e + f_o$, where $f_e \in V_e$ and $f_o \in V_o$. Do you recognize f_e, f_o?*

Exercise 3. *Let $V = (\mathbb{F}_2)^3$, the set of triples (x, y, z) of numbers in $\mathbb{F}_2$, the field with two elements. V is a vector space over $\mathbb{F}_2$.*

1. *Prove that any subspace of V must have either 1, 2, 4, or 8 elements.*
2. *List all the subspaces of V with 1, 2, 4, and 8 elements.*
3. *Give a system of linear equations whose solutions give exactly the linear subspace $\{(0, 0, 0), (1, 1, 1)\}$.*