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STAT 6046 Tutorial Week 12

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Today's plan

- Brief review of course material
- Go through selective tutorial questions

Stochastic interest rate models

- So far we have taken a **deterministic** approach, where it was assumed that interest rates used in a financial transaction have been known in advance.

- Although this is true in some practical situations, such as for loans with a fixed rate of interest, in other situations we will not know what future interest rates will be, for example, for variable interest rate loans. In these cases the rate of interest can be treated as a random variable and is said to be **stochastic**.

Statistic Revision

- For a **discrete random variable** $\widetilde{\boldsymbol{X}}$, with probability function $p(x) = \Pr[\widetilde{\boldsymbol{X}} = \boldsymbol{x}]$.
- Mean: $\mathrm{E}[\widetilde{\boldsymbol{X}}] = \sum_{\boldsymbol{x}} \boldsymbol{x} * \boldsymbol{p}(\boldsymbol{x})$
- Variance: $\mathrm{Var}[\widetilde{\boldsymbol{X}}] = \mathrm{E}[\widetilde{\boldsymbol{X}}^{\boldsymbol{2}}] - (\mathrm{E}[\widetilde{\boldsymbol{X}}])^{\boldsymbol{2}} = \sum_{\boldsymbol{x}} \boldsymbol{x}^{\boldsymbol{2}} * \boldsymbol{p}(\boldsymbol{x}) - (\sum_{\boldsymbol{x}} \boldsymbol{x} * \boldsymbol{p}(\boldsymbol{x}))^{\boldsymbol{2}}$

For a **continuous random variable** $\widetilde{X}$, with probability density function $f(x)$, the probability $P[a < \widetilde{X} < b] = \int_a^b f(x)dx$.

$\widetilde{X}$ has mean: $E[\widetilde{X}] = \int_{-\infty}^{\infty} x \cdot f(x)dx$

and variance: $Var[\widetilde{X}] = E[\widetilde{X}^2] - (E[\widetilde{X}])^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - \left(\int_{-\infty}^{\infty} x \cdot f(x)dx \right)^2$

Statistic Revision

Uniform distribution

$$f(x) = \frac{1}{b-a} \text{ for } a < x < b$$

$$E[\tilde{X}] = \frac{a+b}{2}$$

$$Var[\tilde{X}] = \frac{(b-a)^2}{12}$$

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \text{ for } -\infty < x < \infty$$

$$E[\tilde{X}] = \mu$$

$$Var[\tilde{X}] = \sigma^2$$

Recall that if $\tilde{X}$ is normally distributed with mean and variance as above, then

$$P[a < \tilde{X} < b] = P\left[\frac{a-\mu}{\sigma} < \frac{\tilde{X}-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] = P\left[\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right]$$

where Z has a standard normal distribution (ie. normal distribution with mean 0 and variance 1).

Statistical tables can be used with a standard normal variable to find probabilities.

Single cash flow

$$\tilde{i} = \begin{cases} i_a & prob = a \\ i_b & prob = b \end{cases}$$

The expected value is $E[\tilde{i}] = \sum_i i \cdot p(i) = a \cdot i_a + b \cdot i_b$

To find the variance we can use $Var[\tilde{i}] = E[\tilde{i}^2] - (E[\tilde{i}])^2$

The second moment is $E[\tilde{i}^2] = a \cdot i_a^2 + b \cdot i_b^2$

So, the variance can be written: $Var[\tilde{i}] = (a \cdot i_a^2 + b \cdot i_b^2) - (a \cdot i_a + b \cdot i_b)^2$

Multiple cash flows

– Assuming independence between interest rates:

$$\tilde{S}(n) = (1+\tilde{i}_1)(1+\tilde{i}_2)\cdots(1+\tilde{i}_n)$$
$$E\left[\tilde{S}(n)\right] = E\left[1+\tilde{i}_1\right]\cdot E\left[1+\tilde{i}_2\right]\cdots E\left[1+\tilde{i}_n\right]$$
$$E\left[\tilde{S}(n)^2\right] = E\left[(1+\tilde{i}_1)^2\right]\cdot E\left[(1+\tilde{i}_2)^2\right]\cdots E\left[(1+\tilde{i}_n)^2\right]$$

If the interest rates are independent and identically distributed, with mean $E\left[\tilde{i}\right]$ and variance $Var\left[\tilde{i}\right]$, then the mean and variance of the accumulated value of 1 after n periods are:

$$E\left[\tilde{S}(n)\right] = \left(E\left[1+\tilde{i}\right]\right)^n$$
$$E\left[\tilde{S}(n)^2\right] = \left(E\left[(1+\tilde{i})^2\right]\right)^n$$
$$Var\left[\tilde{S}(n)\right] = E\left[\tilde{S}(n)^2\right] - \left(E\left[\tilde{S}(n)\right]\right)^2 = \left(E\left[(1+\tilde{i})^2\right]\right)^n - \left(E\left[1+\tilde{i}\right]\right)^{2n}$$

Log-Normal

If the annual rate of interest is a random variable, then so is the force of interest $\tilde{\delta} = \ln(1+\tilde{i})$.

Since $\tilde{S}(n) = (1+\tilde{i}_1)(1+\tilde{i}_2)\cdots(1+\tilde{i}_n)$

$$\Rightarrow \ln\left[\tilde{S}(n)\right] = \ln(1+\tilde{i}_1) + \ln(1+\tilde{i}_2) + \ldots + \ln(1+\tilde{i}_n) = \tilde{\delta}_1 + \tilde{\delta}_2 + \ldots + \tilde{\delta}_n$$

By the Central Limit Theorem, the sum of independent, identically distributed random variables is approximately normally distributed for large n.

Therefore, for large n, if the forces of interest $\tilde{\delta}_t$ are independent and identically distributed with mean $E\left[\tilde{\delta}\right]$ and variance $Var\left[\tilde{\delta}\right]$, then $\ln\left[\tilde{S}(n)\right]$ is approximately normally distributed with:

Mean: $E\left[\ln\left[\tilde{S}(n)\right]\right] = E\left[\tilde{\delta}_1 + \tilde{\delta}_2 + \ldots + \tilde{\delta}_n\right] = n \cdot E\left[\tilde{\delta}\right]$

Variance: $Var\left[\ln\left[\tilde{S}(n)\right]\right] = Var\left[\tilde{\delta}_1 + \tilde{\delta}_2 + \ldots + \tilde{\delta}_n\right] = n \cdot Var\left[\tilde{\delta}\right]$

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Annuities

If interest rates are independent and identically distributed with mean $E[\tilde{i}]$, then

$$E\left[\tilde{s}_{\overline{n}|}\right] = 1 + E\left[1+\tilde{i}\right] + \left(E\left[1+\tilde{i}\right]\right)^2 + \ldots + \left(E\left[1+\tilde{i}\right]\right)^{n-1} \Rightarrow$$

$$\boxed{E\left[\tilde{s}_{\overline{n}|}\right] = s_{\overline{n}|}}$$

where $s_{\overline{n}|}$ is evaluated at the interest rate $E[\tilde{i}]$.

Similarly it can be shown that if an annuity-due is expressed as a random variable $\tilde{\ddot{s}}_{\overline{n}|}$ then,

$$\boxed{E\left[\tilde{\ddot{s}}_{\overline{n}|}\right] = \ddot{s}_{\overline{n}|}}$$