§13 - Urysohn's Lemma and the Tietze Extension Theorem

1 Motivation

Our investigation of topology has been bouncing around between studying spaces and their topological properties. We briefly looked at continuous functions, but we haven't focused too much on them. Continuous functions are really important in the study of algebraic topology. In algebraic topology we are concerned with "continuous deformations" of curves into other curves. As a result it is useful to be able to construct various continuous functions.

We have seen some straightforward continuous functions (like compositions of continuous functions, constant functions, projections onto product spaces). We will look at Urysohn's construction of a continuous function in a normal space gives us a great new way to construct a continuous function. This is an example of a theorem whose proof is very interesting.

2 Some Previous Knowledge

The main result will involve constructing a very nice continuous function on a normal space, so let me remind you of the definitions.

Definition. *A topological space* $(X, \mathcal{T})$ *is a* ***Normal Space*** *if it is* T_1 *and for every two disjoint closed set* C *and* D *there are disjoint open sets* U *and* V *such that* $C \subseteq U$ *and* $D \subseteq V$.

Our sound bite was "you can separate disjoint closed sets from each other by disjoint open sets". Urysohn's Lemma will be concerned with showing that "you can separate disjoint closed sets from each other by a continuous function". Let's state Urysohn's lemma and see what this actually means. Recall that the picture says everything about the construction. You are strongly encouraged to follow this proof with a picture! First let us recall an alternate characterization of normality:

Lemma. *A topological space* X *is Normal iff for every open set* U *and every closed set* $C \subseteq U$ *there is an open set* V *such that* $C \subseteq V \subseteq \overline{V} \subseteq U$.

Theorem (Urysohn's Lemma, 1920s)**.** *Let* X *be a* T_1 *space.* X *is normal if and only if for every pair* A, B *of disjoint closed subsets of* X *there is a continuous function* $f : X \longrightarrow [0, 1]$ *such that* $f(a) = 1$ *for all* $a \in A$*, and* $f(b) = 0$ *for all* $b \in B$.

Proof. The [$\Leftarrow$] direction is straightforward. Assume that we can separate disjoint closed sets by a continuous function. Let A, B be disjoint closed sets in X, and find a function

$f : X \longrightarrow [0,1]$ such that $f(a) = 0$ for all $a \in A$, and $f(b) = 1$ for all $b \in B$. Note that $B \subseteq f^{-1}[[0, \frac{1}{2})]$ and $A \subseteq f^{-1}[(\frac{1}{2}, 1]]$. Moreover, they are clearly disjoint and they are open since f is continuous.

Most of the work is in the [$\Rightarrow$] direction. The idea here is to create approximations to our continuous function then take "the limit of these approximations" which, after an argument, turns out to be continuous.

First off, use normality to find an open $U_{\frac{1}{2}}$ such that

$$A \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq X \setminus B.$$

Essentially we have "split the space in two". (Also, for convenience, define $U_1 := X$). Now we can (very poorly) approximate our desired continuous function by taking

$$g(x) := \inf\{t : x \in U_t\}$$

this is a two valued-function that sends elements in A to $\frac{1}{2}$ and elements in B to 1, but it is very much not continuous. So let's approximate better!

Again, by normality, find open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that

$$A \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq X \setminus B.$$

Redefining our g function with these new U_t gives us a 4-valued function that still sends elements of A to $\frac{1}{4}$, but sends elements of B to 1. It still isn't continuous, but it seems a bit better!

Our next step is to find and include $U_{\frac{1}{8}}$, $U_{\frac{3}{8}}$, $U_{\frac{5}{8}}$ and $U_{\frac{7}{8}}$, so that they still have that desired closure property. Then we do the same for things of the form $U_{\frac{n}{16}}$, then $U_{\frac{m}{32}}$, *and so on...*

Strictly speaking, we should use mathematical induction to rigourously construct all $U_{\frac{m}{2^n}}$ for all m, n, but this is really horrible to write down (and fairly impossible to read). You may attempt it as an exercise if you wish though!

Either way, let's suppose that we have a collection of open sets

$$\{U_t : t = \frac{m}{2^n}, n, m \in \mathbb{N}, 0 < t < 1\}$$

such that

$$s < t \Rightarrow \overline{U_s} \subseteq U_t.$$

Moreover assume that $A \subseteq U_t \subseteq X \setminus B$ for all such t. Now comes the part where we define the continuous function $f : X \longrightarrow [0,1]$ ... sort of. The collection of all t of the form $\frac{m}{2^n}$ (called the **Dyadic Numbers**) was convenient to work with above, but now all we care about is the following:

Fact: The collection D of all dyadic numbers between 0 and 1 is a dense subset of $[0,1]$.

(We will skip this proof because it is quite believable.) Now, let us define $f : X \longrightarrow [0,1]$ by

$$f(x) := \inf\{t \in D : x \in U_t\}$$

See, this is really just "the limit" of our approximations.

Claim 1[f does what we want]: $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

This is immediate from density of D and that $A \subseteq U_t \subseteq X \setminus B \subseteq X = U_1$ for all $t \in D$. (The point was that the idea was clever, but the proof is relatively simple. Presumably, historically, this was the idea that Urysohn had, then worked through the details to make it make sense.)

Claim 2: f is continuous.

This part of the proof is annoying, but isn't particularly difficult. To make our lives as simple as possible (but not simpler! -Einstein) we will use the "subbasis version of continuity" (that a function is continuous iff the preimage of every subbasic open set is open) that we saw in §6.3.

<u>Case 1, $[0,a)$ for $0 < a < 1$</u>: Let us show that $f^{-1}([0,a))$ is open in X. Note that

$$f^{-1}([0,a)) = \{x \in X : f(x) < a\},$$

and since D is dense we get that this is also equal to $\bigcup_{t<a} U_t$, which is a union of open sets (in X). Hence $f^{-1}([0,a))$ is open in X. (That was the easy case!)

<u>Case 2, $(a,1]$ for $0 < a < 1$</u>: We will show that $f^{-1}((a,1])$ is open by showing that $X \setminus f^{-1}((a,1])$ is closed. We will do that by showing double containment with a closed set. (This proof works best if you draw a number line of things in $[0,1]$.) First note that

$$X \setminus f^{-1}((a,1]) = \{x \in X : f(x) \leq a\}.$$

Subclaim 2.1: $\{x \in X : f(x) \leq a\} = \bigcap_{a<t} \overline{U_t}$, a closed set in X.

[$\subseteq$] Let $y \in \{x \in X : f(x) \leq a\}$. Since D is dense, there is a $s \in D$ such that $a < s < 1$. Now for any $t \in D$ with $a < t < s$ we get that $\overline{U_t} \subseteq U_s$, and since $y \in U_t \subseteq \overline{U_t}$ (as $f(y) \leq a < t$) we get that $y \in \bigcap_{a<t} \overline{U_t}$.

[$\supseteq$] Let $y \in \bigcap_{a<t} \overline{U_t}$, and let $\epsilon > 0$ (and additionally let $\epsilon < 1 - a$). By density, there is a $t \in D$ such that $a < t < a + \epsilon < 1$. Again by density, there is an $s \in D$ such that $t < s < a + \epsilon$. Thus $\overline{U_t} \subseteq U_s$, and most importantly, this means that $y \in U_s$ so $f(y) < s < a + \epsilon$. Since ϵ was positive and arbitrary we get that $f(y) \leq a$, as desired. $\square$

Phew! That last part there almost got me. Showing Claim 2 is rather tedious. If you want to check your understanding of the proof, check that in subclaim 2.1 we really need an $s \in D$ *and* a $t \in D$. The problem is that tricky 1!

Here's a corollary which should serve as an exercise in manipulating Urysohn's Lemma.

Corollary. *Let X be a T_1 space and let $a < b$ be real numbers. X is normal if and only if for every pair C, D of disjoint closed subsets of X there is a continuous function $f : X \longrightarrow [a, b]$ such that $f(c) = a$ for all $c \in C$, and $f(d) = b$ for all $d \in D$.*

One More Time Exercise: Show that Urysohn's lemma implies that if X is a normal space with A, B, C all mutually disjoint closed sets, then there is a continuous function $f : X \longrightarrow [0, 2]$ such that $f(a) = 0$ ($\forall a \in A$), $f(b) = 1$ ($\forall b \in B$) and $f(c) = 2$, ($\forall c \in C$).

3 Tietze Extension Theorem

In the same family of theorems as Urysohn's Lemma there is a theorem called the Tietze Extension Theorem. It also gives a way to construct a continuous function, but this one assumes that there is already some amount of the function built. It is almost like a topological version of the Direchlet Problem. The proof is omitted here, mostly because I have never seen an application of this theorem (or of the construction in the proof). If you are interested, it is one of the New Ideas problems on Assignment 7.

Theorem (Tietze Extension Theorem)**.** *Let X be a T_1 space. X is normal if and only if whenever A is a closed subset of X and $f : A \longrightarrow [-1, 1]$ is a continuous function, there is a continuous function $F : X \longrightarrow [-1, 1]$ such that $F(a) = f(a)$ for all $a \in A$.*

One Direction Exercise: Prove the [$\Leftarrow$] direction of the Tietze Extension Theorem by using a tiny bit of thought.

4 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

One More Time : Show that Urysohn's lemma implies that if X is a normal space with A, B, C all mutually disjoint closed sets, then there is a continuous function $f : X \longrightarrow [0, 2]$ such that $f(a) = 0$ $(\forall a \in A)$, $f(b) = 1$ $(\forall b \in B)$ and $f(c) = 2$, $(\forall c \in C)$.

One Direction : Prove the $[\Leftarrow]$ direction of the Tietze Extension Theorem by using a tiny bit of thought.