Homework 4 Rui Qiu #999292509

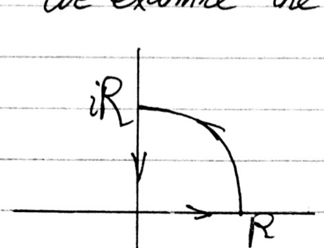
1. 3.1.4.

\# of zeros of f in the first quadrant.

$$f(z) = z^2 + iz + 2 + i$$

Solution:

We examine the function $f(z)$ on the ~~cont~~ contour shown below:



On the segment $0 \le x \le R$, $f(x) = x^2 + ix + 2 + i$ & $|f(x)| \geq |2+i|$

On the quarter-circle, $z = Re^{it}$, $0 \le t \le \frac{\pi}{2}$.

$$f(Re^{it}) = R^2 e^{2it}\left(1 + \frac{i}{Re^{it}} + \frac{2+i}{R^2 e^{2it}}\right) = R^2 e^{2it}(1 + \zeta)$$

which approaches $R^2 e^{2it}$ as $R \nearrow$

Thus $\arg f(Re^{it})$ is approximately $\arg(e^{2it}) = 2t$ for $R \nearrow$
so $\arg f(Re^{it})$ increases from 0 to π as t increases from 0 to $\frac{\pi}{2}$.

On the segment $z = iy$, $R \geq y \geq 0$.

$$f(iy) = -y^2 - y + 2 + i$$

$$\mathrm{Re}(f(iy)) = -y^2 - y + 2 \begin{cases} \geq 0 & \text{when } 0 \le y \le 1 \\ < 0 & \text{when } y > 1 \end{cases}$$

$$\mathrm{Im}(f(iy)) = 1 > 0.$$

Hence as y decreases from R to 0, $f(iy)$ lies in the 4th quadrant & then moves towards the point $w = 2+i$
Consequently, z traverses the contour, $\arg f(z)$ increases exactly by 2π (from $2+i$ to $2+i$).
point

Then by The Argument Principle:

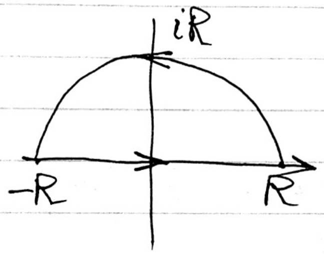
$$\frac{1}{2\pi}\cdot 2\pi = 1 = \#\text{ of zeros of } f \text{ inside 1st quadrant} - \#\text{ of poles of } f \text{ inside 1st quadrant.}$$

Since no poles exist, the # of zeros of f inside 1st quadrant is 1.

2. 3.1.8.

$$f(z) = 2z^4 - 2iz^3 + z^2 + 2iz - 1$$

The upper-half-plane is $\{z : \operatorname{Im} z > 0\}$



Solution:

On the segment $-R \le x \le R$,

$$f(z) = 2z^4 + z^2 - 1 + 2iz(-z^2+1)$$

so we R is very large, $f(R)$ and $f(-R)$ have opposite imaginary part, but the real part is the same i.e. from $-R$ to R, ~~$f(R)$~~ $\arg f(z)$ changes 0.

On the curve (semicircle $z = Re^{i\theta}$ with $\theta \in [0, \pi]$.

$$f(Re^{i\theta}) = 2R^4e^{i4\theta} - 2iR^3e^{3i\theta} + R^2e^{2i\theta} + 2iRe^{i\theta} - 1$$

$$= 2R^4e^{i4\theta}\left(1 - \frac{i}{Re^{i\theta}} + \frac{1}{2R^2e^{2i\theta}} + \frac{i}{R^3e^{3i\theta}} - \frac{1}{2R^4e^{i4\theta}}\right)$$

$$\to 2R^4e^{i4\theta}$$

as $R \to \infty$.

so $\arg f(Re^{i\theta})$ increases from 0 to 4π as θ increases from 0 to π.

So ~~$f(z)$~~ $\arg f(z)$ increases 4π in total.

So $\frac{1}{2\pi}\cdot 4\pi = 2 = \#$ of zeros in the upper half-plane.

3. 3.1.10

Show that there is no entire function F s.t.

$F(x) = 1 - \exp[2\pi i/x]$ for $1 \leq x \leq 2$.

4. 3.1.12

z^3-3z+1 in $1<|z|<2$.

Solution. Let $p(z)=z^3-3z+1$

On the circle $|z|=1$

$$|p(z)+3z|$$
$$=|z^3-3z+1+3z|$$
$$\leq |z^3|+1$$
$$=2<3=|3z|$$

so $p(z)$ and $f(z)=3z$ have the same number of zeros within $|z|=1$, by Rouché's theorem

i.e. $p(z)$ has 1 zero within $|z|=1$.

On the circle $|z|=2$

$|p(z)-z^3|\leq 3(2)+1=7<2^3=|z^3|$

So $p(z)$ and $f(z)=z^3$ have an equal number of zeros within the circle $|z|=2$.

i.e. $p(z)$ has 3 zeros within $|z|=2$.

so 2 zeros lie in $1<|z|<2$. $(3-1=2)$

5. 3.3.4(c)

(c). $(1,0,i)$ onto $(1,0,1+i)$

Solution. Let $T(z)=\dfrac{az+b}{cz+d}$

Plug in $\dfrac{1a+b}{1c+d}=1 \Rightarrow a+b=c+d$

$\dfrac{b}{d}=0 \Rightarrow b=0,\ d\neq 0$

$\dfrac{ai+b}{ci+d}=1+i \Rightarrow ci+d-c+di=ai+b$

$$\left.\begin{matrix} a=c+d \\ (d-c)+(c+d)i=ai \end{matrix}\right\} \begin{matrix} c=d \\ a=2c \end{matrix}$$

So $T(z)=\dfrac{2tz}{tz+t}$ where $z\neq -1$ and $t\in\mathbb{C}\setminus\{0\}$.