§7 - Subspaces

1 Motivation

Many of the topological spaces we have looked at so far have been toys ($\mathbb{R}_{\text{indiscrete}}, \mathbb{R}_{\text{co-finite}}, \mathbb{R}_{\text{co-countable}}$) designed to help us understand the general notions and to help us isolate important properties (Hausdorff, first countable, etc.). These toy examples are great for our understanding, but aren't usually the examples of topological spaces that show up "in the real (mathematical) world".

A wide class of useful topological spaces are the so-called subspaces of $\mathbb{R}^n$. We will look at ways to get new spaces from old spaces by looking at subspaces. The most interesting examples will be subspaces of $\mathbb{R}^n$, but we will also look at subspaces of general topological spaces.

We will see that some of (but not *all* of) the topological invariants we described are actually inherited by subspaces; for example, if a parent space is Hausdorff, then all of its children (i.e. subspaces) will also be Hausdorff spaces. This will tell us that since $\mathbb{R}^n$ has many nice properties, its subspaces will also have those nice properties.

2 The Definition and Some Examples

The idea here will be that given some topological space (like $\mathbb{R}_{\text{usual}}$), we want to be able to get a related topology on a subset (like $\mathbb{Q} \subseteq \mathbb{R}$). This will give rise to a wide, useful class of topological spaces.

Definition. *For $(X, \mathcal{T})$ a topological space, and $Y \subseteq X$ we define $(Y, \mathcal{T}_{subspace})$ as*

$$\mathcal{T}_{subspace} := \{ U \cap Y : U \in \mathcal{T} \}$$

We say that Y "inherits its topology from X." Sometimes we say that "X induces a topology on Y". Sometimes we write $\mathcal{T}_Y$.

It is straightforward to check that this forms a topology on $Y \subseteq X$.

Some Examples:

- For $\mathbb{Z} \subseteq \mathbb{R}$ (given the usual topology), we see that in the subspace topology each point $x \in \mathbb{Z}$ is open as $(x - \frac{1}{2}, x + \frac{1}{2}) \cap \mathbb{Z} = \{ x \}$. So we see that $\mathbb{Z}$ inherits the discrete topology from $\mathbb{R}_{\text{usual}}$.
- For $[0, 1] \subseteq \mathbb{R}$ (given the Sorgenfrey Topology), we see that $[1, 7) \cap [0, 1] = \{ 1 \}$ is open in the subspace topology.
- For $S^1 := \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$, the unit circle in $\mathbb{R}^2$, we see that a basis for the subspace topology are the "open arcs". Given in polar coordinates, the basic open sets are of the form $S_{(a,b)} := \{(1, \theta) : a < \theta < b\}$.

One special class of subspaces are the discrete subspaces. Given a topological space $(X, \mathcal{T})$ and a subspace A, we say that A is a discrete subspace if it inherits the discrete topology from X. For example, $\mathbb{Z}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ both inherit the discrete topology from $\mathbb{R}_{\text{usual}}$, but $\mathbb{Q}$ does not inherit the discrete topology from $\mathbb{R}_{\text{usual}}$.

Large and Discrete Exercise: Show that $\mathbb{R}$ (usual) does not have an uncountable discrete subspace. You may try to do it directly, by using "analytic" properties of $\mathbb{R}$, or you may wish to observe that $\mathbb{R}$ has a topological property which guarantees that it does not have an uncountable discrete subspace.

One very useful fact that we will use constantly, is this one, describing the subspace topology when there is a basis around.

Proposition. *Let $(X, \mathcal{T})$ be a topological space with a basis $\mathcal{B}$, and let $Y \subseteq X$. The collection*

$$\mathcal{B}_Y := \{ B \cap Y : B \in \mathcal{B} \}$$

forms a basis for the subspace topology on Y.

Proof. It is clear that $\mathcal{B}_Y \subseteq \mathcal{T}_{\text{subspace}}$. Let V be an open set in Y containing a point y. Then there is an open set (in X) such that $V = U \cap Y$. So $y \in U$, and since $\mathcal{B}$ is a basis for X, there is a $B \in \mathcal{B}$, containing y such that $y \in B \subseteq U$. Thus

$$y \in B \cap Y \subseteq U \cap Y = V$$

□

3 Subspaces of $\mathbb{R}^n$

Most of the spaces we deal with end up being subspaces of $\mathbb{R}, \mathbb{R}^2$ or $\mathbb{R}^3$. More generally many mathematicians spend their time exclusively studying $\mathbb{R}^n$, and sometimes you will hear people asking "what is your favourite $\mathbb{R}^n$?". So let us take a moment to list some of the important subspaces of various $\mathbb{R}^n$.

Favourite Exercise: Ask some science-y people "What is your favourite $\mathbb{R}^n$?" and see what types of answers you get, and why they answered the way they did.

- We saw $S^1 \subseteq \mathbb{R}^2$ already, but more generally we have the n-sphere:

  $$S^n := \{ (x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \ldots + x_n^2 + x_{n+1}^2 = 1 \}$$

  In $\mathbb{R}^3$ this is just the sphere.

- We can think of the open unit ball $B_1(0, 0, \ldots, 0) \subseteq \mathbb{R}^n$ as a subspace. On assignment 4 you will show that any two open balls (in $\mathbb{R}^n$) are always homeomorphic.
- There is a canonical way of thinking about $\mathbb{R}$ as a subspace of $\mathbb{R}^n$. We look at the "copy of $\mathbb{R}$" as
$$\{ (x, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}) \in \mathbb{R}^n : x \in \mathbb{R} \}$$
It is not hard to show that this subspace is homeomorphic to $\mathbb{R}$. In general, we can think of $\mathbb{R}^n$ as a subspace of $\mathbb{R}^m$ (for $n < m$) by noting that $\mathbb{R}^n \times \{ 0 \}^{m-n} \cong \mathbb{R}^n$.
- The Torus $T^1 := S_1 \times S_1 \subseteq \mathbb{R}^3$ is basically a hollow donut. This is a very important example for those of you who go on to study algebraic or geometric topology.

Where do you live Exercise: In what $\mathbb{R}^n$ does $S^1 \times S^1 \times S^1$ live?

What kind of topological properties do these spaces have? It would be very tedious if we had to show that each of these spaces was, for example, a Hausdorff space. This leads us to the study of Hereditary properties.

4 Hereditary Properties

Some of the properties we have studied so far are inherited through subspaces. This is an extremely powerful technique that allows us to show that a large class of spaces has nice properties. For example, we will show that the Hausdorff property is hereditary, and this will tell us that any subspace of $\mathbb{R}^3$ is also a Hausdorff space.

Definition. *A property ϕ is said to be* ***hereditary*** *if whenever $(X, \mathcal{T})$ is a topological space with property ϕ then all of its subspaces have property ϕ.*

Let us prove the the Hausdorff condition is hereditary, before we list some examples.

Proposition. *The Hausdorff property is a hereditary property.*

Proof. Let $(X, \mathcal{T})$ be a Hausdorff space, and let A be a subspace of X. Let $x, y \in A$ be two distinct point. Since they are in A, they are also in X. Because X is a Hausdorff space, there are disjoint open $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$. So then $x \in U \cap A$ and $y \in V \cap A$, with those sets open in A, and they are disjoint. So A is a Hausdorff space. $\square$

Now let us look at a handful of examples of hereditary properties:

- X is a Hausdorff space;
- X is countable;

- X is first countable;
- X is second countable.

Let us prove one of these:

Proposition. *Every subspace of a first countable space is first countable.*

Proof. Let $(X, \mathcal{T})$ be a first countable space. Let A be a subspace of X, and let $p \in A$. Since p is also in X, there is a countable local basis $\mathcal{B} = \{ B_n : n \in \mathbb{N} \}$, at the point p.

Claim: $\mathcal{B}_A := \{ B_n \cap A : n \in \mathbb{N} \}$ is a countable local basis at p.

It is clear that this family is (1) at most countable, and (2) contains sets which are open in A. So let V be an open set in A, containing p. Then there is an open (in X) set U such that $V = U \cap A$. Since $p \in U$, and $\mathcal{B}$ is a local basis for X, there is a $B_n \in \mathcal{B}$ such that $p \in B_n \subseteq U$. So then $p \in B_n \cap A \subseteq U \cap A = V$, as desired. □

Don't get the impression that all of the nice properties are hereditary. Here are some that are not:

- X is separable;
- X has the countable chain condition (ccc);

Example. *Let us give an example of a space that is separable (hence ccc), but has a subspace that is neither separable nor does is have the ccc.*

Proof. Consider the set $\mathbb{R}$ with open sets $\emptyset$ and $\{ Y \subseteq \mathbb{R} : 7 \in Y \}$ (called "the particular point topology", example 10 in Counterexamples in Topology). Note that this is separable (and hence ccc by assignment 3, C.5), since the closure of $\{ 7 \}$ is $\mathbb{R}$ (as that is the only closed set that contains 7). However, $\mathbb{R} \setminus \{ 7 \}$ inherits the discrete topology (on an uncountable set) which certainly is not separable or ccc. □

Well this looks bad, but all is not yet lost. It turns out that there is a way around this, by just looking at *open* subspaces.

Proposition. *If U is an open subset of X a separable topological space, then U, as a subspace, is separable.*

Proof. We will show that if $D \subseteq X$ is a dense subset of X, then $D \cap U$ is dense in U. This amount to showing that

$$\overline{D \cap U} = U$$

when we compute the closure in U. So let V be an open subset in U. Thus $V = A \cap U$, where A is open in X. So then $A \cap U$ is an open subset of X and must contain a point from D. Thus $D \cap A \cap U \neq \emptyset$. □

Dense in Closed Exercise: Does the previous proposition still hold if we replace "open" by "closed"?

Another way aroung this is to restrict ourselves to $\mathbb{R}^n$, which does have the property that all of its subspaces are separable.

Theorem. *$\mathbb{R}^n$ is hereditarily separable. That is, $\mathbb{R}^n$ is separable and all of its subspaces are separable.*

Proof. We delay this proof until we have started looking at metric spaces. □

5 Facts about subspaces

Here is a dump of some useful lemmas we might need about subspaces. The proofs are all straightforward unwinding of definitions:

Continuous functions and Subspaces:

Proposition (Restriction of domain)**.** *If $f : X \longrightarrow Y$ is a continuous function, and A is a subspace of X, then the restricted function $f \restriction A : A \longrightarrow Y$ is a continuous function.*

Proposition (Expansion of Range)**.** *Let B be a subspace of Y. If $f : X \longrightarrow B$ is continuous, then $f : X \longrightarrow Y$ is continuous.*

Proposition (Inclusion)**.** *Let A be a subspace of X. If $f : A \longrightarrow X$ is the inclusion map ($f(a) = a, \forall a \in A$), then f is continuous.*

Closures and Subspaces:

Proposition. *Let A be a subspace of X. For any $B \subseteq A$, the closure of B (in A) is given by $\overline{B} = A \cap \overline{B}$, where the closure on the right is computed in X.*

For example, if $X = \mathbb{R}_{\text{usual}}$ and $A = (0, 1)$ with $B = (0, \frac{1}{2})$, then (in A) $\overline{B} = (0, \frac{1}{2}] = (0, 1) \cap [0, \frac{1}{2}]$.

Proposition. *Let A be a closed subset of X. If B is closed in the subspace of A, then B is closed in X.*

Proposition. *Let U be an open subset of X. If V is open in the subspace of U, then V is open in X.*

6 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Favourite $\mathbb{R}^n$: Ask some science-y people "What is your favourite $\mathbb{R}^n$?" and see what types of answers you get, and why they answered the way they did.

Large Discrete : Show that $\mathbb{R}$ (usual) does not have an uncountable discrete subspace.

Where do you live? : In what $\mathbb{R}^n$ does $S^1 \times S^1 \times S^1$ live?

Dense in Closed : If D is dense in X, and C is a closed subspace of X, then is D dense in C?