1. Euclidean Space

a) (2 marks) State the Cauchy-Schwarz Inequality for $\mathbb{R}^n$.

$\forall \vec{a}, \vec{b} \in \mathbb{R}^n$, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$

b) (4 marks) Prove for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ that

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

Folland 1.2

c) (4 marks) Rigorously prove from the definition of being open (or a property equivalent to the definition of being open) that $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ is indeed open.

S open $\Leftrightarrow$ every point is an interior pt. (sec 1.4)

Choose $x \in (a,b)$. Let $\varepsilon = \min\{b-x, x-a\}$

Consider $B(\varepsilon, x) = \{y \in \mathbb{R}^n \mid |y-x| < \varepsilon\}$

If $y > x$, $y-x < \varepsilon \leq b-x \Rightarrow y < b$ (and clearly $y > a$)

so $y \in (a,b)$

If $y < x$ $x-y < \varepsilon \leq x-a \Rightarrow a < y$ (and clearly $y < b$)

So $y \in (a,b)$.

Thus $B(\varepsilon, x) \subset (a,b) \Rightarrow (a,b)$ open.

2. Completeness

a) (2 marks) State the Completeness Axiom for $\mathbb{R}$. Let S be a non-empty subset of real numbers. If S has an upperbound, S has a least-upper bound. (denoted Sup S). If S has a lower bound, S has a greatest lower bound.

b) (4 marks) Find and justify the Supremum and Infimum of the range of the following sequence: $\{x_k\}_1^\infty = 1/k^2$.

as $1/k^2 > 0$, 0 is a lower bound. If $l > 0$ was a greater lower bound then for all $k > \sqrt{\frac{1}{l}}$, $\frac{1}{k^2} < l$ so not a lower bound.

$\therefore$ Inf(range($\{x_k\}$)) = 0

As $x_1 = 1$, the lub could not be less than 1.

But 1 is an upper bound as $\frac{1}{k^2}$ decreases from 1.

So Sup (range($\{x_k\}$)) = 1

c) (6 marks) Prove that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. To illustrate the main idea of the proof, it may help (but is not necessary) to sketch a diagram.

See Folland 1.18

3. Continuity and Uniform Continuity

a) (3 marks) Suppose $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$ State the ϵ,δ definitions of both continuity and uniform continuity on S.

Continuity: $\forall \varepsilon > 0 \; \forall \vec{x} \in S \; \exists \delta > 0$ s.t. $\forall \vec{y} \in S \; |x - y| < \delta$
$\Rightarrow |\vec{f}(\vec{x}) - \vec{f}(\vec{y})| < \varepsilon$

Uniform Continuity: $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall \vec{x}, \forall y \in S \; |\vec{x} - \vec{y}| < \delta$
$\Rightarrow |\vec{f}(x) - \vec{f}(\vec{y})| < \varepsilon$

b) (6 marks) Suppose $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$ is continuous at every point of S. If S is compact, prove that $\mathbf{f}$ is uniformly continuous on S.

See Folland 1.33

c) (5 marks) Prove that the following function is uniformly continuous on the set $S = \{(x, y) \mid x^2 + y^2 \leq 2\}$

$$f(x,y) = \frac{x^3 \cos x}{x^2 + y^2},\ x \neq (0,0),\ f(0,0) = 0$$

You may use part b, but be sure to fully justify why every condition is satisfied.

- S is closed (inverse image under the continuous fcn x^2+y^2 of closed set $[0,2]$)

  & bdd (contained in $B(3,0)$)

  $\therefore$ compact.

- Away from $(0,0)$, $f(x,y)$ is a composition of continuous fcns so continuous.

- At $(0,0)$

$$|f(x,y) - f(0,0)| = \left|\frac{x^3 \cos x}{x^2+y^2}\right|$$

but $x^2 \leq x^2+y^2$, $|\cos x| \leq 1$

so $\leq |x| \cdot 1 \cdot \frac{\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = |x| \to 0$ as $(x,y) \to (0,0)$

so cont at $(0,0)$

by b), f is uniformly continuous.

4. Differentiability

a) (2 marks) State the definition of differentiability in several variables.

$f: S \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a} \in S$

if $\exists\ \vec{c} \in \mathbb{R}^n$ (can call $\vec{c} = \nabla f(\vec{a})$) s.t.

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \vec{c}\cdot\vec{h}}{|\vec{h}|} = 0$$

Alternatively: $f(\vec{a}+\vec{h}) = f(\vec{a}) + \vec{c}\cdot\vec{h} + E(\vec{h})$

where $\displaystyle\lim_{h\to 0} \frac{E(\vec{h})}{|\vec{h}|} = 0$

b) (5 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{x^3}{x^2 + y^2},\ x \neq (0, 0),\ f(0, 0) = 0$$

$\vec{x}$ not x

Determine all points where f is and is not differentiable (Hint: treat the cases $x = (0, 0)$ and $x \neq (0, 0)$ separately. You may wish to convert to polar coordinates for the last step).

For $\vec{x} = (x,y) \neq (0,0)$, $\quad \partial_x f = \dfrac{3x^2(x^2+y^2) - x^3(2x)}{(x^2+y^2)^2}$

$\partial_y f = \dfrac{-2yx^3}{(x^2+y^2)^2}$

cont for $(x,y) \neq (0,0)$

$\Rightarrow$ diff for $(x,y) \neq (0,0)$ via 2.19

For $(0,0)$, $\quad \partial_x f(0,0) = \displaystyle\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1$

$\partial_y f(0,0) = \displaystyle\lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\Rightarrow \nabla f(0,0) = (1,0)$

So, $\displaystyle\lim_{\vec{x}\to\vec{0}} \frac{f(x,y) - f(0,0) - (1,0)\cdot(x,y)}{\sqrt{x^2+y^2}} = \lim_{\vec{x}\to\vec{0}} \frac{\frac{x^3}{x^2+y^2} - x}{\sqrt{x^2+y^2}}$

$= \displaystyle\lim_{\vec{x}\to\vec{0}} \frac{-xy^2}{(x^2+y^2)^{3/2}} = \lim_{r\to 0} \frac{-r^3\cos\theta\sin^2\theta}{r^3} = -\cos\theta\sin^2\theta$ DNE

$\Rightarrow f$ not diff at $(0,0)$

c) (3 marks) Compute the directional derivative of $f(x, y) = e^{x^2-y}$ at the point $(2, 4)$ in the direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$\nabla f = (\partial_x f, \partial_y f) = (2xe^{x^2-y}, -e^{x^2-y})$

$\nabla f(2,4) = (4,-1)$

$\partial_{\vec{u}} f(2,4) = \nabla f(2,4) \cdot \vec{u} = (4,-1) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$= \frac{3}{\sqrt{2}}$.

d) (3 marks) The Surface Area of a box with side lengths x,y,z is given by $f(x, y, z) = 2xy + 2xz + 2yz$. Use the method of differentials to estimate the change in Surface Area when changing (x, y, z) from $(1, 1, 1)$ to $(1.1, 0.9, 1)$.

Let $w = f(x,y,z)$

$dw = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz$

$= 2(y+z)dx + 2(x+z)dy + 2(x+y)dz$

as $(1,1,1) \to (1.1, 0.9, 1)$

$dx = 0.1, \quad dy = -0.1, \quad dz = 0$

so $dw = 2 \cdot 2 \cdot 0.1 + 2(2)(-0.1) = 0$

e) (3 marks) Compute the Fréchet derivative Df for the function $f(x, y, z) = (xyz + x^2, z + yx)$.

$Df = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_z f_1 \\ \partial_x f_2 & \partial_y f_2 & \partial_z f_2 \end{pmatrix} = \begin{pmatrix} yz+2x & xz & xy \\ y & x & 1 \end{pmatrix}$