PLEASE HAND IN

# UNIVERSITY OF TORONTO
## Faculty of Arts and Science

## APRIL 2012 EXAMINATIONS

## CSC 236 H1S
Instructor(s): A. Farzan & F. Pitt

Duration — 3 hours

PLEASE HAND IN

**Examination Aids:** *One double*-sided, *hand*written 8.5" × 11" sheet of paper.

**Student Number:** ___________

**Family Name(s):** ___________

**Given Name(s):** ___________

**Lecture Section:** ☐ L0101 (Azadeh Farzan) ☐ L0201 (François Pitt)

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

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This final examination paper consists of 6 questions on 11 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy is complete and fill in the identification section above.*

Answer each question directly on the exam paper, in the space provided, and use the reverse side of the previous page for rough work. If you need more space for one of your solutions, use the reverse side of the previous page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

MARKING GUIDE

| | |
|---|---|
| # 1: | ______/ 5 |
| # 2: | ______/13 |
| # 3: | ______/17 |
| # 4: | ______/ 8 |
| # 5: | ______/12 |
| # 6: | ______/10 |
| BONUS MARKS: | ______/ 4 |
| TOTAL: | ______/65 |



*Good Luck!*

## Question 1. [5 MARKS]

Consider the following modified version of the game of Nim (in the following, we assume that there is an unlimited supply of "tokens"). Two players arrange some number of piles of tokens in a row. By turns each of them takes one token from one of the piles and adds as many tokens as he or she wishes to piles placed to the left of the pile from which the token was taken — except that tokens cannot be added back onto a pile that has become empty. The player that takes the last token wins. Prove that, no matter how they play, the game will eventually end after finitely many steps.
HINT: Use induction on the number of piles.

## Question 2. [13 MARKS]

Consider the iterative algorithm on the right, that evaluates a degree $d$ polynomial $A[d]r^d + \cdots + A[1]r + A[0]$, given a real number $r$, a nonnegative integer $d$, and an array of $d+1$ real numbers $A[0 \ldots d]$.

```
EVAL(A, d, r):
    v ← A[d]
    k ← d
    while k ≠ 0:
        k ← k − 1
        v ← A[k] + v × r
    return v
```

### Part (a) [1 MARK]

Give an expression for the value of $v$ at the end of the first iteration of the while loop and another expression for the value of $v$ at the end of the second iteration of the while loop. Your expressions should depend on $d$ and $r$ and $A$.

### Part (b) [2 MARKS]

Give a loop invariant that expresses $v$ as a function of $d$, $k$, $r$, and $A$.

### Part (c) [4 MARKS]

Prove that algorithm EVAL is partially correct — for this part, assume that your loop invariant is correct (you will prove this on the next page).

## Question 2. (CONTINUED)

**Part (d)** [6 MARKS]

Prove that your loop invariant from Part (b) is correct.

## Question 3. [17 MARKS]

Consider the recursive algorithm on the right, that evaluates a degree $d$ polynomial $A[k+d]r^d + \cdots + A[k+1]r + A[k]$, given a real number $r$, nonnegative integers $k$ and $d$, and an array of $d+1$ real numbers $A[k \ldots k+d]$.

```
RECEVAL(A, k, d, r):
    if d = 0:  return A[k]
    m ← ⌈d/2⌉
    v1 ← RECEVAL(A, k + m, d − m, r)
    v2 ← RECEVAL(A, k, m − 1, r)
    return v1 × r^m + v2
```

### Part (a) [4 MARKS]

Write a recurrence describing the **exact** worst case number of arithmetic operations performed by algorithm RECEVAL$(A, k, d, r)$, as a function of $d$.

Assume that computing $\lceil d/2 \rceil$ takes one arithmetic operation and that, for any positive integer $m$, $E(m)$ is the worst case number of arithmetic operations to compute $r^m$. Do **not** use asymptotic notation. Briefly justify your answer.

### Part (b) [2 MARKS]

Give a good upper bound for your recurrence above, to within a constant factor, if $E(m) = m - 1$. Briefly justify your answer.

### Part (c) [2 MARKS]

Give a good upper bound for your recurrence above, to within a constant factor, if $E(m) = 0$. Briefly justify your answer.

### Part (d) [2 MARKS]

Give a predicate (*i.e.*, a precise, formal statement) that expresses what it means for this algorithm to be correct.

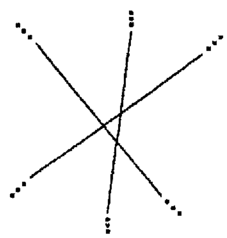
## Question 3. (CONTINUED)

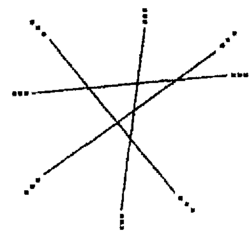
**Part (e)** [7 MARKS]

Now, prove that algorithm RECEVAL is correct.

## Question 4. [8 MARKS]

Consider $n$ lines in the plane such that no two lines are parallel and no three lines cross at a single point (in other words, every pair of lines crosses at a unique point, different from the crossing points for other pairs). Here are examples for $n = 3$ and $n = 4$:





### Part (a) [4 MARKS]

Write a recurrence relation for the number of different regions created by the lines. Justify your answer briefly. (For example, 3 lines create 7 regions and 4 lines create 11 regions.)

### Part (b) [4 MARKS]

Find a closed-form solution for your recurrence, but **do not prove** your solution — just explain what you are doing and show your work.

## Question 5. [12 MARKS]

Give an NFA and a RE for each of the following languages. Justify briefly that your answers are correct, *i.e.*, explain why your NFA accepts every string in the language but no other, and why your RE describes every string in the language but no other.

### Part (a) [6 MARKS]

$L_1 = \{s \in \{a,b\}^* :$ the number of $a$'s in $s$ is a multiple of 3 **and** the number of $b$'s in $s$ is a multiple of $5\}$

## Question 5. (CONTINUED)

**Part (b)** [6 MARKS]

$L_2 = \{s \in \{0,1\}^* : s$ contains *at least* $k$ 0's **or** *at most* $\ell$ 1's$\}$ (where $k, \ell$ are positive integer constants).

## Question 6. [10 MARKS]

A regular expression is said to be in *normal form* if it has the form $R_1 + R_2 + \cdots + R_n$, where none of the $R_i$'s contain an occurrence of the union operator ("+").

Write a detailed, formal proof that, for every regular expression $R$, there is a regular expression $R'$ in normal form such that $L(R') = L(R)$. (HINT: Use the fact that $L\big((A + B)^*\big) = L\big(A^*(BA^*)^*\big)$ for all regular expressions $A, B$, along with other regular expression equivalences from class.)

**Bonus.** [4 MARKS]

**WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer. Please attempt this only *after* you have completed the rest of the Final Examination.**

Consider the operation "sq" that maps any language $L$ to the language $\text{sq}(L) = \{x^{|x|} : x \in L\}$. For example, if $L = \{0, 00, 000\}$ then $\text{sq}(L) = \{0, 0000, 000000000\}$. Prove that there is a *regular* language $L$ such that $\text{sq}(L)$ is **not** regular.