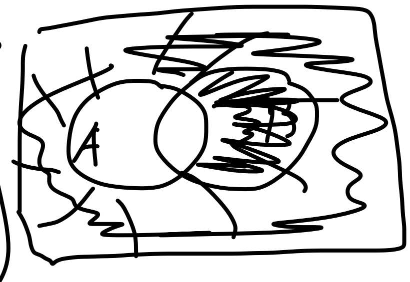
Practice Problems

6 lectures

Ch 1, 2, 4.1, 4.2

1. Given $P(AUB) = 0.7$ and $P(A \cup B^c) = 0.9$, find $P(A)$.

$$P(A) = 1 - P(A^c) = 0.6$$



$$A^c = (A^c \cap B) \cup (A^c \cap B^c)$$

$$P(A^c) = P(A^c \cap B) + P(A^c \cap B^c) = 0.4$$

$$(A^c \cap B) = [A \cup B^c]^c \Rightarrow P(A^c \cap B) = 1 - 0.9 = 0.1$$

$$A^c \cap B^c = (A \cup B)^c \Rightarrow P(A^c \cap B^c) = 1 - 0.7 = 0.3$$

2. Given that A and B are independent with $P(A \cup B) = 0.8$ and $P(B^c) = 0.3$, find $P(A)$.

$$0.8 = P(A) + P(B) - P(A)P(B)$$

$$= P(A)P(B^c) + P(B)$$

$$P(A) = \frac{0.8 - P(B)}{P(B^c)} = \frac{0.8 - 0.7}{0.3} = \frac{1}{3}$$

3. Suppose that customers arrive randomly, during mid-day, at a certain service counter, at an average rate of 20 per hour. What is the probability of at least 2 customers arriving during the next 15 minutes?

$X = \#$ of customers in 15 min

$$X \sim \text{Poisson}\left(\frac{20}{4}\right) = \text{Poisson}(5)$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - \frac{e^{-5} 5^0}{0!} - \frac{e^{-5} 5^1}{1!} = 1 - 6e^{-5}$$

4. A student answers a multiple-choice examination question that offers four possible answers. Suppose that the probability that the student knows the answer to the question is 0.8 and the probability that the student will guess is 0.2. Assume that if the student guesses, the probability of selecting the correct answer is 0.25. If the student correctly answers a question, what is the probability that the student really knew the correct answer?

$A = \{$student guesses$\}$

$B = \{$answers correctly$\}$

$$P(A^c \mid B) = \frac{P(B \mid A^c) P(A^c)}{P(B \mid A^c) P(A^c) + P(B \mid A) P(A)}$$

$$= \frac{1 \cdot 0.8}{1 \cdot 0.8 + 0.25 \cdot 0.2} = \frac{0.8}{0.85}$$

$$= \frac{80}{85} = \frac{16}{17}$$

5. If Y has a probability density function given by $f(y) = 4y^2 e^{-2y}$, y > 0
0, elsewhere,

obtain E(Y) and Var(Y).

$$\boxed{E(Y) = \int_0^\infty y \cdot 4y^2 e^{-2y}\, dy = \text{integration by parts}}$$

$Y \sim \text{Gamma}(3, 2)$

$$\frac{y^{\alpha-1} e^{-\lambda x} \lambda^\alpha}{\Gamma(\alpha)}$$

$$\frac{2^3}{\Gamma(3)} = \frac{8}{2!} = 4$$

$$E(Y) = \frac{\alpha}{\lambda} = \frac{3}{2}, \quad Var(Y) = \frac{\alpha}{\lambda^2} = \frac{3}{4}$$

6. Suppose X is a continuous r.v. With density function $f(x) = 2(1 - x)$, $0 \le x \le 1$
0, otherwise

Find the density of Y = 1/X.

$$h(x) = \frac{1}{x} \searrow \qquad h^{-1}(y) = \frac{1}{y}$$

$$\frac{d}{dy} h^{-1}(y) = -\frac{1}{y^2}$$

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \left| \frac{d}{dy} h^{-1}(y) \right|$$

$$= -2\left(1 - \frac{1}{y}\right)\left(-\frac{1}{y^2}\right), \quad y \geq 1$$

$$= \begin{cases} \frac{2}{y^2}\left(1 - \frac{1}{y}\right), & y \geq 1 \\ 0, & \text{ow} \end{cases}$$

Find E(Y) Var(Y)

7. Suppose that a r.v. X has a strictly increasing cdf F(x). Show that the r.v. Y = F(X) has a uniform distribution on (0, 1).

$$\boxed{f_Y(y) = \begin{cases} 1, & y \in (0,1) \\ 0, & \text{ow} \end{cases}}$$

$y \in (0,1)$

$$F_Y(y) = P(Y \le y) = P(F(X) \le y)$$

$$= P(X \le F^{-1}(y))$$

$\rightarrow$ exists, since $F(X)$ is strictly $\uparrow$

$$= F(F^{-1}(y))$$

$$= y \rightarrow \text{cdf for uniform r.v on } (0,1)$$

8. If Y has distribution function $F(y) = \begin{cases} 0, & y \le 0 \\ y/8, & 0<y<2 \\ y^2/16, & 2 \le y < 4 \\ 1, & y \ge 4, \end{cases}$

find the mean and variance of Y.

$$f_Y(y) = \begin{cases} \frac{1}{8}, & 0<y<2 \\ \frac{y}{8}, & 2 \le y < 4 \\ 0, & \text{ow} \end{cases}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y \cdot \frac{1}{8}\,dy + \int_2^4 y \cdot \frac{y}{8}\,dy$$

$$= \frac{y^2}{16}\Big|_0^2 + \frac{y^3}{24}\Big|_2^4 = \frac{31}{12}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{47}{6} - \left(\frac{31}{12}\right)^2 = 1.16$$

$$E(Y^2) = \int_0^2 y^2 \frac{1}{8}\,dy + \int_2^4 y^2 \frac{y}{8}\,dy$$

$$= \frac{y^3}{24}\Big|_0^2 + \frac{y^4}{32}\Big|_2^4 = \frac{47}{6}$$

$\underline{Ex.}$ $X \sim N(\underset{\mu, \sigma^2}{3, 2})$, $\underline{E[(X-1)^2]} = \underset{\overset{\|}{Var(X)+E(X)^2}}{E(X^2)} - 2E(X) + 1 = 11 - 6 + 1 = 6$

$X-1 \sim N(2, 2)$ $E[(X-1)^2] = 2 + 2^2 = 6$

9. Verify that each of the following are probability functions:

(a) $p(x) = p^x q^{1-x}$, x = 0, 1

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$N(\mu, \sigma^2)$

(b) $p(x) = \binom{n}{x} p^x q^{n-x}$, x = 0, 1, ..., n

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

$exp(\lambda)$

(c) $p(x) = q^{1-x} p$, x = 1, 2,...

$$f(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & ow \end{cases}$$

Uniform(0,1)

(d) $p(x) = e^{-\lambda} \lambda^x / x!$, x = 0, 1, ...

$$f(x) = \frac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}, x \geq 0$$

Gamma(α, λ)

10. Calculate the mean and variance for each of the distributions in # 9.

$$f(x) = \frac{x^{-1/2} e^{-x/2}}{\sqrt{2\pi}}$$

$$\chi^2_{(1)} = Gamma(\tfrac{1}{2}, \tfrac{1}{2})$$