University of Toronto
MAT237Y1Y PROBLEM SET 1
DUE: End of tutorial, Thursday May 23rd, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.

2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.

3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.

4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. Read the posting on 'set operations' found in the optional readings and specifically read practice examples 5.11, 5.14, 5.15 on pages 41,43, 44, respectively, then see their answers on pages 45 and 46. Then for this assignment, complete 5.10 in full on page 48.

2. Read the posting on 'quantifiers' found in the optional readings and do problems 2.10a-d, 2.12, 2.13, 2.15

3. The definition of a ball (see page 9 of Folland) depends on the notion of distance (see page 6) which depends on the definition of the norm (see page 5). If, however, we were to give a different definition of the norm than the 'canonical' norm given in Folland, we would have an analogous definition of distance, and an analogous definition of a ball. This problem compares the normal concept of a ball with different concept of a ball. It is a bit of an odd ball.

   - a) Using the normal definition of norm as in page 5, draw a sketch of the ball $B(1, \mathbf{a})$ for $\mathbf{a} = (0, 2)$

- b) Now we replace the canonical definition of the norm on $\mathbb{R}^n$ with a new norm given by

$$|\mathbf{x}| = |(x_1, \ldots, x_n)| := |x_1| + \cdots + |x_n|$$

Using this norm in the case of $\mathbb{R}^2$ to form an analogous notion of a ball, again sketch $B(1, \mathbf{a})$ for $\mathbf{a} = (0, 2)$ and present a written justification of the validity of your sketch.

Hint: You may want to consider cases like "$x_1 - a_1 > 0, x_2 - a_2 > 0$" or "$x_1 - a_1 > 0, x_2 - a_2 < 0$", etc., seperately to help you draw this ball.

Note that for $\mathbb{R}^2$, in contrast to our normal norm which gives a birds eye view of distance, if you will, this norm gives the distance that a car might travel in downtown Toronto, where they can go only in an East-West direction or a North-South direction but not diagonally.

- c) For $\mathbb{R}^n$ (this can be generalized to an arbitrary Vector Space), a function $|\cdot| : V \to [0, \infty)$ is called a norm if it satisfies three properties:
i) $|\mathbf{x}| = 0$ iff $\mathbf{x} = \mathbf{0}$
ii)$|c\mathbf{x}| = |c||\mathbf{x}|$ for $c \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^n$
iii)$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

In Folland, this last property is demonstrated for the usual norm (as in part a) and is known as the triangle inequality. For this exercise prove that the norm defined in part b satisfies properties i, ii, iii and was thus rightfully called a norm.

4. Identify the interior, boundary, and closure of the following three subsets of $\mathbb{R}$:

$$A = \{1/n |\ n \in \mathbb{Z}^+\}$$

$$B = \{1/n |\ n \in \mathbb{Q}^+\}$$

$$C = \{1/n |\ n \in \mathbb{R}^+\}$$

Note that $0 \notin \mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+$

Hint: Recall the fact from MAT137 that $\mathbb{Q}$ is dense in $\mathbb{R}$, which means that any interval (a,b) in $\mathbb{R}$ contains both rational and irrational numbers.

5. The Cross Product is a binary operation on $\mathbb{R}^3$ that satisfies various algebraic properties such as those given on page 7 of Folland. In particular, it satisfies a quasi-associative law called the Jacobi identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

Derive this identity from the definition of the cross product.

Hint: Can you find a trick that lets you cut the need to repeat a similar calculation three times?

6. Continuity:

- a) Prove directly from the ϵ, δ defintion of continuity (as in do not use the results from Theorems 1.9-1.12 in Folland or similar) that the following function is continuous:

$$f(x, y, z) = 1 + x + 2y$$

- b) Now you may use Theorem 1.9 and the style of proof as in Corollary 1.11 to prove the following function is continuous. In particular, you are encouraged to explicitly use the functions f_1, f_2, f_3, f_4 as defined on page 16 and may additionally use that both the constant function c(x,y)=2 and sin(x) are continuous.

$$f(x, y) = \frac{(xy)^2}{sin(x + y) + 2}$$

- c) Find the limit as (x,y) goes to (0,0) or prove that it does not exist:

$$f(x, y) = \frac{2x + y^2}{\sqrt{(2x)^2 + y^2}}, \quad (x, y) \neq (0, 0)$$

- d) Find the limit as (x,y) goes to (0,0) or prove that it does not exist:

$$f(x, y) = \frac{x^2y - y^3}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

Hint: to solve c) and d) you may want to see the methods described in examples 1,2 and 3 in Section 1.3.