AST121 Basic Newtonian Cosmology

AST121 TAs

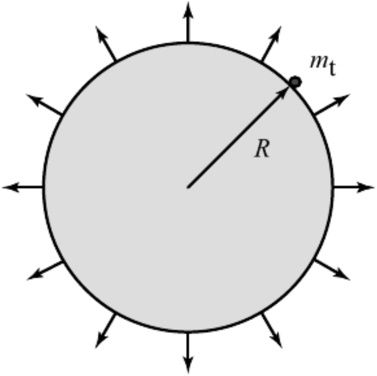
March 22, 2013

OUTLINE

- What is Newtonian cosmology?
- Friedmann's equation
- Why general relativity is needed
- Basic cosmology

What is Newtonian cosmology?

- Assume the density of the whole universe always **uniform**.
- Pick a centre O and a particle m_t at a distance $R(t)$. Let $R(t)$ be the radius of a sphere centred at O. m_t and all other particles in the sphere have **kinetic energy due to radial velocity** and are affected by the **gravity** of other particles in the sphere. External forces do not matter because they all cancel.
- We then have to set how the sphere looks at a certain time (the "initial conditions"), and Newtonian mechanics will then tell us how it evolves.

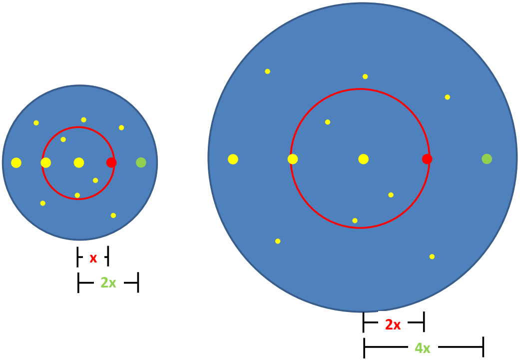


From Hawley and Holcomb 2005.

The energy balance of particle m_t is

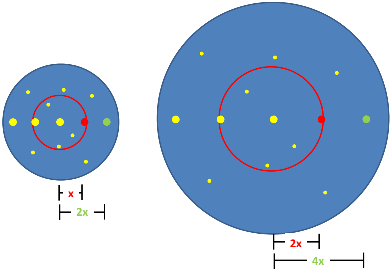
$$E_\infty = -\frac{GMm_t}{R(t)} + \frac{1}{2}m_t v^2 \tag{1}$$

where M is all the mass enclosed within $R(t)$ and v is the velocity of m_t. If $E_\infty < 0$, then m_t will reach a finite radius R_{max}, and then fall back. If $E_\infty = 0$, then m_t will reach $R = \infty$ with $v = 0$. If $E_\infty > 0$ then m_t will reach $R = \infty$ with some positive velocity.



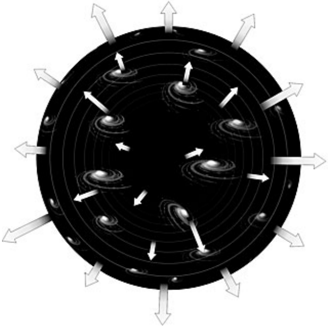
A sphere that is always uniform density is equivalent to one that is being scaled down and up - called a “homologous” expansion by astronomers. This has a number of consequences:

- M is always the same.



- Velocity v must be **radial** and **linearly proportional to** R. This gives us $v(t) = H(t)R(t)$; Hubble observed the same relationship in real life, so this relationship is known as Hubble's Law, with $H(t)$ the Hubble Parameter[1].
- We can represent the evolution of the entire universe with $R(t)$, or a scaling factor $a(t)$.

[1] A consequence of this is that we can pick centre O of our sphere to be anywhere in the universe; see Expansion of the Universe & Dark Energy I in the class notes.



From ScienceBlogs.

- This is a good representation of the universe on large scales if we make two assumptions: that the universe obeys the **cosmological principle** and contains only **“collisionless” material**.
- At the largest scales (as far as we know), the universe obeys the cosmological principle.
- The majority of matter in the universe is dark matter, which only interacts gravitationally and is also “collisionless” at the densities we’re describing.

Friedmann's Equation

We substitute $v = HR$ (not writing (t)) and $M = 4\pi R^3 \rho_m / 3$ into Eqn. 1 to get

$$E_\infty = -\frac{4\pi G R^2 \rho_m m_t}{3} + \frac{1}{2} m_t H^2 R^2 \tag{2}$$

which can be rewritten as

$$H^2 = \frac{8\pi G \rho_m}{3} + 2\frac{E_\infty}{m_t R^2} \tag{3}$$

We could also have gotten this result from integrating the acceleration equation, $\ddot{R} = -GM/R^2$ - this was done in Assignment 2[2].

[2] "Over-dots" indicate time derivatives - $\dot{R} = dR/dt$, and $\ddot{R} = d^2R/dt^2$

We again note that radius R is arbitrary, so we define the **scale factor**, $a(t)$, using the equation

$$a(t) \equiv \frac{a_0}{R_0} R(t) \tag{4}$$

where $a_0 = a(t_0)$ and $R_0 = R(t_0)$. t_0 is some time, which we usually take to be the current age of the universe. We note that

$$H = \frac{\dot{R}}{R} = \frac{\dot{a}}{a} \tag{5}$$

Plugging in $a(t)$ where $R(t)$ was, we get Eqn. 6

$$H^2 = \frac{8\pi G \rho_m}{3} + 2 \frac{E_\infty a_0^2}{m_t a(t)^2 R_0^2} \tag{6}$$

We now note that $2E_\infty/m_t R_0^2$ is a constant. This is because $E_\infty = m_t v_\infty^2/2$, so $2E_\infty/m_t R_0^2 = v_\infty^2/R_0^2$. We don't know what v_∞ is for any given R_0, but we do know that, so long as the universe expands homologously, a particle originally at $2R_0$ should always travel twice as fast as a particle originally at R_0 because one will always be twice as far away from O as the other. Let us then set

$$\frac{2E_\infty}{m_t R_0^2} = -Kc^2 \tag{7}$$

The negative sign and c^2 are for consistency with general relativity, discussed later. We make one more assumption, that $Ka_0^2 = k$, and k only equals -1, 0 or 1. Then

$$\frac{2E_\infty}{m_t R_0^2}\frac{a_0^2}{a(t)^2} = -Kc^2\frac{a_0^2}{a(t)^2} = -\frac{kc^2}{a(t)^2} \tag{8}$$

This means that when K isn't zero, $a_0 = 1/K$, which is **usually not 1**. Therefore, **only in a flat universe can we set a_0 to anything we like**.

We finally rewrite Eqn. 6 to

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{kc^2}{a(t)^2} \tag{9}$$

This is **Friedmann's equation**[3]. If $k = 1$, then the universe will reach a maximum characteristic size $a(t)$ and then fall back. If $k = 0$, then the universe will expand until $a(t) \to \infty$ and $\dot{a} \to 0$. If $k = -1$ then the universe will expand until $a(t) \to \infty$ and $\dot{a} \to A$, where $A > 0$.

[3]Or more accurately the Friedmann-Lemaître equation.

Why General Relativity is Needed

General relativity gives a very different interpretation of cosmology (covered in class). We can use general relativity to derive Friedmann's equation:

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{kc^2}{a(t)^2} + \frac{\Lambda c^2}{3} \tag{10}$$

This is remarkably similar to Eqn. 9, except with one additional term, known as the Λ, cosmological constant or **dark energy** term.

We also find k is a **curvature term** - if $k = 1$ the universe has positive curvature like that of a sphere, if $k = 0$ it is flat and if $k = -1$ it has negative curvature like that of a saddle. From the Einstein field equation, we find that there is a relationship between the density of the universe ρ_m and curvature. There is a critical density ρ_c where $k = 0$; if $\rho_m < \rho_c$, $k = -1$ and if $\rho_m > \rho_c$ $k = 1$.

An exact derivation of Eqn. 10 would also include a ρ_γ term for radiation, since light also curves spacetime. This term is, for most of the history of the universe, negligible, and we won't consider it.

If $\Lambda = 0$, then Eqn. 10 is identical to 9, and how k changes the fate of the universe remains valid. We can then say that **curvature is equivalent to density is equivalent to the fate of the universe**.

If $\Lambda \neq 0$, then this equivalence is **no longer true**!

- **Density ratios (Ω)** are defined as the density of some component of the universe divided by the critical density. Ω_m, for example, is ρ_m/ρ_c, while $\Omega_\Lambda = (\Lambda c^2/8\pi G)/\rho_c = \Lambda c^2/3H^2$. (The $3/8\pi G$ turns $\Lambda c^2/3$ from an energy density into a mass density.) We can actually rewrite the Friedmann equation this way.
- **Redshift (z)** is the lengthening, due to the expansion of the universe, of the wavelengths of photon emitted from distant galaxies on their way to us. A photon's wavelength is proportional to $a(t)$, so $\lambda_{\text{obs}}/\lambda_{\text{emit}} = a(t_{\text{obs}})/a(t_{\text{emit}})$. For things we see, the observation time $t_{\text{obs}} = t_0$. We define then define redshift z to be

$$z = \frac{\Delta\lambda}{\lambda_{\text{emit}}} = \frac{\lambda_0}{\lambda_{\text{emit}}} - 1$$

which means

$$1 + z = \frac{\lambda_0}{\lambda_{\text{emit}}} = \frac{a_0}{a(t_{\text{emit}})} \tag{11}$$

BASIC COSMOLOGY

Derive the evolution of a critical, matter-dominated ($\Lambda = 0$) universe. If we see a galaxy at redshift $z = 10$, how old was the universe when the light we see was emitted by that galaxy?

"Derive the evolution" means find $a(t)$. Here, $\Lambda = 0$ and $k = 0$. We then get:

$$H^2 = \frac{8\pi G}{3}\rho_m$$

Recall earlier that we said M enclosed within our sphere is fixed. This means

$$\begin{aligned}
\frac{4\pi}{3}R(t)^3\rho_m(t) &= \frac{4\pi}{3}R(t_0)^3\rho_m(t_0) \\
\rho_m(t) &= \rho_m(t_0)\frac{R(t_0)^3}{R(t)^3} \\
&= \rho_{m,0}\frac{R_0^3}{R(t)^3} \\
&= \rho_{m,0}\frac{a_0^3}{a(t)^3}
\end{aligned}$$

where we used Eqn. 4 for the last step.

This gives us (using Eqn. 5)

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho_{m,0} \frac{a_0^3}{a^3}$$

$H^2 = 8\pi G \rho_m / 3$ implies $H_0^2 = 8\pi G \rho_{m,0}/3$ (again the 0s mean at t_0, today), so we can write:

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= H_0^2 \frac{a_0^3}{a^3} \\ a\dot{a}^2 &= H_0^2 a_0^3 \\ \sqrt{a} da/dt &= H_0 a_0^{3/2} \\ \int \sqrt{a} da &= \int H_0 a_0^{3/2} dt \end{aligned}$$

Performing the integral we get

$$\frac{2}{3} a^{3/2} = H_0 a_0^{3/2} t + C$$

For this universe, we expect there to be a Big Bang[4], so $a = 0$ when $t = 0$, and C = 0. We could have also done a definite integral with the same initial conditions: $\int_0^a \sqrt{a'}da' = \int_0^t H_0 a_0^{3/2} dt'$. Doing some algebra, we get

$$a(t) = a_0 \left(\frac{3}{2} H_0 t\right)^{2/3}$$

We will plot this function at the end of the section.

We now determine the age of the universe when a $z = 10$ galaxy emitted light to us. Plugging our solution into Eqn. 11 we obtain

$$\begin{aligned} 1 + z &= \frac{t_0^{2/3}}{t_{\mathrm{emit}}^{2/3}} \\ t_{\mathrm{emit}} &= \frac{t_0}{(1+z)^{3/2}} \end{aligned}$$

[4]There are really weird universes in which that is not the case, but we do not consider any in this course.

We could either plug in the known age of the universe for t_0, or, to be self-consistent, use the age of the universe given by a critical, matter-dominated universe. Note that $H(t) = \dot{a}/a$. Plugging in our solution for a, we find that $H(t) = (2/3)(t^{-1/3}/t^{2/3}) = 2/3t$. This gives $t_0 = 2/3H_0$. Now, $H_0 = 70$ km/s/Mpc, which is 7×10^4 m/s/Mpc. 1 Mpc is 3.08×10^{22} m, according to Google, so $7 \times 10^4\,\mathrm{m\,s^{-1}}/(3.08 \times 10^{22}\,\mathrm{m}) = 2.27 \times 10^{-18}\,\mathrm{s^{-1}}$, which gives $t_0 = 2.93 \times 10^{17}\,\mathrm{s} \;=\; 9.3\,\mathrm{Gyr}$.

Plugging this value back into $t_{\mathrm{emit}} = \frac{t_0}{(1+z)^{3/2}}$ gives $t_{\mathrm{emit}} = 0.255\,\mathrm{Gyr}$.

This example illustrates a number of common slip-ups to watch for:

- 0s mean values at t_0, and an equation involving values at t_0 does not have to hold for all times t. **Do not** freely interchange t and t_0.
- Always integrate to turn $\dot{a}$ into a - there are very few situations where doing so is unnecessary. Make sure you don't forget constants of integration, and substitute in initial conditions to remove them.
- Be careful with unit conversions, and ask yourself if your answer makes sense!

Derive the evolution of an empty, matter-dominated ($\Lambda = 0$) universe. If we see a galaxy at redshift $z = 10$, how old was the universe when the light we see was emitted by that galaxy?

Taking Eqn. 10 and noting that $\rho_m = 0$ and $k = -1$ for an empty universe,

$$\begin{aligned} H^2 = \frac{\dot{a}^2}{a^2} &= -\frac{kc^2}{a^2} = \frac{c^2}{a^2} \\ \dot{a} &= c \\ \int da &= \int c\,dt \\ a &= ct + D \end{aligned}$$

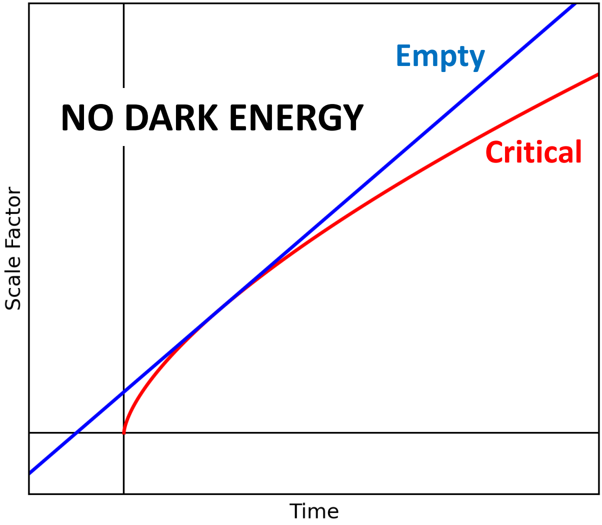
Again we take $a = 0$ when $t = 0$, giving us $a = ct$ (note that $a_0 \neq 0$ in this situation). Using Eqn. 5 and our solution for $a(t)$, $H(t) = 1/t$, and the age $t_0 = 1/H_0$ of this universe is 14.0 Gyr. Using the same procedure as before, we can find $t_{\text{emit}} = t_0/(1+z)$, giving us 1.27 Gyr.

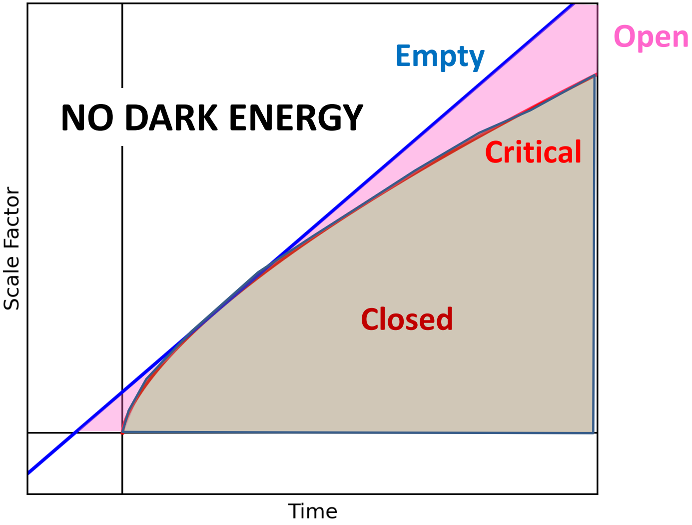
Derive the evolution of an empty, flat, dark energy-dominated universe. How old is the universe?

Taking Eqn. 10 and noting that $\rho_m = 0$ and $k = 0$ for an empty universe,

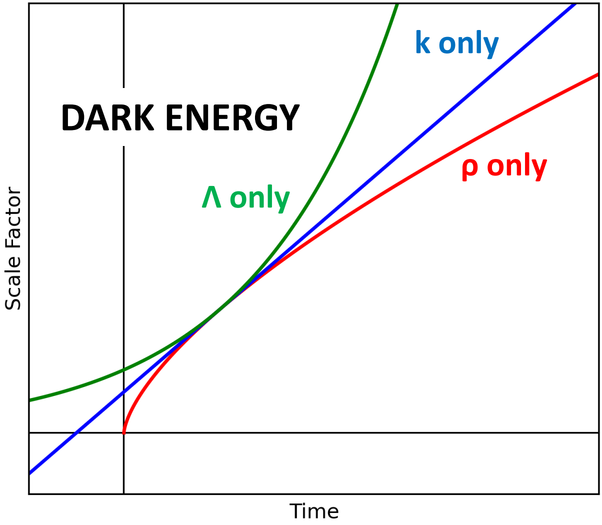
$$\begin{aligned} H^2 = \frac{\dot{a}^2}{a^2} &= \frac{\Lambda c^2}{3} \\ \frac{\dot{a}}{a} &= \sqrt{\Lambda/3}c \\ \int \frac{1}{a} da &= \int \sqrt{\Lambda/3} c dt \\ ln(a) &= \sqrt{\Lambda/3}ct + D \\ a(t) &= A \exp\left(\sqrt{\Lambda/3}ct\right) \end{aligned}$$

When we calculate $H(t)$, we find that $H(t) = \sqrt{\Lambda/3}c$, a constant independent of time. Moreover, if we plug in t = 0 into $a(t)$, we don't get zero. In fact, this universe is infinitely old (it is not possible to perform a reasonable definite integral in this case). Any universe that has a finite age cannot have been dark energy-dominated forever.


Empty
NO DARK ENERGY
Critical
Scale Factor
Time



NO DARK ENERGY
Empty
Open
Critical
Closed
Scale Factor
Time


DARK ENERGY
k only
ρ only
Λ only
Scale Factor
Time

Further Reading

- Hawley & Holcomb , *Foundations of Modern Cosmology*, Ch. 10 - 11
- Carroll & Ostlie, *An Introduction to Modern Astrophysics*, Ch. 29 (on the mathematical side)