Formula Sheet

Effective rate of interest: $i_{u+1} = \dfrac{S(u+1)-S(u)}{S(u)}$

Payment of 1	Compound interest	Simple interest
Accumulated value after t years	$(1+i)^t$	$(1+ti)$
Present value at time 0	$v^t = (1+i)^{-t}$	$(1+it)^{-1}$

i paid at the ***end*** of the period on the balance at the ***beginning*** of the period.
d paid at the ***beginning*** of the period on the balance at the ***end*** of the period.

$$d = \frac{i}{1+i}$$

Present value with simple discount: $(1-d\cdot t)$

Real interest rate: $1+i_{real} = \dfrac{1+i}{1+r}$

The accumulated value of 1 from time 0 to time t under compound interest:

$$S(t) = \left(1+\frac{i^{(m)}}{m}\right)^{mt} = (1+i)^t = v^{-t} = (1-d)^{-t} = \left(1-\frac{d^{(m)}}{m}\right)^{-mt} = e^{\delta t}$$

The present value at time 0 of 1 payable at time t under compound interest is:

$$S(0) = \left(1+\frac{i^{(m)}}{m}\right)^{-mt} = (1+i)^{-t} = v^{t} = (1-d)^{t} = \left(1-\frac{d^{(m)}}{m}\right)^{mt} = e^{-\delta t}$$

Force of interest

For constant force of interest, (under compound interest) $\delta = \ln(1+i)$

	Constant force of interest ($\delta_t = \delta$)	Variable force of interest
Accumulation at time t_2 of an amount $S(t_1)$ invested at t_1	$S(t_2) = S(t_1)\cdot e^{\delta(t_2-t_1)}$	$S(t_2) = S(t_1)\cdot \exp\left(\int_{t_1}^{t_2}\delta_t dt\right)$
Present value at time t_1 of an amount $S(t_2)$ due at time t_2	$S(t_1) = S(t_2)\cdot e^{-\delta(t_2-t_1)}$	$S(t_1) = S(t_2)\cdot \exp\left(-\int_{t_1}^{t_2}\delta_t dt\right)$

Annuities

n payments of 1	Payments made in arrears (at end of each period)	Payments made in advance (at start of each period)
Accumulated value	$s_{\overline{n}\rceil} = \sum_{t=0}^{n-1}(1+i)^t = \dfrac{(1+i)^n-1}{i}$	$\ddot{s}_{\overline{n}\rceil} = \sum_{t=1}^{n}(1+i)^t = \dfrac{(1+i)^n-1}{d}$
Present value	$a_{\overline{n}\rceil} = \sum_{t=1}^{n}v^t = \dfrac{1-v^n}{i}$	$\ddot{a}_{\overline{n}\rceil} = \sum_{t=0}^{n-1}v^t = \dfrac{1-v^n}{d}$

Payments of $\frac{1}{m}$ made each $\frac{1}{m}^{th}$ of a year for n years	Payments made in arrears	Payments made in advance
Accumulated value	$s_{\overline{n}\rvert}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}}$	$\ddot{s}_{\overline{n}\rvert}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}}$
Present value	$a_{\overline{n}\rvert}^{(m)} = \frac{1-v^n}{i^{(m)}}$	$\ddot{a}_{\overline{n}\rvert}^{(m)} = \frac{1-v^n}{d^{(m)}}$

Payments of 1 for perpetuity	Payments made in arrears	Payments made in advance
Present value of 1 per period	$a_{\overline{\infty}\rvert} = \frac{1}{i}$	$\ddot{a}_{\overline{\infty}\rvert} = \frac{1}{d}$
Present value of $\frac{1}{m}$ per period of length $\frac{1}{m}$	$a_{\overline{\infty}\rvert}^{(m)} = \frac{1}{i^{(m)}}$	$\ddot{a}_{\overline{\infty}\rvert}^{(m)} = \frac{1}{d^{(m)}}$

Continuous annuity of 1 per period for n periods	Fixed rate of interest	Variable rate of interest
Accumulated value	$\bar{s}_{\overline{n}\rvert} = \int_0^n (1+i)^{n-t} dt = \frac{(1+i)^n - 1}{\delta}$	$\bar{s}_{\overline{n}\rvert \delta_r} = \int_0^n \exp\left(\int_t^n \delta_r dr\right) dt$
Present value	$\bar{a}_{\overline{n}\rvert} = \int_0^n v^t dt = \frac{1-v^n}{\delta}$	$\bar{a}_{\overline{n}\rvert \delta_r} = \int_0^n \exp\left(-\int_o^t \delta_r dr\right) dt$

Arithmetically increasing annuity of n payments (first payment amount = 1, subsequent payments increase by 1 per period)	Payments made in arrears	Payments made in advance
Accumulated value	$(Is)_{\overline{n}\rvert} = \frac{\ddot{s}_{\overline{n}\rvert} - n}{i}$	$(I\ddot{s})_{\overline{n}\rvert} = \frac{\ddot{s}_{\overline{n}\rvert} - n}{d}$
Present value	$(Ia)_{\overline{n}\rvert} = \frac{\ddot{a}_{\overline{n}\rvert} - nv^n}{i}$	$(I\ddot{a})_{\overline{n}\rvert} = \frac{\ddot{a}_{\overline{n}\rvert} - nv^n}{d}$

Arithmetically decreasing annuity of n payments (first payment amount= n, subsequent payments decrease by 1 per period)	Payments made in arrears	Payments made in advance
Accumulated value	$(Ds)_{\overline{n}\rvert} = \frac{n\cdot(1+i)^n - s_{\overline{n}\rvert}}{i}$	$(D\ddot{s})_{\overline{n}\rvert} = \frac{n\cdot(1+i)^n - s_{\overline{n}\rvert}}{d}$
Present value	$(Da)_{\overline{n}\rvert} = \frac{n - a_{\overline{n}\rvert}}{i}$	$(D\ddot{a})_{\overline{n}\rvert} = \frac{n - a_{\overline{n}\rvert}}{d}$

Increasing annuity	with discrete increases and continuous payments	with continuous increases and continuous payments
Present value	$(I\bar{a})_{\overline{n}\rceil} = \int_0^n \lceil t \rceil v^t dt = \dfrac{\ddot{a}_{\overline{n}\rceil} - nv^n}{\delta}$	$(\bar{I}\bar{a})_{\overline{n}\rceil} = \int_0^n tv^t dt = \dfrac{\bar{a}_{\overline{n}\rceil} - nv^n}{\delta}$

Geometric series summation formula: $1 + x + x^2 + x^3 + ... + x^k = \dfrac{1-x^{k+1}}{1-x} = \dfrac{x^{k+1}-1}{x-1}$

Present value at time 0 of a series of n payments, each of amount 1, commencing at time $k+1$: ${}_{k}|a_{\overline{n}\rceil} = v^k \cdot a_{\overline{n}\rceil} = a_{\overline{n+k}\rceil} - a_{\overline{k}\rceil}$

General formula for increasing annuity

n-payment annuity with first payment A and subsequent payment B larger (or smaller) than the previous one. Payments made in arrears. Accumulated value at time n is

$$S(n) = (A-B)s_{\overline{n}|i} + B\left(Is\right)_{\overline{n}|i}$$

Solving Equations of Value

Quadratic form: $a\left((1+i)^n\right)^2 + b(1+i)^n + c = 0$ solution: $(1+i)^n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Linear interpolation: Given i_1, i_2, $f(i_1)$ and $f(i_2)$: $\dfrac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cong \dfrac{i_0 - i_1}{i_2 - i_1}$

$$\Rightarrow i_0 \cong i_1 + \frac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cdot (i_2 - i_1)$$

Loan calculations

Loan amount = $L = OB_0$

Interest rate per period = i

Amount paid at time t = K_t

Interest charged at the end of the t^{th} period = $I_t = OB_{t-1} \cdot i$

Principal repaid at the end of the t^{th} period = $PR_t = K_t - I_t$

Outstanding balance (principal) just after payment at time t = OB_t

Where $OB_t = OB_{t-1} + OB_{t-1} \cdot i - K_t$

$$OB_t = OB_{t-1} + I_t - K_t = OB_{t-1} - \left(K_t - I_t\right) = OB_{t-1} - PR_t$$

Outstanding balance:

Retrospective method: $OB_t = L(1+i)^t - \sum_{a=1}^{t} (1+i)^{t-a} K_a$

Prospective method: $OB_t = vK_{t+1} + v^2 K_{t+2} + ... + v^{n-t} K_n = \sum_{a=1}^{n-t} v^a K_{a+t}$

Capital Budgeting

Internal Rate of Return – effective rate of interest that makes the net present value of cash flows zero.
Discounted Payback Period – the amount of time it takes for a project to start making money.

Measuring Investment Performance

Money-Weighted Rate of Return – interest rate satisfying the equation of value incorporating the initial and final fund values and the intermediate cash flows.

Time-Weighted Rate of Return – product of the growth factors between successive cash flows.

Bonds

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

Callable bonds

When a bond is to be redeemed at the option of the issuer:

- For a bond bought at a discount, if an investor requires a particular minimum yield, the price paid should be based on the ***latest*** optional redemption date.

- For a bond bought at a premium, if an investor requires a particular minimum yield, the price paid should be based on the ***earliest*** optional redemption date.

Forward Contracts

Forward price with no income payments = $K = S_0 e^{\delta T}$

Forward price with income payments = $K = (S_0 - PV_I)e^{\delta T}$,
where PV_I is the present value at time $t = 0$ of the fixed income payments due during the term of the forward contract.

Value of long forward contract at time r = $\boxed{V_L = (K_r - K_0)e^{-\delta(T-r)}}$
If a security does not pay income, this is equivalent to $V_L = S_r - S_0 e^{\delta r}$
Value of short forward contract at time r = $V_S = -V_L$

Spot rate and forward rates

$$(1+f_{t,T})^{T-t} = \frac{(1+s_T)^T}{(1+s_t)^t}$$

Interest rate risk management

Effective duration (volatility) = $\upsilon = -\dfrac{1}{PV}\dfrac{d}{di}PV = \dfrac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k+1}}{PV}$

Macaulay's duration (discounted mean term or duration) = $\tau = \dfrac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k}}{PV} = (1+i)\upsilon$

Convexity = $c = \dfrac{1}{PV}\dfrac{d^2}{di^2}PV = \dfrac{\sum_{k=1}^{n} C_{t_k} t_k (t_k+1) v_i^{t_k+2}}{PV}$

Macaulay's duration for a bond with n coupon payments = $\dfrac{Fr\cdot\left(Ia\right)_{\overline{n|}j} + n\cdot C\cdot v_j^n}{PV}$

Taylor series

$$f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2!} f''(x) + \ldots$$

$$\frac{PV(i_0+\varepsilon) - PV(i_0)}{PV(i_0)} \cong \varepsilon \frac{PV'(i_0)}{PV(i_0)} + \frac{\varepsilon^2}{2}\frac{PV''(i_0)}{PV(i_0)} = -\varepsilon\frac{\tau}{(1+i_0)} + \frac{\varepsilon^2}{2}c$$

Immunisation

1. $S(i_0) = PV_A(i_0) - PV_L(i_0) = 0$.
2. $\tau_A(i_0) = \tau_L(i_0)$, $\upsilon_A(i_0) = \upsilon_L(i_0)$ or $PV_A'(i_0) = PV_L'(i_0)$
3. $c_A(i_0) \ge c_L(i_0)$ or $PV_A''(i_0) \ge PV_L''(i_0)$

Stochastic interest rate models

Uniform distribution

$f(x) = \dfrac{1}{b-a}$ for $a < x < b$

$E\left[\tilde{X}\right] = \dfrac{a+b}{2}$

$Var\left[\tilde{X}\right] = \dfrac{(b-a)^2}{12}$

Normal distribution

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\exp\left(\dfrac{-(x-\mu)^2}{2\sigma^2}\right)$ for $-\infty < x < \infty$

$E\left[\tilde{X}\right] = \mu$

$Var\left[\tilde{X}\right] = \sigma^2$

Accumulated value of single cash flows

If interest rates are independent and identically distributed, with mean $E[\tilde{i}]$ and variance $Var[\tilde{i}]$, then the mean and variance of the accumulated value of 1 after n periods are:

$$E[\tilde{S}(n)] = (E[1+\tilde{i}])^n$$
$$E[\tilde{S}(n)^2] = \left(E\left[(1+\tilde{i})^2\right]\right)^n$$
$$Var[\tilde{S}(n)] = E[\tilde{S}(n)^2] - (E[\tilde{S}(n)])^2 = \left(E\left[(1+\tilde{i})^2\right]\right)^n - (E[1+\tilde{i}])^{2n}$$

Accumulated value using log-normal

For large n, if the forces of interest $\tilde{\delta}_t$ are independent and identically distributed with mean $E[\tilde{\delta}]$ and variance $Var[\tilde{\delta}]$, then $\ln[\tilde{S}(n)]$ is approximately normally distributed with:

Mean: $E[\ln[\tilde{S}(n)]] = E[\tilde{\delta}_1 + \tilde{\delta}_2 + ... + \tilde{\delta}_n] = n \cdot E[\tilde{\delta}]$
Variance: $Var[\ln[\tilde{S}(n)]] = Var[\tilde{\delta}_1 + \tilde{\delta}_2 + ... + \tilde{\delta}_n] = n \cdot Var[\tilde{\delta}]$

Annuities

$E[\tilde{s}_{\overline{n}|}] = s_{\overline{n}|}$, where $s_{\overline{n}|}$ is evaluated at the interest rate $E[\tilde{i}]$.