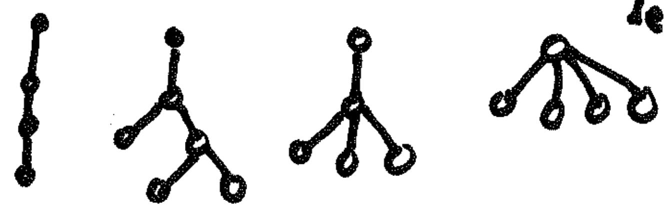
Lecture 6 handout

Trees (§4.1)

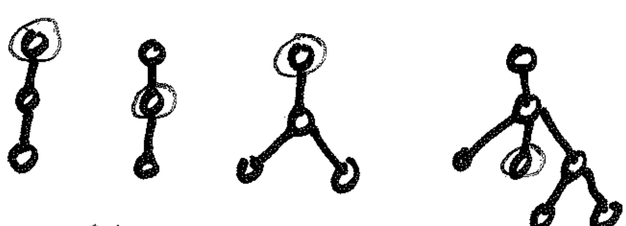
Definition: • A tree is a connected graph without cycles.

• A rooted tree is a tree with a distinguished $\underbrace{\text{1-valent vertex}}_{\text{leaf}}$ called root.

Trees:



Rooted trees:



$$\{\text{trees}\} \underset{\text{forget root}}{\overset{\text{Choose root}}{\rightleftarrows}} \{\text{rooted trees}\}$$

Branching: An oriented rooted tree with $\mathrm{indeg}(v)=1$ for all $v \neq \text{root}$.

• Each non-trivial tree has at least 2 leaves.

- A tree on n vertices has $n-1$ edges.
  (induction)

Definition: A *forest* is an acyclic graph.

 ← forest

Proposition: Let T be a graph with n vertices.
The following are equivalent:

1) T is a tree.
2) There is a unique path between any vertices u, v.
3) Adding any non-parallel edge creates a cycle.
4) Deleting any edge disconnects T.
5) T is connected and has $n-1$ edges.

4.8 **Cayley's Formula:** The number of trees that can be formed from a set of n vertices is n^{n-2}

Proof: Double-count ways to form *rooted* trees.

(see /wiki/Double_counting_(proof_technique)#Counting_trees)

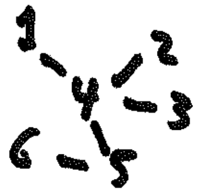
Bonus

Prüfer code

A sequence $(t_1, \ldots, t_{n-2})$ of numbers 1 to n.

Tree to code:

On ith step, remove leaf with smallest label, add its neighbour's label as t_i.



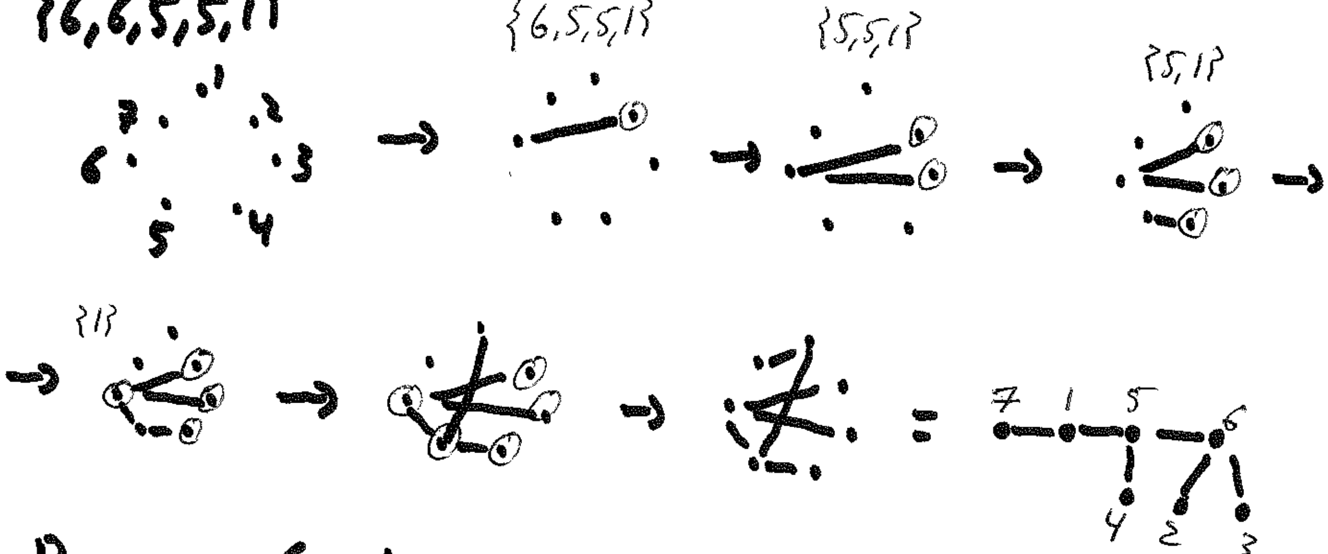
$\rightsquigarrow \{4, 4, 4, 5\}$

Code to tree:

On ith step, add edge between smallest label not in code and t_i.

Finally, add edge between two remaining vertices.

$\{6, 6, 5, 5, 1\}$



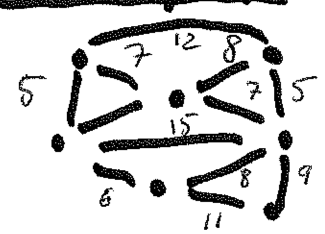
Proves Cayley's formula.

Bonus

Kruskal's algorithm

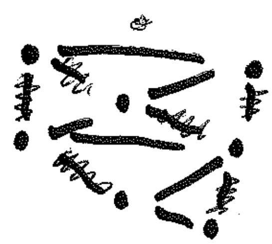
Spanning tree: Spanning subgraph that is a tree.

Weighted graph: Graph equipped with $f: E \to \mathbb{R}$.



(Wikipedia).

Each step, add smallest edge weight not in set which doesn't create a cycle.



Proof: Contradiction.

Choose minimal spanning tree T' that agrees with T for longest time.

$F = T \cap T'$. e in T not in T'. Let $G_0 = T' + e$.

G_0 has a cycle. Break the cycle by an edge not in T. $T_0 = T' + e - e'$ agrees with T longer than T'. contradiction.

Next time: Bonus: Matrix-tree theorem.