Jan 16th, 2013

Recall

Fix prime p

$\mathbb{F}_p = \{\bar{0}, \bar{1}, \cdots, \overline{p-1}\}$

For any $n \in \mathbb{Z}$, $\bar{n} = \bar{c}$ means c is the remainder of the division $\frac{n}{p}$.

If $p=3$ then $\mathbb{F}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$
and $\bar{5} = \bar{2} = \overline{-1}$
$\overline{10} = \bar{1}$

To make $\mathbb{F}_p$ into a field we define $\bar{a} + \bar{b} = \overline{a+b}$, $\bar{a} \cdot \bar{b} = \overline{a \cdot b}$

Again, if $p=3$ then $\bar{5} + \bar{3} = \overline{5+3} = \bar{8} = \bar{2}$
$\bar{5} \cdot \bar{3} = \overline{15} = \bar{0}$

<u>Claim</u>: With these definitions of $+$ and $\cdot$, $\mathbb{F}_p$ is a field.

In $\mathbb{F}_p$ additive identity is $\bar{0}$.
mult. identity is $\bar{1}$.

Note: $\mathbb{F}_p$ has other notations: $\mathbb{Z}_p$, $\mathbb{Z}/p\mathbb{Z}$

<u>Q</u>: In $\mathbb{F}_5$ what's the multi. inverse of $\bar{4}$?
inverse of $\bar{4}$ is $\bar{4}$.

<u>Exs</u> ① $(\mathbb{F}_3)^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, a, b \in \mathbb{F}_3 \right\}$

$(\mathbb{F}_3)^2$ has 9 vectors

<u>Question</u>: Is $\left\{ \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix} \right\}$ a subspace of $(\mathbb{F}_3)^2$.

$\begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix}$ in the set

$\boxed{\{\text{unit vectors}\} \subset \mathbb{R}^2 \text{ NOT a subspace}}$

But $\begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{2} \\ \bar{0} \end{bmatrix}$ not in the set. SO NOT A SUBSPACE

Consider: $V = \left\{ \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{2} \\ \bar{0} \end{bmatrix} \right\}$

Claim: V subspace of $(\mathbb{F}_3)^2$

Question: $\dim V = ? = 1$

$\rightarrow \dim (\mathbb{F}_3)^2 = 2$ cuz $(\mathbb{F}_3)^2$ has basis $\left\{ \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} \\ \bar{1} \end{bmatrix} \right\}$

DEF: Let F be a field and V, W be vector spaces over F. Then a **linear transformation** $T: V \to W$ is a function from V to W s.t.

① for any $v_1, v_2 \in V$ $T(v_1+v_2) = T(v_1) + T(v_2)$

② for any $v \in V$ and $c \in F$, $T(cv) = cT(v)$

Ex1: $A = \begin{bmatrix} 1-i & i \\ 2 & -1+i \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$

$v = \begin{bmatrix} 2 \\ -i \end{bmatrix} \in \mathbb{C}^2$

$$A(v) = \begin{bmatrix} 1-i & i \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} 2 \\ -i \end{bmatrix} = \begin{bmatrix} 2(1-i) - i^2 \\ 4 + i(1+i) \end{bmatrix} = \begin{bmatrix} 3-2i \\ 3+i \end{bmatrix}$$

Q: Is A ~~injective~~ one-to-one? i.e. is $\operatorname{Ker} A = \{0\}$?

to compute $\operatorname{Ker} A$ we look at associated homogeneous equation

$$\left[\begin{array}{cc|c} 1-i & i & 0 \\ 2 & -1+i & 0 \end{array}\right]$$

want: reduce $\begin{bmatrix} 1-i & i \\ 2 & -1+i \end{bmatrix}$

$$\begin{bmatrix} 1-i & i \\ 2 & -1+i \end{bmatrix} \sim \begin{bmatrix} 2 & -1+i \\ i & i \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1+i}{2} \\ 1-i & i \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1+i}{2} \\ 0 & 0 \end{bmatrix}$$

$$i - (1-i)\tfrac{-1+i}{2} = i - i = 0$$

the general solution to homogeneous equation is $y = z$ $x = -\left(\frac{-1+i}{2}\right) z$

$$\operatorname{Ker} A = \operatorname{span}\left\{ \begin{bmatrix} (-1+i)/2 \\ 1 \end{bmatrix} \right\}$$

so A is not one-to-one because $\operatorname{Ker} A \neq 0$.

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* $z = a + ib \in \mathbb{C}$

$\bar{z} = a - ib \in \mathbb{C}$

↑ conjugate of z

Q: $T: \mathbb{C} \to \mathbb{C}$, $T(z) = \bar{z}$

Is T a linear transformation?

i.e. $T(z+w) \stackrel{?}{=} T(z) + T(w)$ True

$T(cz) \stackrel{?}{=} c\,T(z)$, $c, z \in \mathbb{C}$

$\hookrightarrow$ i.e. $\overline{cz} = c\bar{z}$ False

$z = a+ib$, $w = c+id$

$z+w = (a+c) + i(b+d)$

$T(z+w) = (a+c) - i(b+d)$

$T(z) = a - ib$

$T(w) = c - id$

$\therefore T(z+w) = T(z) + T(w)$

$z = a+ib$

$\bar{z} = a - ib$

.... Not True .