May 29th

e.g $S=(0,1)$, $s=1$

#1. Let $S \subseteq \mathbb{R}$ be bounded. Let $s = lub(S)$. Suppose $s \notin S$. Then there is an increasing sequence $(x_n)_{n=1}^{\infty} \subset S$ s.t. $x_n \to s$

$x_1 \quad x_2 \quad \cdots \quad s$

Proof: For each $n = 1, 2, 3, \ldots$ consider $s - \frac{1}{n} < s$. Since $s = lub(S)$, $s - \frac{1}{n}$ is not an upper bound for S.

So $\exists\, x_n \in S$ s.t. $s - \frac{1}{n} < x_n < s$ → ineq. is strict since $s \notin S$

$\Rightarrow$ we get a sequence $(x_n)_{n=1}^{\infty}$. let's check that $x_n \to s$

Fix $\varepsilon > 0$. choose N s.t. $\frac{1}{N} < \varepsilon$

Then when $n \geq N$, $s - x_n < s - (s - \frac{1}{n}) = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Therefore $x_n \to s$

∎

#2 Prove that if $lub(S)$ / $glb(S)$ exists then it's unique.

Pf: Sps b_1 & b_2 are both lub's for the set S

As b_1 is an upper bound, and b_2 is a lub $\Rightarrow b_2 \leq b_1$

b_2 . . . . b_1 . . . $\Rightarrow b_1 \leq b_2$

$\Rightarrow b_1 = b_2$

∎

#3. Let $p > 0$. Consider $\sqrt{p + \sqrt{p + \cdots}}$ as a limit of a sequence. Use Monotone Seq. thm to show it must exist & calculate it.

Seq: $0, \sqrt{p}, \sqrt{p + \sqrt{p}}, \ldots,$

recursive def. $\begin{cases} x_1 = 0 \\ x_n = \sqrt{p + x_{n-1}} \end{cases}$

Idea: So need bounded & monotone ↗.

$\boxed{\text{Increasing}}$ — Proof: Use induction

Base case: $0 \leq \sqrt{p}$ ✓

Inductive step: Sps that $x_{n-2} \leq x_{n-1}$ ← by induction

$x_n = \sqrt{p + x_{n-1}} \geq \sqrt{p + x_{n-2}}$ (*)

(*) follows b/c $f(x) = \sqrt{x}$ is monotone increasing and $p + x_{n-1} \geq p + x_{n-2}$

So $x_n \geq \sqrt{p + x_{n-2}} = x_{n-1}$ ✓

$\boxed{\text{Bounded}}$ — ~~Proof: Sps first that the limit exists~~, $L = \lim_{n\to\infty} x_n$.

$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{p + x_n} = \sqrt{p + \lim_{n\to\infty} x_n}$ b/c $g(x) = \sqrt{p+x}$ is continuous.

$= \sqrt{p + L}$

$\Rightarrow L = \frac{1 \pm \sqrt{4p+1}}{2}$ (take positive) $\checkmark$

$L = \frac{1}{2} + \frac{1}{2}\sqrt{4p+1}$

Check x_n is bounded

Claim: $x_n \leq 2+2p \quad \forall n \quad (*)$

Use induction

Proof: Base: $0 \leq 2+2p$ b/c $p>0$ ✓

Inductive step: Sps $x_{n-1} \leq 2+2p$

Now $x_n \leq 2+2p \iff x_n^2 \leq (2+2p)^2 \longleftarrow$ so it's enough to show this.

$$\begin{aligned}(2+2p)^2 - x_n^2 &= (4+8p+4p^2) - (\sqrt{p+x_{n-1}})^2 = (4+8p+4p^2) - (p+x_{n-1}) \\ &= 4+7p+4p^2 - x_{n-1} \\ &= \underbrace{(2+2p-x_{n-1})}_{>0 \text{ by } (*)} + \underbrace{(2+5p+4p^2)}_{>0 \text{ since } p>0}\end{aligned}$$

So $(2+2p)^2 > x_n^2$

■

#4 S is disconnected if its sets S_1, S_2 (non-empty) s.t. $S = S_1 \cup S_2$ & $S_1 \cap \overline{S_2} = \emptyset = \overline{S_1} \cap S_2$

(a). $\mathbb{Q}$ is disconnected.

Pf: Let $S_1 = (-\infty, \sqrt{2}) \cap \mathbb{Q}$, $S_2 = (\sqrt{2}, +\infty) \cap \mathbb{Q}$.

$\sqrt{2}$ irrational. $\Rightarrow \mathbb{Q} = S_1 \cup S_2$

And $\overline{S_1} = (-\infty, \sqrt{2}]$ clearly $\overline{S_1} \cap S_2 = \emptyset$

Similarly, $\overline{S_2} = \cdots \quad \overline{S_2} \cap S_1 = \emptyset$

■

Skip (b)(c)

#5. Sec1.6 #6. Let $U, V \subseteq \mathbb{R}^n$. $d(U,V) = \inf\{|x-y| : x \in U, y \in V\}$.

(i) $d(U,V) > 0 \Rightarrow U, V$ are disconnected

Pf: We need to show $\overline{U} \cap V = \emptyset = U \cap \overline{V}$.

Let $x \in \overline{U} \Rightarrow$ we can find a sequence $\{x_n\}_{n=1}^{\infty}$ s.t. $x_n \to x$

Since $d(U,V) > 0 \Rightarrow \exists \cdots$ See yesterday's TUT