Lecture 12
Feb 24th, 2015

Define T by:

$$T(n) = \begin{cases} 1 & , n=1 \\ 1+\max(T(\lceil \frac{n}{2} \rceil), T(\lfloor \frac{n}{2} \rfloor)) & , n \geq 2 \end{cases}$$

$$T(1) \leq T(1)$$
$$T(1) \leq T(2)$$
$$T(2) \leq T(2)$$
$$T(1) \leq T(3)$$
$$T(2) \leq T(3)$$
$$T(3) \leq T(3)$$
$$\cdots$$
$$T(123) \leq T(135)$$
$$T(135) = 1 + \max(T(68), T(67))$$
$$T(123) = 1 + \max(T(62), T(61))$$

For $n \in \mathbb{N}$, let $P(n)$ be:

$$T(1) \leq T(n)$$
$$T(2) \leq T(n)$$
$$\cdots$$
$$T(n) \leq T(n)$$

$$P(67) \wedge P(68) \Rightarrow (T(61) \leq T(67)) \wedge (T(62) \leq T(67)) \wedge (T(61) \leq T(68)) \wedge (T(62) \leq T(68))$$

$$\Rightarrow T(123) \leq T(135)$$

Base Case:

IS: Let $n \in \mathbb{N}$, $n \geq 2$

[Prove $T(1) \leq T(n)$ and $\cdots$ and $T(n) \leq T(n)$]

IH: Assume $T(1) \leq T(1) \cdots P(1)$

$T(1) \leq T(2)$ and $T(2) \leq T(2) \cdots P(2)$

$\cdots$

$T(1) \leq T(n-1)$ and $\cdots$ and $T(n-1) \leq T(n-1) \cdots P(n-1)$

Assume $P(n)$ is true for all natural numbers that at least 1 and less than n.

$T(n) = 1 + \max(T(\lceil \frac{n}{2} \rceil), T(\lfloor \frac{n}{2} \rfloor))$ since $n \geq 2 \geq 1$

let $m \in \mathbb{N}$ such that $1 \leq m < n$.

Case: $m=1$, $T(n) = 1 + \max(\cdots) = 1 + T(\lceil \frac{n}{2} \rceil) \geq 1 + T(1) > T(1)$ since $1 \leq \lceil \frac{n}{2} \rceil$ and $P(\lceil \frac{n}{2} \rceil)$

Case: $m \geq 2$, $T(m) = 1 + \max(T(\lceil \frac{n}{2} \rceil), T(\lfloor \frac{n}{2} \rfloor))$

$\lceil \frac{n}{2} \rceil \geq \lceil \frac{n}{2} \rceil = 1$

$\lceil \frac{n}{2} \rceil \in \mathbb{Z}$ so $\lceil \frac{n}{2} \rceil \in \mathbb{N}$

$\frac{n}{2} = n - \frac{n}{2} \leq n-1$ so $n \geq 2$

so $\lceil \frac{n}{2} \rceil \leq n-1$ since $n \in \mathbb{N}$

so $\lceil \frac{n}{2} \rceil \leq n$

so $P(\lceil \frac{n}{2} \rceil)$ from (IH)

Also $1 \leq \lfloor \frac{n}{2} \rfloor \leq \lceil \frac{n}{2} \rceil$, so from $P(\lceil \frac{n}{2} \rceil)$: $T(\lfloor \frac{n}{2} \rfloor) \leq T(\lceil \frac{n}{2} \rceil)$

$1 \leq \lceil \frac{m}{2} \rceil \leq \lceil \frac{n}{2} \rceil$ since $m \leq n$

from $P(\lceil \frac{n}{2} \rceil)$: $T(\lceil \frac{m}{2} \rceil) \leq T(\lceil \frac{n}{2} \rceil)$, so $T(m) \leq T(n)$