Convergence continued...

Let $X_1, \dots, X_n$ be a sequence of r.v's and X be another r.v. Let F_n be cdf of X_n, and F be cdf of X.

Recall: $X_n \xrightarrow{P} X$ if for every $\varepsilon > 0$,

$$P(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$X_n \xrightarrow{d} X \text{ if } \lim_{n\to\infty} F_n(x) = F(x)$$

at all x for which F is continuous.

Def. X_n converges almost surely to X, $X_n \xrightarrow{a.s.} X$

if, for every $\varepsilon > 0$, $P(\lim |X_n - X| < \varepsilon) = 1$

or $\boxed{P(\lim_{n\to\infty} X_n = X) = 1}$

Def. X_n converges to X in quadratic mean,

$X_n \xrightarrow{qm} X$, if $E[(X_n - X)^2] \to 0$ as $n \to \infty$.

Ex. $\Omega = [0,1]$, $X_n(\omega) = \omega + \omega^n$, $X(\omega) = \omega$.

For every $\omega \in [0,1)$, $\omega^n \to 0$ as $n \to \infty$

$X_n(\omega) \to \omega = X(\omega)$

$X_n(1) = 2$ for every n, so $X_n(1) \not\to 1 = X(1)$

But $P([0,1)) = 1 \Rightarrow X_n \xrightarrow{a.s.} X$ (but not pointwise)

Relationships between convergences:

(i) $X_n \xrightarrow{qm} X \Rightarrow X_n \xrightarrow{P} X$

(ii) $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

(iii) $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$

(iv) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$

$$\begin{array}{ccccc} & & qm & & \\ & & \downarrow & & \\ a.s. & \rightarrow & P & \rightarrow & d \end{array}$$

Pf. (i) $X_n \xrightarrow{qm} X$. Fix $\varepsilon > 0$

$$P(|X_n - X| > \varepsilon) = P((X_n - X)^2 > \varepsilon^2) \underset{\text{Markov's}}{\leq} \frac{E[(X_n - X)^2]}{\varepsilon^2}$$

$$\xrightarrow[n \to \infty]{} 0 \Rightarrow X_n \xrightarrow{P} X$$

(ii), (iii) were proven before

(iv) won't be proved here.

Ex. $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{qm} X$

$U \sim Unif(0,1)$, $X_n = \sqrt{n}\, I_{(0, \frac{1}{n})}(U) = \begin{cases} \sqrt{n}, & U \in (0, \frac{1}{n}) \\ 0, & ow \end{cases}$

$$P(|X_n - 0| > \varepsilon) = P(|X_n| > \varepsilon) = P(0 < U < \tfrac{1}{n})$$

$$= \frac{1}{n} \xrightarrow[n \to \infty]{} 0 \Rightarrow X_n \xrightarrow{P} 0.$$

But $E[(X_n - 0)^2] = E(X_n^2) = \int_0^{1/n} (\sqrt{n})^2 \, du$

$$= n \int_0^{1/n} du = n \cdot \frac{1}{n} = 1 \text{ for all } n$$

$$\Rightarrow X_n \overset{qm}{\not\rightarrow} 0.$$

Ex. $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{a.s.} X$

$\Omega = [0,1]$.

Let $X_1(\omega) = \omega + I_{[0,1]}(\omega) = \begin{cases} \omega+1, & \omega \in [0,1] \\ \omega, & \omega \notin [0,1] \end{cases}$

$X_2(\omega) = \omega + I_{[0,\frac{1}{2}]}(\omega)$, $X_3(\omega) = \omega + I_{[\frac{1}{2},1]}(\omega)$

$X_4(\omega) = \omega + I_{[0,\frac{1}{3}]}(\omega)$, $X_5(\omega) = \omega + I_{[\frac{1}{3},\frac{2}{3}]}(\omega)$, $X_6(\omega) = \omega + I_{[\frac{2}{3},1]}(\omega)$

$\vdots$

Let $X(\omega) = \omega$

As $n \to \infty$, $P(|X_n - X| \geq \varepsilon) = P($an interval of ω values whose length is going to $0)$

$\longrightarrow 0 \Rightarrow X_n \xrightarrow{P} X$.

For every ω, $X_n(\omega)$ alternates between ω and $\omega+1$ infinitely often.

If $\omega = \frac{3}{8}$, $X_1(\omega) = \frac{3}{8}+1$, $X_2(\omega) = \frac{3}{8}+1$,

$X_3(\omega) = \frac{3}{8}$, $X_4(\omega) = \frac{3}{8}$, $X_5(\omega) = \frac{3}{8}+1$,

$X_6(\omega) = \frac{3}{8}$, etc

$\Rightarrow X_n \not\rightarrow X$ a.s

Ex. $X_n \xrightarrow{p} c \not\Rightarrow E(X_n) \to c$

$$P(X_n = n^2) = \frac{1}{n}, \quad \underline{P(X_n = 0) = 1 - \frac{1}{n}}$$

$$P(|X_n - 0| < \varepsilon) = P(X_n = 0) = 1 - \frac{1}{n} \xrightarrow[n\to\infty]{} 1$$

$$P(|X_n - 0| \geq \varepsilon) \longrightarrow 0 \Rightarrow X_n \xrightarrow{P} 0$$

But $$E(X_n) = n^2 \cdot \frac{1}{n} + 0\left(1 - \frac{1}{n}\right) = n$$

$$E(X_n) \xrightarrow[n\to\infty]{} \infty .$$

Ex. Let $X_1, \dots, X_n \overset{iid}{\sim} Unif(0,1)$, $X_{(n)} = \max\{X_1, \dots, X_n\}$. Show that $X_{(n)} \xrightarrow{P} 1$.

$$P(|X_{(n)} - 1| \geq \varepsilon) = P(X_{(n)} \geq 1+\varepsilon \text{ or } X_{(n)} \leq 1-\varepsilon)$$

$$= P(X_{(n)} \leq 1 - \varepsilon)$$

$$= P(X_1 \leq 1-\varepsilon, \dots, X_n \leq 1-\varepsilon)$$

$$= \prod_{i=1}^{n} P(X_i \leq 1-\varepsilon) = (1-\varepsilon)^n \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow X_{(n)} \xrightarrow{p} 1 .$$

$n(1 - X_{(n)}) \xrightarrow{d} Exp(1)$

$$P(n(1 - X_{(n)}) \leq t) = P\left(X_{(n)} \geq 1 - \frac{t}{n}\right)$$

$$= 1 - P\left(X_{(n)} \leq 1 - \frac{t}{n}\right) = 1 - \left(1 - \frac{t}{n}\right)^n \xrightarrow[n\to\infty]{} 1 - e^{-t}$$

Ex. $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$.

Let $X_n \sim N(0, \frac{1}{n})$

Let F be a dist. fn for $c = 0$
(dist'n function for a point mass at 0)

$\sqrt{n}\, X_n \sim N(0,1)$

$F_n(t) = P(X_n \leq t) = P(\underbrace{\sqrt{n}\, X_n} \leq \sqrt{n}\, t)$

$= P(Z \leq \underbrace{\sqrt{n}\, t})$, $Z \sim N(0,1)$

$F_n(t) \xrightarrow[n\to\infty]{} 0$, if $t < 0$ $(\sqrt{n}\, t \to -\infty)$

$F_n(t) \xrightarrow[n\to\infty]{} 1$ if $t > 0$ $(\sqrt{n}\, t \to \infty)$

$$F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

$F_n(t) \to F(t) \Rightarrow X_n \xrightarrow{d} 0$

$$P(|X_n - 0| > \varepsilon) = P(|X_n|^2 > \varepsilon^2) \leq \frac{E(X_n^2)}{\varepsilon^2}$$

$$= \frac{1/n}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow X_n \xrightarrow{P} 0$.