March 6th

V Hermitian ips ← inner product space

$T: V \to V$

the <u>adjoint</u> of T is the "operator" $T^*: V \to V$
satisfying $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$

main property: If $\alpha = \{v_1, \cdots, v_n\}$ is an orthonormal basis of V then

$$[T^*]_\alpha = [T]_\alpha^*$$ ← means conjugate transpose

EX: $V = P_1(\mathbb{C})$
$\langle p(x), q(x) \rangle = p(0)\overline{q(0)} + p(i)\overline{q(i)}$

This defines a Hermitian inner product on V (of HW)

$T: V \to V$; $T(p(x)) = p'(x)$

Question: what's T^*? want $T^*(ax+b)$?

$\beta = \{1, x\}$, standard basis of V

$\underset{u_1}{\langle 1,} \underset{u_2}{x \rangle} = -i$ so not orthonormal

Apply GS:
$v_1 = u_1 = 1$
$v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = x - \dfrac{i}{2} 1 = x - \frac{i}{2}$

$\langle v_1, v_1 \rangle = 2$

$\langle u_2, v_1 \rangle = \langle x, 1 \rangle = i$

$\|v_1\| = \sqrt{2}$
$\langle v_2, v_2 \rangle = \frac{1}{2}$
$\|v_2\| = \frac{1}{\sqrt{2}}$

orthonormal basis $\alpha = \{\frac{1}{\sqrt{2}}, \sqrt{2}x - \frac{i}{\sqrt{2}}\}$

$[T]_\alpha = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad T(\frac{1}{\sqrt{2}}) = 0,\ T(\sqrt{2}x - \frac{i}{2}) = \sqrt{2} = 2(\frac{1}{\sqrt{2}})$

$[T^*]_\alpha = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ $\quad \Rightarrow T^*(\frac{1}{\sqrt{2}}) = 2\sqrt{2}x - \sqrt{2}i$
$\quad T^*(\sqrt{2}x - \frac{i}{2}) = 0$

$T^*(ax+b) = \frac{a}{\sqrt{2}} T^*(\sqrt{2}x - \frac{i}{2}) + (\sqrt{2}b + \frac{ia}{2}) T^*(\frac{1}{\sqrt{2}}) = (\sqrt{2}b + \frac{ia}{2})(2\sqrt{2}x - \frac{i}{2})$

$ax + b = \frac{a}{\sqrt{2}}(\sqrt{2}x - \frac{i}{2}) + c(\frac{1}{\sqrt{2}})$

$b = \frac{-ia}{2\sqrt{2}} + \frac{c}{\sqrt{2}} \Rightarrow \sqrt{2}b + \frac{ia}{2} = c$

V Hermitian ips

$T: V \to V$

T normal if $T \cdot T^* = T^* \cdot T$

Spectral theorem for normal operators:

T normal.
Then V has an orthonormal basis of eigenvalues of T. in particular, T diagonalizable.

JORDAN CANONICAL FORM

$\mathbb{Z}$ let's say that $x, y \in \mathbb{Z}$ are equivalent if $x \equiv y$ mod n i.e. n divides $x-y$.

Ex: when $n=2$, $\mathbb{Z} = \mathbb{Z}_{even} \cup \mathbb{Z}_{odd}$
(They have no integers in common.

when $n=3$ ... the set of integers equivalent to 0 is $3\mathbb{Z}$
1 $\quad 3\mathbb{Z}+1$
2 $\quad 3\mathbb{Z}+2$

JORDAN is about to break up these.

$M_n(\mathbb{C})$ recall that A similar to B if there exists an invertible matrix X s.t. $A = XBX^{-1}$
Big picture: $M_n(\mathbb{C})$ get split up into similarity classes and choose the "best" representative for each class.

Will see is that any $A \in M_n(\mathbb{C})$ is similar to a matrix of the form

$$\left[\begin{array}{ccc|ccc} \lambda & 1 & & & & \\ & \lambda & 1 & & 0 & \\ 0 & & \lambda & & & \\ \hline & & & \mu & 1 & \\ & 0 & & & \mu & 1 \\ & & & & & \mu \end{array}\right]$$

Will prove the theorem in 3 major steps
1. triangularizability
2. prove the theorem for "nilpotent" matrices
3. ———— general matrices.

upper triangular matrix is a matrix of the form

$$\begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

recall: evalues of an upper $\triangle$ matrix and its diagonal entries

$T: V \longrightarrow V$; what does it mean for $[T]_\alpha$ to be upper $\triangle$?

$\alpha = \{v_1, \cdots, v_n\}$

$$[T]_\alpha = \begin{bmatrix} a_{11} & a_{12} & & * \\ 0 & a_{22} & & \\ \vdots & 0 & \ddots & \\ 0 & \vdots & \cdots & a_{nn} \end{bmatrix}$$

$$\left.\begin{array}{l} T(v_1) = a_{11} v_1 \\ T(v_2) = a_{22} v_2 + * v_1 \end{array}\right\} \Rightarrow T(\operatorname{span}\{v_1, v_2\}) \subset \operatorname{span}\{v_1, v_2\}$$

$T(v_3) = a_{33} v_3 + * v_2 + * v_1 \qquad T(\operatorname{span}\{v_1, v_2, v_3\}) \subset \operatorname{span}\{v_1, v_2, v_3\}$