Assignment 6 - MAT 327 - Summer 2014

Due July 14th, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

This assignment will have a handful of questions about the idea of a filter on a set X. This will be useful for (1) understanding how Zorn's Lemma works and (2) our proof of Tychonoff's theorem later in the course. Some of you have already seen the notion of a filter on Assignment 2 - NI 1 when you looked at Martin's Axiom.

Definition. *Let X be a non-empty set, and let $\mathcal{F}$ be a collection of subsets of X. We say that $\mathcal{F}$ is a* ***filter on*** *X provided that:*

1. *$X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;*
2. *If $A_1, \ldots, A_N \in \mathcal{F}$ then $A_1 \cap \ldots \cap A_N \in \mathcal{F}$. (This is called "being closed under finite intersections");*
3. *If $A \subseteq B \subseteq X$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$. (This is called "being closed upwards").*

[**C.1**] Let $p \in \mathbb{R}$, and let $\mathcal{T}$ be the usual topology on $\mathbb{R}$. Let $\mathcal{B}_p := \{ U \in \mathcal{T} : p \in U \}$. Show that this is not a filter, but there is a filter $\mathcal{F}$ such that $\mathcal{B}_p \subseteq \mathcal{F}$. Moreover, make sure that $\{ p \} \notin \mathcal{F}$. (Hint: Take $\mathcal{F}_p := \{A \subseteq \mathbb{R} : \exists U \in B_p \text{ such that } U \subseteq A\}$.)

[**C.2**] Use Zorn's Lemma to show that every filter $\mathcal{F}$ on a set X is contained in a maximal filter $\mathcal{U}$ (i.e. $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U}$ is a maximal element in the partial order of all filters on X that contain $\mathcal{F}$, ordered by "$\subseteq$"). Maximal filters are called **Ultrafilters**.

[**C.3**] Go through our Zorn's Lemma proof in §11 and answer the following two questions:

1. Why did we define $\mathbb{P}$ to be the set of all pairwise disjoint open, **countable** subsets of ω_1? What happens if we drop the countable condition? Does the proof still go through?

2. What “type” of set is the chain $\mathcal{C}$? I mean is $\mathcal{C}$ an element of ω_1? A subset of ω_1? A collection of subsets of ω_1? A collection of collections of subsets of ω_1? In that proof find examples of sets of the other types.

[**C.4**] Let C^0 be the collection of all continuous, real-valued functions on the interval $[a,b]$. Define a distance function on C^0 by

$$d(f,g) = \int_a^b |f(x) - g(x)| \; dx$$

Prove that (C^0, d) is a metric space. (For this question you may wish to consult your second-year calculus notes.) If $[a,b] = [0,1]$ describe the ball of radius 1 around the function $f(x) = x$ for all $x \in [0,1]$.

[**C.5**] Let ρ be the usual distance function on $\mathbb{R}$, and let (C^0, d) be defined as above. Define a function

$$F : (C^0, d) \longrightarrow (\mathbb{R}, \rho)$$

by

$$F(f) = \int_a^b f(x) \; dx.$$

Prove that F is continuous.

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] Show that there is a metric space (X, d), with $y \in X$, such that the ball $B_{1.001}(y)$ contains 100 pairwise disjoint open balls of radius 1.

[**A.2**] Let (X, d) be a metric space. Prove that X is separable iff X is second countable iff X is ccc. Give an example of a topological invariant that is not equivalent to these in a metric space.

The next exercise makes use of the notion of a **distance special** subset of $\mathbb{R}$:

Definition. *A set $A \subseteq \mathbb{R}$ is said to be* ***distance special*** *if whenever $a, b \in A$ with $d(a,b) = M > 0$, we have that a and b are the unique elements of A such that $d(a,b) = M$.*

[**A.3**] Using Zorn's Lemma, prove that every uncountable set $A \subseteq \mathbb{R}$ contains an uncountable distance special set. (Hint: A question on a previous assignment might be helpful, or at least of the right flavour.)

New Ideas

You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.

There is only one question this time because it is an important one that we will use later.

[**NI**] Let's investigate the notion of filters and ultrafilters on a set X, which we saw in question C.2. Prove the following facts:

1. If $x \in X$, then $\mathcal{U}_x := \{ A \subseteq X : x \in A \}$ is an ultrafilter. (This is called a **principal ultrafilter**, and is a somewhat trivial example.)

2. A filter $\mathcal{F}$ is an ultrafilter iff for every $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ (but not both!).

3. Let $\mathcal{U}$ be an ultrafilter and let $A \subseteq X$. Prove that $A \in \mathcal{U}$ iff $U \cap A \neq \emptyset$ for all $U \in \mathcal{U}$.

4. Let $\mathcal{F}$ be a collection of subsets of X with the **first two properties** of being a filter, which is called "having the finite intersection property (FIP)". Prove that if $\mathcal{F}$ is maximal (with respect to the FIP) then (1) $\mathcal{F}$ is a filter and (2) $\mathcal{F}$ is an ultrafilter.

Ultrafilters are very neat objects and they measure "largeness" in some sense (Supersets of large sets are large; the intersection of finitely many large sets is large; the full space is large, but the empty set is not.) For those of you who like number theory, there is an amazing proof of Hindman's Theorem that uses ultrafilters.