Chapter 3

Logical Connectives

3.1 Universal quantification and implication again

So far we have considered an implication to be universal quantification in disguise:

Claim 3.1: If an employee is male, then that employee makes less than 55,000.

The English indefinite article "an" signals that this means "Every male employee makes less than 55,000," and this closed sentence is either true or false, depending on the domain of employees. This can be expressed as $\forall x \in E, M(x) \Rightarrow L(x)$, and we can separate the "For all employees," portion from the "if the employee is male, then the employee makes less than 55,000," portion. Symbolically, we can think about $\forall x \in E$ separately from $M(x) \Rightarrow L(x)$, giving us some flexibility about which values we might substitute for x. This allows us to express the unquantified implication:

Claim 3.2: If the employee is male, then that employee makes less than 55,000.

The English definite article "the" often signals an unspecified value, and hence an open sentence. We could transform Claim 3.2 back into Claim 3.1 by prefixing it with "For every employee, . . . "

Claim 3.3: For every employee, if the employee is male, then that employee makes less than 55,000.

Since the claim is about male employees, we are tempted to say $\forall m \in M, L(m)$, which would be correct if the only males we were considering were those in E—$\forall m \in E \cap M, L(m)$ would certainly capture what we mean. Using that approach we would restrict the domain that we are universally quantifying over by intersecting with other domains. However, it is often convenient to restrict in another way: set our domain to the largest universe in which the predicates make sense, and use implication to restrict further. We don't have to avoid reasoning about non-males when we say $\forall e \in E, M(e) \Rightarrow L(e)$, and we get the same meaning as $\forall m \in E \cap M, L(m)$.

It also often happens that the predicate expressed by $M(e)$ doesn't neatly translate into a set that can be intersected with set E, so the universally quantified implication format can be handy. For example, $\forall n \in \mathbb{N}, n > 0 \Rightarrow 1/n \in \mathbb{R}$ means the same things as $\forall n \in \mathbb{N} \setminus \{0\}, 1/n \in \mathbb{R}$, but expressing the set $\mathbb{N} \setminus \{0\}$ seems more awkward than using universally-quantified implication, and there are MUCH worse cases.

How do you feel about verifying Claim 3.2 for all six values in E, which are true/false?[1]

Do you feel uncomfortable saying that the implications with false antecedents are true? Implications are strange, especially when we consider them to involve causality (which we don't in logic). Consider:

Claim 3.4: If it rains in Toronto on June 2, 3007, then there are no clouds.

Is Claim 3.4 true or false? Would your answer change if you could wait the required number of decades? What if you waited and June 2, 3007 were a completely dry day in Toronto, is Claim 3.4 true or false?[2]

3.2 VACUOUS TRUTH

We use the fact that the empty set is a subset of any set. Let $x \in \mathbb{R}$ (the domain is the real numbers). Is the following implication true or false?

CLAIM 3.5: If $x^2 - 2x + 2 = 0$, then $x > x + 5$.

A natural tendency is to process $x > x+5$ and think "that's impossible, so the implication is false." However, there is no real number x such that $x^2 - 2x + 2 = 0$, so the antecedent is false for every real x. Whenever the antecedent is false and the consequent is either true or false, the implication as a whole is TRUE. Another way of thinking of this is that the set where the antecedent is true is empty (vacuous), and hence a subset of every set. Such an implication is sometimes called VACUOUSLY TRUE.

In general, if there are no Ps, we consider $P \Rightarrow Q$ to be true, regardless of whether there are any Qs. Another way of thinking of this is that the empty set contains no counterexamples. Use this sort of thinking to evaluate the following claims:[3]

CLAIM 3.6: All employees making over 80,000 are female.

CLAIM 3.7: All employees making over 80,000 are male.

CLAIM 3.8: All employees making over 80,000 have supernatural powers and pink toenails.

3.3 EQUIVALENCE

Suppose Al quits the domain E. Consider the claim

CLAIM 3.9: Every male employee makes between 25,000 and 45,000.

Is Claim 3.9 true? What is its converse?[4] Is the converse true? Draw a Venn diagram. The two properties describe the same set of employees; they are EQUIVALENT. In everyday language, we might say "An employee is male if and only if the employee makes between 25,000 and 45,000." This can be decomposed into two statements:

CLAIM 3.10: An employee is male if the employee makes between 25,000 and 45,000.

CLAIM 3.11: An employee is male only if the employee makes between 25,000 and 45,000.

Here are some other everyday ways of expressing equivalence:

- P iff Q ("iff" being an abbreviation for "if and only if").
- P is necessary and sufficient for Q.
- $P \Rightarrow Q$, and conversely.

You may also hear

- P [exactly / precisely] when Q

For example, if our domain is $\mathbb{R}$, you might say "$x^2 + 4x + 4 = 0$ precisely when $x = -2$." Equivalence is getting at the "sameness" (so far as our domain goes) of P and Q. We may define properties P and Q differently, but the same members of the domain have these properties (they define the same sets). Symbolically we write $P \Leftrightarrow Q$. So now

An employee is male $\Leftrightarrow$ he makes between 25,000 and 45,000.

Oddly, our (false) Claim 3.5 is an equivalence, since the implications are vacuously true in both directions: $x^2 - 2x + 2 = 0 \Leftrightarrow x > x + 5$.

3.4 Restricting domains

Implication, quantification, conjunction ("and," represented by the symbol $\wedge$), and set intersection are techniques that can be used to restrict domains:

- "Every D that is also a P is also a Q" becomes $\forall x \in D, P(x) \Rightarrow Q(x)$, which we use more commonly than the equivalent $\forall x \in D \cap P, Q(x)$
  (What's the difference between this and $\forall x \in D, P(x) \wedge Q(x)$?)
- "Some D that is also a P is also a Q" becomes $\exists x \in D, P(x) \wedge Q(x)$, which we use more commonly than the equivalent $\exists x \in D \cap P, Q(x)$
  (What's the difference between this and $\exists x \in D, P(x) \Rightarrow Q(x)$?)

3.5 Conjunction (And)

We use $\wedge$ ("and") to combine two sentences into a new sentence that claims that both of the original sentences are true. In our employee database:

Claim 3.12: The employee makes less than 75,000 and more than 25,000.

Claim3.12 is true for Al (who makes 60,000), but false for Betty (who makes 500). If we identify the sentences with predicates that test whether objects are members of sets, then the new $\wedge$ predicate tests whether somebody is in both the set of employees who makes less than 75,000 and the set of employees who make more than 25,000 — in other words, in the intersection. Is it a coincidence that $\wedge$ resembles $\cap$ (only more pointy)?

Notice that, symbolically, $P \wedge Q$ is true exactly when both P and Q are true, and false if only one of them is true and the other is false, or if both are false.

We need to be careful with everyday language where the conjunction "and" is used not only to join sentences, but also to "smear" a subject over a compound predicate. In the following sentence the subject "There" is smeared over "pen" and "telephone:"

Claim 3.13: There is a pen and a telephone.

If we let O be the set of objects, $p(x)$ mean x is a pen, and $t(x)$ mean x is a telephone, then the obvious meaning of Claim 3.13 is:[5] "There is a pen and there is a telephone." But a pedant who has been observing the trend where phones become increasingly smaller and difficult to use might think Claim 3.13 means:[6] "There is a pen-phone."

Here's another example whose ambiguity is all the more striking since it appears in a context (mathematics) where one would expect ambiguity to be sharply restricted.

> The solutions are:
>
> $x < 10$ and $x > 20$
>
> $x > 10$ and $x < 20$

The author means the union of two sets in the first case, and the intersection in the second. We use $\wedge$ in the second case, and disjunction $\vee$ ("or") in the first case.

3.6 Disjunction (Or)

The disjunction "or" (written symbolically as $\vee$) joins two sentence into one that claims that at least one of the sentences is true. For example,

> The employee is female or makes less than 45,000.

This sentence is true for Flo (she makes 20,000 and is female) and true for Carlos (who makes less than 45,000), but false for Al (he's neither female, nor does he make less than 45,000). If we viewed this "or'ed" sentence as a predicate testing whether somebody belonged to at least one of "the set of employees who are female" or "the set of employees who earn less than 45,000," then it corresponds to the union. As a mnemonic, the symbols $\vee$ and $\cup$ resemble each other. Historically, the symbol $\vee$ comes from the Latin word "vel" meaning or.

We use $\vee$ to include the case where more than one of the properties is true; that is, we use an INCLUSIVE-OR. In everyday English we sometimes say "and/or" to specify the same thing that this course uses "or" for, since the meaning of "or" can vary in English. The sentence "Either we play the game my way, or I'm taking my ball and going home now," doesn't include both possibilities and is an exclusive-or: "one or the other, but not both." An exclusive-or is sometimes added to logical systems (say, inside a computer), but we can use negation and equivalence to express the same thing[7] and avoid the complication of having two different types of "or."

3.7 NEGATION

We've mentioned negation a few times already, and it is a simple concept, but it's worth examining it in detail. The negation of a sentence simply inverts its truth value. The negation of a sentence P is written as $\neg P$, and has the value true if P was false, and has the value false if P was true.

Negation gives us a powerful way to check our determination of whether a statement is true. For example, we can check that

CLAIM 3.14: All employees making over 80,000 are female.

is true by verifying that its negation is false. The negation of Claim 3.14 is

CLAIM 3.15: Not all employees making over 80,000 are female.

We cannot find any employees making over 80,000 that are not female (in fact, we cannot find any employees making over 80,000 at all!), so this sentence must be false, meaning the original must be true.
You should feel comfortable reasoning about why the following are equivalent:

- $\neg(\exists x \in D, P(x) \wedge Q(x)) \Leftrightarrow \forall x \in D, (P(x) \Rightarrow \neg Q(x))$.
  In words, "No P is a Q" is equivalent to "Every P is a non-Q."
- $\neg(\forall x \in D, P(x) \Rightarrow Q(x)) \Leftrightarrow \exists x \in D, (P(x) \wedge \neg Q(x))$.
  In words, "Not every P is a Q" is equivalent to "There is some P that is a non-Q."

Sometimes things become clearer when negation applies directly to the simplest predicates we are discussing. Consider

CLAIM 3.16: $\forall x \in D, \exists y \in D, P(x, y)$

What does it mean for Claim 3.16 to be false, *i.e.*, $\neg(\forall x \in D, \exists y \in D, P(x, y))$? It means there is some x for which the remainder of the sentence is false:

CLAIM 3.17: $\neg(\forall x \in D, \exists y \in D, P(x, y)) \Leftrightarrow \exists x \in D, \neg(\exists y \in D, P(x, y))$

So now what does the negated sub-sentence mean? It means there are no y's for which the remainder of the sentence is true:

CLAIM 3.18: $\exists x \in D, \neg(\exists y \in D, P(x, y)) \Leftrightarrow \exists x \in D, \forall y \in D, \neg P(x, y)$

There is some x that for every y makes $P(x, y)$ false. As negation ($\neg$) moves from left to right, it flips universal quantification to existential quantification, and vice versa. Try it on the symmetrical counterpart $\exists x \in D, \forall y \in D, P(x, y)$, and consider

$$\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \neg(\forall y \in D, P(x,y))$$

If it's not true that there exists an x such that the remainder of the sentence is true, then for all x the remainder of the sentence is false. Considering the remaining subsentence, if it's not true that for all y the remainder of the subsentence is true, then there is some y for which it is false:

$$\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \exists y \in D, \neg P(x,y)$$

For every x there is some y that makes $P(x,y)$ false.
Try combining this with implication, using the rule we discussed earlier, plus DeMorgan's law:

$$\neg(\exists x \in D, \forall y \in D, (P(x,y) \Rightarrow Q(x,y))) \Leftrightarrow \neg(\exists x \in D, \forall y \in D, (\neg P(x,y) \vee Q(x,y)))$$

3.8 Symbolic grammar

With connectives such as implication ($\Rightarrow$), conjunction ($\wedge$), and disjunction ($\vee$) added to quantifiers, you can form very complex predicates. If you require these complex predicates to be unambiguous, it helps to impose strict conditions on what expressions are allowed. A syntactically correct sentence is sometimes called a well-formed formula (abbreviated wff). Note that syntactic correctness has nothing to do with whether a sentence is true or false, or whether a sentence is open or closed. The syntax (or grammar rules) for our symbolic language can be summarized as follows:

- Any predicate is a wff.
- If P is a wff, so is $\neg P$.
- If P and Q are wffs, so is $(P \wedge Q)$.
- If P and Q are wffs, so is $(P \vee Q)$.
- If P and Q are wffs, so is $(P \Rightarrow Q)$.
- If P and Q are wffs, so is $(P \Leftrightarrow Q)$.
- If P is a wff (possibly open in variable x) and D is a set, then $\forall x \in D, P$ is a wff.
- If P is a wff (possibly open in variable x) and D is a set, then $\exists x \in D, P$ is a wff.
- Nothing else is a wff.

These rules are recursive, and tell us how we're allowed to build arbitrarily complex sentences in our symbolic language. The first rule is called the base case and specifies the most basic sentence allowed. The rules following the base case are recursive or inductive rules: they tell us how to create a new legal sentence from smaller legal sentences. The last rule is a closure rule, and says we've covered everything.

You should be able to convert a more loosely-structured predicate into a wff, or a wff into a more loosely-structured predicate, whenever it's convenient.

3.9 Truth tables

Predicates evaluate to either true or false once they are completely specified (all unknown values are filled in). If you build complex predicates from simpler ones, using connectives, it's important to know how to evaluate the complex predicate based on the evaluation of fully-specified variants of the simpler predicates it is built out of. A powerful technique for determining the possible truth value of a complex predicate is the use of TRUTH TABLES. In a truth table, we write all possible truth values for the predicates (how many rows do you need?[8]), and compute the truth value of the statement under each of these truth assignments. Each of the logical connectives yield the following truth tables.

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

We often break complex statements into simpler substatements, compute the truth value of the substatements, and combine the truth values back into the more complex statements. For example, we can verify the equivalence

$$(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$$

using the following truth table:

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Since the rightmost column is always true, our statement is a law of logic, and we can use it when manipulating our symbolic statements.

3.10 TAUTOLOGY, SATISFIABILITY, UNSATISFIABILITY

Notice that in the previous section, we didn't specify domains or even meanings for P or Q, nor worry about what values might replace unspecified symbols within P or Q. With truth tables we explored all possible "worlds" (configurations of truth assignments to P and Q). One way of thinking of this is, if $\mathcal{D}$ is the set of domains, and $\mathcal{P}(D)$ is the set of all predicates in domain D, then

$$\forall D \in \mathcal{D}, \forall P \in \mathcal{P}(D), \forall Q \in \mathcal{P}(D), \forall x \in D, (P(x) \Rightarrow Q(x)) \Leftrightarrow (\neg P(x) \vee Q(x))$$

This is a tautology: you can't dream up a domain, or a meaning for predicates P and Q that provides a counter-example, since the truth tables are identical. This is different from, say, $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$, which may be true for some choice of domain, predicates P and Q, or value of domain element x:

$$\exists D \in \mathcal{D}, \exists P \in \mathcal{P}(D), \exists Q \in \mathcal{P}(D), \exists x \in D, (P(x) \Rightarrow Q(x)) \Leftrightarrow (Q(x) \Rightarrow P(x))$$

... so we say this statement is satisfiable. But, there are also choices of domains and/or predicates in which it is false:

$$\exists D \in \mathcal{D}, \exists P \in \mathcal{P}(D), \exists Q \in \mathcal{P}(D), \exists x \in D, \neg((P(x) \Rightarrow Q(x)) \Leftrightarrow (Q(x) \Rightarrow P(x)))$$

... so it is not a tautology. What about something for which no domains, predicates, or values can be chosen to make it true? Such a statement would be unsatisfiable (or a contradiction):

$$\forall D \in \mathcal{D}, \forall P \in \mathcal{P}(D), \forall x \in D, (P(x) \wedge \neg P(x)).$$

3.11 LOGICAL "ARITHMETIC"

If we identify $\wedge$ and $\vee$ with set intersection and union (for the sets where the predicates they are connecting are true), it's clear that they are ASSOCIATIVE and COMMUTATIVE, so

$$P \wedge Q \Leftrightarrow Q \wedge P \quad \text{and} \quad P \vee Q \Leftrightarrow Q \vee P$$
$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R \quad \text{and} \quad P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$$

Maybe a bit more surprising is that we have DISTRIBUTIVE LAWS for each operation over the other:

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

We can also simplify expressions using IDENTITY and IDEMPOTENCY laws:

IDENTITY: $P \wedge (Q \vee \neg Q) \Leftrightarrow P \Leftrightarrow P \vee (Q \wedge \neg Q)$

IDEMPOTENCY: $P \wedge P \Leftrightarrow P \Leftrightarrow P \vee P$

3.12 DeMorgan's Laws

These laws can be verified either by a truth table, or by representing the sentences as Venn diagrams and taking the complement.
Sentence $s_1 \wedge s_2$ is false exactly when at least one of s_1 or s_2 is false. Symbolically:

$$\neg(s_1 \wedge s_2) \Leftrightarrow (\neg s_1 \vee \neg s_2)$$

Sentence $s_1 \vee s_2$ is false exactly when both s_1 and s_2 are false. Symbolically:

$$\neg(s_1 \vee s_2) \Leftrightarrow (\neg s_1 \wedge \neg s_2)$$

By using the associativity of $\wedge$ and $\vee$, you can extend this to conjunctions and disjunctions of more than two sentences.

3.13 Implication, bi-implication, with $\neg, \vee$, and $\wedge$

If we shade a Venn diagram so that the largest possible portion of it is shaded without contradicting the implication $P \Rightarrow Q$, we gain some insight into how to express implication in terms of negation and union. The region that we can choose object x from so that $P(x) \Rightarrow Q(x)$ is $\overline{P} \cup Q$ and this easily translates to $\neg P \vee Q$. This gives us an equivalence:

$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

Now use DeMorgan's law to negate the implication:

$\neg(P \Rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow (\neg\neg P \wedge \neg Q) \Leftrightarrow (P \wedge \neg Q)$

You can use a Venn diagram or some of the laws introduced earlier to show that bi-implication can be written with $\wedge$, $\vee$, and $\neg$:

$(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

DeMorgan's law tells us how to negate this:

$\neg(P \Leftrightarrow Q) \Leftrightarrow \neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \Leftrightarrow \cdots \Leftrightarrow ((\neg P \wedge Q) \vee (P \wedge \neg Q))$

3.14 Transitivity of universally-quantified implication

Consider $\forall x \in D, ((P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x)))$ (I have put the parentheses to make it explicit that the implications are considered before the $\wedge$). What does this sentence imply if considered in terms of P, Q, and R, the subsets of D where the corresponding predicates are true?[9] We can also work this out using the logical arithmetic rules we introduced above: write $((P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))) \Rightarrow (P(x) \Rightarrow R(x))$ using only $\vee, \wedge$, and $\neg$, and show that it is a tautology (always true). Alternatively, use DeMorgan's law, the distributive laws, and anything else that comes to mind to show that the negation of this sentence is a contradiction. Thus, implication is transitive.

A similar transformation is that $\forall x \in D, (P(x) \Rightarrow (Q(x) \Rightarrow R(x))) \Leftrightarrow \forall x \in D, ((P(x) \wedge Q(x)) \Rightarrow R(x))$. Notice this is stronger than the previous result (an equivalence rather than an implication). We'll prove this statement a little later with the help of truth tables.

3.15 Summary of manipulation rules

The following is a summary of the basic laws and rules we use for manipulating formal statements. Try proving each of them using Venn diagrams or truth tables.

identity laws	$P \wedge (Q \vee \neg Q) \iff P$
	$P \vee (Q \wedge \neg Q) \iff P$
idempotency laws	$P \wedge P \iff P$
	$P \vee P \iff P$
commutative laws	$P \wedge Q \iff Q \wedge P$
	$P \vee Q \iff Q \vee P$
	$(P \Leftrightarrow Q) \iff (Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R \iff P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q \iff \neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q \iff \neg P \vee Q$
equivalence	$(P \Leftrightarrow Q) \iff (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
double negation	$\neg(\neg P) \iff P$
DeMorgan's laws	$\neg(P \wedge Q) \iff \neg P \vee \neg Q$
	$\neg(P \vee Q) \iff \neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q) \iff P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q) \iff \neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x)) \iff \exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x)) \iff \forall x \in D, \neg P(x)$
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x) \iff (\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
	$\exists x \in D, P(x) \vee Q(x) \iff (\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

3.16 Multiple quantifiers

Many sentences we want to reason about have a mixture of predicates. For example

CLAIM 3.19: Some female employee makes more than 25,000.

We can make a few definitions, so let E be the set of employees, $\mathbb{Z}$ be the integers, $\mathbf{sm}(e, k)$ be e makes a salary of more than k, and $\mathbf{f}(e)$ be e is female. Now I could rewrite:

CLAIM 3.19 (SYMBOLICALLY): $\exists e \in E, \mathbf{f}(e) \wedge \mathbf{sm}(e, 25000)$.

It seems a bit inflexible to combine e making a salary, and an inequality comparing that salary to 25000, particularly since we already have a vocabulary of predicates for comparing numbers. We can refine the above expression so that we let $\mathrm{s}(e, k)$ be e makes salary k. Now I can rewrite again:

CLAIM 3.19 (REWRITTEN): $\exists e \in E, \exists k \in \mathbb{Z},\ \mathbf{f}(e) \wedge \mathrm{s}(e, k) \wedge k > 25000$.

Notice that the following are all equivalent to Claim 3.19:

$$\exists k \in \mathbb{Z},\ \exists e \in E, \mathbf{f}(e) \wedge \mathrm{s}(e, k) \wedge k > 25000$$
$$\exists e \in E,\ \mathbf{f}(e) \wedge (\exists k \in \mathbb{Z},\ \mathrm{s}(e, k) \wedge k > 25000)$$

This is because $\wedge$ is commutative and associative, and the two existential quantifiers commute.

3.17 MIXED QUANTIFIERS

If you mix the order of existential and universal quantifiers, you may change the meaning of a sentence. Consider the table below that shows who respects who:

	A	B	C	D	E	F
A	⋄					
B		⋄	⋄	⋄	⋄	⋄
C		⋄	⋄	⋄	⋄	⋄
D		⋄	⋄	⋄	⋄	
E		⋄	⋄	⋄		
F		⋄	⋄			

If we want to discuss this table symbolically, we can denote the domain of people by P, and the predicate "x respects y" by $r(x, y)$. Consider the following open sentence:

CLAIM 3.20: $\exists x \in P,\ r(x, y)$ (that is "y is respected by somebody")

If we prepended the universal quantifier $\forall y \in P$ to Claim 3.20, would it be true? As usual, check each element of the domain, column-wise, to see that it is.[10] Symbolically,

CLAIM 3.21: $\forall y \in P, \exists x \in P,\ r(x, y)$

or "Everybody has somebody who respects him/her." You can have different x's depending on the y, so although every column has a diamond in some row, it need not be the same row for each column. What would the predicate be that claims that some row works for each column, that a row is full of diamonds?[11]
Now we have to check whether there is someone who respects everyone:

CLAIM 3.22: $\exists x \in P, \forall y \in P,\ r(x, y)$

You will find no such row. The only difference between Claim 3.21 and Claim 3.22 is the order of the quantifiers. The convention we follow is to read quantifiers from left to right. The existential quantifier involves making a choice, and the choice may vary according to the quantifiers we have already parsed. As we move right, we have the opportunity to tailor our choice with an existential quantifier (but we aren't obliged to).
Consider this numerical example:

CLAIM 3.23: $\forall n \in \mathbb{N}, \exists m_1 \in \mathbb{N}, \exists m_2 \in \mathbb{N}, n = m_1 m_2$.

This says that every natural number has two divisors. What does it mean if you switch the order of the existentially quantified variables with the universally quantified variable? Is it still true? What (if anything) would you need to add to say that every natural number has two distinct divisors?[12]

CHAPTER 3 NOTES

[1]We need to verify the following claims:

- If Al is male, then Al makes less than 55,000.
- If Betty is male, then Betty makes less than 55,000.
- If Carlos is male, then Carlos makes less than 55,000.
- If Doug is male, then Doug makes less than 55,000.

- If Ellen is male, then Ellen makes less than 55,000.
- If Flo is male, then Flo makes less than 55,000.

[2]True, regardless of the cloud situation. In logic $P \Rightarrow Q$ is false exactly when P is true and Q is false. All other configurations of truth values for P and Q are true (assuming that we can evaluate whether P and Q are true or false).

[3]All these claims are true, although possibly misleading. Any claim about elements of the empty set is true, since there are no counterexamples.

[4]Every employee who makes between 25,000 and 45,000 is male.

[5]$\exists x \in O,\, p(x) \wedge \exists y \in O,\, t(y)$, or even $\exists x \in O,\, \exists y \in O,\, p(x) \wedge t(y)$.

[6]$\exists x \in O,\, p(x) \wedge t(x)$

[7]"P exclusive-or Q" is the same as "P not-equivalent-to Q."

[8]If you have n predicates, you need 2^n rows (every combination of T and F).

[9]It implies that P is a subset of R, since $P \subseteq Q$ and $Q \subseteq R$. It is not equivalent, since you can certainly have $P \subseteq R$ without $P \subseteq Q$ or $R \subseteq Q$.

[10]True, there's a diamond in every column.

[11]If we were thinking of the row corresponding to x, then $\forall y \in P,\, r(x, y)$.

[12]$\forall n \in \mathbb{N}, \exists m_1 \in \mathbb{N}, \exists m_2 \in \mathbb{N}, n = m_1 m_2 \wedge m_1 \neq m_2$. Not true for $n = 1$.