Exact ODE's

An exact ODE is of form $M(t,y)dt + N(t,y)dy = 0$ such that there exists $\psi(t,y)$ with

$$M = \frac{\partial \psi}{\partial t} \qquad N = \frac{\partial \psi}{\partial y}$$

In this case, the general solution is $\psi(t,y) = C$.

Theorem: ... is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

Remark: Alternative notation: $\begin{cases} M_y = \frac{\partial M}{\partial y} \\ N_t = \frac{\partial N}{\partial t} \end{cases}$

Example: $\frac{dy}{dt} = \frac{3t(2-ty)}{t^3+2y}$

$$(t^3+2y)dy - 3t(2-ty)dt = 0$$

$$\left.\begin{array}{l} \frac{d}{dt}(t^3+2y) = 3t^2 \\ \frac{d}{dy}(-3t(2-ty)) = 3t^2 \end{array}\right\} \rightarrow \text{exact}$$

Hence, there exists $\psi(x,y)$ with $\frac{d\psi}{dy} = t^3+2y$. $\frac{d\psi}{dt} = -3t(2-ty) = -6t+3t^2y$

Integrate first equation w.r.t. y

$\psi(x,y) = t^3y + y^2 + g(t) \leftarrow$ "constant" of integration

$$\frac{\partial \psi}{\partial t} = 3t^2y - g'(t) \overset{!}{=} -6t + 3t^2y$$

Thus, $g'(t) = -6t$

$g(t) = -3t^2 + C$

Hence, $\psi(t,y) = \underline{t^3y + y^2 - 3t^2 + C}$

is the solution of our ODE.

Example: $\frac{dy}{dt} = -\frac{t+y^2}{2ty\ln(t)}$ $(t>0)$

$$(t+y^2)dt + (2ty\ln(t))dy = 0$$

$$\frac{\partial}{\partial y}(t+y^2) = 2y \qquad \frac{\partial}{\partial t}(2ty\ln(t)) = 2y\ln(t) + 2y$$

<u>it's not exact</u>

But Let's divide by t: $(1+\frac{y^2}{t})dt + 2y\ln(t)dy = 0$

$\frac{\partial}{\partial y}(1+\frac{y^2}{t}) = \frac{2y}{t}$

$\frac{\partial}{\partial t}(2y\ln(t)) = \frac{2y}{t}$ so, now it's exact!

Hence, $\exists\ \psi(x,y)$ with

$\frac{\partial \psi}{\partial t} = (1+\frac{y^2}{t})$

$\frac{\partial \psi}{\partial y} = 2y\ln(t)$

Find: $\psi(t,y) = t + y^2\ln(t)$