Feb. 26th

Practice problems

Let $T: \mathbb{C}^3 \to \mathbb{C}^3$ be the linear transformation defined by the matrix

$$A = \begin{pmatrix} 1 & 2 & i \\ 2 & 3 & 1 \\ 3 & 5 & 1+i \end{pmatrix}$$ Find bases for the image and kernel of T.

(i.e. A is the matrix representative of T with respect to the standard basis of $\mathbb{C}^3$
i.e. $T(x) = Ax \quad \forall x \in \mathbb{C}^3$)

Kernel of T

Recall that $\ker(T) = \{x \in \mathbb{C}^3, T(x) = 0\}$

We want to solve the equation $T(x) = 0$ for $x \in \mathbb{C}^3$

i.e. $Ax = 0$ for $x \in \mathbb{C}^3$

$$\left(\begin{array}{ccc|c} 1 & 2 & i & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 1+i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & i & 0 \\ 0 & -1 & 1-2i & 0 \\ 0 & -1 & 1-2i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2-3i & 0 \\ 0 & -1 & 1-2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2-3i & 0 \\ 0 & 1 & 2i-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solutions are $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1 = (3i-2)x_3$, $x_2 = (1-2i)x_3$, $x_3 \in \mathbb{C}^3$

so $\ker(T) = \text{span}\{(3i-2, 1-2i, 1)\}$

Image of T

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Set: $(0, \infty)$ Field: $\mathbb{R}$

Addition $(+')$ $a +' b = ab \quad a, b \in (0, \infty)$

Scalar Mult. $(\cdot')$ $\lambda \cdot' a = e^\lambda a \quad \lambda \in \mathbb{R}, a \in (0, \infty)$

Is it a vector space?

Exercise: This is a v.s. over $\mathbb{R}$.

Consider $V = \{f \in P_3(\mathbb{C}) : f(1) = 0\} \subseteq P_3(\mathbb{C})$

Prove that V is a subspace of $P_3(\mathbb{C})$.

Proof: (i) The zero vector in $P_3(\mathbb{C})$ is the zero-polynomial, which evaluates at 1 to give 0. Hence $0 \in V$.

(ii) Sps that $f, g \in V$

$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$

So $f + g \in V$

(iii). Suppose that $\lambda \in \mathbb{C}$ and $f \in V$

$$(\lambda f)(1) = \lambda f(1) = \lambda \cdot 0 = 0$$

So, $\lambda f \in V$

Hence, V is a subspace of $P_3(\mathbb{C})$.

There is another solution. Consider $T: P_3(\mathbb{C}) \to \mathbb{C}$ defined by $T(f) = f(1)$

This is a linear transformation

Note that $\ker(T) = \{f \in P_3(\mathbb{C}) : T(f) = 0\} = \{f \in P_3(\mathbb{C}) : f(1) = 0\} = V$

Since the kernel of a linear transformation is always a subspance of the domain. we are done.

What is $\dim V$?

$V = \ker(T)$

$\dim P_3(\mathbb{C}) = \dim(\ker(T) + \dim(\operatorname{im}(T))$