Venue: ______________________________

Student Number: |___|___|___|___|___|___|___|___|

Research School of Finance, Actuarial Studies and Statistics Examinatation

Semester 1 - Mid-semester Exam 2018 **Solutions**

STAT3013/STAT4027/STAT8027 Statistical Inference

Writing Time: 90 minutes
Reading Time: 15 minutes minutes

Exam Conditions:
Central Examination
Students must return the examination paper at the end of the examination
This examination paper is not available to the ANU Library archives

Materials Permitted In The Exam Venue:
(No electronic aids are permitted e.g. laptops, phones)
Two sheets of A4 paper with notes on both sides
Calculator (non programmable)
Unannotated paper-based dictionary (no approval required)

Materials to Be Supplied To Students:
Scribble Paper

Marks

Question 1	Question 2	Question 3	Question 4	Total

Question 1 [**30 marks**]: Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x;\theta) = \frac{2\,x}{\theta}\exp\left(-\frac{x^2}{\theta}\right)$, where $x > 0$, $\theta > 0$. Note:

$$E[X] = \frac{\sqrt{\pi}}{2\sqrt{1/\theta}}.$$

Hint: To find the $E[X^2]$ consider a change of variable in the integration, let $Y = X^2 \;\Rightarrow\; dy = 2x\;dx$. You will recognise a known distribution.

a. [**10 marks**] Derive the maximum likelihood estimator (MLE) for θ. What is the variance of the MLE?

$$\begin{aligned}
L(\theta;\boldsymbol{x}) &= \prod_{i=1}^{n} \frac{2\;x_i}{\theta}\exp\left(-\frac{x_i^2}{\theta}\right) \\
\ell(\theta) &= \sum_{i=1}^{n}\left[log(2) + log(x_i) - log(\theta) - x_i^2/\theta\right] \\
&= nlog(2) + \sum_{i=1}^{n} log(x_i) - nlog(\theta) - \sum_{i=1}^{n} x_i^2/\theta \\
\ell'(\theta) &= -\frac{1}{\theta} + \sum_{i=1}^{n} x_i^2/\theta^2 = 0 \\
\Rightarrow \qquad & \hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{n}
\end{aligned}$$

- Check that this is a maximum:

$$\begin{aligned}
\ell''(\theta) &= \frac{n}{\theta^2} - 2\frac{\sum_{i=1}^{n} x_i^2}{\theta^3} \\
&= \frac{n\theta - 2\sum_{i=1}^{n} x_i^2}{\theta^3}\Big|_{\theta=\hat{\theta}} = \frac{n\frac{\sum_{i=1}^{n} x_i^2}{n} - 2\sum_{i=1}^{n} x_i^2}{\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)^3} \\
&= -\frac{\sum_{i=1}^{n} x_i^2}{\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)^3} < 0
\end{aligned}$$

b. [**10 marks**] Does the MLE of θ attain the Cramér-Rao lower bound? Justify your answer.

- Let's calculate $\text{Var}(\hat{\theta})$:

$$\begin{aligned}
V(\hat{\theta}) &= V\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right) = \frac{1}{n}V(x_i^2) \\
\frac{1}{n}V(x^2) &= \frac{1}{n}\left[E(x^4) - \left(E(x^2)\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
E(x^2) &= \int_0^\infty x^2 \frac{2\ x}{\theta} \exp\left(-\frac{x^2}{\theta}\right) dx \\
\Rightarrow & \quad \int_0^\infty y \frac{2\ x}{\theta} \exp\left(-\frac{y}{\theta}\right) \left[\frac{1}{2x}\right] dy \\
& \quad \int_0^\infty \frac{y}{\theta} \exp\left(-\frac{y}{\theta}\right) dy
\end{aligned}
$$

- I let $y = x^2$ then $dx = (1/2x)dy$.
- We see that we have the kernel for a gamma distribution.

$$
\begin{aligned}
& \frac{1}{\theta} \int_0^\infty y \exp\left(-\frac{y}{\theta}\right) dy \\
& \frac{\Gamma(2)\theta^2}{\theta} \int_0^\infty \frac{1}{\Gamma(2)\theta^2}\ y^{2-1}\ \exp\left(-\frac{y}{\theta}\right) dy \\
& \Gamma(2)\theta \underbrace{\int_0^\infty \frac{1}{\Gamma(2)\theta^2}\ y^{2-1}\ \exp\left(-\frac{y}{\theta}\right) dy}_{=1} = \theta
\end{aligned}
$$

- Notice that this shows that the MLE is unbiased!

$$
\begin{aligned}
E(x^4) &= \int_0^\infty x^4 \frac{2\ x}{\theta} \exp\left(-\frac{x^2}{\theta}\right) dx \\
\Rightarrow & \quad \int_0^\infty y^2 \frac{2\ x}{\theta} \exp\left(-\frac{y}{\theta}\right) \left[\frac{1}{2x}\right] dy \\
& \quad \frac{1}{\theta} \int_0^\infty y^2 \exp\left(-\frac{y}{\theta}\right) dy \\
& \quad \frac{\Gamma(3)\theta^3}{\theta} \int_0^\infty \frac{1}{\Gamma(3)\theta^3} y^{3-1} \exp\left(-\frac{y}{\theta}\right) dy \\
& \quad \frac{\Gamma(3)\theta^3}{\theta} \underbrace{\int_0^\infty \frac{1}{\Gamma(3)\theta^3} y^{3-1} \exp\left(-\frac{y}{\theta}\right) dy}_{=1} = 2\theta^2
\end{aligned}
$$

$$
V(\hat{\theta}) = \frac{1}{n}\left[2\theta^2 - \theta^2\right] = \theta^2/n
$$

- Now let's compare that to the CRLB:

$$\begin{aligned}
\ell''(\theta) &= \frac{n}{\theta^2} - 2\frac{\sum_{i=1}^n x_i^2}{\theta^3} \\
I(\theta) &= -E\left[\ell''(\theta)\right] \\
&= -E\left[\frac{n}{\theta^2} - 2\frac{\sum_{i=1}^n x_i^2}{\theta^3}\right] \\
&= -\frac{n}{\theta^2} + \frac{2}{\theta^3} nE\left[x_i^2\right] \\
&= -\frac{n}{\theta^2} + \frac{2}{\theta^3} n\theta = \frac{n}{\theta^2}
\end{aligned}$$

- So the CRLB $= \frac{\theta^2}{n}$. The MLE attains the CRLB, therefore is MVUE!

c. **[10 marks]** Derive the Cramér-Rao lower bound for an unbiased estimator of $\log(\theta)$.

- Based on work, now let's just apply it for a function of θ:

$$\text{CRLB}(\tau(\theta)) = \frac{\left[\tau'(\theta)\right]^2}{I(\theta)} = \frac{\left[\frac{1}{\theta}\right]^2}{\frac{n}{\theta^2}} = \frac{1}{n}$$

Question 1 **[20 marks]**: A sample of n observations is taken on a random variable X which has a logarithmic series distribution,

$$P(X = x) = \frac{-\theta^x}{x \log(1-\theta)}, \quad x = 1, 2, 3, \ldots,$$

where θ is an unknown parameter in the range $(0, 1)$. Find the MLE of θ.

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{-\theta^{x_i}}{x_i \; \log(1-\theta)} \\
&= \prod_{i=1}^{n} (-1) \frac{\theta^{x_i}}{x_i \; \log(1-\theta)} \\
&= \prod_{i=1}^{n} \frac{\theta^{x_i}}{-\log(1-\theta) x_i} \\
\ell(\theta) &= \sum_{i=1}^{n} \left[x_i \log(\theta) - \log(-\log(1-\theta)) - \log(x_i) \right] \\
&= \log(\theta) \sum_{i=1}^{n} x_i - n \log(-\log(1-\theta)) - \sum_{i=1}^{n} \log(x_i)
\end{aligned}$$

$$\begin{aligned}
\ell'(\theta) = U(\theta) &= \frac{1}{\theta} \sum_{i=1}^{n} x_i + n \frac{1}{(1-\theta)\log(1-\theta)} \\
\ell''(\theta) = H(\theta) &= \frac{-1}{\theta^2} \sum_{i=1}^{n} x_i + n \frac{\log(1-\theta) + 1}{(1-\theta)^2 \log(1-\theta)^2}
\end{aligned}$$

- As we are unable to obtain a closed from analytical solution, we will consider computational solution through the Newton-Raphson procedure:

Algorithm 1 Newton-Raphson

```
let check = 10
let θ_1 = 0.5
let c = 2
while check < 0.00001 do
    θ_c = θ_(c−1) − H^(−1)(θ_(c−1)) U(θ_(c−1))
    calculate check = |θ_c − θ_(c−1)|
    let c = c + 1
return the last value of θ
```

Question 2 [**25 marks**]: Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x; \theta_1, \theta_2) = \sqrt{\frac{\theta_1}{2\pi x^3}} \exp\left\{\frac{-\theta_1(x-\theta_2)^2}{2\theta_2^2 x}\right\}$. These random variables have an inverse Gaussian distribution.

a. [**10 marks**] Is this distribution a member of the two-parameter exponential family?

$$
\begin{aligned}
f(x;\theta_1,\theta_2) &= \sqrt{\frac{\theta_1}{2\pi x^3}} \exp\left\{\frac{-\theta_1(x-\theta_2)^2}{2\theta_2^2 x}\right\} \\
&= \exp\left\{\frac{-\theta_1(x-\theta_2)^2}{2\theta_2^2 x} + log\left(\sqrt{\frac{\theta_1}{2\pi x^3}}\right)\right\} \\
&= \exp\left\{\frac{-\theta_1(x-\theta_2)^2}{2\theta_2^2 x} + (1/2)\left[log(\theta_1) - log(2\pi x^3)\right]\right\} \\
&= \exp\left\{\frac{-\theta_1(x^2-2x\theta_2+\theta_2^2)}{2\theta_2^2 x} + (1/2)\left[log(\theta_1) - log(2\pi x^3)\right]\right\} \\
&= \exp\left\{\frac{-\theta_1 x^2+2x\theta_1\theta_2-\theta_1\theta_2^2}{2\theta_2^2 x} + (1/2)\left[log(\theta_1) - log(2\pi x^3)\right]\right\} \\
&= \exp\left\{\frac{-\theta_1}{2\theta_2^2}x + \frac{\theta_1}{\theta_2} - \frac{\theta_1}{2}\frac{1}{x} + (1/2)\left[log(\theta_1) - log(2\pi x^3)\right]\right\} \\
&= \exp\left\{\frac{-\theta_1}{2\theta_2^2}x - \frac{\theta_1}{2}\frac{1}{x} + \frac{\theta_1}{\theta_2} + (1/2)\left[log(\theta_1) - log(2\pi x^3)\right]\right\}
\end{aligned}
$$

- Identifying the components of an exponential family, we have

$$A_1(\boldsymbol{\theta}) = \frac{-\theta_1}{2\theta_2^2};\ A_2(\boldsymbol{\theta}) = -\frac{\theta_1}{2};\ B_1(x) = x;\ B_2(x) = \frac{1}{x}$$

$$C(x) = -log(2\pi x^3)/2;\ D(\boldsymbol{\theta}) = \frac{\theta_1}{\theta_2} + (1/2)\left[log(\theta_1)\right]$$

b. [**10 marks**] Find a minimal sufficient statistic(s) for (θ_1, θ_2).

As the inverse Gaussian distribution is a member of 2-parameter exponential family, by exponential family theory, $\sum_{i=1}^n x_i, \sum_{i=1}^n \frac{1}{x_i}$ are minimal sufficient (and complete) statistics for θ_1, θ_2.

c. [**5 marks**] Also a find minimal sufficient statistic(s)

i) for θ_2 when θ_1 is known;

When θ_2 is known, again $\sum_{i=1}^n x_i, \sum_{i=1}^n \frac{1}{x_i}$ are minimal sufficient statistics for θ_1. Note, that the number of statistics is greater than the number of parameters, so this is not a member of the 2 parameter exponential family.

ii) for θ_1 when θ_2 is known.

When θ_1 is known, then $\sum_{i=1}^n x_i$ is a complete and minimal sufficient statistic for θ_2.

Question 4 **[25 marks]**: Suppose that $\hat{\theta}$ is an estimator for the parameter θ, and that the $E[\hat{\theta}] - \theta = b(\theta)$, thus the estimator is biased. From base principles show that

$$\mathrm{Var}(\hat{\theta}) \geq \frac{\left(1 + \frac{\partial b(\theta)}{\partial \theta}\right)^2}{I_\theta},$$

where I_θ is Fisher's information.

- Use the Cauchy-Schwarz inequality to prove the result.

$$[Cov(Y,Z)]^2 \leq V(Y)V(Z) \;\Rightarrow\; V(Y) \geq \frac{[Cov(Y,Z)]^2}{V(Z)}$$

- Let $Y = T(\boldsymbol{X})$ and $Z = \frac{\partial}{\partial \theta}\ log\ f(\boldsymbol{x}|\theta)$. Recall,

$$Cov(Y,Z) = E[YZ] - E[Y]\ E[Z]$$

- First note:

$$\begin{aligned}
\frac{\partial}{\partial \theta} E[T(\boldsymbol{X})] &= \frac{\partial}{\partial \theta} \int_{\mathcal{X}} T(\boldsymbol{x}) f(\boldsymbol{x};\theta) dx \\
&= \int_{\mathcal{X}} T(\boldsymbol{x}) \left[\frac{\partial}{\partial \theta} f(\boldsymbol{x};\theta)\right] dx \\
&= \int_{\mathcal{X}} T(\boldsymbol{x}) \left[\frac{\partial}{\partial \theta} f(\boldsymbol{x};\theta) \frac{f(\boldsymbol{x};\theta)}{f(\boldsymbol{x};\theta)}\right] dx \\
&= E\left[T(\boldsymbol{X}) \frac{\frac{\partial}{\partial \theta} f(\boldsymbol{x};\theta)}{f(\boldsymbol{x};\theta)}\right] \\
&= E\left[T(\boldsymbol{X})\ \frac{\partial}{\partial \theta}\ log\ f(\boldsymbol{x};\theta)\right]
\end{aligned}$$

- Second note:

$$\begin{aligned}
E\left[\frac{\partial}{\partial \theta}\ log\ f(\boldsymbol{x};\theta)\right] &= \int_{\mathcal{X}} \left[\frac{\partial}{\partial \theta}\ log\ f(\boldsymbol{x};\theta)\right] f(\boldsymbol{x};\theta) dx \\
&= \int_{\mathcal{X}} \left[\frac{\frac{\partial}{\partial \theta} f(\boldsymbol{x};\theta)}{f(\boldsymbol{x};\theta)}\right] f(\boldsymbol{x};\theta) dx \\
&= \int_{\mathcal{X}} \frac{\partial}{\partial \theta} f(\boldsymbol{x};\theta) dx \\
&= \frac{\partial}{\partial \theta} \int_{\mathcal{X}} f(\boldsymbol{x};\theta) dx \\
&= \frac{\partial}{\partial \theta}(1) = 0
\end{aligned}$$

- Putting these two together

$$\begin{aligned}
Cov\left[T(\boldsymbol{X}),\ \frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right] &= E\left[T(\boldsymbol{X})\ \frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right] \\
&\quad -E\left[\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right]E\left[T(\boldsymbol{X})\right] \\
&= E\left[T(\boldsymbol{X})\ \frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right] - [0]E\left[T(\boldsymbol{X})\right] \\
&= E\left[T(\boldsymbol{X})\ \frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right] \\
&= \frac{\partial}{\partial\theta}E[T(\boldsymbol{X})]
\end{aligned}$$

- Next

$$\begin{aligned}
V\left[\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right] &= E\left[\left(\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right)^2\right] - \left(E\left[\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right]\right)^2 \\
&= E\left[\left(\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right)^2\right] - (0)^2 \\
&= E\left[\left(\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right)^2\right]
\end{aligned}$$

$$\begin{aligned}
V(Y) &\geq \frac{[Cov(Y,Z)]^2}{V(Z)} \\
V\left[T(\boldsymbol{X})\right] &\geq \frac{\left[Cov\left[T(\boldsymbol{X}),\ \frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right]\right]^2}{V\left[\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right]} \\
&\geq \frac{\left[\frac{\partial}{\partial\theta}E[T(\boldsymbol{X})]\right]^2}{E\left[\left(\frac{\partial}{\partial\theta}\ log\ f(\boldsymbol{x};\theta)\right)^2\right]} \\
&\geq \frac{\left[\frac{\partial}{\partial\theta}E[T(\boldsymbol{X})]\right]^2}{I_\theta}
\end{aligned}$$

- Now if

$$E[\hat{\theta}] = \theta + b(\theta) \Rightarrow \frac{\partial}{\partial\theta}E[\hat{\theta}] = 1 + b'(\theta)$$

- We have:

$$V(\hat{\theta}) \geq \frac{[1+b'(\theta)]^2}{I_\theta}$$

End Of Examination