THE AUSTRALIAN NATIONAL UNIVERSITY
SCHOOL OF FINANCE, ACTUARIAL STUDIES AND APPLIED STATISTICS

First Semester Final Examination 2012

FINANCIAL MATHEMATICS

(STAT 2032 / STAT 6046)

Study Period: 15 minutes
Time Allowed: 3 hours

Permitted Material:
Non-Programmable Calculators
Dictionaries (must not contain material added by the student)
Actuarial Tables for examinations (not required)

Students have also been provided with a formula sheet in addition to this exam paper.

Total Marks: 100

Instructions to Candidates:

- ***Attempt ALL 6 questions***
- ***Start your solution to each question on a new page.***
- ***Unless otherwise stated, show all working.***

Question One (16 marks)

(a) Given a nominal rate of discount of 11.5% per annum convertible every 2 months, calculate to the nearest 0.1%:

i. The equivalent force of interest. (2 marks)

ii. The equivalent rate of nominal interest per annum, convertible monthly. (2 marks)

iii. The equivalent rate of nominal discount per annum, convertible half yearly. (2 marks)

(b) Consider the following expression:
$(Da)_{\overline{n|}} + (Da)_{\overline{n-1|}} + (Da)_{\overline{n-2|}} + (Da)_{\overline{n-3|}} + ... + (Da)_{\overline{2|}} + (Da)_{\overline{1|}}$.

With n = 17 and for a particular positive interest rate *i%*, this expression has a present value of 879.68.

(i) Given the above, determine the value at $t = 0$ of the following series of payments (at an interest rate of *i%*):

At time 1 a payment of \$68 + \$64 + \$60 +…+ \$8 + \$4 is made;
At time 2 a payment of \$64 + \$60 + \$56 +…+ \$8 + \$4 is made;
At time 3 a payment of \$60 + \$56 + \$52 +…+ \$8 + \$4 is made;
…and so on until….
At time 15 a payment of \$12 + \$8 + \$4 is made;
At time 16 a payment of \$8 + \$4 is made;
At time 17 a payment of \$4 is made. (3 marks)

(ii) With $i = 0\%$ per annum, which of the 2 expressions has the greater value:

$(Ia)_{\overline{n|}} + (Ia)_{\overline{n-1|}} + (Ia)_{\overline{n-2|}} + ... + (Ia)_{\overline{2|}} + (Ia)_{\overline{1|}}$, or
$(Da)_{\overline{n|}} + (Da)_{\overline{n-1|}} + (Da)_{\overline{n-2|}} + ... + (Da)_{\overline{2|}} + (Da)_{\overline{1|}}$? (1 mark)

(c) Consider the following information relating to an investment fund:

	Calendar year			
	2007	**2008**	**2009**	**2010**
Value of fund at 30 April		$65,600	$72,300	$82,000
Net cashflow received on 1 May		$54,000	$10,000	$350,000
Value of fund at 31 December	$48,000	$82,500	$75,000	$505,000

i. Calculate the annualized time weighted rate of return between 31-December-2007 and 31-December-2010. Give your answer to the nearest 0.1%. (3 marks)

ii. Without doing any calculations, for the same time period as (i) above would you expect the annualized money weighted rate of return to be higher or lower than the annualized time weighted rate of return? Give reasons for your answer. (2 marks)

iii. Describe one disadvantage of using the time weighted rate of return, compared to using the money weighted rate of return. (1 mark)

Question Two (16 marks)

(a) A 15 year bond with nominal value of \$50,000 pays half yearly coupons of 8% per annum and is redeemable at 90% of its nominal value.

An investor (Mrs 'X') is wanting to purchase this bond. She pays income tax of 15%, capital gains tax of 45%, and wishes to achieve a net yield to redemption of 17% per annum (effective).

Assuming there is 2 months until the first coupon payment, calculate the price that Mrs X will pay for this bond. Give your answer to the nearest dollar.

(5 marks)

(b) Six years and two months later, another investor (Mrs 'Y') wants to purchase this bond off Mrs X. Suppose the following assumptions hold for Mrs 'Y'.

- Mrs 'Y' requires a net yield to redemption of 6% per annum
- Mrs 'Y' pays income tax of 14.5% and capital gains tax of 55%

Given these assumptions for Mrs 'Y', determine the actual capital gains tax payable by the original investor, Mrs 'X'. (5 marks)

(c) Another bond has the following structure:

- It has an annual coupon rate of 11.50% with coupons paid half-yearly in arrears for 20 years;
- It is redeemable at 45% of the face value, with the date of redemption at the option of the bond issuer;
- The date of redemption can be anytime between, and including, $t = 15$ years and $t = 20$ years.

Suppose an investor, Mr B, paid \$925.00 for \$1000 nominal of this bond at $t = 0$. He is not subject to income tax or capital gains tax.

Given this, calculate:

(i) The maximum possible yield to redemption he could make by purchasing this bond. Show your workings, and express your answer as an effective annual rate to the nearest 0.5%. (3 marks)

(ii) The minimum yield to redemption that he is guaranteed to make by purchasing this bond. Show your workings, and express your answer as an effective annual rate to the nearest 0.5%. (3 marks)

Question Three (16 marks)

You are considering the purchase of a coffee shop, "MarsBucks". You have determined that the following costs and revenues would apply to this investment:

Set-up Costs that would be incurred at $t = 0$.

1. Health and Safety Licence: $5,000
2. Purchase of building and furniture: $100,000

Ongoing Costs:

1. Staff costs: $300 per day, incurred at the end of each day.
2. Electricity, power and gas: $800 payable at the end of each month

Revenue:

1. $400 at the end of each day.

(a) Assuming that ongoing costs and revenue occur in perpetuity, and that in one year there are 365 days exactly, calculate the NPV of this investment if the risk discount rate is 25% per annum (effective). Start by writing down an appropriate equation of value and give your answer to the nearest dollar. (4 marks)

(b) With the same assumptions as in (a) above, calculate the IRR of this investment. Give your answer to 2 significant figures. (4 marks)

(c) After making the calculations above, you have suddenly realised that you have not accounted for the cost of the coffee. After some research you discover that you will need to buy coffee in bulk, in advance every six months, in perpetuity. The current price for six months' supply of coffee is $2,000. Determine the new NPV of this project, again (as in part (a) above) assuming a risk discount rate of 25% per annum. (2 marks)

(d) However, you now find out that the price of coffee is also subject to changes due to a special type of inflation, which occurs according to the "caffeine inflation index" as follows:

t (years)	Caffeine Index
0	87
0.5	89
1	93
1.5	101
2	99

Assume now that you are going to finance the entire investment from a loan that you take out at t = 0. All subsequent income will be paid against this loan, and all subsequent costs will be paid for from this loan. Assuming that the annual interest rate on the loan is given by $i^{(12)} = 12\%,$ determine the balance of the loan 2 years after taking it out, immediately after all costs and revenue due at that time.

(6 marks)

Question Four (16 marks)

(a) Consider a loan that is paid off exactly after 45 monthly payments. However, the monthly payments are not regular but vary from month to month, as does the applicable interest rate. The monthly payments, interest charges and outstanding balances over time for this loan are given by the following loan schedule:

Time (month)	Monthly Loan Payment	Interest due	Principal Repaid	Outstanding balance
0				$[i]
1	$500	$450	$[ii]	$52,950
2	$7,000	$[iii]	$6,471	$[iv]
3	$[v]	$465	$[vi]	$45,419
…	…	…	…	…
44	$775	$41	$734	$2,513
45	$[vii]	$25	$[viii]	$0

Calculate each of the 8 missing entries [i] – [viii] above. (5 marks)

(b) A loan ("loan 1") of size 2X is taken out at t = 0 (t is measured in years). Annual payments of $1,744.04 (in arrears) are made against this loan, with the interest rate charged on the loan equal to 5% per annum (effective).

Another loan ("loan 2") of size X is taken out at t = 16. The annual payments against this loan are equal to $1,252.45, they are made in arrears, and this loan also incurs interest at 5% per annum effective.

Suppose that the balances of both loans are equal at time t = 22, immediately after the payments due at that time.

As such, at what time is loan 2 paid off? (6 marks)

(c) Suppose a loan is repaid by level payments of \$K per period, for n periods. Also suppose that the periodic interest rate is equal to 0%.

Determine whether the convexity of this loan is equal to $\frac{n^2}{3}+n+\frac{2}{3}$, $\frac{(n+1)^2}{3}$, $\frac{2n+n^2}{3}$, or $\frac{n^2+n}{2}$.

You may find the following results useful:

$$\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$$

$$\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1)$$

Show your workings at all steps. (5 marks)

Question Five (18 marks)

(a) Consider the following three securities, each which has a current price (value) as well as a value in the future dependent on 3 different scenarios for interest rates:

		Future value under interest rate scenario:		
Security	Current Price	One	Two	Three
A	119.6	98.8	119.6	146.6
B	2.3	1.9	2.3	2.8
C	23	19	23	28

(i) Show how an arbitrage opportunity is available from an appropriate selection of buying and selling (taking long and short positions in) combinations of the above securities. (4 marks)

(ii) Explain what you would expect to happen to the current prices of each of the three securities above under the principle of "no arbitrage". (2 marks)

(b) A share in the company "CoolKiwi Ltd" is currently valued at \$45. It produces no income, but you decide to enter into a forward contract on this asset with another party.

The forward contract specifies that you must buy 1,000 shares, for a certain price $\$K_0$, after 6 years.

After 3 years you want to get out of this contract. Another investor is willing to pay you \$500 for the forward contract. Assuming a risk free force of interest of 4%, what was the actual annual growth rate in the value of one share of "CoolKiwi Ltd" over these first three years?

Give your answer to the nearest 0.1%. (4 marks)

(c) Two annuities have identical present values at an annual effective interest rate of 3%.

The first annuity has 22 years of payments and it consists of payments of \$1,500 at the end of every year for the first 12 years, \$2,500 at the end of every year for the next 7 years, and in the final 3 years a continuous payment at a rate of \$3,500 per year is made, for those 3 years.

The second annuity consists of payments of \$$Kt$ at the end of the t^{th} year for 25 years. In addition, each payment for the second annuity increases at 4.00% per annum. That is, the first payment at the end of the 1st year is \$$K$; the payment at the end of the 2nd year is \$$2K(1.04)$; the payment at the end of the 3rd year is \$$3K(1.04)^2$ …., and so on.

Find the value of K. Give your answer to the nearest cent. (8 marks)

Question Six (18 marks)

(a) A financial institution accepts a deposit of $\$X$ from a customer and guarantees to pay the customer the invested amount along with compound interest of 6.5% per annum in ten years.

The institution invests \$15,000 in a two-year zero coupon bond and $\$Y$ in a stock that pays a level annual income in arrears for perpetuity. Both investments yield 6.5% per annum effective.

(i) Determine values for X and Y that allow Redington's first two conditions for immunisation to be met. (5 marks)

(ii) Do you think that the relative values of the assets and liabilities above are immunised against small movements in interest rates? Give a reason for your answer. (2 marks)

(b) Assume that i_t are independent effective rates of interest distributed uniformly on the interval $[-0.02, 0.12]$ for $t = 1, 2, ..., 50$.

Suppose that an investment of \$200 is made at t = 0, and an investment of \$350 is made at t = 27. Find the mean of the total accumulated value at time $t = 50$. (3 marks)

(c) Annual effective returns on an investment in shares in any year are independent and identically distributed. The following distribution specifies all possible returns in any one year:

$$i_t = \begin{cases} -10\% & \text{with probability } (3p)\% \\ +3\% & \text{with probability } (p)\% \\ +4\% & \text{with probability } 30\% \\ +9\% & \text{with probability } (3p)\% \end{cases}$$

You want to invest \$50,000 in the shares given by the above distribution. You want to have an 80% probability that the size of the investment in 20 years time will be greater than some value \$X.

Assuming that the accumulation under this investment has a log-normal distribution, calculate the value of X. (8 marks)

END OF EXAMINATION