Hints and solutions

MAT237Y1Y MIDTERM EXAM

1. Chain rule

   a) (5 marks) Consider the surface described by $F(x, y, z) = 0$, and use chain rule to prove that at any point (a_1, a_2, a_3) a non zero $\nabla F(a_1, a_2, a_3)$ is perpendicular to any tangent line to the surface that passes through any point (a_1, a_2, a_3).

      Solution: This is theorem 2.37

   b) (3 marks) Use the result from (a) to derive an equation of tangent plane to the surface at a point $\boldsymbol{a} = (a_1, a_2, a_3)$. (Present your reasoning.)

      Solution: This is corollary 2.38

   c) (4 marks) Use part (b) to determine the tangent plane to the surface of a sphere of radius 1 centered at the origin, at the point $\boldsymbol{a} = (1/2, 1/2, \sqrt{2}/2)$

      Solution: Surface of the sphere has equation $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ $\nabla F(x, y, z) = (2x, 2y, 2z)$ and therefore $\nabla F(1/2, 1/2, \sqrt{2}/2) = (1, 1, \sqrt{2})$. Now equation of plane is

      $$0 = \nabla F(1/2, 1/2, \sqrt{2}/2) \cdot (\boldsymbol{x} - \boldsymbol{a}) =$$

      $$=$$

      $$(1, 1, \sqrt{2}) \cdot (x - 1/2, y - 1/2, z - \sqrt{2}/2) = x - 1/2 + y - 1/2 + \sqrt{2}(z - \frac{\sqrt{2}}{2} = 0$$

2. Lagrange's method

   a) (3 marks) Explain Lagrange's method for finding maximum and minimum values of a function $f(x, y)$ subject to the constraint $G(x, y) = 0$. (No need for justification.)

      Solution: We must solve the system of equations $\nabla F(x, y) = \lambda \nabla G(x, y, z)$ together with $G(x, y) = 0$ for the unknowns x, y, λ. In that case the values obtained for x and y, will be the point at which max or min takes place, and the corresponding value of the function F at those points will determine the max or min of the function subject to the constraint.

   b) (4 marks) Present an argument (involving chain rule) that justifies your answer in part (a).

      Solution: This is the first paragraph of page 103.

c) (6 marks) Lagrange's method can be extended to deal with two constraints: find maximum and minimum of the function $f(x, y, z) = yz + xy$ subject to two constraints $xy = 1$ and $y^2 + z^2 = 1$.

Solution: This is the idea presented at the bottom of page 104, and for an example you can look at the example 5 page 939 of optional reading for 2.9.

3. Critical points

a) (4 marks) What does it mean for a point (a, b) to be a critical point for a C^2 function $f(x, y)$? Present the second derivative test for classifying the nature of a critical point.

Solution: Definition of critical point is given in page 95, and the second derivative test for two variables is presented in theorem 2.82.

b) (5 marks) Consider the function $f(x, y, z) = 5 - (x^2 - 2y^2 + 3z^2)$. At the critical point $(0, 0, 0)$ form the Hessian matrix $H(0, 0, 0)$ and write the second order Taylor polynomial to express $f(h_1, h_2, h_3)$. Then using the eigenvlaues of $H(0, 0, 0)$ and appropriate choice of the vectors (h_1, h_2, h_3) discuss the nature of this critical point.

Solution: To solve this question you need to read theorem 2.81.

$$H(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

and since the matrix is already diagonal the eigenvlaues of the matrix are on the main diagonal and they are $2, -2$, and 6. According to the definition of the saddle point (the third last line of page 97) since $H(0, 0, 0)$ has positive and negative eigenvalues then the nature of the point $(0, 0, 0)$ is a saddle point.

c) (6 marks) Use your answer in part (a) to determine and classify the nature of all the critical points of the function $f(x, y) = e^y(y^2 - x^2)$

Solution: This question is very similar to the example 1 of the textbook on page 99, also you can find more examples of this idea on page 923 of optional reading 2.8.

4. Taylor's theorem

a) (7 marks) Present the Taylor's polynomial of degree two, in the matrix form, for the function $f(x, y) = e^{x-1}(y - 2) + y$ near the point $\boldsymbol{a} = (1, 2)$, and then use it to approximate the value of $f(1.2, 1.9)$.

Solution:

$$P_{(1,2),2}(x,y) = f(1,2)+[\nabla f(1,2)]^T \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}+\frac{1}{2}[x-1, y-2] \begin{bmatrix} f_{xx}(1,2) & f_{xy}(1,2) \\ f_{yx}(1,2) & f_{yy}(1,2) \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

Now in the above formula place $x = 1.2$ and $y = 1.9$ and calculate the value of $f(1.2, 1.9)$.

b) (4 marks) present the Lagrange form of the remainder $R_{\boldsymbol{a},2}(\boldsymbol{h})$ where $\boldsymbol{h} = (-0.2, 0.3)$. Present your answer in the Multi-index notation, AND in the expanded form.

Solution: see equation 2.72 (page 91): in the Multi-index notation

$$R_{(1,2),2}(-0.2, 0.3) = \sum_{|\alpha|} = 3\partial^\alpha f(1-0.2c, 2+0.3c)\frac{(-0.2, 03)^\alpha}{3!} \text{for some} c \in (0,1)$$

to see the expanded form see page 82 and 83, keeping in mind that sine $|\alpha| = 3$ then α can be decomposed to two components as

$$\alpha = (3,0), (2,1), (1,2), (0,3)$$

and $\boldsymbol{h}^\alpha$ can be in expanded, for example for $\alpha = (2,1)$ as $h_1^2 h_2^1 = (-0.2)^2(0.3)^1 = (0.04)(0.3) = 0.012$. Also $(2,1)! = 2!1! = 2$.

5. Differentiability for functions of several variables

a) (3 marks) What does it mean for a function $f(x,y)$ to be differentiable at a point $\boldsymbol{a} = (a,b)$?

Solution: See page 55

b) (3 marks) Apply this definition to show the function $f(x,y) = xy$ is differentiable at the point $\boldsymbol{a} = (-3, 2)$.

Solution: we know that the $\boldsymbol{c} = \nabla f(-3,2) = (2,-3)$ so that

$$\lim_{\boldsymbol{h} \longrightarrow 0} \frac{f(-3+h_1, 2+h_2) - f(-3,2) - (2h_1 - 3h_2)}{\sqrt{h_1^2 + h_2^2}} =$$

$$\lim_{\boldsymbol{h} \longrightarrow 0} \frac{(-3+h_1)(2+h_2) - (-6) - 2h_1 + 3h_2}{\sqrt{h_1^2 + h_2^2}} == \lim_{\boldsymbol{h} \longrightarrow 0} \frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}} = 0$$

the last equality is because if we write the point (h_1, h_2) in the polar form, that is $h_1 = r\cos\theta$ and $h_2 = r\sin\theta$ then the limit becomes

$$\lim_{r \longrightarrow 0} \frac{r^2 \cos\theta \sin\theta}{r} = \lim_{r \longrightarrow 0} r\cos\theta\sin\theta = 0$$

c) (3 marks) Evaluate $D_u f(\boldsymbol{a})$ where $\boldsymbol{u} = (1, 2)$

Solution: $D_u f(\boldsymbol{a}) = \nabla f(\boldsymbol{a} \cdot \boldsymbol{u}$ ...

d) (3 marks) Determine $df(\boldsymbol{a}; \boldsymbol{h})$, where $\boldsymbol{h} = (0.2, -0.1)$.

Solution: see formula (2.22)

6. Compactness

a) (3 marks) Define what a compact set is, and state Bolzano Weierstrass for compact sets.

Solution: see page 30

b) (4 marks) Demonstrate that the range of the following function is compact.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if}(x, y) \neq (0, 0) \\ 0 & \text{if}(x, y) = (0, 0) \end{cases} \tag{1}$$

Solution: In the polar coordinates the formula $\frac{xy}{x^2+y^2} = \cos\theta \sin\theta = \frac{\sin 2\theta}{2}$ for all $\theta \in [0, 2\pi]$. This formulas is clearly presents us with a bounded range of value, that is in $[-0.5, 0.5]$, which is closed as well.

c) (6 marks) prove that if $\boldsymbol{f} : S \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \boldsymbol{R}^n$ is compact, then the set $\{\boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{x} \in S\}$ is also compact.

Solution: this is theorem 1.22.

7. Connectedness

a) (4 marks) Precisely define the concepts of arcwise connected and that of a disconnection for a set S.

Solution: see pages 34 and 35 for the definitions

b) (3 marks) Demonstrate that any open ball of $\mathbb{R}^n$ is arcwise connected.

Solution: page 71, example 1 proves that the open ball is convex. This implies that the open ball is also path connected (because of existence of a straight line connecting any two points of the open ball.

c) (6 marks) Assume the set $S \subset \mathbb{R}^n$ is open and let $\boldsymbol{a} \in S$. Define the set S_1 to be the set of all $\boldsymbol{x} \in S$ which can be connected to $\boldsymbol{a}$ via a continuous arc, and S_2 be the set of all $\boldsymbol{x} \in S$ which cannot be connected to $\boldsymbol{a}$. Prove that (S_1, S_2) is a disconnection of S.

Solution: This is rewording of theorem 1.30

8. Uniform continuity

a) (3 marks) Present the ϵ-δ definition for a function $\boldsymbol{f} : S \longrightarrow \mathbb{R}^m$ to be uniformly continuous on S? Then negate this statement to give a definition for $\boldsymbol{f}$ NOT to be uniformly continuous on S.

   Solution: the definition of uniform continuity is on page 39 and the negation is at the middle of page 40, in the body of the proof of theorem 1.33.

b) (3 marks) Present an example of a set $S \subset \mathbb{R}^n$ and a continuous function $f : S \longrightarrow \mathbb{R}^m$ which is not uniformly continuous on S. (if you wish you could choose $n = m = 1$.)

   Solution: one such example is given in example 2 of 1.8. Another one was presented in the readings and the problem sets: $f(x) = 1/x$ and the set S is just the interval $(0, \infty)$

c) (5 marks) Prove that if $\boldsymbol{f} : S \longrightarrow \mathbb{R}^m$ is continuous and S is compact, then $\boldsymbol{f}$ must be uniformly continuous on S.

   Solution: This is theorem 1.33