Past Final exam

1. (6 marks) Use an appropriate change of variables to evaluate the double integral $\iint_S \frac{x-2y}{3x-y} dA$, where the region S is bounded by the lines $x - 2y = 0,\ x - 2y = 4,\ 3x - y = 1$ and $3x - y = 8$.

2. (6 marks) State the Divergence theorem and use it to evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dA$, where $\boldsymbol{F}(x, y, z) = (e^y \cos z)\, \boldsymbol{i} + (\sqrt{x^3 + 1} \sin z)\, \boldsymbol{j} + (x^2 + y^2 + 3)\, \boldsymbol{k}$ and S is the graph of $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$ for $z \geq 0$ and oriented upward.
   Note: the surface is not a closed surface.

3. Fubini's theorem and iterated integrals
   a) (5 marks) Use an iterated integral to compute the double integral $\iint_S e^{x^2} dA$ where S is the region bounded by the x-axis, the line $x = 1$ and the line $y = x$.
   b) (7 marks) Consider the function f defined by
   $$f(x, y) = \begin{cases} y^{-2} \text{ if } 0 < x < y < 1 \\ -x^{-2} \text{ if } 0 < y < x < 1 \\ 0 \quad \text{otherwise} \end{cases}$$
   First explain why both iterated integrals on $R = [0, 1] \times [0, 1]$ exist. Then calculate them and show they are not equal. Explain why this does not contradict Fubini's theorem.

4. Implicit function theorem
   a) (8 marks) Give the three representations of a curve in $\mathbb{R}^3$ as presented in the textbook (in the same order), and use the appropriate version of the implicit function theorem to show the implicit (second) representation of a curve is transformable to the graph (first) representation. Make sure to state and use the appropriate regularity condition which guarantees this operation.
   b) (5 marks) Draw the surface S determined by the graph of $x^2 + y + 2z = 4$ in the first octant and oriented outward. Clearly define ∂S as a collection of curves in $\mathbb{R}^3$, with their proper orientations.

5. Surface integrals
   a) (6 marks) For the surface S in question 4(b), use the surface integral to determine the mass of the surface S if the mass density on S is $\rho(x, y, z) = x$. (Note: mass is the total sum of the densities at each and all points of the surface.)
   b) (6 marks) Calculate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dA$ where $\boldsymbol{F}(x, y, z) = (\frac{x}{2},\ y,\ z)$.
   c) (5 marks) Prove that if S is a closed surface, as in the boundary of a solid R in three dimensional space, and $\boldsymbol{F}$ is a C^2 vector field, then $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n} dA = 0$.
   d) (7 marks) Use Stokes' theorem to calculate the surface integral $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n}\, dA$
   where $\boldsymbol{F}(x, y, z) = \boldsymbol{i} + (x - yz)\, \boldsymbol{j} + (xy - \sqrt{z})\, \boldsymbol{k}$ and S is the surface in question 4(b).
   (Hint: use part (c) and replace the surface S from question 4(b) by a union of regions in the coordinate planes.)

6. (12 marks) Give the general formula for the Taylor polynomial of degree two for a function $f(x, y, z)$ (at a general point) and then apply your formula to the function $f(x, y, z) = x + xy + yz + z^2$. Determine the critical point(s) of f, and use the Hessian of f at the critical point(s) to classify them. Explain your reasoning.

7. Conservative vector fields

a) (6 marks) Suppose that $R \subset \mathbb{R}^n$ is an open connected set and let $\boldsymbol{a} \in R$. Show that for any point $\boldsymbol{x} \in R$ there is a curve C that connects $\boldsymbol{x}$ to $\boldsymbol{a}$.

b) (4 marks) Suppose that $\boldsymbol{G}$ is a vector field defined and continuous on an open connected set $R \subset \mathbb{R}^n$. What does it mean for $\boldsymbol{G}$ to be conservative?

c) (8 marks) Prove that $\boldsymbol{G}$ as in part (b) must be the gradient of a C^1 function f on R. (Present your proof for the case $n = 2$.)

d) (7 marks) Consider the vector field

$$\boldsymbol{G}(x, y, z) = (2xy)\,\boldsymbol{i} + (x^2 + \log z)\,\boldsymbol{j} + \frac{y+2}{z}\,\boldsymbol{k}, \;\; z > 0\,.$$

Determine whether $\boldsymbol{G}$ could be the gradient of a scalar valued function; if so determine the function f, and if not explain why.

8. Chain rule

a) (3 marks) State the chain rule for a vector valued function $\boldsymbol{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$, a scalar valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the composition $(f \circ \boldsymbol{g}) : \mathbb{R} \longrightarrow \mathbb{R}$.

b) (3 marks) Prove that the gradient of a C^1 function f is a conservative vector field.

c) (7 marks) Use chain rule (II) and differentiation under the integral sign to calculate $\dfrac{\partial F}{\partial x}$ at the point $\boldsymbol{a} = (1, \pi)$, where $F(x, y) = \displaystyle\int_1^{3x^2} x\cos(x^2 y + \pi t)\,dt$.

9. Green's theorem

a) (2 marks) State Green's theorem for a regular region S in $\mathbb{R}^2$.

b) (6 marks) Use Green's theorem to show $\displaystyle\int_C \frac{\partial f}{\partial n} ds = \iint_S \nabla^2 f dA$ for a function f that is C^2 on $\overline{S}$, where C is the boundary of the region S.

c) (6 marks) Consider $f(x, y) = \ln(x^2 + y^2)$. Let C be the circle of radius 1, and let S be the disc inside C. centered at the origin. Calculate the line integral $\displaystyle\int_C \frac{\partial f}{\partial n} ds$. Calculate $\nabla^2 f$. Why does this not contradict part (b)?

Calculate the line integral $\displaystyle\int_C \nabla f \cdot d\boldsymbol{x}$.

Recall: $\frac{\partial f}{\partial n} = \nabla f \cdot \boldsymbol{n}$.