2017-02-27-lec04

$\forall a \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$
Note that $f : \mathbb{R} \to \mathbb{R}$.

Well-defined mathematical statement: something that is clearly true or false, e.g., $1 + 1 = 3, 1 + 1 < 3, 1 = 5$.

Something like $x + 1 > 3$ is a well-defined statement for any particular choice of $x \in \mathbb{R}$.

Generally, suppose $P(x)$ is a statement which has a well-defined truth value for all choices of $x \in S$.

Then we define

- $\exists x, P(x)$ to mean $P(x)$ is true for at least one $x \in S$.
- $\forall x, P(x)$ to mean $P(x)$ is true for all $x \in S$.
- $\exists x, (x + 1 > 3)$ – TRUE
- $\forall x, (x + 1 > 3)$ – FALSE

More generally, if you have a statement with many free variables, you can get well-defined mathematical statements by **quantifying** all the free variables.

$$P(x, y) := (y = x^3), x, y \in \mathbb{R}$$

- $\forall y, \exists x(y = x^3)$ means every real number has a cube root. (TRUE)
- $\exists x, \forall y(y = x^3)$. It's asserting the existence of a single real number x which is the cube root of all real numbers at once. (FALSE)

In English,
$\forall y, \exists x, y = x^3$ is true for some real number x. (TRUE)
$\exists x$, such that $y = x^3$ is true for all y. (FALSE)
$\exists y, \forall x(y = x^3)$ – (FALSE) for the same reason
$\forall, \exists y(Y = x^3)$ – (TRUE) Every real number has a cube.
$\exists x, \exists y(y = x^3)$ – (TRUE)
$\forall x, \forall y(y = x^3)$ – (FALSE)

Problem: Let a, b be real numbers. $(a, b \in \mathbb{R})$ Prove that the equation $ax^2 + bx = a$ has a real solution.

1. Translate this statement into formal logic.
2. Prove it.

Solution

1. $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \exists x \in \mathbb{R}(ax^2 + bx = a)$
2. Want to solve $ax^2 + bx - a = 0$.

Case 1:
$$x = \frac{-b \pm \sqrt{b^2 - 4a(-a)}}{2a} = \frac{-b \pm \sqrt{b^2 + 4a^2}}{2a}$$
Case 2: If $a = 0$, then this equation is $bx = 0$, so $x = 0$ is a solution.

What did a strawberry say to another strawberry? HOW DID WE GET INTO HIS JAM?

What did a wall say to a ceiling? I'LL MEET YOU AT THE CORNER.

What does $\forall\forall\exists\exists$ mean? For all upside-down A, there exists a backward E.

Logical Connectives: Suppose P, Q well-defined mathematical statements,

- $\neg P$ means not P.
- $P \wedge Q$ means P and Q.
- $P \vee Q$ means P or Q.
- $P \implies Q$ means P implies Q.
- $P \iff Q$ means P if and only if Q.

P	Q	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Claim: $\neg(P \wedge Q))$ is logically equivalent to $(\neg P) \vee (\neg Q)$.

For observation $P \implies Q$ is not logically equivalent to $Q \implies P$ which is the converse, but equivalent to $\neg Q \implies \neg P$ which is the contrapositive.