Assignment 2 - MAT 327 - Summer 2013

Due June 3rd, 2013 at 4:10 PM

Comprehension

[**C.1**] Let $X = \{ 0, 1, 2, 3, 4 \}$. What is the smallest size of a basis that generates the discrete topology on X? What can you say if $X = \{ 0, 1, 2, \ldots, n-1 \}$?

Solution for C.1. The smallest size of a basis that generates the discrete topology on $\{ 0, 1, 2, 3, 4 \}$ is 5. The basis $\mathcal{D} := \{ \{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \} \}$ shows you need *at most* 5 sets.

Claim: If $\mathcal{B}$ is a basis on X that generates the discrete topology, then $\mathcal{D} \subseteq \mathcal{B}$.

From a proposition in class, we see that for each $i \in X$, since $\mathcal{B}$ is a basis for the discrete topology, and $\{ i \}$ is an open set containing i, then there is a basic open $B \in \mathcal{B}$ so that $i \in B \subseteq \{ i \}$. The only possibility is that $\{ i \} = B \in \mathcal{B}$. So we have shown that $\mathcal{D} \subseteq \mathcal{B}$. $\square$

[**C.2**] Let $A \subseteq X$, a topological space. Prove that

$$\overline{A} = \bigcap \{ C \subseteq X : C \text{ is closed and } A \subseteq C \}$$

Solution for C.2. We show both containments:

[$\supseteq$] Let $p \notin \overline{A}$. Then $\overline{A}$ is a closed set containing A that does not contain p. So

$$p \notin \bigcap \{ C \subseteq X : C \text{ is closed and } A \subseteq C \}$$

[$\subseteq$] Let

$$p \notin \bigcap \{ C \subseteq X : C \text{ is closed and } A \subseteq C \}$$

Then there is a closed set C containing A, such that $p \notin C$. Thus $X \setminus C$ is an open set containing p, that is disjoint from A. $\square$

[**C.3**] Find the interior and closure of the following sets (no proof needed):

1. $(2, 3]$ in $\mathbb{R}$ with the usual topology;
2. $(2, 3]$ in $\mathbb{R}$ with the Sorgenfrey Line;
3. $(2, 3]$ in $\mathbb{R}$ with the discrete topology;
4. $B_{100}(0, 0, 0) \setminus B_2(0, 0, 0)$ in $\mathbb{R}^3$ with the usual topology;
5. $\mathbb{Q}$ (as a subset of $\mathbb{R}$ with the usual topology).

Solution for C.3.

	Interior	Closure
$(2,3]$ in $\mathbb{R}_{\text{usual}}$	(2,3)	[2,3]
$(2,3]$ in $\mathbb{R}$ as the Sorgenfrey Line	(2,3)	[2,3]
$(2,3]$ in $\mathbb{R}_{\text{discrete}}$	(2,3]	(2,3]
$B_{100}(0,0,0) \setminus B_2(0,0,0)$ in $\mathbb{R}^3_{\text{usual}}$	$\{\, p \in \mathbb{R}^3 : 2 < \lvert p\rvert < 100 \,\}$	$\{\, p \in \mathbb{R}^3 : 2 \leq \lvert p\rvert \leq 100 \,\}$
$\mathbb{Q}$ (as a subset of $\mathbb{R}_{\text{usual}}$)	$\emptyset$	$\mathbb{R}$

$\square$

[**C.4**] Let $A \subseteq X$ a topological space. Prove that A is open iff $A = \operatorname{int}(A)$. Conclude that

$$\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$$

Solution for C.4. First remark that $\operatorname{int}(A) \subseteq A$ is always true.

[$\Rightarrow$] Suppose that A is open. Then $A \subseteq \operatorname{int}(A)$, by definition of the interior. So then together with our previous observation, $A = \operatorname{int}(A)$.

[$\Leftarrow$] Now, suppose that $A = \operatorname{int}(A)$. It is clear that $\operatorname{int}(A)$ is an open set as it is the union of open sets. So A must be an open set.

The second part follows since $\operatorname{int}(A)$ is an open set. $\square$

[**C.5**] Is it true $\operatorname{int}(\overline{A}) = \overline{(\operatorname{int}(A))}$? Is $\operatorname{int}(\overline{A}) \subseteq \overline{(\operatorname{int}(A))}$? What about $\operatorname{int}(\overline{A}) \supseteq \overline{(\operatorname{int}(A))}$?

Solution for C.5. In general, none of these is true. Taking $A = [2,3]$ in $\mathbb{R}_{\text{usual}}$ shows that

$$\operatorname{int}(\overline{A}) = \operatorname{int}([2,3]) = (2,3) \neq [2,3] = \overline{(2,3)} = \overline{(\operatorname{int}(A))}$$

The same $A = [2,3]$ also shows that

$$\operatorname{int}(\overline{A}) \not\supseteq \overline{(\operatorname{int}(A))}$$

Finally, $A = \mathbb{Q}$ shows

$$\operatorname{int}(\overline{A}) = \operatorname{int}(\mathbb{R}) = \mathbb{R} \not\subseteq \emptyset = \overline{\emptyset} = \overline{(\operatorname{int}(A))}$$

$\square$

Application

[**A.1**] Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{S}$ be a subbasis on X. Along the lines of the proposition we saw in lecture, state and prove a proposition that tells us when the topology generated by $\mathcal{S}$ is $\mathcal{T}$. Use this to prove that

$$\mathcal{S} := \{ (-\infty, b) : b \in \mathbb{R} \} \cup \{ (a, +\infty) : a \in \mathbb{R} \}$$

generates the usual topology on $\mathbb{R}$ (don't forget to check that this is a subbasis!). On your own, write down a "natural subbasis" for the Sorgenfrey line.

Proposition. *A subbasis $\mathcal{S}$ generates $\mathcal{T}$ if and only if $\mathcal{S} \subseteq \mathcal{T}$ and for every open set $U \in \mathcal{T}$ and for every $x \in U$ there is a finite subcollection $\{ S_1, S_2, \ldots, S_n \} \subseteq \mathcal{S}$ such that $x \in \bigcap_{i=1}^{n} S_i \subseteq U$.*

Proof. This follows immediately from the definition of a subbasis. □

[The rest of this question is straightforward.]

[**A.2**] Let's go a bit further than C.1. Let $X = \{ 0, 1, 2, 3, 4 \}$. What is the smallest size of a *subbasis* that generates the discrete topology on X? Write a sentence or two explaining why it is a difficult question to generalize this to $X = \{ 0, 1, 2, \cdots, n-1 \}$.

Solution for A.2. We can see that $\{ \{ 0, 1, 2 \}, \{ 0, 3, 4 \}, \{ 1, 3 \}, \{ 2, 4 \} \}$ is a subbasis with 4 elements that generates the discrete topology on $\{ 0, 1, 2, 3, 4 \}$.

Claim: If $\{ A, B, C \}$ is a collection of subsets of X, then it is not a subbasis for the discrete topology.

Note that there are only 4 possibly non-empty intersections we can make: $A \cap B, A \cap C, B \cap C$ and $A \cap B \cap C$. This shows that at least one of $\{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}$ or $\{ 4 \}$ is not represented.

In general, as we increase n, the number of intersections possible increases (exponentially), and we will not be able to extend our (simple) combinatorial argument. □

[**A.3**] For $A \subseteq X$ a topological space, prove that $X = \operatorname{int}(A) \sqcup \partial(A) \sqcup \operatorname{int}(X \setminus A)$, where '$\sqcup$' means the three sets are mutually disjoint. Give an example of a set A where both $\operatorname{int}(A) \neq \emptyset, \operatorname{int}(X \setminus A) \neq \emptyset$ but $\partial(A) = \emptyset$. What properties must such an A have?

Solution for A.3. We prove this in three parts:

Claim 1: $X \subseteq \operatorname{int}(A) \cup \partial(A) \cup \operatorname{int}(X \setminus A)$, with no claim that the sets are mutually disjoint.

Suppose that $x \in X$, but $x \notin \operatorname{int}(A)$ and $x \notin \operatorname{int}(X \setminus A)$. First we see that $x \in X \setminus \operatorname{int}(A)$ is a closed set containing $X \setminus A$. So we see that $x \in \overline{X \setminus A}$. Analogously, since $x \in X \setminus \operatorname{int}(X \setminus A)$ is a closed set containing A we get that $x \in \overline{A}$. Thus $x \in \overline{A} \cap \overline{X \setminus A} = \partial(A)$.

Claim 2: $\text{int}(A) \cap \text{int}(X \setminus A) = \emptyset$.

This follows from $A \cap (X \setminus A) = \emptyset$ and $\text{int}(A) \subseteq A$ and $\text{int}(X \setminus A) \subseteq X \setminus A$.

Claim 3: $\text{int}(A) \cap \partial(A) = \emptyset$.

Let $x \in \text{int}(A)$, an open set. Since $\text{int}(A) \cap (X \setminus A) = \emptyset$ we see that $x \notin \overline{X \setminus A}$. So $x \notin \partial(A)$.

Completely analogously, we see that $\text{int}(X \setminus A) \cap \partial(X \setminus A) = \emptyset$, and since $\partial(A) = \partial(X \setminus A)$ we are finished.

Moreover...

In the Sorgenfrey line, let $A = [4, 7)$. We can see that $\text{int}(A) = [4, 7), \partial(A) = \emptyset$ and $\text{int}(X \setminus A) = \text{int}((-\infty, 4) \cup [7, +\infty)) = (-\infty, 4) \cup [7, +\infty)$.

In general, we can see that a set has the desired property if and only if it is closed and open. This is from $X = \text{int}(A) \sqcup \text{int}(X \setminus A)$. □

New Ideas

[**NI.1**] Let's think about dense sets... Is it true that the intersection of two dense sets is again dense? What about the intersection of a dense set with an open set? What if both sets are dense and open? What about the intersection of finitely many such sets? Let's now shift our focus to $\mathbb{R}$ (with the usual topology). Is the only open set that contains $\mathbb{Q}$ all of $\mathbb{R}$? What can we say about the intersection of infinitely many sets that are dense and open in $\mathbb{R}$? Does it matter if we only intersect countably many such sets? What if we allow uncountable intersections? Write a couple sentences explaining your thoughts on the following assertion: "There is an uncountable collection of mutually different dense and open sets whose intersection is dense."

Sketch of NI.1. This exercise is, in part, the so called "Baire Category Theorem". A set that is both dense and open is called "dense open"

<u>Is it true that the intersection of two dense sets is again dense?</u>.

No, for example, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}_{\text{usual}}$, but they are clearly disjoint.

<u>What about the intersection of a dense set with an open set?</u>

Again no, taking $\mathbb{Q}$ as the dense set, and $(2, 3)$ as the open set, we see that there is no way that the intersection is dense in $\mathbb{R}$. However, we can see that $\overline{\mathbb{Q} \cap (2, 3)} = \overline{(2, 3)}$, and this isn't true for closed sets. For example,

$$\overline{\mathbb{Q} \cap \{\pi\}} = \emptyset \neq \{\pi\} = \overline{\{\pi\}}$$

<u>What if both sets are dense and open?</u>

Yes! Let A, B be sets that are both dense open in some topological space X. It is clear that the intersection is again open, so we only need to show the following claim:

Claim: $A \cap B$ is dense.

Let U be a non-empty open set in X, we will show that $U \cap A \cap B \neq \emptyset$. Since A is dense, $U \cap A \neq \emptyset$. Moreover, this is a non-empty open set, and since B is dense, $U \cap A \cap B \neq \emptyset$, as required.

<u>What about the intersection of finitely many such sets?</u>

Since the intersection of finitely many open sets is open, the previous exercise, together with induction, gives us that the intersection of finitely many dense open sets is again dense open.

<u>Is the only open set that contains $\mathbb{Q}$ all of $\mathbb{R}$?</u>

No! There is a very counter-intuitive open set that contains all of $\mathbb{Q}$, but avoids many real numbers. Start by enumerating the rational numbers as

$$q_1, q_2, q_3, q_4, q_5, \ldots$$

Then the desired set is

$$A := \bigcup_{n \in \mathbb{N}} B_{2^{-n}}(q_n)$$

This is clearly open, and we can see that it has "length" less than or equal to

$$\sum_{n \in \mathbb{N}} 2 \cdot 2^{-n} = 2 \cdot 1 = 2$$

Since $\mathbb{R}$ has "length" greater than 2, it cannot be that A covers $\mathbb{R}$.

The notion of "length" on $\mathbb{R}$ is covered in more depth in a course on measure theory.

<u>What can we say about the intersection of infinitely many sets that are dense and open in $\mathbb{R}$?</u>

These might be empty! For example, each set $A_x := \mathbb{R} \setminus \{ x \}$ is dense open, for $x \in \mathbb{R}$, but

$$\bigcap_{x \in \mathbb{R}} (\mathbb{R} \setminus \{ x \}) = \mathbb{R} \setminus \mathbb{R} = \emptyset$$

<u>Does it matter if we only intersect countably many such sets?</u>

Theorem (Baire Category Theorem)**.** *The intersection of countably many dense open sets in $\mathbb{R}$ is again dense.*

The proof of this can be found in most analysis textbooks. Note that the intersection of countably many dense open sets might not be open. For example, $\mathbb{R} \setminus \{ q \}$, where $q \in \mathbb{Q}$, is a dense open set, and

$$\bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{ q \}) = \mathbb{R} \setminus \mathbb{Q}$$

which is not open.

What if we allow uncountable intersections?

We already saw that they could be empty. On the other hand, the intersection might be a dense set! For example, each set $A_x := \mathbb{R} \setminus \{ x \}$ is dense open, for $x \in \mathbb{R}$, but

$$\bigcap_{x \in \mathbb{R} \setminus \mathbb{Q}} (\mathbb{R} \setminus \{ x \}) = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{Q}$$

$\square$

[**NI.2**] Write a program that:

1. Allows a user to input a collection of subsets of $X = \{0, 1, 2, 3, 4\}$;

2. Generates the smallest topology $\mathcal{T}$ that contains those sets (always include the full space X);

3. Allows the user to input a subset of $A \subseteq X$, and your program returns the interior, closure and boundary of A (in the topology $\mathcal{T}$).

NI.2 solutions. Soon (with permission) I will post some of the solutions I received for this question. $\square$

[**NI.3**] This is a very famous (and fun!) problem called the **Kuratowski 14-set problem**. Let $A \subseteq X$ a topological space.

1. Prove that there are at most 14 different subsets of X that can be obtained from A by applying the operations of closures and complements successively.

2. Find a subset A of $\mathbb{R}$ (with the usual topology) such that the 14 subsets of $\mathbb{R}$ can be obtained from A by applying the operations of closure and complements successively.

Sketch of NI.3. Like I said, this is a very famous problem. Very good write-ups are available online, and the wikipedia article:

http://en.wikipedia.org/wiki/Kuratowski%27s_closure-complement_problem

should give you a good idea of what is happening here. $\square$