Ex. $X, Y \sim$ iid $Exp(1)$

$Z = \frac{X}{X+Y}$. Find $f_Z(z)$.

Sol'n:

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & ow \end{cases}, \quad f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0; & ow \end{cases}$$

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x,y \geq 0 \\ 0, & ow \end{cases}$$

$$U_1 = \frac{X}{X+Y}, \quad U_2 = X+Y$$

$$h_1(x,y) = \frac{x}{x+y}, \quad h_2(x,y) = x+y$$

$$x = u_1 u_2 = h_1^{-1}(u_1, u_2)$$

$$y = u_2 - u_1 u_2 = u_2(1-u_1) = h_2^{-1}(u_1, u_2)$$

$$J = \det \begin{bmatrix} u_2 & u_1 \\ -u_2 & 1-u_1 \end{bmatrix} = u_2$$

$$f_{U_1,U_2}(u_1,u_2) = e^{-u_1u_2 - u_2(1-u_1)} |u_2| \quad , \quad \begin{matrix} u_1u_2 \geq 0 \\ u_2(1-u_1) \geq 0 \end{matrix}$$

$$0 \leq u_1u_2$$

$$0 \leq u_2(1-u_1) = u_2 - u_1u_2$$

$$0 \leq u_1u_2 \leq u_2 \quad , \quad u_2 \neq 0$$

$$0 \leq u_1 \leq 1$$

$$u_2 > 0$$

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} e^{-u_2} u_2 \ , & \begin{matrix} 0 \leq u_1 \leq 1 \\ u_2 > 0 \end{matrix} \\ 0 \ , & ow \end{cases}$$

$$f_{U_1} = \int_0^{\infty} e^{-u_2} u_2 \, du_2 = 1 , \quad 0 \leq u_1 \leq 1$$

$$U_1 \sim Unif(0,1)$$

$$U_2 \sim Gamma(2,1)$$

$X \sim \chi^2_{(n)}$, $Y \sim \chi^2_{(m)}$

$$Z = \frac{X/n}{Y/m} \sim ? \quad F(n, m)$$