Exchangeability - Example 1

(a) ($i \neq j$) $Cov(y_i, y_j) = Cov(E(y_i|\theta)E(y_j|\theta)) + E(Cov(y_i, y_j|\theta))$

$= Cov(\theta, \theta) + E(0)$

$= Var(\theta) > 0$

So Covariance $\neq 0$, so y_i and y_j are not unconditionally independent

→ Note covariance does not depend on 'i' or 'j'

(b). $f(y) = \int f(y|\theta) f(\theta) d\theta$

$= \int \prod_{i=1}^{n} f(y_i|\theta) f(\theta) d\theta$

$f(y|\theta) = \prod_{i=1}^{n} f(y_i|\theta)$ ⇒ in which order does not matter in computing the conditional density.

→ So in computing the unconditional density $f(y_1 .. y_n)$ → order does not matter either.

Exchangeability example 2

→ Possibilities for non-Bayesian analysis

$\hat{y} = 1$ (assuming $\hat{\theta} = 1$). } assumes 'iid' sampling conditions for future

or assume $\hat{\theta} = 0.1$ and generate a new $\hat{y}$ based on this value. } ignore past data $y_1 \cdots y_n$.

→ ignores unconditional dependence.

→ Bayesian analysis.

→ incorporate uncertainty in θ in prediction for $\tilde{y}$.

$$\Pr(\tilde{y} = 1 | y_1 \cdots y_n) = \int_0^1 \Pr(\tilde{y} = 1 | \theta, y)\, p(\theta | y)\, d\theta$$

$$= \int_0^1 \theta\, p(\theta | y)\, d\theta$$

$$= E(\theta | y) \quad = \frac{1001}{1002} \begin{pmatrix} < 1 \\ > 0.10 \end{pmatrix}.$$

$\theta \sim \text{Unif}(0,1) = \text{Beta}(1,1)$

$\theta | y \sim \text{Beta}(1 + 1000, 1)$

$(\sum y_i = 1000;\ n = 1000)$

Binomial Model - Happiness data

$n = 129 \quad \sum y_i = 118$

(Prior) $\theta \sim \text{Beta}(1,1)$

Posterior $\theta | y \sim \text{Beta}(1+118,\ 1+(129-118))$

$= \text{Beta}(119, 12)$.

Relative posterior probabilities

$$\frac{p(\theta_1 | y)}{p(\theta_2 | y)} = \left(\frac{\theta_1}{\theta_2}\right)^{119-1} \left(\frac{1-\theta_1}{1-\theta_2}\right)^{12-1}.$$

$\sum y_i$ is a sufficient statistic for inference on θ.