MATH6222 week4 lecture 11

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Last time:

- a set A is finite if $|A| = |[n]|$ for some $n \in \mathbb{N}$.
- a set A is countably infinite if $|A| = |\mathbb{N}|$.
- a set A is countable if finite or countably infinite.
- a set A is uncountable if not countable.

Thm: $\mathbb{R}$ is uncountable.

Proof: Cantor's Diag. Argument.
$12.999\ldots$
13 $13.000\ldots$ $13.145\ldots$

$$317.389714\ldots$$

$\mathbb{R} \longleftrightarrow \{\text{Sequences of integers } \{0, \ldots, 9\}\}$

Let S be the set of infinite sequences of 0's and 1's, i.e. an element of S is given by $a_1, a_2, a_3, \ldots$ each $a_i \in \{0, 1\}$ such as

$$11000111\ldots$$

Claim: S is uncountable

We want to show $\nexists$ bijection $f : \mathbb{N} \to S$. Let's assume a bijection exists, and get a contradiction.

$$\begin{array}{llllll} 1 & a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\ 2 & a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\ 3 & a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\ 4 & a_{41} & a_{42} & a_{43} & a_{44} & \ldots \end{array}$$

Let $a_{i1}, a_{i2}, \ldots$ be $f(i)$. i.e. get contradiction, it suffices to produce one element of S which is not in the image of f.

Define $\bar{a_{ij}} = 1 - a_{ij}$

$$\bar{a_{ij}} = \begin{cases} 0 & \text{if } a_{ij} = 1, \\ 1 & \text{if } a_{ij} = 0. \end{cases}$$

Consider the sequence

$$\bar{a_{11}}, \bar{a_{22}}, \bar{a_{33}}, \bar{a_{44}}, \ldots$$

We claim this sequence is not in the image of f.

For any integer, $f(i)$ has a_{ii} in the ith place. Our sequence has $\bar{a_{ii}}$ in the ith place.

Suppose $\exists$ a bijection $f : \mathbb{N} \to \mathbb{Q}$

$$0.070707070707\ldots$$

$$0.13113113113\ldots$$

$$0.299972901010101\ldots$$

So far we have some questions:

1. Infinite sets smaller than $\mathbb{N}$?
2. Does every infinite set A satisfy $|A| = |\mathbb{N}|$ or $|A| = |\mathbb{R}|$?
3. What is a real number?

We would like to say $|A| \leq |B|$ if there exists an injection from $A \to B$.

1. (a) Given an arbitrary infinite set A, define $\mathbb{N} \subset A$
   (b) Given an arbitrary subset $S \subseteq \mathbb{N}$, show $|S| = |\mathbb{N}|$
   Define $f : \mathbb{N} \to S$
   $1 \to$ smallest member of S
   $2 \to$ second smallest member of S

2. Is $\mathbb{R}^2 \xrightarrow{?} \mathbb{R}$
   say $.x_1x_2x_3x_4\ldots$
   and $.y_1y_2y_3y_4\ldots$
   get $.x_1y_1x_2y_2x_3y_3\ldots$
   In fact, we have a bijection. **If we consider $\mathbb{R}^2$ as a set, dimensions are not even well-defined!**