STA305/1004-Class 22

March 23, 2016

Today's Class

- Randomized block designs
  - Linear model and ANOVA
  - Assumptions
- Other Blocking Designs
  - Latin Square
  - Graeco Latin Square
  - hypo-Graeco Latin Square
  - Randomized incomplete block design

Example: penicillin yield

Blocking ≡ Block out variation

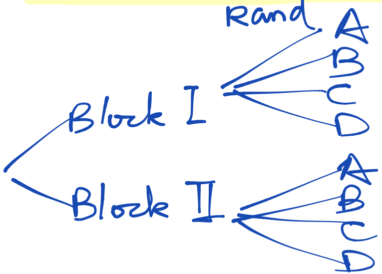
- In this example a process for the manufacture of penicillin was investigated and yield was primary response of interest.
- There were 4 variants of the process (treatments) to be compared.
- An important raw material corn steep liquor varied considerably.
- It was thought that corn steep liquor might causes significant differences in yield.

Block what you can

Randomize what you cannot block

Example: penicillin yield

- Experimenters decided to study 5 blends of corn steep liquor.
- Within each blend the order in which the four treatments were run was random.
- Randomization done separately within each block. Within each blend the order in which the treatments were run were randomized.



Example: penicillin yield

- In a fully randomized one-way design blend differences might not be balanced between the treatments A, B, C, D. This might increase the experimental noise.
- But, by randomly assigning the order in which the four treatments were run within each blend (block), blend differences between the groups were largely eliminated.

A
B
C
D

more of Block I
more of Block II

Example: penicillin yield

The results of the experiment for blend 1

run	blend	treatment	y
1	1	A	89
3	1	B	88
2	1	C	97
4	1	D	94

order
A
C
B
D

The results of the experiment for blend 2

run	blend	treatment	y
4	2	A	84
2	2	B	77
3	2	C	92
1	2	D	79

order
D
B
C
A

Randomization of treatments was done separately within each block.

The ANOVA identity for randomized block designs

The total sum of squares can be re-expressed by adding and subtracting the treatment and block averages as:

trt

Block

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{\cdot\cdot})^2=\sum_{i=1}^{a}\sum_{j=1}^{b}[(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})+(\bar{y}_{\cdot j}-\bar{y}_{\cdot\cdot})+(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{\cdot\cdot})]^2 .$$

After some algebra ... $SS_T=\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{\cdot\cdot})^2$ is equal to

$$b\sum_{i=1}^{a}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2+a\sum_{j=1}^{b}(\bar{y}_{\cdot j}-\bar{y}_{\cdot\cdot})^2+\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{\cdot\cdot})^2$$

So,

$$SS_T=SS_{Treat}+SS_{Blocks}+SS_E$$

ANOVA identity
for block design

Source	SS
Treat	SS_{Treat}
Blocks	SS_B
Error	SS_E
Total	SS_{Total}

Degrees of freedom

- There are N observations so SS_T has $N-1$ degrees of freedom. ✓
- There are a treatments and b blocks so SS_{Treat} and SS_{Blocks} have $a-1$ and $b-1$ degrees of freedom, respectively.
- The sum of squares on the left hand side the equation should add to the sum of squares on the right hand side of the equation. Therefore, the error sum of squares has

$$(N-1)-(a-1)-(b-1)=(ab-1)-(a-1)-(b-1)=(a-1)(b-1)$$

degrees of freedom.

$N = ab$

source	deg f	SS
Treat	$a-1$	SS_{Treat}
Blocks	$b-1$	SS_B
Error	$(a-1)(b-1)$	SS_E
Total	$N-1$	SS_{Total}

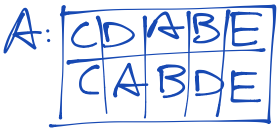
Poll Question

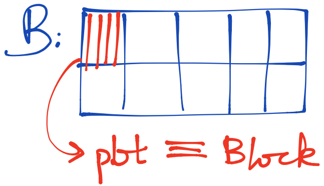
The goal of a certain field experiment is to test the effect of the amount of potash on the strength of cotton. There are 5 levels of potash (the treatments). A large section of a field will receive the treatments. Which of the following is closest to a randomized block design.

Respond at **PollEv.com/nathantaback**

Text **NATHANTABACK** to **37607** once to join, then **A or B**

The field is divided into 10 plots and the 5 treatments are randomly assigned to the plots with each treatment in exactly 2 plots.	**A**	8
The field is divided into 10 plots and 5 smaller sections of each plot is randomly assigned to receive the 5 treatments.	**B**	21





Penicillin Manufacturing Example

The block averages are:

```
block.ave <- sapply(split(tab0404$y,tab0404$blend),mean); block.ave
```

```
 1  2  3  4  5
92 83 85 88 82
```

seems higher than the rest

The treatment averages are:

```
trt.ave <- sapply(split(tab0404$y,tab0404$treatment),mean);trt.ave
```

```
 A  B  C  D
84 85 89 86
```

The grand average is:

```
grand.ave <- mean(tab0404$y);grand.ave
```

```
[1] 86
```

Penicillin Manufacturing Example

The block deviations from the grand average and the sum of squares of block deviations are:

```
block.devs <- block.ave-grand.ave; block.devs; sum(block.devs^2)*4
```

```
 1  2  3  4  5
 6 -3 -1  2 -4
```

```
[1] 264
```

SS_B

The treatment deviations from the grand average and the sum of squares of treatment deviations are:

```
treatment.devs <- trt.ave-grand.ave; treatment.devs; sum(treatment.devs^2)*5
```

```
 A  B  C  D
-2 -1  3  0
```

SS_{Treat}

```
[1] 70
```

Penicillin Manufacturing Example

$H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$

p-values $= P(F_{3,12} > 1.23) = 0.34$ For treatment

$P(F_{4,12} > 3.5) = 0.04$ For blocks

The sum of squares of deviations from the grand average are:

```
all.devs <- tab0404$y-grand.ave; sum(all.devs^2)
```

```
[1] 560
```

SS_T

So, the error sum of squares is:

```
sum(all.devs^2)-sum(treatment.devs^2)*5-sum(block.devs^2)*4
```

```
[1] 226
```

SS_E

ANOVA

	df	SS	MS
Treat	4-1	70	70/3
Block	5-1	264	264/4
Error	12	226	226/12

$F_{treat} = \frac{23.3}{18.83} = 1.23$

$F_{Blocks} = \frac{66}{18.83} = 3.5$

under the assumption of normality (each trt has residual $N(0, \sigma^2)$)

Poll question

If blocking was not incorporated into the design then what would happened to the value of SSE?

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Increase	A	21
Decrease	B	1
Not change	C	0

$SS_T = SS_{Treat} + SS_E$

$SS_T = SS_{Treat} + SS_{Block} + SS_E$

Linear Model for Randomized Block Design

y_{ij} = i^{th} treat in j^{th} block

$i = 1, \cdots, a$

$j = 1, \cdots, b$

- The linear model for the randomized block design is

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij},$$

block effect

treat effect

where $E(\epsilon_{ij}) = 0$.

No Interactions here!

- The model is completely additive.
- It assumes that there is no interaction between blocks and treatments.
- An interaction could occur if an impurity in blend 3 poisoned treatment B and made it ineffective, even though it did not affect the other treatments.

Linear Model for Randomized Block Design

- Another way in which an interaction can occur is when the response relationship is multiplicative
$$E(y_{ij}) = \mu\tau_i\beta_j.$$
- Taking logs and denoting transformed terms by primes, the model then becomes
$$y'_{ij} = \mu' + \tau'_i + \beta'_j + \epsilon'_{ij}$$
- Assuming that ϵ'_{ij} were approximately independent and identically distributed the response $y'_{ij} = log(y_{ij})$ could be analyzed using a linear model in which the interaction would disappear.

Linear Model for Randomized Block Design

Interactions often belong to two categories:

1. **transformable interactions**, which are eliminated by transformation of the original data, and
2. **nontransfromable interactions** such as a treatment -blend interaction that cannot be eliminated via a transformation.

Linear Model for Randomized Block Design

```
pen.model <- lm(y~as.factor(treatment)+as.factor(blend),data=tab0404)
anova(pen.model)
```

```
Analysis of Variance Table

Response: y
                     Df Sum Sq Mean Sq F value  Pr(>F)
as.factor(treatment)  3     70  23.333  1.2389 0.33866
as.factor(blend)      4    264  66.000  3.5044 0.04075 *
Residuals            12    226  18.833
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Calculation of the p-value assumes that

$$\epsilon_{ij} \sim N(0, \sigma^2).$$

So that $MS_{Treat}/MS_E \sim F_{a-1,(a-1)(b-1)}$, $MS_{Blocks} \sim F_{b-1,(a-1)(b-1)}$.

Penicillin example - interpretation

- There is no evidence that the four treatments produce different yields.
- How could this information be used in optimizing yield in the manufacturing process?
- Is one of the treatments less expensive to run? ✓
- If one of the treatments is less expensive to run then an analysis on cost rather than yield might reveal important information.
- The differences between the blocks might be informative.
- In particular the investigators might speculate about why blend 1 has such a different influence on yield.
- Perhaps now the experimenters should study the characteristics of the different blends of corn steep liquor. (Box, Hunter, Hunter, 2005)

Other blocking designs

- Latin square
- Graeco-Latin squares,
- Hyper-Graeco-Latin Squares,
- Balanced incomplete block designs.

The Latin Square Design

- There are several other types of designs that utilize the blocking principle such as The Latin Square design.
- If there is more than one nuisance source that can be eliminated then a Latin Square design might be appropriate.

Latin Square Design - Automobile Emissions

- An experiment to test the feasibility of reducing air pollution.
- A gasoline mixture was modified to produce by changing the amounts of certain chemicals.
- This produced four different types of gasoline: A, B, C, D
- These four treatments were tested with four different drivers and four different cars.

completely randomized design

A	B	C	D
$\bar{Y}_A$	$\bar{Y}_B$	$\bar{Y}_C$	$\bar{Y}_D$

want to eliminate driver-to-driver variability & car-to-car var.

Latin Square Design - Automobile Emissions

- Two blocking factors: cars and drivers.
- The Latin square design was used to help eliminate possible differences between drivers I, II, III, IV and cars 1, 2, 3, 4.
- Randomly allocate treatments, drivers , and cars.

Block #1

block #2

Driver	Car 1	Car 2	Car 3	Car 4
Driver I	A	B	D	C
Driver II	D	C	A	B
Driver III	B	D	C	A
Driver IV	C	A	B	D

each trt appears once in every row & column

"sudoku"

- Trt levels ≡ blocking factor 1 levels ≡ Blocking factor 2 levels

- If one of blocking factor has, say 5 levels, then Latin Square does not work!

Latin Square Design - Automobile Emissions

- The data from the experiment.

Driver	Car 1	Car 2	Car 3	Car 4
Driver I	A	B	D	C
	19	24	23	26
Driver II	D	C	A	B
	23	24	19	30
Driver III	B	D	C	A
	15	14	15	16
Driver IV	C	A	B	D
	19	18	19	16

Latin Square Design - Automobile Emissions

```r
sapply(split(tab0408$y,tab0408$cars), mean)# car means
```

```
 1  2  3  4
19 20 19 22
```

```r
sapply(split(tab0408$y,tab0408$driver), mean)# driver means
```

```
 1  2  3  4
23 24 15 18
```

```r
sapply(split(tab0408$y,tab0408$additive), mean)# additive means
```

```
 A  B  C  D
18 22 21 19
```

```r
mean(tab0408$y) #grand mean
```

```
[1] 20
```

Latin Square Design - Automobile Emissions

- Why not standardize the conditions and make the 16 experimental runs with a single car and single driver for the four treatments?
- Could also be valid but Latin square provides a wider inductive basis.

Latin Square Design - Automobile Emissions

```
latinsq.auto <- lm(y~additive+as.factor(cars)+as.factor(driver),data=tab0408)
anova(latinsq.auto)
```

```
Analysis of Variance Table

Response: y
                  Df Sum Sq Mean Sq F value   Pr(>F)
additive           3     40  13.333     2.5 0.156490
as.factor(cars)    3     24   8.000     1.5 0.307174
as.factor(driver)  3    216  72.000    13.5 0.004466 **
Residuals          6     32   5.333
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$SS_T = \underbrace{SS_{cars} + SS_{drivers}}_{\text{blocking SS}} + \underbrace{SS_{Additives}}_{\text{trt SS}} + SS_E$$

if no blocking then extra SS is in SS_E (just like b4)

Latin Square Design - Automobile Emissions

- Assumming that the residuals are independent and normally distributed and the null hypothesis that there are no treatment differences is true then the ratio of mean squares for treatments and residuals has an $F_{3,6}$ distribution.
- This analysis assumes that treatments, cars, and drivers are additive.
- If the design was replicated then this would increase the degrees of freedom for the residuals and reduce the mean square error.

General Latin Square

$p=4, \; 4 \times 4 = 16$
$p=3, \; 3 \times 3 = 9$
cells

- A Latin square for p factors of a $p \times p$ Latin square, is a square containing p rows and p columns
- Each of the p^2 cells contains one of the p letters that correspond to a treatment.
- Each letter occurs once and only once in each row and column.
- There are many possible $p \times p$ Latin squares.

General Latin Square

Which of the following is a Latin square?

	Col1	Col2	Col3
Row 1	B	A	C
Row 2	A	C	B
Row 3	C	B	A

Latin Square

	Col1	Col2	Col3
Row 1	A	B	C
Row 2	C	A	B
Row 3	B	B	A

Not !

Poll question

An industrial engineer is investigating the effect of four assembly methods (A, B, C, D) on the assembly time of a component. Four operators are selected for the study. The engineer also knows that each assembly method produces fatigue such that the time required for the last assembly might be greater than the time required for the first, regardless of method. The engineer randomly assigns the order that each operator uses the four methods: operator 1 uses the methods in the order: C, A, D, B

Operator	A	B	C	D
I	2	4	1	3
II	4	2	3	1
III	2	1	4	3
IV	3	4	1	2

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The design is:

A. Randomized block design B. Randomized design (without blocking) C. Latin square

21 7 2

Randomized Design

methods A B C D

randomly assign to each operator

Misuse of the Latin Square

- Inappropriate to use Latin square to study factors that can interact.
- Effects of one factor can then be mixed up with interactions of other factors.
- Outliers can occur as a result of these interactions.
- When interactions between factors are likely possible need to use a factorial design.

Graeco-Latin Square

$K=4$

A Graeco-Latin square is a $k \times k$ pattern that permits study of k treatments simultaneously with three different blocking variables each at k levels.

	Car 1	Car 2	Car 3	Car 4
Driver I	A α	B β	C γ	D δ
Driver II	B δ	A γ	D β	C α
Driver III	C β	D α	A δ	B γ
Driver IV	D γ	C δ	B α	A β

Another blocking variable
with levels $\alpha, \beta, \gamma, \delta$

Graeco-Latin Square

- This is a Latin square in which each Greek letter appears once and only once with each Latin letter.
- Can be used to control three sources of extraneous variability (i.e. block in three different directions).

Car	1 Car	2 Car	3 Ca	r 4
Driver I	A α	B β	C γ	D δ
Driver II	B δ	A γ	D β	C α
Driver III	C β	D α	A δ	B γ
Driver IV	D γ	C δ	B α	A β

Graeco-Latin Square

To generate a 3×3 Graeco-Latin square design, superimpose two designs using the Greek letters for the second 3×3 Latin square.

Blocking var #2

Blocking var #1

	Col1	Col2	Col3
Row 1	B α	A β	C γ
Row 2	A γ	C α	B β
Row 3	C β	B γ	A α

Blocking var #3

	Col1	Col2	Col3
Row 1	A	B	C
Row 2	C	A	B
Row 3	B	C	A

relabel
$A \equiv \alpha$
$B \equiv \beta$
$C \equiv \gamma$

hyper-Graeco-Latin Square

These three Latin squares can be superimposed to form a hyper-Graeco-Latin square.
Can be used to control 4 nuisance factors (i.e. block 4 factors).

4 Blocking variables

Row	Col1	Col2	Col3	Col4
Row 1	B $\delta 1$	A $\alpha 4$	D $\gamma 2$	C $\beta 3$
Row 2	C $\alpha 3$	D $\delta 1$	A $\beta 4$	B $\gamma 2$
Row 3	D	B	C	A
Row 4	A	C	B	D

Row	Col1	Col2	Col3	Col4
Row 1	D	A	C	B
Row 2	A	D	B	C
Row 3	B	C	A	D
Row 4	C	B	D	A

$A \equiv \alpha$
$B \equiv \beta$
$C \equiv \gamma$
$D \equiv \delta$

Row	Col1	Col2	Col3	Col4
Row 1	A	D	B	C
Row 2	C	A	D	B
Row 3	B	C	A	D
Row 4	D	B	C	A

$A \equiv 1$
$B \equiv 2$
$C \equiv 3$
$D \equiv 4$

hyper-Graeco-Latin Square

- A machine used for testing the wear on types of cloth.
- Four pieces of cloth can be compared simultaneously on one machine.
- Response is weight loss in tenths of mg when rubbed against a standard grade of emery paper for 1000 revolutions of the machine.

hyper-Graeco-Latin Square

- Specimens of 4 different cloths (A, B,C,D) are compared.
- The wearing qualities can be in any one of 4 positions P_1, P_2, P_3, P_4 on the machine.
- Each emery $(\alpha, \beta, \gamma, \delta)$ paper used to cut into for quarters and each quarter used to complete a cycle C_1, C_2, C_3, C_4 of 1000 revolutions.
- Object was to compare treatments.

hyper-Graeco-Latin Square

Key property is "once & only once"

position ≡ row
& cycle ≡ cols

1. type of specimen holders 1, 2, 3, 4
2. position on the machine P_1, P_2, P_3, P_4.
3. emory paper sheet $\alpha, \beta, \gamma, \delta$.
4. machine cycle C_1, C_2, C_3, C_4.

The design was replicated. The first replicate is shown in the table below.

	P_1	P_2	P_3	P_4
C_1	A α 1 320	B β 2 297	C γ 3 299	D δ 4 313
C_2	C β 4 266	D α 3 227	A δ 2 260	B γ 1 240
C_3	D γ 2 221	C δ 1 240	B α 4 267	A β 3 252
C_4	B δ 3 301	A γ 4 238	D β 1 243	C α 2 290

- symmetric in all factors

i.e. obs. not matter if for example ?

hyper-Graeco-Latin Square

A linear model can be fit so that the ANOVA table and parameter treatment effects can be calculated.

```
wear.hypsq <- lm(y~treatment+as.factor(rep)+as.factor(position)+
                   as.factor(cycle)+as.factor(holder)+
                   as.factor(paper),data=tab0412)
anova(wear.hypsq)
```

```
Analysis of Variance Table

Response: y
                    Df  Sum Sq Mean Sq F value    Pr(>F)
treatment            3  1705.3  568.45  5.3908  0.021245 *
as.factor(rep)       1   603.8  603.78  5.7259  0.040366 *
as.factor(position)  3  2217.3  739.11  7.0093  0.009925 **
as.factor(cycle)     6 14770.4 2461.74 23.3455 5.273e-05 ***
as.factor(holder)    3   109.1   36.36  0.3449  0.793790
as.factor(paper)     6  6108.9 1018.16  9.6555  0.001698 **
Residuals            9   949.0  105.45
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

there is evidence of n treat diff.

Balanced incomplete block design

- Suppose that instead of four samples to be included on each 1000 revolution cycle only three could be included, but the experimenter still wanted to compare four treatments.
- The size of the block is now 3 - too small to accommodate all treatments simultaneously.

Balanced incomplete block design

A balanced incomplete block design has the property that every pair of treatments occurs together in a block the same number of times.

Cycle block			
1	A	B	C
2	A	B	D
3	A	C	D
4	B	C	D

Cycle block	A	B	C	D
1	x	x	x	
2	x	x		x
3	x		x	x
4		x	x	x

AB
AC
AD
BC
BD
CD

each pair
occurs
same # of
times