Eg. (Complementary slackness & degeneracy)

see "A Degenerate Optimal Solution.pdf"

Consider the primial problem

Maximize $z = 3x_1 + 7x_2$ s.t.

$$x_1 + 5x_2 \leq 19$$
$$x_1 - x_2 \leq 1$$
$$-x_1 + 2x_2 \leq 2$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Given: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is optimal

A Degenerate Optimal Solution

Tableau 1:

	x_1	x_2	x_3	x_4	x_5	
x_3	1	5	1	0	0	19
x_4	1	−1	0	1	0	1
x_5	−1	2	0	0	1	2
	−3	−7	0	0	0	0

Tableau 2:

	x_1	x_2	x_3	x_4	x_5	
x_3	$\frac{7}{2}$	0	1	0	$-\frac{5}{2}$	14
x_4	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	2
x_2	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1
	$-\frac{13}{2}$	0	0	0	$\frac{7}{2}$	7

Tableau 3:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{2}{7}$	0	$-\frac{5}{7}$	4
x_4	0	0	$-\frac{1}{7}$	1	$\frac{6}{7}$	0
x_2	0	1	$\frac{1}{7}$	0	$\frac{1}{7}$	3
	0	0	$\frac{13}{7}$	0	$-\frac{8}{7}$	33

Tableau 4 is optimal:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{1}{6}$	$\frac{5}{6}$	0	4
x_5	0	0	$-\frac{1}{6}$	$\frac{7}{6}$	1	0
x_2	0	1	$\frac{1}{6}$	$-\frac{1}{6}$	0	3
	0	0	$\frac{5}{3}$	$\frac{4}{3}$	0	33

(with 3 constraints this problem has 3 basic variables. At $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, there is no slack in any constraint, So at least one basic variable is zero : a degenerate solution)

We will write the dual problem and solve it:

Maximize $z' = 19w_1 + w_2 + 2w_3$ s.t.

$$w_1 + w_2 - w_3 \geq 3$$
$$5w_1 - w_2 + 2w_3 \geq 7$$

$$w_1 \geq 0,\ w_2 \geq 0,\ w_3 \geq 0$$

At primal optimality, $x_1 \neq 0, x_2 \neq 0$ so at dual optimality, both dual constraints are tight. Since there's no slack at primal optimality in any primal constraint, complementary slackness does not say whether w_1, w_2 or w_3 is 0 at dual optimality.

So, complementary slackness <u>only</u>

$$w_1 + w_2 - w_3 = 3$$
$$5w_1 - w_2 + 2w_3 = 7$$

Solution of the system $= \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 5 & -1 & 2 & 7 \end{array}\right] \end{array} \simeq \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -6 & 7 & -8 \end{array}\right]$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{6} & \frac{5}{3} \\ 0 & 1 & -\frac{7}{6} & \frac{4}{3} \end{array}\right]$$

$$\Rightarrow w_1 = \frac{5}{3} - \frac{1}{6}w_3$$
$$w_2 = \frac{4}{3} + \frac{7}{6}w_3$$

Such $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ include each optimal dual solution, but also include some infeasible solution

We have $w_1 \geq 0 \Rightarrow w_3 \leq 10$ (solve $\frac{5}{3} - \frac{1}{6}w_3 \geq 0$)

$w_2 \geq 0$

$w_3 \geq 0$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} - \frac{1}{6}w_3 \\ \frac{4}{3} + \frac{7}{6}w_3 \\ w_3 \end{bmatrix}$ are feasible provided $0 \leq w_3 \leq 10$

But $z = 3x_1 + 7x_2 = 3 \cdot 4 + 7 \cdot 3 = 33$

$z' = 19w_1 + w_2 + 2w_3 = 19\left(\frac{5}{3} - \frac{1}{6}w_3\right) + \left(\frac{4}{3} + \frac{7}{6}w_3\right) + 2w_3 = \frac{95}{3} + \frac{4}{3} = 33$

So both solutions are optimal (weak duality theorem)

Another optimality criterion is:

If x_0 and w_0 are feasible for respective primal and dual problems, and x_0 + w_0 satisfy the conclusion of the complementary slackness theorem, then both solutions are optimal.

These are called Karush–Kuhn–Tucker (or KKT) conditions.