STA457/2202H1S PRACTICE QUESTIONS & SHORT ANSWERS

Question 1 (Random walk): $Y_t = Y_{t-1} + e_t$, $e_t \sim NID(0, \sigma_e^2)$. Show that for $1 \le t \le s$

(1) $E(Y_t) = 0$
(2) $var(Y_t) = t\sigma_e^2$
(3) $\gamma(t,s) = t\sigma_e^2$
(4) $\rho(t,s) = \sqrt{t/s}$
(5) Show that $\lim_{t\to\infty} \rho(h) = 1$ for moderate h.

<u>Answer:</u> WLOG, we can assume that $\mathrm{Y}_1 = \mathrm{e}_1$ (i.e., $Y_0 = 0$) and therefore, $Y_t = Y_{t-1} + e_t = e_1 + e_2 + \cdots + e_t$.

(1) $E(Y_t) = E(e_1 + e_2 + \cdots + e_t) = 0$
(2) $Var(Y_t) = Var(e_1 + e_2 + \cdots + e_t) = \sum_{i=1}^{t} Var(e_i) = t\sigma_e^2$
(3) $\gamma(t,s) = Cov(e_1 + \cdots + e_t, e_1 + \cdots + e_t + e_{t+1} + \cdots + e_s) = \sum_{i=1}^{s}\sum_{j=1}^{t} Cov(e_i, e_j) = t\sigma_e^2$ for $1 \le t \le s$
(4) $\rho(t,s) = \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}} = \frac{t\sigma_e^2}{\sqrt{t\sigma_e^2 \times s\sigma_e^2}} = \sqrt{t/s}$.
(5) $\rho(h) = \rho(t, t+h) = \sqrt{t/(t+h)}$.

Question 2 (Moving average of order 2): $Y_t = 0.5\, e_t + 0.5\, e_{t-1}$, $e_t \sim NID(0, \sigma_e^2)$. Show that

(1) $E(Y_t) = 0$
(2) $var(Y_t) = 0.5\, \sigma_e^2$
(3) $\gamma(t,s) = \begin{cases} 0.5\, \sigma_e^2, & t = 0 \\ 0.25\, \sigma_e^2, & |t-s| = 1 \\ 0, & |t-s| > 1 \end{cases}$
(4) $\rho(t,s) = \begin{cases} 1, & t = 0 \\ 0.5, & |t-s| = 1 \\ 0, & |t-s| > 1 \end{cases}$

<u>Answer:</u>

$$\mu_t = E(Y_t) = E\left\{\frac{e_t + e_{t-1}}{2}\right\} = \frac{E(e_t) + E(e_{t-1})}{2}$$
$$= 0$$

$$Var(Y_t) = Var\left\{\frac{e_t + e_{t-1}}{2}\right\} = \frac{Var(e_t) + Var(e_{t-1})}{4}$$
$$= 0.5\sigma_e^2$$

$$
\begin{aligned}
Cov(Y_t, Y_{t-1}) &= Cov\left\{\frac{e_t + e_{t-1}}{2}, \frac{e_{t-1} + e_{t-2}}{2}\right\} \\
&= \frac{Cov(e_t, e_{t-1}) + Cov(e_t, e_{t-2}) + Cov(e_{t-1}, e_{t-1})}{4} \\
&\qquad\qquad + \frac{Cov(e_{t-1}, e_{t-2})}{4} \\
&= \frac{Cov(e_{t-1}, e_{t-1})}{4} \qquad \text{(as all the other covariances are zero)} \\
&= 0.25\sigma_e^2
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
Cov(Y_t, Y_{t-2}) &= Cov\left\{\frac{e_t + e_{t-1}}{2}, \frac{e_{t-2} + e_{t-3}}{2}\right\} \\
&= 0 \qquad \text{since the } e\text{'s are independent.}
\end{aligned}
$$

Similarly, $Cov(Y_t, Y_{t-k}) = 0$ for $k > 1$, so we may write

$$
\gamma_{t,s} = \begin{cases} 0.5\sigma_e^2 & \text{for } |t-s| = 0 \\ 0.25\sigma_e^2 & \text{for } |t-s| = 1 \\ 0 & \text{for } |t-s| > 1 \end{cases}
$$

$$
\begin{aligned}
\gamma_{t,s} &= E\left\{\cos\left[2\pi\left(\frac{t}{12} + \Phi\right)\right]\cos\left[2\pi\left(\frac{s}{12} + \Phi\right)\right]\right\} \\
&= \int_0^1 \cos\left[2\pi\left(\frac{t}{12} + \phi\right)\right]\cos\left[2\pi\left(\frac{s}{12} + \phi\right)\right]d\phi \\
&= \frac{1}{2}\int_0^1 \left\{\cos\left[2\pi\left(\frac{t-s}{12}\right)\right] + \cos\left[2\pi\left(\frac{t+s}{12} + 2\phi\right)\right]\right\}d\phi \\
&= \frac{1}{2}\left\{\cos\left[2\pi\left(\frac{t-s}{12}\right)\right] + \frac{1}{4\pi}\sin\left[2\pi\left(\frac{t+s}{12} + 2\phi\right)\right]\Big|_{\phi=0}^{1}\right\} \\
&= \frac{1}{2}\cos\left[2\pi\left(\frac{|t-s|}{12}\right)\right]
\end{aligned}
$$

Question 4 Consider a MA(1) process as

$$X_t = a_t + \theta a_{t-1}, \qquad a_t \sim NID(0, \sigma^2).$$

Calculate $var(X_1 + X_2 + X_3)$.

<u>Answer: Skip</u>

Question 5 (General linear process): A general linear process, or $MA(\infty)$ process in class, is a weighted linear combination of present and past white noise terms as

$$Y_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \sum_{j=0}^{\infty} \psi_j a_{t-j},$$

where $\psi_0 = 1, \sum_{j=1}^{\infty} |\psi_j| < \infty$, and $a_t \sim NID(0, \sigma^2)$. Show that

(1) $E(Y_t) = 0$,
(2) $\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+|h|}$ for $|h| = 0,1,2,3, \ldots$.

Answer: Skip

Question 6 ($\mathbf{MA(q)}$ processes): $Y_t = \mu + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$, $a_t \sim NID(0, \sigma^2)$. Show that

(1) $E(Y_t) = \mu$,
(2) $\gamma(0) = \sigma^2(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)$,
(3) $\gamma(h) = Cov(Y_t, Y_{t+h}) = \begin{cases} \sigma^2(-\theta_h + \theta_1\theta_{h+1} + \cdots + \theta_{q-h}\theta_q), h = 1,2, \ldots, q \\ 0 \quad , h > q \end{cases}$.
(4) Suppose that $\{Y_t\}$ is invertible and can be expressed as $Y_t = \sum_1^{\infty} \pi_j X_{t-j} + a_t$. Find π_j for $j = 0,1,2,3,4,5$.

Answer: (1), (2) and (3) skip

Invertible ➔ $\pi(B)X_t = a_t$ ➔ $\frac{\phi(B)}{\theta(B)} X_t = a_t$ ➔ $\pi(B) = \phi(B)/\theta(B)$ ➔ $\pi(B)\theta(B) = \phi(B)$

$$(1 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots)(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) = 1$$

Match coefficients of B^j, $j = 0,1,2,3,4,5$. We can solve for π_j for $j = 0,1,2,3,4,5$.

Question 6 (Stationary $\mathbf{AR(2)}$ processes):
$Y_t - \phi_1 X_{t-1} - \phi_1 X_{t-2} = \mu + a_t, a_t \sim NID(0, \sigma^2)$. Show that

(1) $E(Y_t) = \mu/(1 - \phi_1 - \phi_2)$,
(2) Write down the corresponding Yule-Walker equations.
(3) Calculate the partial autocorrelation functions of $\{Y_t\}$ for lag=1,2,3, …
(4) Suppose that the casual representation of $\{Y_t\}$ is given by $Y_t = \sum_0^{\infty} \psi_j a_{t-j}$. Find ψ_j for $j = 0,1,2,3,4,5$.

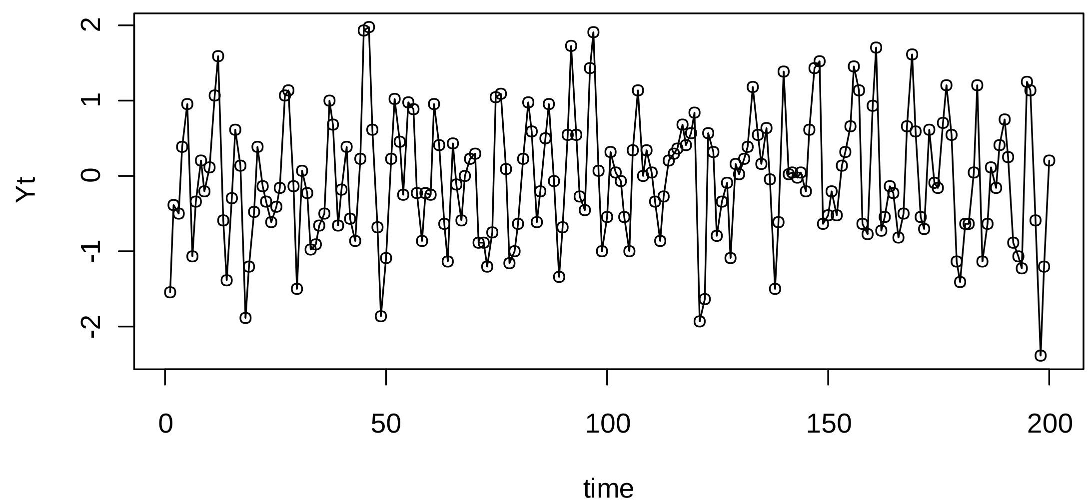
Answer: Skip (See course note).

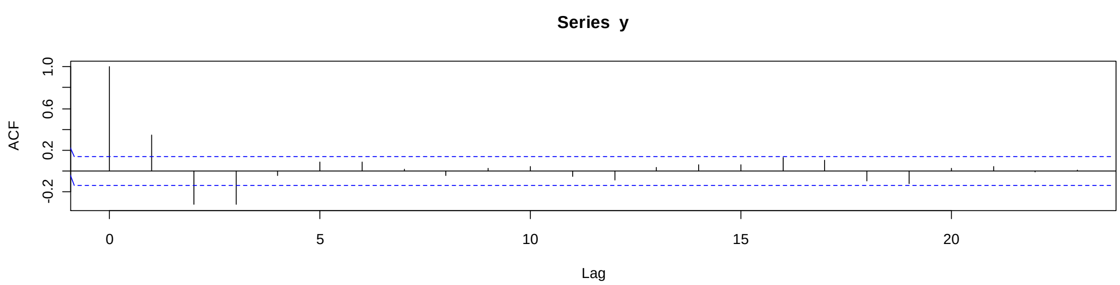
causal ➔ $X_t = \psi(B)a_t$ ➔ $X_t = \frac{\theta(B)}{\phi(B)} a_t$ ➔ $\psi(B) = \theta(B)/\phi(B)$ ➔ $\psi(B)\phi(B) = \theta(B)$

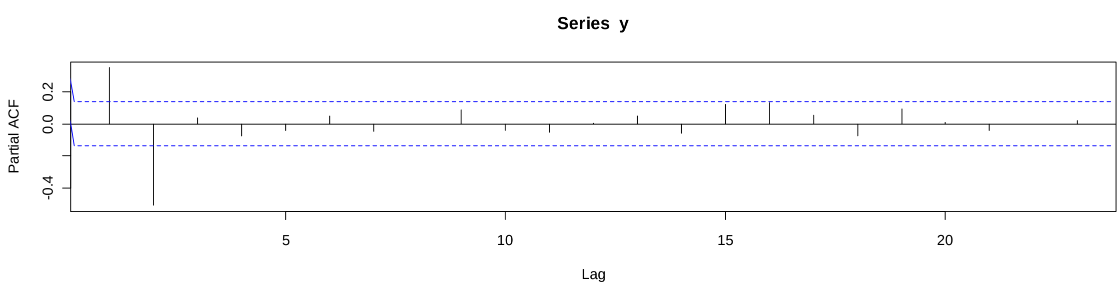
$$(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots)(1 - \phi_1 B - \phi_2 B^2) = 1$$

Match coefficients of B^j, $j = 0,1,2,3,4,5$. We can solve for ψ_j for $j = 0,1,2,3,4,5$.

Question 7 (The method of moment estimation): An analyst decides to find an AR(2) model for this time series by observing the time series plot and correlogram of $\{Y_t\}$ below.







The analyst calculated the sample autocorrelation functions of $\{Y_t\}$ for $\hat{\rho}(h), h = 1,2,3,\ldots 10$ and the results are listed below.

lag	1	2	3	4	5	6	7	8	9	10
rho	-0.78	0.64	-0.53	0.43	-0.36	0.29	-0.24	0.20	-0.16	0.13

(1) Does the analyst make the correct decision to fit an $AR(2)$ model? Why and why not?

(2) Estimate the autoregressive parameters, i.e., ϕ_1 and ϕ_2, using the method of moments. (Hint: Yule-Walker equations)

(3) Is the model stationary?

(4) Suppose the residual autocorrelations functions for lag 1,2,3, ... 10 are

$$\{0.030\ -0.072\ \ 0.013\ \ 0.020\ -0.131\ \ 0.036\ \ 0.057\ -0.063\ \ 0.019\ \ 0.054\}$$

Check the model adequacy using the Ljung-box test for $m = 5, 10$.

Answer:

(1) Yes, PACF cut off at lag 2.

(2) $\phi_1 \approx -0.7, \phi_2 \approx 0.1$ (since the sample ACF in the question contain rounding errors)

(3) Solving $1 - \phi_1 B - \phi_2 B^2 = 0$. The time series is stationary if the roots are outside unit circle.

(4) $Q_{LB}(10) = n(n+2)\sum_{k=1}^{10}(n-k)^{-1}r_k^2 = 200 \cdot 202(\frac{0.03^2}{200-1} + \frac{(-0.072)^2}{200-2} + \cdots + \frac{0.054^2}{200-10})$

Question 8 (Definition)

(1) Define strictly and weakly stationary time series. What is the relationship between them?

(2) Describe the general approach to time series modeling.

(3) Define an autoregressive moving average model of order p and q ($ARMA(p,q)$).

(4) What is the dual relationship between AR and MA models.

(5) Define *Wold Decomposition*. How does this method provide support to the use of $ARMA$ models?

(6) Derive the Yule-Walker equations for an $AR(p)$ process.

(7) Define partial autocorrelation functions.

(8) Describe two methods of model selection that were introduced in class.

Question 10 (Causal/stationary and invertible process): Determine which of the following processes are causal and/or invertible. Assume that $a_t \sim NID(0,1)$.

(1) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = a_t$ **[causal/stationary and invertible]**

(2) $X_t + 1.9X_{t-1} + 0.88X_{t-2} = a_t + 0.2a_{t-1} + 0.7_{t-2}$ **[causal/stationary and invertible]**

(3) $X_t + 0.6X_{t-2} = a_t + 1.2a_{t-1}$ **[causal/stationary but not invertible]**

(4) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = a_t$ **[causal/stationary and invertible]**

(5) $X_t + 1.6X_{t-1} = a_t - 0.4a_{t-1} + 0.04a_{t-2}$ **[not causal/nonstationary but invertible]**