8.13.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x \mid \alpha) = \dfrac{1+\alpha x}{2}$

$-1 \le x \le 1 \; ; \; -1 \le \alpha \le 1.$

a.) From the example we have:

$$E(x) = \mu = \int_{-1}^{1} x \, \frac{1+\alpha x}{2} \, dx = \frac{\alpha}{3}$$

$$\therefore \; \tilde{\alpha} = 3\bar{X}$$

Now, what is $E(\tilde{\alpha}) = E(3\bar{X})$

$$= 3\,E(x) = 3\left(\frac{\alpha}{3}\right) = \alpha.$$

$\therefore \; \tilde{\alpha}$ is an unbiased estimator of α.

b.) $V(X) = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_{-1}^{1} x^2 \, \frac{1+\alpha x}{2} \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{\alpha}{2}\int_{-1}^{1} x^3 dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{\alpha}{2}\left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= 1/3 + 0 = 1/3.$$

$$V(3\bar{X}) = 9\,V(\bar{X}) = 9/n\; V(X)$$

$$= 9/n\left[\tfrac{1}{3} - \frac{\alpha^2}{9}\right]$$

$$= 9/3n - \frac{\alpha^2}{n} = (3-\alpha^2)/n.$$

c.) Based on the CLT:

$$\tilde{\alpha} = 3\bar{X} \;\dot{\sim}\; N\left(\alpha,\ var = (3-\alpha^2)/n\right)$$

If $\alpha = 0$, $n = 25$ then

$$\tilde{\alpha} = 3\bar{x} \;\dot{\sim}\; N(0,\ 3/25)$$

$$P(|\tilde{\alpha}| > 0.5) = P(\tilde{\alpha} < -0.5) + P(\tilde{\alpha} > 0.5)$$

$$= 2\,P\left(\frac{\tilde{\alpha}-0}{\sqrt{3/25}} < \frac{-0.5-0}{\sqrt{3/25}}\right)$$

$$= 2\,P\left(Z < -0.5/\sqrt{3/25}\right)$$

$$= 0.1489.$$

8.17.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x \mid \alpha) = \dfrac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} [x(1-x)]^{\alpha-1}$

$$0 \le x \le 1.$$

Note: This is a beta (a, b) distribution where $a = b = \alpha$.

$\therefore \quad E(x) = \frac{1}{2} \; ; \; V(x) = \dfrac{1}{4(2\alpha+1)}.$

b.) We want a MoM estimator. Setting

$E(x) = \frac{1}{2} = \bar{X}$ does not provide an equation in terms of α!

$$\therefore \quad V(x) = \frac{1}{4(2\alpha+1)} = \frac{1}{n} \sum x_i^2 = (\overline{x^2}) = m_2$$

$$\tilde{\alpha} = \frac{1 - 4m_2}{8m_2}$$

c.) $L(\alpha \mid \underset{\sim}{x}) = \displaystyle\prod_{i=1}^{n} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} x_i^{\alpha-1} (1-x_i)^{\alpha-1}$

$$= \left(\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \right)^n \prod_{i=1}^{n} x_i^{\alpha-1} (1-x_i)^{\alpha-1}$$

$\Rightarrow \ell(\alpha \mid x) = n \left[\log \Gamma(2\alpha) - 2 \log \Gamma(\alpha) \right]$

$$+ (\alpha-1) \left[\sum \log(x_i) + \sum \log(1-x_i) \right]$$

- Now let's differentiate with respect to α:

$$\frac{dl}{d\alpha} = \frac{2n\,\Gamma'(2\alpha)}{\Gamma(2\alpha)} - \frac{2n\,\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log\left(x_i(1-x_i)\right) = 0$$