Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 5

- Due Tues. March 19, 6:10pm sharp. Late assignments will not be accepted.
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Textbook, Section 4.5, **3**.

Solution. For a linear operator to be symmetric, its matrix must be symmetric with respect to an orthonormal basis, the easiest one to chose is the standard basis, for this we need to compute $[T]_\epsilon^\epsilon$ where $\epsilon = \{e_1, e_2, e_3\}$ represents the standard basis of $\mathbb{R}^3$.

$$[T]_\epsilon^\epsilon = [I_{\mathbb{R}^3}]_\epsilon^\beta [T]_\beta^\beta [I_{\mathbb{R}^3}]_\beta^\epsilon$$

where we can easily see that

$$[I_{\mathbb{R}^3}]_\beta^\epsilon = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

To compute the other change of basis matrix we proceed as follows.

$$(1,0,0) = -(1,1,0) + (2,1,0) \Longrightarrow [e_1]_\beta = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly,

$$(0,1,0) = 2(1,1,0) - (2,1,0) \Longrightarrow [e_2]_\beta = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

and

$$(0,0,1) = -(1,1,0) - (1,0,-1) + (2,1,0) \Longrightarrow [e_3]_\beta = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Thus,

$$[I_{\mathbb{R}^3}]^\beta_\epsilon = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

So

$$[T]^\epsilon_\epsilon = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} [T]^\beta_\beta \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Now we can solve for a), b), c).
a) we compute

$$[T]^\epsilon_\epsilon = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

which is not symmetric.
b)

$$[T]^\epsilon_\epsilon = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

This is symmetric.
c)

$$[T]^\epsilon_\epsilon = \begin{bmatrix} 8 & 4 & 8 \\ -2 & -1 & -4 \\ -5 & -3 & -5 \end{bmatrix}$$

and this is not symmetric.
d) We have

$$[T]^\beta_\beta == [I_{\mathbb{R}^3}]^\epsilon_\beta [T]^\epsilon_\epsilon [I_{\mathbb{R}^3}]^\beta_\epsilon.$$

Then

$$[T]^\beta_\beta = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 2 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

which is not a symmetric matrix(though the operator T itself is symmetric here) because the basis β is not an orthonormal basis.

2. Textbook, Section 5.3, **6(b)**.

Solution. We compute the characteristic polynomial

$$p_A(x) = (x-2)(x - \frac{3+\sqrt{5}}{2})(x - \frac{3-\sqrt{5}}{2})$$

The eigenvalues are $\lambda_1 = 2,\ \lambda_2 = \frac{3+\sqrt{5}}{2},\ \lambda_3 = \frac{3-\sqrt{5}}{2}$.
We compute

$$E_{\lambda_1} = null(A - \lambda_1 I) = Span\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Similarly,

$$E_{\lambda 2} = Span\left\{ \begin{bmatrix} 1-\sqrt{5} \\ 0 \\ 2i \end{bmatrix} \right\}$$

and

$$E_{\lambda 3} = Span \begin{bmatrix} 1+\sqrt{5} \\ 0 \\ 2i \end{bmatrix}$$

Now, let

$$\alpha = \{v_1, v_2, v_3\}$$

where $v_1 = (0,1,0),\ v_2 = \frac{1}{||(1-\sqrt{5},0,2i)||}(1-\sqrt{5},0,2i),\ v_3 = \frac{1}{||(1+\sqrt{5},0,2i)||}(1+\sqrt{5},0,2i)$. Then, α is an orthonormal basis of $\mathbb{C}^3$, since it consists of eigenvectors of length 1, each corresponding to a distinct eigenvalue. By the Spectral Theorem, if $A = [T]_\epsilon^\epsilon$ then

$$[T]_\alpha^\alpha = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

In other terms, the spectral decomposition of A is

$$\lambda_1 A_1 + \lambda_2 A_2 +_3 A_3$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Rewrite $(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$ in the form x^TAx, where A is symmetric.

Solution.

$$(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 = \sum_{i,j=1}^{n} a_ia_jx_ix_j$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_1^2 & a_1a_2 & \dots & a_1a_n \\ a_2a_1 & a_2^2 & \dots & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_na_1 & a_na_2 & \dots & a_n^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = X^TAX$$

where $A = (c_{ij}),\ c_{ij} = a_ia_j;\ i,j = 1,\dots,n$ is symmetric.

4. Identify and sketch the conic section given by $7x^2 + 2\sqrt{3}xy + 5y^2 = 1$.

Solution. We write the equation of the conic section in matrix form.

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

where

$$A = \begin{bmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{bmatrix}.$$

We compute $p_A(x) = (x-8)(x-4)$, the eigenvalues are $\lambda_1 = 8,\ \lambda_2 = 4$.

$$E_{\lambda_1} = Span\left\{ \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right\}$$

$$E_{\lambda_2} = Span\left\{ \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \right\}.$$

Now let $\alpha = \{v_1, v_2\}$, where $v_1 = \frac{1}{2}(\sqrt{3}, 1),\ v_2 = \frac{1}{2}(-1, \sqrt{3})$. Then α is an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A. Let

$$Q = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

Then

$$Q^T A Q = \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = Q^T \begin{bmatrix} x \\ y \end{bmatrix}$$

then

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 1$$

in other terms $8x'^2 + 4y'^2 = 1$. This the equation of an ellipse. The graph is attached at the end of the document.

5. Consider the vector spaces $P_2(\mathbb{R})$ with inner product

$$< p(x), q(x) >= p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

Let T: $P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = p'(x)$. Find $T^*(p(x))$ for an arbitrary $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$.

Solution. We need first to construct an orthonormal basis α of $P_2(\mathbb{R})$, then we compute $[T]_{\alpha\alpha}$, to conclude $[T^*]_{\alpha\alpha} = [\overline{T}]^t_{\alpha\alpha}$. Note here that we are working in the real case, so $[T^*]_{\alpha\alpha}$ is simply $[T]^t_{\alpha\alpha}$.
We start we the standard basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$, and use Gram-Shmidt procedure to construct α.
$u_1 = 1$,
$u_2 = x - \frac{<1,x>}{||1||^2}1 = x$,
$u_3 = x^2 - \frac{<1,x^2>}{||1||^2}1 - \frac{<x,x^2>}{||x||^2}x = x^2 - \frac{2}{3}$. We then take

$$\alpha = \{v_1, v_2, v_3\}$$

where

$$v_1 = \frac{u_1}{||u_1||} = \frac{1}{\sqrt{3}},\ v_2 = \frac{u_2}{||u_2||} = \frac{x}{\sqrt{2}},\ v_3 = \frac{u_3}{||u_3||} = \sqrt{\frac{3}{2}}(x^2 - \frac{2}{3})$$

Let us compute $[T]_{\alpha\alpha}$.

$T(v_1) = 0 \Longrightarrow [T(v_1)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(v_2) = \frac{1}{\sqrt{2}} = \sqrt{\frac{3}{2}}v_1 \Longrightarrow [T(v_2)]_\alpha = \begin{bmatrix} \sqrt{\frac{3}{2}} \\ 0 \\ 0 \end{bmatrix}$

$T(v_3) = \sqrt{\frac{3}{2}}2x = 2\sqrt{3}v_2 \Longrightarrow [T(v_3)]_\alpha = \begin{bmatrix} 0 \\ 2\sqrt{3} \\ 0 \end{bmatrix}$.

Thus,

$$[T]_{\alpha\alpha} = \begin{bmatrix} 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$[T^*]_{\alpha\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 2\sqrt{3} & 0 \end{bmatrix}.$$

Let us compute $[p(x)]_\alpha$. We have

$$p(x) = a + bx + cx^2 = (a + \frac{2c}{3})3v_1 + b\sqrt{2}v_2 + c\sqrt{\frac{2}{3}}v_3 \Longrightarrow [p(x)]_\alpha = \begin{bmatrix} (a+\frac{2c}{3})3 \\ b\sqrt{2} \\ c\sqrt{\frac{2}{3}} \end{bmatrix}.$$

Thus

$$[T^*(p(x))]_\alpha = [T^*]_{\alpha\alpha}[p(x)]_\alpha = \begin{bmatrix} 0 \\ \frac{3\sqrt{2}}{2}(a+\frac{2c}{3}) \\ 2b\sqrt{6} \end{bmatrix}.$$

Hence, $T^*(p(x)) = \frac{3\sqrt{2}}{2}(a + \frac{2c}{3})v_2 + 2b\sqrt{6}v_3 = -4b + (\frac{a}{2} + c)x + 6bx^2$.

6. Assume T is a linear operator on $\mathbb{R}^3$, that $\alpha = \{(1,1,1),(1,-1,0),(0,1,-1)\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T and that the corresponding eigenvalues of T are the real numbers a, b, and c. Prove that T is self-adjoint if and only if $b = c$.

Let us first construct an orthonormal basis out of the basis α, by using Gram-Shmidt.
$u_1 = (1,1,1)$,
$u_2 = (1,-1,0) - 0(1,1,1) = (1,-1,0)$,
$u_3 = (0,1,-1) - 0(1,1,1) - \frac{-1}{2}(1,-1,0) = \frac{1}{2}(1,1,0)$.
Let $\beta = \{v_1, v_2, v_3\}$ where

$$v_1 = \frac{u_1}{||u_1||} = \frac{1}{\sqrt{3}}(1,1,1);\ v_2 = \frac{u_2}{||u_2||} = \frac{1}{sqrt2}(1,-1,0);\ v_3 = \frac{u_3}{||u_3||} = \frac{1}{\sqrt{2}}(1,1,0).$$

Using the fact that

$$T((1,1,1)) = a(1,1,1);\ T(1,-1,0) = b(1,-1,0); T(0,1,-1) = c(0,1,-1)$$

together with the relation between the vectors in α, and those in β, we conclude:

$T(v_1) = \frac{a}{\sqrt{3}}(1,1,1) = av_1 \Longrightarrow [T(v_1)]_\beta = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$.

Similarly,$[T(v_2)]_\beta = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$.

$[T(v_3)]_\beta = \begin{bmatrix} 0 \\ \frac{b-c}{2} \\ \frac{c}{2} \end{bmatrix}$.

Then

$$[T]_{\beta\beta} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & \frac{b-c}{2} \\ 0 & 0 & \frac{c}{2} \end{bmatrix}$$

remember that β is an orthonormal basis by construction, then

$$[T^*]_{\beta\beta} = [\overline{T}]^t_{\beta\beta} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & o \\ 0 & \frac{b-c}{2} & \frac{c}{2} \end{bmatrix}$$

T is self-adjoint if and only it $[T^*]_{\beta\beta} = [T]_{\beta\beta}$ if and only if $\frac{b-c}{2} = 0$ if and only if $b = c$.

7. Let V be an n-dimensional inner real product space and let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. Let W be a subspace of V with orthonormal basis $\beta = \{w_1, w_2, \ldots, w_k\}$. Let $A = ([w_1]_\alpha [w_2]_\alpha \ldots [w_k]_\alpha)$ and let P_W be the orthogonal projection onto W.

(a) Show $[P_W]_{\alpha\alpha} = AA^T$.

(b) Show $[P_W]^2_{\alpha\alpha} = [P_W]_{\alpha\alpha}$ and $[P_W]^T_{\alpha\alpha} = [P_W]_{\alpha\alpha}$.

Solution. a) For each $i = 1, \ldots, n$,

$$P_W(v_i) = \sum_{j=1}^{k} < v_i, w_j > w_j \Longrightarrow [P_W(v_i)]_\alpha = \sum_{j=1}^{k} < v_i, w_j > [w_j]_\alpha$$

Thus, we can write [P$_W]_{\alpha\alpha} incolumnblocksas [P_W]_{\alpha\alpha} = \begin{bmatrix}[P_W(v_1)]_\alpha & [P_W(v_2)]_\alpha & \ldots & [P_W(v_n)]_\alpha\end{bmatrix}$

$$= \begin{bmatrix}\sum_{j=1}^{k} < v_1, w_j > [w_j]_\alpha & \sum_{j=1}^{k} < v_2, w_j > [w_j]_\alpha & \ldots & \sum_{j=1}^{k} < v_n, w_j > [w_j]_\alpha\end{bmatrix}$$

$= \begin{bmatrix}[w_1]_\alpha & [w_2]_\alpha & \ldots & [w_k]_\alpha\end{bmatrix} bmatrix < v_1, w_1 >< v_2, w_1 > \ldots < v_n, w_1 > \; < v_1, w_2 >< v_2, w_2 > \ldots < v_n, w_2 > \; \ldots\ldots$

¡v$_1, w_k >< v_2, w_k > \ldots < v_n, w_k >$

.

But for each $j = 1, \ldots, k$

$$w_j = \sum_{i=1}^{n} < w_j, v_i > \Longrightarrow [w_j]_\alpha = \begin{bmatrix} < w_j, v_1 > \\ < w_j, v_2 > \\ \ldots \\ < w_j, v_n > \end{bmatrix}.$$

We then conclude

$$[P_W]_{\alpha\alpha} = AA^T.$$

b) $[P_W]^2_{\alpha\alpha} = (AA^T)(AA^T) = A(A^TA)A^T = AIA^T = AA^T = [P_W]_{\alpha\alpha}$, this is because A is an orthogonal matrix, since its columns are orthonormal.
As for the second part, we have $[P_W]^T_{\alpha\alpha} = (AA^T)^T = (A^T)^TA^T = AA^T = [P_W]_{\alpha\alpha}$.

Suggested Extra Problems (not to be handed in):

- Textbook, Section 4.5 **1, 2, 4, 5, 8**
- Textbook, Section 4.6 **1-5, 10, 14, 16**
- Textbook, Section 5.3 **6(a), 10, 11, 12**
- A linear transformation T: $V \to V$ is said to be **orthogonal** iff for every $x \in V$, $||T(x)|| = ||x||$ (i.e. T is length-preserving).

(a) Prove that T is orthogonal iff $< T(x), T(y) >=< x, y >$ for all $x, y \in V$.
(**Hint**: $||x + y||^2 - ||x||^2 - ||y||^2 = 2 < x, y >$. You must show this though if you use it. Just expand the left hand side.)

(b) Prove T is orthogonal iff T maps an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ to an orthonormal basis $\{T(x_1), T(x_2), \ldots, T(x_n)\}$.

(c) Let $\alpha = \{x_1, x_2, \ldots, x_n\}$ be an arbitrary orthonormal basis for V. Prove that T is orthogonal iff $[T]_{\alpha\alpha}$ is an orthogonal matrix.

Hint: See Textbook, Section 4.6, **4, 5**.