Department of Mathematics, University of Toronto

MAT224H1S - Linear Algebra II

Winter 2013

Problem Set 3

- Due Tues. Feb 12, 6:10pm sharp . Late assignments will not be accepted - even if it's one minute late!
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Let S: $P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $S(p(x)) = xp(x)$, and T: $P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Consider the bases $\alpha = \{1, 1+x, 1+x+x^2\}$ for $P_2(\mathbb{R})$ and $\beta = \{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\}$ for $M_{2\times 2}(\mathbb{R})$.

(a) Find a formula for $TS(p(x))$.

(b) Find $[TS]_{\beta\alpha}$

(b) Use $[TS]_{\beta\alpha}$ to find a basis for the kernel of TS.

(c) Use $[TS]_{\beta\alpha}$ to find a basis for the image of TS.

2. Let $\alpha = \{(-3, 5, 2), (4, 1, 1), v_3\}$ and $\alpha' = \{v_1', (4, 0, -7), v_3'\}$ be bases for $\mathbb{R}^3$, and that the change of basis matrix from α to α' is

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix}.$$

Find v_3, v_1', and v_3'.

3. Suppose the linear transformation $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ has the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

relative to the standard bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$. Find bases α of $P_3(\mathbb{R})$ and β of $P_2(\mathbb{R})$ such the $[T]_{\beta\alpha}$ is the reduced row echelon form of A.

4. Let $T\colon \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$ be defined by

$$T(x, y, z) = (x + 2y, x + y, 2x + z).$$

Find $T^{-1}(x.y.z)$.

5. Let $W = span\{1 + x^2 + x^3, 1 + x + x^2, 3 + x + 3x^2 + 2x^3, -x + x^3\}$. Determine the dimension d of W and find (construct) an isomorphism $T\colon W \to \mathbb{R}^d$.

6(a) Let $T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = p(x) + p'(x)$. Show T is an isomorphism.

6(b) Let $T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = xp'(x)$. Explain why T is not an isomorphism.

6(c) Let $T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = cp(x) - xp'(x)$. For what values of $c \in \mathbb{R}$ is T is an isomorphism. Justify your answer.

7. Let V and W be vector spaces over a field F with bases $\alpha = \{v_1, v_2, \ldots, v_n\}$ and $\beta = \{w_1, w_2, \ldots, w_n\}$ respectively, and let $T\colon V \to W$ be a linear transformation. Prove that T is an isomorphism iff $[T]_{\beta\alpha}$ is an invertible matrix.

Suggested Extra Problems (not to be handed in):

- Textbook, Section 2.5 **8, 12**
- Textbook, Section 2.6 **1, 2, 6, 7, 8, 17**
- Textbook, Section 2.7 **1, 2, 3, 4, 9**
- Textbook, Chapter 2 Supplementary Exercises, **1, 2, 3, 4, 5, 9, 10, 12**