PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2011 EXAMINATIONS

CSC 165 H1F
Instructor(s): F. Pitt & T. Fairgrieve

Duration — 3 hours

PLEASE HAND IN

Examination Aids: *One* 8.5" × 11" sheet of paper, *hand*written on both sides.

Student Number: |_|_|_|_|_|_|_|_|_|

Family Name(s): ____________________

Given Name(s): ____________________

Lecture Section: ☐ L0101 (Tom Fairgrieve) ☐ L5101 (François Pitt)

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This final examination paper consists of 7 questions on 12 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy is complete and fill in the identification section above.*

Answer each question directly on the exam paper, in the space provided, and use the reverse side of the previous page for rough work. If you need more space for one of your solutions, use the reverse side of the previous page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

MARKING GUIDE

# 1:	______/ 8
# 2:	______/12
# 3:	______/20
# 4:	______/12
# 5:	______/ 8
# 6:	______/10
# 7:	______/10
BONUS MARKS:	______/ 4
TOTAL:	______/80

Good Luck!

Question 1. [8 marks]

Part (a) [3 marks]

The definition of the limit of a function, $\lim_{x \to a} f(x) = L$, can be expressed using quantifiers as

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Express the negation of this statement (*i.e.*, the definition of $\lim_{x \to a} f(x) \neq L$) by working the negation "in" — your final answer should *not* contain the negation symbol.

Part (b) [5 marks]

Suppose that variable x does *not* appear in Q. *Without using truth tables*, show that

$$(\neg \exists x, P(x)) \iff \big((\forall x, Q \Rightarrow \neg P(x)) \wedge (\forall x, \neg Q \Rightarrow \neg P(x))\big).$$

Question 2. [12 MARKS]

Consider the domain $D = \{\text{all test and exam questions}\}$, and the predicate symbols $E(x)$: "x is an exam question", $T(x)$: "x is a test question", $H(x, y)$: "x is harder than y", and $L(x)$: "x is long".

Using only these symbols (in addition to appropriate connectives and quantifiers), translate each sentence below. That is, give a natural English sentence that corresponds to each given symbolic sentence, and give a clear symbolic sentence that corresponds to each given English sentence.

Part (a) [2 MARKS]

Every test question is short. # (Assume every question is either "long" or "short".)

Part (b) [2 MARKS]

$\exists x \in D, E(x) \wedge L(x)$

Part (c) [2 MARKS]

Some test question is harder than every exam question.

Part (d) [2 MARKS]

$\neg \exists x \in D, E(x) \wedge \forall y \in D, T(y) \Rightarrow H(x, y)$

Part (e) [2 MARKS]

The easiest exam question is long.

Part (f) [2 MARKS]

$\forall x \in D, E(x) \wedge L(x) \Rightarrow \exists y \in D, T(y) \wedge \neg H(x, y)$

Question 3. [20 MARKS]

For each part of this question, we recommend that you write a *detailed, structured proof*. Part marks *will* be given for having correct elements of the proof structure.

Part (a) [5 MARKS]

Suppose that, for all $x \in \mathbb{R}$, we define $\lceil x \rceil$ as the single value that satisfies:

$$\lceil x \rceil \in \mathbb{Z} \wedge x \leqslant \lceil x \rceil < x + 1.$$

Write a detailed structured proof that $\forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \lceil x \rceil \leqslant y$.
(HINT: You may use the fact that $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m < n + 1 \Leftrightarrow m \leqslant n$.)

Question 3. (CONTINUED)

Part (b) [8 MARKS]

Prove or disprove the following statement:

If x is a real number such that $x^4 + 2x^2 - 2x < 0$, then $0 < x < 1$.

Question 3. (CONTINUED)

Part (c) [7 MARKS]

Prove or disprove the following statement:

$$\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, i \leqslant j \Rightarrow \mathcal{O}(n^i) \subseteq \mathcal{O}(n^j).$$

(Recall that $A \subseteq B$ means $\forall x, x \in A \Rightarrow x \in B$.)

Question 4. [12 MARKS]

For each part, put an "X" in the box next to **each** answer that applies. (No justification required!)

Part (a) [2 MARKS]

$\forall f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \forall g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, f \in \mathcal{O}(g) \Rightarrow \ldots$

- [ ] $\ldots g \in \mathcal{O}(f)$
- [ ] $\ldots f \in \Omega(g)$
- [ ] $\ldots g \in \Omega(f)$
- [ ] $\ldots f \in \Theta(g)$

Part (b) [2 MARKS]

$\exists f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \exists g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, f \notin \mathcal{O}(g) \wedge \ldots$

- [ ] $\ldots g \notin \mathcal{O}(f)$
- [ ] $\ldots f \in \Omega(g)$
- [ ] $\ldots g \in \mathcal{O}(f)$
- [ ] $\ldots g \in \Omega(f)$

Part (c) [2 MARKS]

$\forall f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \forall g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, f \cdot g \in \ldots$ (Where $\forall n \in \mathbb{N}, (f \cdot g)(n) = f(n) \cdot g(n)$.)

- [ ] $\ldots \mathcal{O}(g)$
- [ ] $\ldots \Theta(f)$
- [ ] $\ldots \mathcal{O}(f)$
- [ ] $\ldots \Omega(f)$

Part (d) [2 MARKS]

$3\sqrt{n} + 5 \in \ldots$

- [ ] $\ldots \mathcal{O}(n)$
- [ ] $\ldots \Theta(\sqrt{n})$
- [ ] $\ldots \Omega(n)$
- [ ] $\ldots \Omega(1)$

Part (e) [2 MARKS]

$\dfrac{5n \log n}{n+7} \in \ldots$

- [ ] $\ldots \mathcal{O}(\sqrt{n})$
- [ ] $\ldots \mathcal{O}(\log n)$
- [ ] $\ldots \mathcal{O}(n)$
- [ ] $\ldots \Omega(n)$

Part (f) [2 MARKS]

$4n^3 - 7n + 3 \in \ldots$

- [ ] $\ldots \mathcal{O}(n^2)$
- [ ] $\ldots \mathcal{O}(n^4)$
- [ ] $\ldots \mathcal{O}(n^3)$
- [ ] $\ldots \Omega(n)$

Question 5. [8 MARKS]

Give a bound on the worst-case running time of each algorithm below (write your answer in the space to the right of each algorithm). Make your bound as tight as possible, and show your work (to justify each answer briefly). *Do **not** write detailed proofs!*

Part (a) [2 MARKS]

```
# Precondition: L is a list and n = len(L).
for i = 0, 1, ..., n - 1:
    if L[i] < 0:
        L[i] = -L[i]
```

Part (b) [2 MARKS]

```
# Precondition: L is a list and n = len(L).
i = 0
while i < n:
    L[i] = L[i] - i
    i = i + 1
while i > 0:
    L[i] = L[i] * 2
    i = i - 2
```

Part (c) [2 MARKS]

```
# Precondition: L is a list and n = len(L).
if L[0] is even:
    for i = 0, 1, ..., n^2 - 1:
        L[0] = L[0] + i
else:
    for i = 0, 1, ..., n - 1:
        L[0] = L[0] - i
```

Part (d) [2 MARKS]

```
# Precondition: L is a list and n = len(L).
i = 0
while i < n:
    print L[i]
    i = i * 2
```

Question 6. [10 marks]

Give a tight bound on the worst-case running time of the following algorithm, and write a detailed proof that your bound is correct. (Hint: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.)

```
# Precondition: L is a list that contains n > 0 numbers.
value = 0
step = 1
index = 0
while index < len(L):
    value = L[index] − value
    index = index + step
    step = step + 1
```

Question 7. [10 MARKS]

A *binary sequence* is any sequence (finite or infinite) of 0's and 1's. For example, the sequences "111" and "101001000100001..." are binary sequences, and so is the empty sequence "" (also denoted ε). Let B denote the set of all binary sequences.

Part (a) [5 MARKS]

Prove that the following set is *countable*.

$$B_{<\infty} = \left\{ s \in B : s \text{ is a } \textbf{finite} \text{ binary sequence} \right\}$$

Question 7. (CONTINUED)

Part (b) [5 MARKS]

Prove that the following set is *uncountable*.

$$B_\infty = \{s \in B : s \text{ is an } \textbf{infinite} \text{ binary sequence}\}$$

Bonus. [4 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer. Please attempt this only *after* you have completed the rest of the Final Examination.

Prove that the union of countably many countable sets is countable.