Assignment 7 SOLUTIONS- MAT 327 - Summer 2014

Comprehension

[**C.1**] Determine which of the following sequences converge to $\langle 0,0,0,\ldots\rangle$ when $\mathbb{R}^{\mathbb{N}}$ is given (1) the product topology, (2) the uniform topology, (3) the box topology:

1. $\langle 1,1,1,1,\ldots\rangle$, $\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots\right\rangle$, $\left\langle \frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\ldots\right\rangle$, $\ldots$;
2. $\langle 1,1,1,1,\ldots\rangle$, $\langle 0,1,1,1,\ldots\rangle$, $\langle 0,0,1,1,\ldots\rangle$, $\ldots$;
3. $\langle 1,1,1,1,\ldots\rangle$, $\left\langle 0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots\right\rangle$, $\left\langle 0,0,\frac{1}{3},\frac{1}{3},\ldots\right\rangle$, $\ldots$;
4. $\langle 0,1,0,0,\ldots\rangle$, $\left\langle 0,\frac{1}{2},0,0,\ldots\right\rangle$, $\left\langle 0,\frac{1}{3},0,0,\ldots\right\rangle$, $\ldots$;

	Product	Uniform	Box
Sequence 1	YES	YES	NO
Sequence 2	YES	NO	NO
Sequence 3	YES	YES	NO
Sequence 4	YES	YES	YES

[**C.2**] Let X be a topological space, and let I be a non-empty set. Write down a natural subbasis for the product topology on X^I. Prove that it is a subbasis and that it generates the product topology.

Solution for C.2. In analogy to finite products,

$$\mathcal{S} := \{\pi_\alpha^{-1}(U) : \alpha \in I, U \text{ is open in } X\}$$

It is easy to see that by taking finite intersections of elements of $\mathcal{S}$ we get the standard basis that generates the product topology. It is also easy to see that this is indeed a subbasis. □

[**C.3**] Let X be a topological space, and let I, J be non-empty sets. Show that if $|I| = |J|$, then $X^I \cong X^J$, both with the product topology. Show that the converse is false.

Solution for C.3. Suppose that $b : I \longrightarrow J$ is a bijection. Define a function $F : X^I \longrightarrow X^J$ by sending any function $f \in X^I$ to the function $F(f) \in X^J$ given by $F(f)(j) = f(b^{-1}(j))$ for all $j \in J$. It is an easy exercise to show that F is a bijection. We now show that it is a continuous, open map, which will prove that it is a homeomorphism.

[Continuous] Let $S = \pi_j^{-1}(U)$ be a subbasic open set in X^J. Notice that

$$F^{-1}(S) = F^{-1}(\pi_j^{-1}(U)) = \pi_{b^{-1}(j)}^{-1}(U)$$

since every coordinate (except j) is the entire space. Showing that F is open is similar.

The converse need not be true, because $\mathbb{N}^2 \cong \mathbb{N}^3$ (since they are both countable, infinite discrete spaces), but $2 \neq 3$. $\square$

[**C.4**] With the product topology, is $\mathbb{N}^{\mathbb{N}}$ discrete? What about $\{0,1\}^{\mathbb{N}}$? What about $\mathbb{N}^{\{0,1\}}$? Are these spaces countable or uncountable?

Solution for C.4. $\mathbb{N}^{\mathbb{N}}$ (the collection of all functions from $\mathbb{N}$ to $\mathbb{N}$) and $\{0,1\}^{\mathbb{N}}$ (the collection of all functions from $\mathbb{N}$ to $\{0,1\}$) are uncountable, non-discrete spaces. The set $\mathbb{N}^{\{0,1\}}$ is the collection of all pairs of natural numbers, which is a countable discrete space.

Showing that $\{0,1\}^{\mathbb{N}}$ is non-discrete can be done by showing that each basic open set is infinite, which is obvious. $\square$

Definition. *A T_1 space $(X, \mathcal{T})$ is said to be **completely regular** if whenever $C \subseteq X$ is a closed set, and $p \in X \setminus C$ then there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(c) = 0$ ($\forall c \in C$) and $f(p) = 1$.*

[**C.5**] Let X be a T_1 space. Prove that if X is normal, then it is completely regular. Prove that if X is completely regular, then it is regular. (Both of those should be one or two line proofs). Conclude that calling "complete regularity" the $T_{3\frac{1}{2}}$ property, makes sense.

Solution for C.5. This is just Urysohn's lemma combined with the observation that in a T_1 space normal spaces are regular. Since we called normal spaces T_4 spaces and regular spaces T_3 spaces we see that

$$T_4 + T_1 \Rightarrow T_{3\frac{1}{2}} + T_1 \Rightarrow T_3 + T_1.$$

□

Application

[**A.1**] We saw in class that metrizability is a countably productive property. Prove that metrizability is *not* preserved under arbitrary products. (One approach is to show that $\mathbb{R}^{\mathbb{R}}$ is not first countable.)

Solution for A.1. The minimalist proof of this is to show that $\mathbb{R}^{\omega_1}$ is not first countable. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable collection of open sets each of which contain the constant zero function $z : \omega_1 \longrightarrow \mathbb{R}$. Without loss of generality, we may assume that each U_n is really a basic open set. Thus each U_n has only finitely many coordinates that are not all of $\mathbb{R}$. So there is an $\alpha_n \in \omega_1$ such that if $\beta > \alpha_n$ then $\pi_\beta(U_n) = \mathbb{R}$. By the nice property of ω_1 there is a $\gamma \in \omega_1$ such that $\alpha_n < \gamma$ for all $n \in \mathbb{N}$. Thus $\pi_\gamma(U_n) = \mathbb{R}$ for all $n \in \mathbb{N}$. Thus the open set $\pi_\gamma^{-1}((-1,1))$ does not contain any of the U_n. So $\mathcal{U}$ is not a local basis at z. □

[**A.2**] Let X be an infinite set, and we will see how to make $\mathcal{P}(X)$ into a topological space. We already know that $\{0,1\}^X$ can be given the product topology, and there is a natural correspondence between subsets of X and elements of $\{0,1\}^X$. Identify $A \subseteq X$ with its characteristic function χ_A, which is defined by $\chi_A(x) = 1$ iff $x \in A$. Check that this is a bijection, and then apply C.1 (from Assignment 1!) to get a topology on $\mathcal{P}(X)$. Write down a natural basis for this topology on $\mathcal{P}(X)$. Explain what it means for a sequence of subsets of $\mathbb{N}$, $\langle A_n \rangle$ to converge to a subset A, when $\mathcal{P}(\mathbb{N})$ is given this topology. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. In this topology is $\mathcal{U}$ a closed subset of $\mathcal{P}(\mathbb{N})$?

Solution for A.2. The bijection part is an easy exercise. A natural basis for this topology is

$$\mathcal{B} := \{B_{A,B} : A, B \subseteq X, |A|, |B| < \infty\}$$

where

$$B_{A,B} := \{Y \subseteq \mathbb{N} : A \subseteq Y \subseteq X \setminus B\}$$

If $X = \mathbb{N}$, a sequence $\langle A_n \rangle \longrightarrow A$ provided that the A_n contain longer and longer initial segments of A.

Any principal ultrafilter $\mathcal{U}_n := \{A \subseteq \mathbb{N} : n \in A\}$ is a closed subset of $\mathcal{P}(\mathbb{N})$. Thinking about $\mathcal{P}(\mathbb{N})$ as $\{0,1\}^{\mathbb{N}}$ we see that $\mathcal{U}_n$ can be identified with $\pi_n^{-1}(1)$ which is clearly a closed set. If $\mathcal{U}$ is a non-principal ultrafilter then it contains each set $A_n := [n, +\infty)$ and the sequence $\langle A_n \rangle$ clearly converges to $\emptyset \notin \mathcal{U}$, so it is not closed. □

[**A.3**] Let's expand upon C.5. prove directly (without using Urysohn's Lemma) that every metrizable space is completely regular. Use the idea of the distance from a point to a closed set: $d(C, x) := \inf_{c \in C} d(c, x)$. (You may need to assume that the metric that generates your metrizable space is bounded.) Does this argument *easily* extend to showing that every metrizable space is "completely normal", (where that has the obvious definition)?

Solution for A.3. This appears as Theorem 5.18 in Patty's Foundations of Topology.

Let (X, d) be a metric space, let $C \subseteq X$ be closed and let $p \in X \setminus C$. Define $f : X \longrightarrow [0,1]$ by:

$$f(x) = \min\{\frac{d(x,C)}{d(p,C)}, 1\}$$

We notice that $f(p) = 1$ and $f(x) = 0$ for all $x \in C$. All that remains to show is that f is continuous. In fact by the pasting lemma, it is enough to show that the function $g(x) := d(x, C)$ is continuous.

Let $B_\epsilon(g(x))$ be an open ball around $g(x)$ note that if $y \in B_\epsilon(x)$, then

$$d(y, C) \leq d(y, c) \leq d(y, x) + d(x, c) \leq \epsilon + d(x, c)$$

for all $c \in C$. Thus

$$d(y, C) \leq d(x, C) + \epsilon,$$

and hence $g(y) = d(y, C) \in B_\epsilon(g(x))$.

Note that *this* argument does not easily extend to showing that every metrizable space is "completely normal". The difficulty is that two disjoint closed sets could have distance 0, so we have to be more careful. □

New Ideas

[**NI.1**] The following is an open question: "Is $\mathbb{R}^{\mathbb{N}}$ with the box topology a normal space?". Investigate this problem and provide some evidence for why this problem is difficult. "Investigate" is a word open to interpretation, but I want to be convinced that you understand the problem, why it's difficult, and what are some possible attacks against the problem. Perhaps you want to give me some interesting pairs of disjoint closed sets? It's up to you! (Also, if you do happen to solve this problem, then you will get an A+ in this course.)

[**NI.2**] List the topological properties (of the ones we've studied so far) of $\mathbb{N}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$ (with the product topologies). Prove that they are both separable by exhibiting explicit countable dense sets. Find subspaces $A, B \subseteq \mathbb{R}_{\text{usual}}$ such that $A \cong \mathbb{N}^{\mathbb{N}}$ and $B \cong \{0,1\}^{\mathbb{N}}$.

Idea. We remark that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set. This isn't so tricky to see, because each $f \in \{0,1\}^{\mathbb{N}}$ is really just a string of zeros and ones which correspond to "going left or right" in the Cantor Set. A countable dense set here is the set of all functions f that are eventually zero, that is, for some $N \in \mathbb{N}$, $f(n) = 0$ for $n \geq N$. (This is countable because it is in bijection with the family of finite subsets of $\mathbb{N}$ which we know is countable.)

Now $\mathbb{N}^{\mathbb{N}}$ is a bit trickier. A countable dense subset is given by all functions that are eventually zero. (This set is countable because it is in bijection with $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$). It turns out that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the irrationals, $\mathbb{R} \setminus \mathbb{Q}$! One method to show this is to use the idea of continued fractions. You can read all about it on Wikipedia which provides a great description of them. $\square$

[**NI.3**] State and prove the two common versions of the Tietze extension theorem. Find (or invent!) a question that uses the Tietze extension theorem that could show up on a problem set. Find an application for the Tietze Extension Theorem or Urysohn's Lemma.