Part A: (3 marks) Give the statement of completeness axiom for $\mathbb{R}$ together with the ϵ-characterization of the lub of a set S.

Any Subset $S \subset \mathbb{R}$ which is non-empty & bdd above has a lub.

$u = lub(S)$ if $\forall s \in S$ $s \le u$ and $\forall \epsilon > 0$ $\exists s \in S$ st. $u - \epsilon < s \le u$.

Part B: (3 marks) first determine the lub of the set $S = (1, 2) \cup \{3\}$, and then use the ϵ-characterization of lub to prove your answer.

$lub(S) = 3$. proof given $\epsilon > 0$ $3 - \epsilon < 3 \le 3$ & $3 \in S$.

Part C: (4 marks) Use part (A) to prove any monotone incresasing sequence that is bounded above has a limit.

Let $\{x_n\} = S$, Since S is bdd above The $l = lub(S)$ exists. We show $\lim x_n = l$. proof: given $\epsilon > 0$ $\exists$ $x_N \in S$ st $l - \epsilon < x_N \le l$.

Since $\{x_n\}$ is inc Then $\forall n$ $n > N \Rightarrow x_N < x_n$ so That

$l - \epsilon < x_n \le l$ or $x_n \in B(\epsilon, l)$.

MAT 237, Quiz 2 Name Marking Scheme TUT0201

Part A: (2 marks) Give the statement of the Monotone sequence theorem.

Any monotone increasing sequence which is bounded above must have limit (That is, converging.)

Part B: (4 marks) Apply Monotone sequence theorem to show that the sequence to prove that the limit of the sequence $\{2 - \frac{1}{n} : n = 1, 2, 3, \dots\}$ exists. What is it? Use the completeness and lub ideas to justify your answer.

$\lim 2 - \frac{1}{n} = 2$ b/c The sequence $\{2 - \frac{1}{n}\}$ is increasing and bounded above (by 2). $2 = lub(S)$ b/c $\forall \epsilon > 0$ $2 - \epsilon < 2 - \frac{1}{N} < 2$ where $N > \frac{1}{\epsilon}$

Part C: (4 marks) Consider the sequence of the intervals $I_n = [a_n, b_n]$ such that $I_{n+1} \subset I_n$ and that $(b_n - a_n) \rightarrow 0$. Let l_1 be the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ and l_2 be the limit of the the sequence $\{b_n\}_{n=1}^{\infty}$. Prove that $l_1 = l_2$.

We know $l_1 \leq l_2$ b/c $l_1 = lub(\{a_n\})$ and $l_2 = lub(\{b_n\})$ and all of a_n are less than all of b_m : i.e $a_n \leq l_1 \leq l_2 \leq b_m$.

Now if $l_2 - l_1 > 0$ Then $\forall n$ $b_n - a_n > l_2 - l_1$ b/c but since $b_n - a_n \to 0$ and $l_2 - l_1$ is fixed Then $l_2 - l_1 = 0$.

MAT 237, Quiz 2 Name Marking Scheme TUT0301

Part A: (2 marks) Present the ϵ-δ definition for a function $f : S \longrightarrow \mathbb{R}^k$ to not to be continuous at a point a. $\in S$ (S is a subset of $\mathbb{R}^n$.)

$\exists \epsilon > 0 \quad \forall \delta > 0 \quad \exists x \quad |x - a| < \delta$ but $|f(x) - f(a)| \geq \epsilon$

Part B: (4 marks) Use your definition as in (A) to show that the greatest integer function or step function $f(x) = [x]$ defined on $[0, 2)$ is not continuous at 1.

let $\epsilon = 1$ given $\delta > 0$ let $x = 1 - \frac{\delta}{2}$. Note $|x - 1| = \frac{\delta}{2} < \delta$ but

$f(x) = f(1 - \frac{\delta}{2}) = 0$ while $f(1) = 1$ so $|0 - 1| \geq 1 = \epsilon$.

Part C: (4 marks) Assume $f : S \longrightarrow \mathbb{R}^k$ is not continuous at the point $\boldsymbol{a} \in S$. Use the definition in (A) to construct a sequence $\{\boldsymbol{x}_n\}$ in S which converges to $\boldsymbol{a}$ but $\{f(\boldsymbol{x}_n)\}$ does not converge to $f(\boldsymbol{a})$.

$\exists \epsilon > 0 \quad \forall \delta > 0 \quad \exists x \in S$ st. $|x - a| < \delta$ but $|f(x) - f(a)| \geq \epsilon$

fix $\epsilon > 0$ and let $\delta = 1$, $\exists x_1 \in S$ st $|x_1 - a| < 1$ but $|f(x_1) - f(a)| \geq \epsilon$

$\vdots$

$\delta = \frac{1}{n}$, $\exists x_n \in S$ st $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \epsilon$

$\vdots$

so $\{x_n\} \to a$ but $|f(x_n) - f(a)| \geq \epsilon$ i.e $f(x_n) \not\to f(a)$.

MAT 237, Quiz 2 Name Marking Scheme TUT0401

Part A: (3 marks) Present the ϵ definition for convergence of a sequence $\{x_n\}$ to a point a; also present the ϵ characterization of $a \in \overline{S}$.

$x_n \to a$ if $\forall \epsilon > 0 \ \exists N \ \forall n \ n > N \Rightarrow x_n \in B(\epsilon, a)$

$a \in \overline{S}$ if $\forall \epsilon > 0 \ B(\epsilon, a) \cap S \neq \emptyset$

Part B: (3 marks) Since 1 and 2 both belong to the closure of the set $S = (0, 1) \cup \{2\}$, by theorem 1.14 there must be sequences converging to these two points. Construct two such sequences.

$2 \in \overline{S}$ and The Sequence $x_n = 2 \ \forall n$ Converges to 2.

$1 \in \overline{S}$ and The Sequence $x_n = 1 - \frac{1}{n}$ Converges to 1.

Part C: (4 marks) Prove, using the two definitions in part A, that if $a \in \overline{S}$ then there is a sequence $\{x_n\}$ in S that converges to a.

if $a \in S$ Then let $x_n = a \ \forall n$.

if $a \in \overline{S} \setminus S$ Then $\forall \epsilon > 0 \ \exists x \in S \cap B(\epsilon, a)$.

let $\epsilon = 1 \quad \exists x_1 \in S \cap B(1, a)$

" $\epsilon = \frac{1}{2} \quad \exists x_2 \in S \cap B(\frac{1}{2}, a)$

$\vdots$

$\epsilon = \frac{1}{n}$, Then $x_n \in S$ s.t. $|x_n - a| < \frac{1}{n}$

$\vdots$

$\{x_n\} \to a$ b/c $\forall \epsilon > 0$ let $N > \frac{1}{\epsilon}$ Then $\forall n > N \quad \frac{1}{n} < \frac{1}{N} < \epsilon$

& $|x_n - a| < \frac{1}{n} < \epsilon$.

$\subset S$

MAT 237, Quiz 2 Name Marking Scheme TUT5101

Part A: (3 marks) Present the ϵ-δ definition of continuity of a function $f : S \longrightarrow \mathbb{R}^k$ at a point a. (S is a subset of $\mathbb{R}^n$.) Also give the definition of $\{\boldsymbol{x}_n\}$ converges to a point $\boldsymbol{a}$.

$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \quad |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$

$\forall \delta > 0 \quad \exists K \quad \forall k \quad k > K \implies |x_k - a| < \delta$

Part B: (3 marks) Use part A to show the limit of the sequence $\{\frac{3k+4}{k-5}\}$ is 3.

given $\epsilon > 0$ let $K > 5 + \frac{9}{\epsilon}$. Then $\forall k \quad k > K \implies k - 5 > K - 5 \implies$

So that $K - 5 > \frac{9}{\epsilon}$

and $\frac{9}{K-5} < \epsilon$

$\left|\frac{3k+4}{k-5} - 3\right| = \frac{9}{k-5} < \frac{9}{K-5} < \epsilon$

Part C: (4 marks) Assume $fS \longrightarrow \mathbb{R}^k$ is continuous at the point $\boldsymbol{a} \in S$. Assume that a sequence $\{\boldsymbol{x}_n\}$ in S converges to $\boldsymbol{a}$. Prove that the sequence $\{f(\boldsymbol{x}_n)\}$ also converge to $f(\boldsymbol{a})$.

given $\epsilon > 0$ find $\delta > 0$ s.t. $\forall x \quad |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$ (1)

by cont of f at a

use δ in def of sequences to find a K s.t. $\forall k > K \quad |x_k - a| < \delta$.

together with (1) $|x_k - a| < \delta$ implies $|f(x_k) - f(a)| < \epsilon$

So that $\forall \epsilon > 0 \quad \exists K$ s.t. $\forall k > K \quad |f(x_k) - f(a)| < \epsilon$.

MAT 237, Quiz 2 Name Marking Scheme TUT5201

Part A: (3 marks) Present the sequential equivalent of $a \in \overline{S}$ (Theorem 1.14)

Thm 1.14 Suppose $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x} \in \overline{S}$ iff There is a sequence $\{\mathbf{x}_n\} \subset S$ That Converges to $\mathbf{x}$.

Part B: (4 marks) Determine the closure of the set $S = \{(-1)^n\}_{n=1}^{\infty}$ and then find sequences in S that converge to the points in $\overline{S}$.

$S = \{1, -1\}$ so $\overline{S} = \{1, -1\}$ and The Sequence $x_n = 1$ $\forall n$ Converges to 1

and The Sequence $y_n = -1$ $\forall n$ Converges to -1

Part C: (3 marks) Use part A to show that if the point a is in the closure of the set $S = \{x_n\}_{n=1}^{\infty}$, then there is a subsequence of $\{x_n\}$ that converges to a.

if $a \in \overline{\{x_n\}}$ Then there is a Sequence in $S = \{x_n\}$ That Converges to a (by 1.14). But a Sequence in S will be a sub-sequence of $\{x_k\}$ say $\{x_{k_j}\}_{j=1}^{\infty}$.

Part A: (3 marks) Present the ϵ-δ definition for 'f is continuous at a'. Then give this definition in terms of the open balls and the inverse image of the function f.

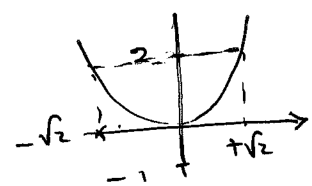
f is Cont at a if $\forall \epsilon>0\ \exists \delta>0\ \forall x\ |x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon$

or $\forall \epsilon>0\ \exists \delta>0\ B(\delta,a)\subset f^{-1}(B(\epsilon,f(a)))$

Part B: (3 marks) For the function $f(x)=x^2$, determine the inverse image $f^{-1}(U)$ for $U=(1,2)$. Repeat with the open interval $U=(-1,2)$ and $U=[0,2)$.

$f^{-1}(U)=\{x\in\mathbb{R} : f(x)\in U\}$. $U=(1,2)$ $f^{-1}((1,2))=(-\sqrt{2},-1)\cup(1,\sqrt{2})$

$f^{-1}((-1,2))=(-\sqrt{2},\sqrt{2})$ $f^{-1}([0,2))=(-\sqrt{2},\sqrt{2})$



Part C: (4 marks) Suppose $f:\mathbb{R}^n\longrightarrow\mathbb{R}^k$ is continuous and U is an open subset of $\mathbb{R}^k$. Prove that the set $S=\{\boldsymbol{x}\in\mathbb{R}^n : f(\boldsymbol{x})\in U\}$ is also open.

Show $S\subset S^{int}$. Pick $a\in S$, $\exists\, b\in U$ st. $f(a)=b$. f is Cont. at a, so $\forall\epsilon>0\ \exists\delta>0\ B(\delta,a)\subset f^{-1}(B(\epsilon,b))$. (*)

now Since U is open $\exists\,\epsilon>0$ st. $B(\epsilon,b)\subset U$, so $f^{-1}(B(\epsilon,b))\subset S$

$\therefore$ by (*) $\exists\,\delta>0$ st $B(\delta,a)\subset f^{-1}(B(\epsilon,b))\subset S$ so $a\in S^{int}$.