TORONTO *LIFE SCIENCES*

Concept Booklet Solutions: PART1

DEC

Term
TEST 2

2008

CONCEPT BOOKLET

Your Key to Success

Our Instructor: **James T. Nguyen**, B.Sc., M.Sc., CFA (IPR)., MBA (IPR)., DDS (IPR)

SOLUTIONS: PART 1

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TERM TEST 2
Study Package Solution's
(PART 1)

ex1) $f(x) = (1 + x + x^4 + x^7 + x^{10})x^{40}$

$f(x) = x^{40} + x^{41} + x^{44} + x^{47} + x^{50}$

* Given what we know, we only consider

$$f(x) = x^{50}$$

$$f'(x) = 50x^{49}$$

$$f''(x) = 50.49x^{48}$$

$$\vdots$$

$$f^{(48)}(x) = 50 \cdot 49 \cdot 48 \cdot \dots \cdot 3x^2$$

$$f^{(48)}(2\sqrt{2}) = 50 \cdot 49 \cdot 48 \cdots 3(2\sqrt{2})^2$$

$$= (50!)4 \longleftarrow \text{answer b}$$

ex2) $f(x) = \dfrac{x^{46} + x^{45} + 2}{1 + x}$, $f^{(46)}(3) = ?$

<u>note</u>: there are different ways to do this.

$$f(x) = \frac{x^{45}(x+1)}{x+1} + \frac{2}{1+x}$$

$$f(x) = x^{45} + 2(1+x)^{-1}$$

answer: $(46!)2^{-93}$

3) $f(x) = \dfrac{x^{65}}{x-1}$, $f^{(63)}(2) = ?$

Complete this by long divison.

$$f(x) = \frac{x^{65}}{x-1} = x^{64} + x^{63} + \dots + x + \frac{1}{x-1}$$

$$f(x) = x^{64} + x^{63} + \dots + x + (x-1)^{-1}$$

Answer : $2(64!)$

4) $f(x) = \dfrac{1}{\underbrace{1+3x+3x^2+x^3}} = \dfrac{1}{(1+x)^3} = (1+x)^{-3}$

knowing Binomial theorem here will make it easier, but not necessary. Can factor the denominator to show it equals the following

$$f'(x) = -3(1+x)^{-4}$$

$$f''(x) = 3 \cdot 4(1+x)^{-5}$$

$$\vdots$$

$$f^{62}(x) = 3 \cdot 4 \cdot 5 \cdots 64(1+x)^{-65}$$

$$f^{62}(x) = \underbrace{\frac{1}{2} \cdot 2} \cdot 3 \cdot 4 \cdot 5 \cdots 64$$

$$f^{(62)}(0) = \frac{1}{2}(64!)$$

ex5) $f(x) = \cos 2x \underbrace{\sin x \cos x}$, $f^{(55)}\left(\frac{\pi}{12}\right) = ?$

note: $f(x) = \frac{1}{2} \, 2\underbrace{\sin x \cos x}\cos 2x$

$= \frac{1}{2}\sin 2x \cos 2x = \frac{1}{2} \cdot \frac{1}{2} 2 \sin 2x \cos 2x$

$f(x) = \frac{1}{4}\sin 4x$

Answer: $f^{55}\left(\frac{\pi}{12}\right) = -2^{107}$

ex6) Answer: $f^{(75)}\left(\frac{\pi}{12}\right) = -2^{73}$

ex8) $f(x) = \sin^4 x - \cos^4 x$

$f(x) = \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)$

$f(x) = \sin^2 - \cos^2 x$

$= -\left(\cos^2 x - \sin^2 x\right)$

$f(x) = -\cos 2x$

ex9) $f(x) = \ln\left(x^4 - 2x^2 + 1\right) - \ln\left(x^2 - 2x + 1\right)$

$f(x) = \ln\left[\frac{x^4 - 2x^2 + 1}{x^2 - 2x + 1}\right]$

$f(x) = \ln\left[\frac{\left(x^2 - 1\right)^2}{(x-1)^2}\right] = \ln\left[\frac{(x^2-1)^2}{(x-1)^2}\right]$

$$f(x) = \ln\left[\frac{(x-1)^2(x+1)^2}{(x-1)^2}\right]$$

$$f(x) = \ln(x+1)^2 = 2\ln(x+1)$$

$$\Rightarrow \quad f'(x) = 2 \cdot \frac{1}{1+x}$$

$$\Rightarrow f'(x) = 2(1+x)^{-1}$$ (now you should be able to finish this question)

L'Hospital's Rule:

ex1) $$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{e^x - 1}{2x} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{e^x}{2} = 1/2$$

ex2) $$\lim_{x \to 0} \frac{x + \sin x}{4x^3 - x} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{1 + \cos x}{12x^2 - 1} = -2$$

ex3) $\lim_{x\to 0} \frac{\sin(3x)}{5x^3-4x} \quad \left(\frac{0}{0}\right)$

$\overset{H}{=} \lim_{x\to 0} \frac{3\cos(3x)}{15x^2-4} = \frac{3}{-4}$

ex4) $\lim_{x\to\infty} (e^x+x)^{\frac{1}{x}}$ (of the form ∞^0)

let $y = \lim_{x\to\infty} (e^x+x)^{\frac{1}{x}}$

$\ln y = \lim_{x\to\infty} \frac{1}{x}\ln(e^x+x)$

$\ln y = \lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \quad \left(\frac{\infty}{\infty}\right)$

$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x+1)}{1}$

$= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} = \lim_{x\to\infty} \frac{1+\frac{1}{e^x}}{1+\frac{x}{e^x}} = 1$

(or use L'Hôp Rule one more time)

Now, $\ln y = 1 \Rightarrow e^{\ln y} = e^1 \Rightarrow \boxed{y=e}$

ex5) Answer: $\ln\left(\frac{a}{b}\right)$

ex6) Given $f(1) = 0$ and $f'(1) = 3$

$$\lim_{x \to 0} \frac{f(1+2x)}{x} \quad \left(\frac{f(1)}{0} = \frac{0}{0} \right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{f'(1+2x) \cdot 2}{1} = f'(1) \cdot 2 = 3 \cdot 2 = 6 \,/\!/$$

ex7)

$$\lim_{x \to 8} \frac{f(x) - f(8)}{x^{1/3} - 2} \quad \left(\text{of the form } \frac{f(8) - f(8)}{0} = \frac{0}{0} \right)$$

$$\overset{H}{=} \lim_{x \to 8} \frac{f'(x)}{\frac{1}{3} x^{-2/3}} = \frac{f'(8)}{\frac{1}{3}(8)^{-2/3}} = \frac{2}{\frac{1}{3}(64)^{-1/3}}$$

$$= 2 \cdot 3 \cdot 4$$

$$= 24$$

ex 10)

$$\lim_{x \to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{2x^2 + 3x^3 + 4x^4} \quad \left(\frac{0}{0} \right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{2ax + b\cos bx + c\cos cx + d\cos dx}{4x + 9x^2 + 16x^3} \quad \left(\frac{b+c+d}{0} \right)$$

In order for the limit to work we need the form $\frac{0}{0}$, therefore

$$\boxed{b + c + d = 0}$$

now,
$$\overset{H}{=} \lim_{x\to 0} \frac{2a - b^2 \sin bx - c^2 \sin cx - d^2 \sin dx}{4 + 18x + 48x}$$

$$= \frac{2a}{4} \Rightarrow \frac{2a}{4} = 3 \Rightarrow \boxed{a = 6}$$

$$\therefore a+b+c+d = 6+0 = 6 //$$

ex 11) $\lim_{x\to\infty} \left(x - x^2 \ln\left(\frac{1+x}{x}\right)\right)$ "need to rewrite" since $\infty - \infty$

$$= \lim_{x\to\infty} x\left[1 - x\ln\left(\frac{1+x}{x}\right)\right]$$

$$= \lim_{x\to\infty} x\left[1 - x\ln\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x\to\infty} \frac{1 - x\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

let $u = \frac{1}{x}$ since $x\to\infty \Rightarrow u\to 0$ and $x = \frac{1}{u}$

$$= \lim_{u\to 0} \frac{1 - \frac{1}{u}\ln(1+u)}{u}$$

$$= \lim_{u\to 0} \frac{u - \ln(1+u)}{u^2} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{u\to 0} \frac{1 - \frac{1}{1+u}}{2u} \quad \left(\frac{0}{0}\right.$$

$$\stackrel{H}{=} \lim_{u \to 0} \frac{\frac{1}{(1+u)^2}}{2} = \lim_{u \to 0} \frac{1}{2(1+u)^2} = 1/2$$

ex 12) $\lim_{x \to \infty} \{(x^5 + 15x^4 + 12x^3 + 9x^2 + 6x + 1)^{1/5} - x\}$

Clearly, this of the form $\infty - \infty$

$$= \lim_{x \to \infty} \left\{ \left[x^5\left(1 + \frac{15}{x} + \frac{12}{x^2} + \frac{9}{x^3} + \frac{6}{x^4} + \frac{1}{x^5}\right)\right]^{1/5} - x \right\}$$

$$= \lim_{x \to \infty} x \left[\left(1 + \frac{15}{x} + \frac{12}{x^2} + \frac{9}{x^3} + \frac{6}{x^4} + \frac{1}{x^5}\right)^{1/5} - 1 \right]$$

$$= \lim_{x \to \infty} \frac{\left(1 + \frac{15}{x} + \frac{12}{x^2} + \frac{9}{x^3} + \frac{6}{x^4} + \frac{1}{x^5}\right)^{1/5} - 1}{1/x}$$

* now it is in the form of $\left(\frac{0}{0}\right)$

$$\stackrel{H}{=} \lim_{x \to \infty} \frac{\frac{1}{5}\left(1 + \frac{15}{x} + \frac{12}{x^2} + \frac{9}{x^3} + \frac{6}{x^4} + \frac{1}{x^5}\right)^{-4/5} \cdot \left(\frac{-15}{x^2} - \frac{24}{x^3} - \frac{27}{x^4} - \frac{24}{x^5} - \frac{5}{x^6}\right)}{x^2}$$

$$= \lim_{x\to\infty} \frac{1}{5}\left(1+\frac{15}{x}+\frac{12}{x^2}+\frac{9}{x^3}+\frac{6}{x^4}+\frac{1}{x^5}\right)^{-4/5}\cdot\left(15+\frac{24}{x}+\frac{27}{x^2}+\frac{24}{x^3}+\frac{5}{x^4}\right)$$

$$= \frac{15}{5} = 3$$

ex15) $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$ note: "many" ways to do this question

$$= \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos^2 x}{\sin^2 x}\right)$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{2\sin x\cos x - 2x\cos^2 x + 2x^2\sin x\cos x}{2x\sin^2 x + x^2\cdot 2\sin x\cos x}$$

$$= \lim_{x\to 0} \frac{2\sin x - 2x\cos x + 2x^2\sin x}{2x\sin^2 x + x^2\sin 2x} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{2\cos x - 2\cos x + 2x\sin x + 4x\sin x + 2x^2\cos x}{2\sin^2 x + 2x\cdot 2\sin x\cos x + 2x\sin 2x + 2x^2\cos 2x}$$

$$= \lim_{x\to 0} \frac{6x\sin x + 2x^2\cos x}{2\sin^2 x + 4x\sin 2x + 2x^2\cos 2x} \quad \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{6\sin x + 6x\cos x + 4x\cos x - 2x^2\sin x}{2\cdot 2\sin x\cos x + 4\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin 2x}$$

$$= \lim_{x \to 0} \frac{6\sin x + 10x\cos x - 2x^2\sin x}{2\sin 2x + 4\sin 2x + 12x\cos 2x - 4x^2\sin 2x}$$

$$= \lim_{x \to 0} \frac{6 \cdot \frac{\sin x}{x} + 10\cos x - 2x\sin x}{2\frac{\sin 2x}{x} + \frac{4\sin 2x}{x} + 12\cos 2x - 4x\sin 2x}$$

$$= \frac{16}{24} = 2/3$$

ex16) We begin the solution for this question:

$$f(x) = \ln\left(\frac{x^4}{\sin^2(x^2)}\right), \quad g(x) = x^4\cos(8x)$$

$$\lim_{x \to 0^-} \frac{f(x)}{g(x)} = \lim_{x \to 0^-} \frac{\ln\left(\frac{x^4}{\sin^2(x^2)}\right)}{x^4\cos(8x)}$$

$$= \lim_{x \to 0^-} \frac{\ln\left(\frac{x^2}{\sin x^2}\right)^2}{x^4}$$

$$= 2\lim_{x \to 0^-} \frac{\ln\left(\frac{x^2}{\sin x^2}\right)}{x^4} \quad \left(\text{of the form } \frac{0}{0}\right)$$

⋮ continue on using L'Hop Rule

18) $\lim_{x\to\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$ (we begin the solution)

$$\ln y = \lim_{x\to\infty} \ln\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{\pi}{2}-\arctan x\right)}{\ln x} \quad \left(\frac{-\infty}{\infty}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\dfrac{1}{\left(\frac{\pi}{2}-\arctan x\right)} \cdot \dfrac{-1}{1+x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{-x}{\left(\frac{\pi}{2}-\arctan x\right)\left[\frac{1}{x}+x\right]}$$

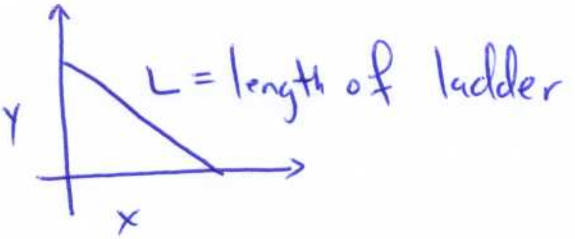
$$= -\lim_{x\to\infty} \frac{-1}{\left(\frac{\pi}{2}-\arctan x\right)\left[\frac{1}{x}+x\right]}$$

⋮ try to form $\left(\frac{0}{0}\right)$ at this point

Solution: $\frac{1}{e}$

§ Related Rates:

ex1)



$\frac{dy}{dt} = -1/4$

at $x = 6$, $\frac{dx}{dt} = 1/3$

note: $x^2 + y^2 = L^2$

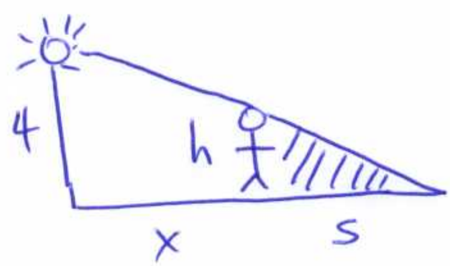
$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow 6 \cdot \frac{1}{3} + y\left(-\frac{1}{4}\right) = 0 \Rightarrow \boxed{y = 8}$$

at $x = 6$: $6^2 + 8^2 = L^2 \Rightarrow 100 = L^2 \Rightarrow \boxed{L = 10}$

ex2)



$\frac{dx}{dt} = \frac{3}{5}$, at $x = 5$

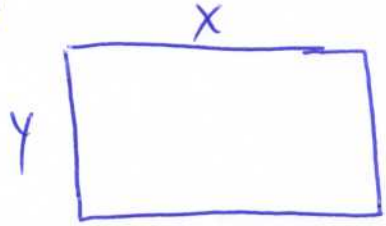
$\frac{ds}{dt} = \frac{4}{3}$

$$\frac{4}{x+s} = \frac{h}{s} \Rightarrow 4s = h(x+s)$$

$$\Rightarrow 4\frac{ds}{dt} = h\frac{dx}{dt} + h\frac{ds}{dt}$$

$$4\left(\frac{4}{3}\right) = h\left(\frac{3}{5}\right) + h\left(\frac{4}{3}\right)$$

now solve for h: $h = \frac{80}{29}$ meters ex3)



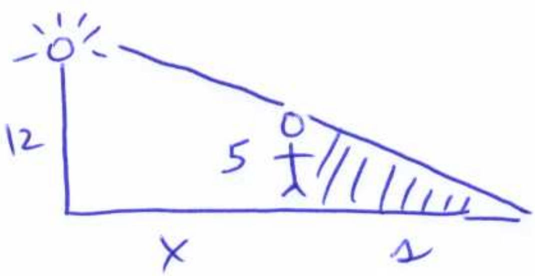
$\frac{dy}{dt} = 2 \text{ m/s}$ at $y = 5$

$\frac{dx}{dt} = 1 \text{ m/s}$ and $x = 3$

$$A = xy \Rightarrow \frac{dA}{dt} = y \cdot \frac{dx}{dt} + x\frac{dy}{dt}$$

$$\frac{dA}{dt} = 5(1) + (2)\cdot 3 = 11 \;//$$

ex4)



$\frac{dx}{dt} = 4$ at $x = 20$?

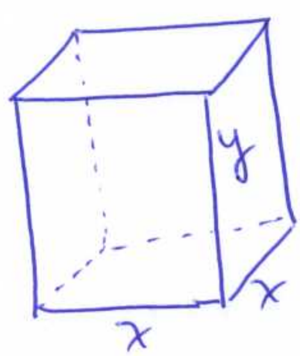
$\frac{ds}{dt} = ?$

know: $\frac{12}{x+s} = \frac{5}{s} \Rightarrow 12s = 5x + 5s$

$$12\frac{ds}{dt} = 5\frac{dx}{dt} + 5\frac{ds}{dt} \Bigg\} \; 7\frac{ds}{dt} = 5 \cdot 4$$

$$\frac{ds}{dt} = \frac{20}{7} \;//$$

ex5)



$\frac{dx}{dt} = 2, \; \frac{dy}{dt} = 3$

at $y = 4$ and $x = 5$, $\frac{dV}{dt} = ?$

$$V = x^2 y$$

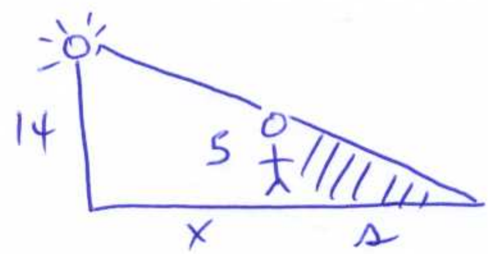
$$\frac{dV}{dt} = 2x\frac{dx}{dt}y + x^2\frac{dy}{dt}$$

$= 2(5)\cdot 2\cdot 4 + 25\cdot(3) = 155$ //

ex 6) $\frac{dA}{dt} = 23$ //

ex 7) $\frac{dA}{dt} = 40$ sq cm/sec

ex 9)



$\frac{dx}{dt} = 3$

$\frac{ds}{dt} = ?$ at $x = 25$

$$\frac{14}{x+s} = \frac{5}{s} \Rightarrow 14s = 5x + 5s$$

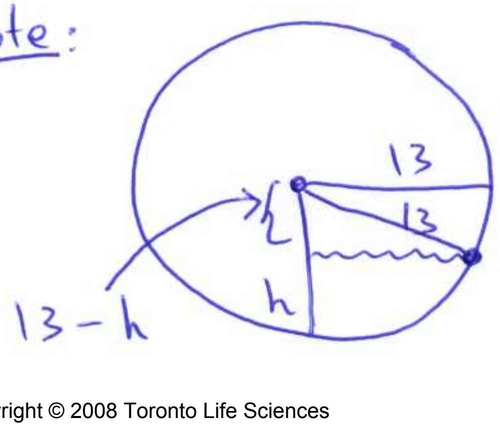
$$\Rightarrow 9s = 5x$$

$$9\frac{ds}{dt} = 5\frac{dx}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{5}{9}(3) = 5/3$$

ex 10) James Will take up this question:

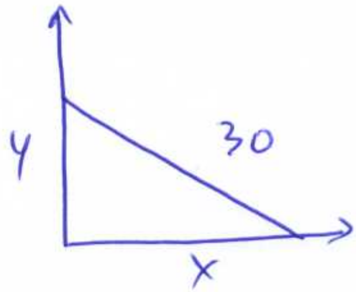
Note:



$\frac{dh}{dt} = -2$

$(13 - h)^2 + r^2 = 169$

ex11)



$$\frac{dy}{dt} = -3$$

$$x^2 + y^2 = 30^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{-y\frac{dy}{dt}}{x} = \frac{3y}{x}$$

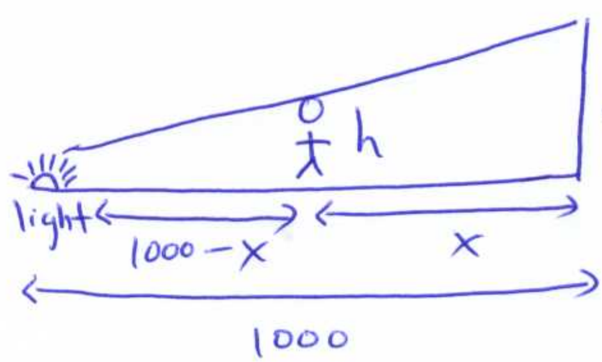
$$\Rightarrow \frac{d^2x}{dt^2} = \frac{x\cdot 3\frac{dy}{dt} - 3y}{x^2}$$

at $x = 30$, $y = 0$

i.e, top of ladder hit's the ground.

$$\frac{d^2x}{dt^2} = \frac{30\cdot 3(-3)}{(30)^2} = \frac{90(-3)}{900} = -\frac{3}{10}$$

ex12)



$$\frac{dx}{dt} = -120$$

$$\frac{ds}{dt} = -46$$

$$\frac{h}{1000-x} = \frac{s}{1000} \Rightarrow \frac{h}{(1000-x)^2}\frac{dx}{dt} = \frac{ds}{dt}\frac{1}{1000}$$

$$h = \frac{-46\cdot(600)^2}{-120\cdot 1000} = 138$$

13) Answer is 2 ft/sec

9/ THE Hyperbolic functions:

ex1) $\lim\limits_{x\to\infty} \dfrac{\sinh}{e^x} = \lim\limits_{x\to\infty} \dfrac{\dfrac{e^x - e^{-x}}{2}}{e^x}$

$= \frac{1}{2} \lim\limits_{x\to\infty} \left[1 - e^{-2x}\right] = 1/2$

ex2) $f(x) = \sinh x$

$f'(x) = \cosh x \longrightarrow f'' = \sinh x$

$f'' = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$f''(\ln 3) = \dfrac{e^{\ln 3} - e^{-\ln 3}}{2}$

$= \dfrac{3 - \frac{1}{3}}{2} = \dfrac{8/3}{2} = 4/3$

ex3) <u>note</u>: $f''(x) = 2\cosh x + x\sinh x$

$f''(x) = 2\left[\dfrac{e^x + e^{-x}}{2}\right] + x\left[\dfrac{e^x - e^{-x}}{2}\right]$

<u>Solution</u>: $f''(\ln 2) = \frac{1}{4}\left[10 + 3\ln 2\right]$

§ Global Max/min & local max/min:

ex1) $f(x) = 3x^2 - 8x + 2$ on $[0, 3]$

$f'(x) = 6x - 8$

1) $f'(x) = 0 \Rightarrow x = 4/3$
2) $f'(x)$ undefined? $\rightarrow$ none

x	$f(x)$
0	3
$4/3$	$\frac{-10}{3} \leftarrow m$
3	$7 \leftarrow M$

So, $M - m = 7 - \left(-\frac{10}{3}\right)$

ex2) $M = 6$ and $m = -10$

$M = 6$, and $M - m = 6 - (-10) = 16$

§ Local max and Min

ex1) at $x = -1$

ex2) none

ex3) 3 in total (2 min and 1 max)

ex4) absolute min at $x = 1/e$ (think locally though!)

ex5) local max at $x=-2$
local min at $x=0$

ex6) local max at $x=0$

ex7) local max at $x=-3$

ex8) local max at $x=e^2$ (please try to show this by hand calculation's)

ex9) local max at $x=-3$
local min at $x=3$

§ Concavity and Points of Inflection:

ex1) concave up from $(0,\infty)$

ex2) 2 points of inflection (at $x=-1$ and $x=0$)

ex3) $x = -\frac{1}{2^{1/3}}$

ex4) $(-2,1)$

ex5) $(2-\sqrt{2},\ 2+\sqrt{2})$

ex6) $-\frac{2}{27}$

ex7) at $x=0$ only

ex8) $(-2,1)$

ex9) $x=-1/2$

ex10) Three

ex11) $(-1, 2)$

ex12) $\left(\frac{3}{2}, \infty\right)$

ex13) none

§ Question's that require manipulation:

2) Answer $\frac{3}{2}(1+\sqrt{3})$