Exerzitien III

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Oct. 6., in your tutorial.

Reading suggestion: **Span, linear independence, Basis**, First two sections of Chapter 2.

Exercise 1. *Let V be a vector space over the field $\mathbb{F}$, and let $v_1, v_2, v_3, v_4 \in V$.*

1. *Prove that if (v_1, v_2, v_3, v_4) spans V, then so does $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.*
2. *Prove that if (v_1, v_2, v_3, v_4) is linearly independent, then so is $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.*
3. *Show that $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_1)$ is linearly dependent.*

Exercise 2. *Determine (and justify) whether the given sequence of vectors is linearly independent or not.*

1. $((-1, 1, 1, 1), (1, -1, 1, 1), (1, 1, -1, 1), (1, 1, 1, -1))$ *in the real vector space* $\mathbb{R}^4$
2. $((1, 0), (i, 0), (0, 1), (0, i))$ *in the complex vector space* $\mathbb{C}^2$
3. $\left(x^2, x^2 + 1, x^2 + 2\right)$ *in the vector space of real polynomials in one variable,* $\mathcal{P}(\mathbb{R})$
4. $\left(x^2, (x + 1)^2, (x + 2)^2\right)$ *in* $\mathcal{P}(\mathbb{R})$
5. $((1, 1, 0), (1, 0, 1), (0, 1, 1))$ *in* $\mathbb{F}_2^3$

Exercise 3. *What is the probability that a list (v_1, v_2, v_3) of three vectors, each chosen at random from $(\mathbb{F}_2)^5$, is linearly independent? Prove your claim. Does this probability increase or decrease as we increase the prime, for example in $(\mathbb{F}_3)^5$ and $(\mathbb{F}_5)^5$?*

Exercise 4. *Let $V = \mathcal{P}_4(\mathbb{R})$ be the vector space of real polynomials of degree ≤ 4 in one variable. Show that*

$$\{f \in \mathcal{P}_4(\mathbb{R}) \ : f(0) = f(1) = 0\}$$

defines a subspace of V, and find a basis for this subspace. Justify your claim and plot graphs of your basis elements.