STA447/STA2006 Stochastic Processes

Gun Ho Jang

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Note

This lecture note is prepared for the course STA447/STA2006 Stochastic Processes. This lecture note may contain flaws. Please consult text book or reference books for confidence.

- Gun Ho Jang

* indicates graduate level. So you may skip those parts.

5 Martingales

Note. Recall that a *stochastic process* is a collection of time indexed random variables, that is, $\{X_t : t \in \mathcal{T}\}$ where $\mathcal{T}$ can be $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ or $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$.

Definition 36. A collection of events $\mathcal{F}$ is called a σ-field if it satisfies
(a) [Sample Space] the sample space $\Omega \in \mathcal{F}$.
(b) [closed under the complement] If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
(c) [closed under the countable union] If $A_1, A_2, \ldots \in \mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Example 50. Let $\mathcal{F}$ be a σ-field.

- From (a) and (b), the empty set $\emptyset = \Omega^c \in \mathcal{F}$.
- $\mathcal{F}$ is *closed under the union*, that is, for any $A, B \in \mathcal{F}$, let $A_1 = A$, $A_2 = B$, $A_n = \emptyset$ for $n \geq 3$, then $A \cup B = \cup_{n=1}^{n} A_n \in \mathcal{F}$.
- $\mathcal{F}$ is closed under the intersection, that is, for any $A, B \in \mathcal{F}$, $A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$.
- $\mathcal{F}$ is closed under the countable intersection, that is, for any $A_n \in \mathcal{F}$, $\cap_{n=1}^{\infty} A_n = (\cup_{n=1}^{\infty} A_n^c)^c \in \mathcal{F}$.

Note. The closedness under the countable union is required to define probability, that is, for a sequence of disjoint events $A_1, A_2, \ldots \in \mathcal{F}$, define $A = \cup_{n=1}^{\infty} A_n$. Then $P(A) = P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$. Hence a countable union of events should be an event.

Definition 37. Let $X, X_1, X_2, \ldots$ be random variables. There exists the smallest σ-field generated by the random variable X denoted by $\sigma(X)$. In general $\sigma(X_1, X_2, \ldots)$ is the smallest σ-field generated by the random variables $X_1, X_2, \ldots$.

Exercise 31. Show that $\sigma(X) \subset \sigma(X, Y)$ for any random variables X and Y.

Note. For convenience, we write $X \in \mathcal{F}$ instead of $\sigma(X) \subset \mathcal{F}$ and read it X *is $\mathcal{F}$-measurable.*

Note. Let $\mathcal{F}_t$ be the collection of events up to time t. It is natural to assume that there are more events as time increases.

Consider a Markov chain X_n. The collection of events up to time n defined by $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ is increasing as n increases.

In an opposite way, we may consider collections of events up to time t before defining a stochastic process.

Definition 38. A sequence of σ-fields $\mathcal{F}_t$ is called a *filtration* if it is increasing, that is, $\mathcal{F}_s \subset \mathcal{F}_t$ if and only if $s \le t \in \mathcal{T}$.
A stochastic process X_t is said to be *adapted to* $\mathcal{F}_t$ if $X_t \in \mathcal{F}_t$ for all t.

Example 51. Let X_t be a stochastic process. Define $\tilde{\mathcal{F}}_t = \sigma(X_s, s \le t)$ the collection of event up to time t. Then $\tilde{\mathcal{F}}_t$ is the smallest filtration to which X_t is adapted. For any filtration $\mathcal{F}_t$ such that $\tilde{\mathcal{F}}_t \subset \mathcal{F}_t$, the process X_t is adapted to $\mathcal{F}_t$. Hence $\tilde{\mathcal{F}}_t$ is called the *natural filtration*.

Definition 39. A stochastic process X_n is said to be a (discrete-time) *martingale* if (a) $\mathbb{E}|X_n| < \infty$, (b) $X_n \in \mathcal{F}_n$, (c) $\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = X_n$ for all n. A stochastic process X_n is said to be a *supermartingale* (or *submartingale*) if (a), (b) and (c$'$) $\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) \le X_n$ (or $\ge X_n$) for all n.

Note. A martingale is both supermartingale and submartingale. If X_n is a submartingale, then $-X_n$ is a supermartingale.

Example 52 (Random walk)**.** Let $X_1, X_2, \ldots$ be an i.i.d. Define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. Then S_n is called a *random walk*. Then S_n is a martingale (or supermartingale or submartingale) if $\mathbb{E}(X_n) = 0$ (or $\mathbb{E}(X_n) \le 0$ or $\mathbb{E}(X_n) \ge 0$).

Example 53. Let S_n be a random walk. Then $S_n - \mathbb{E}(S_n)$ is a martingale.

Theorem 61. Let X_n be a homogeneous Markov chain having transition probability p. If a sequence of functions $f_n : \mathcal{S} \to \mathbb{R}$ satisfying $f_n(x) = \sum_y p(x,y) f_{n+1}(y)$, then $f_n(X_n)$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

Proof. Let $Y_n = f_n(X_n)$. Then $\mathbb{E}(Y_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}(f_{n+1}(X_{n+1}) \,|\, X_0, \ldots, X_n) = \mathbb{E}(f_{n+1}(X_{n+1}) \,|\, X_n) = \sum_{y \in \mathcal{S}} p(X_n, y) f_{n+1}(y) = f_n(X_n) = Y_n$. □

Example 54. Let X_n be a homogeneous Markov chain. Assume that z is an absorbing state define $h(x) = P_x(T_z < \infty)$. Then h satisfies $h(x) = \sum_y p(x,y) h(y)$. Hence $h(X_n)$ is a martingale.

Theorem 62. If X_n is martingale (or supermartingale or submartingale), then $\mathbb{E}(X_n \,|\, \mathcal{F}_m) = X_m$ (or $\le X_m$ or $\ge X_m$) for $m \le n$.

Proof. It is enough to show for a supermartingale $\mathbb{E}(X_n \,|\, \mathcal{F}_m) \le X_m$. Note that $\mathbb{E}(X_n \,|\, \mathcal{F}_{n-1}) \le X_{n-1}$, $\mathbb{E}(X_n \,|\, \mathcal{F}_{n-2}) = \mathbb{E}(\mathbb{E}(X_n \,|\, \mathcal{F}_{n-1}) \,|\, \mathcal{F}_{n-2}) \le \mathbb{E}(X_{n-1} \,|\, \mathcal{F}_{n-2}) \le X_{n-2}$, and $\mathbb{E}(X_n \,|\, \mathcal{F}_m) \le X_m$ by induction. □

Example 55. Let X_n be a martingale with respect to $\mathcal{F}_n$ and φ is a convex function. If $\mathbb{E}|\varphi(X_n)| < \infty$ for all n, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$ because Jensen's inequality, that is, $\mathbb{E}(\varphi(X_{n+1} \,|\, \mathcal{F}_n)) \ge \varphi(\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n)) = \varphi(X_n)$. Further if $X_n \in L^p$ for all n, then $|X_n|^p$ is a submartingale.

Example 56. Let X_n be a submartingale w.r.t. $\mathcal{F}_n$ and φ be an increasing convex function. If $\mathbb{E}|\varphi(X_n)| < \infty$ for all n, then $\varphi(X_n)$ is a submartingale with respect to the same filtration because $\mathbb{E}(\varphi(X_{n+1}) \,|\, \mathcal{F}_n) \ge \varphi(\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n)) \ge \varphi(X_n)$. For any a, $x \mapsto (x-a)^+ = \max(0, x-a)$ is an increasing convex function. Hence $(X_n - a)^+$ is a submartingale.

Exercise 32. Find an submartingale X_n so that X_n^2 is a supermartingale.

Definition 40. A stochastic process H_n is said to be predictable if $H_n \in \mathcal{F}_{n-1}$. Define $(H \cdot X)_n = \sum_{m=1}^n H_m (X_m - X_{m-1})$.

Note. A heuristic definition of an integral is $\int_a^b f(x)\, dx \approx \sum_{k=1}^n f(x_k)\Delta_k = \sum_{k=1}^n f(x_k)(x_k - x_{k-1})$ for $a = x_0 < x_1 < \cdots < x_n = b$. Hence $(H \cdot X) \approx \int_0^t H_s \, dX_s$ is a rough version of a stochastic integral.

Theorem 63. Let X_n be a supermartingale. If $H_n \ge 0$ is predictable and H_n is bounded for each n, then $(H \cdot X)_n$ is a supermartingale.

Proof. It is easy to see that $(H \cdot X)_n \in \mathcal{F}_n$. $\mathbb{E}[(H \cdot X)_{n+1} \,|\, \mathcal{F}_n] = (H \cdot X)_n + \mathbb{E}[H_{n+1}(X_{n+1} - X_n) \,|\, \mathcal{F}_n] = (H \cdot X)_n + H_{n+1}\mathbb{E}(X_{n+1} - X \,|\, \mathcal{F}_n) \le (H \cdot X)_n$. Hence $(H \cdot X)_n$ is a supermartingale. □

5.1 Random Time

Definition 41. A random variable T taking values in $\mathcal{T} \cup \{0, \infty\}$ is called a *random time.* If $\{T \leq t\} \in \mathcal{F}_t$ for all t, T is called a *stopping time.* If $H_A = \inf\{t : X_t \in A\}$ is called the *hitting time of* A.

Example 57. $T_x = \inf\{t : X_t \geq x\}$ is a stopping time and also is a hitting time of $[x, \infty)$.

Example 58. Let b_t is a continuous increasing function. Then the first passage time $T = \inf\{t : X_t \geq b_t\}$ is a stopping time.

Note (Notation $\vee, \wedge$)**.** For convenience, define $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

Example 59. Let S T be stopping times. Then $S \vee T$ and $S \wedge T$ are stopping times. For any $t \in \mathcal{T}$, $\{S \vee T \leq t\} = \{S \leq t\} \cup \{T \leq t\} \in \mathcal{F}_t$ and $\{S \wedge T \leq t\} = \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t$.

Exercise 33. Let S, T, T_n be stopping times and k be a non-negative integer. Show that $S + T, T \wedge k, \sup_n T_n, \inf T_n, \limsup_n T_n, \liminf_n T_n$ are stopping times.

Definition 42. Let T be a stopping time. The σ-field $\mathcal{F}_T$ of events determined to prior to the stopping time T is the collection of events $A \in \mathcal{F}$ for which $A \cap \{T \leq n\} \in \mathcal{F}_n$ for all n.

Exercise 34. Show that $\mathcal{F}_T$ is really a σ-field.

Exercise 35. Let Y_n be $\mathcal{F}_n$-measurable random variable and T be a stopping time. Show that $Y_T \in \mathcal{F}_T$.

Exercise 36. Let S, T be stopping times satisfying $S \leq T$ a.s. Then $\mathcal{F}_S \subset \mathcal{F}_T$.

Theorem 64. Let $X_1, X_2, \ldots$ be i.i.d., $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and T be a stopping time with $P(T < \infty) > 0$. Conditional on $\{T < \infty\}$, $\{X_{T+n}, n \geq 1\}$ is independent of $\mathcal{F}_T$ and has the same distribution as the original sequence $\{X_n, \geq n\}$.

Proof. It is easy to assume that $\mathcal{F} = \sigma(X_1, X_2, \ldots) = \mathcal{F}_0^\infty$ for some $\mathcal{F}_0$. Let $A \in \mathcal{F}_T$ and $B_j \in \mathcal{F}_0$. For a fixed n,

$$P(A, T = n, X_{T+j} \in B_j, 1 \leq j \leq k) = P(A, T = n, X_{n+j} \in B_j, 1 \leq j \leq k) = P(A, T = n) \prod_{j=1}^{k} P(X_j \in B_j).$$

Hence $P(A, T \leq n, X_{T+j} \in B_j, 1 \leq j \leq k) = P(A, T \leq n) \prod_{j=1}^{k} P(X_j \in B_j)$. The theorem follows. $\square$

Note. Strong Markov property is a generalized version of Theorem 64.