1. Consider the following statement:

   If m and n are odd integers, then mn is an odd integer.

   (a) Express the statement using logical notation.

   $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})$
   Alternate: $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, m = 2k+1) \wedge (\exists k \in \mathbb{Z}, n = 2k+1) \Rightarrow (\exists k \in \mathbb{Z}, mn = 2k+1)$

   (b) This statement can be proven using a direct proof. Write a detailed proof *structure* for the statement. **Don't write a complete proof** — for now, focus on the proof structure only and leave out *all* of the actual "content".

   Assume $m, n \in \mathbb{Z}$. # m and n are generic elements of $\mathbb{Z}$
   Assume (m is odd $\wedge$ n is odd) # the antecedent
   $\vdots$
   Then mn is odd. # definition of odd
   Then (m is odd $\wedge$ n is odd) $\Rightarrow$ mn is odd. # introduce $\Rightarrow$
   Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})$ # introduce $\forall$

   (c) Now, complete the proof of the statement.

   Assume $m, n \in \mathbb{Z}$. # m and n are generic elements of $\mathbb{Z}$
   Assume (m is odd $\wedge$ n is odd) # the antecedent
   Then $(\exists k \in \mathbb{Z}, m = 2k+1)$ and $(\exists k \in \mathbb{Z}, n = 2k+1)$. # definition of odd
   Let $i \in \mathbb{Z}$ be such that $m = 2i+1$. # label the quotient $m/2$ by i
   Let $j \in \mathbb{Z}$ be such that $n = 2j+1$. # label the quotient $n/2$ by j
   Then $mn = (2i+1)(2j+1)$ # substitution
   $= 4ij + 2i + 2j + 1$ # algebraic manipulation
   $= 2(2ij + i + j) + 1$
   Then $\exists k \in \mathbb{Z}, mn = 2k+1$. # $k = 2ij + i + j$
   Then mn is odd. # definition of odd
   Then (m is odd $\wedge$ n is odd) $\Rightarrow$ mn is odd. # introduce $\Rightarrow$
   Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})$ # introduce $\forall$

2. Consider the following statement:

   If m and n are integers with mn odd, then m and n are odd.

   (a) Express the statement using logical notation.

   $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (mn \text{ is odd}) \Rightarrow (m \text{ is odd} \wedge n \text{ is odd})$
   Alternate: $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, mn = 2k+1) \Rightarrow (\exists k \in \mathbb{Z}, m = 2k+1) \wedge (\exists k \in \mathbb{Z}, n = 2k+1)$

   (b) This statement can be proven using an indirect proof. Write a detailed proof *structure* for the statement. **Don't write a complete proof** — for now, focus on the proof structure only and leave out *all* of the actual "content".

Assume $m, n \in \mathbb{Z}$. # m and n are generic elements of $\mathbb{Z}$
  Assume (m is even $\vee$ n is even) # the negation of the consequent
    [*Since at least one of m or n is even, let us label one of the even numbers as m and make no assumption about n. The number n could be odd or even. (This argument is often labelled "Without loss of generality, assume m is even." or "WLOG assume m is even.")*]
    WLOG, assume m is even
      $\vdots$
      Then mn is even.
    Then (m is even) $\Rightarrow$ mn is even. # introduce $\Rightarrow$
  Then (m is even $\vee$ n is even) $\Rightarrow$ mn is even. # introduce disjuction in antecedent
  Then mn is odd $\Rightarrow$ (m is odd $\wedge$ n is odd). # apply contrapositive
Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, mn$ is odd $\Rightarrow$ (m is odd $\wedge$ n is odd). # introduce $\forall$

   (c) Now, complete the proof of the statement.

Assume $m, n \in \mathbb{Z}$. # m and n are generic elements of $\mathbb{Z}$
  Assume (m is even $\vee$ n is even) # the negation of the consequent
    WLOG, assume m is even
      Then $\exists k \in \mathbb{Z}, m = 2k$. # definition of even
      Let $i \in \mathbb{Z}$ be such that $m = 2i$. # label the quotient $m/2$ by i
      Then $mn = 2in$ # substitution
      $= 2(in)$ # associativity
      Then $\exists k \in \mathbb{Z}, mn = 2k$. # $k = in$
      Then mn is even. # definition of even
    Then (m is even) $\Rightarrow$ mn is even. # introduce $\Rightarrow$
  Then (m is even $\vee$ n is even) $\Rightarrow$ mn is even. # introduce disjuction in antecedent
  Then mn is odd $\Rightarrow$ (m is odd $\wedge$ n is odd). # apply contrapositive
Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, mn$ is odd $\Rightarrow$ (m is odd $\wedge$ n is odd). # introduce $\forall$