Systems with constant coefficients

$\vec{x}' = A\vec{x}$ $A$ $n\times n$ matrix : $\vec{x}(t) = e^{rt}\vec{\xi}$ is a solution $\Longleftrightarrow$

$r$ is an eigenvalue of $A$, with $\vec{\xi}$ a corresponding eigenvector.

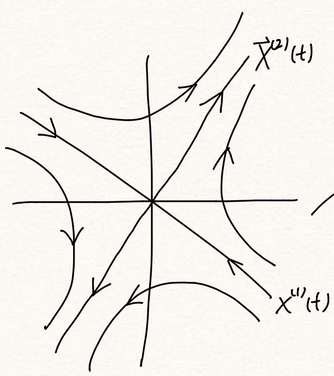
- Consider case $n=2$. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  If $r^{(1)}, r^{(2)}$ are distinct real eigenvalues with eigenvectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}$, get fund. set of solutions

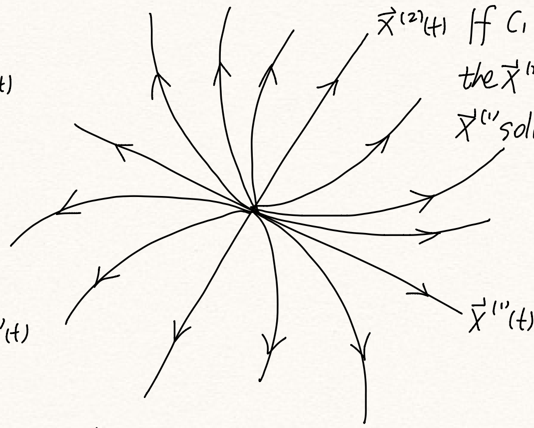
  $\vec{x}^{(1)} = e^{r^{(1)}t}\vec{\xi}^{(1)}$, $\vec{x}^{(2)} = \cdots\cdot$, $\vec{X}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$.

  - Phase portrait for $r^{(1)} < r^{(2)}$, both $\neq 0$.

  - Case 1: $r^{(1)} < 0 < r^{(2)}$

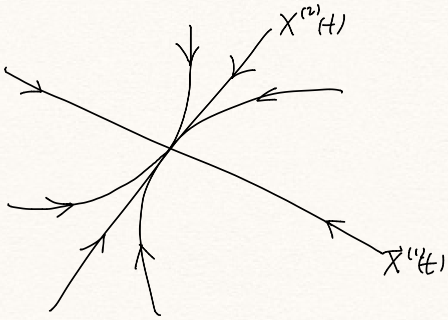



  - Case 2: $0 < r^{(1)} < r^{(2)}$ ( both $\vec{x}^{(1)}, \vec{x}^{(2)}$ go to $\infty$ as $t \to \infty$
    $0$ as $t \to -\infty$)

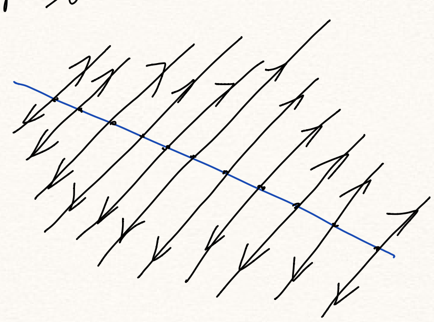



If $c_1, c_2$ are both $\neq 0$ then for $t \to \infty$ the $\vec{x}^{(2)}$ solution dominates, for $t \to -\infty$, $\vec{x}^{(1)}$ solution dominates.

  - Case 3: $r^{(1)} < r^{(2)} < 0$.




If $r^{(1)} = 0$, $r^{(2)} > 0$



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Example: $\vec{x}' = A\vec{x}$ $A = \begin{pmatrix} 1 & 12 \\ 0 & -5 \end{pmatrix}$ solve, and draw phase portrait.

First, find eigenvalue and eigenvector.

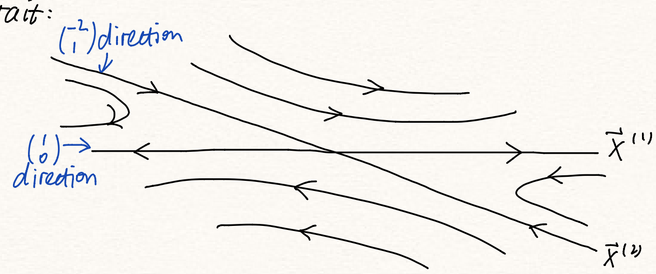
$\det(A - rI) = \begin{vmatrix} 1-r & 12 \\ 0 & -5-r \end{vmatrix} = (1-r)(-5-r)$

$\Rightarrow r^{(1)} = 1,\ r^{(2)} = -5$

$r^{(1)}=1$ $A-I=\begin{pmatrix}0&12\\0&-6\end{pmatrix}$ $(A-I)\begin{pmatrix}1\\0\end{pmatrix}=0$ $\vec{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector.

$r^{(2)}=-5$ $A+5I=\begin{pmatrix}6&12\\0&0\end{pmatrix}$ $\vec{\xi}^{(2)}=\begin{pmatrix}-2\\1\end{pmatrix}$

phase portrait:

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<u>Complex eigenvalues</u>. $\vec{x}'=A\vec{x}$

Suppose $r$ is a <u>complex</u> eigenvalue of $A$.
- then its eigenvector $\vec{\xi}$ is complex.
- $\bar{r}$ is an eigenvalue, with eigenvector $\overline{\vec{\xi}}$

Because $A\vec{\xi}=r\vec{\xi} \Rightarrow A\overline{\vec{\xi}}=\bar{r}\overline{\vec{\xi}}$ ($A$ is real $\bar{A}=A$)

$\vec{x}(t)=e^{rt}\vec{\xi}$ is a complex solution. $\Rightarrow \overline{\vec{x}}(t)=e^{\bar{r}t}\overline{\vec{\xi}}$ is a solution

$\Rightarrow$ both $Re(\vec{x}(t))$, $Im(\vec{x}(t))$ are solutions.

Can use this to replace $\vec{x}, \overline{\vec{x}}$ by pair of <u>real</u> solutions $Re(\vec{x})$, $Im(\vec{x})$.

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<u>Example</u>: $\vec{x}'=A\vec{x}$ $A=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$

$\det(A-rI)=\begin{vmatrix}1-r&-1\\1&1-r\end{vmatrix}=(1-r)^2+1$ $r_1, r_2=1\pm i$

$r^{(1)}$: $A-(1+i)I=\begin{pmatrix}-i&-1\\1&-i\end{pmatrix}$ $\vec{\xi}^{(1)}=\begin{pmatrix}i\\1\end{pmatrix}$ is eigenvector.

$r^{(2)}$: $\vec{\xi}^{(2)}=\begin{pmatrix}-i\\1\end{pmatrix}$

$\vec{x}^{(1)}(t)=e^{(1+i)t}\begin{pmatrix}i\\1\end{pmatrix}=e^t(\cos(t)+i\sin(t))\begin{pmatrix}i\\1\end{pmatrix}=e^t\cos(t)\begin{pmatrix}0\\1\end{pmatrix}-e^t\sin(t)\begin{pmatrix}1\\0\end{pmatrix}+i(\cdots)$ $\vec{x}^{(2)}=\overline{\vec{x}^{(1)}}$

phase portrait: