MATH6222: Homework #9

2017-05-06

Instructor: Dr. David Smyth

Tutor: Mark Bugden (Wednesday 1-2pm)

Rui Qiu u6139152

Problem 1

Let X_1, X_2, X_3 be random variables such that $P(X_i = j) = \frac{1}{n}$ for all $(i, j) \in [3] \times [n]$. Compute the probability that $X_1 + X_2 + X_3 \leq 6$, given that $X_1 + X_2 \geq 4$. You may assume that the random variables are *independent*, i.e.

$$P(X_1 = a_1, X_2 = a_2, X_3 = a_3) = P(X_1 = a_1)P(X_2 = a_2)P(X_3 = a_3).$$

Solution: As $X_1 + X_2 \geq 4$, the possible combinations of X_1 and X_2 that fails this are $(1, 1), (1, 2), (2, 1)$. So

$$P(X_1 + X_2 \geq 4) = 1 - \frac{1}{n} \cdot \frac{1}{n} \cdot 3 = 1 - \frac{3}{n^2}$$

Now we consider the possible combinations of X_1, X_2, X_3 when $X_1 + X_2 + X_3 \leq 6$. Note that since $X_1 + X_2 \geq 4$, then

$$X_3 \leq 6 - (X_1 + X_2) \leq 2$$

So $X_3 = 1$ or 2.

- When $X_3 = 2$, $(X_1, X_2) \in \{(1, 3), (2, 2), (3, 1)\}$. The total probability here is $\frac{1}{n} \cdot 3 \cdot \frac{1}{n^2} = \frac{3}{n^3}$.
- When $X_3 = 1$, $(X_1, X_2) \in \{(1, 3), (2, 2), (3, 1), (1, 4), (2, 3), (3, 2), (4, 1)\}$. The total probability here is $\frac{1}{n} \cdot \frac{7}{n^2} = \frac{7}{n^3}$.

Therefore,

$$P(X_1 + X_2 + X_3 \leq 6 | X_1 + X_2 \geq 4) = \frac{\frac{3}{n^3} + \frac{7}{n^3}}{1 - \frac{3}{n^2}} = \frac{10}{n^3 - 3n} \leq 1$$

Also note that $n^3 - 3n \geq 10$, so $n \geq 3$ makes this meaningful.

Problem 2

You hold a bag of ten coins, all superficially similar, but nine are fair, and one is foul (it shows heads with probability $\frac{9}{10}$). You draw out a coin and begin flipping it.

(a) The first five tosses are $HHHTH$. What is the probability that you are flipping one of the fair coins?

Solution: Suppose A be the event that first five tosses are $HHHTH$, B be the event that we are flipping one of the fair coins.

Problem 2 continued on next page. . .

We are interested in the conditional probability $P(B|A)$, which is:

$$\begin{aligned}
P(B|A) &= \frac{P(A\cap B)}{P(A)} = \frac{P(A\cap B)}{P(A|B)P(B) + P(A|\neg B)P(\neg B)} \\
&= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\neg B)P(\neg B)} \\
&= \frac{(\frac{1}{2})^5 \cdot \frac{9}{10}}{(\frac{1}{2})^5 \cdot \frac{9}{10} + (\frac{9}{10})^4 \cdot \frac{1}{10} \cdot \frac{1}{10}} \\
&= \frac{3125}{3854} \\
&\simeq 0.81085
\end{aligned}$$

(b) The next five tosses are $HHHHH$. Now what is the probability that you are flipping one of the fair coins?

Let C be the event that the first ten tosses are $HHHTHHHHHH$. And similarly,

$$\begin{aligned}
P(B|C) &= \frac{P(C\cap B)}{P(C)} \\
&= \frac{P(C\cap B)}{P(C|B)P(B) + P(C|\neg B)P(\neg B)} \\
&= \frac{(\frac{1}{2})^{10} \cdot \frac{9}{10}}{(\frac{1}{2})^{10} \cdot \frac{9}{10} + (\frac{9}{10})^9 \cdot \frac{1}{10} \cdot \frac{1}{10}} \\
&\simeq 0.18491
\end{aligned}$$

Problem 3

Suppose that a collection of $2n$ insects is randomly divided into n pairs. If the collection consists of n males and n females, what is the expected number of male-female pairs?

Solution: Suppose X is a random variable that indicates the number of male-female pairs in such randomization. Suppose again

$$X_{i,j} = \begin{cases} 1, & \text{if the i-th male insect is paired with the j-th female insect.} \\ 0, & \text{otherwise.} \end{cases}$$

Then by linearity of expectation,

$$E(X) = \sum_{(i,j)\in[n]\times[n]}^{(n,n)} E(X_{i,j}) = n^2 E(X_{i,j})$$

Problem 3 continued on next page...

Since for each certain insect, it has the same probability to pair with any other $2n-1$ insect, so that a particular (i, j) pair has the expectation:

$$E(X_{i,j}) = 0 \cdot P(i,1) + \cdots + 1 \cdot P(i,j) + \cdots + 0 \cdot P(i,n) = \frac{1}{2n-1}$$

Therefore,

$$E(X) = \frac{n^2}{2n-1}.$$

Problem 5

Recall that in the finger game, player A and B show 1 or 2 fingers, and A then receives a payoff according to the following chart (a negative number indicates that A pays B).

	B shows 1	B shows 2
A shows 1	-2	$+3$
A shows 2	$+3$	-4

We considered a scenario where A shows 1 finger with probability x and B shows 1 finger with probability y, and showed that $x = \frac{7}{12}$ gives an expected payoff of $\frac{1}{12}$ for A, and that this strategy is optimal. Here, *optimal* means that for any other choice of x, there exists a $y \in [0, 1]$ such that the expected payoff is lower than 1/12.

(a) For what range of values $x \in [0, 1]$ can A guarantee a positive expected payoff, no matter how B plays?

Solution:
Suppose A has probability x to show 1 finger. Then the expected payoff when B shows 1 finger is $-2x + 3(1-x) = 3 - 5x$. And the expected payoff when B shows 2 fingers is $3x - 4(1-x) = 7x - 4$.
Let both expected payoff be greater than 0, so we have $3 - 5x > 0, 7x - 4 > 0$. Solve these we get $\frac{4}{7} < x < \frac{3}{5}$.
So when $x \in \left(\frac{4}{7}, \frac{3}{5}\right)$, A can guarantee a positive expected payoff, no matter how B plays.

(b) Prove that $y = 7/12$ is the optimal strategy for B.

Proof:

The expected payment from A to B is $-2y + 3(1-y) = 3 - 5y$ and $3y - 4(1-y) = 7y - 4$ when A shows 1 or 2 finger(s) respectively.

Problem 5 continued on next page...

$$x(3-5y)+(1-x)(7y-4) = 3x-5xy+7y-4-7xy+4x = 7x+7y-12xy-4 = x(7-12y)+7y-4$$

This is the amount of money A pays B, so B wants to maximize the minimum of them, this happens when the effect of x is totally eradicated as

$$7 - 12y = 0 \implies y = \frac{7}{12}$$

The maximized minimum is therefore, $7 \cdot \frac{7}{12} - 4 = \frac{1}{12}$.

To show it is *optimal*, we assume $y \neq \frac{7}{12}$, such that we have a smaller payoff than $\frac{1}{12}$:

$$\begin{aligned}
x(7-12y) + 7y - 4 &< \frac{1}{12} \\
x(7-12y) + 7y &< \frac{49}{12} \\
\text{If } y < \frac{7}{12},\ x &< \frac{\frac{49}{12} - 7y}{7 - 12y} = \frac{7}{12} \\
\text{If } y > \frac{7}{12},\ x &> \frac{\frac{49}{12} - 7y}{7 - 12y} = \frac{7}{12}
\end{aligned}$$

Therefore, as long as $y \neq \frac{7}{12}$, we can always find a x in $[0, 1]$ such that the expected payoff of B is smaller than $\frac{1}{12}$. Thus the strategy with $y = \frac{7}{12}$ is *optimal*.

(c) Assuming that both players play their optimal strategy, what proportion of the games do A and B actually win.

Solution: In this case, $x = y = \frac{7}{12}$.

- A wins when the sum is odd, $P(A \text{ wins}) = x(1-y) + (1-x)y = \frac{7}{12} \cdot \frac{5}{12} + \frac{5}{12} \cdot \frac{7}{12} = \frac{35}{72}$.
- B wins when the sum is even, $P(B \text{ wins}) = xy + (1-x)(1-y) = 1 - x - y + 2xy = 1 - \frac{7}{12} - \frac{7}{12} + 2 \cdot \frac{7}{12} \cdot \frac{7}{12} = \frac{37}{72}$

So B wins $\frac{37}{72}$ of the games, while A wins $\frac{35}{72}$ of the games.