University of Toronto

MAT237Y1Y PROBLEM SET 2

DUE: End of tutorial, Thursday June 6th, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.

2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.

3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.

4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. Read Exercise 1 on page 40 of Folland (solutions found on Portal) which describes a notion called Holder Continuity which is a stronger condition than uniform continuity. For each of the functions in a) and b) determine if they are or are not each of the following on the specified domains: Continuous, Uniformly Continuous, and Holder Continuous for some $\lambda > 0$. You may use the result of Exercise 1 and Theorem 1.33.

   a) $\sqrt{x}$ for $x \in [0, 1]$
   Hint: Postulate a good candidate for λ and then try to find the algebraic trick to manipulate the inequalities into what you want.

   b) $f(x) = 1/ln(x)$ on $0 < x \leq 1/2$ and 0 at 0. I.e., the domain is [0,1/2].
   You may use your 1st year calculus knowledge of how limits of ln(x) or $x^\lambda ln(x)$ behave.

2. Theorem 1.33 demonstrates that continuity implies uniform continuity on compact sets. However for $f : [0, \infty) \to \mathbb{R}$ while the domain is not compact it is enough that the limit as $x \to \infty$ exists. That is, prove (via a rigorous $\epsilon - \delta$ argument) that if f is continuous on $[0, \infty)$ and $lim_{x \to \infty} f(x)$ exists, then f is uniformly continuous on $[0, \infty)$. Hint: you may wish to break up $[0, \infty)$ into two regions, one which is compact, and one consisting of sufficiently large values only.

3. Prove that x^n is differentiable with derivative nx^{n-1} for all $x \in \mathbb{R}$, $n \in \mathbb{Z}^+$ from the definition of the derivative given in equation 2.1 on page 44. Hint: Induction

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = xsin(1/x)\ if\ x \neq 0,\ f(0) = 0$$

   - a) Show that f is continuous at every point, and differentiable at every point except x = 0.
   - b) Let g(x)=xf(x). Show that g is differentiable at every point including x = 0.
   - c) Let h(x) = g(x) + x/2. Show that $h'(0) > 0$ but there is no neighborhood of 0 on which h is increasing. (Note the comparison between this example and Theorem 2.8c)

   Note: Unlike question 3, for question 4 and 5 you no longer need to work directly from the definitions. You may compute derivatives using rules like the quotient rule.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{xy^2}{x^2+y^2}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$

   - a) Is f continuous on $\mathbb{R}^2$? Justify.
   - b) Is f differentiable on $\mathbb{R}^2$? Justify.
   - c) Show that all the directional derivatives of f at (0,0) exist, and compute them.

   If you wish, you may use polar coordinates $x = rcos(\theta)$ and $y = rsin(\theta)$ to evaluate your final limits.

   Enjoy!