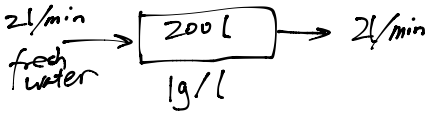
Mixing problems

① A tank contains 200 l of dye solution at concentration of 1g/l. The tank is — with fresh water at rate of 2l/min. The well-stirred mixture flows out at same rate. After ~~that~~ what time the concentration of dye reached 1% of its initial value?



Solution: Let $x(t)$ amount of dye in grams. Initial amount: $x(0) = 1 \cdot \frac{g}{l} \cdot 200\,l = 200\,g$.

$$\frac{dx}{dt} = \underbrace{\frac{x(t)}{200\,l}}_{\text{concentration at time } t} \cdot \underbrace{2\frac{l}{min}}_{\text{flow rate}} = -\frac{x(t)}{100} \cdot \frac{1}{min}$$

$\frac{dx}{dt} = \frac{-x}{100\,min}$ has solution $x(t) = x(0)e^{-\frac{t}{100\,min}}$

Let T be the time at which

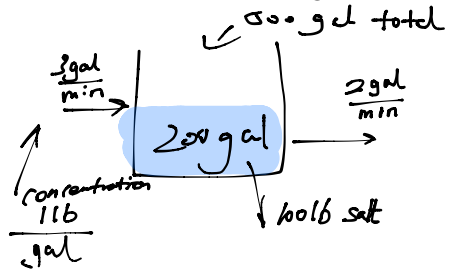
$$x(T) = \frac{1}{100}x(0)$$

Thus $x(0)e^{-\frac{T}{100\,min}} = \frac{1}{100}x(0)$

$e^{\frac{T}{100\,min}} = 100$

$T = (\ln 100)\,100\,min = 100 \ln 100\ min = 200 \ln 10\ min \approx 2.3 \times 200 = 460\ min$

② A tank of 500 gal capacity, contains 200 gal of water with 100 lb of salt. A solution of concentration $1\frac{lb}{gal}$ enters at rate $3\frac{gal}{min}$, mixture flows out at rate of 2 gal/min. What is the concentration at time T of overflow.



Solution: Let $x(t)$ amount of salt at time t. Let $w(t)$ amount of water

Thus $w(t) = 200\,gal + (\frac{3\,gal}{min} - \frac{2\,gal}{min}) \cdot t = 200\,gal + \frac{1\,gal}{min} \cdot t$

Overflows at time $T = 300$ min

$(w(T) = 500\ \text{gal})$.

$$\frac{dx}{dt} = 3\frac{\text{gal}}{\text{min}} \cdot 1\frac{\text{lb}}{\text{gal}} - 2\frac{\text{gal}}{\text{min}}\frac{x(t)}{200\text{gal} + t\cdot\frac{\text{gal}}{\text{min}}}$$

$$\frac{dx}{dt} = 3 - \frac{2x}{200+t}$$ this is linear

$$\frac{dx}{dt} = \left(\frac{2}{200+t}\right)x = 3$$ integrating factor

$X(0) = 100$ $\int \exp(p(t))\,dt$

After calculation:

$$X(t) = \frac{200^2}{(200+t)^2}x(0) + (200+t) - \frac{200^3}{(200+t)^2}$$

put $X(0) = 100$, $t = T = 300$, find $x(T) = \ldots = 484$ $\rightsquigarrow$ concentration at time of overflow is

$$\frac{484\,\text{lb}}{500\,\text{gal}} = 0.968\frac{\text{lb}}{\text{gal}}$$