Lecture 21

itinenary $S(x_0)=(s_0 s_1 \cdots)$ $s_i=\begin{cases} 0 & \text{if } Q_c^i(x_0)\in I_0 \\ 1 & \text{if } Q_c^i(x_0)\in I_1 \end{cases}$

sequence space $\Sigma=\{(s_0 s_1 s_2 \cdots): s_i\in\{0,1\}\}$

distance $d[s,t]=\sum_{i=0}^{\infty}\frac{|s_i-t_i|}{2^i}$

Examples: let $s=(0000\cdots)$
$t=(1111\cdots)$
$u=(010101\cdots)$
$v=(101010\cdots)$

$$d[s,t]=\sum_{i=0}^{\infty}\frac{1}{2^i}=2$$

$$d[u,v]=\sum_{i=0}^{\infty}\frac{1}{2^i}=2$$

$$d[s,v]=1+\frac{0}{2}+\frac{1}{2^2}+\frac{0}{2^3}+\frac{1}{2^4}+\cdots$$
$$=\sum_{i=0}^{\infty}\frac{1}{2^{2i}}=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$$
$$d[s,u]=0+\frac{1}{2}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\cdots=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\cdots\right)=\frac{2}{3}$$

Definition: A function $d:\Sigma\to\mathbb{R}$ is called a distance or metric if for all $s,t,u\in\Sigma$
1. $d[s,t]\geq 0$ and $=0$ iff $s=t$ (nonnegativity)
2. $d[s,t]=d[t,s]$ (symmetry)
3. $d[s,u]\leq d[s,t]+d[t,u]$ (triangle inequality)

The space Σ with a distance d is called a metric space.

Proposition: The function d defined earlier is a distance on Σ

Proof: (i) $\frac{|s_i-t_i|}{2^i}\geq 0$ for any $s_i,t_i\in\{0,1\}$. $i\in\{0,1,2,\cdots\}$
so $d[s,t]\geq 0$

Also, $d[s,t]=0 \Leftrightarrow \sum_{i=0}^{\infty}\frac{|s_i-t_i|}{2^i}=0 \Leftrightarrow \frac{|s_i-t_i|}{2^i}=0$
$\Leftrightarrow s_i=t_i \quad \forall i=0,1,2,\cdots$
$\Leftrightarrow s=t$

(ii) ✓
(iii) $d[s,t]+d[t,u]=\sum\frac{|s_i-t_i|}{2^i}+\sum\frac{|t_i-u_i|}{2^i}\geq\sum\frac{|s_i-t_i+t_i-u_i|}{2^i}=d[s,u]$ ∎

Q: What does it mean when two sequences are close together, when the distance is small?

Proximity theorem: Let $s, t \in \Sigma$, s.t. $s_i = t_i$ for all $i = 0, 1, \cdots, n$.
Then $d[s,t] \le \frac{1}{2^n}$.
Conversely, if $d[s,t] < \frac{1}{2^n}$, then $s_i = t_i$ for $i = 0, 1, \cdots, n$.

Pf: assume $s_i = t_i$ for $i = 0, \cdots, n$

Then
$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \le \sum_{i=n+1}^{\infty} \frac{1}{2^i} \underset{\downarrow \; i=n+1+j}{=} \frac{1}{2^{n+1}} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{2}{2^{n+1}} = \frac{1}{2^n}$$

// first part

For the second part, assume that $d[s,t] < \frac{1}{2^n}$

assume by contradict'n, that $\exists$ (at least) one $0 \le j < n$ s.t. $s_j \ne t_j$

Then
$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \ge \frac{|s_i - t_i|}{2^i} = \frac{1}{2^j} \underset{\downarrow \; j \le n}{\ge} \frac{1}{2^n} \quad \Rightarrow\Leftarrow$$

∎