Duration: **60 minutes**
Aids Allowed: **none**

Student Number: ____________________

Family Name(s): ____________________

Given Name(s): ____________________

Do ***not*** *turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 3 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use one of the "blank" pages for rough work. If you need more space for one of your solutions, use a "blank" page and *indicate clearly the part of your work that should be marked*.

In your answers, you may use without proof any theorem or result covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do — part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

Marking Guide

# 1:	______/12
# 2:	______/12
# 3:	______/12
Bonus Marks:	______/ 3
TOTAL:	______/36

Good Luck!

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each answer with the appropriate question and part number.

Question 1. [12 marks]

Part (a) [1 mark]

Write the *converse* of the following statement: "If I am tired, then I don't drive."

Part (b) [1 mark]

Write the *contrapositive* of the following statement: "If I am tired, then I don't drive."

Part (c) [4 marks]

Draw a Venn diagram showing a situation where the following statement is true. Clearly label each part of your diagram and explain what each part represents.

"Every student who studied and is rested will do well on this test."

Part (d) [3 marks]

Is the statement below True or False? Justify your response.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$$

Part (e) [3 marks]

Is the statement below True or False? Justify your response.

$$\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n > 1 \Rightarrow (m < n \land m \mid n)$$

(The notation "$x \mid y$" means "x exactly divides y" — or equivalently, "y is an integer multiple of x".)

Question 2. [12 MARKS]

Consider domain $D = \{\text{all programs and all programmers}\}$, and predicate symbols $C(x)$: "x is a program", $P(x)$: "x is a programmer", $K(x, y)$: "x knows y", $W(x, y)$: "x wrote y", and $G(x)$: "x is good".

Using only these symbols (in addition to appropriate connectives and quantifiers), translate each sentence below. That is, give a natural English sentence that corresponds to each given symbolic sentence, and give a clear symbolic sentence that corresponds to each given English sentence.

Part (a) [2 MARKS]

Every program was written by some programmer.

Part (b) [2 MARKS]

$\exists x \in D, \neg G(x) \wedge (P(x) \vee C(x))$

Part (c) [2 MARKS]

Every programmer knows some programmer who has written a bad program.

Part (d) [2 MARKS]

$\forall x \in D, P(x) \Rightarrow \exists y \in D, C(y) \wedge W(x, y)$

Part (e) [2 MARKS]

Some good program was written by an unknown programmer (one known by no programmer).

Part (f) [2 MARKS]

$\exists x \in D, P(x) \wedge G(x) \wedge \neg \exists y \in D, C(y) \wedge W(x, y)$

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each answer with the appropriate question and part number.

Question 3. [12 MARKS]

For each equivalence below:

- if the equivalence is always true, provide a derivation from one side to the other (justify each step of your derivation from the list of standard equivalences given at the end of this question);
- otherwise, provide an interpretation that makes the equivalence false (along with a brief explanation/ justification that the equivalence is false under your interpretation).

NOTE: In this question, we use a simplified quantifier notation where we omit the domain — *e.g.*, we write "$\forall x$" instead of "$\forall x \in D$". You should also use this simplified form in your answers to this question.

Part (a) [4 MARKS]

$\forall x, P(x) \iff \exists y, P(y)$

Part (b) [4 MARKS]

$P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow R$

Question 3. (CONTINUED)

Part (c) [4 MARKS]

$(\forall x, P(x)) \Rightarrow (\exists x, Q(x)) \iff \exists x, P(x) \Rightarrow Q(x)$

Standard Equivalences (where P, Q, $P(x)$, $Q(x)$, *etc.* are arbitrary sentences)

- *Commutativity*
  $P \wedge Q \iff Q \wedge P$
  $P \vee Q \iff Q \vee P$
  $P \Leftrightarrow Q \iff Q \Leftrightarrow P$
- *Associativity*
  $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R$
  $P \vee (Q \vee R) \iff (P \vee Q) \vee R$
- *Identity*
  $P \wedge (Q \vee \neg Q) \iff P$
  $P \vee (Q \wedge \neg Q) \iff P$
- *Absorption*
  $P \wedge (Q \wedge \neg Q) \iff Q \wedge \neg Q$
  $P \vee (Q \vee \neg Q) \iff Q \vee \neg Q$
- *Idempotency*
  $P \wedge P \iff P$
  $P \vee P \iff P$
- *Double Negation*
  $\neg\neg P \iff P$
- *DeMorgan's Laws*
  $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
  $\neg(P \vee Q) \iff \neg P \wedge \neg Q$
- *Distributivity*
  $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- *Implication*
  $P \Rightarrow Q \iff \neg P \vee Q$
- *Biconditional*
  $P \Leftrightarrow Q \iff (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- *Renaming* (where $P(x)$ does not contain variable y)
  $\forall x, P(x) \iff \forall y, P(y)$
  $\exists x, P(x) \iff \exists y, P(y)$
- *Quantifier Negation*
  $\neg\forall x, P(x) \iff \exists x, \neg P(x)$
  $\neg\exists x, P(x) \iff \forall x, \neg P(x)$
- *Quantifier Commutativity*
  $\forall x, \forall y, S(x,y) \iff \forall y, \forall x, S(x,y)$
  $\exists x, \exists y, S(x,y) \iff \exists y, \exists x, S(x,y)$
- *Quantifier Distributivity* (where S does not contain variable x)
  $S \wedge \forall x, Q(x) \iff \forall x, S \wedge Q(x)$
  $S \vee \forall x, Q(x) \iff \forall x, S \vee Q(x)$
  $S \wedge \exists x, Q(x) \iff \exists x, S \wedge Q(x)$
  $S \vee \exists x, Q(x) \iff \exists x, S \vee Q(x)$
  $(\forall x, P(x)) \wedge (\forall x, Q(x)) \iff \forall x, P(x) \wedge Q(x)$
  $(\exists x, P(x)) \vee (\exists x, Q(x)) \iff \exists x, P(x) \vee Q(x)$

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each answer with the appropriate question and part number.

Bonus. [3 MARKS]

WARNING! This question is difficult and will be marked harshly: credit will be given only for making *significant* progress toward a correct answer. Please attempt this only *after* you have completed the rest of the test.

Give a derivation to show that

$$P \iff P \wedge (P \vee Q)$$

using the equivalences at the end of question 3 to justify **each** step of your derivation — every change in the expression must be justified. *You may use no other equivalence as justification, and you will receive no credit for using truth-tables.*

On this page, please write nothing except your name.

Family Name(s): ____________________

Given Name(s): ____________________

Total Marks = 36