UNIVERSITY OF TORONTO
Faculty of Arts and Science
APRIL EXAMINATIONS 2015
CSC336H1 LEC 5101 S (20151)
Duration - 2 hours
Aids allowed: pocket calculators

Last (family) name :

First (given) name :

Student id :

Question	Marks
1	/ 15
2	/ 30
3	/ 15
4	/ 20
5	/ 10
6	/ 10
Total	/ 100

As noted in the syllabus of the course, you must achieve at least 33% in the final exam in order to pass the course.

Please write legibly. Unreadable answers are worthless.

You can use both sides of all eight (8) sheets for your answers, except the front (cover) page.
You must return all 8 sheets.

1. [*15 points*] Consider

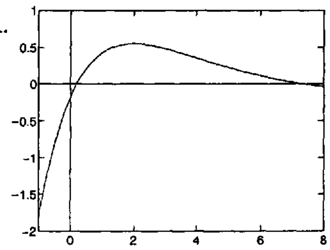
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 2 & 10 & 4 \end{bmatrix}, \quad A^{-1} = \frac{1}{14} \cdot \begin{bmatrix} 16 & 2 & -4 \\ -12 & 2 & 3 \\ 22 & -6 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix}.$$

(a) [*7 points*] What is the condition number of A in the infinity norm? What is the condition number of A in the 1-norm? What is the condition number of A^{-1} in the infinity norm?

(b) [*8 points*] Assume that, on a computer system with two decimal digits floating-point arithmetic (sign separate), we obtain an (approximate) solution $\hat{x}$ to $Ax = b$, such that $\|b - A\hat{x}\|_\infty \le 0.1$. Find an upper bound, as sharp as you can, for the relative error $\frac{\|x - \hat{x}\|_\infty}{\|x\|_\infty}$, *without* calculating x.
You can use the following excerpt from the notes without proof, but you must explain how you use it:

$$\frac{\|x - \hat{x}\|_a}{\|x\|_a} \le \kappa_a(A)\left(\frac{\|b - \hat{b}\|_a}{\|b\|_a} + \frac{\|A - \hat{A}\|_a}{\|A\|_a} + \frac{\|r\|_a}{\|b\|_a}\right)$$

2. Consider computing the root(s) of the function $f(x) \equiv xe^{-\frac{x}{2}} - \frac{e^{-1}}{2}$, defined in $\mathbf{R}$.
(Note that $e = \exp(1) \approx 2.7183$ and $e^{-1} = \exp(-1) = \frac{1}{e} \approx 0.3679$.)
This function has the graph shown to the right.
We are interested in computing the smallest root of $f(x)$.



(a) [*5 points*] Using mathematical arguments, show that there exists exactly one root of $f(x)$ in $[0,2]$.
Using mathematical arguments, show that there exists exactly one root of $f(x)$ in $(-\infty, 6]$ (i.e. there are no roots in $(-\infty, 0)$ and no roots in $(2, 6]$). You can use the fact that $f(6) > 0$.
Let r denote the root of $f(x)$ in $(-\infty, 6]$.

(b) [*5 points*] Using $x^{(0)} = 0$ as initial guess, apply (by hand) one Newton iteration to compute an approximation $x^{(1)}$ to the root. Indicating how $x^{(1)}$ is computed, simplify as much as you can, and write the result in terms of e^{-1}.

(c) [*5 points*] The equation $f(x) = 0$ can be equivalently written as $x = \frac{e^{-1}}{2} e^{\frac{x}{2}}$, which gives rise to the fixed-point iteration scheme $x^{(k+1)} = g(x^{(k)})$, with $g(x) = \frac{e^{-1}}{2} e^{\frac{x}{2}}$.
Using $x^{(0)} = 0$ as initial guess, apply (by hand) one fixed-point iteration to compute an approximation $x^{(1)}$ to the root. Indicating how $x^{(1)}$ is computed, simplify as much as you can, and write the result in terms of e^{-1}.

(d) [*10 points*] Show that the fixed-point iteration scheme converges to r, if started anywhere in $[0,2]$.

(e) [*5 points*] What is the convergence rate (order) of Newton's if started close enough to r? Explain.
What is the convergence rate (order) of the fixed-point iteration in (c)-(d)? Explain.

3. Consider the system of two nonlinear equations with respect to the two unknowns x and y

$$4x^2 + y^2 = 25$$

$$xye^{-\frac{x+y}{4}} = 0.2$$

(a) [*8 points*] Formulate the Jacobian matrix for the above nonlinear system. The entries of the Jacobian should be given in terms of the variables x and y.

(b) [*7 points*] Assume one wants to apply Newton's method to the above system with starting guess $[x^{(0)}, y^{(0)}]^T$. Find conditions on $x^{(0)}$ and $y^{(0)}$, so that Newton's method is applicable and explain. (Simplify the condition(s) as much as you can.)
Is Newton's method applicable to the above system with starting guess $[1, 1]^T$? Explain.

4.

(a) [*5 points*] Using any of the methods taught in class, construct the polynomial $p_2(x)$ of degree at most 2, that interpolates the function $f(x) = e^x$, at the points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Bring the polynomial into monomial form, if it is not already in that form. The coefficients of the polynomial are to be given either in terms of e and e^{-1}, or as (approximate) numerical values.

(b) [*3 points*] Give the error formula for this interpolation problem (i.e. for *this* f and *these* data points). The formula should involve an unknown point ξ. Any other functions involved in the formula should be given explicitly in terms of x.
Excerpt from notes:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j),$$

where ξ is an unknown point in ospr $\{x_0, x_1, \cdots, x_n, x\}$, that depends on x.

(c) [*12 points*] Using the polynomial interpolation error formula, give an upper bound for each of $\max_{-1 \le x \le 1} |e^x - p_2(x)|$, $\max_{-1 \le x \le 2} |e^x - p_2(x)|$, $\max_{-2 \le x \le 1} |e^x - p_2(x)|$. Explain how you got the bounds. The bounds are to be given as (approximate) numerical values.

If you don't have a calculator, you can use the following approximate values: $e^{-1} = 0.378$, $e = 2.72$, $e^2 \approx 7.39$, $\frac{2}{3\sqrt{3}} \approx 0.3849$.

5. [*10 points*] Write down all conditions on a_i, b_i and c_i, $i = 1, 2$, which guarantee that the following function $Q(x)$ is a continuous ($\mathbb{C}^0$) quadratic piecewise polynomial in $\mathbb{R}$ with knots $0, 1, 2$. Describe what each condition means. (Do not try to solve for a_i, b_i and c_i, $i = 1, 2$.)

$$Q(x) = \begin{cases} q_1(x) \equiv a_1 x^2 + b_1 x + c_1 & \text{for } 0 \le x \le 1 \\ q_2(x) \equiv a_2 x^2 + b_2 x + c_2 & \text{for } 1 \le x \le 2 \\ q_3(x) \equiv 0 & \text{elsewhere} \end{cases}$$

6. *[10 points]* Assume that A and b are already defined matrix and vector, respectively, of appropriate dimensions. What is approximately the output of the following piece of MATLAB code? Explain.

```
x = A\b; norm(b-A*x)
```

What is approximately the output of the following piece of MATLAB code? Explain.

```
x = linspace(1, 5, 1000); y = x.^2;
xi = [1 2 4 5]; yi = xi.^2;
polyfit(xi, yi, 3)
max(abs(polyval(polyfit(xi, yi, 3), x) - y))
```

If the output of the piece of MATLAB code

```
x = linspace(0, 5, 1000); y = exp(x);
xi = linspace(0, 5, 11); yi = exp(xi);
max(abs(spline(xi, yi, x) - y))
```

is

```
ans =
   0.157554067726011
```

what is approximately the output of the following piece of MATLAB code? Explain.

```
xi = linspace(0, 5, 21); yi = exp(xi);
max(abs(spline(xi, yi, x) - y))
```


Continued...

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