Lecture 27

Chaotic system : A dynamic system $F: X \to X$ is chaotic if
① The set of all periodic points is dense in X
② F is transitive.
③ F is sensitive to initial data.

Example : The shift map $\sigma : \Sigma \to \Sigma$ is chaotic
we already proved conditions ① & ②
Now we prove sensitivity.
choose $\beta = 1$, Let $s \in \Sigma$, so $s = (s_0 s_1 s_2 \cdots)$
Let $\varepsilon > 0$, choose $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \varepsilon$
Suppose that $t \in \Sigma$ with $d[s,t] < \frac{1}{2^n} < \varepsilon$ and $t \neq s$
$(t_0 t_1 \cdots)$
Since $t \neq s$, we know that $t_i \neq s_i$ for some $i \in \{0,1,2,\cdots\}$
So

$$\sigma^i(t) = (t_i t_{i+1} \cdots)$$
$$\sigma^i(s) = (s_i s_{i+1} \cdots)$$

and $$d[\sigma^i(s), \sigma^i(t)] = \sum_{n=0}^{\infty} \frac{|s_{i+n} - t_{i+n}|}{2^n} \geq \frac{|s_i - t_i|}{2^0} = 1$$

This proves that σ is chaotic.

Remark : We proved more than just sensitivity : we proved that if $t \neq s$, then the orbits will eventually be separated by at least distance 1.

Q. How do we relate this to Q_c?

Density Proposition : Sps $F: X \to Y$ is continuous and onto, and suppose that D is dense in X. Then $F(D)$ is dense in Y.

Thm : The map $Q_c : \Lambda \to \Lambda$ is chaotic for $c \leq -\frac{5+2\sqrt{5}}{4}$

Proof:

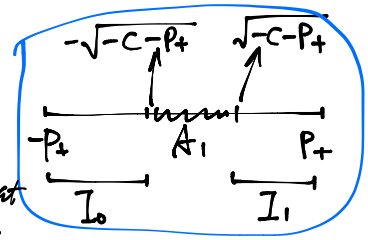
density : $S : \Lambda \to \Sigma$ is a homeomorphism, so $S^{-1} : \Sigma \to \Lambda$ is also a homeomorphism.

$$\begin{tikzcd} \Lambda \arrow[r, "Q_c"] & \Lambda \\ \Sigma \arrow[u, "S^{-1}"] \arrow[r, "\sigma"] & \Sigma \arrow[u, "S^{-1}"'] \end{tikzcd}$$

we also know that the periodic points for σ are dense in Σ.
So by the density proposition, $S^{-1}(\{\text{periodic points of } \sigma\})$ is dense in Λ.
and if s is periodic for σ, then $S^{-1}(s)$ is periodic for Q_c, so the periodic pts for Q_c are dense in Λ.

transitivity : we have $\hat{s} \in \Sigma$ which has a dense orbit under σ, so $S^{-1}(\{\text{orbit of } \hat{s}$ under $\sigma\})$ is dense in Λ.
And $S^{-1}(\hat{s}) = \hat{x} \in \Lambda$ and S^{-1} maps the orbit of $\hat{s}$ under σ into the orbit of $\hat{x}$ under Q_c. So the orbit of $\hat{x}$ under Q_c is dense in Λ.

Sensitivity: Let $\beta = \text{length}(A_1) = 2\sqrt{-c-p_+}$



so for all $x, y \in \Lambda$ s.t. $x \neq y$.

Then $S(x) \neq S(y)$, so there is $k \in \mathbb{N}$ s.t. the kth element of $S(x)$ and $S(y)$ are different, which means that $Q_c^k(x)$ and $Q_c^k(y)$ are not in the same interval I_0 or I_1

Thus, $|Q_c^k(x) - Q_c^k(y)| \geq$ length of $A_1 = \beta$

∎

§10.2 Other chaotic maps

The Vmap is defined as $V: [-2,2] \to [-2,2]$

$$V(x) = 2|x| - 2$$

Claim: V is a chaotic dynamic system

(we can relate Vmap to prove other maps are chaotic)