Exerzitien V

Submit your *concise* solutions in the correct order and no later than 4:10 pm on Oct. 27., in your tutorial.

Reading suggestion: Axler Chapter 3, parts 1 (linear maps) and 2 (null space and range).

Exercise 1. *Determine (using row reduction) in each case whether the linear system has solutions, and if so, describe the solution set parametrically. For example, if we have one equation $x+y=3$, it certainly has many solutions in $\mathbb{R}^2$ – one way of describing them all is to let y be any $t \in \mathbb{R}$ and then $x = t(-1)+3$. so we would say that the solution set is*

$$\{t(-1,1)+(3,0) \ : \ t \in \mathbb{R}.\}$$

a)
$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 1 \\ x_1 & & & + & 2x_3 & = & 1 \\ x_1 & - & 3x_2 & + & 4x_3 & = & 2 \end{array}$$

b)
$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & 1 \\ x_1 & + & x_2 & - & x_3 & + & x_4 & = & 2 \\ x_1 & + & 7x_2 & - & 5x_3 & - & x_4 & = & 3 \end{array}$$

c)
$$\begin{array}{rcrcrcr} x & + & 2y & & & = & -2 \\ x & & & + & 2z & = & -3 \\ & & y & - & z & = & 2 \end{array}$$

d)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 6x_3 & = & -1 \\ 2x_1 & + & x_2 & + & x_3 & = & 8 \\ 3x_1 & + & x_2 & - & x_3 & = & 15 \\ x_1 & + & 3x_2 & + & 10x_3 & = & -5 \end{array}$$

Exercise 2. *Let $P : V \longrightarrow V$ be a linear operator, and suppose that $P^2 = P$. Let $N = \mathrm{Null}(P)$ and $R = \mathrm{Range}(P)$, note that these are both subspaces of V. Set $Q = \mathrm{Id}_V - P$, where Id_V is the identity operator on V.*

1. *Show that $Q^2 = Q$.*
2. *Show that $PQ = QP = 0$.*
3. *Show that $N \cap R = \{0\}$.*
4. *Show that $v = Pv + Qv$, for any $v \in V$.*
5. *Conclude from the above facts that $V = N \oplus R$.*

Exercise 3. *Let $(x,y) \in \mathbb{R}^2$ and define the Fibonacci map $F \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ as follows:*

$$F(x,y) = (y, x+y).$$

1. *Let $v_0 = (0,1)$. Prove that the n^{th} number in the Fibonacci sequence $1, 1, 2, 3, 5, \dots$ is given by the first component of $F^n(v_0) \in \mathbb{R}^2$.*
2. *Write the matrix $A = [F]_e^e$ of F using the standard basis $e = ((1,0),(0,1))$. Compute the matrix powers A^2, A^3, A^4, A^5.*
3. *Write the matrix $B = [F]_b^b$ of F using a different basis $b = ((1, \frac{1+\sqrt{5}}{2}), (-\frac{1+\sqrt{5}}{2}, 1))$. Now compute B^k explicitly for all $k \in \mathbb{N}$.*
4. *Express v_0 in terms of the basis b and compute $F^n(v_0)$ explicitly for all $n \in \mathbb{N}$. Conclude with a formula for the n^{th} Fibonacci number.*