PRACTICE QUESTIONS FOR FINAL EXAM

(1) State and prove Fermat's Little Theorem.

(2) (a) Prove that the sum of the digits in the base 10 expansion of a natural number n is a multiple of 3 if and only if n is a multiple of 3.
(b) Prove that $6|x$ when $x+1$ and $x-1$ are prime, with one exception.
(c) Suppose that $x+1$ and $x-1$ are prime. Form a new number by concatenating the digits of one with the digits of the other. Thus $\{11, 13\}$ can become 1113 or 1311. Prove that the resulting number is not prime, with one exception.

(3) The fraction of games that a tennis player wins against each of her four opponents is .6, .5, .45, .4, respectively. Suppose that she plays 30 matches against each of the first two and 20 matches against each of the last two. Given that she wins a particular match, what is the probability that she was playing against the i^{th} opponent for $i \in \{1, 2, 3, 4\}$.

(4) Consider a dial with a pointer that is equally likely to point to each of 5 regions, numbered $\{1, 2, \ldots, 5\}$ in cyclic order. When the dial points to region k, the gambler receives 2^k dollars.
(a) What is the expected payoff per spin of the dial?
(b) Suppose that the gambler has the following option. After the spin, the gambler can accept that payoff or flip a fair coin to change it. If the coin shows heads, the the pointer moves one spot counterclockwise. If the coin shows tails, the pointer moves one spot counterclockwise. When should the gambler flip the coin? What is the expected payoff under the optimal strategy?

(5) A math department has n professors and $2n$ courses, each professor teaching two courses per semester.
(a) How many ways are there to assign the courses in Semester 1?
(b) How many ways are there to assign courses in Semester 2 so that no professor teaches the same pair of courses in Semester 2 as in Semester 1?

(6) Let S be a set of $n+1$ numbers in $[2n]$. Prove that S contains a pair of numbers, one of which divides the other.

(7) Prove that any tree has a vertex of degree one.

(8) Let G be a simple planar graph with no 3-cycle. Prove that $\chi(G) \leq 4$.