**THE AUSTRALIAN NATIONAL UNIVERSITY**
**RESEARCH SCHOOL OF FINANCE, ACTUARIAL STUDIES AND APPLIED STATISTICS**

## First Semester Final Examination 2012 - ***SOLUTIONS***

# FINANCIAL MATHEMATICS

## (STAT 2032 / STAT 6046)

**Study Period: 15 minutes**
**Time Allowed: 3 hours**

**Permitted Material:**
**Non-Programmable Calculators**
**Dictionaries (must not contain material added by the student)**
**Actuarial Tables for examinations (not required)**

**Students have also been provided with a formula sheet in addition to this exam paper.**

**Total Marks: 100**

***Instructions to Candidates:***

- ***Attempt ALL 6 questions***
- ***Start your solution to each question on a new page.***
- ***Unless otherwise stated, show all working.***

**Question One (16 marks)**

(a)

i. $(1-\frac{d^{(6)}}{6})^{-6}=(1-\frac{0.115}{6})^{-6}=1.12313=e^{\delta}\rightarrow\delta=11.6\%$

ii. $1.12313=(1+\frac{i^{(12)}}{12})^{12}\rightarrow i^{(12)}=11.7\%$

iii. $1.12313=(1-\frac{d^{(2)}}{2})^{-2}\rightarrow d^{(2)}=11.3\%$

*for each part: 1 mark for correct method, 1 mark for correct answer.*

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[b]

[i]

This payment stream is equal to

$4(Da_{\overline{17}|}+Da_{\overline{16}|}+Da_{\overline{15}|}+Da_{\overline{14}|}+...+Da_{\overline{2}|}+Da_{\overline{1}|})=4(879.68)=3518.72$

*3 marks for correct answer, 1 for some reasonable attempt, 0 otherwise.*

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[ii]

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With $i = 0$, they have the same value. *0 marks or 1 mark*

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[c]

(i)

$$(1+TWRR)^3=\left(\frac{65,600}{48,000}\right)\left(\frac{72,300}{65,600+54,000}\right)\left(\frac{82,000}{72,300+10,000}\right)\left(\frac{505,000}{82,000+350,000}\right)=0.96226$$

$\rightarrow TWRR=0.96226^{1/3}-1=-1.3\%$

*3 marks for correct answer, 1 for some reasonable attempt, 0 otherwise.*

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(iii)

Expect to be greater. Overall, all 'new' money = \$472,000, grows overall to \$505,000 so analyzing by MWRR will mean the overall rate is positive. (In fact the MWRR = +6.1%)

The major contributor to the MWRR is the final net cashflow of \$350,000 received on 1 May 2010. This amount plus the value of the fund at 30 April 2010 (\$82,000) clearly then grew at a positive rate to the closing balance at 31 December, so the overall MWRR will be weighted towards that positive earning rate.

*1 mark for saying "greater", 1 mark for reasonable explanation.*

(iii)

More information is required for the calculation of the TWRR - that being the fund balances at times just prior to new money into or out of a particular fund.

*0 marks or 1 mark*

Alternatively *1 mark* can be awarded for saying that the TWRR does not give the actual return for the investor whereas the MWRR does (i.e. this is a disadvantage from the investors point of view, rather than from a calculation point of view)

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## Question Two (16 marks)

[a]
Excluding capital gains, Mrs X pays a price P of:

$$P = 1.17^{4/12}\left(50,000(0.08)(1-0.15)a^{(2)}_{\overline{15}|17\%} + (0.9)50,000v^{15}_{17\%}\right) = 1.17^{4/12}\left(18,841.4 + 4269.97\right) = 24,353$$

A capital gain exists so the price paid is:

$$P' = 24,353 - \left((0.9)50,000 - P'\right)(0.45)v_{17\%}^{(15-\frac{2}{6})}$$

$$\rightarrow P'(1-0.044994) = 24,353 - 2,024.72 = 22,328.28$$

$$\rightarrow P' = 23,380$$

*5 marks for correct answer, minus 1 mark for each separate mistake made (income tax = 1, capital gains tax = 1, adjusting for 2 months to first coupon payment = 1, correct redemption amount = 1, correct workings = 1). Maximum of 4/5 if answer not exactly correct.*

[b]
Excluding capital gains tax, the price paid by Mrs 'Y' is given by:

$$P = 50,000(0.08)a^{(2)}_{\overline{8.5}|6\%}(1-0.145) + 45,000v^{8.5}_{6\%} = 50,016.34$$

So no capital gains tax is applicable for Mr Y and price = $50,016.34$

Hence Mrs 'X' is selling for $50,016.34 something which cost her $23,380 → amount of capital gains tax paid is 0.45 x (50,016 – 23,380) = 11,986

*5 marks for correct answer, minus 1 mark for each separate mistake made.*
*Obtaining correct price for Mrs Y = 3 marks,*
*obtaining correct capital gains tax for Mrs X = 2 marks. No marks to be deducted for errors carried forward from part (a) in this part of the calculation.*

[c]
i.
ii.
Two equations of value required are:

$$925 = 115a^{(2)}_{\overline{15}|} + 450v^{15} - - - - - (1)$$

$$925 = 115a^{(2)}_{\overline{20}|} + 450v^{20} - - - - - (2)$$

Solving (2) by linear interpolation gives 12.09% so 12.00% → the solution for (i)
Solving (1) by linear interpolation gives 11.32% so 11.50% → the solution for (ii)

*For each part – 3 marks if correct, 2 if correct but not rounded to the nearest 0.5% or if not given in terms of an annual effective rate, 1 if some reasonable attempt made (or if given in term of a semi- annual effective rate that itself is not rounded to the nearest 0.5%), 0 otherwise.*

## <u>Question Three (16 marks)</u>

[a]

Equation of value is:

$$-105k-(365)(300)a_{\overline{\infty}|}^{(365)}-(12)800a_{\overline{\infty}|}^{(12)}+(365)(400)a_{\overline{\infty}|}^{(365)}=0\rightarrow -105k+(365)(100)a_{\overline{\infty}|}^{(365)}-(12)800a_{\overline{\infty}|}^{(12)}=0$$

Working exactly in terms of days:

$i=25\%:i^{(12)}=12(1.25^{1/12}-1)=0.22523$ and $i^{(365)}=365(1.25^{1/365}-1)=0.22321$

$$\text{NPV} = -105,000+\frac{(365)(100)}{i^{(365)}}-\frac{(12)800}{i^{(12)}}=15,900$$

Alternatively, working in terms of continuous approximation:

$i=25\%:i^{(12)}=12(1.25^{1/12}-1)=0.22523$ and $i^{(\infty)}=\delta=\ln(1.25)=0.22314$

$$\text{NPV} = -105,000+\frac{(365)(100)}{\delta}-\frac{(12)800}{i^{(12)}}=15,949$$

Either of these approaches is ok.

*4 marks for correct answer, minus 1 mark for first error (3/4 = maximum mark if not done correctly).*
*Equation of value worth 2 marks and correct calculations overall worth 2 marks.*

[*b*]

25% gave an NPV of 15,900 so try a RDR a little higher - say 30%.

Working exactly in terms of days:

$i=30\%:i^{(12)}=12(1.30^{1/12}-1)=0.26525$ and $i^{(365)}=365(1.30^{1/365}-1)=0.26246$

$$\text{NPV} = -105,000+\frac{(365)(100)}{i^{(365)}}-\frac{(12)800}{i^{(12)}}=-2,123$$

Alternatively, working in terms of continuous approximation:

$i=30\%:i^{(12)}=0.26525$ and $i^{(\infty)}=\delta=\ln(1.30)=0.26236$

$$\text{NPV} = -105,000+\frac{(365)(100)}{\delta}-\frac{(12)800}{i^{(12)}}=-2,073$$

Interpolate either of the [exact day]/[exact day] or [continuous]/[continuous] combinations to get an IRR of 29% to 2 s.f.

*4 marks for correct answer, minus 1 mark for first error (3/4 = maximum mark if not done correctly).*
*Credit to be given for reasonable guesses or improvements from initial guesses. If incorrect equation of value carried through from part [a], maximum mark in this section = 3 out of 4.*

(c)

Additional cost of coffee = $\frac{(2)(2000)}{d^{(2)}}$ where $d^{(2)} = 2\left(1-1.25^{-0.5}\right) = 0.21115$

Hence cost of coffee = $\frac{(2)(2000)}{0.21115} = 18,944$ and NPV = 15,900 - 18,944 = -3,044

*1 mark for correct calculation of additional cost of coffee, 1 mark for adding this to the NPV calc.*

(d)

Loan accumulates to $105,000(1.01)^{36} = 133,322.14$ after 2 years

Additional income minus costs (other than coffee) against this loan are given by

$$100 s_{\overline{730}|j\%} - 800 s_{\overline{24}|1\%} = 100\frac{(1+j)^{730}-1}{j} - 800\frac{(1.01)^{24}-1}{0.01} = 82,440.34 - 21,578.77 = 60,861.57$$

where $j = 1.01^{12/365} - 1 = 0.0327188\%$

Real cost of coffee accumulated to time 2 is given by:

$$2000\left((1.01)^{24}\frac{87}{87} + (1.01)^{18}\frac{89}{87} + (1.01)^{12}\frac{93}{87} + (1.01)^{6}\frac{101}{87} + (1.01)^{0}\frac{99}{87}\right) = 2000(6.068187) = 12,136.37$$

So loan at $t = 2$ is equal to $133,322.14 - 60,861.57 + 12,136.37 = 84,597$

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*6 marks if correct, minus 1 mark for each mistake made, with*
*3 marks allocated to valuing accumulation of indexed cost of coffee correctly, and*
*3 marks allocated to the other components of the loan balance at t=2.*

*Loan balance at t = 2 can only be done retrospectively – any prospective attempt gives a maximum mark of 3/6.*

**Question Four (16 marks)**

[a]

[i] 53,000
[ii] 50
[iii] 529
[iv] 46,480
[v] 1,525
[vi] 1,060
[vii] 41
[viii] 2,538
[ix] 2,513

*1 mark for at least 1 correct, 0.5 marks for each correct answer thereafter.*

[b]

Loan 1 at t = 22: $2X(1.05)^{22} - 1744.04 s_{\overline{22}|5\%}$

Loan 2 at t = 22: $X(1.05)^{6} - (12)(102.05) s^{(12)}_{\overline{6}|5\%}$

$\rightarrow 2X(1.05)^{22} - 1744.04 s_{\overline{22}|5\%} = X(1.05)^{6} - (12)(102.05) s^{(12)}_{\overline{6}|5\%}$

$\rightarrow X\left(2(1.05)^{22} - (1.05)^{6}\right) = -(12)(102.05) s^{(12)}_{\overline{6}|5\%} + 1744.04 s_{\overline{22}|5\%}$

$\rightarrow X\left(2(1.05)^{22} - (1.05)^{6}\right) = \dfrac{58635.80}{4.51043} = 13,000$

So for loan 2 = 13,000, monthly payments in arrears of 102.05 pay this off:

$13,000 = (12)(102.05)\dfrac{1-v^n}{i^{(12)}} \rightarrow v^n = 0.48100 \rightarrow n = \dfrac{-\ln 0.48100}{\ln 1.05} = 15.$

So loan 2 is paid off at t=31.

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*4 marks to establish the correct value of X.*
*1 mark for determining 15 payments are required.*
*1 mark for stating t = 31.*

[c]

$$PV = K\left(v + v^2 + \ldots + v^n\right) = K\sum_{j=1}^{n} v^j$$

$$PV' = -K\sum_{j=1}^{n} j v^{j+1}$$

$$PV'' = K\sum_{j=1}^{n} j(j+1) v^{j+2}$$

$$c(i)=\frac{PV''}{PV}=\frac{K\sum_{j=1}^{n} j(j+1)v^{j+2}}{K\sum_{j=1}^{n} v^{j}}$$

When $i=0$, $v=1$

$$c(0)=\frac{\sum_{j=1}^{n} j(j+1)}{\sum_{j=1}^{n} 1}=\frac{\sum_{j=1}^{n} j^2+\sum_{j=1}^{n} j}{n}=\frac{n(n+1)(2n+1)}{6n}+\frac{n(n+1)}{2n}$$

$$=\frac{(n+1)(2n+1)+3(n+1)}{6}=\frac{(n+1)\left[(2n+1)+3\right]}{6}=\frac{(n+1)(n+2)}{3}=\frac{n^2}{3}+n+\frac{2}{3}.$$

*6 marks for correct derivation and answer.*
*max 3 marks if correct answer not derived.*
*1,2 or 3 marks to be awarded as appropriate for reasonable attempts.*

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**Question Five (18 marks)**

[a]
(i) Sell 2 "B" securities and 3 "C" securities, and purchase one "A" security.
This has a net cost of 0 at time 0.
If scenario 1 eventuates, the total value of 2B + 3C = 98.8 which matches 1A.
If scenario 2 eventuates, the total value of 2B + 3C = 119.6 which matches 1A.
If scenario 3 eventuates, the total value of 2B + 3C = 145.6 which is less than 1A's value of 146.6.
So for no cost now, there is a non-zero probability of a future gain (with no probability of future loss) → arbitrage exists.

*2 marks for choice of assets which works.*
*2 marks for explaining how it works (no marks awarded for this part if incorrect selection of and position in assets).*
*So marks awarded will be 0,2, 3 or 4 for this overall part [a].*

(ii) The arbitrage opportunity should mean that the price of B and C should decrease slightly, and the price of A should increase slightly (demand for A goes up and demand for B/C goes down).

*1 mark for correct statement about price of A.*
*1 mark for correct statement about prices of B and C.*

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[b]

At t = 3, $V_S = -V_L = 500 = S_0 e^{0.04(3)} - S_3$ →
$S_3 = S_0 e^{0.04(3)} - 500 = (45)(1000)e^{0.04(3)} - 500 = 50,237.36$

So annual growth rate in the share value was $\left(\frac{50,237.36}{45,000}\right)^{1/3} - 1 = 3.74\% = 3.7\%.$

*2 marks for correct derivation of $S_5$, 1 for correct subsequent growth rate.*

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[c]

$PV(1) = 1500a_{\overline{12|}} + 2500v^{12}a_{\overline{7|}} + 3500v^{19}\bar{a}_{\overline{3|}} = 31,585.68.$

$PV(2) = Kv + 2K(1.04)v^2 + 3K(1.04)^2 v^3 + 4K(1.04)^3 v^4 + ... + 25K(1.04)^{24} v^{25}$

$Let\ \text{w} = \dfrac{1.04}{1.03} and\ \text{j}=\dfrac{1.03}{1.04} - 1 = -0.00962$

$\rightarrow PV(2) = Kv(1 + 2w + 3w^2 + 4w^3 + ...25w^{24}) = Kv_{3\%}(I\ddot{a})_{25:j\%} = Kv\left(\dfrac{\ddot{a}_{25} - 25v^{25}}{d}\right)$

$= Kv(379.9643)$

$= K(368.8974)$

$\rightarrow$ K = 85.62

*2 marks for correct calculation of PV(1)*
*5 marks for correct calculation of PV(2)*
*1 mark for equating these expressions and solving for K.*

**Question Six (18 marks)**

[a]
[i]
Condition 1:
$P_A(i_0) = P_L(i_0)$

Let $A_1$ be the redemption value of the two-year bond.
Let $A_2$ be the annual installment of the perpetuity.
Let $L$ be the accumulated liability paid to the customer.

$P_A(i_0) = 15,000 + Y = A_1 v_{0.065}^2 + A_2 a_{\overline{\infty}|} = A_1 v_{0.065}^2 + A_2 i^{-1}$
$P_L(i_0) = X = L v_{0.065}^{10}$
$\Rightarrow 15,000 + Y = X \qquad (1)$

Condition 2:
$P_A'(i_0) = P_L'(i_0)$
$P_A'(i_0) = -2A_1 v_{0.065}^3 - A_2 i^{-2} = -2v_{0.065}^1(15,000) - Yi^{-1} = -28,169.01 - Y(0.065)^{-1}$
$P_L'(i_0) = -10Lv_{0.065}^{11} = -10v_{0.065}X = -(9.38967)\mathrm{X}$
$\Rightarrow 28,169.01 + Y(0.065)^{-1} = (9.38967)\mathrm{X} \qquad (2)$

Solving for $X$ and $Y$:
Multiply equation (1) by $0.065^{-1}$ and subtract equation (2)
$202,600.22 = (5.99495)X \Rightarrow X \cong 33,795$
$\Rightarrow Y = X - 15,000 = 18,795$

*1 mark for establishing equation 1*
*2 marks for establishing equation 2 or its equivalent if terms of volatility.*
*2 marks for subsequent correct working to get correct answers.*

[ii]
The assets are more spread out than the liabilities, so the fund (relative values of assets and liabilities) is immunised at 6.5%.

*0.5 marks for saying they are immunized, 1.5 marks for reasonable explanation.*
*note: just saying that the convexity of assets is greater than that of liabilities, without referring to any reason why that can be said, = 0 marks out of 1.5.*
*Note: calculations to show convexity (A) > convexity (L) are ok for full marks too, but not necessary.*

[b]

$$E\left[1+\tilde{i}\,\right]=\frac{0.98+1.12}{2}=1.05$$

$$E\left[200\tilde{S}(50)+350\tilde{S}(23)\right]$$

$$=200\left(E\left[1+\tilde{i}\,\right]\right)^{50}+350\left(E\left[1+\tilde{i}\,\right]\right)^{23}$$

$$=200\left(1.05\right)^{50}+350\left(1.05\right)^{23}$$

$$\cong 3,368.51$$

*0.5 marks for saying or using somewhere* $E\left[1+\tilde{i}\,\right]=1.05$

*0.5 marks for correct equation for accumulated value of each separate investment (1 mark total)*
*0.5 marks for correct accumulation of each separate investment (1 mark total)*
*0.5 marks for correct final answer*

[c]

Firstly, $3p+p+0.3+3p=1\rightarrow p=0.1$

$$\delta_t=\begin{cases}\ln(0.9) & \text{with probability } 0.3\\ \ln(1.03) & \text{with probability } 0.1\\ \ln(1.04) & \text{with probability } 0.3\\ \ln(1.09) & \text{with probability } 0.3\end{cases}$$

$$\rightarrow E(\delta)=(0.3)\ln(0.9)+(0.1)\ln(1.03)+(0.3)\ln(1.04)+(0.3)\ln(1.09)=0.008967$$

$$\rightarrow E(\delta^2)=(0.3)\left(\ln(0.9)\right)^2+(0.1)\left(\ln(1.03)\right)^2+(0.3)\left(\ln(1.04)\right)^2+(0.3)\left(\ln(1.09)\right)^2=0.0061071$$

$$\rightarrow \text{var}(\delta)=0.0061071-0.008967^2=0.006027$$

$$E\left[\ln\left[\tilde{S}(20)\right]\right]=20\cdot 0.008967=0.17934$$

$$\left(50,000E\left[\ln\left[\tilde{S}(20)\right]\right]=50,000\cdot 20\cdot E\left[\tilde{\delta}\right]=50,000\cdot 20\cdot 0.008967=8,967\right)$$

$$Var\left[\ln\left[\tilde{S}(20)\right]\right]=20\cdot Var\left[\tilde{\delta}\right]=0.120533$$

$$\left(50,000^2 Var\left[\ln\left[\tilde{S}(20)\right]\right]=50,000^2\cdot 20\cdot Var\left[\tilde{\delta}\right]=301,333,506\right)$$

$$\Pr\left[Z>-0.8416\right]=0.8$$

$$\Pr\left[\tilde{S}(20)>X\right]=\Pr\left[\tilde{Z}>\frac{\ln(X)-0.17934}{\sqrt{0.120533}}\right]$$

$$\rightarrow\frac{\ln(X)-0.17934}{\sqrt{0.120533}}=-0.8416\rightarrow\ln(X)=-0.11285$$

$$\rightarrow X=\exp(-0.11285)\rightarrow 50,000X=44,664$$

Answers in the range $(44{,}534, 44{,}689)$ are sufficient (these relate to $Z>-0.84$ & $Z>-0.85$)

*0.5 marks for calculating p*
*1 mark for* $E(\delta)=0.0089672$
*1 mark for* $E(\delta^2)=0.0061071$
*0.5 marks for* $\text{var}(\delta)=0.0060267$
*1 mark for* $E\left[\ln\left[\tilde{S}(20)\right]\right]$
*1 mark for* $Var\left[\ln\left[\tilde{S}(20)\right]\right]$
*0.5 marks for correct Z value at 80%*
*2 marks for correct probability statement at workings*
*0.5 marks for correct X.*
*To a maximum of 6 marks, if X not correct.*