MATH6222: Homework #3

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Problem 1

Find and prove a formula for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

Proof:

First we observe that:

$$\begin{aligned}
&\text{when } n=1, \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \\
&\text{when } n=2, \sum_{i=1}^{2} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\
&\text{when } n=3, \sum_{i=1}^{3} \frac{1}{i(i+1)} = \frac{2}{3} + \frac{1}{3(3+1)} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4} \\
&\text{when } n=4, \sum_{i=1}^{4} \frac{1}{i(i+1)} = \frac{3}{4} + \frac{1}{4(4+1)} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5} \\
&\quad \cdots
\end{aligned}$$

We guess the formula could be

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Base step: When $n = 1, \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$, proved.

Inductive step: Suppose the formula holds for $n = k > 1, k \in \mathbb{Z}$. We want to show it also holds for $n = k + 1$.

$$
\begin{aligned}
\sum_{i=1}^{k} \frac{1}{i(i+1)} &= \frac{k}{k+1} \\
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+1+1)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}
$$

Therefore, when $n = k+1$, our formula still holds. Thus by inductive hypothesis, the formula $\sum\limits_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ is true.

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Problem 3

Determine the set of positive real number x such that

$$x^n + x < x^{n+1}$$

Solution:

In this problem, we suppose the number n is a positive integer (i.e. $n = 1, 2, 3, \dots$)

Let $n = 1$ and observe:

$$\begin{aligned} x^1 + x &< x^2 \\ (x-2)x &> 0 \end{aligned}$$

Since x is positive, so $x > 2$.
Now we want to check if $x > 2$ holds for $n = 2, 3, \dots$

Base step: We can prove this by induction where base step $n = 1$ is proved already.

Inductive step: Suppose this is true for $n = k$ where k is a non-negative integer greater than 1, then we want to show $n = k + 1$ holds.

Since $x^k + x < x^{k+1}$, both sides multiplied by x:

$$x^{k+1} + x^2 < x^{k+2}$$

Also, $x > 2$, then $x(x-2) > 0$, so $x^2 > 2x > x$. Hence

$$x^{k+1} + x < x^{k+1} + x^2 < x^{k+2}.$$

So we proved the case for $n = k+1$. By inductive hypothesis, we have the set for $x^n + x < x^{n+1}$ to be $\{x > 2, x \in \mathbb{R}\}$.

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Problem 4

Starting from 0, two players take turns adding $1, 2$, or 3 to a single running total. The first player who brings the total to 1000 or more wins. Prove that the second player has a winning strategy for this game.

Proof:

Quick thinking (backward): what is the scenario that the first player make a move that is "very close to winning" but cannot win? The idea is somehow player 2 gets the score 996, so that player 1 could not win the game no matter what value he adds to the total. Then player 2 only needs to add the difference between 4 and player 1's value, then player 2 wins in the end.

This means, if player 2 gets 996 first, player 2 wins (although need one more move). And we think about this recursively:

How can we guarantee that player 2 gets 996 first? Somehow we should let player 2 get 992 first!

What about letting player 2 get the multiples of 4 **all the way**? Let's prove this.

Suppose the goal of the game is to hit $4n$ scores, where n is an non-negative integer, and player 2 has a strategy to win the game. (We call this $P(n)$)

Base step: For $n = 1$, player 1 could add $1, 2, 3$ to the total. But player 2 could add $3, 2, 1$ correspondingly to win the game. $P(1)$ is true.

Inductive step: Suppose it is true for $n = k$, for k is an non-negative integer greater than 1. This means player 2 can hit a score of $4k$ first. Now we aim for $4(k + 1)$ total score. Similarly, again, player 1 could add $1, 2, 3$ to the total, resulting a total score of $4k + 1, 4k + 2$, or $4k + 3$. Then player 2 could add $3, 2, 1$ to the total, and the result would always be $4(k + 1)$. Player 2 wins again. $P(k + 1)$ is proved too.

Therefore, player 2 always has a strategy to win the game.

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Problem 5

Recall that an L-tile is just a tile with three squares shaped like an L. We say a board admits an L-tiling if it is possible to completely cover it with L-tiles, such that each tile lies completely on the board, and no two tiles overlap.

(a) Prove that a $2^k \times 2^k$ chessboard with a single square in the lower left corner deleted admits an L-tiling, for any $k \in \mathbb{N}$.

(b) Prove that a $2^k \times 2^k$ chessboard with *any* single square deleted admits an L-tiling, for any $k \in \mathbb{N}$.

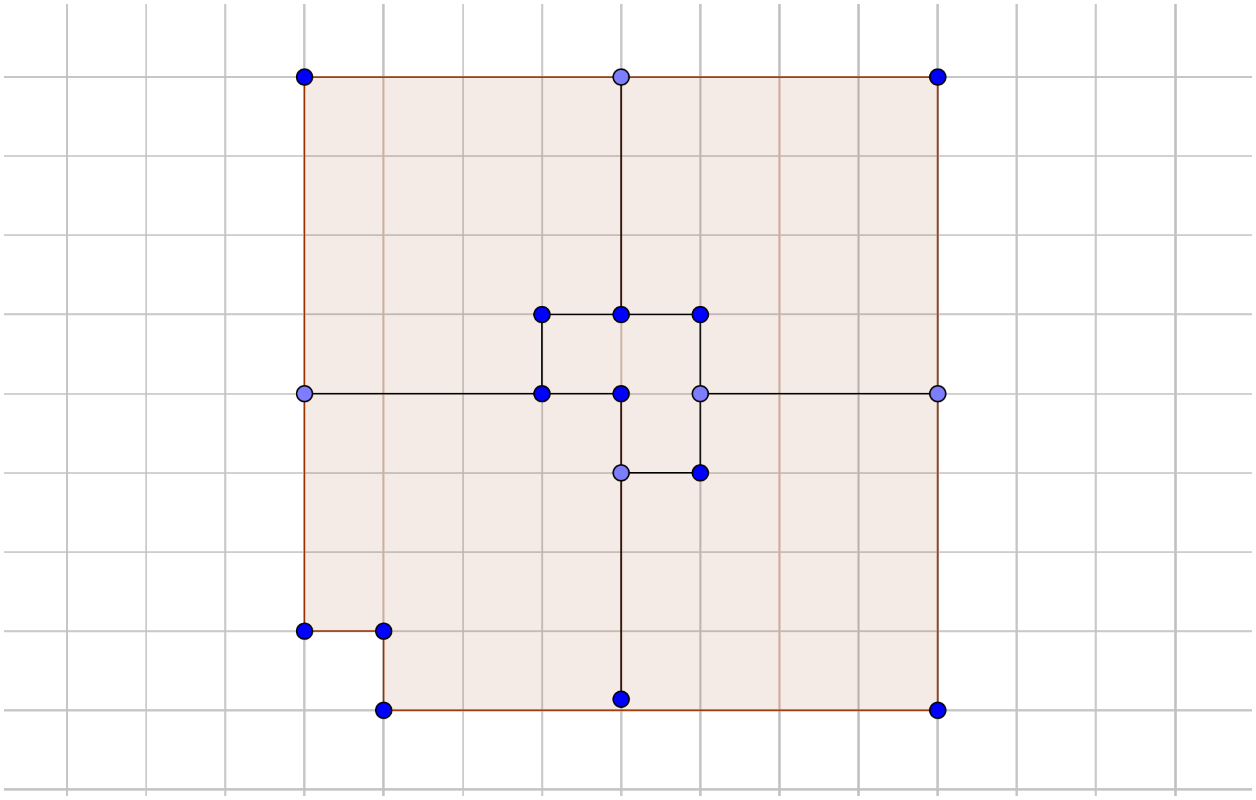
(a) **Proof:**

We can prove this by induction.

Base step: $k = 1$, the 2×2 chessboard with a single square in the lower left corner deleted itself is a L-tile.

Inductive step: Suppose we can cover a $2^n \times 2^n$ chessboard, we want to show we can cover a $2^{n+1} \times 2^{n+1}$ chessboard as well.

Problem 5 continued on next page. . .



We can divide the $2^{n+1} \times 2^{n+1}$ chessboard in the way shown above, so that the separate 5 parts of chessboard are 4 $2^n \times 2^n$ (with one corner unit removed) chessboard, and 1 L-tile. It is obvious that L-tile can be covered by one L-tile, and according to inductive hypothesis, the 4 $2^n \times 2^n$ chessboard can be covered by certain number of L-tiles too. Hence, the $2^{n+1} \times 2^{n+1}$ chessboard can be covered by L-tiles. And we are done.

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(b) **Proof:**

Similarly, we use induction to prove this as well.

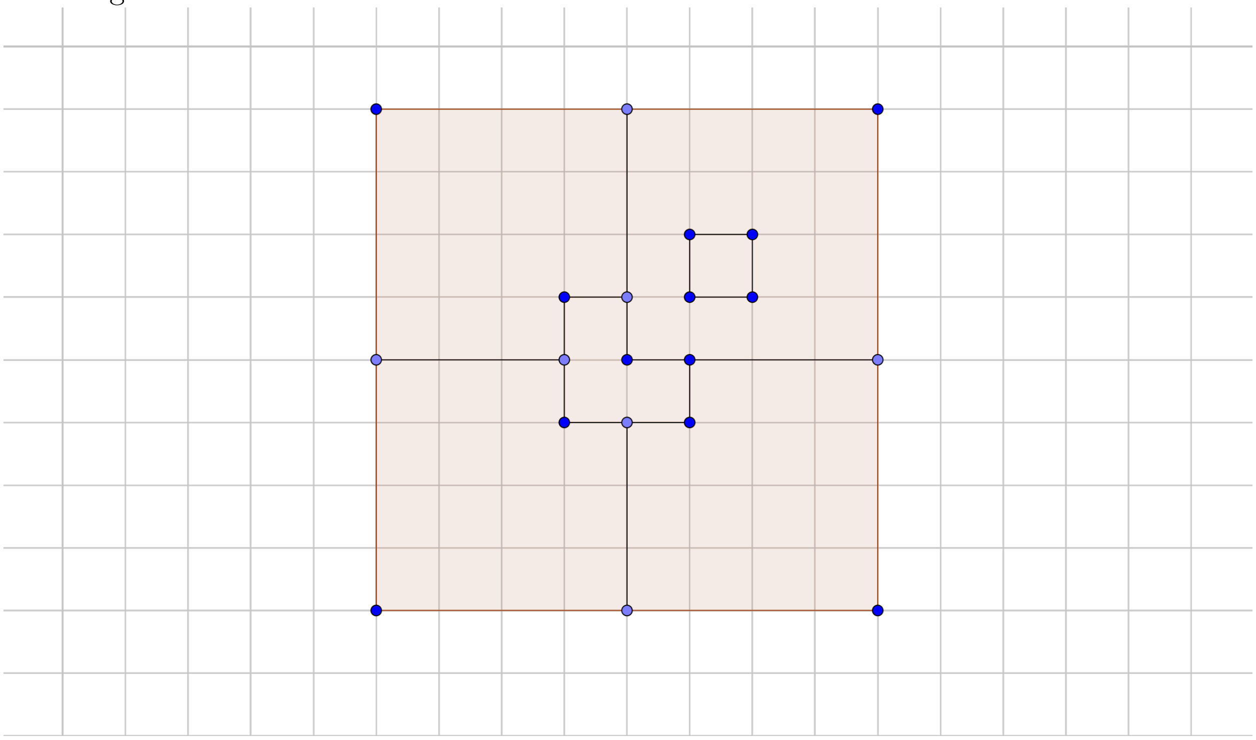
Base step: when $k = 1$, the 2×2 chessboard with a single square in the lower left corner deleted is still a L-tile.

Inductive step: Suppose when $k = n$, the $2^n \times 2^n$ chessboard can be covered by L-tiles. We want to show when $k = n + 1$, the $2^{n+1} \times 2^{n+1}$ chessboard can be covered by L-tiles, too.

The strategy is that we find the midpoint of edges of this large square, connect the opposite two midpoints, separating the large square into 4 small squares each with edge length 2^n.

Problem 5 continued on next page. . .

Now we observe which part contains the deleted 1 unit of square. And we treat it as the deleted lower left corner in part (a). And similarly, we cut the large square into 5 parts as following:



So we have 1 L-tile part, 3 $2^n \times 2^n$ chessboards missing a corner square, and a $2^n \times 2^n$ chessboard missing a random square inside it. Again, by inductive hypothesis, the $2^n \times 2^n$ chessboards (missing one square) can be covered by L-tile, and the L-tile can be covered by an L-tile directly.

Hence a $2^k \times 2^k$ chessboard with *any* single square deleted admits an L-tiling, for any $k \in \mathbb{N}, k > 0$.

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