Practice problems solutions

Question 1 solution

a) For an induced subgraph $H \leq G$, consider the indicator function $1_H : V(G) \to \{0,1\}$ defined as $1_H(v) = \begin{cases} 1 & \text{if } v \in V(H); \\ 0 & \text{otherwise.} \end{cases}$

Indicator functions are in bijective correspondence with induced subgraphs, so G has 2^n induced subgraphs, including G itself and including the null graph.

b) Let C be a shortest cycle in G, and consider the induced subgraph on $V(C)$, denoted H. Each vertex in H has valence ≥ 2, and H is connected. If H contains an edge $e \notin E(C)$ then e is a chord. Because G is simple, for $\psi(e) = \{u, v\}$, $C = uP_1vP_2u$, P_1 and P_2 have length >1, so uP_1veu has length $|P_1|+1 < |P_1|+|P_2|$ which is the length of C, contradicting C being a shortest cycle. So $V(H) = V(C)$, $E(H) = E(C)$, therefore $H = C$.

Q.E.D.

Question 2 solution

a) Let $u, v \in V(G)$ be vertices such that the distance from u to v is d, the diameter of G.

Partition $V(G)$ into sets $S_0 = \{u\}, S_1, S_2, \dots S_d$ so that $d(u,x) = i$ iff $x \in S_i$. For $|i - j| > 1$ there is an edge in $\bar{G}$ between each vertex in S_i and each vertex in S_j. For $|i-j| = 1$ there is an edge from $x \in S_i$ to some $y \in S_{(i \pm 3) \bmod d}$ and from there to each $z \in S_j$, because $|(i \pm 3) \bmod d - j| > 1$ for $d > 3$. $((i+3)$ if $j > i$, $(i-3)$ otherwise$)$. So the diameter of $\bar{G}$ is less than 3. Q.E.D.

b) If $\mathrm{diam}(G) > 3$ then $\mathrm{diam}(\bar{G}) < 3$ by (a), so $G \ncong \bar{G}$.

If $\mathrm{diam}(G) \leq 3$ then we are OK.

Question 3 solution

a) **Lemma**: The centre c of T is not a leaf.

Proof: Assume c were a leaf.

$m_c := \max\{d(c,v) \mid v \in V(T)\} \geq 2$ because vertex u adjacent to c is not a leaf. For each $v \neq c$, $v \in V(T)$, we have $d(u,v) = d(c,v) - 1$, and $d(u,c) = 1$. This contradicts minimality of m_c. $\square$

Thus, $m_c := \max\{d(c,v) \mid v \in V(T)\}$ is realized for v a leaf (if v were not a leaf, the path could be extended, contradicting maximality.

For every non-leaf vertex u, set

$S(u) := \{d(u,v) \mid v \text{ is a leaf of } T\}$.

Deleting all leaves of T reduces every element of $S(u)$ by 1 for all u. Thus the centre is unchanged.

b) Induction, on number of vertices

Base: $|V(T)| = 1$ that vertex is the centre.

$|V(T)| = 2$ vertices are adjacent centres.

Induction hypothesis: True for $|V(T)| \leq n$

Induction: $V(T) = n+1$. Delete all leaves. Centres are unchanged, and we are finished by induction.

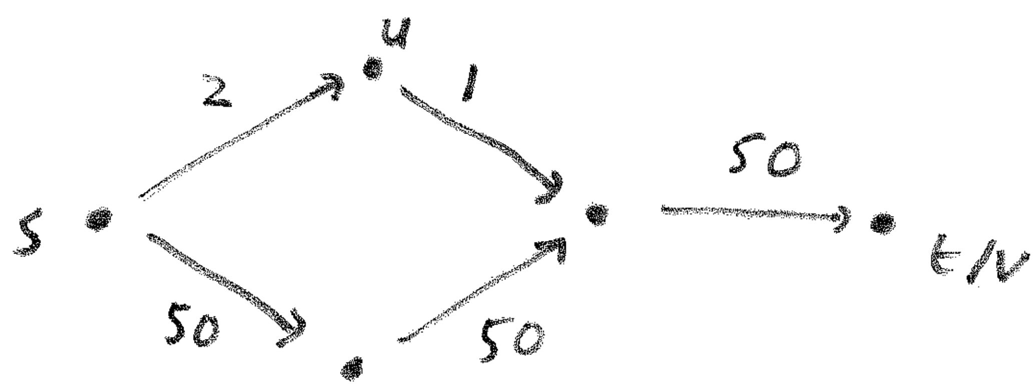
Question 4 solution

For each pair of vertices $x_1, x_2 \in X$, delete all edges between x_1, x_2 and joint neighbours of both x_1 and x_2. This subtracts an even number from $\deg(x_1)$ and from $\deg(x_2)$, and from each of their joint neighbours y, deletes both $\{y, x_1\}$ and $\{y, x_2\}$. Thus the condition is preserved. Repeat until no two vertices in X have joint neighbours. Then match arbitrarily.

Question 5 Solution

There is a problem with the question.

Counterexample:



Question 6 solution

If G has girth 6, then

$$2e = \sum \deg(f_i) \geq 6f \implies f \leq \frac{e}{3}.$$

By Euler's formula, if G were planar then

$$2 \leq v - e + \frac{e}{3} = v - \frac{2e}{3} \implies 6 \leq 3v - 2e$$

But for a cubic graph, $3v = 2e$ by the handshake lemma, so this is impossible.

Question 7 solution

Form a tree T with a vertex for each block of G, and an edge for each common vertex between blocks, and choose a root r. For each block B_v $(v \in V(T))$, colour B_v with the minimal number of colours. $f_v : V(B_v) \to C_v$.

Starting from the coloured B_r, for each of its coloured neighbours (in T) B_v, identify the relevant vertex, and permute the colours of B_v so as to agree with colours of B_r – if $C_v = \{1, 2, \dots, \chi(B_v)\} \subseteq \mathbb{N}$ and the vertex is coloured k in B_r, permute the image of f_v so that the colour of B_v agrees. Continue until G is coloured.