map of 1.7 (Connectedness)

a disconnection for a set S is a pair (S_1, S_2) such that $S \subset S_1 \cup S_2$, $\overline{S_1} \cap S_2 = \emptyset$ & $S_1 \cap \overline{S_2} = \emptyset$

$S \subseteq \mathbb{R}^n$ is disconnected if it has a disconnection

S is connected if it has no disconnection

1.25 Connected subsets of $\mathbb{R}$ are precisely the intervals

Pf: if $c \notin I$ then
$S_1 = (-\infty, c) \cap S$
$S_2 = S \cap (c, +\infty)$
forms a disconnection for S

1.26 Continuous functions map connected sets to connected sets; that is if $f : \mathbb{R}^n \to \mathbb{R}^m$ is cont. & $S \subset \mathbb{R}^n$, then $f(S) = \{f(x) : x \in S\} \subset \mathbb{R}^m$ is also connected

IVT: $f : S \to \mathbb{R}$, $V \subset S$ (connected),
if $a, b \in V$ and $f(a) < t < f(b)$
Then $\exists c \in V$ such that $f(c) = t$

proof:
- by 1.26 $f(V)$ is a connected subset of $\mathbb{R}$
- by 1.25 $f(V)$ is an interval
- by definition of interval $t \in f(V)$
- by definition of $f(V)$. $\exists c \in V : f(c) = t$.

$I \subset \mathbb{R}$ is said to be an interval if
$\forall a, b \in I \cdot \forall c \in \mathbb{R}$ $a < c < b \Rightarrow c \in I$

A set $S \subseteq \mathbb{R}^n$ is called arc connected or path connected if for each two pts a, b there is a map $f : [0\ 1] \to \mathbb{R}^n$ $f(0) = a$, $f(1) = b$ & $f(t) \in S$ for all $t \in [0\ 1]$

1.28: if $S \subset \mathbb{R}^n$ is arc connected then it is connected

1.29 connected $\not\Rightarrow$ arc connected

* Open balls are path connected.

1.30 if S is open & connected then it is path connected.