Classes on next M, T & W.

Inhomogeneous linear systems

$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \qquad \vec{x}(t_0) = \vec{\xi} \quad (*)$

Sps $\vec{x}^{(1)}, \vec{x}^{(2)}$ are fund. set of solution of $\vec{x}' = P(t)\vec{x}$, and

$\Psi(t) = (\vec{x}^{(1)}(t), \vec{x}^{(2)}(t))$ the fund. matrix

Find solution for $(*)$

$\vec{x}(t) = \Psi(t)\vec{u}(t)$

$\vec{x}'(t) = \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$

$\quad = P(t)\Psi(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$

$\quad = P(t)\vec{x}(t) + \Psi(t)\vec{u}'(t)$

This should be equal to $P\vec{x} + \vec{g}$ by $(*)$

$\Rightarrow \Psi(t)\vec{u}'(t) = \vec{g}(t)$

$\vec{u}'(t) = \Psi(t)^{-1}\vec{g}(t)$

$\vec{u}(t) - \vec{u}(t_0) = \int_{t_0}^{t} \Psi(s)^{-1}\vec{g}(s)\,ds$

$$\vec{x}(t) = \underset{\overset{\parallel}{\vec{\xi}}}{\vec{x}(t_0)} + \Psi(t)\int_{t_0}^{t}\Psi(s)^{-1}\vec{g}(s)\,ds$$ variation of parameters

Example: $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\vec{x} + \begin{pmatrix} t^{-1} \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$ has $r = 0$ repeated eigenvalue

$\cdots - \quad \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{x}^{(2)} = \begin{pmatrix} 2t \\ 4t-1 \end{pmatrix}$

$\Psi(t) = \begin{pmatrix} 1 & 2t \\ 2 & 4t-1 \end{pmatrix}$

$\Psi(t)^{-1}\vec{g}(t) = -\begin{pmatrix} 4t-1 & -2t \\ -2 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} t^{-1}-4 \\ 2t^{-1} \end{pmatrix}$

$\int_{t_0}^{t} \Psi(s)^{-1}\vec{g}\,s\,ds = \begin{pmatrix} \ln(t) - 4t - \ln(t_0) + 4t_0 \\ 2\ln(t) - 2\ln(t_0) \end{pmatrix}$

multiply by $\Psi(t)$, and done.

Another method is undetermined coefficients

Apply to $\vec{x}' = A\vec{x} + \vec{g}(t)$

If components of $\vec{g}(t)$ are products of polynomial, $\sin(\mu t)$, $\cos(\mu t)$, exponentials.

Example.

$$\vec{x}' = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 2t \end{pmatrix}$$

Find solution: $\vec{x}(t) = \vec{a} + t\vec{b}$

Plug into $\vec{x}' = A\vec{x} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Get $\underbrace{\vec{b}}_{\vec{x}'} = A\vec{a} + tA\vec{b} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Compare coefficients:

$$\vec{b} = A\vec{a} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\vec{0} = A\vec{b} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Second equation gives $\vec{b} = A^{-1}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$$= -\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

Now solve this equation for $\vec{a}$:

$$\vec{a} = A^{-1}\left(\vec{b} - \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 17 \\ -25 \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = \vec{a} + t\vec{b} = \begin{pmatrix} 17 + 4t \\ -25 - 6t \end{pmatrix}$$

autonomous $\longrightarrow$ means the RHS does not involve t

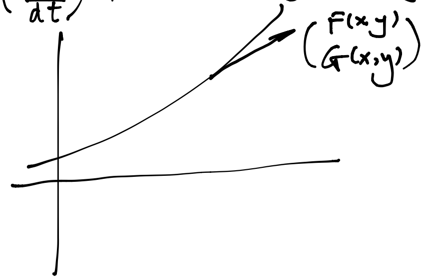
$$\vec{x}' = \vec{F}(\vec{x}, \cancel{t})$$

Nonlinear systems

$$(*) \begin{cases} \dfrac{dx}{dt} = F(x, y) \\ \dfrac{dy}{dt} = G(x, y) \end{cases}$$

If $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution of $(*)$. then the curve it traces in x-y plane is the trajectory ("solution curve")

$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ plays role of velocity vector (tangent to solution curve)



Get equation for solution curves: $y = \varphi(x)$
given by differential equation

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{G(x,y)}{F(x,y)}.$$

Suppose (x_0, y_0) satisfies

$$F(x_0, y_0) = G(x_0, y_0) = 0.$$

Then $x(t) = x_0$, $y(t) = y_0$ is a solution of $(*)$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is called critical point, and $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is called equilibrium solution.

Example: $x' = x - y = F(x,y)$
$y' = 1 - x^2 = G(x,y)$

critical point $= ?$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

(to be continued)