Research School of Finance, Actuarial Studies and Statistics

EXAMINATION

Semester 1 - End of Semester, 2016

STAT2032/STAT6046 Financial Mathematics

Examination/Writing Time Duration: 180 minutes
Reading Time: 15 minutes

Exam Conditions:
Central Examination
Students must return the examination paper at the end of the examination
This examination paper is not available to the ANU Library archives

Materials Permitted in the Exam Venue:
(No electronic aids are permitted e.g. laptops, phones)
Calculator (non-programmable)
Paper-based dictionary (no approval required)
Prescribed formula sheet and tables as supplied by lecturer

Materials to be Supplied to Students:
1×20 page plain
Scribble paper

Total Marks Available: 100

Instructions to Students:

- Attempt ALL questions.
- Start your solution to each question on a new page.
- Unless otherwise stated, show all working steps.

Question 1 (11 marks)

Consider two bank accounts A and B, where account A earns **compound** interest and account B earns **simple** interest. A series of deposits are to be made annually in arrears by Anna in either one of these two accounts. The first annual deposit is \$4,000 and each subsequent annual deposit is \$400 more than the previous deposit amount. When the amount of annual deposit hits \$10,000 for the very first time, Anna will stop depositing further amount into the account.

(a) If the constant effective annual rate of interest for account A is 4% per annum, calculate the future accumulated value (rounded to the nearest cent) at 4 years after the last deposit payment if Anna deposits into account A. (4 marks)

$$\begin{aligned} FV &= \Big(3600 s_{\overline{16}|} + 400(Is)_{\overline{16}|}\Big)(1+i)^4 \\ &= \Big(3600\big(\frac{1.04^{16}-1}{0.04}\big) + 400\big(\frac{\frac{1.04^{16}-1}{\frac{0.04}{1.04}} - 16}{0.04}\big)\Big)(1.04^4) \\ &= 170265.2345 = 170265.23 \end{aligned}$$

(b) Following from part (a), if Anna intends to withdraw her accumulated amount at 4 years after the last deposit payment, calculate the **constant** simple interest rate per annum (rounded to 2 decimal places in %) for account B such that Anna would be indifferent between both accounts.

You may find the following result useful:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(5 marks)

$$\begin{aligned} &\sum_{t=1}^{16} \big(3600 + 400t\big)\big(1 + (20-t)i\big) \\ &= \sum_{t=1}^{16} 3600 + 400t + 72000i + 4400ti - 400t^2 i \\ &= 3600(16) + 400\frac{16(17)}{2} + 72000(16)i + 4400i\frac{16(17)}{2} - 400i\frac{16(17)(33)}{6} \\ \Rightarrow 170265.23 &= 112000 + 1152000i \\ \Rightarrow i &= 0.050577456 = 5.06\% \end{aligned}$$

(c) Suppose that the simple interest rate offered by account B is the solution found in part (b), **without** performing any calculations, comment on which account would have a higher accumulated amount at the time when the last deposit is made. Justify your answer. (2 marks)

Account B will have a higher accumulated amount. Account A that earns compound interest will produce more interest payments for the 4-year period than account B that earns simple interest. Given that they have identical accumulated value 4 years later, account B will have a higher amount at the time the last deposit is made.

Question 2 (11 marks)

(a) "The summation of interest payment and capital repayment (in terms of dollar amount) for **each** loan repayment is always constant." True or false? Justify your answer. (1 mark)

False. Interest payment + capital repayment = loan repayment amount, but loan repayment amount is not necessarily constant.

A loan of $\$Z$ is to be fully repaid by $4n$ level annual repayments of $\$X$ made in arrears at a constant effective interest rate. Shortly after the $(2n)$-th repayment, the loan outstanding amount at that time is 80% of the original loan amount Z.

(b) Calculate the **proportion** of the capital repayment in the $(3n+1)$-th repayment, that is, the value of $\frac{\text{capital repayment}_{3n+1}}{X}$. (5 marks)

$$\begin{aligned}
L_0 &= X a_{\overline{4n}|} \\
L_{2n} &= X a_{\overline{2n}|} = 0.8 L_0 = 0.8 X a_{\overline{4n}|} \\
\Rightarrow \frac{a_{\overline{4n}|}}{a_{\overline{2n}|}} &= \frac{1-v^{4n}}{1-v^{2n}} = 1 + v^{2n} = \frac{5}{4} \\
v^{2n} &= 0.25 \\
\frac{C_{3n+1}}{X} &= \frac{X - iL_{3n}}{X} \\
&= \frac{X - iXa_{\overline{n}|}}{X} = v^n = 0.5
\end{aligned}$$

(c) Given that the total capital repayment of the first $3n$ level annual repayments is \$28,000, calculate the value of Z and the total capital repayment from the $(n+1)$-th annual repayment until the $(3n)$-th annual repayment. (5 marks)

$$\begin{aligned}
L_0 - L_{3n} &= X a_{\overline{4n}|} - X_{\overline{n}|} \\
&= X\Big(\frac{1-v^{4n}-(1-v^n)}{i}\Big) \\
&= X\Big(\frac{v^n - v^{4n}}{i}\Big) = 0.4375\frac{X}{i} = 28000 \quad \Rightarrow \frac{X}{i} = 64000 \\
Z = L_0 &= \frac{X}{i}\big(1-v^{4n}\big) = 60000 \\
L_n - L_{3n} &= X a_{\overline{3n}|} - X_{\overline{n}|} \\
&= X\Big(\frac{1-v^{3n}-(1-v^n)}{i}\Big) \\
&= \frac{X}{i}\big(v^n - v^{3n}\big) = 24000
\end{aligned}$$

Question 3 (10 marks)

Jimmy invested continuously in a fund for 10 years. The payment rate **during** year t is \$$100t$. 10 years later, the continuous investment stream stopped but Jimmy waited for another 5 years before he decided to withdraw the fund balance. The total accumulated value at that time was \$9,121.42.

(a) Calculate the money-weighted rate of return quoted on an annual basis for Jimmy's continuous investment, rounded to the nearest **integer** %. (4 marks)

$$100(I\bar{s})_{\overline{10|}}(1+i)^5 = 100(I\bar{a})_{\overline{10|}}(1+i)^{15} = 9121.42$$

$$100\Big(\frac{\frac{1-(1+i)^{-10}}{\frac{i}{1+i}} - 10(1+i)^{-10}}{\ln(1+i)}\Big)(1+i)^{15} = 9121.42$$

$$i = 0.06 = 6\%$$

(b) 10 years before the continuous investment, Jimmy purchased a special investment product for a price of \$2,000. The sole purpose of this special product was to generate the entire continuous payment contribution into the fund. By treating the special investment product as a project, calculate its net present value (rounded to the nearest cent) at a risk discount rate of 4% per annum. (3 marks)

$$\begin{aligned} NPV &= -2000 + \frac{1}{1.04^{10}}(100)(I\bar{a})_{\overline{10|}i=0.04} \\ &= -2000 + \frac{1}{1.04^{10}}(100)\Big(\frac{\frac{1-1.04^{-10}}{\frac{0.04}{1.04}} - 10(1.04^{-10})}{\ln 1.04}\Big) \\ &= 893.2118644 = 893.21 \end{aligned}$$

(c) Following from part (b), calculate the discounted payback period (expressed in terms of **integer** years) for the special investment product. (3 marks)

$$AP(DPP) \geq 0 \Leftrightarrow NPV^{DPP}(0.04) \geq 0$$

$$NPV^{t^*+10}(0.04) = -2000 + \frac{1}{1.04^{10}}(100)\Big(\frac{\frac{1-1.04^{-t^*}}{\frac{0.04}{1.04}} - t^*(1.04^{-t^*})}{\ln 1.04}\Big) \geq 0$$

By trial and error $t^* = 9 \Rightarrow DPP = 19$

Question 4 (10 marks)

Firm XYZ claims that its stock is inflation-protected because it promises to pay annual dividend in arrears perpetually after a deferred period of 5 years. The first dividend payment (to be paid 6 years from now) is set to be \$1 and the subsequent dividend amount in year t will grow according to the inflation rate during year t. Stock XYZ's cost of equity, which is the constant value used to discount the future cash flows to obtain its fair value, is known to be 8% per annum.

Firm ABC pays continuous dividend at a dividend yield equals to the "force of inflation" – the continuous version of the discrete annual inflation rate. Stock ABC is currently trading at \$50 per share.

Assume that dividends are always reinvested to increase the number of shares held.

(a) If stock XYZ is fairly valued at \$13.61 per share right now, calculate the implied **constant** discrete annual inflation rate that is applicable perpetually, rounded to 2 decimal places in %. (3 marks)

$$
\begin{aligned}
13.61 &= \frac{1}{1.08^5}\sum_{t=1}^{\infty}\frac{(1+j)^{t-1}}{1.08^t} \\
&= \frac{1}{1.08^5}\frac{1}{1.08}\frac{1}{1-\frac{1+j}{1.08}} = \frac{1}{1.08^5}\frac{1}{0.08-j} \\
\Rightarrow j &= 0.029993887 = 3.00\%
\end{aligned}
$$

(b) Jack is buying a forward contract on stock ABC right now in order to purchase 100 units of shares 2 years later at a pre-determined forward price. Given a 2% constant risk-free force of interest and if the constant force of inflation is calculated according to the solution obtained in part (a), calculate the forward price of the contract, rounded to the nearest cent. (2 marks)

$$
\begin{aligned}
\delta = 0.02, \delta_j &= \ln 1.03 \\
K &= 100(50e^{(0.02-\ln 1.03)(2)}) \\
&= 4905.31989 = 4905.32
\end{aligned}
$$

(c) Suppose that you are interested to invest in 100 shares of stock ABC and the **market** forward price for 100 shares to be transacted 2 years later is \$4,800. Would you purchase the share right now or would you follow Jack's strategy to buy the forward contract? Justify your answer by considering the number of shares held 2 years later under both strategies. Describe the two strategies clearly. (5 marks)

Buy the forward contract right now and pay zero initial cost. Invest $4800e^{-0.02(2)}$ in riskless investment right now. The amount will grow to \$4,800 2 year later, pay the forward price of \$4,800 and obtain 100 shares of stock ABC.

The same amount of $4800e^{-0.02(2)}$ can be used to purchase $\frac{4800e^{-0.04}}{50}$ shares right now, and will grow to $\frac{4800e^{-0.04}}{50}(1.03^2) = 97.85294554$ shares of stock ABC.

Decision: buy the forward contract.

Question 5 (16 marks)

On 1 January 2016, Cathy buys a 10-year semi-annual coupon bond that pays coupons in arrears at a nominal rate of 8% per annum. This bond is redeemed at the par value of \$1,000 and its current price P is determined such that Cathy would obtain a **gross** effective redemption yield of at least 10% per annum.

On 1 January 2016, Alan buys a 20-year bond that pays annual coupon in arrears at a coupon rate of 6% per annum. The bond issuer can choose to redeem the bond at 120% if it is redeemed on any coupon paying dates of 31 December between year 2026 and year

2030. Alternatively, the bond issuer can redeem the bond at 105% on any 31 December between year 2031 and year 2035.

Taxes are payable at the same time as the income and capital gains (if applicable).

(a) Calculate the maximum value of P, rounded to the nearest cent. (2 marks)

Maximum value of P occurs at the minimum gross yield of 10%.

$$\begin{aligned} P &= 0.08(1000)\Big(\frac{1-1.1^{-10}}{2(1.1^{0.5}-1)}\Big) + \frac{1000}{1.1^{10}} \\ &= 889.1050276 = 889.11 \end{aligned}$$

(b) John, who is liable to pay an income tax of 20% and a capital gains tax of 30%, purchases the same bond bought by Cathy at the maximum price of P calculated in part (a). Calculate the **net** effective redemption yield i for John from the bond purchase, rounded to 2 decimal places in %. (4 marks)

$$889.11 = (1-0.2)0.08(1000)\Big(\frac{1-(1+i)^{-10}}{2\big((1+i)^{0.5}-1\big)}\Big) + \big(1000-0.3(1000-889.11)\big)(1+i)^{-10}$$

$$\begin{aligned} f(i) &= 32\Big(\frac{1-(1+i)^{-10}}{(1+i)^{0.5}-1}\Big) + 966.733(1+i)^{-10} - 889.11 \\ f(0.079) &= 2.588745557, \quad f(0.080) = -3.45668831 \\ i &\approx \frac{0.08(2.588745557) - 0.079(-3.45668831)}{2.588745557-(-3.45668831)} = 0.079428215 = 7.94\% \end{aligned}$$

(c) Alan is subjected to an income tax of 10% and a capital gains tax of 15%. If he intends to achieve a minimum net redemption yield of 5%, determine the corresponding expected redemption date for the redemption rates of 120% and 105% **respectively**. Explain the rationales clearly. (3 marks)

$$\begin{aligned} &\text{year } 11-15: \quad (1-t_I)g = (1-0.1)\frac{0.06(1)}{1.2} = 0.045 < i = 0.05 \Rightarrow \text{capital gains} \\ &\text{year } 16-20: \quad (1-t_I)g = (1-0.1)\frac{0.06(1)}{1.05} = 0.051428571 > i = 0.05 \Rightarrow \text{capital loss} \end{aligned}$$

For the optional redemption from 31 December 2026 to 2030, Alan will have capital gains. The bond issuer would choose to redeem at the latest possible date (31 December 2030) such that Alan would obtain the minimum yield.

For the optional redemption from 31 December 2031 to 2035, Alan will have capital loss. The bond issuer would choose to redeem at the earliest possible date (31 December 2031) such that Alan would obtain the minimum yield.

(d) Calculate Q (rounded to the nearest cent), the price to be paid by Alan on 1 January 2016 to attain the minimum net redemption yield of 5% if the face value of the bond is \$10,000. (4 marks)

$$
\begin{aligned}
P_{\text{redeemed at 2030}} &= \frac{(1-0.1)0.06(10000)\left(\frac{1-1.05^{-15}}{0.05}\right)+(1-0.15)(1.2)(10000)(1.05^{-15})}{1-0.15(1.05^{-15})}\\
&= 11328.79107\\
P_{\text{redeemed at 2031}} &= (1-0.1)0.06(10000)\left(\frac{1-1.05^{-16}}{0.05}\right)+1.05(10000)(1.05^{-16})\\
&= 10662.56654\\
Q &= \min(11328.79107, 10662.56654) = 10662.56654 = 10662.57
\end{aligned}
$$

(e) Suppose Alan paid the price Q found in part (d). Without performing **further** calculations, explain why Alan would obtain at least 5% net redemption yield regardless of whether the bond issuer chooses to redeem at the two expected redemption dates determined in part (c) or other possible redemption dates. (3 marks)

If the bond is redeemed at 2031, Alan would obtain 5% yield. If the bond is redeemed later than 2031, the capital loss would be delayed and hence Alan would attain a higher yield.

If the bond is redeemed at 2030, the fair value should be higher (11328.79107), hence by paying Q Alan would get a higher yield. If the bond is redeemed earlier than 2030, the capital gains would be accelerated and resulting in a higher yield.

Question 6 (13 marks)

A fund has liabilities of \$$P$ million (due at time t_1) and \$$Q$ million (due at time t_2), where $t_1 \neq t_2$. The fund is planning to invest in a portfolio of assets consisting of zero-coupon bonds to exactly match the modified durations of its assets and liabilities.

(a) "The Macaulay's duration of a zero-coupon bond is independent of any spot or forward rates of interest". True or false? Justify your answer. (1 mark)

True. The Macaulay's duration captures the discounted mean term, so its value for a n-year zero-coupon bond is n, hence independent from any spot/forward rates.

(b) Consider a **single** n-year zero-coupon bond as the only available asset to perform the exact matching of modified durations. Show that the convexity of asset cannot be greater than or equal to the convexity of liabilities (as described above) for any notional amount X of the n-year zero-coupon bond valued at any constant discrete spot rate i. (6 marks)

$$\begin{aligned}
\frac{Xnv^{n+1}}{Xv^n} &= \frac{Pt_1v^{t_1+1}+Qt_2v^{t_2+1}}{Pv^{t_1}+Qv^{t_2}} \\
\Rightarrow nv\big(Pv^{t_1}+Qv^{t_2}\big) &= Pt_1v^{t_1+1}+Qt_2v^{t_2+1} \\
\Rightarrow n &= \frac{Pt_1v^{t_1}+Qt_2v^{t_2}}{Pv^{t_1}+Qv^{t_2}} \\
c_A(i) &= \frac{Xn(n+1)v^{n+2}}{Xv^n} = n(n+1)v^2 \\
&= \Big(\frac{Pt_1v^{t_1}+Qt_2v^{t_2}}{Pv^{t_1}+Qv^{t_2}}\Big)\Big(\frac{P(t_1+1)v^{t_1}+Q(t_2+1)v^{t_2}}{Pv^{t_1}+Qv^{t_2}}\Big)v^2 \\
c_L(i) &= \frac{Pt_1(t_1+1)v^{t_1+2}+Qt_2(t_2+1)v^{t_2+2}}{Pv^{t_1}+Qv^{t_2}}
\end{aligned}$$

Suppose $c_A(i) \geq c_L(i)$

$$\Rightarrow \big(Pt_1v^{t_1}+Qt_2v^{t_2}\big)\big(P(t_1+1)v^{t_1}+Q(t_2+1)v^{t_2}\big) \geq \big(Pt_1(t_1+1)v^{t_1}+Qt_2(t_2+1)v^{t_2}\big)\big(Pv^{t_1}+Qv^{t_2}\big)$$

$$\begin{aligned}
PQv^{t_1+t_2}\big(t_1(t_2+1)+t_2(t_1+1)\big) &\geq PQv^{t_1+t_2}\big(t_1(t_1+1)+t_2(t_2+1)\big) \\
&\Rightarrow 2t_1t_2 > t_1^2+t_2^2
\end{aligned}$$

The last inequality above is not true because $(t_1-t_2)^2 \geq 0 \Leftrightarrow t_1^2+t_2^2 \geq 2t_1t_2$ and the equality only holds when $t_1 = t_2$. Hence $c_A(i) < c_L(i)$.

(c) Given $P = 4, Q = 6, t_1 = 19, t_2 = 21$ and a constant discrete spot rate of 7%, calculate the required notional amounts (rounded to the nearest cents) of 15-year zero-coupon bond and 25-year zero-coupon bond so that the fund is immunised against small changes in interest rates. Also, verify that the third condition of the Redington's immunisation is satisfied. (6 marks)

Let X and Y be the notional amounts (in terms of million) of 15-year and 25-year zero-coupon bonds, respectively.

$$\begin{aligned}
V_A(0.07) = V_L(0.07): &\quad Xv^{15}+Yv^{25} = 4v^{19}+6v^{21} \\
\nu_A(0.07) = \nu_L(0.07): &\quad 15Xv^{16}+25Yv^{26} = 4(19)v^{20}+6(21)v^{22} \\
&\Rightarrow X = \frac{24v^{20}+24v^{22}}{10v^{16}} = 3.430169846 \\
&\quad Y = \frac{4v^{19}+6v^{21}-Xv^{15}}{v^{25}} = 7.120034199
\end{aligned}$$

$$\begin{aligned}
c_A(0.07) &= \frac{15(16)Xv^{17}+25(26)Yv^{27}}{Xv^{15}+Yv^{25}} = 393.4881594 \\
c_L(0.07) &= \frac{19(20)4v^{21}+21(22)6v^{23}}{4v^{19}+6v^{21}} = 372.52563
\end{aligned}$$

Since $c_A(0.07) > c_L(0.07)$, the third condition of the Redington's immunisation is satisfied.

Question 7 (17 marks)

It is known that the one-year par yield is 5% per annum and the two-year par yield is 6% per annum. A 3-year annual coupon bond with a coupon rate of 4% and redeemed at par provides a yield to maturity of 7%. Assuming that there are no taxes and coupon payments are payable in arrears,

(a) Calculate the 1-year, 2-year, and 3-year discrete spot rates, rounded to 2 decimal places in % throughout the calculation process. (6 marks)

$$
\begin{aligned}
1 &= \frac{0.05+1}{1+s_1} && \Rightarrow s_1 = 0.05 = 5\% \\
1 &= \frac{0.06}{1.05} + \frac{0.06+1}{(1+s_2)^2} && \Rightarrow s_2 = 0.060302987 = 6.03\% \\
P_{3 \text{ year bond}} &= \frac{0.04}{0.07}\left(1 - \frac{1}{1.07^3}\right) + \frac{1}{1.07^3} \\
&= 0.921270518 = \frac{0.04}{1.05} + \frac{0.04}{1.0603^2} + \frac{1.04}{(1+s_3)^3} && \Rightarrow s_3 = 0.070569552 = 7.06\%
\end{aligned}
$$

(b) Calculate the 1-year discrete and continuous forward rates of interest starting in 2 years' time, rounded to 2 decimal places in %. (2 marks)

$$
\begin{aligned}
f_{2,1} &= \frac{1.0706^3}{1.0603^2} - 1 = 0.091501142 = 9.15\% \\
F_{2,1} &= \ln 1.091501142 = 0.087553943 = 8.76\%
\end{aligned}
$$

(c) Calculate the 3-year par yield, rounded to 2 decimal places in %. (2 marks)

$$
\begin{aligned}
1 &= c_3\left(\frac{1}{1.05} + \frac{1}{1.0603^2} + \frac{1}{1.0706^3}\right) + \frac{1}{1.0706^3} \\
\Rightarrow c_3 &= 0.069660415 = 6.97\%
\end{aligned}
$$

(d) Describe the use of no arbitrage assumption in the valuation of securities. (2 marks)

Under this assumption, any two securities or combination of securities that have identical cash flows structure at any time point must have the same value/price at any time point.

For the purpose of valuing securities, we create two portfolio strategies such that their payoffs are identical. By this replicating portfolio approach, we utilize the no arbitrage assumption to equate the values of these two strategies and hence we can determine the value of securities under consideration.

(e) One year later, the 3-year par yield remains the same as the value found in part (c). At that time, a 3-year zero-coupon bond with a face value of \$1 is trading at a price of \$0.88, whereas a 4-year zero-coupon bond is selling at a price of \$0.85 per \$1 face value. Assuming that there are no arbitrage opportunities, calculate the price of a 4-year annual coupon bond at that time with a coupon rate of 7% and a face value of $1,000. (5 marks)

$$0.0697\Big(\frac{1}{1+s_1}+\frac{1}{(1+s_2)^2}+\frac{1}{(1+s_3)^3}\Big)+0.88=1$$
$$\Rightarrow \frac{1}{1+s_1}+\frac{1}{(1+s_2)^2}+\frac{1}{(1+s_3)^3}=\frac{1-0.88}{0.0697}$$

$$\begin{aligned}P_{4\text{ year bond}}&=70\Big(\frac{1}{1+s_1}+\frac{1}{(1+s_2)^2}+\frac{1}{(1+s_3)^3}\Big)+1070(0.85)\\&=1030.016499=1030.02\end{aligned}$$

Question 8 (12 marks)

The random annual returns $\tilde{i}_t$, for $t=1,2,\ldots,20$ are independently and identically distributed. $(1+\tilde{i}_t)$ is lognormally distributed with unknown parameter μ (expressed in 2 decimal places) and $\sigma^2=0.01$. The standard deviation of $\tilde{i}_t$ is known to be 0.10592.

(a) If cash flows of $500 are payable annually in arrears for 20 years, calculate the expected accumulated value (rounded to the nearest cent) at the time when the last payment is made. (5 marks)

$$\begin{aligned}\mathbb{V}(1+\tilde{i}_t)&=e^{2\mu+\sigma^2}\big(e^{\sigma^2}-1\big)=0.10592^2=\mathbb{V}(\tilde{i}_t)\\ \because \sigma^2=0.01,\Rightarrow \mu&=0.05\\ \mathbb{E}(\tilde{i}_t)&=e^{\mu+\frac{\sigma^2}{2}}-1=0.056540614\\ \mathbb{E}\big(500\tilde{s}_{\overline{20|}}\big)&=500s_{\overline{20|}0.056540614}\\&=500\big(\frac{1.0565460614^{20}-1}{0.0565460614}\big)=17723.24227=17723.24\end{aligned}$$

(b) Suppose the 20 annuity cash flows of $500 in part (a) is invested as a single lump sum at the end of year 8, calculate the probability that the random accumulated amount of this investment at the end of year 20 is greater than the expected accumulated value of the 20-year annuity obtained in part (a). (5 marks)

$$\ln\tilde{S}(12)\sim N(12\mu,12\sigma^2)\Rightarrow \ln\tilde{S}(12)\sim N(0.6,0.12)$$
$$\begin{aligned}\Pr\big(10000\tilde{S}(12)>17723.24\big)&=\Pr\big(\ln\tilde{S}(12)>0.572291679\big)\\&=\Pr\big(Z>\frac{0.572291679-0.6}{\sqrt{0.12}}\big)=\Pr(Z>-0.07998703)\\&\approx\Pr(Z>-0.08)=Pr(Z<0.08)=0.5319\end{aligned}$$

(c) Consider a single investment of $1 invested at time 0, determine the target of the accumulated value at the end of year 20 such that there is a 50% chance of obtaining a random accumulated amount less than this target. (2 marks)

$$\begin{aligned}\Pr(\tilde{S}(20)<k)&=\Pr(\ln\tilde{S}(20)<\ln k)=0.5\\ \Rightarrow \ln k&=\mathbb{E}\big(\ln\tilde{S}(20)\big)=20\mu=1\\ \Rightarrow k&=e=2.718281828\end{aligned}$$

END OF EXAMINATION

Formula Sheet for Final Exam

1.
$$A(t_1, t_2) = \exp\Big(\int_{t_1}^{t_2} \delta(t)\, dt\Big)$$

2.
$$1 + i = \Big(1 + \frac{i^{(p)}}{p}\Big)^p = \Big(1 - \frac{d^{(p)}}{p}\Big)^{-p} = (1 - d)^{-1}$$

3.
$$PV_t = \sum_{j: t_j \geq t} c_{t_j} v(t, t_j)$$

4.
$$PV(t, T_2) = \int_t^{T_2} \rho(s) \exp\Big(-\int_t^s \delta(u)\, du\Big)\, ds$$

5.
$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

6.
$$a_{\overline{n}|}^{(p)} = \frac{1 - v^n}{i^{(p)}}$$

7.
$$\overline{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$$

8.
$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

9.
$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

10.
$$(\overline{I}\overline{a})_{\overline{n}|} = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta}$$

11.
$$(I\overline{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

12.

$$i \approx \frac{i_2 f(i_1) - i_1 f(i_2)}{f(i_1) - f(i_2)}$$

13.

$$\text{IRR} = \{i^* : NPV(i^*) = 0\}$$

14.

$$\text{DPP} = \{\min t \in \mathbb{Z} : AP(t) \geq 0\}$$

15.

$$\text{MWRR} = \{i : (1+i)^{t_n - t_0} F_{t_0} + (1+i)^{t_n - t_1} N_{t_1} + (1+i)^{t_n - t_2} N_{t_2} + \cdots + (1+i)^{t_n - t_{n-1}} N_{t_{n-1}} = F_{t_n}\}$$

16.

$$\text{TWRR} = \{i : (1+i)^{t_n - t_0} = \left(\frac{F_{t_1 -}}{F_{t_0} + N_{t_0}}\right)\left(\frac{F_{t_2 -}}{F_{t_1 -} + N_{t_1}}\right)\left(\frac{F_{t_3 -}}{F_{t_2 -} + N_{t_2}}\right) \cdots \left(\frac{F_{t_n}}{F_{t_{n-1} -} + N_{t_{n-1}}}\right)\}$$

17.

$$g = \frac{cF}{R}, \quad (1 - t_I)g \begin{cases} < i^{(p)} & \Leftrightarrow P < R \\ = i^{(p)} & \Leftrightarrow P = R \\ > i^{(p)} & \Leftrightarrow P > R \end{cases}$$

18.

$$P = \begin{cases} \big(1 - t_I\big) cF a_{\overline{n}|}^{(p)} + Rv^n, & \text{for } i^{(p)} \leq (1 - t_I)g \\ \frac{\left(1 - t_I\right) cF a_{\overline{n}|}^{(p)} + \left(1 - t_C\right) Rv^n}{1 - t_C v^n}, & \text{for } i^{(p)} > (1 - t_I)g \end{cases}$$

19.

$$K_t = S_t e^{\delta(T-t)}$$

20.

$$K_t = \big(S_t - I\big) e^{\delta(T-t)} \quad \text{or} \quad K_t = S_t e^{(\delta - D)(T-t)}$$

21.

$$V_{l,t} = -V_{s,t} = \big(K_t - K_0\big) e^{-\delta(T-t)}$$

22.

$$(1 + f_{t,r})^r = \frac{(1 + s_{t+r})^{t+r}}{(1 + s_t)^t}$$

23.

$$\tau(i) = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

24.

$$\nu(i) = -\frac{A'(i)}{A(i)} = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k+1}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

25.

$$c(i) = \frac{A''(i)}{A(i)} = \frac{\sum_{k=1}^{n} t_k(t_k+1) C_{t_k} v_i^{t_k+2}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

26.

$$\text{Redington's immunisation:} \begin{cases} V_A(i_0) = V_L(i_0) & \\ V_A'(i_0) = V_L'(i_0) & \text{or} \quad \nu_A(i_0) = \nu_L(i_0) \\ V_A''(i_0) \geq V_L''(i_0) & \text{or} \quad c_A(i_0) \geq c_L(i_0) \end{cases}$$

27.

$$\text{independence:} \begin{cases} \mathbb{E}\big(\tilde{S}(n)\big) = \prod_{t=1}^{n} \mathbb{E}\big(1+\tilde{i}_t\big) \\ \mathbb{V}\big(\tilde{S}(n)\big) = \prod_{t=1}^{n} \mathbb{E}\big((1+\tilde{i}_t)^2\big) - \Big(\prod_{t=1}^{n} \mathbb{E}\big(1+\tilde{i}_t\big)\Big)^2 \end{cases}$$

28.

$$i.i.d.: \begin{cases} \mathbb{E}\big(\tilde{S}(n)\big) = \big(1+\mu\big)^n \\ \mathbb{V}\big(\tilde{S}(n)\big) = \Big(1+2\mu+\sigma^2+\mu^2\Big)^n - \big(1+\mu\big)^{2n} \end{cases}$$

29.

$$\text{independence:} \begin{cases} \mathbb{E}(\tilde{\ddot{s}}_{\overline{n}|}) = \mathbb{E}\big(1+\tilde{i}_n\big)\mathbb{E}\big(1+\tilde{\ddot{s}}_{\overline{n-1}|}\big) \\ \mathbb{E}(\tilde{s}_{\overline{n}|}) = 1 + \mathbb{E}\big(1+\tilde{i}_n\big)\mathbb{E}\big(\tilde{s}_{\overline{n-1}|}\big) \end{cases}$$

30.

$$i.i.d.: \begin{cases} \mathbb{E}(\tilde{\ddot{s}}_{\overline{n}|}) = \ddot{s}_{\overline{n}|\mu} \\ \mathbb{E}(\tilde{s}_{\overline{n}|}) = s_{\overline{n}|\mu} \end{cases}$$

31.

$$\text{lognormal model } i.i.d.: \begin{cases} \ln \tilde{S}(n) \sim N(n\mu, n\sigma^2) \\ \mathbb{E}\big(\tilde{S}(n)\big) = e^{n\mu + \frac{n\sigma^2}{2}} \\ \mathbb{V}\big(\tilde{S}(n)\big) = e^{2n\mu + n\sigma^2}\big(e^{n\sigma^2} - 1\big) \end{cases}$$

32.

$$\mathbb{E}(X) = \sum_{j=1}^{m} p_j X_{\omega_j}$$
$$\mathbb{V}(X) = \mathbb{E}(X^2) - \Big(\mathbb{E}(X)\Big)^2 = \sum_{j=1}^{m} p_j X_{\omega_j}^2 - \Big(\sum_{j=1}^{m} p_j X_{\omega_j}\Big)^2$$

33.

$$\begin{aligned} X &\sim N(\mu, \sigma^2) \\ \mathbb{E}(X) &= \mu \\ \mathbb{V}(X) &= \sigma^2 \\ \Pr(X > a) &= \Pr(Z > \frac{a-\mu}{\sigma}) \end{aligned}$$

34.

$$\begin{aligned} \ln X &\sim N(\mu, \sigma^2) \\ \mathbb{E}(X) &= e^{\mu + \frac{\sigma^2}{2}} \\ \mathbb{V}(X) &= e^{2\mu+\sigma^2}\Big(e^{\sigma^2} - 1\Big) \end{aligned}$$

35.

$$\begin{aligned} X &\sim U[c, d] \\ \mathbb{E}(X) &= \frac{c+d}{2} \\ \mathbb{V}(X) &= \frac{(d-c)^2}{12} \end{aligned}$$