Department of Mathematics **University of Toronto** **MAT332F, 2011**	**Midterm Test** **Tuesday October 25, 2:00 p.m.**

Read the following instructions carefully!

- Please turn off all cellular phones. **Calculators, scrap paper, electronic devices, and other aids are prohibited.**
- Please write your name and student number on the exam booklet.
- This test contains 5 questions worth a total of 100 points, and 2 bonus questions worth a total of 15 bonus points.
- You will receive 1 point on any question or part of a question for writing simply "*I don't know*" and nothing else.
- Please write **complete proofs**, in complete sentences with all the main steps justified. As with problem sets, please remember that when you are asked to find or calculate something you must justify that what you have found is correct and complete. You may use results from your lecture to help in your justification.

Core Problems.

(1) Prove that a simple graph with at least 2 vertices contains two vertices of equal degree. (*10pt*)

(2)
  (a) List all the 2–regular graphs with 6 vertices. (*10pt*)
  (b) Prove your answer in part (a). (*10pt*)

(3) Prove that a subgraph $H \subseteq G$ is contained in a spanning tree of the connected graph G if and only if H is acyclic. (*18pt*)

(4)
  (a) State and prove the Handshake Lemma. (*12pt*)
  (b) Find a simple graph on 10 vertices which is isomorphic to its complement, or prove that no such graph can exist. (*13pt*)

(5) Let G be a strongly connected tournament on $n \geq 3$ vertices.
  (a) Write down the definition of "*strongly connected tournament*". (*2pt*)
  (b) Prove that G contains a directed triangle through each vertex u in G (a directed triangle is a triple of edges $\{(u,v),(v,w),(w,u)\}$ for some pair of vertices v,w in G). (*9pt*)
  (c) Using Part (a) as the base case for induction, prove *Moon's Theorem*, which states that for any $k \in \{3,4,\ldots,n\}$, there exists a directed k–cycle in G. (*Hint*: Starting from a directed k-cycle, construct a directed $(k+1)$-cycle by replacing an edge in the cycle by a directed path of length 2, or a path of length 2 in the cycle by a directed path of length 3.) (*11pt*)
  (d) Deduce *Camion's Theorem*, which states that a tournament is Hamiltonian if and only if it is strongly connected. (*5pt*)

Bonus Problems.

B (1): Draw the tree corresponding to Prüfer code: $(2,8,9,4,7,7,2)$. (*5 bonus points*)

B (2): For the vertex u of the 6 vertex graph G below, and for each vertex v in G, how many $u-v$ walks of length 4 are there in G? (*10 bonus points*)