STA257 – Term Test – October 22, 2012

Last Name:__________________ First Name(s):________________Student #:___________________

TA's Name:_______________ Tutorial Room: ____________

Solutions

Time allowed: 100 minutes. Total marks = 30. Marks shown in brackets.

Check that you have all the consecutively numbered pages of this test.

Aids allowed: one-sided handwritten aid sheet + non-programmable calculator

As a general rule, ***best marks*** go to ***best (clear, succinct, complete) answers***.

Show your work and answer in the space provided (or indicate clearly where to look), in ink. **Pencil may be used, but then remarks will not be allowed.** Use backs of pages for rough work.

Proportion your time carefully among the questions and limit your time spent on any single question, particularly if worth very few marks.

Good luck!

1. Given that A and B are independent with $P(A) = 2P(B)$ and $P(A \cap B) = 0.08$, find $P(A^c \cap B^c)$. **[4]**

$$P(A \cap B) = P(A)P(B) = 2P(B)^2 = 0.08$$

$$P(B)^2 = 0.04 \Rightarrow P(B) = 0.2 \Rightarrow P(A) = 0.4$$

$$P(A^c \cap B^c) = P(A^c)P(B^c) = (1-P(A))(1-P(B))$$

$$= (1-0.4)(1-0.2) = 0.6 \cdot 0.8 = 0.48$$

2. Toss a fair coin. If we observe H, we select a ball from box # 1. If we observe T, we select a ball from box # 2. Box # 1 contains three red balls and four blue balls. Box # 2 contains five red balls and two blue balls. Let A = {a red ball is selected}. Calculate P(H|A). **[3]**

$$P(H|A) = \frac{P(A|H)P(H)}{P(A|H)P(H) + P(A|T)P(T)}$$

$$= \frac{\frac{3}{7} \cdot \frac{1}{2}}{\frac{3}{7} \cdot \frac{1}{2} + \frac{5}{7} \cdot \frac{1}{2}} = \frac{3/7}{8/7} = \frac{3}{8}$$

3. A particular sale involves four items randomly selected from a large lot that is known to contain 10% defectives. Let Y denote the number of defectives among the four sold. The purchaser of the items will return the defectives for repair, and the repair cost is given by $C = 3Y^2 + Y + 2$. Find the expected repair cost. **[4]**

$E(C) = 3E(Y^2) + E(Y) + 2$

$Y \sim Bin(4, 0.1)$

$E(Y) = np = 0.4$, $Var(Y) = npq = 4 \cdot 0.1 \cdot 0.9 = 0.36$

$E(Y^2) = Var(Y) + E(Y)^2 = 0.36 + 0.16 = 0.52$

$E(C) = 3 \cdot 0.52 + 0.4 + 2 = 3.96$

4. Let P(A) = P(B) = 1. Show that $P(A \cap B) = 1$. **[2]**

$P(A \cup B)$ $\quad P(A) = 1 \Rightarrow P(A \cup B) = 1$

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1 + 1 - 1 = 1$

5. Let Y ~ Poisson (λ). Find Var(Y). Show your complete work. **[5]**

$$P_Y(y) = \frac{\lambda^y e^{-\lambda}}{y!}$$

$$E(Y) = \lambda$$

$$E(Y(Y-1)) = \sum_{y=0}^{\infty} \frac{y(y-1)\lambda^y e^{-\lambda}}{y!} = \lambda^2 \sum_{y=2}^{\infty} \frac{\lambda^{y-2} e^{-\lambda}}{(y-2)!} = (y-2=z)$$

$$= \lambda^2 \underbrace{\sum_{z=0}^{\infty} \frac{\lambda^z e^{-\lambda}}{z!}}_{=1} = \lambda^2$$

$$E(Y^2) = E(Y(Y-1)) + E(Y) = \lambda^2 + \lambda$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

6. Let X ~ Unif(0,1), Y = -log X (natural logarithm). Find the pdf of Y, $f_Y(y)$. **[4]**

$$f_X(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & ow \end{cases}$$

$$h(x) = -\log x \searrow, \quad h^{-1}(y) = e^{-y}, \quad \frac{d}{dy} h^{-1}(y) = -e^{-y}$$

$$f_Y(y) = -1 \cdot (-e^{-y}) = \begin{cases} e^{-y}, & 0 < y < \infty \\ 0, & ow \end{cases}$$

7. Let Y have the density function given by $f(y) = \begin{cases} cy^2 + y, 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}$.

(a) Find c **[2]**
(b) Find cdf *F(y)* **[2]**
(c) Find $P(0 \le Y \le 0.5)$ **[1]**

(a) $1 = \int_0^1 (cy^2+y)\,dy = \left(c\frac{y^3}{3} + \frac{y^2}{2}\right)\Big|_0^1 = \frac{c}{3} + \frac{1}{2} \Rightarrow \frac{c}{3} = \frac{1}{2} \Rightarrow c = \frac{3}{2}$

(b) $F_Y(y) = \int_0^y \left(\frac{3}{2}t^2 + t\right)dt = \left(\frac{t^3}{2} + \frac{t^2}{2}\right)\Big|_0^y = \begin{cases} 0, & y < 0 \\ \frac{y^3}{2} + \frac{y^2}{2}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$

(c) $P(0 \le Y \le 0.5) = F(0.5) - F(0) = \frac{1}{16} + \frac{1}{8} = \frac{3}{16}$.

8. Let Z ~ N(2, 4), X = -Z+2, and $Y = X^2$. Find E(Y). **[3]**

$X \sim N(0, 4)$, $E(Y) = E(X^2) = Var(X) + E(X)^2 = 4 + 0 = 4$

or

$E(Y) = E(Z^2 - 4Z + 4) = E(Z^2) - 4E(Z) + 4$

$E(Z^2) = Var(Z) + E(Z)^2 = 4 + 4 = 8$

$E(Y) = 8 - 4 \cdot 2 + 4 = 4$.

Problem	1	2	3	4	5	6	7	8	
Max	4	3	4	2	5	4	5	3	Total = 30
Mark									

THE END