University of Toronto - Faculty of Arts and Sciences - Department of Mathematics

MAT 244H1Y

Introduction to Ordinary Differential Equations

AUGUST 2010 EXAMINATIONS

August 18, 2010

PLEASE HAND IN

Instructor: T. Tzaneteas

Duration: 3 hours

No calculators or other aids are allowed.

All questions are of equal value.

Family Name: ______________________

Given Name: ______________________

Student Number: ______________________

Question	Mark	Question	Mark
1		6	
2		7	
3		8	
4		9	
5		10	
		Total	

Question 1.

Find the general solution of the differential equation

$$y^{(7)} - y^{(3)} = 0.$$

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Question 2.

Find the general solution of equation

$$y''' - 6y'' + 12y' - 8y = 2e^{2t} - 1.$$

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Question 3.

Solve the differential equation

$$(1-x)y'' + y = 0$$

by means of a power series about the point $x = 0$: find the recurrence relation for the coefficients, and the find the first four terms of two independent solutions y_1 and y_2. Prove that they are independent solutions.

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Question 4.

Find the general solution of the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}.$$

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Question 5.

Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

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Question 6.

Consider the differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where A is a 2×2 matrix (independent of t). We have seen that $\mathbf{x} = 0$ is an equilibrium solution of this equation for any A.

(a) Give an example of a matrix A so that 0 is a saddlepoint.

(b) Give an example of a matrix A so that 0 is stable but not asymptotically stable.

(c) Give an example of a matrix A so that there are infinitely many equilibrium solutions.

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Question 7.

Consider the system of equations

$$\begin{cases} x' = x - x^2 - xy, \\ y' = 3y - xy - 2y^2. \end{cases}$$

Find all equilibrium points of the system and for each one find the eigenvalues of the linearized system and determine its stability (with respect to the nonlinear system).

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Question 8.

Consider the equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & \alpha \\ 1 & 2 \end{pmatrix} \mathbf{x},$$

where α is a real number.

(a) Find the critical values at which the qualitative behaviour of the system changes (i.e., the phase portrait changes).

(b) Draw the phase portrait of the system for $\alpha = -2$ and $\alpha = 3$.

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Question 9.

In this question you will prove Abel's Theorem for the second order equation

$$y'' + p(t)y' + q(t)y = 0, \ \ t \in I,$$

where I is some open interval.

(a) State Abel's Theorem.

(b) Let y_1 and y_2 be two solutions of the differential equation above. Differentiate the Wronksian of y_1 and y_2 (with respect to t) 1to find a first order differential equation that it satisfies.

(c) Solve the equation you found in part (b) for W to complete the proof of Abel's Theorem.

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Question 10.

(a) Consider the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Suppose that the quantity

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{yN - xM}$$

depends only on xy. Show that the differential equation has an integrating factor of the form $\mu = f(xy)$ and find an explicit formula for μ. (Hint: use the chain rule. It may also be useful to introduce the variable $z = xy$.)

(b) Use part (a) to find the (implicit) general solution of the differential equation

$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + \frac{3y}{x}\right)\frac{dy}{dx} = 0.$$

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