April 2nd

Problem Set 7

Q3 $\quad T:\mathbb{R}^4 \to \mathbb{R}^4$ linear operator given by $A=\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$

Find a basis $\alpha \in \mathbb{R}^4$ s.t. $[T]_\alpha^\alpha$ is in JCF.

Also, find $[T]_\alpha^\alpha$

STRATEGY
- Find the characteristic polynomial of A
- Find "canonical basis" for each of the generalized eigenspaces
- Combine these bases to obtain a basis of $\mathbb{R}^4$.

Solution: $P_A(\lambda)=\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ -1 & 0 & 2 & 1-\lambda \end{vmatrix}$

$$=(1-\lambda)\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \end{vmatrix}$$

$$=(1-\lambda)(1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1-\lambda & 1 \end{vmatrix}$$

$$=\lambda^2(\lambda-2)^2$$

Eigenvalue of A: $0, 2$

dim of gen eigenspace for eigenvalue $0 = 2$

The dim of the generalized eigenspace for the eigenvalue λ = the algebraic multiplicity of λ as a root of the characteristic polynomial.

Recall: $T:V\to V$ linear transf
V f.d. vector space over a field F
$\lambda \in F$ eigenvalue of T

Def: The eigenspace of T for λ is $E_\lambda = \{v\in V : T(v)=\lambda v\} = \{v\in V : (T-\lambda I_V)(v) = 0\}$
$= \ker(T-\lambda I_V)$

The generalized eigenspace of T for λ is $E_\lambda^{gen} = \{v \in V : (T-\lambda I_V)^n(v) = 0$ for some $n_v \in \mathbb{Z}_{>0}\}$
A vector $v\in V$ belongs to $E_\lambda^{gen} \iff$ Some power of $T-\lambda I_V$ sends v to 0.

Find the generalized eigenspaces for 0 and 2

$\underline{E_0}$: $E_0 = \ker(A)$ $\dim(\ker A) = 1$

$\ker(A^2)$ is 2-dim STOP

Find a basis for (A^2)

$\alpha = \{v_1, v_2\}$

Compute : $Av_1 = v_3$, $Av_2 = v_4$

- If $v_3 = v_4 = 0$, then $\{v_1\}, \{v_2\}$ are cycles of length 1 and together give a "canonic-al" basis for the generalized eigenspace for $\lambda = 0$.

- If one of v_3, v_4 (say v_3) is non-zero. then $\{v_1, v_3\}$ is a cycle of length 2.

this occurs in our example

The same occurs for the gen. eigenspace for $\lambda = 2$.

→ obtain 2 cycles of length 2.

SO
$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

tells me there're 2 , 2×2 blocks