Assignment 3 - MAT 327 - Summer 2014

Due June 9, 2014 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Prove $\mathrm{FIN}(\mathbb{N}) := \{ F \subseteq \mathbb{N} : F \text{ is finite} \}$ is a countable set. (Hint: You *may* want to write $\mathrm{FIN}(\mathbb{N})$ as a countable union of countable sets.)

[**C.2**] Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces with $\overline{D} = X$ and $\overline{E} = Y$. Show that $D \times E$ is a dense subset of $X \times Y$, (taken, of course, with the product topology). Conclude that the product of two separable spaces is again separable.

[**C.3**] Prove or disprove that the product of two Hausdorff Spaces is again a Hausdorff space.

[**C.4**] Suppose that $(X, \mathcal{T})$ and $(X, \mathcal{U})$ are topological spaces with $\mathcal{T} \subseteq \mathcal{U}$. Prove or disprove the following two statements:

1. If $(X, \mathcal{T})$ is a Hausdorff space, then so is $(X, \mathcal{U})$; and
2. If $(X, \mathcal{U})$ is a Hausdorff space, then so is $(X, \mathcal{T})$

Definition. *A topological space is said to* ***satisfy the countable chain condition*** *(or* **ccc***) if whenever $\mathcal{A}$ is an uncountable collection of open subsets of X, there are distinct $A, B \in \mathcal{A}$ such that $A \cap B \neq \emptyset$.*

In "English", this says "A space is ccc if it doesn't contain an uncountable collection of mutually disjoint open sets."

[**C.5**] Prove that every separable space satisfies the ccc. (First *think* about $\mathbb{R}_{\text{usual}}$, and why that space is ccc.)

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

[**A.1**] Let A be a countable subset of $\mathbb{R}$. Show that there is a number $x \in \mathbb{R}$ such that $(x + A) \cap A = \emptyset$, where

$$x + A := \{ x + a : a \in A \}.$$

[**A.2**] Let $(X, \mathcal{T})$ be a topological space where X is countable. Is this space necessarily separable? Is it necessarily second countable? What about first countable? What about ccc?

Prove which properties do follow from X being countable, and find or invent a counterexample for the properties that don't follow. If you don't come up with an example on your own (e.g. you found your example in Counterexamples in Topology) please cite your source and prove all assertions you make about the counterexample.

[**A.3**] Fill out the following table with "YES", "NO" or "MOUNTAIN". No proof needed.

	Separable	2nd Countble	1st Countable	ccc
$\mathbb{R}_{\text{usual}}$				
$\mathbb{R}_{\text{co-countable}}$				
$\mathbb{R}_{\text{co-finite}}$				
Everest				
K2				
Matterhorn				
$\mathbb{R}_{\text{discrete}}$				
$\mathbb{N}_{\text{discrete}}$				
Kilimanjaro				

New Ideas

For this section please work on and sumbit ***at least one*** *of the following problems. You may consult other students, texts, online resources or other* *professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] This is the story of a rabbit and a rabbit catcher, which both live on $\mathbb{Z} \times \mathbb{Z}$, the so-called "integer lattice". The catcher lives at the origin $(0, 0)$ and every day may place the single trap he owns somewhere on the integer lattice. If, in the night, the rabbit hops on his trap then the catcher has caught the rabbit and can then make soup; otherwise, the catcher can try again the next day.
Now, the catcher knows only a little about how the rabbit moves each night. The rabbit's hole is on one of the lattice points (but the catcher doesn't know which one), and the rabbit has picked a line of integer slope that goes through his hole (but the catcher doesn't know which line). On the first night the rabbit hops along the line until she gets to the next integer lattice point on the line, then she rests there. The following night she hops down the line again (in the same direction as before) until she reaches a lattice point, then she rests there. She continues on in this way indefinitely.
For example, let's say the rabbit hole is at $(0, 5)$ and the rabbit has picked the line $2x + 5$. On night 0, the rabbit will be at the point $(0, 5)$. On night 1, the rabbit will be at point $(1, 7)$. On night 2, the rabbit will be at point $(2, 9)$.
Show that even though the catcher doesn't know (1) where the rabbit hole is, (2) what line the rabbit is jumping along, and (3) along which direction the rabbit is jumping, the catcher can *still* catch the rabbit in a finite amount of days.
When attempting this problem, **don't get overwhelmed**. Break it down to its core by tearing away extra layers of complexity, then build it back up again.

[**NI.2**] Let's push the idea of the countable chain condition that we saw in C.5... Show that there is an uncountable collection of mutually disjoint circles in the plane. Is there an uncountable collection of mutually disjoint "figure-8"s in the plane? For an extra challenge, replace "figure-8" by "Y-shaped" in the previous question, (where "Y-shaped" means a union of three line segments at a common point). If you're still hungry for more, show that if you have a collection $\mathcal{C}$ of circles in the plane, such that no two circles cross each other, then the collection of tangent points

$$\{ p \in \mathbb{R}^2 : \exists C, D \in \mathcal{C}, C \cap D = \{ p \} \},$$

is countable.