1.) $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$; $\varepsilon_i \overset{iid}{\sim}$ normal$(0, \sigma^2)$

$$RSS = \sum_{i=1}^{n} \left(y_i - (b_0 + b_1 x_i) \right)^2$$

Residual Sum of Squares

$$\frac{\partial RSS}{\partial b_0} = 2 \sum_{i=1}^{n} \left(y_i - (b_0 + b_1 x_i) \right) (-1) \qquad (1)$$

$$\frac{\partial RSS}{\partial b_1} = 2 \sum_{i=1}^{n} \left(y_i - (b_0 + b_1 x_i) \right) (-x_i) \qquad (2)$$

$$(1) \Rightarrow \quad -2 \left[\Sigma y_i - n b_0 - b_1 \Sigma x_i \right] = 0$$

$$\bar{y} - b_0 - b_1 \bar{x} = 0$$

$$b_0 = \hat{\beta}_0 = \bar{y} - b_1 \bar{x}$$

$$(2) = -2 \left[\Sigma y_i x_i - b_0 \Sigma x_i - b_1 \Sigma x_i^2 \right] = 0$$

$$\frac{\Sigma y_i x_i}{n} - b_0 \bar{x} - b_1 \frac{\Sigma x_i^2}{n} = 0$$

$$\frac{\Sigma y_i x_i}{n} - (\bar{y} - b_1 \bar{x}) - b_1 \frac{\Sigma x_i^2}{n} = 0$$

$$\frac{\Sigma y_i x_i}{n} - \bar{y} \bar{x} + b_1 \bar{x}^2 - b_1 \frac{\Sigma x_i^2}{n} = 0$$

$$b_1 = \hat{\beta}_1 = \frac{\frac{\Sigma y_i x_i}{n} - \bar{y}\bar{x}}{\Sigma x_i^2 - \bar{x}^2} = \frac{\Sigma (y_i - \bar{y})(x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$E(\hat{\beta}_1) = E\left(S_{xy}/S_{xx} \right) = \frac{1}{S_{xx}} E(S_{xy}) = \frac{1}{S_{xx}} E\left(\sum (x_i - \bar{x})(y_i - \bar{y}) \right)$$

$$= \frac{1}{S_{xx}} E\left(\sum (x_i - \bar{x}) y_i - \sum (x_i - \bar{x}) \bar{y} \right)$$

$$= \frac{1}{S_{xx}} E\left(\sum (x_i - \bar{x}) y_i - \bar{y} \underbrace{\sum (x_i - \bar{x})}_{= 0} \right)$$

$$= \frac{1}{S_{xx}} E\left(\sum (x_i - \bar{x}) y_i \right) = \frac{1}{S_{xx}} \sum (x_i - \bar{x}) E(y_i)$$

$$= \frac{1}{S_{xx}} \sum (x_i - \bar{x}) (\beta_0 + \beta_1 x_i)$$

$$= \frac{1}{S_{xx}} \left[\sum (x_i - \bar{x}) \beta_0 + \sum (x_i - \bar{x}) \beta_1 x_i \right]$$

$$= \frac{1}{S_{xx}} \left[\beta_0 \underbrace{\sum (x_i - \bar{x})}_{= 0} + \beta_1 \sum (x_i - \bar{x}) x_i \right]$$

$$= \frac{1}{S_{xx}} \beta_1 \sum (x_i - \bar{x}) x_i = \frac{1}{S_{xx}} \beta_1 S_{xx} = \beta_1 .$$

Note: $S_{xx} = \sum (x_i - \bar{x})^2 = \sum (x_i - \bar{x}) x_i$.

$$E(\hat{\beta}_0) = E(\bar{y} - \hat{\beta}_1 \bar{x})$$

$$= E(\bar{y}) - E(\hat{\beta}_1)\bar{x} = E(\bar{y}) - \beta_1 \bar{x}$$

$$= E\left(\frac{1}{n}\sum y_i\right) - \beta_1 \bar{x}$$

$$= \frac{1}{n}\sum E(y_i) - \beta_1 \bar{x}$$

$$= \left(\frac{1}{n}\right)\sum(\beta_0 + \beta_1 x_i) - \beta_1 \bar{x}$$

$$= \left(\frac{1}{n}\right)\left[n\beta_0 + \beta_1 \sum x_i\right] - \beta_1 \bar{x}$$

$$= \beta_0 + \beta_1 \bar{x} - \beta_1 \bar{x} = \beta_0 .$$

$$V(\hat{\beta}_1) = V\left(\frac{S_{xy}}{S_{xx}}\right) = \frac{1}{S_{xx}^2} V(S_{xy})$$

$$= \frac{1}{S_{xx}^2} V\left(\sum (x_i - \bar{x})(y_i - \bar{y})\right)$$

$$= \frac{1}{S_{xx}^2} V\left(\sum (x_i - \bar{x}) y_i - \bar{y} \underbrace{\sum (x_i - \bar{x})}_{0}\right)$$

$$= \frac{1}{S_{xx}^2} V\left(\sum (x_i - \bar{x}) y_i\right) = \frac{\sum (x_i - \bar{x})^2 V(y_i)}{S_{xx}^2}$$

$$= \frac{\sum (x_i - \bar{x})^2 \sigma^2}{S_{xx}^2} = \sigma^2 \frac{\sum (x_i - \bar{x})^2}{S_{xx}^2} = \sigma^2 \frac{S_{xx}}{S_{xx}^2}$$

$$= \sigma^2 / S_{xx}.$$

$$V(\hat{\beta}_0) = V(\bar{y} - \hat{\beta}_1 \bar{x})$$

$$Cov(\bar{y}, \hat{\beta}_1) = Cov\left(\frac{1}{n}\sum_i y_i, \frac{1}{S_{xx}}\sum_j (x_j - \bar{x}) y_j\right)$$

$$= \frac{1}{n}\frac{1}{S_{xx}}\sum_i \sum_j Cov(y_i, y_j)$$

$\Rightarrow$ if $i \neq j \Rightarrow Cov(y_i, y_j) = 0$.

if $i = j \Rightarrow Cov(y_i, y_j) = V(y_i) = \sigma^2$

$$= \frac{1}{n}\frac{1}{S_{xx}}\sum_{i=1}^{n}(x_i - \bar{x})\sigma^2$$

$$= \frac{\sigma^2}{n S_{xx}} \underbrace{\sum_{i=1}^{n}(x_i - \bar{x})}_{= 0} = 0.$$

$$\therefore V(\bar{y} - \hat{\beta}_1 \bar{x}) = V(\bar{y}) + V(\hat{\beta}_1)\bar{x}^2$$

$$= V\left(\frac{1}{n}\sum y_i\right) + \sigma^2/S_{xx}\,(\bar{x}^2)$$

$$= \frac{1}{n^2}\sum V(y_i) + \frac{\sigma^2}{S_{xx}}(\bar{x}^2)$$

$$= \sigma^2/n + \frac{\sigma^2}{S_{xx}}(\bar{x}^2) = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)$$

2.) $y_{ij} = \mu_j + \varepsilon_{ij}$ for $i = 1, \dots, n$
$j = 1, \dots, J$

- Let's find the least-squares estimates for μ_j.

$$SSE = \sum_i \sum_j (y_{ij} - \mu_j)^2 = \sum_i \sum_j (y_{ij} - m_j)^2$$

$$\Rightarrow \quad \frac{\partial SSE}{\partial m_j} = -2 \sum_i (y_{ij} - m_j) = 0$$

$$\Rightarrow \sum_i y_{ij} = n m_j$$

$$\therefore m_j = \hat{\mu}_j = \frac{\sum_i y_{ij}}{n} = \frac{y_{\cdot j}}{n} = \bar{y}_{\cdot j} .$$

- As you might expect we just take the sample mean in each group.