UNIVERSITY OF TORONTO
Faculty of Arts and Science

AUGUST 2011 EXAMINATIONS

MAT301H1Y

Duration – 3 hours
No Aids Allowed

LAST NAME: ________________________________

FIRST NAME: ________________________________

STUDENT NUMBER: ________________________________

INSTRUCTIONS: PLEASE READ

Please check that this test has 12 numbered pages. Do not tear out any pages. Scrap paper is not allowed; use the backs of pages for rough work. If you want the back of a page marked, please indicate this clearly on the front of the page.

Unless otherwise mentioned, you are REQUIRED TO COMPLETELY JUSTIFY YOUR ANSWERS. The correct answer without computation or justification is worth no credit.

Question:	1	2	3	4	5	6	Total
Points:	15	20	10	15	20	20	100
Score:							

Question 1 [15 marks]

Determine whether the following statements are true or false. If they're true, prove them; if they're false, provide a counter-example. G and G' always denote groups.

(1–a) [3 marks] If $\varphi\colon G \to G'$ is a homomorphism and $\operatorname{im}(\varphi)$ is cyclic, then G is cyclic.

(1–b) [3 marks] Every abelian group has a non-trivial normal proper subgroup.

(1–c) [3 marks] A group of order 1000 has a subgroup of order 25.

(1–d) [3 marks] If G and G' are abelian, then $G \oplus G'$ is also abelian.

(1–e) [3 marks] The centraliser of a in G, denoted $C(a)$, is a normal subgroup of G.

Question 2 [20 marks]

Provide an example of each of the following and explain your reasoning.

(2–a) [5 marks] A non-trivial homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_8$.

(2–b) [5 marks] A non-trival proper normal subgroup of D_n.

(2–c) [5 marks] A pair of subgroups $H, K \leq G$, where $G = D_5 \oplus \mathbb{Z}_6$, such that $G = H \times K$. (Recall this means that G is an internal direct product of H and K.)

(2–d) [5 marks] A 2-Sylow subgroup of Q, the quaternion group.

Question 3 [10 marks]

(3–a) [5 marks] Give an example of an element of order 30 in A_{12}.

(3–b) [5 marks] What are all the possible cycle types of elements in A_7?

Question 4 [15 marks]

Let G be a group and let H be a subgroup of G

(4–a) [6 marks] If $[G : H] = 2$, prove that H is normal in G.

(4–b) [9 marks] Let $H = \{aba^{-1}b^{-1} : a, b \in G\}$. Prove that $H \trianglelefteq G$ and that G/H is abelian.

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Question 5 [20 marks]

(5–a) [6 marks] List all the isomorphism classes of abelian groups of order 96.

(5–b) [5 marks] To which of the above groups is $\mathbb{Z}_{24} \oplus \mathbb{Z}_4$ isomorphic? (Provide a proof.)

(5–c) [9 marks] Prove that an abelian group of order 2^n must have an odd number of elements of order 2.

Question 6 [20 marks]
Let G be a group of order 60.

(6–a) [5 marks] What are the possibilities for the number of Sylow p-subgroups in G, for each prime p which divides 60.

(6–b) [7 marks] Let H be a group of order 20. Prove that H has a normal subgroup of order 5.

(6–c) [8 marks] Prove that if G has a normal subgroup of order 3, then it must also have a normal subgroup of order 5. (Hint: use parts (a) and (b))

TOTAL PAGES: 12