STA302/1001: Methods of Data Analysis

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Chapter 2: Simple Linear Regression (Part II)

Comparing Models

- known as Analysis of Variance (ANOVA)
- a simple example: comparing two regression models

$$E(Y|X = x) = \beta_0 \text{ v.s. } E(Y|X = x) = \beta_0 + \beta_1 x$$

- which one to use?
- first model: a horizontal line
  - it says the slope is zero, or
  - cannot help predict Y given X, or
  - X and Y are not related ...

The First Model

- The model is assumed as $\mathrm{E}(Y|X = x) = \beta_0$
- β_0 can be estimated by minimizing $\sum(y_i - \beta_0)^2$, that is, by OLS with only the intercept parameter
- thus $\hat{\beta}_0 = \overline{y}$, the sample mean of $\{y_1, \ldots, y_n\}$.
- residual sum of squares is

$$\sum(y_i - \hat{\beta}_0)^2 = \sum(y_i - \overline{y})^2 = SYY$$

with $n - 1$ degrees of freedom

Which One to Use?

- call $\widehat{\mathrm{E}(Y|X)} = \hat{\beta}_0$ *fitted model 1*
  call $\widehat{\mathrm{E}(Y|X)} = \hat{\beta}_0 + \hat{\beta}_1 x$ *fitted model 2*
- use *fitted model 1* or *fitted model 2*?
- one method is to compare RSS's from two models
- $RSS_1 = SYY$, $RSS_2 = SYY - \frac{(SXY)^2}{SXX}$
- we know $RSS_2 \leq RSS_1$
- the idea is, if adding the slope β_1 does not help much, then RSS_2 should not be much smaller than RSS_1.

Which One to Use? (cont...)

- key question: how small is small?
- we calculate the difference between RSS_1 and RSS_2, called “sum of squares due to regression” ($SSreg$):

$$\begin{aligned} SSreg &= RSS_1 - RSS_2 \\ &= SYY - \left(SYY - \frac{(SXY)^2}{SXX} \right) \\ &= \frac{(SXY)^2}{SXX} \end{aligned}$$

- $$\begin{aligned} df \text{ for } SSreg &= df \text{ for } RSS_1 - df \text{ for } RSS_2 \\ &= (n-1) - (n-2) = 1 \end{aligned}$$

The ANOVA Table

- essentially we compare the "standardized version of $SSreg$" v.s. "standardized version of RSS_2"
- we will summarize our comparison in an ANOVA table

Source	df	SS	MS	F	p-value
Regression	1	$SSreg$	$SSreg/1$	$MSreg/\hat{\sigma}^2$	
Residual	$n-2$	RSS	$\hat{\sigma}^2 = RSS/(n-2)$		
Total	$n-1$	SYY			

- SS: sum of squares
  MS: mean squares

F-test For Regression

- if the slope β_1 is "useful", then

$$RSS_2 \ll RSS_1 \quad \Rightarrow \quad SSreg \text{ will be relatively large}$$
$$\Rightarrow \quad F = \frac{SSreg/1}{RSS/(n-2)} \text{ will be large}$$

- F is a rescaled version of $SSreg = RSS_1 - RSS_2$

  key assumption for F-test: e_i are i.i.d. $N(0, \sigma^2)$, then

$$\frac{SSreg}{\sigma^2} \sim \chi^2_1 \text{ (if } \beta_1 = 0), \quad \frac{RSS}{\sigma^2} \sim \chi^2_{n-2}, \quad SSreg \perp RSS$$

- recall F-distribution: $F \sim F_{(1,n-2)}$, given $\beta_1 = 0$

- what we are doing is a statistical test

  NH : $\mathrm{E}(Y|X = x) = \beta_0$ v.s. AH : $\mathrm{E}(Y|X = x) = \beta_0 + \beta_1 x$

F-test For Regression (cont...)

- we compare "the observed value of F" calculated from the sample to the critical value, $F_{(\alpha,1,n-2)}$, the upper-α quantile or $100(1-\alpha)$th percentile of $F_{(1,n-2)}$
- if $F_{obs} > F_{(\alpha,1,n-2)}$, reject NH, use model 2.
- if $F_{obs} \leq F_{(\alpha,1,n-2)}$, don't reject NH (don't say accept)
- Forbe's data, use R function qf(0.95, 1, 15) to find $F_{0.05,1,15} = 4.543$

Source	df	SS	MS	F	p-value
Regression on $Temp$	1	425.639	425.639	2962.79	≈0
Residual	15	2.155	0.144		

- conclusion?

p-value and Interpretation

- What does it mean? Assuming the NH is true, the probability that the test statistic is at lease as extreme as was observed in the sample, e.g., in the previous F-test, p-value$= P(F \geq F_{obs}|\beta_1 = 0) \approx 0$
- a measure of the strength of the evidence against NH in favor of AH, not the probability that NH is true
- compare p-value with significance level α, say $\alpha = 0.05$
- statistical significance v.s. scientific significance
- latter needs the former to confirm

Coefficient of Determination, R^2

- definition

$$R^2 = \frac{SSreg}{SYY}$$

- scale-free one number summary
- measure the strength of the relationship between x_i and y_i
- to see this, notice that
- SYY: variability in the data
- $SSreg$: variability in the data explained by the slope

Coefficient of Determination, R^2 (cont...)

- Forbes' data
$$R^2 = \frac{425.63910}{427.79402} = 0.995$$
- it means that the straight line model explains 99.5% of the variability in the data
- another way to look at R^2:
$$R^2 = \frac{SSreg}{SYY} = \frac{(SXY)^2}{SXX\ SYY} = r_{xy}^2$$
- the square of sample correlation between X and Y

Confidence Intervals and Tests

- for "simple problems", if $\hat{\theta}$ is an estimate for θ, then a $100(1-\alpha)\%$ confidence interval (C.I.) for θ is

$$(\hat{\theta} - t_{(\frac{\alpha}{2},d)}\, se(\hat{\theta}), \quad \hat{\theta} + t_{(\frac{\alpha}{2},d)}\, se(\hat{\theta}))$$

  where $se(\hat{\theta})$ is the standard error for $\hat{\theta}$, and $t_{(\frac{\alpha}{2},d)}$ is the value that cuts off $\frac{\alpha}{2} \cdot 100\%$ in the upper tail of the t-distribution with df$= d$

- when to use t-distribution or normal?
- what is the correct way to interpret “a 95% C.I. for θ is $(3.5, 5.6)$?

Confidence Intervals and Tests for β_0

- key assumption: e_i's are i.i.d. $N(0, \sigma^2)$
- for the intercept β_0 the C.I. is

$$(\hat{\beta}_0 - t_{(\frac{\alpha}{2}, n-2)}\, se(\hat{\beta}_0), \quad \hat{\beta}_0 + t_{(\frac{\alpha}{2}, n-2)}\, se(\hat{\beta}_0))$$

  where $se(\hat{\beta}_0) = \hat{\sigma}(\frac{1}{n} + \frac{\bar{x}^2}{SXX})^{\frac{1}{2}}$

- Hypothesis test: for a pre-fixed β_0^*, say $\beta_0^* = 0$
  NH: $\beta_0 = \beta_0^*$, β_1 arbitrary
  AH: $\beta_0 \neq \beta_0^*$, β_1 arbitrary
- t-statistic $t = \frac{\hat{\beta}_0 - \beta_0^*}{se(\hat{\beta}_0)}$ and compare to $t_{(\frac{\alpha}{2}, n-2)}$

Confidence Intervals and Tests for β_1

- for the slope β_1

$$\text{C.I.} : \hat{\beta}_1 \pm t_{(\frac{\alpha}{2}, n-2)} \, se(\hat{\beta}_1)$$

$$= \hat{\beta}_1 \pm t_{(\frac{\alpha}{2}, n-2)} \, \frac{\hat{\sigma}}{\sqrt{SXX}}$$

- Hypothesis test: similar to β_0
- a special case of NH: $\beta_1 = 0$ v.s. AH: $\beta_1 \neq 0$
- same as comparing "$y = \beta_0$" and "$y = \beta_0 + \beta_1 x$"

Confidence Intervals and Tests – t and F

- doing the t-test

  NH: $\beta_1 = 0$ vs AH: $\beta_1 \neq 0$

  is the same as comparing "$y = \beta_0$" and "$y = \beta_0 + \beta_1 x$" with an F-test
- t-statistic: $t = \frac{\hat{\beta}_1 - 0}{se(\hat{\beta}_1)} = \frac{\hat{\beta}_1}{\hat{\sigma}/\sqrt{SXX}}$
- $t^2 = \frac{\hat{\beta}_1^2}{\hat{\sigma}^2/SXX} = \frac{\hat{\beta}_1^2 SXX}{\hat{\sigma}^2} = F$-statistic from ANOVA Table
- that is, there is a one-to-one correspondence here
- from the fact that the square of t_d is $F_{(1,d)}$
- (then why do we study both the t and the F tests?)

Prediction and Fitted Values

- first, a simple question
- if $X_1, X_2, \cdots, X_m \sim$ i.i.d. $N(\mu, \sigma^2)$, what is $\mathrm{Var}(\bar{X})$?
- should it be smaller or larger than $\mathrm{Var}(X_i)$?
- prediction: predict the value of y given a new value of x
- denote the new values: x_*, y_*
- x_* is known but y_* is not
- e.g., "income" $= 10 + 20\times$ "year of education"
- You have done 16 years of education. How much are you expected to earn?

Prediction

- $\tilde{y}_* = \hat{\beta}_0 + \hat{\beta}_1 x_*$
- $x_* = 16$, $\tilde{y}_* = 100 + 200 \times 16 = 3300$
- You are expected to earn $3300 a month
- $\tilde{y}_*$ predicts unbiasedly the unobserved y_* (verify)

$$\begin{aligned}\mathrm{Var}(\tilde{y}_* - y_* | \mathbb{X}, x_*) &= \mathrm{Var}(y_* | x_*) + \mathrm{Var}(\tilde{y}_* | \mathbb{X}, x_*) \\ &= \sigma^2 + \sigma^2 \left(\frac{1}{n} + \frac{(x_* - \bar{x})^2}{SXX}\right)\end{aligned}$$

$$\mathrm{sepred}(\tilde{y}_* - y_* | \mathbb{X}, x_*) = \hat{\sigma}\left(1 + \frac{1}{n} + \frac{(x_* - \bar{x})^2}{SXX}\right)^{\frac{1}{2}}$$

- we can construct a prediction interval for y_*:

$$\tilde{y}_* \pm t_{(\frac{\alpha}{2}, n-2)} \,\mathrm{sepred}(\tilde{y}_* | \mathbb{X}, x_*)$$

Fitted Values

- same "income - years of education" example
- what is the average income of <u>all</u> people who have done 16 years of education?
- this is an estimation problem, not prediction
- estimated by the fitted value

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x \quad \text{with } x = 13$$

- its standard error is $\mathrm{sefit}(\hat{y}|\mathbb{X}, x) = \hat{\sigma}(\frac{1}{n} + \frac{(x-\bar{x})^2}{SXX})^{\frac{1}{2}}$
- compare $\mathrm{sefit}(\hat{y}|\mathbb{X}, x)$ with $\mathrm{sepred}(\tilde{y}_*|\mathbb{X}, x_*)$
- notation in text is a bit confusing

Fitted Values (cont...)

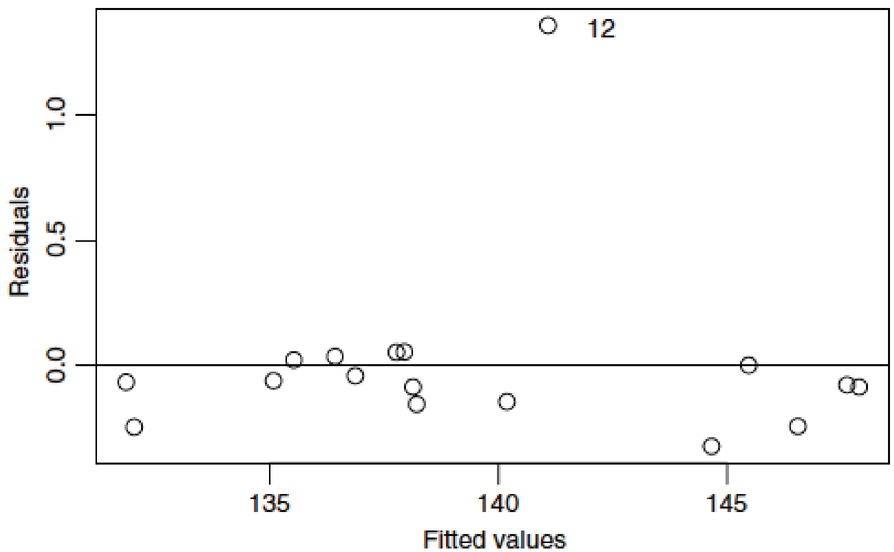
- confidence interval:

$$(\hat{\beta}_0 + \hat{\beta}_1 x) \pm \mathrm{sefit}(\hat{y}|\mathbb{X}, x)[2F(\alpha; 2, n-2)]$$

- note: we are using a F-distribution, not t
- why? another course will tell you...

The Residuals

- definition: $\hat{e}_i = y_i - \hat{y}_i$
- plots can show problems in our modeling
- a useful plot: residuals v.s. fitted values
- Forbes' data



The Residuals (cont...)

- Case 12: possible outlier
- remove Case 12 and re-do the regression
- Summary Statistics for Forbes' Data with All Data and with Case 12 deleted

Quantity	All Data	Delete Case 12
$\hat{\beta}_0$	-42.138	-41.308
$\hat{\beta}_1$	0.895	0.891
se($\hat{\beta}_0$)	3.340	1.001
se($\hat{\beta}_1$)	0.016	0.005
$\hat{\sigma}$	0.379	0.113
R^2	0.995	1.000

A “Good” Residual Plot from Heights Data