UNIVERSITY OF TORONTO
Faculty of Arts and Sciences

AUGUST EXAMINATIONS

MAT237Y1Y

Duration - 3 hours

No Aids Allowed

Instructions: There are 7 questions and 18 pages including the cover page. There is a total of 125 marks which include 25 bonus marks. There is one item in question 3 which is marked toward your term test. Try to answer as many questions as you can. Note that the number of questions is more than you are expected to answer in a 3 hour exam, so please make a careful selection and answer those questions whose answers you are most confident about, within the space provided; (please clearly specify if you use back of a sheet to answer a question.) A complete answer consists of clear and sufficient reasoning, and in each case a large portion of the mark is dedicated to reasoning.

NAME: (last, first) ______________________

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

MARKER'S REPORT:

Question	MARK
Q1 /27	
Q2 /14	
Q3 /25	
Q4 /13	
Q5 /12	
Q6 /10	
Q7 /24	
TOTAL /125	

1. Answer to the following questions and make sure to explain your answers by referring to necessary, yet short calculations based on relevant theorems while briefly quoting or naming the theorem and presenting and checking the necessary details.

   a) (2 marks) Is it possible to solve the equation $\ln y - \sin(xy) - \cos(\frac{\pi y}{2}) = 0$ for y in terms of x near the point $(0, 1)$.

   b) (5 marks) Consider the transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $f(x, y) = (x^2 - 2xy, x \ln y)$. Is f invertible near the point $f(-1, 1)$? If it is, find the frechet derivative of the inverse at the given point.

c) (5 marks) Does $\boldsymbol{f}(x, y) = (x^2 - 2xy, x \ln y, x - y)$ define a smooth surface at the point $\boldsymbol{f}(1, 1)$? If yes, find the equation of the tangent plane to the surface at the given point.

d) (5 marks) Fill in the bounds of the triple integral on the right: (to justify your answer you need to refer to the diagram as well)

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\,dy\,dx = \int \int \int dx\,dy\,dz$$

e) (5 marks) Could the following system be solved for the unknowns u and v in terms of the variables x, y and z near the point $(1,1,1,1,1)$? If yes, then calculate $\frac{\partial u}{\partial x}$.

$$\begin{cases} xy^2 + xzu + yv^2 = 3 \\ u^3yz + 2xv - u^2v^2 = 2 \end{cases}$$

f) (5 marks) Is the vector field $\boldsymbol{G}(x, y, z) = (xe^z, 1, e^z)$ conservative? If not explain why, and if yes find a function f such that $\boldsymbol{G} = \nabla f$.

2. Green's theorem

a) (2 marks) State Green's theorem for a vector field $\mathbb{F}$ on a region S of the plane.

b) (5 marks) Use Green's theorem to show that ∇f satisfies the independence of path property where f is a C^2 function defined on $\mathbb{R}^2$.

c) (7 marks) verify Green's theorem for the vector field $\boldsymbol{F}(x, y) = (x^2y + y, y^2 - x)$ on the region S bounded by the curve of $y = x^2$, the line $x = 1$ and the x-axis.

3. integrals

a) (4 marks) What does it mean for a set $S \subset \mathbb{R}^2$ to be measurable, and what is the relevance of measurability in definition of a double integral $\iint_S f(x,y)dA$?

b) (6 marks) Use an appropriate double integral to calculate $\displaystyle\int_0^1 (e^{\sqrt{x}} - 1)dx$.

c) (12 marks toward your term test) Give statements of both Extreme value theorem and Intermediate value theorems and then present the proof of only one of them.

d) (8 marks) (MVT for double integrals) Prove if S is compact, connected, measurable subset of $\mathbb{R}^2$ and f and g are continuous on S and g is positive on S, then there exists a $\boldsymbol{a} \in S$ such that

$$\iint_S fg dA = f(\boldsymbol{a}) \iint_S g dA$$

Make sure to clearly present the reason behind every assupltion made in the statement of the theorem.

e) (8 marks) (FTC for one variable) Let F be a continuous function on $[a, b]$ that is differentiable except perhaps as finitely many points in $[a, b]$, and let f be a function on $[a, b]$ that agrees with F' at all points where the later is defined. If f is integrable on $[a, b]$, prove $\int_a^b f(t)dt = F(b) - F(a)$.

4. Chain rule

a) (3 marks) State the chain rule for a vector valued function $\boldsymbol{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$, a scalar valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the composition $(f \circ \boldsymbol{g}) : \mathbb{R} \longrightarrow \mathbb{R}$.

b) (3 marks) Prove that the gradient of a C^1 function f is a conservative vector field.

c) (8 marks) Use chain rule (II) and differentiation under the integral sign to calculate $\dfrac{\partial F}{\partial x}$ at the point $\boldsymbol{a} = (0.5, 1)$, where $F(x, y) = \displaystyle\int_1^{\cos^{-1} x} \frac{y^2}{t} \sin(t + xt)dt$.

5. Smooth curves and the IFT:

a) (4 marks) Give the three representations of the smooth curve in $\mathbb{R}^3$ in the order they appeared in the textbook.

b) (8 marks) Choose one of the representations (ii) or (iii), and state a regularity condition that guarantees this representation can be translated to representation (i). Then prove your claim.

6. Transformations of $\mathbb{R}^2$

a) (3 marks) Present the change of variable formula for double integrals for a linear transformation $\boldsymbol{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $(x, y) = \boldsymbol{G}(u, v)$, a measurable region S and an integrable function $f(x, y)$ on S.

b) (7 marks) Apply your formula from part (a) to integrate $\displaystyle\iint_S (2x + y)^2 e^{(x-y)} dA$ where the region S is the region bounded by $2x + y = 1, 2x + y = 4x - y = -1$ and $x + y = 1$.

7. Stokes' and Divergence theorems

a) (9 marks) Use Stokes' theorem to evaluate the surface integral $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n} dA$, where $\boldsymbol{F}(x, y, z) = (y, -x, e^{xz})$ and S is the surface of the portion of the solid sphere $x^2 + y^2 + (z-1)^2 = 2$ above the xy-plane cut off by the plane $x + y = 1$ (orients up and outward.) (Note the boundary of the surface, on the $z = 0$ level, consists of three quarters of a circle and a straight line, and it surrounds the origin.)

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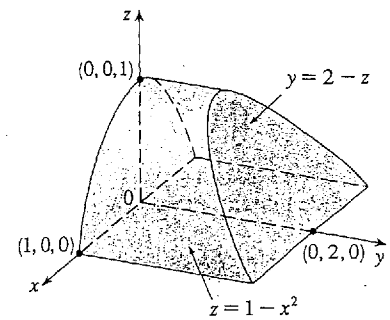
b) (7 marks) Use divergence theorem to compute the surface integral of $\boldsymbol{F}(x, y, z) = (xy, y^2 + e^{xz^2}, \sin(xy))$, on T, the surface of the region E of $\mathbb{R}^3$ bounded by parabolic cylinder $z = 1 - x^2$ and the planes $z = 0,\ y = 0$, and $y + z = 2$.



c) (8 marks) Prove that as long as the vector field $\boldsymbol{F}$ is C^1 on a closed surface S (not necessarily C^1 in the inside of S) the total flux of $\nabla \times \boldsymbol{F}$ across S vanishes, that is the surface integral of $\nabla \times \boldsymbol{F}$ on S is 0.