PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 5 SOLUTIONS

(1) The summation identity for binomial coefficients states that:

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Give two proofs of this identity, one using the bug-path model for binomial coefficients, and one using induction.

Solution: First let's prove this by induction. For a fixed non-negative integer k, let $P(n)$ denote the claim

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

For $n = 0$ this is simply $\binom{0}{k} = \binom{1}{k+1}$. If $k = 0$ then both sides are equal to 1, while if $k > 0$ both sides are equal to 0; so $P(0)$ holds. Now assume $P(n)$ is true, and note that

$$\sum_{i=0}^{n+1} \binom{i}{k} = \sum_{i=0}^{n} \binom{i}{k} + \binom{n+1}{k}.$$

Thus, by the inductive hypothesis we have

$$\sum_{i=0}^{n+1} \binom{i}{k} = \binom{n+1}{k+1} + \binom{n+1}{k}.$$

Note that the RHS is equal to $\binom{n+2}{k+1}$, since $[n+2]$ has $\binom{n+1}{k}$ subsets of size k that contain 0 and $\binom{n+1}{k+1}$ that don't contain 0. Thus we have shown that $P(n) \implies P(n+1)$ for all $n \in \mathbb{N}_0$, so $P(n)$ holds for all $n \in \mathbb{N}_0$.

Now let's use the bug-path model. Let $N(w, h) = \binom{w+h}{h}$ denote the number of bug-paths from $(0, 0)$ to (w, h). Notice that if $w = n - k$ and $h = k$ we can write $\binom{n+1}{k+1} = N(w, h + 1)$, so we can interpret the RHS of the identity as the number of paths from $(0, 0)$ to $(w, h + 1)$.

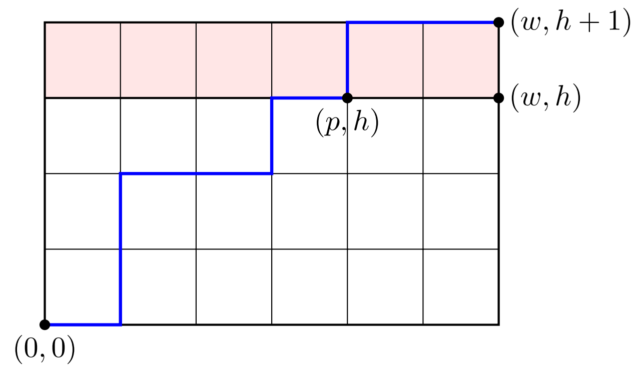

FIGURE 1. Bug-path from $(0,0)$ to $(w, h+1)$ crossing strip at p

Any such path must cross the strip between h and $h+1$ (shaded red in Figure **??**) at exactly one place - let p denote the x-coordinate of this place (so that the line segment from (p, h) to $(p, h+1)$ is part of the path). Since there is no choice but to travel horizontally from $(p, h+1)$ to $(w, h+1)$ after you have crossed the strip, such a path is determined by the path it took from $(0,0)$ to (p, h); so the number of bug-paths from $(0,0)$ to $(w, h+1)$ that cross the strip at p is exactly $N(p, h)$. Thus adding up the contributions from each p we get the identity

$$N(w, h+1) = \sum_{p=0}^{w} N(p, h);$$

which (converting back to binomial coefficients in terms of $n = w + h$, $k = h$ and $i = p + k$) can be written

$$\binom{n+1}{k+1} = \sum_{i=k}^{n} \binom{i}{k}.$$

Since the binomial coefficients $\binom{i}{k}$ are zero for $i < k$, we can start the sum at $i = 0$ without changing the value.

(2) Give short, insightful proofs of the following formulae:

(a) $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$,

Solution: This is a generalization of the "Chairperson Identity" from the last tutorial sheet: both sides of this formula count the number of ways of choosing a committee of k members, with j designated executive members, from a pool of n people. (The chairperson identity is the case $j = 1$.) On the LHS we first choose the k committee members from the n people, and then choose j of the k committee members to be executives. On the RHS we first choose the j executives from the n people, and then choose the remaining $k - j$ committee members from the remaining $n - j$ people.

(b) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

Solution: Once again we can interpret this as counting committees - this time it's the number of committees (with single chairperson) of *any* size that can be formed from a pool of n people.
Recall that the number of committees with exactly k members is $k\binom{n}{k}$, since there are $\binom{n}{k}$ subsets of k people, and once this subset has been chosen there are k choices for chairperson. Thus we can get the number of committees of arbitrary size by adding this up over all possible values of k, which gives the expression on the LHS.
On the RHS, we instead choose the chairperson first from the pool of n people, which leaves $n-1$ people, each of whom can independently either be in the committee or not; so we have one choice with n options and $n-1$ choices with 2 options, yielding $n2^{n-1}$ overall possibilities.

(3) Count the sets of six cards from a standard deck of 52 cards that have at least one card in every suit.

Solution: We can divide the set of such hands in to two categories - those that have three cards of a single suit (and thus one of each of the others), and those that have two cards of one suit, two of another and one each of the remaining two suits.

First let's describe a procedure to choose a set from the first category. First we need choose the suit that will appear on three cards, for which we have 4 options. Then we choose the values of the three cards in that suit, for which we have $\binom{13}{3}$ options. Finally choose the values for the cards in the three remaining suits, which each independently have 13 options; so overall we have $4\binom{13}{3}13^3$ possibilities.

For the second category, we first choose which two suits will occur on two cards each, for which we have $\binom{4}{2}$ options. For each of these two suits we have $\binom{13}{2}$ options for the values of the cards, and for each of the two other suits we have 13 options; so overall there are $\binom{4}{2}\binom{13}{2}^2 13^2$ options.

Thus we have a grand total of

$$4\binom{13}{3}13^3 + \binom{4}{2}\binom{13}{2}^2 13^2 = 8682544$$

possible sets.

(4) Count the number of ways to group $2n$ people into n distinct pairs. (For example, the answer is 3 when $n = 2$).

Solution: First let's instead count the number of ways to divide the people into *ordered* pairs - that is, we're going to divide the group in to n pairs and then give one member of each pair a silly hat. To do this, we will first choose the n lucky hat recipients, which gives us $\binom{2n}{n}$ options; and then decide how to pair up the n hat recipients with the remaining n people, for which there are $n!$ choices. Thus there are $\binom{2n}{n}n!$ possible divisions of the group into ordered pairs.

Now, note that for each division of the group in to pairs there are exactly 2^n corresponding divisions in to ordered pairs, since for each pair there are two choices for whom gets the hat. Thus we can find the number of divisions in to pairs by dividing the number of divisions in to ordered pairs by 2^n, yielding the final answer

$$\frac{n!}{2^n}\binom{2n}{n} = \frac{(2n)!}{n!2^n}.$$

(5) (a) Count the solutions in *positive* integers to the equation $x_1 + \ldots + x_k = n$.

Solution: This is equivalent to counting the number of ways to place n indistinguishable balls in to k distinguishable bins, with the requirement that every bin contains at least one ball: x_i is just the number of balls that we put in the i^{th} bin. We might as well start by just putting a ball in every bin, reducing the problem to counting the number of ways to put $n - k$ balls in k bins[1].

But we already know how to count the number of ways to do this using the idea of *balls and walls*[2]: we can represent each configuration as a string of $n - 1$ symbols, $k - 1$ of which are walls (the boundaries between the bins) and $n - k$ of which are the balls. A configuration is thus uniquely determined by the positions of the walls; so there are

$$\binom{n-1}{k-1}$$

possible configurations.

(b) Count the solutions in *non-negative* integers to the equation $x_1 + \ldots + x_k \leq n$.

Solution: This is a similar problem, with two differences. The fact that the x_k are now only required to by non-negative means we get to choose where to put the full n balls, rather than just $n - k$ of them; and we now have an arbitrary number m of balls between 0 and n, rather than a fixed number n.

We can deal with this second issue by starting with n balls and introducing the *rubbish bin*, in which we will place the remaining $n - m$ balls. But the rubbish bin is just a bin, so we are really just counting the number of ways of putting n balls in $k + 1$ bins. Following the same balls-and-walls argument with n balls and k walls gives the answer

$$\binom{n+k}{k}.$$

[1] Equivalently, we can define $y_i = x_i - 1$, and note that y_i must be *non-negative* integers adding to $n - k$.

[2] c.f. *stars and bars*, *frogs and logs*.