Solns

THE FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2004

MAT 246Y
Concepts in Abstract Mathematics

Examiners: J. Korman and P. Rosenthal
Duration: 3 hours

LAST NAME: ______________________

FIRST NAME: ______________________

STUDENT NUMBER: ______________________

- There are ten questions, each of which is worth 10 marks.
- This paper has a total of 11 pages, including this cover page.
- **No calculators, scrap paper, or other aids permitted.**
- Write your answer in the space provided. Use the back sides of the pages for scrap work.
- **DO NOT tear any pages from this test.**

FOR MARKERS ONLY	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

1. (a) Does the following congruence have an integral solution: $x^5 \equiv 3 \pmod 4$?
Prove that your answer is correct.

check 0, 1, 2, 3 mod 4

3 works :

$$3^5 \equiv (-1)^5 \equiv -1 \equiv 3 \quad \text{mod } 4$$

(b) Show that there is no digit a such that the number $2794a1$ is divisible by 8.

$$279400 = 279 \times 1000 + 400$$

div. by 8

So enough to show $10a+1$ not divisible by 8 :

$$10a+1 \equiv 0 \quad \text{mod } 8$$
$$2a+1 \equiv 0 \quad \text{mod } 8$$
$$8a+4 \equiv 0 \quad \text{mod } 8$$
$$4 \equiv 0 \quad \text{mod } 8$$

2. (a) Let $f(x)$ be a polynomial with integer coefficients and let a, k, and m be integers. Suppose that $f(a) \equiv k \pmod{m}$. Prove that $f(a+m) \equiv k \pmod{m}$.

$$a+m \equiv a \mod m$$

$$(a+m)^k \equiv a^k \mod m \quad \forall k \geq 1$$

$$f(a+m) \equiv f(a) \equiv K \mod m$$

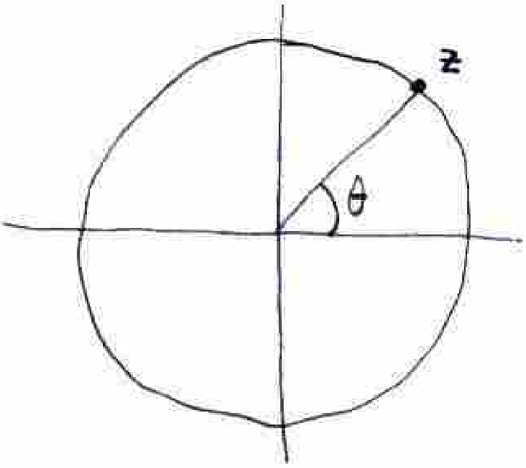
(b) Prove that for all primes $p > 3$, $2(p-3)! \equiv -1 \pmod{p}$.

Wilson : $(p-1)! \equiv -1 \mod p$

$$\begin{array}{c} (p-1)! \\ \parallel \\ (p-3)!\,(p-2)(p-1) \end{array}$$

But $(p-2)(p-1) \equiv (-2)(-1) \equiv 2 \mod p$

3. (a) Find $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{10}$. Show your work.



$$z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$z = 1 \cdot e^{i\theta} \qquad \theta = \pi/4$$

$$z^{10} = e^{i\,10\theta} \qquad 10\theta = \frac{\pi}{2} + 2\pi$$

$$z^{10} = e^{i\pi/2} = i$$

(Basically, we multiply θ by 10).

(b) Find all the cube roots of 2. Show your work.

$$(re^{i\theta})^3 = 2$$

$$r^3 e^{i3\theta} = 2e^{i0}$$ zero

$$\begin{cases} r = \sqrt[3]{2} \\ \theta = \dfrac{0 + 2\pi k}{3} \qquad k \in \mathbb{Z} \end{cases}$$

$k=0$: $\theta_0 = 0$

$k=1$: $\theta_1 = 2\pi/3$

$k=2$: $\theta_2 = 4\pi/3$

4. (a) Is $2^{598} + 3$ divisible by 15 ? Show that your answer is correct.

div. by 15 $\Rightarrow$ div. by 3

but $2^{598} + 3 \equiv 2^{598} \not\equiv 0 \mod 3$

(b) Prove that $\sqrt[3]{5} + \sqrt{3}$ is irrational.

$$\sqrt[3]{5} + \sqrt{3} = \frac{m}{n} \qquad (m,n) = 1$$

$$\sqrt[3]{5} = \frac{m}{n} - \sqrt{3}$$

$$\sqrt{5} = \left(\frac{m}{n} - \sqrt{3}\right)^3 = \left(\frac{m}{n}\right)^3 + 3\left(\frac{m}{n}\right)\sqrt{3}^2 - 3\left(\frac{m}{n}\right)^2\sqrt{3} - 3\sqrt{3}$$

$$= \underbrace{\left(\frac{m}{n}\right)^3 + 9\left(\frac{m}{n}\right)}_{A} - \sqrt{3}\underbrace{\left(3\left(\frac{m}{n}\right)^2 + 3\right)}_{B}$$

$$5 = (A - \sqrt{3}B)^2 = A^2 - 2AB\sqrt{3} + 3B^2$$

onal $\rightarrow$ $\frac{5 - A^2 - 3AB^2}{-2AB} = \sqrt{3}$ $\leftarrow$ irrational

5. Let t be a transcendental number.

(a) Prove that $\{a+bt | a,b \in \mathbb{Q}\}$ is not a number field.

Enough to show $t^2 \notin \{a+bt \mid a,b \in \mathbb{Q}\}$

Suppose

$$t^2 = a + bt \quad \text{some } a,b \in \mathbb{Q}$$

$\Rightarrow$ t is a solⁿ of the polynomial

$$x^2 - bx - a = 0$$

$\Rightarrow$ t is algebraic

X.

(b) Prove that $t^4 + 7t + 2$ is transcendental.

Suppose its algebraic

$\Rightarrow$ $\exists$ polyⁿ $p(x)$ with rational coeffs.

s.t. $\underbrace{p(t^4 - 7t + 2)}_{\substack{\text{polynomial in } t \\ \text{with rational} \\ \text{coeffs.}}} = 0$

$\Rightarrow$ t is algebraic

X

6. Define the n^{th} *Fermat number* $F_n = 2^{2^n} + 1$ for $n \in \mathbb{N}$. The first few Fermat numbers are $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257$.

(a) Prove by induction that $F_0 \cdot F_1 \cdots F_{n-1} + 2 = F_n$ for $n \geq 1$.

$n = 1$: $F_0 + 2 = F_1$ ✓

Assume true for n, show true for $n+1$:

$$\underbrace{F_0 \cdot F_1 \cdots F_{n-1}}_{F_n - 2} F_n + 2 \stackrel{?}{=} F_{n+1}$$

$$(F_n)^2 - 2F_n + 2 \stackrel{?}{=} F_{n+1}$$

Note: $F_{n+1} = 2^{2^{n+1}} + 1 = 2^{2^n \cdot 2} + 1 = \left(2^{2^n}\right)^2 + 1$

$= (F_n - 1)^2 + 1 = (F_n)^2 - 2F_n + 2$

(b) Use part (a) above, to prove that each pair of distinct Fermat numbers is relatively prime. (You might note that this gives another proof that there are infinitely many primes.)

$F_i \mid F_n \qquad 0 \leq i \leq n$

$\Rightarrow F_i \mid 2$

but $F_i \not> 2$

7. For each of the following, answer 'true' or 'false' and justify your answer.

alse (a) An angle of 92.5° is constructible.

$\Rightarrow 2.5^\circ$ constructible X

true (b) The number $tan\frac{\pi}{4}$ is constructible.

$= 1$

false (c) The number $2^{1/6}$ is constructible.

$\Rightarrow 2^{1/3}$ constructible

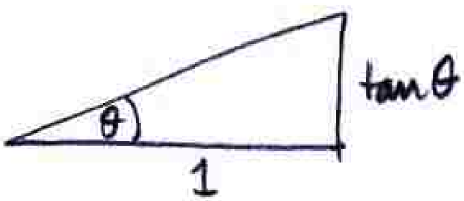
$\Rightarrow x^3 - 2$ has constructible roots

hence rational roots X

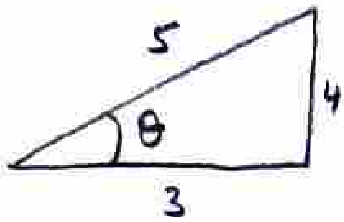
rue (d) The number $2^{3/2}$ is constructible.

$= \sqrt{8}$

lse (e) There is an angle θ such that $tan\theta$ is a constructible number, but θ is not a constructible angle.



8. Prove that the acute angle whose cosine is $\frac{3}{5}$ cannot be trisected with a straightedge and compass.



Hint: $\cos\theta = 4\left(\cos\frac{\theta}{3}\right)^3 - 3\cos\frac{\theta}{3}$

$\left.\begin{array}{l}\theta \text{ constructible} \\ \text{Assume trisectable}\end{array}\right\} \Rightarrow \frac{\theta}{3}$ constructible

$\Rightarrow \cos\frac{\theta}{3}$ const.

$\Rightarrow \frac{3}{5} = 4x^3 - 3x$ has a constructible root

$\Rightarrow$ has a rational root...

9. (a) Let $S := \{f : \{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}\} \to \mathbb{Q}\}$ be the set of all functions mapping the set $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}\}$ into the rational numbers. What is the cardinality of S ? Prove that your answer is correct.

Any such f is determined by a 4-tuple of rational numbers.

so $S \leftrightarrow \mathbb{Q}^4$

so $|S| = |\mathbb{Q}^4| = \aleph_0$

(b) Let $T := \{g : S \to \{0,1\}\}$ be the set of all functions mapping the set S from part (a) above into the set $\{0,1\}$. What is the cardinality of T ? Prove that your answer is correct.

$|T| = 2^{|S|} = 2^{\aleph_0} = c$.

10. Recall that a *tower* is a finite sequence of number fields, the first of which is $\mathbb{Q}$, such that the other number fields are obtained from their predecessors by adjoining square roots. Is the set of all towers countable ? Prove that your answer is correct.

tower $\mathbb{Q} = F_0 \subset \mathbb{Q}(\sqrt{r_0}) = F_1 \subset \cdots \subset F_n$

$$\mathcal{T} = \bigcup_{k \geq 1} \mathcal{T}_k \leftarrow \text{towers of length } k$$

$$\mathcal{T}_k := \left\{ \mathbb{Q} = F_0 \subseteq F_0(\sqrt{r_0}) \subseteq \cdots \subseteq F_{k-1} \;\middle|\; \begin{array}{l}\text{some} \\ \text{conditions} \\ \text{on } r_i\end{array} \right\}$$

$$\subseteq \left\{ \mathbb{Q} = F_0 \subseteq F_0(\sqrt{r_0}) \subseteq \cdots \subseteq F_{k+1} \right\}$$

$$\updownarrow$$

$$\mathbb{Q}^k$$

$\aleph_0 = |\mathbb{Q}| \leq |\mathcal{T}_k| \leq |\mathbb{Q}^k| = \aleph_0$

so $|\mathcal{T}_k| = \aleph_0$

so $|\mathcal{T}| = \aleph_0$ ($\because$ Countable union of countable sets).