TUTORIAL 5

(1) The least common multiple of a two numbers is the smallest number divisible by both. Prove that

$$\gcd(a, b)\operatorname{lcm}(a, b) = ab.$$

Hint: Use prime factorizations of a and b.

(2) Prove that 3 divides $4^n - 1$ for every positive integer n.

(3) A prime of the form $2^n - 1$ is called a *Mersenne prime*. For example, 3 and 7 are Mersenne primes, but 5 and 11 are not. A positive integer n is called *perfect* if it is the sum of all the smaller positive integers that divide it. For example, 6 and 28 are the first two perfect numbers.

Prove that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect. In fact, Euclid conjectured that every perfect number is of this form, and this is still an open problem.

(Hint: You may need to use the geometric summation formal which states that for any real number $q \neq 1$,

$$1 + q + q^2 + \ldots + q^k = \frac{q^{k+1} - 1}{q - 1}.$$

Do you see why this is true?)