University of Toronto
Department of Mathematics
FACULTY OF ARTS AND SCIENCE
MAT224H1Y
Linear Algebra II

Final Exam
August 18, 2010

Duration: 3 hours

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

Signature: ______________________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

1. Let $(\cdot, \cdot)$ be the inner product on $P_2(\mathbb{C})$ defined by

$$(p(x), q(x)) := \int_{-1}^{1} \overline{p(t)} q(t)\, dt.$$

(a) Using the Gram-Schmidt process with the basis $\{1, x, x^2\}$, find an orthonormal basis for $P_2(\mathbb{C})$.

(b) Find the distance between $p(x) = 1$ and $q(x) = x^2$.

EXTRA PAGE FOR QUESTION 1 - please do not remove.

2. Consider c to be a fixed real number. Define the real quadratic form $q_c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$q_c(x_1, x_2) := x_1^2 + (2 + 2c)x_1x_2 + x_2^2.$$

(a) Find the matrix for q_c relative to the standard basis of $\mathbb{R}^2$.

(b) Find variables y_1, y_2 that diagonalize q_c, and give the formula for q_c with respect to these new variables. (Note: you are allowed to have y_1, y_2 depend on c.)

(c) For what values of c is q_c a positive definite real quadratic form? For what values is it positive semidefinite?

EXTRA PAGE FOR QUESTION 2 - please do not remove.

3. Let W be a subspace of an inner product space V and let $W^{\perp}$ denote its orthogonal complement.

(a) Show that $W^{\perp}$ is also a subspace of V.

(b) Show that $W \cap W^{\perp} = \{0\}$.

4. Let $V = \mathbb{R}_{2\times 2}$. Define subspaces W_1, W_2 of V by

$$W_1 := span\left\{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right\},$$
$$W_2 := \{A \in \mathbb{R}_{2\times 2} \mid A = A^T\}.$$

Then V is the direct sum of W_1 and W_2 (you do not need to show this).

(a) Give a formula for the projection P onto W_1 along W_2.

(b) Give a formula for the projection Q onto W_2 along W_1.

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5. **(a)** State the definition of a unitary matrix.

(b) Let $U \in \mathbb{C}_{n \times n}$ be unitary. Is U diagonalizable? Why or why not?

(c) Show that if λ is an eigenvalue of a unitary matrix U then $|\lambda| = 1$.

6. Let

$$A := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ i & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{C}_{4\times 4}.$$

(a) Determine the Jordan canonical form J of A.

(b) Find an invertible matrix $P \in \mathbb{C}_{4\times 4}$ such that $J = P^{-1}AP$.

EXTRA PAGE FOR QUESTION 6 - please do not remove.