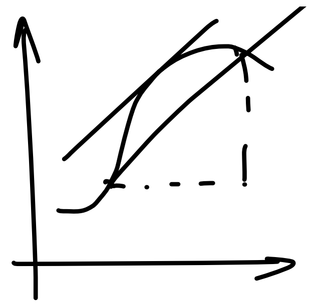
Lecture 6

§5.4 Why is this true?

RECALL:

Mean Value Thm. Let $F(x)$ be a continuous function on $[a,b]$ and differentiable in (a,b). Then $\exists\, c \in (a,b)$ s.t.

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$



Thm: (Attractive fixed pts)
Let p be an attractive fixed pt for F. Then $\exists$ an open interval I with $p \in I$ s.t. if $x \in I$ then $x_n = F^n(x) \in I$ for all $n \in \mathbb{N}$
And $x_n \longrightarrow p$

Proof: We know that p is an attractive fixed point so $F(p)=p$ and $|F'(p)|<1$ (By definition)

Then $\exists\, 0<\lambda<1$ s.t. $|F'(p)|<\lambda<1$
So $\exists\, \delta>0$ st. $|F'(x)|<\lambda$ for all $x \in \boxed{(p-\delta, p+\delta)}$ $\rightarrow$ This is the "I"

Define $I=(p-\delta, p+\delta)$ which is an open interval.
Let $x_0 \in I$ and $x_n = F^n(x_0)$.
By the MVT, $\exists\, c$ between x_0 & p s.t.

$$\frac{F(x_0)-F(p)}{x_0-p} = F'(c)$$

so $$\left|\frac{F(x_0)-F(p)}{x_0-p}\right| = |F'(c)| < \lambda \text{ because } c \in I$$

Thus $|F(x_0)-F(p)| < \lambda\, |x_0-p|$

$$|x_1-p| < \lambda\, |x_0-p|$$

So we can prove by induction (exercise) that

$$|x_n-p| < \lambda^n\, |x_0-p|$$

Then $\begin{cases} |x_n-p| < |x_0-p| \Rightarrow x_n \in I \\ x_n \to p \quad \text{b/c} \quad \lambda^n \longrightarrow 0 \end{cases}$

∎

$\boxed{\text{Remark}}$: to choose/find the interval I, we need I to satisfy
- $|F'(x)| < \lambda < 1$ for all $x \in I$
- I is symmetric around p.

Thm: (Repelling fixed pts) Assume F is differentiable. Let p be an repelling fixed pt for F. Then $\exists$ an open interval I with $p \in I$ s.t. if $x \in I$, $x \neq p$, then $|F(x_0)-p| > |x_0-p|$

Proof: We know that p is a repelling fixed pt, so $F(p)=p$ and $|F'(p)|>1$ (By defition)
Then $\exists\ \lambda > 1$ s.t. $|F'(p)| > \lambda > 1$ and $\exists\ \delta > 0$ satisfying $|F'(x)| > \lambda$ for all $x \in \underbrace{(p-\delta, p+\delta)}_{I}$. Define $I=(p-\delta, p+\delta)$ and let $x_0 \in I$, $x_0 \neq p$

By MVT $\exists$ c between x_0 and p s.t.

$$\frac{|F(x_0)-F(p)|}{|x_0-p|} = |F'(c)| > \lambda$$

Thus $|F(x_0)-F(p)| > \lambda |x_0-p|$

so $|F(x_0)-p| > |x_0-p|$ ∎

$\boxed{\text{Remark}}$: From the proof we actually have

$$|x_n-p| > \lambda^n |x_0-p| \quad \text{if } x_0, x_1, \cdots, x_{n-1} \in I$$

$\Rightarrow$ The orbit will escape I.
i.e. $\exists N$ s.t. $x_N \notin I$

Ex: $F(x) = 2x(2-x)$ has 2 fixed pts
$x=0$ has $x=3/2$
and $F'(0) = 4 > 1$ so 0 is a repelling fixed point.