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MAT 337
Midterm Exam
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NO AIDS ALLOWED

Total: 250 points, not including a bonus problem

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Problem 1 [20 points]

Determine which of the following sequences converge:

(a) $(a_n)_{n=2}^{\infty}$, where $a_n = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})...(1 - \frac{1}{n^2})$.

(b) $(b_n)_{n=1}^{\infty}$, where $b_n = x^{\frac{1}{n}}$ and $x > 0$.

Explain.

$a_n = \frac{n+1}{2n}$

(a). $\frac{a_{n+1}}{a_n} = \frac{(1-\frac{1}{2^2})\cdots(1-\frac{1}{n^2})(1-\frac{1}{(n+1)^2})}{(1-\frac{1}{2^2})\cdots(1-\frac{1}{n^2})} = 1 - \frac{1}{(n+1)^2} < 1$ ×

So ~~By ratio test~~, $\lim_{n\to\infty} a_n = 0$ $\lim_{n\to\infty} a_n = \frac{1}{2}$

it converges

(b). ① if $0<x<1$, $\frac{x^{\frac{1}{n+1}}}{x^{\frac{1}{n}}} = x^{\frac{1}{n+1}-\frac{1}{n}}$ since $\frac{1}{n+1}-\frac{1}{n}<0$

So $x^{\frac{1}{n+1}-\frac{1}{n}} > 1$, ~~therefore~~ b_n is increasing

but $b_n = x^{\frac{1}{n}} < 1$ is bounded,

by monotone sequence thm, b_n converges to 1.

② if $x=1$, $b_i = 1$ $\forall i$, converges to 1.

③ if $x>1$, $x^{\frac{1}{n+1}-\frac{1}{n}} < 1$ therefore b_n is decreasing,

but b_n is also $b_n > 0$ since $x>0$.

So by monotone seq. thm, ~~b_n converges to 0~~

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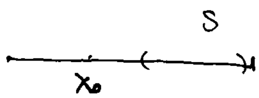
Problem 2 [30 points] The distance $d(x_0, S)$ between a real number x_0 and a non-empty set S of real numbers is defined by $d(x_0, S) = \inf_{x \in S} |x_0 - x|$. If S is bounded below and $x_0 = \inf S$, prove that $d(x_0, S) = 0$.



Proof: S bounded below, by definition,

~~Use~~ $\forall x \in S,\ x \geq \inf S = x_0$

let $\inf_{x \in S} |x_0 - x| = d(x_0, S) = z$

$\forall x \in S,\ |x_0 - x| \geq z$

We want to show $z = 0$

Suppose not, $z = \varepsilon > 0$ for some ε. (z not negative since it's an absolute value)

$\Rightarrow |x_0 - x| \geq \varepsilon$

$\Rightarrow |\inf S - x| = |x - \inf S| \geq \varepsilon$

$\Rightarrow \inf S \leq x - \varepsilon$ ①

By the defintion of infimum,

$\inf + \varepsilon > x \quad \forall x \in S$ ②

① ② contradiction.

Thus $z = 0$

i.e. $d(x_0, S) = \inf_{x \in S} |x_0 - x| = 0$.

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Problem 3 [30 points]

(a) State the rearrangement theorem for conditionally convergent series.

(b) Consider an infinite series $\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. Prove that the series converges conditionally.

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(c) Show that if $\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = s$ then the sum of the rearranged series $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\ldots$ converges to $\frac{s}{2}$.

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(a). ~~the~~ There exicts one rearrangement of a conditionally convergent series converges absolutely.

(a). If $\Sigma_{n=1}^{\infty} a_n$ is a conditionally convergent series, then for every ~~for~~ real number L, ~~there is~~ a rearrangement that converges to L.

(b) Let every term be $a_n = \frac{(-1)^{n-1}}{n}$

~~$|a_n| = \frac{1}{n}$~~ since $|a_n| = \frac{1}{n}$

~~$\lim \frac{1}{n}$~~ $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$ conditionally by Lebiniz's Alternating convergence theorem.

So the serives converges ~~absolutely~~.

~~So it converges conditionally as well.~~

(b)

Let $b_n = \frac{1}{n}$, then b_n is a non-negative ~~and~~ monotone decreasing sequence, with $\lim_{n\to\infty} b_n = 0$, this by Leibinz Alternating Series Test, $\Sigma_{n=1}^{\infty}(-1)^n b_n$ converges.

So $\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = -\Sigma_{n=1}^{\infty}(-1)^n b_n$ converges.

By 1) & 2) we know the series converges conditionally.

(c). $\lim_{n\to\infty} S_{3n} = \frac{S}{2}$

(need to prove)

(c). $1-\frac{1}{2}-\frac{1}{4} = \frac{1}{2}-\frac{1}{4}$

$\frac{1}{3}-\frac{1}{6}-\frac{1}{8} = \frac{1}{6}-\frac{1}{8}$

$\frac{1}{5}-\frac{1}{10}-\frac{1}{12} = \frac{1}{10}-\frac{1}{12}$

rearranged $S' = \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots$

$S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$

So $S' = S\cdot\frac{1}{2} = \frac{S}{2}$

~~Therefore~~

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Problem 4 [60 points] Let $S_0 = [0,1]$. Construct S_{i+1} from S_i by removing an open middle interval from each interval in S_i. That is $S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$; $S_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc.

18 (a) Is $C = \cap_{i \geq 1} S_i$ a closed set? Is it compact?

18 (b) Prove that $C = \cap_{i \geq 1} S_i$ is not empty;

0 (c) Consider $\Sigma_{i=1}^{\infty} \frac{y_i}{3^i}$, where $y_i \in \{0, 2\}$ for all natural numbers i. Prove that this series converges. Let $x = \Sigma_{i=1}^{\infty} \frac{y_i}{3^i}$. Is x an element in C?

8 (d) What is the cardinality of C?

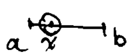
(a). $C = \cap_{i \geq 1} S_i = S_i$

S_i is a union of many many closed sets,

So S_i is closed

$\Rightarrow C$ is closed.

since ~~the~~ $\forall x \in C$, ~~lim (x...)~~



you draw a ball at x, with radius smaller than the distance from x to the closer endpoint of the interval where x is located in.

~~as the distance (radius of the ball) → 0~~

~~...~~ So x cannot get out of ~~...~~ the small interval as desired.

~~...~~ So x is bounded

By Heine-Borel thm, C is closed & bounded

so C is compact.

(b). since ~~...~~ every process we delete ~~...~~ some part of the previous S_i, So $S_0 \supset S_1 \supset S_2 \supset \cdots$ is ~~a~~ decreasing ~~...~~,

By Canter intersection theorem,

$C = \cap_{i \geq 1} S_i \neq \emptyset$.

c). This is equivalent with to show that the length of S_i, since every point on S_i can be represented as $\frac{y_i}{3^i}$, $y_i \in \{0, 2\}$.

So $x = \Sigma \frac{y_i}{3^i} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \notin$ ~~...~~ C

x is not an element of C.

(d). since all the points in $[0,1]$ can be represented by tenary expassion

$(0.x_1x_2 \cdots x_3)_{base_3} =$ ~~...~~ $\Sigma x_k \frac{1}{3^k}$ $\Rightarrow$ this means there is a one-to-one and onto map from $[0,1]$ to C

So $\|[0,1]\| = \|C\|$

and $\|[0,1]\| = \|\mathbb{R}\|$

So $\|C\| = \|\mathbb{R}\|$

The cardinality of C is the cardinality of real #s.

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Problem 5 [40 points]

Determine whether the following series converge or diverge.

(a)$\Sigma_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n}$ 0

(b)$\Sigma_{n=1}^{\infty}e^{-n^2}$ 10

(a). let $a_n=\sqrt{n+1}-\sqrt{n}$

(a_n) is bounded, since $n+1>n$

$\Rightarrow \sqrt{n+1} > \sqrt{n}$

$\Rightarrow \sqrt{n+1} - \sqrt{n} > 0$

So a_n has lower bound 0.

(a_n) is decreasing.

want to show

~~$\sqrt{n+1}-\sqrt{n}$~~

base case: $\sqrt{1+1}-\sqrt{1}=2-1=1$

$\sqrt{2+1}-\sqrt{2}=\sqrt{3}-\sqrt{2}<1$

...

Suppose true for $k=n$. then,

prove true for $k=n+1$.

i.e. $\sqrt{(n+1)+1}-\sqrt{n+1} < \sqrt{n+1}-\sqrt{n}$

$\sqrt{t+2}-\sqrt{t+1}<\sqrt{t+1}-\sqrt{t}$ (let $t=n$)

$t+2+t+1-2\sqrt{(t+2)(t+1)} < t+1+t-2\sqrt{t(t+1)}$

$1-\sqrt{(t+2)(t+1)} < -\sqrt{t(t+1)}$

$1-2\sqrt{(t+2)(t+1)}+(t+2)(t+1) > t(t+1)$

$1-2\sqrt{(t+2)(t+1)} > (t+1)(t-t-2)$

$1-2\sqrt{(t+2)(t+1)} > -2(t+1)$

$-2\sqrt{(t+2)(t+1)} > -2t-3$

$4(t^2+3t+2) < 4t^2+12t+9$

$4t^2+12t+8<4t^2+12t+9$

$8<9$. ✓ done.

So (a_n) is bounded below & decreasing.

By monotone sequence thm, it's convergent.

$\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}\to 0$

so $\sum_{n=1}^{\infty} a_n$ converges.

(b). let $b_n=e^{-n^2}$

$b_1=e^{-1}$

$b_2=e^{-4}$

...

$\frac{b_{n+1}}{b_n}=\frac{e^{-(n+1)^2}}{e^{-n^2}}=e^{-(n+1)^2+n^2}$

$=e^{(n+n+1)(n-n-1)}$

$=e^{-2n-1}$

since $-2n-1$ decreases as n increases

So e^{-2n-1} decreases.

~~but $e^t>0$ $\forall t\in\mathbb{R}$.~~

~~So e^{-2n-1} decreases & is bounded below. similarly by monotone sequence theorem,~~

~~(b_n) is convergent.~~

$e^{-2n-1}\to 0$ as $n\to\infty$ ✓

so (b_n) is convergent.

Then $\lim_{n\to\infty} b_n=0$

So $\sum_{n=1}^{\infty} b_n$ is also convergent.

(33)

Problem 6 [**40 points**] Prove that every convergent sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, ρ) is a Cauchy sequence.

Proof: $(x_n)_{n=1}^{\infty}$ convergent to , say L

$\Rightarrow$ ~~$\lim_{n\to\infty} x_n$~~ $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$

s.t. ~~$|x_n - L| < \varepsilon$~~ whenever $n \geq N$

$\varphi(x_n, L) \leq \varepsilon$

Let ~~m, n be two natural numbers~~. $\varepsilon' = \frac{1}{2}\varepsilon$

~~then $\forall \varepsilon'>0, \exists$ ... $\in \mathbb{N}$ s.t. $\varphi(x_m, n)$,~~

~~$\forall \varepsilon'>0, \exists m \in \mathbb{N}$ s.t. $\varphi(x_n, x_m) \leq \frac{1}{2}\varepsilon'$ whenever $n \geq N$~~

~~$\forall \varepsilon'>0, \exists m \in \mathbb{N}$ st. $\varphi(x_m, L) < \varepsilon'$ whenever $m \geq N$,~~

~~$\varepsilon = \varepsilon' + \varepsilon' \geq \varphi(x_n, x_m) + \varphi(x_m, L) \geq \varphi(x_n, L)$ by def'n of metric space.~~

whe

$\forall \varepsilon'>0, \exists N$ s.t $\varphi(x_m, L) \leq \varepsilon'$ whenever $m \geq N$

$\forall \varepsilon'>0, \exists N$ s.t. $\varphi(x_n, L) \leq \varepsilon'$ whenever $n \geq N$

$\varphi(x_m, L) + \varphi(x_n, L) \leq \varepsilon' + \varepsilon' = \varepsilon$ whenever $m, n \geq N$

(by triangle inequality)

Thus $(x_n)_{n=1}^{\infty}$ is Cauchy by definition

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Problem 7 [30 points]

Let ρ be a discrete metric on a nonempty set X. Describe all of the open and closed subsets of X.

open subsets.

$\forall x \in X$, $\exists \varepsilon > 0$ $\varphi(x, \varepsilon) < r$, r is a positive real number

So all the satisfying open ball are the open subsets of X.

$\forall x \in X, \exists \varepsilon > 0, \varphi(x, \varepsilon) \leq r$. r is a non-negative real number.

so all the satisfying ~~closed~~ ball (if $r=0$, then points) in X are such closed subsets of X.

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Bonus Problem [50 points] A sequence $(x_n)_{n=1}^{\infty}$ satisfies $0 < x_1 \leq x_2$ and $x_{n+2} = (x_{n+1}x_n)^{\frac{1}{2}}$ for $n = 1, 2, \ldots$. Prove that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and hence converges.

Proof. By induction. $x_{n+2} = \sqrt{x_{n+1} x_n}$

$x_3^2 = x_2 x_1$ $\qquad x_3 = \sqrt[2]{x_2 x_1}$

$x_4 = \sqrt{x_3 x_2} = \sqrt{\sqrt{x_2 x_1} \cdot x_2} = \sqrt[4]{x_2^3 x_1}$

$x_5 = \sqrt{x_4 \cdot x_3} = \sqrt{\sqrt[4]{x_2^3 x_1} \cdot \sqrt[4]{x_2^2 x_1^2}} = \sqrt[8]{x_2^5 x_1^3}$

$$\boxed{x_n = \sqrt[2^{n-2}]{x_2^n x_1^{n-2}}}$$

~~test~~ for $k = n+1$ case.

$\cancel{x_{n+2} = \sqrt[2^n]{x_2^{n+2} x_1^{n}}}$

$\cancel{x_{n+1} = \sqrt[2^{n-1}]{x_2^{n+1} x_1^{n-1}}}$

$\cancel{x_n = \sqrt[2^{n-2}]{x_2^{n} x_1^{n-2}}}$

$\cancel{x_{n+2} = \sqrt{x_{n+1} x_n}}$

$\cancel{RHS = \sqrt{\sqrt[2^{n-1}]{x_2^{n+1} x_1^{n-1}} \cdot x_2^{n} x_1^{n-1}}}$

$= \cancel{\sqrt[2^n]{x_2^{n}}}$

(see next page)

3	1	1
4	1	3
5	5	7
6	7	13

$$x_{n+2} = \sqrt{x_{n+1} x_n}$$

$$x_3 = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$x_4 = x_2^{\frac{1}{2}} x_3^{\frac{1}{2}} = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}+\frac{1}{2}} = x_1^{\frac{1}{4}} x_2^{\frac{3}{4}}$$

$$x_5 = x_3^{\frac{1}{2}} x_4^{\frac{1}{2}} = (x_1^{\frac{1}{4}} x_2^{\frac{3}{4}})^{\frac{1}{2}} x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$= x_1^{\frac{1}{8}+\frac{1}{2}} x_2^{\frac{3}{8}+\frac{1}{2}}$$

$$= x_1^{\frac{5}{8}} x_2^{\frac{7}{8}}$$

$$x_6 = x_5^{\frac{1}{2}} x_4^{\frac{1}{2}} = (x_1^{\frac{5}{8}} x_2^{\frac{7}{8}})^{\frac{1}{2}} (x_1^{\frac{1}{4}} x_2^{\frac{3}{4}})^{\frac{1}{2}} = (x_1^{\frac{5}{16}+\frac{1}{8}})(x_2^{\frac{7}{16}+\frac{3}{8}})$$

$$\cancel{= (x_1^{\frac{5}{16}+\frac{1}{8}})(x_2^{\frac{5}{8}+\frac{7}{8}})} = x_1^{\frac{7}{16}} x_2^{\frac{13}{16}}$$

$$\cancel{x_1^{\frac{7}{16}} x_2^{\frac{\cdot}{16}}}$$

So $x_n = x_1^{\frac{\cdots}{2^{n-2}}} x_2^{\frac{\cdots}{2^{n-2}}}$ (don't have time to get the expression here)

* Basic idea is that, we find the explict expression of x_n with x_1 & x_2 by induction.

Then we prove that $\lim_{n\to\infty} x_n = 0$ by ratio test

so when n ~~and~~ large enough, ~~m, n~~ x_m ~~and x_n are~~ is close enough to x_n ~~close enough~~, hence Cauchy.

Then by Cauchy Criterion, (x_n) converges.