Lecture 16 §7.3

Finite dimensional normal space behaves like $\mathbb{R}^n$

Thm: Let V be a normed vector space of dimension n, $\exists$ an isomorphism $T: \mathbb{R}^n \to V$ and $0 < c < C$, s.t. $\forall a \in \mathbb{R}^n$

$$c\|a\|_E \leq \|T(a)\| < C\|a\|_E$$

$\|a\|_E = (\sum a_i^2)^{1/2}$ Euclidean : put 2 i.e. $\|\cdot\|_2$

Proof: ① Choose a minimal basis for V
$\beta = \{V_1, \cdots, V_n\}$

Define $T: \mathbb{R}^n \to V$ $(a_1, \cdots, a_n) \mapsto \sum a_i V_i$

① $\|T(a)\| = \|\sum a_i V_i\| \underset{\text{triangle inequality}}{\leq} \sum |a_i| \|V_i\| \leq [\sum (a_i)^2]^{1/2} \cdot \underbrace{(\sum \|V_i\|)^{1/2}}_{C}$

$$\leq C \cdot (\sum |a_i|^2)^{1/2} = C\|a\|_E$$

② How small can $\|T(a)\|$ be ?

Consider $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$

S^{n-1} is closed & bdd subset of $\mathbb{R}^n \Rightarrow S^{n-1}$ is compact

$f: S^{n-1} \to \mathbb{R}$

$f(x) = \|T(x)\|$

- $f(x) \geq 0$
- Is $f(x) = 0$ for $\forall x$?

Can it happen that $T(x) = 0$ for $x \in S^{n-1}$

$T(x) = \sum x_i V_i$ No

f takes on its minimum on S^{n-1} (by compactness)

Let $x_0 \in S^{n-1}$

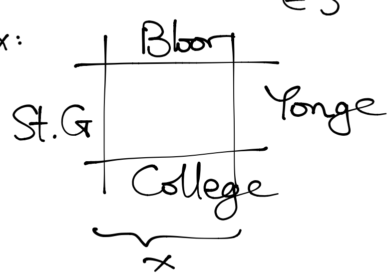
$f(x) \geq f(x_0)$ $\forall x \in S^{n-1}$

$c = f(x_0) = \|T(x_0)\|$

Let $a \in \mathbb{R}^n$

$$\|T(a)\| = \|T(\|a\|) \cdot \underbrace{\frac{a}{\|a\|}}_{\in S^{n-1}} \| = \|a\|_E \cdot \|T(\frac{a}{\|a\|})\| \geq \|a\|_E \cdot c$$

Ex:



$\sqrt{x^2+y^2}$

$V = (\mathbb{R}^2, \|\cdot\|_1)$

$\|(x,y)\|_1 = |x| + |y|$

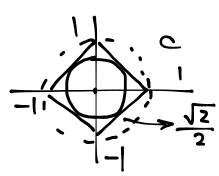
$T: (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, \|\cdot\|_1)$

$(*)$ $c\|(x,y)\|_2 \leq \|T(x,y)\|_1 \leq C\|(x,y)\|_2$

What is the largest value of c

Smallest C

s.t. $(*)$ is true



c can be 1

Thm: $T: \mathbb{R}^n \to V$ is Lipschitz (with L. constant C).

$T^{-1}: V \to \mathbb{R}^n$ is Lipschitz (with constant $1/c$).

$\|T(x,y)\| \leq C\|x-y\|$.

$\quad \|$

$\|T(x)-T(y)\|$

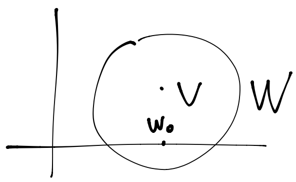
Cor: ~~Thm:~~ A subset $A \subseteq V$ is $\begin{pmatrix} \text{open} \\ \text{closed} \\ \text{bdd} \\ \text{compact} \end{pmatrix}$ iff $T^{-1}(A)$ is $\begin{pmatrix} \text{open} \\ \text{closed} \\ \text{bdd} \\ \text{compact} \end{pmatrix}$

a subset of finite-dimensional normed vector space

Cor: $A \subseteq V$ is compact iff it's closed & bdd

Cor: $A \subseteq V$ is complete, and in particular, it is closed.

Thm: $(V, \|\cdot\|)$ and $W \subseteq V$ is finite dimensional vector space. $\forall v \in V, \exists w^* \in W$ (one closest point to v) s.t. $\|w^* - v\| = \inf\{\|w-v\| : w \in W\}$



Proof: $M = \inf\{\|w-v\| : w \in W\}$

$B_{2M}(V)$