6.9) We know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$.

- Let $W = \frac{(n-1)S^2}{\sigma^2}$

from the table.

$$E(W) = E\left(\frac{(n-1)S^2}{\sigma^2}\right) = (n-1)$$

$$\Rightarrow \quad E\left(\frac{(n-1)S^2}{\sigma^2}\right) = (n-1)$$

$$\frac{(n-1)}{\sigma^2} E(S^2) = (n-1)$$

$$E(S^2) = \sigma^2 \frac{(n-1)}{(n-1)} = \sigma^2.$$

$\therefore$ Unbiased.

from the table

- $V(W) = 2(n-1)$

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

$$\frac{(n-1)^2}{\sigma^4} V(S^2) = 2(n-1)$$

$$V(S^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{(n-1)}$$

8.5) $X = \begin{cases} 1 \\ 2 \end{cases}$ $P(X=1) = \theta$; $P(X=2) = 1-\theta$

Data: $x_1 = 1,\ x_2 = 2,\ x_3 = 2.$ $n = 3.$

a.) MoM:

$$E(x) = \theta(1) + (1-\theta)(2)$$
$$= 2 - 2\theta + \theta = 2 - \theta.$$

Set $E(x) = \bar{X} \Rightarrow 2 - \theta = \bar{X}$

$$-\theta = \bar{X} - 2$$
$$\tilde{\theta} = 2 - \bar{X}$$

In this case we have: $\tilde{\theta} = 2 - \frac{5}{3} = \frac{1}{3}.$

- $E(\tilde{\theta}) = E(2 - \bar{X}) = 2 - E(\bar{X})$

$$= 2 - E(x) = 2 - (2-\theta) = \theta.$$

$\therefore$ $\tilde{\theta}$ is an unbiased estimator of θ!

$$\bullet \quad V(\tilde{\theta}) = V(2 - \bar{X})$$

$$= (-1)^2 V(\bar{X}) = V(X)/n$$

$$= \frac{E(x^2) - [E(x)]^2}{n}$$

$$\Rightarrow \quad E(x^2) = \theta(1^2) + (1-\theta)(2^2)$$

$$= \theta + (1-\theta)4$$

$$= 4 - 4\theta + \theta = 4 - 3\theta$$

$$V(\hat{\theta}) = \frac{4 - 3\theta - (2-\theta)^2}{n} = \frac{4 - 3\theta - [4^2 - 4\theta + \theta^2]}{n}$$

$$= \frac{\theta(1-\theta)}{n}.$$

b.) $L(\theta \mid \underset{\sim}{x}) = \prod_{i=1}^{n} \theta^{I(x_i=1)} (1-\theta)^{I(x_i=2)}$

$$= \theta^{\overbrace{\Sigma I(x_i=1)}^{n_1}} (1-\theta)^{\overbrace{\Sigma I(x_i=2)}^{n_2}}$$

$n_1 = \#$ of $x_i = 1$

$n_2 = \#$ of $x_i = 2$

$$= \theta^{n_1} (1-\theta)^{n_2}$$

$$\ell(\theta) = n_1 \log(\theta) + n_2 \log(1-\theta)$$

$$\ell'(\theta) = \frac{n_1}{\theta} - \frac{n_2}{1-\theta} = 0.$$

$$\frac{n_1}{\theta} = \frac{n_2}{1-\theta} \Rightarrow \hat{\theta} = \frac{n_1}{n_1+n_2}$$

$\therefore$ If $Y = \#$ of $x_i = 1$ out of $n = n_1+n_2$

then $\hat{\theta} = \frac{Y}{n}$; $Y \sim \text{binomial}(n, \theta)$

$\therefore$ For the specific case $\hat{\theta} = 1/3$.

- Check: $\ell''(\theta) = -\frac{n_1}{\theta^2} - \frac{n_2}{(1-\theta)^2} < 0$ ✓

- $E(\hat{\theta}) = E(Y/n) = \frac{1}{n} E(Y) = \frac{n\theta}{n} = \theta.$

$\therefore$ $\hat{\theta}$ is an unbiased estimator for θ.

- $V(\hat{\theta}) = V(Y/n) = \frac{1}{n^2} V(Y) = \frac{1}{n^2} n(\theta)(1-\theta)$

$$= \frac{\theta(1-\theta)}{n}$$

$\therefore$ Both estimators are unbiased and have the same verience.

8.52.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x \mid \theta) = (\theta+1)x^{\theta}$; $0 \le x \le 1$.

a.) MoM:

$$E(X) = \int_0^1 x\,(\theta+1)x^{\theta}\,dx$$

$$= \int_0^1 (\theta+1)\,x^{\theta+1}\,dx$$

$$= (\theta+1)\,\frac{x^{\theta+2}}{\theta+2}\Big|_0^1 = \frac{(\theta+1)}{(\theta+2)}$$

$$\Rightarrow \frac{\theta+1}{\theta+2} = \bar{X} \Rightarrow \tilde{\theta} = \frac{2\bar{X}-1}{(1-\bar{X})}$$

b.) $$L(\theta \mid \underset{\sim}{x}) = \prod_{i=1}^{n} (\theta+1)\,x_i^{\theta}$$

$$= (\theta+1)^n \prod_{i=1}^{n} x_i^{\theta}$$

$$\ell(\theta \mid \underset{\sim}{x}) = n\log(\theta+1) + \theta \sum_{i=1}^{n} \log(x_i)$$

$$\ell'(\theta \mid \underset{\sim}{x}) = \frac{n}{\theta+1} + \sum_{i=1}^{n} \log(x_i) = 0$$

$$\hat{\theta} = \frac{n}{-\sum \log(x_i)} - 1$$

$$\ell''(\theta \mid \underset{\sim}{x}) = -\frac{n}{(\theta+1)^2} < 0 \checkmark$$