PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 10
DUE FRIDAY, MAY 19, 4PM.

Warm-up problems. These are completely optional.

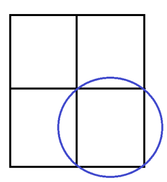
(1) Each year the grievance committee consists of three professors. How many professors must there be in the department to avoid having the same committee in a period of 11 years.

(2) How many integers less than 252 are relatively prime to 252.

Problems to be handed in. Solve four of the following five problems.

(1) Prove that every set of five points in the square of area 1 has two points separated by distance at most $\sqrt{2}/2$. Prove that this is the best possible by exhibiting five points with no pair less than $\sqrt{2}/2$ apart.

If we divide the square into 4 quadrants of equal size, then if two points are both in the same quadrant, the maximum distance they can be away from each other[1] is $\sqrt{2}/2$.



Since there are five points and only four quadrants, the pigeonhole principle says that at least one quadrant must contain two points. It follows that there must be at least two points separated by a distance of no more than $\sqrt{2}/2$.

The configuration consisting of a point on each corner, and a point at the center of the square gives a configuration with no pair less than $\sqrt{2}/2$ apart.

(2) A private club has 90 rooms and 100 members. Keys must be given to members such that each set of 90 members can be assigned 90 distinct rooms whose doors they can open. Each key opens one door. The management wants to minimize the total number of keys. Prove that the minimum number of keys is 990. (Hint: Consider the scheme where 90 of the members have one key, and the remaining 10 members have keys to all 90 rooms. Prove that this works, and that no scheme with fewer keys works.)

[1]You can see this by noting that both points x and z lie within a circle of radius $\sqrt{2}/4$, and then applying the triangle inequality to the pairs (x, y) and (y, z), where y is the center of the circle.

Let 10 of the members be distinguished, VIP members, and give to each of them a full set of 90 keys (so each one of them could open any door they choose). Give the remaining 90 members a single key each, with each of these keys opening a different door. To show that this scheme works, consider an arbitrary selection of 90 members from the original 100. Any such selection will contain some number, n, of VIP members, where $n \in \{0, 1, 2, \dots, 10\}$, and the remaining $90 - n$ members will not be VIPs.The non-VIPs can then be assigned to the door they have a key for, and the n VIPs can be assigned the remaining n doors. The VIPs will be able to open whatever door they are assigned because they can open every door. Thus the minimum number of keys is ≤ 990.

To prove that this is the minimum, we first note the following fact: if every door is opened by at least 11 keys, then the total number of keys must be at least 990 (since each key opens only one door and $90 \times 11 = 990$). The contrapositive of this statement says that if the total number of keys is less than 990, then there exists a door which is opened by only 10 keys (or fewer). Assume then, for the sake of a contradiction, that there is a scheme where we can give members keys in the required way such that the total number of keys is less than 990. We then know that there exists a door which is only opened by 10 keys (or fewer). The number of people who can open this door is less than or equal to 10, and we can therefore choose a subset of 90 people from the complement, none of whom have a key to this door. This gives us the required contradiction.

(3) Given $n, k \in \mathbb{N}$, determine the maximum size of a subset of $[n]$ that has no two numbers differing by exactly k.

If $k \geq n$ then no two elements differ by k and the answer is n. Suppose then, that $k < n$. We write $n = pk + r$, and then consider two cases: p even and p odd.

p **odd:**

Since p is odd, we write $n = (2m+1)k + r$, with $r < n$. Split the set $[n]$ into the following subsets:

$$\begin{aligned} S_1 &:= \{1, 2, \dots, 2k\} \\ S_2 &:= \{2k+1, \dots, 4k\} \\ &\vdots \\ S_m &:= \{(2m-2)k, \dots, 2mk\} \end{aligned}$$

plus the extra stuff:

$$E_1 := \{2mk+1, \dots, 2mk+k\} \tag{1}$$
$$E_2 := \{2mk+k+1, \dots, 2mk+k+r\} \tag{2}$$

A bit messy, but we have $[n] = S_1 \cup S_2 \cup \dots \cup S_m \cup E_1 \cup E_2$. Notice that each S_i has exactly $2k$ elements. Now consider the subset which takes the first k elements from each S_i, and then all the elements from E_1. This set contains $(m+1)k$ elements, no two of which differ by exactly k. To prove that this is the maximum, suppose for the sake of contradiction that we could choose a subset $\mathcal{A}$ which contained more than $(m+1)k$ elements, no two of which differed by exactly k. Since at most k elements of $\mathcal{A}$ may come from $E_1 \cup E_2$,[2] it follows that more than mk of the elements must be

[2]Consider the k pigeonholes $\{1, k+1\}, \{2, k+2\}, \dots, \{r, k+r\}, \{r+1\}, \dots \{k\}$ of $E_1 \cup E_2$. If we choose more than k elements, at least one of these sets will contain two, and therefore our selection will contain elements that differ by k.

chosen from $\bigcup_{i=1}^{m} S_i$. By the pigeonhole principle, this means that at least one of the S_i must have more than k elements chosen from it, and therefore $\mathcal{A}$ contains elements which differ by exactly k, giving us our contradiction.

p **even:**

The case for p even is similar: we write $n = 2mk + r$, and split $[n]$ into the same sets of $2k$ elements. This time, however, the extra stuff is just a single set of size r:

$$E := \{2mk + 1, \ldots, 2mk + r\} \tag{3}$$

Taking the first k elements from each S_i and then the remaining elements from E gives us a set of size $mk + r$. The proof that this is maximum follows the same logic as the p odd case.

Summary

Write $n = pk + r$. If p is odd, then the maximum size is $\frac{p+1}{2}$. If p is even, then the maximum size is $\frac{p}{2} + r$.

(4) Given five types of coins (5 cent, 10 cent, 20 cent, 50 cent, 1 dollar), give a formula for the number of possible collections of n coins which contain no more than four coins of any one type.

First note that when $n > 20$, then the pigeonhole principle says that the answer is zero, and if $n = 20$ then the answer is 1. Now consider the number of ways to select n coins from 5 types. By the balls and walls theorem the number of such selections is $\binom{n+4}{4}$. Let these selections be our universe, U, and denote by A_i the set of selections where coin type i is picked more than 4 times. The number of collections of n coins with no more than four of each type is then given by $|U - (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)|$. We can guarantee to be in a set A_i by first selecting 5 coins of type i, and then choosing the rest as we see fit. Thus we have

$$|A_i| = \binom{(n-5)+5-1}{5-1}$$

Similarly, we can guarantee we are in the set A_i and A_j by choosing 5 coins from A_i and then 5 coins from A_j. So

$$|A_i \cap A_j| = \binom{(n-10)+5-1}{5-1}$$

and so on. Note already that if $n < 10$ then this formula is nonsense. There is no way to select 9 coins which has at least 5 coins of two different types. The inclusion/exclusion principle then tells us that the number we are looking for is

$$\sum_{k=0}^{\lfloor \frac{n}{5} \rfloor} (-1)^k \binom{5}{k} \binom{n-5k+4}{4},$$

where the sum ranges over the integers which make combinatorial sense. We could also simply have our sum going from $k = 0$ to 5, provided we define $\binom{n}{k}$ appropriately for $n < k$.

(5) Consider a set of $2n$ insects, n male and n female. In each situation below, derive formulas for the number of of ways to partition them into pairs so that the i^{th} largest male is not paired with the i^{th} largest female. (Leave answers as summations)

(a) Same sex pairs are allowed.

(b) Each pair has one insect of each sex.

Recall from a previous assignment that the total number of ways to group $2n$ objects into n distinct pairs is

$$\frac{(2n)!}{2^n \cdot n!}.$$

This is the size of our universe, and we let A_i denote the set of pairings where the ith largest male is paired with the ith largest female. For k intersections of the A_i, we have already determined k pairs of insects that have been paired, so the size of the set is the number of ways of determining the remaining $(n-k)$ pairs. This is

$$\frac{(2n-2k)!}{2^{n-k} \cdot (n-k)!}.$$

The answer then follows from the inclusion/exclusion principle:

$$|U - A_1 \cup \cdots \cup A_n| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2n-2k)!}{2^{n-k} \cdot (n-k)!}$$

For part (b), the number of ways of choosing pairs in the universe is now $n!$, and for k-fold intersections there are $(n-k)!$ choices. The number of ways to partition them into pairs is therefore

$$|U - A_1 \cup \cdots \cup A_n| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

References