Tutorial 4

STAT 3013/4027/8027

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1. Answer the following questions from SI: 2.13, 2.14.
2. Let $U \sim \text{uniform}(0,1)$.
   a. Show that both $-\log(U)$ and $-\log(1-U)$ are exponential random variables.
   b. Show that $\log\left(\frac{u}{1-u}\right)$ is a logistic(0,1) random variable.
   c. Show how to generate samples from a logistic(μ, β).

2.13 Find minimal sufficient stats for samples of size n from

a). unif dist. on $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$

b). unif dist. on $[-\theta, \theta]$

2.14. Obs. made on rvs $X_1, \dots, X_n$ are i.i.d each with a beta dist. whose p.d.f. is

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1.$$

α, β are pos. parameters & $B(\alpha,\beta)$ is a beta function. Write down minimal sufficient stats for (α,β).

2.13) a) $X_1, \dots, X_n \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$f(x) = \frac{1}{\theta + \frac{1}{2} - \theta + \frac{1}{2}} = 1, \quad \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2}$$

$$= 1 \cdot I_{[\theta - \frac{1}{2} \le x \le \theta + \frac{1}{2}]}$$

$$L(\theta | \vec{X}) = \prod_{i=1}^{n} 1 \cdot I_{[\theta-\frac{1}{2} \le x \le \theta+\frac{1}{2}]} = 1^n \cdot I_{[\theta-\frac{1}{2} \le x_1, \dots, x_n \le \theta + \frac{1}{2}]} = 1^n \cdot I_{[\theta - \frac{1}{2} \le x_{(1)} \le x_{(2)} \le \dots \le x_{(n)} \le \theta + \frac{1}{2}]}$$

$$\Downarrow$$

$$= 1^n I_{[x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}]}$$

$\therefore \theta - \frac{1}{2} \le x_{(1)}$ & $\theta + \frac{1}{2} \ge x_{(n)}$

$\Rightarrow \theta \le x_{(1)} + \frac{1}{2}$ & $\theta \ge x_{(n)} - \frac{1}{2} \Rightarrow x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$

$$R(\theta) = \frac{L(\theta|\vec{x})}{L(\theta|\vec{y})}$$

$$= \frac{1^n \cdot I_{[x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}]}}{1^n \cdot I_{[y_{(n)} - \frac{1}{2} \le \theta \le y_{(1)} + \frac{1}{2}]}}$$

① Sps. $y_{(n)} - \frac{1}{2} < \theta < x_{(n)} + \frac{1}{2}$ then $R(\theta) = \frac{0}{1} = 0$.

② When $x_{(n)} = y_{(n)}$ & $x_{(1)} = y_{(1)}$ we have $R(\theta) = 1$.

b). $X_1, \cdots, X_n \sim U(-\theta, \theta)$

$$f(x) = \frac{1}{\theta + \theta} = \frac{1}{2\theta}, \quad -\theta \le x \le \theta$$

$$= \frac{1}{2\theta} I_{[-\theta \le x \le \theta]}$$

$$L(\theta|\vec{x}) = \prod_{i=1}^{n} \frac{1}{2\theta} I_{[-\theta \le x \le \theta]}$$

$$= \left(\frac{1}{2\theta}\right)^n I_{\{-\theta \le x_1, \cdots, x_n \le \theta]}$$

$$= \left(\frac{1}{2\theta}\right)^n I_{\{-\theta \le x_{(1)} \le \cdots \le x_{(n)} \le \theta]} \qquad (*)$$

$x_{(1)} \ge -\theta$ & $x_{(n)} \le \theta$

$\Rightarrow \theta \ge -x_{(1)}$ & $\theta \ge x_{(n)}$

$\Rightarrow \theta \ge \max\{-x_{(1)}, x_{(n)}\}$

$\Rightarrow \theta \ge \max\{|x_{(1)}|, |x_{(n)}|\} \Rightarrow \theta \ge \max\{|x_i|\}$

$$(*) = \left(\frac{1}{2\theta}\right)^n I_{\{\theta \ge \max\{|x_i|\}]}$$

$$R(\theta) = \frac{L(\theta|\vec{x})}{L(\theta|\vec{y})} = \frac{\left(\frac{1}{2\theta}\right)^n I_{[\theta \ge \max\{|x_i|\}]}}{\left(\frac{1}{2\theta}\right)^n I_{[\theta \ge \max\{|y_i|\}]}}$$

setting $\max\{|x_i|\} = \max\{|y_i|\}$

$R(\theta) = 1 \Rightarrow \max\{|x_i|\}$ is minimal sufficient.

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2.a). $U \sim Unif(0,1)$
$Y = -\log(u) \sim Exp(1)$

$$F_Y(y) = P[Y \le y] \le P[-\log u \le y]$$
$$= P[\log u \ge -y]$$
$$= P[u \ge e^{-y}]$$
$$= 1 - P[u < e^{-y}] \quad \int_0^{e^{-y}} 1\, dy$$
$$= 1 - e^{-y}$$

$$f_Y(y) = F_Y'(y) = e^{-y} ; \; y > 0$$
$$Y \sim Exp(1)$$

$$-\log(1-u) \sim Exp(1)$$

let $v = 1-u$, $\Rightarrow u = 1-v$
$$|J| = \left|\frac{du}{dv}\right| = |-1|$$

$$\therefore f_V(v) = 1 \cdot |J|$$
$$= 1 \cdot |-1| = 1$$

$$\therefore -\log(1-u) \sim Exp(1)$$

2.14 $X_1, \dots, X_n \sim Beta(\alpha, \beta)$

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} ; 0 < x < 1$$

$$L(\vec{\theta}|\vec{x}) = \prod_{i=1}^{n} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} ; \; 0 < x < 1$$

$$= \left(\frac{1}{B(\alpha,\beta)}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x_i)^{\beta-1}$$

$$R(\theta) = \frac{L(\vec{\theta}|\vec{x})}{L(\vec{\theta}|\vec{y})} = \frac{\left(\frac{1}{B(\alpha,\beta)}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x_i)^{\beta-1}}{\left(\frac{1}{B(\alpha,\beta)}\right)^n \prod_{i=1}^{n} y_i^{\alpha-1} \prod_{i=1}^{n} (1-y_i)^{\beta-1}}$$

setting $\prod(\cdot) = \prod(\cdot)$ & $\prod(\;) = \prod(\;)$
$\therefore R(\theta) = 1$