PROBLEMS FOR THE LECTURE 4

Problem 1.
Given an acute triangle ABC, prove that the perimeter of the inscribed triangle $A'B'C'$ connecting the intersections of the 3 altitudes of ABC with opposite sides is less than twice any altitude.

Problem 2.
Let ABC be a regular triangle $(\angle\hat{A} = \angle\hat{B} = \angle\hat{C} = 60^\circ)$. Prove that for every point P inside this triangle the sum of distances from P to sides AB, BC and CA is the same.

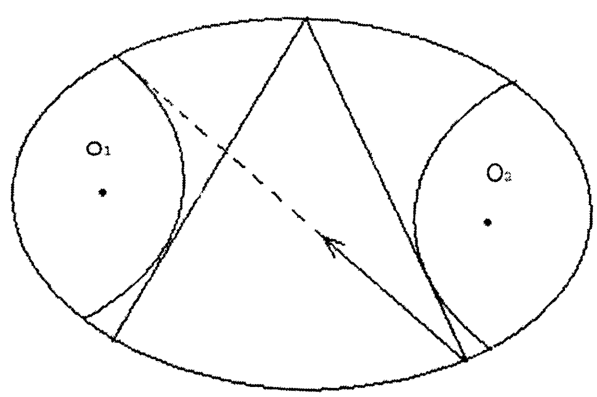
Problem 3.
Let $f_A(x, y)$ be a distance between (x, y) and $A = (a, b)$,

$$f(x, y) = \sqrt{(x-a)^2 + (y-b)^2}\,.$$

Prove that grad f is a vector which belong to a line joining (x, y) and A, which is "looking away from point A" and which has length equal to 1, $\|\text{grad}\, f\| = 1$.

Problem 4.
Assume that x, y, z are vectors on plane such that $x + y + z = 0$ and $\|x\| = \|y\| = \|z\| = 1$. Prove that angle between any two such vectors is equal to 120°.

Problem 5.
Prove that an ellipse has a following optical property. A ray of light which intersects a segment joining focuses O_1, O_2 forever after reflections will be tangent to a same hyperbola with focuses O_1, O_2 (see picture 1).



Picture 1.

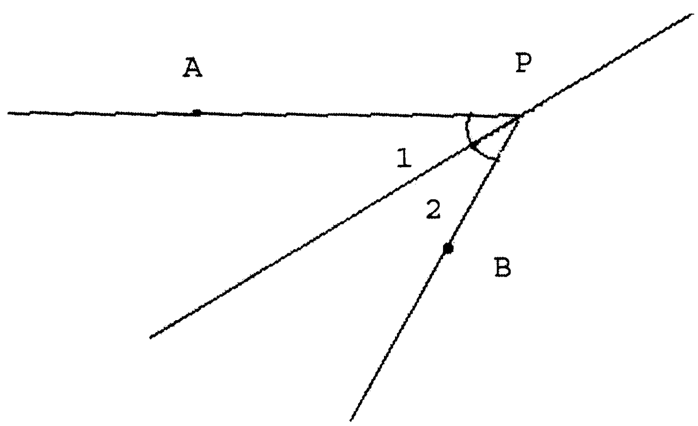
Problem 6.

(1) Let l be a line and let A and B be two points which lay at different sides of l. Assume that $P \in l$ and $|PA - PB|$ has maximal possible value. Prove that $\angle 1 = \angle 2$ (see picture 2).

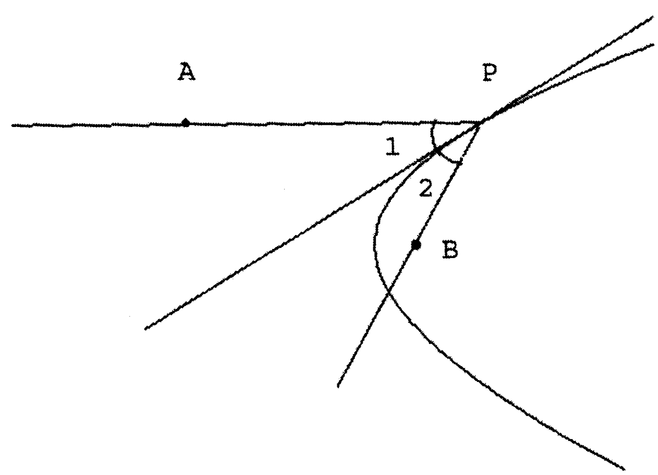
(2) Using (1) explain why for a tangent line to a branch of hyperbola given by relation

$$(PA) - (PB) = \text{constant}$$

the following holds: $\angle 1 = \angle 2$ (see picture 3).



Picture 2.



Picture 3.