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$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 2 & \text{if } x \notin \mathbb{Q} \end{cases}$

$x_0 \notin \mathbb{Q}$ $\lim_{x \to x_0} (x_0) = 2$

MAT 337
Midterm Exam 2
March 12, 2014

NAME

NO AIDS ALLOWED

Total: 250 points, not including a bonus problem

Problem 1 [30 points]

(a) Give an example of a function $f : [0, 1] \longrightarrow \mathbb{R}$ that is continuous at the irrational numbers of $[0, 1]$, but is discontinuous at the rational numbers. (10)

(b) Show that $f(x, y) = \max\{x, y\}$ is continuous on $\mathbb{R}^2$. (20)

Solution: (a). ~~$f(x) = \{$ $f(x) = x$~~

0

~~$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ with } x = \frac{p}{q}, \gcd(p,q)=1, p,q \in \mathbb{N}^+ \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$~~

~~since $\lim f(x)$~~

~~$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 2 & \text{if} \end{cases}$~~

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(b). Want to show $\forall \varepsilon > 0$, $\forall$ point $(a,b) \in \mathbb{R}^2$ $\exists r > 0$ s.t. $|f(x,y) - f(a,b)| < \varepsilon$ whenever $|(x,y) - (a,b)| < r$

$r > |(x,y) - (a,b)| = |(x-a)^2 + (y-b)^2| \geq |x-a|$

$r > |(x,y) - (a,b)| = |(x-a)^2 + (y-b)^2| \geq |y-b|$

$|f(x,y) - f(a,b)| = |\max\{x,y\} - \max\{a,b\}|$

① if $x \geq y$, $a \geq b$ or $x \leq y$, $a \leq b$. WLOG, say $x \geq y$, $a \geq b$.

Then $|f(x,y) - f(a,b)| = |x - a| < \varepsilon$

Then simply we take $r = \varepsilon$. we can have $f(x,y)$ continuous

② if $x \geq y$, $a < b$ or $x \leq y$, $a > b$ WLOG say $x \geq y$, $a < b$

Then $|f(x,y) - f(a,b)| = |x - b| < \varepsilon$

(Then (somehow we can choose $r = ?\varepsilon$ s.t. f is continuous for this case as well).

Problem 2 [45 points] (15×3)

0 (a) Is it true that any uniformly continuous function is Lipschitz?

13 (b) Show that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (2x + 3y, -3x + 4y)$ is uniformly continuous.

13 (c) Show that $\sin x$ is Lipschitz on $\mathbb{R}$.

Solution:

(a). No. Counter-example:

~~f(x)~~ $\forall \varepsilon>0, \exists r>0$ s.t. $|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<r$ (uniformly continuous)

this is not an example

But not Lipschitz, so $\frac{|f(x)-f(a)|}{|x-a|}$ is not bounded up.

so ε is much larger than r, or to say, ε increases faster than r.

this is poorly stated

(b). Since the cordinates of $f(x,y)$ is linear,

so the coordinates of f are uniformly continuous on $\mathbb{R}^2$.

i.e. ~~2x+3y~~ $g(x,y)=2x+3y$

$h(x,y)=-3x+4y$

are uniformly continuous on $\mathbb{R}^2$.

Hence $f(x,y)=(g(x,y), h(x,y))$ is uniformly continuous on $\mathbb{R}^2$.

(c). Let $f(x)=\sin x$, $f'(x)=\cos x$, $|\cos x| \in [0,1]$ $\forall x,y$ $\exists c$

$$\frac{|f(x)-f(y)|}{|x-y|} \leq |f'(c)| = 1 \quad \text{where } c \in [x,y].$$

Hence $|f(x)-f(y)| \leq 1 \cdot |x-y|$ $\forall x,y \in \mathbb{R}$.

Therefore $\sin x$ is Lipschitz on $\mathbb{R}$.

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Problem 3 **[30 points]** Let $f : (X, \rho) \longrightarrow (Y, \sigma)$ be a continuous map from a metric space (X, ρ) to a metric space (Y, σ). Let C be a compact subset of X. Show that $f(C)$ is compact.

to prove $f(C)$ compact need to start with arbitrary sequence in $f(C)$.

Proof: f is continuous from X to Y. $C \subseteq X$

C is compact, show $f(C)$ is compact.

Since compact is equivalent to "sequentially compact" (Borel-Lebesgue)

Then C has a sequence (x_n) such that it has a subsequence (x_{n_k}) converges to x and $x \in C$.

By theorem, since f is continuous,

if $\lim_{k\to\infty} x_{n_k} = a$ then $\lim_{k\to\infty} f(x_{n_k}) = f(a)$

Hence for the image of (x_{n_k}),

we have a sequence $f(x_{n_k})$ converges to $f(x)$.

and $f(x_{n_k}) \in f(C) \subseteq Y$ (So far, we have seq. $f(x_n)$ whose subsequence $f(x_{n_k})$ converges)

~~And we need to show $f(x) \in f(C)$~~

By the definition of sequential compactness,

$f(C)$ is sequentially compact,

therefore $f(C)$ is compact.

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Problem 4 [40 points] If f is a continuous one-to-one function of a compact metric space X onto Y, show that f^{-1} is continuous.

Y is a metric space.

Proof: X is compact ~~from X~~ $f: X \to Y$. continuous.

Then X is complete and totally bounded.

So $\forall \varepsilon > 0$, $\exists$ finitely many points $x_1, \cdots, x_n$ such that

$\{B_\varepsilon(x_i) : 1 \le i \le n\}$ is an open cover of X.

Since f is one-to-one, so there are exactly n many points $f(x_1), \cdots, f(x_n)$ in Y.

Y has infinitely many points in it

So f maps an open cover in X to exactly an open cover in Y.

Problem 5 [**45 points**] Let $S^1 = \{(x,y) \in \mathbb{R}^2, x^2+y^2=1\}$. Let $f : S^1 \longrightarrow \mathbb{R}$ be a continuous function. Show that f cannot be one-to-one.

Solution: W.t.s.

$\forall \varepsilon > 0, \exists r > 0$. s.t.

$|f(x,y) - f(a,b)| < \varepsilon$ whenever $|(x,y) - (a,b)| < r$.

$\cancel{|f(x,y) - f(a,b)| = x^2+y^2-a^2-b^2|}$

Let $f = x+y$

so $|f(x,y) - f(a,b)| = |(x+y)-(a+b)| < \varepsilon$ ①

$|(x-a)^2 + (y-b)^2| < r$ ②

① $\Rightarrow$ $|(x-a)+(y-b)| < \varepsilon$

$|(x-a)^2 + (y-b)^2 - 2(x-a)(y-b)| < \varepsilon^2$ ③

$\cancel{\varepsilon^2 - 2(x-a)(y-b)} = |(x-a)^2 + (y-b)^2| < \varepsilon^2 + 2(x-a)(y-b)$

so we know $\varepsilon^2 + 2(x-a)(y-b)$

reaches ~~(maximum when $x-a=2, y-b=2$~~ ~~or $x-a=-2, y-b=-2$)~~

~~So $\varepsilon^2 + 2\times2\times2 = \varepsilon^2+8$ is a maximum~~

minimum when $x=a$ or $y=b$

i.e. $\min(③) = \varepsilon^2$

so we when we know ② , we want to infer ③

so $\cancel{\ldots}$ $r \leq \varepsilon^2$.

just take $r = \varepsilon^2$. then f is continuous.

Show f is not one-to-one, since no matter what function

~~First we know to function $g(x,y) = x^2+y^2-1$.~~

~~This is not 1-1 since for $x^2+y^2=1$~~

f is, we project the kernel S^1 onto $\mathbb{R}$, multiple points are mapped to a single point.

Take $f(x,y) = x+y$ as an example, $(-1,0)$ and $(0,-1)$ are both in S^1, but they are mapped to same value -1 on $\mathbb{R}$ by f. Hence f is not one-to-one.

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Problem 6 [30 points] Let f be continuous on the closed inverval $I = [a, b]$. Suppose that for each $x \in I$, there exists a point $y \in I$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove the existence of $q \in I$ for which $f(q) = 0$.

Proof: Since $I = [a,b]$ is closed and bounded,

~~By theorem~~ EVT $\exists\, c, d \in [a,b]$ s.t. $\forall x \in [a,b]$

$f(c) \leq f(x) \leq f(d)$.

i.e. there must exists extremum. Take each absolute value of result,

~~Let $f(c) = f(x)$, i.e. $c = x$.~~

~~So $|f(y)| \leq \frac{1}{2}|f(x)| = \frac{1}{2}|f(c)|$~~

~~Since $f(y) \leq f(c)$~~

~~So $|f(c)|$~~

suppose we have $|f(a)|$ which is the smallest one.

Then $\exists y$ $|f(y)| \leq \frac{1}{2}|f(a)|$

for such an y, $|f(y)| \geq |f(a)|$

So $|f(y)| = |f(a)|$

Hence $|f(a)| = 0$

and $f(a)$ is such a $f(q) = 0$

we are looking for.

Problem 7 [30 points]

Let $f : [0, 1] \longrightarrow (X, \rho)$ be a continuous map, where (X, ρ) is a metric space. Let $U = \{U_\lambda, \lambda \in \Lambda\}$ be an open cover of (X, ρ). Prove that there exists a subdivision of $[0, 1]$, $s_0, ..., s_n$, where $0 = s_0 < s_1 < ... < s_{n-1} < s_n = 1$, such that for each $i \in \{1, ..., n\}$ the set $f([s_{i-1}, s_i]) \subset U_\lambda$ for some $\lambda \in \Lambda$.

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Solution:

Bonus Problem [50 points] Let I be a closed interval and let $0 < \alpha < 1$. Let $f : I \longrightarrow I$ satisfy the inequality $|f(x) - f(y)| \leq \alpha|x - y|$ for each $x \in I$ and $y \in I$. Let $x_1 \in I$ and define $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \ldots$. Prove that the sequence $(x_n)_{n=1}^{\infty}$ converges and that its limit satisfies $l = f(l)$.

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- seems that we are going to use the relation between uniformly continuous and Lipschitz function here.

Since $|f(x) - f(y)| \leq \alpha|x-y|$ $\forall x \in I$ & $y \in I$.