Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 6

- Due Tues. March 26, 6:10pm sharp. Late assignments will not be accepted.
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Suppose V is an inner product space, and T: $V \mapsto V$ a linear operator. We already know the dimension theorem:

$$dim(V) = dim(im(T)) + dim(ker(T)).$$

When V is an inner product space, more can be said. The following question describes the relationship among the four subspaces defined by a linear operator T: $ker(T)$, $im(T)$, $ker(T^*)$, $im(T^*)$ - we'll call these four subspaces the *four fundamental subspaces* of T.

Show

(a) $ker(T) = im(T^*)^\perp$, and $ker(T)^\perp = im(T^*)$.

(b) $im(T) = ker(T^*)^\perp$, and $im(T)^\perp = ker(T^*)$.

Notice that all you need to show is $ker(T) = im(T^*)^\perp$, since then $ker(T)^\perp = (im(T^*)^\perp)^\perp = im(T^*)$. Also, part (b) follows by applying (a) to T^* in place of T.

Solution. Suppose that $v \in V$. Note that

$$v \in ker(T) \Leftrightarrow T(v) = 0$$

$$\Leftrightarrow < T(v), w >= 0 \ \forall w \in V$$

$$\Leftrightarrow < v, T^*(w) >= 0 \ \forall w \in V$$

$$\Leftrightarrow < v, u >= 0 \ \forall u \in im(T^*)$$

$$\Leftrightarrow v \in im(T^*)^\perp.$$

It follows that $ker(T) = im(T^*)^\perp$.

Furthermore, note that $ker(T)^\perp = (im(T^*)^\perp)^\perp = im(T^*)$. As was suggested in the question, (b) then follows from replacing each instance of T with T^* (noting that $(T^*)^* = T$). Explicitly, $ker(T^*) = im((T^*)^*)^\perp = im(T)^\perp$ and $ker(T^*)^\perp = im((T^*)^*) = im(T)$.

2. Consider $P_2(\mathbb{R})$ together with inner product $< p(x), q(x) >= \int_0^1 p(x)q(x)\,dx$. Let T: $P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = p'(x)$. Find a basis for each of the *four fundamental subspaces* of T

Solution. Suppose that $p(x) = a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R})$. Note that $T(p(x)) = a_1 + 2a_2x$, so that $p(x) \in ker(T)$ if and only if $a_1 + 2a_2x$ is the zero polynomial (ie. $a_1 = a_2 = 0$). Therefore, $ker(T) = \{a_0 : a_0 \in \mathbb{R}\}$, of which $\{1\}$ is a basis.

Secondly, note that $1 = T(x)$ and $x = T(\frac{1}{2}x^2)$. We conclude that $1, x \in im(T)$. Since 1 and x are linearly independent, and since $dim(im(T)) = dim(P_2(\mathbb{R})) - dim(ker(T)) = 3 - 1 = 2$, we conclude that $\{1, x\}$ is a basis of $im(T)$.

By virtue of Question 1, $ker(T^*) = im(T)^\perp$. Furthermore, $p(x) \in im(T)^\perp$ if and only if $p(x)$ is orthogonal to each element of a basis of $im(T)$. Hence, $ker(T^*) = im(T)^\perp = \{p(x) \in P_2(\mathbb{R}) :< 1, p(x) >= 0 =< x, p(x) >\}$. If $p(x) = a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R})$, then

$$< 1, p(x) >= \int_0^1 (a_0 + a_1x + a_2x^2)\,dx = a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2,$$

and

$$< x, p(x) >= \int_0^1 (a_0x + a_1x^2 + a_2x^3)\,dx = \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2.$$

Accordingly, we solve the system

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 0$$
$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = 0$$

for $(a_0, a_1, a_2) \in \mathbb{R}^3$. The solutions space is $\{(a_0, -6a_0, 6a_0) : a_0 \in \mathbb{R}\}$. Therefore, $p(x) \in im(T)^\perp$ if and only if it is of the form $a_0 - 6a_0x + 6a_0x^2$, $a_0 \in \mathbb{R}$. Hence, $\{1 - 6x + 6x^2\}$ is a basis of $im(T)^\perp = ker(T^*)$.

We again invoke Question 1 in noting that $im(T^*) = ker(T)^\perp$. Recalling that $ker(T)$ has basis $\{1\}$, we conclude that for $p(x) = a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R})$, $p(x) \in ker(T)^\perp$ if and only if $< 1, p(x) >= 0$, ie. $a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 0$. The associated solution space is $\{(a_0, a_1, -3a_0 - \frac{3}{2}a_1) : a_0, a_1 \in \mathbb{R}\}$. From this, it follows that $p(x) \in ker(T)^\perp$ if and only if it is of the form $a_0 + a_1x + (-3a_0 - \frac{3}{2}a_1)x^2$, $a_0, a_1 \in \mathbb{R}$. We conclude that $\{1 - 3x^2, x - \frac{3}{2}x^2\}$ is a basis of $ker(T)^\perp = im(T^*)$.

3. Suppose V is an inner product space, and T: $V \mapsto V$ a linear operator. Show that $dim(im(T)) = dim(im(T^*))$.

Hint: Use question 1 together with that fact that for any subspace W of V, $V = W \oplus W^\perp$.

Solution Note that $V = im(T) \oplus im(T)^\perp$, meaning that $dim(V) = dim(im(T)) + dim(im(T)^\perp)$. Hence,

$$dim(im(T)) = dim(V) - dim(im(T)^\perp).$$

By virtue of Question 1, $im(T)^\perp = ker(T^*)$. Therefore,

$$dim(im(T)) = dim(V) - dim(ker(T)^*).$$

By applying the Rank-Nullity Theorem to the linear transformation $T^* : V \to V$, we see that $dim(V) - dim(ker(T)^*) = dim(im(T)^*)$. Hence,

$$dim(im(T)) = dim(V) - dim(ker(T)^*) = dim(im(T)^*).$$

4. Let $N\colon \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$N = \begin{bmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Show that N is nilpotent and find the smallest k such that $N^k = 0$.

Solution. By direct computation, we find that $N^2 = 0$. Hence, N is a nilpotent matrix. Since $N^1 = N \neq 0$, 2 is the smallest positive integer k for which $N^k = 0$.

(b) Find the canonical form of N and a canonical basis.

Solution. Let us first find a collection of non-overlapping cycles. Since 2 is the smallest positive integer k for which $N^k = 0$, we begin by finding $im(N^1) = im(N)$. For this, note that each column of N is a scalar multiple of the first column, $e_1 + e_2 - e_3$. Since the image of N is the span of the columns of N, we conclude that $im(N) = span\{e_1 + e_2 - e_3\}$. Accordingly, we will construct a cycle containing $e_1 + e_2 - e_3$. To do this, we must find a solution $x \in \mathbb{R}^4$ to $e_1 + e_2 - e_3 = Nx$. One solution is $x = e_1$, so that $\alpha_1 := \{e_1 + e_2 - e_3, e_1\}$ is a cycle of length 2. Furthermore, $N(2e_1 + e_2) = 0$ and $N(4e_1 + e_4) = 0$, so that $\alpha_2 := \{2e_1 + e_2\}$ and $\alpha_3 := \{4e_1 + e_4\}$ are cycles of length 1. Since the set $\{e_1 + e_2 - e_3, 2e_1 + e_2, 4e_1 + e_4\}$ is linearly independent, we conclude that the cycles α_1, α_2, and α_3 are non-overlapping. Hence, $\{e_1 + e_2 - e_3, e_1, 2e_1 + e_2, 4e_1 + e_4\}$ is a canonical basis of $\mathbb{R}^4$ for N. Since α_1, α_2, and α_3 have respective lengths 2, 1, and 1, the canonical form of N consists of a 2×2 nilpotent Jordan block and two 1×1 nilpotent Jordan blocks (arranged along the diagonal in this order). Therefore, the canonical form of N is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Suppose A is a real 4×4 matrix that is nilpotent of order (index) k, $1 \leq k \leq 4$. Make a list of all the possible dimensions of $null(A)$, $null(A^2), \ldots, null(A^{k-1})$ (note that $dim(null(A^k)) = 4$) and the corresponding canonical forms of A.

Solution. Before proceeding, we make a few general observations. Suppose that F is a field. An $n \times n$ nilpotent matrix N over F is in canonical form if and only if N is constructed by placing nilpotent Jordan blocks along the diagonal, such that if one moves from left-to-right along the matrix and records the sequence of widths of the Jordan blocks, the sequence is non-increasing. With this understanding, we note that the nilpotent matrices in canonical form correspond to the sequences $k_1 \geq k_2 \geq \ldots \geq k_m$ of positive integers satisfying $k_1 + \ldots + k_m = n$ (m is the number of Jordan blocks appearing, k_1 is the width of the first Jordan block, k_2 is the width of the second Jordan block, etc.) So, to find all canonical forms is to find all such sequences.

Let us specialize our findings in the context of the given problem. Here, $n = 4$, and the possible values of m are 1, 2, 3, and 4. For $m = 1$, the only sequence is 4. This corresponds to the 4×4 nilpotent Jordan block, namely

$$N_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that N_1 is nilpotent of order 4. Furthermore, by direct calculation, we find that $dim(null(N_1)) = 1$, $dim(null(N_1^2)) = 2$, and $dim(null(N_1^3)) = 3$.

For $m = 2$, the sequences are $2 \geq 2$ and $3 \geq 1$. The respective associated canonical forms are

$$N_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$N_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that N_2 is nilpotent of order 2, while N_3 is nilpotent of order 3. Also, $dim(null(N_2)) = 2$, while $dim(null(N_3)) = 2$ and $dim(null(N_3^2)) = 3$.

For $m = 3$, the only sequence is $2 \geq 1 \geq 1$. The associated canonical form is

$$N_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that N_4 is nilpotent of order 2. Furthermore, $dim(null(N_4)) = 3$.

For $m = 4$, the only sequence is $1 \geq 1 \geq 1 \geq 1$. The associated canonical form is the zero matrix. The zero matrix is nilpotent of order 1.

6. Let $T\colon P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear operator defined by $T(p(x)) = p''(x) + p(x)$. Find a basis α for $P_3(\mathbb{R})$ such that $[T]_{\alpha\alpha}$ is in canonical form and determine $[T]_{\alpha\alpha}$.

Solution. Note that $T - I_{P_3(\mathbb{R})} : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ is given by $(T - I_{P_3(\mathbb{R})})(p(x)) = p''(x)$. Since taking a second derivative twice of a polynomial in $P_3(\mathbb{R})$ yields the zero polynomial, it follows that $(T - I_{P_3(\mathbb{R})})^2 = 0$. Now, note that $(T - I_{P_3(\mathbb{R})})(x^2) = 2$ and $(T - I_{P_3(\mathbb{R})})(x^3) = 6x$. Since 2 and $6x$ are linearly independent, it follows that $\alpha_1 := \{2, x^2\}$ and $\alpha_2 := \{6x, x^3\}$ are non-overlapping cycles. Noting that $dim(P_3(\mathbb{R})) = 4$, it follows that $\alpha := \{2, x^2, 6x, x^3\}$ is a canonical basis of $P_3(\mathbb{R})$ for the nilpotent transformation $T - I_{P_3(\mathbb{R})}$. Since this basis was constructed by taking a union of two non-overlapping cycles of length two, the canonical form of $T - I_{P_3(\mathbb{R})}$ consists of two 2×2 nilpotent Jordan blocks along the diagonal. In other words,

$$[T - I_{P_3(\mathbb{R})}]_\alpha^\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $[T - I_{P_3(\mathbb{R})}]_\alpha^\alpha = [T]_\alpha^\alpha - [I_{P_3(\mathbb{R})}]_\alpha^\alpha = [T]_\alpha^\alpha - I_4$, it follows that

$$[T]_\alpha^\alpha = I_4 + [T - I_{P_3(\mathbb{R})}]_\alpha^\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution. There is a more computational alternative to the solution given above. Let $\beta = \{1, x, x^2, x^3\}$ be the standard basis of $P_3(\mathbb{R})$. We find that

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Our task is then to find a canonical basis γ of $\mathbb{R}^4$ for $[T]_\beta^\beta$. To this end, we observe that

$$N := [T]_\beta^\beta - I_4 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

a nilpotent matrix. Secondly, $N^2 = 0$, so that 2 is the smallest k for which $N^k = 0$. Hence, we note that $im(N) = span\{2e_1, 6e_2\}$. Note that $2e_1 = Ne_3$ and $6e_2 = Ne_4$, meaning that $\gamma_1 := \{2e_1, e_3\}$ and $\gamma_2 := \{6e_2, e_4\}$ are two cycles of length two. Since $\{2e_1, 6e_2\}$ is a linearly independent set, these cycles are non-overlapping. Therefore, $\gamma := \{2e_1, e_3, 6e_2, e_4\}$ is a canonical basis of $\mathbb{R}^4$ for N. Hence, γ is a canonical basis of $\mathbb{R}^4$ for $[T]_\beta^\beta$. Now, think of $[T]_\beta^\beta$ as a linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^4$, noting that $[S]_\omega^\omega = [T]_\beta^\beta$, where ω is the standard basis of $\mathbb{R}^4$. We have shown that $[S]_\gamma^\gamma$ is in canonical form. Also,

$$\begin{aligned} [S]_\gamma^\gamma &= ([I_{\mathbb{R}^4}]_\gamma^\omega)^{-1}[S]_\omega^\omega[I_{\mathbb{R}^4}]_\gamma^\omega \\ &= ([I_{\mathbb{R}^4}]_\gamma^\omega)^{-1}[T]_\beta^\beta[I_{\mathbb{R}^4}]_\gamma^\omega, \end{aligned}$$

so that $([I_{\mathbb{R}^4}]_\gamma^\omega)^{-1}[T]_\beta^\beta[I_{\mathbb{R}^4}]_\gamma^\omega$ is in canonical form. So, if α is a basis of $P_3(\mathbb{C})$ satisfying $[I_{P_3(\mathbb{R})}]_\alpha^\beta = [I_{\mathbb{R}^4}]_\gamma^\omega$, then

$$\begin{aligned} [T]_\alpha^\alpha &= ([I_{P_3(\mathbb{R})}]_\alpha^\beta)^{-1}[T]_\beta^\beta[I_{P_3(\mathbb{R})}]_\alpha^\beta \\ &= ([I_{\mathbb{R}^4}]_\gamma^\omega)^{-1}[T]_\beta^\beta[I_{\mathbb{R}^4}]_\gamma^\omega, \end{aligned}$$

a matrix in canonical form (so that α is a canonical basis). Note that $\alpha := \{2, x^2, 6x, x^3\}$ satisfies this property, since

$$[I_{\mathbb{R}^4}]_\gamma^\omega = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As with the earlier solution, we then find that

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Suggested Extra Problems (not to be handed in):

- Textbook, Section 6.2 **1-5, 7, 12, 13**
- Let N: $\mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$N = \begin{bmatrix} 6 & 2 & 1 & -1 \\ -7 & -1 & -1 & 2 \\ -9 & -7 & -2 & -1 \\ 13 & 3 & 2 & -3 \end{bmatrix}$$

(a) Show that N is nilpotent and find the order (index) of N (i.e. the smallest k such that $N^k = 0$).

(b) Find the canonical form of N and a canonical basis.