$\underline{Ex}$. $m_X(t) = \frac{1}{6}e^{t} + \frac{2}{6}e^{4t} + \frac{3}{6}e^{9t}$

Find $E(X)$, $E(\sqrt{X})$, $Var(X)$.

$$m_X(t) = E(e^{tX}) = \sum_x e^{tx} p_x$$

$$\Rightarrow \quad X: \; 1 \quad 4 \quad 9$$

$$p_x: \; 1/6 \quad 2/6 \quad 3/6$$

$$E(X) = 1 \cdot 1/6 + 4 \cdot 2/6 + 9 \cdot 3/6 = \frac{1+8+27}{6}$$

$$= 36/6 = 6$$

$$E(\sqrt{X}) = \sqrt{1} \cdot 1/6 + \sqrt{4} \cdot 2/6 + \sqrt{9} \cdot 3/6 = \frac{14}{6} = \frac{7}{3}$$

$$E(X^2) = 1^2 \cdot 1/6 + 4^2 \cdot \frac{2}{6} + 9^2 \cdot \frac{3}{6} = \frac{1+32+243}{6} = \frac{276}{6} = 46$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= 46 - 36 = 10$$

$\underline{Ex}$. $m_X(t) = \frac{1}{2}(1 + e^{t})$

Find $Var(X)$.

$$E(X) = m'_X(0) = \frac{1}{2}e^{t}\Big|_{t=0} = \frac{1}{2}$$

$$E(X^2) = m''_X(0) = \frac{1}{2}$$

$$Var(X) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$\underline{Ex}$. $X \sim Gamma(\alpha, \lambda)$. Find $m_X(t)$.

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}, & x > 0 \\ 0, & ow \end{cases}$$

$$m_X(t) = \underline{E}(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)} dx$$

$$= \int_0^\infty \frac{x^{\alpha-1} e^{-x(\lambda-t)}}{\Gamma(\alpha)} \lambda^{\alpha} dx$$

$$\left(\begin{array}{l} y = x(\lambda - t) \\ dy = (\lambda - t) dx \end{array} \right) = \int_0^\infty \left(\frac{y}{\lambda - t}\right)^{\alpha-1} \frac{e^{-y} \lambda^{\alpha}}{\Gamma(\alpha)} \frac{dy}{\lambda - t}$$

$$= \int_0^\infty \left[\frac{\lambda}{\lambda - t}\right]^{\alpha} \underbrace{\frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)}}_{Gamma(\alpha, 1)} dy =$$

$$= \left[\frac{\lambda}{\lambda - t}\right]^{\alpha} = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$$

$$m_X(t) = \left(1-\frac{t}{\lambda}\right)^{-\alpha}$$

$$E(X) = m_X'(0) = -\alpha\left(1-\frac{t}{\lambda}\right)^{-\alpha-1} \cdot \left(-\frac{1}{\lambda}\right)\Bigg|_{t=0} = \frac{\alpha}{\lambda}$$

$$E(X^2) = m_X''(0) = -\alpha(-\alpha-1)\left(1-\frac{t}{\lambda}\right)^{-\alpha-2} \cdot \frac{1}{\lambda^2}\Bigg|_{t=0} =$$

$$= \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$Var(X) = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

<u>Ex</u>. Let $X_1 \sim Gamma(\alpha_1, \lambda), \ldots, X_n \sim Gamma(\alpha_n, \lambda)$

$Y = X_1 + \ldots + X_n$, X_i's are independent

$$m_Y(t) = m_{X_1}(t) \cdot \ldots \cdot m_{X_n}(t)$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_n} =$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1+\ldots+\alpha_n}$$

$\longrightarrow$ mgf for $Gamma(\alpha_1+\ldots+\alpha_n, \lambda)$

$\Rightarrow Y \sim Gamma(\alpha_1+\ldots+\alpha_n, \lambda)$