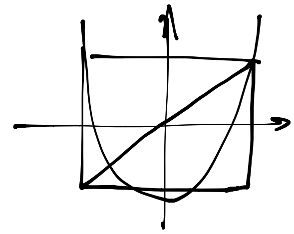
Lecture 16

$\Lambda_n = \{x \in I : Q_c^n(x) \notin I \ \& \ Q_c^k(x) \in I$ for all $k < n$.

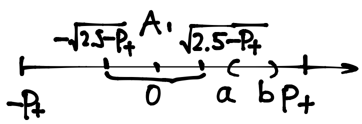
$\Lambda = I - \bigcup_{n=1}^{\infty} A_n$



- Λ contains no open intervals

Proof: Sps $(a,b) \subset \Lambda$
(for $c = -2.5$)

(a,b) on right or left of A
if (a,b) on right, $0 < a < b$, Q_c increasing on (a,b)



$Q_c((a,b)) = (Q_c(a), Q_c(b)) = (a^2-2.5, b^2-2.5)$
So length of $Q_c(a,b)$ is $b^2 - a^2 = (b-a)(a+b) \geq 2\sqrt{2.5-P_+}\,(b-a) \approx \underbrace{1.16}_{>1}(b-a)$

and $Q_c(a,b) \subset \Lambda$ & $|Q_c(a,b)| \geq 1.16|(a,b)|$

Repeat this process with new interval $Q_c(a,b)$ k times to get
$|Q_c^k(a,b)| \geq (1.16)^k |(a,b)|$ for k large enough, $(1.16)^k(b-a)$ is

Larger than the interval of I which contradicts the fact that $\Lambda \subset I$.

Remark: for this proof to work, we needed $2\sqrt{-c-P_+} > 1$ which is true for $c < -\frac{5+2\sqrt{5}}{4} \approx -2.368$
For $-\frac{5+2\sqrt{5}}{4} \leq c < -2$, the result is still valid, but harder to prove.

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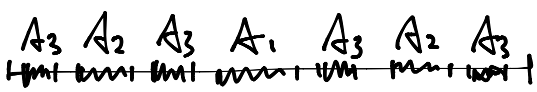
§7.3 Cantor Set



Definition of a Cantor Set

1. Start with $[0,1]$
2. Remove the middle third $(1/3, 2/3)$, leaving two closed intervals $[0,1/3]$ and $[2/3, 1]$
   each of length $1/3$
3. repeat step 2 with every closed interval remaining.

- The set K is the set of points in $[0,1]$ remaining after this process is repeated again & again without end.

- $K \neq \emptyset$. the end points of each closed interval at each step are in K.
  $0, 1/3, 2/3, 1/9, 2/9, \cdots \in K$

• The sets K and Λ are constructed in a similar way, since K has 'nicer' points, we study K.

Properties of the Cantor Set:

① K is a closed subset of $[0,1]$.

② K is completely disconnected: it doesn't contain any open intervals (like Λ)

③ $a \in K$ iff there is a Ternary expansion (base 3) of a such that

$$a = 0.s_1 s_2 s_3 s_4 \cdots \text{ with } s_i \in \{0, 2\}$$

This means that if we write a in base 3
$a \in K$ iff it has a base 3 expansion has No 1's

④ K is uncountable