Lecture 5

Now that we have some integration tools are our disposal, we can consider some integrals for moments and statistics of the MP distribution.

Moments of the MP distribution

Proposition: For the standard MP distribution F_y with index $y>0$ and $\sigma^2=1$, it holds for any analytic function f on a domain containing the interval $[a,b]=[(1\pm\sqrt{y})^2]$

$$\int f(x)\,dF_y(x) = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{f(|1+\sqrt{y}z|^2)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})}\,dz$$

Proof: (We will prove a stronger case later).

Let's look at some applications.

Example 1: Logarithms of eigenvalues are often used in multivariate analysis. Set

$$f(x) = \log(x).$$

Assume $0<y<1$ so that we dont get zero eigenvalues.

$$\int \log(x)\,dF_y(x) = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{\log(|1+\sqrt{y}z|^2)\,(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})}\,dz$$

$z \in \mathbb{C}$.

$$|1+\sqrt{y}z|^2 = (1+\sqrt{y}z)(\overline{1+\sqrt{y}z}) = (1+\sqrt{y}z)(1+\sqrt{y}\bar{z})$$

$$= -\frac{1}{4\pi i}\oint_{|z|=1} \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})}\,dz$$

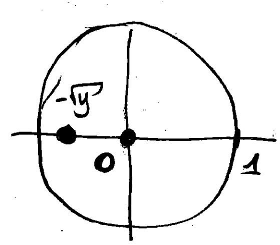
$$-\frac{1}{4\pi i}\oint_{|z|=1} \frac{\log(1+\sqrt{y}\bar{z})(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})}\,dz$$

$$=: I_1 + I_2$$

Q: When do these integrals have singularities?

There is one at the point $z=0$, due to $\underbrace{\text{the } \frac{1}{z^2} \text{ term}}_{\text{"order 2 pole"}}$.

Another at $z=-\sqrt{y}$.



both within contour $|z|=1$

By Cauchy residue theorem,

$$\int_C f(z)\,dz = 2\pi i \sum_{a\in C} \mathrm{Res}(f;a)$$

where a are points of singularity.

We need to find the residues at the points $z=0$ and $z=-\sqrt{y}$. We could expand and find the Laurent series but there is an easier way.

<u>Proposition</u>: If f has a pole of order $n \geq 1$ at a. Define $g(z) = (z-a)^n f(z)$ then

$$\mathrm{Res}(f;a) = \frac{1}{(n-1)!} \lim_{z\to a} g^{(n-1)}(z).$$

<u>Proof</u>: Remember that the residue is the term c_{-1} in the Laurent series expansion of $f(z)$:

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + a_0 + \cdots$$

So $g(z) = c_{-n} + \cdots + c_{-1}(z-a)^{n-1} + c_0(z-a)^n + \cdots$

and $g^{(n-1)}(z) = (n-1)!\,c_{-1} + n(n-1)\cdots 2\cdot c_0(z-a) + \cdots$

Hence, $\lim_{z\to a} g^{(n-1)}(z) = g^{(n-1)}(a) = (n-1)!\,c_{-1}$. ∎

Applying this proposition at $a = -\sqrt{y}$

$$\lim_{z \to -\sqrt{y}} \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{z^2(1+\sqrt{y}z)\cancel{(z+\sqrt{y})}} \cancel{(z-(-\sqrt{y}))} = \frac{\log(1-y)(1-y)^2}{y(1-y)}$$

$$= \log(1-y)\,\frac{(1-y)}{y}$$

The singularity at $a=0$ is of order 2, so

$$g(z) = \cancel{z^2}\, \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{\cancel{z^2}(1+\sqrt{y}z)(z+\sqrt{y})}$$

$$= \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{(1+\sqrt{y}z)(z+\sqrt{y})}.$$

$$g'(z) = \frac{\sqrt{y}(1-z^2)^2}{(\sqrt{y}+z)(1+\sqrt{y}z)^2} - \frac{4z(1-z^2)\log(1+\sqrt{y}z)}{(\sqrt{y}+z)(1+\sqrt{y}z)} -$$

$$\frac{\sqrt{y}(1-z^2)^2\log(1+\sqrt{y}z)}{(\sqrt{y}+z)(1+\sqrt{y}z)^2} - \frac{(1-z^2)^2\log(1+\sqrt{y}z)}{(\sqrt{y}+z)^2(1+\sqrt{y}z)}$$

$$g'(0) = \frac{\sqrt{y}}{\sqrt{y}} - 0 - 0 - 0 = 1.$$

So by the residue theorem $I_1 = -\frac{1}{4\pi i}\left[2\pi i \cdot \left(\log(1-y)\frac{(1-y)}{y} + 1\right)\right]$

$$= -\frac{1}{2}\left(\log(1-y)\frac{(1-y)}{y} + 1\right)$$

Now for I_2 we have

$$I_2 = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{\log(1+\sqrt{y}\,\bar{z})(1-z^2)^2}{z^2(1+\sqrt{y}\,z)(z+\sqrt{y})}\,dz.$$

we shall make the change of variable $s=\bar{z}$ and notice that since $|z|=1$, we have

$$\frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta} = \bar{z}$$

$$\text{So } I_2 = -\frac{1}{4\pi i}\oint_{|s|=1} \frac{\log(1+\sqrt{y}\,s)\left(1-\left(\frac{1}{s}\right)^2\right)^2}{\left(\frac{1}{s}\right)^2\left(1+\sqrt{y}\left(\frac{1}{s}\right)\right)\left(\frac{1}{s}+\sqrt{y}\right)}\left(-\frac{1}{s^2}\right)ds.$$

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and this can be shown to be

$$I_2 = I_1.$$

$$\text{hence } I = -\log(1-y)\frac{(1-y)}{y} - 1.$$

Example 2. We can calculate the mean of the MP distribution.

For all $y>0$,

$$\int x \, dF_y(x) = 1.$$

Proof: This can be shown in the same way as Example 1.

For any monomial function $f(x) = x^k$ for $k \in \mathbb{N}$, the residue approach becomes tedious. There is a direct proof as well. (Bai & Silverstein 2010; Lemma 3.1).

Proposition: The moments of the standard MP distribution

$$\beta_k := \int x^k \, dF_y(x) = \sum_{r=0}^{k-1} \frac{1}{r+1} \binom{k}{r} \binom{k-1}{r} y^r$$

Proof: $p_y(x) = \begin{cases} \frac{1}{2\pi x y} \sqrt{(b-x)(x-a)}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$ density.

$a = (1-\sqrt{y})^2$
$b = (1+\sqrt{y})^2$

$$\beta_k = \frac{1}{2\pi y} \int_a^b x^{k-1} \sqrt{(b-x)(x-a)} \, dx$$

set $x = 1+y+z$, $dx = dz$

$x = a \Rightarrow (1-\sqrt{y})^2 = 1+y+z$.
$z = (1-\sqrt{y})^2 - 1 - y$
$= -2\sqrt{y}$.

$(b-x)(x-a) = (2\sqrt{y} - z)(2\sqrt{y} + z)$
$= (4y - z^2)$

$x = b \Rightarrow z = 2\sqrt{y}$.

So $\beta_k = \dfrac{1}{2\pi y}\displaystyle\int_{-2\sqrt{y}}^{2\sqrt{y}} (1+y+z)^{k-1}\sqrt{4y-z^2}\,dz$

Recall $(a+b)^{\alpha} = \displaystyle\sum_{k=0}^{\alpha} \binom{\alpha}{k} a^k b^{\alpha-k}$ $\qquad \binom{\alpha}{k} := \dfrac{\alpha!}{k!(\alpha-k)!}$

$a = 1+y$

$b = z$

$\alpha = k-1$

$$= \frac{1}{2\pi y}\int_{-2\sqrt{y}}^{2\sqrt{y}} \sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l} z^l\sqrt{4y-z^2}\,dz$$

$$= \frac{1}{2\pi y}\sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l}\int_{-2\sqrt{y}}^{2\sqrt{y}} z^l\sqrt{4y-z^2}\,dz.$$

set $z = 2\sqrt{y}\,u$, $dz = 2\sqrt{y}\,du$. $\qquad z = -2\sqrt{y} \Rightarrow u = -1.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad z = 2\sqrt{y} \Rightarrow u = 1.$

$\Rightarrow 4y - z^2 = 1-u^2$

$$= \frac{1}{2\pi y}\sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l}\, 2\sqrt{y}\,(2\sqrt{y})^l\int_{-1}^{1} u^l\sqrt{1-u^2}\,du$$

$$= \frac{1}{2\pi y}\sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l}(4y)^{\frac{l+1}{2}}\int_{-1}^{1} u^l\sqrt{1-u^2}\,du.$$

$$= \frac{1}{2\pi y}\sum_{l=0}^{[(k-1)/2]}\binom{k-1}{2l}(1+y)^{k-1-2l}(4y)^{l+1}\int_{-1}^{1} u^{2l}\sqrt{1-u^2}\,du.$$

set $u = \sqrt{w}$, $du = \frac{1}{2}\frac{1}{\sqrt{w}}dw$. $\qquad u=-1,\ w=1$

$$= \frac{1}{2\pi y}\sum_{l=0}^{[(k-1)/2]}\binom{k-1}{2l}(1+y)^{k-1-2l}(4y)^{l+1}\int_{0}^{1} w^{l-1/2}\sqrt{1-w}\,dw.$$

As $\int_0^1 w^{\ell-\frac12}\sqrt{1-w}\,dw = \dfrac{\sqrt{\pi}\,\Gamma(\ell+\frac12)}{2\Gamma(2+\ell)}$ if $\ell > -\frac{1}{2}$.

$\Gamma(\ell+\frac12) = \dfrac{(2\ell)!}{4^\ell \ell!}\sqrt{\pi}$

$\Gamma(n) := (n-1)!$

$\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}\,dx.$

$$= \sum_{\ell=0}^{[(k-1)/2]} \frac{1}{\cancel{2\pi y}} \frac{(k-1)!}{\cancel{(2\ell)!}\,((k-1)-2\ell)!} \frac{\sqrt{\pi}}{\cancel{2}} \frac{\cancel{(2\ell)!}\sqrt{\pi}}{\cancel{4^\ell}\ell!\,(\ell+1)!} \cancel{4^{\ell+1}} y^{\ell+1}(1+y)^{k-1-2\ell}$$

$$= \sum_{\ell=0}^{[(k-1)/2]} \frac{(k-1)!}{\ell!\,(\ell+1)!\,(k-1-2\ell)!} y^\ell (1+y)^{k-1-2\ell}.$$

As $(1+y)^{k-1-2\ell} = \sum_{s=0}^{k-1-2\ell} \binom{k-1-2\ell}{s} y^s = \sum_{s=0}^{k-1-2\ell} \frac{(k-1-2\ell)!}{s!\,(k-1-2\ell-s)!} y^s$

$$= \sum_{\ell=0}^{[(k-1)/2]} \frac{(k-1)!}{\ell!\,(\ell+1)!\,\cancel{(k-1-2\ell)!}} y^\ell \sum_{s=0}^{k-1-2\ell} \frac{\cancel{(k-1-2\ell)!}}{s!\,(k-1-2\ell-s)!} y^s$$

$$= \sum_{\ell=0}^{[(k-1)/2]} \sum_{s=0}^{k-1-2\ell} \frac{(k-1)!}{\ell!\,(\ell+1)!\,s!\,(k-1-2\ell-s)!} y^{\ell+s}$$

Subst. $r = \ell + s$ $\quad s=0 \Rightarrow r=\ell \quad s = k-1-2\ell \Rightarrow r = k-1-\ell$

$$= \sum_{\ell=0}^{[(k-1)/2]} \sum_{r=\ell}^{k-1-\ell} \frac{(k-1)!}{\ell!\,(\ell+1)!\,s!\,(k-1-r-\ell)!} y^r$$

$$= \frac{1}{k} \sum_{r=0}^{k-1} \binom{k}{r} y^r \sum_{\ell=0}^{\min(r,k-1-r)} \binom{r}{\ell}\binom{k-r}{k-r-\ell-1}$$

$$= \frac{1}{k} \sum_{r=0}^{k-1} \binom{k}{r}\binom{k}{r+1} y^r = \sum_{r=0}^{k-1} \frac{1}{r+1}\binom{k}{r}\binom{k-1}{r} y^r.$$

Fubini theorem for sequences: If $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|<\infty$ then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}.$$

How did I use that?

$$\sum_{\ell=0}^{[(k-1)/2]}\sum_{r=\ell}^{k-1-\ell}\frac{(k-1)!}{\ell!\,(\ell+1)!\,(r-\ell)!\,(k-1-r-\ell)!}y^r$$

$$=\sum_{\ell=0}^{\infty}1(\ell\le[(k-1)/2])\sum_{r=0}^{\infty}1(r\ge\ell)\,1(r\le k-1-\ell)\quad\sim$$

$$=\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty}1(\ell\le[(k-1)/2])\,1(\ell\le r)\,1(\ell\le k-1-r)\,1(r\le k-1).\quad\sim$$

$$=\sum_{r=0}^{k-1}\sum_{\ell=0}^{\min(r,k-1-r)}\blacksquare$$

$$\blacksquare=\frac{k!}{r!(k-r)!}y^r\frac{\cancel{(k-1)!}\,\cancel{r!}\,(k-r)!}{\cancel{k!}\,\ell!\,(\ell+1)!\,(r-\ell)!\,(k-1-r-\ell)!}$$

$$\blacksquare=\binom{k}{r}y^r\frac{1}{k}\cdot\frac{r!}{\ell!(r-\ell)!}\cdot\frac{(k-r)!}{(\ell+1)!\,(k-1-r-\ell)!}$$

$k-r-(k-1-r-\ell)$
$=\ell+1.$

$$=\frac{1}{k}\binom{k}{r}y^r\binom{r}{\ell}\binom{k-r}{k-1-r-\ell}$$

$$=\sum_{r=0}^{k-1}\frac{1}{k}\binom{k}{r}y^r\sum_{\ell=0}^{\min(r,k-1-r)}\binom{r}{\ell}\binom{k-r}{k-1-r-\ell}$$

Generalised MP distribution

Previously, we've seen the case where the population covariance matrix has the simple form $\Sigma = \sigma^2 I_p$.

We can consider a slightly more general case if we make the assumption that the observation vectors $\{y_k\}_{1 \leq k \leq n}$ can be represented as

$$y_k := \Sigma^{\frac{1}{2}} x_k \qquad x_k \text{ iid}, \ \Sigma^{\frac{1}{2}} \text{ nonneg. sqroot of } \Sigma.$$

This gives the associated covariance matrix

$$\tilde{B}_n = \frac{1}{n}\sum_{k=1}^{n} y_k y_k^* = \Sigma^{\frac{1}{2}}\left(\frac{1}{n}\sum_{k=1}^{n} x_k x_k^*\right)\Sigma^{\frac{1}{2}}$$

$$= \Sigma^{\frac{1}{2}} S_n \Sigma^{\frac{1}{2}}.$$

S_n is the sample covariance matrix with iid components

The eigenvalues of $\tilde{B}_n$ are the same as $S_n\Sigma$.

The following result holds for $B_n = S_n T_n$ for general nonnegative definite matrix T_n. ($T_n = \Sigma$ is a special case)

<u>Theorem</u> Let S_n be the sample covariance matrix $S_n = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^*$ with IID components and let (T_n) be a sequence of nonnegative definite Hermitian matrices of size $p \times p$. Define $B_n = S_n T_n$ and assume:

(1) The entries (x_{jk}) of the data matrix $X = (X_1, \dots X_n)$ are IID with mean zero and variance 1.

(2) The data dimension to sample size ratio $p/n \to y > 0$ as $n \to \infty$.

(3) The sequence (T_n) is either deterministic or independent of (S_n)

(4) Almost surely, the sequence $(H_n = F^{T_n})$ of the ESD of (T_n) weakly converges to a non-random probability measure H.

Then, almost surely, F^{B_n} weakly converges to a non-random probability measure $F_{y,H}$. Its Stieltjes transform is given by

$$s(z) = \int \frac{1}{t(1-y-yzs(z)) - z} \, dH(t), \quad z \in \mathbb{C}_+ . \qquad (*)$$

Notice that the ST of $F_{y,H}$ is <u>implicitly</u> defined.

It can be shown that a unique solution exists but, unfortunately, no closed-form solution exists.

(see Silverstein & Combettes 1992.)

There is a better way to present the ST of $F_{y,H}$.

Consider for B_n a <u>companion matrix</u>

$$\underline{B_n} = \frac{1}{n} \mathbb{X}^* T \mathbb{X} \qquad \text{size } n \times n.$$

Both matrices share the same nonzero eigenvalues so their ESD satisfy

$$n F^{\underline{B_n}} - p F^{B_n} = (n-p)\delta_0$$

<u>Note</u>: Given two matrices $A_{p\times q}$ and $B_{q\times p}$ where $p \geq q$, eigenvalues of AB is that of BA augmented by $p-q$ zeros.

$$B_n = S_n T_n = \frac{1}{n} \mathbb{X}\mathbb{X}^* T_n . \qquad \underline{B_n} = \frac{1}{n} \mathbb{X}^* T \mathbb{X}$$

$\mathbb{X}$ $p \times n$ matrix

When $p/n \to y > 0$, F^{B_n} has limit $F_{c,H}$ if and only if $F^{\underline{B_n}}$ has limit $\underline{F}_{c,H}$. In this case, the limit satisfies

$$\underline{F}_{c,H} - yF_{c,H} = (1-y)\delta_0 .$$

and their ST are related by

$$\underline{s}(z) = -\frac{1-y}{z} + y\,s(z).$$

Now substituting $\underline{s}$ for s in (*) yields

$$\underline{s}(z) = \left(z - y\int \frac{t}{1+ts(z)}\,dH(t)\right)^{-1}$$

solving in z gives

$$z = -\frac{1}{\underline{s}(z)} + y\int \frac{t}{1+t\underline{s}(z)}\,dH(t). \qquad (**)$$

which defines the inverse function of $\underline{s}$.

(*) is called the <u>Marcenko-Pastur equation</u> and (**) is the <u>Silverstein equation</u>.

Limiting spectral distribution for Random Fisher matrices

In the univariate case, when we need to test equality between the variances of 2 Gaussian populations, a Fisher statistic of the form S_1^2/S_2^2 is used where S_i^2 are estimators of the unknown variances in the two populations.

The equivalent for the multivariate setting is:

Take two independent samples $\{\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_{n_1}\}$ and $\{\mathbb{Y}_1, \mathbb{Y}_2, \dots, \mathbb{Y}_{n_2}\}$ both from p-dimensional population with iid components and finite second moment.

$$\mathbb{S}_1 = \frac{1}{n_1}\sum_{k=1}^{n_1} \mathbb{X}_k \mathbb{X}_k^*$$

$$\mathbb{S}_2 = \frac{1}{n_2}\sum_{k=1}^{n_2} \mathbb{Y}_k \mathbb{Y}_k^*$$

Then $\mathbb{F}_n := \mathbb{S}_1 \mathbb{S}_2^{-1}$ is called a Fisher matrix. $n=(n_1,n_2)$

(Note: need $p \leq n_2$ so that $\mathbb{S}_2$ invertible)

Let $s>0$ and $0<t<1$. The Fisher LSD $F_{s,t}$ is the distribution with density function

$$p_{s,t}(x) = \frac{1-t}{2\pi x(s+tx)}\sqrt{(b-x)(x-a)} \qquad a\leq x\leq b.$$

with $a=a(s,t)=\dfrac{(1-h)^2}{(1-t)^2}$, $b=b(s,t)=\dfrac{(1+h)^2}{(1-t)^2}$,

$$h=h(s,t)=(s+t-st)^{1/2}.$$

When $s>1$, $F_{s,t}$ has a mass at $x=0$ of value $1-1/s$ with the total mass of the rest of the distribution for $x>0$ is equal to $1/s$.

The Fisher LSD has many similarities to the standard MP distribution. This is not a coincidence as the MP LSD F_y is the Fisher LSD $F_{y,0}$ (ie. $s,t=y,0$)

Also note $t\to 1_-$, $a(s,t)\to\frac{1}{2}(1-s)^2$, $\underbrace{b(s,t)\to\infty}_{\text{supp}(F_{s,t})\text{ becomes unbounded}}$.

<u>Theorem:</u> For an analytic function f on a domain containing $[a,b]$ (as above). We have

$$\int_a^b f(x)\,dF_{s,t}(x) = -\frac{h^2(1-t)}{4\pi i}\oint_{|z|=1}\frac{f\left(\frac{|1+hz|^2}{(1-t)^2}\right)(1-z^2)^2\,dz}{z(1+hz)(z+h)(tz+h)(t+hz)}$$

<u>Proof:</u> Using the density $p_{s,t}(x)$

$$I = \int_a^b f(x)\,dF_{s,t}(x) = \int_a^b f(x)\frac{1-t}{2\pi x(s+xt)}\sqrt{(x-a)(b-x)}\,dx.$$

Make change of variable $x = \dfrac{1+h^2+2h\cos(\theta)}{(1-t)^2}$ $\quad\theta\in(0,\pi)$

$$x-a = \frac{1+h^2+2h\cos(\theta)}{(1-t)^2} - \frac{(1-h)^2}{(1-t)^2} = \frac{2h+2h\cos(\theta)}{(1-t)^2}$$

$$b-x = \frac{(1+h)^2}{(1-t)^2} - \frac{1+h^2+2h\cos(\theta)}{(1-t)^2} = \frac{2h-2h\cos(\theta)}{(1-t)^2}$$

$$\sqrt{(x-a)(b-x)} = \sqrt{\frac{(2h)^2}{(1-t)^4}(1-\cos(\theta))(1+\cos(\theta))}$$

$$= \frac{2h}{(1-t)^2}\sin(\theta)$$

$$x=a \Rightarrow \cos(\theta) = \frac{a(1-t)^2-(1+h^2)}{2h}$$

$$=0.$$

$$\Rightarrow \theta = 0$$

$$x=b \Rightarrow \theta=\pi$$

$$dx = \frac{-2h\sin(\theta)}{(1-t)^2}\,d\theta.$$

hence,

$$I = \frac{2h^2(1-t)}{\pi}\int_0^{\pi} \frac{f\left(\frac{1+h^2+2h\cos(\theta)}{(1-t)^2}\right)\sin^2(\theta)\,d\theta}{(1+h^2+2h\cos(\theta))(s(1-t)^2+t(1+h^2+2h\cos(\theta)))}$$

$$= \frac{\cancel{2}h^2(1-t)}{\pi}\frac{1}{\cancel{2}}\int_0^{2\pi} \frac{f\left(\frac{1+h^2+2h\cos(\theta)}{(1-t)^2}\right)\sin^2(\theta)\,d\theta}{(1+h^2+2h\cos(\theta))(s(1-t)^2+t(1+h^2+2h\cos(\theta)))}$$

Now let $z = e^{i\theta}$,

$$1+h^2+2h\cos(\theta) = |1+hz|^2$$

$$\sin(\theta) = \frac{z-z^{-1}}{2i}$$

$$\log(z) = i\theta \Rightarrow \theta = \frac{1}{i}\log(z) \Rightarrow d\theta = \frac{1}{iz}dz.$$

$$I = -\frac{h^2(1-t)}{4\pi i}\oint_{|z|=1} \frac{f\left(\frac{|1+hz|^2}{(1-t)^2}\right)(1-z^2)^2\,dz}{z^3|1+hz|^2(s(1-t)^2+t|1+hz|^2)}$$

On $|z|=1$, we have $|1+hz|^2 = (1+hz)(1+hz^{-1})$.

So expanding denominator and simplifying we have result.

Example: Take $(s,t)=(y,0)$ and we get the result for MP distribution

Example 2: The first two moments are

$$\int x \, dF_{s,t}(x) = \frac{1}{1-t}, \quad \int x^2 \, dF_{s,t}(x) = \frac{h^2+1-t}{(1-t)^3}.$$

Hence the variance equals $h^2/(1-t)^3$.