5.1.) $X_1, \dots, X_n \overset{\text{indep}}{\sim} f(x_i)$; $E(X_i) = \mu$; $V(X_i) = \sigma_i^2$

$$E(\bar{X}) = E\left(\frac{1}{n} \sum X_i\right) = \frac{1}{n} \sum E(X_i)$$

$$= \frac{1}{n} \sum \mu = \frac{1}{n} n \mu = \mu.$$

$$V(\bar{X}) = V\left(\frac{1}{n} \sum X_i\right) = \frac{1}{n^2}\left[V(X_1) + \cdots V(X_n) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j)\right]$$

$$= \frac{1}{n^2}\left[V(X_i) + \cdots + V(X_n) + 0\right]$$

$$= \frac{1}{n^2} \sum \sigma_i^2$$

$$P(|\bar{X} - \mu| > \varepsilon) = P(|\bar{X} - \mu|^2 > \varepsilon^2) \le \frac{E[(\bar{X} - \mu)^2]}{\varepsilon^2}$$

$$= \frac{V(\bar{X})}{\varepsilon^2} = \frac{\sum \sigma_i^2}{n^2 \varepsilon} \to 0 \text{ as } n \to \infty.$$

$$\therefore \bar{X} \xrightarrow{P} \mu$$

5.13.) Let $X_i = \begin{cases} 50, & \text{with prob. } 1/2 \text{ (North step)} \\ -50, & \text{with prob } 1/2 \text{ (South step)} \end{cases}$

$$E(X_i) = 50\,(1/2) + (-50)\,(1/2) = 0$$

$$E(X_i^2) = 50^2\,(1/2) + (-50)^2\,(1/2) = 2500$$

$$V(X_i) = E(X_i^2) + [E(X_i)]^2$$
$$= 2500 + 0^2 = 2500$$

- That was for a single step, Now let's Consider 60 steps:

$$W_{60} = X_1 + X_2 + \cdots + X_{60} = \sum_{i=1}^{60} X_i$$

$$E(W_{60}) = \sum_{i=1}^{60} E(X_i) = n(0) = 0.$$

$$V(W_{60}) = \sum_{i=1}^{n} V(X_i) = 60\,(2500) = 150,000.$$

By the CLT: $W_{60} \,\dot{\sim}\, N(0,\ 150,000)$

$\therefore$ After 1 hour we expect the walker to be in the same place (with a lot of variability around that guess).

5.16.) $X_1, \ldots, X_{20} \overset{iid}{\sim} f(x) = 2x \qquad 0 \le x \le 1$

$$S = \sum_{i=1}^{20} X_i$$

- Let's get the mean and varience for a single X.

$$E(x) = \int_0^1 x \, 2x \, dx = \int_0^1 2x^2 \, dx = \frac{2x^3}{3}\Big|_0^1$$
$$= \frac{2}{3}$$

$$E(x^2) = \int_0^1 x^2 \, 2x \, dx = \int_0^1 2x^3 \, dx = \frac{2}{4} x^4 \Big|_0^1 = \frac{1}{2}$$

$$\therefore V(x) = E(x^2) - [E(x)]^2 = \frac{1}{18}$$

- Now let's consider $S = \sum_{i=1}^{20} X_i$.

$$E(S) = n \, E(x) = 20 \left(\tfrac{2}{3}\right)$$
$$V(S) = n \, V(S) = 20 \left(\tfrac{1}{18}\right)$$

Based on the CLT, $S \mathrel{\dot\sim} \text{Normal}\left(20(\tfrac{2}{3}), 20(\tfrac{1}{18})\right)$

$$P(S \le 10) = P\left(\frac{S - 40/3}{\sqrt{20/18}} \le \frac{10 - 40/3}{\sqrt{20/18}}\right)$$

$$= P(Z \le -3.16) = 0.00078.$$