$\boxed{\text{Relating multiple linear model with \& without the ith case}}$

Key relation: $(X_{(i)}'X_{(i)})^{-1} = (X'X)^{-1} + \dfrac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1-h_{ii}}$ (1)

where $h_{ii} = x_i'(X'X)^{-1}x_i$

Note $X = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$, $X'X = (x_1 \cdots x_n)\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \sum_{i=1}^{n} x_i x_i'$

$$X_{(i)}'X_{(i)} = \sum_{j\neq i} x_j x_j' = X'X - x_i x_i'$$

To verify (1), only need to show (r.h.s.)$\times(X'X - x_i x_i') = I_{p\times p}$

$$\left((X'X)^{-1} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1-h_{ii}}\right)(X'X - x_i x_i')$$

$$= I_{p\times p} - (X'X)^{-1}x_i x_i' + \frac{(X'X)^{-1}x_i x_i'\cancel{(X'X)^{-1}(X'X)}}{1-h_{ii}} - \frac{(X'X)^{-1}x_i\overbrace{(x_i'(X'X)^{-1}x_i)}^{\to h_{ii}}x_i'}{1-h_{ii}}$$

$$= I_{p\times p} - (X'X)^{-1}x_i x_i' + \frac{(X'X)^{-1}x_i x_i' - (X'X)^{-1}x_i x_i' h_{ii}}{1-h_{ii}}$$

$$= I_{p\times p} - (X'X)^{-1}x_i x_i' + (X'X)^{-1}x_i x_i' = I_{p\times p}$$

More general result: $(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$

We then can deduce some important relations, for instance.

$$\hat\beta_{(i)} = \hat\beta - \frac{(X'X)^{-1}x_i\hat e_i}{1-h_{ii}}, \quad \left(\hat e_i = y_i - \hat y_i = y_i - x_i'\hat\beta\right) \qquad (2)$$

show: $\hat\beta_{(i)} = (X_{(i)}'X_{(i)})^{-1}X_{(i)}'y_{(i)} = (X_{(i)}'X_{(i)})^{-1}\left(\sum_{j\neq i} x_j y_j\right) = (X_{(i)}'X_{(i)})^{-1}(X'y - x_i y_i)$

(perturbed $\hat\beta$)

$$= \left[(X'X)^{-1} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1-h_{ii}}\right](X'y - x_i y_i) \qquad \left(\underbrace{\hat\beta = (X'X)^{-1}X'y}\right)$$

$$= \hat\beta + \frac{(X'X)^{-1}x_i x_i'\hat\beta}{1-h_{ii}} - (X'X)^{-1}x_i y_i - \frac{(X'X)^{-1}x_i\overbrace{(x_i'(X'X)^{-1}x_i)}^{\to h_{ii}}y_i}{1-h_{ii}}$$

$$= \hat\beta + \frac{(X'X)^{-1}x_i x_i'\hat\beta}{1-h_{ii}} - (X'X)^{-1}x_i y_i\underbrace{\left(1 + \frac{h_{ii}}{1-h_{ii}}\right)}_{\to \frac{1}{1-h_{ii}}}$$

$$= \hat\beta - \frac{(X'X)^{-1}x_i(y_i - x_i'\hat\beta)}{1-h_{ii}} = \hat\beta - \frac{(X'X)^{-1}x_i\hat e_i}{1-h_{ii}}$$

From (2), it's easy to derive $\hat{e}_{(i)} = y_i - \hat{y}_{(i)} = \dfrac{\hat{e}_i}{1-h_{ii}}$ (3)

$$y_i - x_i'\hat{\beta}_{(i)} = y_i - x_i'\left(\hat{\beta} - \frac{(X'X)^{-1}x_i\hat{e}_i}{1-h_{ii}}\right) = y_i - x_i'\hat{\beta} + \frac{x_i'(X'X)^{-1}x_i\hat{e}_i}{1-h_{ii}}$$

$$= \hat{e}_i + \frac{h_{ii}}{1-h_{ii}}\hat{e}_i = \frac{\hat{e}_i}{1-h_{ii}}$$

"Deleted residual formula".

Also easy to derive Cook's distance:

$$D_i = (\hat{\beta}_{(i)} - \hat{\beta})'(X'X)(\hat{\beta}_{(i)} - \hat{\beta}) / p'\hat{\sigma}^2 \qquad \text{plug in } \hat{\beta}_{(i)} = \hat{\beta} - \frac{(X'X)^{-1}x_i\hat{e}_i}{1-h_{ii}}$$

$$= \frac{1}{p'\hat{\sigma}^2}\frac{\hat{e}_i x_i'(X'X)^{-1}\cancel{(X'X)}\cancel{(X'X)^{-1}}x_i\hat{e}_i}{(1-h_{ii})^2}$$

$$= \frac{1}{p'\hat{\sigma}^2}\frac{\hat{e}_i^2 h_{ii}}{(1-h_{ii})^2} \qquad \text{Note } r_i^2 = \frac{\hat{e}_i^2}{\hat{\sigma}^2(1-h_{ii})}$$

$$= \frac{1}{p'} r_i^2 \frac{h_{ii}}{1-h_{ii}}$$

It's a bit involved to establish $t_i = r_i\left(\dfrac{n-p'-1}{n-p'-r_i^2}\right)^{\frac{1}{2}}$ with $r_i = \dfrac{\hat{e}_i}{\hat{\sigma}\sqrt{1-h_{ii}}}$

$$t_i = \frac{y_i - \hat{y}_{(i)}}{\hat{\sigma}_{(i)}\sqrt{1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i}}, \quad \text{recall } (X_{(i)}'X_{(i)})^{-1} = X'X + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1-h_{ii}}$$

$$1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i = 1 + x_i'\left[(X'X)^{-1} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1-h_{ii}}\right]x_i$$

$$= 1 + h_{ii} + \frac{h_{ii}^2}{1-h_{ii}} = \frac{1}{1-h_{ii}}$$

$$y_i - \hat{y}_{(i)} = \hat{e}_{(i)} = \frac{\hat{e}_i}{\cancel{\sqrt{}}1-h_{ii}}, \text{ then } t_i = \frac{1}{\hat{\sigma}_{(i)}}\frac{\hat{e}_i}{1-h_{ii}}\frac{1}{\sqrt{\frac{1}{1-h_{ii}}}} = \frac{\hat{e}_i}{\hat{\sigma}_{(i)}\sqrt{1-h_{ii}}}$$

Another relation (verify next page): $\hat{\sigma}_{(i)}^2 = \hat{\sigma}^2\left(\dfrac{n-p'-1}{n-p'-r_i^2}\right)^{-1}$ (4)

$\downarrow$ "Deleted MSE"

Combine the above:

$$t_i = \frac{\hat{e}_i}{\hat{\sigma}\left(\frac{n-p'-1}{n-p'-r_i^2}\right)^{-\frac{1}{2}}\sqrt{1-h_{ii}}} = r_i\left(\frac{n-p'-1}{n-p'-r_i^2}\right)^{\frac{1}{2}}$$

Lastly we show (4) i.e. $(n-p'-1)\hat{\sigma}_{(i)}^2 = (n-p'-r_i^2)\hat{\sigma}^2$

Note $(n-p'-r_i^2)\hat{\sigma}^2 = \left[n-p' - \dfrac{\hat{e}_i^2}{\hat{\sigma}^2(1-h_{ii})}\right]\hat{\sigma}^2 = (n-p')\hat{\sigma}^2 - \dfrac{\hat{e}_i^2}{1-h_{ii}}$

$$(n-p'-1)\hat{\sigma}_{(i)}^2 = \sum_{j\neq i}\left(y_j - x_j'\hat{\beta}_{(i)}\right)^2 = \sum_{j=1}^{n}\left(y_j - x_j'\hat{\beta}_{(i)}\right)^2 - \left(y_i - x_i'\hat{\beta}_{(i)}\right)^2$$

$$= \sum_{j=1}^{n}\left(y_j - x_j'\hat{\beta} + \frac{x_j'(X'X)^{-1}x_i\hat{e}_i}{1-h_{ii}}\right)^2 - \frac{\hat{e}_i^2}{(1-h_{ii})^2}$$

$$= \sum_{j=1}^{n}\left(y_j - x_j'\hat{\beta}\right)^2 + \frac{\left(\sum_{j=1}^{n} h_{ij}^2\right)\hat{e}_i^2}{(1-h_{ii})^2} + 2\left(\sum_{j=1}^{n}\hat{e}_j h_{ij}\right)\frac{\hat{e}_i}{1-h_{ii}} - \frac{\hat{e}_i^2}{(1-h_{ii})^2}$$

$$= (n-p')\hat{\sigma}^2 + \frac{h_{ii}\hat{e}_i^2}{(1-h_{ii})^2} + 0 - \frac{\hat{e}_i^2}{(1-h_{ii})^2} = (n-p')\hat{\sigma}^2 - \frac{\hat{e}_i^2}{1-h_{ii}}$$

$\left(\sum_{j=1}^{n} h_{ij}^2 = h_{ii},\ \sum_{j=1}^{n} h_{ij}\hat{e}_j = 0 \text{ due to } H\hat{e} = H(I-H)Y = 0\right)$