Oct. 9th

Autonomous differential equations

Recall: A general nth order ODE has the form $F(t, y, y', \dots, y^{(n)}) = 0$. It is called autonomous if t does not explicitly appear. i.e. $\frac{dF}{dt} = 0$

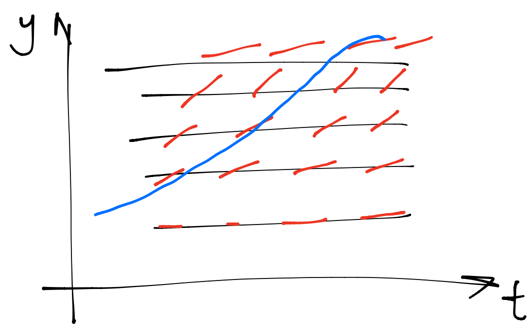
Example: $m\frac{d^2x}{dt^2} = mg - \alpha\frac{dx}{dt}$ is autonomous, but $m\frac{d^2x}{dt^2} = \beta\sin(t)$ is not autonomous.

First order autonomous eqn: $\boxed{y' = f(y)}$ $(*)$

Features: • Separable:

$$(*) \Rightarrow \frac{1}{f(y)}dy = dt \Rightarrow \int\frac{dy}{f(y)} = \int dt$$

• Isoclines are horizontal ($f(y) = \text{const.}$ means $y = \text{constant.}$)



• If $y_1(t)$ is a solution, and $c \in \mathbb{R}$ then $y_2(t) = y_1(t+c)$ is a solution.

• If $a \in \mathbb{R}$ is a zero of f ($f(a) = 0$) then $y = a$ is a solution

One calls the zeroes of f the critical points of $y' = f(y)$, and the corresponding solutions $y = a$ the equilibrium solutions.

Example:

① Verhulst equation (logistic equation)

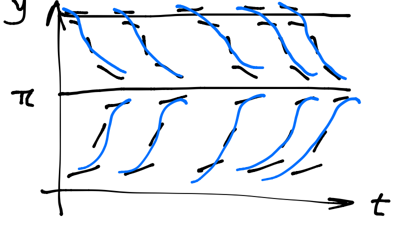
$$y' = \underbrace{r(1 - \frac{y}{k})y}_{f(y)}$$

Equilibrium solution: $y = 0$ (unstable)

$y = k$ (stable)

② $y' = \sin(y)$

Critical points: $0, \pm\pi, \pm 2\pi, \cdots k\pi$



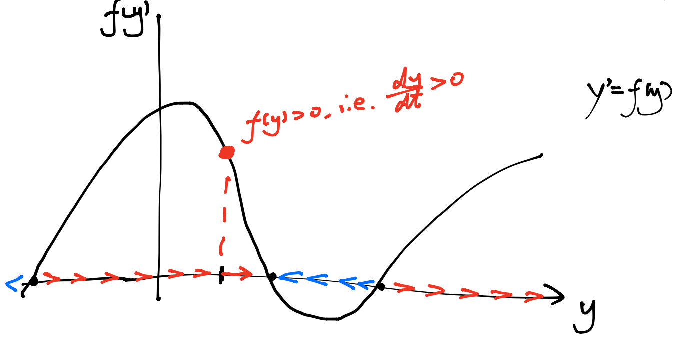
$0, \pm 2\pi, \pm 4\pi \dots$ unstable

$\pm\pi, \pm 3\pi, \pm 5\pi \cdots$ stable

Definition: An equilibrium solution $y(t)=a$ is (asymptotically) stable if for all nearby initial condition, the solution satisfy $\lim_{t\to\infty} y(t)=a$

(I.e. there exists $\varepsilon>0$ s.t. for all $y_0 \in \mathbb{R}$ with $|y_0-a|<\varepsilon$, the solution of $y'=f(y), y(t_0)=y_0$ satisfies $\lim_{t\to\infty} y(t)=a$.)

Easier way of deciding stability: consider graph of



Thus: f changes from $+$ to $-$ $\Rightarrow$ stable

changes from $-$ to $+$ $\Rightarrow$ unstable

In particular:

$f'(a)<0 \Rightarrow$ stable

$f'(a)>0 \Rightarrow$ unstable

Example:

① $y'=r(1-\frac{y}{k})y=f(y)$ $\quad (f'(y)=r(1-2\frac{y}{k}))$

$f(0)=0,\ f'(0)=r>0$ unstable

$f(k)=0,\ f'(k)=-r<0$ stable

② $y'=\sin(y)=f(y)$ $\quad f'(y)=\cos(y)$

$f(0)=0,\ f'(0)=\cos(0)=1>0$ unstable

$f(\pi)=0,\ f'(\pi)=\cos(\pi)=-1<0$ stable

$f(k\pi)=0,\ f'(k\pi)=(-1)^k$ unstable if k is even

stable if k is odd.

③ falling object in medium with viscosity.

$$m\frac{d^2x}{dt^2}=mg-\underbrace{k\frac{dx}{dt}}_{\text{"drag force"}}$$

Can regard this as 1st order ODE for $v=\frac{dx}{dt}$

$m\frac{dv}{dt}=mg-kv$

$\frac{dv}{dt}=g-\frac{k}{m}v=f(v)$

Equilibrium solution: $v=\frac{gm}{k}=a$

$f'(v)=-\frac{k}{m}<0 \Rightarrow$ stable