Feb 5th

Problem Set 3

Q7: V, W vector spaces over F

$\alpha = \{v_1, v_2, \cdots, v_n\}$ and $\beta = \{w_1, w_2, \cdots, w_n\}$

$T: V \to W$ linear transformation

Prove that T is an isomorphism iff $[T]_{\beta\alpha}$ is an invertible matrix.

($\Rightarrow$) T is an iso

Let $T^{-1}: W \to V$ denote the inverse linear transformation.

$T \cdot T^{-1} = Id_W$ and $T^{-1} \cdot T = Id_V$

$$[T \cdot T^{-1}]_\beta^\beta = \underset{\underset{I_n}{\|}}{[Id_W]_\beta^\beta} \qquad [T^{-1} \cdot T]_\alpha^\alpha = \underset{\underset{I_n}{\|}}{[Id_V]_\alpha^\alpha}$$

[NOT YET DONE]

($\Leftarrow$) Assume that $[T]_\alpha^\beta$ is an invertible matrix

Try to prove directly that $T: V \to W$ is $\underset{\text{injective}}{\underline{1-1}}$ and $\underset{\text{surjective}}{\underline{\text{onto}}}$

__Injective__: Suppose that $v \in V$ and $T(v) = 0$

$$0 = [T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha \Rightarrow [v]_\alpha = 0$$

$$\Rightarrow v = 0 \Rightarrow \ker(T) = \{0\}$$

__Surjective__: Suppose $w \in W$

Q5: $W = \text{span}\{1+x^2+x^3, 1+x+x^2, 3+x+3x^2+2x^3, -x+x^3\}$ Determine the dimension d of W and find an isomorphism $T: W \to \mathbb{R}^d$

We would like to "modify" the whole spanning set to obtain another spanning set consisting of linearly independent vectors, since the new spanning set would then be a basis of W.

$1+x^2+x^3 = (1\,0\,1\,1)$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$W = \text{span}\{1+x^2+x^3, x-x^3\}$

Note that $1+x^2+x^3$ and $x-x^3$ are linearly indep

So these vectors form a basis of W.