Today: §0.3 (matrix inversion)

Eg: (as before) was

Matrix ①

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & -3 & -4 & 11 \\ 3 & -1 & -1 & 1 & 2 & -13 \end{array}\right]$$

This lead to Matrix

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} \frac{1}{3} & -\frac{1}{2} & 0 & \frac{2}{3} & 1 & -4 \\ -\frac{7}{3} & 0 & 1 & \frac{1}{3} & 0 & 5 \end{array}\right]$$

giving the solution set by solving x_5 and x_3 in terms of for x_1, x_2, x_4

Now let $A_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, A_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, A_4 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, A_5 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

So the coefficients matrix of the system is $[A_1 | A_2 | A_3 | A_4 | A_5]$. To get matrix ③, $[A_5 | A_3]$ was replaced by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Inversion of $[A_5 | A_3]$:

$$\left[\begin{array}{cc|cc} -4 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \simeq \left[\begin{array}{cc|cc} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} & 1 \end{array}\right] \simeq \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

$R_1 \to -\frac{1}{4} R_1$, $R_2 \to R_2 - 2 \text{ new } R_1$

$R_2 \to -\frac{2}{3} R_2$, $R_1 \to R_1 - \frac{1}{4} \text{ new } R_2$

Check: $\begin{pmatrix} -4 & -1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$!!!

In fact, $\begin{pmatrix} -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \cdot$ Matrix ① = Matrix ③

In detail: 1st row of Matrix 3: $-\frac{1}{6} \times R_1$ of matrix ① $+ \frac{1}{6} R_2$ of matrix ①

2nd row of .... $-\frac{1}{3} \times R_1$ .... $-\frac{2}{3} R_2$ of matrix ①.

Another Goal: solve for x_2 and x_5 in term of x_1, x_3, x_4.

Matrix ①, pivot in A_2: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ..... Matrix : $\left(\begin{array}{ccccc|c} \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{3}{2} & -2 & \frac{11}{2} \\ \frac{7}{2} & 0 & -\frac{3}{2} & -\frac{1}{2} & 0 & -\frac{15}{2} \end{array}\right)$

Matrix ①: $\left(\begin{array}{ccccc|c} \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{3}{2} & -2 & \frac{11}{2} \\ \frac{7}{2} & 0 & -\frac{3}{2} & -\frac{1}{2} & 0 & -\frac{15}{2} \end{array}\right)$ ← pivot, $\frac{1}{2} \times$ old R_1 ← old R_2 + ~~new~~ new R_1

An attempt to invert $(A_2 | A_5) = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ will lead to $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

and so $(A_2 | A_5)$ has no inverse!

The columns A_2 and A_5 are Linearly Dependent (§0.5)

Remark: Linear dependence of n vectors in $\mathbb{R}^m$ ($n \neq m$) is defend in most lin-algebra courses. ~~have~~ here we restrict ~~to the~~ case: $n = m$.

Def'n: Given vectors $A_1, \dots, A_m$ in $\mathbb{R}^m$, a relation of linear dependence for $A_1, \dots, A_m$ is a true equation: $x_1 A_1 + x_2 A_2 + \cdots + x_m A_m = 0 \in \mathbb{R}^m$ for certain scalars $x_1, x_2, \dots, x_m$ which are not all zero. $A_1, \dots, A_m$ are linear dependant, $\rightarrow$ provided there exists a relation of linear dependence for $A_1, \dots, A_m$.