

Assignment 5 SOLUTIONS- MAT 327 - Summer 2013

Comprehension

[C.1] Let $C, D \subseteq \omega_1$ be closed (in the order topology), unbounded subsets of ω_1 . Prove that $C \cap D$ is a closed unbounded subset of ω_1 .

Proof. There are a handful of proofs for this, but they all have the same basic idea. Note that $C \cap D$ is closed since the intersection of two closed sets is closed. We only need to show that $C \cap D$ is unbounded.

Let $\alpha \in \omega_1$, and let us construct (recursively) a $\beta \in C \cap D$ such that $\alpha < \beta$. Suppose that we already have chosen

$$c_1 < d_1 < c_2 < d_2 < \dots < c_n < d_n$$

where $c_i \in C$ and $d_i \in D$ for all $1 \leq i \leq n$. Since C is unbounded, there is a $c_{n+1} \in C$ such that $d_n < c_{n+1}$. Since D is unbounded there is a d_{n+1} such that $c_{n+1} < d_{n+1}$.

By induction we get two “interweaved” sequences $\{c_i : i \in \mathbb{N}\}$ and $\{d_i : i \in \mathbb{N}\}$. Since the union of the two sequences is a countable set, there is a *least* $\beta \in \omega_1$ that is above $\{c_i : i \in \mathbb{N}\} \cup \{d_i : i \in \mathbb{N}\}$.

Claim: $\beta \in C \cap D$.

Since we chose β to be the least element above $\{c_i : i \in \mathbb{N}\} \cup \{d_i : i \in \mathbb{N}\}$ we can see that $\{c_i : i \in \mathbb{N}\} \rightarrow \beta$ and $\{d_i : i \in \mathbb{N}\} \rightarrow \beta$. Since C is closed, $\beta \in C$. Since D is closed, $\beta \in D$. Hence we have the claim. \square

[C.2] Let (L, \leq) be a linear order. Prove that (L, \leq) is a well-order if and only if L does not contain any infinite decreasing chains. (A chain $C = \{x_n : n \in \mathbb{N}\}$ is decreasing provided that $n < m$ implies $x_m < x_n$.)

Proof. First $[\Rightarrow]$, by proving the contrapositive. Suppose that L has an infinite decreasing chain C . Consider the collection

$$I := \{x \in C : \text{there are infinitely many } y \in C \text{ with } y \leq x\}$$

If we show that I is non-empty, then clearly I will have no minimal element, so L will not be a well order. This is clear though, since C is a decreasing chain we have $C \subseteq I$.

[\Leftarrow] Again we prove it by contrapositive. Suppose that (L, \leq) is not a well-order. So there is a non-empty $S \subseteq L$ such that S has no least element. If S was finite, then its minimum would exist (as non-empty finite subsets of a linear always have a minimal element), so S must be infinite. \square

[C.3] Suppose that ω_1 is in bijective correspondence with \mathbb{R} . Is \mathbb{R} with its usual order a well-order? Why or why not?

Proof. Any such bijection has nothing to do with the usual order on \mathbb{R} , (which is clearly not a well-order). Some students like to claim (for whatever bizarre reason) that \mathbb{R} with its usual order is a well order if such a bijection were to exist. I don't know why. \square

[C.4] By exhibiting an open cover that doesn't have a countable subcover, prove that ω_1 with the order topology is not a Lindelöf space. (Conclude that first countable does not imply Lindelöf.)

Proof. For $\alpha \in \omega_1$ let $U_\alpha := \{x \in \omega_1 : x < \alpha\}$, which is a basic open set (which is also countable). Clearly $\{U_\alpha : \alpha < \omega_1\}$ is a cover of ω_1 . If $A \subset \omega_1$ is countable, then $\bigcup_{\alpha \in A} U_\alpha$ is a countable union of countable sets, so is definitely not all of ω_1 . \square

[C.5] Prove that every second countable space is a Lindelöf space.

Proof. Let \mathcal{B} be a countable basis of a topological space X , and let $\mathcal{U} := \{U_\alpha : \alpha \in I\}$ be a collection of open sets such that $X \subseteq \bigcup_{\alpha \in I} U_\alpha$.

For each point $x \in X$ there is a $U_{\alpha_x} \in \mathcal{U}$ such that $x \in U_{\alpha_x}$. Since \mathcal{B} is a basis there is a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_{\alpha_x}$.

Now, clearly $X \subseteq \bigcup_{x \in X} B_x$ (and clearly there are only countably many such B_x), but, who cares? We wanted a subcollection of \mathcal{U} that covers X . So for each B_x , choose one U'_{α_x} such that $B_x \subseteq U'_{\alpha_x}$. This (countable) collection of U'_{α_x} is exactly the countable collection that we wanted. \square

Application

[A.1] Suppose that (L, \leq) is a linear order. Prove that L , with the order topology, has no non-trivial clopen subsets if and only if L is Dedekind complete and has no gaps.

Proof. The strategy here is to prove $\neg \Leftarrow \neg$ by showing that a gap gives a non-trivial clopen set, and any set that witnesses the failure of Dedekind completeness is a non-trivial clopen set. Then we show $\neg \Rightarrow \neg$ by showing that if L has a non-trivial clopen set and it is Dedekind complete, then it has a gap. //

$[\neg \Leftarrow \neg]$ Suppose that L has a gap given by $x < y$. Clearly, $(-\infty, y) = (-\infty, x]$ is a non-trivial clopen set.

So now suppose instead that L is not Dedekind complete. That means that there is a non-empty set $S \subseteq L$ that is bounded above by M , but has no least upper bound. Let

$$A := \{b \in L : s < b, \forall s \in S\}$$

which is the set of all upper bounds for S . This is non-empty (since $M \in A$) and it is not all of L since S is non-empty. To show that A is open, note that if $b \in A$ then b is *not* a least upper bound for S , thus there is a $b' \in A$ such that $b' < b$. Clearly $b \in (b', \infty) \subseteq A$, so A is open. To show that $L \setminus A$ is open, note that if $x \in L \setminus A$ then it is not an upper bound for S . Thus there is an $s \in S \subseteq L \setminus A$ such that $x < s$. Thus $x \in (-\infty, s) \subseteq L \setminus A$, so $L \setminus A$ is open. Together this shows that A is a non-trivial clopen set.

$[\neg \Rightarrow \neg]$ Let $C \subseteq L$ be a non-trivial clopen set and suppose that L is Dedekind complete. Since C is non-trivial, let $M \in L \setminus C$ and, without loss of generality, assume that there is a point $c \in C$ such that $c < M$. Now let $x = \sup\{a \in C : a < M\}$, which exists since L is Dedekind complete. Since C is closed, $x \in C$. Since C is open, there is an open set $(a, y) \subseteq C$ such that $x \in (a, y)$. Here we see that $(a, y) = (a, x]$, which clearly tells us that x, y form a gap. \square

[A.2] Prove that $\omega_1 + 1$ is a Lindelöf space. (Conclude that Lindelöf does not imply first countable or second countable.) If you are hungry, then prove that $\omega_1 + 1$ is actually *compact*, (that is, every open cover has a *finite* subcover).

Proof. Let \mathcal{U} be an open cover of $\omega_1 + 1$. There is an open set $U_\Omega \in \mathcal{U}$ such that $\Omega \in U_\Omega$, where Ω is the largest element in $\omega_1 + 1$. Since U_Ω is open it contains a basic open set of the form $(\alpha, \Omega]$, where $\alpha \in \omega_1$. Now to show that \mathcal{U} has a countable subcover it suffices to observe that $[0, \alpha]$ is a countable set, and for each $x \in [0, \alpha]$ we can pick an $U_x \in \mathcal{U}$ such that $x \in U_x$. Thus $\{U_x : x \in [0, \alpha] \cup \{\Omega\}\}$ is the desired countable subcover.

To show compactness, we start of the same way by choosing a U_Ω that contains Ω and covers $(\alpha_1, \Omega]$. Now we cover α_1 by a set $U_{\alpha_1} \in \mathcal{U}$. If U_{α_1} contains an interval of the form $(\alpha_2, \alpha_1]$ then repeat this process with α_2 . If U_{α_1} does not contain such an interval, then $\{\alpha_1\}$ has an immediate predecessor, α_2 . So repeat this process with α_2 .

Either way we see that $\alpha_1 > \alpha_2 > \alpha_3 > \dots$ and since $\omega_1 + 1$ is a well order, this descending chain must be finite. The corresponding U_{α_i} (together with U_Ω) will produce a finite subcover. \square

[A.3] Prove that every regular Lindelöf space is normal.

Proof. This proof is identical to the proof that regular second countable spaces are normal (in §9), except that we produce a countable cover of the closed sets using the Lindelöf property instead of second countability.

Let A, B be closed disjoint sets in X , a regular Lindelöf space. For each point $a \in A$ fix an open set U_a such that $a \in U_a$ and $\overline{U_a} \cap B = \emptyset$, which we can do since X is regular. Notice that $A \subseteq \bigcup_{a \in A} U_a$, and so $X = (X \setminus A) \cup \bigcup_{a \in A} U_a$. Since X is a Lindelöf space there is a countable set $\{U_n : n \in \mathbb{N}\}$ such that

$$A \subseteq \bigcup_{a \in A} U_a = \bigcup_{n \in \mathbb{N}} U_n$$

(Notice that $X \setminus A$ is disjoint from A , so once we have a countable cover of X , if that included the set $X \setminus A$ then we throw it away and still have a cover of A .) Similarly, find $\{V_n : n \in \mathbb{N}\}$ such that

$$B \subseteq \bigcup_{n \in \mathbb{N}} V_n$$

with the additional requirement that $\overline{V_n} \cap A = \emptyset$.

From here, repeat the proof in the notes line by line to ensure that $\bigcup_{n \in \mathbb{N}} U_n$ and $\bigcup_{n \in \mathbb{N}} V_n$ are disjoint. \square

New Ideas

[NI.1] Prove that every continuous function $f : \omega_1 \rightarrow \mathbb{R}$ has a countable range (with both spaces given their order topologies). Conclude that every such function is eventually constant. (There are many different ways to prove this!)

Here is Ivan's proof:

Definition 1. Let (X, \mathcal{T}) be an uncountable topological space, and $S \subseteq X$ an uncountable subset. A point $x \in X$ is called a condensation point of S if every open neighbourhood of x contains uncountably many points of S . That is, for every U such that $x \in U \in \mathcal{T}$, $S \cap U$ is uncountable.

Lemma 2. Let $S \subseteq \mathbb{R}$ be uncountable. Then there exists $s \in S$ such that s is a condensation point of S .

Proof. We work in the subspace topology on S . Second countability is hereditary, so fix a countable basis $\mathcal{B} = \{ B_n \mid n \in \mathbb{N} \}$ of this topology on S .

Assume for a contradiction that S contains no condensation points of S . This means that for every $s \in S$, there exists an open neighbourhood U containing s such that U is countable. Since \mathcal{B} is a basis, there is an $n(s) \in \mathbb{N}$ such that $s \in B_{n(s)} \subseteq U$.

But then the collection $\{ B_{n(s)} \mid s \in S \} \subseteq \mathcal{B}$ is a countable collection of countable sets covering S , contradicting the fact that S is uncountable. \square

Corollary 3. *Let $S \subseteq \mathbb{R}$ be uncountable. Then S contains infinitely many condensation points of S*

Proof. This is an easy induction using the lemma. Let $s \in S$ be a condensation point of S . Then $S \setminus \{s\}$ is uncountable, so apply the lemma again. Continue in this way. \square

Proposition 4. *Let $f : \omega_1 \rightarrow \mathbb{R}$ be continuous. Then the image of f is countable.*

Proof. Assume for a contradiction that the image of f , which we'll call S , is uncountable. Then by the corollary there exist two condensation points of S in S , which we'll denote s_1 and s_2 . Using the regularity of S , find two open sets U_1 and U_2 (in the subspace topology on S) containing s_1 and s_2 respectively, such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Since the s_i are condensation points of S , the sets U_i (and in particular the sets $\overline{U_i}$) are uncountable.

Then, since f is continuous, $f^{-1}(\overline{U_1})$ and $f^{-1}(\overline{U_2})$ are two disjoint closed subsets of ω_1 , which are uncountable and therefore unbounded. But this is impossible, since any two closed and unbounded subsets of ω_1 must intersect, by Problem C.1 on Assignment 5. \square

[NI.2] Give a clear description of the so called “long-line”. Prove that it is “locally Euclidean” and list the topological properties that it has (of the 10 or so that we have seen so far), and prove a couple of them. What are some interesting modifications you can make to the construction?

I won't post “solutions” for these two questions since the point was to investigate the question on your own, not read someone else's solution.

[NI.3] Many students like to claim that \mathbb{R} is in bijection with ω_1 . This assertion is called the “Continuum Hypothesis” (CH). Read the wikipedia article for the Continuum Hypothesis and at least one other article about CH. (I might try Gödel's 1947 “What is Cantor's Continuum Problem?” or Juliette Kennedy's 2011 “Can the Continuum Hypothesis be Solved?”.)

- In your own words, state the Continuum Hypothesis in language that a student in this course would understand.
- In your own words, state the Continuum Hypothesis in language that a general math student would understand.

- State a question in mathematics that is answerable if we assume that CH is true.
- State a question in mathematics that is answerable if we assume that CH is false.
- Who were the important people in proving that CH is independent from the usual axioms of set theory; what were their contributions?
- Ask a working mathematician how they feel about the Continuum Hypothesis; do they “think it’s true”?