We'll skip section : 2.7 "numerical methods")

2nd order linear equation.

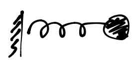
A second order equation $F(t, y, y', y'') = 0$ is called linear if it is of the form

$y'' + p(t)y' + g(t)y = g(t)$ (*)

(or, sometimes $P(t)y'' + Q(t)y' + R(t)y = C(t)$)

Example: Newton's equation $m\frac{d^2x}{dt^2} = F(x, t)$ is a second order ODE

Mass-spring system.



$$m\frac{d^2x}{dt^2} + \underbrace{\gamma\frac{dx}{dt}}_{\text{friction}} + \underbrace{kx}_{\text{spring force}} = \overbrace{F}^{\text{extended force}}$$

"Physically", it's "clear" that (*) should have a unique solution for given initial conditions.
$y(t_0), y'(t_0)$
(We'll get back to this).

The equation (*) is linear homogeneous. if $g(t) = 0$:

$$y'' + p(t)y' + g(t)y = 0$$

(Has nothing to do with homogeneous 1st ODE $y' = f(\frac{y}{t})$.)

We'll start by looking at linear homogeneous equation with constant coefficients:

$$ay'' + by' + cy = 0 \quad (**)$$

(a, b, c are constant).

Possibly with initial condition $y(t_0), y'(t_0)$ . .

$y'' - y = 0$ $\quad y(t) = e^t$
$y(t) = 0$
$y(t) = \cosh(t) = \frac{e^t + e^{-t}}{2}$
$y(t) = e^{-t}$
. . . .

Note: If $y_1(t), y_2(t)$ are solution, then $y_1(t) + y_2(t)$ is a solution.

$$(y_1 + y_2)'' - (y_1 + y_2) = y_1'' + y_2'' - y_1 - y_2 = (y_1'' - y_1) + (y_2'' - y_2) = 0$$

Note: If y_1, y_2 are solutions of $(**)$ and A_1, A_2 are constants, then $y(t)=A_1 y_1(t)+A_2 y_2(t)$ is again a solution.

$ay''+by'+cy=a(A_1 y_1''+A_2 y_2'')+b(A_1 y_1'+A_2 y_2')+\cdots$

General Solution : $y(t)=A_1 e^{t}+A_2 e^{-t}$ $\Rightarrow$ for $y''-y=0$

Back to general eqn $(**)$

Let's try $y(t)=e^{rt}$ (name $r\in\mathbb{R}$).

$0=ay''+by'+cy=ar^2e^{rt}+bre^{rt}+ce^{rt}=e^{rt}(ar^2+br+c)$

This works if $ar^2+br+c=0$.

The equation $\boxed{ar^2+br+c=0 \quad (***)}$ is called the characteristic equation of $(**)$

Solutions: $r_1, r_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

If $b^2-4ac>0$, there are two solutions.

Letting r_1, r_2 be the two roots of char. equ $(***)$, get two solutions $e^{r_1 t}, e^{r_2 t}$ of $(**)$.

General solution:

$$y(t)=A_1 e^{r_1 t}+A_2 e^{r_2 t}$$

Example: $y''+5y'+6y=0$

$$y(0)=2,\ y'(0)=3$$

Solution: Char. eqn: $r^2+5r+6=0$

$$r_1=2,\ r_2=3$$

General solution: $y(t)=A_1 e^{-3t}+A_2 e^{-2t}$

To solve the IVP, have to choose A_1, A_2 s.t.

$$y(0)=A_1+A_2 \overset{!}{=} 2$$

$$y'(0)=-3A_1-2A_2 \overset{!}{=} 3$$

$$A_1=-7,\ A_2=9$$

Thus $y(t)=-7e^{-3t}+9e^{-2t}$

Example: $4y''-y=0$, $y(-2)=1$, $y'(-2)=-1$

Solution: $4r^2-1=0$, $r=\pm\frac{1}{2}$

Gen Solution: $y(t)=A_1 e^{t/2}+A_2 e^{-t/2}$

$y(-2)=A_1 e^{-1}+A_2 e \overset{!}{=} 1$

$$y'(-2) = \frac{1}{2}A_1 e^{-1} - \frac{1}{2}A_2 e^{1} \overset{!}{=} -1$$

$$A_1 = -\frac{1}{2}e$$

$$A_2 = -\frac{3}{2}e^{-1}$$

So $\underline{y(t) = -\frac{1}{2}e^{t/2+1} + -\frac{3}{2}e^{-(t/2+1)}}$