Symbolic Logic

1. Consider the following statements:

   (S1) Programs that passed test 1 also passed test 2.
   (S2) Programs passed test 2 unless they failed test 1.
   (S3) Programs passed test 2 only if they passed test 1.

   (a) Rewrite statements (S1), (S2), and (S3) using precise symbolic notation.
   (b) Which of the three statement have the same meaning?

2. Consider the statement:

   (S4) All Java programs passed test 1.

   (a) Rewrite (S4) using implication but no quantification.
   (b) Rewrite (S4) using precise symbolic notation.
   (c) Write the contrapositive of (S4), symbolically and in English.
   (d) Write the converse of (S4), symbolically and in English.

The next three questions are based on the following database.

program	language	test 1	test 2
1	Java	fail	pass
2	Java	fail	pass
3	C	pass	fail
4	Java	pass	fail
5	C	fail	fail
6	C	pass	pass

3. Draw a Venn diagram with sets to represent "programs written in C", "programs that passed test 1", and "programs that passed test 2" (make sure that your sets overlap to divide the diagram into eight regions). Then, for each program, write the program number in the appropriate region of your diagram (based on the information in the database above).

   Solution:
   Let P denote the set of all programs.
   Let C denote the set of programs written in C.
   Let T_1 denote the set of programs that passed test 1.
   Let T_2 denote the set of programs that passed test 2.
   The diagram is given in the figure below (or at the top of the next page).

4. Draw three copies of your diagram from the preceding question. On the first copy, shade the region(s) that corresponds to "programs that have passed every test". On the second copy, shade the region(s) that corresponds to "programs that have passed some text". On the third copy, share the region(s) that corresponds to "programs that have passed no test".

5. State whether each statement below is true or false. When appropriate, justify your answer by citing a specific counter-example.

   (S5) Every Java program passed some test.
   (S6) Some Java program passed no test.
   (S7) No C program passed every test.

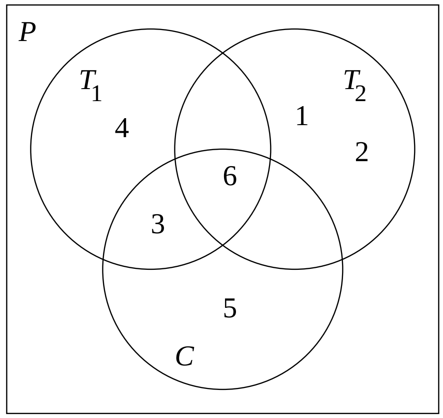

Figure 1: Venn diagram for Question 3.

6. Let P denote the sentence "x is even". Let Q denote the sentence "x^2 is even". Write the sentence $P \to Q$ in English:
   (a) Using the words "if, then".
   (b) Using the word "implies".
   (c) Using the words "only if".
   (d) Using the words "is necessary for".
   (e) Using the words "is sufficient for".

7. Consider the following sentence about integers a, b, c:

   If a divides bc, then a divides b or a divides c.

   For each sentence below, state whether it is the same as, the negation of, the converse of, the contrapositive of, or unrelated to the statement above. Justify each of your answers briefly (*e.g.*, by writing both statements in symbolic notation).
   (a) If a divides b or a divides c, then a divides bc.
   (b) If a does not divide b or a does not divide c, then a does not divide bc.
   (c) a divides bc and a does not divide b and a does not divide c.
   (d) If a does not divide b and a does not divide c, then a does not divide bc.
   (e) a does not divide bc or a divides b or a divides c.
   (f) If a divides bc and a does not divide c, then a divides b.
   (g) If a divides bc or a does not divide b, then a divides c.

8. For each quantified statement below, rewrite the statement in English and state whether it is true or false. When appropriate, justify your answer with an example or counter-example.
   (a) $\exists m \in \mathbb{N}\, \forall n \in \mathbb{N}, m > n$
   (b) $\forall m \in \mathbb{N}\, \exists n \in \mathbb{N}, m > n$
   (c) $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}, m > n$

9. In calculus, a function f with domain $\mathbb{R}$ (the real numbers) is defined to be *strictly increasing* provided that for all real numbers x and y, $f(x) < f(y)$ whenever $x < y$. Complete each of the following sentences using the appropriate symbolic notation.

(a) A function f with domain $\mathbb{R}$ is strictly increasing provided that ...

(b) A function f with domain $\mathbb{R}$ is not strictly increasing provided that ...

10. Find three sets A, B, C with as few elements as possible so that statement $(S1)$ below is true but statement $(S2)$ is false.

$$(S1) \quad \forall x \in A \, \exists y \in B, x + y \in C$$

$$(S2) \quad \exists y \in B \, \forall x \in A, x + y \in C$$

11. At a murder trial, four witnesses give the following testimony.

Alice: If either Bob or Carol is innocent, then so am I.

Bob: Alice is guilty, and either Carol or Dan is guilty.

Carol: If Bob is innocent, then Dan is guilty.

Dan: If Bob is guilty, then Carol is innocent; however, Bob is innocent.

(a) Is the testimony consistent, *i.e.*, is it possible that everyone is telling the truth?

(b) If every innocent (not guilty) person tells the truth and every guilty person lies, determine (if possible) who is guilty and who is innocent.

12. Consider the following statement: (1)

"If a program has a syntax error, then the program will not compile."

(a) Define the domain and predicates necessary to translate the statement into precise symbolic notation.

(b) Translate (1) into precise symbolic notation.

(c) Give the converse of (1) first in English, then in precise symbolic notation.

(d) Give the contrapositive of (1) first in English, then in precise symbolic notation.

(e) Give the contrapositive of your answer to 18c in precise symbolic notation.

13. Assume you are given the following predicate symbols and your domain is $\mathbb{N}$, the set of natural numbers (we assume that $0 \in \mathbb{N}$).

$g(x, y)$: x *is greater than* y

$e(x, y)$: x *equals* y

$sum(x, y, z)$: $x + y = z$

$prod(x, y, z)$: $x \cdot y = z$

Translate the following statements into *idiomatic* English:

(a) $\forall x \in \mathbb{N}, g(x, 0)$

(b) $\forall x \in \mathbb{N}, \exists z \in \mathbb{N}, prod(x, 2, z)$

(c) $\forall a \in \mathbb{N}, \forall b \in \mathbb{N}, \forall c \in \mathbb{N}, (g(x, y) \land g(y, z)) \to g(x, z)$

(d) $\neg(\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, g(m, n))$

Translate the following English statements into precise symbolic notation. Only use the predicates and domain defined above. Make sure you specify the domain of your variables in your solution and that your predicates are boolean.

(e) Every positive multiple of 5 is greater than 7.

(f) There is a smallest odd integer.

(g) If $x + y = z$ then $y + x = z$.

(h) Not all integers are multiples of 2.

14. Logic Puzzle: There are many brain teasers involving deserted islands and the people who inhabit them. One such puzzle, involves an island consisting of two different races. The Truth Tellers and the Liars. The Truth Tellers always tell the truth and the Liars, falsehoods. Suppose you meet three people U, V and W from this island. The first person U does not speak your language however V offers to translate. For each case, determine (if possible) from their responses to the following question, which race they each belong. If it is not possible, clearly show why it is not possible to determine which race at least one of U, V or W belong to.

How many of you are Truth Tellers?

Responses:

(a) V: "U said, '*Exactly one of us is a Truth Teller*.'"
W: "Don't believe V. He is a Liar".

(b) V: "U said, '*Exactly one of us is a Liar*.'"
W: "V's statement is true."

15. Consider our example from class about rainy days.

(2) *Every rainy day I bring an umbrella.*

(3) *If I bring an umbrella, then I stay dry.*

(a) Define predicates and a domain. Write statements (2) and (3) in precise symbolic notation.

(b) For each of the following statements, determine whether it has the same meaning as (2). If it has a different meaning, make a small alteration to the statement so that it has the same meaning.

i. I bring an umbrella, if it is a rainy day.
ii. If it is a rainy day, I bring an umbrella.
iii. I bring an umbrella only if it is a rainy day.
iv. A rainy day is sufficient for me to bring an umbrella.
v. A rainy day is necessary for me to bring an umbrella.

(c) Assume that it is a rainy day. What conclusions can you draw given statements (2) and (3). Explain your reasoning.

(d) Now assume that I forgot my umbrella. What conclusions can you draw? Explain your reasoning.

16. Recall the table of hockey stats from class.

Number	Pos.	Player	Team	GP	G	A	PTS
1	C	Alexei Zhamnov	PHI	11	4	8	12
2	RW	Jarome Iginla	CAL	13	6	6	12
3	C	Joe Sakic	COL	11	7	5	12
4	C	Vincent Damphousse	SJ	11	5	6	11
5	RW	Martin St. Louis	TB	9	4	7	11
6	LW	Fredrik Modin	TB	9	5	6	11
7	C	Saku Koivu	MON	11	3	8	11
8	C	Peter Forsberg	COL	11	4	7	11
9	RW	Alexei Kovalev	MON	11	6	4	10

(a) Draw a Venn diagram with sets that show "players with 5 or more goals", "players who have played at least 11 games", "players who have more points than games played". Using the information in the table, enter each player's number into the appropriate region of the diagram.

(b) Make a copy of the diagram in (16a)and shade in the region that corresponds to the statement "All players who have played less than 11 games yet scored more points than games played".

(c) Make another copy of the diagram in (16a) and shade in the region that corresponds to the statement "Every player that has more points than games played and has scored at least 5 goals".

17. The following was heard on TV recently:

"Product X is so good, it won an award!"

The makers of product X want you to believe certain (possibly implicit) hypothesis/hypotheses and conclusion(s).

(a) Write these down explicitly.

(b) Formalize them with appropriate domains and predicates.

18. There is a set P, of problems that are "polynomial time solvable" and a set NP, of problems that are only "non-deterministically polynomial time solvable". Consider the following statement:

(S1) *"If $P = NP$, then the problem SAT is polynomial time solvable."*

(a) Define the domain and predicates necessary to translate the statement into precise symbolic notation.
(b) Translate (S1) into precise symbolic notation.
(c) Give the converse of (S1) first in English, then in precise symbolic notation.
(d) Give the contrapositive of (S1) first in English, then in precise symbolic notation.
(e) Give the contrapositive of your answer to 18c in precise symbolic notation.

Now consider the same sentence expressed using quantification:

(S2) *If every problem is polynomial time solvable if and only if it is non-deterministically polynomial time solvable, then the problem SAT is polynomial time solvable."*

(f) Define the domain and predicates necessary to translate the statement into precise symbolic notation.

(g) Translate (S2) into precise symbolic notation. Simplify your answer such that *only predicates* are negated (not entire sentences).

(h) Give the converse of (S2) first in English, then in precise symbolic notation. Simplify your answer such that *only predicates* are negated.

(i) Give the contrapositive of (S2) first in English, then in precise symbolic notation. Simplify your answer such that *only predicates* are negated.

(j) Give the contrapositive of your answer to 18h in precise symbolic notation. Simplify your answer such that *only predicates* are negated.

19. Consider the following database D of programs that test inputs. A program in this database may return a certificate when the input to the algorithm is accepted, rejected or both, and may or may not be linear in running time.

Program	Certificate	Linear
1	reject	yes
2	accept	no
3	accept	yes
4	reject	no
5	both	no
6	neither	yes

(a) Draw a Venn diagram with a region for the programs that return a certificate when the input is accepted, a region for the programs that return a certificate when the input is rejected and a region for the programs that are linear. Add programs 1 to 6 into your diagram.

(b) Define the following predicates:

$A(x)$ represents *"program x returns a certificate if the input is accepted."*

$R(x)$ represents *"program x returns a certificate if the input is rejected."*

$L(x)$ represents *"program x runs in linear time."*

For each statement redraw your Venn diagram from part (a) and shade in the region which makes the statement **false**.

(S1) $A(x) \rightarrow L(x)$

(S2) $\neg L(x) \rightarrow A(x)$

(c) Are the following statements true? if yes, explain why, if not, give a counterexample.

(S3) $\forall x \in D, L(x) \rightarrow \neg(A(x) \wedge R(x))$

(S4) Some program is linear and returns a certificate on rejected input.

(S5) $\forall x \in D, (A(x) \wedge R(x)) \rightarrow \neg L(x)$.

(d) Rewrite (S4) using "*every*" instead of "*some*" such that the meaning remains the same.

20. Now that we can write statements precisely, we can draw logical conclusions from a set of statements and *prove* that the conclusion is a consequence of the statements. These questions will help prepare you for learning to write proofs. For each set of statements, define a domain and set of predicates. Rewrite the statements and conclusions in precise symbolic notation. Assuming that the statements are true, determine which one of the possible conclusions can be drawn from the statements. Justify your choice of conclusion by explaining how the two statements imply the conclusion.

(a) (S1) "*All politicians are powerful people.*"

(S2) "*No powerful people are easily forgotten.*"

Possible conclusions:

i. *People who are easily forgotten are politicians.*

ii. *Politicians are not easily forgotten.*

iii. *No powerful people are politicians.*

iv. *Some easily forgotten people are politicians.*

v. *All politicians are easily forgotten.*

(b) (S3) "*If Bart gets in trouble then either Homer or Milhouse get in trouble.*"
(S4) "*Homer does not get in trouble.*
Possible conclusions:

i. *If Milhouse gets in trouble then Bart gets in trouble.*

ii. *If Homer does not get in trouble then Milhouse does not get in trouble.*

iii. *Milhouse gets in trouble if Bart gets in trouble.*

iv. *Bart gets in trouble whenever Homer gets in trouble.*

v. *Either Bart gets in trouble or Milhouse gets in trouble.*

21. Assume you are given the following predicate symbols and your domain is $\mathbb{N}$, the set of natural numbers (we assume that $0 \in \mathbb{N}$).

$g(x, y)$: x *is greater than* y
$e(x, y)$: x *equals* y
$sum(x, y, z)$: $x + y = z$
$prod(x, y, z)$: $x \cdot y = z$

Each of the following statements is a mathematical property of the natural numbers. Translate the following statements into English and state the property.

(a) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, prod(x, y, x)$.

(b) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, sum(x, y, x)$.

(c) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, (sum(x, y, z) \leftrightarrow sum(y, x, z))$

(d) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, g(x, y) \wedge g(y, z) \rightarrow g(x, z)$

(e) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, \forall t \in \mathbb{N}, [sum(x, y, z) \wedge prod(z, t, w)$
$\leftrightarrow (\exists u \in \mathbb{N}, \exists v \in \mathbb{N}, prod(x, t, u) \wedge prod(y, t, v) \wedge sum(u, v, w))]$

Now consider the following statements. Using the above predicates rewrite each statement in precise symbolic notation.

(f) x divides m. Recall that *divides* means that $xy = m$ for some number y.

(g) m is the smallest number that x divides.

(h) $LCM(x, y, m)$: m is the smallest number that both x and y divide. [Hint: Do you need to quantify the variables x, y and m?]

(i) $GCD(a, b, c)$: a is the greatest common divisor of b and c.

22. Consider the following sentence:

    (S1) *If $m = 2^n - 1$ is a prime number, then n is prime.*

    [Note: these types of prime numbers are called *Mersenne Primes*.]

    Rewrite (S1) without using "If . . ., then . . ." but using:

    (a) "implies"

    (b) "is sufficient for"

    (c) "is necessary for"

    (d) "whenever"

    (e) "only if"

    (f) "requires"

23. Determine whether $\exists$ can be factored from an implication. In other words, is

$$\exists x \in X, (p(x) \to q(x)) \quad \Leftrightarrow \quad (\exists x \in X, p(x)) \to (\exists x \in X, q(x))$$

    true? Explain your reasoning. Marks will only be given for your *explanation*.

24. For each set of sentences, define the domain X, the value of $a \in X$ (for part b), and the predicates $A(x)$ and $B(x)$ such that the last sentence is false and the other sentences are true.

    (a)

$$\begin{array}{ll} (T) & \forall x \in X, A(x) \to B(x) \\ (F) & \exists x \in X, A(x) \wedge B(x) \end{array}$$

    (b)

$$\begin{array}{ll} (T) & \forall x \in X, A(x) \to B(x) \\ (T) & \neg A(a) \\ (F) & \neg B(a) \end{array}$$

25. Let $p(n)$ and $q(n)$ represent the following predicates:

$$p(n) : n \text{ is odd} \quad q(n) : n^2 \text{ is odd}$$

    where the domain is the set of integers. Determine which of the following statements are logically equivalent to each other.

(a) If the square of any integer is odd, then the integer is odd.

(b) $\forall n \in \mathbb{Z}, (p(n)$ is necessary for $q(n))$.

(c) The square of any odd integer is odd.

(d) There are some integers whose squares are odd.

(e) Given any integer whose square is odd, that integer is likewise odd.

(f) $\forall n \in \mathbb{Z}, \neg p(n) \rightarrow \neg q(n)$.

(g) Every integer with an odd square is odd.

(h) Every integer with an even square is even.

(i) $\forall n \in \mathbb{Z}, p(n)$ is sufficient for $q(n)$.

26. (a) Determine whether $\exists$ can be factored from an implication. In other words, is

$$\exists x \in X, p((x) \rightarrow q(x)) \quad \Leftrightarrow \quad (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))$$

true?

(b) Consider

$$\forall y \in D, \exists x \in D, p(x, y) \rightarrow \exists x \in D, \forall y \in D, p(x, y).$$

i. Define D and $p(x, y)$ such that the statement is true.

ii. Define D and $p(x, y)$ such that the statement is false.

(c) Consider the following statement:

$$(\forall x \in X, p(x) \rightarrow \neg\forall x \in X, q(x)) \leftrightarrow (\exists x \in X, \forall y \in X(p(x) \rightarrow \neg q(y)))$$

i. Define D, $p(x)$ and $q(y)$ such that the statement is true.

ii. Define D, $p(x)$ and $q(y)$ such that the statement is false.

HINT: You may need to use the logical equivalence laws defined in class to simplify the statement first. If so, ***show and justify each step***.

27. Consider the statement

(1) *A number is rational if it can be written as $\frac{a}{b}$ where a and b are integers.*

(a) Define a domain and a set of predicates and rewrite the statement in precise symbolic notation.

(b) Rewrite (1) using

- "sufficient"
- "only if"
- "is necessary"
- conjunction *instead* of implication.

(c) Write the converse and the contrapositive of the converse of (1). Is the converse true? if so, how can we alter (1) to reflect this. If not, give a counter example.