MAT 337
Sample Midterm Exam

NAME

NO AIDS ALLOWED

Total: 250 points, not including a bonus problem

Problem 1 [20 points]

Determine which of the following sequences converge.

(a) $(a_n)_{n=1}^{\infty}$, where $a_n = \frac{2^{n^2}}{n!}$.

(b) $(b_n)_{n=1}^{\infty}$, where $b_1 = \sqrt{2}; b_{n+1} = \sqrt{2\sqrt{b_n}}$.

Explain.

Problem 2 [30 points]

(a) Suppose that S is bounded above and that $S_o \subset S$. Prove that $\sup S_0 \leq \sup S$.

(b) Suppose that S is bounded above and that α is any real number. Prove that $\sup_{x \in S}(x + \alpha) = \sup_{x \in S} x + \alpha$.

Problem 3 [30 points] Decide which of the following series converge absolutely, conditionally, or not at all.

(a) $\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n \log(n+1)}$

(b) $\Sigma_{n=1}^{\infty} \frac{(-1)^n \sin \frac{1}{n}}{n}$

Problem 4 [60 points] Let $S_0 = [0,1]$. Construct S_{i+1} from S_i by removing an open middle interval from each interval in S_i. That is $S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$; $S_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc.

(a) Is $C = \cap_{i\geq 1} S_i$ a perfect set?

(b) Prove that Lebesgue measure of $C = \cap_{i\geq 1} S_i$ is zero;

(c) Let $x = \Sigma_{i=1}^{\infty} \frac{1}{3^i}$. Is x an element in C?

(d) Prove that C is nowhere dense.

Problem 5 [40 points]

Let X be a metric space with metric d. Define $\bar{d} : X \times X \longrightarrow \mathbb{R}$ by the formula

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

Show that $\bar{d}$ is a metric on X and that every subset of X is bounded.

Problem 6 [40 points] Prove that every Cauchy sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, ρ) is bounded.

Problem 7 [30 points] Which of the following subsets of $\mathbb{R} \times \mathbb{R}$ are complete:

(a) The set of points (x, y), where both x and y are rational.

(b) A unit metric ball centered at the origin $B_r(0) = \{(x, y) | x^2 + y^2 < r\}$.

(c) $\overline{B_r(0)}$.