University of Toronto
Faculty of Arts and Science

MAT224H1F
Linear Algebra II

Final Examination
December 2011

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Duration: 3 hours

PLEASE HAND IN

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

[10] **1.** Let W be the one dimensional subspace of $\mathbb{C}^3$ spanned by $w = (1 - i, 1 + i, 1 + 2i)$. Find the matrix of the orthogonal projection onto $W^{\perp}$ with respect to the standard basis of $\mathbb{C}^3$.

EXTRA PAGE FOR QUESTION 1 - please do not remove.

[10] **2.** Consider the vector spaces $P_1(\mathbb{R})$, and $P_2(\mathbb{R})$ both with inner product

$$< p(x), q(x) >= \int_0^1 p(x)q(x)\,dx.$$

Let T: $P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be defined by $T(p(x)) = p'(x)$. Find $T^*(p(x))$ for an arbitrary $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$.

EXTRA PAGE FOR QUESTION 2 - please do not remove.

[10] **3.** Let $A = \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}$. Find the spectral decomposition of A.

EXTRA PAGE FOR QUESTION 3 - please do not remove.

[10] **4.** Let $N\colon \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$N = \begin{bmatrix} 6 & 2 & 1 & -1 \\ -7 & -1 & -1 & 2 \\ -9 & -7 & -2 & -1 \\ 13 & 3 & 2 & -3 \end{bmatrix}$$

(a) Show that N is nilpotent and find the smallest k such that $N^k = 0$.

(b) Find the canonical form of N and a canonical basis.

EXTRA PAGE FOR QUESTION 4 - please do not remove.

[10] **5.** Let $T\colon P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear operator defined by $T(p(x)) = p''(x) + p(x)$. Find a basis α for $P_3(\mathbb{R})$ such that $[T]_{\alpha\alpha}$ is in Jordan canonical form.

EXTRA PAGE FOR QUESTION 5 - please do not remove.

[10] **6.** Let V and W be finite dimensional inner product spaces over $\mathbb{R}$ and $T\colon V \to W$ a linear transformation. Let $x_1, x_2, \ldots, x_k$ be vectors in V such that $\{T(x_1), T(x_2), \ldots, T(x_k)\}$ is a basis for the image of T.

(a) Prove that $\{T^*(T(x_1)), T^*(T(x_2)), \ldots, T^*T(x_k))\}$ is a linearly independent subset of the image of T^* and explain why this implies that $rank(T) \leq rank(T^*)$.

(b) Since part (a) holds for all linear transformations from V into W, explain why this implies that $rank(T) = rank(T^*)$.