Tutorial Problems - Sections 8 to 10 - MAT 327 - Summer 2014

8 Finite Products

1. Let $\mathcal{T} := \{U \in \mathcal{P}(\mathbb{R}) : 0 \in U\} \cup \{\emptyset\}$, and let $\mathcal{U} := \{U \in \mathcal{P}(\mathbb{R}) : 1 \in U\} \cup \{\emptyset\}$. Describe the product topology on $\mathbb{R} \times \mathbb{R}$ determined by $\mathcal{T}$ and $\mathcal{U}$.

2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $a \in X$, and let $b \in Y$. Prove that the functions $f : X \longrightarrow X \times Y$ and $g : Y \longrightarrow X \times Y$ defined by $f(x) = (x, b)$ and $g(y) = (a, y)$ are embeddings. (An embedding is an injective, continuous, open map.)

3. Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{U}$ denote the product topology on $X \times X$, let $\Delta := \{(x, x) : x \in X\}$, and let $\mathcal{U}_\Delta$ be the subspace topology on Δ determined by $\mathcal{U}$. Prove that $(X, \mathcal{T})$ is homeomorphic to $(\Delta, \mathcal{U}_\Delta)$. (The set Δ is called the **diagonal**.)

4. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and suppose $X_1 \times X_2$ has the product topology. For $i = 1, 2$, let A_i be a subset of X_i. Prove that $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

5. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and suppose $X_1 \times X_2$ has the product topology. For $i = 1, 2$, let A_i be a subset of X_i. Prove that

$$\operatorname{int}(A_1 \times A_2) = \operatorname{int}(A_1) \times \operatorname{int}(A_2).$$

9 Separation Axioms

Definition 1. *A space $(X, \mathcal{T})$ is T_0 if for all $x \neq y$, there is an open set such that $x \in U$ and $y \notin U$ or there is an open set V such that $y \in V$ and $x \notin V$.*

1. Let $(X, \mathcal{T})$ be a topological space. Prove that $(X, \mathcal{T})$ is a T_0 space if and only if for each pair a and b of distinct members of X, $\overline{\{a\}} \neq \overline{\{b\}}$.

2. Let X be a set and let $D \subseteq X$. Define a topology $\mathcal{T}$ on X by saying that a subset of X is closed whenever $C = C \cup D$ and a subset U of X belongs to $\mathcal{T}$ whenver $X \setminus U$ is closed. For each $i = 0, 1, 2$, under what conditions on D is $(X, \mathcal{T})$ a T_i space?

3. Let $(X, \mathcal{T})$ be a T_1-space, let $(Y, \mathcal{U})$ be a topological space, and let f be a closed map of X onto Y. Prove that $(Y, \mathcal{U})$ is a T_1 space.

10 Partial Orders

1. Prove (directly) that ω_1 is not ccc.

2. The previous exercise might get you mistakenly thinking that somehow ω_1 is discrete; this is very much not true! Show that ω_1 has infinitely many points that are not open (hint: look at what we did for $\omega + 1$). Then show that ω_1 has uncountably many points which are not open. (Is this true for $\mathbb{R}_{\text{usual}}$?)

3. Prove that $A \subseteq \omega_1$ is bounded iff it is countable. Conclude that $C \subseteq \omega_1$ is uncountable iff it is unbounded. (Is this true for $\mathbb{R}$?)

4. Prove that $\{ x \in \omega_1 : \{ x \}$ is not open $\}$ is a closed subset of ω_1. Conclude from a previous exercise that it is a closed and unbounded set.

5. A subset $C \subseteq \omega_1$ is said to be a **club** subset of ω_1 if it is closed and unbounded. Note that final segments (i.e. everything to the *right* of a point α) are club. Prove that the intersection of countably many club sets is again club.