Assignment 3 - Solutions - MAT 327 - Summer 2014

Comprehension

[**C.1**] Prove $\mathrm{FIN}(\mathbb{N}) := \{ F \subseteq \mathbb{N} : F \text{ is finite} \}$ is a countable set.

Proof 1 of C.1. Note that

$$\mathrm{FIN}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} \mathcal{P}(\{ 0, 1, 2, \ldots, n \})$$

and each of those power sets is finite, so we have written $\mathrm{FIN}(\mathbb{N})$ as the countable union of finite sets, so it is countable. $\square$

Proof 2 of C.1. We first note that each $\mathbb{N}^n$ is a countable set, since it is the product of finitely many countable sets. There is a natural injection

$$f : \mathrm{FIN}(\mathbb{N}) \longrightarrow \{\emptyset\} \cup \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$$

taken by mapping $\{ x_1, x_2, \ldots, x_n \}$, written in increasing order, to $(x_1, x_2, \ldots, x_n)$ and mapping $\emptyset$ to $\emptyset$. Since the range of this injection is a countable union of countable sets, it is countable, thus the domain is also countable. $\square$

[**C.2**] Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces with $\overline{D} = X$ and $\overline{E} = Y$. Show that $D \times E$ is a dense subset of $X \times Y$, (taken, of course, with the product topology). Conclude that the product of two separable spaces is again separable.

Proof. Let A be a non-empty open set in $X \times Y$. It must contain a (non-empty) basic open set $U \times V$. Here U is open in X and V is open in Y. It suffices to show that $U \times V$ intersects $D \times E$. So there is a $d \in U \cap D$ and an $e \in V \cap E$. So then $(d, e) \in (U \times V) \cap (D \times E)$.
We conclude that if both D and E were countable (and dense), then $D \times E$ is countable and dense in $X \times Y$. $\square$

[**C.3**] Prove or disprove that the product of two Hausdorff Spaces is again a Hausdorff space.

Proof. This is true, which will show that the Hausdorff property is finitely productive. Note that there are many tedious ways to prove this fact, but it is really quite simple.
Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be Hausdorff spaces. Let $(x_1, x_2) \neq (y_1, y_2) \in X_1 \times X_2$. Since the points are distinct, they must be be different in at least one of their coordinates. Suppose that they disagree in coordinate 1 (the other case is analogous). Then $x_1 \neq y_1$. So there are disjoint open sets $U, V \in \mathcal{T}_1$ such that $x_1 \in U$ and $y_1 \in V$. So $(x_1, x_2) \in U \times X_2$ and $(y_1, y_2) \in V \times X_2$, and these are (disjoint) open sets in $X_1 \times X_2$. $\square$

[**C.4**] Suppose that $(X, \mathcal{T})$ and $(X, \mathcal{U})$ are topological spaces with $\mathcal{T} \subseteq \mathcal{U}$. Prove or disprove the following two statements:

1. If $(X, \mathcal{T})$ is a Hausdorff space, then so is $(X, \mathcal{U})$; and

2. If $(X, \mathcal{U})$ is a Hausdorff space, then so is $(X, \mathcal{T})$

Solution to (i). This is true. (The sound bite is "Adding more open sets to a topology makes it *easier* to be a Hausdorff space.") Suppose that $x \neq y$ in X, so there are $U, V \in \mathcal{T}$ that are disjoint sets such that $x \in U$ and $y \in V$. So $U, V \in \mathcal{U}$, and they are still disjoint, and still $x \in U$ and $x \in V$. $\square$

Solution to (ii). This is not always true. (The sound bite is "We may remove the witnesses to the Hausdorff property when we remove open sets.") For example, $\mathbb{R}_{\text{discrete}}$ is a Hausdorff space, but $\mathbb{R}_{\text{co-finite}}$ (or $\mathbb{R}_{\text{indiscrete}}$, if you like) is not a Hausdorff space. Clearly, $\mathbb{R}_{\text{indiscrete}} \subseteq \mathbb{R}_{\text{co-finite}} \subseteq \mathbb{R}_{\text{discrete}}$. $\square$

[**C.5**] Prove that every separable space satisfies the ccc.

Proof. We will prove the contrapositive, that "not ccc implies not separable". Suppose that $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ is an uncountable collection of mutually disjoint open sets (so here I is uncountable). Note that at most one of the U_α is the empty set, so without loss of generality, assume that none of the U_α is empty. (The "without loss of generality" is because we will still be left with uncountably many open sets).

Let D be a dense set, and we will show that D is not countable. For each $\alpha \in I$ choose an $x_\alpha \in U_\alpha \cap D$ (which can be done since each U_α is non-empty), and notice that $x_\alpha \neq x_\beta$ if $\alpha \neq \beta \in I$ (because U_α is disjoint from U_β). Thus $\{x_\alpha : \alpha \in I\}$ is uncountable. $\square$

Application

[**A.1**] Let A be a countable subset of $\mathbb{R}$. Show that there is a number $x \in \mathbb{R}$ such that $(x + A) \cap A = \emptyset$, where

$$x + A := \{\, x + a : a \in A \,\}$$

Proof. The thing to look at here is the "set of all distances from A" which is countable, then take x to be a distance not in this set.
So let

$$\mathcal{D}(A) := \{\, a - b : a, b \in A \,\}$$

and notice that there is an obvious injection from $\mathcal{D}(A)$ into $A \times A$, which is countable, so $\mathcal{D}(A)$ is countable. This tells us that

$$\mathbb{R} \setminus \mathcal{D}(A) \neq \emptyset$$

So chose an $x \in \mathbb{R} \setminus \mathcal{D}(A)$.
Claim: $(x + A) \cap A = \emptyset$.
Suppose not. Let $a \in (x+A) \cap A$, so $a \in A$ and $a \in (x+A)$. Thus $a = x + b$, with $b \in A$. Thus $a - b = x$, a contradiction. $\square$

[**A.2**] Let $(\, X, \mathcal{T} \,)$ be a topological space where X is countable. Is this space necessarily separable? Is is necessarily second countable? What about first countable? What about ccc?

Proof. A countable space is always separable, since $\overline{X} = X$. This automatically implies (by C.5) that the space is ccc.
A countable space does not have to be first countable (which also tells us that that a countable space does not have to be second countable). A classic

example is the Arens-Fort space, a topology on $\{0\} \cup (\mathbb{N} \times \mathbb{N})$. You can read about this example in Counterexamples in Topology, it is example 26.
Another example can be constructed on $\{0\} \cup (\mathbb{N} \times \mathbb{N})$ by using the tutorial problems 4.8 and 4.9.
The main thing to draw from this exercise is that some topological properties care about "how many points are in the space" (like separable and ccc), but some other properties care about "how many open sets are in the space" (like first and second countability). These are not hard rules, but it should help you understand that the difference between having a small number of points, and having a small number of open sets. $\square$

[**A.3**] Fill out the following table with "YES", "NO" or "PANTS". No proof needed.

Proof. Here's the complete table:

	Separable	2nd Countble	1st Countable	ccc
$\mathbb{R}_{\text{usual}}$	YES	YES	YES	YES
$\mathbb{R}_{\text{co-countable}}$	NO	NO	NO	YES
$\mathbb{R}_{\text{co-finite}}$	YES	NO	NO	YES
Everest	MOUNTAIN	MOUNTAIN	MOUNTAIN	MOUNTAIN
K2	MOUNTAIN	MOUNTAIN	MOUNTAIN	MOUNTAIN
Metterhorn	MOUNTAIN	MOUNTAIN	MOUNTAIN	MOUNTAIN
$\mathbb{R}_{\text{discrete}}$	NO	NO	YES	NO
$\mathbb{N}_{\text{discrete}}$	YES	YES	YES	YES
Kilimanjaro	MOUNTAIN	MOUNTAIN	MOUNTAIN	MOUNTAIN

Question C.5 tells us that "Separable $\Rightarrow$ ccc" and it is pretty easy to see that "second countable $\Rightarrow$ separable" and "second countable $\Rightarrow$ first countable". $\square$

New Ideas

[**NI.1**] The rabbit catching problem. Show that even though the catcher doesn't know (1) where the rabbit hole is, (2) what line the rabbit is jumping along, and (3) along which direction the rabbit is jumping, the catcher can *still* catch the rabbit in a finite amount of days. [See the problem set for the full statement.]

Sketch of NI.1. The main idea here is that the rabbit only has countably many paths (path 1, path 2, ...) she can go down (taking into account that she can start anywhere), and on day (1) we will assume that the rabbit is on path 1, and set a trap where we think she will be (one day into that path). On day 2 we will assume the rabbit is on path 2 and set a trap where we think the rabbit will be (two days into that path). It continues on in this way. Since we know that the rabbit started on some path n, we will catch her after n days (of course we don't know *which* n, but we know that it must be some n).
Let us note that there are only countably many paths the rabbit can go down. Call this set P, and note that

$$P = \underbrace{(\mathbb{Z} \times \mathbb{Z})}_{\text{Rabbit Hole}} \times \underbrace{\mathbb{Z}}_{\text{slope}} \times \underbrace{\{-1, 1\}}_{\text{direction}}$$

is countable. $\square$

[**NI.2**] Let's push the idea of the countable chain condition that we saw in C.5... Show that there is an uncountable collection of mutually disjoint circles in the plane. Is there an uncountable collection of mutually disjoint "figure-8"s in the plane? For an extra challenge, replace "figure-8" by "Y-shaped" in the previous question, (where "Y-shaped" means a union of three line segments at a common point). If you're still hungry for more, show that if you have a collection $\mathcal{C}$ of circles in the plane, such that no two circles cross each other, then the collection of tangent points $\{p \in \mathbb{R}^2 : \exists C, D \in \mathcal{C}, C \cap D = \{p\}\}$, is countable.

Proof. You can check that the collection of (concentric) circles in the plane centred at the origin, is an uncountable collection of disjoint circles. For the figure-8 and "Y-shaped figues" you can find an almost complete solution at:

http://math.stackexchange.com/a/185514/50277

Remember that if you choose to use other resources to help you with the New Ideas problems you must fully understand the solution you present. It is unacceptable to *copy and paste* a Math Stack Exchange Solution, even if you leave me a note saying "I found this on MSE". Copy-and-pasting does not prove to me that you understand your solution. $\square$