Formula Sheet

Effective rate of interest: $i_{u+1} = \dfrac{S(u+1) - S(u)}{S(u)}$

Payment of 1	Compound interest	Simple interest
Accumulated value after t years	$(1+i)^t$	$(1+ti)$
Present value at time 0	$v^t = (1+i)^{-t}$	$(1+it)^{-1}$

i paid at the ***end*** of the period on the balance at the ***beginning*** of the period.
d paid at the ***beginning*** of the period on the balance at the ***end*** of the period.

$$d = \frac{i}{1+i}$$

Present value with simple discount: $(1 - d \cdot t)$

Real interest rate: $1 + i_{real} = \dfrac{1+i}{1+r}$

The accumulated value of 1 from time 0 to time t under compound interest:

$$S(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = (1+i)^t = v^{-t} = (1-d)^{-t} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt} = e^{\delta t}$$

The present value at time 0 of 1 payable at time t under compound interest is:

$$S(0) = \left(1 + \frac{i^{(m)}}{m}\right)^{-mt} = (1+i)^{-t} = v^t = (1-d)^t = \left(1 - \frac{d^{(m)}}{m}\right)^{mt} = e^{-\delta t}$$

Force of interest

For constant force of interest, (under compound interest) $\delta = \ln(1+i)$

	Constant force of interest ($\delta_t = \delta$)	Variable force of interest
Accumulation at time t_2 of an amount $S(t_1)$ invested at t_1	$S(t_2) = S(t_1) \cdot e^{\delta(t_2 - t_1)}$	$S(t_2) = S(t_1) \cdot \exp\left(\int_{t_1}^{t_2} \delta_t dt\right)$
Present value at time t_1 of an amount $S(t_2)$ due at time t_2	$S(t_1) = S(t_2) \cdot e^{-\delta(t_2 - t_1)}$	$S(t_1) = S(t_2) \cdot \exp\left(-\int_{t_1}^{t_2} \delta_t dt\right)$

Annuities

n payments of 1	Payments made in arrears (at end of each period)	Payments made in advance (at start of each period)
Accumulated value	$s_{\overline{n}\rceil} = \sum_{t=0}^{n-1} (1+i)^t = \dfrac{(1+i)^n - 1}{i}$	$\ddot{s}_{\overline{n}\rceil} = \sum_{t=1}^{n} (1+i)^t = \dfrac{(1+i)^n - 1}{d}$
Present value	$a_{\overline{n}\rceil} = \sum_{t=1}^{n} v^t = \dfrac{1 - v^n}{i}$	$\ddot{a}_{\overline{n}\rceil} = \sum_{t=0}^{n-1} v^t = \dfrac{1 - v^n}{d}$

Payments of $\frac{1}{m}$ made each $\frac{1}{m}^{th}$ of a year for n years	Payments made in arrears	Payments made in advance
Accumulated value	$s_{\overline{n}\rvert}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}}$	$\ddot{s}_{\overline{n}\rvert}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}}$
Present value	$a_{\overline{n}\rvert}^{(m)} = \frac{1-v^n}{i^{(m)}}$	$\ddot{a}_{\overline{n}\rvert}^{(m)} = \frac{1-v^n}{d^{(m)}}$

Payments of 1 for perpetuity	Payments made in arrears	Payments made in advance
Present value of 1 per period	$a_{\overline{\infty}\rvert} = \frac{1}{i}$	$\ddot{a}_{\overline{\infty}\rvert} = \frac{1}{d}$
Present value of $\frac{1}{m}$ per period of length $\frac{1}{m}$	$a_{\overline{\infty}\rvert}^{(m)} = \frac{1}{i^{(m)}}$	$\ddot{a}_{\overline{\infty}\rvert}^{(m)} = \frac{1}{d^{(m)}}$

Continuous annuity of 1 per period for n periods	Fixed rate of interest	Variable rate of interest
Accumulated value	$\bar{s}_{\overline{n}\rvert} = \int_0^n (1+i)^{n-t} dt = \frac{(1+i)^n - 1}{\delta}$	$\bar{s}_{\overline{n}\rvert \delta_r} = \int_0^n \exp\left(\int_t^n \delta_r \, dr\right) dt$
Present value	$\bar{a}_{\overline{n}\rvert} = \int_0^n v^t dt = \frac{1-v^n}{\delta}$	$\bar{a}_{\overline{n}\rvert \delta_r} = \int_0^n \exp\left(-\int_o^t \delta_r \, dr\right) dt$

Arithmetically increasing annuity of n payments (first payment amount = 1, subsequent payments increase by 1 per period)	Payments made in arrears	Payments made in advance
Accumulated value	$(Is)_{\overline{n}\rvert} = \frac{\ddot{s}_{\overline{n}\rvert} - n}{i}$	$(I\ddot{s})_{\overline{n}\rvert} = \frac{\ddot{s}_{\overline{n}\rvert} - n}{d}$
Present value	$(Ia)_{\overline{n}\rvert} = \frac{\ddot{a}_{\overline{n}\rvert} - nv^n}{i}$	$(I\ddot{a})_{\overline{n}\rvert} = \frac{\ddot{a}_{\overline{n}\rvert} - nv^n}{d}$

Arithmetically decreasing annuity of n payments (first payment amount= n, subsequent payments decrease by 1 per period)	Payments made in arrears	Payments made in advance
Accumulated value	$(Ds)_{\overline{n}\rvert} = \frac{n \cdot (1+i)^n - s_{\overline{n}\rvert}}{i}$	$(D\ddot{s})_{\overline{n}\rvert} = \frac{n \cdot (1+i)^n - s_{\overline{n}\rvert}}{d}$
Present value	$(Da)_{\overline{n}\rvert} = \frac{n - a_{\overline{n}\rvert}}{i}$	$(D\ddot{a})_{\overline{n}\rvert} = \frac{n - a_{\overline{n}\rvert}}{d}$

Increasing annuity	with discrete increases and continuous payments	with continuous increases and continuous payments
Present value	$(I\bar{a})_{\overline{n}\rvert} = \int_0^n \lceil t \rceil v^t dt = \dfrac{\ddot{a}_{\overline{n}\rvert} - nv^n}{\delta}$	$(\bar{I}\bar{a})_{\overline{n}\rvert} = \int_0^n t v^t dt = \dfrac{\bar{a}_{\overline{n}\rvert} - nv^n}{\delta}$

Geometric series summation formula: $1 + x + x^2 + x^3 + ... + x^k = \dfrac{1 - x^{k+1}}{1 - x} = \dfrac{x^{k+1} - 1}{x - 1}$

Present value at time 0 of a series of n payments, each of amount 1, commencing at time $k+1$: ${}_{k}|a_{\overline{n}\rvert} = v^k \cdot a_{\overline{n}\rvert} = a_{\overline{n+k}\rvert} - a_{\overline{k}\rvert}$

General formula for increasing annuity

n-payment annuity with first payment A and subsequent payment B larger (or smaller) than the previous one. Payments made in arrears. Accumulated value at time n is

$$S(n) = (A - B)s_{\overline{n}\rvert i} + B\left(Is\right)_{\overline{n}\rvert i}$$

Solving Equations of Value

Quadratic form: $a\left((1+i)^n\right)^2 + b(1+i)^n + c = 0$ solution: $(1+i)^n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Linear interpolation: Given i_1, i_2, $f(i_1)$ and $f(i_2)$: $\dfrac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cong \dfrac{i_0 - i_1}{i_2 - i_1}$

$$\Rightarrow i_0 \cong i_1 + \frac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cdot (i_2 - i_1)$$

Loan calculations

Loan amount = $L = OB_0$

Interest rate per period = i

Amount paid at time t = K_t

Interest charged at the end of the t^{th} period = $I_t = OB_{t-1} \cdot i$

Principal repaid at the end of the t^{th} period = $PR_t = K_t - I_t$

Outstanding balance (principal) just after payment at time t = OB_t

Where $OB_t = OB_{t-1} + OB_{t-1} \cdot i - K_t$

$$OB_t = OB_{t-1} + I_t - K_t = OB_{t-1} - \left(K_t - I_t\right) = OB_{t-1} - PR_t$$

Outstanding balance:

Retrospective method: $OB_t = L(1+i)^t - \sum_{a=1}^{t} (1+i)^{t-a} K_a$

Prospective method: $OB_t = vK_{t+1} + v^2 K_{t+2} + ... + v^{n-t} K_n = \sum_{a=1}^{n-t} v^a K_{a+t}$