11.10.11

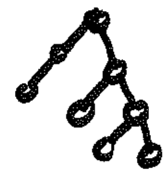
Lecture 7 handout

bonus

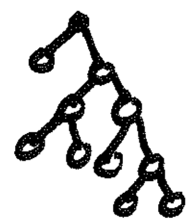
Catalan numbers

/wiki/Catalan_number/

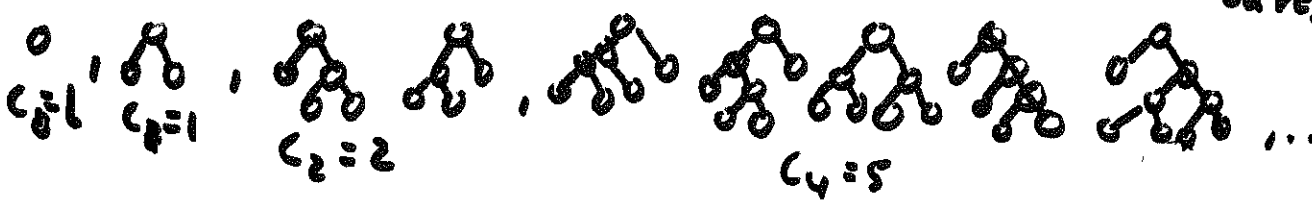
Binary tree: rooted tree in which every vertex has ≤ 2 children



Full binary tree: binary tree in which each vertex has 0 or 2 children



Q: How many full binary trees have $n+1$ leaves?



$c_0=1$, $c_1=1$, $c_2=2$, $c_4=5$, ...

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Formula: $C_n = \frac{1}{n+1}\binom{2n}{n}$

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Proof: Step 1: C_n is #{up-right paths on $n \times n$ grid which stay below the diagonal}

Proof: right = split
up: close

Step 2: Reflection argument:

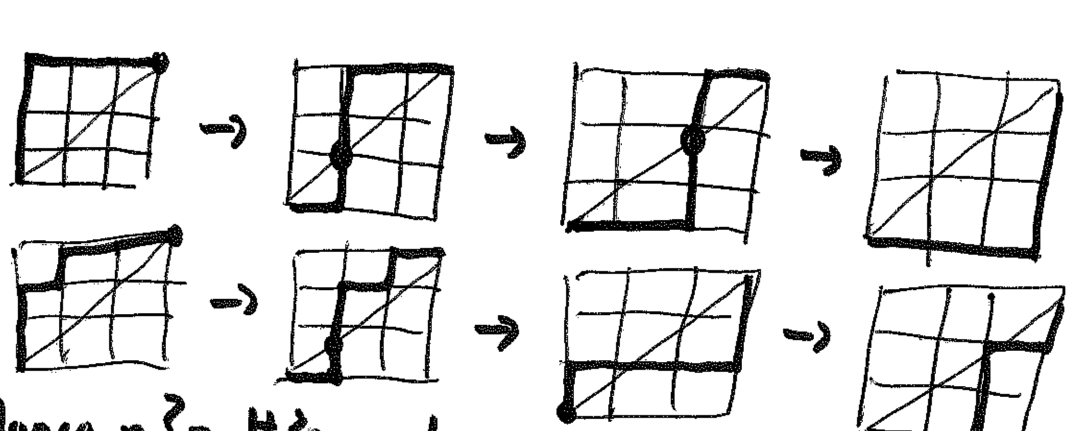
Total up-right paths: $\binom{2n}{n}$

Exceedance = #{edges above diagonal}

1st diagonal encounter: dot

Last (horizontal) edge before 2nd encounter: black.

switch before and after black.



#{exceedance n} = #{exceedance $n-1$} = .. = #{exceedance 0}

So divide by $n+1$.

Laplacian matrix (Ch. 20).

$L = D - A$ $\qquad$ $L = B^T B$ ← signed incidence matrix.

1) Graphs as spring networks

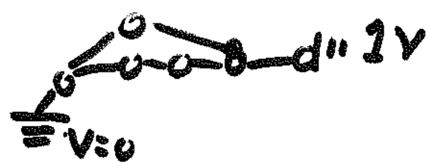
spring constant $= 1$ (for now)

potential energy: $\frac{l^2}{2}$

Physics: position minimizes potential energy

$$\frac{1}{2} \sum_{(a,b) \in E} (x(a) - x(b))^2$$ subject to constraints

2) Graphs as resistor networks:



apply voltages at some vertices. Measure induced voltages and current flow.

Minimize $\sum_{(a,b) \in E} (v(a) - v(b))^2$ subject to voltages.

Laplacian quadratic form: $\sum_{(a,b) \in E} (x(a) - x(b))^2 = x^T L x$

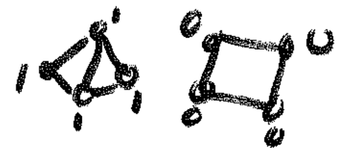
minimize subject to boundary constraints

$\rightsquigarrow$ Solve $Lx = b$.

eigenvalues: $Lx = \lambda_i x$.

First eigenvalue: $x = (1,1,\ldots,1)$ $Lx = 0$ $\boxed{\lambda_1 = 0}$

Second eigenvalue: $\lambda_2 > 0 \iff G$ is connected

Proof: G not connected 

G connected, $x \perp (1,1,\ldots,1)$ means $\sum x(u) = 0$
So must be different values 

Complexity: number of spanning trees in G, $\kappa(G)$.

Matrix-Tree Theorem:

$$\boxed{\kappa(G) = \frac{1}{n} \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n}$$

Proof 1: Deletion-contraction

$$\kappa(G) = \kappa(G \setminus e) + \kappa(G/e).$$

Proof 2: See Srivastava notes.

Next time: Flows. Sections 7.1-7.2.