(1) Predicate: $\cancel{\forall} n \in \mathbb{N}$, each set with n elements has 2^{n-1} subsets with odd size.

$P(n): \forall S, |S| = n, n \geq 1 \Rightarrow |P_{odd}(S)| = 2^{n-1}$

Base case: $P(1)$ holds

IS: $P(n) \rightarrow P(n+1)$

$P(1): [\forall n \in \mathbb{N}, n \geq 1 \wedge P(n) \Rightarrow P(n+1)] \Rightarrow \forall n \in \mathbb{N}, n \geq 1. P(n)$

IH: We assume for a general natural # $n \rightarrow P(n)$

$|P_{odd}(S)| = 2^{n-1}, \quad |S| = n$

CSC236 tutorial exercise #1

want $|P_{odd}(S_{n+1})| = 2^n, \quad |S_{n+1}| = n+1$
odd

Winter 2015

15 January 2015

1. Use a variation of simple induction to prove that for most natural numbers n, any set of n elements has 2^{n-1} subsets with an odd number of elements.

$|P_{odd}(S)| = \frac{1}{2}|P(S)| = \frac{1}{2} \cdot 2^n = 2^{n-1} = |P_{even}(S)|, \quad S_{n+1} = S_n \cup \{a_{n+1}\}$

$P(S_{n+1}) = P(S_n) \cup \{X \cup \{a_{n+1}\} \mid X \in P(S_n)\}$

$P(S_n) = P_{odd}(S_n) \cup P_{even}(S_n), \quad P_{odd}(S_n) \cap P_{even}(S_n) = \emptyset$

$P_{odd}(S_{n+1}) = P_{odd}(S_n) \cup \underbrace{\{X \cup \{a_{n+1}\} \mid X \in P_{even}(S_n)\}}_{*}$

$|P_{odd}(S_{n+1})| = |P_{odd}(S_n)| + |*|$

$2^n = 2^{n-1} + 2^{n-1}$ done. ✓

2. We proved in class that for all natural numbers n, $3^n \geq n^3$. Your task is to complete the following alternative proof.

Define $P(n) :=$ "$3^n \geq n^3$".

As before, we prove $\forall n. P(n)$ by a variation of simple induction.

Base case: you decide what base cases you need. We used 0, 1, 2, 3 in class, but this is a different proof, so perhaps you will need a smaller or larger number of base cases.

Let n be an arbitrary natural number that is at least as large as your largest base case. Assume $P(n)$.

Goal: $3^{n+1} \geq (n+1)^3$, or equivalently $(n+1)^3 \leq 3^{n+1}$.

Expanding $(n+1)^3$ gives $n^3 + 3n^2 + 3n + 1$. Hence, the Goal is equivalent to:

NewGoal: $n^3 + 3n^2 + 3n + 1 \leq 3^{n+1}$.

Prove NewGoal!

(2) $P(n): \cancel{\forall} n \in \mathbb{N}, 3^n \geq n^3$

Base cases: $n=0, \ 1 \geq 0$

$n=1, \ 3 \geq 1$

$n=2, \ 9 \geq 8$

$n=3, \ 27 \geq 27$

IH: Assume for general $n \in \mathbb{N}, n \geq 3, P(n) \rightarrow P(n+1)$

$P(n): 3^n \geq n^3$

$P(n+1): 3^{n+1} \geq (n+1)^3 = n^3 + 3n^2 + 3n + 1$

$$\begin{aligned} 3^{n+1} = 3 \cdot 3^n = 3^n + 3^n + 3^n \geq n^3 + n^3 + n^3 \geq n^3 + n(n^2) + n^2 \cdot n &\geq n^3 + 3n^2 + 9n \\ &\geq n^3 + 3n^2 + 6n + 3n \\ &\geq n^3 + 3n^2 + 3n + 1 \\ &= (n+1)^3 \end{aligned}$$
because $n \geq 3$

CSC236 tutorial exercise #2
Winter 2015

22 January 2015

1. Finish any lingering questions about last tutorial's exercises.

2. Prove by induction that, for any natural number n, the sum of the naturals from 0 to n (i.e. $0 + 1 + 2 + \ldots + n$) is $\frac{n(n+1)}{2}$.
   Clearly and explicitly structure your inductive proof:

   - Define a predicate P whose domain is the natural numbers such that you are proving P holds for every natural number. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
   - Label your base case or base cases. $n=0,\ 0=\frac{0 \cdot 1}{2} \rightarrow P(0)$ ✓
   - Label your inductive hypothesis (IH) and every place where you use it.
   - Label your inductive step.

IS: $P(n) \rightarrow P(n+1)$

$\sum_{i=1}^{n} i = \frac{n(n+2)}{2}$

IH ✓

$\sum_{i}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \longrightarrow P(n+1)$ ✓

3. Quiz will be closely related to one of the tutorial exercises from last week or this week.