University of Toronto
FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, DECEMBER 2011

APM 236H1F
Applications of Linear Programming

Examiner: P. Kergin
Duration: 3 hours

PLEASE HAND IN

FAMILY NAME ____________________

GIVEN NAME(S) ____________________

STUDENT NO ____________________

SIGNATURE ____________________

INSTRUCTIONS:

NO calculators or other aids allowed. There are 6 questions, each worth 20 marks. Questions 2, 3, 5, and 6 have part-questions, whose values are stated within the part-questions themselves. Total marks = 120.

This exam consists of 12 pages, printed on both sides of the paper. Write solutions in spaces provided. Pages 3 and 12 are blank and may be used for the solution(s) of any of the problems, or for rough work. Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

GRADER'S REPORT	
1	
2	
3	
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5	
6	
TOTAL	

1. **Use artificial variables to write a linear programming problem** in canonical form with non-negative resource vector whose solution will determine whether there exists (and, if so, find) non-negative reals x_1 , x_2 , x_3 , and x_4 such that $x_1 + x_2 + x_3 + x_4 = 1$ and $x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. After you set the problem up, use the **simplex method** to solve it.

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2.(a)(10 marks) Find an **optimal solution**, in $\mathbb{R}^3$, of the problem:

Maximize $z = x_1 - 4x_2 + 7x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & \leq & 3 \\ 4x_1 & - & x_2 & - & 2x_3 & \geq & 0 \\ x_1 & - & 2x_2 & + & x_3 & \leq & 4 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Note that the second constraint has "$\geq$". This is not a typographical error.

2.(b)(10 marks) Find **all optimal solutions**, <u>in $\mathbb{R}^3$</u>, of the problem of question 2.(a):

Maximize $z = x_1 - 4x_2 + 7x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & \leq & 3 \\ 4x_1 & - & x_2 & - & 2x_3 & \geq & 0 \\ x_1 & - & 2x_2 & + & x_3 & \leq & 4 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

3.(a)(10 marks) **State** the **weak duality theorem**.

3.(b)(10 marks) **State** the **strong duality theorem**.

4. **Solve** the problem:

Minimize $z = 4x_1 + 5x_2 + x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} -x_1 & + & x_2 & + & 3x_3 & \geq & 3 \\ x_1 & + & 2x_2 & + & x_3 & \geq & 4 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

5. Consider the primal problem:

Maximize $z = 2x_1 - x_2$ subject to the constraints

$$\begin{array}{rcrcl} x_1 & - & x_2 & \leq & 1 \\ 2x_1 & + & 3x_2 & \geq & 2 \\ x_1 & + & 2x_2 & = & 4 \\ x_1 & + & x_2 & \leq & 3 \end{array}, \quad x_1 \geq 0, x_2 \geq 0.$$

5.(a)(4 marks) **Solve** the primal problem **graphically**.

5.(b)(4 marks) For $i = 1$, 2, 3, and 4, let w_i denote the dual variable associated with the i^{th} primal constraint. Write the dual of the primal problem, using only the decision variables w_1, w_2, w_3, and w_4.

5.(c)(12 marks) Let S denote the set of points, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ in $\mathbb{R}^4$, which are optimal for the dual problem. **Find the extreme points** of S.

6. Consider the problem:

Maximize $z = 9x_1 + 8x_2 + 6x_3 + 9x_4 + 11x_5$ subject to the constraints

$$\begin{array}{ccccccccccc} 3x_1 & + & 2x_2 & + & x_3 & + & x_4 & + & x_5 & \leq & 5 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & \leq & 3 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.$$

6.(a)(5 marks) The fourth tableau of the simplex solution of this problem has basic variables $\{x_1, x_3\}$ (in that order). **Use this information** to find the matrix $\mathbf{B}^{-1}$ which corresponds to the fourth tableau.

6.(b)(15 marks) **Beginning from the fourth tableau**, use the **revised simplex method to solve** the problem on page 10:

Maximize $z = 9x_1 + 8x_2 + 6x_3 + 9x_4 + 11x_5$ subject to the constraints

$$\begin{array}{ccccccccccc} 3x_1 & + & 2x_2 & + & x_3 & + & x_4 & + & x_5 & \leq & 5 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & \leq & 3 \end{array}, \; x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.$$

Note that there in no need to write the fourth tableau.

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