P72 B, D, E, F, G, J, L
P76 H
P183 B, C, F, J
P87 D, H, I

P89 A, B, C
P90. F, H
~~P117 C, D, E, F, I~~
~~P119 B, D, I, J~~

MAT337 Midterm 2 Review

extra P81 ABCDEF JL
P87 A
P89 DE I
P93 A

Coverage:
§ 9.1, 9.2, 5.1–5.7
& materials covered in class, such as the Lebesgue # lemma
suggested problems on Page 183, 87, 89.

Chapter 9 Metric Spaces
§ 9.1 Definitions & Examples.

Def. X be a set, a <u>metric</u> on a set X is a function ρ defined on $X \times X$ taking values in $[0, +\infty)$ with the following properties.
① positive definiteness $\rho(x, y) = 0$ iff $x = y$
② symmetry $\rho(x, y) = \rho(y, x)$ $\forall x, y \in X$.
③ triangle inequality $\rho(x, y) \leq \rho(x, y) + \rho(y, z)$ $\forall x, y, z \in X$
A <u>metric space</u> is a set X with a metric ρ, denoted by (X, ρ)
If the metric is understood, we use <u>X alone</u>.

Def. The <u>ball</u> $B_r(x)$ of radius $r > 0$ about a point x is defined as $\{y \in X : \rho(x, y) < r\}$ We write $B_r^{\rho}(x)$ if the metric is ambiguous.
A subset U is <u>open</u> if $\forall x \in U$, $\exists r > 0$ s.t. $B_r(x)$ is contained in U and the <u>interior</u> of a set A, $\int A$ is the largest open set contained in A.
A sequence (x_n) is said to <u>converge to x if</u> $\lim_{x \to \infty} \rho(x, x_n) = 0$.
A set C is <u>closed</u> if it contains all limit points of sequences of points in C and the <u>closure</u> of a set A, $\overline{A}$ is the set of all limit <u>points</u> of A.

Def. A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, ρ) is a <u>Cauchy sequence</u> if $\forall \varepsilon > 0$, $\exists$ an integer N s.t. $\rho(x_i, x_j) < \varepsilon$ $\forall i, j \geq N$
A metric space X is <u>complete</u> if $\forall$ Cauchy sequence converges (in X).

Def: A function f from a metric space (X, ρ) into a metric space (Y, σ) is continuous if $\forall x_0 \in X$ & $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\sigma(f(x), f(x_0)) < \varepsilon$ whenever $\rho(x, x_0) < \delta$

Thm: f map a metric space (X, ρ) into a metric space (Y, σ).
TFAE:
(1) f is continuous on X
(2). $\forall (x_n)$ with $\lim_{n\to\infty} x_n = a \in X$, we have $\lim_{n\to\infty} f(x_n) = f(a)$;
(3). $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for $\forall$ open set U in Y

~~Thm: The space C_b~~

$\boxed{\text{§9.2 Compact Metric Spaces}}$

old "compact" $\Rightarrow$ sequential compact.
new compact actually = sequential compact in norm space/metric space
but $\neq$ in topological spaces

Def. A collection of open sets $\{U_\alpha : \alpha \in A\}$ in X is an <u>open cover</u> of $Y \subseteq X$ if $Y \subseteq \bigcup_{\alpha\in A} U_\alpha$. A <u>subcover</u> of Y in $\{U_\alpha : \alpha \in A\}$ is just a subcollection $\{U_\alpha : \alpha \in B\}$ for some $B \subseteq A$ that is still a cover of Y. In particular, it is a finite subcover if B is finite, that is, a finite collection of the U_α that covers Y.
A collection of closed sets $\{C_\alpha : \alpha \in A\}$ has the finite intersection property is every finite subcollection has nonempty intersection.

Def. A metric space X is compact if every open cover of X has a finite subcover. A metric space X is sequentially compact if every sequence of points in X has a convergent subsequence~~s~~.

A metric space X is ~~totally bounded~~ if for every $\varepsilon > 0$, there are finitely many points $x_1, \dots, x_k \in X$ s.t. $\{B_\varepsilon(x_i) : 1 \le i \le k\}$ is an open cover

Thm Borel-Lebesgue thm
For a metric space X, the FAE:
(1) X is compact
(2). Every collection of closed subsets of X with ~~the~~ finite intersection property has nonempty intersection.
(3). X is sequentially compact.
(4). X is complete and totally bounded.

The Lebesgue number lemma.
Let $\mathcal{A}$ be an open covering of the metric space (X, ρ)
If X is compact, there is a $\delta > 0$ s.t. for each subset of X having diameter $< \delta$, $\exists$ exists an element A containing it.

Chapter 5 Functions

§5.1 Limits and Continuity

Def. limit of a function:

Let $S \subset \mathbb{R}^n$ & f be a function from S to $\mathbb{R}^m$.
If $\vec{a}$ is a limit point of $S\setminus\{\vec{a}\}$, then a point $\vec{v} \in \mathbb{R}^m$ is the limit of f at $\vec{a}$ if $\forall \varepsilon > 0$, $\exists\, r > 0$ s.t.

$$\|f(\vec{x}) - \vec{v}\| < \varepsilon \text{ whenever } 0 < \|\vec{x} - \vec{a}\| < r \text{ and } \vec{x} \in S$$

We write $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{v}$

Def: Let $S \subset \mathbb{R}^n$ and let f be a function from S into $\mathbb{R}^m$, f is continuous at $\vec{a} \in S$ if $\forall \varepsilon > 0$, $\exists\, r > 0$ s.t. $\forall \vec{x} \in S$ with $\|\vec{x} - \vec{a}\| < r$, we have $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$.
Moreover, f is continuous on S if it's continuous at each point $\vec{a} \in S$.
If f is not continuous at $\vec{a}$, we say f is discontinuous at $\vec{a}$.

Def: A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is called a Lipschitz function if $\exists\, C \in \mathbb{R}$ s.t.

$$\|f(\vec{x}) - f(\vec{y})\| \le C\|\vec{x} - \vec{y}\| \quad \forall \vec{x}, \vec{y} \in S$$

The Lipschitz constant of f is the smallest C for which this condition holds.

Prop. Every Lipschitz function is continuous.

Cor: Every linear map A from $\mathbb{R}^n$ to $\mathbb{R}^m$ is Lipschitz, and therefore is continuous.

§5.2 Discontinuous functions

- removable singularity

- Heaviside Function

Def: The limit of f as x approaches a from the right exists and equals L if $\forall \varepsilon>0\ \exists r>0$ s.t.

$$|f(x)-L|<\varepsilon \quad \forall\ a<x<a+r$$

Write $\lim_{x\to a^+} f(x)=L$, define limits from the left similarity, writing $\lim_{x\to a^-} f(x)=L$.

When $\lim_{x\to a^-} f(x)\neq \lim_{x\to a^+} f(x)$, jump discontinuity

- piecewise continuous

Def: The limit of a function $f(x)$ as x approaches a is $+\infty$ if $\forall N>0$ $\exists r>0$ s.t. $f(x)>N\ \forall\ 0<|x-a|<r$

We write $\lim_{x\to a} f(x)=+\infty$, we define $\lim_{x\to a} f(x)=-\infty$ similarly.

characteristic function.

- def: $V\subset S\subset \mathbb{R}^n$ is open in S or relatively open (w.r.t. S) if $\exists$ an open set U in $\mathbb{R}^n$ s.t. $U\cap S=V$.

§5.3 Properties of Continuous Functions

Def: $f:(X,\rho)\to(Y,\sigma)$ is continuous if $\forall x_0\in X$ & $\varepsilon>0$, $\exists\delta>0$ s.t. $\sigma(f(x),f(x_0))<\varepsilon$ whenever $\rho(x,x_0)<\delta$.

(X,ρ), (Y,σ) are metric spaces.

Thm: $f: S\subset\mathbb{R}^n\to\mathbb{R}^m$, the FAE:

(1). f is continuous on S

(2). $\forall$ convergent sequence $(\vec{x}_k)_{k=1}^{\infty}$ with $\lim_{k\to\infty}\vec{x}_k=\vec{a}$ in S, $\lim_{k\to\infty} f(\vec{x}_k)=f(\vec{a})$

(3). $\forall$ open set U in $\mathbb{R}^m$, the set $f^{-1}(U)=\{\vec{x}\in S: f(\vec{x})\in U\}$ is open in S.

Thm: f, g are functions from a common domain $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and $\vec{a} \in S$ s.t. $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{u}$ and $\lim_{\vec{x} \to \vec{a}} g(\vec{x}) = \vec{v}$,

then

(1). $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) + g(\vec{x}) = \vec{u} + \vec{v}$

(2). $\lim_{\vec{x} \to \vec{a}} \alpha f(\vec{x}) = \alpha \vec{u}$ for any $\alpha \in \mathbb{R}$

When the range is contained in $\mathbb{R}$, say $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = u$ and $\lim_{\vec{x} \to \vec{a}} g(\vec{x}) = v$, then

(3). $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) g(\vec{x}) = uv$, and

(4). $\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{u}{v}$ provided that $v \neq 0$

Thm: If f, g are functions from a common domain S into $\mathbb{R}^m$ that are continuous at $\vec{a} \in S$, and $\alpha \in \mathbb{R}$, then.

(1). $f+g$ is continuous at $\vec{a}$

(2). αg is continuous at $\vec{\alpha}$

and when the range is contained in $\mathbb{R}$

(3). fg is continuous at $\vec{a}$, and

(4). f/g is continuous at $\vec{a}$ provided that $g(\vec{a}) \neq 0$.

Thm: Sps that f maps a domain S contained in $\mathbb{R}^n$ into a subset T of $\mathbb{R}^m$ and g maps T into $\mathbb{R}^1$, if f is continuous at $\vec{a} \in S$ and g is continuous at $f(\vec{a}) \in T$, then the function $g \circ f$ is continuous at $\vec{a}$. Thus if f and g are continuous, then so is $g \circ f$.

§5.4 Compactness and Extreme Values

Thm: Let C be a compact subset of $\mathbb{R}^n$, and let f be a continuous function from C into $\mathbb{R}^m$. then the image set $f(C)$ is compact.

Thm: (extreme value theorem)
Let C be a compact subset of $\mathbb{R}^n$, let f be a continuous function from C to into $\mathbb{R}$. Then there are points $\vec{a}$ and $\vec{b}$ in C attaining the minimum and maximum values of f on C. That is

$$f(\vec{a}) \le f(\vec{x}) \le f(\vec{b}) \quad \text{for all } \vec{x} \in C$$

§5.5 Uniform~~ly~~ Continuity

def: A function $f : S \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly ~~continuity~~ continuous if for every $\varepsilon > 0$, there is a positive real number $r > 0$ s.t. $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$ whenever $\|\vec{x} - \vec{a}\| < r$, $\vec{x}, \vec{a} \in S$.

Prop: Every Lipschitz ~~co~~ function is uniformly continuous.

Cor: Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous.

Cor: Let f be a differentiable real-valued function on $[a, b]$ with a bdd derivative. Then f is unif. continuous on $[a, b]$.

Thm: Sps that $C \subset \mathbb{R}^n$ ~~that~~ is compact ~~that~~ and $f : C \to \mathbb{R}^m$ is continuous. Then f is uniformly continuous on C.

§5.6 Intermediate Value Theorem

Thm: (IVT)
If f is a continuous real-valued function on $[a, b]$ and $z \in \mathbb{R}$ satisfies $f(a) < z < f(b)$ then $\exists$ a point $c \in (a, b)$ s.t. $f(c) = z$.

Cor: If f is a continuous real-valued function on $[a, b]$, then $f([a, b])$ is a closed interval.

Def: A path in $S \subset \mathbb{R}^n$ from $\vec{a}$ to $\vec{b}$ both pts in S is the image of a continuous function γ from $[0, 1]$ into S s.t. $\gamma(0) = \vec{a}$ and $\gamma(1) = \vec{b}$.

Cor: Sps that $S \subset \mathbb{R}^n$ and f is a continuous real-valued function on S. If $\exists$ a path from $\vec{a}$ to $\vec{b}$ in S and $z \in \mathbb{R}$ with $f(\vec{a}) < z < f(\vec{b})$, then there is a point $\vec{c}$ on the path s.t. $f(\vec{c}) = z$.

Def: A subset A of $\mathbb{R}^n$ is not connected if there are disjoint open sets U and V s.t. $A \subset U \cup V$ and $A \cap U \neq \emptyset \neq A \cap V$. o.w. A is connected.

§5.7 Monotone Functions

def: monotone (strictly) function

Prop: If $f \nearrow$ on (a,b), then one-sided limits of f exist at each $c \in (a,b)$ and $\lim_{x\to c^-} f(x) \leq f(c) \leq \lim_{x \to c^+} f(x)$

for decreasing $\downarrow$, inequalities reversed.

Cor. The only type of discontinuity that a monotone function on an interval can have is a jump discontinuity.

Cor. If f is a monotone on $[a,b]$ and the range of f intersects every nonempty open interval ~~in~~ in $[f(a), f(b)]$ then f is continuous.

Thm. A ~~mo~~ monotone func. on $[a,b]$ has at most countably many discontinuities.

Thm. Let f be a continuous strictly increasing function on $[a,b]$. Then f maps $[a,b]$ one-to-one and onto $[f(a), f(b)]$. The inverse function f^{-1} is also continuous and strictly increasing.

Suggested Problems

~~P113~~

§9.2 P183

(B.) Proof: Y is closed & totally bdd, $Y \subset X$, X is complete.
$\Rightarrow Y$ is compact.

~~X comple: $\rho(a_m, a_n) \leq \delta$, $\forall m,n \geq N$~~
~~$\Rightarrow Y$ is complete $\Rightarrow$ every Cauchy in Y converges~~
~~Y closed $\Rightarrow Y$ contains all limit pts in Y~~
~~Y totally bdd $\Rightarrow \forall \varepsilon > 0, \exists$ finitely many pts $x_1, \dots, x_k \in Y$~~
~~s.t. $\{B_\varepsilon(x_i) : 1 \leq i \leq k\}$ is an open cover.~~

Show $Y \subset X$, X complete
Y compact $\Leftrightarrow Y$ closed & totally bdd. ✓
($\Rightarrow$) Sps Y compact $\Rightarrow Y$ ~~closed~~ complete & totally bdd (Borel-Lebesgue)
(Show closed) Sps $(x_n) \in Y$, $x_n \to x \notin Y$
(x_n) converges $\Rightarrow (x_n)$ is Cauchy by def'n of Completeness., $x \in Y$

($\Leftarrow$). ~~Show Y~~
Sps Y closed & totally bdd (w.t.s. Y compact)
Y closed $\Rightarrow \forall$ convergent $(x_n) \in Y$ we have $x_n \to x \in Y$ as well.
convergent $\Rightarrow$ Cauchy.
$\Rightarrow \forall$ Cauchy seq. $\to$ some point in Y $\Rightarrow Y$ complete.
Y complete + Y totally bdd $\Rightarrow Y$ compact.

C. Sps $S \subseteq X$. X is compact
S is closed

Prove S is compact.

Proof: consider $\forall (x_n) \in S$, $(x_n) \in X$ as well.

X is compact $\Rightarrow \exists$ subseq $(x_{n_k}) \to x \in X$

$(x_{n_k}) \in S$, S closed $\Rightarrow x \in S$

$\Rightarrow S$ is sequentially compact

$\Rightarrow S$ is compact.

F. A decreasing sequence of nonempty compact subsets $A_1 \supset A_2 \supset \cdots$ of a metric space (X, ρ) has nonempty intersection.

Pick $a_i \in A_i$

All of A_i's compact $\Rightarrow$ sequentially compact

So consider A_1

$\Rightarrow$ the sequence (a_i) in A_1 has convergent subseq (a_{i_k}) s.t.

$a_{i_k} \to a \in A_1$

on the other hand, $a_{i_k} \in A_i$

if $k \geq i$ for sure $\Rightarrow \forall A_i$ we can just delete several first terms (a_{i_k}) and say $a \in A_i \Rightarrow a \in \bigcap A_i$

~~I. Let S_n for $n \geq 1$ be a finite union of disjoint~~ closed ~~balls in $\mathbb{R}^k$ of radius at most 2^{-n} s.t. $S_{n+1} \subset S_n$ and S_{n+1} has at least 2 balls inside each ball of S_n. Prove $C = \bigcap_{n \geq 1} S_n$ is a ~~pef~~ perfect, nowhere dense compact subset of $\mathbb{R}^k$.~~

J. If f is a continuous 1-1 func of a compact metric space X onto Y, show f^{-1} is continuous.

Proof: Sps f^{-1} not cont.

$\Rightarrow \exists\ b \in Y$ & $(y_n) \in Y$

s.t. $y_n \to b$ but

$x_n = f^{-1}(y_n) \not\to a = f^{-1}(b)$

$\Rightarrow \exists\ \varepsilon > 0\ \forall N .\ \rho(x_n, a) \geq \varepsilon$ if $n \geq N$

$\Rightarrow$ ~~X~~ not compact

in Y, $\exists\ (x_{n_k}) \to a' \in X$

$\Rightarrow f(x_{n_k}) = y_{n_k} \to f(a')$

know $y_n \to f(a)$ & $y_n \to b$

$\Rightarrow y_{n_k} \to b$

$\Rightarrow f(a') = f(a)$

$\Rightarrow a = a'$ (1-1)

P87

§ 5.5

D. $f(x)=x^p$ is not uniformly cont. on $\mathbb{R}$ if $p>1$.

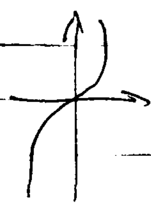
Proof: def: $\forall \varepsilon>0, \exists r>0$ s.t.

$\|f(\vec{x})-f(\vec{a})\| < \varepsilon$ whenever $\|\vec{x}-\vec{a}\| < r$, $\vec{x}, \vec{a} \in S$

So ~~when~~ $\cancel{\|x-a\|<r}$

~~see~~ $\cancel{\|f(x)-f(a)\| = \|x^p - a^p\|}$

Let $p=3$, so we have.



$= x=-\frac{r}{2}, a=\frac{r}{2}$

sps $\|\vec{x}-\vec{a}\|<r$, x & a are symmetry about origin,

then $\|f(\vec{x})-f(\vec{a})\| < \varepsilon = 2\cdot\left(\frac{r}{2}\right)^3 = \frac{r^3}{4}$

but when $x=0, a=r$,

$\|f(x)-f(a)\| < r^3 > \frac{r^3}{4}$

~~namely, set. $\|f(x$~~

So not uniformly continuous.

H. f cont. on (a,c). $a<b<c$. if f unif. cont. on $(a,b]$ & $[b,c)$ then on (a,c).

Proof: $\forall \varepsilon>0, \exists r>0$. s.t.

$\|f(x)-f(m)\|<\varepsilon$ whenever $\|x-m\|<r\leq b-a$ $x, m \in (a,b]$

$\forall \varepsilon>0, \exists r'>0$ s.t.

$\|f(x')-f(n)\|<\varepsilon$ whenever $\|x'-n\|<r'\leq c-b$, $x', n \in [b,c)$

$a < x \leq m \leq b \leq x' < n < c$

So $\|x-n\| < r+r' < c-a$

$\Rightarrow \|f(x)-f(b)\| + \|f(b')-f(n)\|$

$\geq \|f(x)-f(n)\|$

$\|f(x)-f(n)\| < 2\varepsilon$

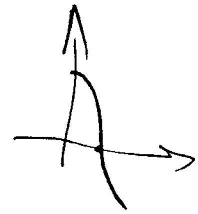
So $\|f(x)-f(n)\| < 2\varepsilon$ whenever $\|x-n\| < c-a$

I. $f(x)$ cont. on $(0,1]$. Show f is uniformly cont. iff $\lim_{x\to 0^+} f(x)$ exists.

Proof: $\forall \varepsilon>0, \exists$ ~~$\delta>0$~~ $r>0$, s.t. $\forall$ $x\in (0,1]$
$\|x-a\|<r$, we have $\|f(x)-f(a)\|<\varepsilon$

~~if $\lim_{x\to 0^+} f(x)$ exists~~

~~$\Rightarrow$~~ $f(x)$ cont. on $(0,1] \Rightarrow f(x)$ diff. on $(0,1]$
lim exists $\Rightarrow$ diff on $[0,1]$
$\Rightarrow f$ uniformly cont. on $[a,b]$.



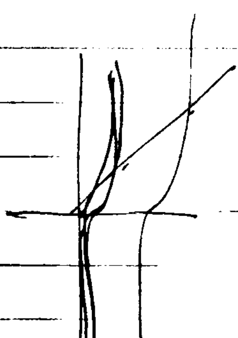
P89 § 5.6 The IVT

A. ~~Show~~ $\exists x \in (0, \frac{\pi}{2})$, s.t. $\cos x = x$.

(a). ~~If f~~ $f(x) = \cos x$ continuous on $[0, \frac{\pi}{2}]$ and $x \in \mathbb{R}$
s.t. $f(0) > x > f(\frac{\pi}{2})$, then $\exists\, c \in (0, \frac{\pi}{2})$ s.t. $f(c) = x$

(b). Sps there are more than 1 solution.
Say two solutions are a and b
then the path from a to b in $\cos x$ and $z \in \mathbb{R}$
with $f(a) < z < f(b)$.
Then $\exists\, c$ on path s.t. $f(c) = z$ ~~cosc~~
since actually $f(a) = f(b)$ so $f(c) = \cos c = z = f(a) = f(b)$.
so $f(c) = f(a) = f(b)$
same point.
$c = a = b$.

B. $\tan x = x$ in $[0, 11]$. 2 solutions.



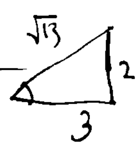
$\frac{11}{2\pi} \approx$ [illegible]. 2

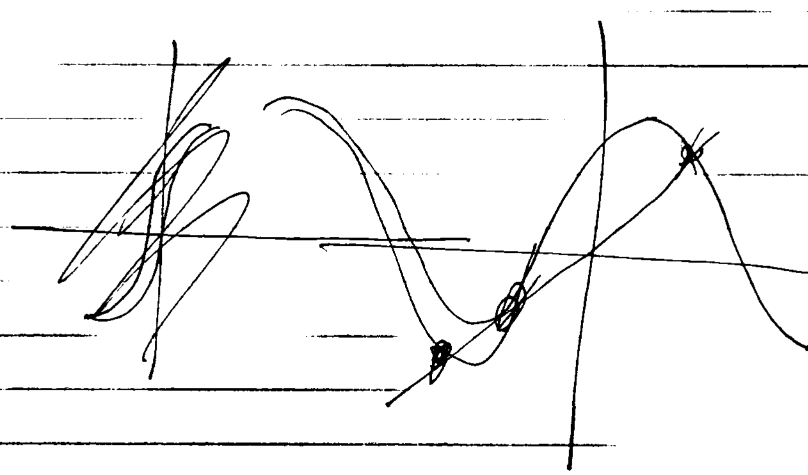
C. $2\sin x + 3\cos x = x$

$2\cos x - 3\sin x = 0$

$2\cos x = 3\sin x$

$\tan x = \frac{\sin x}{\cos x} = \frac{2}{3}$





§ 5.1 Limits & Continuity

B. $f(x) = \frac{x}{\sin x}$ for $0<|x|<\frac{\pi}{2}$ & $f(0)=1$.

Show f cont. at 0.

Find $r>0$ s.t. $|f(x)-1| < 10^{-6}$ $\forall\ |x|<r$ } same problem de facto.

want to pick r to decided

know $\sin x < x < \cancel{\tan x} = \frac{\sin x}{\cos x}$

$\Rightarrow \cos x < \frac{\sin x}{x} < 1$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ as $x \to 0$

$\Rightarrow \lim_{x\to 0} \frac{x}{\sin x} \to 1$ i.e. $\forall \varepsilon>0, \exists \delta$ s.t. $\left|\frac{x}{\sin x} - 1\right| < \varepsilon$ whenever $|x-0|<\delta$

D. Prove f cont. at $(0, y_0)$ where f on $\mathbb{R}^2$

$$f(x,y) = \begin{cases} (1+xy)^{1/x} & \text{if } x\neq 0 \\ e^y & \text{if } x=0 \end{cases}$$

know $(1+\frac{1}{x})^x \to e^1$ as $x\to\infty$

$\Rightarrow (1+xy)^{\frac{1}{x}} \to e^y$ as $\frac{1}{x}\to\infty$ i.e. $x\to 0$

Show $e^{y_0} \to e^y$ as $y_0 \to y$

take r $r = \ln(\varepsilon\cdot e^{-y_0}+1)$

if $|y-y_0| < \ln(\varepsilon\cdot e^{-y_0}+1)$

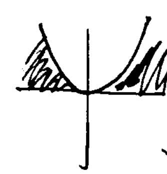
$y-y_0 < \ln(\varepsilon\cdot e^{-y_0}+1)$

~~$f(y)-f$~~

$f(y-y_0) < \varepsilon\cdot e^{-y_0}+1$

$e^{y-y_0} < \varepsilon\cdot e^{-y_0}+1$

$e^y - e^{y_0} < \varepsilon$ ✓



level set

$y = \frac{1}{2}x^2$

E. $f(x,y) = \begin{cases} 0 & \text{if } y \le 0 \text{ or if } y \ge x^2 \\ \sin\left(\frac{\pi y}{x^2}\right) & \text{if } 0 < y < x^2 \end{cases}$

(a). Show f not ~~a~~ cont. at origin

(b). Show the restriction ~~be~~ of f to any straight line through origin is continuous.

a). $f(0,0) = 0$

$\sin\left(\frac{\pi y}{x^2}\right) \to \sin\frac{\pi}{2}$?

$y < x^2$ $\quad y = \frac{1}{2}x^2 \to \sin\frac{\pi}{2} \to 1$

$y = \frac{1}{3}x^2 \to \sin\frac{\pi}{3} \to \frac{\sqrt{3}}{2}$

wts. ~~Take~~ $\exists \varepsilon > 0, \forall r > 0,$

$\exists (x,y)$ s.t.

$\|(x,y)\| < r$ but

$|f(x,y)| > \varepsilon$

Take $\varepsilon = \frac{1}{2}$

$f(x, \frac{1}{2}x^2) = \sin\left(\frac{\pi \frac{1}{2}x^2}{x^2}\right) = \sin\frac{\pi}{2} = 1 > \frac{1}{2}$

wts $\exists x$ s.t $\|(x, \frac{1}{2}x^2)\| < r^2$

wts $x^2 + \frac{1}{4}x^4 - r^2 < 0$

has soln

$\Delta = 1 + r^2 > 0$ ✓

b). Show $y = kx$

$\lim_{x \to 0} \sin\left(\frac{k\pi}{x}\right) \to 0$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\left|\sin\frac{k\pi}{x} - 0\right| < \varepsilon$ whenever $|x| < \delta$ ✓

F. (a). A function $f: S \subset \mathbb{R}^n \to \mathbb{R}^m$ has limit $\vec{v}$ as $\vec{x} \to \vec{a}$ provided that $\forall \varepsilon > 0, \exists r > 0$ s.t. $f(B_r(\vec{a}) \cap S \setminus \{a\}) \subset B_\varepsilon(\vec{v})$

Proof: $\lim_{x \to a} f(x) = v \Rightarrow \forall \varepsilon > 0, \exists \delta$, $|f(x) - f(a)| < \varepsilon$ whenever $0 < |x - a| < r$

and $x \in S$.

(b). $\forall \varepsilon > 0, \exists r > 0$ s.t $|x - a| < r$ & $x \in S \Rightarrow |f(x) - v| < \varepsilon$

$f(B_a(r) \cap S) \subseteq B_\varepsilon(v)$

G. Sps $f: \mathbb{R}^n \to \mathbb{R}$ cont. If there are $\vec{x} \in \mathbb{R}^n$ and $C \in \mathbb{R}$ s.t. $f(\vec{x}) < C$, then prove that $\exists r$ s.t. $\forall \vec{y} \in B_r(\vec{x})$, $f(\vec{y}) < C$.

Proof. Sps $\forall r$ $\exists \vec{y} \in B_r(\vec{x})$ s.t. $f(\vec{y}) \geq C$

f cont. $\Rightarrow \forall \varepsilon > 0, \exists r > 0$ s.t. $|f(y) - f(x)| < \varepsilon$ whenever $|y-x| < r$ $(y \in B_r(x))$

take $\varepsilon = C - f(x)$

$\Rightarrow |f(y) - f(x)| < \varepsilon = C - f(x)$

$f(y) < C$ contradiction.

J. Show $f: [a,b] \to \mathbb{R}$ is a diff. func. s.t. $|f'(x)| \leq M$ on $[a,b]$, then f is Lipschitz.

Pf: $\dfrac{f(b) - f(a)}{b-a} = f'(c) \leq M$

L.

Show linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ with matrix $[a_{ij}]$ can be written as $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \varepsilon_i \pi_j$

~~map~~ coordinate functions:

$\pi_j(x_1, \cdots, x_n) = x_j$ of $\mathbb{R}^n$ into $\mathbb{R}$ reads off jth coordinate

$\varepsilon_i(t) = t\vec{e_i}$ maps $\mathbb{R}$ into $\mathbb{R}^m$ by sending $\mathbb{R}$ onto the ith coordinate axis.

$\pi_j(\vec{x}) = x_j$

$\varepsilon_i \pi_j(\vec{x}) = \varepsilon_i(x_j) = x_j \vec{e_i} = (0 \ldots x_j \cdots 0)$

$\Rightarrow \|\varepsilon_i \pi_j(x) - \varepsilon_i \pi_j(y)\| = \|(0, \cdots, x_j - y_j, \cdots, 0)\|$

$\|x - y\| = \|(x_1 - y_1), (x_2 - y_2), \cdots, (x_j - y_j), \cdots \|$

clearly, $\|\varepsilon_i \pi_j(x) - \varepsilon_i \pi_j(y)\| \leq \|x - y\|$

So. $a_{ij} \varepsilon_i \pi_j(x) = A(x)$

$\|A(x) - A(y)\| = \|a_{ij}(0, \cdots, x_j - y_j, \cdots 0)\|$

$= |a_{ij}| \cdot \|0, \cdots, x_j - y_j, \cdots, 0\|$

$\leq |a_{ij}| \|x - y\|$

§ 5.2 Discontinuous Functions.

H. $f(x) = x\chi_{\mathbb{Q}}(x)$ on $\mathbb{R}$. Show f is cont. at 0 & this is the only continuous pt.

$$f(x) = \begin{cases} x & \text{when } x \in \mathbb{Q} \\ 0 & \text{when } x \notin \mathbb{Q} \end{cases}$$

① $f(x)$ cont. at 0 :

$\forall \varepsilon > 0, \exists r$, s.t. $|f(x) - f(0)| < \varepsilon$ whenever $|x-0| < r$

$\Rightarrow$ $|f(x) - 0| < \varepsilon$ whenever $|x| < r$

$|f(x)| < \varepsilon$ whenever $|x| < r$

$|x| < \varepsilon$ whenever $|x| < r$

So we only need to take $\varepsilon = r$.

done.

② only cont. at 0. (not cont. at other pts)

sps $f(x)$ cont. at $x = a \neq 0$

~~$\exists \varepsilon > 0$, s.t. $\forall r > 0$ $|f(x) - f(a)| \geq \varepsilon$ whenever~~ $\exists |x-a| < r$

if ~~$a \in \mathbb{Q}$~~

§ 5.5

A. $g(x)=\sqrt{x}$ is unif. cont. on $[0,+\infty)$

$\forall \varepsilon>0, \exists \delta>0$. s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$

$|\sqrt{x}-\sqrt{y}|<\varepsilon$ whenever $|x-y|<\delta$

$|x-y| = |\sqrt{x}+\sqrt{y}||\sqrt{x}-\sqrt{y}|$

$|\sqrt{x}-\sqrt{y}| \leq |\sqrt{x-y}| \leq |\sqrt{x+y}| \leq |\sqrt{x}+\sqrt{y}|$

$\delta > |x-y| = |\sqrt{x}+\sqrt{y}||\sqrt{x}-\sqrt{y}| \geq |\sqrt{x+y}||\sqrt{x}-\sqrt{y}| \geq |\sqrt{x-y}||\sqrt{x}-\sqrt{y}| \geq |\sqrt{x}-\sqrt{y}|^2$

$\Rightarrow |\sqrt{x}-\sqrt{y}|^2 \leq |x-y| < \delta$

so $\forall \varepsilon$ pick $\delta = \varepsilon^2$

$\Rightarrow$ if $|x-y|<\varepsilon^2$ we have $|f(x)-f(y)|<\varepsilon$.

D. Show $f(x)=x^p$ is not unif cont. on $\mathbb{R}$ if $p>1$.

$\exists \varepsilon>0, \forall r>0$, $|f(x)-f(y)|\geq\varepsilon$ for some $x,y\in S$ & $|x-y|<r$

$|f(x)-f(y)|=|x^p-y^p|$

H. given $\varepsilon>0$, $\exists \delta_1>0$ s.t. $\forall y\in[a,b]$, $(y-b)<\delta_1$

$\Rightarrow |f(y)-f(b)|<\frac{\varepsilon}{2}$

$\exists \delta_2>0$ s.t. $\forall y\in[b,c)$

$|y-b<\delta_2 \Rightarrow |f(y)-f(b)|<\frac{\varepsilon}{2}$

given $\varepsilon>0$, $\exists \delta=\delta_1+\delta_2$

$|f(x)-f(y)| \leq |f(x)-f(b)|+|f(b)-f(y)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

$|x-y| = |x-b+b-y| \leq |x-b|+|y-b| = \delta_1+\delta_2=\delta$

I. let $f(x)$ be cont. on $(0,1]$. Show f is unif. cont. iff $\lim_{x\to 0^+} f(x)$ exists

$\lim_{x\to 0^+} f(x) = L$, $\forall \varepsilon > 0, \exists r$ s.t $|f(x)-L| < \frac{\varepsilon}{2}$ whenever $0<x<r$.

Show unif. cont. on $(0,1]$

consider 2 interval $[r,1]$ and $(0,r)$

① $[\varepsilon,1]$: Compact interval

$\Rightarrow$ unif cont. on $[r,1]$

② on $(0,1)$

$\lim_{x\to 0^+} f(x) = L \Rightarrow \forall \varepsilon>0, \exists r$ s.t.

$|f(x)-L| < \frac{\varepsilon}{2} \quad \forall 0<x<r$

so far $\forall \varepsilon > 0$ we have $|f(x)-f(y)| < |f(x)-L+L-f(y)|$

$\leq |f(x)-L| + |f(y)-L|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

guaranteed when $0<x<r$ & $0<y<r$

$\Rightarrow \forall \varepsilon>0, \exists \delta = 2r > 0$

s.t. $|f(x)-f(y)| < \varepsilon$

whenever $|x-y| < \delta = 2r$

$\Rightarrow f$ uni cont. on $(0,1]$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x)-f(y)| < \varepsilon$ wherever

$|x-y| < \delta$ & $x,y \in (0,1]$.

$\Rightarrow \forall \varepsilon > 0 \; \exists \delta$ s.t. $f(x) \in B_\varepsilon(f(y))$

$\forall x \in B_\delta(y) \cap (0,1]$

Consider $y_k = \frac{1}{k} \to 0$

So $\forall x \in B_\delta(\frac{1}{k}) \cap (0,1]$

we have $f(x) \in B_\varepsilon(f(y_k)) \Rightarrow f(B_\delta(\frac{1}{k}) \cap (0,1]) \subset B_\varepsilon(f(y_k))$

§ 5.6 IVT

A. show $\exists x \in (0, \frac{\pi}{2})$, s.t. $\cos x = x$

(a). $f(x) = \cos x - x$

$f(0) = 1 > 0$, $f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$

$\Rightarrow$ IVT

(b). prove $\exists$ 1 solution.

$f'(x) = -\sin x - 1$. decreasing

B. How many Sol'n ~~in~~ $\tan x = x$ in $[0, 11]$.

~~tan~~ 0 is a solution.

let $f(x) = \tan x - x$

no solution on $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi)$

1 solution on $(\pi, \frac{3}{2}\pi)$,

no on $(\frac{3}{2}\pi, 2\pi)$

1 on $(2\pi, \frac{5}{2}\pi)$

1 on $(3\pi, \frac{7}{2}\pi)$ ✓

4 solutions.

C. $2\sin x + 3\cos x = x$ has 3 solutions

let $f(x) = 2\sin x + 3\cos x - x$

$-\sqrt{13} \leq 2\sin x + 3\cos x \leq \sqrt{13}$

Then ?

D. odd degree $\Rightarrow$ at least 1 real sol'n.

$P(x) = a_n x^n + \cdots + a_1 x + a_0$

$\lim_{x \to +\infty} \frac{P(x)}{a_n x^n} = 1$, $\lim_{x \to -\infty} \frac{P(x)}{a_n x^n} = 1$ $a_n x^n$ has different sign for $+, -\ x$

WLOG, let $a_n > 0$.

we know $P(x) > 0$ when $x \to +\infty$

$P(x) < 0$ when $x \to -\infty$

So by IVT. $\exists c$, s.t. $x = c$, $P(c) = 0$.

I. Show $\mathbb{Q}$ is not connected.

r is some irrational

$A = (-\infty, r) \cap \mathbb{Q}$

$B = (r, +\infty) \cap \mathbb{Q}$

$A \cap B = \emptyset$, $A \cup B = \mathbb{Q}$

A, B ~~is~~ open w.r.t. $\mathbb{Q}$

$\Rightarrow A, B$ separations $\Rightarrow$ disconnected.

E. $T(\vec{x})$ at $\vec{x}$ cont. Show $\exists \vec{x}$ s.t. $T(\vec{x}) = T(-\vec{x})$

$f(x) = T(x) - T(-x)$

$f(-x) = T(-x) - T(x)$

So $f(x) + f(-x) = 0$

if $f(x) = 0 \Rightarrow T(x) - T(-x) = 0$

if $f(x) > 0$ then $f(-x) < 0$. IVT.

H. (a). Show cont. on $(-\infty, \infty)$ cannot take every real value exactly twice

Sps. $f(a) = f(b) = 0$, $a < b$. WLOG for a, $c \in (a, b)$. $f(c) > 0$.

$\cancel{\Rightarrow \forall \varepsilon < f(c)}$

(b).

§5.7 Monotone

A. ~~$x_1 \le x_2 \Rightarrow g(x_1$~~

g: $x_1 \le x_2$ $\quad g(x_1) \ge g(x_2)$

f: $\quad f(g(x_1)) \le f(g(x_2))$

B. f: $x_1 \le x_2$ $\quad f(x_1) \ge f(x_2)$

$g(x_1) \ge g(x_2)$

$f(x_1) + g(x_1) \ge f(x_2) + g(x_2)$

C. $x_1 \le x_2$

$f(x_1) \ge f(x_2)$

$g(x_1) \ge g(x_2)$

$f(x_1) \cdot g(x_1) \ge f(x_2)(x_2)$ not sure about ~~state~~.

have to discuss sign.

§5.4 Compactness & Extreme

A. A noncompact subset of $\mathbb{R}^n$.

① A not closed.

$(x_n) \in A$ but $x_n \to x \notin A$

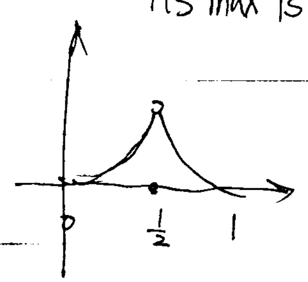
So construct $f = \|x_k - x\|$ whose domain is (x_n) & gets maximum on $f(x)$

② A not bdd

$\forall A \not\subset B_r(0)$ ~~condi~~ consider $f(x) = \frac{-1}{\|\vec{x}\|}$

its max is 0. when $\|\vec{x}\| \to \infty$.

B.



bdd in $[0,1]$. dis. no sup reached.