

University of Toronto
Department of Mathematics

MAT246H1S
Concepts in Abstract Math

Final Examination
APRIL 2010

Examiners: F. Murnaghan, P. Rosenthal

Duration: 3 hours

PLEASE HAND IN

Last Name: _____

Given Name: _____

Student Number: _____

Section: (Day or Evening) _____

No calculators or other aids are allowed. Show your work and justify your answers. Write your answer in the space provided after the question.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
TOTAL	/100

[5] 1.a) Let S be the set of pairs (m, n) of natural numbers with the property that

(i) $(1, 1)$ belongs to S ;

(ii) S contains $(m + 1, 1)$ whenever S contains $(m, 1)$; and

(iii) whenever, for each n , S contains (m, n) for all m , S contains $(m, n + 1)$ for all m .

Prove that S is the set of all pairs of natural numbers.

[5] 1.b) Use part a) to prove that $(m + 1)^n > mn$ for all natural numbers m and n .

- [10] 2. Suppose that x , y and z are integers satisfying $x^2 + y^2 = z^2$. Prove that at least one of x and y is divisible by 3.

- [5] 3. a) Suppose that m and n are relatively prime natural numbers. Prove that there exist integers x and y such that

$$\frac{1}{mn} = \frac{x}{m} + \frac{y}{n}.$$

- [5] 3. b) Give an example to show that part a) need not hold if m and n are not relatively prime.

[10] 4. Find an integer x that simultaneously satisfies both of the following congruences:

$$x \equiv 46 \pmod{51}$$

$$x \equiv 23 \pmod{52}$$

- [10] 5. Assume that a natural number n and its fifth power n^5 are represented by digits in the usual way. Prove that n and n^5 have the same final (i.e. units) digit.

[10] 6. Prove that $\sqrt{7} + \sqrt{11} + \sqrt{5}$ is irrational.

- [10] 7. Determine which (if any) of the following numbers are constructible and prove that your answers are correct.

a) $\sqrt{7} - \frac{(\sqrt[3]{5} + 4)}{18}$

b) $\sin(5^\circ)$

- [10] 8. Suppose that the sets S and T have the same cardinality as each other and that the sets U and V have the same cardinality as each other. Let

$$S \times U = \{ (s, u) \mid s \in S, u \in U \}$$

$$V \times T = \{ (v, t) \mid v \in V, t \in T \}$$

Prove that the sets $S \times U$ and $V \times T$ have the same cardinality.

- [10] 9. Let \mathbb{Q} denote the set of rational numbers. Determine the cardinality of the set of all functions mapping \mathbb{Q} into the set $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}\}$. Prove that your answer is correct.

[5] **10.a)** Is there a line in the x - y plane such that every point on it is constructible? Prove that your answer is correct.

[5] **10.b)** Is there a line in the x - y plane such that no point on it is constructible? Prove that your answer is correct.

