MAT240 Final Review

Def. complex number is an ordered pair (a,b), where $a,b\in\mathbb{R}$ but we will ~~be~~ write this as $a+bi$. The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C}=\{a+bi : a,b\in\mathbb{R}\}$$

1.1 $(x_1,\cdots,x_n)+(y_1,\cdots,y_n)=(x_1+y_1,\cdots,x_n+y_n)$

Def. A vectorspace is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:
commutativity, associativity, additive identity, additive inverse, multiplicative identity, distributive properties.

1.2 Proposition: A vector space has a unique additive identity.
$$v+0=v$$

1.3 Proposition. Every element in a vector space has a unique additive inverse.
$$v+v'=0$$

1.4. Proposition. $0v=0$ for every $v\in V$.
1.5 Proposition. $a0=0$ for every $a\in\mathbb{F}$.
1.6 Proposition: $(-1)v=-v$ for every $v\in V$.

Def. A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V)
If U is a subset of V, then to check that U is a subspace of V we need only check that U satisfies the following:
additive identity $0\in U$;
closed under addition $u,v\in U$ implies $u+v\in U$;
closed under scalar multiplication $a\in\mathbb{F}$ and $u\in U$ implies $au\in U$.

Def. Suppose $U_1,\cdots U_m$ are subspaces of V. The sum of $U_1,\ldots,U_m$, denoted $U_1+U_2+\cdots+U_m$, is defined to be the set of all possible sums of elements of $U_1,\ldots,U_m$. More precisely $U_1+U_2+\cdots+U_m=\{u_1+u_2+\cdots+u_m : u_1\in U_1,\cdots,u_m\in U_m\}$

$U=\{(x,0,0)\in\mathbb{F}^3 : x\in\mathbb{F}\}$ and $W=\{(0,y,0)\in\mathbb{F}^3 : y\in\mathbb{F}\}$
Then 1.7 $U+W=\{(x,y,0)\in\mathbb{F}^3 : x,y\in\mathbb{F}\}$

Suppose $U_1,\cdots,U_m$ are subspaces of V s.t. $V=U_1+\cdots+U_m$. Thus every element of V can be written in the form $u_1+\cdots+u_m$, $u_j\in U_j$
V is the direct sum of subspaces $U_1,\cdots,U_m$, written $V=U_1\oplus\cdots\oplus U_m$, if each element of V can be written uniquely as a sum $u_1+\cdots+u_m$ where each $u_j\in U_j$.

1.8 Proposition: Suppose that $U_1, \dots, U_n$ are subspaces of V. Then $V = U_1 \oplus \cdots \oplus U_n$ if and only if both the following conditions hold:

(a) $V = U_1 + \cdots + U_n$

(b). the only way to write 0 as a sum $u_1 + \cdots + u_n$, where each $u_j \in U_j$, is by taking all the u_j's equal to 0.

1.9 Proposition: Suppose that U and W are subspaces of V. Then $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.

Def. A linear combination of a list $(v_1, \dots, v_m)$ of vectors in V is a vector of the form

2.1 $\quad a_1 v_1 + \cdots + a_m v_m$

where $a_1, \dots, a_m \in \mathbb{F}$. The set of all linear combinations of $(v_1, \dots, v_n)$ is called the span of $(v_1, \dots, v_m)$, denoted span $(v_1, \dots, v_m)$. In other words,

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbb{F}\}$$

Span of the empty list $()$ equals $\{0\}$.

If span $(v_1, \dots, v_n)$ equals V, we say that $(v_1, \dots, v_m)$ spans V. A vector space is called finite dimensional if some list of vectors in it spans the space.

Def. A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have degree m if $\exists$ scalars $a_0, a_1, \dots a_m \in \mathbb{F}$ with $a_m \neq 0$ s.t.

2.2 $\quad p(z) = a_0 + a_1 z + \cdots + a_m z^m$, for all $z \in \mathbb{F}$

The polynomial that is identically 0 is said to have degree $-\infty$.

Def. A vector space that is not finite dimensional is called infinite dimensional

Def. A list $(v_1, \dots, v_m)$ of vectors in V is called linearly independent if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.

Def. A list $(v_1, \dots, v_m)$ of vectors in V is called linearly dependent ~~iff~~ if $\exists\, a_1, \dots, a_m \in \mathbb{F}$, not all 0, s.t. $a_1 v_1 + \cdots + a_m v_n = 0$.

2.4 Linear Dependence Lemma: If $(v_1, \dots, v_m)$ is linearly dependent in V and $v_1 \neq 0$, then $\exists\, j \in \{2, \dots, m\}$ s.t. the following hold:

(a). $v_j \in \text{span}(v_1, \dots, v_{j-1})$; (b) if the j^{th} term is removed from $(v_1, \dots, v_m)$, the span of the remaining list equals span $(v_1, \dots, v_m)$

2.6 Theorem: In a ~~particular~~ finite-dimensional vector space, the length of every linearly indep. list of vectors is less than or equal to the length of every spanning list of vectors.

2.7 ~~Theorem~~ Proposition: Every subspace of a finite-dim vector space is finite-dim!

Def. A basis of V is a list of vectors in V that is linearly indep. and spans V. For example, $((1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,\ldots,0,1))$ is a basis of $\mathbb{F}^n$, called the standard basis of $\mathbb{F}^n$.

2.8 Proposition: A list $(v_1,\cdots,v_n)$ of vectors in V is a basis of V if and only if every $v\in V$ can be written uniquely in the form

2.9 $$v=a_1v_1+\cdots+a_nv_n$$

where $a_1,\cdots,a_n\in\mathbb{F}$.

2.10 Theorem: Every spanning list in a vector space can be reduced to a basis of the vector space.

2.11 Corollary: Every finite-dim. vector space has a basis.

2.12 Theorem: Every linearly indep. list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

2.13 Proposition: Suppose V is finite dim and U is a subspace of V. Then $\exists$ a subspace W of V s.t. $V=U\oplus W$.

Def: Define the dimension as the length of a basis.

2.14 Theorem: Any two bases of a finite-dim. vector space have the same length.

2.15 Proposition: If V is finite dim. and U is a subspace of V, then $\dim U\le\dim V$

2.16 Proposition: If V is finite dim, then every spanning list of vectors in V with length $\dim V$ is a basis of V.

2.17 Proposition: If V is finite dimensional, then every linearly indep. list of vectors in V with length $\dim V$ is a basis of V.

2.18 Theorem: If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1+U_2)=\dim U_1+\dim U_2-\dim(U_1\cap U_2)$$

2.19. Proposition: Suppose V is finite-dim and $U_1,\cdots,U_m$ are subspaces of V s.t.

2.20 $$V=U_1+\cdots+U_m$$

and

2.21 $$\dim V=\dim U_1+\cdots+\dim U_m$$

Then $V=U_1\oplus\cdots\oplus U_m$

Def. A linear map from V to W is a function $T: V \to W$ with the following properties:
additivity: $T(u+v) = Tu + Tv$ for all $u, v \in V$;
homogeneity: $T(av) = a(Tv)$ for all $a \in \mathbb{F}$ and all $v \in V$.

Def: For $T \in \mathcal{L}(V,W)$, the null space of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\}.$$

3.1 Proposition: If $T \in \mathcal{L}(V,W)$, then ~~until~~ null T is a subspace of V.

Def: A linear map $T: V \to W$ is called injective if whenever $u, v \in V$ and $Tu = Tv$, we have $u = v$.

3.2 Proposition: Let $T \in \mathcal{L}(V,W)$. Then T is injective if and only if null $T = \{0\}$.

Def: For $T \in \mathcal{L}(V,W)$, the range of T, denoted range T, is the subset of W consisting of those vectors that are of the form Tv ~~or~~ for some $v \in V$:

$$\text{range } T = \{Tv : v \in V\}$$

3.3 Proposition: If $T \in \mathcal{L}(V,W)$, then range T is a subspace.

Def. A linear map $T: V \to W$ is called surjective if its range equals W.

3.4 Theorem: If V is finite dim and $T \in \mathcal{L}(V,W)$, then range T is a finite dim. subspace of W and $\dim V = \dim \text{null } T + \dim \text{range } T$.

3.5 Corollary: If V and W are finite-dim vector spaces such that $\dim V > \dim W$, then no linear map from V to W is injective.

3.6 Corollary: If V and W are finite-dim vector spaces such that $\dim V < \dim W$, then no linear map from V to W is surjective.

Def: An m-by-n matrix is a rectangular array with m rows and n columns that looks like this:

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

~~3.8~~ 3.9. Matrix addition: $\mathcal{M}(T+S) = \mathcal{M}(T) + \mathcal{M}(S)$

3.10 Matrix scalar multiplication: $\mathcal{M}(cT) = c\mathcal{M}(T)$

3.11 $\mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S)$

3.14 Proposition: Suppose that $T \in \mathcal{L}(V,W)$ and $(v_1, \dots, v_n)$ is a basis of V and $(w_1, \dots, w_m)$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v) \quad \text{for every } v \in V.$$

Def: A linear map $T \in \mathcal{L}(V,W)$ is called invertible if there ~~is~~ exists a linear map $S \in \mathcal{L}(W,V)$ such that ST equals the identity map on V and TS equals the identity map on W. A linear map $S \in \mathcal{L}(W,V)$ satisfying $ST = I$ and $TS = I$ is called an inverse of T (note that the first I is the identity map on V

3.17 Proposition: A linear map is invertible iff it is injective & surjective.

Def: Two vector spaces are called isomorphic if there is an invertible linear map from one vector ~~to~~ space onto the other one.

3.18 Theorem: Two finite-dim vector spaces are isomorphic iff they have the same dimension.

3.19 Proposition: Suppose that $(v_1, \cdots, v_n)$ is a basis of V and $(w_1, \cdots, w_m)$ is a basis of W. Then $\mathcal{M}$ is an invertible linear map between $\mathcal{L}(V, W)$ and $\mathrm{Mat}(m, n, \mathbb{F})$

3.20 Proposition: If V and W are finite dim, then $\mathcal{L}(V, W)$ is finite dim and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

3.21 Theorem: Suppose V is finite dim. If $T \in \mathcal{L}(V)$, then the following are equivalent:

(a). T is invertible;

(b) T is injective;

(c). T is surjective.

Def: We say that U is invariant under T if $u \in U$ implies $Tu \in U$. In other words, U is invariant under T if $T|_U$ is an operator on U.

Def: A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of $T \in \mathcal{L}(V)$ if there exists a nonzero vector $u \in V$ s.t. $Tu = \lambda u$.

Def: A vector $u \in V$ is called an ~~eigenvalue~~ eigenvectors of T (corresponding to λ) if $Tu = \lambda u$.

5.6 Theorem: Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T and $v_1, \cdots, v_m$ are corresponding nonzero eigenvectors. Then $(v_1, \cdots, v_m)$ is linearly independent

5.9 Corollary: Each operator on V has at most $\dim V$ distinct eigenvalues.

5.10 Theorem: Every operator on a finite-dim, nonzero, complex vector space has an

Suppose $(v_1, \cdots, v_n)$ is a basis of V. For each $k = 1, \cdots, n$. we can write

$$Tv_k = a_{1,k} v_1 + \cdots + a_{n,k} v_n.$$

where $a_{j,k} \in \mathbb{F}$ for $j = 1, \cdots, n$. The n-by-n matrix

5.11
$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}$$

is called the matrix of T with respect to the basis $(v_1, \cdots, v_n)$;

Def: The diagonal of a square matrix consists of the entries along the straight line from the upper left corner to the bottom right corner.

5.12 Proposition: Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V. The following are equivalent.

(a) the matrix of T with respect to $(v_1, \cdots, v_n)$ is upper triangular.

(b) $Tv_k \in \operatorname{span}(v_1, \cdots, v_k)$ for each $k = 1, \ldots, n$;

(c). $\operatorname{span}(v_1, \cdots, v_k)$ is invariant under T for each $k = 1, \cdots, n$.

5.13 Theorem: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

5.16 Proposition: Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then T is invertible iff all the entries on the diagonal of that upper-triangular matrix are nonzero.

5.18 Proposition: Suppose $T \in \mathcal{L}(V)$ has an upper-tri matrix with respect to some basis of V. Then the eigenvalues of T consist precisely of the entries on the diagonal of that upper-trian... matrix.

Def: A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.

5.20. Proposition: If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, ~~there~~ then T has a diagonal matrix with respect to some basis of V.

5.21 Proposition: Suppose $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a). T has a diagonal matrix with respect to some basis of V;

(b) V has a basis consisting of eigenvectors of T;

s.t. $V = U_1 \oplus \cdots \oplus U_n$

(c). there $\exists$ one-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T,

(d). $V = \operatorname{null}(T - \lambda_1 I) \oplus \cdots \oplus \operatorname{null}(T - \lambda_m I)$

(e). $\dim V = \dim \operatorname{null}(T - \lambda_1 I) + \cdots + \dim \operatorname{null}(T - \lambda_m I)$

Def: Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. A vector $v \in V$ is called a generalized ~~eigenvalue~~ eigenvector of T corresponding to λ if

8.3 $\quad (T-\lambda I)^j v = 0 \quad$ for some positive integer j.

8.5 Proposition: If $T \in \mathcal{L}(V)$ and $\underline{m}$ is a nonnegative integer such that $\text{null } T^m = \text{null } T^{m+1}$, then

$$\text{null } T^0 \subset \text{null } T^1 \subset \cdots \subset \text{null } T^m = \text{null } T^{m+1} = \text{null } T^{m+2} = \cdots$$

8.6 Proposition: If $T \in \mathcal{L}(V)$, then

$$\text{null } T^{\dim V} = \text{null } T^{\dim V+1} = \text{null } T^{\dim V+2} = \cdots$$

8.7 ~~Corall~~ Corollary: Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. Then the set of generalized eigenvectors of T corresponding to λ ~~the definition~~ equals $\text{null } (T-\lambda I)^{\dim V}$.

Def: An operator is called nilpotent if some power of it equals 0. For example, the operator $N \in \mathcal{L}(\mathbf{F}^4)$ defined by

$$N(z_1, z_2, z_3, z_4) = (z_3, z_4, 0, 0)$$

8.8 Corollary: Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $N^{\dim V} = 0$.

8.9 Proposition: If $T \in \mathcal{L}(V)$, then

$$\text{range } T^{\dim V} = \text{range } T^{\dim V+1} = \text{range } T^{\dim V+2} = \cdots$$

8.10 Theorem: Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then for every basis of V with respect to which T has an upper-tri matrix, λ appears on the diagonal of the matrix of T precisely $\dim \text{null } (T-\lambda I)^{\dim V}$ times.

8.18 Proposition: If V is a complex vector space and $T \in \mathcal{L}(V)$, then the sum of the multiplicities of all the eigenvalues of T equals $\dim V$.

Def: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Let d_j denote the multiplicity of λ_j as an eigenvalue of T. The polynomial

$$(z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$$

is called the characteristic polynomial.

[redacted]

8.20 Cayley-Hamilton Theorem: Suppose that V is a complex vector space and $T \in \mathcal{L}(V)$. Let q denote the characteristic poly

8.22 Proposition: If $T \in \mathcal{L}(V)$ and $p(\mathbb{F})$, then null $p(T)$ is invarient under T.

8.23 Theorem: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ be the distinct eigenvalues of T, and let $U_1, \cdots, U_m$ be the corresponding subspaces of generalized ~~eigenvalues~~ eigenvectors. Then

(a) $V = U_1 \oplus \cdots \oplus U_m$

(b) each U_j is invariant under T,

(c) each $(T - \lambda_j I)|U_j$ is nilpotent.

8.25 Corollary: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

8.26 Lemma: Suppose N is a nilpotent operator ~~of~~ on V. Then there is a basis of V with respect to which the matrix of N has the form

$$\begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix};$$

here all entries on and below the diagonal are 0's.

8.28 Theorem: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ be the distinct eigenvalues of T. Then there is a basis of V with respect to which T has a block diagonal matrix of the form.

$$A_j = \begin{bmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}$$

Def A linear combination of $(I, T, T^2, \cdots, T^{m-1})$, scalars $a_0, a_1, a_2, \cdots, a_{m-1} \in \mathbb{F}$ s.t.

$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{m-1} T^{m-1} + T^m = 0$$

The polynomial

$a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + z^m$ is called the minimal polynomial of T. It is the monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree s.t. $p(T) = 0$.

e.g. poly min of Identity operator I is $z - 1$. The min poly of operator on $\mathbb{F}^2$ whose matrix $\begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$ is $20 - 9z + z^2$.

8.34 THM: Let $T \in \mathcal{L}(V)$ and let $q \in P(F)$. The $q(T) = 0$ iff the min. poly of T ~~di~~ divides q.

8.36 THM: Let $T \in \mathcal{L}(V)$, Then the roots of the min. poly of T are precisely the eigenvalues of T.

8.~~3~~47 THM: Suppose V is a complex vector space. If $T \in \mathcal{L}(V)$ then $\exists$ a basis of V that is a Jordan basis for T.