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$10 + 3 = \frac{13}{20}$

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Q1:

1(a). Solution:

Note: Hermitian inner product $\langle u, v\rangle = \overline{\langle v, u\rangle}$

$W = \text{span}\{(0, -i, 1), (1+i, 2, 1)\}$

Perform Gram-Schmidt Process

Let $u_1 = (0, -i, 1)$, $u_2 = (1+i, 2, 1)$

$v_1 = u_1 = (0, -i, 1)$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$

$= (1+i, 2, 1) - \frac{(1+i)0 + 2i + 1}{0 + (-i\cdot i) + 1}(0, -i, 1)$

$= (1+i, 2, 1) - (i + \frac{1}{2})(0, -i, 1)$

$= (1+i, 2, 1) - (0, 1 - \frac{1}{2}i, \frac{1}{2} + i)$

$= (1+i, 1 + \frac{1}{2}i, \frac{1}{2} - i)$

Therefore $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{0^2 + 1^2 + 1^2}}(0, -i, 1) = \frac{1}{\sqrt{2}}(0, -i, 1)$

$w_2 = \frac{v_2}{\|v_2\|} = \frac{(1+i, 1+\frac{1}{2}i, \frac{1}{2} - i)}{\sqrt{(1+i)(1-i) + (1+\frac{1}{2}i)(1-\frac{1}{2}i) + (\frac{1}{2} - i)(\frac{1}{2} + i)}}$

$= \frac{1}{\sqrt{(1+1) + (1+\frac{1}{4}) + (\frac{1}{4} + 1)}}(1+i, 1+\frac{1}{2}i, \frac{1}{2} - i)$

$2 +$
$4 + \frac{1}{2} = \frac{9}{2}$

$= \frac{1}{\sqrt{\frac{9}{2}}}(1+i, 1+\frac{1}{2}i, \frac{1}{2} - i)$

$= \frac{\sqrt{2}}{3}(1+i, 1+\frac{1}{2}i, \frac{1}{2} - i)$

Hence $\{\frac{1}{\sqrt{2}}(0, -i, 1), \frac{\sqrt{2}}{3}(1+i, 1+\frac{1}{2}i, \frac{1}{2} - i)\}$ is such an orthonormal basis.

$w_1 = \frac{1}{\sqrt{2}}(0, -i, 1)$

$w_2 = \frac{\sqrt{2}}{3}(1+i, 1+\frac{1}{2}i, \frac{1}{2}-i)$

(2). Solution

since $P_W(\vec{x}) = \sum_{i=1}^{k} \langle x, w_i \rangle w_i$ for $x = (x_1, x_2, x_3) \in \mathbb{C}^3$

$$\boxed{Tx_i = \langle x_i, w_1 \rangle w_1 + \langle x_i, w_2 \rangle w_2}$$

$$T(1,0,0) = \frac{1}{2} 0 \cdot (0, -i, 1) + \frac{2}{9}(1 \cdot (1-i))(1+i, 1+\frac{1}{2}i, \frac{1}{2}-i)$$

$$= \frac{2}{9}(2, \frac{3}{2} - \frac{1}{2}i, \frac{3}{2} - \frac{3}{2}i)$$

$$= (\frac{4}{9}, \frac{1}{3} - \frac{1}{9}i, -\frac{1}{9} - \frac{1}{3}i)$$

$$T(0,1,0) = \frac{1}{2}(i)(0, -i, 1) + \frac{2}{9}(1 - \frac{1}{2}i)(1+i, 1+\frac{1}{2}i, \frac{1}{2}-i)$$

$$= (0, \frac{1}{2}, \frac{i}{2}) + (\frac{1}{3} + \frac{1}{9}i, \frac{5}{18}, -\frac{5}{18}i)$$

$$= (\frac{1}{3} + \frac{1}{9}i, \frac{7}{9}, \frac{2}{9}i)$$

$$T(0,0,1) = \frac{1}{2}(1)(0, -i, 1) + \frac{2}{9}(\frac{1}{2} + i)(1+i, 1+\frac{1}{2}i, \frac{1}{2}-i)$$

$$= (0, -\frac{1}{2}i, \frac{1}{2}) + (-\frac{1}{9} + \frac{1}{3}i, \frac{5}{18}i, \frac{5}{18})$$

$$= (-\frac{1}{9} + \frac{1}{3}i, -\frac{2}{9}i, \frac{7}{9})$$

Hence $(x_1, x_2, x_3)[T] = \langle x_i, w_1 \rangle w_1 + \langle x_i, w_2 \rangle w_2$

$$T = \begin{bmatrix} \frac{4}{9} & \frac{1}{3} + \frac{1}{9}i & -\frac{1}{9} + \frac{1}{3}i \\ \frac{1}{3} - \frac{1}{9}i & \frac{7}{9} & -\frac{2}{9}i \\ -\frac{1}{9} - \frac{1}{3}i & \frac{2}{9}i & \frac{7}{9} \end{bmatrix}$$

is the matrix of the orthogonal projection onto W with respect to the standard basis of $\mathbb{C}^3$.

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Q2.

Proof: Since $\langle p(x), q(x)\rangle = p(x_0)\overline{q(x_0)} + \cdots + p(x_n)\overline{q(x_n)}$ for $p(x), q(x) \in P_n(\mathbb{C})$

① For $p(x), q(x)$ and $r(x) \in P_n(\mathbb{C})$ and $a, b \in \mathbb{C}$

$\langle ap(x)+bq(x), r(x)\rangle$

$= (ap(x_0)+bq(x_0))\overline{r(x_0)} + (ap(x_1)+bq(x_1))\overline{r(x_1)} + \cdots + (ap(x_n)+bq(x_n))\overline{r(x_n)}$

$= ap(x_0)\overline{r(x_0)} + ap(x_1)\overline{r(x_1)} + \cdots + ap(x_n)\overline{r(x_n)} + bq(x_0)\overline{r(x_0)} + \cdots + bq(x_n)\overline{r(x_n)}$

$= a(p(x_0)\overline{r(x_0)} + p(x_1)\overline{r(x_1)} + \cdots + p(x_n)\overline{r(x_n)}) + b(q(x_0)\overline{r(x_0)} + q(x_1)\overline{r(x_1)} + \cdots + q(x_n)\overline{r(x_n)})$

$= a\langle p(x), r(x)\rangle + b\langle q(x), r(x)\rangle$ verified.

② For $p(x), q(x) \in P_n(\mathbb{C})$,

$\langle p(x), q(x)\rangle = p(x_0)\overline{q(x_0)} + p(x_1)\overline{q(x_1)} + \cdots + p(x_n)\overline{q(x_n)}$

$= \overline{\overline{p(x_0)}q(x_0)} + \overline{\overline{p(x_1)}q(x_1)} + \cdots + \overline{\overline{p(x_n)}q(x_n)}$

$= \overline{\langle p(x), q(x)\rangle}$ verified.

③ For all $p(x) \in P_n(\mathbb{C})$, suppose $p(x_k) = a_k + b_k i$ for all $a_k, b_k \in \mathbb{R}$ and $k = 0, 1, 2, \cdots, n$

$\langle p(x), p(x)\rangle = p(x_0)\overline{p(x_0)} + p(x_1)\overline{p(x_1)} + \cdots + p(x_n)\overline{p(x_n)}$

$= (a_0+b_0i)(a_0-b_0i) + (a_1+b_1i)(a_1-b_1i) + \cdots + (a_n+b_ni)(a_n-b_ni)$

$= a_0^2+b_0^2 + a_1^2+b_1^2 + \cdots + a_n^2+b_n^2 \geq 0$ verified.

If $\langle p(x), p(x)\rangle = a_0^2+b_0^2+a_1^2+b_1^2+\cdots+a_n^2+b_n^2 = 0$

Since $a_k, b_k \in \mathbb{R}$

$a_0 = a_1 = \cdots = a_n = b_0 = b_1 = \cdots = b_n = 0$.

Thus $p(x) = (0+0i)(0-0i) + \cdots + (0+0i)(0-0i) = 0$ verified.

Hence the three properties of a Hermitian inner product have been proved

i.e. $\langle p(x), q(x)\rangle = p(x_0)\overline{q(x_0)} + \cdots + p(x_n)\overline{q(x_n)}$ defines an inner product on $P_n(\mathbb{C})$.

Q3.

Solution: We want to make S be orthogonal.

Let $f(x)=3x^2-2x-1$

$g(x)=cx^2+x-1$

$h(x)=5x^2+cx-9$

then $\langle f(x), g(x)\rangle=0$

$\langle f(x), h(x)\rangle=0$

$\langle g(x), h(x)\rangle=0$

Since $\langle p(x), q(x)\rangle=p(-1)q(-1)+p(0)q(0)+p(2)q(2)$

then $\langle f(x), g(x)\rangle=f(-1)g(-1)+f(0)g(0)+f(2)g(2)$

$$=(3(-1)^2-2\times(-1)-1)(c(-1)^2-1-1)$$
$$+(-1)\cdot(-1)+(3\times2^2-2\times2-1)(4c+2-1)$$
$$=4(c-2)+1+7(4c+1)$$
$$=4c-8+1+28c+7$$
$$=28c+4c$$
$$=32c=0$$

So c is set to be 0.

Check: $\langle f(x), h(x)\rangle=f(-1)h(-1)+f(0)h(0)+f(2)h(2)$

$$=4(5-9)+(-1)(-9)+7(20-9)$$
$$=-16+9+77$$
$$=70\neq0$$

Therefore this is a contradiction to $\langle f(x), h(x)\rangle=0$.

\# Hence this problem doesn't have a c such that S is orthogonal.

[redacted]

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Q4.

Solution: $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{C})$. Let $u_1=1$, $u_2=x$, $u_3=x^2$

Gram-Schmidt Process

$v_1 = u_1 = 1$

Note: $\langle p(x), q(x)\rangle = p(0)\overline{q(0)} + p(i)\overline{q(i)} + p(2i)\overline{q(2i)}$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$= x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}\cdot 1$$

$$= x - \frac{0\cdot 1 + i\cdot 1 + 2i\cdot 1}{1\cdot 1 + 1\cdot 1 + 1\cdot 1}\cdot 1$$

$$= x - \frac{3i}{3}$$

$$= x - i$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}\cdot 1 - \frac{\langle x^2, x-i\rangle}{\langle x-i, x-i\rangle}\cdot(x-i)$$

$$= x^2 - \frac{0\cdot 1 + (-1)\cdot 1 + (-4)\cdot 1}{1\cdot 1 + 1\cdot 1 + 1\cdot 1}\cdot 1 - \frac{0\cdot i + (-1)\cdot 0 + (-4)(-i)}{(-i\cdot i) + 0 + i\cdot(-i)}(x-i)$$

$$= x^2 + \frac{5}{3} - 2i(x-i)$$

$$= x^2 - 2ix - 2 + \frac{5}{3} = x^2 - 2ix - \frac{1}{3}$$

Since $\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{\langle 1, 1\rangle} = \sqrt{3}$

$$\|v_2\| = \sqrt{\langle v_2, v_2\rangle} = \sqrt{\langle x-i, x-i\rangle}$$

$$= \sqrt{(0-i)\overline{(0-i)} + (i-i)\overline{(i-i)} + (2i-i)\overline{(2i-i)}}$$

$$= \sqrt{(-i)i + 0 + i(-i)}$$

$$= \sqrt{1+0+1}$$

$$= \sqrt{2}$$

$$\|v_3\| = \sqrt{\langle v_3, v_3\rangle} = \sqrt{\langle x^2-2ix-\tfrac{1}{3},\ x^2-2ix-\tfrac{1}{3}\rangle}$$

$$= \sqrt{(0-0-\tfrac{1}{3})(0-0-\tfrac{1}{3}) + (i^2-2i^2-\tfrac{1}{3})(i^2-2i^2-\tfrac{1}{3}) + (4i^2-4i^2-\tfrac{1}{3})(4i^2-4i^2-\tfrac{1}{3})}$$

$$= \sqrt{\tfrac{1}{9}+\tfrac{4}{9}+\tfrac{1}{9}}$$

$$= \sqrt{\tfrac{6}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

Hence $\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\} = \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}(x-i), \frac{3}{\sqrt{6}}(x^2-2ix-\frac{1}{3})\right\}$

is an orthonormal basis for $P_2(\mathbb{C})$.

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Q5.

Proof:

$(i) \Rightarrow (ii)$

Suppose $v \in W_1 + W_2$ can be expressed in two ways

Then $v = w_1 + w_2 = w_1' + w_2'$

hence $w_1 - w_1' = w_2' - w_2$

Note that $w_1 - w_1' \in W_1$ and $w_2' - w_2 \in W_2$

Since W_1 and W_2 are two subspaces.

So $v \in W_1 \cap W_2$

By (i) $V = W_1 \oplus W_2 \Rightarrow W_1 \cap W_2 = \{0\}$

so $w_1 - w_1' = w_2' - w_2 = 0$

Therefore $w_1 = w_1'$, $w_2 = w_2'$

Contradiction

Hence $\forall v \in V$ can be written uniquely as $v = w_1 + w_2$ for $w_1 \in W_1$, $w_2 \in W_2$.

$(ii) \Rightarrow (iii)$

If $w_1 + w_2 = 0 = 0 + 0$ this is unique

then $w_1 = w_2 = 0$

And since $\forall v \in V$, $v = w_1 + w_2$ for some w_1, w_2.

So $V = W_1 + W_2$.

$(iii) \Rightarrow (i)$ ~~Suppose α_1 is a basis for W_1, α_2 is a basis for W_2.~~

$V = W_1 + W_2$ for $w_1 \in W_1$, $w_2 \in W_2$ if $w_1 + w_2 = 0$ then $w_1 = w_2 = 0$

Since $v = w_1 + w_2 = 0$ only if $w_1 = w_2 = 0$

So $W_1 \cap W_2 = \{0\}$

Then $V = W_1 \oplus W_2$ i.e. $(iii) \Rightarrow (i)$ ~~for V.~~

~~Therefore $\dim V = \dim W_1 + \dim W_2$ and $\alpha = \alpha_1 \cup \alpha_2 = \alpha_1 + \alpha_2$ for α is a basis~~

~~Check $\langle \alpha, c \rangle = 0$.~~

~~$\langle \alpha_1 + \alpha_2, c \rangle = \langle \alpha_1, c \rangle + \langle \alpha_2, c \rangle = 0$~~

~~Since $w_1 + w_2 = 0 \Rightarrow w_1 = w_2 = 0$~~

So ~~$\langle \alpha_1, c \rangle = \langle \alpha_2, c \rangle = 0 \Rightarrow c = 0$ so α is a basis for V.~~

So far we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), it means whenever we have one of them we can derive the other two statements So they are equivalent.

Then we only need (i) $\Leftrightarrow$ (iv) to say four statements are equivalent.

(i) $\Rightarrow$ (iv)

Since (i) $\Rightarrow$ (ii) $V = W_1 + W_2$ uniquely.

suppose $\alpha_1 = \{x_1, \cdots, x_n\}$, $\alpha_2 = \{y_1, \cdots, y_n\}$ which are basis for W_1 and W_2 respectively (why $\dim W_1 = \dim W_2$? Because they are two subspaces of V).

Then $w_1 = a_1x_1 + \cdots + a_nx_n$

$w_2 = b_1y_1 + \cdots + b_ny_n$ for $a_i, b_i \in \mathbb{R}$

$w_1 + w_2 = a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_ny_n$

know that $\alpha = \alpha_1 \cup \alpha_2 = \{x_1, \cdots, x_n, y_1, \cdots, y_n\}$

if $a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_ny_n = 0$

then $a_1x_1 + \cdots + a_nx_n = -b_1y_1 - \cdots - b_ny_n$

so $a_1x_1 + \cdots + a_nx_n$, $-b_1y_1 - \cdots - b_ny_n \in W_1 \cap W_2 = \{0\}$ (by definition of (i))

as $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ are linearly independent

therefore $a_1 = \cdots = a_n = -b_1 = \cdots = -b_n = 0$

Hence $\alpha = \alpha_1 \cup \alpha_2$ is a basis for $W_1 + W_2 = V$ (since $W_1 \oplus W_2 = V \Rightarrow W_1 + W_2 = V$)

(iv) $\Rightarrow$ (i)

Conversely, $\alpha = \alpha_1 \cup \alpha_2$ is a basis for V, α_1 is a basis for W_1, α_2 is a basis for W_2

Then $V = W_1 + W_2$

Say $a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots b_ny_n = 0$

only if $a_1 = \cdots = a_n = 0 = b_1 = \cdots = b_n$

then $w_1 = a_1x_1 + \cdots + a_nx_n$

$w_2 = b_1y_1 + \cdots + b_ny_n$ are linearly independent $w_1 = w_2$ only if they both are 0.

Hence $W_1 \cap W_2 = \{0\}$

Therefore $V = W_1 \oplus W_2$

($\rightarrow$) So we now have 4 statements equivalent. [redacted]

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Q 6.

Proof: $W_1 \cap W_2 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = A^T = -A^T\}$

Hence $A^T = -A^T = 0 \Rightarrow A = 0$

So $W_1 \cap W_2 = \{0\}$

If we have now 2 skew-symmetric matrices A, B such that $A = -A^T$, $B = -B^T$

Then $(A+B)^T = +A^T + B^T = -A - B = -(A+B)$

$(\alpha A)^T = -(\alpha A)$

Now $\forall$ matrix $A \in M_{n\times n}(\mathbb{R})$ can be written as a sum of $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Since $A + A^T = (A + A^T)^T = A^T + A$

hence B is symmetric.

Since $A - A^T = -(A - A^T)^T = -A^T + A = A - A^T$

hence C is skew-symmetric

Therefore $M_{n\times n}(\mathbb{R}) = W_1 + W_2$

So $M_{n\times n}(\mathbb{R}) = W_1 \oplus W_2$

[redacted]

Q7 Claim: $T: V \to V$ only two distinct λ_1 & λ_2, then T is diagonalizable iff $V = E_{\lambda_1} \oplus E_{\lambda_2}$

Proof:

($\Rightarrow$) Suppose T is diagonalizable.

Then $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = \dim(V)$

so $E_{\lambda_1} + E_{\lambda_2} = V$.

Need $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$

Suppose α is a basis for V.

α_1 is a basis for E_{λ_1} $(\alpha_1 = (a_1, a_2, \cdots, a_n)$

α_2 is a basis for E_{λ_2} $\alpha_2 = (b_1, b_2, \cdots, b_n)$

By the proof in Q5.

$\alpha = \alpha_1 \cup \alpha_2$ is a basis for V.

Say $E_{\lambda_1} = \text{span}\{\alpha_1\}$

$E_{\lambda_2} = \text{span}\{\alpha_2\}$

?

Now suppose $x_i = \{x_1^{\lambda_i}, x_2^{\lambda_i}, \cdots, x_n^{\lambda_i}\}$ for $i = 1, 2$

$v \in E_{\lambda_1} \cap E_{\lambda_2}$

then $(a_1 x_1^{\lambda_1} + a_2 x_2^{\lambda_1} + \cdots + a_n x_n^{\lambda_1}) - (b_1 x_1^{\lambda_2} + \cdots + b_n x_n^{\lambda_2}) = 0$

so $a_1 = a_2 = \cdots = a_n = b_1 = b_2 = \cdots = b_n = 0$

Hence $v = 0$?

so $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$ proved.

Therefore $E_{\lambda_1} \oplus E_{\lambda_2} = V$.

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Converse?