University of Toronto

MAT237Y1Y PROBLEM SET 3

DUE: End of tutorial, Thursday June 13th, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.

2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.

3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.

4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. Suppose that F(x,y,z) = 0 is an equation that can be solved to yield any of the three variables as a function of the other two. Use implicit differentiation to show that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

2. The wave equation:

   a) Show that $u(x,t) = f(x-ct) + g(x+ct)$, where c is a constant, is a solution of the wave equation in one dimension, ie $\partial_t^2 u = c^2 \partial_x^2 u$. For your interest, note that this solution to the wave equation consists of two functions who keep the same shape but travel to the left and right with speed c.

b) For $\mathbf{x} = (x, y, z) \neq (0, 0, 0)$ and $t \in \mathbb{R}$, show that $u(\mathbf{x}, t) = r^{-1}g(ct - r)$, where $r = |\mathbf{x}|$ and $g : \mathbb{R} \to \mathbb{R}$ is a C^2 function, is a solution to the wave equation in three dimensions, i.e., $\partial_t^2 u = c^2(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u)$.

Hint: Read the solutions to exercises 4 and 5 from 2.2 to see examples of functions satisfying partial differential equations.

3. Let $w = f(x, y, s, t) = x^2y + s^2x + t$ where $x = g_1(s, t) = s + t$, $y = g_2(s, t) = t^2$. Use the method of differentials (and chain rule, not substitution) to estimate the change in w going from $(s, t) = (1, 2)$ to $(s, t) = (1.03, 1.98)$. Now suppose we were to only change one of the independent variables, which should we change to result in the largest change in w?

4.

a) Sketch the surface formed by $F(x, y, z) = 0$ for

$$F(x, y, z) = x^2 + y^2 + z^2 - 1$$

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b) Consider two paths $\mathbf{g_1} : [0, \pi] \subset \mathbb{R} \to \mathbb{R}^3$, $\mathbf{g_2} : [0, \pi] \subset \mathbb{R} \to \mathbb{R}^3$ given by

$$\mathbf{g_1}(t) = (cos(t), 0, sin(t))$$

$$\mathbf{g_2}(t) = (0, cos(t), sin(t))$$

Prove that the image of $\mathbf{g_1}$ and $\mathbf{g_2}$ is contained on the surface for all $t \in [0, \pi]$ and prove that the intersection point of the image of these two paths is at (0,0,1). Now sketch the image of $\mathbf{g_1}$ and $\mathbf{g_2}$, clearly labelling the two paths and their intersection point.

c) Compute $\mathbf{g_1'}(t^*)$ and $\mathbf{g_2'}(t^*)$ where t^* is the value of t where the image of the paths intersect (and which you should compute). Sketch these two vectors on your surface.

d) Compute $\nabla F(0, 0, 1)$ and write down a formula for the tangent plane at (0,0,1). Now sketch both the plane and $\nabla F(0, 0, 1)$ on our surface.

e) Verify that $\mathbf{g_1'}(t^*)$, $\mathbf{g_2'}(t^*)$, and $\nabla F(0, 0, 1)$ are all orthogonal to each other.

Note: Please draw several sketches as one goes along so as to keep everything clear, don't try to get it all on one.

5. The Minkowski Sum of of two subsets S_1, S_2 of $\mathbb{R}^n$ (or, more generally, any vector space) is defined to be

$$S_1 + S_2 := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in S_1,\ \mathbf{y} \in S_2\}$$

For

$$S_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$$

$$S_2 = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 0, 1 \leq y \leq 2\}$$

First sketch S_1, S_2, and $S_1 + S_2$. Then, prove that $S_1 + S_2$ is convex. Finally prove (Hint: one line each) that $S_1 + S_2$ is pathwise connected and connected.

6. When we get to Section 2.8, we will need to use some more Linear Algebra that wasn't covered in the original Linear Algebra Quiz. This question is intended as a review of Linear Algebra you should have already been taught in preparation for 2.8. Please consult your Linear Algebra Textbook.

   a) For a matrix A, state what it means for λ to be an eigenvector of A, and for $\mathbf{x}$ to be a corresponding eigenvector. Determine all eigenvalues and their corresponding eigenvectors for the the following matrix

   $$\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

   b) Now normalize the eigenvectors (by dividing by their lengths) and construct a matrix P whose columns consist of the normalized eigenvectors. Two matrices B and C are said to be similar if there exists an invertible matrix P such that $P^{-1}BP = C$ and it can be shown such matrices have similar properties; in particular they share the same eigenvalues. Let D be the diagonal matrix whose entries along the main diagonal are the eigenvalues of A, appearing in the same order as the columns of P.

   $$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   Use the Matrix P above to show that A (from part a)) and D are similar. We will thus say that A is diagonalizable.

Enjoy!