March 8th

if $P_1(z), P_2(z)$ — polynomials then can divide P_1 by P_2 with a remainder
i.e. there exist polynomials $Q(z)$ and $R(z)$

s.t. $P_1(z) = P_2(z)Q(z) + R(z)$ $\boxed{\deg(R(z)) < \deg P_2(z)}$

Ex: divide $P_1(z) = z^4 - 3z^2 + 2$ by $P_2(z) = z^2 + z - 1$

$$
\begin{array}{r}
z^2 - z - 1 \\
z^2+z-1 \,\big)\, \overline{z^4 - 3z^2 + 2} \\
\underline{z^4 + z^3 - z^2} \\
-z^3 - 2z^2 + 2 \\
\underline{-z^3 - z^2 + z} \\
-z^2 - z + 2 \\
\underline{-z^2 - z + 1} \\
1
\end{array}
$$

$z^4 - 3z^2 + 2 = (z^2 + z - 1)(z^2 - z - 1) + 1$

If $P(z) = a_n z^n + \dots + a_1 z + a_0$ $a_n \neq 0$, $a_i \in \mathbb{C}$
then $P(z)$ has a complex root z_1

i.e. $P(z_1) = 0$

divide $P(z)$ by $z - z_1$ with remainder
$P(z) = Q(z)(z - z_1) + R(z)$

$\deg(R(z)) < 1$
$\deg(R(z)) = 0$
$\Rightarrow R(z) = c$ — constant

$P(z) = Q(z)(z - z_1) + c$

Claim: $c = 0$. plug in $z = z_1$

$0 = P(z_1) = Q(z_1)(z_1 - z_1) + c = c$

$\Rightarrow c = 0$

$\Rightarrow P(z) = Q(z)(z - z_1)$

$\deg(z - z_1) = 1$

Ex: $P(z) = z^2 - 3z + 2$, $z = 1 \Rightarrow P(1) = 0$?
divide $z^2 - 3z + 2$ by $z - 1$

$z^2 - 3z + 2 = (z-1)(z-2)$

Ex:

$P(z) = z^3 + 2z^2 + 4z + 3$

$z_1 = -1$ is a root

$P(-1) = 0$

divide $z^3 + 2z^2 + 4z + 3$ by $z+1$

$\longrightarrow z^3 + 2z^2 + 4z + 3 = (z+1)(z^2 + z + 3) + 0$

$P(z) = a_n z^n + \cdots + a_1 z + a_0$

z_1 is a root iff $P(z) = (z - z_1) Q(z) + 0$

if $P(z) = (z - z_1) Q(z) \Rightarrow$

$z = z_1 \Rightarrow P(z_1) = 0$

$P(z) = (z - z_1) Q(z)$

$\deg P = n \Rightarrow \deg Q = n-1$

$\Rightarrow$ by the Fundamental theorem of algebra, it has a root z_2

$\Rightarrow Q(z) = (z - z_2) Q_2(z) \leftarrow \deg = n-2$ etc.

$\Rightarrow P(z) = (z - z_1)(z - z_2) Q_2(z)$

$\Rightarrow$ after n steps we get

$P(z) = a(z - z_1)(z - z_2) \cdots (z - z_n)$ complex number

ex: $z^2 - 3z + 2 = (z-1)(z-2)$

$z^2 - 4z + 4 = (z-2)(z-2)$

$\Rightarrow$ the roots of $P(z)$ are $z_1, \cdots, z_n$ (and nothing else)

$\Rightarrow P(z)$ has at most n roots

$P(z) = a_n (z - z_1)^{k_1} (z - z_2)^{k_2} \cdots (z - z_l)^{k_l}$

z_i's are distinct

$k_1 + \cdots + k_l = n$

$P(z) = (z-1)(z-1)(z+2)(z+2)(z+2)$

$= (z-1)^2 (z+1)^3$

ex: $z^6 - 2z^2 - 3 = 0$

Solve over $\mathbb{C}$ we'll have 6 roots

$z^3 = y \quad y^2 - 2y - 3 = (y-3)(y+1) = 0$

(2) $z^3 = 3$ or (1) $z^3 = -1$

(1) $-1 = 1(\cos\pi + i\sin\pi)$

$z_k = \cos\left(\frac{\pi + 2\pi k}{3}\right) + i\sin\left(\frac{\pi + 2\pi k}{3}\right)$

$k = 0, 1, 2$

$k=0, z_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{3} i$

$k=1, z_1 = \cos\pi + i\sin\pi = -1$

$k=2, z_2 = \frac{1}{2} - \frac{\sqrt{3}}{3} i$

so $z^3+1 = (z-(\frac{1}{2}+\frac{\sqrt{3}}{2}i))(z+1)(z-(\frac{1}{2}-\frac{\sqrt{3}}{2}i))$

2) $z^3-3=0$ $\quad z = 3(\cos 0 + i\sin 0)$

z_4, z_5, z_6 $\quad 3^{\frac{1}{3}}(\cos\frac{0+2\pi k}{3} + i\sin\frac{0+2\pi k}{3})$

$k=0,1,2$

$z_4 = \sqrt[3]{3}(\cos 0 + i\sin 0) = \sqrt[3]{3}$

$z_5 = \sqrt[3]{3}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = \sqrt[3]{3}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

$z_6 = \sqrt[3]{3}(\cos\frac{4\pi}{3} + i\sin\frac{4}{3}\pi) = \sqrt[3]{3}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$

HW: if $p(z) = \cdots$ if all a_i's are real then if z_1 is a root of $P(z)$ then $\overline{z_1}$ is also a root.

$z^6 - 2z^3 - 3$

$(1+i)z^4 - (2-3i)z + 6 = 0 \Rightarrow$ not true that roots come in conjugate pairs