Lecture 6

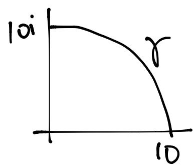
Line Integral & Green's Thm

Let γ be piecewise smooth curve and $f: D \to \mathbb{C}$ (D - domain) continuous, so that $\gamma \subseteq D$. Define the line integral

$$\boxed{\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt}$$

For γ, $z=\gamma(t)$, $\frac{dz}{dt} = \gamma'(t) \Rightarrow dz = \gamma'(t)dt$

<u>Ex</u>: Let $f(z)=z^2$, γ = quarter of circle (radius 10, centered at 0, joining 10 to 10i)



$$\gamma(t) = 10e^{it} \quad 0 \le t \le \frac{\pi}{2}$$

$$\gamma'(t) = i\,10e^{it}$$

$$\int_\gamma z^2 dz = \int_0^{\pi/2} (\gamma(t))^2 \cdot \gamma'(t)dt = \int_0^{\frac{\pi}{2}} (10e^{it})^2 \cdot (i10e^{it})dt$$

$$= \int_0^{\frac{\pi}{2}} 1000\, i e^{i3t} dt = \frac{1000 i \cdot e^{i3t}}{3i} \Big|_0^{\frac{\pi}{2}}$$

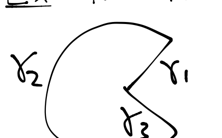
$$= 1000/3 \cdot e^{i\,3\pi/2} - \frac{1000}{3} e^{i\cdot 0}$$

$$= -\frac{1000}{3} i - \frac{1000}{3} = \frac{1000}{3}(-1-i)$$

Note: If $\gamma = \gamma_1 - \gamma_2$ is p.w. smooth, then

$$\int_\gamma f(z)dz = \int_{\gamma_1 \cup \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

<u>Ex</u>: for "Pac-man" Example



$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$$

i.e. $\int_a^b = -\int_b^a$

Some useful ESTIMATES

- $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$
- $\text{length}(\gamma) = \int_a^b |\gamma'(t)|\,dt$

important ☆ • $\left|\int_\gamma f(z)dz\right| \le \text{length}(\gamma) \cdot \max_{z \text{ on } \gamma} |f(z)|$

USEFUL FACT:

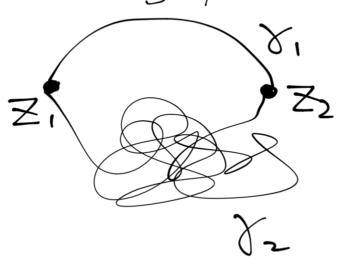
Lets look at $\int_\gamma z^m dz$:

Parameterize γ: $\gamma : [a,b] \Rightarrow \mathbb{C}$

$$\int_\gamma z^m dz = \int_a^b (\gamma(t))^m \cdot \gamma'(t)dt$$
$$= \frac{\gamma^{m+1}(t)}{m+1}\Big|_a^b$$
$$= \frac{1}{m+1}(\underbrace{\gamma^{m+1}(b)}_{\text{end pt}} - \underbrace{\gamma^{m+1}(a)}_{\text{initial pt}})$$
$$= \frac{1}{m+1} z^{m+1} \Big|_{\text{initial pt}}^{\text{end pt}}$$

answer only depends on end pts of γ, not γ itself.

i.e.



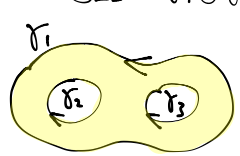
$$\Longrightarrow \int_{\gamma_1} z^m dz = \frac{1}{m+1}(z_1^{m+1} - z_2^{m+1})$$
$$= \int_{\gamma_2} z^m dz$$

F.T.C $= F(b) - F(a)$ $\quad (F' = f)$

Green's THM

vectors Field version: Ω = domain in $\mathbb{R}^2$

$\partial\Omega = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ $\quad \gamma$'s p.w. smooth simple closed



$\vec{F} = (F_1(x,y), F_2(x,y))$ smooth vector field

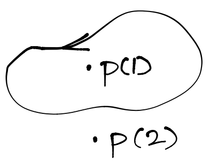
$$\boxed{\int_{\partial\Omega} \vec{F}(x,y)\,d\vec{\gamma} = \iint_\Omega \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy}$$

Complex Version:

Ω: domain with boundary $\partial\Omega = \gamma_1 \cup \cdots \cup \gamma_n$
γ's p.w. smooth, simple, closed curves oriented positively.
Let f be a cts function.

$$\boxed{\int_{\partial\Omega} f(z)\,dz = i \iint_{\Omega} \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy \qquad \text{where } \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},\quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \qquad \text{note: } f = u + iv}$$

Ex: Let γ be p.w. smooth simple closed curve oriented pos.
Find $\displaystyle\int_\gamma \frac{1}{z-p}\,dz$ for $p \notin \gamma$



two cases (1). p inside γ
(2) p outside γ

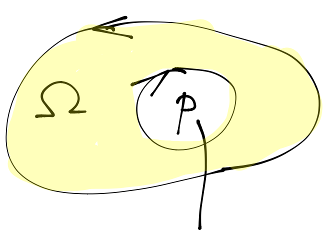
Case (2): Ω = inside of γ
Then $\frac{1}{z-p}$ is cts inside Ω

$$\frac{\partial f}{\partial x} = \frac{-1}{(z-p)^2}, \quad \frac{\partial f}{\partial y} = \frac{-i}{(z-p)^2}$$

So then $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{-1}{(z-p)^2} + \frac{1}{(z-p)^2} = 0$

So $\displaystyle\int_\gamma \frac{1}{z-p}\,dz = i\iint_\Omega 0\,dx\,dy = 0$

Case (1): $\displaystyle i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy = \int_{\partial\Omega} \frac{1}{z-p}\,dz = \int_{\gamma \cup C} \frac{1}{z-p}\,dz$



$$= \int_\gamma \frac{1}{z-p}\,dz + \int_C \frac{1}{z-p}\,dz$$

$$\Rightarrow \int_\gamma \frac{1}{z-p} = \int_{-C} \frac{1}{z-p}\,dz$$

we can calculate this

An easy calculation shows:

counter clockwise $\rightarrow$ $\displaystyle\int_{-C} \frac{1}{z-p}\,dz = 2\pi i$