AST121 Problem Set 2 Solutions

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1 Gravity

1.1 Acceleration as a differential

Acceleration: $a = \frac{-GM}{r^2}$.

But $a = \frac{dv}{dt}$, so $\frac{dv}{dt} = \frac{-GM}{r^2}$. This is a differential equation describing variation of velocity with respect to gravitational acceleration.

1.2 Conservation of Energy

Now we have to show that $\frac{dv}{dt} = \frac{-GM}{r^2} = \frac{\frac{d}{dt}(\frac{GM}{r})}{v}$. We can approach this two ways - doing it 'forwards' with integration, or working 'backwards' from the answer by evaluating $\frac{d}{dt}(\frac{GM}{r})$. The latter approach is easier but the former is arguably more correct.

Using integration:

$\frac{dv}{dt} = \frac{-GM}{r^2}$. But $\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = v \cdot \frac{dv}{dr}$, so we can write $v\frac{dv}{dr} = \frac{-GM}{r^2}$ and then $vdv = \frac{-GMdr}{r^2}$. Then taking the (indefinite) integral of both sides:

$\int vdv = \int \frac{-GMdr}{r^2}$

$\frac{v^2}{2} + c = \frac{GM}{r} + c$, where c is some constant. You can see we now have the GM/r term we were looking for and just need to take the derivative of both sides again:

$\frac{d}{dt}(\frac{v^2}{2} + c) = \frac{d}{dt}(\frac{GM}{r} + c) = \frac{d}{dt}(\frac{GM}{r}) + \frac{d}{dt}(c)$, where we note that the derivative of a sum is the sum of the derivatives.

Now recall that the derivative of a constant is zero. As well, the time derivative of a function of time (which both r and v are) requires using the chain rule:

$\frac{d}{dt}(\frac{v^2}{2}) = \frac{d}{dv}(\frac{v^2}{2})\frac{dv}{dt} = (\frac{2v}{2})\frac{dv}{dt} = a \cdot v$. So we have shown that:

$a \cdot v = \frac{d}{dt}(\frac{GM}{r})$, so $a = \frac{\frac{d}{dt}(\frac{GM}{r})}{v}$. But $a = -\frac{GM}{r^2}$, so:

$-\frac{GM}{r^2} = \frac{\frac{d}{dt}(\frac{GM}{r})}{v}$, which is exactly what we set out to prove.

Using derivatives:

We can simply evaluate $\frac{d}{dt}(\frac{GM}{r})$, working backwards from the answer. While this is generally not very useful if you don't have the answer to begin with, it is an approach one can take if the alternatives aren't working out for whatever reason.

$\frac{d}{dt}(\frac{GM}{r}) = GM\frac{d}{dt}(\frac{1}{r}) = GM \cdot (-\frac{1}{r^2}) \cdot \frac{dr}{dt} = -\frac{GM}{r^2} \cdot v$. If we plug this in to the right side of the equation we are trying to prove:

$\frac{\frac{d}{dt}(\frac{GM}{r})}{v} = \frac{-\frac{GM}{r^2} \cdot v}{v} = -\frac{GM}{r^2}$, which is exactly what we were attempting to show.

Now we will prove conservation of energy, which requires us to show that:

$\frac{d}{dt}(\frac{v^2}{2} - \frac{GM}{r}) = 0$.

The term $\frac{v^2}{2}$ is simply the kinetic energy of the object divided by its mass, while $-\frac{GM}{r}$ is the potential energy (again divided by the mass of the object). As long as the mass of the object remains constant and gravity is the only force acting on the object, the sum of these two terms will remain constant. We can see this by recalling that the derivative of a constant is zero, so:

$\frac{v^2}{2} - \frac{GM}{r} = constant.$

This constant is called the specific orbital energy and determines the nature of the orbit - elliptical if negative, parabolic if zero and hyperbolic. Elliptical orbits are typical for motions of bound objects such as planets around a star or stars around the center of a galaxy. Parabolic orbits orbits are also referred to as zero-energy orbits and are relevant for objects barely escaping from a surface (as we will see shortly) and in many cases orbits of galaxies around each other.

To prove conservation of energy, note:

$-\frac{GM}{r^2} = \frac{\frac{d}{dt}(\frac{GM}{r})}{v}$, so $-\frac{GMv}{r^2} = \frac{d}{dt}(\frac{GM}{r})$ and $-\frac{GMv}{r^2} - \frac{d}{dt}(\frac{GM}{r}) = 0$, which is halfway to the solution. We now only need to prove that:

$-\frac{GMv}{r^2} = \frac{d}{dt}(\frac{v^2}{2})$, where $-\frac{GMv}{r^2} = av$. In fact, we already showed above that:

$\frac{d}{dt}(\frac{v^2}{2}) = \frac{d}{dv}(\frac{v^2}{2})\frac{dv}{dt} = (\frac{2v}{2})\frac{dv}{dt} = a \cdot v$.

Or equivalently, one could write that $\int \frac{v^2 dt}{2} = a \cdot v + c$.

1.3 Maximum Height

Here we will use conservation of energy to determine the maximum height of an object thrown from the surface of a planet (or really, any other massive object).

$\frac{v^2}{2} - \frac{GM}{r} = constant$ is a statement of conservation of energy.

For the case of throwing an apple upwards from the surface of the Earth, there are two 'special' values of r (the distance from the apple to the centre of the Earth) and v that we can consider.

As we throw the apple from the surface, $r = R_0$ (the radius of the Earth) and $v = v_0$ (the speed we throw the apple at). So:

$\frac{v_0^2}{2} - \frac{GM}{R_0} = constant$.

At some point, the apple will reach its maximum height (which we will call h) and begin falling again. At that point, $r = R_0 + h$ and $v = 0$. So:

$\frac{0}{2} - \frac{GM}{R_0+h} = constant$.

Since energy is conserved, these two constants are equal to each other:

$\frac{v_0^2}{2} - \frac{GM}{R_0} = -\frac{GM}{R_0+h}$. We can rearrange this equation by multiplying by $R_0 + h$ and dividing by the left hand side:

$(R_0 + h) = -\frac{GM}{\frac{v_0^2}{2} - \frac{GM}{R_0}}$, which gives:

$h = -\frac{GM}{\frac{v_0^2}{2} - \frac{GM}{R_0}} - R_0$. This can be further rearrange into neater but otherwise equivalent expressions relating the maximum height to the initial velocity of the apple and the mass and radius of the Earth. Note that this does not in any way depend on the mass of the apple.

1.4 Escape Velocity

Finally, we can define a third 'special' value of r at infinity. In this case, there is a minimum speed one must throw the apple at in order to escape the gravitational pull of the Earth or equivalently never turn around and fall back down. That speed is the escape velocity v_{esc}. If we again use energy conservations:

$\frac{v_0^2}{2} - \frac{GM}{R_0} = -\frac{GM}{R_0+h}$, we have:

$\frac{v_{esc}^2}{2} - \frac{GM}{R_0} = -\frac{GM}{R_0+\infty} = 0$, which is exactly the case of a parabolic or zero-energy orbit. So:

$\frac{v_{esc}^2}{2} - \frac{GM}{R_0} = 0$, $v_{esc}^2 = \frac{2GM}{R_0}$ and $v_{esc} = \sqrt{\frac{2GM}{R_0}}$. For the Earth this is about 11.2 km/s.

2 Kepler's Law

Kepler's third law of planetary motion relates the period P of a planet around the sun with its semi-major axis a:

$P^2 = a^3$, where the period is in years and the semi-major axis in AU. This is true for planets around a star with the mass of the sun; it is more generally from Newton's laws that:

$P^2 = \frac{4\pi^2 a^3}{GM_{star}}$.

Eris' semi-major axis is 67 AU (note: the text of the assignment says this is its mean distance, which was a typo). So its period is:

$P = a^{3/2} = (67)^{3/2} = 549$ years.

Now how big does Eris appear from Earth? This is asking for an angle, which has units of radians or degrees. Note a circle has 2π radians = 360 degrees, so 1 degree = 0.01745.

To find the angle, draw a triangle with a base equal to the distance r from Earth to Eris (which varies during the year but can be approximated as 67 AU) and the height equal to the diameter d of Eris, which is given as 2400 km.

Now $tan(\theta) = d/r$.[1] We can use the approximation that $tan(\theta) = sin(\theta) = \theta$ for $\theta \ll 1$, so $\theta = 2400km/67AU = 2.39 \cdot 10^{-7} radians = 1.37 \cdot 10^{-5} degrees = 0.0494 arcseconds$.

For reference, the angular sizes of the sun and moon are roughly equal at about 31 arcminutes (half of a degree) each. The human eye cannot resolve objects smaller than 3-4 arcseconds in optimal conditions, while the Hubble space telescope can barely resolve Eris with a diffraction limit of 0.05 arcseconds for violet light.

Any object smaller than this size will be unresolved, i.e. it will look like a point source to a naked eye or a telescope. Point sources like stars appear to twinkle (a phenomenon called scintillation) due to turbulence in the atmosphere, while larger objects like planets do not - although one may notice the edges of the moon shifting with a good telescope or camera. A (dwarf) planet like Eris appears so small that it would twinkle like a star... if it was bright enough for us to see from the ground in the first place.

3 Fundamental Forces

In this question, we will calculate the relative strengths of three of the four fundamental forces (electromagnetic, gravitational and strong nuclear) within a proton.

To calculate the forces, place one down quark (charge $q = -e/3$, mass $m \sim 4.9 MeV/c^2 = 8.7 \cdot 10^{-30}$ kg) at a distance of $10^{-15} m$ from two down quarks (combined $q = 2 \cdot 2/3$, $m \sim 2 \cdot 2.4 MeV/c^2 = 8.6 \cdot 10^{-30}$ kg). This is only a rough estimate since the down quarks are not in the exact same position but nonetheless gives a reasonable answer.

3.1 Electrostatic force

This force keeps negatively charged electrons orbiting around protons. It also attracts oppositely charged quarks inside a proton.

$F_e = -k_e q_1 q_2 / r^2 = -\frac{1}{4\pi\epsilon_0} \cdot \frac{(-e/3)(4e/3)}{(10^{-15} m)^2} = 102N$. This is an extremely large force for such tiny particles, equivalent to the force of gravity on a very fat cat or small dog.

The energy is the force times the distance - or more specifically, this is a quantity usually called 'work' in classical physics.

$E = F \cdot d = 102N/10^{-15} m = 10^{-13} J = 0.64 MeV$.

3.2 Gravitational force

$F_e = -GmM/r^2 = -G \cdot \frac{8.7 \cdot 10^{-30} kg \cdot 8.6 \cdot 10^{-30} kg}{(10^{-15} m)^2} = 4.99 \cdot 10^{-39} N$. This is $5 \cdot 10^{-41}$ times smaller than the electrostatic force and contains only a measly

$3.1 \cdot 10^{-35} eV$. Gravity is far too weak to keep atoms together. However, the gravitational force is still important in nature because unlike the electrostatic force, it can only attract and does not repel. Furthermore, the universe has been neutral for billions of years now.

3.3 Strong force

We don't need to know how the strong force operates to estimate its energy - simply assume that the $1 GeV$ of energy holding a proton together comes from the strong force, since it clearly does not originate from the electrostatic or gravitational forces.

$F = E/d = 1 GeV/10^{-15} m = 1.6 \cdot 10^5 N$, over a thousand times stronger than the electrostatic force. The strong force binds atomic nuclei together, preventing protons from repelling each other. However, its range is very short, so it is not important outside of atoms. In fact, it is so short that it cannot hold nuclei much larger than lead together, which is why very massive nuclei like uranium are unstable and tend to decay radioactively over time.

[1] Technically, one should use $tan(\theta/2) = (d/2)/r$, but the difference is negligible for small angles.