STAT6046 Assignment

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Question 1

(1)

Before the calculation, we need to note that:

- The effective monthly interest rate of bank saving is $\frac{0.04}{12}$.
- The salary is updated every year with `salary` $\cdot 1.04$.
- The property price is updated every year with `property` $\cdot 1.05$.
- The expense is updated every month with `expense` $\cdot 1.002$.
- The bank balance is updated with interest compounding as `balance` $\cdot (1 + \frac{0.04}{12})$, then plus the salary of current month, minus the expense of current month.
- The stopping condition is that the bank balance is greater or equal to 20% of the current property price.

I created a data frame by month, with property price, the salary, expense and balance of that month in a row. The simulation loop will run until the stopping condition is satisfied.

```
##    month year property  salary     expense     balance
## 43    43    4   694575 5624.32 2718.844146 122644.1305
## 44    44    4   694575 5624.32 2724.281834 125952.9824
## 45    45    4   694575 5624.32 2729.730398 129267.4153
## 46    46    4   694575 5624.32 2735.189859 132587.4368
## 47    47    4   694575 5624.32 2740.660239 135913.0547
## 48    48    4   694575 5624.32 2746.141559 139244.2767
```

```
## Duration: 48 months.
```

Therefore, **48 months** after the starting date (1/1/2018), which is now **31/12/2021**, I can purchse the home.

(2)

(i)

The periods number of 30 years home loan is $n = 12 \times 30 = 360$.

The amount of loan to pay is the 80% percent of property price on 31/12/2021, which is

```
## [1] 555660
```

The effective monthly interest rate for home loan is $i = \frac{0.045}{12}$:

```
## [1] 0.00375
```

Correspondingly, $v = (1 + i)^{-1}$ is equal to

```
## [1] 0.99626401
```

Then by the formula: $L = Ka_{\overline{360}|}$, the **monthly level payment** K is

```
## [1] 2815.447593
```

```
## The monthly level payment K is 2815.45
```

(ii)

The **total interest** is calculated as

$$K \cdot (n - a_{\overline{360|}})$$

```
## [1] 457901.1333
```

```
## The total interest we need to pay is 457901.13
```

And the **interest portion of the 15th payment** is calculated as

$$K \cdot (1 - v^{15-1})$$

```
## [1] 2044.358996
```

```
## The interset portion of the 15th payment we need to pay is 2044.36
```

(iii)

The basic rules of cash flows are similar to the ones we set in part (1). But we should notice that there are some other differences:

- The starting date is now the end of month 48. The amount of downpayment has already been deducted from bank balance.
- The cash flows affecting bank balance every month are salary, expense, bank interests and monthly level payment which is the K we calculated in the previous question.

$$\text{new balance} = \text{last balance} \times (1 + \text{interest rate}) + \text{salary} - \text{expense} - \text{level payment}$$

```
##     month.new year.new  salary.new expense.new balance.new
## 355       355       30 18241.90548 5581.668814 2119127.683
## 356       356       30 18241.90548 5592.832152 2136025.068
## 357       357       30 18241.90548 5604.017816 2152967.592
## 358       358       30 18241.90548 5615.225851 2169955.382
## 359       359       30 18241.90548 5626.456303 2186988.569
## 360       360       30 18241.90548 5637.709216 2204067.279
```

```
## The balance after paying off is 2204067.28
```

(3)

(i)

The quickest way is to **not save any money in bank**, i.e. we just pay for the loan with all salary excluding monthly expense. Why is this? Because the nominal intereset rate of loan (4.5% p.a.) is greater than the nominal interest rate of investment (4% p.a.). In other words, if we save $1 in bank for a month, the interest we gain can never cover the interest generated by loan.

(ii)

We collect some information from the end of month 48 (31/12/2021), and we are paying for an outstanding balance which is equal to 80% of property price at that time.

Meanwhile, for each month, we don't pay the level payment K. Instead, we pay as much as we can. The amount we paid consists of two parts:

- One part used to cover the interest of outstanding balance generated monthly.
- One part as principle payment.

The first part can be calculated by

$$\text{outstanding balance from last month} \times \text{loan interest}$$

The second part can be calculated as

$$\text{salary} - \text{expense} + \text{bank balance from last month} - \text{loan interest}$$

The stopping condition is when the outstanding balance hits zero, i.e. when we pay off.

```
##     month.quick year.quick salary.quick expense.quick payment.quick
## 173         173         15  10129.08258   3880.062717   6249.019860
## 174         174         15  10129.08258   3887.822842   6241.259735
## 175         175         15  10129.08258   3895.598488   6233.484089
## 176         176         15  10129.08258   3903.389685   6225.692892
## 177         177         15  10129.08258   3911.196464   6217.886112
## 178         178         15  10129.08258   3919.018857   6210.063720
##     principle.payment interest.payment      ob.loan
## 173       6127.041760     121.97810031 26400.451656
## 174       6142.258041      99.00169371 20258.193615
## 175       6157.515863      75.96822606 14100.677752
## 176       6172.815350      52.87754157  7927.862402
## 177       6188.156628      29.72948401  1739.705773
## 178       6203.539823       6.52389665 -4463.834050
```

```
## We pay off in month 178
```

Thus, 178 **months** after we pay the downpayment, we can finally pay off the home loan. And the date will be **31/10/2036**.

(iii)

Note that the **last payment** is the outstanding balance the one month ago compounded with loan interest for one month extra, i.e.

$$\text{OB}_{177} \cdot (1 + i_{\text{loan}})$$

```
## The last payment is 1746.23
```

Question 2

(1)

First we need to get the basic information at the time we invest (either) two bonds, which is 2 years after we purchased our home, i.e. 31/12/2023.

```
##    month.new year.new  salary.new expense.new balance.new
## 24        24        2 6083.264512  2881.03303 8675.930286
```

The Bond price P is the bank balance at that time.

The **nominal amount of Bond A** is

$$P \cdot (1 + 6\%)^{15}$$

```
## [1] 20792.37181
```

```
## The nominal amount of Bond A is 20792.37
```

The half-yearly coupon rate is $\frac{0.04}{2} = 0.02$.

And the half-yearly redemption rate is $j = 1.09^{1/2} - 1$.

```
## [1] 0.04403065089
```

With the value of j, we can calculate **the nominal amount of Bond B**:

$$\frac{P}{r \cdot a_{\overline{360}|j} + v_j^{30}}$$

```
## [1] 14362.60289
```

```
## The nominal amount of Bond B is 14362.6
```

(2)

This is the case when income tax and capital gain tax are considered.

Bond A

For Bond A which matures in 15 years, on the date 31/12/2038, i.e. on that day, a large amount of mony is added to my balance. After that, we still have 13 years (30-2-15=13) of loans to pay (K). So now we set year 2024 as year 1, and we end the procedure by the end of year 2051 which is year 28.

```
##     month.A year.A    salary.A   expense.A   balance.A
## 331     331     28 18241.90548 5581.668814 2121383.200
## 332     332     28 18241.90548 5592.832152 2138288.103
## 333     333     28 18241.90548 5604.017816 2155238.170
## 334     334     28 18241.90548 5615.225851 2172233.530
## 335     335     28 18241.90548 5626.456303 2189274.310
## 336     336     28 18241.90548 5637.709216 2206360.639
```

```
## If we buy Bond A, at the end of the 30-year loan, we have $ 2206360.64 in our balance.
```

Bond B

Bond B matures in 10-15 years, but we know that $P < C$ so there is capital gain. Thus the issuer would prefer a longer maturity. Therefore, the length of Bond B should also be 15 years, which are equivalent to 30 periods because it is compounded half-yearly).

Also, we have to note that the gross yield of Bond B haven't changed, so the nominal amount of Bond B is the still the value we calculated without taxes.

```
##     month.B year.B    salary.B   expense.B   balance.B
## 331     331     28 18241.90548 5581.668814 2129393.296
## 332     332     28 18241.90548 5592.832152 2146324.899
## 333     333     28 18241.90548 5604.017816 2163301.755
## 334     334     28 18241.90548 5615.225851 2180323.993
## 335     335     28 18241.90548 5626.456303 2197391.741
## 336     336     28 18241.90548 5637.709216 2214505.129
```

```
## If we buy Bond B, at the end of the 30-year loan, we have $ 2214505.13 in our balance.
```

So far, we have already calculated the balances after buying Bond A ($2206360.64) and Bond B ($2214505.13) and paying off all the loans. So our strategy would be choose Bond B.

Appendix

Code

```
# ---------------- Q1 ----------------
# ---------------- (1) ----------------

options(digits=10)

df <- data.frame()
month <- 1
year <- 1
property <- 600000
salary <- 5000
expense <- 2500
balance <- 0
inv.rate <- 0.04/12

while (balance < property * 0.2) {
    if (month %% 12 == 1 && year != 1) {
        salary <- salary * 1.04
        property <- property * 1.05
    }
    if (month != 1) {
        expense <- expense * 1.002
    }
    balance <- balance * (1 + inv.rate)
    balance <- balance + salary - expense
    df <- rbind(df, c(month, year, property, salary, expense, balance))
    month <- month + 1
    if (month %% 12 == 1) {
        year <- year + 1
```

```
    }
}
colnames(df) <- c('month', 'year', 'property',
                  'salary', 'expense', 'balance')
end.month <- month - 1
tail(df)
cat("Duration:", end.month, "months.")
# --------------- (2) ---------------
(loan <- df[end.month,]$property * 0.8) # BIG L

n <- 12 * 30
(i.loan.rate <-  0.045/12)
(v.loan.rate <- 1 / (1 + i.loan.rate))
(K <- loan / ((1 - v.loan.rate ** n) / i.loan.rate))
cat("The monthly level payment K is", round(K, 2))
(total.interest <- K * (n - (1 - v.loan.rate ** n) / i.loan.rate))
cat("The total interest we need to pay is", round(total.interest, 2))
(interest15 <- K * (1 - v.loan.rate ** (n - (15-1))))
cat("The interset portion of the 15th payment we need to pay is", round(interest15, 2))
df2 <- data.frame()
month.new <- 1
year.new <- 1
payment <- K
salary.new <- df[end.month,]$salary # starting salary in month 48
expense.new <- df[end.month,]$expense # starting expense month 48
balance.new <- df[end.month,]$balance - df[end.month,]$property * 0.2
# starting balance in month 48

while (month.new <= 360) {
    if (month.new %% 12 == 1) {
        salary.new <- salary.new * 1.04
    }
    expense.new <- expense.new * 1.002
    balance.new <- balance.new * (1 + inv.rate)
    balance.new <- balance.new + salary.new - expense.new - payment
    df2 <- rbind(df2, c(month.new, year.new, salary.new, expense.new, balance.new))
    month.new <- month.new + 1
    if (month.new %% 12 == 1) {
        year.new <- year.new + 1
    }
}
colnames(df2) <- c('month.new', 'year.new', 'salary.new', 'expense.new', 'balance.new')
tail(df2)

cat("The balance after paying off is", round(df2[360,]$balance.new, 2))
# --------------- (3) ---------------
df3 <- data.frame()
ob.loan <- loan
month.quick <- 1
year.quick <- 1
salary.quick <- df[end.month,]$salary # salary in month 48
expense.quick <- df[end.month,]$expense # expense in month 48
balance.quick <- df[end.month,]$balance - df[end.month,]$property * 0.2 # balance in month 48
```

```
payment.quick <- salary.quick - expense.quick + balance.quick

while(ob.loan > 0) {
    if (month.quick %% 12 == 1) {
        salary.quick <- salary.quick * 1.04
    }
    expense.quick <- expense.quick * 1.002
    balance.quick <- balance.quick * (1 + inv.rate)
    payment.quick <- salary.quick - expense.quick + balance.quick
    balance.quick <- 0
    interest.payment <- i.loan.rate * ob.loan
    principle.payment <- payment.quick - interest.payment
    ob.loan <- ob.loan - principle.payment
    df3 <- rbind(df3, c(month.quick, year.quick, salary.quick, expense.quick,
                        payment.quick, principle.payment, interest.payment, ob.loan))
    month.quick <- month.quick + 1
    if (month.quick %% 12 == 1){
        year.quick <- year.quick + 1
    }
}
colnames(df3) <- c('month.quick', 'year.quick', 'salary.quick', 'expense.quick',
                   'payment.quick', 'principle.payment', 'interest.payment', 'ob.loan')
tail(df3)
cat("We pay off in month", tail(df3, 1)$month.quick)
cat("The last payment is", round(df3[tail(df3,1)$month.quick-1,]$ob.loan*(1+i.loan.rate), 2))
# --------------- Q2 ---------------
# --------------- (1) ---------------
df2[24,]
# No tax considered
P <- df2[24, ]$balance.new
(nominal.A <- P * ((1.06) ** 15))
cat("The nominal amount of Bond A is", round(nominal.A, 2))
r <- 0.04 / 2 # half-yearly coupon rate
(j <- 1.09 ** (1/2) - 1) # half-yearly 'redemption rate'
v_j <- 1 / (1+j)
a_30.j <- (1-v_j**30)/j
(nominal.B <- P / (r * a_30.j + v_j ** 30))
cat("The nominal amount of Bond B is", round(nominal.B, 2))
# --------------- (2) ---------------
coupon.tax <- 0.2
capital.gain.tax <- 0.3
# --------------- Bond A ---------------

df4 <- data.frame()
month.A <- 1
year.A <- 1
# level payment is still K
salary.A <- df2[24,]$salary.new # starting salary on 31/12/2023
expense.A <- df2[24,]$expense.new # starting expense on 31/12/2023
balance.A <- 0 # starting balance on 31/12/2023

while (month.A <= 12 * 28) {
    if (month.A %% 12 == 1) {
```

```
        salary.A <- salary.A * 1.04
    }
    expense.A <- expense.A * 1.002
    balance.A <- balance.A * (1 + inv.rate)
    balance.A <- balance.A + salary.A - expense.A - K
    if (month.A == 12 * 15) {
        balance.A <- balance.A + nominal.A - (nominal.A - P) * (capital.gain.tax)
        # NOTE: the an amount of capital gain tax charged
    }
    df4 <- rbind(df4, c(month.A, year.A, salary.A, expense.A, balance.A))
    month.A <- month.A + 1
    if (month.A %% 12 == 1) {
        year.A <- year.A + 1
    }
}
colnames(df4) <- c('month.A', 'year.A', 'salary.A', 'expense.A', 'balance.A')
tail(df4)

cat('If we buy Bond A, at the end of the 30-year loan, we have $',
    round(df4[336,]$balance.A, 2), 'in our balance.')

# --------------- Bond B ---------------

df5 <- data.frame()
month.B <- 1
year.B <- 1
# level payment is still K
salary.B <- df2[24,]$salary.new # starting salary on 31/12/2023
expense.B <- df2[24,]$expense.new # starting expense on 31/12/2023
balance.B <- 0 # starting balance on 31/12/2023

while (month.B <= 12 * 28) {
    if (month.B %% 12 == 1) {
        salary.B <- salary.B * 1.04
    }
    expense.B <- expense.B * 1.002
    balance.B <- balance.B * (1 + inv.rate)
    balance.B <- balance.B + salary.B - expense.B - K
    if (month.B %% 6 == 0 && year.B <= 15) {
        balance.B <- balance.B + nominal.B * r * (1-coupon.tax)
        # NOTE: an amount of coupon tax charged
    }
    if (month.B == 12 * 15 && year.B == 15) {
        balance.B <- balance.B + (nominal.B - (nominal.B - P) * capital.gain.tax)
        # NOTE: an amount of capital gain tax charged
    }
    df5 <- rbind(df5, c(month.B, year.B, salary.B, expense.B, balance.B))
    month.B <- month.B + 1
    if (month.B %% 12 == 1) {
        year.B <- year.B + 1
    }
}
colnames(df5) <- c('month.B', 'year.B', 'salary.B', 'expense.B', 'balance.B')
```

```r
tail(df5)

cat('If we buy Bond B, at the end of the 30-year loan, we have $',
    round(df5[336,]$balance.B, 2), 'in our balance.')
```