# **DERIVATIONS: NATURAL DEDUCTION** Part 1

#### 3.2 E1

Which inference rule justifies the following arguments? (mp, mt, dn or none)

a) 
$$\sim R \rightarrow P$$
  
 $\sim R$ 

∴P

b) 
$$\sim\sim S \rightarrow T$$

 $\therefore S \rightarrow T$ 

c) 
$$P \rightarrow \sim Q$$

Q ∴ ~P

d) 
$$(P \rightarrow \sim R) \rightarrow \sim S$$

 $\therefore \sim (P \rightarrow \sim R)$ 

MP

NONE! DN cannot be used on a

sentential part

NONE! First you must use DN, then

vou can use MT

e) ~(~P →Q) ∴P→Q

 $P \rightarrow (P \rightarrow \sim P)$ 

 $\therefore P \rightarrow \sim P$ 

g)  $S \rightarrow R$ ~P

h)  $Q \rightarrow (S \rightarrow P)$  $\sim (S \rightarrow P)$ ∴~Q

None!

MP

NONE!

∴ ~S

MT

MT

#### 3.2 E2

What can you infer (if anything) in one step from the following? What rule of inference are you using? (mp, mt, dn)

∴?

b) 
$$\sim \sim (V \rightarrow W)$$
  
 $\sim W$ 

.. ?

c)  $\sim$ S  $\rightarrow \sim\sim$ T ~S

·: ?

d) 
$$\sim Y \rightarrow \sim Z$$
  
 $\sim Z$   
 $\therefore$ ?

nothing with MP/MT

nothing in one step with MP/MT.

After DN on the first

~~T MP

nothing with MP/MT

e)  $P \rightarrow (Q \rightarrow R)$  $\sim Q \rightarrow R$ 

∴ ?

premise, MT yields ~V. f)  $P \rightarrow (Q \rightarrow R)$  $\sim (Q \rightarrow R)$ 

∴ ?

g)  $\sim \sim (\sim P \rightarrow \sim \sim \sim Q)$  h)  $\sim Z \rightarrow \sim X$  $\therefore$ ?

nothing with MP/MT

~P MT

 $(P \rightarrow Q) \rightarrow R$  j)  $X \rightarrow \sim Y$  $P \rightarrow Q$ 

∴ ?

*:* ?

 $\begin{array}{cccc} k) & \sim W \to (Z \to \sim X) & I) & (\sim P \to R) \to \sim Q \\ & \sim \sim X & & \sim \sim Q \end{array}$ ∴ ?

R MP nothing in one step. After DN on the second premise, MT gets you ~X.

nothing

 $\sim$ ( $\sim$ P $\rightarrow$ R)

In all of these, you can infer the double negated premises (premise with two ~ in front) with DN. For example, a)  $\sim (\sim S \rightarrow \sim \sim T)$  dn, or  $\sim \sim \sim S$  dn.

### 3.3 E1:

Check the work in the following derivations. Does each line follow from available lines using the rule cited?

(a) 
$$\sim T \rightarrow \sim S$$
.  $R \rightarrow \sim \sim T$ .  $S$ .  $\therefore R$ 

1 <del>Show</del> ~R		ERROR.	show line incorrect
2	S	pr3	
3	~T → ~S	pr1	
4	Т	2 3 mt	ERROR. You need the negated consequent to use MT.
5	R → ~~T	pr2	
6	$ \begin{array}{c} R \to \sim T \\ R \to T \end{array} $	5 dn	ERROR. dn cannot be used on a sentential component.
7	R	4 6 mp	ERROR. mp moves from a conditional and the antecedent to the consquent.
8		7 dd	·

This cannot be fixed. It is not valid, so the conclusion cannot be derived from the premises.

(b) 
$$\sim (P \rightarrow \sim Q) \rightarrow \sim \sim S$$
.  $Q \rightarrow \sim S$ .  $Q \therefore \sim P$ 

1 Sho	<del>&gt;₩</del> ~P		
2	~S	 pr2 pr3 r	np
3	$\sim (P \rightarrow \sim Q) \rightarrow S$	pr1 dn	ERROR. dn cannot be used on a sentential component.
4	~~(P → ~Q)	2 3 mt	·
5	$ \begin{array}{l}     \sim (P \to \sim Q) \\     P \to \sim Q \\     \sim Q \\     \sim P \end{array} $	4 dn	
6	~~Q	pr3 dn	
7	~P	5 6 mt	
8		7 dd	

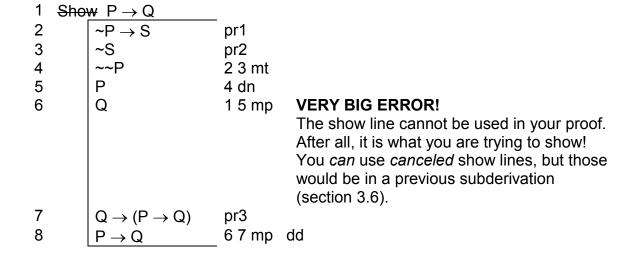
To fix this one:  $\sim$ (P  $\rightarrow$   $\sim$ Q)  $\rightarrow$   $\sim$ ~S. Q  $\rightarrow$   $\sim$ S. Q  $\therefore$   $\sim$ P

1 Show ~P					
2	~S ~~~S	pr2 pr3 mp			
3	~~~S	2 dn			
4	$\sim\sim$ (P $\rightarrow\sim$ Q)	pr1 3 mt			
5	$ \begin{array}{l} \sim \sim S \\ \sim \sim (P \to \sim Q) \\ P \to \sim Q \\ \sim \sim Q \\ \sim P \end{array} $	4 dn			
6	~~Q	pr3 dn			
7	~P	5 6 mt			
8		7 dd			

(c) 
$$Z \rightarrow (X \rightarrow \sim W)$$
.  $\sim Z \rightarrow \sim X$ .  $X$ .  $\therefore \sim W$ 

1 Show 
$$\sim$$
W pr3  
3  $\sim \sim$ X 2 dn  
4  $\sim Z \rightarrow \sim$ X pr2  
5  $\sim \sim$ Z 3 4 mt  
6 Z 5 dn  
7  $Z \rightarrow (X \rightarrow \sim W)$ . pr1  
8  $X \rightarrow \sim$ W 6 7 mp  
9  $\sim$ W 2 8 mp dd

(d) 
$$\sim P \rightarrow S$$
.  $\sim S$ .  $Q \rightarrow (P \rightarrow Q)$ .  $\sim (\sim S \rightarrow Q) \rightarrow \sim P$ .  $\therefore P \rightarrow Q$ 



#### 3.3 E2

Construct direct derivations for the following, showing that the conclusion can be validly inferred from the premises.

(a) 
$$P \to Q$$
.  $R \to \sim Q$ .  $\sim S \to R$ .  $P$ .  $\therefore S$ 

1 Show S show conclusion

2  $P \to Q$  pr1

3  $P$  pr4

4  $Q$  2 3 mp (or pr1 pr4 mp)

5  $\sim \sim Q$  4 dn

6  $R \to \sim Q$  pr2

7  $\sim R$  5 6 mt (or 5 pr2 mt)

8  $\sim S \to R$  pr3

9  $\sim \sim S$  7 8 mt (or 7 pr3 mt)

10  $S$  9 dn

11  $S \to S \to R$  9 dn

10 dd

(b) Y. 
$$X \to (Y \to Z)$$
.  $\sim X \to \sim W$ . W.  $\therefore \sim \sim Z$ 

1 Show  $\sim \sim Z$  show conclusion pr4

2 dn

4  $\sim \times W$  2 dn

4  $\sim \times X \to \sim W$  pr3

5  $\sim \sim X$  3 4 mt (or 3 pr3 mt)

6  $X$  5 dn

7  $X \to (Y \to Z)$  pr2

8  $Y \to Z$  6 7 mp (or 6 pr2 mp)

9  $Y$  pr1

10  $Z$  8 9 mp (or 9 pr1 mp)

11  $\sim \sim Z$  10 dn

12 11 dd

(c) 
$$P \rightarrow (Q \rightarrow (R \rightarrow S))$$
.  $\sim \sim P$ .  $(R \rightarrow S) \rightarrow \sim P$ .  $\therefore \sim Q$ 

(d) 
$$(P \rightarrow \sim Q) \rightarrow (\sim R \rightarrow S)$$
.  $\sim S$ .  $\sim (P \rightarrow \sim Q) \rightarrow T$ .  $T \rightarrow S$ .  $\therefore R$ 

1 Sho	₩R	show conclusion
2	~S	pr2
3	$T \rightarrow S$	pr4
4	~T	2 3 mp (or pr2 pr4 mp)
5	$\sim$ (P $\rightarrow$ $\sim$ Q) $\rightarrow$ T	pr3
6	$\sim\sim$ (P $\rightarrow\sim$ Q)	5 4 mt (or 4 pr3 mt)
7	$P \rightarrow \sim Q$	6 dn
8	$(P \rightarrow \sim Q) \rightarrow (\sim R \rightarrow S)$	pr1
9	$\sim$ R $\rightarrow$ S	7 8 mp (or 7 pr1 mp)
10	~~R	2 9 mt (or pr2 9 mt)
11	R	10 dn
12		11 dd

### 3.4 E1 Now, try a few:

(a) 
$$W \rightarrow (X \rightarrow \sim Y)$$
.  $Z \rightarrow Y$ .  $Z$ .  $\therefore W \rightarrow \sim X$ 

1 Sh	ow W→~X	show conclusion
2	W	ass CD
3	$X \rightarrow \sim Y$	2 pr1 mp
4	Y	pr2 pr3 mp
5	~~Y ~X	4 dn
6	~X	3 5 mt
7		5 cd

(b) 
$$P \rightarrow (R \rightarrow T)$$
.  $P \rightarrow \sim T$ .  $W \rightarrow R$ .  $\therefore P \rightarrow \sim W$ 

1 Show  $P \rightarrow \sim W$  SHOW CONC.

2  $P$  ASS CD

3  $P \rightarrow (R \rightarrow T)$  PR1

4  $P \rightarrow \sim T$  PR2

5  $W \rightarrow R$  PR3

6  $R \rightarrow T$  2 3 MP

7  $\sim T$  2 4 MP

8  $\sim R$  6 7 MT

8 5 MT CD

## 3.5 E1 Now, try a few:

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(a) 
$$P \rightarrow R$$
.  $Q \rightarrow \sim R$ .  $P$ .  $\therefore \sim (P \rightarrow Q)$ 

~W

1	1 Show ~ $(P\rightarrow Q)$ show conc.				
2		$P \rightarrow Q$	ass ID		
3		$P \rightarrow R$	pr1		
4		$Q \rightarrow \sim R$	pr2		
5		P	pr3		
6		Q	2 5 mp		
7		R	3 5 mp		
8		~~R	7 dn		
9		~Q	8 4 mt		
10			6 9 ID		

$$(b) \quad \begin{array}{cccc} \sim (\sim Y \rightarrow Y) \rightarrow Y. & \therefore Y \\ & 1 & \text{Show Y} & \text{Show conc.} \\ & 2 & \sim Y & \text{ass ID} \\ & 3 & \sim \sim (\sim Y \rightarrow Y) & 2 \text{ pr1 mt} \\ & 4 & \sim Y \rightarrow Y & 3 \text{ dn} \\ & 5 & Y & 2 \text{ 4 mp} \\ & 6 & 2 \text{ 5 id} \end{array}$$

(c)  $P \rightarrow R$ .  $R \rightarrow S$ . P.  $S \rightarrow \sim P$ .  $\therefore Z$ 

1 8	Show Z	show conc.
2	~Z	ass id
2 3	$P \rightarrow R$	pr1
4	$R \rightarrow S$	pr2
5	P	pr3
6	S → ~P	pr4
7	R	3 5 mp
8	S	4 7 mp
9	~P	6 8 mp
10		5 9 id

## 3.6 EG1 Let's try one:

(a)  $P \rightarrow Q$ .  $S \rightarrow \sim Q$ .  $(P \rightarrow \sim S) \rightarrow T$ .  $\therefore P \rightarrow T$ 

1 Sh	$ow P \rightarrow T$		show conclusion
2	P	ass cd	
3	$show P \rightarrow \sim S$		show ant. pr3
4	P	ass cd	
5	Q	4 pr1 mp	alternately, 2 pr1 mp
6	~~Q	5 dn	
7	~S	6 pr2 mt	
8		7 cd	
9	T	3 pr3 mp	
10		9 cd	

# 3.6 EG2 Let's try one:

$$\sim P \rightarrow R$$
.  $P \rightarrow S$ .  $T \rightarrow \sim S$ .  $(\sim R \rightarrow \sim \sim S) \rightarrow (T \rightarrow P)$ . T.  $\therefore$  U

SHO	₩U	SHOW CONC			
	~U				ASS ID
	~S				PR5 PR3 MP
	~P				PR2 3 MT
	R			PR1 4 MP	
	<del>SHOW</del> ~R→~~S				SHOW ANT. PR4
		~R			ASS CD
	<del>SHOW</del> ~~S				
			~S		ASS ID
			R		5 R
			~R		7 R 10 ID
					8 CD
	T→P				6 PR4 MP
	P				PR5 13 MP
					4 14 ID
	SHO	~S ~P R <del>SHO</del>	~U ~S ~P R <del>SHOW</del> ~R <del>~R</del> <del>SHO</del>	~U ~S ~P R SHOW ~R→~~S ~R SHOW ~~S  R ~R ~R T→P	~U ~S ~P R SHOW ~R→~~S ~R SHOW ~~S  ~R R ~S R ~R ~R

# 3.6 E 2 Provide a derivation to prove that the following arguments are valid.

a) $S \rightarrow \sim P$ . $P \rightarrow R$ . $R \rightarrow (\sim S \rightarrow Q)$ . $\therefore P \rightarrow Q$				
1 SHOW $P \rightarrow Q$ SHOW				
2	P	ass cd		
3	R	pr2 2 mp		
4	~~P	2 dn		
5	~S	4 pr1 mt		
6	$\sim$ S $\rightarrow$ Q	pr3 3 mp		
7	Q	5 6 mp		
8		7 cd		

b) 
$$(P \to Q) \to R$$
.  $S \to \sim P$ .  $\sim S \to (T \to Q)$ .  $\therefore T \to (P \to R)$   
1 SHOW  $T \to (P \to R)$  SHOW CONC  
2 T ass cd  
3 show  $P \to R$   
4 P ass cd  
5  $\sim \sim P$  4 dn  
6  $\sim \sim S$  pr2 5 mt  
7 T  $\to Q$  pr3 6 mp  
8 Show  $P \to Q$   
9 Q 27 mp  
10 cd  
12 R 8 pr1 mp  
13 12 cd  
14 3 cd

c) 
$$P \rightarrow \sim W$$
.  $S \rightarrow W$ .  $P \rightarrow (T \rightarrow R)$ .  $(\sim R \rightarrow \sim T) \rightarrow S$ .  $\therefore \sim P$ 

1 <del>SH</del>	1 SHOW ~P SHOW CONC					
2	P	P				
3	~W			2 pr1 mp		
4	~S			3 pr2 mt		
5	Show ~R	→ ~T		Show pr4 ant		
6	~R	-		ass cd		
7	T -	$\rightarrow$ R		2 pr3 mp		
8	~T			6 7 mt		
9				8 cd		
10	S			5 pr4 mp		
11				4 10 id		

d) 
$$\sim (P \rightarrow S)$$
.  $R \rightarrow \sim P$ .  $\sim R \rightarrow (T \rightarrow S)$ .  $P \rightarrow T$ .  $\therefore W$ 

1	1 SHOW W			SHOW CONC
2		~W		ass id
3		~(P -	$\rightarrow$ S)	pr1
4		$R \rightarrow$	~P	pr2
5		~R —	$\rightarrow$ $(T \rightarrow S)$	pr3
6		$P \rightarrow$	T	pr4
7		Show	$ \downarrow P \rightarrow S $	show opposite of line 3/pr1. Why? Because it will give you a contradiction! Plus, it should be easy to show. If you need it and can show it, put in a show line!
8			P	ass cd
9			Т	8 6 mp
10			~~P	8 dn
11			~R	4 10 mt
12			$T \rightarrow S$	5 11 mp
13			S	9 12 mp
14				13 cd
15				3 7 id

3 cd

## 3.8 EG1 Let's try another.

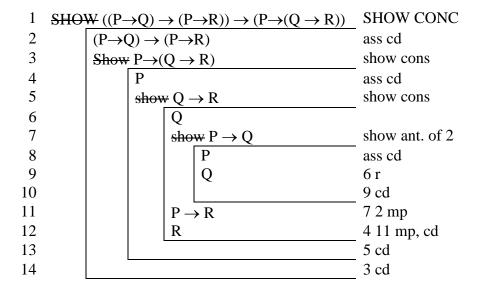
T4

11

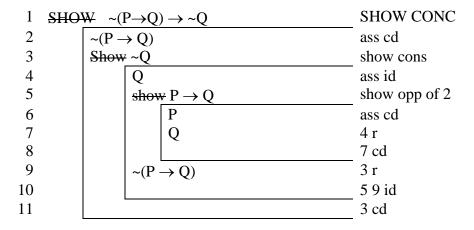
 $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$ 

#### 3.8 E 1 Prove the following theorems:

a) 
$$\therefore ((P \rightarrow Q) \rightarrow (P \rightarrow R)) \rightarrow (P \rightarrow (Q \rightarrow R))$$



b) 
$$\therefore \sim (P \rightarrow Q) \rightarrow \sim Q$$



c) 
$$\therefore$$
 (P  $\rightarrow$  S)  $\rightarrow$  ((T  $\rightarrow$  P)  $\rightarrow$  ( $\sim$ S  $\rightarrow$   $\sim$ T))

