UNIVERSITY OF TORONTO

Faculty of Arts and Science

August 2011 Examination

STA347H1S

Duration - 3 hours

Aids Allowed: non-programmable calculator

First Name: ______________________________

Last name: ______________________________

Student number: ______________________________

Instructions:

- There are 18 pages including this page. The last 2 pages are formulae that may be useful. Please check that you are not missing any page.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need **not** be expressed in decimal form.
- Show all your work and answer in the space provided. Use back of pages for rough work.
- If you do not understand a question, or are having some other difficulty, do not hesitate to ask your T.A for clarification.
- Marks are shown in brackets at the beginning of each question. Total marks: 100

Good luck ☺ !!!

Question	1	2	3	4	5	6	7	8	9	10	Total
Max	7	7	8	13	10	10	10	12	7	16	100
Score											

Question 1

A hat contains three cards as follows: one card is red on both sides, one is black on both sides, and one is red on one side and black on the other. Suppose one card is chosen at random and is placed flat on the table, so we can see one side only.

a) (3 marks) What is the probability that this one side we can see is red?

b) (4 marks) Given that the one side we can see is red, what is the probability that the card chosen is red on both sides?

Question 2

Let X be a discrete random variable with probability mass function given by:

$$p_X(x) = \begin{cases} 2^{-x} & x = 1, 2, 3, \ldots. \\ 0 & \text{otherwise} \end{cases}$$

a) (3 points) Let $Y = X^2$. Find the probability mass function of Y.

b) (4 points) Let $Z = X - 1$. Find the probability mass function of Z. Identify the distribution of Z by name and specify all parameter values.

Question 3

(8 points) Suppose $\Omega = \{1, 2, 3\}$, with $P(\Phi) = 0$ and $P(\{1,2,3\}) = 1$. Let $x = P(\{1,2\})$, $y = P(\{2,3\})$ and $z = P(\{1,3\})$. Find (with proof) necessary conditions on the values of x, y and z such that P is countably additive.

Question 4

Suppose (X, Y) have a Dirichlet($\alpha_1, \alpha_2, \alpha_3$) distribution. The joint density of (X, Y) is:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x^{\alpha_1-1} y^{\alpha_2-1} (1-x-y)^{\alpha_3-1} & x \geq 0,\ y \geq 0 \text{ and } x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) (6 marks) Find the marginal densities of X and Y. Are X and Y independent?

b) (7 marks) Let $U = \frac{X}{X+Y}$. Find the density of U. Identify the distribution of U by name and specify all parameter values.

Question 5

a) (5 points) Let X and Y be discrete random variables defined on the sample probability space. Suppose that $X \leq Y$. Prove that $E(X) \leq E(Y)$.

b) (5 points) Suppose X is discrete random variable, such that $E(\min(X, M)) = E(X)$. Prove that $P(X > M) = 0$.

Question 6

The Canadian Mint produces dimes with an average diameter of 1 centimeter and standard deviation of 0.01.

a) (2 point) Find an upper bound for the probability that a dime has a diameter of more then 2.

b) (4 points) Find a lower bound for the number of coins in a lot of 100 coins that are expected to have a diameter between 0.95 and 1.05.

c) (4 points) Let $\overline{X}_{72} = \frac{1}{72}(X_1 + \cdots + X_{72})$ be the average diameter of 72 coins produced by this company. Find $P(0.91 \leq \overline{X}_{72} \leq 1.08)$. Justify your answer!

Question 7

Suppose X and Y are jointly distributed discrete random variables with probability mass function

$$p_{X,Y}(x,y) = k\frac{(1-\theta)^x \theta^y}{y!} \quad , \qquad x, y = 0,1,2, \ldots.., \quad 0 < \theta < 1$$

a) (3 points) Determine the value of the constant k.

b) (2 points) Are X and Y independent? Why or why not?

c) (5 points) Find $E\left(X^{Y}\right)$.

Question 8

Suppose $X_1, X_2, \ldots$ are independent each having exponential distribution with parameter λ and N has a Poisson(λ) distribution and is independent of the X_i's.
To answer this question, you may find it useful to know that the moment generating function of a Gamma(α, λ) random variable is $m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$

a) (5 Points) Determine the moment generating function of $S_N = X_1 + X_2 + \cdots + X_N$.

b) (3 Points) Use your answer in (a) to find the mean of S_N.

c) (4 points) Find the distribution of $Y = \sum_{i=1}^{n} 2\lambda X_i$. Identify the distribution of Y by name and specify all parameter values.

Question 9

Suppose X denote the number of bacteria per cubic centimeter in a particular liquid and that X has a Poisson distribution with parameter λ. Further, suppose that λ varies from location to location and has a Gamma(2, 5) distribution. If we randomly select a location,

a) (3 points) What is the expected number of bacteria per cubic centimeter?

b) (4 points) What is the standard deviation of the number of bacteria per cubic centimeter?

Question 10

(16 points) For each statement below, explain carefully why it is true or why it is false.

a) Suppose X and Y are indicator variables for the events A and B respectively. Then $X+Y$ is always an indicator variable for $A \cup B$.

b) Suppose X_n is a continuous random variable with density

$$f_{X_n}(x) = \begin{cases} \dfrac{1 + x/n}{1 + 1/2n} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\{X_n\}$ converges in distribution to $X \sim \text{Uniform}[0,1]$.

c) Let X and Y be discrete random variables, with $P(X=1)>0$ and $P(X=2)>0$. Suppose that $P(Y=1 \mid X=1)=\frac{3}{4}$ and $P(Y=2 \mid X=2)=\frac{3}{4}$. Then X and Y can not be independent.

d) Suppose $X_1, X_2, \ldots, X_n$ are independent each with Exponential(λ_i) distribution. Then, the distribution of $\min(X_1, X_2, \ldots, X_n)$ is Exponential$\left(\sum_{i=1}^{n} \lambda_i\right)$.

The Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

The Beta Function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Some Important Discrete Probability Distributions

Distribution	Probability Function	Mean	Variance
Binomial(n, p)	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$	np	$np(1-p)$
Bernoulli(p)	same as Binomial$(1, p)$		
Poisson(λ)	$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, \ldots$	λ	λ
Geometric(p)	$p(x) = p(1-p)^x$ for $x = 0, 1, 2, \ldots$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$

Some Important Continuous Probability Distributions

Distribution	Density Function	Mean	Variance
Uniform(a, b)	$f(x) = \frac{1}{b-a}$ for $a < x < b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ for $x \in \mathbb{R}$	μ	σ^2
Standard Normal	same as Normal$(0, 1)$		
Exponential(λ)	$f(x) = \lambda e^{-\lambda x}$ for $x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(α, λ)	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ for $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Chi-square(n)	same as Gamma $\left(\frac{n}{2}, \frac{1}{2}\right)$		

END!

Total Pages = (18)
Total Marks = (100)