11.10 To actually construct the rationals $\mathbb{Q}$ from the integers $\mathbb{Z}$, let $S = \{(a, b): a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Define an equivalence relation "$\sim$" on S by $(a, b) \sim (c, d)$ iff $ad = bc$. We then define the set $\mathbb{Q}$ of rational numbers to be the set of equivalence classes corresponding to $\sim$. The equivalence class determined by the ordered pair (a, b) we denote by $[a/b]$. Then $[a/b]$ is what we usually think of as the fraction a/b. For $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$, we define addition and multiplication in $\mathbb{Q}$ by

$$[a/b] + [c/d] = [(ad + bc)/bd],$$
$$[a/b] \cdot [c/d] = [ac/bd].$$

We say that $[a/b]$ is *positive* if $ab \in \mathbb{N}$. Since $a, b \in \mathbb{Z}$ with $b \neq 0$, this is equivalent to requiring $ab > 0$. The set of positive rationals is denoted by $\mathbb{Q}^+$, and we define an order "$<$" on $\mathbb{Q}$ by

$$x < y \quad \text{iff} \quad y - x \in \mathbb{Q}^+.$$

(a) Verify that $\sim$ is an equivalence relation on S.
(b) Show that addition and multiplication are well-defined. That is, suppose that $[a/b] = [p/q]$ and $[c/d] = [r/s]$. Show that $[(ad + bc)/bd] = [(ps + qr)/qs]$ and $[ac/bd] = [pr/qs]$.
(c) For any $b \in \mathbb{Z}\backslash\{0\}$, show that $[0/b] = [0/1]$ and $[b/b] = [1/1]$.
(d) For any $a, b \in \mathbb{Z}$ with $b \neq 0$, show that $[a/b] + [0/1] = [a/b]$ and $[a/b] \cdot [1/1] = [a/b]$. Thus $[0/1]$ corresponds to zero and $[1/1]$ corresponds to 1.
(e) For any $a, b \in \mathbb{Z}$ with $b \neq 0$, show that $[a/b] + [(-a)/b] = [0/1]$ and $[a/b] \cdot [b/a] = [1/1]$.
(f) Verify that the set $\mathbb{Q}$ with addition, multiplication, and order as given above satisfies the axioms of an ordered field.

11.11 Construct the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$ in a method similar to that used in Exercise 11.10 by defining an appropriate equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Section 12 THE COMPLETENESS AXIOM

In the preceding section we presented the field and order axioms of the real numbers. Although these axioms are certainly basic to the real numbers, by themselves they do not characterize $\mathbb{R}$. That is, we have seen that there are other mathematical systems that also satisfy these 15 axioms. In particular, the set $\mathbb{Q}$ of rational numbers is an ordered field. The one additional axiom that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ (and from other ordered fields) is called the completeness axiom. Before presenting this axiom, let us look briefly at why it is needed—at why the rational numbers by themselves are inadequate for analysis.

Consider the graph of the function $f(x) = x^2 - 2$, shown in Figure 12.1. It appears that the graph crosses the horizontal axis at a point between 1 and 2. But does it really? How can we be sure? In other words, how can we be certain that there is a "number" x on the axis such that $x^2 - 2 = 0$? It turns out that if the x-axis consists only of rational numbers, then no such number exists. That is, there is no rational number whose square is 2. In fact, we can easily prove the more general result that $\sqrt{p}$ is irrational (not rational) for any prime number p. (Recall that an integer $p > 1$ is prime iff its only divisors are 1 and p.)

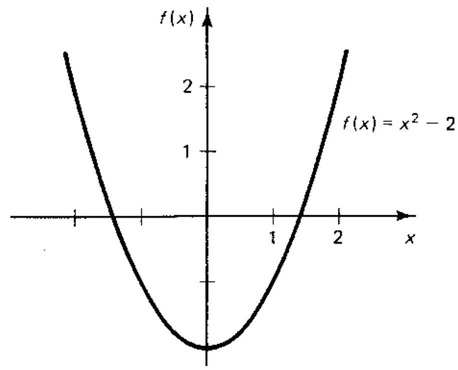

Figure 12.1

12.1 THEOREM Let p be a prime number. Then $\sqrt{p}$ is not a rational number.

Proof: We suppose that $\sqrt{p}$ is rational and obtain a contradiction. If $\sqrt{p}$ is rational, then we can write $\sqrt{p} = m/n$, where m and n are integers with no common factors. Then $pn^2 = m^2$, so m^2 must be a multiple of p. Since p is prime, this implies that m is also a multiple of p. That is, $m = kp$ for some integer k. But then $pn^2 = k^2p^2$, so that $n^2 = k^2p$. Thus n^2 is a multiple of p, and as above we conclude that n is also. Hence m and n are both multiples of p, contradicting the fact that they have no common factors. ▮

There are, of course, many other irrational numbers besides $\sqrt{p}$ for p prime. We saw in Section 8 that there are, in fact, more irrational numbers than there are rational. Thus, if we were to restrict our analysis to rational numbers, our "number line" would have uncountably many "holes" in it. It is these holes in the number line that the completeness axiom fills. To state this final axiom for $\mathbb{R}$, we need some preliminary definitions.

Upper Bounds and Suprema

12.2 DEFINITION Let S be a nonempty subset of $\mathbb{R}$. If there exists a real number m such that $m \geqslant s$ for all $s \in S$, then m is called an **upper bound** for S, and we say that S is bounded above. If $m \leqslant s$ for all $s \in S$, then m is a **lower bound** for S and S is bounded below. The set S is said to be **bounded** if it is bounded above and bounded below.

A set may have upper or lower bounds, or it may have none. If m is an upper bound for S, then any number greater than m is also an upper bound. If an upper bound m for S is a member of S, then m is called the **maximum** (or largest element) of S, and we write

$$m = \max S.$$

Similarly, if a lower bound of S is a member of S, then it is called the **minimum** (or least element) of S, denoted by min S. While a set may have many upper and lower bounds, if it has a maximum or a minimum, then those values are unique. Thus we speak of *an* upper bound and *the* maximum.

12.3 EXAMPLES (a) The set $S = \{2, 4, 6, 8\}$ is bounded above by 8, 9, $8\frac{1}{2}$, π^2, and any other real number greater than or equal to 8. Since $8 \in S$, we have max $S = 8$. Similarly, S has many lower bounds, including 2, which is the largest of the lower bounds and the minimum of S. It is easy to see that any finite set is bounded and always has a maximum and a minimum.

(b) The interval $[0, \infty)$ is not bounded above. It is bounded below by any nonpositive number, and of these lower bounds, 0 is the largest. Since $0 \in [0, \infty)$, 0 is the minimum of $[0, \infty)$.

(c) The interval $(0, 1]$ has a maximum of 1, and this is the smallest of the upper bounds. It is bounded below by any nonpositive number, and of these lower bounds, 0 is the largest. Since $0 \notin (0, 1]$, the set has no minimum.

12.4 PRACTICE Find upper and lower bounds, the maximum, and the minimum of the set $T = \{q \in \mathbb{Q} : 0 \leqslant q \leqslant \sqrt{2}\}$, if they exist.

Since any number larger than an upper bound is also an upper bound, we have found it useful to identify the smallest or least upper bound in our examples. It is also helpful to know the greatest of the lower bounds.

12.5 DEFINITION Let S be a nonempty set. If S is bounded above, then the least upper bound of S is called its **supremum** and is denoted by sup S. Thus $m =$ sup S iff

(a) $m \geqslant s$, for all $s \in S$, and
(b) if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

If S is bounded below, then the greatest lower bound of S is called its **infimum** and is denoted by inf S.

12.6 PRACTICE Characterize inf S in a way analogous to that given for sup S.

It is easy to show (Exercise 12.4) that if a set has a supremum, then it is unique. What may not be clear is whether a set that is bounded above must have a least upper bound. Indeed, the set $T = \{q \in \mathbb{Q}: 0 \leqslant q \leqslant \sqrt{2}\}$ in Practice 12.4 does not have a supremum when considered as a subset of $\mathbb{Q}$. The problem is that sup $T = \sqrt{2}$, and $\sqrt{2}$ is one of the "holes" in $\mathbb{Q}$.

When considering subsets of $\mathbb{R}$, it has been true that each set bounded above has had a least upper bound. This supremum may be a member of the set, as in the interval $[0, 1]$, or it may be outside the set, as in the interval $[0, 1)$, but in both cases the supremum *exists* as a real number. This fundamental difference betweeen $\mathbb{Q}$ and $\mathbb{R}$ is the basis for our final axiom of the real numbers, the **completeness axiom**:

> Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, sup S exists and is a real number.

While the completeness axiom refers only to sets that are bounded above, the corresponding property for sets bounded below follows readily. Indeed, suppose that S is a nonempty subset of $\mathbb{R}$ that is bounded below. Then the set $-S = \{-s: s \in S\}$ is bounded above and the completeness axiom implies the existence of a supremum, say m. It follows (Exercise 12.5) that $-m$ is the infimum of S. Thus every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

To illustrate the techiques of working with suprema, we include the following two theorems. It goes without saying that analogous results hold for infima.

12.7 THEOREM Given nonempty subsets A and B of $\mathbb{R}$, let C denote the set

$$C = \{x + y: x \in A \text{ and } y \in B\}.$$

If each of A and B has a supremum, then C has a supremum and

$$\sup C = \sup A + \sup B.$$

Proof: Let $\sup A = a$ and $\sup B = b$. If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$. Thus $z = x + y \leqslant a + b$, so $a + b$ is an upper bound for C. By the completeness axiom, C has a least upper bound, say $\sup C = c$. We must show that $c = a + b$. Since c is the *least* upper bound for C, we have $c \leqslant a + b$.

To see that $a + b \leqslant c$, choose any $\varepsilon > 0$. Since $a = \sup A$, $a - \varepsilon$ is not an upper bound for A and there must exist x in A such that $a - \varepsilon < x$. Similarly, since $b = \sup B$, there exists y in B such that $b - \varepsilon < y$. Combining these inequalities, we have

$$a + b - 2\varepsilon < x + y \leqslant c.$$

That is, $a + b < c + 2\varepsilon$ for every $\varepsilon > 0$. Thus by Theorem 11.6, $a + b \leqslant c$.

Finally, since $c \leqslant a + b$ and $c \geqslant a + b$, we conclude that $c = a + b$. ∎

12.8 THEOREM Suppose that D is a nonempty set and that $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$. If for every $x, y \in D$, $f(x) \leqslant g(y)$, then $f(D)$ is bounded above and $g(D)$ is bounded below. Furthermore, $\sup f(D) \leqslant \inf g(D)$.

Proof: Given any $y_0 \in D$, we have $f(x) \leqslant g(y_0)$, for all $x \in D$. Thus $f(D)$ is bounded above by $g(y_0)$. It follows that the *least* upper bound of $f(D)$ is no larger than $g(y_0)$. That is, $\sup f(D) \leqslant g(y_0)$. Since this last inequality holds for all $y_0 \in D$, $g(D)$ is bounded below by $\sup f(D)$. Thus the *greatest* lower bound of $g(D)$ is no smaller than $\sup f(D)$. That is, $\sup f(D) \leqslant \inf g(D)$. ∎

The Archimedean Property

One of the important consequences of the completeness axiom is called the Archimedean property. It states that the natural numbers $\mathbb{N}$ are not bounded above in $\mathbb{R}$. Although this property may seem obvious at first, its proof actually depends on the completeness axiom. In fact, there are other ordered fields in which it does not hold. (See Exercise 12.14.)

12.9 THEOREM (Archimedean Property of $\mathbb{R}$) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Proof: If $\mathbb{N}$ were bounded above, then by the completeness axiom it would have a least upper bound, say $\sup \mathbb{N} = m$. Since m is a *least* upper bound, $m - 1$ is not an upper bound for $\mathbb{N}$. Thus there exists an n_0 in $\mathbb{N}$ such that $n_0 > m - 1$. But then $n_0 + 1 > m$, and since $n_0 + 1 \in \mathbb{N}$, this contradicts m being an upper bound for $\mathbb{N}$. ▮

There are several equivalent forms of the Archimedean property that are useful in different contexts. We establish their equivalence in the following theorem.

12.10 THEOREM Each of the following is equivalent to the Archimedean property.

(a) For each $z \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > z$.
(b) For each $x > 0$ and for each $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx > y$.
(c) For each $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < x$.

Proof: We shall prove that Theorem 12.9 $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ Theorem 12.9, thereby establishing their equivalence.

If (a) were not true, then for some $z_0 \in \mathbb{R}$ we would have $n \leqslant z_0$ for all $n \in \mathbb{N}$. But then z_0 would be an upper bound for $\mathbb{N}$, contradicting Theorem 12.9. Thus the Archimedean property implies (a).

To see that (a) $\Rightarrow$ (b), let $z = y/x$. Then there exists $n \in \mathbb{N}$ such that $n > y/x$, so that $nx > y$.

(c) follows from (b) by taking $y = 1$ in (b). Then $nx > 1$, so that $1/n < x$. Since $n \in \mathbb{N}$, $n > 0$ and also $1/n > 0$.

Finally, suppose that $\mathbb{N}$ were bounded above by some real number, say m. That is, $n < m$ for all $n \in \mathbb{N}$. But then $1/n > 1/m$, for all $n \in \mathbb{N}$, and this contradicts (c) with $x = 1/m$. Thus (c) implies the Archimedean property. ▮

In Theorem 12.1 we showed that $\sqrt{p}$ is not rational when p is prime. We are now in a position to prove there is a positive *real* number whose square is p, thus illustrating that we actually have filled in the "holes" in $\mathbb{R}$.

12.11 THEOREM Let p be a prime number. Then there exists a positive real number x such that $x^2 = p$.

Proof: Let $S = \{r \in \mathbb{R}: r > 0 \text{ and } r^2 < p\}$. Since $p > 1$, $1 \in S$ and S is nonempty. Furthermore, if $r \in S$, then $r^2 < p < p^2$, so $r < p$. Thus S is bounded above by p, and by the completeness axiom, $\sup S$ exists as a real number. Let $x = \sup S$. It is clear that $x > 0$, and we claim that $x^2 = p$. To prove this, we shall show that neither $x^2 < p$ nor $x^2 > p$ is consistent with our choice of x.

Suppose first that $x^2 < p$. Then $(p - x^2)/(2x + 1) > 0$, so that Theorem 12.10(c) implies the existence of some $n \in \mathbb{N}$ such that†

$$\frac{1}{n} < \frac{p - x^2}{2x + 1}.$$

But then we have

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} = x^2 + \frac{1}{n}\left(2x + \frac{1}{n}\right)$$
$$\leqslant x^2 + \frac{1}{n}(2x + 1) < x^2 + (p - x^2) = p.$$

It follows that $x + 1/n \in S$, which contradicts our choice of x as an upper bound for S.

Now suppose that $x^2 > p$. Then $(x^2 - p)/(2x) > 0$. Again using Theorem 12.10(c), there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{x^2 - p}{2x}.$$

But then we have

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$
$$> x^2 - (x^2 - p) = p.$$

This implies that $x - 1/m > r$, for all $r \in S$, so $x - 1/m$ is an upper bound of S. Since $x - 1/m < x$, this contradicts our choice of x as the *least* upper bound of S.

† In the formal proof above we have mysteriously introduced the inequality $(1/n) < [(p - x^2)/(2x + 1)]$. It is instructive to see how we might come up with such a requirement.

If $x^2 < p$, then we somehow want to contradict the fact that x is an upper bound for S. That is, we want to find some y in S with $y > x$. A simple way to get $y > x$ is to add something positive to x. But if x^2 is close to p, that "something" will have to be small. By taking $y = x + 1/n$, we guarantee that $y > x$, we can make y as close to x as we want, and y is still fairly easy to work with.

Now to make $(x + 1/n) \in S$, we must have $(x + 1/n)^2 = x^2 + 2x/n + 1/n^2 < p$. We can control the size of $1/n$ (by choosing n carefully), so we rewrite the inequality to emphasize the contribution of $1/n$:

$$x^2 + \frac{1}{n}\left(2x + \frac{1}{n}\right) < p.$$

If it were not for the $1/n$ inside the parentheses, we could easily solve for $1/n$. But we can get rid of that $1/n$ by observing that $1/n \leqslant 1$ for all $n \in \mathbb{N}$. Thus we want to choose n so that

$$x^2 + \frac{1}{n}\left(2x + \frac{1}{n}\right) \leqslant x^2 + \frac{1}{n}(2x + 1) < p.$$

Solving the last inequality for $1/n$, we obtain the requirement that appears in the proof.

The inequality that is used in the case $x^2 > p$ is found in a similar manner.

Finally, since neither $x^2 < p$ nor $x^2 > p$ are possibilities, we conclude by the trichotomy law that in fact $x^2 = p$. ▮

The Density of the Rational Numbers

We conclude this section with another property of $\mathbb{R}$ that is probably familiar to the reader: Between any two real numbers there is a rational number. More precisely, we say that the set $\mathbb{Q}$ is **dense** in $\mathbb{R}$. Once again, our proof will ultimately depend on the completeness axiom.

12.12 THEOREM (Density of $\mathbb{Q}$ in $\mathbb{R}$) If x and y are real numbers with $x < y$, then there exists a rational number r such that $x < r < y$.

Proof: We begin by supposing that $x > 0$. Using the Archimedean property 12.10(a), there exists $n \in \mathbb{N}$ such that $n > 1/(y - x)$. That is, $nx + 1 < ny$. Since $nx > 0$, it is not difficult to show (Exercise 12.7) that there exists $m \in \mathbb{N}$ such that $m - 1 \leqslant nx < m$. But then $m \leqslant nx + 1 < ny$, so that $nx < m < ny$. It follows that the rational number $r = m/n$ satisfies $x < r < y$.

Finally, if $x \leqslant 0$, choose an integer k such that $k > |x|$. Then apply the argument above to the positive numbers $x + k$ and $y + k$. If q is a rational satisfying $x + k < q < y + k$, then the rational $r = q - k$ satisfies $x < r < y$. ▮

Using Theorem 12.12, we can easily show that between any two real numbers there is also an irrational number. (Thus the irrationals are also dense in $\mathbb{R}$.) We pause first for you to prove the following preliminary result.

12.13 PRACTICE Let x be a nonzero rational number and let y be irrational. Prove that xy is irrational.

12.14 THEOREM If x and y are real numbers with $x < y$, then there exists an irrational number w such that $x < w < y$.

Proof: Apply Theorem 12.12 to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$ to obtain a rational number $r \neq 0$ such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

It follows from Practice 12.13 that $w = r\sqrt{2}$ is irrational, and $x < w < y$. ▮

ANSWERS TO PRACTICE PROBLEMS

12.4 Any real number x such that $x^2 \geqslant 2$ is an upper bound for T. The smallest of these upper bounds is $\sqrt{2}$, but since $\sqrt{2} \notin \mathbb{Q}$, set T has no maximum. The minimum of T is 0.

12.6 $m = \inf S$ iff (i) $m \leqslant s$, for all $s \in S$, and (ii) if $m' > m$, then there exists $s' \in S$ such that $s' < m'$.

12.13 Since x is rational and $x \neq 0$, we have $x = m/n$ for some nonzero integers m and n. If xy were rational, then we could write $xy = p/q$ for some p, $q \in \mathbb{Z}$. But then

$$y = \frac{xy}{x} = \frac{p/q}{m/n} = \frac{pn}{mq},$$

so y would have to be rational too, a contradiction.

EXERCISES

12.1 For each subset of $\mathbb{R}$, give its supremum if it has one. Otherwise, write "no sup."

(a) $\{1, 3\}$ (b) $\{\pi, 3\}$

(c) $[0, 4]$ (d) $(0, 4)$

(e) $\left\{\frac{1}{n}: n \in \mathbb{N}\right\}$ (f) $\left\{1 - \frac{1}{n}: n \in \mathbb{N}\right\}$

(g) $\left\{\frac{n}{n+1}: n \in \mathbb{N}\right\}$ (h) $\left\{(-1)^n\left(1 + \frac{1}{n}\right): n \in \mathbb{N}\right\}$

(i) $\left\{n + \frac{(-1)^n}{n}: n \in \mathbb{N}\right\}$ (j) $(-\infty, 4)$

(k) $\bigcap_{n=1}^{\infty}\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ (l) $\bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 2 - \frac{1}{n}\right]$

(m) $\{r \in \mathbb{Q}: r < 5\}$ (n) $\{r \in \mathbb{Q}: r^2 < 5\}$

12.2 Repeat Exercise 12.1 for the infimum of each set.

12.3 Let S be a nonempty bounded subset of $\mathbb{R}$ and let $m = \sup S$. Prove that $m \in S$ iff $m = \max S$.

12.4 Let S be a nonempty bounded subset of $\mathbb{R}$. Prove that $\sup S$ is unique.

***12.5** Let S be a nonempty bounded subset of $\mathbb{R}$ and let $k \in \mathbb{R}$. Define $kS = \{ks: s \in S\}$. Prove the following:

(a) If $k \geqslant 0$, then $\sup(kS) = k \cdot \sup S$ and $\inf(kS) = k \cdot \inf S$.

(b) If $k < 0$, then $\sup(kS) = k \cdot \inf S$ and $\inf(kS) = k \cdot \sup S$.

12.6 Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf T \leqslant \inf S \leqslant \sup S \leqslant \sup T$.

12.7 (a) Prove: If $y > 0$, then there exists $n \in \mathbb{N}$ such that $n - 1 \leqslant y < n$.

(b) Prove that the n in part (a) is unique.

12.8 (a) Prove: If x and y are real numbers with $x < y$, then there are infinitely many rational numbers in the interval $[x, y]$.
(b) Repeat part (a) for irrational numbers.

12.9 Let y be a positive real number. Prove that for every $n \in \mathbb{N}$ there exists a unique positive real number x such that $x^n = y$.

***12.10** Let D be a nonempty set and suppose that $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$. Define the function $f + g: D \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$.

(a) If $f(D)$ and $g(D)$ are bounded above, then prove that $(f + g)(D)$ is bounded above and $\sup[(f + g)(D)] \leqslant \sup f(D) + \sup g(D)$.
(b) Find an example to show that a strict inequality in part (a) may occur.
(c) State and prove the analog of part (a) for infima.

12.11 Let $x \in \mathbb{R}$. Prove that $x = \sup \{q \in \mathbb{Q}: q < x\}$.

12.12 Let a/b be a fraction in lowest terms with $0 < a/b < 1$.

(a) Prove that there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} \leqslant \frac{a}{b} < \frac{1}{n}.$$

(b) If n is chosen as in part (a), prove that $a/b - 1/(n + 1)$ is a fraction that in lowest terms has a numerator less than a.
(c) Use part (b) and the principle of strong induction (Exercise 10.11) to prove that a/b can be written as a finite sum of distinct unit fractions:

$$\frac{a}{b} = \frac{1}{n_1} + \cdots + \frac{1}{n_k},$$

where $n_1, \ldots, n_k \in \mathbb{N}$. (As a point of historical interest, we note that in the ancient Egyptian system of arithmetic all fractions were expressed as sums of unit fractions and then manipulated using tables.)

12.13 Prove Euclid's division algorithm: If a and b are natural numbers, then there exist unique numbers q and r, each of which is either 0 or a natural number, such that $r < a$ and $b = qa + r$.

12.14 Let F be the ordered field of rational functions as given in Example 11.5, and note that F contains both $\mathbb{N}$ and $\mathbb{R}$ as subsets.

(a) Show that F does not have the Archimedean property. That is, find a member z in F such that $z > n$ for every $n \in \mathbb{N}$.
(b) Show that the property in Theorem 12.10(c) does not apply. That is, find a positive member z in F such that, for all $n \in \mathbb{N}$, $0 < z \leqslant 1/n$.
(c) Show that F does not satisfy the completeness axiom. That is, find a subset B of F such that B is bounded above but B has no least upper bound. Verify your answer.

12.15 We have said that the real numbers can be characterized as a complete ordered field. This means that any other complete ordered field F is essentially the same as $\mathbb{R}$ in the sense that there exists a bijection $f: \mathbb{R} \to F$ with the following properties for all $a, b \in \mathbb{R}$:

(1) $f(a + b) = f(a) + f(b)$,
(2) $f(a \cdot b) = f(a) \cdot f(b)$,
(3) $a < b$ iff $f(a) < f(b)$.

2.6 Exponential Functions

Many sequences have the property that each term is greater than or equal to the terms that have come before it. Similarly, it is often the case that each term of a sequence is less than or equal to the terms that have come before it. This section introduces some terminology to describe these properties.

Definition A sequence $\{a_n\}$ is said to be *increasing* if $a_n \leq a_{n+1}$ for every n. A sequence $\{b_n\}$ is said to be *decreasing* if $b_n \geq b_{n+1}$ for every n. Either type of sequence is said to be *monotone* or *monotonic* (see Figure 1).

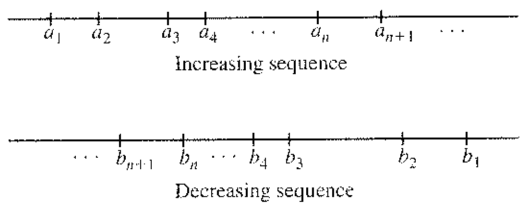

Figure 1

The Monotone Convergence Property of the Real Numbers

Recall from Chapter 1 that every irrational number has an infinite nonrepeating decimal expansion. What exactly is meant by this? Consider the irrational number π. When we write $\pi = 3.141592653\ldots$, we mean that π is the limit of the sequence

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415, 3.14159,\ 3.141592,\ 3.1415926, \ldots.$$

The terms of this sequence are increasing. They do not, however, grow arbitrarily large. In fact, every term in the sequence is less than 3.2. Here is a definition for a sequence that does not have arbitrarily large terms.

Definition A sequence $\{a_n\}$ is *bounded above* if there is a real number U such that $a_n \leq U$ for all n. We refer to U as an *upper bound* for the sequence $\{a_n\}$ and say that $\{a_n\}$ is *bounded above* by U. Similarly, a sequence $\{a_n\}$ is *bounded below* if there is a real number L such that $L \leq a_n$ for all n. We refer to L as a *lower bound* for the sequence $\{a_n\}$ and say that $\{a_n\}$ is *bounded below* by L. If $\{a_n\}$ is bounded both above and below, then we simply say that $\{a_n\}$ is *bounded*. In this final case, there is a number $M > 0$ such that $|a_n| \leq M$ for all n. We say that $\{a_n\}$ is *bounded* by M.

Every bounded monotonic sequence $\{a_n\}_{n=1}^{\infty}$ is contained in a closed interval: If $\{a_n\}$ is *increasing* and bounded above by M, then $\{a_n\}$ is contained in $[a_1, M]$. If $\{a_n\}$ is *decreasing* and bounded below by M, then $\{a_n\}$ is contained in $[M, a_1]$. Our next theorem guarantees that a bounded increasing sequence actually converges to a real number, even though we may not be able to specify in advance what that number is.

Theorem 1 *Monotone Convergence Property* If $\{a_n\}_{n=1}^{\infty}$ is monotonic and bounded, then $\{a_n\}_{n=1}^{\infty}$ converges to some real number ℓ. If $\{a_n\}_{n=1}^{\infty}$ lies in a closed interval I, then ℓ belongs to I.

The monotone convergence property is one way of expressing the completeness of the real number system. Equivalent versions of the completeness property may be found in the Genesis & Developments for Chapters 1 and 2. It is important to understand this property's many consequences that are crucial for calculus. In Section 2.3, we learned the Intermediate Value Theorem and the Extreme Value Theorem for continuous functions: Both of these theorems rely on the completeness property of $\mathbb{R}$. Now we see that it is the completeness property (in the form of the monotone convergence property) that tells us that there is a point on the real number line for every irrational number (that is, for every infinite nonrepeating decimal expansion).

Example 1 Let $\{a_n\}$ be defined by $a_1 = 1$ and $a_n = a_{n-1} + 1/n^n$ for $n \geq 2$. Show that this sequence converges.

Solution According to the recursive definition of the sequence $\{a_n\}$, we have $a_n > a_{n-1}$ for each n. In other words, the sequence $\{a_n\}$ is increasing. The table provides the first several terms to five decimal places.

n	1	2	3	4	5	6	7
a_n	1.00000	1.25000	1.28704	1.29094	1.29126	1.29129	1.29129

The tabulated values of $\{a_n\}$ indicate that $\{a_n\}$ is increasing and bounded. To *prove* that $\{a_n\}$ is bounded, notice that $1/n^n \leq 1/2^n$ for $n \geq 2$. It follows that

$$a_2 = 1 + \frac{1}{2^2} < 1 + \frac{1}{2} + \frac{1}{4}$$
$$a_3 = a_2 + \frac{1}{3^3} < a_2 + \frac{1}{2^3} < 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$
$$a_4 = a_3 + \frac{1}{4^4} < a_3 + \frac{1}{2^4} < 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

and, in general,

$$a_n < 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}.$$

Therefore, by Example 9 from Section 2.5,

$$a_n < 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

for every n. The sequence $\{a_n\}$ is bounded by 2 and, therefore, converges to a number $\ell \leq 2$. ■

The monotone convergence property allows us to deduce that $\{a_n\}$ converges to some real number ℓ, but it does not tell us what the number ℓ is. We can, however, compute ℓ to any required accuracy. A practical rule of thumb in numerical work is to terminate computation when several successive calculations agree to the required accuracy. For example, if we require three decimal places of ℓ, then we can be reasonably sure that $\ell = 1.291\ldots$, because the computed values a_5, a_6, and a_7 agree on those digits.

If $b_n = f(b_{n-1})$ for some continuous function f and if $\{b_n\}$ converges to a point b that is in the domain of f, then the limit must be a root of the equation $f(x) = x$:

$$f(b) = f\left(\lim_{n\to\infty} b_n\right) = f\left(\lim_{n\to\infty} b_{n-1}\right) \overset{\text{Continuity}}{=} \lim_{n\to\infty} f(b_{n-1}) = \lim_{n\to\infty} b_n = b.$$

A root of the equation $f(x) = x$ is called a *fixed point* of f. The following example illustrates how we can use this idea to evaluate a limit.

Let $b_1 = 1$. For $n \geq 2$, let $b_n = \sqrt{2 + b_{n-1}}$. The sequence $\{b_n\}$ is bounded and increasing. Evaluate its limit.

The monotone convergence property asserts that $b_n \to \ell$ for some number ℓ. Let $f(x) = \sqrt{2+x}$. Since $b_n = f(b_{n-1})$, we know that ℓ is a fixed point of f: $\ell = \sqrt{2+\ell}$. Thus, $\ell^2 = 2 + \ell$, or $\ell^2 - \ell - 2 = 0$. The two roots of this quadratic equation are -1 and 2. Because $\{b_n\}$ is a sequence of positive terms, we may rule out -1 as a limit. We conclude that $b_n \to 2$. Figure 2 illustrates how we may use the plots of the equations $y = f(x)$ and $y = x$ to understand the limiting behavior of the sequence $\{b_n\}$.

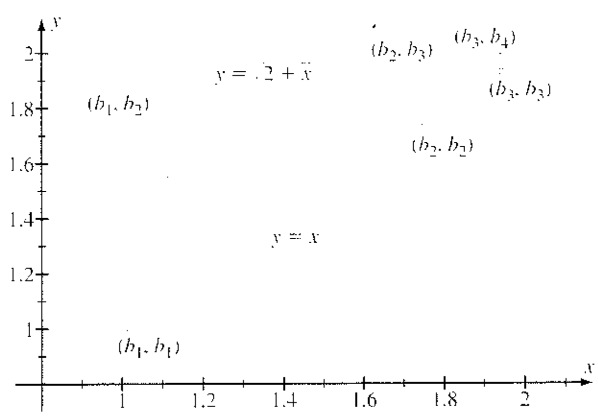

Figure 2

quickquiz

1. What is a monotone sequence? Do all monotone sequences converge?
2. What do we mean by 4^{π}?
3. What is the value of $\lim_{n\to\infty}(1+1/n)^n$?
4. Sketch the graphs of $x \mapsto e^x$ and $x \mapsto (e/\pi)^x$.

EXERCISES

1–17 21–30

Problems for Practice

In Exercises 1–6, simplify the expression.

1. $\sqrt{2}^{\sqrt{3}} \cdot \sqrt{2}^{\sqrt{3}}$
2. $4^{\pi} \cdot 4^{e}$
3. $(1/8)^{-\pi/3}$
4. $(8^{\sqrt{3}} \cdot 4^{\sqrt{7}})/2^{\pi}$
5. $\left(\sqrt{11}^{\sqrt{2}}\right)^{\sqrt{2}}$
6. $\left(\sqrt{e}^{\sqrt{2}}\right)^{2}$

In Exercises 7–12, make a rough sketch of the function. Label salient points.

7. $f(x) = 3^x + 1$
8. $x \mapsto 2^{-x}$
9. $g(x) = 3 - e^x$
10. $x \mapsto |e^x - 1|$
11. $x \mapsto e^{|x|}$
12. $f(x) = 1 - 1/2^{x-1}$

In Exercises 13–16, find the limit.

13. $\lim_{x\to\infty}(e^{2x} - e^{-2x})/(e^{2x} + e^{-2x})$
14. $\lim_{x\to e^-} \pi^{1/(x-e)}$
15. $\lim_{x\to e^+} \pi^{1/(x-e)}$
16. $\lim_{x\to(\pi/2)^-}(1/\sqrt{2})^{\tan(x)}$
17. Find all asymptotes of each equation.
    a. $y = (e^{7x} + e^{-7x})/(e^{7x} - e^{-7x})$
    b. $y = e^{-x^2}$
    c. $y = 3x/(x - e)$
    d. $y = (3^x + 2^x)/(3^x - 3)$
18. An annual interest rate of 6% is paid on an initial investment of \$1000. How much is the investment worth in 1 year under the following conditions?
    a. Simple interest
    b. Semiannual compounding
    c. Quarterly compounding
    d. Daily compounding
    e. Continuous compounding
19. An annual interest rate of 7% is paid on an initial investment of \$1000. How much is the investment worth in 5 years under the following conditions?
    a. Simple interest
    b. Semiannual compounding
    c. Quarterly compounding
    d. Daily compounding
    e. Continuous compounding
20. If an amount of income A is to be received at a future date, the *present value* of that payment is the amount P, which will grow to A under continuous compounding at the current interest rate by the time the payment is received. Mr. Woodman wants to give \$100,000 to his son Chip when Chip turns 25 years old. Given that current interest rates are 5% and Chip has just turned 18, what is the present value of the gift?

Use Theorem 4 to calculate the limits in Exercises 21–24.

21. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$
22. $\lim_{n\to\infty}\left(1-\frac{e}{n}\right)^n$
23. $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$
24. $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$

In Exercises 25–28, a convergent sequence $\{a_n\}$ is defined recursively. Calculate its limit by using the method from Example 2.

25. $a_1 = 1,\ a_n = \sqrt{2 + 3a_{n-1}}$
26. $a_1 = 3/5,\ a_n = \sin(\pi a_{n-1}/3)$
27. $a_1 = 1,\ a_n = (a_{n-1} + 6)/(a_{n-1} + 2)$
28. $a_1 = 1,\ a_n = (a_{n-1}/2) + (1/a_{n-1}^2)$

Further Theory and Practice

29. Suppose $P(t) = Ae^{kt}$ where k is a positive constant. Show that there is a number τ (known as the doubling time) in the open interval $(0, 1/k)$ such that
$$P(t+\tau) = 2P(t)$$
for every t. Find a formula involving k and τ.
30. Suppose $P(t) = mt + b$. Can P have a doubling time (in the sense of Exercise 29)? Explain.