Tutorial 1

P40 1.6

(a). $X_t = \underbrace{\beta_1 + \beta_2 t}_{\text{signal}} + \underbrace{w_t}_{\text{noise}}$, $w_t \sim WN(0, \sigma_w^2)$

$E(X_t) = E(\beta_1 + \beta_2 t + w_t) = \beta_1 + \beta_2 t + E(w_t) = \beta_1 + \beta_2 t$

if $\beta_2 \neq 0$, then non-stationary

if $\beta_2 = 0$, stationary

(b). $Y_t = X_t - X_{t-1}$ is stationary

$= (\beta_1 + \beta_2 t + w_t) - (\beta_1 + \beta_2 (t-1) + w_{t-1})$

$= \beta_2 + w_t - w_{t-1}$

$E(Y_t) = \beta_2$

$\gamma(0) = E[(y_t - \beta_2)(y_t - \beta_2)] = E[(w_t - w_{t-1})^2] = E(w_t^2 - 2w_t w_{t-1} + w_{t-1}^2) = 2\sigma_w^2$

$\parallel$

$\gamma(t, t+0)$

$\gamma(1) = E[(w_{t+1} - w_t)(w_t - w_{t-1})]$

$= E(w_{t+1}w_t - w_{t+1}w_{t-1} - w_t^2 + w_t w_{t-1}] = -\sigma_w^2$

$\gamma(2) = E[(w_{t+2} - w_{t+1})(w_t - w_{t-1})] = 0$

$$\gamma(h) = \begin{cases} 2\sigma^2 & h=0 \\ -\sigma^2 & h=1 \\ 0 & o.w. \end{cases}$$

(c). $V_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}$ to show $E(V_t) = \beta_1 + \beta_2 t$

$E(aX+bY) = aE(X) + bE(Y)$
Linearity of expectation

$E(V_t) = E(\frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j})$

$= \frac{1}{2q+1} \sum_{j=-q}^{q} E(X_{t-j})$

$= \frac{1}{2q+1} \sum_{j=-q}^{q} (\beta_1 + \beta_2 t)$

$= \beta_1 + \beta_2 t$

Fact: X_t & $X_t - f(t)$ have the same $\gamma(s,t)$

$V_t' = V_t - \beta_1 - \beta_2 t = (\frac{1}{2q+1} \sum_{j=-q}^{q} [\beta_1 + \beta_2(t-j) + w_{t-j}]) - \beta_1 - \beta_2 t$

$= \frac{1}{2q+1} \sum_{j=-q}^{q} w_{t-j}$

$\gamma(0) = E[(\frac{1}{2q+1} \sum_{j=-q}^{q} w_{t-j})^2]$

$= (\frac{1}{2q+1})^2 E[(\sum_{j=-q}^{q} w_{t-j})^2]$

$= (\frac{1}{2q+1})^2 (2q+1) \sigma_w^2$

$= \sigma_w^2 / (2q+1)$ $\qquad$ $E(w_t w_s) = 0$ if $t \neq s$

WN

$t-q \quad t \quad t+1 \quad t+q \quad t+q+1 \quad t+2q+1$

$V'_t \quad V'_{t+1} \quad V_{t+2q+1} = \frac{1}{2q+1}\sum_{j=-q}^{q} X_{t+2q+1-j}$

$V_t \qquad t+q+1$

$\gamma(2q+1) = cov(V'_t, V'_{t+2q+1}) = 0$

$\gamma(h) = 0$ if $h \geq 2q+1$

$\gamma(1) = (\frac{1}{2q+1})^2 E[(\sum_{j=-q}^{q} w_{-j})(\sum_{j=-q}^{q} w_{-j})] = (\frac{1}{2q+1})^2 \cdot 2q \cdot \sigma_w^2$

$-q \quad 1-q \quad 0 \quad q \quad 1+q$

$$\gamma(h) = \begin{cases} (\frac{1}{2q+1})^2(2q+1-h)\sigma_w^2 & \text{for } 0 \leq h \leq 2q \\ 0 & \text{o.w.} \end{cases}$$

1.8

$X_t = \delta + X_{t-1} + w_t$ with $X_0 = 0$

(a). Show that $X_t = \delta t + \sum_{k=1}^{t} w_k$

$X_t = \delta + (\delta + X_{t-2} + w_{t-1})$
$= \cdots\cdots$
$= \delta t + \sum_{k=1}^{t} w_k$

(b). $\mu_x(t) = E(X_t) = E(\delta t + \sum_{k=1}^{t} w_k) = \delta t + \sum_{k=1}^{t} E(w_k) = \delta t$

$\gamma(s,t) = E[(X_t - \delta t)(X_s - \delta s)]$
$= E[(\sum_{k=1}^{t} w_k)(\sum_{j=1}^{s} w_j)]$
$= \min(s,t)\sigma^2$ depends on t

(d). show $\rho(t-1,t) = \sqrt{\frac{t-1}{t}} \to 1$ as $t \to \infty$

$\rho(t-1,t) = \frac{\gamma(t-1,t)}{\sqrt{\gamma(t-1,t-1)} \cdot \sqrt{\gamma(t,t)}} = \frac{\sqrt{t-1}\,\sigma^2\sqrt{t-1}}{\sqrt{(t-1)\sigma^2 t\sigma^2}} = \sqrt{\frac{t-1}{t}}$

(e). $X_t = \delta + X_{t-1} + w_t$

$E(X_t) = \delta t$ suggest to take 1st order diff op.

$y_t = x_t - x_{t-1} = x_{t-1} + w_t - x_{t-1} = w_t$
$-\delta$

$E(y_t) = 0 \qquad \gamma(0) = \sigma^2,\ \gamma(h) = 0$ if $h > 0$.