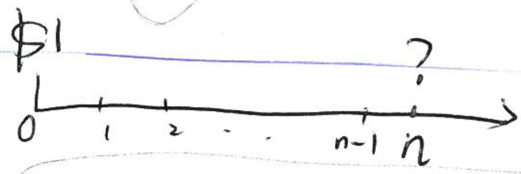
LN 12.1

$\tilde{i}$: $E[\tilde{i}]$, $Var[\tilde{i}]$

$\tilde{S}(n)$: $E[\tilde{S}(n)]$, $Var[\tilde{S}(n)]$

<u>CLT</u> : $\boxed{\text{for large } n}$, $\tilde{X}_i$ i.i.d, $i=1,2,\cdots n$

$\Rightarrow \sum_i \tilde{X}_i$ Approximately $\sim N(\quad , \quad)$



$\tilde{S}(n) = (1+\tilde{i}_1)(1+\tilde{i}_2)\cdots(1+\tilde{i}_n)$

$$\boxed{\ln(\tilde{S}(n))} = \sum_{k=1}^{n} \ln(1+\tilde{i}_k) \overset{\text{Approximately}}{\sim} N\Big(\overset{\mu}{\overset{\|}{n\cdot E[\tilde{\delta}]}}, \overset{\sigma^2}{\overset{\|}{n\cdot Var[\tilde{\delta}]}}\Big)$$

$\tilde{i}_k$: i.i.d. $k=1,2,\cdots n$. $\Rightarrow$ $\boxed{\ln(1+\tilde{i}_k)}$, i.i.d.

$\ln(1+\tilde{i}_k) = \tilde{\delta}_k$

$\tilde{\delta}_k \to E[\tilde{\delta}_k]=E[\tilde{\delta}]$

$\tilde{\delta}_k \to Var[\tilde{\delta}_k]=Var[\tilde{\delta}]$

$\boxed{\tilde{S}(n) \overset{\text{Appro.}}{\sim} \text{Logrormal Distribution.}}$

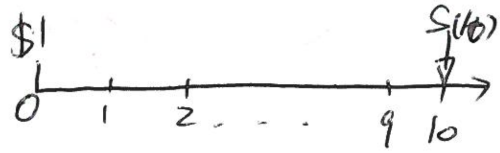
$$E[\ln[\tilde{S}(n)]] = E[\tilde{\delta}_1+\tilde{\delta}_2+\cdots+\tilde{\delta}_n] = n\cdot E[\tilde{\delta}]$$

$$Var[\ln[\tilde{S}(n)]] = Var[\tilde{\delta}_1+\tilde{\delta}_2+\cdots+\tilde{\delta}_n] = n\cdot Var[\tilde{\delta}]$$

$$\Pr[a < \tilde{S}(n) < b] = \Pr[\ln a < \ln \tilde{S}(n) < \ln b]$$

$$= \Pr\left[\frac{\ln a - \mu}{\sigma} < \underbrace{\boxed{\frac{\ln \tilde{S}(n) - \mu}{\sigma}}}_{Z} < \frac{\ln b - \mu}{\sigma}\right] \quad \sim N(0,1)$$

Ex: $\Pr[\tilde{S}(10) > 3.247321]$, $\tilde{i}_t$, $t = 1, 2, \cdots, 10$ i.i.d.



$$\tilde{i} = \begin{cases} 0.10, & prob = 0.5 \\ 0.15, & prob = 0.5 \end{cases}$$

① Use log-normal Approx. for $\tilde{S}(10)$; ② Exact value

Sol ①: $\ln[\tilde{S}(10)] \overset{\text{Approx.}}{\sim} N(n \cdot E[\tilde{\delta}],\ n \cdot Var[\tilde{\delta}])$

$$\tilde{\delta} = \ln(1 + \tilde{i}) = \begin{cases} \ln(1.1), & prob = 0.5 \\ \ln(1.15), & prob = 0.5 \end{cases}$$

$$E[\tilde{\delta}] = \ln(1.1) \cdot 0.5 + \ln(1.15) \cdot 0.5 = 0.117536$$

$$Var[\tilde{\delta}] = E[\tilde{\delta}^2] - (E[\tilde{\delta}])^2 = \left[(\ln 1.1)^2 \cdot 0.5 + (\ln(1.15))^2 \cdot 0.5\right] - (0.117536)^2$$

$$= 0.00049399$$

③

$\Rightarrow$ mean $= n \cdot E[\tilde{S}] = 10 \cdot \cancel{E[\tilde{S}]}\ 0.117536 = 1.17536$

Variance $= n \cdot Var[\tilde{S}] = 10 \cdot 0.00049399 = 0.0049399$

$\Rightarrow \ln[\tilde{S}(10)] \overset{\text{Approx.}}{\sim} N(1.17536, 0.0049399)$

$\Rightarrow Pr[\tilde{S}(10) > 3.247321]$

$= Pr[\ln[\tilde{S}(10)] > \ln(3.247321)]$

$= Pr\left[\underbrace{\boxed{\frac{\ln[\tilde{S}(10)] - 1.17536}{\sqrt{0.0049399}}}}_{\sim N(0,1)} > \frac{\ln(3.247321) - 1.17536}{\sqrt{0.0049399}}\right)$

$= Pr[\ Z > 0.035]$

$= 1 - Pr[Z \le 0.035]$

$= 0.486$

② $\tilde{S}(10) = (1+\tilde{i}_1)(1+\tilde{i}_2) \cdots (1+\tilde{i}_{10})$

Trial & Error: # of 0.15 & # of 0.10.

Test ⑤, $i=0.15$; ⑤, $i=0.10$; $\Rightarrow \tilde{S}(10) = (1+0.15)^5 \cdot (1+0.10)^5$

$= 3.239311 < 3.247321$

Test ⑥, $i=0.15$, ④, $i=0.10$; $\Rightarrow \tilde{S}(10) = (1+0.15)^6 \cdot (1+0.10)^4$

$= 3.386552 > 3.247321$

$$\Pr[\tilde{S}(10) > 3.247321] = \sum_{i=6}^{10} C_{10}^{i} \cdot 0.5^{10-i} \cdot 0.5^{i}$$

$$C_{10}^{i} = \boxed{\frac{10 \times 9 \times 8 \times \cdots (10-i+1)}{i \times (i-1) \times \cdots 2 \cdot 1}}$$

$$C_{10}^{6} = C_{10}^{4}$$

$$= 0.5^{10}\left(\boxed{C_{10}^{6} + C_{10}^{7} + C_{10}^{8} + C_{10}^{9} + C_{10}^{10}}\right)$$

$$\left(C_{10}^{4} + C_{10}^{3} + C_{10}^{2} + C_{10}^{1} + C_{10}^{0}\right) = \boxed{C_{10}^{6} + C_{10}^{7} + C_{10}^{8} + C_{10}^{9} + C_{10}^{10}}$$

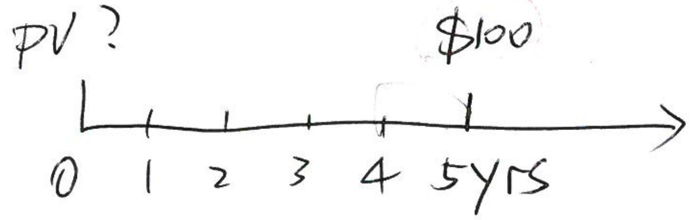
$$= 0.5^{10}\left(\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} + \frac{10 \times 9 \times 8}{3 \times 2} + 10 + 1\right)$$

$$= 0.5^{10}\,(210 + 120 + 45 + 10 + 1)$$

$$= 0.377$$

Present Values: $\tilde{i}$ $\boxed{\tilde{S}_n}$

Ex:



$\tilde{i}_t$, $t = 1, 2, \cdots 5$, i.i.d $\quad \boxed{\tilde{i} = \begin{cases} 0.10, & \text{prob} = 0.5 \\ 0.15, & \text{prob} = 0.5 \end{cases}}$

Sol

$$\widetilde{PV} = 100\,(1+\tilde{i}_5)^{-1}(1+\tilde{i}_4)^{-1} \cdots (1+\tilde{i}_1)^{-1}$$

$$E[\widetilde{PV}] = E\left[100 \cdot (1+\tilde{i}_5)^{-1}(1+\tilde{i}_4)^{-1} \cdots (1+\tilde{i}_1)^{-1}\right]$$

$$= 100 \cdot E\left[(1+\tilde{i}_5)^{-1}\right] \cdot E\left[(1+\tilde{i}_4)^{-1}\right] \cdots E\left[(1+\tilde{i}_1)^{-1}\right]$$

$$= 100 \cdot \left(E\left[\frac{1}{1+\tilde{i}}\right]\right)^{5}$$

$$\Rightarrow \frac{1}{1+\tilde{i}} \sim \begin{cases} \frac{1}{1.1}, & prob = 0.5 \\ \frac{1}{1.15}, & prob = 0.5 \end{cases}$$

$$\Rightarrow E\left[\frac{1}{1+\tilde{i}}\right] = \frac{1}{1.1} \cdot 0.5 + \frac{1}{1.15} \cdot 0.5 = (\quad)$$

$$\Rightarrow E[\tilde{PV}] = 100 \cdot (\quad)^5 = \$55.63$$

A.V. = F.V.

Wrong sol :

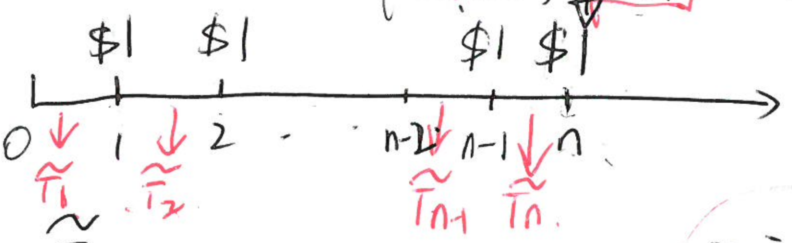
$$\underbrace{100}_{E[A.V.]} = PV \cdot E[(1+\tilde{i}_1)(1+\tilde{i}_2)\cdots(1+\tilde{i}_5)]$$

$$100 = PV \cdot (E[1+\tilde{i}])^5$$

$$100 = PV \cdot (1.125)^5$$

$$\Rightarrow PV = \frac{100}{(1.125)^5} = \$55.49.$$

Annuities : $\boxed{A.V}$ $\boxed{s_{\overline{n}|i}}$ $E[A.V.]$



$\tilde{i}_t$, $t = 1, 2, \cdots n$, i.i.d

$$\boxed{E[\tilde{i}_t] = E[\tilde{i}] \quad t=1,2,\cdots n}$$

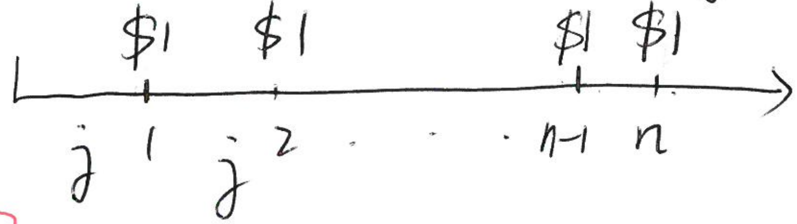
$$\tilde{S}(n) = \$1 + (1+\tilde{i}_n) + (1+\tilde{i}_n)(1+\tilde{i}_{n-1}) + \cdots + (1+\tilde{i}_n)(1+\tilde{i}_{n-1})\cdots(1+\tilde{i}_2)$$

$$E[\tilde{S}(n)] = 1 + E[1+\tilde{i}] + \left(E[1+\tilde{i}]\right)^2 + \cdots + \left(E[1+\tilde{i}]\right)^{n-1}$$

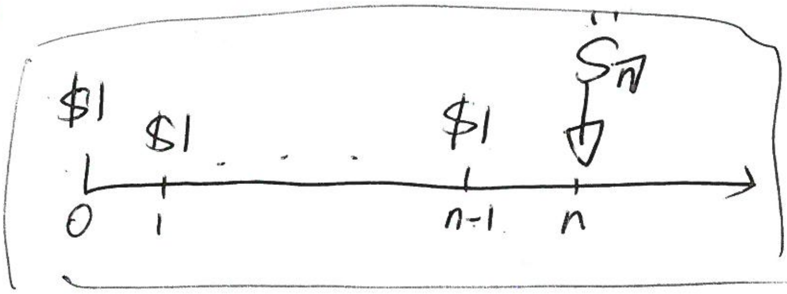
$E[\tilde{i}] = j \Rightarrow$

$$= 1 + \left(1+E[\tilde{i}]\right) + \left(1+E[\tilde{i}]\right)^2 + \cdots + \left(1+E[\tilde{i}]\right)^{n-1}$$

$$= 1 + (1+j) + (1+j)^2 + \cdots (1+j)^{n-1}.$$



$$\boxed{E[\tilde{S}(n)] = s_{\overline{n}|j}}$$

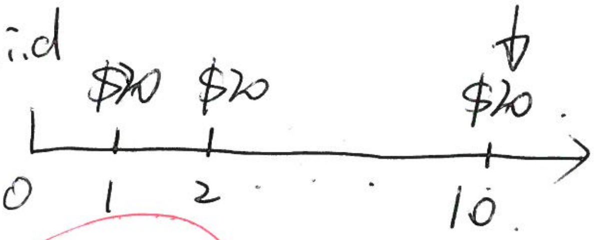


$\tilde{\ddot{s}}(n) =$

$$(1+\tilde{i}_n) + (1+\tilde{i}_n)\cdot(1+\tilde{i}_{n-1}) \cdots\cdots + (1+\tilde{i}_n)\cdots(1+\tilde{i}_1)$$

$$\Rightarrow \boxed{E[\tilde{\ddot{s}}(n)] = \ddot{s}_{\overline{n}|j}}, \quad j = E[\tilde{i}]$$

Ex $\tilde{i} = \begin{cases} 0.10 & , \text{prob} = 0.5 \\ 0.15 & \text{prob} = 0.5 \end{cases}$, i.i.d



$$E[A.V.] = E[20\cdot \tilde{s}(10)] = 20\, E[\tilde{s}(10)]$$

$$= 20\cdot s_{\overline{10}|j}$$

$$j = E[\tilde{i}] = 0.1\times 0.5 + 0.15\times 0.5 = 0.125 \Rightarrow E[A.V.] = 359.57$$