University of Toronto
Department of Mathematics

MAT224H1S
Linear Algebra II

Midterm Examination I
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Duration: 1 hour 30 minutes

Last Name: ____________________

Given Name: ____________________

Student Number: ____________________

Tutorial Code: ____________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/6
6	/4
TOTAL	/50

1. Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by

$$T(p(x)) = p(x-1).$$

(a) Show that T is a linear operator.

(b) Find the matrix of T relative to the basis $\alpha = \{1, 1+x, 1+x+x^2\}$ of $P_2(\mathbb{R})$.

Solution:

(a) Let $p, q \in P_2(\mathbb{R})$ and $a, b \in \mathbb{R}$. Then we have for every $x \in \mathbb{R}$:

$$\begin{aligned} T((a \cdot p + b \cdot q)(x)) &= (a \cdot p + b \cdot q)(x-1) \\ &= a \cdot p(x-1) + b \cdot q(x-1) \\ &= a \cdot T(p(x)) + b \cdot T(q(x)). \end{aligned}$$

This shows that T is a linear operator.

(b) We compute the images of the basis elements of α under T and represent them as linear combinations of basis elements in α:

$$T(1) = 1 = 1 \cdot 1$$

$$T(1+x) = 1 + (x-1) = -1 \cdot 1 + 1 \cdot (1+x)$$

$$T(1+x+x^2) = 1 + (x-1) + (x-1)^2 = 1 - x + x^2 = 2 - 2 \cdot (1+x) + (1+x+x^2)$$

Hence the matrix of T relative to the basis α is given by

$$[T]_{\alpha\alpha} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Is the set $\{(i,1,2i),(1,1+i,i),(1,3+5i,-4+3i)\}$ a basis for $\mathbb{C}^3$? Justify your answer.

Solution:

We row reduce to row echolon form:

$$\begin{pmatrix} i & 1 & 2i \\ 1 & 1+i & i \\ 1 & 3+5i & -4+3i \end{pmatrix} \xrightarrow{i\cdot I} \begin{pmatrix} -1 & i & -2 \\ 1 & 1+i & i \\ 1 & 3+5i & -4+3i \end{pmatrix} \xrightarrow{\tilde{II}=II+I} \begin{pmatrix} -1 & i & -2 \\ 0 & 1+2i & -2+i \\ 1 & 3+5i & -4+3i \end{pmatrix}$$

$$\xrightarrow{I\tilde{II}=III+I} \begin{pmatrix} -1 & i & -2 \\ 0 & 1+2i & -2+i \\ 0 & 3+6i & -6+3i \end{pmatrix} \xrightarrow{I\tilde{II}=III-3II} \begin{pmatrix} -1 & i & -2 \\ 0 & 1+2i & -2+i \\ 0 & 0 & 0 \end{pmatrix}$$

Since there exist only two leading ones, the linear span of the three vectors has only two dimensions. Therefore they are not linear independent and can not form a basis for $\mathbb{C}^3$.

3. Let $T\colon \mathbb{R}_{2\times 2} \to \mathbb{R}_{2\times 2}$ be the linear transformation be defined by

$$T(A) = A\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A.$$

(a) Find a basis for the kernel of T.

(b) Find a basis for the range of T.

Solution:

Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a, b, c, d \in \mathbb{R}$. Then

$$T(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix} - \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$= \begin{pmatrix} b-c & a-d \\ d-a & c-b \end{pmatrix}$$

(a) $Ker(T) = \{A \in \mathbb{R}_{2\times 2}, T(A) = 0\}$.

Now $T(A) = 0 \Leftrightarrow \begin{pmatrix} b-c & a-d \\ d-a & c-b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow b = c$ and $a = d$.

$\Rightarrow Ker(T) = \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix}, a, b \in \mathbb{R}\right\}$.

It follows that a basis α of $Ker(T)$ is given by

$$\alpha = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

(b) We know that $Range(T) = span\left\{T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$$= span\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$$

$$= span\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

.

But these two matrices are linear independent, so we get that a basis β for the range of T is given by

$$\beta = \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, cx_2, x_1 + x_3)$$

where $c \in \mathbb{R}$ is a constant. For what values of c does there exist a basis α such that $[T]_{\alpha\alpha}$ diagonal? Justify your answer.

Solution:

We know that such a basis exists (in other words T is diagonalizable) if and only if the algebraic multiplicities of all eigenvalues of T equal the corresponding geometric multiplicities. In particular, if T has three distinct eigenvalues, then T is diagonalizable (since all geometric and algebraic multiplicities must be equal to 1).

Represent T as a matrix A with respect to the standard basis $\mathcal{E}_3$:

$$[T]_{\mathcal{E}_3\mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & c & 0 \\ 1 & 0 & 1 \end{pmatrix} =: A.$$

Now we can compute the eigenvalues of T by solving $det(xI - A)$ for x:

$$det \begin{pmatrix} x-1 & -1 & -1 \\ 0 & x-c & 0 \\ -1 & 0 & x-1 \end{pmatrix} = (x-1) \cdot det \begin{pmatrix} x-c & 0 \\ 0 & x-1 \end{pmatrix} - det \begin{pmatrix} -1 & -1 \\ x-c & 0 \end{pmatrix}$$

$$= (x-1)^2(x-c) - (x-c) = x^3 - (2+c)x^2 - 2cx = x(x-2)(x-c).$$

So the eigenvalues of T are given by $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = c$.

By the comments above, we conclude that T is diagonalizable if $c \neq 2$ and $c \neq 0$. The other two cases must be examined separately:

$c = 0$

Then the eigenvalue $\lambda = 0$ has algebraic multiplicity 2, and we need to check whether there are two linear independent vectors with eigenvalue 0. That is we want to know whether the kernel of T is two-dimensional. But

$$[T]_{\mathcal{E}_3\mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and this matrix has two linear independent rows, so $Ker([T]_{\mathcal{E}_3\mathcal{E}_3})$ is only 1-dimensional. It follows that the geometric multiplicity of the eigenvalue 0 is strictly smaller than the algebraic one, and T is not diagonalizable for $c = 0$.

$\underline{c = 2}$

Then $[T]_{\mathcal{E}_3\mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $2 \cdot I - [T]_{\mathcal{E}_3\mathcal{E}_3} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

The matrix $[T]_{\mathcal{E}_3\mathcal{E}_3}$ has two linear independent rows, so $Ker(2 \cdot I - [T]_{\mathcal{E}_3\mathcal{E}_3})$ is only 1-dimensional. It follows that the geometric multiplicity of the eigenvalue 2 is strictly smaller than its algebraic multiplicity, and T is not diagonalizable for $c = 2$

We conclude that T is diagonalizable if and only if $c \in \mathbb{R} \setminus \{0, 2\}$.

5. Let V and W be vector spaces over a field F, let $\alpha = \{v_1, \ldots, v_n\}$ and $\beta = \{w_1, \ldots, w_n\}$ be bases for V and W respectively, and let $T\colon V \to W$ be a linear transformation. Prove that T is an isomorphism iff $[T]_{\beta\alpha}$ is an invertible matrix.

Solution:
" $\Rightarrow$ ":
Assume that T is an isomorphism. Then there exists a linear transformation $T^{-1} : W \to V$ such that $T^{-1} \circ T = id_V$ and $T \circ T^{-1} = id_W$. Then

$$I = [id_V]_{\alpha\alpha} = [T^{-1} \circ T]_{\alpha\alpha} = [T^{-1}]_{\alpha\beta}[T]_{\beta\alpha}$$

and

$$I = [id_W]_{\beta\beta} = [T \circ T^{-1}]_{\beta\beta} = [T]_{\beta\alpha}[T^{-1}]_{\alpha\beta}.$$

It follows that the matrix $[T]_{\beta\alpha}$ is invertible with inverse matrix $[T^{-1}]_{\alpha\beta}$.

" $\Leftarrow$ ":
Conversely suppose that the matrix $[T]_{\beta\alpha}$ is invertible. Then there exists a matrix $A = (a_{i,j})$, such that $[T]_{\beta\alpha} \cdot A = A \cdot [T]_{\beta\alpha}$=I.
Define a linear operator $S : W \to V$ on the basis elements in β by

$$S(w_j) = \sum_{i=1}^{n} a_{i,j} v_i.$$

Then by definition $[S]_{\alpha\beta} = A$. Now

$$[S \circ T]_{\alpha\alpha} = [S]_{\alpha\beta}[T]_{\beta\alpha} = A \cdot [T]_{\beta\alpha} = I$$

and

$$[T \circ S]_{\beta\beta} = [T]_{\beta\alpha}[S]_{\alpha\beta} = [T]_{\beta\alpha} \cdot A = I.$$

This implies that S is an inverse for T, and T is an isomorphism.

6. Let $V = M_{22}$, the set of all 2×2 matrices. Let the operation of vector addition in V be the usual matrix addition but let scalar multiplication in V be defined by

$$c \cdot A = cA^T.$$

Is V a vector space? Justify your answer.

<u>*Solution:*</u>

V is not a vector space, because the condition $1 \cdot v = v$ does not hold for all vectors $v \in V$. For example take

$$v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then

$$1 \cdot v = v^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq v.$$