Lecture 16
March 10th, 2015

A2 solution
$m, n \in \mathbb{N}, m, n \geq 1$
$a_0 = m, b_0 = n, k_0 = 1, l_0 = 1$

loop condition $a_i \neq b_i$

$$a_{i+1} = \begin{cases} a_i + m & , a_i < b_i \\ a_i & , a_i > b_i \end{cases}$$ (since $a_i \neq b_i$

$$b_{i+1} = \begin{cases} b_i & , a_i < b_i \\ b_i + n & , a_i > b_i \end{cases}$$

$$k_{i+1} = \begin{cases} k_i + 1 & , a_i < b_i \\ k_i & , a_i > b_i \end{cases}$$

$$l_{i+1} = \begin{cases} l_i & , a_i < b_i \\ l_i + 1 & , a_i > b_i \end{cases}$$

POST-CONDITION: IF THERE IS A LAST ITERATION t, THEN $k_t m = \boxed{l_t n = b_t}$

This part is true always

Since $a_t = b_t$,
$b_t = k_t m$ would be true if $a_t = k_t m$

For $i \in \mathbb{N}$, let $I(i)$ be: $\exists k_i, l_i \in \mathbb{N}$ $a_i = k_i m \wedge b_i = l_i n$
$I(0)$: $a_0 = m = k_0 m, \leq mn$ $b_0 = n = 1 \cdot n = l_0 n$ $\leq mn$
IS: Let $i \in \mathbb{N}$, assume $a_i = k_i m \wedge b_i = l_i n$ (IH)
Case $a_i < b_i$
$a_{i+1} = a_i + m = k_i m + m$ by (IH)
$= (k_i + 1)m = k_{i+1} m$
$b_{i+1} = b_i = l_i n$ by (IH)
$= l_{i+1} n$
Case $a_i > b_i$
...
If terminates at t, then $a_t = b_t$
and $I(t)$ i.e. $a_t = k_t m$, $b_t = l_t n$
so $k_t n = a_t = b_t = l_t n$

$6, 15$ (m, n)
$a_i \leq mn$
$b_i \leq mn$
and ith iteration then $a_i \neq b_i$
CASE $a_i < b_i$: $a_i < b_i \leq mn$, so $a_i < mn$
so $a_i = k_i m < n \cdot m$, so $k_i \leq n - 1$
$a_{i+1} = km + m \leq (n-1)m + m = mn$
$b_{i+1} = b_i \leq mn$

VARIANT: $mn - \min(a_i, b_i) \geq 0$ by $a_i, b_i \leq mn \in \mathbb{Z}$ by $m, n \in \mathbb{N}$
$a_i = k_i m \in \mathbb{N}$, $b_i = l_i n \in \mathbb{N}$

case $a_i < b_i$
Let $c = \alpha m = \beta n$, then $a_i < b_i \leq \alpha m$, so $a_i < \alpha m$
so $a_i = km < \alpha m$, so $k \leq \alpha - 1$
$a_{i+1} = km + m < (\alpha - 1)m + m = \alpha m$
$b_{i+1} = b_i \leq m\alpha = c$