UNIT 4
SEMANTICS: FUN WITH TRUTH-TABLES

4.1 MEANING AND TRUTH

We've mastered the syntax for sentential logic: how to construct grammatical complex sentences out of the atomic sentences using the logical connectives. The syntax also tells us a lot about the relations between those sentences. We can use the rules of inference to move from a set of sentences to related sentences – that's what we are doing when we do derivations. Thus, we can determine whether or not an argument is valid.

A valid argument is one in which the truth of the conclusion follows from the truth of the premises. A sound argument also has true premises. Thus, we need to know which sentences are true and which are false. We have to interpret those sentences to decide whether they are true or false. That's semantics.

The semantic value of a sentence is its truth-value (whether it is true or false). When symbolizing complex English sentences, we used an abbreviation scheme so that every atomic sentence stood for some simple sentence. Each of those simple sentences is either true or false. Now we need to know how to determine the truth-values of the molecular sentences from the truth-values of the atomic sentences.

What you might be wondering: How are meaning and truth related?

Isn't semantics the study of meaning? Why are we talking about truth-values?

Whether a sentence is true or false depends on whether or not the sentence picks out some actual state of affairs (a fact, perhaps). Whether or not the sentence states a fact (picks out a state of affairs) depends on what the sentence means. Thus, truth and meaning are intimately connected – one might want to say that to know what a sentence means is to know what conditions make it true or false.

> Suppose a man says (earnestly, neither joking nor lying), "Snakes have wings." This is clearly a false sentence. Yet the man appears to think it is true. We have to wonder whether or not he means what we mean by 'snake' or 'wing' – or whether he knows what the sentence means (perhaps he was just learning the language, and was trying to say that birds have wings.) But, if he knows what it means to say, "Snakes have wings," then he knows what would make it true – snakes having wings. Since snakes don't have wings, he can't mean what we mean *and* say something true.

Wittgenstein's *Tractatus* begins 'The world is all that is the case.' Anything that is the case is a fact – a true state of affairs. Propositions express these facts – they are logical representations of those states of affairs. And complex propositions are truth-functions of simpler propositions. Thus, if we know which basic propositions are true and false, we can determine the truth or falsity of any proposition. (Wittgenstein gave us the first truth-tables in the *Tractatus*.)

4.2 DETERMINING THE TRUTH-VALUE OF COMPLEX SENTENCES

Truth-value assignment (TVA): an assignment of truth-values to the atomic sentences.

Truth-table: a table of all possible truth-value assignments for a sentence or set of sentences.

When the connectives were introduced, they were defined in terms of the truth-values of the molecular sentences based on the truth-values of the sentential components. We can understand more complex sentences in terms of their truth-tables as well.

Since the truth-table gives the truth-values for all possible truth-value assignments (TVAs), one of those TVAs will represent the actual state of affairs (if each atomic sentence has a truth-value.)

The truth-tables for the logical connectives show us how to determine the truth-value of a complex sentence from the truth-values of the sentential components. The negation is straightforward – if a sentence, P, is true, then ~P is false and if P is false then ~P is true. The other four logical operators are binary connectives, connecting two sentences. Thus we have to consider all the different possible truth-value combinations for the two sentential components. Since each sentential component can be either true or false, there are four possible truth-value assignments for each connective.

These are the truth-tables for the 5 connectives. They show how the truth-value of the molecular sentence can be determined from the truth-values of the sentential components:

P	**Q**	**P → Q**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

P	**~P**
T	**F**
F	**T**

P	**Q**	**P ∧ Q**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

P	**Q**	**P ∨ Q**
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

P	**Q**	**P ↔ Q**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Given the truth-values of the atomic sentences, we can use these truth-tables to determine the truth-value of the molecular sentences, (P → Q), ~P, (P ∧ Q), (P ∨ Q) and (P ↔ Q).

But we can also use the truth-tables to determine the truth-value of *all* possible sentences if we use it recursively – doing it over and over, treating more and more complex sentences as the immediate sentential components of the main connectives.
Let's try it out, recursively.

P and Q are atomic sentences. Suppose that P is false and Q is true. How do we figure out the truth-value of $\sim(P \rightarrow Q) \vee (Q \leftrightarrow P)$?

Here is the truth-table for the connectives. We are interested in the line that has P false and Q true:

P	Q	$P \rightarrow Q$	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$\sim P$
T	T	T	T	T	T	F
T	F	F	F	T	F	F
F	T	T	F	T	F	T
F	F	T	F	F	T	T

We can use the truth-table for $\rightarrow$ to determine the truth-value of $(P \rightarrow Q)$ given the truth-values of P and Q. The third line of the truth-table for $(P \rightarrow Q)$ tells us that when the antecedent (P) is false and the consequent (Q) is true, then $(P \rightarrow Q)$ is true.

Now we can use the truth-table for ~ to determine the truth-value of $\sim(P \rightarrow Q)$. It tells us that when a sentence is true, the negation of the sentence is false. We need to know the truth-value of $\sim(P \rightarrow Q)$ when P is false and Q is true. Since $(P \rightarrow Q)$ is true on this TVA, $\sim(P \rightarrow Q)$ is false.

Now we want to use the truth-tables to determine the truth-value of $\sim(P \rightarrow Q) \vee (Q \leftrightarrow P)$. We know that the first disjunct, $\sim(P \rightarrow Q)$, is false. But we need to know the truth value of the second disjunct, $(Q \leftrightarrow P)$. We can use the truth-table for $\leftrightarrow$ to determine the truth-value of $(Q \leftrightarrow P)$. The second line of the truth-table for $\leftrightarrow$ tells us that the sentence is false if the left side is true and the right side is false. Since the left side, Q, is true and the right side, P, is false, $(Q \leftrightarrow P)$, is false.

Now we can use the truth-table for $\vee$ to determine the truth-value of $\sim(P \rightarrow Q) \vee (Q \leftrightarrow P)$. The fourth line of the truth-table for $\vee$ tells us that if the first sentence, $\sim(P \rightarrow Q)$, is false and the second sentence, $(Q \leftrightarrow P)$, is false, then the resulting disjunction is false. Thus, $\sim(P \rightarrow Q) \wedge (Q \leftrightarrow P)$ is false on this TVA (P is false and Q is true).

We can keep going, building up more complex sentences and determining their truth-values by using the truth-tables recursively. We can also consider more atomic sentences.

Now we have a method for determining the truth-value of every possible sentence that can be constructed in sentential logic.

In the example above we considered only one truth-value assignment (the third row in the table below). The full truth-table shows the truth-value of the sentence for all the possible truth-value assignments. The column below the main connective ($\vee$), gives the truth-value of the sentence for any TVA. The column below $\rightarrow$ gives the truth-value of $(P \rightarrow Q)$ for any TVA; that below ~ gives the truth-value of $\sim(P \rightarrow Q)$ for any TVA and the column below $\leftrightarrow$ gives the truth-value of $(Q \leftrightarrow P)$ for any TVA.

P	**Q**	**~**	**(P**	**$\rightarrow$**	**Q)**	$\downarrow$ $\vee$	**(Q**	**$\leftrightarrow$**	**P)**
T	T	F	T	T	T	**T**	T	T	T
T	F	T	T	F	F	**T**	F	F	T
F	T	F	F	T	T	**F**	T	F	F
F	F	F	F	T	F	**T**	F	T	F

4.3 HOW DO I SET UP A TRUTH-TABLE?

- Begin to set up a table. Draw a horizontal line.
- Above the line, list the atomic sentences in alphabetical order, then write the compound sentence(s) leaving plenty of space between the components.
- Figure out how many rows the truth-table will have. Each row represents one TVA. Every atomic sentence has two possible TVA's. Thus, the number of possible TVA's is 2^n, where n is the number of atomic sentences.
- Draw a vertical line going down that many rows after the atomic sentences.
- Assign truth-values for the atomic sentences

  Atomic sentence #1 will assigned 'T' for the 1st half of the TVA's and 'F' for the 2nd half.

  Atomic sentence #2 will be assigned 'T' for the 1st and 3rd quarters of the TVA's and 'F' for the 2nd and 4th quarters.

  Atomic sentence #3 will be assigned 'T' for the 1st, 3rd, 5th and 7th eighths of the TVA's and 'F' for the 2nd, 4th, 6th and 8th eighths of the TVA's. …

Note: in setting up the truth-table, it is important to leave enough space between symbols so that you will be able to distinguish different columns of 'T's and 'F's below them.

Let's make a truth-table for the sentence: $\sim R \vee S$

Here I have listed the two atomic sentences in alphabetical order and the compound (molecular) sentence above the horizontal line.

R	S	~	R	∨	S

There are 2 atomic sentences, so there will be 2^2 possible TVA's. That's 4 rows.

I draw a vertical line going down 4 rows and assign truth-values for R and S.

R	S	~	R	∨	S
T	T				
T	F				
F	T				
F	F				

The next step is to figure out what the compound sentence's truth-value is for every row, or for each possible TVA.

DETERMINING THE TRUTH-VALUE OF THE COMPOUND SENTENCE:

1. Assign truth-values to each occurrence of an atomic sentence on the right side of the vertical line in accordance to the TVA's on the left side of the vertical line.

2. Using the truth-values of the atomic sentences and the truth-tables for the five connectives, determine truth-values for the compound sentence, by working from the atomic sentences up to bigger and bigger molecular sentences. The truth-value of a molecular sentence is shown with 'T' or 'F' below the main connective of that sentence. For every row of the truth-table (for every TVA), put a 'T' or 'F' in a column directly below each connective.

3. The last column you complete should be below the main connective. Mark that connective with an arrow ↓ . The truth-value under the main connective is the truth-value of the compound sentence for each truth-value assignment.

REMEMBER:

Work from the atomic sentences up to bigger and bigger molecules, one column at a time.

Across each row, every occurrence of an atomic sentence letter must have the same truth-value.

Be neat!

4.3 EG1: Let's try it:

R	S	~	R	∨	S
T	T		T		T
T	F		T		F
F	T		F		T
F	F		F		F

- Start with the truth-values of the atomic sentences, R and S

- Next use the truth-table for ~ to calculate the truth-values of ~R.

- Finally, use the truth-table for ∨ to calculate the truth-values of ~R ∨ S (using the columns under '~' and 'S' as the arguments for ∨).

How would the truth-table differ if there were brackets: ~(R ∨ S)? In this case, we would have to calculate the truth-values of of R ∨ S considering the negation, ~.

4.3 EG2

Let's do a truth-table for the sentence: $(Q \vee \sim R) \wedge \sim (P \rightarrow Q)$

Since there are 3 atomic sentences, there will be 2^3 possible TVA's. That's 8 rows.

Truth-table for: $(Q \vee \sim R) \wedge \sim (P \rightarrow Q)$

P	Q	R	(Q	∨	~	R)	↓ ∧	~	(P	→	Q)
T	T	T	T			T			T		T
T	T	F	T			F			T		T
T	F	T	F			T			T		F
T	F	F	F			F			T		F
F	T	T	T			T			F		T
F	T	F	T			F			F		T
F	F	T	F			T			F		F
F	F	F	F			F			F		F

- Start with the truth-values of the atomic sentences, P, Q and R.
- Next use the truth-table for ~ to calculate the truth-values of ~R.
- Now, use the truth-table for ∨ (and the columns under Q and ~) to calculate the truth-values for $(Q \vee \sim R)$.
- Next use the truth-table for → to calculate the truth-values of $(P \rightarrow Q)$
- Use the truth-table for ~ to calculate the truth-values of $\sim(P \rightarrow Q)$
- Finally, use the truth-table for ∧ (and the columns under ∨ and the second ~) to calculate the truth-values of the whole sentence.

We can look under the main connective (marked by the down arrow) to find the truth-value of the sentence for every truth-value assignment.

Which truth-value assignments make the sentence true?

Which truth-value assignments make the sentence false?

The truth-table can tell us a lot about the relations between sentences. For instance:

We see that only one TVA yields the truth-value T for: $(Q \vee \sim R) \wedge \sim (P \rightarrow Q)$

Since that is the only TVA that makes it true, we know that ***if*** the molecular sentence is true ***then*** P must be true and Q and R must both be false.

$$[(Q \vee \sim R) \wedge \sim(P \rightarrow Q)] \rightarrow (P \wedge \sim Q \wedge \sim R)$$

Now we know that the following is a valid argument:

$$(Q \vee \sim R) \wedge \sim (P \rightarrow Q). \;\; \therefore \;\; (P \wedge \sim Q \wedge \sim R).$$

4.4 TRUTH-TABLES AND PROPERTIES OF SENTENCES & ARGUMENTS

We can use truth-tables to learn a lot about sentences and the logical relations between sentences – for instance, to discover whether a sentence is a tautology (a logical truth), a contradiction (a logical falsehood), or whether a set of sentences is inconsistent (cannot all be true) or whether a set of sentences entails a further sentence (whether an argument is valid).

TAUTOLOGIES, CONTRADICTIONS AND CONTINGENT SENTENCES:

We can use truth-tables to determine whether a compound sentence is a tautology, a contradiction, or a contingent sentence.

A sentence ϕ is a **tautology** if and only if ϕ is true on every truth-value assignment.

A sentence ϕ is a **contradiction** if and only if ϕ is false on every truth-value assignment.

A sentence ϕ is a **contingent sentence** if and only if ϕ is true on some truth-value assignments and false on some truth-value assignments.

Although these concepts are defined here in terms of truth-value assignments, they are essentially the same concepts as their 'logical' or 'philosophical' counterparts.

	Logical /Philosophical	**Truth-functional**
Tautology	cannot be false an analytic or logical truth. true by definition	true for every TVA truth-functionally true
Contradiction	cannot be true an analytic or logical falsehood false by definition	false for every TVA truth-functionally false
Contingent sentence	can be true or false logically indeterminate logically contingent synthetic	true on some TVA's and false on some TVA'. truth-functionally indeterminate truth-functionally contingent

In *The Critique of Pure Reason,* Kant introduced the terms 'analytic' and 'synthetic' to make this distinction. For Kant, an analytic sentence such as, "Any triangle has three sides," is one in which the predicate concept is contained in the subject concept. One can see why such sentences would be necessarily true. In a synthetic sentence, such as, "Creatures with hearts have kidneys," the predicate concept is not contained in the subject concept. For the most part, this means that it takes observation or experience with the world to determine whether such sentences are true. Kant thought that some synthetic sentences are known prior to experience, but there is no general agreement as to whether there are synthetic *a priori* truths. For our purposes, true synthetic sentences are empirical facts about the world.

The distinction was very important in the first half of the twentieth century for philosophers such as Frege, Russell, Wittgenstein, Carnap and Ayer – working on problems in philosophy of language, such as meaning, as well as on the foundations of mathematics and logic, and their relations to language. Quine, in "Two Dogmas of Empiricism", argued that the analytic/synthetic distinction was untenable.

Is the sentence tautologous, contradictory or contingent?

Look in the column under the main connective (which you might want to mark with '↓') in a completed truth-table.

Tautology:	there are only T's
Contradiction:	there are only F's
Contingent sentence:	there are T's and F's

4.4 EG1: Let's try it out.

Are the following sentences tautologous, contradictory or contingent?

a) $(\sim P \vee Q) \leftrightarrow (P \rightarrow Q)$

P	Q	(~	P	∨	Q)	↓ ↔	(P	→	Q)
T	T		T		T		T		T
T	F		T		F		T		F
F	T		F		T		F		T
F	F		F		F		F		F

b) $\sim(\sim P \vee \sim Q) \wedge (Q \rightarrow \sim P)$

P	Q	~	(~	P	∨	~	Q)	↓ ∧	(Q	→	~	P)
T	T			T			T		T			T
T	F			T			F		F			T
F	T			F			T		T			F
F	F			F			F		F			F

c) $\sim(P \leftrightarrow (P \rightarrow Q))$

P	Q	~	(P	↔	(P	→	Q))
T	T		T		T		T
T	F		T		T		F
F	T		F		F		T
F	F		F		F		F

Equivalency, Consistency and Inconsistency:

We can use truth-tables to determine whether or not two or more sentences are equivalent or consistent:

> Sentence ϕ and ψ are **logically equivalent** if and only if there is no truth-value assignment on which ϕ and ψ different truth-values. (They are logically equivalent if and only if ϕ and ψ have the same truth-value for every truth-value assignment.)
>
> A set of sentences is **consistent** if and only if there is at least one truth-value assignment on which all the members of the set are true.
>
> A set of sentences is **inconsistent** if and only if there is no truth-value assignment on which all members of the set are true (it is not consistent).

To determine whether a set of sentences has any of these properties, we need only complete the truth-table and analyze the columns under the main connective for each sentence.

Are sentences logically equivalent, consistent or inconsistent?

Look in the column under the main connective (which you might want to mark with '↓') for each sentence in a completed truth-table.

Logical Equivalency	same truth-value for every TVA.
Consistency	all sentences T for at least one TVA
Inconsistency	no TVA on which every sentence is T

Note: For logical equivalency and inconsistency, one must examine ***every*** TVA. For consistency, just one TVA is required (provided all sentences are true on that TVA).

4.4 EG2

a) Are these sentences equivalent?

$(P \vee \sim Q)$ $\quad$ $(Q \rightarrow P)$

b) Are these sentences consistent?

$\sim(P \vee Q)$ $\quad$ $(P \leftrightarrow Q)$

c) Are these sentences consistent, equivalent or neither?

$(\sim P \wedge Q)$ $\quad$ $(Q \rightarrow P)$

VALIDITY

We can also use truth-tables to determine whether an argument is valid.

An argument is valid if and only if the conclusion is true if all the premises are true. Thus, we can show that an argument is valid if the truth-table that lists all possible truth-value assignments for the premises and conclusion is such that whenever there is a 'T' for each of the premises, there is a 'T' for the conclusion.

Another way to think about it: any valid argument can be expressed as a theorem such that the conjunction of the premises forms the antecedent and the conclusion forms the consequent.

Theorems are sentences that can be derived from the empty set – they are always true. Thus, a theorem would be true on any truth-value assignment. That sounds like a tautology! Indeed, all theorems *are* tautologies. They are true on every truth-value assignment. A valid argument can be expressed as a tautological implication – an implication or conditional sentence that is a tautology.

Thus, a valid argument is one in which the premises tautologically imply the conclusion.

> A set of sentences **tautologically implies** a sentence ϕ if and only if there is no truth-value assignment for which all the sentences in the set are true and ϕ is false.
>
> An argument is **valid** if and only if there is no truth-value assignment for which all the premises are true and the conclusion is false.

Tautological Implication $\{\psi, \chi, \ldots\}$ implies ϕ	No TVA on which all sentences $\{\psi, \chi, \ldots\}$ are T and ϕ is F.
Validity	No TVA on which all premises are T and the conclusion is F.

4.4 EG3: Is this argument valid? $(P \wedge {\sim}Q) \vee R$. $\sim R \vee Q$. $\therefore {\sim}P \rightarrow Q$

Does $\{ ((P \wedge {\sim}Q) \vee R),\ ({\sim} R \vee Q) \}$ tautologically imply $({\sim}P \rightarrow Q)$?

P	Q	R	(P	∧	~	Q)	∨	R	~	R	∨	Q	~	P	→	Q
T	T	T														
T	T	F														
T	F	T														
T	F	F														
F	T	T														
F	T	F														
F	F	T														
F	F	F														

4.4 E1: Construct a full truth-table for each of the following sentences. Determine whether each sentence is a tautology, a contradiction or a contingent sentence.

a) $Q \rightarrow (S \rightarrow Q)$

b) $(T \leftrightarrow \sim T) \rightarrow \sim (T \leftrightarrow \sim T)$

c) $(P \leftrightarrow Q) \rightarrow (\sim P \rightarrow \sim Q)$

d) $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \wedge (P \wedge \sim R)$

e) $\sim P \rightarrow ((P \vee Q) \rightarrow Q)$

f) $(W \wedge X) \rightarrow ((Y \wedge \sim Y) \wedge W)$

g) $\sim S \rightarrow ((T \wedge S) \rightarrow U)$

h) $((P \wedge Q) \vee R) \leftrightarrow ((P \vee Q) \wedge (\sim P \rightarrow R))$

i) $(S \rightarrow (Q \rightarrow V)) \leftrightarrow (\sim (V \vee \sim Q) \wedge S)$

4.4 E2: Construct a full truth-table for each of the following pairs of sentences. Determine whether each pair is equivalent.

a) $\sim (P \wedge Q)$ $\quad$ $\sim P \wedge \sim Q$

b) $P \rightarrow (Q \rightarrow P)$ $\quad$ $(R \wedge \sim R) \vee (Q \rightarrow Q)$

c) $T \leftrightarrow (S \vee R)$ $\quad$ $\sim T \leftrightarrow (\sim S \wedge \sim R)$

d) $P \wedge (Q \vee R)$ $\quad$ $(P \wedge Q) \vee R$

e) $(P \vee \sim(S \wedge T)) \rightarrow \sim S$ $\quad$ $(S \vee \sim(P \wedge T)) \rightarrow \sim P$

f) $(W \wedge X) \vee \sim(W \vee X)$ $\quad$ $W \leftrightarrow X$

g) $P \vee \sim(W \vee \sim Y)$ $\quad$ $(Y \leftrightarrow \sim P) \vee W$

4.4 E3: Construct a full truth-table for each of the following sets of sentences. Determine whether each set is consistent or inconsistent.

a) $P \wedge (R \vee \sim S)$. $\quad$ $\sim(P \vee \sim(S \rightarrow R))$

b) $P \rightarrow Q$. $\quad$ $R \rightarrow P$. $\quad$ $R \wedge \sim Q$.

c) $W \leftrightarrow \sim Y$. $\quad$ $(W \vee Z) \wedge (\sim Y \vee Z)$. $\quad$ $Z \leftrightarrow \sim W$.

d) $P \leftrightarrow (Q \vee R)$. $\quad$ $R \rightarrow (\sim Q \vee P)$. $\quad$ $Q \leftrightarrow \sim R$.

e) $S \rightarrow (R \vee Q)$. $\quad$ $Q \leftrightarrow \sim T$. $\quad$ $T \rightarrow (\sim R \wedge S)$.

f) $(P \wedge (Q \rightarrow \sim S)) \rightarrow (S \rightarrow \sim P)$. $\quad$ $\sim(Q \leftrightarrow \sim P)$.

4.4 E4: Construct a full truth-table for each of the following arguments. Determine whether it is valid.

a) $P \wedge Q$. $Q \rightarrow R$. $\therefore \sim P \vee R$.

b) $S \rightarrow (T \vee W)$. $\sim T$. $\therefore \sim (S \vee T)$.

c) $\sim (P \vee (\sim S \wedge Q))$. $S \rightarrow (P \rightarrow Q)$. $\therefore \sim P \leftrightarrow \sim S$.

d) $\sim R \vee (S \leftrightarrow \sim T)$. $S \rightarrow R$. $\sim R \vee T$. $\therefore \sim S$.

e) $R \rightarrow Q$. $\sim(S \wedge T) \leftrightarrow R$. $\therefore \sim T \vee \sim Q$

4.5 SHORTENED TRUTH-TABLES

A shortened truth-table is one which does not include all possible truth-value assignments. It usually contains only one or two truth-value assignments (chosen because of their significance).

Shortened truth-tables are often useful timesavers – especially when there are a lot of atomic sentences to consider. (A truth-table with 6 atomic sentences has 64 rows!)

Shortened truth-tables can only demonstrate certain things; but it's very efficient if you know what you are using your truth-table to show and you know that one or two TVAs can demonstrate it.

> A doctor can give you a full physical, run the full sweep of tests. But to do this every time you visit would be costly and time consuming. Often when you go to the doctor with a complaint, he/she narrows the problem down, thinks about what to look for (strep throat), what it looks like (red, sore throat and fever) and runs a single test (throat swab.)
>
> Like the doctor, we can avoid the time consuming full examination -- writing out the entire truth table. Instead, we test the sentence(s) in one fell swoop. This method is more efficient, but you need to know exactly what you're looking for. Once you know what you're looking for, you can run the appropriate test.

The Shortened Truth-Table Shows:	**TVA requirements:**	**Test**
A set of sentences is consistent:	All sentences of the set T on one TVA.	Can we make the sentences true?
A sentence is **not** a tautology:	Sentence F on one TVA.	Can we make the sentence false?
A sentence is **not** a contradiction:	Sentence T on one TVA.	Can we make the sentence true?
Two sentences are **not** equivalent:	One sentence T and the other F on one TVA.	Can we make one sentence true and the other false?
An argument is **not** valid:	All premises T and the conclusion F on one TVA.	Can we make the premises true and the conclusion false?
A set of sentences $\{\psi, \chi, \ldots\}$ does **not** tautologically entail another sentence, ϕ:	All sentences in set $\{\psi, \chi, \ldots\}$ T and ϕ F on one TVA.	Can we make all sentences in the set true and ϕ false?
A sentence is contingent:	Sentence T on one TVA. Sentence F on one TVA.	Can we make the sentence true and can we make it false?

Even when constructing a full truth-table, it is worth considering the requirements for showing the properties we are interested in and using it to reduce the number of TVAs that you calculate. If you reach a TVA (or two) that decides the question you are answering, your work is done!

The trick in constructing shortened truth-tables is working from the truth-value of the molecular sentences to the truth-values of the atomic sentences. It is important to have full understanding of the connectives, since you have to move backwards from the connectives to the components.

The following chart gives the component sentences' truth-values for each molecular sentence when true and when false. Sometimes when you know whether a sentence is true or false, you can immediately infer the truth-value of the sentential components. Other times, you might have to consider a few possibilities.

Molecular Sentence	**When Molecular Sentence is true:**	**When Molecular sentence is false**
$\sim \phi$	Negated sentence is false ϕ : F	Negated sentence is true ϕ : T
$\phi \rightarrow \psi$	Antecedent false or consequent true (3 possibilities) ϕ : T ψ : T ϕ : F ψ : T ϕ : F ψ : F	Antecedent true, consequent false ϕ : T ψ : F
$\phi \wedge \psi$	Both conjuncts are true ϕ : T ψ : T	At least one conjunct is false (3 possibilities) ϕ : T ψ : F ϕ : F ψ : T ϕ : F ψ : F
$\phi \vee \psi$	At least one disjunct is true (3 possibilities) ϕ : T ψ : T ϕ : T ψ : F ϕ : F ψ : T	Both disjuncts false ϕ : F ψ : F
$\phi \leftrightarrow \psi$	Both true, or both false. (2 possibilities) ϕ : T ψ : T ϕ : F ψ : F	One is true, one is false (2 possibilities) ϕ : T ψ : F ϕ : F ψ : T

How to construct a shortened truth-table:

Start the same way as with a full truth-table.

Determine the main connectives of all the molecular sentences.

Under each of the main connectives, put T or F according to your desired results.

If you want to show:	**What to put under the main connectives …**
A set of sentences is consistent:	T for all sentences in the set.
A sentence is **not** a tautology:	F
A sentence is **not** a contradiction:	T
Two sentences are **not** equivalent:	T for one sentence; F for the other sentence. (You might need to try this both ways!)
An argument is **not** valid:	T for all premises; F for the conclusion.
A set of sentences does **not** tautologically entail another sentence, ϕ:	T for all sentences in the set; F for ϕ.
A sentence is contingent. You need two lines (two TVAs):	T on one line. F on one line.

4.5 EG1

Show that this sentence is not a contradiction: $((P \wedge Q) \rightarrow R) \wedge (Q \wedge \sim R)$

P	Q	R	((P	$\wedge$	Q)	$\rightarrow$	R)	$\downarrow$ $\wedge$	(Q	$\wedge$	~	R)
								T				

I've marked the main connective and put 'T' below it. I want to find a TVA that makes the sentence true since that will show that it is not a contradiction. Now, I can work with my understanding of the connectives to find the truth-values of the atomic sentences.

Conjunctions are true only if both conjuncts are true, so I can put 'T' below the $\rightarrow$ in the first conjunct and below the $\wedge$ in the second. Since the second conjunct $(Q \wedge \sim R)$ is true, I can put 'T' below each conjunct (Q and ~R), and thus, I can put 'F' below R.

This gives me the truth-value of two atomic sentences. I put them below R and Q on the left, and below all other occurrences of R and Q. Now it is easy to determine the truth-value of P. Since the conditional is true, and R, the consequent, is false, the antecedent must be false as well. To make the antecedent false, at least one conjunct must be false. Since Q is true, P must be false.

This gives me a truth-value assignment that makes the sentence true. P=F, Q=T, R=F.

4.5 EG2: Let's try a few:

a) Show that the following sentence is not a tautology:

$$[\sim (S \vee P) \vee (Q \leftrightarrow R)] \rightarrow [\sim(R \leftrightarrow P) \vee (S \rightarrow Q)]$$

P	Q	R	S

[~	(S	∨	P)	∨	(Q	↔	R)]	→	[~	(R	↔	P)	∨	(S	→	Q)]

b) Show that these sentences are consistent:

$R \rightarrow (P \vee Q)$ $\quad$ $R \leftrightarrow (P \rightarrow S)$ $\quad$ $\sim (Q \vee S)$

P	Q	R	S

R	→	(P	∨	Q)	R	↔	(P	→	S)	~	(Q	∨	S)

c) Show that this argument is invalid:

$S \rightarrow P$ $\quad$ $Q \rightarrow (R \vee S)$ $\quad$ $\therefore\ \sim P \rightarrow (Q \wedge R)$

P	Q	R	S

S	→	P	Q	→	(R	∨	S)	~	P	→	(Q	∧	R)

4.5 E1: Construct a shortened truth table for each of the following that shows what is asked.

Show that each of the following is not a tautology:

a) $((P \wedge Q) \vee {\sim}S) \rightarrow (P \vee (Q \wedge {\sim}S))$

b) $[{\sim}P \wedge (Q \rightarrow (S \vee P))] \rightarrow ((S \wedge {\sim}P) \rightarrow Q)$

c) $[(W \leftrightarrow {\sim}X) \wedge ({\sim}(W \vee Y) \rightarrow Z)] \rightarrow ({\sim}Z \rightarrow {\sim}X)$

Show that each of the following is not a contradiction:

d) $[(P \rightarrow Q) \wedge (P \rightarrow R)] \leftrightarrow (Q \leftrightarrow {\sim}R)$

e) $((P \wedge Q) \rightarrow (R \wedge {\sim}R)) \wedge (P \vee Q)$

f) $[{\sim} (S \wedge T) \rightarrow (U \vee S)] \wedge {\sim} [{\sim}U \rightarrow {\sim} (T \vee S)]$

Show that each of the following is a contingent sentence:

g) $(S \rightarrow T) \wedge (T \rightarrow R) \leftrightarrow (R \leftrightarrow S)$

h) $(P \vee ({\sim}Q \leftrightarrow R)) \vee (R \rightarrow {\sim}(P \vee Q))$

i) ${\sim} [(P \vee {\sim}S) \vee ((R \wedge T) \rightarrow (T \leftrightarrow P))]$

Show that the following pairs are not equivalent:

j) ${\sim}({\sim}W \vee {\sim}(X \wedge Y))$. $(X \vee Y) \wedge {\sim}W$.

k) ${\sim}(P \vee Q) \rightarrow (R \rightarrow Q)$. $R \rightarrow (P \wedge Q)$.

l) $({\sim}P \wedge Q) \wedge {\sim}(R \vee S)$. ${\sim} (R \vee P) \wedge {\sim} (Q \wedge S)$.

Show that the following sets are consistent:

m) $P \rightarrow {\sim}Q$. $P \leftrightarrow R$. $R \vee Q$.

n) ${\sim} (S \vee T) \leftrightarrow (U \wedge W)$. $U \leftrightarrow T$. ${\sim}S \leftrightarrow {\sim}W$.

o) $(P \vee Q) \vee {\sim}(S \vee Q)$. $S \rightarrow {\sim}Q$. ${\sim}P \rightarrow T$. ${\sim}T \wedge S$.

p) $(Q \wedge R) \rightarrow (S \wedge {\sim}T)$. $(T \vee Q) \leftrightarrow (S \rightarrow {\sim}R)$. ${\sim}(P \vee {\sim}(T \wedge R))$.

Show that the following are not valid arguments:

q) $P \vee (Q \vee S)$. ${\sim}S$. $\therefore {\sim}P$

r) ${\sim}(T \vee {\sim}(R \vee {\sim}S))$. $R \rightarrow (T \rightarrow S)$. $\therefore {\sim}T \leftrightarrow {\sim}R$.

s) $(P \rightarrow Q) \wedge (R \rightarrow S)$. $(Q \leftrightarrow {\sim}S) \wedge ({\sim}S \rightarrow P)$ $\therefore P$

t) ${\sim}((P \leftrightarrow T) \rightarrow (S \rightarrow W))$. ${\sim}(T \leftrightarrow {\sim}Q)$. $\therefore (P \vee Q) \rightarrow (R \wedge S)$

Tautology: A sentence ϕ is a **tautology** if and only if ϕ is true on every truth-value assignment.

Contradiction: A sentence ϕ is a **contradiction** if and only if ϕ is false on every truth-value assignment.

Contingency: A sentence ϕ is a **contingent sentence** if and only if ϕ is true on some truth-value assignments and false on some truth-value assignments.

Equivalence: Sentence ϕ and ψ are **logically equivalent** if and only if there is no truth-value assignment on which ϕ and ψ have different truth-values.

Consistency: A set of sentences is **consistent** if and only if there is at least one truth-value assignment on which all the members of the set are true.

Inconsistency: A set of sentences is **inconsistent** if and only if there is no truth-value assignment on which all members of the set are true (it is not consistent).

Tautological Implication: A set of sentences **tautologically implies** a sentence ϕ if and only if there is no truth-value assignment on which all the sentences in the set are true and ϕ is false.

Validity: An argument is **valid** if and only if there is no truth-value assignment on which all the premises are true and the conclusion is false.

These concepts are closely related to each other. A valid argument can be understood as a tautological implication and can be expressed as a tautology. Two logically equivalent sentences can be expressed as a tautologous biconditional (each side being one of the equivalent sentences.) Indeed, many theorems take that form, e.g. $(\sim P \vee Q) \leftrightarrow (P \rightarrow Q)$.

All of these concepts can be understood in terms of consistency/inconsistency:

A sentence ϕ is a tautology if and only if $\{\sim\phi\}$ is inconsistent.

A sentence ϕ is a contradiction if and only if $\{\phi\}$ is inconsistent.

A sentence ϕ is contingent if and only if both $\{\phi\}$ and $\{\sim\phi\}$ are consistent.

Sentences ϕ and ψ are equivalent if and only if $\{\sim\phi \leftrightarrow \psi\}$ is inconsistent.

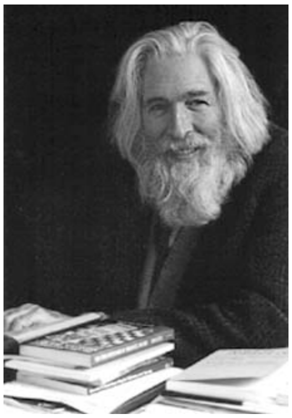
A set of sentences tautologically implies a sentence ϕ if and only if the set formed by the original set of sentences together with $\sim\phi$ forms is inconsistent.

An argument is valid if and only if the set of sentences consisting of the premises and the negation of the conclusion is inconsistent.

4.7 FUN WITH TRUTH-TABLES

Logic Puzzles: Knights and Knaves

Raymond Smullyan is a mathematician, logician and philosopher (also novelist, concert pianist and magician!) He has written many popular books on logic and mathematics – on topics such as Gödel's theorem, set theory and paradoxes. Many of his popular books incorporate puzzles.



Knights and Knaves Puzzles are logic puzzles created by Raymond Smullyan that share the following background assumptions[1]:

In the land of knights and knaves, every inhabitant is either a knight or a knave. Knights always tell the truth and knaves always lie. (Every statement is either a lie or it is the truth.)

← Raymond Smullyan

Many of the puzzles involve trying to figure out how to get correct information from an inhabitant if you don't know whether he/she is a knight or a knave.

> **4.7 EG1:** A classic Knights and Knave puzzle:
>
> You are walking to town and the road forks. Two people are standing by the road – and you know that one is a knight and the other is a knave. You are allowed to ask one of the two people exactly one question. Can you think of a question to ask that will let you discover the road to town?

Other puzzles start with statements that the inhabitants make. You must use their statements to determine which of the speakers are knights and which are knaves (and sometimes other information).

Truth-tables can make it very easy to solve these types of puzzles – provided that you can symbolize their statements!

> **4.7 EG2:** Let's start with an easy one.
>
> Patty says: Sam is a knave.
>
> Sam says: Neither of us are knaves

This is fairly easy to reason through – assume Patty is a knave. If Patty is a knave then what she says is untrue. Thus, Sam is not a knave, he's a knight. But if Sam is a knight then what he says is true – and neither Sam nor Patty are knaves. That is a contradiction, thus the assumption is false and Patty is a knight. Since Patty is a knight, what she says is true, and Sam is a knave. What Sam says is false – one of the two of them is a knave (namely Sam).

---

[1] Smullyan, Raymond. What is the name of this book? The riddle of Dracula and other logical puzzles. (Englewood Cliffs, NJ: Prentice-Hall, 1978)

Using truth-tables to solve Knights and Knaves Puzzles:

We can also use a shortened truth table to solve many of these puzzles.

The trick is symbolizing the statements.

What a person says is true ***if and only if*** that person is a knight.

Thus, if we use an atomic sentence, P, to abbreviate: " 'so-and-so' is a knight.", then we can symbolize the statements as biconditionals. P ↔ (what P says)

Symbolization scheme:

P: Patty is a knight. S: Sam is a knight.

1. Patty says: Sam is a knave.
2. Sam says: Neither of us are knaves.

P ↔ ~S Patty is a knight if and only if [Sam is a knave]

This is a symbolization of 1. If the sentence in brackets is true then Patty is telling the truth, and she is a knight. If Patty is a knight then the sentence in brackets is true.

S ↔ ~(~P ∨ ~S) Sam is a knight if and only if [Neither Patty nor Sam are knaves]

This is a symbolization of 2. If the sentence in brackets is true then Sam is telling the truth and he is a knight. If Sam is a knight then the sentence in brackets is true.

Now the shortened truth-table: The two biconditionals are true. So use that information to determine the truth-values of P and S. The first row started with the assumption that P was false. This led to a contradiction. Thus P must be true. The second row could be completed without contradiction.

			T				**T**						
P	**S**	**P**	**↔**	**~**	**S**	**S**	**↔**	**~**	**(~**	**P**	**∨**	**~**	**S)**
F		F	T	F	T	T	**F**	F	T	F	T	F	T
T	F	T	T	T	F	F	T	F	F	T	T	T	F

Note: I've suggested you do it with shortened truth tables. But of course you can also do a derivation. Start with the biconditionals as your premises and then derive the atomic sentences or their negations.

4.7 EG3: Now for some more complicated ones.

a) Rhonda says: Will is knight and Sam is a knave.
Will says: Sam is not a knave.
Patty says: Will and Sam are either both knaves or they are both knights.
Sam says: Rhonda is a knave.

R: Rhonda is a knight. W: Will is a knight. P: Patty is a knight. S: Sam is a knight.

P R S W

---

b)

	Peter says:	Sarah would say that I am a knight.*
	Randy says:	Of Peter and myself, exactly one is a knight.
	Sarah says:	Randy is not a knave.

* Peter is a knight if and only if [Sarah would say Peter is a knight] .
We symbolize “Sarah would say Peter is a knight”: $S \leftrightarrow P$
(Sarah is a knight if and only if what she says is true.)
Thus * line is symbolized: $P \leftrightarrow (S \leftrightarrow P)$ (Peter is a knight if and only if $[S \leftrightarrow P]$)

P R S

---

c) Peggy says: Zoe would say that Quinton is a knight.

Quinton says: Shawna and I are not the same.

Ryan says: Shawna is a knave.

Shawna says: Either I am a knight or Zoe is a knave.

Zoe says: Peggy is a knave unless Ryan is.

Again, you need to symbolize Peggy's statement as a biconditional with the right side a further biconditional.

Quinton's statement is that Quinton and Shawna are not the same. Quinton is a knight if and only if Shawna is a knave. This can be symbolized: $Q \leftrightarrow \sim S$ OR $\sim (Q \leftrightarrow S)$. Thus, Quinton is a knight if and only if [Quinton and Shawna are not the same].

$Q \leftrightarrow (Q \leftrightarrow \sim S)$

P	Q	R	S	Z

$P \leftrightarrow (Z \leftrightarrow Q)$	$Q \leftrightarrow (S \leftrightarrow \sim Q)$	$R \leftrightarrow \sim S$

$S \leftrightarrow (S \vee \sim Z)$	$Z \leftrightarrow (\sim P \vee \sim R)$

d) Poppy: Qasim is a knave and Ralph is a knight.
Qasim: Poppy would say I am a knave.
Ralph: Qasim and I aren't the same.

e) Rianna: Ursula would say that Waldo is a knave.
Stuart: Vinnie would tell you that Trixie is a knave.
Trixie: Rianna and Waldo are both knights or both knaves.
Ursula: Trixie is a knight or I am a knight.
Vinnie: Stuart is a knight or Rianna is a knight, but not both.
Waldo: I am a knight and Vinnie is a knave.

f) Paul says: If Rory is a knight then so is Walter.
Queenie says: Uri is a knave or Walter is a knight.
Rory says: Queenie and Uri are both knights.
Suzy says: Val's a knave
Uri says: Walter's a knave but Rory is a knight.
Val says: Paul and Rory are the same.
Walter says: Either I'm a knight or Val's a knave.

Remember: you may have to try two possibilities.

More Knights and Knaves?

You can find more puzzles online:

http://www.hku.hk/cgi-bin/philodep/knight/puzzle

And in logic books by: Raymond Smullyan

Some puzzles introduce 'Spies', who sometimes tell the truth and sometimes lie. Can you figure out how to symbolize the sentences if some people are spies? (Hint: a person who tells the truth is a knight or a spy.)

You can also derive solutions to these problems with our natural deduction system. Give it a try!