APM462H1S, Winter 2014, About the Final
first installment

The Final Exam for APM462 will take place on Tuesday April 22, 9am-noon, in EX 310.

It will cover material from the entire term, but will heavily emphasize the second half of the term. In particular, it will cover material from the list below.

(1) in principle, any topic from the first part of the course, including:
   (a) unconstrained minimization problems in E^n or in a subset Ω of E^n
   (b) convex sets and convex functions
   (c) algorithms for finding minima (steepest descent etc.)

   These will not be emphasized however. Also, almost everything we learned about unconstrainted minimization problems in E^n has a counterpart in the harder minimization problems from the second half of the course. Similarly, convexity also appeared in our discussion of optimal control.

(2) minimization problems with equality constraints
   (a) first-order conditions and Lagrange multipliers
   (b) regular points, tangent spaces
   (c) second-order conditions

(3) minimization problems with both equality and inequality constraints, and in particular the Karush-Kuhn-Tucker Theorem

(4) the Calculus of Variations – that is, Section 4.1.1 of the notes of Evans, also covered in assignment # 4.

(5) systems of ordinary differential equations. In particular, you should know the definition of e^{tM}, when M is a matrix (note, e^{tM} is often denoted $\mathbf{X}(t)$ in the notes of Evans), and you should know how to use it to write down the formula for the solution of a homogeneous or nonhomogeneous ODE. (See Theorem 2.1 in the notes of Evans.) This is not something we are particularly interested in for its own sake, in this class, but it is an important tool in linear control problems.

(6) controllability of linear systems
   (a) In particular, you should be able to use Theorems 2.2, 2.3 and 2.5 from the notes of Evans to solve problems about controllability.
   (b) Existence of bang-bang controls – Theorem 2.8 in the notes of Evans. Just know what it says.

(7) Linear time-optimal control
   (a) in particular, Pontrygin's Maximum Principle for linear time-optimal control: what it says and how to use it in practice.

The Pontryagin Maximum Principle for general (i.e. not necessarily linear) optimal control problems (Chapter 4 from the lecture notes of Evans) will not appear on the Final Exam.

You also should know linear algebra and calculus at the level of this class (see the notes on Blackboard), especially including topics that have appeared on any of the homework assignments.

What kind of questions to expect?

In general, it is the instructor's job to design test questions in such a way that a well-prepared student can solve them in the available time. So if the instructor does his job correctly, you will not be asked to find the eigenvalues and eigenvectors of a 6 by 6 matrix or to minimize a complicated function of 10 variables.

A large portion of the test will consist of routine questions designed to test your grasp of basic material. These could include questions like the ones below.

- Solve the problem

$$\begin{aligned} &\text{minimize } f(x) := \dots \\ &\text{subject to } h(x) := \dots = 0 \end{aligned} \tag{1}$$

  or
- Solve the problem

$$\begin{aligned} &\text{minimize } f(x) := \dots \\ &\text{subject to } h(x) := \dots = 0, \quad g(x) = \dots \le 0. \end{aligned}$$

- For the minimization problem (1), use the second-order necessary and/or sufficient conditions to check whether a point is a local minimum. (Section 11.5, 11.6 of Luenberger and Ye, see also below)
- Let $I[x(\cdot)] =$ ...., and answer some question about minimizers, such as: find a minimizer, given boundary values $x(a) = x^0, x(b) = x^1$.
- Consider the system of ODEs

$$\frac{d}{dt}x(t) = Mx(t) + N\alpha(t), \qquad \alpha(t) \in [-1,1]^m \text{ for all } t \tag{2}$$

  where

$$M = \dots, \qquad N = \dots.$$

  Is it controllable?

  What is the controllability matrix G, and what is its rank?

  Is the origin in the interior of the "reachable set" $\mathcal{C}$, where

$$\begin{aligned} \mathcal{C} := \{x^0 \in \mathbb{R}^n : \ &\text{there exists a control steering} \\ &\text{the system from } x^0 \text{ to the origin in finite time}\} \ ? \end{aligned}$$

- For the system (2), what does the Pontryagin Maximum Principle tell us about an optimal control $\alpha^*(\cdot)$ (*i.e.* a control that steers the system from a point s^0 to the origin in the shortest possible time)?

practice questions

One way to practice for the test is just to make up concrete instances of the kinds of basic questions described above.

For example:

minimization with equality constraints

- Solve the problem
$$\begin{aligned}&\text{minimize } f(x) := \frac{1}{2}x^TQx - b^Tx\\&\text{subject to } h(x) := d^Tx + e = 0\end{aligned}$$
where Q is positive definite and b, d are column vectors.
- Solve the problem, if possible:
$$\begin{aligned}&\text{minimize } f(x) := d^Tx + e\\&\text{subject to } h(x) := \frac{1}{2}x^TQx - b^Tx = 0\end{aligned}$$
where Q is positive definite and b, d are column vectors. When does a solution exist? (This one is harder.)
- Solve the problem, if possible:
$$\begin{aligned}&\text{minimize } f(x) := \frac{1}{2}(x^2 + y^2) - z\\&\text{subject to } h(x) := \frac{1}{2}(y^2 + z^2) - x = 0\end{aligned}$$
- etc.... You can make up problems of this sort, if you like, (but some of them up being difficult or impossible to solve. You can also probably find examples of this kind of question in some calculus textbooks.

minimization with inequality constraints (and maybe equality constraints as well)

- Solve the problem
$$\begin{aligned}&\text{minimize } f(x) := \frac{1}{2}x^TQx - b^Tx\\&\text{subject to } h(x) := d^Tx = 0, \qquad g(x) = e^Tx \le 0.\end{aligned}$$
where Q is positive definite and b, d, e are column vectors.
- Solve the problem
$$\begin{aligned}&\text{minimize } f(x) := x^2 + 2y^2 + z^2\\&\text{subject to } g(x) = x + 3y \ge 5.\end{aligned}$$
- Solve the problem
$$\begin{aligned}&\text{minimize } f(x) := x^2 + 2y^2 + z^2\\&\text{subject to } h(x) := x + 2y + z = 5 \quad \textit{and} \quad g(x) = x + 3y \ge 5.\end{aligned}$$
- etc.... You can make up problems of this sort, if you like, (but some of them up being difficult or impossible to solve.

second-order conditions for constrained minimization problems.

- Consider the problem

$$\begin{aligned}&\text{minimize } f(x,y,z) := \frac{1}{2}(x^2+y^2-z^2)\\&\text{subject to } h(x) := x+2y+3z-4=0.\end{aligned}$$

  Find a point satisfying the first-order necessary conditions, and determine whether or not it is a local minimum.
- Consider the problem

$$\begin{aligned}&\text{minimize } f(x,y,z) := x+2y+3z\\&\text{subject to } h(x) := \frac{1}{2}(x^2+y^2-z^2)=0.\end{aligned}$$

  Find a point satisfying the first-order necessary conditions, and determine whether or not it is a local minimum.
- Consider the problem

$$\begin{aligned}&\text{minimize } f(x,y,z) := \frac{1}{2}(x^2+y^2-z^2)\\&\text{subject to } h(x) := 9x^2+6xy+y^2-16=0.\end{aligned}$$

  Find all points satisfying the first-order necessary conditions, and use the second-order conditions to determine whether or not they are local minima. Is there a global minimum?
- etc.

more questions on other topics to follow.....