that $k\ell$ is nearly n). Thus, we have almost exactly our γn^2 additional edges left to accommodate elsewhere in the graph: either in ϵ-regular pairs of density less than d, or in some exceptional way, i.e. in irregular pairs, inside a partition set, or with an end in V_0. Now the number of edges in low-density ϵ-regular pairs is less than

$$\tfrac{1}{2}k^2 d\ell^2 \leqslant \tfrac{1}{2}dn^2,$$

and hence less than half of our extra edges if $d \leqslant \gamma$. The other half, the remaining $\frac{1}{2}\gamma n^2$ edges, are more than can be accommodated in exceptional ways, provided we choose m large enough and ϵ small enough (giving an additional upper bound for ϵ). It is now a routine matter to compute the values of m and ϵ that will work.

Exercises

hint: It is not difficult to determine an upper bound for ex(n, K_1,r). What remains to be proved is that this bound can be achieved for all r and n.

1.$^-$ Show that $K_{1,3}$ is extremal without a P^3.

2.$^-$ Given $k > 0$, determine the extremal graphs of chromatic number at most k.

3.$^-$ Is there a graph that is edge-maximal without a K^3 minor but not extremal?

4. Determine the value of $\operatorname{ex}(n, K_{1,r})$ for all $r, n \in \mathbb{N}$.

$\frac{(r-1)n}{2}$, r odd/n even

$\frac{(r-1)n-1}{2}$ o.w

5.$^+$ Given $k > 0$, determine the extremal graphs without a ~~matching~~ of size k.

(Hint. Theorem 2.2.3 and Ex. 18, Ch. 2.)

6. Without using Turán's theorem, show that the maximum number of edges in a triangle-free graph of order $n > 1$ is $\lfloor n^2/4 \rfloor$.

7. Show that

$$t_{r-1}(n) \leqslant \tfrac{1}{2}n^2 \,\frac{r-2}{r-1}\,,$$

with equality whenever $r-1$ divides n.

The bounds given in the hint are the sizes of two particularly simple Tur´an graphs—which ones?

8. Show that $t_{r-1}(n)/\binom{n}{2}$ converges to $(r-2)/(r-1)$ as $n \to \infty$.

(Hint. $t_{r-1}\big((r-1)\lfloor\frac{n}{r-1}\rfloor\big) \leqslant t_{r-1}(n) \leqslant t_{r-1}\big((r-1)\lceil\frac{n}{r-1}\rceil\big)$.)

9. Show that deleting at most $(m-s)(n-t)/s$ edges from a $K_{m,n}$ will never destroy all its $K_{s,t}$ subgraphs.

10. For $0 < s \leqslant t \leqslant n$ let $z(n,s,t)$ denote the maximum number of edges in a bipartite graph whose partition sets both have size n, and which does not contain a $K_{s,t}$. Show that $2\operatorname{ex}(n, K_{s,t}) \leq z(n,s,t) \leq \operatorname{ex}(2n, K_{s,t})$.

11.$^+$ Let $1 \leqslant r \leqslant n$ be integers. Let G be a bipartite graph with bipartition $\{A, B\}$, where $|A| = |B| = n$, and assume that $K_{r,r} \not\subseteq G$. Show that

$$\sum_{x \in A} \binom{d(x)}{r} \leqslant (r-1)\binom{n}{r}.$$

Using the previous exercise, deduce that $\mathrm{ex}(n, K_{r,r}) \leqslant cn^{2-1/r}$ for some constant c depending only on r.

12. The *upper density* of an infinite graph G is the infimum of all $\alpha \in \mathbb{R}$ such that G has only finitely many (isomorphism types of) finite subgraphs of edge density $> \alpha$. Use the Erdős-Stone theorem to show that this number always takes one of the countably many values $0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$.

13. Given a tree T, find an upper bound for $\mathrm{ex}(n, T)$ that is linear in n and independent of the structure of T, i.e. depends only on $|T|$.

Proposition 1.2.2 and Corollary 1.5.4

14. Show that the Erdős-Sós conjecture is best possible in the sense that, for every k and infinitely many n, there is a graph on n vertices and with $\frac{1}{2}(k-1)n$ edges that contains no tree with k edges.

15.$^-$ Prove the Erdős-Sós conjecture for the case when the tree considered is a star.

16. Prove the Erdős-Sós conjecture for the case when the tree considered is a path.

(Hint. Use Exercise 8 of Chapter 1.)

17. Can large average degree force the chromatic number up if we exclude some tree as an induced subgraph? More precisely: For which trees T is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, every graph of average degree at least $f(k)$ either has chromatic number at least k or contains an induced copy of T?

18. Given two numerical graph invariants i_1 and i_2, write $i_1 \leqslant i_2$ if we can force i_2 to be arbitrarily high on some subgraph of G by assuming that $i_1(G)$ is large enough. (Formally: write $i_1 \leqslant i_2$ if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, given any $k \in \mathbb{N}$, every graph G with $i_1(G) \geqslant f(k)$ has a subgraph H with $i_2(H) \geqslant k$.) If $i_1 \leqslant i_2$ as well as $i_1 \geqslant i_2$, write $i_1 \sim i_2$. Show that this is an equivalence relation for graph invariants, and sort the following invariants into equivalence classes ordered by $<$: minimum degree; average degree; connectivity; arboricity; chromatic number; colouring number; choice number; $\max\{\, r \mid K^r \subseteq G \,\}$; $\max\{\, r \mid TK^r \subseteq G \,\}$; $\max\{\, r \mid K^r \preccurlyeq G \,\}$; $\min \max d^+(v)$, where the maximum is taken over all vertices v of the graph, and the minimum over all its orientations.

19.$^+$ Prove, from first principles and without using average or minimum degree arguments, the existence of a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph of chromatic number at least $f(r)$ has a K^r minor.

(Hint. Use induction on r. For the induction step $(r-1) \to r$ try to find a connected set U of vertices whose neighbours induce a subgraph that needs enough colours to contract to K^{r-1}. If no such set U exists, show that the given graph can be coloured with fewer colours than assumed.)

Imitate the proof of Proposition 7.2.2.

20. Given a graph G with $\varepsilon(G) \geqslant k \in \mathbb{N}$, find a minor $H \preccurlyeq G$ such that $\delta(H) \geqslant k \geqslant |H|/2$.

21.$^+$ Find a constant c such that every graph with n vertices and at least $n + 2k(\log k + \log\log k + c)$ edges contains k edge-disjoint cycles (for all $k \in \mathbb{N}$). Deduce an edge-analogue of the Erdős-Pósa theorem (2.3.2).

(Hint. Assuming $\delta \geqslant 3$, delete the edges of a short cycle and apply induction. The calculations are similar to the proof of Lemma 2.3.1.)

22. Simplify the proof of Theorem 7.2.1 by using Exercise 23 of Chapter 3.

23.$^+$ Show that any function h as in Lemma 3.5.1 satisfies the inequality $h(r) > \frac{1}{8}r^2$ for all even r, and hence that Theorem 7.2.1 is best possible up to the value of the constant c.

24. Characterize the graphs with n vertices and more than $3n - 6$ edges that contain no $TK_{3,3}$. In particular, determine $\mathrm{ex}(n, TK_{3,3})$.

(Hint. You may use the theorem of Wagner that every edge-maximal graph without a $K_{3,3}$ minor can be constructed recursively from maximal planar graphs and copies of K^5 by pasting along K^2s.)

25.$^-$ Derive the four colour theorem from Hadwiger's conjecture for $r = 5$.

26.$^-$ Show that Hadwiger's conjecture for $r+1$ implies the conjecture for r.

Consider a suitable supergraph.

27.$^-$ Deduce the following weakening of Hadwiger's conjecture from known results: given any $\epsilon > 0$, every graph of chromatic number at least $r^{1+\epsilon}$ has a K^r minor, provided that r is large enough.

28.$^-$ Show that any graph constructed as in Proposition 7.3.1 is edge-maximal without a K^4 minor.

29. Prove the implication $\delta(G) \geqslant 3 \Rightarrow G \supseteq TK^4$.

(Hint. Proposition 7.3.1.)

30. A multigraph is called *series-parallel* if it can be constructed recursively from a K^2 by the operations of subdividing and of doubling edges. Show that a 2-connected multigraph is series-parallel if and only if it has no (topological) K^4 minor.

31. Without using Theorem 7.3.8, prove Hadwiger's conjecture for all graphs of girth at least 11 and r large enough. Without using Corollary 7.3.9, show that there is a constant $g \in \mathbb{N}$ such that all graphs of girth at least g satisfy Hadwiger's conjecture, irrespective of r.

32.$^+$ Prove Hadwiger's conjecture for $r = 4$ from first principles.

33.$^+$ Prove Hadwiger's conjecture for line graphs.

34. Prove Corollary 7.3.5.

35.$^-$ In the definition of an ϵ-regular pair, what is the purpose of the requirement that $|X| > \epsilon\,|A|$ and $|Y| > \epsilon\,|B|$?

36.$^-$ Show that any ϵ-regular pair in G is also ϵ-regular in $\overline{G}$.

Can you colour the edges of K_5 red and green without creating a red or a green triangle? Can you do the same for a K_6 ?

Exercises

1.$^-$ Determine the Ramsey number $R(3)$. 6 (5 is impossible)

2.$^-$ Deduce the case $k = 2$ (but c arbitrary) of Theorem 9.1.3 directly from Theorem 9.1.1.

Construct a partition such that neither class contains an infinite arith- metic progression. Construct the classes inductively and simultane- ously.

3. An *arithmetic progression* is an increasing sequence of numbers of the form $a, a+d, a+2d, a+3d \ldots$. *Van der Waerden's theorem* says that no matter how we partition the natural numbers into two classes, one of these classes will contain arbitrarily long arithmetic progressions. Must there even be an infinite arithmetic progression in one of the classes?

4. Can you improve the exponential upper bound on the Ramsey number $R(n)$ for perfect graphs?

5.$^+$ Construct a graph on $\mathbb{R}$ that has neither a complete nor an edgeless induced subgraph on $|\mathbb{R}| = 2^{\aleph_0}$ vertices. (So Ramsey's theorem does not extend to uncountable sets.)

6.$^+$ Prove the edge version of the Erdős-Pósa theorem (2.3.2): there exists a function $g \colon \mathbb{N} \to \mathbb{R}$ such that, given $k \in \mathbb{N}$, every graph contains either k edge-disjoint cycles or a set of at most $g(k)$ edges meeting all its cycles.

(Hint. Consider in each component a normal spanning tree T. If T has many chords xy, use any regular pattern of how the paths xTy intersect to find many edge-disjoint cycles.)

7.$^+$ Use Ramsey's theorem to show that for any $k, \ell \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every sequence of n distinct integers contains an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell+1$. Find an example showing that $n > k\ell$. Then prove the theorem of Erdős and Szekeres that $n = k\ell + 1$ will do.

Use the fact that n >=4 points span a convex polygon if and only if every four of them do.

8. Sketch a proof of the following theorem of Erdős and Szekeres: for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that among any n points in the plane, no three of them collinear, there are k points spanning a convex k-gon, i.e. such that none of them lies in the convex hull of the others.

9. Prove the following result of Schur: for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for every partition of $\{1, \ldots, n\}$ into k sets, at least one of the subsets contains numbers x, y, z such that $x + y = z$.

Translate the given k-partition of {1, 2, . . . , n} into a k-colouring of the edges of K_n .

10. Let $(X, \leqslant)$ be a totally ordered set, and let $G = (V, E)$ be the graph on $V := [X]^2$ with $E := \{(x,y)(x',y') \mid x < y = x' < y'\}$.

   (i) Show that G contains no triangle.

   (ii) Show that $\chi(G)$ will get arbitrarily large if $|X|$ is chosen large enough.

11. A family of sets is called a Δ-*system* if every two of the sets have the same intersection. Show that every infinite family of sets of the same finite cardinality contains an infinite Δ-system.

12. Prove that for every $r \in \mathbb{N}$ and every tree T there exists a $k \in \mathbb{N}$ such that every graph G with $\chi(G) \geqslant k$ and $\omega(G) < r$ contains a subdivision of T in which no two branch vertices are adjacent in G (unless they are adjacent in T).

Imitate the proof of Proposition 9.2.1.

13. Let $m, n \in \mathbb{N}$, and assume that $m-1$ divides $n-1$. Show that every tree T of order m satisfies $R(T, K_{1,n}) = m+n-1$.

14. Prove that $2^c < R(2, c, 3) \leqslant 3c!$ for every $c \in \mathbb{N}$.

    (Hint. Induction on c.)

15.$^-$ Derive the statement $(*)$ in the first proof of Theorem 9.3.1 from the theorem itself, i.e. show that $(*)$ is only formally stronger than the theorem.

The lower bound is easy. Given a colouring for the upper bound, con- sider a vertex and the neighbours joined to it by suitably coloured edges.

16. Show that, given any two graphs H_1 and H_2, there exists a graph $G = G(H_1, H_2)$ such that, for every *vertex*-colouring of G with colours 1 and 2, there is either an induced copy of H_1 coloured 1 or an induced copy of H_2 coloured 2 in G.

17. Show that the Ramsey graph G for H constructed in the second proof of Theorem 9.3.1 does indeed satisfy $\omega(G) = \omega(H)$.

18.$^-$ Show that any Kuratowski set $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ of a given collection $\mathcal{C}$ of non-trivial graph properties is unique up to equivalence.

19. Deduce Theorem 9.4.5 (ii) from Proposition 9.4.2, and vice versa.

Notes

Due to increased interaction with research on random and pseudo-random[4] structures (the latter being provided, for example, by the regularity lemma), the Ramsey theory of graphs has recently seen a period of major activity and advance. Theorem 9.2.2 is an early example of this development.

For the more classical approach, the introductory text by R.L. Graham, B.L. Rothschild & J.H. Spencer, *Ramsey Theory* (2nd edn.), Wiley 1990, makes stimulating reading. This book includes a chapter on graph Ramsey theory, but is not confined to it. Surveys of finite and infinite Ramsey theory are given by J. Nešetřil and A. Hajnal in their chapters in the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995. The Ramsey theory of infinite sets forms a substantial part of combinatorial set theory, and is treated in depth in P. Erdős, A. Hajnal, A. Máté & R. Rado, *Combinatorial Set Theory*, North-Holland 1984. An attractive collection of highlights from various branches of Ramsey theory, including applications in algebra, geometry and point-set topology, is offered in B. Bollobás, *Graph Theory*, Springer GTM 63, 1979.

Theorem 9.2.2 is due to V. Chvátal, V. Rödl, E. Szemerédi & W.T. Trotter, The Ramsey number of a graph with bounded maximum degree, *J. Comb.*

[4] Concrete graphs whose structure resembles the structure expected of a random graph are called *pseudo-random*. For example, the bipartite graphs spanned by an ϵ-regular pair of vertex sets in a graph are pseudo-random.

$e \in E$, to obtain a Hamilton cycle H of G^2. Since we did not mark Q, and since by (5) no edge of Q was lifted at its other end, $\overline{H}$ contains the edges of Q. By (1), these lie in $E \cup \overline{E}$. Hence the edges of H at x lie in E, as desired. □

Fleischner's theorem has a natural extension to locally finite graphs, which is much harder to prove:

Theorem 10.3.3. (Georgakopoulos 2009)
The square of a 2-connected locally finite graph contains a Hamilton circle.

We close the chapter with a far-reaching conjecture generalizing Dirac's theorem:

Conjecture. (Seymour 1974)
Let G be a graph of order $n \geqslant 3$, and let k be a positive integer. If G has minimum degree

$$\delta(G) \geqslant \frac{k}{k+1}\, n\,,$$

then G has a Hamilton cycle H such that $H^k \subseteq G$.

For $k = 1$, this is precisely Dirac's theorem. The conjecture was proved for large enough n (depending on k) by Komlós, Sárközy and Szemerédi (1998).

Exercises

Induction.

1. An oriented complete graph is called a *tournament*. Show that every tournament contains a (directed) Hamilton path.

2. Show that every uniquely 3-edge-colourable cubic graph is hamiltonian. ('Unique' means that all 3-edge-colourings induce the same edge partition.)
Consider the union of two colour classes.

3. Given an even positive integer k, construct for every $n \geqslant k$ a k-regular graph of order $2n + 1$.

4.$^-$ Prove or disprove the following strengthening of Proposition 10.1.2: 'Every k-connected graph G with $|G| \geqslant 3$ and $\chi(G) \geqslant |G|/k$ has a Hamilton cycle.'

5. (i)$^-$ Show that hamiltonian graphs are 1-tough.
(ii) Find a graph that is 1-tough but not hamiltonian.

6. Prove the toughness conjecture for planar graphs. Does it hold with $t = 2$, or even with some $t < 2$?