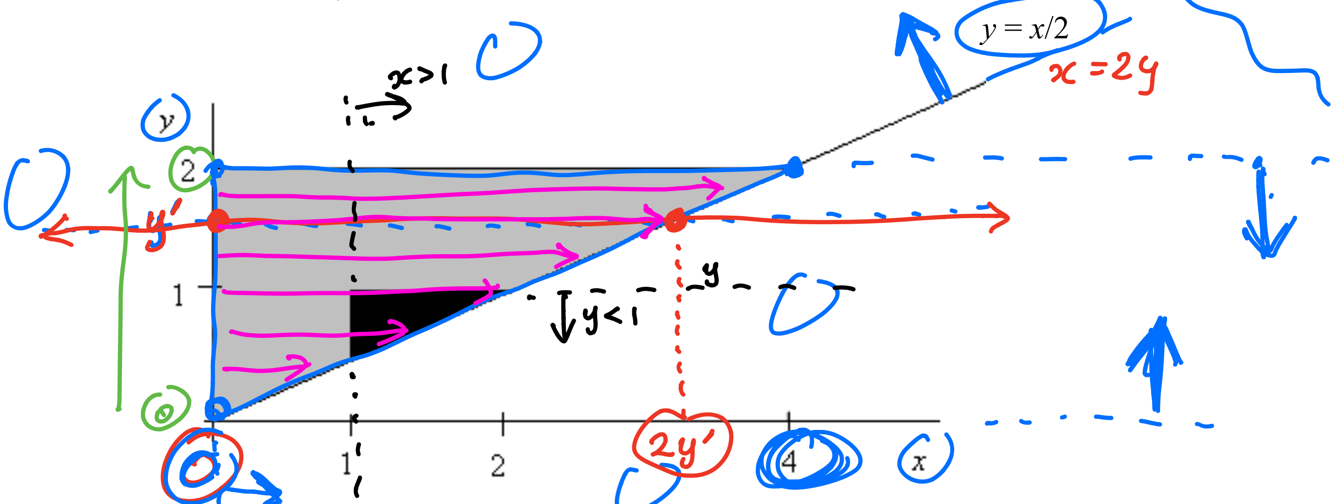
Example 6 Suppose that X and Y are two continuous random variables with joint pdf

$$f(x,y) = cxy, \quad 0 < x < 2y < 4.$$

$x > 0$
$y < 2$
$0 < y$
$x < 2y$
$y > x/2$

Find: **(a)** $P(X > 1, Y < 1)$
(b) EY
(c) ρ.

(a) We first draw the region where the joint pdf is positive.



We then find the value of c.

$1 = \iint f(x,y)\,dx\,dy$

$$1 = c\int_{[y=]0}^{2} y\left(\int_{[x=]0}^{2y} x\,dx\right)dy.$$ The inner integral here equals $I = \left[\frac{x^2}{2}\right]_{x=0}^{2y} = 2y^2$.

$\frac{(2y)^2}{2} - \frac{0^2}{2}$

So $1 = c\int_0^2 y 2y^2\,dy = 8c$. Therefore, $c = \frac{1}{8}$.

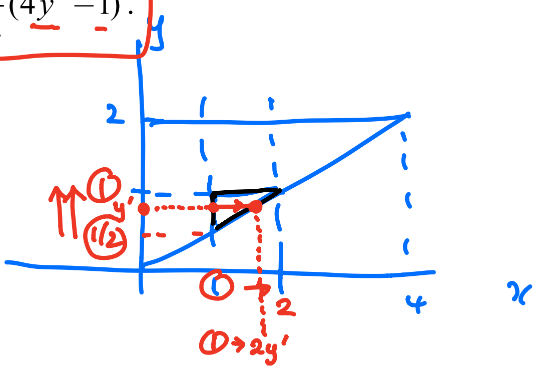
(Alternatively, $1 = c\int_{x=0}^{4} x\left(\int_{y=x/2}^{2} y\,dy\right)dx = \ldots = 8c \Rightarrow c = 1/8$.)

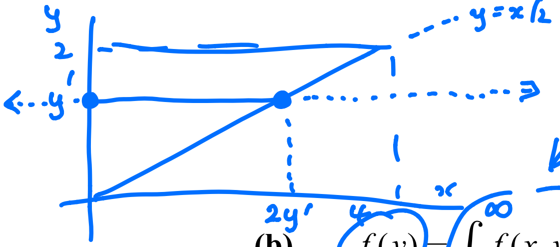
$\iint f(x,y)\,dx\,dy$

The required probability is $p = P(X > 1, Y < 1) = \frac{1}{8}\int_{1/2}^{1} y\left(\int_{1}^{2y} x\,dx\right)dy$.

The inner integral here equals $I = \left[\frac{x^2}{2}\right]_{x=1}^{2y} = \frac{1}{2}(4y^2 - 1)$.

So $p = \frac{1}{8}\int_{1/2}^{1} y\,\frac{1}{2}(4y^2 - 1)\,dy = \frac{9}{256}$.





$\int \frac{1}{8} xy\,dx$

(b) $f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \frac{1}{8} y \int_0^{2y} x dx = \frac{1}{4} y^3, \ 0 < y < 2.$

So $EY = \int_0^2 y \frac{1}{4} y^3 dy = \frac{8}{5}.$ $= 1.6$

$EY^2 - (EY)^2$

(c) $EY^2 = \int_0^2 y^2 \left(\frac{1}{4} y^3 \right) dy = \frac{8}{3}$ $\qquad \sigma_Y^2 = \frac{8}{3} - \left(\frac{8}{5} \right)^2 = \frac{8}{75} \ \left(= \frac{72}{675} \right)$

$EX = \int_0^2 \int_{x=0}^{2y} x \left(\frac{1}{8} xy \right) dxdy = \frac{32}{15}$ $\qquad EX^2 = \int_0^2 \int_{x=0}^{2y} x^2 \left(\frac{1}{8} xy \right) dxdy = \frac{16}{3}$

$\sigma_X^2 = \frac{16}{3} - \left(\frac{32}{15} \right)^2 = \frac{528}{675}$

$E(XY) = \int_0^2 \int_{x=0}^{2y} xy \left(\frac{1}{8} xy \right) dxdy = \frac{32}{9}$ $\qquad Cov(X,Y) = \frac{32}{9} - \frac{32}{15} \left(\frac{8}{5} \right) = \frac{96}{675}$

$\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{96/675}{\sqrt{528/675}\sqrt{72/675}} = \frac{4}{\sqrt{66}} = 0.4924.$

NB: Large values of X are associated with large values of Y.
This is reflected in the fact that $\rho > 0$.

Conditional expectations

$E(X|y) \equiv$

$$E(X \mid Y = y) = \begin{cases} \sum_x x\, p(x \mid y), & \text{if } X \text{ is discrete} \\ \int x\, f(x \mid y) dx, & \text{if } X \text{ is continuous} \end{cases}$$

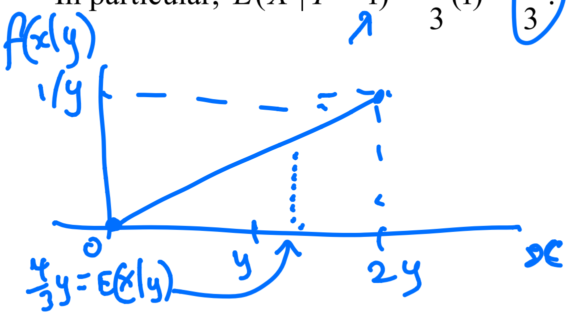
In Example 6, what is the expected value of X given that $Y = 1$?

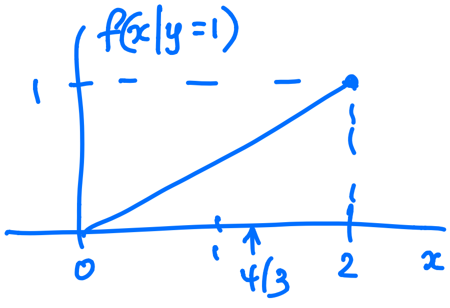
$f(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{xy/8}{y^3/4} = \frac{x}{2y^2}, \quad 0 < x < 2y.$

Therefore $E(X \mid Y = y) = \int_0^{2y} x \frac{x}{2y^2} dx = \frac{1}{2y^2} \left[\frac{x^3}{3} \right]_0^{2y} = \frac{1}{2y^2} \frac{(2y)^3}{3} = \frac{4}{3} y.$

$E(X|y) =$

In particular, $E(X \mid Y = 1) = \frac{4}{3}(1) = \frac{4}{3}.$





fn of y

"Random" expectations

By $E(X|Y)$ we denote the function $E(X|Y=y)$ with y replaced by Y.

What is $E(X|Y)$ in Example 6?

Recall that $E(X|Y=y)=\frac{4}{3}y$. Therefore $E(X|Y)=\frac{4}{3}Y$. r.v.

The law of iterated expectation LIE

$EX = EE(X|Y)$.

fn of Y

Proof: Assuming that X and Y are both continuous,

$$EE(X|Y)=\int E(X|Y=y)f(y)dy = \int\left(\int xf(x|y)dx\right)f(y)dy$$

$$=\int\int xf(x,y)dxdy = \cancel{EX} = \int x\left(\int f(x,y)dy\right)dx$$

$$= \int x f(x)\,dx = EX.$$

(This proof can be easily modified for the case where X or Y or both are discrete.)

In Example 6, find EX using the law of iterated expectation.

$EX = EE(X|Y)=E\left(\frac{4}{3}Y\right)=\frac{4}{3}EY=\frac{4}{3}\times\frac{8}{5}=\frac{32}{15}$ (as before, but done more easily).

Related definitions and results

1. $E(g(X)|Y=y)=\begin{cases}\sum_x g(x)p(x|y), & \text{if } X \text{ is discrete}\\ \int g(x)f(x|y)dx, & \text{if } X \text{ is continuous}\end{cases}$

   $E(g(X)|Y) = E(g(X)|Y=y)$ with y changed to Y.

   $Var X = E\{[X-EX]^2\}$

2. $Eg(X) = EE\{g(X)|Y\}$.

3. $Var(X|y) \equiv Var(X|Y=y)=E\{[X-E(X|Y=y)]^2\,|Y=y\}$. $(E=EE)$

LIV

4. $\boxed{VarX = EVar(X|Y)+VarE(X|Y).}$ $(V=EV+VE)$

5. $Cov(X,Z|Y=y)=E\{[X-E(X|Y=y)][Z-E(Z|Y=y)]|Y=y\}$.

   $(Cov(X,Z) = E\{[X-EX][Z-EZ]\})$

LIC

6. $Cov(X,Z)=ECov(X,Z|Y)+Cov\{E(X|Y),E(Z|Y)\}$. $(C = EC + CEE)$

LIP 7. $\boxed{P(A) = E\,P(A|Y)}$ Need $P(A|Y=y) = \ldots\ldots$ (fn of y)

follows from the LIE by considering the fact that $P(A)=E(U)$ where $U=I(A)=\begin{cases}1 & \text{if } A\\ 0 & \text{if } \bar{A}\end{cases}$, etc.

LIE & LIV

Example 7 Twenty bolts have just been randomly sampled from the production line in a factory. You are now going to count the number of defectives amongst them.

From experience, you know that the proportion of defective bolts produced in the factory is constant throughout any given day, but varies from day to day in a uniform manner between 0.1 and 0.3.

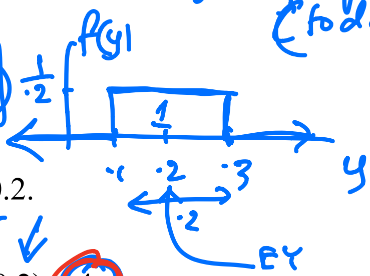
(a) How many defective bolts do you expect to find?

(b) What is the variance of the number of defective bolts?

Y = probability of any given bolt being defective (today)

(a) Let X be the number of defectives amongst the 20, and let Y be the proportion of defectives amongst all bolts produced in the factory today.

Then $\boxed{(X|Y=y) \sim Bin(20,y)}$, and $\boxed{Y \sim U(0.1,0.3)}$.



So $E(X|Y=y) = 20y$ (np), $E(X|Y) = 20Y$, and $EY = 0.2$.

Therefore $\boxed{EX = EE(X|Y)} = E(20Y) = 20EY = 20(0.2) = 4.$

(b) First, $Var(X|Y=y) = 20y(1-y)$. (np(1-p))

Therefore $Var(X|Y) = 20Y(1-Y) = 20(Y-Y^2)$.

So $VarX = EVar(X\,|\,Y) + VarE(X\,|\,Y)$

$$= E\{20(Y-Y^2)\} + Var\{20Y\}$$

$$= 20(EY - EY^2) + 400VarY.$$

Now $VarY = \dfrac{(0.3-0.1)^2}{12} = \dfrac{1}{300}$,

and $EY^2 = VarY + (EY)^2 = \dfrac{1}{300} + 0.2^2 = \dfrac{13}{300}$.

So $VarX = 20\left(\dfrac{1}{5} - \dfrac{13}{300}\right) + 400\dfrac{1}{300} = \dfrac{67}{15} = 4.4667.$

O/w → $EX = \sum_x x f(x) = \boxed{}$?

$f(x) = \int f(x,y)\,dy = \int f(y) f(x|y)\,dy$

$$= \int_{.1}^{.3} \left(\frac{1}{.2}\right)\binom{20}{x} y^x (1-y)^{20-x} \, dy$$

Beta pdf ?

$= ??$

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