UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER 2002 EXAMINATIONS

STA 257H1F

Duration - 3 hours

No Aids Allowed

NAME: ______________________________

STUDENT NUMBER: ______________________________

- There are 15 pages including this page. The last two pages are formulae that may be useful.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total marks: 110

1	2	3	4	5	6	7	8

9	10	11	12	13	14	15

1. (10 marks) A, B, and C are events in the probability space $(S, \mathcal{F}, P)$.

   (a) Prove that $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

$$P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$$
$$\leq P(A \cup B) + P(C)$$
$$= P(A) + P(B) - P(A \cap B) + P(C)$$
$$\leq P(A) + P(B) + P(C)$$

   (b) If A is a subset of B, is it possible for A and B to be independent events? If impossible, explain why. If it is possible, give an example.

$A \subset B$

$$P(A \cap B) = P(A)$$

If A, B are indep,

$$P(A \cap B) = P(A)\,P(B)$$

$\rightarrow$ possible if $P(B) = 1$ or $P(A) = 0$

Continued

2. (5 marks) Suppose X is a continuous random variable with density function

$$f(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density of $Y = 1/X$.

$y = \frac{1}{x} = h(x)$; $h^{-1}(y) = \frac{1}{y}$, $\frac{d}{dy} h^{-1}(y) = -\frac{1}{y^2}$

$y \geq 1$

$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|$

$= 2\left(1 - \frac{1}{y}\right) \cdot \left| \frac{-1}{y^2} \right| = \frac{2}{y^2}\left(1 - \frac{1}{y}\right)$

$f_Y(y) = \begin{cases} \frac{2}{y^2}\left(1 - \frac{1}{y}\right), & y \geq 1 \\ 0, & \text{ow} \end{cases}$

3. (5 marks) Suppose that a random variable X has a strictly increasing cumulative distribution function $F(x)$. Show that the random variable $Y = F(X)$ has a uniform distribution on $(0, 1)$.

$0 \leq Y \leq 1$

$F_Y(y) = P(Y \leq y) = P(F(X) \leq y)$

$= P(F^{-1}(F(X)) \leq F^{-1}(y))$

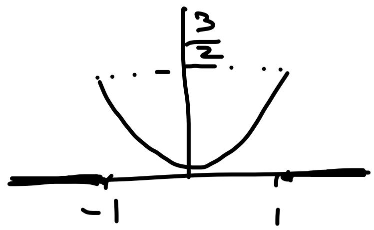
$= P(X \leq F^{-1}(y))$

$= F(F^{-1}(y)) = y$

$\Rightarrow Y \sim \text{Unif}(0, 1)$

4. (12 marks) Let X be a continuous random variable with density function $f(x) = \frac{3}{2}x^2$, $-1 \le x \le 1$ and 0 otherwise. Sketch the following functions, clearly indicating at least three important points on each horizontal axis.

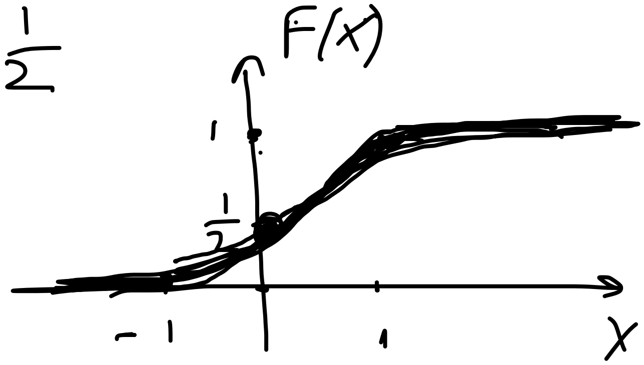
(a) the density function for X



(b) the cumulative distribution function for X

$$F(x) = \int_{-1}^{x} \frac{3}{2} t^2 \, dt = \frac{x^3}{2} + \frac{1}{2}$$

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x^3}{2} + \frac{1}{2}, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



(c) the approximate density function of $(X_1 + X_2 + \cdots + X_{100})/100$ where the X_i are independent random variables with density f

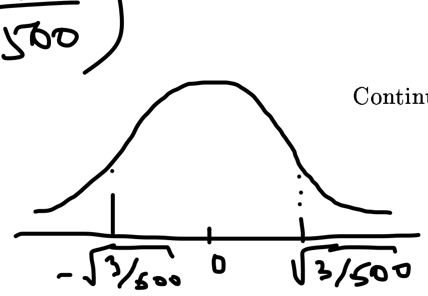
$$S = \frac{X_1 + \dots + X_{100}}{100}$$

$$E(S) = \frac{1}{100} \sum E(X_i) = 0, \quad Var(S) = \frac{1}{100^2} \frac{3}{5} \cdot 100 = \frac{3}{500}$$

$$E(X_i) = \int_{-1}^{1} \frac{3}{2} x^3 \, dx = 0$$

$$E(X_i^2) = \int_{-1}^{1} \frac{3}{2} x^4 \, dx = \frac{3}{5} \Rightarrow Var(X_i) = \frac{3}{5}$$

By CLT, $S \sim N\left(0, \frac{3}{500}\right)$



5. (10 marks) Suppose that X and Y are jointly distributed discrete random variables with probability function

$$p(x,y) = kq^2p^{x+y}, \quad x,y = 0,1,2,\ldots, \quad 0 < p < 1, \quad q = 1-p$$

(a) Determine the value of the constant k.

$$\sum_{y=0}^{\infty}\sum_{x=0}^{\infty} kq^2p^{x+y} = \sum_{y=0}^{\infty} kq^2p^y \sum_{x=0}^{\infty} p^x$$

$$= \sum kq^2p^y \cdot \frac{1}{1-p}$$

$$= kq \sum_{y=0}^{\infty} p^y = kq \cdot \frac{1}{1-p} = \boxed{k=1}$$

(b) Are X and Y independent? Why or why not? Yes

$$p(x,y) = p(x)\,p(y), \quad p(x) = qp^x$$
$$p(y) = qp^y$$

(c) Find $P(X+Y=t)$.

$$T = X+Y$$

$$P(T=t) = P(X+Y=t) = \sum_{x=0}^{t} q^2p^xp^{t-x}$$

$$= \sum_{x=0}^{t} q^2p^t = q^2p^t\sum_{x=0}^{t} 1$$

$$= (t+1)q^2p^t, \quad t = 0,1,2,\ldots$$

Continued

6. (5 marks) A fair coin is flipped 20 times. Given that the total number of heads is 12, what is the probability function for the number of heads in the first 10 flips?

X = # of heads in the 1st 10 flips

Y = # of heads in the 2nd 10 flips

$$P(X=x \mid X+Y=12) = \frac{P(X=x \text{ and } Y=12-x)}{P(X+Y=12)}$$

$$= \frac{\binom{10}{x}\left(\frac{1}{2}\right)^{10} \cdot \binom{10}{12-x}\left(\frac{1}{2}\right)^{10}}{\binom{20}{12}\left(\frac{1}{2}\right)^{20}} = \frac{\binom{10}{x}\binom{10}{12-x}}{\binom{20}{12}}, \quad x = 2, 3, \ldots$$

7. (5 marks) Prove that, for a Poisson random variable X, if the parameter λ is not fixed and is itself an exponential random variable with parameter 1, then

$$P(X = x) = \left(\frac{1}{2}\right)^{x+1}$$

$\lambda \sim \exp(1)$

$$P(X=x) = \int_0^\infty P_{x,\lambda} \; d\lambda$$

$$= \int_0^\infty P(X=x \mid \lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!} e^{-\lambda}\, d\lambda$$

$$= \frac{1}{x!}\int_0^\infty \lambda^x e^{-2\lambda}\, d\lambda = \left(\begin{matrix} u = 2\lambda \\ du = 2\,d\lambda \end{matrix} \right)$$

$$= \frac{1}{x!}\frac{1}{2}\int_0^\infty \frac{u^x}{2^x} e^{-u}\, du = \frac{\Gamma(x+1)}{x!\,2^{x+1}} \underbrace{\int_0^\infty \frac{u^x e^{-u}\, du}{\Gamma(x+1)}}_{=1}$$

Continued

$$= \frac{x!}{x!}\frac{1}{2^{x+1}} = \left(\frac{1}{2}\right)^{x+1}$$

8. (10 marks) Let X and Y be two independent random variables.

(a) Show that $\text{Cov}(X, XY) = E(Y)V(X)$.

$$\text{Cov}(X, XY) = E(X^2Y) - E(X)E(XY)$$
$$= E(X^2)E(Y) - E(X)^2E(Y)$$
$$= E(Y)[E(X^2) - E(X)^2]$$
$$= E(Y)\,\text{Var}(X)$$

(b) Prove that

$$\rho(X+Y, X-Y) = \frac{V(X) - V(Y)}{V(X) + V(Y)}$$

$$\rho(X+Y, X-Y) = \frac{\text{Cov}(X+Y, X-Y)}{\sqrt{\text{Var}(X+Y)}\sqrt{\text{Var}(X-Y)}}$$

$$= \frac{\text{Cov}(X,X) - \cancel{\text{Cov}(X,Y)} + \cancel{\text{Cov}(X,Y)} - \text{Cov}(Y,Y)}{\sqrt{\text{Var}(X) + \text{Var}(Y)}\sqrt{\text{Var}(X) + \text{Var}(Y)}}$$

$$= \frac{\text{Var}(X) - \text{Var}(Y)}{\text{Var}(X) + \text{Var}(Y)}$$

9. (7 marks) Let X be a nonnegative random variable with $E(X) = 5$ and $E(X^2) = 42$. Find an upper bound for $P(X \geq 11)$ using

(a) Markov's inequality

$$P(X \geq 11) \leq \frac{E(X)}{11} = \frac{5}{11}$$

(b) Chebyshev's inequality

$$\underbrace{P(|X-\mu| \geq a)}_{P(X \geq 11)} \leq \frac{\mathrm{Var}\, X}{a^2}$$

$$P(|X-5| \geq 6) = P(X-5 \geq 6 \text{ or } X-5 \leq -6$$

$$= P(X-5 \geq 6) = P(X \geq 11)$$

$$P(X \geq 11) = P(|X-5| \geq 6) \leq \frac{\mathrm{Var}\, X}{36}$$

$$= \frac{42-25}{36} = \frac{17}{36}$$

10. (10 marks) Let X and Y be independent Gamma random variables with parameters (α_1, λ) and (α_2, λ), respectively. Let $U = X + Y$ and $V = X/(X+Y)$.

(a) Find the joint density function of U and V.

$$x = uv \quad , \quad y = u - uv$$

$$J = \det \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -uv - u + uv = -u$$

$$f_{X,Y}(x,y) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} y^{\alpha_2-1} e^{-\lambda(x+y)} \quad , \; x,y > 0$$

$u > 0 \; , \quad 0 < v < 1$

$$f_{U,V}(u,v) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (uv)^{\alpha_1-1} (u-uv)^{\alpha_2-1} e^{-\lambda u} \, |-u|$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \underbrace{u^{\alpha_1+\alpha_2-1}} \, v^{\alpha_1-1} (1-v)^{\alpha_2-1} e^{-\lambda u}$$

(b) Identify the marginal distributions of U and V.

$$f_U(u) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} u^{\alpha_1+\alpha_2-1} e^{-\lambda u} \quad , \; u > 0$$

$$U \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda)$$

$$f_V(v) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1-1} (1-v)^{\alpha_2-1} \quad , \; 0 < v < 1$$

$$V \sim \text{Beta}(\alpha_1, \alpha_2)$$

Continued

11. (7 marks) $\pi(t)$ is the probability generating function of a non-negative integer-valued random variable X.

$$\pi_X(t) = \sum_{i=0}^{\infty} p_i t^i$$
$$= p_0 + p_1 t + p_2 t^2 + \ldots$$

(a) What is $\pi(1)$?

$$= \sum_{i=1}^{\infty} p_i = 1$$

(b) What is $\pi(0)$?

$$= P(X=0) = p_0$$

(c) What is $\frac{1}{2}(\pi(1) + \pi(-1))$?

$$= \frac{1}{2}\left[\sum_{i=0}^{\infty} p_i(1 + (-1)^i)\right] = \frac{1}{2}\left[2p_0 + 2p_2 + 2p_4 + \ldots\right]$$
$$= p_0 + p_2 + p_4 + \ldots = P(X=0) + P(X=2) + P(X=4) + \ldots$$
$$= P(X \text{ is even})$$

(d) If $0 < p < 1$ and $q = 1 - p$ then $p/(1 - qt)$ is a probability generating function. But $\pi(t) = p/(1+qt)$ is not a probability generating function; for one thing, $\pi(1)$ does not have the right value. However, $\pi(t) = \alpha/(1 + qt)$ does have the right value at $t = 1$ if α is chosen correctly. Why is it still not a probability generating function?

$$\pi(t) = \frac{\alpha}{1+qt} \quad \text{is not pgf}$$

$$= \alpha(1 \underbrace{- qt}_{<0} + q^2t^2 - \ldots)$$

12. (6 marks) For a random variable X, its moment generating function is $m_X(t) = (1/81)(e^t + 2)^4$.

(a) Find $P(X \leq 2)$.

$$m_X(t) = \frac{1}{81}\left(e^{4t} + 8e^{3t} + 24e^{2t} + 32e^{t} + 16\right)$$

$$P(X \leq 2) = p_0 + p_1 + p_2 = \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$$

$$= \frac{72}{81} = \frac{8}{9}$$

(b) Find EX.

$$m_X'(t) = \frac{1}{81} \cdot 4(e^t + 2)^3(e^t)$$

$$m_X'(0) = \frac{4}{81} 27 = \frac{108}{81}$$

13. (5 marks) Let X_1, X_2, X_3, X_4 be independent and identically distributed exponential random variables with parameter λ. Let $X_{(4)} = \max\{X_1, X_2, X_3, X_4\}$. Find $P(X_{(4)} \geq 3\lambda)$.

$$P(X_{(4)} \geq 3\lambda) = 1 - P(X_{(4)} < 3\lambda)$$

$$= 1 - \left(1 - e^{-3\lambda^2}\right)^4$$

Continued

14. (5 marks) Suppose the random variable X has a $N(3,9)$ distribution and the random variable Y has a $N(1,4)$ distribution and X and Y are independent.

(a) Give an expression for $P(X+2Y \leq 6)$ in terms of Φ, the cumulative distribution function for the standard normal distribution.

$$E(X+2Y) = 3+2\cdot 1 = 5$$

$$Var(X+2Y) = 9+4\cdot 4 = 25$$

$$X+2Y \sim N(5,25) \Rightarrow \frac{X+2Y-5}{5} \sim N(0,1)$$

$$P(X+2Y \leq 6) = P\left(Z \leq \frac{6-5}{5}\right) = P\left(Z \leq \frac{1}{5}\right)$$

$$= \Phi\left(\frac{1}{5}\right)$$

(b) Find a random variable Z that is a function of both X and Y such that Z has a Chi-square distribution with parameter 2.

$$\frac{X-3}{3} \sim N(0,1), \quad \frac{Y-1}{2} \sim N(0,1)$$

$$Z = \underbrace{\left(\frac{X-3}{3}\right)^2}_{\sim \chi^2_1} + \underbrace{\left(\frac{Y-1}{2}\right)^2}_{\sim \chi^2_1} \sim \chi^2_{(2)}$$

15. (8 marks) $X_1, X_2, \ldots, X_n$ is a random sample of a Bernoulli random variable with parameter p.

(a) Find the value of the constant a that makes

$$a(X_1 + X_1^2 + X_2 + X_2^2 + \ldots + X_n + X_n^2)$$

an unbiased estimator for p.

(b) Is the estimator in part (a) consistent? Explain.

The Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

The Beta Function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Some Important Discrete Probability Distributions

Distribution	Probability Function	Mean	Variance
Binomial(n, p)	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$	np	$np(1-p)$
Bernoulli(p)	same as Binomial$(1, p)$		
Poisson(λ)	$p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, \ldots$	λ	λ
Geometric(p)	$p(x) = p(1-p)^x$ for $x = 0, 1, 2, \ldots$	$\dfrac{1-p}{p}$	$\dfrac{1-p}{p^2}$

Continued

Some Important Continuous Probability Distributions

Distribution	Density Function	Mean	Variance
Uniform(a, b)	$f(x) = \frac{1}{b-a}$ for $a < x < b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ for $x \in \mathbb{R}$	μ	σ^2
Standard Normal	same as Normal$(0, 1)$		
Exponential(λ)	$f(x) = \lambda e^{-\lambda x}$ for $x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(α, λ)	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ for $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Chi-square(n)	same as Gamma $\left(\frac{n}{2}, \frac{1}{2}\right)$		

Total pages 15
Total marks 110