Constant coefficient systems

$\vec{x}' = A\vec{x}$ (A $n\times n$ matrix) If r eigenvalues of A with eigenvector $\vec{\xi}$ then $\vec{x} = e^{rt}\vec{\xi}$ is a solution.

If A has n distinct eigenvalues $r^{(1)}, \dots, r^{(n)}$ get fundamental set of solutions $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$. What if there are repeated eigenvalues?

I.e. sps the root r of $\det(A - rI) = 0$ has multiplicity $k > 1$.

Can happen that one has k linearly independent eigenvalues. Then we're OK.

Ex: $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ here $r = 3$ has mult. 2 but every vector is eigenvector

But sometimes there are less.

<u>Ex:</u> $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

$$\det(A - rI) = \begin{vmatrix} 3-r & 1 \\ 0 & 3-r \end{vmatrix} = (3-r)^2$$

$$A - 3I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\vec{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the only eigenvector, up to multiple.

Get $\vec{x}^{(1)} = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{x}^{(2)} = ?$

Suppose A $n\times n$ matrix, r igenvalue of multiplicity 2 with just one eigenvector $\vec{\xi}$ (up to multiple) $\rightsquigarrow \vec{x} = e^{rt}\vec{\xi}$ is a solution

Try $\vec{x} = e^{rt}(t\vec{\xi} + \vec{\eta})$

$$\vec{x} = e^{rt}(t\vec{\xi} + \vec{\eta})$$

$$\vec{x}' = e^{rt}(tr\vec{\xi} + r\vec{\eta} + \vec{\xi})$$

$$A\vec{x} = e^{rt}(tA\vec{\xi} + A\vec{\eta})$$

$\vec{x}' = A\vec{x} \rightsquigarrow r\vec{\xi} = A\vec{\xi}\ ,\ r\vec{\eta} + \vec{\xi} = A\vec{\eta}$

I.e., $(A - rI)\vec{\xi} = 0$

$(A - rI)\vec{\eta} = \vec{\xi}$

Apply $A - rI$ to second equation, get $(A - rI)^2\vec{\eta} = 0$

<u>Method</u>: Find nonzero solution to $(A - rI)^2\vec{\eta} = 0$ such that

$\underbrace{(A - rI)\vec{\eta}}_{\vec{\xi}}$ is non-zero

$$\vec{x}^{(1)}(t) = e^{rt}\vec{\xi},$$

$$\vec{x}^{(2)} = e^{rt}(t\vec{\xi} + \vec{\eta})$$

are linearly independent solutions

<u>Example</u>: $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$ $\vec{x}' = A\vec{x}$

One finds $r = 4$ is eigenvalue of multiplicity 2.

$A - 4I = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}$ has eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A - 4I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Every non-zero $\vec{\eta}$ satisfies $(A - 4I)^2\vec{\eta} = 0$

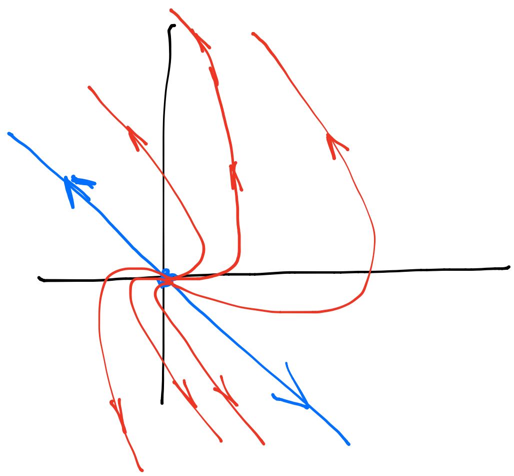
Take $\vec{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (not a eigenvector)

$\vec{\xi} = (A - 4I)\vec{\eta} = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$\vec{x}^{(1)}(t) = e^{4t}\begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $\vec{x}^{(2)} = e^{4t}\left(t\begin{pmatrix} -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$\vec{x}^{(1)} = e^{4t}\begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $\vec{x}^{(2)} = e^{4t}\begin{pmatrix} 1 - 3t \\ 3t \end{pmatrix}$

Phase portrait



$\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)}$,