PLEASE HAND IN

UNIVERSITY OF TORONTO
ST. GEORGE

DECEMBER 2008 EXAMINATIONS

CSC 165H1 F
Instructor: M. David

Duration — 3 hours

Examination Aids: None.

PLEASE HAND IN

Student Number: ___________

Last (Family) Name(s): ___________________

First (Given) Name(s): ___________________

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*Do **not** turn this page until you have received the signal to start.*

This final examination consists of 7 questions on 12 pages (including this one), printed on one side of the paper.

Answer each question directly on the examination paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part that should be marked.*

In your answers, you may use without proof any result or theorem covered during the course in lectures, tutorials, assignments, or term tests. You must justify all other facts required for your solution.

It is an academic offence for students to possess the following items at their examination desks: cell phones, pagers of any kind, IPODs, MP3 players, wristwatch computers, personal digital assistants (e.g., palm pilots) or any other device that is electronic. If any of these items are in your possession now, please ensure they are turned off and put them with your belongings at the front (or side) of the room before the examination begins - no penalty will be imposed. A penalty WILL BE imposed if any of these items are kept at your examination desk.

Please note, students are NOT allowed to petition to REWRITE a final examination.

MARKING GUIDE

# 1:	____/ 16
# 2:	____/ 18
# 3:	____/ 24
# 4:	____/ 20
# 5:	____/ 20
# 6:	____/ 20
# 7:	____/ 12
TOTAL:	____/130

Good Luck!

Question 1. [16 MARKS]

Part (a) [6 MARKS]

Consider the following statement:

There are infinitely many primes that, when added 1, become a power of 2.

Transcribe this statement in formal predicate notation. You may use: quantifiers, logical operators, intermediate variables, comparisons ($\leq$, $<$, $>$, $\geq$, $=$, $\neq$), basic mathematical operators ($+$, $-$, $\times$, and $/$, but *no exponentiation*), and the predicate Prime(x) (that returns true if and only if x is a prime).

Part (b) [6 MARKS]

Let S be a set of numbers. We define the "distance" between two numbers x and y to be $|x-y|$. Consider the following statement:

There is a unique pair of numbers in the set S that are closest to each other.

Transcribe this statement in formal predicate notation. In addition to the operators listed in the previous question, you may use the absolute value function (for a number z, you may write $|z|$), and you may test equality between sets of 2 elements (for numbers a, b, c, d, you may write $\{a, b\} \neq \{c, d\}$).

Part (c) [4 MARKS]

Find two sets S_1 and S_2, of exactly 5 numbers each, for which the statement in Part (b) is *true* and *false*, respectively.

Question 2. [18 MARKS]

For each of the statements below, determine whether they are true for all domains D and predicates P, Q: in *both* directions, in *one* direction, or in *neither* direction. For directions that are true, explain why they are true. For directions that are false, give examples of D, P, Q where they are false.

Part (a) [6 MARKS]

$$(\exists x \in D, P(x) \vee Q(x)) \iff (\exists x \in D, P(x)) \vee (\exists x \in D, Q(x)).$$

Part (b) [6 MARKS]

$$(\forall x \in D, \exists y \in D, P(x,y)) \iff (\exists y \in D, \forall x \in D, P(x,y)).$$

Part (c) [6 MARKS]

$$(\exists x \in D, P(x) \Rightarrow Q(x)) \iff (\exists x \in D, P(x)) \Rightarrow (\exists x \in D, Q(x)).$$

Question 3. [24 MARKS]

Part (a) [10 MARKS]

For natural numbers a, b, define the predicate $d(a,b) : \exists n \in \mathbb{N}, a = n \cdot b$. Consider the following statement:

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, (d(x,z) \wedge d(y,z)) \Rightarrow d(x+y,z).$$

Using the phrase "is a multiple of" (possibly several times), explain in English what the statement says. Then, prove the statement using our proof structure.

Part (b) [4 MARKS]

Consider the following statement:

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, d(x \cdot y, z) \Rightarrow (d(x,z) \wedge d(y,z)).$$

First, explain in English what it says. Then, give natural numbers x, y, z for which the inner implication is false.

Part (c) [10 MARKS]

Consider the sequence a_i defined by $\forall i \in \mathbb{N}, a_i = \lfloor i/2 \rfloor$. You can check that the sequence looks like: $0, 0, 1, 1, 2, 2, \ldots$. Using our proof structure, prove the following statement:

$$\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, \forall k \in \mathbb{N}, k > j \Rightarrow a_k \neq a_i.$$

Hint. It might be helpful to think of it this way: let $P(i) : \exists j \in \mathbb{N}, \forall k \in N, k > j \Rightarrow a_k \neq a_i$. What does $P(i)$ say? (Answering this is not part of the question, but it might help you solve it.)

Question 4. [20 marks]

Part (a) [10 marks]

Prove that:

$$\frac{n-8}{2n^2+2} \in O(1).$$

Part (b) [10 MARKS]
Let $T : \mathbb{N} \to \mathbb{R}^{\geq 0}$ be the function defined by:

$$T(n) = \begin{cases} 1, & n = 0 \\ T(n-1) + \log(n^2), & n > 0 \end{cases}$$

Prove that $T(n) \in O(n \log n)$. (Here, $\log = \log_2$.) *Hint.* Use induction.

Question 5. [20 MARKS]

Consider the following program.

```
def mult(m,n):
   # Pre: m,n natural numbers
   # Post: returns m * n
   z = 0
   a = m
   b = n
   # loop invariant: z = m*n - a*b
   while (a > 0 and b > 0):
      if a > b:
         z = z + b
         a = a - 1
      elif a < b:
         z = z + a
         b = b - 1
      else:
         z = z + a + b - 1
         a = a - 1
         b = b - 1
   return z
```

Prove that this program is correct. (You may assume it eventually terminates.)

(*continued*)

Question 6. [20 MARKS]

Consider the following algorithm:

```
findTwo(A,n)
   # Pre: A is an array of n elements, A[1] ... A[n]
   # Post: return true iff A contains two equal elements
   i <- 1
   while i < n do
      j <- i+1
      while j <= n do
         if A[j] = A[i] then
            return true
         endif
         j <- j+1
      endwhile
      i <- i+1
   endwhile
   return false
```

Let $B(n)$ be the *best-case* running time of this algorithm on inputs of size n (we say that an input is of size n if the array A contains n elements). Let $W(n)$ be the *worst-case* running time of this algorithm on inputs of size n.

Part (a) [4 MARKS]

Show that $B(n) \in O(1)$.

Part (b) [8 MARKS]
Show that $W(n) \in \mathcal{O}(n^2)$.

Part (c) [8 MARKS]
Show that $W(n) \in \Omega(n^2)$.

Question 7. [12 MARKS]

Part (a) [6 MARKS]

Suppose we have a floating-point system with base $\beta = 10$, one sign bit, emin $= -7$, emax $= 8$, and $t = 2$ digits in the mantissa. The system is normalized, meaning that the left-most bit is 0 if and only if we are representing the number 0. In general, $(d_1 d_0, e)$ represents $d_1.d_0 \times \beta^e$. The system uses *truncate-to-zero* (or *round-down*) for numbers that cannot be represented exactly.

Let $P(x) = x^2 - (0.14)x$. We want to evaluate P at $x = 0.13$. Give two different ways in which we can perform this evaluation, and for each, compute the relative error in the result. Show your work.

Part (b) [6 MARKS]

Let $f(x) = e^x$. What is the *condition number* of f? What does this tell you about implementing f for $0 < x < 2$? (Recall, $\frac{d}{dx}(e^x) = e^x$).

Total Marks = 130