CSC165H1 S - Exercise 6
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Question 1:

(a)

Proof:

Assume that the precondition holds: A is a list and len(A) > 0.

Then just before the loop condition is evaluated for the first time, we know: x = A[0] and i = 1.

Then $\exists \mathrm{k} = 0 \in \mathrm{N}$ such that $A[k] = x$. # since x = A[0]

Then $\forall j \in N$, $j < i \Rightarrow j = 0$

Then $\mathrm{A[j]} = \mathrm{A[0]} \leq \mathrm{x}$.

Then if the precondition holds, the loop invariant also holds just before the loop condition is evaluated for the first time.

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(b)

Proof:

Assume that the loop invariant holds:

$\exists \mathrm{k} \in \mathrm{N}$ such that $A[k] = x$ and $\forall j \in N$, $\mathrm{j} < \mathrm{i} \Rightarrow \mathrm{A[j]} \leq \mathrm{x}$.

Assume that the loop terminates,

i.e., the loop condition evaluates to False: $\neg(i < len(A))$

Then $i \geq len(A)$

Then $i \leq len(A)$

Then $i = len(A)$ # when line 6 is executed

Then $j < len(A)$ # since $\mathrm{j} < i$ and $i = len(A)$

Then $\forall j \in N$, $\mathrm{j} < \mathrm{i} = \mathrm{len(A)} \Rightarrow \mathrm{A[j]} \leq \mathrm{x}$.

Then x is the largest value of A[j] where $j \in N$ and $\mathrm{j} < \mathrm{i}$

Then $\exists \mathrm{k} \in \mathrm{N}$ such that $A[k] = x$ # since the loop invariant is true

Then x is in the list A.

Then x is the largest value found in the list A.

Then if the loop terminates, the postcondition holds.

Then if the loop invariant holds and the loop terminates, the postcondition holds.

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(c)
Proof:
Assume that the precondition holds.
Assume the loop invariant is true ($\exists k \in N$ such that $A[k] = x$ and $\forall j \in N$, $j < i \Rightarrow A[j] \le x$) and the loop carries out at least one more iteration.
Then $i < len(A)$ # by the loop condition
Then $A[i] > x$ or $A[i] \le x$
Case 1: $A[i] > x$
Then the algorithm sets $x' = A[i]$
so by the end of the iteration,
* $\exists k \in N$ such that $A[k] = x'$ # since k = i
* $\forall j \in N$, $j < i \Rightarrow A[j] \le x'$ # since $x' = A[i]$
Hence, the loop invariant still holds at the end of the iteration.
Case 2: $A[i] \le x$
Then the algorithm sets $i' = i + 1$, and x is unchanged
so by the end of the iteration,
* $\exists k \in N$ such that $A[k] = x$ # by the loop invariant;
* $\forall j \in N$, $j < i' \Rightarrow A[j] \le x$ #because $j < i \Rightarrow A[j] \le x$ and $i' = i + 1$
Hence, the loop invariant still holds at the end of the iteration.
In both cases, the loop invariant is true at the end of the iteration.
Hence, the loop invariant is true at the end of every iteration such that it was true at the beginning of the iteration.
Hence, the loop invariant holds if the precondition was true.

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(d)
Proof:
Each time the loop body is executed, *i is increased by 1.*
Since len(A) is a finite natural number, i will finally equal to len(A)
Then i' = len(A) for some i'.
Then the loop condition "i < len(A)" is eventually false.

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Question 2:
(a)
Proof:
Assume that the precondition holds: x is a natural number , y is a positive natural number.
Then just before the loop condition is evaluated for the first time, we know r = x. # by the statement in the algorithm
Then $x \ge 0$ # x is a natural number
Then $r \ge 0$ # since r = x
Then $\exists q \in N$ such that $x = yq + r$ # when $q = 0$, $x = 0 + r = r = x$ since $x = r$
Then if the precondition holds, the loop invariant also holds just before the loop condition is evaluated for the first time.

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(b)
Proof:
Assume the loop invariant holds: $r \geq 0$ and $\exists q \in N$ such that $x = yq + r$

Assume that the loop terminates,

i.e. the loop condition evaluates to False: $\neg(r \geq y)$

Then $r < y$ # negation of the loop condition

Then $x < y$ # since $r = x$ and $r < y$

Then $\exists q \in N$ such that $x = yq + r$

Then $yq + r < y$ # since $x = yq + r$ and $x < y$

Then q = 0 # q, r are positive natural numbers and y is a positive integer

Then $x = r < y$

Then $r = x\%y$ for natural number x and positive integer y.

Then, if the loop terminates, the postcondition holds.

Then, if the loop invariant holds and the loop terminates, the postcondition holds.

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(c)
Proof:
Assume that the precondition holds.

Assume the loop invariant is true ($r \geq 0$ and $\exists q \in N$ such that $x = yq + r$) and the loop carries out at least one more iteration.

Then $r \geq y$

Then $x = r \geq y$

Then the algorithm sets $r' = r - y$

Then $r' \geq 0$

Then $x = yq + r' = yq + r - y = y(q-1) + r$

Then $\exists q \in N$ such that $x = yq + r'$

when $q = 1$, $x = yq + r' = yq + r - y = y(q-1) + r = 0 + r = r$

Hence, the loop invariant still holds at the end of the iteration.

Hence, the loop invariant is true at the end of every iteration such that it was true at the beginning of the iteration.

Hence, the loop invariant holds if the precondition was true.

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(d)
Proof:
Each time the loop body is executed, set new r to the difference between the previous r and y (i.e. $r' = r - y$)

Then r is decreasing during each iteration of the loop.

Then when r is small enough, there exists a condition that r < y.

Then the loop condition is false and the loop eventually terminates.

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