UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2009

Math 240H1 Algebra I — Final Exam

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Solve all of the following 5 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Good Luck!

Problem 1. Let V be some vector space over a field F and let W_1 and W_2 be subspaces of V.

1. Prove that $W_1 \cap W_2$ is also a subspace of V.

2. Denote the set of all sums $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$ by $W_1 + W_2$. Prove that $W_1 + W_2$ is a subspace of V.

3. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Hint. You may want to start with a basis β of $W_1 \cap W_2$ and extend it to a basis β_1 of W_1 and a basis β_2 of W_2.

Tip. Quote any theorem you use!

Problem 2. State and prove the "dimension theorem", also known as the "rank-nullity theorem", for a given linear transformation $T : V \to W$.
Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.

Problem 3.

1. Let $L : P_3(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation given by $L(p) = \begin{pmatrix} p(-2) \\ p(1) \\ p(3) \end{pmatrix}$. Find the matrix A representing L relative to the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and the standard basis of $\mathbb{R}^3$.

2. Let $w = a + bi$ be a complex number and let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = w \cdot z$. Considering $\mathbb{C}$ as a vector space over $\mathbb{R}$, find the matrix B representing T relative to the basis $\{1, i\}$ of $\mathbb{C}$.

Tip. Neatness, cleanliness and organization count!

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Problem 4. Find all the solutions (if any exist) of the following two systems of linear equations:

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Tip. Show all intermediate steps!

Problem 5. In a very clean, neat and tidy table on your solutions notebook, indicate for each of the following statements if it is true or false.

1. The function $\det : M_{n\times n}(F) \to F$ is a linear transformation.
2. The determinant of a square matrix can be evaluated using an expansion along any row or column.
3. If two rows of a square matrix A are identical then $\det(A) = 0$.
4. If B is a matrix obtained from a square matrix A by interchanging any two columns, then $\det(B) = -\det(A)$.
5. If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.
6. If B is a matrix obtained from a square matrix A by adding c times one row to another row, then $\det(B) = c\det(A)$.
7. If $A \in M_{n\times n}(F)$ has rank n, then $\det A \neq 0$.
8. The determinant of an upper triangular matrix (a matrix A for which $A_{ij} = 0$ whenever $i > j$) is equal to the product of its diagonal entries.
9. For any square matrix A, $\det(A) = \det(A^T)$.
10. For any invertible square matrix A, $\det(A) = \det(A^{-1})$.

Tips. No need to copy the statements to your notebook, and no need to justify your answers. Note that a statement is considered "true" only if it is **always** true, and not just **sometimes**.

Good Luck!