2nd order homogeneous ODE with constant coeff?

$\boxed{ay''+by'+cy=0}$ (*)

Putting $y(t)=e^{rt}$ gives condition

$\boxed{ar^2+br+c=0}$ (**) characteristic equation.

Solution of (**)

$r_{1/2}=r_1,r_2=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}$

If $b^2-4ac>0$ ...

$b^2-4ac=0$, get repeated root of (**) $r=-\frac{b}{2a}$

$b^2-4ac<0$, get two complex roots (see below).

Repeated root case $b^2-4ac \Rightarrow r=-\frac{b}{2a}$

get one solution $y_1(t)=e^{rt}$

Need another solution ...

Fact. In the repeated root case: $b^2=4ac$, $r=-\frac{b}{2a}$, $y_1(t)=e^{rt}$, $y_2(t)=te^{rt}$ are both solution of (*)

Check, $y_2(t)=te^{rt}$

$y_2'(t)=tre^{rt}+e^{rt}$

$y_2''(t)=t^2re^{rt}+2re^{rt}$

$\Rightarrow ay''+by'+cy$

$=te^{rt}\underbrace{(ar^2+br+c)}_{=0 \text{ since } r \text{ root}}+e^{rt}\underbrace{(a\cdot 2r+b)}_{=0 \text{ since } r=\frac{b}{-2a}}=0$

Example: $y''-4y'+4y=0$

$y(1)=1,\ y'(1)=0$

Char. eqn: $r^2-4r+4=0$

repeated root: $r=2 \Rightarrow$ general solution

$y(t)=A_1e^{2t}+A_2te^{2t}$

Initial condition given: $1=y(1)=A_1e^2+A_2e^2=(A_1+A_2)e^2$

$0=y'(1)=2A_1e^2+3A_2e^2=(2A_1+3A_2)e^2$

$A_1=3e^{-2}$

$A_2=-2e^{-2}$

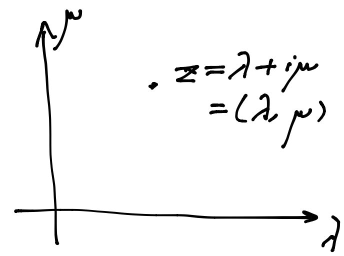
Thus, $y(t)=3e^{2(t-1)}-2te^{2(t-1)}$

Complex numbers (review)

Idea: Introduce "imaginary unit" i with $i^2=-1$. Complex number is expression $z=\lambda+mi$ where $\lambda, m \in \mathbb{R}$.

One pitches such number in complex plane $\mathbb{C}=\mathbb{R}^2$

$\lambda=\mathrm{Re}(z)$ "real part"; $\mu=\mathrm{Im}(z)$ "imaginary part"



One defines, for $z_1=\lambda_1+i\mu_1$
$z_2=\lambda_2+i\mu_2$

Sum: $z_1+z_2=(\lambda_1+\lambda_2)+i(\mu_1+\mu_2)$

Product: $z_1 z_2=(\lambda_1\lambda_2-\mu_1\mu_2)+i(\lambda_1\mu_2+\mu_1\lambda_2)$

Has usual properties, e.g. $z_1 z_2 = z_2 z_1$, distribute law.

Complex numbers:

For $z=\lambda+i\mu$ one defines complex conjugate $\bar{z}=\lambda-i\mu$

absolute value $|z|=\sqrt{\lambda^2+\mu^2}$

Note: $z\cdot\bar{z}=|z|^2$

Example: $\frac{1}{3+4i}$

General: $\frac{1}{z}=\frac{\bar{z}}{z\cdot\bar{z}}=\frac{1}{|z|^2}\cdot\bar{z}$

$$\frac{1}{3+4i}=\frac{3-4i}{(3-4i)(3+4i)}=\frac{3-4i}{25}=\frac{3}{25}-\frac{4}{25}i$$

Complex exponentials $e^z=\exp(z)$

Recall: Taylor series of e^x:

$$\boxed{e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n}$$

e^z should have property $e^{z_1}\cdot e^{z_2}=e^{z_1+z_2}$

Thus we should have $e^z=e^{\lambda+i\mu}=e^\lambda\cdot e^{i\mu}$

$$e^{i\mu}=\sum_{n=0}^{\infty}\frac{1}{n!}i^n\mu^n$$
$$=\sum_{n=0}^{\infty}\frac{1}{(2m)!}(-1)^m\mu^{2m}+i\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}(-1)^m\mu^{2m+1}$$
$$\boxed{=\cos(\mu)+i\sin(\mu)}$$

$\Rightarrow e^{i\mu}=\cos(\mu)+i\sin(\mu)$ EULER'S FORMULA

Special case: $e^{i\pi}=-1$

For general complex $z=\lambda+i\mu$, can define $e^z=e^\lambda(\cos(\mu)+i\sin(\mu))$

Homework: Check $e^{z_1+z_2}=e^{z_1}\cdot e^{z_2}$ Check: $\frac{d}{dt}e^{tz}=ze^{tz}$