Vectors and matrices -- review of terminology

Matrix: a rectangular array of (possibly complex) numbers arranged in rows and columns.
Order or **size**: a matrix A with m rows and n columns is said to be of **order** or **size** $m \times n$ ($A \in \mathbb{C}^{m \times n}$).

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}$$

Square matrix of order or size n (or $n \times n$): $n = m$

Row vector: $1 \times n$ matrix, $x = (x_1, x_2, \cdots, x_n)$
Column vector $m \times 1$ matrix, $y = (y_1, y_2, \cdots, y_m)^T$.
A **vector** is a column vector, unless otherwise stated.

Equal matrices/vectors: of same order and the corresponding elements are equal.

Vectors and matrices -- review of terminology -- properties

Properties of addition and multiplication of matrices:
Let A, B and C be matrices of appropriate order and α and β be scalars.
$A + B = B + A$ and in general $A \cdot B \neq B \cdot A$
$(A + B) + C = A + (B + C)$
$\alpha(A + B) = \alpha A + \alpha B$, $\quad (\alpha + \beta)A = \alpha A + \beta A$
$(A + B)C = AC + BC$, $\quad A(B + C) = AB + AC$
$\alpha(AB) = (\alpha A)B = A(\alpha B)$

Zero (null) matrix (0): a matrix with all elements equal to zero. $A + 0 = 0 + A = A$.

Identity matrix **I** of order n: a square matrix of order n, with 1's on the diagonal and 0's anywhere else.

Inverse A^{-1} of a square matrix A of order n: a square matrix $B = A^{-1}$ of order n with the property $AB = BA = \mathbf{I}$.
Invertible or **non-singular** matrix: A^{-1} exists.

singular -> not invertible, so it is "single"

Singular or **non-invertible** matrix: A^{-1} does not exist.

Properties of inversion of matrices:
Let A and B be invertible matrices and α be non-zero scalar.

$$(A^{-1})^{-1} = A, \qquad (AB)^{-1} = B^{-1}A^{-1}, \qquad (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}.$$

Vectors and matrices -- review of terminology

Sum of two matrices $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times n}$: a matrix $C \in \mathbb{C}^{m \times n}$ with elements $c_{ij} = a_{ij} + b_{ij}$.
Product of a matrix $A \in \mathbb{C}^{m \times n}$ by a scalar $\alpha \in \mathbb{C}$: a matrix $C \in \mathbb{C}^{m \times n}$ with elements $c_{ij} = \alpha a_{ij}$.
Product of two matrices $A \in \mathbb{C}^{m \times l}$, $B \in \mathbb{C}^{l \times n}$: a matrix $C \in \mathbb{C}^{m \times n}$ with elements $c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}$.

Vectors and matrices -- review of terminology

Transpose $B = A^T$ of $A \in \mathbb{R}^{m \times n}$: the $n \times m$ matrix for which $b_{ij} = a_{ji}$, $i = 1, \cdots, n$, $j = 1, \cdots, m$.
Properties of transposition of matrices: $(A^T)^T = A$, $(AB)^T = B^T A^T$

Symmetric matrix A: $A^T = A$.

Orthogonal matrix: a real matrix A for which $A^T A = \mathbf{I}$; if the matrix is square orthogonal, its transpose and its inverse are equal.

A^T = A^{-1}

Conjugate transpose $B = A^H = \bar{A}^T$ of $A \in \mathbb{C}^{m \times n}$: the $n \times m$ matrix for which $b_{ij} = \bar{a}_{ji}$, $i = 1, \cdots, n$, $j = 1, \cdots, m$.

overbar denotes a scalr complex conjugate: the complex conjugate of a+bi is a-bi.

Properties of conjugate transposition of matrices:
$(A^H)^H = A$, $(AB)^H = B^H A^H$

Hermitian matrix A: $A^H = A$.

because real number's conjugate is itself

For all real matrices we have $A^H = A^T$.

Unitary matrix: a complex matrix A for which $A^H A = \mathbf{I}$; if the matrix is square unitary, its conjugate transpose and its inverse are equal.

Normal matrix A: $A^H A = AA^H$.

Vectors and matrices -- review of terminology

Permutation matrix P: a square matrix, whose elements are all "0" or "1" and there is exactly one "1" in every row and column, i.e. P is $\mathbf{I}$ with its rows and columns rearranged. 每行列各1个1

Elementary permutation matrix P: a permutation matrix, arising from interchanging only two rows (or only two columns) of $\mathbf{I}$.

- The product of permutation matrices is a permutation matrix.
- Permutation matrices are orthogonal.

Linearly independent vectors $v_1, v_2, \cdots, v_n \in \mathbb{C}^m$: for any linear combination of them for which $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$, $c_i \in \mathbb{C}$, $i = 1, \cdots, n$, this implies $c_1 = 0, c_2 = 0, \cdots, c_n = 0$.

Inner product (x, y) of vectors x and y: $(x, y) \equiv x^T \cdot y$. ($x^H \cdot y$ for complex vectors).

Orthogonal vectors $v_1, v_2, \cdots, v_n \in \mathbb{R}^m$: $i \neq j \Rightarrow v_i^T \cdot v_j = 0$. ($v_i^H \cdot v_j = 0$ for $v_i \in \mathbb{C}^m$).

Orthonormal vectors $v_1, v_2, \cdots, v_n \in \mathbb{R}^m$: orthogonal and $v_i^T \cdot v_i = 1$. ($v_i^H \cdot v_i = 1$ for $v_i \in \mathbb{C}^m$).

- Orthogonal vectors are linearly independent.

Vectors and matrices -- review of terminology

Diagonal of a square matrix of order n: the set of elements $\{a_{ii}, i = 1, \cdots, n\}$.

Diagonal matrix: a square matrix with zero off-diagonal elements, i.e., $a_{ij} = 0$ for $i \neq j$.

Lower triangular matrix: a square matrix with zero super-diagonal elements, i.e., $a_{ij} = 0$ for $i < j$. (**Strictly lower triangular**: $a_{ij} = 0$ for $i \leq j$.)

Upper triangular matrix: a square matrix with zero sub-diagonal elements, i.e., $a_{ij} = 0$ for $i > j$. (**Strictly upper triangular**: $a_{ij} = 0$ for $i \geq j$.)

Unit lower triangular matrix: a lower triangular matrix with 1's on the diagonal, i.e., $a_{ij} = 0$ for $i < j$, $a_{ii} = 1$, $i = 1, \cdots, n$.

Unit upper triangular matrix: an upper triangular matrix with 1's on the diagonal, i.e., $a_{ij} = 0$ for $i > j$, $a_{ii} = 1$, $i = 1, \cdots, n$.

Dense matrix: most of its elements are non-zero.
Density of a matrix: the ratio of the number of non-zero elements over the total.
Sparse matrix: most of its elements are zero.
Sparsity of a matrix: the ratio of the number of zero elements over the total.

Vectors and matrices -- review of terminology

Banded matrix: a sparse matrix A that has all its non-zero elements near the diagonal. Its **lower bandwidth** is l and its **upper** bandwidth u, if all elements below the l-th subdiagonal and above the u-th superdiagonal are zero. In other words, $a_{ij} = 0$, if $i - j > l$ or $j - i > u$.

(Full or **total) Bandwidth**: $l + u + 1$.

Symmetrically banded matrix: a banded matrix with $l = u$.

Semi-bandwidth of a symmetrically banded matrix: l or u.

Tridiagonal matrix: a symmetrically banded matrix with semi-bandwidth 1.

Pentadiagonal matrix: a symmetrically banded matrix with semi-bandwidth 2.

(Row) **Diagonally dominant** matrix: a square matrix A for which $|a_{ii}| \geq \sum_{j=1\ j\neq i}^{n} |a_{ij}|$ for all $i = 1, \cdots, n$.

Strictly (row) **diagonally dominant** matrix: a square matrix A for which $|a_{ii}| > \sum_{j=1\ j\neq i}^{n} |a_{ij}|$ for all $i = 1, \cdots, n$.

- Strictly diagonally dominant matrices are nonsingular.

Vectors and matrices -- review of terminology

Positive definite matrix: a square matrix A for which, for any vector $x \neq 0$, $x^T A x > 0$.

- A symmetric matrix is positive definite (SPD), iff the diagonal elements of the U factor in the LU factorisation are positive.

Non-negative (positive, non-positive, negative) matrix: $a_{ij} \geq 0$ ($a_{ij} > 0$, $a_{ij} \leq 0$, $a_{ij} < 0$, respectively), for all i, j. *Notation*: $A \geq 0$ ($A > 0$, $A \leq 0$, $A < 0$)

Monotone matrix: a real square non-singular matrix A for which $A^{-1} \geq 0$.

M-matrix: a real square non-singular matrix A for which $a_{ij} \leq 0$, for $i \neq j$, and $A^{-1} \geq 0$.

Determinant $det(A)$ of a $n \times n$ matrix A: a scalar defined as follows:
If A is 1×1 then $\det(A) = a_{11}$. Else $\det(A) =$
$= a_{11} \det(A'_{11}) - a_{12} \det(A'_{12}) + \cdots + (-1)^{n+1} a_{1n} \det(A'_{1n})$
$= a_{11} \det(A'_{11}) - a_{21} \det(A'_{21}) + \cdots + (-1)^{n+1} a_{n1} \det(A'_{n1})$,
where A'_{ij} is the $(n-1) \times (n-1)$ submatrix formed by deleting the i-th row and the j-th column of A.
Properties of determinants: $\det(AB) = \det(A)\det(B) = \det(BA)$.

Linear systems -- review of some properties

System of m linear equations with n unknowns:

$$
\begin{array}{ccccccccc}
a_{11}\cdot x_1 & + & a_{12}\cdot x_2 & + & . & . & + & a_{1n}\cdot x_n & = & b_1\\
a_{21}\cdot x_1 & + & a_{22}\cdot x_2 & + & . & . & + & a_{2n}\cdot x_n & = & b_2\\
. & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & .\\
a_{m1}\cdot x_1 & + & a_{m2}\cdot x_2 & + & . & . & + & a_{mn}\cdot x_n & = & b_m
\end{array}
$$

where the coefficients a_{ij}, $i = 1, \cdots, m$, $j = 1, \cdots, n$, and the right side b_i, $i = 1, \cdots, m$, are given and x_j, $j = 1, \cdots, n$, are the unknowns.
A linear system can be written in a matrix form as $Ax = b$, where A is an $m \times n$ matrix, x is a $n \times 1$ vector and b is a $m \times 1$ vector.

- $Ax = b$ has at most one solution, iff $Ax = 0$ has only the trivial solution.

- If A is $m \times n$ and $m < n$, then $Ax = 0$ has non-trivial solutions.
- If A is $m \times n$ and $Ax = b$ has a solution for every b, then $m \leq n$.

Linear systems -- review of some properties

- If A is square, the following are equivalent:

− A is singular
− $Ax = 0$ has a non-trivial solution
− $\det(A) = 0$
− The columns of A are linearly dependent
− If $Ax = b$ has a solution, then it has infinitely many solutions.

- If A is square, the following are equivalent:

− A is invertible
− $Ax = 0$ has only the trivial solution
− $\det(A) \neq 0$
− The columns of A are linearly independent
− $Ax = b$ has a (unique) solution for every right side b.

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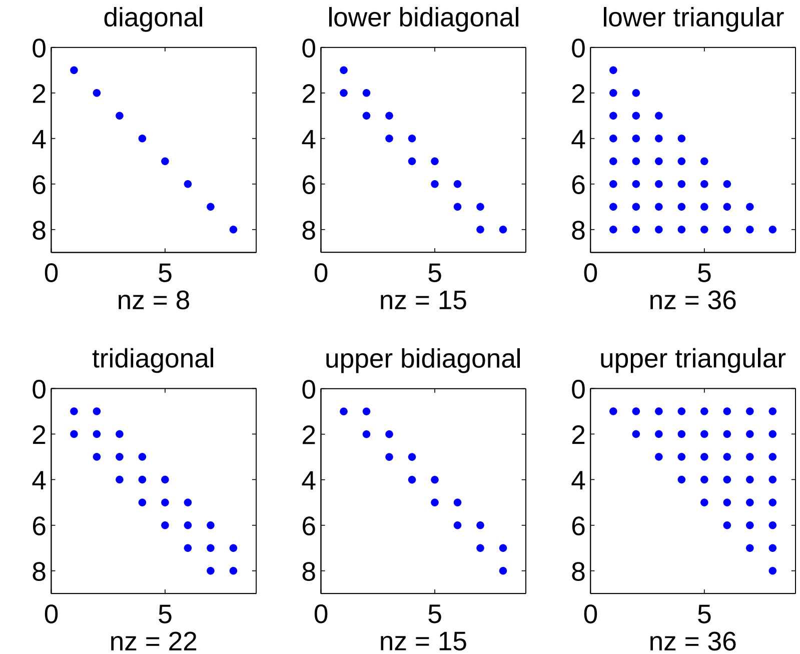

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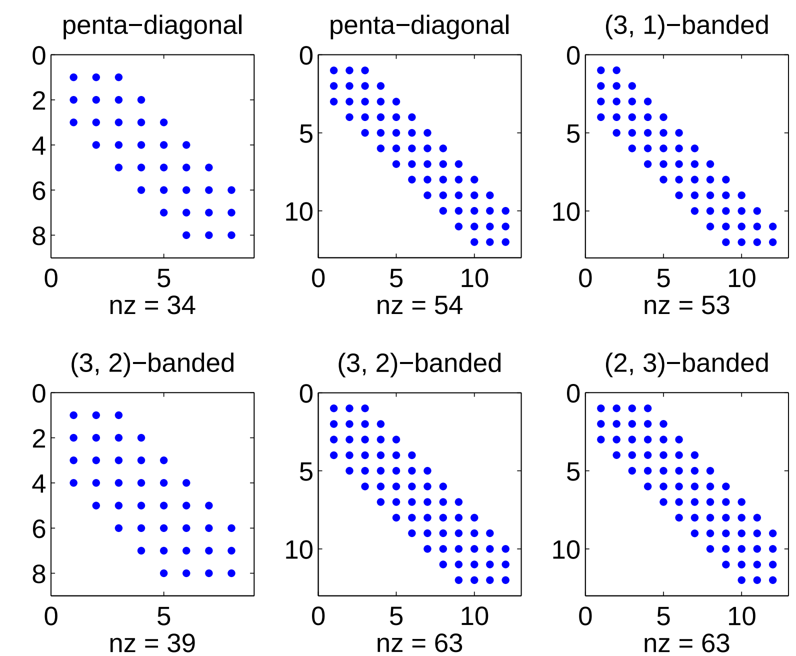

Null space, range space and rank of a matrix

Column (row) rank of an $m \times n$ matrix A: the number of linearly indepedent columns (rows) of A, i.e., the dimension of the subspace spanned by the columns (rows) of A, (the dimension of $R(A)$).

Rank: the column and row ranks of a matrix A are equal and their common value is the rank of A.

- An invertible matrix of order n has rank n.
- If P is a nonsingular square matrix, then PA has the same rank as A.
- $Ax = b$ has a solution, iff the rank of the augmented matrix $(A\ b)$ is the same as the rank of A.

Null space or **kernel**, $N(A)$ or $ker(A)$, of an $m \times n$ matrix A: the set of all vectors x such that $Ax = 0$.
Nullity: the dimension of the null space.

- An invertible matrix has nullity 0.

Range space $R(A)$ of an $m \times n$ matrix A: the set of all vectors x such that for some vector y we have $Ay = x$.

- $Ax = b$ has a solution, iff $b \in R(A)$.
- If A is $n \times n$, then rank + nullity = order (n).

