ABOUT OUR MIDTERM

The problems for the midterm will include the problems from the exercises you should have solved before the first two quizzes and the following problems that appeared on past midterms and exams (see examples below).

I. Prove the following theorems:

Ceva's theorem.

Menelaus's theorem.

Optical property of ellipse/hyperbola.

Separation theorem for compact convex sets (one can separate a point outside a compact convex set from the set by a hyperplane).

Radon's theorem.

Helly's theorem.

Dehn-Somerville relations.

II. Solve the following problems:

a. *Problems on Ceva's and Menelaus' theorems:*

1) Term test 2008, problem 1.

2) Show that the internal bisectors of the three angles of a triangle pass through one point.

b. *Problems with center of mass:*

3) Term test 2010, problem 2.
4) Term test 2008, problem 3.

5) Take a 4-gon. Consider three segments: two of them join the midpoints of opposite sides and the third one joins the midpoints of the diagonals. Show that the three segments intersect at one point and that this point is the midpoint of each one of them.

6) Consider 6 points $A_1, \dots, A_6$ in three-dimensional space R^3. Let $B_1, \dots, B_6$ be the midpoints of the segments $A_1A_2, \dots, A_5A_6, A_6A_1$ respectively. Show that the point of intersection of the medians in triangle $B_1B_3B_5$ coincides with the point of intersection of the medians in the triangle $B_2B_4B_6$.

c. *Extremal problems:*

7) Sum of distances to some segments with some coefficients.

8) Term test 2010, problem 1.

9) Term test 2009, problem 1.

10) Consider a regular triangle ABC. Find all points O for which the sum $4O_{AB} + O_{BC} + O_{CD}$ is the smallest possible. Here O_{AB}, O_{BC} and O_{CD} are the distances from point O to the sides AB, BC and CD respectively.

11) Consider an angle $AOB = \frac{2\pi}{n}$ as a billiard with two infinite sides.Take a billiard trajectory such that its first piece is a segment starting inside the angle and going in a way parallel to AO towards the side OB and intersecting it at a point $C \in OB$. Find the shortest distance from the trajectory to the point O assuming that the length of the segment OC is c.

d. *Problems on Dehn–Somerville relations:*

12) Term test 2010, problem 4.

13) Take a convex polyhedron in $\mathbf{R}^3$. Denote by f_0, f_1 and f_2 the number of its vertices, edges and faces, respectively. Prove:

(1) $3f_0 \leq 2f_1$. Hint: at least 3 edges meet at each vertex of the polyhedron.

(2) $2f_1/f_2 < 6$ – the average number of edges on faces of the polyhedron is strictly less that 6. Hint: use 1) and Euler formula $f_0 - f_1 + f_2 = 2$.

14) Find the F-polynomial and H-polynomial of a prism having a convex 2012-gon as a base. How many of the vertices of the prism have index one with respect to a linear function that is not constant on any of its edges?

e. *Inversion:*

15) Term test 2008, problem 5.

16) Is there an inversion that takes the points $(2, 0), (2, 0), (0, 2), (0, 1)$ into vertices of a square?

17) For three given lines in the plane find a point O such that after an inversion centered at O the three lines become circles of equal radii.

18) Describe the image under inversion of the family of lines passing through the point $(0, 1)$ after inversion in the unit circle centered at the origin.

19) Consider a circle inscribed in a square. Describe the image of the square after inversion in this circle.