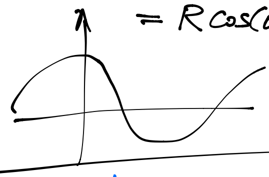
Review: Free oszillator  (Mass-spring system)

$mx'' + kx = 0$

Gen. Solution

$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$; $\omega_0 = \sqrt{\frac{k}{m}}$, $T = \frac{2\pi}{\omega_0}$

$= R\cos(\omega_0 t - \delta)$, $R = \sqrt{A^2 + B^2}$



Damped oscillator

$mx'' + \beta x' + kx = 0$

Char. equation

$mr^2 + \beta r + k = 0$

$r_1, r_2 = -\frac{\beta}{2m} \pm \frac{1}{2m}\sqrt{\beta^2 - 4mk}$

Case 1: Small damping $\beta^2 < 4mk$

Get $r_1, r_2 = \lambda \pm i\mu$ where

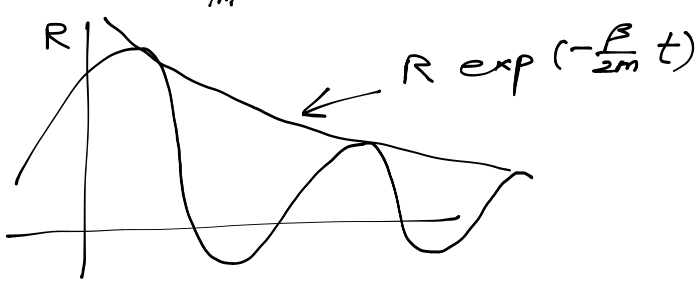
$\lambda = \frac{-\beta}{2m}$, $\mu = \frac{1}{2m}\sqrt{4mk - \beta^2}$

Gen. Solution:

$x(t) = e^{\lambda t}(A\cos(\mu t) + B\sin(\mu t)) = e^{\lambda t} R\cos(\omega t - \delta)$

$x(t) = \exp(\frac{-\beta}{2m} t)\cos(\omega t - \delta)$

$\omega = \sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}$ Note: $\omega < \omega_0$

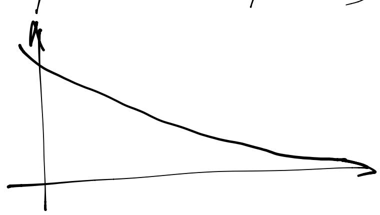


Case 2:

Strong damping: $\beta^2 > 4mk$

Then r_1, r_2 are both real, both < 0.

$x(t) = A\exp(-r_1 t) + B\exp(-r_2 t)$



No oscillators!

Case 3: $\beta^2 = 4mk$

$r_1 = r_2 = -\frac{\beta}{2m}$

$A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$= R\cos(\omega_0 t - \delta)$

$R = \sqrt{A^2+B^2}$, $\tan(\delta) = \frac{B}{A}$

Fundamental Solutions

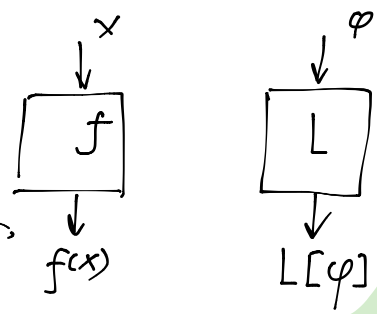
$$\boxed{y'' + p(t)y' + g(t)y = g(t)}$$

General second order linear ODE.

Introduce notation:

$L[y] = y'' + p(t)y' + g(t)$ (*)

L is called an "operator", i.e. a machine where you put in a function, and it produces a new function. (General operator, could take derivatives, integrate, ...)



Note: (*) is a linear operator, meaning $L[c_1y_1 + c_2y_2] = c_1[y_1] + c_2[y_2]$

y_1, y_2 functions, $c_1, c_2 \in \mathbb{R}$

$$\left.\begin{aligned} (c_1y_1+c_2y_2)'' &= c_1y_1'' + c_2y_2'' \\ p(t)(c_1y_1+c_2y_2)' &= c_1p(t)y_1' + c_2p(t)y_2' \\ g(t)(c_1y_1+c_2y_2) &= c_1g(t)y_1 + c_2g(t)y_2 \end{aligned}\right\} \Longrightarrow L[c_1y_1+c_2y_2] = c_1L[y_1] + c_2L[y_2]$$

The original ODE now reads

$L[y] = g(t)$

Homogeneous ODE:

$L[y] = 0$.

Thm (Principle of superposition).

If y_1, y_2 are solutions of $L[y]=0$, and $c_1, c_2 \in \mathbb{R}$ then $c_1y_1 + c_2y_2$ is again a solution.

$\longrightarrow L[y] = y'' + p(t)y' + g(t)y$

Thm (Existence and uniqueness).

Suppose p, g, g are continuous on open interval I, containing t_0. Then the ~~VIP~~ IVP

$L[y] = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$

has a unique solution, defined for all $t \in I$.