DIFFERENTIAL EQUATIONS
(MATH 242.01)
TEST 2 – OCTOBER 23, 2002

1	(10*pts*)
2	(15*pts*)
3	(20*pts*)
4	(17*pts*)
5	(18*pts*)
6	(20*pts*)

Name: ____________________

Directions:
Answer all questions in the space provided and **box your answer**. You can also use the back of the facing opposite page if you need more room. You must show intermediate work for partial credit. Calculators are allowed.

1. A substance N_1 decays with a half life of 25 seconds to substance N_2. N_2 has a half-life of 3 minutes and decays to the stable substance N_3. Model the system (but do not solve the equations) if initially, there are 10 grams of N_1, 5 grams of N_2, and 3 grams of N_3. Let $x_j(t) =$ amount (grams) of N_j at time t.

$$\boxed{\begin{aligned} x_1' &= -\frac{\ln 2}{25\text{ sec}} x_1, & x_1(0) &= 10 \text{ grams} \\ x_2' &= \frac{\ln 2}{25\text{ sec}} x_1 - \frac{\ln 2}{3\text{ min.}} x_2, & x_2(0) &= 5 \text{ grams} \\ x_3' &= \frac{\ln 2}{3\text{ min}} x_2, & x_3(0) &= 3 \text{ grams} \end{aligned}}$$

2. Consider the differential equation $y'' - y' - 2y = 0$.
   a.) Determine the auxiliary equation and corresponding solutions.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = -1, 2$$

$$\boxed{\begin{aligned} y_1(t) &= e^{-t} \\ y_2(t) &= e^{2t} \end{aligned}}$$

   b.) Compute the associated Wronskian for these solutions.

$$W(x) = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^{t} > 0, \text{ all } t.$$

   c.) Determine the general solution for the equation.

y_1 & y_2 are linearly independent by part b)

so $\boxed{y(t) = c_1 e^{-t} + c_2 e^{2t}}$ is the general solution.

3. For each of the following equations, provide the general solution:

a.) $y''' + 6y'' + 9y' = 0$

$\lambda^3 + 6\lambda^2 + 9\lambda = 0 \Rightarrow \lambda(\lambda+3)^2 = 0 \qquad \lambda = 0, -3, -3$

$$\boxed{y(t) = c_1 + c_2 e^{-3t} + c_3 t e^{-3t}}$$

b.) $(D+2)(D^2 - 4D + 13)^2[y] = 0$

$\lambda = -2$; $\lambda = 2 \pm 3i$ [double roots]

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{2t} \cos 3t + c_3 t e^{2t} \cos 3t + c_4 e^{2t} \sin 3t + c_5 t e^{2t} \sin 3t}$$

4. a.) Write the differential equation

$$y''' - 3y'' - 4y' = 5x + 3xe^{4x}$$

in operator form. $\boxed{(D^3 - 3D^2 - 4D)y = 5x + 3xe^{4x}}$

b.) Determine the homogeneous solution.

$D(D-4)(D+1)y_h = 0$

$$\boxed{y_h(x) = c_1 e^{4x} + c_2 e^{-x} + c_3}$$

c.) Determine the annihilator for the right hand side.

$$\left.\begin{array}{l} D^2(5x) = 0 \\ (D-4)^2(3xe^{4x}) = 0 \end{array}\right\} \Rightarrow \boxed{D^2(D-4)^2 \text{ is the annihilator}}$$

d.) Find the *form* of the general solution to the equation.

$$\boxed{y(x) = \underbrace{c_1 e^{4x} + c_2 e^{-x} + c_3}_{y_h} + \underbrace{Ax + Bx^2 + Cxe^{4x} + Dx^2 e^{4x}}_{y_p}}$$

5. For the differential equation $y'' - y' - 12y = 20e^{2t}$,

a.) determine the homogeneous solution.

$$(D^2 - D - 12)y_h = (D-4)(D+3)y_h = 0$$

$$\boxed{y_h(t) = c_1 e^{4t} + c_2 e^{-3t}}$$

b.) compute a particular solution.

$(D-2)(20e^{2t}) = 0$ term $D-2$ does not appear in the homogeneous equation, so $y_p(t) = Ae^{2t}$. Back substitute to get

$$(4A - 2A - 12A)e^{2t} = 20e^{2t} \Rightarrow A = -2$$

$$\boxed{y_p(t) = -2e^{2t}}$$

c.) determine the general solution for the equation.

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-3t} - 2e^{2t}}$$

d.) find the solution to the initial value problem when $y(0) = 1$ and $y'(0) = -1$.

$$1 = y(0) = c_1 e^0 + c_2 e^0 - 2e^0 \Rightarrow c_1 + c_2 = 3$$

$$-1 = y'(0) = 4c_1 e^0 - 3c_2 e^0 - 4e^0 \Rightarrow 4c_1 - 3c_2 = 3$$

Solving these simultaneous equations gives $c_1 = \frac{12}{7}, c_2 = \frac{9}{7}$

$$\boxed{y(t) = \frac{12}{7} e^{4t} + \frac{9}{7} e^{-3t} - 2e^{2t}}$$

6. For the variable coefficient ODE

$$t^2y'' + 5ty' + 4y = 0$$ Cauchy-Euler Eqn.

a.) determine a solution by making an educated guess.

Let $y(t) = t^m$, then $m(m-1) + 5m + 4 = 0$ or $m^2 + 4m + 4 = 0$,

$m = -2$ a double root.

$$\boxed{y_1(t) = t^{-2}}$$

b.) find an additional solution.

Use the method of §4.2 $\quad y'' + \frac{5}{t}y' + \frac{4}{t^2}y = 0$

$$y_2(t) = y_1(t)\left[\int \frac{e^{-\int p\,ds}}{y_1^2(s)}\,ds\right] = t^{-2}\left[\int \frac{e^{-\int \frac{5}{s}ds}}{(t^{-2})^2}\,dt\right]$$

$$= t^{-2}\int \frac{t^{-5}}{t^{-4}}\,dt = \boxed{(\ln t)\,t^{-2}}$$

c.) determine the Wronskian for your two solutions.

$$W(t) = \begin{vmatrix} t^{-2} & t^{-2}\ln t \\ -2t^{-3} & -2t^{-3}\ln t + t^{-2}\left(\frac{1}{t}\right) \end{vmatrix} = t^{-5} - 2t^{-5}\ln t + 2t^{-5}\ln t = t^{-5} \neq 0, \text{ if } t>0.$$

$\therefore$ y_1 & y_2 are linearly independent

d.) determine the general solution to the ODE.

$$\boxed{y(t) = c_1 t^{-2} + c_2 t^{-2}\ln t}$$