Lecture 6.

More about Normal Distributions.

$X \sim N(0,1)$ - St. Normal dist'n

$$\varphi(x) = f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

Normal = Gaussian

$$\underbrace{a e^{-\frac{(x-b)^2}{2c^2}}}_{\text{Gaussian functions}}, \quad a, b, c \in \mathbb{R}$$

Q: Is $\varphi(x)$ a valid df? ✓

cdf of X:

$$\Phi(x) = F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$E(X) = \int_{-\infty}^{\infty} \underbrace{x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{\text{odd}} dx = 0$$

$$Var(X) = E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \left[\begin{array}{l} u = x \\ du = dx \\ x e^{-x^2/2} dx = dv \\ v = -e^{-x^2/2} \end{array}\right] = \Big[\underbrace{-x e^{-x^2/2}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} e^{-x^2/2} dx\Big]\frac{2}{\sqrt{2\pi}} = \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{=1}$$

General Normal Distribution.

$Z \sim N(0,1)$

$X = aZ + b, \quad a, b \in \mathbb{R}, \; a \neq 0$

$h(z) = az + b$

$h^{-1}(x) = \frac{x-b}{a}, \quad \frac{d}{dx} h^{-1} = \frac{1}{a}$

$$f_X(x) = \varphi(h^{-1}(x)) \cdot \frac{d}{dx} h^{-1}$$

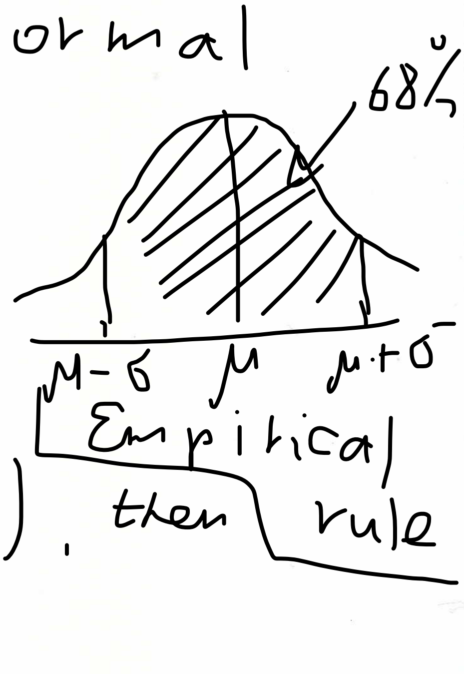
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2a^2}} \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(x-b)^2}{2a^2}} \Rightarrow \text{non-standard normal}$$

$E(X) = aE(Z) + b = b$

$Var(X) = a^2 Var(Z) = a^2$

$X \sim N(b, a^2)$



Empirical rule

In general, if $Y \sim N(\mu, \sigma^2)$, then

$Z = \frac{Y - \mu}{\sigma} \sim N(0,1)$

Claim: $\underline{Y \sim N(\mu, \sigma^2)}$, $X = aY + b \sim N(a\mu + b, a^2\sigma^2)$

Pf: hw problem #1

$$P(|Y-\mu| \leq \sigma) = P\left(\left|\overbrace{\frac{Y-\mu}{\sigma}}^{Z}\right| \leq 1\right) = P(-1 \leq Z \leq 1)$$

$$= P(Z \leq 1) - P(Z \leq -1) = 0.8413 - 0.1587$$

$$= .6826$$

The Chi-Square Distribution.

$Z \sim N(0,1)$

$X = Z^2 \sim ?$

HW problem #2

$P(|X-\mu| \leq 2\sigma) = ?$

$P(\quad\quad \leq 3\sigma) = ?$

$X \sim N(\mu, \sigma)$

$$F_X(x) = P(X \leq x) = P(Z^2 \leq x)$$

$$= P(-\sqrt{x} \leq Z \leq \sqrt{x}) = F_Z(\sqrt{x}) - F_Z(-\sqrt{x})$$

$$f_X(x) = \begin{cases} f_Z(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}, & x > 0 \\ 0, & ow, \end{cases}$$

$$= \begin{cases} \cancel{2} \frac{1}{\cancel{2}\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-x/2}, & x > 0 \\ 0, & ow \end{cases}$$

$$= \frac{e^{-x/2} x^{-1/2} \left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}}, \quad x > 0$$

λ^α

$$\Rightarrow X \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right), \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$X \sim \chi^2_{(1)}$ - chi-square dist'n with $\underbrace{\text{parameter}}_{\text{degree of freedom}}$ 1