Worth: 2% **Due:** *Before* 10pm on Tuesday 24 January 2012.

Remember to write your *full name* and *student number* prominently on your submission.
Please read and understand the policy on Collaboration given on the Course Information Sheet. Then, to protect yourself, list on the front of your submission ***every*** *source of information you used to complete this homework (other than your own lecture and tutorial notes, and materials available directly on the course webpage). For example, indicate clearly the* ***name*** *of every student with whom you had discussions, the* ***title*** *of every additional textbook you consulted, the* ***source*** *of every additional web document you used,* etc.
For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks **will** be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.

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1. Consider the following solitaire game, played with a collection of n identical coins.

   - At any time during the game, the coins are divided into a number of *groups*. At the beginning, all the coins are in the same group.
   - The game is played in *rounds*. During each round, you pick one of the existing groups (any one) that contains *more* than one coin and you split it into two smaller groups (in any way you like).
   - Your gain in that round is the *product* of the sizes of the two new groups, in dollars!
   - The game is over once every group contains exactly one coin.

   For example, suppose you start with 13 coins. For your first round, you may decide to split the group into a group of 6 and a group of 7: this pays $6 \times 7 = 42$ dollars. In the next round, if you break the 7-coin group into a group of 2 and a group of 5, you will get 10 more dollars (for a total of 52 dollars so far). And so on.

   Prove that for any natural number n, if you start the game with a single group of n coins, then *regardless of how you play the game* (*i.e.*, no matter how you choose to split up the groups), you always win a total of *exactly* $n(n-1)/2$ dollars.

   Hint: This is easy to do using complete induction...

2. Define a set $M \subseteq \mathbb{Z}^2$ as follows:

   (a) $(3, 2) \in M$,
   (b) for all $(x, y) \in M$, $(3x - 2y, x) \in M$,
   (c) nothing else belongs to M.

   Use structural induction to prove that $\forall (x, y) \in M, \exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$.

   Note: The point of this problem is to practice structural induction, so **you will get no credit if you prove it using another form of induction**.