FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 8

CALCULATING YIELDS

If the investor is not subject to taxation the yield is referred to as a **gross yield**. If the investor is subject to taxation the yield is referred to as a **net yield.**

As stated previously, the internal rate of return on a security is sometimes referred to as the **yield to redemption** or the **redemption yield**.

The redemption yield on a security can be found by using the methods described in earlier lectures (solving a quadratic equation, linear interpolation, etc…). There is a direct relationship between the price and yield of a bond.

If the redemption amount is equal to the face value or nominal amount of the bond, then $C = F$ and the following results hold:

$P = F \;\Leftrightarrow$ the effective half-yearly yield equals the coupon rate per half-year $j = r$
$P > F \;\Leftrightarrow\; j < r$
$P < F \;\Leftrightarrow\; j > r$

Remember that the lower the interest rate, the higher the present value, and vice versa. Since the price of a bond is the present value of the future payments, the higher the price, the lower the yield.

Note: the formulae above assume that no income tax is payable. If we assume income tax is payable on the coupons at the rate t_I, then

$$P = F \Leftrightarrow j = r(1-t_I)$$
$$P > F \Leftrightarrow j < r(1-t_I)$$
$$P < F \Leftrightarrow j > r(1-t_I)$$

If $P = F$, the bond is said to be bought at par.
If $P > F$, the bond is said to be bought at a premium.
If $P < F$, the bond is said to be bought at a discount.

In the cases where the redemption amount is not equal to the face value it is useful to define a modified coupon rate g where $g = \dfrac{Fr}{C}$. This effectively expresses the coupon rate as a percentage of the redemption amount rather than the face value.

In this situation the following results hold:

$$P = C \iff j = g$$
$$P > C \iff j < g$$
$$P < C \iff j > g$$

Again, if income tax is included, g in the formulae above must be multiplied by $(1 - t_I)$.

Note that when $C = F$, $g = \frac{Cr}{C} = r$, and we get the earlier formulae relating P and F.

EXAMPLE

Find the gross nominal yield convertible half-yearly obtained by an investor who pays $12,000 for $10,000 nominal of a six-year bond with half-yearly coupon payments of 13% pa. Assume that the bond is redeemed at par.

Solution

We need to find the nominal interest rate $i^{(2)}$ that satisfies the equation of value:

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n$$

$$12{,}000 = 10{,}000(0.065) \cdot a_{\overline{12}|j} + 10{,}000 \cdot v_j^{12}$$

where $j = \frac{i^{(2)}}{2}$.

Since j is the half-yearly effective interest rate, it follows that $2j$ is the nominal yield convertible half-yearly.

If the purchase price was $10,000, the half-yearly redemption yield would be $j = 6.5\% = r$ ($P = F \iff j = r$). This corresponds to a nominal yield convertible half yearly of 13%, and a redemption yield of just over 13%.

Since the investor has paid $12,000 which is more than the face value (ie. $P > F$), the half-yearly redemption yield j will be less than the coupon rate r =0.065.

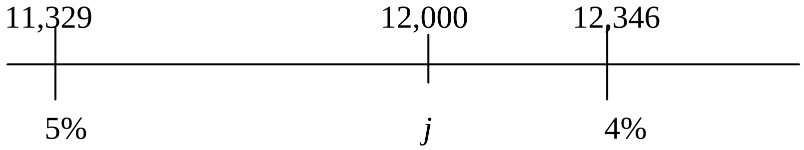
Starting with an initial guess of $j = 5\%$ gives:

At 5%: $10{,}000(0.065) \cdot a_{\overline{12}|j} + 10{,}000 \cdot v_j^{12} = \$11{,}329$

Since the investor is paying more than this, we need to try a lower rate:

At 4%: $10{,}000(0.065) \cdot a_{\overline{12}|j} + 10{,}000 \cdot v_j^{12} = \$12{,}346$

We can approximate j by interpolating (linearly).



The half-yearly redemption yield is

$$j \cong 5\% + \frac{12{,}000 - 11{,}329}{12{,}346 - 11{,}329}(4\% - 5\%) = 4.34\%$$

The corresponding gross nominal yield $i^{(2)}$ is $2j = 8.68\%$

In this example, the half-yearly redemption yield of 4.34% is less than the half-yearly coupon rate of 6.5% because the investment was bought at \$12,000, which is a higher price than the redemption payment of \$10,000 $(P > F)$.

THE EFFECT OF THE TERM TO REDEMPTION ON THE YIELD

When the price is not equal to the redemption value ($P \neq C$), the yield depends on the value of n. In the previous example, the yield would increase as n increases, since the loss of capital (corresponding to the extra \$2,000 in the purchase price over the nominal amount of \$10,000) would spread over a longer period.

If n decreased, the yield in this example would decrease, since the loss of capital is spread over a shorter period.

When the purchase price is more than the redemption price $(P > C)$:
• as n increases the yield increases.
• as n decreases the yield decreases.

Conversely, if we had a purchase price less than the redemption price ($P < C$), then the opposite would apply. We are making a capital gain (corresponding to a savings in the purchase price), so:

When the purchase price is less than the redemption price $(P < C)$:
• as n increases the yield decreases.
• as n decreases the yield increases.

For example, if an investor was choosing between two bonds based on redemption yield, and if the bonds were identical except for their dates of maturity, then:

- If $P > C$ the investor is making a capital loss, and will obtain a higher yield by purchasing the bond which is redeemed later.
- If $P < C$ the investor is receiving a capital gain on redemption, and will obtain a higher yield by purchasing the bond which is redeemed earlier.

EXAMPLE
An investor purchases a $100 zero-coupon bond for $80. Calculate the redemption yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution
If the purchase price is $80, find i that solves the equation:
$80 = 100v^n$

(a) If $n = 5$, $i = 4.6\%\,pa$
(b) If $n = 10$, $i = 2.3\%\,pa$

Since there is a capital gain, the sooner that the bond is redeemed, the higher the redemption yield.

<u>FINDING YIELDS WHEN TAX IS PAYABLE</u>

Finding the net yield when an investor is liable for capital gains tax and income tax is best achieved by writing down the equation in terms of the net receipts.

If the price P' and the redemption amount C of the bond are known, then the net yield can be found by solving the equation:

$$\begin{aligned} P' &= Fr(1-t_I)\cdot a_{\overline{n}|j} + C\cdot v_j^n - t_C(C-P')v_j^n \\ &= Fr(1-t_I)\cdot a_{\overline{n}|j} + \left[C - t_C(C-P')\right]v_j^n \end{aligned}$$

EXAMPLE
A bond with face value $1,000 will be redeemed at par after 10 years. The bond pays half-yearly coupons of 6% pa. An investor purchases the bond for $800. The investor pays income tax and capital gains tax at the rates of 40% and 30% respectively. What is the investor's net redemption yield?

Solution

We have a half-yearly coupon rate of $r = \dfrac{0.06}{2} = 0.03$ and $n = 20$ coupon payments. Since $P < C$ there is a capital gain. The equation of value is:

$$P' = Fr(1-t_I)\cdot a_{\overline{n|}j} + \left[C - t_C(C-P')\right]v_j^n$$
$$800 = 1{,}000(0.03)(1-0.4)\cdot a_{\overline{20|}j} + \left[1{,}000 - 0.3(1{,}000-800)\right]v_j^{20}$$
$$800 = 18\cdot a_{\overline{20|}j} + 940v_j^{20}$$

We could either solve this equation for j and then find $i = (1+j)^2 - 1$, or alternatively, we can write the equation in terms of i and solve:

$$800 = 36\cdot a_{\overline{10|}i}^{(2)} + 940v_i^{10}$$

Note that the income each year of \$36 is 4.5% of the purchase price. Since there is a gain on redemption, the net redemption yield is clearly greater than 4.5%.

Try $i = 5\%$: $36\cdot a_{\overline{10|}i}^{(2)} + 940v_i^{10}$ = \$858.49

Try $i = 6\%$: $36\cdot a_{\overline{10|}i}^{(2)} + 940v_i^{10}$ = \$793.77

Interpolate to find the redemption yield i:

$$i \cong 6\% + \frac{800-793.77}{858.49-793.77}(5\%-6\%) = 5.90\%$$

CALLABLE BONDS - OPTIONAL REDEMPTION DATES

Sometimes a bond is issued without a fixed redemption date. In such cases the terms of issue may provide that the borrower or lender can redeem the security at the borrower's or lender's option at any interest date on or after a specified date, or between two specified dates.

If the interest rate payable on the security is high relative to market rates, it will be cheaper for the borrower to repay the loan and borrow from elsewhere. Conversely, if the interest rate payable is relatively low, it will be cheaper for the borrower to allow the loan to continue.

For a lender, if the interest received on the loan is high relative to market rates, it will be less profitable for the lender to redeem the loan and invest elsewhere. Conversely, if the interest rate received is relatively low, it will be more profitable for the lender to redeem the loan and invest elsewhere.

The discussion that follows assumes that a bond is redeemable at the option of the borrower.

When bonds are issued with optional redemption dates it is prudent for an investor to assume that the bond will be redeemed at the time considered to be most favourable by the bond issuer (the borrower). In other words, the bond should be assumed to be redeemed at the time when the bond issuer will obtain the greatest yield, which corresponds to the time at which the lender will obtain the lowest yield. The time depends on how the market moves in the future.

EXAMPLE
A bond with a face value of $1,000 and coupons of 10% pa payable half-yearly is issued with the condition that redemption at par can take place at the option of the issuer (borrower) on any coupon date between 12 and 15 years from the issue date. An investor wishes to generate a nominal yield of at least 12% convertible half-yearly.

How much should the investor offer to pay for the bond?

Solution
To generate a nominal yield of 12% convertible half-yearly the price that should be paid is:

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n = \$1000(0.050) \cdot a_{\overline{n}|0.06} + \$1000 \cdot v_{0.06}^n$$

where n is the number of coupon payments. n can range from 24 to 30.

For $n = 24$, $P = \$1000(0.050) \cdot a_{\overline{24}|0.06} + \$1000 \cdot v_{0.06}^{24} = 874.5$

For $n = 25$, $P = \$1000(0.050) \cdot a_{\overline{25}|0.06} + \$1000 \cdot v_{0.06}^{25} = 872.2$

…

For $n = 29$, $P = \$1000(0.050) \cdot a_{\overline{29}|0.06} + \$1000 \cdot v_{0.06}^{29} = 864.1$

For $n = 30$, $P = \$1000(0.050) \cdot a_{\overline{30}|0.06} + \$1000 \cdot v_{0.06}^{30} = 862.4$

The prices range from $862.4 to $874.5.

Since the bond issuer can redeem this bond at any time between the 24th and 30th coupon repayment, ***the investor should offer the lowest price***. Support for this is shown below.

If the investor pays the maximum price of $874.5 (based on redemption at $n = 24$) and the bond is redeemed by the issuer at $n > 24$, then the actual nominal yield will be less than 12% pa. For example, if the bond is redeemed at $n = 25$, then the yield to the investor can be found by solving:

$\$874.5 = \$1000(0.050) \cdot a_{\overline{25}|j} + \$1000 \cdot v_j^{25} \Rightarrow i^{(2)} = 11.96\%$. ie. the investor will receive a nominal yield of 11.96%, which is less than the required 12%.

In fact, if the investor pays *any* amount greater than the minimum price, then there is a chance that the investment will not produce the required yield. For example, if the investor pays the second-lowest price of $864.1 (based on redemption at $n = 29$) and the bond is redeemed by the issuer at $n = 30$, then the actual nominal yield to the investor will be less than 12%:

$$\$864.1 = \$1000(0.050) \cdot a_{\overline{30}|j} + \$1000 \cdot v_j^{30} \Rightarrow i^{(2)} = 11.97\% .$$

If the investor pays the minimum price and the actual redemption date is other than the one on which the minimum price is based, then the investor will earn a yield greater than the minimum desired yield. For this example, if the investor pays the minimum price of $862.4 (based on redemption at $n = 30$) and the bond is redeemed by the issuer at $n < 30$, then the actual nominal yield will be greater than 12%. For example, if the bond is redeemed by the issuer at $n = 29$, then

$$\$862.4 = \$1000(0.050) \cdot a_{\overline{29}|j} + \$1000 \cdot v_j^{29} \Rightarrow i^{(2)} = 12.03\%$$

Maximum yield for the issuer corresponds to minimum yield for the investor. In the above example there is a capital gain on redemption ($P < F$) so the minimum yield for the investor (and the maximum yield for the bond issuer) is obtained if redemption takes place at the latest possible date. Since the investor should assume that the bond issuer is after the maximum possible yield, which corresponds with redemption at the latest possible date, the investor should price the bond as if it were to be redeemed at the latest possible date.

Conversely if a bond is bought at a premium and there is a capital loss on redemption (ie. $P > F$) then the minimum yield to the investor (and the maximum yield to the bond issuer) will be obtained if redemption takes place at the earliest possible date. In this case, the investor should assume that the bond will be redeemed at the earliest date.

In summary,

When a bond is to be redeemed at the option of the issuer:

- For a bond bought at a discount, if an investor requires a particular minimum yield, the price paid should be based on the ***latest*** optional redemption date.
- For a bond bought at a premium, if an investor requires a particular minimum yield, the price paid should be based on the ***earliest*** optional redemption date.

In both cases, the minimum price is paid by the investor. If something more than the minimum price is paid, the investor runs the risk of having redemption occur at a time which is to the investor's disadvantage.

EXAMPLE
A fixed interest bond with a coupon of 8% pa payable half yearly in arrears can be redeemed at par at the option of the issuer at any time between 10 and 15 years from the date of issue. What price should an investor subject to tax at 25% on income, who wishes to obtain a net nominal yield of at least 7% convertible half-yearly, pay for $100 nominal of this bond?

Solution
We first need to determine whether or not there is a gain on redemption. ie. whether the bond is purchased at a discount or at a premium.

Using the formula introduced earlier in these lecture notes, $P < F \iff j > r(1-t_1)$.
$j = \dfrac{0.07}{2}$ and $r = \dfrac{0.08}{2} = 0.04 \Rightarrow r(1-t_1) = 0.04(1-0.25) = 0.03$.
Since $0.035 > 0.03 \Rightarrow j > r(1-t_1)$

It follows that $P < F$ and, therefore, in order to achieve a net nominal yield of 7%, the bond must be purchased at a discount.

Since there is a gain on redemption, the amount paid should be based on the latest optional redemption date. Therefore, the bond price is:

$$P = Fr(1-t_1)\cdot a_{\overline{n}|j} + C\cdot v_j^n = 100(0.04)(0.75)\cdot a_{\overline{30}|0.035} + 100\cdot v_{0.035}^{30} = \$90.80.$$

BONDS WITH INFLATION LINKED PAYMENTS

We now consider the case when bond payments are subject to inflation.

If we have a constant annual future rate of inflation y, then the bond equation reduces to:

$$P = \sum_{p=1}^{n} Fr(1+y)^{p/2} v_j^p + C(1+y)^{n/2} v_j^n,$$

(where we have n half-year periods and j is an effective half-yearly yield)

or in terms of an annual effective redemption yield i,

$$P = \sum_{p=1}^{n} Fr(1+y)^{p/2} v_i^{p/2} + C(1+y)^{n/2} v_i^{n/2} = \sum_{p=1}^{n} Fr\cdot v_{i'}^{p/2} + C\cdot v_{i'}^{n/2}$$

where, $v_{i'} = (1+y)v_i \Rightarrow i' = \dfrac{i-y}{1+y}$, where the real rate of interest i' is the rate of interest after allowing for the effect of inflation.

Using a half-yearly effective *real* rate of interest $j' = (1+i')^{1/2} - 1$, the bond equation can be written

$$\boxed{P = Fr \cdot a_{\overline{n}|j'} + C \cdot v_{j'}^{n}}$$

If we write the bond equation in terms of the annual *real* rate of interest i', then:

$$\boxed{P = 2Fr \cdot a_{\overline{n/2}|i'}^{(2)} + C \cdot v_{i'}^{n/2}}$$

EXAMPLE
Find the price that an investor should pay for an inflation-linked bond of \$100 nominal redeemable at par, if future inflation is assumed to be 5.25% per annum. Assume that the investor requires a gross redemption yield of 10% per annum and coupons of 4.2% per annum are payable half-yearly for 16 years.

Solution
The gross redemption yield $i = 10\%$ and we have a constant future rate of inflation $y = 5.25\%$. Therefore, the real rate of interest is $i' = \dfrac{i-y}{1+y} = \dfrac{0.0475}{1.0525} = 4.5131\%$.

The present value of the payments is:

$$P = Fr \cdot a_{\overline{n}|j'} + C \cdot v_{j'}^{n} = \$100\left(\frac{0.042}{2}\right) \cdot a_{\overline{32}|j'} + \$100 \cdot v_{j'}^{32}$$

where $j' = (1+i')^{1/2} - 1 = (1.045131)^{1/2} - 1 = 0.022316$

$$\Rightarrow P = \$100(0.021) \cdot a_{\overline{32}|j'} + \$100 \cdot v_{j'}^{32} = \$97.01$$

Alternatively, we could have used $P = 2Fr \cdot a_{\overline{n/2}|i'}^{(2)} + C \cdot v_{i'}^{n/2}$:

$$P = 2Fr \cdot a_{\overline{n/2}|i'}^{(2)} + C \cdot v_{i'}^{n/2} = \$100(0.042) \cdot a_{\overline{16}|i'}^{(2)} + \$100 \cdot v_{i'}^{16} = \$97.01$$

VALUATION OF SHARES AND PROPERTY

The valuation of the price to be paid for shares and property follows much the same procedure as for fixed interest securities; ie. it is calculated by reference to the present value of expected future cash flows.

EXAMPLE
The current rent of a property is \$5,000 per month payable in arrears. This is expected to increase by 5% per annum in future, commencing from the 13th rent payment. If the property is assumed to be held indefinitely, calculate the purchase price for a buyer who wishes to obtain an effective yield of 7% p.a.

Solution
Since the property is assumed to be held indefinitely, we don't need to make any assumptions about the future sale price of the property. The price will simply be the present value of the rental payments:

$$P = 5000(v^{1/12} + v^{2/12} + ... + v^{12/12}) + 5000(1.05)(v^{13/12} + v^{14/12} + ... + v^{24/12}) + ...$$
$$= 5000(v^{1/12} + v^{2/12} + ... + v^{12/12})(1 + 1.05v + 1.05^2 v^2 + ...)$$
$$= 5000(12)a^{(12)}_{\overline{1}|0.07}\ddot{a}_{\overline{\infty}|i'}$$

where $v' = \dfrac{1.05}{1.07}$ and $i' = \dfrac{1.07}{1.05} - 1$

$$P = 60,000 \times 0.964215 \times \frac{\frac{1.07}{1.05}}{\frac{1.07}{1.05} - 1} = \$3.095\text{m}$$