Homework Assignment #5

MAT 335 – Chaos, Fractals, and Dynamics – Fall 2013

PARTIAL SOLUTION

Chapter 9.5. If $\mathbf{s} \in \Sigma$ satisfies $d[\mathbf{s}, \mathbf{0}] = \frac{1}{2}$, then

$$\sum_{i=0}^{\infty} \frac{s_i}{2^i} = \frac{1}{2},$$

so $s_0 = 0$. We then have two cases:

Case 1. If $s_1 = 1$, then $s_i = 0$, for all $i = 2, 3, \dots$

Case 2. If $s_1 = 0$, then

$$\frac{1}{2} = \sum_{i=2}^{\infty} \frac{s_i}{2^i} \leqslant \sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{1}{4} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2},$$

so $s_i = 1$ for all $i = 2, 3, \dots$.

We conclude that there are only two sequences at distance $\frac{1}{2}$ from $\mathbf{0}$, $\mathbf{s} = (01000 \cdots)$ and $\mathbf{t} = (00111 \cdots)$.

Chapter 10.6. The set T_1 is not dense in Σ. For example, the sequence $\mathbf{1} = (1111 \cdots) \in \Sigma$ and a sequence $\mathbf{s} \in T_1$ satisfies $\mathbf{s} = (s_0 s_1 s_2 s_3 0 s_5 s_6 \cdots)$, so

$$d[\mathbf{s}, \mathbf{1}] = \sum_{i=0}^{\infty} \frac{|s_i - 1|}{2^i} \geqslant \frac{|s_4 - 1|}{2^4} = \frac{1}{2^4} = \frac{1}{16}.$$

So a sequence of elements of T_1 cannot converge to $\mathbf{1}$, since elements of T_1 are always at a distance of at least $\frac{1}{16}$ from $\mathbf{1}$.

Chapter 10.8. The set T_3 is dense in Σ. Let $\mathbf{s} = (s_0 s_1 s_2 s_3 \cdots) \in \Sigma$ be arbitrary.
Then construct a sequence of elements of T_3:

$$\mathbf{s}_n = (s_0 s_1 \cdots s_n 0000 \cdots) \in T_3.$$

Since $\mathbf{s}$ and $\mathbf{s}_n$ have the same first $n+1$ elements, the Proximity Theorem states that

$$d[\mathbf{s}, \mathbf{s}_n] \leqslant \frac{1}{2^n} \to 0.$$

This proves that T_3 is dense in Σ.

Chapter 10.21. To prove that the tent map is chaotic in $[0,1]$, we show that it is conjugate to the V map, which we proved in lectures to be chaotic in $[-2,2]$.
Consider the conjugacy $h:[0,1]\to[-2,2]$ defined* by $h(x)=2-4x$. This function is linear, so it is one-to-one, it is easy to prove that it is onto and it is continuous with continuous inverse.
Moreover,

$$(h\circ T)(x)=\begin{cases} h(2x) & \text{if } x\leqslant \frac{1}{2} \\ h(2-2x) & \text{if } x>\frac{1}{2}\end{cases} = \begin{cases} 2-8x & \text{if } x\leqslant \frac{1}{2} \\ 8x-6 & \text{if } x>\frac{1}{2}\end{cases}$$

$$(V\circ h)(x)=2\big|h(x)\big|-2=|4-8x|-2=\begin{cases} 2-8x & \text{if } 4-8x\geqslant 0 \\ 8x-6 & \text{if } 4-8x<0\end{cases} = \begin{cases} 2-8x & \text{if } x\leqslant \frac{1}{2} \\ 8x-6 & \text{if } x>\frac{1}{2}\end{cases}$$

So $h\circ T=V\circ h$.
This proves that T and V are conjugates, and since V is chaotic, the tent map T is chaotic in $[0,1]$.

Chapter 10.22. Similar to the previous question with the conjugacy $h(x)=\cos(\pi x)$.

*The simplest functions that map $[0,1]$ into $[-2,2]$ are linear functions: one is $y=4x-2$, the other $y=2-4x$. The second is our conjugacy because it "inverts" the tent map into the V map.