CSC236 2015 Winter, Assignment 1

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1.Prove by Induction that $1 + mn \leqslant (1 + m)^n$ for all natural numbers m and n.

PROOF:
Let $P(m, n)$ be the predicate defined as follows:

$$P(m, n) : 1 + mn \leqslant (1 + m)^n$$

We are going to prove that $P(m, n)$ is true for all natural numbers m and n.

BASE CASE: $m = 0$, $n = 0$, $1 + 0 \times 0 = 1 \leqslant (1 + 0)^0 = 1$. Thus $P(0, 0)$ holds.

INDUCTIVE STEP: Note that this problem should be done with a *multidimensional induction*. We want to do induction on m and n separately.

INDUCTION ON n: Let $n = j$ be arbitrary natural number, and assume that $P(m, j)$ holds, and we want to show $P(m, j + 1)$ also holds.

Note that we have inductive hypothesis: $1 + mj \leqslant (1 + m)^j$.

$1 + m(j + 1) = 1 + mj + m \leqslant (1 + m)^j + m \leqslant (1 + m)^j + m(1 + m)^j = (1 + m)^j (1 + m) = (1 + m)^{j+1}$

Hence $P(m, j + 1)$ holds.

INDUCTION ON m: LeT $m = i$ be arbitrary natural number, and assume that $P(i, n)$ holds, and we want to show $P(i + 1, n)$ also holds. But now the inductive hypothesis is $1 + in \leqslant (1 + i)^n$.

We need to use *Binomial Theorem*[1] down in the proof:

So,

$$1 + (i + 1)n = 1 + in + n \leqslant (1 + i)^n + n \leqslant 1 + n(i + 1) + \ldots + (i + 1)^n = (1 + (i + 1))^n$$

Therefore, $P(i + 1, n)$ holds. Then we can combine the two inductions we made previously, predicate $P(m, n)$ holds for all natural numbers m, n.

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2.Let the sequence r be defined by:

$$r_1 = 1,$$

$$r_n = 1 + r_{\lfloor\sqrt{n}\rfloor}, n \geqslant 2.$$

Prove by Induction that r_n is $O(log_2(log_2(n)))$.

PROOF:
Firstly, we do the following observation:

n	$\log_2(\log_2(n))$	r_n	$r_n \leqslant 4\log_2(\log_2(n))$
2	0	$1 + r_{\lfloor\sqrt{2}\rfloor} = 1 + r_1 = 1 + 1 = 2$	/
3	0.6644...	$1 + r_{\lfloor\sqrt{3}\rfloor} = 1 + r_1 = 1 + 1 = 2$	$2 \leqslant 2.6576$
4	1	$1 + r_{\lfloor\sqrt{4}\rfloor} = 1 + r_2 = 1 + 2 = 3$	$3 \leqslant 4$
5	1.2153...	$1 + r_{\lfloor\sqrt{5}\rfloor} = 1 + r_2 = 1 + 2 = 3$	$3 \leqslant 4.8612$
6	1.3701...	$1 + r_{\lfloor\sqrt{6}\rfloor} = 1 + r_2 = 1 + 2 = 3$	$3 \leqslant 5.4804$
...	...	...	...
15	1.9660...	$1 + r_{\lfloor\sqrt{15}\rfloor} = 1 + r_3 = 1 + 2 = 3$	$3 \leqslant 7.864$
16	2	$1 + r_{\lfloor\sqrt{16}\rfloor} = 1 + r_4 = 1 + 3 = 4$	$4 \leqslant 8$
...	...	...	
255	2.9990...	$1 + r_{\lfloor\sqrt{255}\rfloor} = 1 + r_{15} = 1 + 3 = 4$	$4 \leqslant 11.996$
256	3	$1 + r_{\lfloor\sqrt{256}\rfloor} = 1 + r_{16} = 1 + 4 = 5$	$5 \leqslant 12$
257	3.0010...	$1 + r_{\lfloor\sqrt{257}\rfloor} = 1 + r_{16} = 1 + 4 = 5$	$5 \leqslant 12.004$
...	...	...	...
65536	4	$1 + r_{\lfloor\sqrt{65536}\rfloor} = 1 + r_{256} = 1 + 5 = 6$	$6 \leqslant 16$
...	...	...	...
4294967295	4.9999...	$1 + r_{\lfloor\sqrt{4294967295}\rfloor} = 1 + r_{65535} = 1 + 5 = 6$	$6 \leqslant 20$
4294967296	5	$1 + r_{\lfloor\sqrt{4294967296}\rfloor} = 1 + r_{65536} = 1 + 6 = 7$	$7 \leqslant 20$

Since r_1 is fixed, and $\log_2(\log_2 1)$ is meaningless. So we consider the following predicate $P(n)$:

$$\forall n_0 \in \mathbb{N}, \exists c \in \mathbb{R}^+, \forall n \geqslant n_0, \text{ such that } r_n \leqslant c\log_2(\log_2(n))$$

BASE CASE: Let $c = 4$, when $n = 3$, $r_3 = 2 \leqslant 4\log_2(\log_2(3)) = 2.6576$. So the predicate $P(n)$

holds.

Why 3? Instead of 2? This is because we notice that if $n = 2$, $\log_2(\log_2(1)) = 0$, so no matter what value c is, $c\log_2(\log_2(n))$ is 0.

INDUCTIVE STEP: Let $n = k \geqslant 3$ be an arbitrary natural number, and assume $P(k)$ holds, which is our inductive hypothesis, then we want to show $P(k+1)$ holds as well, i.e.
$r_{k+1} \leqslant 4\log_2(\log_2(k+1))$.

Since $P(k)$ holds, .

CASE I: if $r_{k+1} = r_k + 1$.
First we notice that $r_{k+1} = r_k + 1$ only happens when $k+1 = 2^{2^m} = 4^m, m = 2^j$, where j is a natural number.

Assume $r_{k+1} = r_{4^m} = 1 + r_{2^m} = 1 + r_k$

$$r_k = r_{2^m} \leqslant 4\log_2\log_2(2^m) = 4\log_2(m) \text{ by IH.}$$

Since $1 \leqslant 4\log_2(2m) - 4\log_2 m = 4\log_2 2 = 4$, then

$$1 + 4\log_2 m \leqslant 4\log_2(2m)$$

$$r_{4^m} = 1 + 4\log_2 m \leqslant 4\log_2(2m) = 4\log_2\log_2(2^{2m}) = 4\log_2\log_2(4^m)$$

$$\text{i.e., } r_{k+1} \leqslant 4\log_2\log_2(k+1), P(k+1) \text{ holds.}$$

CASE II: if $r_{k+1} = r_k$, then:

$$r_{k+1} = r_k \leqslant 4\log_2(\log_2(k)) \leqslant 4(\log_2(\log_2(k+1))),$$

since logarithm function is monotonically increasing. Thus $P(k+1)$ holds.

Therefore, in both cases, $P(k+1)$ holds, it is true for all natural number $k \geqslant 3$.

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3.Consider the number of binary trees of heigh h, where we measure height by number of levels. For example, the empty tree is the only tree of height 0, a single-node tree is the only tree of height 1, and there are 3 trees of height 2.

- (a) Give a recursive algebraic formula for a sequence b, and prove for all natural numbers h that b_h is the number of binary trees of height h.
- (b) Let the sequence a be defined by:

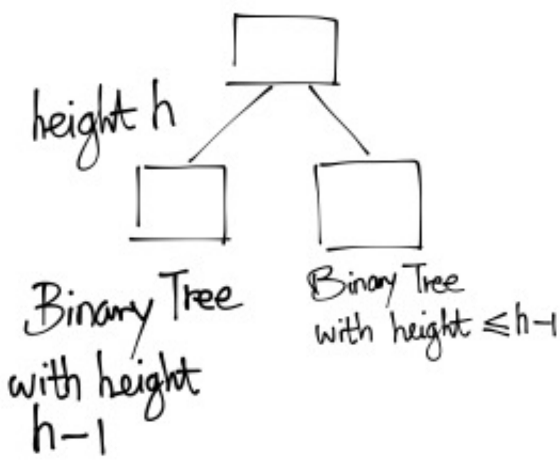
$$a_0 = 0,$$

$$a_{n+1} = a_n^2 + 1, n \in \mathbb{N}.$$

Prove that $b_{h+1} = a_{h+1}^2 - a_h^2$ for all natural numbers h.

SOLUTION:

(a) **GENERAL IDEA**: [PROBLEM DO WE NEED TO REALLY USE INDUCTION HERE]



binary tree with height h

Since we already know the number of binary trees of height $h-1$, which is b_{h-1}, for a binary tree, say if the left subtree has b_{h-1} possibilities, then right tree could be a subtree of height $h-1$, or any natural number no greater than $h-1$, i.e., the right subtree has b_{h-1} possibilities to be a subtree with height $h-1$, b_{h-2} possibilities to be a subtree with height $h-2$,... and even as an empty subtree with height 0. So in total the right subtree has $b_0 + b_1 + \ldots + b_{h-1}$ possibilities. And we can always swap left subtree with right subtree, so there are $2b_{h-1}(b_0 + b_1 + \ldots + b_{h-1})$ possibilities. But note that for the scenario in which both left and right subtrees are of height $n-1$, we double-count this in the previous formula, so we have to subtract one here, which is b_{h-1}^2.

So finally we claim that for a certain height h, the number of possible binary trees b_h is given by:

$$b_h = b_{h-1}(2(b_0 + b_1 + \ldots + b_{h-2}) + b_{h-1}) = 2b_{h-1}\sum_{i=0}^{h-1} b_i - b_{h-1}^2 \text{ where natural number } n \geqslant 2.$$

Check by different h values, and note that $b_0 = b_1 = 1$

- When $h = 2$, $b_2 = 2 \times b_1(b_0 + b_1) - b_1^2 = 3$
- When $h = 3$, $b_3 = 2 \times b_2(b_0 + b_1 + b_2) - b_2^2 = 30 - 9 = 21$
- When $h = 4$, $b_4 = 2 \times b_3(b_0 + b_1 + b_2 + b_3) - b_3^2 = 2 \times 21(1 + 1 + 3 + 21) - 21^2 = 651$

(b) Observe:

h	a_h	a_{h+1}	b_{h+1}
0	0	1	$1 = 1^2 - 0^2$
1	1	2	$3 = 2^2 - 1^2$
2	2	5	$21 = 5^2 - 2^2$
3	5	26	$651 = 26^2 - 5^2$
4	26	677	$457653 = 677^2 - 26^2$
…	…	…	…

We have the following as a predicate $P(h)$: for natural number h, $b_{h+1} = a_{h+1}^2 - a_h^2$ where b_{h+1} is given by the formula in problem (a) and a_h is given by $a_h = a_h^2 + 1$.

BASE CASE: when $h = 0$, $b_{0+1} = a_{0+1}^2 - a_0^2 = 1 - 0 = 1$. Predicate $P(0)$ holds.

INDUCTIVE STEP: let $h = k$ is a natural number, assume $P(k)$ holds and we want to sure $P(k + 1)$ holds, i.e $b_{k+2} = a_{k+2}^2 - a_{k+1}^2$.

$$RHS = (a_{k+1}^2 + 1)^2 - a_{k+1}^2 = a_{k+1}^4 + 2a_{k+1}^2 + 1 - a_{k+1}^2 = a_{k+1}^4 + a_{k+1}^2 + 1$$

Use the formula from (a):
$LHS = 2b_{k+1} \sum_{i=0}^{k+1} b_i - b_{k+1}^2 = 2(a_{k+1}^2 - a_k^2)(1 + a_1^2 - a_0^2 + a_2^2 - a_1^2 + \ldots + a_{k+1}^2 - a_k^2) - (a_{k+1}^2 - a_k^2)^2$
$= 2(a_{k+1}^2 - a_k^2)(1 + a_{k+1}^2) - (a_{k+1}^2 - a_k^2) = a_{k+1}^4 + a_{k+1}^2 + (a_{k+1}^2 - 2a_k^2 - a_k^4)$
$= a_{k+1}^4 + a_{k+1}^2 + 1$ # by squaring both sides of $a_{k+1} = a_k^2 + 1$

Hence $RHS = LHS$, i.e. $P(k + 1)$ holds as desired.
Therefore, $b_{h+1} = a_{h+1}^2 - a_h^2$ holds for all natural numbers h.

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1. Binomial Theorem. For $x, y \in \mathbb{R}(x + y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$. ↩