Mike's questions - MAT 327 - Summer 2013

When I was an undergrad and took this course we had weekly assignments. Here is that list of problems.

I have also marked the questions I did not submit a solution for. Sometimes I didn't have time to complete them, sometimes I didn't understand the question and sometimes I just couldn't solve the problem. Some of these are really embarassing.

These questions are all taken from C. Wayne Patty's "Foundations of Topology".

1 Topological Spaces

1. Let d be the usual metric and let ρ be the square metric on $\mathbb{R}^2$. Prove that:
   (a) $\rho(a, b) \leq d(a, b)$ for all $a, b \in \mathbb{R}^2$;
   (b) $d(a, b) \leq \sqrt{2} \cdot \rho(a, b)$ for all $a, b \in \mathbb{R}^2$.
2. Let X be an infinite set and let $\mathcal{T} := \{ U \in \mathcal{P}(X) : U = \emptyset \text{ or } X \setminus U \text{ is finite} \}$. Prove that $\mathcal{T}$ is a topology on X.
3. Give an example of a set X and topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X such that $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X.
4. Let $\{ \mathcal{T}_\alpha : \alpha \in \Lambda \}$ be a collection of topologies on a set X. Prove that there is a unique topology $\mathcal{T}$ on X such that: (1) for each $\alpha \in \Lambda$, $\mathcal{T}$ is finer than $\mathcal{T}_\alpha$, and (2) if $\mathcal{T}'$ is a topology on X that is finer than $\mathcal{T}_\alpha$ for each $\alpha \in \Lambda$, then $\mathcal{T}$ is coarser than $\mathcal{T}'$.
5. Let $\{ \mathcal{T}_\alpha : \alpha \in \Lambda \}$ be a collection of topologies on a set X. Prove that there is a unique topology $\mathcal{T}$ on X such that: (1) for each $\alpha \in \Lambda$, $\mathcal{T}$ is coarser than $\mathcal{T}_\alpha$, and (2) if $\mathcal{T}'$ is a topology on X that is coarser than $\mathcal{T}_\alpha$ for each $\alpha \in \Lambda$, then $\mathcal{T}$ is finer than $\mathcal{T}'$.

2 Basis for a Topology

1. Let $\mathcal{T}$ be the usual topology on $\mathbb{R}$. Prove that $\mathcal{B} = \{ (a, b) : a < b \text{ and } a, b \in \mathbb{Q} \}$ is a countable basis for $\mathcal{T}$.
2. Let $\mathcal{T}$ be the Sorgenfrey topology on $\mathbb{R}$. Prove that $(\mathbb{R}, \mathcal{T})$ is first countable, but not second countable.

3. Let $\mathcal{B}$ be the collection of all intervals of the form $[a, b)$, where $a < b$ and a and b are rational. Prove that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$. Is $\mathcal{T}$ the Sorgenfrey topology on $\mathbb{R}$?

4. Let X be the set of all functions that map $[0, 1]$ into $[0, 1]$. For each subset A of $[0, 1]$, let $B_A = \{ f \in X : f(x) = 0, \forall x \in A \}$. Prove that $\mathcal{B} = \{ B_A : A \subseteq [0, 1] \}$ is a basis for a topology on X.

5. Let $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ be topological spaces. Let $\mathcal{B} = \{ U \times V : U \in \mathcal{T}_1, V \in \mathcal{T}_2 \}$. Prove that $\mathcal{B}$ is a basis for a topology on $X \times Y$.

3 Closed Sets, Closures, and Interiors of Sets

1. Let $\mathcal{T}$ be the usual topology on $\mathbb{R}$ and let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b)$ is neither open nor closed.

2. Let $\mathcal{T}$ be the Sorgenfrey topology on $\mathbb{R}$ and let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b)$ is both open and closed.

3. Let $X = \{ 1, 2, 3 \}$ and let $\mathcal{T} = \{ X, \emptyset, \{1\}, \{1, 2\}, \{1, 3\} \}$. Then $\mathcal{T}$ is a topology on X.
   (a) List the closed subsets of $(X, \mathcal{T})$.
   (b) Find $\overline{\{1\}}$.
   (c) Find $\overline{\{2\}}$.
   (d) Find $\mathrm{int}(\{2, 3\})$.
   (e) Find $\mathrm{boundary}(\{2, 3\})$.

4. Let $\mathcal{T}$ be the finite complement topology on $\mathbb{R}$, and let $A = [0, 1]$. Find $\overline{A}$ and $\mathrm{int}(A)$ and prove your answers.

5. Let $\mathcal{T} = \{ U \in \mathcal{P}(\mathbb{R}) : 0 \notin U \text{ or } U = \mathbb{R} \}$.
   (a) Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$.
   (b) Describe the closed subsets of $\mathbb{R}$.
   (c) Find $\overline{\{1\}}$.

6. Let A be a subset of a topological space $(X, \mathcal{T})$. Prove that:
   (a) $\mathrm{int}(A), \mathrm{int}(X \setminus A)$ and $\mathrm{boundary}(A)$ are pairwise disjoint sets whose union is X.
   (b) $\mathrm{boundary}(A)$ is a closed set.
   (c) $\overline{A} = \mathrm{int}(A) \cup \mathrm{boundary}(A)$.

(d) boundary$(A) = \emptyset$ if and only if A is both open and closed.

7. Let X be a set, and let $\mathrm{cl} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be a function such that the following conditions hold:

   (a) For each $A \in \mathcal{P}(X)$, $A \subseteq \mathrm{cl}(A)$.

   (b) For each $A \in \mathcal{P}(X)$, $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

   (c) $\mathrm{cl}(\emptyset) = \emptyset$.

   (d) If $A, B \in \mathcal{P}(X)$, then $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

   Let $\mathcal{T} = \{ U \in \mathcal{P}(X) : \text{there is a subset } C \text{ of } X \text{ such that } \mathrm{cl}(C) = C \text{ and } U = X \setminus C \}$. Prove that $\mathcal{T}$ is a topology on X. Properties (a)-(d) are called the **Kuratowski Closure Properties** in honour of K. Kuratowski (1896-1980).

8. Let X be a set and let $D \subseteq X$. Define a function $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ by $f(A) = A \cup D$ for each $A \in \mathcal{P}(X)$.

   (a) Prove that f satisfies the Kuratowski Closure Properties.

   (b) Describe the members of $\mathcal{T}$, where $\mathcal{T}$ is the topology defined in the previous exercise.

   (c) What is the topology $\mathcal{T}$ when $D = \emptyset$?

   (d) What is the topology $\mathcal{T}$ when $D = X$?

4 Convergence

1. Let X be a set and let d be the discrete metric on X. Prove that (X, d) is complete.

2. Let A be a sounded subset of a metric space (X, d). Prove that $\overline{A}$ is bounded.

3. Give an example of a set X and metrics d and ρ on X such that the topology induced by d is the same as the topology induced by ρ, but (X, d) is complete, while (X, ρ) is not.

4. Let (X, d) be a metric space and let A be a dense subset of X such that every Cauchy sequence in A converges in X. Prove that (X, d) is complete.

5. Let $(X, \leq)$ be a linearly ordered set, and let $\mathcal{T}$ denote the order topology on X. Prove that $(X, \mathcal{T})$ is a Hausdorff space.

5 Continuous Functions and Homeomorphisms

1. Let $(X, \mathcal{T})$ be a separable space, let $(Y, \mathcal{U})$ be a topological space, and let $f : X \longrightarrow Y$ be a continuous function that maps X onto Y. Prove that $(Y, \mathcal{U})$ is separable.

2. Give examples of topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and a function $f : X \longrightarrow Y$ such that

   (a) f is open but not closed.

   (b) f is closed but not open.

3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $f : X \longrightarrow Y$ be a bijection. Prove that the following statements are equivalent:

   (a) f is a homeomorphism.

   (b) f is open and continuous.

   (c) f is closed and continuous

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and define $g : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $g(x) = (x, f(x))$. Prove that g is continuous.

6 Subspaces

1. Let $X = \{1, 2, 3\}$ and let $A = \{2, 3\}$.

   (a) If $\mathcal{T} = \{\emptyset, \{1\}, X\}$, what is $\mathcal{T}_A$?

   (b) If $\mathcal{T} = \{\emptyset, \{1, 2\}, X\}$, what is $\mathcal{T}_A$?

2. Prove that Hausdorff is hereditary.

3. Prove that the axiom of first countability is hereditary.

4. Give an example of a topological space $(X, \mathcal{T})$, a subspace $(A, \mathcal{T}_A)$ of $(X, \mathcal{T})$, and a closed set in $(A, \mathcal{T}_A)$ that is not closed in $(X, \mathcal{T})$.

5. Let $(X, \mathcal{T}), (Y, \mathcal{U})$, and $(Z, \mathcal{V})$ be topological spaces such that there is an embedding of X in Y and an embedding of Y in Z. Prove that there is an embedding of X in Z. (X is embedded in Y if X is homeomorphic to a subspace of Y.)

6. Let $(X, \mathcal{T})$ be a topological space, and let $B \subseteq A \subseteq X$. Show that the boundary of B, considered as a subset of A, is a subset of the boundary of B, considered as a subset of X, intersected with A.

7. Let A be an open subset of a separable space $(X, \mathcal{T})$. Prove that $(A, \mathcal{T}_A)$ is separable.

8. Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(X, \mathcal{T})$. Prove that the inclusion map $i : A \longrightarrow X$ defined by $i(a) = a$ for each $a \in A$ is continuous.

9. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $(A, \mathcal{T}_A)$ be a subspace of $(X, \mathcal{T})$, and let $f : X \longrightarrow Y$ be a continuous function. Prove that $f|_A : A \longrightarrow Y$ is continuous.

10. Let $\mathcal{B}'$ be the collection of all open disks in $\mathbb{R}^2$ with a finite number of straight lines through the center removed, and let

$$\mathcal{B} = \{\, B \cup \{c\} : B \in \mathcal{B}' \text{ and } c \text{ is the center of } B \,\}$$

    (a) Show that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2$.
    (b) Compare $\mathcal{T}$ with the usual topology $\mathcal{U}$ on $\mathbb{R}^2$.
    (c) Let A denote a striaght line in $\mathbb{R}^2$. Describe $\mathcal{T}_A$.
    (d) Let A denote a circle in $\mathbb{R}^2$. Compare $\mathcal{T}_A$ and $\mathcal{U}_A$.

11. Let $\mathcal{T}$ denote the subspace topology on $[0, 1)$ determined by the usual topology on $\mathbb{R}$, and let $\mathcal{U}$ denote the subspace topology on $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ determined by the usual topology on $\mathbb{R}^2$. Define $f : [0, 1) \longrightarrow (S^1, \mathcal{U})$ by

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

    (a) Prove that f is a bijection.
    (b) Prove that f is continuous.
    (c) Prove that f^{-1} is not continuous.

7 Product Topology on $X \times Y$

1. Let $X = \{1, 2, 3\}$, $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, X\}$, $Y = \{4, 5\}$ and $\mathcal{U} = \{\emptyset, \{4\}, Y\}$. Find a basis $\mathcal{B}$ for the product topology on $X \times Y$.

2. Let X and Y be infinite sets, let $\mathcal{T}$ be the discrete topology on X and $\mathcal{U}$ be the indiscrete topology on Y. Describe the product topology on $X \times Y$.

3. Let $\mathcal{T} := \{U \in \mathcal{P}(\mathbb{R}) : 0 \in U\} \cup \{\emptyset\}$, and let $\mathcal{U} := \{U \in \mathcal{P}(\mathbb{R}) : 1 \in U\} \cup \{\emptyset\}$. Describe the product topology on $\mathbb{R} \times \mathbb{R}$ determined by $\mathcal{T}$ and $\mathcal{U}$.

4. Let $\mathcal{T}$ be the Sorgenfrey topology on $\mathbb{R}$, and let $\mathcal{U}$ be the product topology on $(\mathbb{R}, \mathcal{T}) \times (\mathbb{R}, \mathcal{T})$. Identify the subspace topology on the line $L := \{(x, -x) : x \in \mathbb{R}\}$.

5. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $a \in X$, and let $b \in Y$. Prove that the functions $f : X \longrightarrow X \times Y$ and $g : Y \longrightarrow X \times Y$ defined by $f(x) = (x, b)$ and $g(y) = (a, y)$ are embeddings.

6. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be separable spaces, and let $\mathcal{T}$ denote the product topology on $X = X_1 \times X_2$. Prove that $(X, \mathcal{T})$ is a separable space.

7. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be first countable spaces, and let $\mathcal{T}$ denote the product topology on $X = X_1 \times X_2$. Prove that $(X, \mathcal{T})$ is a first countable space.

8. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be second countable spaces, and let $\mathcal{T}$ denote the product topology on $X = X_1 \times X_2$. Prove that $(X, \mathcal{T})$ is a second countable space.

9. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be first countable spaces, and let $\mathcal{T}$ denote the product topology on $X_1 \times X_2$, and let $\mathcal{U}$ denote the product topology on $X_2 \times X_1$. Prove that $(X_1 \times X_2, \mathcal{T})$ is homeomorphic to $(X_2 \times X_1, \mathcal{U})$.

10. Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{U}$ denote the product topology on $X \times X$, let $\Delta := \{(x, x) : x \in X\}$, and let $\mathcal{U}_\Delta$ be the subspace topology on Δ determined by $\mathcal{U}$. Prove that $(X, \mathcal{T})$ is homeomorphic to $(\Delta, \mathcal{U}_\Delta)$. (The set Δ is called the **diagonal**.)

11. (**I didn't solve this.**) Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{U}$ denote the product topology on $X \times X$. Prove that $(X, \mathcal{T})$ is Hausdorff if and only if the diagonal is a closed subset of $(X \times X, \mathcal{U})$.

12. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and suppose $X_1 \times X_2$ has the product topology. For $i = 1, 2$, let A_i be a subset of X_i. Prove that $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

13. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and suppose $X_1 \times X_2$ has the product topology. For $i = 1, 2$, let A_i be a subset of X_i. Prove that

$$\mathrm{int}(A_1 \times A_2) = \mathrm{int}(A_1) \times \mathrm{int}(A_2).$$

14. Let X, Y_1 and Y_2 be sets, for each $i = 1, 2$, let $U_i \subseteq Y_i$ and let $f_i : X \longrightarrow Y_i$ be a function, and define $f : X \longrightarrow Y_1 \times Y_2$ by $f(x) = (f_1(x), f_2(x))$. Prove that $f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$.

15. (**I didn't solve this.**) Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), (Y_1, \mathcal{U}_1), (Y_2, \mathcal{U}_2)$ and $(Z, \mathcal{V})$ be topological spaces, and let $f : X_1 \longrightarrow Y_1, g : X_2 \longrightarrow Y_2$ and $F : Y_1 \times Y_2 \longrightarrow Z$ be continuous functions. Prove that the function $G : X_1 \times X_2 \longrightarrow Z$ defined by $G((x_1, x_2)) = F(f(x_1), g(x_2))$ is continuous.

16. (**I didn't solve this.**) Prove that the function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$f(x) = (x^2 - 5, \frac{1}{x^2 + 1})$$

is continuous.

8 Quotient Maps

We didn't cover quotient maps, but here are the questions anyway. Don't worry about these.

1. Define an equivalence relation $\sim$ on $X = \mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_1 = y_2$. Let $(X/\sim, \mathcal{U})$ be the identification space, and let $\mathcal{T}$ denote the usual topology on $\mathbb{R}$. Prove that $(X/\sim, \mathcal{U})$ is homeomorphic to $(\mathbb{R}, \mathcal{T})$.

2. Let $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ or } y = 0\}$, and let $\mathcal{T}$ be the subspace topology on X induced by the usual topology on $\mathbb{R}^2$, and let $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Define $f : X \longrightarrow \mathbb{R}$ by $f((x, y)) = x$ for all $(x, y) \in X$. Prove that f is a quotient map and show that it is neither open nor closed.

3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{V})$ be topological spaces, and let f be a function that maps X onto Y, and let $\mathcal{U}$ be the quotient topology on Y induced by f. Prove that if f is continuous and closed, then $\mathcal{U} = \mathcal{V}$.

4. Prove that the composition of two quotient maps is a quotient map.

5. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{D}$ be a partition of X. Let $p : X \longrightarrow \mathcal{D}$ be the natural map, and let $\mathcal{U}$ be the quotient topology on $\mathcal{D}$ induced by p. Prove that a subset $\mathcal{E}$ of $\mathcal{D}$ is open if and only if $\bigcup\{E : E \in \mathcal{E}\}$ is open in X.

6. Let $Y = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$. Let $\mathcal{T}$ be the usual topology on $\mathbb{R}^2$, and deifne $f : \mathbb{R}^2 \longrightarrow Y$ by $f((0, y)) = (0, y)$ and $f((x, y)) = (x, 0)$ if $x \neq 0$. Let $\mathcal{U}$ be the quotient topology on Y induced by f. Show that $(Y, \mathcal{U})$ is not Hausdorff.

7. Let $Y = \{a, b, c\}$ and define $f : \mathbb{R} \longrightarrow Y$ by:

$$f(x) = \begin{cases} a & : x < 0 \\ b & : x = 0 \\ c & : x > 0 \end{cases}$$

   Describe the quotient topology on Y induced by f.

8. Let $X = [0, 1] \cup (2, 3]$, let $Y = [0, 2]$, and suppose X and Y have the usual topologies. Define $f : X \longrightarrow Y$ by:

$$f(x) = \begin{cases} x & : x \in [0, 1] \\ x - 1 & : x \in (2, 3] \end{cases}$$

   Is f a quotient map? Prove your answer.

9. Let $X := \bigcup_{n\in\mathbb{N}}(\mathbb{R} \times \{n\})$, and let $Y := \bigcup_{n\in\mathbb{N}}\{(x,y) \in \mathbb{R}^2 : y = nx\}$. Suppose both X and Y have the subspace topology induced by the usual topology on $\mathbb{R}^2$. Define $p : X \longrightarrow Y$ by $p((x,n)) = (x,nx)$ for each $x, n \in X$.

   (a) Show that p maps X onto Y.

   (b) Show that p is not a quotient map.

9 Connectedness

1. Prove that the product of two connected spaces is connected.

2. Prove that no two of the intervals $[0,1], (0,1)$ and $[0,1)$ are homeomorphic. (Hint: Use cut-points.)

3. Are any of the following subspaces of $\mathbb{R}^2$ homeomorphic? A "+" shape, a "P" shape, a "Y" shape and a square.

4. (**I didn't solve this.**) Let p be a cut-point of a connected space $(X, \mathcal{T})$ and suppose A and B form a separation of $X \setminus \{p\}$. Prove that $A \cup \{p\}$ is connected.

5. (**I didn't solve this.**) Let $(L, \leq)$ be a linearly ordered set and let $\mathcal{T}$ be the order topology on X. Prove that $(X, \mathcal{T})$ is connected if and only if $(X, \leq)$ is Dedekind complete and has no gaps.

10 Path Connectedness

1. Show that the topologist's comb is pathwise connected but not locally connected.

2. Let $\leq$ denote the dictionary order relation on $I \times I$ determined by less than or equal to on I, and let $\mathcal{T}$ denote the order topology on $I \times I$. Prove that $(I \times I, \mathcal{T})$ is locally connected, but not locally pathwise connected.

3. Prove that the continuous image of a pathwise connected space is pathwise connected.

4. Prove that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus A$ is pathwise connected.

5. The **deleted comb space** is the space obtained from the topologist's comb by deleting the open interval $\{0\} \times (0,1)$. Prove that the deleted comb space is connected and has two path components.

11 Compactness in Metric Spaces

1. Prove that every subset of a totally bounded metric space is totally bounded.
2. (**I didn't solve this.**) Let $\mathcal{O}$ be a collection of open intervals such that

$$I \subseteq \bigcup \{O : O \in \mathcal{O}\}.$$

   Prove that there is a finite subset $\{O_1, O_2, \ldots, O_N\}$ of $\mathcal{O}$ such that $I \subseteq \bigcup_{n=1}^{N} O_N$.
3. (**I didn't solve this.**) Let (X, d) be a compact metric space, and let ρ be any metric on X such that the topology induced by ρ is the topology induced by d. Prove that (X, ρ) is bounded.
4. (**I didn't solve this.**) Let $(X, \mathcal{T})$ be a metrizable space such that every metric that generates $\mathcal{T}$ is bounded. Prove that X is compact.
5. Let (X, d) be a totally bounded metric space. Prove that X is separable.
6. Give an example of a comapct metric space $(X, \mathcal{T})$, a topological space $(Y, \mathcal{U})$ that is not Hausdorff, and a continuous function f that maps X onto Y.
7. (**I didn't solve this.**) Let $\{A_\alpha : \alpha \in I\}$ be a family of closed subsets of a compact metric space (X, d) such that $\bigcap_{\alpha \in I} A_\alpha = \emptyset$. Prove that there is a positive number ϵ such that if B is any subset of X of diameter less than ϵ, then there exists $\beta \in I$ such that $B \cap A_\beta = \emptyset$.
8. (**I didn't solve this.**) Prove that every comapct metric space is second countable.
9. (**I didn't solve this.**) Prove that every compact subset of a metric space is closed and bounded.
10. Give an example of a bounded metric space that is not compact.

12 T_0, T_1 and T_2 spaces.

1. Let $(X, \mathcal{T})$ be a topological space. Prove that $(X, \mathcal{T})$ is a T_0 space if and only if for each pair a and b of distinct members of X, $\overline{\{a\}} \neq \overline{\{b\}}$.
2. Let $(X, \mathcal{T})$ be a topological space, let R be an equivalence relation on X, and let $\mathcal{U}$ be the quotient topology on X/R induced by the natural map. Prove that $(X/R, \mathcal{U})$ is a T_1 space if and only if for each $x \in X$, $[x]$ is a closed subset of X.
3. For each $i = 0, 1$, prove that every subspace of a T_i space is a T_i space.

4. For each $i = 0, 1$, prove that the product of T_i spaces is a T_i space.

5. Let $I = [0, 1]$, and define $x \sim y$ provided $x - y$ is rational. This is clearly an equivalence relation on I. Let $p : I \longrightarrow I/\sim$ be the natural map.

   (a) Prove that $I/\sim$ is not Hausdorff;

   (b) Prove that p is an open map.

6. Let X be a set and let $D \subseteq X$. Define a topology $\mathcal{T}$ on X by saying that a subset of X is closed whenever $C = C \cup D$ and a subset U of X belongs to $\mathcal{T}$ whenver $X \setminus U$ is closed. For each $i = 0, 1, 2$, under what conditions on D is $(X, \mathcal{T})$ a T_i space?

7. Let $(X, \mathcal{T})$ be a T_1-space, let $(Y, \mathcal{U})$ be a topological space, and let f be a closed map of X onto Y. Prove that $(Y, \mathcal{U})$ is a T_1 space.

8. Let $(X, \mathcal{T})$ be a T_1-space and let $(\mathcal{D}, \mathcal{U})$ be a decomposition space of X. Prove that $(\mathcal{D}, \mathcal{U})$ is T_1 if and only if each member of $\mathcal{D}$ is a closed subset of X.

9. We know that we can define a topology in terms of closed sets. Define a basis for the closed sets of a topology.