University of Toronto

MAT237Y1Y PROBLEM SET 8

DUE: End of tutorial, Thursday August 8th, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.

2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.

3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.

4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. Line integral:

   a) Let $f : D \to \mathbb{R}$ be a C^1 function defined on an open subset D of $\mathbb{R}^n$, and let C be a C^1 curve in D. Prove that

   $$\int_C \nabla f \cdot d\mathbf{x} = f(B) - f(A),$$

   where A and B are the initial and terminal points of the curve C. Then present an argument to show that if C is a closed C^1 curve then the line integral will equal to zero.

   b) Using part (a), show that the vector field

   $$\mathbf{H}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$$

   is not the gradient of a C^1 function f (i.e. it cannot be written $H = \nabla f$ for any f).

   (Hint: Choose C as a unit circle.)

c) Using 5.30 on page 238, show that the vector field

$$\mathbf{H}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$$

is not the gradient of a C^2 function f.

d) Consider the vector field

$$\mathbf{G}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

and calculate the expression

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

for this vector field. Consider the region S of the plane bounded by the square with vertices at $(2, 2), (2, -2), (-2, 2)$ and $(-2, -2)$, and the unit circle (centered at the origin). Verify the conclusion of Green's theorem for this function and the region S and ∂S. (that is, show the equality 5.14 on page 223 holds.)

e) Consider the region S as in part (d) and consider the slice of S that corresponds to the region of plane between the angles $\theta = -\pi/4$ and $\theta = \pi/4$ (as in the polar coordinate). Draw this region (name it as S_1). Then show that the vector field $\mathbf{G}$ is the gradient of the function $\tan^{-1}(y/x)$. Calculate the line integral of $\mathbf{G}$ on ∂S_1.

2. Surface integrals:

a) Consider the ice cream cone T bounded below by the cone $z = 2\sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 1$. (region T described in figure 4.10 on page 185.) Present a formula for the $\mathbf{n}$ for each of the two pieces that form the surface of T. Then use the surface integral to find the surface area of T.

b) Let the surface S be the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane. Calculate the surface integral of the function

$$f(x, y, z) = 4 - z$$

on the surface S.

c) Evaluate the surface integral of the vector field

$$\mathbf{F}(x, y, z) = (x, y, z - 2y)$$

on the surface S parameterized by $\mathbf{G}(u, v) = (u \cos v, u \sin v, v)$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

3. A spiral and a torus

   a) Let (r, θ) denote the polar coordinates and let $g(t) = (x(t), y(t))$ be a C^1 curve in $\mathbb{R}^2$. Show that

$$|g'(t)| = \sqrt{|\frac{dr}{dt}|^2 + r^2|\frac{d\theta}{dt}|^2}. \tag{1}$$

   b) Draw a picture of the spiral

$$r(t) = t, \quad \theta(t) = t,$$

   where $0 \leq t \leq c$. Use equation (1) to write an integral expression for the length of this arc (as a function of c, and you don't need to find the value of this integration).

   c) Consider the surface T defined in cylindrical coordinates (r, θ, z) by

$$\begin{aligned} r(u, v) &= \sin u + 2, \\ \theta(u, v) &= v, \\ z(u, v) &= \cos u, \end{aligned}$$

   where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$. Draw a picture of this shape, which is called as a torus.

   d) Compute the surface area of T.

4. Differential Operators:

   a) Do question 4 on page 239.

   b) Do question 5 on page 239. (Hint: they are all in the lecture notes, but I want you to practice those calculations.)