D7. Consider a clock with vectors drawn from the center to each hour as shown in the accompanying figure.

(a) What is the sum of the 12 vectors that result if the vector terminating at 12 is doubled in length and the other vectors are left alone?

(b) What is the sum of the 12 vectors that result if the vectors terminating at 3 and 9 are each tripled and the others are left alone?

(c) What is the sum of the 9 vectors that remain if the vectors terminating at 5, 11, and 8 are removed?

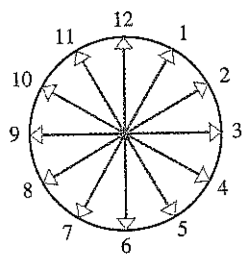


Figure Ex-D7

D8. Draw a picture that shows four nonzero vectors in the plane, one of which is the sum of the other three.

D9. Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) If $\mathbf{x}+\mathbf{y}=\mathbf{x}+\mathbf{z}$, then $\mathbf{y}=\mathbf{z}$.

(b) If $\mathbf{u}+\mathbf{v}=\mathbf{0}$, then $a\mathbf{u}+b\mathbf{v}=\mathbf{0}$ for all a and b.

(c) Parallel vectors with the same length are equal.

(d) If $a\mathbf{x}=\mathbf{0}$, then either $a=0$ or $\mathbf{x}=\mathbf{0}$.

(e) If $a\mathbf{u}+b\mathbf{v}=\mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors.

(f) The vectors $\mathbf{u}=(\sqrt{2},\sqrt{3})$ and $\mathbf{v}=\left(\frac{1}{\sqrt{2}},\frac{1}{2}\sqrt{3}\right)$ are equivalent.

Working with Proofs

P1. Prove part (*e*) of Theorem 1.1.5.

P2. Prove part (*f*) of Theorem 1.1.5.

P3. Prove Theorem 1.1.6 without using components.

Technology Exercises

T1. (***Numbers and numerical operations***) Read how to enter integers, fractions, decimals, and irrational numbers such as π and $\sqrt{2}$. Check your understanding of the procedures by converting π, $\sqrt{2}$, and $1/3$ to decimal form with various numbers of decimal places in the display. Read about the procedures for performing the operations of addition, subtraction, multiplication, division, raising numbers to powers, and extraction of roots. Experiment with numbers of your own choosing until you feel you have mastered the techniques.

T2. (***Drawing vectors***) Read how to draw line segments in two- or three-dimensional space, and draw some line segments with initial and terminal points of your choice. If your utility allows you to create arrowheads, then you can make your line segments look like geometric vectors.

T3. (***Operations on vectors***) Read how to enter vectors and how to calculate their sums, differences, and scalar multiples. Check your understanding of these operations by performing the calculations in Example 4.

T4. Use your technology utility to compute the components of $\mathbf{u}=(7.1,-3)-5(\sqrt{2},6)+3(0,\pi)$ to five decimal places.

Section 1.2 Dot Product and Orthogonality

In this section we will be concerned with the concepts of length, angle, distance, and perpendicularity in R^n. We will begin by discussing these concepts geometrically in R^2 and R^3, and then we will extend them algebraically to R^n using components.

NORM OF A VECTOR The length of a vector $\mathbf{v}$ in R^2 or R^3 is commonly denoted by the symbol $\|\mathbf{v}\|$. It follows from the theorem of Pythagoras that the length of a vector $\mathbf{v}=(v_1,v_2)$ in R^2 is given by the formula

$$\|\mathbf{v}\|=\sqrt{v_1^2+v_2^2} \tag{1}$$

(Figure 1.2.1a). A companion formula for the length of a vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 can be obtained using two applications of the theorem of Pythagoras (Figure 1.2.1b):

$$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (QR)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2$$

Thus,

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{2}$$

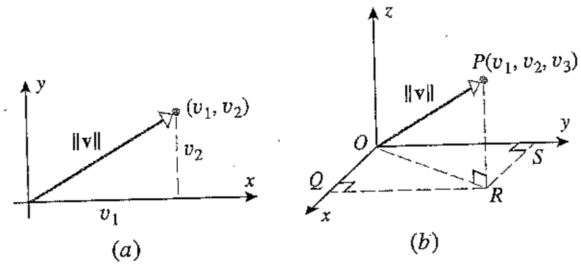


Figure 1.2.1 (a) (b)

Motivated by Formulas (1) and (2), we make the following general definition for the length of a vector in R^n.

Definition 1.2.1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***length*** of $\mathbf{v}$, also called the ***norm*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$, is denoted by $\|\mathbf{v}\|$ and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

EXAMPLE 1 Calculating Norms

From (3), the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$ ■

Since lengths in R^2 and R^3 are nonnegative numbers, and since $\mathbf{0}$ is the only vector that has length zero, it follows that $\|\mathbf{v}\| \geq 0$ and that $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$. Also, multiplying $\mathbf{v}$ by a scalar k multiplies the length of $\mathbf{v}$ by $|k|$, so $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$. We will leave it for you to prove that these three properties also hold in R^n.

Theorem 1.2.2 *If $\mathbf{v}$ is a vector in R^n, and if k is any scalar, then:*

(a) $\|\mathbf{v}\| \geq 0$

(b) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(c) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

UNIT VECTORS

A vector of length 1 is called a ***unit vector***. If $\mathbf{v}$ is a nonzero vector in R^n, then a unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v}$ is given by the formula

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \tag{4}$$

In words, Formula (4) states that *a unit vector with the same direction as a vector* $\mathbf{v}$ *can be obtained by multiplying* $\mathbf{v}$ *by the reciprocal of its length*. This process is called ***normalizing*** $\mathbf{v}$. The vector $\mathbf{u}$ has the same direction as $\mathbf{v}$ since $1/\|\mathbf{v}\|$ is a positive scalar; and it has length 1 since part (c) of Theorem 1.2.2 with $k = 1/\|\mathbf{v}\|$ yields

$$\|\mathbf{u}\| = \|k\mathbf{v}\| = |k|\|\mathbf{v}\| = k\|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

Sometimes you will see Formula (4) expressed as

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

This is just a more compact way of writing the scalar product in (4).

EXAMPLE 2 Normalizing a Vector

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from (4)

$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

As a check, you may want to confirm that $\mathbf{u}$ is in fact a unit vector. ■

CONCEPT PROBLEM Unit vectors are often used to specify directions in 2-space or 3-space. Find a unit vector that describes the direction that a bug would travel if it walked from the origin of an xy-coordinate system into the first quadrant along a line that makes an angle of 30° with the positive x-axis. Also, find a unit vector that describes the direction that the bug would travel if it walked into the third quadrant along the line.

THE STANDARD UNIT VECTORS

When a rectangular coordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinate axes are called the ***standard unit vectors***. In R^2 these vectors are denoted by

$$\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1) \tag{5}$$

and in R^3 they are denoted by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1) \tag{6}$$

(Figure 1.2.2).

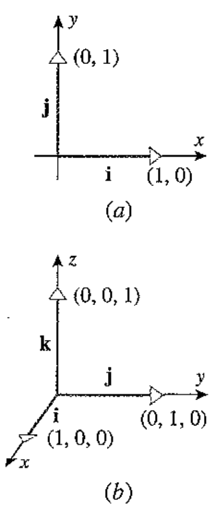


Figure 1.2.2

Observe that every vector $\mathbf{v} = (v_1, v_2)$ in R^2 can be expressed in terms of the standard unit vectors as

$$\mathbf{v} = (v_1, v_2) = v_1(1, 0) + v_2(0, 1) = v_1\mathbf{i} + v_2\mathbf{j}$$

and every vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 can be expressed in terms of the standard unit vectors as

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

For example,

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

REMARK The $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation for vectors in R^2 and R^3 is common in engineering and physics, but it will be used only occasionally in this text.

More generally, we define the ***standard unit vectors in R^n*** to be

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1) \tag{7}$$

We leave it for you to verify that every vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in R^n can be expressed in terms of the standard unit vectors as

$$\mathbf{v} = (v_1, v_2, \dots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n \tag{8}$$

DISTANCE BETWEEN POINTS IN R^n

If P_1 and P_2 are points in R^2 or R^3, then the length of the vector $\overrightarrow{P_1P_2}$ is equal to the distance d between the two points (Figure 1.2.3). Specifically, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in

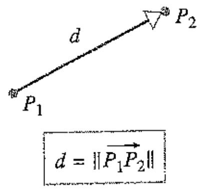


Figure 1.2.3

R^2, then Theorem 1.1.1(a) implies that

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{9}$$

This is the familiar distance formula from analytic geometry. Similarly, the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space is

$$d(\mathbf{u}, \mathbf{v}) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{10}$$

Motivated by Formulas (9) and (10), we make the following definition.

Definition 1.2.3 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

For example, if

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

We leave it for you to use Formula (11) to show that distances in R^n have the following properties.

Theorem 1.2.4 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are points in* R^n, *then:*

(a) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(b) $d(\mathbf{u}, \mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

This theorem states that distances in R^n behave like distances in visible space; that is, distances are nonnegative numbers, the distance between distinct points is nonzero, and the distance is the same whether you measure from $\mathbf{u}$ to $\mathbf{v}$ or from $\mathbf{v}$ to $\mathbf{u}$.

DOT PRODUCTS

We will now define a new kind of multiplication that will be useful for finding angles between vectors and determining whether two vectors are perpendicular.

Definition 1.2.5 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** of $\mathbf{u}$ and $\mathbf{v}$, also called the ***Euclidean inner product*** of $\mathbf{u}$ and $\mathbf{v}$, is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by the formula

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{12}$$

In words, *the dot product is calculated by multiplying corresponding components of the vectors and adding the resulting products.* For example, the dot product of the vectors $\mathbf{u} = (-1, 3, 5, 7)$ and $\mathbf{v} = (5, -4, 7, 0)$ in R^4 is

$$\mathbf{u} \cdot \mathbf{v} = (-1)(5) + (3)(-4) + (5)(7) + (7)(0) = 18$$

REMARK Note the distinction between scalar multiplication and dot products—in scalar multiplication one factor is a scalar, the other is a vector, and the result is a vector; and in a dot product both factors are vectors and the result is a scalar.

EXAMPLE 3 An Application of Dot Products to ISBNs

Most books published in the last 25 years have been assigned a unique 10-digit number called an ***International Standard Book Number*** or ISBN. The first nine digits of this number are split into three groups—the first group representing the country or group of countries in which the book originates, the second identifying the publisher, and the third assigned to the book title itself. The tenth and final digit, called a ***check digit***, is computed from the first nine digits and is used to ensure that an electronic transmission of the ISBN, say over the Internet, occurs without error.

To explain how this is done, regard the first nine digits of the ISBN as a vector $\mathbf{b}$ in R^9, and let $\mathbf{a}$ be the vector

$$\mathbf{a} = (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

Then the check digit c is computed using the following procedure:

1. Form the dot product $\mathbf{a} \cdot \mathbf{b}$.
2. Divide $\mathbf{a} \cdot \mathbf{b}$ by 11, thereby producing a remainder c that is an integer between 0 and 10, inclusive. The check digit is taken to be c, with the proviso that $c = 10$ is written as X to avoid double digits.

For example, the ISBN of the brief edition of *Calculus*, sixth edition, by Howard Anton is

0-471-15307-9

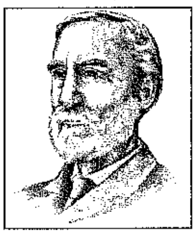
which has a check digit of 9. This is consistent with the first nine digits of the ISBN, since

$$\mathbf{a} \cdot \mathbf{b} = (1, 2, 3, 4, 5, 6, 7, 8, 9) \cdot (0, 4, 7, 1, 1, 5, 3, 0, 7) = 152$$

Dividing 152 by 11 produces a quotient of 13 and a remainder of 9, so the check digit is $c = 9$. If an electronic order is placed for a book with a certain ISBN, then the warehouse can use the above procedure to verify that the check digit is consistent with the first nine digits, thereby reducing the possibility of a costly shipping error. ■

Linear Algebra in History

The dot product notation was first introduced by the American physicist and mathematician J. Willard Gibbs in a pamphlet distributed to his students at Yale University in the 1880s. The product was originally written on the baseline, rather than centered as today, and was referred to as the *direct product*. Gibbs's pamphlet was eventually incorporated into a book entitled *Vector Analysis* that was published in 1901 and coauthored by Gibbs and one of his students. Gibbs made major contributions to the fields of thermodynamics and electromagnetic theory and is generally regarded as the greatest American physicist of the nineteenth century.



Josiah Willard Gibbs
(1839–1903)

ALGEBRAIC PROPERTIES OF THE DOT PRODUCT

In the special case where $\mathbf{u} = \mathbf{v}$ in Definition 1.2.5, we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 \tag{13}$$

This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{14}$$

Dot products have many of the same algebraic properties as products of real numbers.

Theorem 1.2.6 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

We will prove parts (*c*) and (*d*) and leave the other proofs as exercises.

Proof (*c*) Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Then

$$k(\mathbf{u} \cdot \mathbf{v}) = k(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = (ku_1)v_1 + (ku_2)v_2 + \cdots + (ku_n)v_n = (k\mathbf{u}) \cdot \mathbf{v}$$

Proof (*d*) The result follows from parts (*a*) and (*b*) of Theorem 1.2.2 and the fact that

$$\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + \cdots + v_nv_n = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2$$ ■

The following theorem gives some more properties of dot products. The results in this theorem can be proved either by expressing the vectors in terms of components or by using the algebraic properties already established in Theorem 1.2.6.

Theorem 1.2.7 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

We will show how Theorem 1.2.6 can be used to prove part (*b*) without breaking the vectors down into components. Some of the other proofs are left as exercises.

Proof (*b*)

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) && \textbf{[By symmetry]}\\ &= \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} && \textbf{[By distributivity]}\\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} && \textbf{[By symmetry]}\end{aligned}$$

■

Formulas (13) and (14) together with Theorems 1.2.6 and 1.2.7 make it possible to manipulate expressions involving dot products using familiar algebraic techniques.

EXAMPLE 4 Calculating with Dot Products

$$\begin{aligned}(\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v})\\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v})\\ &= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2\end{aligned}$$

■

ANGLE BETWEEN VECTORS IN R^2 AND R^3

To see how dot products can be used to calculate angles between vectors in R^2 and R^3, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^2 or R^3, and define the ***angle*** between $\mathbf{u}$ and $\mathbf{v}$ to be the smallest nonnegative angle θ through which one of the vectors can be rotated in the plane of the vectors until it coincides with the other (Figure 1.2.4). Algebraically, the radian measure of θ is in the interval $0 \le \theta \le \pi$, and in R^2 the angle θ is generated by a counterclockwise rotation.

The following theorem provides an effective way to calculate the angle between vectors in both R^2 and R^3.

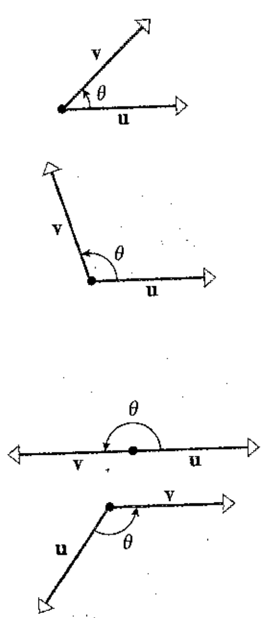


Figure 1.2.4

Theorem 1.2.8 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors in* R^2 *or* R^3, *and if* θ *is the angle between these vectors, then*

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \quad \textit{or equivalently,} \quad \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) \tag{15–16}$$

Proof Suppose that the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} - \mathbf{u}$ are positioned to form the sides of a triangle, as shown in Figure 1.2.5. It follows from the law of cosines that

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{17}$$

Using Formula (13) and the properties of the dot product in Theorems 1.2.6 and 1.2.7, we can rewrite the left side of this equation as

$$\begin{aligned}\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u})\\ &= (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u}\\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}\\ &= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2\end{aligned}$$

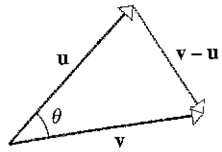


Figure 1.2.5

Substituting the last expression in (17) yields

$$\|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

which we can simplify and rewrite as

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Finally, dividing both sides of this equation by $\|\mathbf{u}\|\|\mathbf{v}\|$ yields (15). ■

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3 and $\mathbf{u}\cdot\mathbf{v} = 0$, then it follows from Formula (16) that $\theta = \cos^{-1} 0 = \pi/2$. Conversely, if $\theta = \pi/2$, then $\cos\theta = 0$ and $\mathbf{u}\cdot\mathbf{v} = 0$. Thus, *two nonzero vectors in R^2 or R^3 are perpendicular if and only if their dot product is zero.*

CONCEPT PROBLEM What can you say about the angle between the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^2 or R^3 if $\mathbf{u}\cdot\mathbf{v} > 0$? What if $\mathbf{u}\cdot\mathbf{v} < 0$?

EXAMPLE 5 An Application of the Angle Formula

Find the angle θ between a diagonal of a cube and one of its edges.

Solution Assume that the cube has side a, and introduce a coordinate system as shown in Figure 1.2.6. In this coordinate system the vector

$$\mathbf{d} = (a, a, a)$$

is a diagonal of the cube, and the vectors $\mathbf{v}_1 = (a, 0, 0)$, $\mathbf{v}_2 = (0, a, 0)$, and $\mathbf{v}_3 = (0, 0, a)$ run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between $\mathbf{d}$ and $\mathbf{v}_1$. From Formula (15), the cosine of this angle is

$$\cos\theta = \frac{\mathbf{v}_1\cdot\mathbf{d}}{\|\mathbf{v}_1\|\|\mathbf{d}\|} = \frac{a^2}{a(\sqrt{3a^2})} = \frac{1}{\sqrt{3}}$$

Thus, with the help of a calculating utility,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$

■

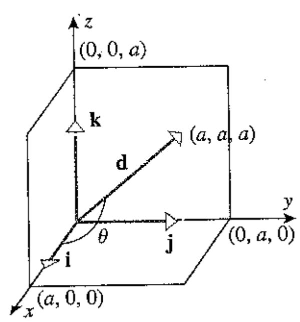


Figure 1.2.6

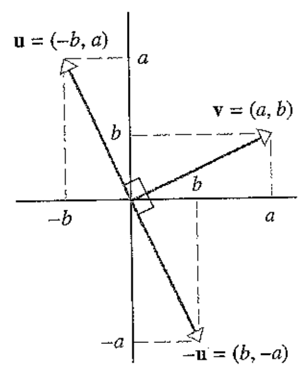


Figure 1.2.7

EXAMPLE 6 Finding a Vector in R^2 That Is Perpendicular to a Given Vector

Find a nonzero vector in R^2 that is perpendicular to the nonzero vector $\mathbf{v} = (a, b)$.

Solution We are looking for a nonzero vector $\mathbf{u}$ for which $\mathbf{u}\cdot\mathbf{v} = 0$. By experimentation, $\mathbf{u} = (-b, a)$ is such a vector, since

$$\mathbf{u}\cdot\mathbf{v} = (-b, a)\cdot(a, b) = -ba + ab = 0$$

The vector $-\mathbf{u} = (b, -a)$ is also perpendicular to $\mathbf{v}$, as is any scalar multiple of $\mathbf{u}$ (Figure 1.2.7). ■

ORTHOGONALITY

To generalize the notion of perpendicularity to R^n we make the following definition.

Definition 1.2.9 Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** if $\mathbf{u} \cdot \mathbf{v} = 0$, and a nonempty set of vectors in R^n is said to be an ***orthogonal set*** if each pair of distinct vectors in the set is orthogonal.

REMARK Note that we do not require $\mathbf{u}$ and $\mathbf{v}$ to be nonzero in this definition; thus, two vectors in R^2 and R^3 are orthogonal if and only if they are either nonzero and perpendicular or if one or both of them are zero.

EXAMPLE 7 An Orthogonal Set of Vectors in R^4

Show that the vectors

$$\mathbf{v}_1 = (1, 2, 2, 4), \quad \mathbf{v}_2 = (-2, 1, -4, 2), \quad \mathbf{v}_3 = (-4, 2, 2, -1)$$

form an orthogonal set in R^4.

Solution Because of the symmetry property of the dot product, we need only confirm that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \quad \mathbf{v}_1 \cdot \mathbf{v}_3 = 0, \quad \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$$

We leave the computations to you. ■

If S is a nonempty set of vectors in R^n, and if $\mathbf{v}$ is orthogonal to every vector in S, then we say that $\mathbf{v}$ is ***orthogonal to the set S***. For example, the vector $\mathbf{k} = (0, 0, 1)$ in R^3 is orthogonal to the xy-plane (Figure 1.2.2b).

EXAMPLE 8 The Zero Vector Is Orthogonal to R^n

Part (a) of Theorem 1.2.7 states that if $\mathbf{0}$ is the zero vector in R^n, then $\mathbf{0} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in R^n. Thus, $\mathbf{0}$ is orthogonal to R^n. Moreover, $\mathbf{0}$ is the only vector in R^n that is orthogonal to R^n, since if $\mathbf{v}$ is a vector in R^n that is orthogonal to R^n, then, in particular, it would be true that $\mathbf{v} \cdot \mathbf{v} = 0$; this implies that $\mathbf{v} = \mathbf{0}$ by part (d) of Theorem 1.2.6. ■

REMARK Although the result in Example 8 may seem obvious, it will prove to be useful later in the text, since it provides a way of using the dot product to show that a vector $\mathbf{w}$ in R^n is the zero vector—just show that $\mathbf{w} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in R^n.

ORTHONORMAL SETS

Orthogonal sets of unit vectors have special importance, and there is some terminology associated with them.

Definition 1.2.10 Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthonormal*** if they are orthogonal and have length 1, and a set of vectors is said to be an ***orthonormal set*** if every vector in the set has length 1 and each pair of distinct vectors is orthogonal.

EXAMPLE 9 The Standard Unit Vectors in R^n Are Orthonormal

The standard unit vectors in R^2 and R^3 form orthonormal sets, since these vectors have length 1 and run along the coordinate axes of rectangular coordinate systems (Figure 1.2.2). More generally, the standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

in R^n form an orthonormal set, since

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \;\text{ if } i \neq j \quad \text{and} \quad \|\mathbf{e}_1\| = \|\mathbf{e}_2\| = \cdots = \|\mathbf{e}_n\| = 1$$

(verify). ■

In the following example we form an orthonormal set of three vectors in R^4.

EXAMPLE 10 An Orthonormal Set in R^4

The vectors

$$\mathbf{q}_1 = \left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{4}{5}\right), \quad \mathbf{q}_2 = \left(-\tfrac{2}{5}, \tfrac{1}{5}, -\tfrac{4}{5}, \tfrac{2}{5}\right), \quad \mathbf{q}_3 = \left(-\tfrac{4}{5}, \tfrac{2}{5}, \tfrac{2}{5}, -\tfrac{1}{5}\right)$$

form an orthonormal set in R^4, since

$$\|\mathbf{q}_1\| = \|\mathbf{q}_2\| = \|\mathbf{q}_3\| = 1$$

and

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = 0, \quad \mathbf{q}_1 \cdot \mathbf{q}_3 = 0, \quad \mathbf{q}_2 \cdot \mathbf{q}_3 = 0$$

(verify). ■

EUCLIDEAN GEOMETRY IN R^n

Formulas (3) and (11) are sometimes called the ***Euclidean norm*** and ***Euclidean distance*** because they produce theorems in R^n that reduce to theorems in Euclidean geometry when applied in R^2 and R^3. Here are just three examples:

1. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides (theorem of Pythagoras).
2. The sum of the lengths of two sides of a triangle is at least as large as the length of the third side.
3. The shortest distance between two points is along a straight line.

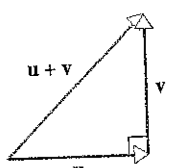


Figure 1.2.8

To extend these theorems to R^n, we need to state them in vector form. For example, a right triangle in R^2 or R^3 can be constructed by placing orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ tip to tail and using the vector $\mathbf{u}+\mathbf{v}$ as the hypotenuse (Figure 1.2.8). In vector notation, the theorem of Pythagoras now takes the form

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

The following theorem is the extension of this result to R^n.

Theorem 1.2.11 (***Theorem of Pythagoras***) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in* R^n, *then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \tag{18}$$

Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\end{aligned}$$ ■

We have seen that the angle between nonzero vectors in R^2 and R^3 is given by the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) \tag{19}$$

Since this formula involves only the dot product and norms of the vectors $\mathbf{u}$ and $\mathbf{v}$, and since the notions of dot product and norm are applicable to vectors in R^n, it seems reasonable to use Formula (19) as the *definition* of the angle between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n. However, this plan would only work if it were true that

$$-1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \le 1 \tag{20}$$

for all nonzero vectors in R^n. The following theorem gives a result, called the ***Cauchy–Schwarz inequality***, which will show that (20) does in fact hold for all nonzero vectors in R^n.

Linear Algebra in History

The Cauchy–Schwarz inequality is named in honor of the French mathematician Augustin Cauchy and the German mathematician Hermann Schwarz. Variations of this inequality occur in many different settings and under various names. Depending on the context in which the inequality occurs, you may find it called Cauchy's inequality, the Schwarz inequality, or sometimes even the Bunyakovsky inequality, in recognition of the Russian mathematician who published his version of the inequality in 1859, about 25 years before Schwarz.

Augustin Louis Cauchy (1789–1857)

Hermann Amandus Schwarz (1843–1921)

Viktor Yakovlevich Bunyakovsky (1804–1889)

Theorem 1.2.12 (***Cauchy–Schwarz Inequality in R^n***) *If* **u** *and* **v** *are vectors in R^n, then*

$$(\mathbf{u}\cdot\mathbf{v})^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \quad (21$$

or equivalently (by taking square roots),

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\| \quad (22$$

Proof Observe first that if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then both sides of (21) are zero (verify), so equali holds in this case. Now consider the case where **u** and **v** are nonzero. As suggested by Fi ure 1.2.9, the vector **v** can be written as the sum of some scalar multiple of **u**, say $k\mathbf{u}$, and a vect **w** that is orthogonal to **u**. The appropriate scalar k can be computed by setting $\mathbf{w} = \mathbf{v} - k\mathbf{u}$ a using the orthogonality condition $\mathbf{u}\cdot\mathbf{w} = 0$ to write

$$0 = \mathbf{u}\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v} - k\mathbf{u}) = (\mathbf{u}\cdot\mathbf{v}) - k(\mathbf{u}\cdot\mathbf{u})$$

from which it follows that

$$k = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}} \quad (2$$

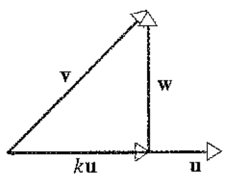


Figure 1.2.9

Now apply the theorem of Pythagoras to the vectors in Figure 1.2.9 to obtain

$$\|\mathbf{v}\|^2 = \|k\mathbf{u}\|^2 + \|\mathbf{w}\|^2 = k^2\|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 \quad (2$$

Substituting (23) for k and multiplying both sides of the resulting equation by $\|\mathbf{u}\|^2$ yields (verif

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 = (\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\|^2\|\mathbf{w}\|^2 \quad (2$$

Since $\|\mathbf{u}\|^2\|\mathbf{w}\|^2 \ge 0$, it follows from (25) that

$$(\mathbf{u}\cdot\mathbf{v})^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$$

This establishes (21) and hence (22).

REMARK The Cauchy–Schwarz inequality now allows us to use Formula (19) as a definiti of the angle between nonzero vectors in R^n.

There is a theorem in plane geometry, called the ***triangle inequality***, which states that t sum of the lengths of two sides of a triangle is at least as large as the third side. The followi theorem is a generalization of that result to R^n (Figure 1.2.10).

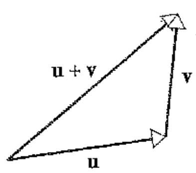


Figure 1.2.10

Theorem 1.2.13 (***Triangle Inequality for Vectors***) *If* **u**, **v**, *and* **w** *are vectors in R^n, then*

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \quad (26$$

Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2\\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 && \textbf{[Property of absolute value]}\\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \textbf{[Cauchy–Schwarz inequality]}\\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

Formula (26) now follows by taking square roots.

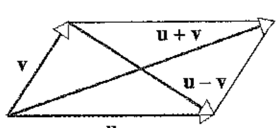


Figure 1.2.11

There is a theorem in plane geometry which states that for any parallelogram the sum of t squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of t four sides. The following theorem is a generalization of that result to R^n (Figure 1.2.11).

Theorem 1.2.14 (*Parallelogram Equation for Vectors*) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n, *then*

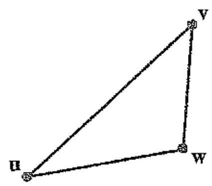
$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \tag{27}$$

Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v})\\ &= 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)\end{aligned}$$ ■

Finally, let $\mathbf{u}$ and $\mathbf{v}$ be any two points in R^2 or R^3. To say that the shortest distance from $\mathbf{u}$ to $\mathbf{v}$ is along a straight line implies that if we choose a third point $\mathbf{w}$ in R^2 or R^3, then

$$d(\mathbf{u},\mathbf{v}) \le d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$$



$d(\mathbf{u},\mathbf{v}) \le d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$

Figure 1.2.12

(Figure 1.2.12). This is called the ***triangle inequality for distances***. The following theorem is the extension to R^n.

Theorem 1.2.15 (*Triangle Inequality for Distances*) *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are points in* R^n, *then*

$$d(\mathbf{u},\mathbf{v}) \le d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}) \tag{28}$$

Proof

$$\begin{aligned}d(\mathbf{u},\mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\|\\ &= \|(\mathbf{u}-\mathbf{w}) + (\mathbf{w}-\mathbf{v})\| && \text{[Add and subtract } \mathbf{w}.]\\ &\le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\| && \text{[Triangle inequality for vectors]}\\ &= d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}) && \text{[Definition of distance]}\end{aligned}$$ ■

LOOKING AHEAD The notions of length, angle, and distance in R^n can all be expressed in terms of the dot product (which, you may recall, is also called the *Euclidean inner product*):

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} \tag{29}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\sqrt{\mathbf{u}\cdot\mathbf{u}}\sqrt{\mathbf{v}\cdot\mathbf{v}}}\right) \tag{30}$$

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})} \tag{31}$$

Thus, it is the algebraic properties of the Euclidean inner product that ultimately determine the geometric properties of vectors in R^n. However, the most important algebraic properties of the Euclidean inner product can all be derived from the four properties in Theorem 1.2.6, so this theorem is really the foundation on which the geometry of R^n rests. Because R^n with the Euclidean inner product has so many of the familiar properties of Euclidean geometry, it is often called ***Euclidean n-space*** or ***n-dimensional Euclidean space***.

Exercise Set 1.2

In Exercises 1 and 2, find the norm of $\mathbf{v}$, a unit vector that has the same direction as $\mathbf{v}$, and a unit vector that is oppositely directed to $\mathbf{v}$.

1. (a) $\mathbf{v} = (4, -3)$ (b) $\mathbf{v} = (2, 2, 2)$
(c) $\mathbf{v} = (1, 0, 2, 1, 3)$

2. (a) $\mathbf{v} = (-5, 12)$ (b) $\mathbf{v} = (1, -1, 2)$
(c) $\mathbf{v} = (-2, 3, 3, -1)$

In Exercises 3 and 4, evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$.

3. (a) $\|\mathbf{u} + \mathbf{v}\|$ (b) $\|\mathbf{u}\| + \|\mathbf{v}\|$
(c) $\|-2\mathbf{u} + 2\mathbf{v}\|$ (d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$

4. (a) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$ (b) $\|\mathbf{u} - \mathbf{v}\|$
(c) $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$ (d) $\|\mathbf{u}\| - \|\mathbf{v}\|$

In Exercises 5 and 6, evaluate the given expression with $\mathbf{u} = (-2, -1, 4, 5)$, $\mathbf{v} = (3, 1, -5, 7)$, and $\mathbf{w} = (-6, 2, 1, 1)$.

5. (a) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$ (b) $\|3\mathbf{u}\| - 5\|\mathbf{v}\| + \|\mathbf{u}\|$
(c) $\|-\|\mathbf{u}\|\mathbf{v}\|$

6. (a) $\|\mathbf{u}\| - 2\|\mathbf{v}\| - 3\|\mathbf{w}\|$ (b) $\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$
(c) $\big\|\,\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\big\|$

7. Let $\mathbf{v} = (-2, 3, 0, 6)$. Find all scalars k such that $\|k\mathbf{v}\| = 5$.

8. Let $\mathbf{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|k\mathbf{v}\| = 4$.

In Exercises 9 and 10, find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$.

9. (a) $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (2, 2, -4)$
(b) $\mathbf{u} = (1, 1, 4, 6)$, $\mathbf{v} = (2, -2, 3, -2)$

10. (a) $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$
(b) $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

In Exercises 11 and 12, find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$.

11. (a) $\mathbf{u} = (3, 3, 3)$, $\mathbf{v} = (1, 0, 4)$
(b) $\mathbf{u} = (0, -2, -1, 1)$, $\mathbf{v} = (-3, 2, 4, 4)$
(c) $\mathbf{u} = (3, -3, -2, 0, -3, 13, 5)$, $\mathbf{v} = (-4, 1, -1, 5, 0, -11, 4)$

12. (a) $\mathbf{u} = (1, 2, -3, 0)$, $\mathbf{v} = (5, 1, 2, -2)$
(b) $\mathbf{u} = (2, -1, -4, 1, 0, 6, -3, 1)$, $\mathbf{v} = (-2, -1, 0, 3, 7, 2, -5, 1)$
(c) $\mathbf{u} = (0, 1, 1, 1, 2)$, $\mathbf{v} = (2, 1, 0, -1, 3)$

13. Find the cosine of the angle between the vectors in each part of Exercise 11, and then state whether the angle is acute, obtuse, or a right angle.

14. Find the cosine of the angle between the vectors in each part of Exercise 12, and then state whether the angle is acute, obtuse, or a right angle.

15. A vector $\mathbf{a}$ in the xy-plane has a length of 9 units and points in a direction that is 120° counterclockwise from the positive x-axis, and a vector $\mathbf{b}$ in that plane has a length of 5 units and points in the positive y-direction. Find $\mathbf{a} \cdot \mathbf{b}$.

16. A vector $\mathbf{a}$ in the xy-plane points in a direction that is 47° counterclockwise from the positive x-axis, and a vector $\mathbf{b}$ in that plane points in a direction that is 43° clockwise from the positive x-axis. What can you say about the value of $\mathbf{a} \cdot \mathbf{b}$?

17. Solve the equation $5\mathbf{x} - 2\mathbf{v} = 2(\mathbf{w} - 5\mathbf{x})$ for $\mathbf{x}$, given th $\mathbf{v} = (1, 2, -4, 0)$ and $\mathbf{w} = (-3, 5, 1, 1)$.

18. Solve the equation $5\mathbf{x} - \|\mathbf{v}\|\mathbf{v} = \|\mathbf{w}\|(\mathbf{w} - 5\mathbf{x})$ for $\mathbf{x}$ with and $\mathbf{w}$ being the vectors in Exercise 17.

In Exercises 19 and 20, determine whether the expression makes sense mathematically. If not, explain why.

19. (a) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
(c) $\|\mathbf{u} \cdot \mathbf{v}\|$ (d) $(\mathbf{u} \cdot \mathbf{v}) - \|\mathbf{u}\|$

20. (a) $\|\mathbf{u}\| \cdot \|\mathbf{v}\|$ (b) $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{w}$
(c) $(\mathbf{u} \cdot \mathbf{v}) - k$ (d) $k \cdot \mathbf{u}$

In Exercises 21 and 22, verify that the Cauchy–Schwarz inequality holds.

21. (a) $\mathbf{u} = (3, 2)$, $\mathbf{v} = (4, -1)$
(b) $\mathbf{u} = (-3, 1, 0)$, $\mathbf{v} = (2, -1, 3)$
(c) $\mathbf{u} = (0, 2, 2, 1)$, $\mathbf{v} = (1, 1, 1, 1)$

22. (a) $\mathbf{u} = (4, 1, 1)$, $\mathbf{v} = (1, 2, 3)$
(b) $\mathbf{u} = (1, 2, 1, 2, 3)$, $\mathbf{v} = (0, 1, 1, 5, -2)$
(c) $\mathbf{u} = (1, 3, 5, 2, 0, 1)$, $\mathbf{v} = (0, 2, 4, 1, 3, 5)$

In Exercises 23 and 24, show that the vectors form an orthonormal set.

23. $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, -\frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$,
$\mathbf{v}_3 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, -\frac{5}{6}\right)$, $\mathbf{v}_4 = \left(\frac{1}{2}, \frac{1}{6}, -\frac{5}{6}, \frac{1}{6}\right)$

24. $\mathbf{v}_1 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$, $\mathbf{v}_2 = \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$,
$\mathbf{v}_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$

25. Find two unit vectors that are orthogonal to the nonzer vector $\mathbf{u} = (a, b)$.

26. For what values of k, if any, are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?
(a) $\mathbf{u} = (2, k, k)$, $\mathbf{v} = (1, 7, k)$
(b) $\mathbf{u} = (k, k, 1)$, $\mathbf{v} = (k, 5, 6)$

27. For which values of k, if any, are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?
(a) $\mathbf{u} = (k, 1, 3)$, $\mathbf{v} = (1, 7, k)$
(b) $\mathbf{u} = (-2, k, k)$, $\mathbf{v} = (k, 5, k)$

28. Use vectors to find the cosines of the interior angles of th triangle with vertices $A(0, -1)$, $B(1, -2)$, and $C(4, 1)$.

29. Use vectors to show that $A(3, 0, 2)$, $B(4, 3, 0)$, an $C(8, 1, -1)$ are vertices of a right triangle. At which verte is the right angle?

30. In each part determine whether the given number is a vali ISBN by computing its check digit.
(a) 1-56592-170-7 (b) 0-471-05333-5

31. In each part determine whether the given number is a vali ISBN by computing its check digit.
(a) 0-471-06368-1 (b) 0-13-947752-3

It will be convenient to have a more compact way of writing expressions such as $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ and $x_1^2 + x_2^2 + \cdots + x_n^2$ that arise in working with vectors in R^n. For this purpose we will use ***sigma notation*** (also called ***summation notation***), which uses the Greek letter Σ (capital sigma) to indicate that a sum is to be formed. To illustrate how the notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form k^2, where k is an integer between 1 and 5, inclusive. This sum can be written in sigma notation as

$$\sum_{k=1}^{5} k^2$$

This directs us to form the sum of the terms that result by substituting successive integers for k, starting with $k = 1$ and ending with $k = 5$. In general, if $f(k)$ is a function of k, and if m and n are integers with $m \leq n$, then

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \cdots + f(n)$$

This is the sum of the terms that result by substituting successive integers for k, starting with $k = m$ and ending with $k = n$. The number m is called the ***lower limit of summation***, the number n the ***upper limit of summation***, and the letter k the ***index of summation***. It is not essential to use k as the index of summation; any letter can be used, though we will generally use i, j, or k. Thus,

$$\sum_{k=1}^{n} a_k = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the norm of $\mathbf{u}$ and the dot product of $\mathbf{u}$ and $\mathbf{v}$ can be expressed in sigma notation as

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{\sum_{k=1}^{n} u_k^2}$$

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{k=1}^{n} u_kv_k$$

32. (***Sigma notation***) In each part, write the sum in sigma notation.
(a) $a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$
(b) $c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2$
(c) $b_3 + b_4 + \cdots + b_n$

33. (***Sigma notation***) Write Formula (11) in sigma notation.

34. (***Sigma notation***) In each part, evaluate the sum for

$$c_1 = 3, c_2 = -1, c_3 = 5, c_4 = -6, c_5 = 4$$
$$d_1 = 6, d_2 = 0, d_3 = 7, d_4 = -2, d_5 = -3$$

(a) $\sum_{k=1}^{4} c_k + \sum_{k=2}^{5} d_k$ (b) $\sum_{j=1}^{5} (2c_j - d_j)$

(c) $\sum_{k=1}^{5} (-1)^k c_k$

35. (***Sigma notation***) In each part, confirm the statement by writing out the sums on the two sides.

(a) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(b) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(c) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

Discussion and Discovery

D1. Write a paragraph or two that explains some of the similarities and differences between visible space and higher-dimensional spaces. Include an explanation of why R^n is referred to as *Euclidean space*.

D2. What can you say about k and $\mathbf{v}$ if $\|k\mathbf{v}\| = k\|\mathbf{v}\|$?

D3. (a) The set of all vectors in R^2 that are orthogonal to a nonzero vector is what kind of geometric object?
(b) The set of all vectors in R^3 that are orthogonal to a nonzero vector is what kind of geometric object?
(c) The set of all vectors in R^2 that are orthogonal to two noncollinear vectors is what kind of geometric object?
(d) The set of all vectors in R^3 that are orthogonal to two noncollinear vectors is what kind of geometric object?

D4. Show that $\mathbf{v}_1 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\mathbf{v}_2 = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ are orthonormal vectors, and find a third vector $\mathbf{v}_3$ for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set.

D5. Something is wrong with one of the following expressions. Which one is it, and what is wrong?

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}), \quad \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}, \quad (\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$$

D6. Let $\mathbf{x} = (x, y)$ and $\mathbf{x}_0 = (x_0, y_0)$. Write down an equality or inequality involving norms that describes
(a) the circle of radius 1 centered at $\mathbf{x}_0$;
(b) the set of points inside the circle in part (a);
(c) the set of points outside the circle in part (a).

D7. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in R^n such that $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 1$, then $d(\mathbf{u}, \mathbf{v}) =$ ________. Draw a picture to illustrate your result in R^2.

where the symbol Π (capital Greek pi) directs you to form the product of all factors $(x_j$ whose subscripts satisfy the specified inequalities. As in the 3×3 case, this product is n‹ if $x_1, x_2, \ldots, x_n$ are distinct, which proves Theorem 2.3.1.

CROSS PRODUCTS

A basic problem in the study of rotational motion in 3-space is to find the axis of rotat a spinning object and to identify whether the rotation is clockwise or counterclockwise i specified point of view along the axis. To formulate this problem in vector terms, suppo: some rotation about an axis through the origin of R^3 causes a nonzero vector **u** to rotate nonzero vector **v**. Since the axis of rotation must be perpendicular to the plane of **u** and nonzero vector **w** that is orthogonal to the plane of **u** and **v** will serve to identify the orie of the rotational axis (Figure 4.3.4). Moreover, if the direction of **w** can be chosen so the r of **u** into **v** appears counterclockwise looking toward the origin from the terminal point then the vector **w** will carry all of the information needed to identify both the orientation axis and the direction of rotation. Accordingly, our goal in this subsection is to define kind of vector multiplication that will produce **w** when **u** and **v** are known.

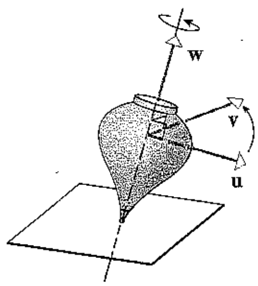


Figure 4.3.4

Definition 4.3.7 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors i then the ***cross product of* u *with* v**, denoted by $\mathbf{u} \times \mathbf{v}$, is the vector i defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

or equivalently,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Linear Algebra in History

The cross product notation $A \times B$ was introduced by the American physicist and mathematician J. Willard Gibbs in a series of unpublished lecture notes for his students at Yale University. It appeared in a published work for the first time in the second edition of the book *Vector Analysis*, by Edwin Wilson (1879–1964), a student of Gibbs. Gibbs originally referred to $A \times B$ as the "skew product."

REMARK Note that the cross product of vectors is a vector, whereas t product of vectors is a scalar.

A good way to remember Formula (10) is to express $\mathbf{u} \times \mathbf{v}$ in terms standard unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ and to wri in the form of a 3×3 determinant as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

You should confirm that the cross product formula on the right side resu expanding the 3×3 determinant* by cofactors along the first row.

EXAMPLE 9 Calculating a Cross Product

Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$. Find

(a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$ (c) $\mathbf{u} \times \mathbf{u}$

Solution (a)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k} = (2, -7$$

*This is not a determinant in the usual sense, since true determinants have scalar entries. Thus, you shou of this formula as a convenient mnemonic device.

Solution (b) We could proceed as in part (a), but a simpler approach is to observe that interchanging **u** and **v** in a cross product interchanges the rows of the 2×2 determinants on the right side of (12), and hence reverses the sign of each component. Thus, it follows from part (a) that

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}) = (-2, 7, 6)$$

Solution (c) If $\mathbf{u} = \mathbf{v}$, then each of the 2×2 determinants on the right side of (11) is zero because its rows are identical. Thus,

$$\mathbf{u} \times \mathbf{u} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = (0, 0, 0) = \mathbf{0}$$ ■

The following theorem summarizes some basic properties of cross products that can be derived from properties of determinants. Some of the proofs are given as exercises.

Theorem 4.3.8 *If* **u**, **v**, *and* **w** *are vectors in* R^3 *and* k *is a scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Recall that one of our goals in defining the cross product of **u** with **v** was to create a vector that is orthogonal to the plane of **u** and **v**. The following theorem shows that $\mathbf{u} \times \mathbf{v}$ has this property.

Theorem 4.3.9 *If* **u** *and* **v** *are vectors in* R^3, *then:*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ **[u × v is orthogonal to u]**

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ **[u × v is orthogonal to v]**

We will prove part (*a*); the proof of part (*b*) is similar.

Proof (a) If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then it follows from Formula (10) that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$$ ■

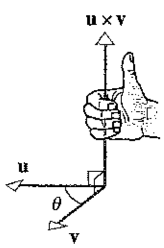


Figure 4.3.5

In general, if **u** and **v** are nonzero and nonparallel vectors, then the direction of $\mathbf{u} \times \mathbf{v}$ in relation to **u** and **v** can be determined by the following right-hand rule:* If the fingers of the right hand are cupped so they curl in the direction of rotation that takes **u** into **v** in at most 180°, then the thumb will point (roughly) in the direction of $\mathbf{u} \times \mathbf{v}$ (Figure 4.3.5). It should be evident from this rule that the direction of rotation from **u** to **v** will appear to be counterclockwise to an observer looking toward the origin from the terminal point of $\mathbf{u} \times \mathbf{v}$. We will not prove this fact, but we will illustrate it with the six possible cross products of the standard unit vectors **i**, **j**, and **k**:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array} \tag{13}$$

*Recall that we agreed to consider only right-handed coordinate systems in this text. Had we used left-handed coordinate systems instead, then a left-hand rule would apply here.

These products are easy to derive; for example,

$$\mathbf{i} \times \mathbf{j} = (1, 0, 0) \times (0, 1, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \mathbf{k}$$

As predicted by the right-hand rule, the 90° rotation of $\mathbf{i}$ into $\mathbf{j}$ appears to be counterclockwise looking toward the origin from the terminal point of $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and the 90° rotation of $\mathbf{j}$ into $\mathbf{k}$ appears to be counterclockwise looking toward the origin from the terminal point of $\mathbf{j} \times \mathbf{k} = \mathbf{i}$.

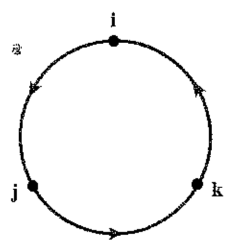


Figure 4.3.6

CONCEPT PROBLEM Confirm that the remaining four cross products in (13) satisfy the right-hand rule.

REMARK A useful way of remembering the six cross products in (13) is to use the diagram in Figure 4.3.6. In this diagram, the cross product of two consecutive vectors in the counterclockwise direction is the next vector around, and the cross product of two consecutive vectors in the clockwise direction is the negative of the next vector around.

WARNING We can write a product of three real numbers as abc and the product of three matrices as ABC because the associative laws $a(bc) = (ab)c$ and $A(BC) = (AB)C$ ensure that the same result is obtained no matter how parentheses are inserted. However, the associative law does *not* hold for cross products; for example,

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0} \quad \text{whereas} \quad (\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

so $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$. Accordingly, expressions such as $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ should never be used because they are ambiguous.

The next theorem is concerned with the length of a cross product.

Theorem 4.3.10 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in* R^3, *and let* θ *be the angle between these vectors.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$

(*b*) *The area A of the parallelogram that has* $\mathbf{u}$ *and* $\mathbf{v}$ *as adjacent sides is*

$$A = \|\mathbf{u} \times \mathbf{v}\| \tag{14}$$

Proof (*a*) Since $0 \le \theta \le \pi$, it follows that $\sin\theta \ge 0$ and hence that

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

Thus,

$$\begin{aligned}
\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1 - \cos^2\theta} \\
&= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \qquad \textbf{[Theorem 1.2.8]} \\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\
&= \sqrt{\left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\
&= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2} \\
&= \|\mathbf{u} \times \mathbf{v}\| \qquad \textbf{[Formula (10)]}
\end{aligned}$$

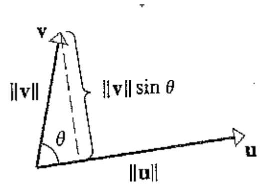


Figure 4.3.7

Proof (*b*) Referring to Figure 4.3.7, the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides can be viewed as having a base of length $\|\mathbf{u}\|$ and an altitude of length $\|\mathbf{v}\| \sin\theta$. Thus, its area A is

$$A = \text{(base)(altitude)} = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\| \qquad \blacksquare$$

EXAMPLE 10 Area of a Triangle in 3-Space

Find the area of the triangle in R^3 that has vertices $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution The area A of the triangle is half the area of the parallelogram that has adjacent sides $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ (Figure 4.3.8). Thus,

$$A = \tfrac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\|$$

We will leave it for you to show that $\overrightarrow{P_1P_2} = (-3, -2, 2)$ and $\overrightarrow{P_1P_3} = (-2, 2, 3)$ and also that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = -10\mathbf{i} + 5\mathbf{j} - 10\mathbf{k} = (-10, 5, -10)$$

Thus,

$$A = \tfrac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\| = \tfrac{1}{2}\sqrt{225} = \tfrac{15}{2}$$

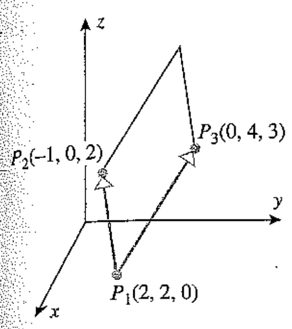


Figure 4.3.8

Exercise Set 4.3

> In Exercises 1–4, find the adjoint of A, and then compute A^{-1} using Theorem 4.3.3.

1. $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$

> In Exercises 5–10, solve the equations by Cramer's rule, where applicable.

5. $\begin{aligned} 7x_1 - 3x_2 &= 3 \\ 3x_1 + x_2 &= 5 \end{aligned}$

6. $\begin{aligned} 4x + 5y &= -8 \\ 11x + y &= 29 \end{aligned}$

7. $\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$

8. $\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ 2x_1 - x_2 &= -2 \\ 4x_1 - 3x_3 &= 0 \end{aligned}$

9. $\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 11 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{aligned}$

10. $\begin{aligned} 2x_1 + 2x_2 - x_3 + x_4 &= 4 \\ 4x_1 + 3x_2 - x_3 + 2x_4 &= 6 \\ 8x_1 + 5x_2 - 3x_3 + 4x_4 &= 12 \\ 3x_1 + 3x_2 - 2x_3 + 2x_4 &= 6 \end{aligned}$

11. Find x without solving for y and z:

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x - 2y - z &= 2 \\ 3x + y + z &= 4 \end{aligned}$$

12. Find y without solving for x and z:

$$\begin{aligned} x + 2y + 3z &= -2 \\ 3x - y + z &= 1 \\ -x + 4y - 2z &= -3 \end{aligned}$$

13. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible for all values of θ, and use Theorem 4.3.3 to find A^{-1}.

14. Show that the matrix

$$A = \begin{bmatrix} \tan\alpha & -1 & 0 \\ 1 & \tan\alpha & 0 \\ 0 & 0 & \cos^2\alpha \end{bmatrix}$$

is invertible for $\alpha \neq \frac{\pi}{2} + n\pi$, where n is an integer. Then use Theorem 4.3.3 to find A^{-1}.

> In Exercises 15 and 16, use determinants to find all values of k for which the system has a unique solution, and for those cases use Cramer's rule to find the solution.

15. $\begin{aligned} 3x + 3y + z &= 1 \\ 4x + ky + 2z &= 2 \\ 2kx + 2ky + kz &= 1 \end{aligned}$

16. $\begin{aligned} 2x + 3ky - kz &= 1 \\ x - y + 2z &= -1 \\ 3kx + 2y - z &= 3 \end{aligned}$

> In Exercises 17 and 18, state when the inverse of the matrix A exists in terms of the parameters that the matrix contains.

17. $A = \begin{bmatrix} x & y & 0 \\ 0 & x & y \\ y & x & 0 \end{bmatrix}$

18. $A = \begin{bmatrix} s & 0 & t & 0 \\ 0 & s & 0 & t \\ t & 0 & s & 0 \\ 0 & t & 0 & s \end{bmatrix}$

In Exercises 19 and 20, find the area of the parallelogram determined by the columns of A.

19. $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ **20.** $A = \begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix}$

In Exercises 21 and 22, find the volume of the parallelepiped determined by the columns of A.

21. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ **22.** $A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

In Exercises 23 and 24, find the area of the parallelogram with the given vertices.

23. $P_1(1,2)$, $P_2(4,4)$, $P_3(7,5)$, $P_4(4,3)$

24. $P_1(3,2)$, $P_2(5,4)$, $P_3(9,4)$, $P_4(7,2)$

In Exercises 25 and 26, find the area of the triangle with the given vertices.

25. $A(2,0)$, $B(3,4)$, $C(-1,2)$

26. $A(1,1)$, $B(2,2)$, $C(3,-3)$

In Exercises 27 and 28, find the volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

27. $\mathbf{u} = (2,-6,2)$, $\mathbf{v} = (0,4,-2)$, $\mathbf{w} = (2,2,-4)$

28. $\mathbf{u} = (3,1,2)$, $\mathbf{v} = (4,5,1)$, $\mathbf{w} = (1,2,4)$

In Exercises 29 and 30, determine whether $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie in the same plane when positioned so that their initial points coincide.

29. $\mathbf{u} = (-1,-2,1)$, $\mathbf{v} = (3,0,-2)$, $\mathbf{w} = (5,-4,0)$

30. $\mathbf{u} = (5,-2,1)$, $\mathbf{v} = (4,-1,1)$, $\mathbf{w} = (1,-1,0)$

31. Find all unit vectors parallel to the yz-plane that are orthogonal to the vector $(3,-1,2)$.

32. Find all unit vectors in the plane determined by $\mathbf{u} = (3,0,1)$ and $\mathbf{v} = (1,-1,1)$ that are orthogonal to the vector $\mathbf{w} = (1,2,0)$.

33. Use the cross product to find the sine of the angle between the vectors $\mathbf{u} = (2,3,-6)$ and $\mathbf{v} = (2,3,6)$.

34. (a) Find the area of the triangle having vertices $A(1,0,1)$, $B(0,2,3)$, and $C(2,1,0)$.
(b) Use the result of part (a) to find the length of the altitude from vertex C to side AB.

In Exercises 35 and 36, let $\mathbf{u} = (3,2,-1)$, $\mathbf{v} = (0,2,-3)$, and $\mathbf{w} = (2,6,7)$. Compute the indicated vectors.

35. (a) $\mathbf{v} \times \mathbf{w}$ (b) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
(c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

36. (a) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$ (b) $\mathbf{u} \times (\mathbf{v} - 2\mathbf{w})$
(c) $(\mathbf{u} \times \mathbf{v}) - 2\mathbf{w}$

In Exercises 37 and 38, find a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

37. (a) $\mathbf{u} = (-6,4,2)$, $\mathbf{v} = (3,1,5)$
(b) $\mathbf{u} = (-2,1,5)$, $\mathbf{v} = (3,0,-3)$

38. (a) $\mathbf{u} = (1,1,-2)$, $\mathbf{v} = (2,-1,2)$
(b) $\mathbf{u} = (3,3,1)$, $\mathbf{v} = (0,4,2)$

In Exercises 39–42, show that the given identities hold for any $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in R^3, and any scalar k.

39. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

40. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

41. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

42. $\mathbf{u} \times \mathbf{0} = \mathbf{0}$ and $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

In Exercises 43 and 44, find the area of the parallelogram determined by the given vectors $\mathbf{u}$ and $\mathbf{v}$.

43. (a) $\mathbf{u} = (1,-1,2)$, $\mathbf{v} = (0,3,1)$
(b) $\mathbf{u} = (2,3,0)$, $\mathbf{v} = (-1,2,-2)$

44. (a) $\mathbf{u} = (3,-1,4)$, $\mathbf{v} = (6,-2,8)$
(b) $\mathbf{u} = (1,1,1)$, $\mathbf{v} = (3,2,-5)$

In Exercises 45 and 46, find the area of the triangle in 3-space that has the given vertices.

45. $P_1(2,6,-1)$, $P_2(1,1,1)$, $P_3(4,6,2)$

46. $P(1,-1,2)$, $Q(0,3,4)$, $R(6,1,8)$

47. Show that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are any vectors in 3-space, then

$$(\mathbf{a} + \mathbf{d}) \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$$

48. Simplify $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

49. Find a vector that is perpendicular to the plane determined by the points $A(0,-2,1)$, $B(1,-1,-2)$, and $C(-1,1,0)$.

50. Consider the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ in R^3. The expression $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
(a) Show that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

(b) Give a geometric interpretation of $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ (vertical bars denote absolute value).

51. Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{bmatrix}$$

(a) Evaluate A^{-1} using Theorem 4.3.3.
(b) Evaluate A^{-1} using the method of Example 3 in Section 3.3.
(c) Which method involves less computation?

52. Suppose that A is nilpotent, that is, $A^k = 0$ for some k. Use properties of the determinant to show that A is not invertible.

53. Show that if **u**, **v**, and **w** are vectors in R^3, with no two of them collinear, then $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane determined by **v** and **w**.

54. Show that if **u**, **v**, and **w** are vectors in R^3, with no two of them collinear, then $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane determined by **u** and **v**.

55. Use Formula (2) to show that the inverse of an invertible upper triangular matrix is upper triangular. [*Hint:* Examine which terms in the adjoint matrix of an upper triangular matrix must be zero.]

56. Use Formula (2) to show that the inverse of an invertible lower triangular matrix is lower triangular. [*Hint:* Examine which terms in the adjoint matrix of a lower triangular matrix must be zero.]

57. Use Cramer's rule to find a polynomial of degree 3 that passes through the points $(0, 1)$, $(1, -1)$, $(2, -1)$, and $(3, 7)$.

Discussion and Discovery

D1. Suppose that **u** and **v** are noncollinear vectors with their initial points at the origin in 3-space.
(a) Make a sketch that illustrates how $\mathbf{w} = \mathbf{v} \times (\mathbf{u} \times \mathbf{v})$ is oriented in relation to **u** and **v**.
(b) What can you say about the values of $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$? Explain your reasoning.

D2. If $\mathbf{u} \neq \mathbf{0}$, is it valid to cancel **u** from both sides of the equation $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and conclude that $\mathbf{v} = \mathbf{w}$? Explain your reasoning.

D3. Something is wrong with one of the following expressions. Which one is it and what is wrong?

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}), \quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}), \quad (\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$$

D4. What can you say about the vectors **u** and **v** if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$?

D5. Give some examples of other algebraic rules that hold for multiplication of real numbers but not for the cross product of vectors.

D6. Solve the following system by Cramer's rule:

$$\begin{aligned} cx_1 - (1-c)x_2 &= 3 \\ (1-c)x_1 + \qquad cx_2 &= -4 \end{aligned}$$

For what values of c is this solution valid? Explain.

D7. Let $A\mathbf{x} = \mathbf{b}$ be the system in Exercise 12.
(a) Solve by Cramer's rule.
(b) Solve by Gauss–Jordan elimination.
(c) Which method involves less computation?

D8. Indicate whether the statement is true (T) or false (F). Justify your answer.
(a) If A is a square matrix, then $A\,\mathrm{adj}(A)$ is a diagonal matrix.
(b) Cramer's rule can be used to solve any system of linear equations if the number of equations is the same as the number of unknowns.
(c) If A is invertible, then $\mathrm{adj}(A)$ must also be invertible.
(d) If A has a row of zeros, then so does $\mathrm{adj}(A)$.
(e) If **u**, **v**, and **w** are vectors in R^3, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

Working with Proofs

P1. Prove that if **u** and **v** are nonzero, nonorthogonal vectors in R^3, and θ is the angle between them, then

$$\tan\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{(\mathbf{u} \cdot \mathbf{v})}$$

P2. Prove that if **u** and **v** are nonzero vectors in R^2 and α and β are the angles in the accompanying figure, then

$$\cos(\alpha - \beta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

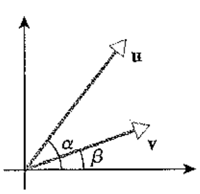


Figure Ex-P2