TORONTO *LIFE SCIENCES*

2007 Test Year: Test 2

Nov

MAT135Y1 | 2007

Solutions

Your Key to Success

Our Instructor: **James T. Nguyen**, B.Sc., M.Sc., CFA (IPR)., MBA (IPR)., DDS (IPR)

www.TLifeSci.com

Code: 0286

PLEASE READ CAREFULLY:
Each of the following 20 multiple-choice questions has exactly one correct answer. **Indicate your answer to each question by completely filling in the appropriate circle in the ANSWER BOX on the front page. Use a sharp dark pencil!**

MARKING SCHEME: 5 marks for a correct answer, 0 for no answer, a wrong answer or an unclear answer or indicating more than one answer. You are not required to justify your answers.

ADVICE: Once you have done a question, you should indicate your answer on the front page immediately. Don't wait till the end of the test to transfer your answers from the inside pages to the front page!

WARNING: Your computations and answers indicated on these inside pages will NOT count. **Only the final answers indicated in the ANSWER BOX on the front page will count.** If you have done a question correctly but have indicated a wrong answer on the front page due to carelessness (or whatever reason), you will get a zero for that question.

1. Find the value of $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$. $\left(\frac{0}{0}\right)$

Ⓐ 1

Ⓑ undefined

Ⓒ 0

Ⓓ $\sqrt{2}$

✓ Ⓔ $\frac{1}{2}$

Use L'Hôp Rule:

$$\stackrel{H}{=} \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$$

or use L'Hôp Rule again

2. Find the value of $\lim_{x\to\infty} \dfrac{x^4 - x + \sin x}{-x^3 - 3x^4 - 2\cos x}$.

Ⓐ 1

✓ Ⓑ $-\frac{1}{3}$

Ⓒ $-\frac{1}{2}$

Ⓓ -1

Ⓔ $-\infty$

(just use the coefficient theorem!)

$$\lim_{x\to\infty} \frac{1 - \frac{1}{x^3} + \frac{\sin x}{x^4}}{-\frac{1}{x} - 3 - \frac{2\cos x}{x^4}}$$

INDICATE YOUR ANSWERS ON THE FRONT PAGE IMMEDIATELY.

For your own record, you may also want to indicate your answers on these inside pages.

3. Let

$$f(x) = \begin{cases} 4(x-2) - \dfrac{\sin(8x)}{kx} & \text{if} \quad x < 0 \\ 2(x+k) & \text{if} \quad x \geq 0. \end{cases}$$

Find the value of the constant k so that f is continuous everywhere.

Ⓐ ✓ -2

Ⓑ $-\frac{1}{4}$

Ⓒ 0

Ⓓ $\frac{1}{2}$

Ⓔ $\frac{1}{4}$

Since $f(x)$ is continuous $\Rightarrow$ Limit exist

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-} 4(x-2) - \frac{\sin(8x)}{kx} = \lim_{x\to 0^+} 2(x+k)$$

$$-8 - \frac{8}{k} = 2k$$

$$-8k - 8 = 2k^2 \Rightarrow -4k - 4 = k^2$$

$$\Rightarrow k^2 + 4k + 4 = 0$$

$$\Rightarrow (k+2)^2 = 0 \Rightarrow \boxed{k = -2}$$

4. The graph of $f(x) = \dfrac{x}{(x+6)^2}$ has a horizontal tangent line at $x =$

Ⓐ 4

Ⓑ 3

Ⓒ 2

Ⓓ 5

Ⓔ ✓ 6

$$y = \frac{x}{(x+6)^2} \Rightarrow y' = \frac{(x+6)^2 \cdot 1 - x\, 2(x+6)}{(x+6)^4}$$

$$y' = 0 \Rightarrow (x+6)^2 + 2x(x+6) = 0$$

$$\Rightarrow x + 6 + 2x = 0$$

$$\Rightarrow \boxed{x = 6}$$

INDICATE YOUR ANSWERS ON THE FRONT PAGE IMMEDIATELY.

5. A ball is being thrown vertically upward so that its height (above ground) t seconds after it is thrown is $(25+16t-16t^2)$ feet. What is the maximum height (above ground) attained by the ball?

   Ⓐ 32 feet

   Ⓑ 31 feet

   Ⓒ 30 feet

   Ⓓ 28 feet

   Ⓔ 29 feet

$h(t) = 25 + 16t - 16t^2$

$\Rightarrow v(t) = 16 - 32t$

$\Rightarrow v(t) = 0 \Rightarrow 16 - 32t = 0 \Rightarrow \boxed{t = 1/2}$

$h(1/2) = 25 + 16\left(\frac{1}{2}\right) - 16\left(\frac{1}{2}\right)^2$

$= 29$ feet

6. If $xy^3 + xy = 6$, find the value of $\dfrac{dy}{dx}$ at the point where $x = 3,\ y = 1$.

   Ⓐ $\frac{1}{2}$

   Ⓑ 0

   Ⓒ $\frac{1}{3}$

   Ⓓ $-\frac{1}{6}$

   Ⓔ $-\frac{1}{4}$

$xy^3 + xy = 6$

$\Rightarrow y^3 + x \cdot 3y^2 y' + y + xy' = 0$

$1 + 9y' + 1 + 3y' = 0$

$\Rightarrow y' = -1/6$

INDICATE YOUR ANSWERS ON THE FRONT PAGE IMMEDIATELY.

7. Find the value of $\lim\limits_{x\to\infty} \dfrac{2\sinh x + \cosh x}{e^x}$.

Ⓐ 3

Ⓑ 0

Ⓒ $\dfrac{3}{2}$ ✓

Ⓓ -1

Ⓔ undefined

$$= \lim_{x\to\infty} \frac{2\cdot\frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x})}{e^x}$$

$$= \lim_{x\to\infty} (1 - e^{-2x}) + \tfrac{1}{2}(1 + e^{-2x})$$

$$= \left(1 - \frac{1}{e^\infty}\right) + \frac{1}{2}\left(1 + \frac{1}{e^\infty}\right)$$

$$= 1 + 1/2 = 3/2$$

8. A rectangular box has a square base. If the height of the box is increasing at 3 cm/min and each edge of its base is increasing at 2 cm/min, how fast will the volume of the box be increasing when the height of the box is 4 cm and the area of its base is 9 sq cm?

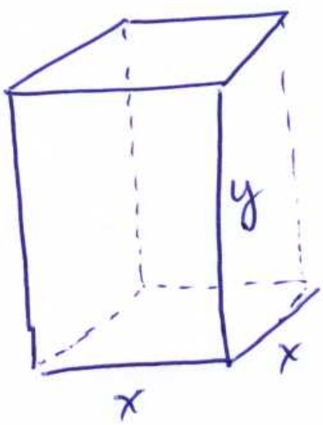
Ⓐ 85 cc/min.

Ⓑ 65 cc/min.

Ⓒ 80 cc/min.

Ⓓ 75 cc/min. ✓

Ⓔ 70 cc/min.



$$V = x^2 y$$

$$\frac{dV}{dt} = 2x\frac{dx}{dt}y + x^2\frac{dy}{dt}$$

$$= 2(3)(2)(4) + 9(3)$$

$$= 75$$

$$\frac{dy}{dt} = 3 \qquad \frac{dx}{dt} = 2$$

(at $y = 4$ and $x^2 = 9$
$\Rightarrow x = 3$)

INDICATE YOUR ANSWERS ON THE FRONT PAGE IMMEDIATELY.

9. Find the number c which satisfies the conclusion of the Mean Value Theorem for the function $f(x) = \frac{1}{x}$ on $[1, 3]$.

Ⓐ $\frac{3}{2}$

Ⓑ $\sqrt{3}$ ✓

Ⓒ 2

Ⓓ $\frac{5}{2}$

Ⓔ $\frac{5}{4}$

By MVT there's a "c" $\in [1,3]$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

$f'(c) = \frac{f(3) - f(1)}{3-1}$

$f'(c) = \frac{\frac{1}{3} - \frac{1}{1}}{2} = \frac{-2/3}{2} = -1/3$

note: $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$\Rightarrow -\frac{1}{c^2} = -1/3$

$\Rightarrow c^2 = 3$

$\Rightarrow c = \sqrt{3}$ (since $c > 0$)

10. The function $f(x) = (x^2 - 4)^2$ has a local maximum at $x =$

Ⓐ 0 ✓

Ⓑ 2

Ⓒ $-\sqrt{2}$

Ⓓ -2

Ⓔ $\sqrt{2}$

$f'(x) = 2(x^2-4) \cdot 2x$

$f'(x) = 4x(x-2)(x+2)$

$f'(x) = 0 \Rightarrow x = 0, -2, 2$

− + − +

−2 0 2

local max at $x=0$

INDICATE YOUR ANSWERS ON THE FRONT PAGE IMMEDIATELY.

11. How many points of inflection does the graph of $y = x^6 + x^4 + x^2 - 5x - 4$ have?

- Ⓐ two
- Ⓑ one
- Ⓒ none ✓
- Ⓓ three
- Ⓔ more than three

$f'(x) = 6x^5 + 4x^3 + 2x - 5$

$f''(x) = 30x^4 + 12x^2 + 2$

$f''(x) = 2\underbrace{[15x^4 + 6x^2 + 1]}$

$\Rightarrow$ note always positive **

Can never change sign's

$\Rightarrow$ no pts of Inflection

12. The graph of $y = x^4 - 6x^3 - 3x + 4$ is concave downward on

- Ⓐ $(-\infty, 0) \bigcup (3, \infty)$
- Ⓑ $(0, 3)$ ✓
- Ⓒ $(2, 5)$
- Ⓓ $(-2, 1)$
- Ⓔ $(1, 4)$

$f'(x) = 4x^3 - 18x^2 - 3$

$f''(x) = 12x^2 - 36x$

$f''(x) = 12x(x-3)$

$f''(x) = 0 \Rightarrow x = 0, 3$

+   −   +

0   3

concave down!

on $(0,3)$

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13. The graph of $y = \dfrac{3x^3 + x^2 - 7x - 4}{x^2 + 2x + 1}$ has one vertical asymptote and one other asymptote. This other asymptote is the line

(By long divison always)

Ⓐ $y = 3x + \dfrac{1}{2}$

Ⓑ $y = 3x + 1$

Ⓒ $y = 3x$

Ⓓ $y = 3x - 5$ ✓

Ⓔ $y = 3x - 4$

$$
\begin{array}{r}
3x - 5 \\
x^2 + 2x + 1 \;\big)\; 3x^3 + x^2 - 7x - 4 \\
-\,3x^3 + 6x^2 + 3x \\
\hline
-5x^2 - 10x - 4
\end{array}
$$

done! (no need to continue!)

$$\boxed{y = 3x - 5}$$

14. The sum of two positive real numbers is 12. What is the smallest possible value of the sum of their squares?

Ⓐ 72 ✓

Ⓑ 74

Ⓒ 68

Ⓓ 76

Ⓔ 70

given: $x + y = 12 \iff y = 12 - x$

$F = x^2 + y^2$

$F = x^2 + (12 - x)^2$

$F' = 2x + 2(12 - x)(-1)$

$F' = 0 \implies 2x - 2(12 - x) = 0$

$\implies x - 12 + x = 0 \implies \boxed{x = 6}$

$\implies \boxed{y = 6}$

$F = x^2 + y^2 = 6^2 + 6^2 = 72$

15. If $f(x) = \ln(\ln x)$, find the value of $f'(\frac{1}{e})$.

Ⓐ $\dfrac{1}{e}$

Ⓑ $\sqrt{e}$

Ⓒ $2e$

Ⓓ $-\dfrac{1}{e}$

Ⓔ undefined ✓

note: $\frac{1}{e}$ not in domain of $f(x) = \ln(\ln x) \implies$ undefined!

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16. Given that the tangent line to the graph of f at $(0,0)$ has equation $2y = x$ and that f has a horizontal asymptote at ∞ with equation $y = 2$, find the value of

$\lim_{x\to 0+} \left\{ \frac{\sin(2x)}{f(x)} - x^2 f(\frac{1}{x}) \right\}$. Tangent line $y = \frac{x}{2}$ at $(0,0) \Rightarrow \lim_{x\to 0} \frac{f(x)}{x} = f'(0) = 1/2$

Ⓐ $\frac{1}{2}$ Horizontal Asymptote $y = 2 \Rightarrow \lim_{x\to 0^+} x^2 f(\frac{1}{x})$

Ⓑ 4 ✓ $= \left(\lim_{x\to 0^+} x^2\right)\left(\lim_{x\to 0^+} f(\frac{1}{x})\right)$

Ⓒ 2 $= 0 \cdot \lim_{u\to\infty} f(u) = 0 \cdot 2 = 0$

Ⓓ 0

Ⓔ not determinable due to insufficient information Now, $\lim_{x\to 0^+} \left\{ \frac{\sin(2x)}{f(x)} - x^2 f(\frac{1}{x}) \right\}$

$= \lim_{x\to 0^+} \left\{ \frac{\sin(2x)}{x} \Big/ \frac{f(x)}{x} - 0 \right\} = \frac{2}{1/2} = 4$

17. Let $f(x) = 2^{x+1} - 2^{-x}$. If $g(x) = f^{-1}(x)$, i.e. if g is the inverse function of f, what is the value of $g'(1)$?

Ⓐ $\frac{3}{2\ln 2}$ $f(0) = 2 - 1 = 1 \Rightarrow$ So $f^{-1}(1) = 0$

Ⓑ $\frac{2}{3\ln 2}$ Hence, $g'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)}$

Ⓒ $\frac{1}{2\ln 2}$

Ⓓ $\frac{1}{3\ln 2}$ ✓ $= \frac{1}{(\ln 2)(2 + 2^0)}$

Ⓔ $\frac{2}{9\ln 2}$ $= \frac{1}{3\ln 2}$

(note: $f'(x) = 2^{x+1}\ln 2 + 2^{-x}\ln 2$
$= (\ln 2)(2^{x+1} + 2^{-x})$)

18. Find the value of $\lim_{x\to\frac{\pi}{4}} \frac{\tan 2x}{\cot(\frac{\pi}{4} - x)}$.

Ⓐ 1 $= \lim_{x\to\pi/4} \frac{\tan 2x}{\cot(\frac{\pi}{4} - x)} \overset{H}{=} \lim_{x\to\pi/4} \frac{2\sec^2(2x)}{\csc^2(\pi/4 - x)}$

Ⓑ $\frac{1}{4}$

Ⓒ $\frac{1}{2}$ ✓ $= \lim_{x\to\pi/4} \frac{2\sin^2(\pi/4 - x)}{\cos^2(2x)} \overset{H}{=} \lim_{x\to\pi/4} \frac{-4\sin(\frac{\pi}{4}-x)\cos(\frac{\pi}{4}-x)}{-4\cos(2x)\sin(2x)}$

Ⓓ 2

Ⓔ undefined $= \lim_{x\to\pi/4} \frac{\sin(\pi/4 - x)}{\cos(2x)} \overset{H}{=} \lim_{x\to\pi/4} \frac{-\cos(\frac{\pi}{4} - x)}{-2\sin(2x)} = -1/-2 = 1/2$

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19. If $f = \log_x 2$ (i.e. logarithm of 2 with base x), find the value of $\frac{d^2y}{dx^2}$ at $x = 2$.

Ⓐ $\frac{2+\ln 2}{4(\ln 2)^2}$

Ⓑ $\frac{4+\ln 2}{2(\ln 2)^2}$

Ⓒ $\frac{2+\ln 2}{2(\ln 2)^2}$

Ⓓ undefined

Ⓔ $\frac{4+\ln 2}{4(\ln 2)^2}$

$y = \log_x 2 \quad \therefore \; 2 = x^y$

$\Rightarrow \ln 2 = \ln(x^y) = y \ln x$

$0 = \frac{y}{x} + (\ln x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-y}{x \ln x} = -\frac{\log_x 2}{x \ln x}$

$\frac{dy}{dx} = \frac{-\ln 2}{x(\ln x)^2}$

Now,

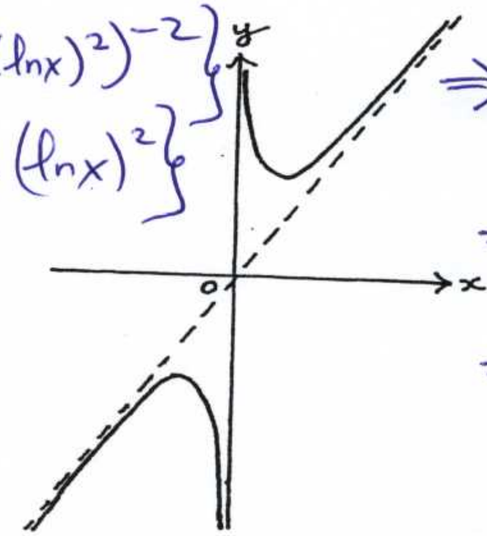
$\frac{d^2y}{dx^2} = (-\ln 2)\left\{-(x(\ln x)^2)^{-2}\right\} \cdot \left\{\frac{x\, 2\ln x}{x} + (\ln x)^2\right\}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\ln 2}{4(\ln 2)^4}\left(2\ln 2 + (\ln 2)^2\right)$

$= \frac{(\ln 2)^2(2+\ln 2)}{4(\ln 2)^4}$

$= \frac{2+\ln 2}{4(\ln 2)^2}$

20.



To which one of the following functions does the above graph correspond?

Ⓐ $f(x) = \frac{x^3+1}{x^3}$

Ⓑ $f(x) = \frac{x^3+1}{x^2}$

Ⓒ $f(x) = \frac{x^4+1}{x^3}$

Ⓓ $f(x) = \frac{x^5+1}{x^3}$

Ⓔ $f(x) = \coth x$

Note: Just Use the limit and approach 0 on both sides