Problem 6 Assign 2

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$$p(\theta \mid y) = \int_0^\infty p(\theta, \sigma^2 \mid y)\, d\sigma^2$$

$$= \int_0^\infty \frac{(\sigma_0^2 \nu_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)} (\sigma^2)^{-\nu_0/2-1} e^{-\frac{\sigma_0^2\nu_0}{2\sigma^2}} \left(\frac{\kappa_0}{2\pi\sigma^2}\right)^{-1/2} e^{-\frac{\kappa_0}{2\sigma^2}(\theta-\mu_0)^2} (2\pi\sigma^2)^{-n/2}$$

$$\times e^{-\frac{1}{2\sigma^2}\sum(y_i-\theta)^2}\, d\sigma^2$$

$$\propto \int_0^\infty \sigma^{2(-\nu_0/2 - \frac{1}{2} - n/2 - 1)} e^{-\frac{1}{2\sigma^2}\left(\sigma_0^2\nu_0 + \kappa_0(\theta-\mu_0)^2 + \sum_{i=1}^n (y_i-\theta)^2\right)}\, d\sigma^2$$

Let $a = \dfrac{\nu_0 + 1 + n}{2}$

and $b = \dfrac{\sigma_0^2\nu_0 + \kappa_0(\theta-\mu_0)^2 + \sum_{i=1}^n (y_i-\theta)^2}{2}$

$$\propto \frac{\Gamma(a)}{(b^a)} \underbrace{\int_0^\infty \frac{b^a}{\Gamma(a)} (\sigma^2)^{-a-1} e^{-\frac{b}{2\sigma^2}}\, d\sigma^2}_{\text{pdf of Inv Gamma}(a,b)}$$

$$= \frac{\Gamma(a)}{(b^a)} \propto b^{-a} \quad \text{(since } a \text{ does not involve } \theta\text{)}$$

$$= \left(\sigma_0^2\nu_0 + \kappa_0(\theta-\mu_0)^2 + (n-1)s^2 + n(\bar{y}-\theta)^2\right)^{-\left(\frac{\nu_0+n+1}{2}\right)}$$

Let $\boxed{A} = \sigma_0^2 \nu_0 + \kappa_0(\theta - \mu_0)^2 + (n-1)s^2 + n(\bar{y}-\theta)^2$

$$\boxed{A} = \left(\sigma_0^2\nu_0 + \kappa_0(\theta^2 - 2\theta\mu_0 + \mu_0^2) + (n-1)s^2 + n\bar{y}^2 - 2n\bar{y}\theta + n\theta^2\right)$$

$$= \left(\left(\underbrace{\sigma_0^2\nu_0 + (n-1)s^2}_{B} + n\bar{y}^2 + \kappa_0\mu_0^2\right) + \underbrace{(n+\kappa_0)\theta^2 - 2(\kappa_0\mu_0 + n\bar{y})\theta}_{\text{'complete the square'}}\right)$$

$$= \boxed{B} + (n+\kappa_0)\left[\theta^2 - 2\frac{(\kappa_0\mu_0 + n\bar{y})}{(\kappa_0+n)}\theta + \left(\frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0+n}\right)^2\right] - \frac{(n\bar{y} + \kappa_0\mu_0)^2}{\kappa_0+n}.$$

$$= \left(\underbrace{\sigma_0^2\nu_0 + (n-1)s^2 + n\bar{y}^2 + \kappa_0\mu_0^2 - \frac{(n\bar{y} + \kappa_0\mu_0)^2}{\kappa_0+n}}_{\boxed{C.}} + (n+\kappa_0)(\theta - \mu_n)^2\right)$$

$$\left(\mu_n = \frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0+n}\right).$$

$$\boxed{C} = \sigma_0^2\nu_0 + (n-1)s^2 + \frac{n\bar{y}^2(\kappa_0+n) + \kappa_0\mu_0^2(\kappa_0+n) - n^2\bar{y}^2 - 2n\bar{y}\kappa_0\mu_0 - \kappa_0^2\mu_0^2}{\kappa_0+n}$$

$$= \sigma_0^2\nu_0 + (n-1)s^2 + \frac{n\kappa_0\bar{y}^2 + \cancel{n^2\bar{y}^2} + \cancel{\kappa_0^2\mu_0^2} + n\kappa_0\mu_0^2 - \cancel{n^2\bar{y}^2} - 2n\bar{y}\kappa_0\mu_0 - \cancel{\kappa_0^2\mu_0^2}}{\kappa_0+n}.$$

$$= \sigma_0^2\nu_0 + (n-1)s^2 + \frac{n\kappa_0(\bar{y}^2 - 2\bar{y}\mu_0 + \mu_0^2)}{\kappa_0+n}$$

$$= \sigma_0^2\nu_0 + (n-1)s^2 + n\kappa_0(\bar{y} - \mu_0)^2 / \kappa_0+n.$$

Let $\boxed{\kappa_n} = \kappa_0 + n$

$\boxed{\nu_n} = \nu_0 + n.$

$\boxed{C} = \boxed{\nu_n \sigma_n^2} = \nu_0 \sigma_0^2 + (n-1)s^2 + \left(\frac{\kappa_0 n}{\kappa_0 + n}\right)(\bar{y} - \mu_0)^2.$

So.

$$p(\theta \mid y) \propto \left[\nu_n \sigma_n^2 + \kappa_n(\theta - \mu_n)^2\right]^{-(\nu_n+1)/2}$$

$$= \left[1 + \frac{\kappa_n(\theta - \mu_n)^2}{\nu_n \sigma_n^2}\right]^{-(\nu_n+1)/2}$$

$$= \boxed{t_{\nu_n}\left(\theta \mid \mu_n, \sigma_n^2/\kappa_n\right)}.$$

That is a t distribution with

- ν_n degrees of freedom
- location parameter $= \mu_n = \frac{\kappa_0 \mu_0}{\kappa_0 + n} + \frac{n\bar{y}}{\kappa_0 + n}$ $= E(\theta \mid y)$
- scale parameter $\frac{\sigma_n^2}{\kappa_n}.$