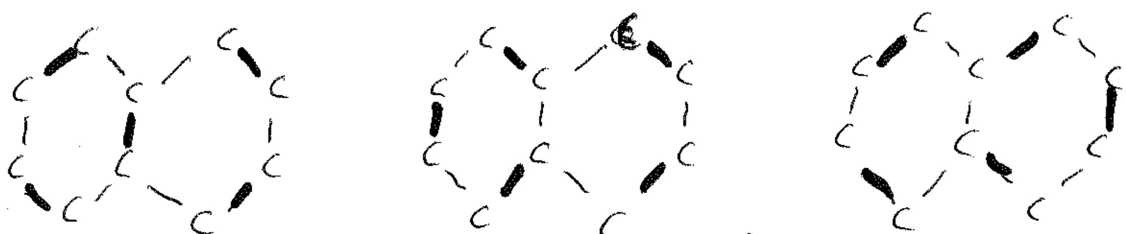
Lecture 9 handout

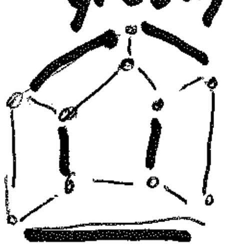
16.1 Let G be a loopless graph.

A <u>matching</u> of G is a set of pairwise nonadjacent edges. A <u>perfect matching</u> covers every vertex of G.



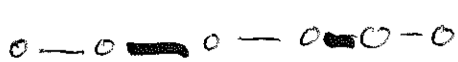
perfect matchings of naphthalene

A <u>maximal matching</u> arises by choosing edges in a greedy fashion until no more edges can be chosen.



maximal matching

<u>Berge's Theorem</u>: M is a maximal matching $\Longleftrightarrow$ G contains no M-augmenting path.

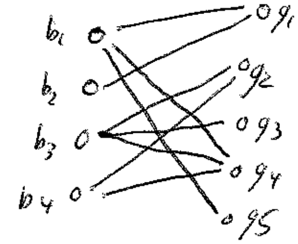


M-augmenting path.

(16.2) Hall's marriage theorem (1935)

$B = \{b_1, \ldots, b_n\}$ boys

$G = \{g_1, \ldots, g_m\}$ girls.



Edge between b_i and g_j if they fancy one another. How can we marry off all the boys/girls?

Marriage Condition: Every set of k boys collectively fancies at least k girls (or vice versa) $1 \le k \le n$.

Marriage Theorem: The marriage problem can be solved iff the marriage condition holds.

Proof 1: Induction.

(i) Every k boys ($k<n$) collectively fancies at least $k+1$ girls. marry one off, finish by induction

(ii) Else, marry off k boys, and the $n-k$ remaining boys can be married off by the induction hypothesis.

Everyone is happy and the proof is complete.

Proof 2:

E = a finite set. $\mathcal{S} = (S_1, \dots, S_m)$ a family of subsets of S.

Transversal: A set of m distinct elements of S, one from each S_i.

Marriage Theorem (again): $\mathcal{S}$ has a transversal $\Leftrightarrow$ the union of any k of the subsets S_i contains at least k elements. $1 \le k \le m$.

Proof: If $|S_1| > 1$, remove an element from S_1 without altering the condition. Repeat until each subset contains one element.

Validity of reduction procedure:

Removal of $x, y \in S_1$ invalidates the condition

$\Rightarrow \exists$ subsets A, B s.t. $|P| \le |A|$ and $|Q| \le |B|$ where

$$P := \bigcup_{i \in A} S_i \cup \{S_1 - \{x\}\} \qquad Q := \bigcup_{j \in B} S_j \cup (S_1 - \{y\})$$

But
$$|A| + |B| \ge |P| + |Q| = |P \cup Q| + |P \cap Q| \ge$$

$$\ge \Big| \bigcup_{i \in A \cup B} S_i \cup S_1 \Big| + \Big| \bigcup_{j \in A \cap B} S_j \Big| \ge$$

condition $\Rightarrow$ $\ge |A \cup B| + 1 + |A \cap B| = |A| + |B| + 1$ contradiction.

Corollary – König Egerváry Theorem

Covering of a graph: Subset K of $V(G)$ such that every edge of G is incident to a vertex in K.

Minimal covering: Covering of smallest cardinality. Size: $\beta(G)$.

Term rank: Largest number of 1's of an (0,1)-matrix, no two of which lie on the same row or column.

Maximal number of edges in a matching: $\alpha'(G)$

König-Egevary: $\alpha'(G) = \beta(G)$ for bipartite G.

The term rank equals the minimum number μ of rows and columns which together contain all 1's of M.

Proof: Put M in form
$$\begin{array}{cc} \begin{array}{cc} \overbrace{}^{n-s} & \overbrace{}^{s} \end{array} & \\ \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right) & \begin{array}{l} \}r \\ \}m-r \end{array} \end{array} \qquad S_i := \{\text{integers } j \le n-s \text{ s.t. } a_{ij} = 1\}$$

Marriage condition holds, so $\mathcal{S} := \{S_1, \dots, S_r\}$ has a transversal. So A contains r 1's, no two of which lie on the same row or column. Same for B. So term rank $\geq r+s = \mu$. But clearly term rank $\leq \mu$.

Next time: Planarity. 10.1–10.3.