**The moment generating function method (Thm 6.1)**

Recall that the *moment generating function (mgf)* of a random variable *X* is
$m_X(t) = Ee^{Xt}$.

Mgf's can be used to identify distributions as follows:

If the mgf of a rv $X$ is the same as that of another rv $U$,
we may conclude that $X$ has the same distribution as $U$.

(Ie, if $m_X(t) = m_U(t)$, then $F_X(k) = F_U(k)$ and $f_X(k) = f_U(k)$ for all *k*.)

Let us now tackle the problem in Example 8. ($Z \sim N(0,1)$. Find the dsn of $X = Z^2$.)

$$m_X(t) = Ee^{Xt} = Ee^{Z^2t} = \int_{-\infty}^{\infty} e^{z^2t}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2(1-2t)}dz$$

$$= c\int_{-\infty}^{\infty} \frac{1}{c\sqrt{2\pi}}e^{-\frac{1}{2c^2}z^2}dz, \qquad \text{where } c^2 = \frac{1}{1-2t}$$

$= c$. (The integral must equal 1.)

Thus $m_X(t) = (1-2t)^{-1/2}$.
But $(1-2t)^{-1/2}$ is the mgf of $U \sim \text{Gam}(1/2,2)$.
(Recall that if $W \sim Gam(a,b)$ then $m(t) = (1-bt)^{-a}$.)
It follows that $X \sim \text{Gam}(1/2,2)$.
Equivalently, $X \sim \chi^2(1)$. (Recall that if $R \sim \text{Gam}(k/2,2)$ then $R \sim \chi^2(k)$.)

Therefore the pdf of *X* is $f(x) = \dfrac{x^{\frac{1}{2}-1}e^{-x/2}}{2^{1/2}\Gamma(1/2)} = \dfrac{1}{\sqrt{2\pi xe^x}}, x > 0$.

Another solution: Let $Y = |Z|$. Then $f(y) = \dfrac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$, $y > 0$.

(This follows by symmetry about $z = 0$. It can also be proved using the cdf method.)
Now $x = y^2$ is a strictly increasing function, since *y* can't be negative.

So by the transformation method, $X = Y^2$ has pdf

$$f(x) = f(y)\left|\frac{dy}{dx}\right| = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}x}\left|\frac{1}{2}x^{-\frac{1}{2}}\right| = \frac{1}{\sqrt{2\pi xe^x}}, x > 0., \text{ as before.}$$

**Two useful results when applying the mgf technique**

**1.** If $X = a + bY$, then
$$m_X(t) = e^{at} m_Y(bt).$$ (Prove this as an exercise.)

**2.** If $Y_1, \ldots, Y_n$ are independent random variables and $X = Y_1 + \ldots + Y_n$, then
$$m_X(t) = m_{Y_1}(t) \ldots m_{Y_n}(t).$$ (This is Thm 6.2.)

**Example 9** $Y \sim N(0,1)$. Find the dsn of $X = a + bY$. (This is an earlier exercise.)

$m_Y(t) = e^{\frac{1}{2}t^2}$. (This is proved in Tutorial 7.)

Therefore $m_X(t) = e^{at} m_Y(bt) = e^{at} e^{\frac{1}{2}(bt)^2} = e^{at + \frac{1}{2}b^2 t^2}$, which is the mgf of $U \sim N(a, b^2)$.

It follows that $X \sim N(a, b^2)$.

**Example 10** Suppose that $Y_1, \ldots, Y_n$ are independent gamma rv's, such that the $i$th one has parameters $a_i$ and $b$.
Find the distribution of $X = Y_1 + \ldots + Y_n$.

$$\begin{aligned} m_X(t) &= m_{Y_1}(t) \ldots m_{Y_n}(t) \\ &= (1 - bt)^{-a_1} \ldots (1 - bt)^{-a_n} \\ &= (1 - bt)^{-\dot{a}}, \quad \text{where } \dot{a} = a_1 + \ldots + a_n. \end{aligned}$$

Hence $X \sim \text{Gam}(\dot{a}, b)$.

*Corollary:* If $Y_1, \ldots, Y_n \sim \text{iid } \chi^2(1)$, then $Y_1 + \ldots + Y_n \sim \chi^2(n)$.
(NB: $\chi^2(r) = Gam(r/2, 2)$.)

**Exercise** Suppose that $Y_1, \ldots, Y_n$ are independent normally distributed rv's such that the $i$th one has mean $a_i$ and variance $b_i^2$.
Let $X = \sum_{i=1}^{n} k_i Y_i$. Show that $X \sim N\left(\sum_{i=1}^{n} k_i a_i, \sum_{i=1}^{n} k_i^2 b_i^2\right)$.

$$\begin{aligned} m_X(t) = Ee^{\left(\sum_{i=1}^{n} k_i Y_i\right)t} &= E\prod_{i=1}^{n} e^{k_i Y_i t} = \prod_{i=1}^{n} Ee^{Y_i(k_i t)} = \prod_{i=1}^{n} m_{Y_i}(k_i t) \\ &= \prod_{i=1}^{n} e^{a_i(k_i t) + \frac{1}{2}b_i^2 (k_i t)^2} = e^{\left(\sum_{i=1}^{n} k_i a_i\right)t + \frac{1}{2}\left(\sum_{i=1}^{n} k_i^2 b_i^2\right)t^2} \end{aligned}$$ (see Thm 6.3).

**Order statistics**

Suppose that $Y_1,\ldots,Y_n$ are iid rv's.

Let: $U_1$ be the smallest of these (ie, $U_1 = \min(Y_1,\ldots,Y_n)$)
$U_2$ be the second smallest
........................................
$U_n$ be the largest (ie, $U_n = \max(Y_1,\ldots,Y_n)$)

(Thus $U_1 \le U_2 \le \ldots \le U_n$.)

We call $U_k$ the $k$th *order statistic*. (Recall Problem 1 in Tutorial 6.)

**Example 11** Suppose that $Y_1, Y_2$ ~ iid *Expo*($b$).
Find the pdf of the second order statistic, $U_2 = \max(Y_1, Y_2)$.

$$
\begin{aligned}
F_{U_2}(u) = P(U_2 < u) = P\{\max(Y_1,Y_2) < u\} &= P(Y_1 < u, Y_2 < u) \\
&= P(Y_1 < u)P(Y_2 < u) \quad \text{(by independence)} \\
&= P(Y_1 < u)^2 \\
&= (1 - e^{-u/b})^2, \quad u > 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{So } f_{U_2}(u) = F_{U_2}'(u) &= 2(1 - e^{-u/b})^1(-e^{-u/b})(-1/b) \\
&= 2(1 - e^{-u/b})\frac{1}{b}e^{-u/b}, \quad u > 0.
\end{aligned}
$$

*Exercise*: Show that $EU_2 = 3b/2$ (NB: $EU_2 > EY_i = b$, as one would expect.)

$$EU_2 = 2\int_0^\infty u\frac{1}{b}e^{-u/b}du - \int_0^\infty u\frac{1}{b/2}e^{-u/(b/2)}du \quad = 2b - b/2 \quad = 3b/2.$$

If $Y_1,\ldots,Y_n$ are continuous and iid, then the pdf of the $k$th order statistics $U_k$ is

$$f_{U_k}(u) = \frac{n!}{(k-1)!(n-k)!}F(u)^{k-1}[1-F(u)]^{n-k}f(u),$$

where $f(y)$ and $F(y)$ are the pdf and cdf of $Y_1$, respectively. (See Thm 6.5.)

Note that this formula is in agreement with $f_{U_2}(u)$ in Example 11, where $n = k = 2$.

**Range restricted distributions**

**Example 12** Suppose that the number of accidents which occur each year at a certain intersection follows a Poisson distribution with mean $\lambda$.

Find the pdf of the number of accidents at this intersection last year if it is known that at least one accident occurred there during that year.

Let $Y$ be the number of accidents at the intersection last year.

Then $X = (Y \mid Y > 0)$ has pdf

$$\begin{aligned}
p(x) &= P(X = x) \\
&= P(Y = x \mid Y > 0) \\
&= \frac{P(Y = x, Y > 0)}{P(Y > 0)} \\
&= \frac{P(Y = x)}{1 - P(Y = 0)} \quad \text{for} \quad x > 0 \\
&= \frac{e^{-\lambda}\lambda^x / x!}{1 - e^{-\lambda}}, \quad x = 1,2,3,...
\end{aligned}$$

For example, if $\lambda = 3.2$ then $p_X(4) = \dfrac{e^{-3.2}3.2^4/4!}{1-e^{-3.2}} = 0.186$,

which we note is slightly higher than $p_Y(4) = e^{-3.2}3.2^4/4! = 0.178$.

What is the expected number of accidents last year?

$$\begin{aligned}
E(Y \mid Y > 0) = EX &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x / x!}{1 - e^{-\lambda}} \\
&= \frac{1}{1 - e^{-\lambda}} \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{(where the first term in the sum is zero)} \\
&= \frac{\lambda}{1 - e^{-\lambda}},
\end{aligned}$$

which we note is higher that $EY = \lambda$.

For example, if $\lambda = 3.2$ then $EX = 3.336 > 3.2 = EY$.