UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2010 EXAMINATIONS

STA257H1F – Section L0101

Duration – 3 hours

Examination aids: Non-programmable Calculators

Last Name: ______________________
First Name: ______________________
Student #: ______________________

Instructions:
1. There are 10 questions and 14 pages in total (including this cover sheet), each worth 10 marks.
2. The last two pages contain lists of useful formulas.
3. Answer all questions directly on the examination paper. Use the backs of the pages or the third-to-last page if more space is needed, and provide clear pointers to your work.
4. Show your intermediate work, and write clearly and legibly.
5. Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical expressions need not necessarily be expressed in decimal format.
6. Read the questions carefully and answer the question that is being asked.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	Total

1.

Let A, B, and C be any 3 events in some sample space S.

a. (4 marks)

If $P(A \mid C) \geq P(B \mid C)$ and $P(A \mid \bar{C}) \geq P(B \mid \bar{C})$, show that $P(A) \geq P(B)$.

b. (6 marks)

If $P(A) = 1/3$, $P(B) = 1/5$, and $P(A \mid B) + P(B \mid A) = 2/3$, find $P(\bar{A} \cap \bar{B})$.

2.
A fair coin is tossed repeatedly and X is the number of tosses before the first head appears. You independently repeat the experiment, and Y is the number of tosses before the first head appears in the second sequence of tosses.

a. (2 marks)
Give the probability mass function of X.

b. (2 marks)
Find $P(X > n)$, for $n \geq 1$.

c. (3 marks)
Find $P(X = Y)$.

d. (3 marks)
Find $P(X > Y)$.

3.

X_1, X_2 are positive continuous random variables, with joint probability density function

$$f(x_1, x_2) = \begin{cases} 2 \cdot x_1 \cdot \exp\{-x_1(2 + x_2)\}, & \text{for } x_1, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}.$$

a. (2 marks)

Find the marginal probability density function of X_1.

b. (2 marks)

Find $P(X_1 > 1)$.

c. (6 marks)

Find the marginal probability density function of X_2.

4.

Let X be an arbitrary random variable with $E\left(X^4\right) < \infty$.

a. (3 marks)

Show that $E\left(X^2\right) \ge \left[E(X)\right]^2$.

b. (4 marks)

For $\mu = E(X)$, show that $E\left[(X-\mu)^4\right] \ge \sigma^4$.

c. (3 marks)

If $\mu = E(X) = 1$, $E\left(X^2\right) = 2$, and $E\left(X^3\right) = 5$, find $E\left[(X-\mu)^3\right]$.

5.

X_1, X_2 are two independent Exponential(β) random variables.

a. (3 marks)

Find the probability density function of $Y = \min(X_1, X_2)$.

b. (2 marks)

Use the moment generating function method to identify the distribution of $Z = X_1 + X_2$.

c. (5 marks)

For constant $c > 0$, find $P(X_1 > c \cdot X_2)$.

6.

X_1, X_2 are two independent Gamma random variables, with parameters $(a_1, 1)$ and $(a_2, 1)$, respectively. Define the random variables $Y = X_1 + X_2$ and $Z = X_1 / X_2$.

a. (2 marks)

Find $Cov(Y, X_1)$.

b. (3 marks)

Find the joint probability density function of X_1 and Z.

c. (5 marks)

Find the marginal probability density function of Z.

7.

X_1, X_2 are continuous random variables with joint probability density function

$$f(x_1, x_2) = \begin{cases} c \cdot x_1^2, & \text{for } 0 \le x_1 \le 1 \text{ and } -x_1 \le x_2 \le x_1 \\ 0, & \text{otherwise} \end{cases}.$$

a. (3 marks)

Find the value of c.

b. (3 marks)

Find $Cov(X_1, X_2)$.

c. (4 marks)

Find the marginal probability density function of X_2.

8.

X_1, X_2, X_3 independently follow Gamma distributions with shape parameters $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$ and scale parameters $\beta_1 = \beta_2 = \beta_3 = 1$, and define $Y = X_1 + X_2 + X_3$.

a. (2 marks)
Find the mean and variance of *Y*.

b. (3 marks)
Using Markov's inequality, find an upper bound for $P(Y \geq 30)$.

c. (5 marks)
Find $E(Y / X_3)$.

9.

X_1, X_2 are Normal random variables with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation coefficient ρ. Let $Y = X_1 + X_2$ and $Z = X_1 - X_2$, where

$E(Y) = 5, \quad V(Y) = 19$

$E(Z) = 1, \quad V(Z) = 7, \quad Cov(Y,Z) = -5$

a. (2 marks)
Find μ_1, μ_2.

b. (4 marks)
Find ρ.

c. (4 marks)
Find $E\left(e^{Y+Z}\right)$.

10.

X_1, X_2 are two continuous random variables. The marginal probability density function of X_1 is $f_1(x_1) = \begin{cases} 12x_1^2(1-x_1), & \text{for } 0 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$, and the conditional probability density function of X_2 given $X_1 = x_1$ is $f_{2|1}(x_2 \mid x_1) = \begin{cases} 1/x_1, & \text{for } 0 < x_2 < x_1 \\ 0, & \text{otherwise} \end{cases}$.

a. (5 marks)

Find the marginal probability density function of X_2.

b. (5 marks)

Find the marginal probability density function of $Y = \log(X_2)$.

Extra Space -- Use if needed and indicate clearly which questions you are answering

Some Useful Formulas

Distribution	**Probability Mass Function**	**Mean**	**Variance**	**Moment Generating Function** ***m*(*t*)**
Binomial	$p(x)=\binom{n}{x}p^x(1-p)^{n-x}$ $x=0,1,\ldots,n$	np	$np(1-p)$	$\left[pe^t+(1-p)\right]^n$
Geometric	$p(x)=p(1-p)^{x-1}$ $x=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(x)=\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x=0,1,\ldots,\min(n,r)$	$\frac{nr}{N}$	$\frac{nr}{N}\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(x)=\frac{\lambda^x e^{-\lambda}}{x!}$ $x=0,1,2,\ldots$	λ	λ	$\exp\left[\lambda\left(e^t-1\right)\right]$
Negative Binomial	$p(x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$ $x=r,r+1,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$

Some Useful Formulas

Distribution	**Probability Density Function**	**Mean**	**Variance**	**Moment Generating Function *m(t)***
Uniform	$f(x)=\frac{1}{\theta_2-\theta_1}$ for $\theta_1 \le x \le \theta_2$; where $\theta_1, \theta_2 \in \mathbb{R}$	$\frac{\theta_1+\theta_2}{2}$	$\frac{(\theta_2+\theta_1)^2}{12}$	$\frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$ for $x \in \mathbb{R}$; where $\mu \in \mathbb{R}, \sigma > 0$	μ	σ^2	$\exp\left\{\mu t+\frac{\sigma^2 t^2}{2}\right\}$
Exponential	$f(x)=\frac{1}{\beta}e^{-x/\beta}$ for $x > 0$; where $\beta > 0$	β	β^2	$(1-\beta t)^{-1}$
Gamma	$f(x)=\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}$ for $x > 0$; where $\alpha, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha}$
Chi-square	$f(x)=\frac{x^{(\nu/2)-1}e^{-x/2}}{2^{\nu/2}\Gamma(\nu/2)}$ for $x > 0$; where $\nu \ge 1$ $(\nu \in \mathbb{N})$	ν	2ν	$(1-2t)^{-\nu/2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$; where $\alpha, \beta > 0$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	no closed form expression

End of Exam
Total pages = 14

Total Marks = 100