

1. **THREESMALLEST**( $A, n$ ):
 

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      if  $n = 3$ :    return SOLVE_THREE( $A[0], A[1], A[2]$ )
      if  $n = 4$ :    return SOLVE_FOUR( $A[0], A[1], A[2], A[3]$ )
      if  $n = 5$ :    return SOLVE_FIVE( $A[0], A[1], A[2], A[3], A[4]$ )
       $m = \lfloor n/2 \rfloor$ 
       $(a, b, c) = \text{THREESMALLEST}(A[0 \dots m-1])$ 
       $(d, e, f) = \text{THREESMALLEST}(A[m \dots n-1])$ 
      # Perform a “partial merge” of  $(a, b, c)$  and  $(d, e, f)$  to obtain the three smallest elements.
      if  $a < d$ :
        if  $b < d$ :
          if  $c < d$ :    return  $(a, b, c)$ 
          else:        return  $(a, b, d)$ 
        else: #  $d < b$ 
          if  $b < e$ :    return  $(a, d, b)$ 
          else:        return  $(a, d, e)$ 
      else: #  $d < a$ 
        if  $a < e$ :
          if  $b < e$ :    return  $(d, a, b)$ 
          else:        return  $(d, a, e)$ 
        else: #  $e < a$ 
          if  $a < f$ :    return  $(d, e, a)$ 
          else:        return  $(d, e, f)$ 
      
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This satisfies the recurrence for  $C(n)$  exactly, since the first recursive call is made on an input of size  $\lfloor n/2 \rfloor$ , the second on an input of size  $\lceil n/2 \rceil$ , and the algorithm performs 3 more comparisons between list elements during its “partial merge” phase at the end.

2. The Master Theorem applies to the recurrence for  $C(n)$ , with  $a = 2$  (two recursive calls are made),  $b = 2$  (each recursive call is on an input roughly half the size), and  $d = 0$  (the algorithm carries out a constant amount of work outside the recursive calls).

Since  $a = 2 > 1 = b^d$ , the Master Theorem allows us to conclude that  $C(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 2}) = \Theta(n)$ .

3. CLAIM:  $\forall n \geq 3, C(n) = 2n - 3$ .

PROOF: By complete induction on  $n \geq 3$ .

**Base Cases:** We show that every initial value of  $C(n)$  satisfies the statement:

$$C(3) = 3 = 6 - 3 = 2(3) - 3$$

$$C(4) = 5 = 8 - 3 = 2(4) - 3$$

$$C(5) = 7 = 10 - 3 = 2(5) - 3$$

**Ind. Hyp.:** Assume  $n \geq 6$  and  $C(k) = 2k - 3$  for  $k \in \{3, 4, \dots, n-1\}$ .

$$\begin{aligned}
 \text{Ind. Step: } C(n) &= C(\lceil n/2 \rceil) + C(\lfloor n/2 \rfloor) + 3 && (\text{since } n \geq 6) \\
 &= (2\lceil n/2 \rceil - 3) + (2\lfloor n/2 \rfloor - 3) + 3 && (\text{by the I.H.}) \\
 &= 2(\lceil n/2 \rceil + \lfloor n/2 \rfloor) - 3 - 3 + 3 \\
 &= 2n - 3
 \end{aligned}$$

**Conclusion:** By induction,  $\forall n \geq 3, C(n) = 2n - 3$ .

This means our divide-and-conquer algorithm is better than the naive algorithm, performing  $n - 3$  fewer comparisons on inputs of size  $n$ .