Feb 4th
Week 4 notes
Frog couple example.
To prove Markov Chain Convergence Thm. need to define a new MC $\{(X_n, Y_n)\}_{n=0}^{\infty}$
. . . . . .

$$\tau = \inf\{n \geq 0, X_n = Y_n = 0\} \quad \text{for sure } \tau < \infty$$

$P_{(ij)}(\tau < \infty) = 1$

$P_{(ij)}(\tau = m, X_n = k)$ where $m = 0, 1, \ldots, m \leq n$

$= P_{(ij)}(\tau = m) \, P_{i_0 k}^{(n-m)}$

$= P_{(ij)}(\tau = m, Y_n = k)$

Then $|P_{ik}^{(n)} - P_{jk}^{(n)}|$

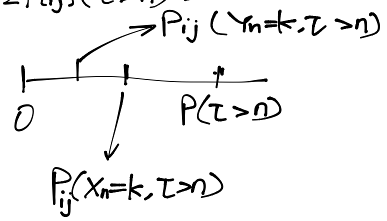
$= |P_{(ij)}(X_n = k) - P_{(ij)}(Y_n = k)|$

$= |\sum_{m=0}^{\infty} P_{(ij)}(X_n = k, \tau = m) - \sum_{m=0}^{\infty} P_{(ij)}(Y_n = k, \tau = m)|$

$= |\sum_{m=0}^{n} P_{(ij)}(X_n = k, \tau = m) + P_{(ij)}(X_n = k, \tau > n) - \left(\sum_{m=0}^{n} P_{(ij)}(Y_n = k, \tau = m)\right) - P_{(ij)}(Y_n = k, \tau > n)|$

$= |P_{(ij)}(X_n = k, \tau > n) - P_{(ij)}(Y_n = k, \tau > n)|$

$\leq 2 P_{(ij)}(\tau > n) \longrightarrow 0$ as $n \to \infty$ since $P_{(ij)}(\tau < \infty) = 1$



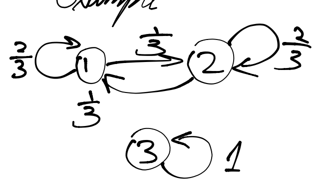
Then $|P_{ij}^{(n)} - \pi_j| = |\sum_{k \in S} \pi_k (P_{ij}^{(n)} - P_{kj}^{(n)})| \leq \sum_{k \in S} \pi_k \underbrace{|P_{ij}^{(n)} - P_{kj}^{(n)}|}_{\to 0 \text{ by above}}$

The whole thing $\to 0$ as $n \to \infty$ by M-test

i.e. $\lim_{n \to \infty} P_{ij}^{(n)} = \pi_j$

Then for any initial prob. $\{\nu_i\}$, $P(X_n = j) = \sum_{i \in S} \nu_i \underbrace{P_{ij}^{(n)}}_{\downarrow \pi_j} \longrightarrow \pi_j$ as $n \to \infty$ by M-test

example.



$\pi_1 = 0$ or $\pi_1 = \frac{1}{2}$ or $\pi_1 = \frac{1}{3}$

$\pi_2 = 0$ $\quad \pi_2 = \frac{1}{2}$ $\quad \pi_2 = \frac{1}{3}$

$\pi_3 = 1$ $\quad \pi_3 = 0$ $\quad \pi_3 = \frac{1}{3}$

not irreducible
so not unique!

Urn's example

X_n

$\pi_j = 2^{-d}\binom{d}{j}$

Periodic Convergence Thm

Suppose MC. is irreducible, stationary dist $\{\pi_j\}$, and period $b \geq 2$

$$\lim_{n\to\infty} \frac{1}{b}\left(P_{(ij)}^{(n)} + P_{(ij)}^{(n+1)} + \cdots + P_{(ij)}^{(n+b-1)}\right) = \pi_j$$

$P_{ij}^{(n)} \to \pi_j$

e.g. if $b=2$: $\lim_{n\to\infty} \frac{1}{2}[P_{ij}^{(n)} + P_{ij}^{(n+1)}] = \pi_j$

Ehrenfest's Urn: $= 2^{-d}\binom{d}{2}$

and $\forall \{\nu_i\}$, $\frac{1}{b}[P(X_n=j) + P(X_{n+1}=j) + \cdots + P(X_{n+b-1}=j)] = \pi_j$

If n is even, $P_{0j}^{(n)} > 0$ then j is even

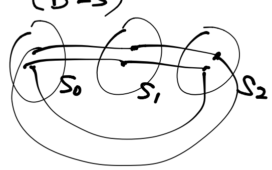
$S_0 = \{j : j \text{ is even}\}$

$S_1 = \{j : j \text{ is odd}\}$

Fix $i \in S$, let $S_r = \{j : P_{ij}^{(bm+r)} > 0 \text{ for some } m \in \mathbb{N}\}$ for $r = 0, 1, \cdots, b-1$,

Then $\{S_r\}_{r=0,\cdots,b-1}$ form a partition

$(b=3)$



$P^{(3)}$ is markov chain on S_0, irreducible, aperiodic

and if $p_i = b\pi_i$ $\forall i \in S_0$,

then p is stat. dist for $P^{(3)}$ on S_0.

---

s.r.w. $P_{ii}^{(n)} \to 0$

$\forall i,j$, $\exists m$ s.t. $P_{ji}^{(m)} > 0$

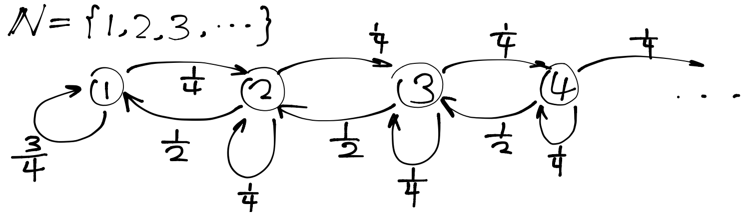
Then $P_{ii}^{(n+m)} \geq P_{ij}^{(n)} P_{ji}^{(m)}$.

So $P_{ij}^{(n)} \leq \frac{P_{ii}^{(n+m)}}{P_{ji}^{(m)}} \to 0$

But $\frac{1}{2}[P_{ij}^{(n)} + P_{ij}^{(n+1)}] \to \pi_j$ so $\pi_j = 0$ $\forall j$

$\searrow 0$ so $\sum_j \pi_j = 0$, impossible

$S = \mathbb{N} = \{1, 2, 3, \cdots\}$



state space infinite' irreducible, aperiodic.
Is there stat dist π ?
Is chain reversible w.r.t. π ?

Need $\pi_1 P_{12} = \pi_2 P_{21}$ i.e. $\pi_1(\frac{1}{4}) = \pi_2(\frac{1}{2})$
i.e. $\pi_2 = \pi_1/2$

And $\pi_2 P_{23} = \pi_3 P_{32}$
i.e. $\pi_2(\frac{1}{4}) = \pi_3(\frac{1}{2})$
i.e. $\pi_3 = \pi_2/2$, etc.
So $\pi_i = \pi_i / 2^{i-1}$, so $\pi_i = \frac{1}{2^i}$
Then $\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i} \quad \forall i$

Application: MCMC
$S = \mathbb{Z}$, $(\pi_i)_{s\in S}$, $\forall$ prob dist with $\sum_{s\in S} \pi_s = 1$ and $\pi_s > 0$. Can we find MC transition (P_{ij}) which make (π_s) stationary?

Metropolis Algorithm

- Is it reducible w.r.t. π ?
  $\pi_i P_{ij} \stackrel{?}{=} \pi_j P_{ji}$
  ($i=j$, of course $j=i+2$, both zero etc)
  $j=i+1$: $\pi_i P_{i,i+1} \stackrel{?}{=} \pi_{i+1} P_{i+1,i}$
  $\pi_i \frac{1}{2} \min[1, \pi_{i+1}/\pi_i] \quad \pi_{i+1} \frac{1}{2} \min[1, \pi_i/\pi_{i+1}]$ 先有日哥 后有天
  $\frac{1}{2} \min[\pi_i, \pi_{i+1}] = \frac{1}{2} \min[\cdots]$
  So $\cdots$