MATH6222 week 5 lecture 14

Rui Qiu

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Last time we have:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Claim that $\binom{n}{k} :=$ coefficient of $x^k y^{n-k}$ in $(x+y)^n$.

The reasoning is: pick k number of x's from n number of $(x+y)$'s, and automatically pick $n-k$ number of y's from n number of $(x+y)$'s.

Pascal's Formula:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof:

1. Formula (direct proof)

$$\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(n-1)!(n-1-k)!} + \frac{(n-1)!}{(n-1)!(n-k)!}$$

2. Recall $\binom{n}{k}$ is the number of bug paths to the coordinate $(n-k, k)$. Interpret it.

3. $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = y^n + nxy^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots$

   Recall $(x+y)^n = (x+y)(x+y)^{n-1}$ where $(x+y)^{n-1}$ again is equal to $\sum\limits_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$

$$\sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$$

We claim that the first term is equal to:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k}$$

So the original formula becomes:

$$\sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k} = \sum_{k=0}^{n} \left[\binom{n-1}{k-1} + \binom{n-1}{k} \right] x^k y^{n-k}$$

Define $\binom{n}{k} = 0$ if n, k do not satisfy $0 \leq k \leq n$.

Probability: Given n equally likely outcomes, an event is defined to be a subset of these outcomes, and the probability of an event is just by definition $\frac{|A|}{n}$.

What is the probability of being dealt (for 5 cards):

- a pair
- a straight
- a flush

We have $\binom{52}{5}$ possible hands. (in fact 2598960)

To get a pair, we make following choices:

- choose a rank for which we have two cards: 13.
- choose two suits for the pair: $\binom{4}{2}$.
- choose three ranks from my remaining set of 12 ranks: $\binom{12}{3}$.
- choose a suit three times: 4^3.

Multiply them together, about 1098240, so probability is around 42%.

To get a straight,

- choose the lowest rank: 9 (consider A, 2, 3, 4, 5 as a non-straight)
- choose one of 4 suits 5 times: 4^5.
- eliminate the number of straight flushes 4.

$9 \times (4^5 - 4)$, probability around 0.35%.