Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 1

- Due Tues. Jan 29, 6:10pm sharp. Late assignments will not be accepted - even if it's one minute late!
- Be sure to clearly write your name, student number, your tutorial group and the name of your TA on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. In the first class we discussed fields and showed that, in addition to the real numbers, the complex numbers form a field. There are of course many others. One of the more important fields in number theory and algebra is $\mathbb{Z}_p$ where p is prime. This field has only p numbers $0, 1, 2, \ldots, (p-1)$ and in this field one evaluates the ordinary sum and product and then takes the remainder after division by p. For example, consider $\mathbb{Z}_2$ one of the smallest and simplest fields. It has only two elements 0 and 1. $1+1=0$ in $\mathbb{Z}_2$ since $1+1=2$ and after dividing 2 by 2 the remainder is 0. In $\mathbb{Z}_2$ then, all possible sums and products are:

$$0+0=0,\ 0+1=1,\ 1+0=1,\ 1+1=0,$$
$$0\cdot 0=0,\ 0\cdot 1=0,\ 1\cdot 0=0,\ 1\cdot 1=1.$$

Write out all possible sums and products for both $\mathbb{Z}_3$ and $\mathbb{Z}_5$. Record the operations of addition and multiplication in a table (see Section 5.1, #11).

2 (a) Consider the subspace $S = span\{(1,2,0,1),(2,0,1,2)\}$ of $\mathbb{Z}_3^4$. Does the vector $(1,1,1,1)$ belong to S? How about the vector $(1,0,1,1)$?

2 (b) Find a basis for the subspace $S = span\{(1,2,1,2,1),(1,1,2,2,1),(0,1,2,0,2)\}$ of $\mathbb{Z}_3^4$.
(**Note**: $\mathbb{Z}_p^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{Z}_p\}$.)

3 (a) Consider the subspace $S = span\{(3,2,4,1),(1,0,3,2),(2,2,0,4)\}$ of $\mathbb{Z}_5^4$. Find the dimension of S.

3 (b) Find the dimension of $P_n(\mathbb{Z}_3)$ for all $n \geq 1$.

4. Let $A = \begin{bmatrix} 1 & i & -1+i & -1 \\ 2 & 1+2i & -2+3i & -2 \\ 1+i & i & -2+i & -1-i \end{bmatrix}$. Find a basis for the row space of A and the column space of A.

5. Let $T\colon V \to W$ be a linear transformation. Let U be a subspace of W. Show that its pre-image $T^{-1}(U) = \{v \in V \mid T(v) \in U\}$ is a subspace of V.

6. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that has the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

relative to the bases $\alpha = \{(1,-1,1),(0,1,0),(1,0,0)\}$ of $\mathbb{R}^3$ and $\beta = \{(3,2),(2,1)\}$ of $\mathbb{R}^2$. Find $T(x,y,z)$ for any $(x,y,z) \in \mathbb{R}^3$.

7. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that has the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

relative to the bases $\{(1,2,0),(1,1,1),(1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,1),(1,-1)\}$ of $\mathbb{R}^2$. Find the matrix of T relative to the bases $\{(2,3,0),(1,1,1),(2,3,1)\}$ of $\mathbb{R}^3$ and $\{(3,-1),(1,-1)\}$ of $\mathbb{R}^2$.

8. Let β be a basis for the n-dimensional vector space V over the field F and let $v_1, v_2, \ldots, v_n$ be vectors in V. Prove that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V if and only if $\{[v_1]_\beta, [v_2]_\beta, \ldots, [v_n]_\beta\}$ is a basis for F^n.

Suggested Extra Problems (not to be handed in):

- Textbook, Section 2.1 **3, 4, 5, 10, 11**
- Textbook, Section 2.2 **3, 4, 12**
- Textbook, Section 5.1 **1, 3, 4, 8**
- Textbook, Section 5.2 **1, 2, 3, 4, 5, 6, 7, 8, 14**
- How many vectors belong to $\mathbb{Z}_p^n$?
- List all the vectors in $\mathbb{Z}_2^2$ and compute the sum of each pair of vectors including the sum of each vector with itself.
- Consider the vector space $\mathbb{Z}_2^3$. Which of the following subsets of $\mathbb{Z}_2^3$ are subspaces of $\mathbb{Z}_2^3$? Justify your answer.

(i) $\{(0,0,0),(1,0,0)\}$

(ii) $\{(0,0,0),(0,1,0),(1,0,1),(1,1,1)\}$

(iii) $\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$

- Let $T\colon P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = xp(x).$$

Consider the bases $\alpha = \{1-x, 1-x^2, x\}$ for $P_2(\mathbb{R})$ and $\beta = \{1, 1+x, 1+x+x^2, 1-x^3\}$ for $P_3(\mathbb{R})$. Find $[T]_{\beta\alpha}$.