Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 3

- Due Tues. Feb 12, 6:10pm sharp . Late assignments will not be accepted - even if it's one minute late!
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Let S: $P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $S(p(x)) = xp(x)$, and T: $P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Consider the bases $\alpha = \{1, 1+x, 1+x+x^2\}$ for $P_2(\mathbb{R})$ and $\beta = \{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\}$ for $M_{2\times 2}(\mathbb{R})$.

(a) Find a formula for $TS(p(x))$.

(b) Find $[TS]_{\beta\alpha}$

(c) Use $[TS]_{\beta\alpha}$ to find a basis for the kernel of TS.

(d) Use $[TS]_{\beta\alpha}$ to find a basis for the image of TS.

Solution

(a) If $p(x) = a + bx + cx^2$, then $TS(p(x)) = TS(a + bx + cx^2) = T(ax + bx^2 + cx^3) = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$. $\square$

(b) Let a_i be the ith vector of α, and b_i the ith vector in β (ordered as given). Abusing notation slightly, let $\varepsilon = \{e_i(x) = x^i\}_{i=0}^n$ denote the standard basis of $P_n(\mathbb{R})$ for any n. Then we know $[TS]_{\beta\alpha} = [I]_{\beta\varepsilon}[TS][I]_{\varepsilon\alpha}$, where $[TS]$ is the standard matrix of TS, so we need only compute each of $[I]_{\beta\varepsilon}$, $[TS]$, and $[I]_{\varepsilon\alpha}$. We have

$$[I]_{\varepsilon\alpha} = \begin{bmatrix} [a_1]_\varepsilon & [a_2]_\varepsilon & [a_3]_\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[TS] = \begin{bmatrix} TS(1) & TS(x) & TS(x^2) \end{bmatrix} = \begin{bmatrix} T(x) & T(x^2) & T(x^3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[I]_{\beta\varepsilon} = [I]_{\varepsilon\beta}^{-1} = \begin{bmatrix} [b_1]_\varepsilon & [b_2]_\varepsilon & [b_3]_\varepsilon & [b_4]_\varepsilon \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{-1}.$$

Row reducing $[[I]_{\varepsilon\beta}|I]$, we find

$$[I]_{\beta\varepsilon} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

It follows that

$$[TS]_{\beta\alpha} = [I]_{\beta\varepsilon}[TS][I]_{\varepsilon\alpha} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix}. \quad \square$$

(c) We know $x \in \ker TS \Leftrightarrow TSx = 0 \Leftrightarrow [TS]_{\beta\alpha}[x]_\alpha = 0$. Row reducing $[TS]_{\beta\alpha}$, we get

$$\frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that the only solution to $[TS]_{\beta\alpha}[x]_\alpha = 0$ is the trivial solution, so that the kernel of TS is trivial. $\square$

(d) The row reduced echelon form of $[TS]_{\beta\alpha}$ has a leading entry in each column, implying that all the columns of $[TS]_{\beta\alpha}$ are members of a basis for the column space of $[TS]_{\beta\alpha}$. Multiply each of these vectors by $[I]_{\varepsilon\beta}$ to get a basis for $M_2(\mathbb{R})$ (or rather, for $\mathbb{R}^4$, which naturally corresponds to $M_2(\mathbb{R})$):

$$\left\{ \frac{1}{2}[I]_{\varepsilon\beta}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}_\beta, \frac{1}{2}[I]_{\varepsilon\beta}\begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix}_\beta, \frac{1}{2}[I]_{\varepsilon\beta}\begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}_\beta \right\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

$\square$

2. Let $\alpha = \{(-3,5,2),(4,1,1),v_3\}$ and $\alpha' = \{v_1', (4,0,-7), v_3'\}$ be bases for $\mathbb{R}^3$, and that the change of basis matrix from α to α' is

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix}.$$

Find v_3, v_1', and v_3'.

Solution Let $v_1 = (-3, 5, 2)$, $v_2 = (4, 1, 1)$, and $v_2' = (4, 0, -7)$. We have

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix} = [I]_{\alpha'\alpha} = \begin{bmatrix} [v_1]_{\alpha'} & [v_2]_{\alpha'} & [v_3]_{\alpha'} \end{bmatrix},$$

so

$$v_1 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = -3v_1' + v_2' + 2v_3' = -3v_1' + \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix} + 2v_3',$$

$$v_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = v_1' - v_2' - v_3' = v_1' - \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix} - v_3'$$

$$v_3 = 5v_1' + 2v_2' - v_3' = 5v_1' + 2\begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix} - v_3'$$

Solving the second equation for v_1' gives $v_1' = v_3' + (8, 1, -6)$; substituting this into the first equation yields $v_3' = (-17, -8, 9)$, and thus $v_1' = (-9, -7, 3)$. Finally, substituting these values into the third equation above shows $v_3 = (-20, -27, -8)$. □

3. Suppose the linear transformation $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ has the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

relative to the standard bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$. Find bases α of $P_3(\mathbb{R})$ and β of $P_2(\mathbb{R})$ such the $[T]_{\beta\alpha}$ is the reduced row echelon form of A.

Solution Row reduce A to get $[T]_{\beta\alpha}$:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 + R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2/3 \end{bmatrix} \xrightarrow[R_2 - 2R_3]{R_1 + 4R_3} \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 2/3 \end{bmatrix} = [T]_{\beta\alpha}.$$

Let $E_1, \ldots, E_6$ be the elementary matrices corresponding to the elementary row operations performed above (in order). Then

$$E_6 \cdots E_1 A = E_6 \cdots E_1 [T][I] = [T]_{\beta\alpha},$$

where $[I]$ is of course the standard matrix of I. Thus we may interpret $E_6 \cdots E_1$ as a change of basis matrix $[I]_{\beta\varepsilon}$, and take $\alpha = \{1, x, x^2, x^3\}$ to be the standard basis of $P_3(\mathbb{R})$ (we think of $[I] = [I]_{\varepsilon\alpha}$). To identify the members of β, we need only invert $E_6 \cdots E_1$ to get

$$E_1^{-1} \cdots E_6^{-1} = [I]_{\varepsilon\beta} = \begin{bmatrix} [b_1]_\varepsilon & [b_2]_\varepsilon & [b_3]_\varepsilon \end{bmatrix}.$$

We have

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

So that

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$[I]_{\varepsilon\beta} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

and the members of β are given by the columns of $[I]_{\varepsilon\beta}$; that is

$$\beta = \left\{1 + x^2, 2 + x + x^2, 2x + x^2\right\}.$$

$\square$

4. Let $T\colon \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$ be defined by

$$T(x, y, z) = (x + 2y, x + y, 2x + z).$$

Find $T^{-1}(x, y, z)$.

Solution We have

$$[T] = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \left[T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then

$$[[T]|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - R_2}$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right],$$

where we note that in $\mathbb{Z}_3^3$,

$$2 \cdot 0 = 0, \quad 2 \cdot 1 = 2, \quad 2 \cdot 2 = 1,$$

so that

$$0 = \frac{0}{2}, \quad 1 = \frac{2}{2}, \quad 2 = \frac{1}{2},$$

(ie. division by 2 is tantamount to multiplication by 2). Thus

$$T^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [T^{-1}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 2y \\ x + 2y \\ 2x + 2y + z \end{bmatrix}$$

$\square$

5. Let $W = span\{1 + x^2 + x^3, 1 + x + x^2, 3 + x + 3x^2 + 2x^3, -x + x^3\}$. Determine the dimension d of W and find (construct) an isomorphism $T\colon W \to \mathbb{R}^d$.

Solution Row reduce the matrix whose columns (naturally) correspond to the members of the spanning set of W

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are leading ones in precisely the first and second columns, so $\{1 + x^2 + x^3, 1 + x + x^2\}$ is a basis for W, and $\dim W = 2$.

To construct an isomorphism $T : W \to \mathbb{R}^2$, we need only define a surjective map from the basis $\{1 + x^2 + x^3, 1 + x + x^2\}$ of W to a basis of $\mathbb{R}^2$, and extend linearly—this will automatically be linear and surject onto $\mathbb{R}^2$, hence be an isomorphism. For example, we may define T on $\{1+x^2+x^3, 1+x+x^2\}$ by

$$T(1 + x^2 + x^3) = (1, 0), \quad T(1 + x + x^2) = (0, 1).$$

Then by extending linearly, for any $a(1+x^2+x^3)+b(1+x+x^2) \in W = span\{1+x^2+x^3, 1+x+x^2\}$, we have

$$T(a(1 + x^2 + x^3) + b(1 + x + x^2)) = aT(1 + x^2 + x^3) + bT(1 + x + x^2) = (a, b).$$

□

6(a) Let $T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = p(x) + p'(x)$. Show T is an isomorphism.

6(b) Let $T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = xp'(x)$. Explain why T is not an isomorphism.

6(c) Let $T\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = cp(x) - xp'(x)$. For what values of $c \in \mathbb{R}$ is T is an isomorphism. Justify your answer.

Solution Since T is linear, it is enough to show that T is either injective or surjective to prove that it is an isomorphism; in the case that T is not an isomorphism, we need only show that either injectivity or surjectivity fails. We will proceed by checking whether or not T is injective, which will be accomplished by verifying whether or not it has trivial kernel. Let $k \le n$ and set $p(x) = a_0 + a_1x + \cdots + a_kx^k$.

6(a) Suppose $T(p(x)) = 0$, so that $0 = p(x) + p'(x)$. thus we have

$$0 = (a_0+a_1x+\cdots+a_kx^k)+(a_1+2a_2x+\cdots+ka_kx^{k-1}) = (a_0+a_1)+(a_1+2a_2)x+\cdots+(a_{k-1}+ka_k)x^{k-1}+a_kx^k.$$

Since the x^i are independent, it follows that each coefficient must be zero. Hence $a_k = 0$, so that $0 = a_{k-1} + ka_k = a_{k-1}$, and inductively, $a_i = 0$ for each $i = 1, \ldots, k$. Thus $p(x) = 0$, and it follows that T is injective, hence an isomorphism. □

6(b) Let $p(x) = 1 \neq 0$. Then $T(p(x)) = xp'(x) = x \cdot 0 = 0$. So T is not injective. □

6(c) Similarly to our computation in part (a), we find that $T(p(x)) = 0$ if and only if

$$0 = ca_0+(ca_1-1a_1)x+(ca_2-2a_2)x^2+\cdots+(ca_k-ka_k)x^k = ca_0+(c-1)a_1x+(c-2)a_2x^2+\cdots+(c-k)a_kx^k,$$

which occurs if and only if all the coeffients $(c - i)a_i$ are zero. Thus if $c = i$ for some $i \in \{0, 1, \ldots, k\}$, then we may find a nonzero polynomial such as $p(x) = x^i$ such that $T(p(x)) = (c - i)a_ix^i = 0$, so that T is not injective. Otherwise, if $c \notin \{0, 1, \ldots, k\}$, it must be that each $a_i = 0$ if $T(p(x)) = 0$, so that $p(x) = 0$, and T is injective. □

7. Let V and W be vector spaces over a field F with bases $\alpha = \{v_1, v_2, \ldots, v_n\}$ and $\beta = \{w_1, w_2, \ldots, w_n\}$ respectively, and let $T\colon V \to W$ be a linear transformation. Prove that T is an isomorphism iff $[T]_{\beta\alpha}$ is an invertible matrix.

Solution First assume that T is an isomorphism. Thus T is invertible with inverse map $T^{-1} : W \to V$, and we may take the matrix $[T^{-1}]_{\alpha\beta}$ of this map with respect to the bases β and α. We verify that this is the inverse matrix for $[T]_{\beta\alpha}$, so that $[T]_{\beta\alpha}$ is invertible:

$$[T]_{\beta\alpha}[T^{-1}]_{\alpha\beta}[w]_\beta = [T]_{\beta\alpha}[T^{-1}w]_\alpha = [TT^{-1}w]_\beta = [w]_\beta \quad \forall w \in W.$$

Conversely, assume that $[T]_{\beta\alpha}$ is invertible. Note that from Problem Set 2, we know the maps $\iota_\alpha : V \to [F^n]_\alpha$ and $\iota_\beta : W \to [F^n]_\beta$ given by $\iota_\alpha(v) = [v]_\alpha$ and $\iota_\beta(w) = [w]_\beta$ are isomorphisms. Now define $S : W \to V$ by

$$S(w) = \iota_\alpha^{-1}\left([T]_{\beta\alpha}^{-1}\left(\iota_\beta(w)\right)\right).$$

Now observe that for any $v \in V$,

$$\iota_\beta^{-1}\left([T]_{\beta\alpha}(\iota_\alpha(v))\right) = \iota_\beta^{-1}\left([T]_{\beta\alpha}[v]_\alpha\right) = \iota_\beta^{-1}\left([Tv]_\beta\right) = Tv,$$

so that

$$\begin{aligned} STv &= S\left(\iota_\beta^{-1}\left([T]_{\beta\alpha}(\iota_\alpha(v))\right)\right) \\ &= \iota_\alpha^{-1}\left([T]_{\beta\alpha}^{-1}\left(\iota_\beta\left(\iota_\beta^{-1}\left([T]_{\beta\alpha}(\iota_\alpha(v))\right)\right)\right)\right) \\ &= \iota_\alpha^{-1}\left([T]_{\beta\alpha}^{-1}[T]_{\beta\alpha}(\iota_\alpha(v))\right) \\ &= \iota_\alpha^{-1}\left(\iota_\alpha(v)\right) \\ &= v. \end{aligned}$$

Similarly, $TSw = w$ for each $w \in W$, and it follows that $S = T^{-1}$. $\square$

Suggested Extra Problems (not to be handed in):

- Textbook, Section 2.5 **8, 12**
- Textbook, Section 2.6 **1, 2, 6, 7, 8, 17**
- Textbook, Section 2.7 **1, 2, 3, 4, 9**
- Textbook, Chapter 2 Supplementary Exercises, **1, 2, 3, 4, 5, 9, 10, 12**