Question 1. [10 MARKS]

Prove by simple induction that for all positive natural numbers n, if $x_1, ..., x_{2n}$ is a sequence of $2n$ numbers, and at least $n+1$ of the numbers are positive, then there is at least one pair of adjacent positive numbers in the sequence (i.e. x_i and x_{i+1} are both positive for some $i \in \{1, \ldots, 2n\}$).

State your Inductive Hypothesis (IH) and mention everywhere you use it.

Let $P(n)$ be the predicate "if $x_1, ..., x_{2n}$ is a sequence of $2n$ numbers, and at least $n+1$ of the numbers are positive, then there is at least one pair of adjacent positive numbers."

Proof 1

This is essentially the same argument as written up by some of you on Piazza. That writeup is very good. `https://piazza.com/class/i4insg4g5q26kd?cid=137`

We prove $\forall n \geq 1. P(n)$ by induction on n.

Base case $n = 1$, then x_1 and x_2 are both positive.

Assume IH $P(n)$ for an arbitrary $n \geq 1$. Let $x_1, \ldots, x_{2n+2}$ be a sequence with $n+2$ positive numbers. If x_{2n+1} and x_{2n+2} are both positive, then we are done. Otherwise, at least one is non-positive, and so at least $n+1$ positive numbers are in $x_1, \ldots, x_{2n}$. Then IH for n tells us there are two adjacent positive numbers in $x_1, \ldots, x_{2n}$, and so obviously by extension there are two adjacent positive numbers in $x_1, \ldots, x_{2n+2}$.

Proof 2

Let $P'(n)$ be the predicate "if $x_1, ..., x_{2n}$ is a sequence of $2n$ numbers with no adjacent positive numbers, then it has at most n positive numbers." $P'(n)$ is true iff $P(n)$ is true for all $n \geq 1$; to see this, write $P(n)$ in the form $(A \wedge B) \implies C$, and then note that $P'(n)$ is $(A \wedge \neg C) \implies \neg B$. We prove $\forall n \geq 1. P'(n)$ by induction on n.

Base case $n = 1$. By the hypothesis of $P'(1)$, the sequence x_1, x_2 has either 0 or 1 positive numbers, hence it has at most 1 positive number.

Assume IH $P'(n)$ for an arbitrary $n \geq 1$. Let $x_1, \ldots, x_{2n+2}$ be a sequence with no adjacent positive numbers. The IH applies to the first $2n$ elements, so there are at most n positive numbers in there. Morever, since the full length $2n+2$ sequence has no adjacent positive numbers, at most 1 of x_{2n+1} and x_{2n+2} is positive. In total, there are at most $n+1$ positive numbers in the length $2n+2$ sequence.

Proof 3

This is a proof from the pigeonhole principle, which can be proved by induction. We'll be giving that to you as an exercise.

Let $n \geq 1$ be arbitrary and let $x_1, ..., x_{2n}$ be any sequence with at least $n+1$ positive numbers. Break up the sequence into n pairs $[x_1, x_2], [x_3, x_4], \ldots, [x_{2n-1}, x_{2n}]$. Each of the positive numbers in the sequence must be in one of those pairs. Since there are at least $n+1$ positive numbers and only n pairs, by the pigeonhole principle one of the pairs must have two positive numbers.

Question 2. [10 MARKS]

Consider the function:

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ [f(\lfloor \frac{n}{2} \rfloor)]^2 + 2f(\lfloor \frac{n}{2} \rfloor) & \text{if } n \geq 1 \end{cases}$$

Part (a) [8 MARKS]

For $n \in \mathbb{N}$ let $P(n)$ be: "$f(n)$ is divisible by 10." Prove by complete induction that $P(n)$ is true for all natural numbers $n \geq 2$.

As usual, label your Inductive Hypothesis (IH), and when you use your IH, mention which numbers you're using it for and why this is valid.

Hint: thinking about part (b) on the next page first might help.

Proof

Note: $f(1) = [f(\lfloor \frac{1}{2} \rfloor)]^2 + 2f(\lfloor \frac{1}{2} \rfloor) = [f(0)]^2 + 2f(0) = 2^2 + 2 \cdot 2 = 8$.

Base Cases $P(2)$ and $P(3)$.
$f(2) = [f(\lfloor 2/2 \rfloor)]^2 + 2f(\lfloor 2/2 \rfloor) = [f(1)]^2 + 2f(1) = 8^2 + 2 \cdot 8 = 64 + 16 = 80$, which is a multiple of 10.
$f(3) = [f(\lfloor 3/2 \rfloor)]^2 + 2f(\lfloor 3/2 \rfloor) = [f(1)]^2 + 2f(1) = 8^2 + 2 \cdot 8 = 64 + 16 = 80$, which is a multiple of 10.

Inductive Step.
Let n be a natural number that is **at least** 4.
(IH) Assume P is true for each **natural number** that is **at least** 2 and **less than** n.

Since n is at least 4: n is also at least 1 so using the formula for f:

$$f(n) = [f(\lfloor \frac{n}{2} \rfloor)]^2 + 2f(\lfloor \frac{n}{2} \rfloor)$$

Since n is at least 4: $\lfloor \frac{n}{2} \rfloor$ is **at least** $\lfloor \frac{4}{2} \rfloor = 2$.
Also, the floor makes $\lfloor \frac{n}{2} \rfloor$ an integer, so being at least 2 makes $\lfloor \frac{n}{2} \rfloor$ a **natural number**.
And, $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2}$ which is **less than** n since n is positive.
So: $\lfloor \frac{n}{2} \rfloor$ is a **natural number** that is **at least** 2 and **less than** n.

So the (IH) includes that P is true for $\lfloor \frac{n}{2} \rfloor$, i.e. $f(\lfloor \frac{n}{2} \rfloor)$ is a multiple of 10.

Let k be the integer such that $f(\lfloor \frac{n}{2} \rfloor) = 10 \cdot k$. Then

$$f(n) = [10 \cdot k]^2 + 2 \cdot 10 \cdot k = 10 \cdot (10k^2 + 2k)$$

which is a multiple of 10 since $10k^2 + 2k$ is an integer, since k is an integer.

Part (b) [2 MARKS]

Your proof in part (a) contains, perhaps implicitly, proofs of $P(2), P(3), P(4), P(5), \ldots$
Some of those rely on an induction hypothesis.

For each of the following pairs of statements, circle one of the two options (i) or (ii), and if you choose option (ii), then fill in the blank space with a specific number.

$P(2)$ was proved without using an/the (IH).
$P(3)$ was proved without using an/the (IH).
$P(4)$ was proved from the assumption $P(2)$ from the (IH): 4 is a natural number that is at least 4, and so from the (IH) we used P for $\lfloor \frac{4}{2} \rfloor = 2$.
$P(5)$ was proved from the assumption $P(2)$ from the (IH): 5 is a natural number that is at least 4, and so from the (IH) we used P for $\lfloor \frac{5}{2} \rfloor = 2$.
$P(6)$ was proved from the assumption $P(3)$ from the (IH): 6 is a natural number that is at least 4, and so from the (IH) we used P for $\lfloor \frac{6}{2} \rfloor = 3$.
$P(236)$ was proved from the assumption $P(118)$ from the (IH): 236 is a natural number that is at least 4, and so from the (IH) we used P for $\lfloor \frac{236}{2} \rfloor = 118$.

Question 3. [10 MARKS]

An almost-balanced binary tree is a binary tree such that for every internal (i.e. non-leaf) node u, the height of one of u's subtrees is 1 more than the height of the other (possibly empty) subtree.

height is defined as in Assignment 1: the number of nodes on the tree's longest root-to-leaf path.

For example: The empty tree, with no nodes, is the unique height 0 almost-balanced binary tree. A single isolated node is the unique height 1 almost-balanced binary tree. A height 2 tree whose root has exactly one child leaf is almost-balanced.

Prove that every almost-balanced binary tree of height $h \in \mathbb{N}$ has at least $(1.4)^h - 1$ nodes.

You might find it useful that $1.4^2 = 1.96 < 2.4$.

As usual, label your Inductive Hypothesis (IH), and when you use your IH, mention which numbers you're using it for and why this is valid.

Let $P(h)$ be "Every almost-balanced binary tree of height $h \in \mathbb{N}$ has at least $(1.4)^h - 1$ nodes". We prove $\forall h \in \mathbb{N}.P(h)$ by complete induction.

Let $h \in \mathbb{N}$ be arbitrary. Assume:
IH: For all natural numbers $h' < h$, $P(h')$.

Case $h = 0$. The unique height 0 almost-balanced binary tree has $0 = 1.4^0 - 1$ nodes.
Case $h = 1$. The unique height 1 almost-balanced binary tree has 1 node and $1 > 1.4^1 - 1 = .4$.

Case $h \geq 2$. Let T be an arbitrary height h tree. Since $h \geq 1$, the root of T has two subtrees, T_1, T_2, one of them possibly empty. By the definition of almost-balanced, they are also almost-balanced. By the definition of height and almost-balanced, one of the subtrees has height $h - 1$ and the other height $h - 2$. Without loss of generality T_1 has height $h - 1$ and T_2 has height $h - 2$. The IH applies for both, so T_1 has at least 1.4^{h-1} nodes and T_2 has at least 1.4^{h-2} nodes. Since $\#nodes(T) = 1$ (for the root) $+ \#nodes(T_1) + \#nodes(T_2)$, we have that T has at least $1 + (1.4^{h-1} - 1) + (1.4^{h-2} - 1) = 1.4^{h-1} + 1.4^{h-2} - 1$ nodes. Rewrite to $1.4^{h-2}(1.4^1 + 1.4^0) - 1 = 1.4^{h-2}(2.4) - 1$. Using the hint $2.4 > 1.4^2$ next, have $\#nodes(T) \geq 1.4^{h-2}(1.4^2) - 1 = 1.4^h - 1$. So $P(h)$ is proved.

0: ______ / ??

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TOTAL: ______ /30