PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 11 SOLUTIONS

(1) Let G be a simple graph with n vertices.

(a) Let x and y be nonadjacent vertices of degree at least $(n+k-2)/2$. Prove that x and y have at least k common neighbors.

Solution: Since x and y are not adjacent, their neighbours form sets $N_x, N_y \subset V(G) \setminus \{x,y\}$. The given condition on the degrees means that N_x, N_y have at least $(n+k-2)/2$ elements each from a pool of $n-2$ possibilities; so applying the formula $|N_x \cap N_y| = |N_x| + |N_y| - |N_x \cup N_y|$ along with $N_x \cup N_y \subset V(G) \setminus \{x,y\}$ we get $|N_x \cap N_y| \geq k$.

(b) Prove that if every vertex has degree at least $\lfloor n/2 \rfloor$, then G is connected. Show that this bound is the best possible whenever $n \geq 2$ by exhibiting a disconnected n-vertex graph where every vertex has at least $\lfloor n/2 \rfloor - 1$ neighbors.

Solution: We will show the contrapositive. If G was disconnected, then we could divide it in to two connected components A, B such that there are no edges from A to B. The smaller (by vertex count) of A, B would have at most $\lfloor n/2 \rfloor$ vertices, and thus each of its vertices can only be connected to at most $\lfloor n/2 \rfloor - 1$ others. To show this is the best possible, consider the disjoint union $K_{\lfloor n/2 \rfloor} \sqcup K_{\lceil n/2 \rceil}$ of complete graphs.

(2) Let G be a connected graph with $m \geq 2$ vertices of odd degree. (Recall from the previous tutorial that m is even.) Prove that the minimum number of trails that together traverse each edge of G exactly once is $m/2$. (Hint: Transform G into a new graph G' by adding edges and/or vertices.)

Solution: Since m is even, we can pair up the vertices of odd degree. Let G' be the graph obtained from G by adding one edge joining the members of each of these $m/2$ pairs. Since each odd-degree vertex in G is given exactly one new edge and each even-degree vertex is untouched, we know G' has all even degrees and thus admits an Eulerian circuit. If we restrict back to G, we remove exactly $m/2$ edges from this circuit, thus dividing it in to $m/2$ trails in G. Since the circuit was Eulerian for G', in particular it traversed every edge in G, and thus every edge in G is traversed by one of the $m/2$ trails.
To show this is the smallest number of trails possible, note that if we have a covering of G by t trails, every odd-degree vertex must be the endpoint of at least one trail. Since each trail has two endpoints, this implies $t \geq m/2$.

(3) Let G be a simple graph with n vertices and no cycles of length three. Prove that G has at most $n^2/4$ edges. (Hint: Consider the subgraph consisting of neighbors of a vertex of maximum degree and the edges among them.)

Solution: Let k denote the maximum degree of G and a choose a vertex v with this maximal degree. Let A be the set of vertices neighbouring v and $B = V(G) \setminus A$ the rest of the vertices. If we compare G to the complete bipartite graph $H = K_{A,B}$ constructed from the vertex sets A and B, we see that each vertex has degree in H at least as high as in G:

- If $w \in A$ then w cannot be G-adjacent to any another vertex in A (since this would form a triangle with v), while in H it is adjacent to *every* vertex not in A.
- If $w \in B$ then w has degree $|A| = k$ in H, and its degree in G cannot exceed this because k was taken to be the maximal degree.

Thus by the degree-sum formula we can conclude that the number of edges in G is at most the number of edges in H, which we know is $k(n-k)$. Completing the square, this is $-(k-n/2)^2 + n^2/4$, which is at most $n^2/4$.

(4) Suppose that every vertex of a graph G has degree at most k. Prove that $\chi(G) \leq k+1$. Show that this bound is the best possible by exhibiting (for every k) a graph with maximum degree k and chromatic number $k+1$.

Solution: We prove this by induction on n, the number of vertices in the graph. Since the one-vertex graph can be one-coloured, the proposition holds for $n = 1$. Now assume the proposition holds for $n-1$, take a graph G_n with n vertices and maximal degree at most k, and remove a single vertex v (along with its j connecting edges) to get a smaller graph G_{n-1}. Since removing edges cannot increase the maximal degree, we know G_{n-1} can be $k+1$-coloured by the induction hypothesis, so we just need to colour v in a compatible way. Since $j \leq k$, there are at most k colours represented in the neighbours of v, so at least one of the $k+1$ colours must not be represented. By choosing this colour for v we have constructed a colouring of G_n.
To show this is the best possible bound, consider the complete graph K_{k+1}, which has chromatic number $k+1$ since its vertices are pairwise connected and thus must have pairwise distinct colours.

(5) Let T be a tree with m edges, and let G be a simple graph in which every vertex has degree at least m. Prove that G contains T as a subgraph. (Hint: Induction on m.)

Solution: For the base case, let T be a tree with 0 edges; i.e. a lone vertex. Since any (non-empty) graph has a vertex, the proposition holds for $m = 0$. Now assume the proposition holds for $m-1$, and let T, G satisfy the hypotheses for m. Choose any leaf v in T and let T' be the tree obtained from T by removing v. Since G has minimal degree at least m, it certainly has degrees greater than $m-1$; so by the induction hypothesis we know that G contains T' as a subgraph. Consider w, the branch connected to v, which is a vertex of T' and thus is now identified as a vertex of G. Since there are only $m-1$ other vertices in T', the minimal degree hypothesis on G tells us that w has a neighbour not contained in T'; so identifying this neighbour with v we have embedded T in G.