FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 9

ARBITRAGE AND FORWARD CONTRACTS

Arbitrage is the simultaneous buying and selling of two economically equivalent but differentially priced portfolios so as to a make risk-free profit. For example, purchasing a good for a price of \$5 and then selling it elsewhere for a price of \$7.

An assumption is often made that arbitrage does not exist.

We will show how this assumption can be used to value forward contracts which will be defined later in these notes.

ARBITRAGE

Arbitrage in financial mathematics is generally described as a risk-free trading profit. More accurately, an arbitrage opportunity exists if either:

a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss; or
b) an investor can make a deal that has zero initial cost, no risk of future loss, and a possibility of a future profit.

EXAMPLES

i) The Raiders are playing the Tigers in a rugby league match. A friend says that he will pay you \$1,000 if the Raiders win.

This is an arbitrage opportunity because there is no initial cost, no risk of future loss, and a possibility of earning \$1,000.

ii) The Raiders are playing the Tigers in a rugby league match. A friend says that he will pay you \$1,000 if the Raiders win. If the Tigers win you agree to pay him \$1.

This is not an arbitrage opportunity because there is a risk that you will have to pay \$1 (ie. there is a risk of future loss).

iii) Manchester United agrees to pay \$1,000,000 for an Italian defender. Chelsea is happy to pay \$30,000,000 for the same player.

This is an arbitrage opportunity for Manchester United because they can purchase the player and then immediately sell the player to Chelsea for a profit of \$29,000,000 with no risk of loss.

The concept of arbitrage is very important because we generally assume that in modern developed financial markets arbitrage opportunities don't exist. This assumption is referred to as the "No-Arbitrage" assumption, and is fundamental to modern financial mathematics.

If we assume that there are no arbitrage opportunities in a market, then it follows that any two securities or combinations of securities that give exactly the same payments must have the same price. This is sometimes called the "Law of One Price".

If two securities with identical payments have different prices then an arbitrage opportunity exists. Arbitrageurs would spot this opportunity and buy the cheaper security and sell the more expensive security so as to make an immediate, risk-free profit. The arbitrage position would soon disappear however because of the increased demand for the cheaper security and the lack of demand for the more expensive security. This would force the security prices back into line.

EXAMPLE
Consider the following two securities and the following scenario:

Security	Current Price P_0	Price Increase $P_1(u)$	Price Decrease $P_1(d)$
A	6	7	5
B	11	14	10

At time 1 there are two possible outcomes. Either the prices go up, in which case the securities pay amounts $P_1(u)$, or the prices go down, in which case the securities pay amounts $P_1(d)$.

Investors can either:
- buy the securities in which case they pay the time 0 price (P_0) and receive the time 1 income; or
- sell the securities, in which case they receive the time 0 price and pay the time 1 outgo (ie. they can promise to pay a future amount at time 1 to an investor in return for receiving the price at time 0).

Is there an arbitrage opportunity in this example?

Solution
An investor could promise to pay the future payments for two units of security A in exchange for the purchase price according to the schedule above. ie. the investor sells two units of security A in exchange for $12. At time 0 the investor also buys one unit of security B for $11. The table below shows the net result of this transaction at time o and time 1:

Security	Initial Income/(Outgo) – Time 0	Final Income/(Outgo) if prices increase – Time 1	Final Income/(Outgo) if prices decrease – Time 1
A	12	(14)	(10)
B	(11)	14	10
Total	**1**	**0**	**0**

Thus, the investor makes a profit at time 0 of $1, with no risk of future loss.

It is clear that investment A is not as attractive as investment B. There will be excessive demand for B and no demand for A, leading to a reduction in the price of A and an increase in the price of B. Ultimately a balance would be achieved, when $P_0^A = \frac{P_0^B}{2}$ (since $P_1^A = \frac{P_1^B}{2}$), at which time the prices are consistent and the arbitrage opportunity is eliminated. This satisfies the "law of one price".

In practice, in major developed securities markets when arbitrage opportunities arise they are very quickly eliminated as investors spot them and trade on them. Such opportunities are so fleeting in nature, according to the empirical evidence, that it is sensible, realistic and prudent to assume that they do not exist.

The "No -Arbitrage" assumption enables us to find the price of complex instruments by "replicating" the payoffs. This means that if we can construct a portfolio of assets with exactly the same payments as the investment that we are interested in, then the price of the investment must be the same as the price of the "replicating portfolio".

In the following section we use these ideas to price forward contracts.

FORWARD CONTRACTS

A **forward contract** is an agreement made at some time $t = 0$ between two parties under which one agrees to buy from the other a specified amount of an asset (denoted by S) at a specified price on a specified future date. The investor agreeing to sell the asset is said to hold a ***short*** forward position in the asset, and the buyer is said to hold a ***long*** forward position.

We will define some notation and terms used for this section:

- S_0 the price of the underlying asset (for example, a unit of equity stock) at time $t = 0$ (ie. the price of the asset when the agreement is made).
- S_r the price of the underlying asset at time r, where $r > 0$. We do not know with certainty what the price will be at any future date.
- T the specified future date at which the forward contract matures (that is, when the sale actually happens).
- K the **forward price**. This is the price of the asset agreed at time $t = 0$ to be paid at time $t = T$ (ie. the price paid when the contract matures).
- δ the force of interest that is available on a risk-free investment over the term of the contract. In this context this is known as the "risk-free" force of interest.

We assume that this is known. The risk-free force of interest is likely to be based on the yield obtainable by borrowing or lending a security issued by a developed country's government for the same term as the forward contract.

There are three possible scenarios that can arise at the time of maturity of a forward contract:

- if the forward price is greater that the actual asset worth at maturity, $K > S_T$, then the seller of the asset has received K for an asset worth S_T and has made a profit at time T of $K - S_T$.(ie. the short forward position will purchase the asset at time T for an amount S_T and then immediately sell it to the long forward position for an amount K).

- if the forward price is less that the actual asset worth at maturity, $K < S_T$, then the buyer has paid K for an asset worth S_T and has made a profit at time T of $S_T - K$. (ie. the buyer purchases the asset at time T for an amount K and then can immediately sell it for S_T).

- if the forward price is equal to the actual asset worth, $K = S_T$, then neither the buyer nor seller have made a profit or loss. This is equivalent to the trade happening on the open market.

EXAMPLE
Investor A agrees to sell 1,000 Lend Lease shares in six months to Investor B at a price of $11 per share. The current price is $10.50 per share.

$\Rightarrow S_0 = \$10.50 \times 1000 = \$10,500$

$11 is the forward price per share. $\Rightarrow K = \$11 \times 1000 = \$11,000$

Six months later the share price is $10.70. $\Rightarrow S_T = \$10,700$

Investor A (the short forward position) sells 1,000 shares, which are worth $10,700 in six months, for $11,000.

He therefore makes a profit from the contract of $K - S_T = \$300$

Similarly, Investor B (the long forward position) has to buy the shares for a loss of $300.

CALCULATING THE FORWARD PRICE OF A SECURITY

To determine the forward price K we can use the replication argument that comes from the "no arbitrage" assumption.

We assume that there are no payments or costs associated with holding the stock.

We will first find the forward price for a security with no income, and then we will consider securities with income, such as coupon-paying bonds.

Securities with no income

Consider the following two investment portfolios:

Portfolio A: Enter into a long forward contract to buy one unit of an asset S, with forward price K, maturing at time T; simultaneously invest an amount $Ke^{-\delta T}$ in the risk-free investment. (Remember that an amount invested in the risk-free investment is assumed to have a constant force of interest of δ.)

Portfolio B: Buy one unit of the asset S, at the current price S_0

The cash flows from each portfolio are given below.

At time $t = 0$, outgo is:

A $\quad Ke^{-\delta T}$

B $\quad S_0$

At time $t = T$, value of investments is:

A $\quad$ From the risk-free investment the investor receives an amount of $Ke^{-\delta T}e^{\delta T} = K$.

From the forward contract the investor owns an amount of S_T and pays the forward price K.

$\Rightarrow$ Value = $K + S_T - K = S_T$

B $\quad S_T$

Therefore, the future cash flows of portfolio A are identical to those of portfolio B - both give a return of one unit of the underlying asset S_T. The no-arbitrage assumption states that when the future cash flows of two portfolios are identical, the price must also be the same - that is:

$$Ke^{-\delta T} = S_0 \Rightarrow \boxed{K = S_0 e^{\delta T}}$$

So, we have found the price for the forward contract with no need to model how the asset price S_t will actually move over the term of the contract.

EXAMPLE
A six-month forward contract exists in a zero-coupon corporate bond with a current price of $50. The yield available on six-month government securities is 9% pa effective. Calculate the forward price.

Solution
We have been given $S_0 = \$50$, and $\delta = \ln(1+i) = \ln(1.09)$. Using the formula derived above,

$$K = S_0 e^{\delta T} \Rightarrow K = 50\left(e^{\frac{6}{12}\ln(1.09)} \right) = 50(1.09)^{6/12} = \$52.20$$

<u>Securities with income</u>

Assume now that at some time t_1, $0 \le t_1 \le T$, the security underlying the forward contract provides a fixed amount c to the holder of the contract. For example, if the security is a government bond, there will be a fixed coupon payment due every six months.

Consider the following two portfolios:

Portfolio A: Enter a forward contract to buy one unit of an asset S, with forward price K, maturing at time T; simultaneously invest an amount $Ke^{-\delta T} + ce^{-\delta t_1}$ in the risk-free investment.

Portfolio B: Buy one unit of the asset, at the current price of S_0. At time t_1 invest the coupon of c that is received from the asset in the risk-free investment.

The cashflows from each portfolio are given below.

At time $t = 0$, outgo is:

A $\quad Ke^{-\delta T} + ce^{-\delta t_1}$

B $\quad S_0$

At time $t = T$, value of investments is:

A From the risk-free investment the investor receives an amount of
$\left(Ke^{-\delta T} + ce^{-\delta t_1}\right)e^{\delta T} = K + ce^{\delta(T-t_1)}$

From the forward contract the investor owns an amount of S_T and pays the forward price K
$\Rightarrow$ Value = $K + ce^{\delta(T-t_1)} + S_T - K = S_T + ce^{\delta(T-t_1)}$

B The payout is the asset value at time T plus the accumulated value of the invested coupon $\Rightarrow S_T + ce^{\delta(T-t_1)}$.

Therefore, the future cashflows of portfolio A are identical to those of portfolio B - both give a net portfolio value of $S_T + ce^{\delta(T-t_1)}$. Under the "no-arbitrage assumption" the prices must also be the same - that is:

$$Ke^{-\delta T} + ce^{-\delta t_1} = S_0 \Rightarrow \boxed{K = S_0 e^{\delta T} - ce^{\delta(T-t_1)}}$$

So, the forward price is equal to the accumulated value at time T of the current price, less the accumulated value at time T of the income payment, which will not be received by the buyer of the forward contract.

EXAMPLE
A fixed interest security pays half-yearly coupons of 10% per annum in arrears and is redeemable at par. Two months before the next coupon payment is due, an investor negotiates a forward contract in which he agrees to buy \$20,000 nominal of the security in six-months' time. The current price of the security is \$50.20 per \$100 nominal and the risk-free force of interest is 7% pa.

Calculate the forward price.

Solution
Using the formula given above, the forward price is equal to:

$$K = S_0 e^{\delta T} - ce^{\delta(T-t_1)}$$

$S_0 = 20,000 \times \frac{50.20}{100} = \$10,040$ = security value at time 0 (time of agreement of contract).

$T = \frac{6}{12}$ = time to maturity of the contract.

$t_1 = \frac{2}{12}$ = time from contract agreement to the coupon payment.

$c = 20,000 \times \frac{0.1}{2} = \$1000$ = amount of coupon payment.

$\delta = 0.07$ = risk-free force of interest.

$$\Rightarrow K = S_0 e^{\delta T} - ce^{\delta(T-t_1)} = 10,040e^{0.07(0.5)} - 1,000e^{0.07(\frac{4}{12})} = \$9,374$$

For a forward contract on a fixed interest security there may be more than one coupon. If we let PV_I denote the present value at time $t = 0$ of the fixed income payments due during the term of the forward contract, then the forward price at time $t = 0$ for security S is:

$$\boxed{K = (S_0 - PV_I)e^{\delta T}}$$

From the example above, this would be equivalent to:

$$K = \left(10,040 - 1,000e^{-0.07(\frac{2}{12})}\right)e^{0.07(0.5)} = 10,040e^{0.07(0.5)} - 1,000e^{0.07(\frac{4}{12})}$$

EXAMPLE
Consider the previous example. Two months before the next coupon payment is due, a different investor negotiates a forward contract to purchase $50,000 nominal of the stock in 16 months' time. Calculate the forward price of the contract.

Solution

$$S_0 = 50,000 \times \frac{50.20}{100} = \$25,100$$

$$T = \frac{16}{12}$$

$$c = 50,000 \times \frac{0.1}{2} = \$2,500$$

The coupon payments due during the term of the contract occur after 2 months, 8 months, and 14 months time. The present value of these coupon payments (at force of interest $\delta = 0.07$) is:

$$PV_I = 2,500\left(e^{-(2/12)\delta} + e^{-(8/12)\delta} + e^{-(14/12)\delta}\right) = \$7,160.96$$

$$\Rightarrow K = (S_0 - PV_I)e^{\delta T} = (25,100 - 7,160.96)e^{0.07(16/12)} = \$19,693.97$$

THE VALUE OF A FORWARD CONTRACT

It may be of interest to find the value of a forward contract at intermediate times, between the time of agreement and the time of maturity.

At the time of agreement, $t = 0$, the value of the contract to the buyer and seller is 0. At the time of maturity, $t = T$, the value of the contract to the seller of the asset is $K - S_T$ and the value of the contract to the buyer is $S_T - K$.

Suppose at time $r > 0$ an investor holds a ***long forward contract*** - that is the investor holds a contract agreeing to buy an asset S at a specified maturity date for a forward price K_0. The investor decides to sell this contract for an amount V_L at time r (ie. the long forward contract will be held by another party – removing the requirement for the investor to purchase the asset S at time T).

We wish to find the value of V_L at time r.

Consider the following two portfolios:

Portfolio A: Buy the existing long forward contract for price V_L at time r. Invest $K_0 e^{-\delta(T-r)}$ at time r in the risk-free investment for $T - r$ years.

Portfolio B: Buy a new long forward contract maturing at the same date, forward price K_r. Invest $K_r e^{-\delta(T-r)}$ in the risk-free investment for $T - r$ years.

The cashflows from each portfolio are given below.

At time $t = r$, outgo is:

A $\qquad V_L + K_0 e^{-\delta(T-r)}$

B $\qquad K_r e^{-\delta(T-r)}$

At time $t = T$, value of investments is:

A $\qquad$ From the risk-free investment the investor receives an amount $K_0 e^{-\delta(T-r)} e^{\delta(T-r)} = K_0$

From the forward contract the investor receives an amount S_T and pays the forward price K_0

$\Rightarrow$ Value = $K_0 + S_T - K_0 = S_T$

B $\qquad$ From the risk-free investment the investor receives an amount $K_r e^{-\delta(T-r)} e^{\delta(T-r)} = K_r$

From the forward contract the investor receives an amount S_T and pays the forward price K_r

$\Rightarrow$ Value = $K_r + S_T - K_r = S_T$

By the no arbitrage assumption we have $V_L + K_0 e^{-\delta(T-r)} = K_r e^{-\delta(T-r)}$ which can be written as:

$$\boxed{V_L = \left(K_r - K_0\right) e^{-\delta(T-r)}}$$

This formula is independent of whether the security pays income or not (which will be reflected in the calculation of the forward prices K_0 and K_r). In the special case of a security which does not pay income, since:

$$K_0 = S_0 e^{\delta T} \Rightarrow K_0 e^{-\delta(T-r)} = S_0 e^{\delta T} e^{-\delta(T-r)} = S_0 e^{\delta r} \text{ and}$$
$$K_r = S_r e^{\delta(T-r)} \Rightarrow K_r e^{-\delta(T-r)} = S_0 e^{\delta(T-r)} e^{-\delta(T-r)} = S_r$$

we can re-write this as:

$$\boxed{V_L = S_r - S_0 e^{\delta r}}$$

This is the value of a long forward contract at time r. ie. a forward contract under which the investor concerned agrees to buy an asset in the future. In this case, if the return on S over the period $t = 0$ to $t = r$ is greater than the risk free return, then the contract to buy the asset at $t = T$ has a positive value as the investor is looking likely to be purchasing the asset at a discount to its true value at $t = T$ under the forward contract.

Using a similar argument, it can be shown that the value of a ***short forward contract*** at time r is:

$$\boxed{V_S = -V_L}$$

EXAMPLE
Consider the previous example where the duration of the contract is 16 months and all other factors are held constant. Calculate the value of the long forward contract 6 months before the end of the contract if the security price at this date is $45.60 per $100 nominal.

Solution
We now know that $K_0 = 19{,}693.97$ from the previous example. All other factors are the same. Since this is a coupon paying bond we need to use the general equation $V_L = \left(K_r - K_0\right)e^{-\delta(T-r)}$ and not the simplified form. The only thing we are missing is K_r. This is the forward price calculated at the valuation date. Since there is 6 months until the contract finishes, there is still one coupon due in 4 months time.

$$K_r = \left(S_r - PV_I\right)e^{\delta(T-r)}$$
$$= \left(\frac{45.60}{100}\times 50{,}000 - 2{,}500e^{-0.07\times 4/12}\right)e^{0.07\times 6/12}$$
$$= 21{,}082.79$$

$$V_L = \left(21{,}082.79 - 19{,}693.97\right)e^{-0.07\times 6/12} = \$1{,}341.05$$

THE YIELD CURVE AND THE TERM STRUCTURE OF INTEREST RATES

So far we have priced bonds using the same discount rate for all coupon payments and the redemption amount.

In practice the interest rate offered on an investment usually varies according to the term of the investment.

Bonds with different terms to maturity usually have different yield rates due to different economic conditions, default risks and investors' objectives.

Variation occurs because interest rates are influenced by:

- Supply and demand
  - Interest rates are determined by market forces. eg. Little demand for finance will force interest rates down.
- Base rates
  - Investors will have a view on how central banks will change the base rate in the future.
- Interest rates in other countries
  - The cost of borrowing in other countries will influence the demand for finance in the local country.

- Expected future inflation
  - Lenders will expect interest rates to exceed inflation.
- Tax rates
  - Investors will require a certain level of return after tax.
- Risks associated with changes in interest rates
  - The risk of loss due to a change in interest rates is greater for longer-term investments, so in general, interest rates will be higher for longer-term investments.

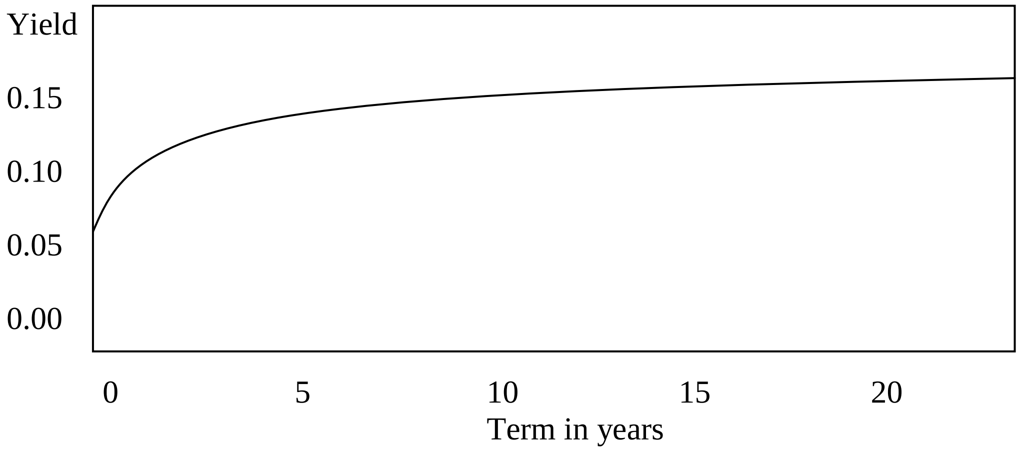
<u>THE YIELD CURVE</u>

The relationship between the term-to-maturity and yield-to-maturity is called the term structure of interest rates, and can be represented by a ***yield curve***, which is a graph of bond yields against their time until maturity.

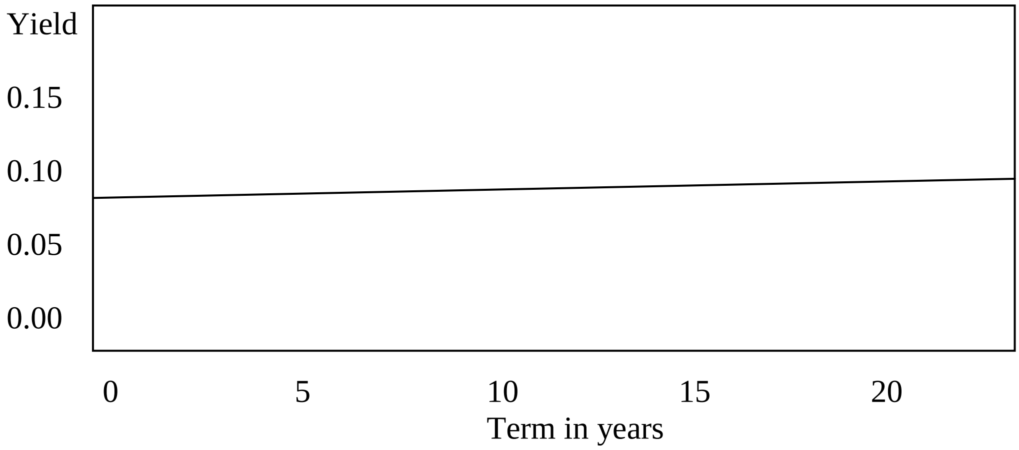
The shapes that the yield curve can take include:

- Upward sloping or normal
- Flat (or almost flat)
- Downward sloping or inverted
- Humped

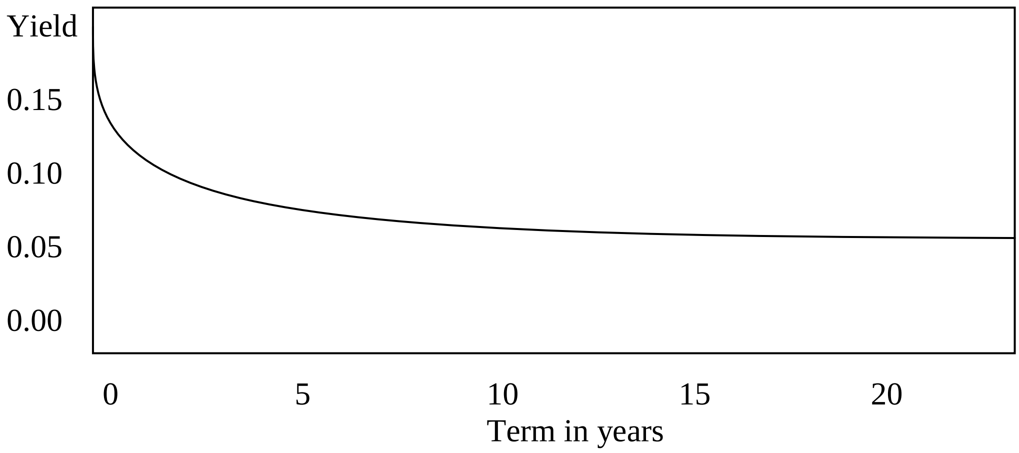
<u>Upward sloping</u>



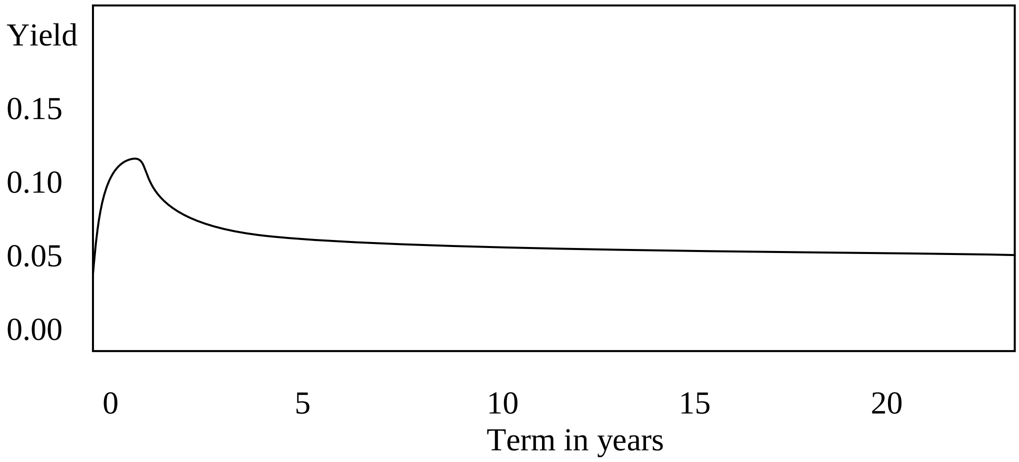
<u>Flat</u>



Downward sloping



Humped



There are three main theories that are used to explain why interest rates vary according to the term of the investment:

1. **Expectations theory**
   The demand for short and longer-term investments will vary according to expectations of future interest rate movements. If the market expects a rise in interest rates there will be a decreased demand for (and therefore the price of) long-term investments relative to short-term investments. This is because a rise in interest rates corresponds to a decrease in the price of a bond (as the present value of future cash flows will be lower), thus decreasing the immediate capital value of the bond. Conversely, expectations of a future drop in interest rates will lead to a higher demand and, therefore, higher prices and lower yields for long-term bonds.

   The expectations theory can be considered to be a self fulfilling prophecy, as the expectation of future interest rate changes results in the actual change occurring.

2. **Liquidity preference**
   Longer term bonds are more sensitive to interest rate movements than short term bonds (we will cover this in detail in the section on interest rate risk). Under this theory it is assumed that the resulting increased risk of loss for long term bonds may explain some of the excess returns on long term bonds as shown in the increasing yield curve.

3. **Market segmentation**
   Different investment terms are required by different investors. For example, superannuation funds with long-term liabilities are interested in longer-term bonds, whereas bank liabilities are short-term, so banks require more short-term bonds. The relative demands of investors will affect the prices and yields of short-term and long-term securities. Similarly, the supply of bonds will vary, as governments and company borrowing requirements change.

These theories and the discussion above of the different yield curves are NOT EXAMINABLE, but are included here to help you understand why we need to consider interest rate variation when valuing fixed interest securities.

The remainder of the material below is examinable.

SPOT RATES

The redemption yield (internal rate of return) on a bond is usually a messy quantity to use analytically. It is a convenient summary measure of a bond's expected return, and therefore a popular tool to compare different bonds, but the use of a single interest rate to discount multiple cash flows is only appropriate when the yield curve is flat.

Since the yield curve is not usually flat, the present value of payments made or received at different times should be calculated using different discount rates.

A more useful concept than the redemption yield is the ***spot rate of interest***, which is the yield to maturity on a zero coupon bond, or a single cash flow. A spot interest rate is an interest rate on an investment that is available now and holds until some specified future time.

Let the effective yield per period on a zero coupon bond with a term of t periods be denoted by s_t. This is called the t-period spot rate of interest. The t-period spot rate is a measure of the average interest rate over the period from now until t periods time.

The accumulation at time t of an investment of 1 at time 0 is $(1+s_t)^t$.

EXAMPLE
Calculate the annual spot rates for the following unit ($1) zero coupon bonds with the following terms and prices:

1 year = \$0.94
5 years = \$0.70
10 years = \$0.47

Solution

1 year: $\$0.94 = v_{s_1}^{1} = (1+s_1)^{-1} \Rightarrow s_1 = 6.4\%$

5 years $\$0.70 = v_{s_5}^{5} = (1+s_5)^{-5} \Rightarrow s_5 = 7.4\%$

10 years: $\$0.47 = v_{s_{10}}^{10} = (1+s_{10})^{-10} \Rightarrow s_{10} = 7.8\%$

Spot rates can be used to value coupon bonds. A coupon bond can be valued as the sum of zero-coupon bonds of different maturities. Each coupon payment and the redemption payment on a coupon-paying bond can be treated as separate zero-coupon bonds. The present value of each payment can be found separately at the corresponding spot interest rate.

Since coupon bonds usually have coupons payable half-yearly, it is convenient to work with effective half-yearly spot rates. We now redefine s_p as the spot rate of interest corresponding to period p where p is measured in half-years.

Recall that the equation of value at t=0 for a bond with half-yearly coupon payments at half-yearly effective interest j can be written:

$$P = \sum_{p=1}^{n} Fr \cdot v_j^p + C \cdot v_j^n$$

In terms of **half-yearly spot rates**, the bond price can be written:

$$P = \sum_{p=1}^{n} Fr \cdot v_{s_p}^{p} + C \cdot v_{s_n}^{n} = Fr \cdot \left(v_{s_1} + v_{s_2}^{2} + \ldots + v_{s_n}^{n} \right) + C \cdot v_{s_n}^{n}$$

or,

$$\boxed{P = \frac{Fr}{(1+s_1)} + \frac{Fr}{(1+s_2)^2} + \ldots + \frac{Fr+C}{(1+s_n)^n}}$$

The price of a coupon-bond can be written as the sum of the prices of corresponding zero-coupon bonds. If we let P_p equal the price of a unit zero coupon bond maturing in p half-years, then:

$$P = Fr \cdot \left(P_1 + P_2 + \ldots + P_n \right) + C \cdot P_n$$

EXAMPLE
Find the price of a $100 bond maturing in 4 years at par that pays half-yearly coupons of 10% per annum. Assume that the term structure of annual nominal spot rates convertible half-yearly is given as below:

Time to maturity (yrs)	spot rate (% per annum convertible half-yearly)
0.5	7.50
1.0	7.75
1.5	8.00
2.0	8.00
2.5	8.25
3.0	8.50
3.5	8.50
4.0	9.00

Calculate the gross redemption yield of this bond.

Solution
The price of the bond in terms of half-yearly spot rates is:

$$P = Fr \cdot \left(v_{s_1} + v_{s_2}^2 + \ldots + v_{s_n}^n \right) + C \cdot v_{s_n}^n$$

We need to convert the nominal spot rates convertible half-yearly to half-yearly effective spot rates:

$$P = 5\left(\frac{1}{\left(1+\frac{0.075}{2}\right)} + \frac{1}{\left(1+\frac{0.0775}{2}\right)^2} + \ldots + \frac{1}{\left(1+\frac{0.09}{2}\right)^8} \right) + \frac{100}{\left(1+\frac{0.09}{2}\right)^8} = 103.72$$

The gross redemption yield of this bond is the effective yield i that solves:

$103.72 = 5 \cdot a_{\overline{8}|j} + 100 \cdot v_j^8$, where $i = (1+j)^2 - 1$,

By interpolation we can find that $i = 9.07$ %.

FORWARD RATES

Spot rates tell us about interest rates over a period that starts now. ***Forward rates*** are rates agreed today, for an investment in a future period. Multi-period spot rates can be decomposed into the product of one-period forward rates.

The discrete time forward rate $f_{t,T}$ (per period) is the interest rate agreed at time 0 for an investment made at time $t > 0$ until time T .

For example, if an investor agrees at time 0 to invest \$100 at time t, the accumulated investment value at time T is:

$$100(1+f_{t,T})^{T-t}$$

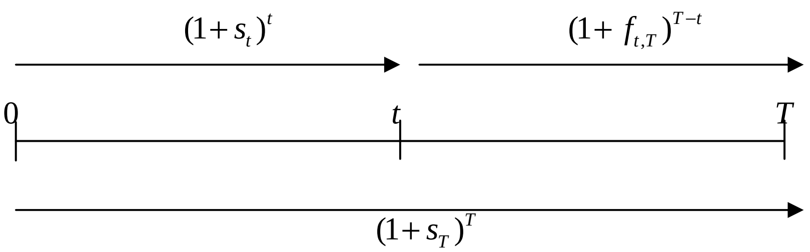
The forward rate $f_{t,T}$ can be considered a measure of the average interest rate per period between times t and T.

Forward rates and spot rates are related. As shown above, the accumulation at time t of an investment of 1 at time 0 is $(1+s_t)^t$. If we agree at time 0 to invest this accumulated amount at time t, the rate per period earned by time T will be $f_{t,T}$. The total accumulated amount is then $(1+s_t)^t(1+f_{t,T})^{T-t}$. This is equivalent to 1 invested at time 0 for T periods at the spot rate s_T. Therefore,

$$(1+s_t)^t(1+f_{t,T})^{T-t}=(1+s_T)^T \text{, or}$$

$$\boxed{(1+f_{t,T})^{T-t}=\frac{(1+s_T)^T}{(1+s_t)^t}}$$

The relationship between spot rates and forward rates can be represented on a time line.



The one-period forward rate $f_{t,t+1}$ is the rate of interest per period from time t to time $t+1$ and can be expressed in terms of spot rates as below:

$$(1+f_{t,t+1})=\frac{(1+s_{t+1})^{t+1}}{(1+s_t)^t}$$

The t-period spot rate s_t can be written in terms of one-period spot rates and a series of one-period forward rates:

$$(1+s_t)^t=(1+s_1)(1+f_{1,2})(1+f_{2,3})(1+f_{3,4})\ldots(1+f_{t-1,t})$$

In other words, the t-period spot rate s_t is the geometric average of the one-period spot rate and a series of one-period forward rates, each commencing one period later in the future.

$$(1+s_1) \longrightarrow (1+f_{1,2}) \longrightarrow (1+f_{2,3}) \longrightarrow \cdots\cdots \longrightarrow (1+f_{t-1,t})$$

$$\xrightarrow{\hspace{10em}} (1+s_t)^t$$

Working with periods of half-years and defining $f_{t,t+1}$ as the forward rate per half-year t between t and $t+1$, the bond price can be written as:

$$\boxed{P = \frac{Fr}{(1+f_{0,1})} + \frac{Fr}{(1+f_{0,1})(1+f_{1,2})} + \dots + \frac{Fr+C}{(1+f_{0,1})(1+f_{1,2})\dots(1+f_{n-1,n})}}$$

EXAMPLE

If the one-period spot rate equals 6% and the two-period spot rate equals 8.08%, find the forward rate that holds between period 1 and 2 $\left(f_{1,2}\right)$.

Solution

$$(1+f_{1,2}) = \frac{(1+s_2)^2}{(1+s_1)^1} = \frac{(1.0808)^2}{(1.06)^1} = 1.1020 \Rightarrow f_{1,2} = 10.20\%$$

EXAMPLE

Find the effective half-yearly forward rates for each future half-year corresponding to the nominal spot rates convertible half-yearly given below.

Time to maturity (yrs)	spot rate (% per annum convertible half-yearly)
0.5	7.50
1.0	7.75
1.5	8.00
2.0	8.00

Solution

Working in periods of half years and reducing the % per annum nominal rates to effective convertible half-yearly rates:

$$f_{0,1} = s_1 = 0.075/2 = 0.0375$$

For the second half-year:

$$f_{1,2} = \frac{(1+s_2)^2}{(1+f_{0,1})} - 1 = \frac{(1+0.0775/2)^2}{(1.0375)} - 1 \cong 0.04$$

For the third half-year:

$$f_{2,3} = \frac{(1+s_3)^3}{(1+s_2)^2} - 1 = \frac{(1+0.08/2)^3}{(1+0.0775/2)^2} - 1 \cong 0.0425$$

For the last half-year:

$$f_{3,4} = \frac{(1+s_4)^4}{(1+s_3)^3} - 1 = \frac{(1+0.08/2)^4}{(1+0.08/2)^3} - 1 = 0.04$$