STA257H1F Fall, 2008

Term Test

October 22, 2008

Name: Solution

Student Number:______________________________________

Tutorial: (circle one)

- o Tutorial A (A – Ka) Avery (SS2106)
- o Tutorial B (Ki-Se) Panpan (SS2108)
- o Tutorial C (Sh – Z) Alex (SS1087)

Instructions:

- Time: 90 minutes.
- No aids allowed except a nonprogrammable calculator.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need **not** be expressed in decimal form.
- If you do not understand a question, or are having some other difficulty, do not hesitate to ask your instructor for clarification.
- There are 12 pages including this page. Please check that you are not missing any page.
- Show your work and answer in the space provided, in ink. Pencil may be used, but then remarks will not be allowed. Use back of pages for rough work.
- Total point: 70.

Good luck!!

Av Av Av Ax Ax P P

Question	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Max	10	7	10	14	14	15	70
Score							

Question 1

a) (5 points) Let P be probability measure defined on the sample space Ω. For any event A define $Q(A) = \frac{1}{P(A)}$. Is Q a probability measure on Ω? Why or why not?

1 No since it does not satisfy the conditions of additivity of the union of disjoint events.

Let A and B be disjoint events with $P(A) > 0$, $P(B) > 0$ then

$$Q(A \cup B) = \underset{1}{\frac{1}{P(A\cup B)}} = \underset{1}{\frac{1}{P(A)+P(B)}} \neq \underset{1}{\frac{1}{P(A)} + \frac{1}{P(B)}} = \underset{1}{Q(A) + Q(B)}$$

b) (5 points) Show that if A, B, C are (mutually) independent, then B and $A \cup C$ are independent..

$$P(B \cap (A \cup C)) = P((B\cap A) \cup (B \cap C))$$

$$= P(B \cap A) + P(B\cap C) - P(A \cap B \cap C)$$

$$= P(B)\cdot P(A) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$$

$$= P(B)\left[P(A) + P(C) - P(A) \cdot P(C)\right]$$

$$= P(B) \cdot P(A \cup C)$$

1 mark for each line

Question 2

10 people are getting into an elevator in a building that has 20 floors.

a) (4 points) What is the probability that each person gets off on a different floor?

1.5 $N(\Omega) = 20^{10}$ ← all possible ways in which 10 people can get off.

1.5 $N(A) = P^{20}_{10} = \binom{20}{10} \cdot 10!$ ← all the ways in which 10 people get off on a different floor

$$P(A) = \frac{N(A)}{N(\Omega)} = \frac{\frac{20!}{10!}}{20^{10}} = 0.0655$$

1

b) (3 points) What is the probability that all of them get off on the same floor?

1 $N(A) = \binom{20}{1}$ ← only need to choose one floor out of 20 in which they will all get off.

$$P(A) = \frac{\binom{20}{1}}{20^{10}} = \frac{1}{20^9}$$

2

Question 3

Suppose a couple of your friends go to 'Sushi on Bloor' on either Monday or Friday each week, not both. 26% of the time they go on Monday. On Mondays, the probability of receiving a good service is 0.72. On Fridays, probability of receiving a good service is only 0.13.

a) (3 points) What is the probability that they went to that restaurant on Monday and received a good service?

$P(\text{Monday and good service}) = P(\text{good service} \mid \text{Monday}) \cdot P(\text{Monday})$

$\wedge$ 1

$= 0.72 \times 0.26 = 0.1872$

1

b) (3 points) What is the probability that they received good service at that restaurant last week?

1/2 1.5

$P(\text{good service}) = P(\text{good service} \mid M) \cdot P(M) + P(\text{good service} \mid F) \cdot P(F)$

$= 0.72 \times 0.26 + 0.13 \times 0.74 = 0.2834$

1

c) (4 points) Suppose that you don't know which day they went last week, but they tell you they received good service. What is the probability that they went on Monday?

$$P(M \mid \text{good service}) = \frac{P(M \text{ and good service})}{P(\text{good service})} = \frac{0.72 \cdot 0.26}{0.2834} = 0.66$$

1 1 2

Question 4

The number of students arriving at the ATM machine in Sidney Smith hall is a Poisson random variable with mean of 3 students per minutes.

a) (2 points) What is the probability that in a given minute exactly 4 students arrive at the ATM?

X- # of students arriving at the ATM per minute

1

$X \sim Poisson(3)$

$$P(X=4) = \frac{e^{-3} 3^4}{4!} = 0.168$$

1

b) (3 points) What is the probability that in a given minute at least one student arrive at the ATM?

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0) = 1 - \frac{e^{-3} \cdot 3^0}{0!} = 0.9502$$

1 1 1

c) (2 points) What is the expected number of students arriving at the ATM in a given hour?

Y - # of students arriving at the ATM per hour

1

$$Y \sim Poisson(\underbrace{180}_{60 \text{ minuts} \times 3})$$

1 $E(Y) = 180$.

d) (3 points) What is the probability that during half an hour there were exactly 2 minutes in which no student arrive at the ATM?

P(in a given minute no student arrive at the ATM) = $P(X=0) = e^{-3}$ 1

Let Z = # of minutes in half an hour in which no cars crossed the intersection.

1

$Z \sim Bin(30, e^{-3})$

$P(Z=2) = \binom{30}{2}(e^{-3})^2 \cdot (1-e^{-3})^{28}$

1

d) (4 points) Starting at 8:00 AM, what is the probability that the first minute in which no student arrive at the ATM was after 8:26 AM?

Let T = # of minutes until the first minute in which no student arrive at the ATM.

1

1 $T \sim Geometric(e^{-3})$

$P(T > 26) = (1-e^{-3})^{26}$

2

Question 5

The length of time to failure (in hundreds of hours) for a transistor is a random variable X with a distribution function given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x^2} & x \geq 0 \end{cases}$$

a) (4 points) Show that $F_x(x)$ has the properties of a distribution function (Note: F is known to be right continuous function).

(1) • $F_X(-\infty) = \lim_{x \to -\infty} 0 = 0$

(1) • $F_X(\infty) = \lim_{x \to \infty} 1 - e^{-2x^2} = 1$

• Need to show that $F_X(x)$ is non-decreasing. So if $x_1 \leq x_2$

(2) then $x_1^2 \leq x_2^2 \Rightarrow -2x_1^2 \geq -2x_2^2 \Rightarrow e^{-2x_1^2} \geq e^{-2x_2^2}$

$\Rightarrow 1 - e^{-2x_1^2} \leq 1 - e^{-2x_2^2} \Rightarrow F_X(x_1) \leq F_X(x_2)$

b) (2 points) Find the probability density function of X.

(1) (1)

$$f_X(x) = \begin{cases} 4x e^{-2x^2} & x > 0 \\ 0 & \text{o.w} \end{cases}$$

c) (4 points) Find the probability that the transistor operates for at least 260 hours if it is known that it can operate no more then 300 hours.

(1)

$$P(X \geq 2.6 \mid X \leq 3) = \frac{P(X \geq 2.6 \cap X \leq 3)}{P(X \leq 3)} = \frac{P(2.6 \leq X \leq 3)}{P(X \leq 3)}$$

(1)

$$= \frac{F_X(3) - F_X(2.6)}{F_X(3)} = \frac{e^{-13.52} - e^{-18}}{1 - e^{-18}}$$

(1) (1)

d) (4 points) Find the mean and variance of *X*.

$$E(X) = \int_0^\infty x \cdot 4x e^{-2x^2} dx = \int_0^\infty 4x^2 e^{-2x^2} dx = \int_0^\infty 2u^{1/2} e^{-2u} du$$

(2)

let $u = x^2$
$du = 2x\,dx$

$$= \frac{2\Gamma(\frac{3}{2})}{2^{3/2}} \cdot \underbrace{\int_0^\infty \frac{e^{-2u}\, 2^{3/2} \cdot u^{1/2}}{\Gamma(\frac{3}{2})} du}_{=1} = \frac{1}{\sqrt{2}} \Gamma\left(\frac{3}{2}\right)$$

since integrand is gamma$(\frac{3}{2}, 2)$ density

$$E(X^2) = \int_0^\infty x^2 4x e^{-2x^2} dx = \int_0^\infty 2u e^{-2u} du = \frac{1}{2}$$

(1)

let $u = x^2$
$du = 2x\,dx$

Note: the above integral is the $E(u)$ where $u \sim \exp(2)$

$$\Rightarrow Var(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \left(\frac{1}{\sqrt{2}} \Gamma\left(\frac{3}{2}\right)\right)^2$$

(1)

Question 6

Let X have the density function give by

$$f_X(x) = \begin{cases} 0.2 & -1 < x \le 0 \\ 0.2 + cx & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) (3 points) Find the constant c that makes it a valid density.

We must have $\int_{-\infty}^{\infty} f_X(x) = 1$ 1

$$\int_{-1}^{0} 0.2\,dx + \int_{0}^{1} (0.2 + cx)\,dx = 0.2 \cdot x \Big|_{-1}^{0} + \left(0.2x + \frac{cx^2}{2}\right)\Big|_{0}^{1}$$

1

$$= 0.2 + 0.2 + \frac{c}{2} = 1$$

$$\Rightarrow \quad c = 2 \cdot (1 - 0.4) = 1.2 \quad 1$$

b) (3 points) Find the cumulative distribution function of X, i.e. $F_X(x)$

2 1

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \int_{-1}^{x} 0.2\,dx = 0.2(x+1) & -1 \le x < 0 \\ 0.2 + \int_{0}^{x} (0.2 + 1.2x)\,dx = 0.2 + 0.2x + \frac{1.2x^2}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

c) (4 points) Find the third moment of X.

$$E(X^3) = \int_{-1}^{0} x^3\, 0.2\, dx + \int_{0}^{1} x^3 (0.2 + 1.2x)\, dx$$

(1) (1)

$$= 0.2 \frac{x^4}{4}\Big|_{-1}^{0} + \left(0.2\frac{x^4}{4} + \frac{1.2 \cdot x^5}{5}\right)\Big|_{0}^{1}$$

(1)

$$= -\frac{0.2}{4} + \frac{0.2}{4} + \frac{1.2}{5} = 0.24$$

(1)

d) (5 points) Find the variance of $C = 3X + 7$.

First we need to find the variance of X.

$$V(X) = E(X^2) - (E(X))^2.$$

(1) $E(X) = \int_{-1}^{0} x \cdot 0.2\, dx + \int_{0}^{1} x \cdot (0.2 + 1.2x)\, dx = 0.2\frac{x^2}{2}\Big|_{-1}^{0} + \left(0.2\frac{x^2}{2} + 1.2\frac{x^3}{3}\right)\Big|_{0}^{1} = 0.4$

$$E(X^2) = \int_{-1}^{0} x^2\, 0.2\, dx + \int_{0}^{1} x^2 (0.2 + 1.2x)\, dx = 0.2\frac{x^3}{3}\Big|_{-1}^{0} + \left(0.2\frac{x^3}{3} + 1.2\frac{x^4}{4}\right)\Big|_{0}^{1}$$

(1)

$$= \frac{0.2}{3} + \frac{0.2}{3} + \frac{1.2}{4} = 0.4333$$

(1) $V(X) = 0.4333 - (0.4)^2 = 0.27333$

$\Rightarrow$ $Var(C) = Var(3X + 7) = 9 \cdot Var(X) = 9 \times 0.27333 = 2.46.$

(2)