Midterm test solutions

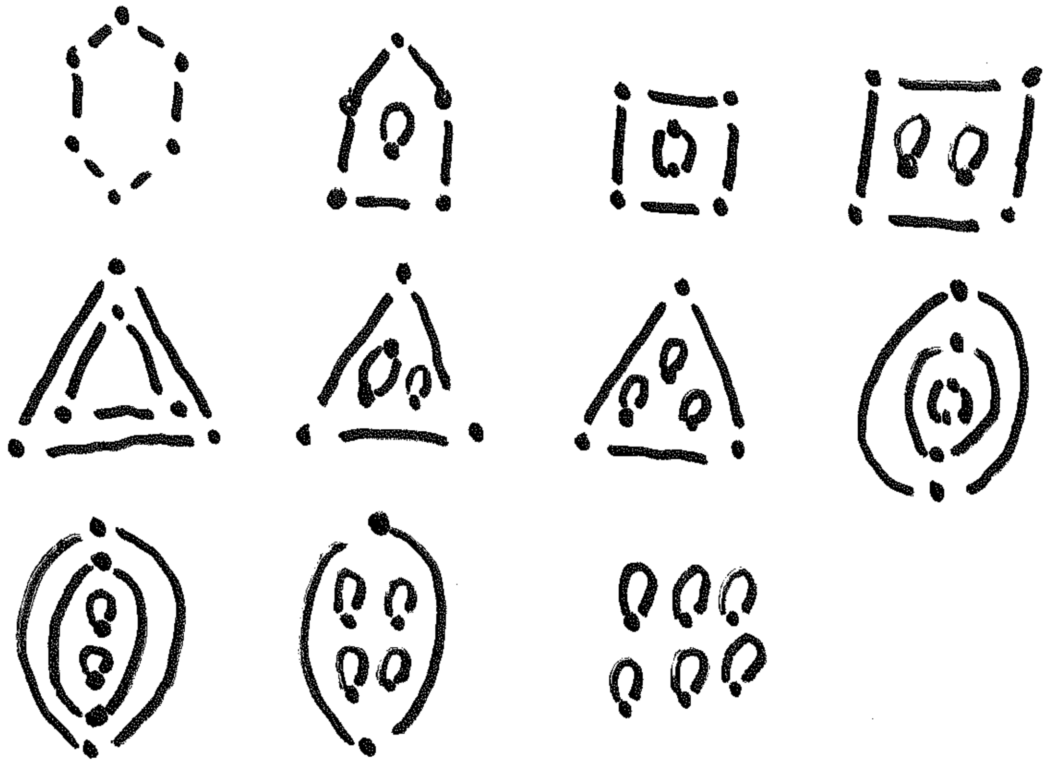
Question 1 solution

Let G be a simple graph with n vertices. If G contains an isolated vertex, then it contains no vertex of valence $n-1$, so the set of possible valences of vertices of G is $\{0,1,\dots,n-2\}$, a set of size $n-1$. Hence by the pidgeonhole principle, G must have two vertices of equal valence.

If G does not have an isolated vertex, the set of possible valences of its vertices is $\{1,\dots,n-1\}$, a set of size $n-1$. Again by the pidgeonhole principle, G must have two vertices of equal valence.

Question 2 solution

a)



b) We must prove:

Lemma: A connected 2-regular graph on k vertices is a k-cycle.

Proof: Base case: 

Induction hypothesis: true for n vertices.

Induction: Contract an edge of a 2-regular graph on $n+1$ vertices to obtain a 2-regular graph on n vertices, which is an n-cycle by IH.

Question 3 solution

($\Rightarrow$) If H contained a cycle $v_1 v_2 \ldots v_k v_1$ then any graph T containing H would contain the cycle $v_1 v_2 \ldots v_k v_1$. In particular, H could not be a subgraph of a spanning tree.

($\Leftarrow$) Contract all edges in H to form a graph G'. Take a spanning tree T' of G'. Reexpand the edges of H. $T' \cup H$ is a spanning tree of G (T' is the tree in G corresponding to $T' \subseteq G'$).

Question 4 solution

a) **Handshake Lemma**

For a graph G with vertex set $V(G) := \{v_1, \dots v_n\}$ and edge set $E(G) := \{e_1, \dots, e_m\}$,

$$\sum_{i=1}^{n} \deg v_i = 2|E|.$$

Proof: $\deg v_i$ is the number of edges incident to v_i. Each edge is counted twice in $\sum_{i=1}^{n} \deg v_i$, once for each of its incident vertices.

b) No such graph can exist.

Assume G was a graph on 10 vertices with

$G \cong \bar{G} = K_n - E(G)$

$|E(K_n)| = \frac{9 \cdot 10}{2} = 45$

So $|E(G)| + |E(G')| = 45$, which is impossible because $|E(G)| = |E(G')|$.

Question 5 solution

a) A strongly connected tournament is an oriented complete graph such that for any two vertices $u, v \in V(G)$ there is a directed path from u to v in G.

b) Because G is strongly connected, u has a predecessor and a successor. Let V and W be the set of predecessors and successors of u correspondingly. Because G is strongly connected, there exist $v \in V$, $w \in W$ such that (w,v) is an edge in G. uwvu is a directed triangle in G.

c)

Let $C_k = v_1 v_2 \ldots v_k v_1$ be a directed k-cycle in G. Let U be the set of vertices in G which are predecessors to $\{v_1, \ldots, v_k\}$, and V be the set of successors of all of $\{v_1, \ldots, v_k\}$.

<u>Case 1</u>: $G - (U \cup V \cup C_k) \neq \emptyset$

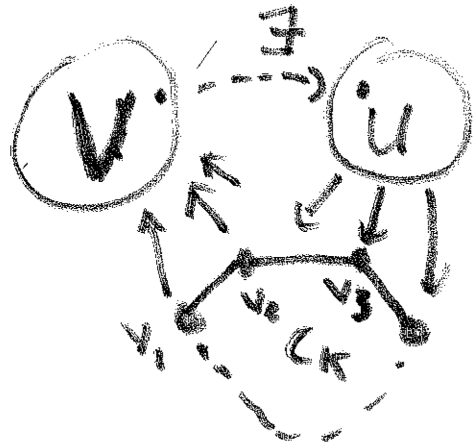
Let x be a vertex in $G - (U \cup V \cup C_k)$. Then $\exists\, i, j \in \{1, \ldots, k\}$ s.t. (x, v_i) and (v_j, x) are edges in G. This implies the existence of $l \in \{1, \ldots, k\}$ s.t. (x, v_l) and $(v_{(l-1) \bmod k}, x)$ are edges in G. Replace $(v_{(l-1) \bmod k}, v_l)$ by this pair of edges to obtain a directed $(k+1)$-cycle.

<u>Case 2</u>: $G - (U \cup V \cup C_k) = \emptyset$.

U, V are non-empty as in Part (b). There exist $u \in U$, $w \in V$ s.t. (w, u) is an edge in G by strong connectedness.

(continued)

c) Replace $v_1 v_2 v_3$ by $v_1 w u v_3$ to obtain a $k+1$-cycle.



Moon's Theorem follows by induction.

d) ($\Leftarrow$) By Part (c), a strongly-connected tournament contains an n-cycle, that is a Hamiltonian cycle.

($\Rightarrow$) Any vertices $u, v \in V(G)$ are connected by directed paths along the Hamiltonian cycle, one of which goes from u to v, and the other from v to u.