Eg. To find (if possible) a relation of linear dependence for $A_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $A_5 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

we seek x_2 and x_5 (not both 0), so that $x_2 A_2 + x_5 A_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

That is
$$\begin{aligned} 2x_2 - 4x_5 &= 0 \\ -x_2 + 2x_5 &= 0 \end{aligned}$$

(A homogeneous system, with 0's only on the right hand sides. The trivial solution is $x_2 = 0, x_5 = 0$.
we seek a non-trivial solution)

Matrix ①

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow \frac{1}{2}R_1 \\ \leftarrow R_2 + \text{new} R_1 \end{array}$$

The original system has the same solution of
$$\begin{aligned} x_2 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

For example, $x_5 = 1$. $x_2 = 2$.

$x_2 A_2 + x_5 A_5 = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ A_2 and A_5 are linearly dependent.

Eg. To determine whether $A_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $A_5 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ are linearly independent by finding (if possible) a relation of linear dependence.

$x_3 A_3 + x_5 A_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$$\begin{aligned} -x_3 - 4x_5 &= 0 \\ -x_3 + 2x_5 &= 0 \end{aligned}$$

Matrix ①

$$\left[\begin{array}{cc|c} -1 & -4 & 0 \\ -1 & 2 & 0 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 6 & 0 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

The system represented, $x_3 = 0, x_5 = 0$.
has the same solution as $x_3 A_3 + x_5 A_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, only the trivial solution.
So A_3 and A_5 are linearly independent.

Thm : Let $Ax=b$ where the vector of unknowns is $x=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $b=\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, and A is an $m\times n$ matrix.

Now Let $A_{i_1}, \dots, A_{i_m}$ be columns of A where $i_1, \dots, i_m$ are column indices in $1, \dots, n$.

The following are equivalent

① The $m\times m$ matrix $[A_{i_1} | \cdots | A_{i_m}]$ is invertible.

② The system has a unique solution for $x_{i_1}, \dots, x_{i_m}$ in terms of all the other $n-m$ variable.

③ $A_{i_1}, \dots, A_{i_m}$ is linearly independent.

Remark : If ①, ②, and ③ hold, then $A_{i_1}, \dots, A_{i_m}$ are distinct, so $n \geq m$.

Definition : In case ①, ② and ③ hold (that is $A_{i_1}, \dots, A_{i_m}$ are linearly independent), the basic solution having basic variables $x_{i_1}, \dots, x_{i_m}$ is the unique solution where all other variables (x_j, $j\neq i_1, \dots, j\neq i_m$) are 0.

Eg. To find the basic solution (if any) having x_3 & x_5 basic in

$$x_1+2x_2-x_3-3x_4-4x_5=11$$
$$3x_1-x_2-x_3+x_4+2x_5=-13$$

Set $x_1=0, x_2=0, x_4=0$:

$$\left.\begin{aligned} -x_3-4x_5&=11 \\ -x_3+2x_5&=-13 \end{aligned}\right\}$$

$$\left[\begin{array}{cc|c} -1 & -4 & 11 \\ -1 & 2 & -13 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & -11 \\ 0 & 6 & -24 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -4 \end{array}\right]$$

$\Rightarrow$ unique solution is the solution just found is basic :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ -4 \end{bmatrix}$$

Remark : In any basic solution, each non-basic variable is 0.