PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 2 SOLUTIONS

(1) Consider the equation

$$x^4y + ay + x = 0. \tag{1}$$

(a) Show that the following statement is false. "For all $a, x \in \mathbb{R}$, there is a unique y such that $x^4y + ay + x = 0$."

Solution: Since the statement claims a fact about *all* a, x, to show it is false we just need to produce a single counterexample. Choosing $x = 1, a = -1$, we find $x^4y + ay + x = 1$ for *any* y, and thus there no y for which it equals 0.

(b) Find the set of real numbers a such that the following statement is true. "For all $x \in \mathbb{R}$, there is a unique y such that $x^4y + ay + x = 0$."

Solution: Fix $a, x \in \mathbb{R}$ and rewrite (1) as $y(x^4 + a) + x = 0$. If $x^4 + a \neq 0$ then this equivalent to

$$y = -\frac{x}{x^4 + a},$$

so there is exactly one solution y. If $x^4 + a = 0$ there are two possibilities: either $x = 0$ and every y is a solution, or $x \neq 0$ and no y is a solution. Either way, there is not a *unique* solution y, so we have shown that for $x, a \in \mathbb{R}$, there is unique y satisfying (1) if and only if $x^4 + a \neq 0$.

We now just need to identify the $a \in \mathbb{R}$ such that this is true for *every* x; i.e. the a such that the equation $x^4 + a = 0$ has no real solution x. This is true exactly when $a > 0$, so the set we're looking for is $\{a \in \mathbb{R} : a > 0\}$.

(2) Let $P(x)$ be the assertion "x is odd" and let $Q(x)$ be the assertion "$x^2 - 1$ is divisible by 8." Determine whether the following statements are true.

(a) $(\forall x \in \mathbb{Z})\ (P(x) \implies Q(x))$,

Solution: Let $x \in \mathbb{Z}$ satisfy $P(x)$; i.e. $x = 2k + 1$ for some $k \in \mathbb{Z}$. Then we have

$$x^2 - 1 = (x - 1)(x + 1) = 2k(2k + 2) = 4k(k + 1).$$

Since k and $k + 1$ are consecutive integers, at least one of them must be even; so $k(k + 1)$ is divisible by 2 and thus $x^2 - 1$ is divisible by 8, which is exactly $Q(x)$. Thus $P(x) \implies Q(x)$.

(b) $(\forall x \in \mathbb{Z})\ (Q(x) \implies P(x))$.

Solution: Let $x \in \mathbb{Z}$ satisfy $Q(x)$; i.e. $x^2 - 1 = 8k$ for some $k \in \mathbb{Z}$. Then $x^2 = 8k + 1$ is odd. Since the product of even numbers is even, the fact that x^2 is odd implies that x is odd, which is the definition of $P(x)$. Thus $Q(x) \implies P(x)$.

(3) Using statements about set membership, prove the statements below, where A, B, C are any sets. Use a picture to illustrate the results and guide the proofs.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution: Assume $x \in A \cup (B \cap C)$. By the definition of the union, this means either $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ *and* $x \in A \cup C$, so by the definition of the intersection $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$ then $x \in B$ and $x \in C$; so $x \in A \cup B$ and $x \in A \cup C$ and thus $x \in (A \cup B) \cap (A \cup C)$.

Now assume $x \in (A \cup B) \cap (A \cup C)$, i.e. $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$ then $x \in A \cup E$ for any E, so in particular $x \in A \cup (B \cap C)$. If $x \notin A$ then $x \in A \cup B$ and $x \in A \cup C$ imply $x \in B$, $x \in C$ respectively, and thus $x \in B \cap C$ and therefore $x \in A \cup (B \cap C)$.

Since we have shown set inclusions in both directions, we have the desired equality of sets.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution: Assume $x \in A \cap (B \cup C)$, so $x \in A$, and $x \in B$ or $x \in C$. If $x \in B$ then $x \in A$ implies $x \in A \cap B$, and likewise if $x \in C$ then we have $x \in A \cap C$. Since one of these holds, x is in at least one of $A \cap B$ or $A \cap C$ and is thus in their union. Conversely, assume $x \in (A \cap B) \cup (A \cap C)$, so $x \in A \cap B$ or $x \in A \cap C$. Either way we can conlude $x \in A$; and either $x \in B$ or $x \in C$, so $x \in B \cup C$. Combining these conclusions we get $x \in A \cap (B \cup C)$.

(4) Consider tokens that have some letter written on one side and some integer written on the other, in unknown combinations. The tokens are laid out, some with the letter side up, some with number side up. Explain which tokens must be turned over to determine whether these statements are true.

(a) Whenever the letter side is a vowel, the number side is odd.

Solution: Remember that $P \implies Q$ is true unless P is true and Q is false, so if we know P is false we can conclude $P \implies Q$ without needing to check Q, and likewise if Q is true we can conclude $P \implies Q$ without needing to check P. Thus we only need to check the other statement (i.e. flip a token) in the cases where we know P is true or we know Q is false.
To make this correspond to the first case, we let P denote "is a vowel" and Q denote "is odd". By the reasoning above, we need to turn over the tokens that have vowels (P) or even numbers ($\neg Q$) showing.

(b) The letter side is a vowel if and only if the number side is odd.

Solution: The statement $P \iff Q$ is false if either $P \wedge \neg Q$ or $\neg P \wedge Q$, so knowing the value of either P or Q alone cannot determine the value of $P \iff Q$. Thus in order to determine whether or not this statement is true, we must turn over every token.