STAT2001 Tutorial 12 Solutions

Problem 1

Let p be the proportion of all items produced that are defective.
Then $r = p/(1 - p)$, and therefore $p = r/(1 + r)$.

Now $X \sim \text{Bin}(n,p)$, so that $p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x}\left(\frac{r}{1+r}\right)^x \left(\frac{1}{1+r}\right)^{n-x}$, $x = 0,\ldots,n$.

Therefore the likelihood function is $L(r) = \frac{r^x}{(1+r)^n}$, $r > 0$.

So the loglikelihood function is $l(r) = \log L(r) = x \log r - n \log(1+r)$.

The derivative of the loglikelihood is $l'(r) = \frac{x}{r} - \frac{n}{1+r}$.

Setting $l'(r) = 0$ yields the MLE, $\hat{r} = x/(n-x)$.

(This estimate makes sense because it is the ratio of the number of defective to nondefective items in the sample. That ratio is an obvious estimate of the same ratio in the population.)

Note: r's ML *estimate* is $\hat{r} = x/(n-x)$, and r's ML *estimator* is $\hat{r} = X/(n-X)$.
The acronym for both quantities is "MLE".

Alternative working

The MLE of p is $\hat{p} = \bar{x} = x/n$ (the proportion of defective items in the sample).

Therefore the MLE of $r = \frac{p}{1-p}$ is $\hat{r} = \frac{\hat{p}}{1-\hat{p}} = \frac{x/n}{1-x/n} = \frac{x}{n-x}$. This follows by:

The invariance property of MLE's
Suppose that $\hat{\theta}$ is the MLE of θ, and $\phi = g(\theta)$ is a continuous function. Then the MLE of ϕ is $\hat{\phi} = g(\hat{\theta})$.

Problem 2

(a) The first raw moment is $\mu_1' = EY^1 = \lambda$. The first sample moment is $m_1' = \bar{y}$.
Setting $\mu_1' = m_1'$ automatically yields λ's MOME, $\hat{\lambda} = \bar{Y}$.

$$p(y_1,\dots,y_n)=\prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{y_i}}{y_i\,!}=\frac{e^{-n\lambda}\lambda^{\dot{y}}}{\prod_{i=1}^{n}y_i\,!},\quad y_i>0 \text{ for all } i,\ \text{where } \dot{y}=\sum_{i=1}^{n}y_i\,.$$

So the likelihood function is $L(\lambda)=e^{-n\lambda}\lambda^{\dot{y}},\ \lambda>0$.
So the loglikelihood is $l(\lambda)=\log L(\lambda)=-n\lambda+\dot{y}\log\lambda$.
Then, $l'(\lambda)=-n+\dot{y}/\lambda$.
Setting $l'(\lambda)=0$ yields $\lambda=\dot{y}/n=\bar{y}$.

So λ's MLE is $\hat{\lambda}=\bar{Y}$ (same as the MOME).

(b) $E\hat{\lambda}=E\bar{Y}=\lambda$.
So $B(\hat{\lambda})=E\hat{\lambda}-\lambda=\lambda-\lambda=0$ (thus the MLE is unbiased).

$$\begin{aligned}MSE(\hat{\lambda})&=Var\hat{\lambda}+B(\hat{\lambda})^2\\&=Var\hat{\lambda}\quad\text{since } \hat{\lambda} \text{ is unbiased}\\&=Var\bar{Y}=\lambda/n\,.\end{aligned}$$

(c) λ's MLE is the sample mean $\bar{Y}$.
Now $EY_1=\lambda$ and $VarY_1=\lambda<\infty$.
It follows by the law of large numbers that λ's MLE is consistent for λ.

(d) $\hat{\lambda}=\bar{y}=(1 + 0 + 3 + 1 + 0 + 1)/6 = 1$.
The probability of interest is $p=P(Y=0)=e^{-\lambda}\lambda^0/0!=e^{-\lambda}$.
By the invariance property of MLE's (see Problem 1), the MLE of p is

$$\hat{p}=e^{-\hat{\lambda}}=e^{-1}=0.368.$$

Problem 3

$$\mu_1'=EY=\int_0^1 y(\theta+1)y^{\theta}dy=\frac{\theta+1}{\theta+2}\,.$$

$m_1'=\bar{y}$ = (1/4)(0.31 + 0.76 + 0.29 + 0.97) = 0.5825.

Setting $\mu_1'=m_1'$ leads to the MOME,

$$\hat{\theta}_{\text{MOM}}=\frac{2\bar{y}-1}{1-\bar{y}}=\frac{2(0.5825)}{1-0.5825}=0.395\,.$$

$f(y_1,\ldots,y_n) = \prod_{i=1}^{n}(\theta+1)y_i^{\theta}$, $0 < y_i < 1$ for all *i*.

So the likelihood is $L(\theta) = (\theta+1)^n \tilde{y}^{\theta}, \theta > -1$,

where $\tilde{y} = \prod_{i=1}^{n} y_i$ = 0.31(0.76)0.29(0.97) = 0.066274.

So the loglikelihood is $l(\theta) = \log L(\theta) = n\log(\theta+1) + \theta\log\tilde{y}$.

Then, $l'(\theta) = \dfrac{n}{\theta+1} + \log\tilde{y}$.

Setting $l'(\theta) = 0$ yields the MLE,

$$\hat{\theta}_{\text{ML}} = -1 - \frac{n}{\log\tilde{y}} = -1 - \frac{4}{\log 0.066274} = 0.474.$$

Problem 4

(a) $H_0: p = 1/6$

$H_1: p \neq 1/6$

Test statistic: $Z = \dfrac{\hat{p}-1/6}{\sqrt{(1/6)(1-1/6)/600}} \mathrel{\dot\sim} \text{N}(0,1)$ if H_0 is true

Rejection region: $|Z| > z_{0.025} = 1.96$

$\hat{p} = 123/600 = 0.205$

$z = \dfrac{0.205-1/6}{\sqrt{(1/6)(1-1/6)/600}}$ = 2.52, which is in the rejection region.

So reject H_0.

We conclude that the die is not fair.

(b) The *p*-value is $P(|Z| \geq |z|) \approx 2P(Z \geq 2.52) \approx 2(0.0059) = 0.0118$.

(Note that *Z* here has the standard normal distribution only approximately. Also, *Z*'s distribution is not perfectly symmetric.)

(c) The acceptance region is $-1.96 \leq \dfrac{(y/600)-1/6}{\sqrt{(1/6)(1-1/6)/600}} \leq 1.96$,

or equivalently, $a \leq y \leq b$,

where $(a,b) = (\,600(1/6) \pm 1.96\sqrt{600(1/6)(1-1/6)}\,) = (100 \pm 17.89)$

$= (82.11, 117.89)$.

Thus our test involves the decision to accept H_0 if and only if $83 \leq y \leq 117$.

Therefore $\beta = \beta(p)$ = P(Type II error)

= P(Accept H_0 | H_0 is false)

= $P(83 \le Y \le 117)$ where Y ~ Bin(n, p) and n = 600

$\approx P\left(\frac{83-np}{\sqrt{np(1-p)}} < U < \frac{117-np}{\sqrt{np(1-p)}}\right)$ where U ~ N(0,1).

If p = 105/600 = 0.175, we find that $\beta \approx P(-2.36 < U < 1.29)$

= $P(U > -2.36) - P(U > 1.29)$

= $1 - P(U > 2.36) - P(U > 1.29)$

= 1 – 0.0091 – 0.0995

= 0.891.

Discussion

Note that β is approximately $1 - \alpha$ = 1 – 0.05 = 0.95 for values of p near 1/6.

The probability in (c) can be evaluated exactly, using a computer,
according to

$$\beta(p) = \sum_{y=83}^{117} \binom{600}{y} p^y (1-p)^{600-y} .$$

We find that $\beta(0.175)$ = 0.902 (not 0.891).

We also find that the true significance level of the test is

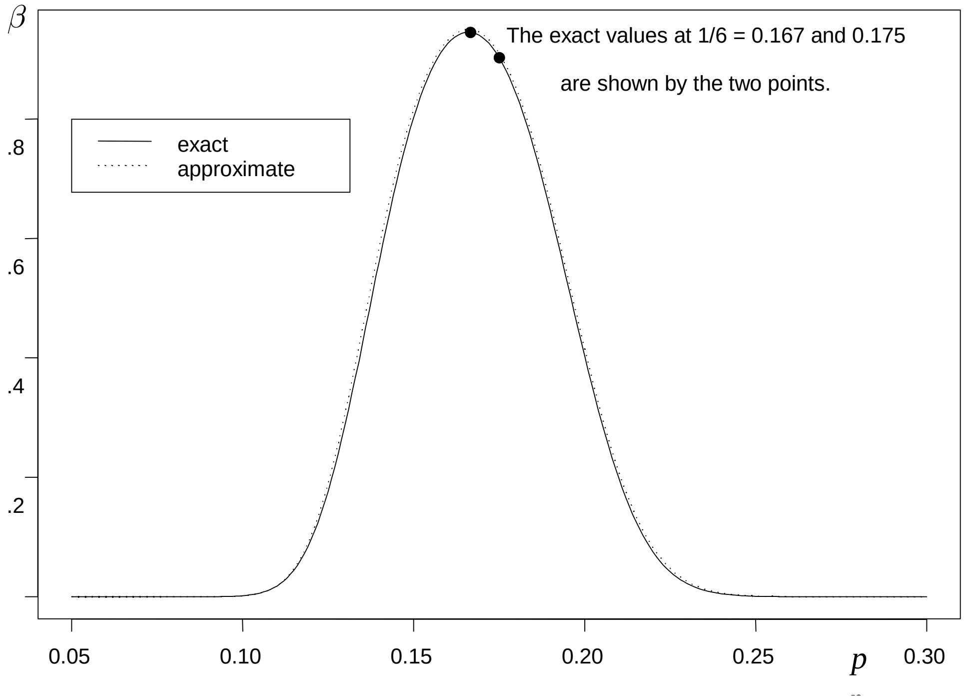
$\alpha = 1 - \beta(1/6) = 1 - 0.945 = 0.045$ (not 0.050).

Furthermore, the p-value in (b) is exactly 0.0136 (not 0.0118).

A graph showing the function $\beta(p)$ is shown on the next page.

The approximations made in this problem could be improved slightly by incorporating a continuity correction. However, it would still be impossible to construct a test with significance level *exactly* equal to 0.05. This is because the data Y has a discrete distribution.

Figure 1 **Probabilities of a Type II error**



Splus computing code (non-assessable)

```
pbinom(117,600,0.175)-pbinom(82,600,0.175)     # 0.9024
pbinom(117,600,1/6)-pbinom(82,600,1/6)         #  0.945
1-(pbinom(122,600,1/6)-pbinom(77,600,1/6))     #  0.01363

pvec <- seq(0.05,0.3,0.001)
beta <- pbinom(117,600,pvec)-pbinom(82,600,pvec)
means <- 600* pvec; sds <- sqrt(600*pvec*(1-pvec))
beta2 <- pnorm(117.89,means,sds) - pnorm(82.11,means,sds)

plot(pvec,beta,type=”l”,xlab=””,ylab=””)             # exact
lines(pvec,beta2,lty=2)                              # approximate
points(1/6,0.945);  points(0.175,0.9024)             # exact
legend(0.05,0.8,c(“exact”,”approximate”),lty=1:2)
text(0.25,0.8,
 ”The exact values at 1/6 = 0.167 and 0.175\n are shown by the two points.”)
text(0.25,0.6,”blank“); text(0.25,0.5,”blank“) # for editing purposes
```