1. Prove or disprove that the set $S_1 = \{(a,b) : a \in \mathbb{N}, b \in \mathbb{N}\}$ is countable.

   *(*This is basically just a review of the argument used to show that $\mathbb{Q}$ is countable, which was done in class.*)*
   Intuitively, each element of the set is a pair of integers, i.e., *a finite amount of information, so the set should be countable.*
   Come up with a systematic way to list every element in S_1. An idea similar to the counting of $\mathbb{Q}^+$ will work. First, write down every pair (a,b) in a 2-dimensional table:

$$
\begin{array}{c|ccccccc}
a\backslash b & 0 & 1 & 2 & 3 & \dots & k & \dots \\
\hline
0 & (0,0) & (0,1) & (0,2) & (0,3) & \dots & (0,k) & \dots \\
1 & (1,0) & (1,1) & (1,2) & (1,3) & \dots & (1,k) & \dots \\
2 & (2,0) & (2,1) & (2,2) & (2,3) & \dots & (2,k) & \dots \\
3 & (3,0) & (3,1) & (3,2) & (3,3) & \dots & (3,k) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\
k & (k,0) & (k,1) & (k,2) & (k,3) & \dots & (k,k) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

   Next, start at the upper-left corner and list elements in an increasing "triangle" pattern, as follows (using extra spaces between the "bands" of the triangle pattern):

$$(0,0), \quad (1,0),(0,1), \quad (2,0),(1,1),(0,2), \quad (3,0),(2,1),(1,2),(0,3), \quad \dots$$

   This is equivalent to organizing the list into sub-lists, where each sub-list has a constant value of $a+b$ and elements within a sublist are ordered by increasing value of b:

   - sub-list 0: $(0,0)$
   - sub-list 1: $(1,0),(0,1)$
   - sub-list 2: $(2,0),(1,1),(0,2)$
   - sub-list 3: $(3,0),(2,1),(1,2),(0,3)$
   - $\dots$

   (Note: we could have ordered sub-lists by increasing value of a instead; this was an arbitrary choice and both possibilities are fine.)
   The list above defines a function $f_1 : \mathbb{N} \to S_1$ that is onto: $\forall a \in \mathbb{N}, \forall b \in \mathbb{N}$, the element $(a,b) \in S_1$ appears in sub-list number $a+b$, at position number b (counting from 0 in both cases), i.e., *there is some $n \in \mathbb{N}$ such that $f_1(n) = (a,b)$.*
   Alternatively, we could also try to define a function $f_1' : S_1 \to \mathbb{N}$ that is one-to-one. We do not need to do both! *The argument above is sufficient to show that S_1 is countable. We show the alternative argument here only for your reference.*
   One possibility would be $f_1'((a,b)) = 2^a 3^b$. Clearly, $f_1'((a,b)) \in \mathbb{N}$ for all $(a,b) \in S_1$. Moreover, f_1' is one-to-one — though proving this requires the use of the Fundamental Theorem of Arithmetic (that every natural number can be decomposed into a product of prime factors in a unique way*).* *For your reference, here is the argument.*

   Assume $(a_1,b_1) \in S_1$, $(a_2,b_2) \in S_1$.
       Assume $f_1'((a_1,b_1)) = f_1'((a_2,b_2))$.
           Then $2^{a_1}3^{b_1} = 2^{a_2}3^{b_2}$.
           Then $a_1 = a_2$ *and* $b_1 = b_2$. # *by the Fundamental Theorem of Arithmetic*
       Then $f_1'((a_1,b_1)) = f_1'((a_2,b_2)) \Rightarrow (a_1,b_1) = (a_2,b_2)$.
   Then f_1' *is one-to-one.*

2. Prove or disprove that the set $S_2 = \mathcal{P}(\mathbb{N})$ is countable.

   Recall that the power set of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A. That is $\mathcal{P}(A) = \{X : X \subseteq A\}$.

What do the elements of $\mathcal{P}(\mathbb{N})$ look like?

- $\{1\} \in S_2$
- $\{108, 148, 165\} \in S_2$
- $\{e \in \mathbb{N} : \exists k \in \mathbb{N}, e = 2k\}$ *(The even numbers.)*
- $\{o \in \mathbb{N} : \exists k \in \mathbb{N}, o = 2k + 1\}$ *(The odd numbers.)*
- $\{o \in \mathbb{N} : \exists k \in \mathbb{N}, o = 2k + 1 \wedge o > 165\}$ *(The odd numbers that are greater than 165.)*

Since there are an infinite number of natural numbers, there are an infinite number of sets in the power set. Some of the elements in the power set contain a finite number of natural numbers. But some of the elements in the power set contain an infinite number of natural numbers.

Our intuition, then, tells us then that S_2 is uncountable, as there are an infinity of elements in the set, and some of those elements require an infinite amount of information to describe.

We can prove this by proving that there is no function $f : \mathbb{N} \to S_2$ that is onto. That is, we need to prove the statement $\forall f : \mathbb{N} \to S_2, \exists x \in S_2, \forall n \in \mathbb{N}, x \neq f(n)$. We will do this by constructing an element of S_2 that is not mapped on to by any function $f : \mathbb{N} \to S_2$.

Assume S_2 is countable.
  Then $\exists f : \mathbb{N} \to S_2$ that is onto.
  Let $f_0 : \mathbb{N} \to S_2$ be onto.
  Then $\forall D \in S_2, \exists n \in \mathbb{N}, D = f_0(n)$.
  # D is a set of natural numbers.
  # We can think about the value of $f_0(n)$, $\forall n \in \mathbb{N}$
  # construct a special element of S_2
  Let $D = \{m \in \mathbb{N} : m \notin f_0(m)\}$.
  # $f_0(m)$ is a set of natural numbers (since $f_0 : \mathbb{N} \to S_2$)
  # D is the set of natural numbers that are not in $f_0(m)$

  Then $D \in S_2$.   # since D is a set of natural numbers

  # try to find the natural number that f_0 maps to D.
  # that is, try to find n such that $f_0(n) = D$.
  Assume $n \in \mathbb{N}$
    Then either $n \in f_0(n)$ or $n \notin f_0(n)$.
    Case 1: Assume $n \in f_0(n)$
      Then $n \notin D$.   # since $n \in f_0(n)$
      Then $D \neq f_0(n)$.   # since $n \in f_0(n)$ and $n \notin D$
    Case 2: Assume $n \notin f_0(n)$
      Then $n \in D$.   # since $n \notin f_0(n)$
      Then $D \neq f_0(n)$.   # since $n \notin f_0(n)$ and $n \in D$
    Then, in either case, $D \neq f_0(n)$.
  Then $\forall n \in \mathbb{N}, D \neq f_0(n)$
  # pull out the negation
  Then $\neg\exists n \in \mathbb{N}, D = f_0(n)$.
  Then this contradicts f_0 being onto.
  But that followed from assuming that S_2 is countable.
Then S_2 is uncountable.
Then $\mathcal{P}(\mathbb{N})$ is uncountable.