Tutorial 3

January 29, 2015

1. Use inductions to prove that $(1 + \frac{1}{n})^n \leqslant n$ for most natural numbers n.

Proof:

Predicate $P(n)$: $n \in \mathbb{N}, (1 + \frac{1}{n})^n \leqslant n$.

Base Case: when $n = 3, (1 + \frac{1}{3})^3 \leqslant 3$, so $P(3)$ holds.

Inductive Step: suppose we have $P(k)$ for k is a natural number, then we want to show it also holds for case $k + 1$.

Assume $(1 + \frac{1}{k})^k \leqslant k$,
then $(1 + \frac{1}{k+1})^k \leqslant (1 + \frac{1}{k})^k \leqslant k$,
so $(1 + + \frac{1}{k+1})^{k+1} \leqslant k(1 + \frac{1}{k+1}) = k + \frac{k}{k+1} < k + 1$ since $\frac{k}{k+1} < 1$.
Therefore, $P(k + 1)$ holds as well.

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1. Prove that any full binary trees with more than zero nodes has exactly one more leaf than internal nodes. A full binary tree is one in which every node is either a leaf or has both children.

Proof:

Leaf - a node with no children
Internal node - a node with at least one child

Suppose the number of leaves to be $f(x)$ and the number of internal nodes to be $t(x)$ where x is the number of nodes in the tree.

Predicate $P(x)$: ...

Base Case: $f(1) - t(1) = 1 - 0 = 1$, so $P(1)$ holds.

Note that x the number of nodes in the tree has its own restriction, which has to be an odd number like $1, 3, 5, 7, 9, \ldots$

Inductive Step: suppose $P(2k + 1)$ holds where k is a natural number, we want to show $P(2k + 3)$ also holds.

Since $P(2k + 1)$ holds, $f(2k + 1) - t(2 + 1) = 1$.

Now if $2k + 3$ case, we have to add two leaves to a previous leaf, which makes it an internal node so:

$$f(2k + 3) = f(2k + 1) - 1 + 2 = f(2k + 1) + 1;$$

$$t(2k + 3) = t(2k + 1) + 1.$$

Therefore $f(2k + 3) - t(2k + 3) = f(2k + 1) + 1 - t(2k + 1) - 1 = 1.$

So $P(2k + 3)$ holds as well.

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NOTE: you can use **height** instead of number of nodes to prove as well.