§6 - Continuous Functions and Homeomorphisms

1 Motivation

There is an old "joke" (that isn't particularly funny) that goes: "A topologist is a person who can't tell the difference between a doughnut and a coffee cup." The idea here is that a topologist thinks that two spaces are the same if "one can be transformed into the other without ripping or glueing, but allowing stretching and bending." So to a topologist a coffee cup is the same as a doughnut is the same as an elastic band is the same as a bucket, but all of these are different from a marble or a pair of pants.

Alphabet Exercise: Separate the capital letters A through Z (without serifs) into equivalence classes, where the relation is "can be transformed into the other by reflecting, stretching, bending or flipping, but not cutting and glueing." For example, the letters 'H' and 'K' are equivalent, but 'Y' and 'X' are not.

Here we will define what it means for two spaces to be "topologically equivalent". Morally, two spaces will be topologically equivalent if there is a bijection from one space to the other *that preserves the topological structure* of each space. Remember that the topology and topological structure of a space codes closeness of points and sets. What is the topological structure of a space? It is merely the topology itself, the collection of all open sets.

Our notion of topologically equivalent will depend on the general notion of the continuity of a function between topological spaces. It will turn out that we can define "continuity of a function" in general topological spaces, and this will be of fundamental importance. As such we will look at many ways of checking that a function is continuous, and we will see many methods for constructing continuous functions. In Section 4 on closures, we hinted at how the definition might go...

2 Continuous Functions and Examples

Here we go!

Definition. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a function. We say that f is continuous provided that $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{U}$.*

The sound bite for this definition is that "**for continuous functions, the preimage of every open set is open**".

Before we dive into properties of continuous functions, let's check that this definition lines up with the one we had in first-year calculus.

Example. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function (in the first-year calculus sense) then f is a continuous function from $\mathbb{R}_{usual}$ to $\mathbb{R}_{usual}$.*

Proof. Let $V \subseteq \mathbb{R}$ be open, and let $y \in V$. Then there is an $\epsilon > 0$ such that $B_\epsilon(y) \subseteq V$. It will be enough to prove the following claim.

Claim: $f^{-1}(B_\epsilon(y))$ is open.

Let $x \in f^{-1}(B_\epsilon(y))$. Find a $\gamma > 0$ such that $B_\gamma(f(x)) \subseteq B_\epsilon(y)$.

Since f is continuous there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \gamma$. In other words, if $a \in B_\delta(x)$ then $f(a) \in B_\gamma(f(x)) \subseteq B_\epsilon(y)$. So $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$. $\square$

Almost all of the continuous functions you encounter in mathematics are of the form $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, or possibly $f : S \longrightarrow \mathbb{R}$ where S is a subspace of $\mathbb{R}^n$, (which we will discuss later). It is good to really understand these types of continuous functions. We will look at continuous functions from general topological spaces, but it is good to keep $\mathbb{R}^n$ in mind.

Some Examples:

- As we have already seen, $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ is a continuous function (when $\mathbb{R}$ has the usual topology).
- The identity function $\mathrm{id} : \mathbb{R}_{\text{Sorgenfrey}} \longrightarrow \mathbb{R}_{\text{usual}}$, given by $\mathrm{id}(x) = x$, is continuous because if V is an open set in the usual topology, then $\mathrm{id}^{-1}(V) = V$ is an open set in the Sorgenfrey Line.

- The identity function $\mathrm{id} : \mathbb{R}_{\text{usual}} \longrightarrow \mathbb{R}_{\text{Sorgenfrey}}$, given by $\mathrm{id}(x) = x$, is *NOT* continuous because $V = [1, 2)$ is an open set in the Sorgenfrey line, but $\mathrm{id}^{-1}(V) = V = [0, 1)$ is not open in the usual topology.

- A function $f : X_{\text{discrete}} \longrightarrow Y$ is always continuous, regardless of the topology on Y.

- A function $f : X \longrightarrow Y_{\text{indiscrete}}$ is always continuous, regardless of the topology on X.

3 Equivalent Conditions to being Continuous

The proof that our first year calculus version of continuous is the same as our topology version of continuous is good because it suggests the following idea:

Definition. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a function, and let $a \in X$. We say that f* ***is continuous at*** *a provided that for every open set V containing $f(a)$, there is an open set U containing a such that $f(U) \subseteq V$.*

In fact, this gives us our first equivalent definition of a continuous function:

Proposition. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a function. TFAE:*

i. f is continuous;

ii. f is continuous at a, for all $a \in X$;

iii. $f^{-1}(C)$ is closed for every C closed in Y.

Proof. The proof will go $ii \Rightarrow i \Rightarrow iii \Rightarrow ii$.

[$ii \Rightarrow i$] Suppose [ii]. Let V be open in Y, and let $a \in f^{-1}(V)$. So there is an open (in X) set U containing a such that $f(U) \subseteq V$. Thus $U \subseteq f^{-1}(V)$.

[$i \Rightarrow iii$] Suppose [i]. Let C be closed in Y. So $Y \setminus C$ is open in Y. Thus $f^{-1}(Y \setminus C)$ is open in X and

$$f^{-1}(Y \setminus C) = f^{-1}(Y) \setminus f^{-1}(C) = X \setminus f^{-1}(C)$$

So $f^{-1}(C)$ is closed.

[$iii \Rightarrow ii$] Suppose [iii]. Let $a \in X$, and let V be an open set (in Y) containing $f(a)$. Then $Y \setminus V$ is a closed set in Y that does not contain $f(a)$. So $f^{-1}(Y \setminus V)$ is a closed set in X that does not contain a. Define $U := X \setminus f^{-1}(Y \setminus V) = f^{-1}(V)$, which is an open set containing a. And we can see that $f(U) \subseteq V$. (See "Things You Should Know", Section 3, Fact 9.) $\square$

Now we present more equivalent conditions for being continuous. Some of these talk about closures and some talk about bases. We will use many of these at different times, depending on the situation we find ourselves in. The theorem about continuity and bases will be extremely useful for us.

Theorem. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a function. TFAE:*

i. f is continuous;

ii. If $A \subseteq X$ then $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. First we show $i \Rightarrow ii$.

[$i \Rightarrow ii$] Suppose [i]. Let $f(x) \in f(\overline{A})$, (so $x \in \overline{A}$) we want to show that $f(x) \in \overline{f(A)}$. So let V be open (in Y) and contain $f(x)$. Then $f^{-1}(V)$ is an open set (in X) containing x. So there is an $a \in A \cap f^{-1}(V)$. So then $f(a) \in f(A) \cap V$.

[$ii \Rightarrow i$] Suppose [ii]. We will show that the preimage of any closed set is closed. Let C be a closed set in Y. We want to show that $f^{-1}(C)$ is closed in X. To simplify notation, let $A := f^{-1}(C)$, (so $f(A) \subseteq C$) and we need to show $\overline{A} \subseteq A$. Take any $x \in \overline{A}$. Then

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C$$

since C is closed. Thus $f(x) \in C$, or equivalently $x \in f^{-1}(C) = A$. $\square$

Another Way Exercise: This proof is clean, but it doesn't suggest much intuition for "why" this should be true. Try using an equivalent defintion of continuity to prove [$ii.$] $\Rightarrow$ [$i.$]. Can you find a proof that is more intuitive, but possibly longer?

Theorem. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be topological spaces, with $\mathcal{B}$ a basis for $\mathcal{U}$, and $\mathcal{S}$ a subbasis for $\mathcal{U}$. Let $f : X \longrightarrow Y$ be a function. TFAE:*

*i. f is continuous. ("**The preimage of open sets are open.**");*

*ii. $f^{-1}(B) \in \mathcal{T}$ for every $B \in \mathcal{B}$. ("**The preimage of basic open sets are open.**");*

*iii. $f^{-1}(S) \in \mathcal{T}$ for every $S \in \mathcal{S}$ ("**The preimage of subbasic open sets are open.**").*

Proof. It is clear that $i \Rightarrow ii$ and $i \Rightarrow iii$.

[$ii \Rightarrow i$] Let U be an open set in Y. We know that because $\mathcal{B}$ is a basis for $\mathcal{U}$, there is a subcollection $\mathcal{C} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{C} = U$. Since we are assuming [ii], we know that $f^{-1}(B)$ is open in X, for each $B \in \mathcal{C}$. So

$$\bigcup_{B \in \mathcal{C}} f^{-1}(B) = f^{-1}(\bigcup_{B \in \mathcal{C}} B) = f^{-1}(U)$$

is open in X, because it is a union of sets open in X. (The first equality above is a simple set identity.)

[$iii \Rightarrow i$] This is similar and is left as an exercise. □

Some Examples:

- We are now in a position to check that the addition function $\text{PLUS} : \mathbb{R}^2 \longrightarrow \mathbb{R}$, (where each has its usual topology), given by $\text{PLUS}(x, y) = x + y$ is a continuous function. This would be tedious if we tried to check the usual definition of continuity, but we will check continuity by checking the subbasis version of continuity. On assignment 2 we saw that

$$\mathcal{S} := \{ (-\infty, b) : b \in \mathbb{R} \} \cup \{ (a, +\infty) : a \in \mathbb{R} \}$$

  is a subbasis for the usual topology on $\mathbb{R}$. Letting $S = (a, +\infty)$, we see that $\text{PLUS}^{-1}(S) = \{ (x, y) \in \mathbb{R}^2 : y > -x + a \}$ is definitely an open set in $\mathbb{R}^2$ (it is everything strictly above a line), and if $(x, y) \in \text{PLUS}^{-1}(S)$, then $y + x > a$. Analogously, we can check that $\text{PLUS}^{-1}((-\infty, b)) = \{ (x, y) \in \mathbb{R}^2 : y < -x + b \}$.

- Similarly, you can check that multiplication is a continuous function from $\mathbb{R}_{\text{usual}}$ to $\mathbb{R}_{\text{usual}}$.

4 Open Functions

There is an important notion that students often conflate with continuity: the notion of an open function. We introduce this idea now, mostly to prevent misunderstandings.

Definition. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a function. We say that f is an* ***open function*** *provided that $f(U) \in \mathcal{U}$ for every $U \in \mathcal{T}$.*

In English we say **"a function is open if it sends open sets to open sets."**

Do not confuse open functions with continuous functions. On assignment 1 (C.5) you gave an example of a continuous function (from $\mathbb{R}_{\text{usual}}$ to $\mathbb{R}_{\text{usual}}$) that wasn't an open function.

Some examples:

- Any function $f : X \longrightarrow Y_{\text{discrete}}$ is open, regardless of the topology on X.
- The function f : $\mathbb{R}_{\text{Sorgenfrey}} \longrightarrow \mathbb{R}_{\text{Sorgenfrey}}$ defined by $f(x) := |x|$ is not open because $f([-10, -5)) = (5, 10]$, which is not an open set in the image space.

We can also state a related property of functions between topological spaces which is sometimes useful.

Definition. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a function. We say that f is an* ***closed function*** *provided that for every $C \subseteq X$ that is closed (in X), we have that $f(C)$ is closed (in Y).*

5 Various Facts about Continuous Functions

The following is a useful list of facts about continuous functions, some of which we have already seen. This is part of a theorem taken from Munkres', on page 107: **Rules for constructing continuous functions**.

Theorem. *Let X, Y, and Z be topological spaces.*

Constant Functions : If $f : X \longrightarrow Y$ maps all of X into the single point y_0 of Y, then f is continuous.

Composites : If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, then the map $g \circ f : X \longrightarrow Z$ is continuous.

Proof. For the first property, let $f : X \longrightarrow Y$ be such that $f(x) = y_0$ for all $x \in X$. Let V be an open set in Y. If $y_0 \in V$ then $f^{-1}(V) = X$, which is open in X. Otherwise, $f^{-1}(V) = \emptyset$, which is open in X.

For the second property, let V be open in Z. Then $g^{-1}(V)$ is open in Y and so $f^{-1}(g^{-1}(V))$ is open in X. Finally, it remains to note that $f^{-1} \circ g^{-1}(V) = (g \circ f)^{-1}(V)$, by simple set identities. $\square$

6 Homeomorphisms

Now we shift our attention to homeomorphisms, which is the fundamental "morphism" of topology. Two topological spaces will be "topologically the same" is there is a homeomorphism from one space to the other. (In group theory we say that two groups are "the same" if they are "isomorphic".) A homeomorphism will be a function that preserves all *topological* structure of a space.

Definition. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$ be a bijection. We say that f is a* ***homeomorphism*** *provided that both f and its inverse function f^{-1} are continuous functions. In this case we say X* ***is homeomorphic to*** *Y, and write $X \cong_{homeo} Y$, or if there is no confusion $X \cong Y$.*

Some Examples:

Example. *: The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x + 7$ is a homeomorphism (where $\mathbb{R}$ is given it's usual topology).*

Always Exercise: Check that the function above is still a homeomorphism if both domain and range are given the Sorgenfrey topology. Are there any other topologies for which this is true? (Is it true for *all* topologies on $\mathbb{R}$?)

Example. $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ *is a homeomorphism, with both spaces given their usual topologies.*

Proof. It is clear that on $(-\frac{\pi}{2}, \frac{\pi}{2})$ this function is a bijection. We know from first-year calculus that tan is a continuous function, and we also know that $\tan^{-1} =: \arctan$ is also a continuous function. $\square$

Example. *$f : X_{discrete} \longrightarrow Y_{discrete}$ is a homeomorphism if and only if f is a bijection. In particular, two finite discrete spaces are homeomorphic if and only if they have the same number of elements.*

Example. *$\mathbb{Q}$ is not homeomorphic to $\mathbb{R}$ (with any topologies) because they can never be put in bijection.*

Some Facts:

Proposition. *Let X, Y, and Z be topological spaces.*

Identity : The identity function $e_X : X \longrightarrow X$ defined by $e_X(x) = x$ for all $x \in X$, is a homeomorphism

Symmetry : If $f : X \longrightarrow Y$ is a homeomorphism, then $f^{-1} : Y \longrightarrow X$ is a homeomorphism.

Inverse : If $f : X \longrightarrow Y$ is a homeomorphism, then $f^{-1} \circ f : X \longrightarrow X$ is the identity function e_X. Similarly, $f \circ f^{-1} : Y \longrightarrow Y$ is the identity function e_Y.

Composites : If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are homeomorphisms, then the map $g \circ f : X \longrightarrow Z$ is a homeomorphism.

This tells us that $\text{Homeo}(X) := \{ f : X \longrightarrow X | f \text{ is a homeomorphism } \}$ is a **group** under function composition. For those of you who like topology *and* group theory, this should be a very interesting example! Later on in the course we will give this group a topology and we will have a very interesting topological space.

Group Exercise: (For those of you who have seen groups). What sort of algebraic properties does Homeo(X) have? Is it Abelian? Does it have elements of finite order? What does Homeo($\mathbb{R}$) "look like"? That is, describe 3 or 4 "different" homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$.

We leave the following fact as an exercise which gives an alternate way of checking that a function is a homeomorphism.

Proposition. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$ be a bijection. TFAE:*

- *f is a homeomorphism*
- *f is continuous and an open function*
- *f is continuous and a closed function*

7 Topological Invariants

In general, we will be concerned with showing that two spaces are homeomorphic or not. Showing that two spaces are homeomorphic amounts to giving a homeomorphism between the spaces. Showing that two spaces are not homeomorphic involves showing that *no possible function* between the spaces is a homeomorphism. This is not always easy, so we will spend a lot of time describing **topological invariants**. These will be properties that are preserved under homeomorphisms. For example, the Hausdorff property is a topological invariant. This will give us a way to show that two spaces are not homeomorphic, namely by showing that one space has the property ($\mathbb{R}_{\text{usual}}$ has the Hausdorff property) and showing that the other space does not have the property ($\mathbb{R}_{\text{co-finite}}$ does not have the Hausdorff property).

Definition. *A property ϕ is a **topological invariant** if whenever X and Y are homeomorphic topological spaces, then X has property ϕ iff Y has property ϕ.*

Proposition. *The following properties are topological invariants.*

- *X is Countable;*
- *X is a Hausdorff Space;*
- *X contains a point x such that $\{ x \}$ is open.*
- *X is Separable;*
- *X is First Countable;*

- *X is Second Countable;*
- *X has the countable chain condition (ccc).*

Proof. These are all straightforward, so we will just show one of them, and the rest are left as exercises.

Claim: Second Countability is a topological invariant.

Let $f : X \longrightarrow Y$ be a homeomorphism, and let X be second countable. So let $\mathcal{B} := \{ B_n : n \in \mathbb{N} \}$ be a countable basis for X. Let us show that $\mathcal{C} := \{ f(B_n) : n \in \mathbb{N} \}$ is a countable basis for Y. We first observe that $\mathcal{C}$ is countable, and has an obvious ennumeration, and each $f(B_n)$ is open (since f^{-1} is continuous).

Now let $y \in Y$ and $y \in V \subseteq Y$ be an open set. We see that $f^{-1}(y) \in f^{-1}(V) \subseteq X$ an open set (since f is continuous), so there is a B_n such that $f^{-1}(y) \in B_n \subseteq f^{-1}(V)$. Thus $y \in f(B_n) \subseteq V$. $\square$

More Invariants? Exercise: Go through your Analysis notes and see if you can find some properties that were stated in the language of open and closed sets in $\mathbb{R}^n$ that are actually topological invariants.

We can now use this to observe some facts:

Proposition.

- *$\mathbb{R}_{usual}$ is not homeomorphic to $\mathbb{R}_{indiscrete}, \mathbb{R}_{\text{co-finite}}$ or $\mathbb{R}_{\text{co-countable}}$, because $\mathbb{R}_{usual}$ is a Hausdorff space, but the rest are not.*
- *$\mathbb{R}_{usual}$ is not homeomorphic to the Sorgenfrey Line, because $\mathbb{R}_{usual}$ is second countable, but the Sorgenfrey line is not.*
- *For X an infinite set, $X_{\text{co-finite}}$ is not homeomorphic to $X_{\text{co-countable}}$ because $X_{\text{co-countable}}$ contains an infinite closed set that isn't X, but $X_{\text{co-finite}}$ contains no such set.*

What have you done for me lately - Exercise: Look at some of the topological spaces we have defined so far and decide which ones are homeomorphic and which are not.

Doing that exercise you will get into the awkward situation that it is hard to show that $\mathbb{R}$ is different from $\mathbb{R}^2$. With some ingenuity we will be able to distinguish between these spaces. Telling the difference between $\mathbb{R}^2$ and $\mathbb{R}^3$ will be much harder.

8 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Alphabet : Separate the capital letters A through Z (without serifs) into equivalences classes, where the relation is "can be transformed into the other by relfecting, stretching, bending or flipping, but not cutting and glueing." For example, the letters 'H' and 'K' are equivalent, but 'Y' and 'X' are not.

Another Way : Try using an equivalent defintion of continuity to prove $f(\overline{A}) \subseteq \overline{f(A)} \Rightarrow f$ is continuous. Can you find a proof that is more intuitive, but possibly longer?

Always : Check that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = x + 7$ is a homeomorphism if both domain and range are given the Sorgenfrey topology. Are there any other topologies for which this is true? (Is it true for *all* topologies on $\mathbb{R}$?)

Group : What sort of algebraic properties does Homeo(X) have? Is it Abelian? Does it have elements of finite order? What does Homeo($\mathbb{R}$) "look like"? That is, describe 3 or 4 "different" homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$.

More? : Go through your Analysis notes and see if you can find some properties that were stated in the language of open and closed sets in $\mathbb{R}^n$ that are actually topological invariants.

WHYDFML : Look at some of the topological spaces we have defined so far and decide which ones are homeomorphic and which are not.