University of Toronto

MAT237Y1Y PROBLEM SET 6

DUE: End of tutorial, Thursday July 18th, no exceptions

Instructions:

1. Please clearly identify your **Tutorial** (TUT5101 with Boris Lishak, TUT5102 with James Mracek, or TUT5103 with Nan Wu), your full **Name** and your **Student Number** at the top of your problem set. If you do not indicate these, you may not have your problem sets returned.

2. Please write clearly and present polished, perfected solutions; all rough work and initial drafts should be done separately. You are being graded not just for the correctness of the final solution, but for your skill at presenting a cogent argument.

3. This problem set contains practice exercises that are complementary to the textbook material and as such they are designed to help with reflection on the subject. It is extremely important that each student have a chance to individually think about the problems.

4. By participating in this problem set you agree that you are aware of the university regulations regarding plagiarism. You are encouraged to discuss the problem set and to help one another with the ideas and concepts, but may not copy from each others' solutions. The markers are carefully monitoring your solutions, and any incidence of plagiarism will be dealt with according to the university regulations.

Problems:

1. Curves in space: representation (ii) of a curve in the space is the intersection of two surfaces: $F(x, y, z) = 0 = G(x, y, z)$. This is a system of two equations and three unknowns. One can eliminate any one of the variables x, y or z and obtain a relationship between the other two variables. If we repeat this process of elimination twice, we obtain the representation (i). This exercise demonstrate how conic sections (the curves which are born as intersection of a cone $F(x, y, z) = x^2 + y^2 - z^2 = 0$ with any plane). In each of the following case, a plane is given, first describe the relation between the plane and the cone, and then give the representation (i) of the curve of intersection, and finally draw the curve (together with the cone and each plane).

   a) $z = 3$.

   b) $y = 2$.

   c) $z + y = 2$.

   d) $2z + y = 4$.

2. Carefully draw the surface represented by the parametric representation $\mathbf{f}(u, v) = (u \cos v, u \sin v, \sqrt{u})$. Make sure to first establish a relationship between x, y, and z. Then draw a wire frame, that is, draw a few level curves (curves with $z = c$ for $c = 0, 1, 2, 3$) and then draw the curve of intersection of the surface with the xz-plane and the yz-plane to better understand the general shape. Then repeat this process and this time draw the curves $u = 0, 1, 4, 9$ and $v = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

3. Transformations of $\mathbb{R}^2$.

   a) Consider the transformation $(u, v) = \mathbf{f}(x, y) = (x - y, xy)$. Demonstrate the effect of this transformation on the lines

   $$x - y = constant, \quad x + y = 0,$$

   and the curves

   $$xy = constant.$$

   In particular demonstrate the effect of this transformation of the region of the xy-plane bounded by the curves $y = 1/x, y = 2/x, x - y = 1$ and $x - y = 2$.

   b) Could any point of the xy-plane be mapped to the point $(2, -2)$ of the uv-plane? Try to determine the range of the function $\mathbf{f}$.

   c) Investigate the possibility of finding an inverse for this transformation near the generic point $\mathbf{x} = (x, y)$. Determine the points (u, v) where the conditions of invertibility fail. Continue to determine the Fréchet derivative of the (local) inverse of the transformation $\mathbf{f}$ near the point $(u, v) = (1, 2)$.

4. Implicit Function Theorem and Inverse Mapping Theorem:

   a) Apply the Implicit Function Theorem to show that no C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ can be one to one near any point of its domain.

   b) Assume $\frac{\partial f}{\partial x}(a, b) \neq 0$ and define $\mathbf{g}(x, y) = (f(x, y), y)$. Prove that $\mathbf{g}$ is a one to one function near the point (a, b).

   c) Repeat part (a) by using the Inverse Mapping Theorem instead of the Implicit Function Theorem. (Hint: try using part (b)).

5. Use the Fundamental Theorem of Calculus part (a) and Theorem 2.8 in §2.1, to prove that for $0 \leq a < b$ and any function f continuous on the interval $[a, b]$, we have

   $$\int_a^b 2xf(x^2)dx = \int_{a^2}^{b^2} f(x)dx.$$

   Then use a similar argument to prove that if $a < b \leq 0$ then

   $$\int_a^b 2xf(x^2)dx = -\int_{b^2}^{a^2} f(x)dx.$$