**59.** Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

**60.** (a) Find a parametric representation for the torus obtained by rotating about the $z$-axis the circle in the $xz$-plane with center $(b, 0, 0)$ and radius $a < b$.   [*Hint:* Take as parameters the angles $\theta$ and $\alpha$ shown in the figure.]

(b) Use the parametric equations found in part (a) to graph the torus for several values of $a$ and $b$.

(c) Use the parametric representation from part (a) to find the surface area of the torus.

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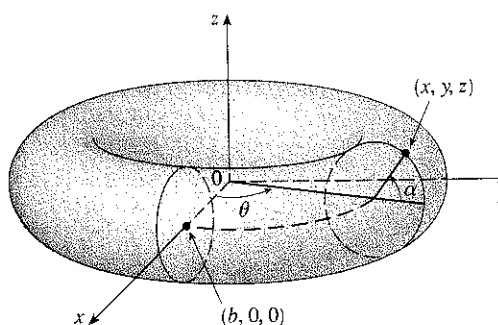
## 16.7 SURFACE INTEGRALS

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose $f$ is a function of three variables whose domain includes a surface $S$. We will define the surface integral of $f$ over $S$ in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of $S$. We start with parametric surfaces and then deal with the special case where $S$ is the graph of a function of two variables.

### PARAMETRIC SURFACES

Suppose that a surface $S$ has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

We first assume that the parameter domain $D$ is a rectangle and we divide it into subrectangles $R_{ij}$ with dimensions $\Delta u$ and $\Delta v$. Then the surface $S$ is divided into corresponding patches $S_{ij}$ as in Figure 1. We evaluate $f$ at a point $P_{ij}^*$ in each patch, multiply by the area $\Delta S_{ij}$ of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

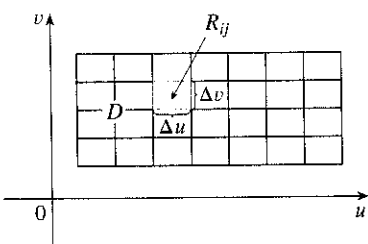Then we take the limit as the number of patches increases and define the **surface integral of $f$ over the surface $S$** as

$$\boxed{1} \qquad \iint_S f(x, y, z)\, dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

Notice the analogy with the definition of a line integral (16.2.2) and also the analogy with the definition of a double integral (15.1.5).

To evaluate the surface integral in Equation 1 we approximate the patch area $\Delta S_{ij}$ by the area of an approximating parallelogram in the tangent plane. In our discussion of surface area in Section 16.6 we made the approximation

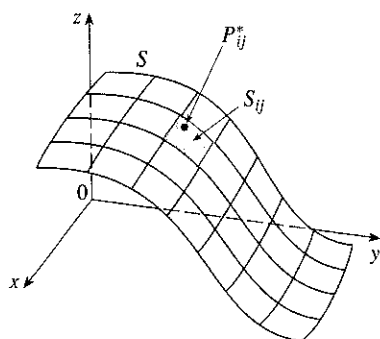$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$$

**FIGURE 1**

where $\qquad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\,\mathbf{i} + \dfrac{\partial y}{\partial u}\,\mathbf{j} + \dfrac{\partial z}{\partial u}\,\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\,\mathbf{i} + \dfrac{\partial y}{\partial v}\,\mathbf{j} + \dfrac{\partial z}{\partial v}\,\mathbf{k}$

are the tangent vectors at a corner of $S_{ij}$. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of $D$, it can be shown from Definition 1, even when $D$ is not a rectangle, that

> We assume that the surface is covered only once as $(u, v)$ ranges throughout $D$. The value of the surface integral does not depend on the parametrization that is used.

$$\boxed{2} \qquad \iint_S f(x, y, z)\,dS = \iint_D f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$$

Observe also that

$$\iint_S 1\,dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain $D$. When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

**EXAMPLE 1** Compute the surface integral $\iint_S x^2\,dS$, where $S$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 4 in Section 16.6, we use the parametric representation

$$x = \sin\phi\,\cos\theta \quad y = \sin\phi\,\sin\theta \quad z = \cos\phi \quad 0 \leqslant \phi \leqslant \pi \quad 0 \leqslant \theta \leqslant 2\pi$$

that is, $\qquad \mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$

As in Example 10 in Section 16.6, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Therefore, by Formula 2,

> Here we use the identities
> $$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$
> $$\sin^2\phi = 1 - \cos^2\phi$$
> Instead, we could use Formulas 64 and 67 in the Table of Integrals.

$$\iint_S x^2\,dS = \iint_D (\sin\phi\,\cos\theta)^2\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA$$

$$= \int_0^{2\pi}\!\int_0^{\pi} \sin^2\phi\,\cos^2\theta\,\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^{\pi} \sin^3\phi\,d\phi$$

$$= \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \int_0^{\pi} (\sin\phi - \sin\phi\,\cos^2\phi)\,d\phi$$

$$= \tfrac{1}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi} \Big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{4\pi}{3} \qquad \square$$

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface

$S$ and the density (mass per unit area) at the point $(x, y, z)$ is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z) \, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \iint_S x \rho(x, y, z) \, dS \qquad \bar{y} = \frac{1}{m} \iint_S y \rho(x, y, z) \, dS \qquad \bar{z} = \frac{1}{m} \iint_S z \rho(x, y, z) \, dS$$

Moments of inertia can also be defined as before (see Exercise 39).

## GRAPHS

Any surface $S$ with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right) \mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right) \mathbf{k}$$

Thus

$$\boxed{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

$$\boxed{4} \qquad \iint_S f(x, y, z) \, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

Similar formulas apply when it is more convenient to project $S$ onto the $yz$-plane or $xz$-plane. For instance, if $S$ is a surface with equation $y = h(x, z)$ and $D$ is its projection on the $xz$-plane, then

$$\iint_S f(x, y, z) \, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1} \, dA$$

**EXAMPLE 2** Evaluate $\iint_S y \, dS$, where $S$ is the surface $z = x + y^2, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2$. (See Figure 2.)

SOLUTION Since

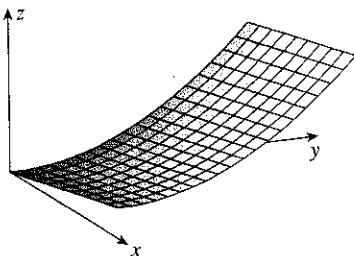$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$



**FIGURE 2**

Formula 4 gives

$$\iint_S y\, dS = \iint_D y\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx$$

$$= \int_0^1 dx \sqrt{2} \int_0^2 y\sqrt{1 + 2y^2}\, dy$$

$$= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}(1 + 2y^2)^{3/2}\big]_0^2 = \frac{13\sqrt{2}}{3} \qquad \square$$

If $S$ is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots,$ $S_n$ that intersect only along their boundaries, then the surface integral of $f$ over $S$ is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

**EXAMPLE 3** Evaluate $\iint_S z\, dS$, where $S$ is the surface whose sides $S_1$ are given by the cylinder $x^2 + y^2 = 1$, whose bottom $S_2$ is the disk $x^2 + y^2 \le 1$ in the plane $z = 0$, and whose top $S_3$ is the part of the plane $z = 1 + x$ that lies above $S_2$.

SOLUTION The surface $S$ is shown in Figure 3. (We have changed the usual position of the axes to get a better look at $S$.) For $S_1$ we use $\theta$ and $z$ as parameters (see Example 5 in Section 16.6) and write its parametric equations as

$$x = \cos\theta \qquad y = \sin\theta \qquad z = z$$

where

$$0 \le \theta \le 2\pi \qquad \text{and} \qquad 0 \le z \le 1 + x = 1 + \cos\theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus the surface integral over $S_1$ is

$$\iint_{S_1} z\, dS = \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\, dA$$

$$= \int_0^{2\pi} \int_0^{1+\cos\theta} z\, dz\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos\theta)^2\, d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2}$$



$z$

$S_3\ (z = 1 + x)$

$y$

$S_1\ (x^2 + y^2 = 1)$
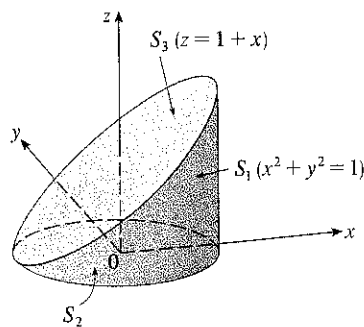
$x$

$0$

$S_2$

**FIGURE 3**

Since $S_2$ lies in the plane $z = 0$, we have

$$\iint_{S_2} z \, dS = \iint_{S_2} 0 \, dS = 0$$

The top surface $S_3$ lies above the unit disk $D$ and is part of the plane $z = 1 + x$. So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have

$$\iint_{S_3} z \, dS = \iint_D (1 + x) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + r \cos \theta) \sqrt{1 + 1 + 0} \, r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 (r + r^2 \cos \theta) \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{3} \cos \theta\right) d\theta$$

$$= \sqrt{2} \left[\frac{\theta}{2} + \frac{\sin \theta}{3}\right]_0^{2\pi} = \sqrt{2}\, \pi$$

Therefore

$$\iint_S z \, dS = \iint_{S_1} z \, dS + \iint_{S_2} z \, dS + \iint_{S_3} z \, dS$$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\, \pi = \left(\tfrac{3}{2} + \sqrt{2}\right)\pi \qquad \square$$

## ORIENTED SURFACES

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5. If an ant were to crawl along the Möbius strip starting at a point $P$, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point $P$ without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore a Möbius strip really has only one side. You can graph the Möbius strip using the parametric equations in Exercise 32 in Section 16.6.
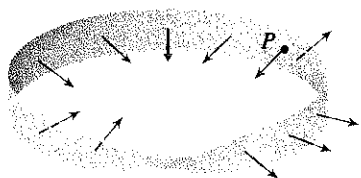


**FIGURE 4**
A Möbius strip

**TEC** Visual 16.7 shows a Möbius strip with a normal vector that can be moved along the surface.
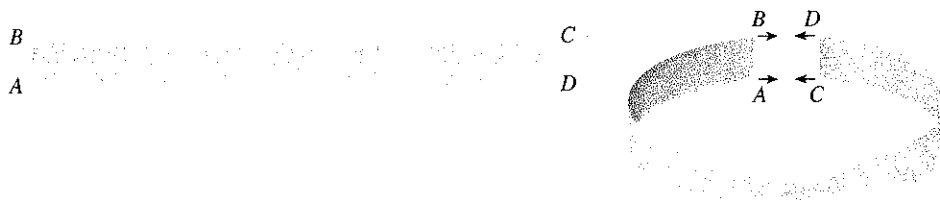


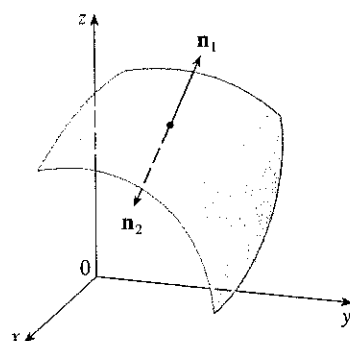**FIGURE 5**
Constructing a Möbius strip

**FIGURE 6**

From now on we consider only orientable (two-sided) surfaces. We start with a surface $S$ that has a tangent plane at every point $(x, y, z)$ on $S$ (except at any boundary point). There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at $(x, y, z)$. (See Figure 6.)

If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point $(x, y, z)$ so that $\mathbf{n}$ varies continuously over $S$, then $S$ is called an **oriented surface** and the given choice of $\mathbf{n}$ provides $S$ with an **orientation**. There are two possible orientations for any orientable surface (see Figure 7).
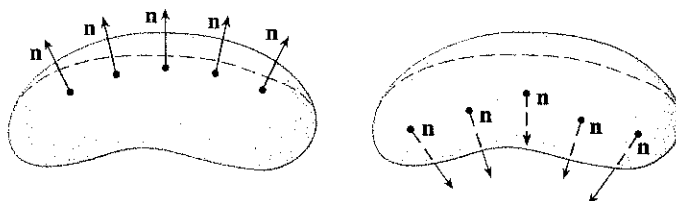


**FIGURE 7**
The two orientations
of an orientable surface

For a surface $z = g(x, y)$ given as the graph of $g$, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

If $S$ is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\boxed{6} \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 4 in Section 16.6 we found the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta\, \mathbf{i} + a \sin \phi \sin \theta\, \mathbf{j} + a \cos \phi\, \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 10 in Section 16.6 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi \cos \theta\, \mathbf{i} + a^2 \sin^2\phi \sin \theta\, \mathbf{j} + a^2 \sin \phi \cos \phi\, \mathbf{k}$$

and $$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$$



**FIGURE 8**
Positive orientation

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin \phi \cos \theta\, \mathbf{i} + \sin \phi \sin \theta\, \mathbf{j} + \cos \phi\, \mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

For a **closed surface**, that is, a surface that is the boundary of a solid region $E$, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from $E$, and inward-pointing normals give the negative orientation (see Figures 8 and 9).
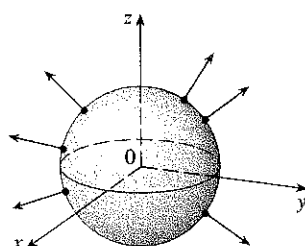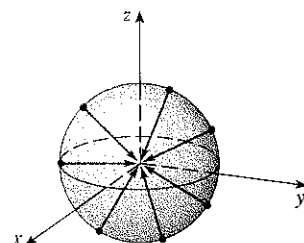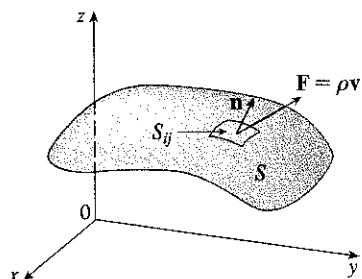


**FIGURE 9**
Negative orientation

## SURFACE INTEGRALS OF VECTOR FIELDS



**FIGURE 10**

Suppose that $S$ is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through $S$. (Think of $S$ as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide $S$ into small patches $S_{ij}$, as in Figure 10 (compare with Figure 1), then $S_{ij}$ is nearly planar and so we can approximate the mass of fluid crossing $S_{ij}$ in the direction of the normal $\mathbf{n}$ per unit time by the quantity

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where $\rho$, $\mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on $S_{ij}$. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector $\mathbf{n}$ is $\rho\mathbf{v} \cdot \mathbf{n}$.) By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over $S$:

$$\boxed{7} \qquad \iint\limits_S \rho\mathbf{v} \cdot \mathbf{n} \, dS = \iint\limits_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z) \, dS$$

and this is interpreted physically as the rate of flow through $S$.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS$$

A surface integral of this form occurs frequently in physics, even when $\mathbf{F}$ is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of $\mathbf{F}$ over $S$.

---

$\boxed{8}$ **DEFINITION** If $\mathbf{F}$ is a continuous vector field defined on an oriented surface $S$ with unit normal vector $\mathbf{n}$, then the **surface integral of $\mathbf{F}$ over $S$** is

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across $S$.

---

In words, Definition 8 says that the surface integral of a vector field over $S$ is equal to the surface integral of its normal component over $S$ (as previously defined).

If $S$ is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \, dS$$

$$= \iint\limits_D \left[ \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

where $D$ is the parameter domain. Thus we have

※ Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 16.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$\boxed{9} \qquad \iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

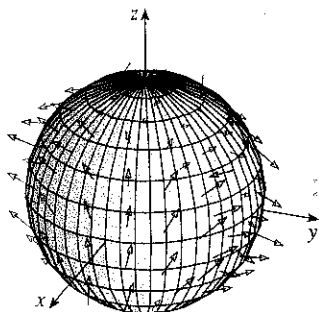Figure 11 shows the vector field **F** in Example 4 at points on the unit sphere.



**FIGURE 11**

**EXAMPLE 4** Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION Using the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin\phi \cos\theta\,\mathbf{i} + \sin\phi \sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \qquad 0 \leq \phi \leq \pi \qquad 0 \leq \theta \leq 2\pi$$

we have

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\,\mathbf{i} + \sin\phi \sin\theta\,\mathbf{j} + \sin\phi \cos\theta\,\mathbf{k}$$

and, from Example 10 in Section 16.6,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi \cos\theta\,\mathbf{i} + \sin^2\phi \sin\theta\,\mathbf{j} + \sin\phi \cos\phi\,\mathbf{k}$$

Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi \sin^2\phi \cos\theta + \sin^3\phi \sin^2\theta + \sin^2\phi \cos\phi \cos\theta$$

and, by Formula 9, the flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (2\sin^2\phi \cos\phi \cos\theta + \sin^3\phi \sin^2\theta)\, d\phi\, d\theta$$

$$= 2\int_0^{\pi} \sin^2\phi \cos\phi\, d\phi \int_0^{2\pi} \cos\theta\, d\theta + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= 0 + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \qquad \left( \text{since } \int_0^{2\pi} \cos\theta\, d\theta = 0 \right)$$

$$= \frac{4\pi}{3}$$

by the same calculation as in Example 1. □

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface $S$ given by a graph $z = g(x, y)$, we can think of $x$ and $y$ as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left( -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right)$$

Thus Formula 9 becomes

$$\boxed{10} \qquad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

This formula assumes the upward orientation of $S$; for a downward orientation we multiply by $-1$. Similar formulas can be worked out if $S$ is given by $y = h(x, z)$ or $x = k(y, z)$. (See Exercises 35 and 36.)

**✓ EXAMPLE 5** Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and $S$ is the boundary of the solid region $E$ enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

SOLUTION $S$ consists of a parabolic top surface $S_1$ and a circular bottom surface $S_2$. (See Figure 12.) Since $S$ is a closed surface, we use the convention of positive (outward) orientation. This means that $S_1$ is oriented upward and we can use Equation 10 with $D$ being the projection of $S_1$ on the $xy$-plane, namely, the disk $x^2 + y^2 \le 1$. Since

$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

on $S_1$ and

$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

we have

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

$$= \iint_D \left[ -y(-2x) - x(-2y) + 1 - x^2 - y^2 \right] dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left( \tfrac{1}{4} + \cos\theta \sin\theta \right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}$$

The disk $S_2$ is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)\, dA = \iint_D 0\, dA = 0$$

since $z = 0$ on $S_2$. Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces $S_1$ and $S_2$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2} \qquad \square$$

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if $\mathbf{E}$ is an electric field (see Example 5 in Section 16.1), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of $\mathbf{E}$ through the surface $S$. One of the important laws of electro-
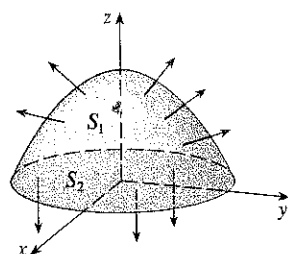


FIGURE 12

statics is **Gauss's Law**, which says that the net charge enclosed by a closed surface $S$ is

$$\boxed{11} \qquad\qquad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where $\varepsilon_0$ is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}$ $C^2/N \cdot m^2$) Therefore, if the vector field $\mathbf{F}$ in Example 4 represents an electric field, we can conclude that the charge enclosed by $S$ is $Q = 4\pi\varepsilon_0/3$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point $(x, y, z)$ in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K \nabla u$$

where $K$ is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface $S$ in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

**EXAMPLE 6** The temperature $u$ in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere $S$ of radius $a$ with center at the center of the ball.

SOLUTION Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where $C$ is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K \nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$$

where $K$ is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point $(x, y, z)$ is

$$\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and so

$$\mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

But on $S$ we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F} \cdot \mathbf{n} = -2aKC$. Therefore the rate of heat flow across $S$ is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \iint_S dS$$

$$= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3 \qquad \square$$

## 16.7 EXERCISES

1. Let $S$ be the boundary surface of the box enclosed by the planes $x = 0$, $x = 2$, $y = 0$, $y = 4$, $z = 0$, and $z = 6$. Approximate $\iint_S e^{-0.1(x+y+z)} \, dS$ by using a Riemann sum as in Definition 1, taking the patches $S_{ij}$ to be the rectangles that are the faces of the box $S$ and the points $P_{ij}^*$ to be the centers of the rectangles.

2. A surface $S$ consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that $f$ is a continuous function with

$$f(\pm 1, 0, 0) = 2 \qquad f(0, \pm 1, 0) = 3 \qquad f(0, 0, \pm 1) = 4$$

   Estimate the value of $\iint_S f(x, y, z) \, dS$ by using a Riemann sum, taking the patches $S_{ij}$ to be four quarter-cylinders and the top and bottom disks.

3. Let $H$ be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose $f$ is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing $H$ into four patches, estimate the value of $\iint_H f(x, y, z) \, dS$.

4. Suppose that $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$, where $g$ is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z) \, dS$, where $S$ is the sphere $x^2 + y^2 + z^2 = 4$.

**5–18** Evaluate the surface integral.

5. $\iint_S x^2 yz \, dS$,
   $S$ is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

6. $\iint_S xy \, dS$,
   $S$ is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

7. $\iint_S yz \, dS$,
   $S$ is the part of the plane $x + y + z = 1$ that lies in the first octant

8. $\iint_S y \, dS$,
   $S$ is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

9. $\iint_S yz \, dS$,
   $S$ is the surface with parametric equations $x = u^2$, $y = u \sin v$, $z = u \cos v$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

10. $\iint_S \sqrt{1 + x^2 + y^2} \, dS$,
    $S$ is the helicoid with vector equation
    $\mathbf{r}(u, v) = u \cos v \, \mathbf{i} + u \sin v \, \mathbf{j} + v \, \mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

11. $\iint_S x^2 z^2 \, dS$,
    $S$ is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$

12. $\iint_S z \, dS$,
    $S$ is the surface $x = y + 2z^2$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$

13. $\iint_S y \, dS$,
    $S$ is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$

14. $\iint_S y^2 \, dS$,
    $S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy$-plane

15. $\iint_S (x^2 z + y^2 z) \, dS$,
    $S$ is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

16. $\iint_S xz \, dS$,
    $S$ is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$

17. $\iint_S (z + x^2 y) \, dS$,
    $S$ is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant

18. $\iint_S (x^2 + y^2 + z^2) \, dS$,
    $S$ is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

**19–30** Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface $S$. In other words, find the flux of $\mathbf{F}$ across $S$. For closed surfaces, use the positive (outward) orientation.

19. $\mathbf{F}(x, y, z) = xy \, \mathbf{i} + yz \, \mathbf{j} + zx \, \mathbf{k}$, $S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

20. $\mathbf{F}(x, y, z) = y \, \mathbf{i} + x \, \mathbf{j} + z^2 \, \mathbf{k}$,
    $S$ is the helicoid of Exercise 10 with upward orientation

21. $\mathbf{F}(x, y, z) = xze^y \, \mathbf{i} - xze^y \, \mathbf{j} + z \, \mathbf{k}$,
    $S$ is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation

22. $\mathbf{F}(x, y, z) = x \, \mathbf{i} + y \, \mathbf{j} + z^4 \, \mathbf{k}$,
    $S$ is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation

23. $\mathbf{F}(x, y, z) = x \, \mathbf{i} - z \, \mathbf{j} + y \, \mathbf{k}$,
    $S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation toward the origin

24. $\mathbf{F}(x, y, z) = xz \, \mathbf{i} + x \, \mathbf{j} + y \, \mathbf{k}$,
    $S$ is the hemisphere $x^2 + y^2 + z^2 = 25$, $y \geqslant 0$, oriented in the direction of the positive $y$-axis

25. $\mathbf{F}(x, y, z) = y \, \mathbf{j} - z \, \mathbf{k}$,
    $S$ consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

26. $\mathbf{F}(x, y, z) = xy \, \mathbf{i} + 4x^2 \, \mathbf{j} + yz \, \mathbf{k}$, $S$ is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, with upward orientation

**27.** $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
$S$ is the cube with vertices $(\pm 1, \pm 1, \pm 1)$

**28.** $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$,   $S$ is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

**29.** $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$,   $S$ is the boundary of the solid half-cylinder $0 \le z \le \sqrt{1 - y^2}, 0 \le x \le 2$

**30.** $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
$S$ is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

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**[CAS] 31.** Evaluate $\iint_S xyz\,dS$ correct to four decimal places, where $S$ is the surface $z = xy, 0 \le x \le 1, 0 \le y \le 1$.

**[CAS] 32.** Find the exact value of $\iint_S x^2yz\,dS$, where $S$ is the surface in Exercise 31.

**[CAS] 33.** Find the value of $\iint_S x^2y^2z^2\,dS$ correct to four decimal places, where $S$ is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the $xy$-plane.

**[CAS] 34.** Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2 e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the $xy$-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

**35.** Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where $S$ is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

**36.** Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where $S$ is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

**37.** Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$, if it has constant density.

**38.** Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}, 1 \le z \le 4$, if its density function is $\rho(x, y, z) = 10 - z$.

**39.** (a) Give an integral expression for the moment of inertia $I_z$ about the $z$-axis of a thin sheet in the shape of a surface $S$ if the density function is $\rho$.
(b) Find the moment of inertia about the $z$-axis of the funnel in Exercise 38.

**40.** Let $S$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If $S$ has constant density $k$, find
(a) the center of mass and (b) the moment of inertia about the $z$-axis.

**41.** A fluid has density $870 \text{ kg/m}^3$ and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where $x$, $y$, and $z$ are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \le z \le 1$.

**42.** Seawater has density $1025 \text{ kg/m}^3$ and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where $x$, $y$, and $z$ are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \ge 0$.

**43.** Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \le a^2$, $z \ge 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

**44.** Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

**45.** The temperature at the point $(x, y, z)$ in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \le x \le 4$.

**46.** The temperature at a point in a ball with conductivity $K$ is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere $S$ of radius $a$ with center at the center of the ball.

**47.** Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(r) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant $c$, where $r = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere $S$ with center the origin is independent of the radius of $S$.

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## 16.8  STOKES' THEOREM

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region $D$ to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface $S$ to a line integral around the boundary curve of $S$ (which is a space curve). Figure 1 shows