① Recall $f(a+h) = \sum_{|\alpha| \le k} \frac{\partial^\alpha f}{\alpha!} h^\alpha + R_{a,k}(h)$ $\quad \alpha = (\alpha_1, \cdots, \alpha_n)$

1. a) Consider $f(x,y) = x^3y^2$. With multi-index notation as in the text, compute $\partial^\alpha f(x,y)$ for all $|\alpha| \le 3$ and use this to write the Taylor polynomial of degree 3 (k=3) for f at the point (x,y) as per Equation 2.69.

b) Now use the expansion for f(x+h, y+k) and Equation 2.69 to determine the remainder $R_{(x,y),3}(h,k)$. Now verify Equation 2.72 by considering all α such that $|\alpha| = 4$ and computing all $\partial^\alpha f(x,y)$.

c) Determine the Taylor polynomial of degree 2 of $g(x,y) = x^2 + y$ at $(1,2)$.

d) Present the 3rd-order Taylor polynomial for $\frac{1}{2-x^2-y}$ near (0,1). (See example 2 on page 93).

$\partial^{1,0} f = \partial_x f = 3x^2y^2$, $\partial^{2,0} f = \partial_x^2 f = 6xy^2$

$\partial^{3,0} f = 6y^2$, $\partial^{0,1} f = 2x^3y$, $\partial^{1,1} f = 6x^2y$, $\cdots$

$P_{(x,y),3}(h_1,h_2) = x^3y^2 + (3x^2y^2h_1 + 2x^3yh_2) + (3xy^2h_1^2 + 6x^2yh_1h_2 + x^3h_2^2) + (y^2h_1^3 + 6xyh_1^2h_2$
$+ 3x^2h_1h_2^2)$

Look at $|\alpha| = 3$

$$\sum_{|\alpha|=3} \frac{\partial^\alpha f}{\alpha!} h^\alpha = \frac{\partial^{3,0} f}{3!0!} h_1^3 h_2^0 + (2,1) + (1,2) + (0,3)$$
$$= \frac{6y^2}{6} h_1^3 + \frac{12xy}{2} h_1^2 h_2 + \frac{6x^2}{2} h_1 h_2^2 + 0$$

b). $R_{(x,y),3}(h_1,h_2) = f(x+h_1, y+h_2) - P_{(x,y),3}(h_1,h_2)$
$= (x+h_1)^3(y+h_2)^2 - P_{(x,y),3}(h_1,h_2)$
$= \cdots$
$= 3xh_1^2h_2^2 + 2yh_1^3h_2 + h_1^3h_2^2$

From Taylor's Thm (eqn 2.72) - Lagrange form)

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(a+ch)}{\alpha!} h^\alpha \quad , c \in (0,1)$$

Return to $f(x,y) = x^3y^2 \to R_{(x,y),3}(h_1,h_2) = \sum_{|\alpha|=4} \frac{\partial^\alpha f(a+ch)}{\alpha!} h^\alpha$

$\partial^{4,0} f = 0$, $\partial^{3,1} f = 12y$, $\partial^{2,2} f = 12x$, $\partial^{1,3} f = \partial^{0,4} f = 0$

$$R_{(x,y),3}(h_1,h_2) = \frac{\partial^{4,0} f(x+ch_1, y+ch_2)}{4!\,0!} h_1^4 h_2^0 + \frac{\partial^{3,1} f(x+ch_1, y+ch_2)}{3!\,1!} h_1^3 h_2$$
$$+ \cdots$$

$$= 0 + \frac{12(y+ch_2)}{6} h_1^3 h_2 + \cdots \qquad = 2yh_1^3h_2 + 3xh_1^2h_2^2 + 5c\,h_1^3h_2^2$$

This matches the expression above for $R_{(x,y),3}(h_1,h_2)$ if we put $c=\frac{1}{5}$ (this demonstrates Taylor's thm for this particular function $f(x,y)=x^3y^2$).

c). $g(x,y)=x^2+y$, $P_{(1,2),2}(h_1,h_2)=3+2h_1+h_2+h_1^2$

d). $f(x,y)=\dfrac{1}{2-x^2-y}$ near $(0,1)$. $a=(0,1)$, $h=(h_1,h_2)$. $f(a+h)=\dfrac{1}{2-(0+h_1)^2-(1+h_2)}$

$$= \frac{1}{1-(h_1^2+h_2)}$$

Recall: $\dfrac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$ valid $|x|<1$

Apply this to $\dfrac{1}{1-(h_1^2+h_2)} = 1+\cdots$

To get $P_{3,(0,0)}(h_1,h_2)$, just keep terms of total deg ≤ 3

$P_{3,(0,1)}(h_1,h_2)=1+h_2+(h_1^2+h_2^2)+(2h_1^2h_2+h_2^3)$

---

2. a) Determine and classify all the critical points of $f(x,y)=x^3y^2$ according to theorem 2.82.

b) At the point (0,1) determine ∇f and the Hessian. Use your third degree expansion from question 1 to see if you can draw any conclusions about the behaviour of f near the point (0,1). If the degree is zero then you must move to the 4th degree polynomial.

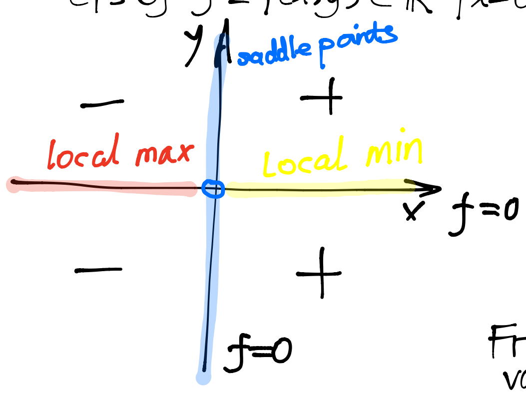
c) Repeat b) for (0,0). This time you may need to go all the way to the 5th degree.

d) Use your expansion from 1a to write the degree 2 Taylor polynomial in the form $\nabla f\cdot\mathbf{h}+1/2\mathbf{h}^T H\mathbf{h}$ as in 2.80

② Critical points $f(x,y)=x^3y^2$

a). $\nabla f=\vec{0}=(3x^2y^2, 2x^3y) \iff$ either x or y or both equal zero

CPs of $f=\{(x,y)\in\mathbb{R}^2 \mid x=0 \text{ or } y=0 \text{ (or both)}\}$



b). $(0,1)$. $\nabla f(0,1)=0$.

$H=\begin{pmatrix}\partial_1^2 f & \partial_2\partial_1 f\\ \partial_1\partial_2 f & \partial_2^2 f\end{pmatrix} \underset{\text{at }(0,1)}{=} \begin{pmatrix}0&0\\0&0\end{pmatrix}$

From 1 a). $P_{(0,1),3}(h_1,h_2)=h_1^3$ (other terms vanish b/c we set $x=0$.

$\begin{cases} +\text{ ve} & \text{when } h_1>0\\ -\text{ ve} & h_1<0\end{cases}$

this is another way of seeing that $(0,1)$ is a saddle point.