University of Toronto

Faculty of Arts and Science

APRIL EXAMINATIONS 2012

STA 305 H1S/ STA 1004 HS

Duration - 2 hours

Examination aids: cheat-sheet, non-programmable calculators

Name: _____ Student I.D. _____

Section: _____

1. Please **print** your name and student id. number in the above space.

2. Please provide the answers in the corresponding space.

3. There are three parts to the exam (A, B and C) for a total of 100 points. Partial grades are indicated.

4. Tables of the standard distributions used are appended.

A	B	C	Total

Consider a two-way model

$$y_{ijk} = \mu + \tau_{ij} + \epsilon_{ijk}$$

in which i represents the levels of treatment factor A and j represents the level of treatment factor B. The analysis of variance obtained with the statistical package SAS for this model is given below.

Source	DF	Sum of Squares	Mean Square	F Value
Model	11	820.3257241	74.5750658	57.38
Error	24	31.1940776	1.2997532	
Corrected Total	35	851.5198018		

Suppose *we do not know* whether there is interaction between factors A and B. The two-way complete model can also be written as

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

a) [10 pts] The Type III analysis of variance for this model is shown below with some parts missing. Please fill in the missing parts with the correct quantities.

Source	DF	Type III SS	Mean Square	F Value
A	3	279.5847104	93.1949035	78.05
B	–	536.1158928	–	–
A*B	–	–	–	–

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b) [10 pts] Do you accept or reject (at a 95% level of significance) the hypothesis stating that there is no interaction between factors A and B?

c) [10 pts] Use the information given in this problem so far and the information given below to construct a 95% confidence interval (without any corrections) for the interaction contrast

$$[(\alpha\beta)_{11} - (\alpha\beta)_{13}] - [(\alpha\beta)_{21} - (\alpha\beta)_{23}].$$

Explain your answer.

```
ESTIMATE '11-13-21+23' ... ;
                                        Standard
Parameter               Estimate        Error
11-13-21+23            -1.13629534       --
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Part B [35 points]

Consider a balanced and incomplete block design for comparing 4 treatments. Available are four blocks, each with 3 units. If we denote the treatment levels are denoted "1", "2","3" and "4" the table below shows the allocation of treatments to each block.

Block 1	Block 2	Block 3	Block 4
1	2	3	4
2	3	4	1
3	4	1	2

The model considered is

$$y_{hi} = \mu + \tau_i + \theta_h + \epsilon_{hi}$$

and we assume that ϵ_{hi} are independent noise terms, but their variance changes from block to block, so that $\epsilon_{hi} \sim N(0, \sigma_h^2)$. We assume that we *know* $\sigma_1^2 = 1$,$\sigma_2^2 = \sigma_3^2 = 0.5$ and $\sigma_4^2 = 0.1$.

a) [10 pts] Find four unbiased estimators for $\tau_3 - \tau_2$ and compute their variances.

b) [15 pts] For this question assume that all variances σ_i are known to be equal to 0.5. Find the unbiased estimator of $\tau_3 - \tau_2$ with the smallest variance. Compute its variance.

c) [10 pts] Compute the length of the confidence intervals constructed from the estimators you established in parts a) and b).

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Part C [35 points]

We are interested in studying the effects of 2 treatments, treatment A with 3 levels and treatment B with 4 levels. In order to study them, we run one experiment where each treatment combination is applied r times. Unfortunately, we do not account for a covariate that should have been considered at the time the experiment was planned. Therefore we must consider the following 2-way model that accounts for covariate X:

$$y_{ijk} = \mu + \alpha_i + \gamma_j + \beta(x_{ijk} - \bar{x}...) + \epsilon_{ijk},$$

where α_i is the effect of treatment A at level i, γ_j is the effect of treatment B at level j, with independent errors $\epsilon_{ijk} \sim N(0, \sigma_1^2)$, for all $1 \leq i \leq 3$ and $1 \leq j \leq 4$ and $1 \leq k \leq r$.

a) [15 points] Find the least squares estimator of $\Delta = \alpha_2 - \alpha_1$ and show the formula for its variance.

b) (10 pts) In a second experiment we have applied treatments A and B on a sample from *the same population* but this time we have not measured the covariate X. So, for this second experiment we can fit the simple 2-way model

$$y_{ijk} = \tilde{\mu} + \tilde{\alpha}_i + \tilde{\gamma}_j + \eta_{ijk},$$

where $\eta_{ijk} \sim N(0, \sigma_2^2)$ independently for all $1 \leq i \leq 3$ and $1 \leq j \leq 4$ and $1 \leq k \leq r$.

Find the least squares estimator for $\tilde{\alpha}_2 - \tilde{\alpha}_1$ and its variance.

c) (10 pts) Is the estimator found at part b) unbiased for the parameter Δ defined in part a)? Explain your answer in as much detail as you can.

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Table A.3 Standard normal distribution:* Upper α critical coefficients, z_α, and upper-tail probabilities, $\alpha = P(Z > z_\alpha)$

α	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005	0.00025	0.0001
z_α	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291	3.481	3.719

z_α	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005

* Values were generated using the SAS statements "z_alpha = probit(1-alpha);" and "alpha = 1 - probnorm(z_alpha);".

Table A.4 Student's t-distribution:* Upper α critical coefficients, $t_{df,\alpha}$, where $\alpha = P(t_{df} > t_{df,\alpha})$

df	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005	0.0001
1	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6	3183
2	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60	70.70
3	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92	22.20
4	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610	13.03
5	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869	9.678
6	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959	8.025
7	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408	7.063
8	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041	6.442
9	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781	6.010
10	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587	5.694
11	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437	5.453
12	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318	5.263
13	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221	5.111
14	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140	4.985
15	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073	4.880
16	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015	4.791
17	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965	4.714
18	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922	4.648
19	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883	4.590
20	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850	4.539
21	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819	4.493
22	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792	4.452
23	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768	4.415
24	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745	4.382
25	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725	4.352
26	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707	4.324
27	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690	4.299
28	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674	4.275
29	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659	4.254
30	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646	4.234
35	1.306	1.690	2.030	2.438	2.724	2.996	3.340	3.591	4.153
40	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551	4.094
45	1.301	1.679	2.014	2.412	2.690	2.952	3.281	3.520	4.049
50	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496	4.014
55	1.297	1.673	2.004	2.396	2.668	2.925	3.245	3.476	3.986
60	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460	3.962
70	1.294	1.667	1.994	2.381	2.648	2.899	3.211	3.435	3.926
80	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416	3.899
90	1.291	1.662	1.987	2.368	2.632	2.878	3.183	3.402	3.878
100	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390	3.862
110	1.289	1.659	1.982	2.361	2.621	2.865	3.166	3.381	3.848
120	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373	3.837
∞	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291	3.719

* Values $t_{df,\alpha}$ were generated using the SAS statements "t = tinv(1-alpha,df);" for $df < \infty$ and "t = probit(1-alpha)" for $df = \infty$.

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Table A.6 F-distribution:* Upper α critical coefficients, $F_{\nu_1,\nu_2,\alpha}$, where $\alpha = P(F_{\nu_1,\nu_2} > F_{\nu_1,\nu_2,\alpha})$

ν_2	α	ν_1 1	2	3	4	5	6	7	8	9
1	0.100	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9
	0.050	161	200	216	225	230	234	237	239	241
	0.010	4052	5000	5403	5625	5764	5859	5928	5981	6022
2	0.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	0.050	18.5	19.0	19.2	19.3	19.3	19.3	19.4	19.4	19.4
	0.010	98.5	99.0	99.2	99.3	99.3	99.3	99.4	99.4	99.4
	0.001	999	999	999	999	999	999	999	999	999
3	0.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	0.050	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	0.010	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.4
	0.001	167	149	141	137	135	133	132	131	130
4	0.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	0.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	0.010	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7
	0.001	74.1	61.3	56.2	53.4	51.7	50.5	49.7	49.0	48.5
5	0.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	0.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	0.010	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2
	0.001	47.2	37.1	33.2	31.1	29.8	28.8	28.2	27.7	27.2
6	0.100	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	0.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	0.010	13.8	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	0.001	35.5	27.0	23.7	21.9	20.8	20.0	19.5	19.0	18.7
7	0.100	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	0.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	0.010	12.3	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	0.001	29.3	21.7	18.8	17.2	16.2	15.5	15.0	14.6	14.3
8	0.100	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
	0.050	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
	0.010	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
	0.001	25.4	18.5	15.8	14.4	13.5	12.9	12.4	12.1	11.8
9	0.100	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
	0.050	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
	0.010	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
	0.001	22.9	16.4	13.9	12.6	11.7	11.1	10.7	10.4	10.1
10	0.100	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
	0.050	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
	0.010	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
	0.001	21.0	14.9	12.6	11.3	10.5	9.93	9.52	9.20	8.96

* Values $F_{\nu_1,\nu_2,\alpha}$ were generated using the SAS statement
"f = round(finv(1-alpha,df1,df2),0.01);".

(continued) F-distribution: Upper α critical coefficients, $F_{\nu_1, \nu_2, \alpha}$

ν_2	α	ν_1 1	2	3	4	5	6	7	8	9
11	0.100	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27
	0.050	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
	0.010	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
	0.001	19.7	13.8	11.6	10.4	9.58	9.05	8.66	8.35	8.12
12	0.100	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
	0.050	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
	0.010	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
	0.001	18.6	13.0	10.8	9.63	8.89	8.38	8.00	7.71	7.48
13	0.100	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
	0.050	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
	0.010	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
	0.001	17.8	12.3	10.2	9.07	8.35	7.86	7.49	7.21	6.98
14	0.100	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
	0.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	0.010	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
	0.001	17.1	11.8	9.73	8.62	7.92	7.44	7.08	6.80	6.58
15	0.100	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
	0.050	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
	0.010	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
	0.001	16.6	11.3	9.34	8.25	7.57	7.09	6.74	6.47	6.26
16	0.100	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
	0.050	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
	0.010	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
	0.001	16.1	11.0	9.01	7.94	7.27	6.80	6.46	6.19	5.98
17	0.100	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
	0.050	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
	0.010	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
	0.001	15.7	10.7	8.73	7.68	7.02	6.56	6.22	5.96	5.75
18	0.100	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
	0.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
	0.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
	0.001	15.4	10.4	8.49	7.46	6.81	6.35	6.02	5.76	5.56
19	0.100	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
	0.050	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
	0.010	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
	0.001	15.1	10.2	8.28	7.27	6.62	6.18	5.85	5.59	5.39
20	0.100	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
	0.050	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
	0.010	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
	0.001	14.8	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24

Total pages = 12