PLEASE HAND IN

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UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

FINAL EXAM

CSC 236H1F
DURATION — 3 HOURS

AIDS ALLOWED: HANDWRITTEN 8.5"x11" AID SHEET

STUDENT NUMBER: |_|_|_|_|_|_|_|_|_|

LAST NAME: _____

FIRST NAME: _____

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This exam consists of 6 questions on 16 pages (including this one). *When you receive the signal to start, please make sure that your copy of the exam is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-16 of this exam.

1: _____/ 8

2: _____/ 8

3: _____/15

4: _____/10

5: _____/ 6

6: _____/15

TOTAL: _____/62

Good Luck!

QUESTION 1. [8 MARKS]

Let $\mathcal{F}$ be the SMALLEST set such that:

1. Any tree consisting of a single node is an element of $\mathcal{F}$
2. If $t_1, t_2 \in \mathcal{F}$, then so is a binary tree consisting of a root with t_1 and t_2 as subtrees.

Recall that leaves are nodes without children, whereas interior nodes have at least one child. Use structural induction to prove that every $t \in \mathcal{F}$ has exactly one more leaf than interior nodes.

QUESTION 2. [8 MARKS]

The greatest common divisor (GCD) of a pair of natural numbers is the largest natural number that divides them both with no remainder, or 0 if they are both 0. Some examples: the GCD of $(3, 8) = 1$, and the GCD of $(6, 15) = 3$.

You may assume without proof that for any natural numbers m, n, if $m \neq 0$, then GCD of (m, n) = GCD of $(n\,\%m\,, m)$.

Prove the given gcd algorithm correct with respect to its pre- and post-condition:

```
def gcd(m, n) :
  if m == 0 : return n
  else : return gcd(n % m, m)
```

PRECONDITION:

- m and n are natural numbers

POSTCONDITION:

- gcd terminates and returns the GCD of (m, n)

HINT: You should carry out induction on m.

QUESTION 3. [15 MARKS]

The algorithm for GCD in the last question suggests the following recurrence for the number of steps required to calculate the GCD of a pair of natural numbers (m, n):

$$T(m,n) = \begin{cases} 1 & \text{if } m = 0 \\ 1 + T(n \,\%\, m, m) & \text{if } m > 0 \end{cases}$$

Although it may seem daunting to analyze a recurrence in two variables, the Fibonacci numbers $F(n)$ and $F(n+1)$ provide a special case. In what follows you may assume without proof that $F(n+1) > F(n) > F(n+1)/2$ for all $n > 2$.

PART (A) [4 MARKS]

Find $T(F(5), F(6))$ by repeated substitution. Be sure to carry out all the steps, and note that for any natural number k, $k\%1 = 0$.

PART (B) [6 MARKS]

Use repeated substitution to make a conjecture about a closed form for $T(F(n), F(n+1))$, where $F(k)$ is the kth Fibonacci number.

PART (C) [5 MARKS]

Use induction to prove your conjecture from the previous part.

QUESTION 4. [10 MARKS]

Consider the loop:

```
def f(x) :
  y = x*x
  while y != 0 :
    x = x - 2
    y = y - 4*x - 4
```

PRECONDITION: x is an even natural number.

Devise a loop invariant, and use it to prove that the precondition implies termination.

QUESTION 5. [6 MARKS]

Let $R = (00^*(0+1)^*1^*1 + 11^*(0+1)^*0^*0)$.

PART (A) [2 MARKS]

What can you say about the number of substrings from $\{01, 10\}$ in strings from $L(R)$? Justify your statement (no formal proof required).

PART (B) [4 MARKS]

What can you say about the number of strings from $\{01, 10\}$ in strings from $L(RR)$? Justify your statement (no formal proof required).

QUESTION 6. [15 MARKS]

Consider the language:

$$L = \{x \in \{0,1\}^* : \text{ second-last symbol of } x \text{ is } 0\}$$

PART (A) [10 MARKS]

What is the minimum number of states for a DFSA that accepts L? Prove that no DFSA with fewer states can accept L.

PART (B) [5 MARKS]

Draw a DFSA that accepts L using a minimum number of states.

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Total Marks = 62