PLEASE HAND IN

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UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

AUGUST 2010 EXAMINATIONS

FINAL EXAM

CSC 165H1Y
DURATION — 3 HOURS

NO AIDS ALLOWED

LAST NAME: ______________________________

FIRST NAME: ______________________________

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 10 questions on 16 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question,". You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-16 of this test.

Marking breakdown (Total = 108 marks).

Question 1	10 marks	Question 6	10 marks
Question 2	16 marks	Question 7	8 marks
Question 3	15 marks	Question 8	8 marks
Question 4	6 marks	Question 9	10 marks
Question 5	15 marks	Question 10	10 marks

Good Luck!

Here are some definitions and results that will be useful throughout the exam. (note: you may also use any result from X = {notes, assignments, tutorials, lectures} by saying "from the x", where $x \in X$)

1. Let $\mathbb{N}$= the set of natural numbers (i.e {0, 1, 2, 3, ...})
2. Let $\mathbb{R}$= the set of real numbers and $\mathbb{R}^+$ = the set of positive real numbers
3. Let $\mathbb{F}$= $\{\mathbb{N} \rightarrow \mathbb{R}^{\geq 0}\}$
4. $\forall f, g \in \mathbb{F}$: $f \in O(g) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c * g(n)$
5. $\forall f, g \in \mathbb{F}$: $f \in \Omega(g) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq c * g(n)$
6. $\forall f, g \in \mathbb{F}$: $f \in \Theta(g) \Leftrightarrow \exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 * g(n) \leq f(n) \leq c_2 * g(n)$
7. $\forall f, g \in \mathbb{F}, \forall z \in \mathbb{R}^+, f = z * g \Rightarrow f \in \Theta(g)$
8. $\forall m, n, r \in \mathbb{N}, r = m\%n \Leftrightarrow (0 \leq r < n) \wedge (\exists q \in \mathbb{N}, m = q * n + r)$
9. $W_P(n) \in \Omega(f(n)) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, \exists x \in I, size(x) = n \wedge n \geq B \Rightarrow t_P(x) \geq c * f(n)$
10. $y = log_b(x) \Leftrightarrow b^y = x$
11. $log_b(xy) = log_b(x) + log_b(y)$
12. $log_b(x/y) = log_b(x) - log_b(y)$

commutative laws	$P \wedge Q$	$\Leftrightarrow$	$Q \wedge P$
	$P \vee Q$	$\Leftrightarrow$	$Q \vee P$
	$(P \Leftrightarrow Q)$	$\Leftrightarrow$	$(Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R$	$\Leftrightarrow$	$P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R$	$\Leftrightarrow$	$P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R)$	$\Leftrightarrow$	$(P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R)$	$\Leftrightarrow$	$(P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q$	$\Leftrightarrow$	$\neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q$	$\Leftrightarrow$	$\neg P \vee Q$
equivalence	$(P \Leftrightarrow Q)$	$\Leftrightarrow$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
double negation	$\neg(\neg P)$	$\Leftrightarrow$	P
DeMorgan's laws	$\neg(P \wedge Q)$	$\Leftrightarrow$	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	$\Leftrightarrow$	$\neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q)$	$\Leftrightarrow$	$P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q)$	$\Leftrightarrow$	$\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x))$	$\Leftrightarrow$	$\exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x))$	$\Leftrightarrow$	$\forall x \in D, \neg P(x)$
identity	$P \vee (Q \wedge \neg Q)$	$\Leftrightarrow$	P
	$P \wedge (Q \vee \neg Q)$	$\Leftrightarrow$	P
idempotence	$P \vee P$	$\Leftrightarrow$	P
	$P \wedge P$	$\Leftrightarrow$	P
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x)$	$\Leftrightarrow$	$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
	$\exists x \in D, P(x) \vee Q(x)$	$\Leftrightarrow$	$(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

QUESTION 1. [10 MARKS]

Symbolic Representations of Ideas.

PART (A) [5 MARKS]

Consider the following predicates:

MP(x) : x is a Mersenne prime
Prime(x) : x is prime.

Using the above predicates, provide an equivalent symbolic statement for the statement below:

(S1A) A natural number n is a Mersenne prime if and only if n is a prime number that can be written in the form $2^k - 1$ for some positive integer k.

PART (B) [5 MARKS]

Provide an equivalent symbolic statement for the following statement:

(S1B) Conjecture: There is an infinite number of Mersenne primes.

QUESTION 2. [16 MARKS]

Distributive laws for quantifiers: For each of the following determine if they are true in both directions, one direction, or false in both directions. Briefly explain your answers.

PART (A) [4 MARKS]

$\forall x \in D, P(x) \wedge Q(x) \Leftrightarrow (\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$

PART (B) [4 MARKS]

$\exists x \in D, P(x) \wedge Q(x) \Leftrightarrow (\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$

PART (C) [4 MARKS]

$\forall x \in D, P(x) \vee Q(x) \Leftrightarrow (\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$

PART (D) [4 MARKS]

$\exists x \in D, P(x) \vee Q(x) \Leftrightarrow (\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

QUESTION 3. [15 MARKS]

PART (A) [5 MARKS]

Consider the following predicate:

$$P(n): \; \exists k \in \mathbb{N}, n = 7k + 4.$$

Using the proof structure from this course, prove the following statement:

(S2) $\forall n \in \mathbb{N}, P(n) \Rightarrow P(3n + 6)$

PART (B) [10 MARKS]

Let $\mathbb{F}_{\mathbb{R}}$ be the set of functions mapping the real numbers to the real numbers (i.e. $f \in \mathbb{F}_{\mathbb{R}} \Leftrightarrow f\colon \mathbb{R} \to \mathbb{R}$). Consider the following predicates regarding functions in $\mathbb{F}$:

$P(f,g)$: $\forall w \in \mathbb{R}, \forall z \in \mathbb{R}, (w = z) \vee (f(w) \neq f(z) \wedge g(w) \neq g(z))$

$Q(f,g)$: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, f(g(x)) \neq f(g(y)) \vee (x = y)$

Prove the following statement: $\forall f, g \in \mathbb{F}_{\mathbb{R}}, P(f,g) \Rightarrow Q(f,g)$
Hint: You may find it easier to prove an equivalent statement.

QUESTION 4. [6 MARKS]

Using equivalence transformations (see pg 1), prove or disprove the following:
$((A \wedge B) \vee (B \wedge (\neg A \vee \neg C))) \Leftrightarrow ((C \vee \neg B) \Rightarrow B)$

QUESTION 5. [15 MARKS]

PART (A) [5 MARKS]

Prove the following statement about the asymptotic behaviour of the log function:

$$\forall c \in \mathbb{R}^+, log(n^c) \in \Theta(logn)$$

Let $\mathbb{F} = \{\mathbb{N} \rightarrow \mathbb{R}^{\geq 0}\}$. Prove or disprove each of (s3a) and (s3b) below:

PART (B) [5 MARKS]

(S3A) $\exists f \in \mathbb{F}, \exists g \in \mathbb{F}, log(f(n) \cdot g(n)) \in O(log(f(n)))$

PART (C) [5 MARKS]

(S3B) $\exists f \in \mathbb{F}, \exists g \in \mathbb{F}, log(f(n) \cdot g(n)) \in O(1)$

QUESTION 6. [10 MARKS]

Recall: $\mathbb{F} = \{\mathbb{N} \to \mathbb{R}^{\geq 0}\}$

PART (A) [5 MARKS]

Disprove the following statement: $\forall f \in \mathbb{F}, \forall g \in \mathbb{F}, f(n) \in \mathcal{O}(g(n)) \Rightarrow 2^{f(n)} \in \mathcal{O}(2^{g(n)})$

PART (B) [5 MARKS]

Prove that for any natural number k, $\sum_{i=1}^{n} i^k \in \mathcal{O}(n^{k+1})$.

QUESTION 7. [8 MARKS]

Prove correctness of the following Python function. Think of an appropriate loop invariant (one that easily leads you to the post-condition), and don't forget to prove termination.

```
#pre-condition: A is a non-empty list of natural numbers
#post-condition: return the largest element of A
DEF maxNum(A):
  i = 0
  max = 0
  WHILE (NOT i == len(A))
    # invariant: ???
    IF A[i] > max : max = A[i]
    i = i + 1
  RETURN max
```

QUESTION 8. [8 MARKS]

PART (A) [4 MARKS]

Suppose f(x) = $ln(x)$ (the natural log, i.e. log_e, of x). Explain how the condition number of f is related to the relative error of f's input versus the relative error of f's output. Explain what this tells you about implementing f for x $\in$ (1, 3)?

PART (B) [4 MARKS]

Suppose you have a floating-point number system with base $\beta = 3$, one sign bit, emin = -2 and emax = 4, t = 4 digits in the mantissa (significand), with the convention that the significand for non-zero values begins with a left-most non-zero digit (i.e. normalized), which is the unit value to the left of the radix point. Round-to-nearest is used for numbers that cannot be represented exactly. How many distinct numbers are there in this system in the range (-27, 27)?

QUESTION 9. [10 MARKS]

```
# Pre-condition: A is an array of constant time comparable objects
""" selectionSort(A) sorts the elements of A in non-decreasing order """
DEF selectionSort(A):
  n = len(A)
  i = 0
  WHILE i < n-1 :
    min = i
    j = i + 1
    WHILE j < n :
      IF A[j] < A[min] :
        min = j
      j = j + 1
    swap A[i] AND A[min]
    i = i + 1
  # post-condition: A is sorted in non-decreasing order
  RETURN A
```

Let $t(A)$ be the number of lines executed by selectionSort on the Array A and $W(n)$ be the worst-case number of lines executed over all arrays of length n. Prove that $W(n) \in \Omega(n^2)$. (i.e. prove $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \exists A \in I, length(A) = n \wedge t(A) \geq cn^2$)

QUESTION 10. [10 MARKS]

Prove the following iterative program is correct, no proof of termination required.

```
#Pre: A is a sorted array,
#     x is a value which is comparable with the elements of A
#Post: The index of x in A is returned, or -1 is returned when x ∉ A.
DEF BS(A,x):
    first = 0
    last = len(A) -1
    #invariant: x ∈ A ⇔ x ∈ A[first_i : last_i]
    WHILE last-first ≥ 0:
      IF last == first:
        IF A[last] == x:
          RETURN last
      ELSE:
        mid = (first+last)/2
        IF A[mid] < x:
          first = mid +1
        ELSE:
          last = mid
    RETURN -1
```

This page left (nearly) blank for things that don't fit elsewhere.

# 1:	______/ 10
# 2:	______/ 16
# 3:	______/ 15
# 4:	______/ 6
# 5:	______/ 15
# 6:	______/ 10
# 7:	______/ 8
# 8:	______/ 8
# 9:	______/ 10
# 10:	______/ 10
TOTAL:	______/108