UNIVERSITY OF TORONTO
Faculty of Arts and Sciences

APRIL EXAMINATIONS
2012
MAT237Y1Y

Duration - 3 hours

No Aids Allowed

Instructions: There are 9 questions and 18 pages including the cover page. There is a total of 125 marks which include 25 bonus marks. Try to answer as many questions as you can. Note that the number of questions is more than you are expected to answer in a 3 hour exam, so please make a careful selection and answer those questions whose answers you are most confident about, within the space provided; (please clearly specify if you use back of a sheet to answer a question.)

NAME: (last, first) ____________________________

STUDENT NUMBER: ____________________________

SIGNATURE: ____________________________

MARKER'S REPORT:

Question	MARK
Q1 /6	
Q2 /6	
Q3 /12	
Q4 /13	
Q5 /24	
Q6 /12	
Q7 /25	
Q8 /13	
Q9 /14	
TOTAL /125	

1. (6 marks) Use an appropriate change of variables to evaluate the double integral $\iint_S \frac{x-2y}{3x-y}\,dA$, where the region S is bounded by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$ and $3x-y=8$.

2. (6 marks) State the Divergence theorem and use it to evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dA$, where $\boldsymbol{F}(x, y, z) = (e^y \cos z)\, \boldsymbol{i} + (\sqrt{x^3+1}\sin z)\, \boldsymbol{j} + (x^2 + y^2 + 3)\, \boldsymbol{k}$ and S is the graph of $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$ for $z \geq 0$ and oriented upward.

Note: the surface is not a closed surface.

3. Fubini's theorem and iterated integrals

a) (5 marks) Use an iterated integral to compute the double integral $\iint_S e^{x^2}\,dA$ where S is the region bounded by the x-axis, the line $x = 1$ and the line $y = x$.

b) (7 marks) Consider the function f defined by

$$f(x,y) = \begin{cases} y^{-2} \text{ if } 0 < x < y < 1 \\ -x^{-2} \text{ if } 0 < y < x < 1 \\ 0 \quad \text{otherwise} \end{cases}$$

First explain why both iterated integrals on $R = [0,1] \times [0,1]$ exist. Then calculate them and show they are not equal. Explain why this does not contradict Fubini's theorem.

4/18

4. Implicit function theorem

a) (8 marks) Give the three representations of a curve in $\mathbb{R}^3$ as presented in the textbook (in the same order), and use the appropriate version of the implicit function theorem to show the implicit (second) representation of a curve is transformable to the graph (first) representation. Make sure to state and use the appropriate regularity condition which guarantees this operation.

b) (5 marks) Draw the surface S determined by the graph of $x^2 + y + 2z = 4$ in the first octant and oriented outward. Clearly define ∂S as a collection of curves in $\mathbb{R}^3$, with their proper orientations.

5. Surface integrals

a) (6 marks) For the surface S in question 4(b), use the surface integral to determine the mass of the surface S if the mass density on S is $\rho(x, y, z) = x$. (Note: mass is the total sum of the densities at each and all points of the surface.)

b) (6 marks) Calculate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dA$ where $\boldsymbol{F}(x, y, z) = (\frac{x}{2}, y, z)$.

c) (5 marks) Prove that if S is a closed surface, as in the boundary of a solid R in three dimensional space, and $\boldsymbol{F}$ is a C^2 vector field, then $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n} dA = 0$.

d) (7 marks) Use Stokes' theorem to calculate the surface integral $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n}\, dA$ where $\boldsymbol{F}(x, y, z) = \boldsymbol{i} + (x - yz)\,\boldsymbol{j} + (xy - \sqrt{z})\,\boldsymbol{k}$ and S is the surface in question 4(b).
(Hint: use part (c) and replace the surface S from question 4(b) by a union of regions in the coordinate planes.)

6. (12 marks) Give the general formula for the Taylor polynomial of degree two for a function $f(x, y, z)$ (at a general point) and then apply your formula to the function $f(x, y, z) = x + xy + yz + z^2$. Determine the critical point(s) of f, and use the Hessian of f at the critical point(s) to classify them. Explain your reasoning.

7. Conservative vector fields

a) (6 marks) Suppose that $R \subset \mathbb{R}^n$ is an open connected set and let $\boldsymbol{a} \in R$. Show that for any point $\boldsymbol{x} \in R$ there is a curve C that connects $\boldsymbol{x}$ to $\boldsymbol{a}$.

b) (4 marks) Suppose that $\boldsymbol{G}$ is a vector field defined and continuous on an open connected set $R \subset \mathbb{R}^n$. What does it mean for $\boldsymbol{G}$ to be conservative?

c) (8 marks) Prove that $\boldsymbol{G}$ as in part (b) must be the gradient of a C^1 function f on R. (Present your proof for the case $n = 2$.)

d) (7 marks) Consider the vector field

$$\boldsymbol{G}(x,y,z) = (2xy)\,\boldsymbol{i} + (x^2 + \log z)\,\boldsymbol{j} + \frac{y+2}{z}\,\boldsymbol{k}, \; z > 0 .$$

Determine whether $\boldsymbol{G}$ could be the gradient of a scalar valued function; if so determine the function f, and if not explain why.

8. Chain rule

a) (3 marks) State the chain rule for a vector valued function $\boldsymbol{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$, a scalar valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the composition $(f \circ \boldsymbol{g}) : \mathbb{R} \longrightarrow \mathbb{R}$.

b) (3 marks) Prove that the gradient of a C^1 function f is a conservative vector field.

c) (7 marks) Use chain rule (II) and differentiation under the integral sign to calculate $\dfrac{\partial F}{\partial x}$ at the point $\boldsymbol{a} = (1, \pi)$, where $F(x, y) = \displaystyle\int_1^{3x^2} x\cos(x^2 y + \pi t)\,dt$.

9. Green's theorem

a) (2 marks) State Green's theorem for a regular region S in $\mathbb{R}^2$.

b) (6 marks) Use Green's theorem to show $\int_C \frac{\partial f}{\partial n} ds = \iint_S \nabla^2 f dA$ for a function f that is C^2 on $\overline{S}$, where C is the boundary of the region S.

c) (6 marks) Consider $f(x,y) = \ln(x^2 + y^2)$. Let C be the circle of radius 1, and let S be the disc inside C. centered at the origin. Calculate the line integral $\displaystyle\int_C \frac{\partial f}{\partial n} ds$. Calculate $\nabla^2 f$. Why does this not contradict part (b)?

Calculate the line integral $\displaystyle\int_C \nabla f \cdot d\boldsymbol{x}$.

Recall: $\frac{\partial f}{\partial n} = \nabla f \cdot \boldsymbol{n}$.