University of Toronto

FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, DECEMBER 2011

MAT 246H1F - CONCEPTS IN ABSTRACT MATH

Instructor: F. Murnaghan

Duration - 3 hours

Total marks: 100

No calculators or other aids allowed.

[10] 1. Prove that $\dfrac{(3n)!}{(3!)^n}$ is a natural number for every natural number n.

(*Note*: If m is a natural number, then $m! = m(m-1)(m-2)\cdots 4\cdot 3\cdot 2\cdot 1$.)

[10] 2. Prove that if a and b are integers and 8 divides a^2+3b^2, then both a and b are even.

[13] 3. Let n be a natural number. Prove that 5 divides 4^n-3^n if and only if 4 divides n.

[10] 4. Let p be a prime number and let a and b be nonzero integers such that $\gcd(a,b)=1$. Prove that if p^2 divides ab then p^2 divides a or p^2 divides b.

[10] 5. a) Prove that if n is an integer, then $\gcd(6n-1, 2n-4) = 1$ or 11.

b) Find all integers n such that $\gcd(6n-1, 2n-4) = 11$.

[15] 6. In each case below, please explain your answer fully.

a) Is the angle 3.75° constructible?

b) Is the number $\dfrac{\sqrt[4]{12}}{\cos(29^\circ)}$ constructible?

c) Is every real root of the polynomial x^3-4x^2+3x+2 constructible?

[10] 7. Suppose that t is a nonzero real number. Prove that t is a constructible number if and only if $\dfrac{t}{\sqrt{t^2+1}}$ is a constructible number.

[12] 8. Let $S = \{\, x \in \mathbb{R} \mid x^3 \in \mathbb{N} \,\}$.

a) Find the cardinality $|S|$ of S. Justify your answer.

b) Let T be the collection of closed intervals in the real line $\mathbb{R}$ whose lengths belong to S. Find the cardinality $|T|$ of T. Justify your answer.

[10] 9. Let S, T and U be sets. Assume that $|U| \leq |S|$ and $|S| = |T|$.
Let $X = \{\, f : U \to S \mid f \text{ is one-to-one} \,\}$ and $Y = \{\, f : U \to T \mid f \text{ is one-to-one} \,\}$.
Prove that $|X| = |Y|$.