Term Test 1 is on Wed. 10. October. See the website

Today: - § 1.1 example

§ 1.2 matrix notation

§ 1.3 convexity

Eg. (standard and canonical form)

Minimize $z = 8x_1 + 9x_2$ s.t.

$$x_1 + 2x_2 \geq 3$$
$$4x_1 + 5x_2 = 0$$
$$6x_1 + 7x_2 \leq -5$$
$$x_1 \geq 0, x_2 \text{ unrestricted}$$

To get standard form: Maximize $z = -8x_1 - 9x_2$ s.t.

$$-x_1 - 2x_2 \leq -3$$
$$4x_1 + 5x_2 \leq 10$$
$$-4x_1 - 5x_2 \leq -10$$
$$6x_1 + 7x_2 \leq -5$$
$$x_1 \geq 0, \; x_2 \text{ unrestricted}$$

Standard form: Let $x = x^+ - x^-$

Maximize $z = -8x_1 - 9x_2^+ + 9x_2^-$ s.t

$$-x_1 - 2x_2^+ + 2x_2^- \leq -3$$
$$4x_1 + 5x_2^+ - 5x_2^- \leq 10$$
$$-4x_1 - 5x_2^+ + 5x_2^- \leq -10$$
$$6x_1 + 7x_2^+ - 7x_2^- \leq -5$$
$$x_1 \geq 0, \; x_2^+ \geq 0, \; x_2^- \geq 0$$

Could continue with the standard problem to get the canonical form: get unnecessarily many constraints. We start from the given problem. The slack variables are x_3 and x_4.

Canonical form (again, $x_2 = x_2^+ - x_2^-$)

Maximize $z = -8x_1 - 9x_2^+ + 9x_2^-$ s.t.

$$x_1 + 2x_2^+ - 2x_2^- - x_3 = 3$$
$$4x_1 + 5x_2^+ - 5x_2^- = 10$$
$$6x_1 + 7x_2^+ - 7x_2^- + x_4 = -5$$
$$x_1 \geq 0, x_2^+ \geq 0, x_2^- \geq 0, x_3 \geq 0, x_4 \geq 0.$$

§1.2 Matrix notation

$\leq$ and $\geq$ for vectors

<u>Definition</u> If $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$, we say $x \leq y$ provided $x_1 \leq y_1, \cdots, x_m \leq y_m$

$x \geq y$ provided $x_1 \geq y_1, \cdots, x_m \geq y_m$

<u>Eg</u>. If $x \in \mathbb{R}^n$ and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ with $x \geq 0 \in \mathbb{R}^n$, then:

$x_1 \geq 0, \cdots, x_n \geq 0$.

<u>Remarks</u>: In chapter 3, we use $\leq$ and $\geq$ for new vectors. Then $x \leq y$ (new vectors) provided $x^T \leq y^T$ (in the sense of the definition).

This is not a total order: In $\mathbb{R}^2$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \nleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \ngeq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

In matrix notation standard and canonical problems are presented as:

$$\text{Maximize } z = c^T x \quad \text{s.t.}$$
$$Ax \leq b$$
$$x \geq 0 \in \mathbb{R}^n$$

standard form

$$\text{Maximize } z = c^T x \text{ s.t.}$$
$$Ax = b$$
$$x \geq 0 \in \mathbb{R}^n$$

canonical form

(where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$).

§ 1.3 Geometry

__Definitions__: If $a \in \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$, $a \neq 0 \in \mathbb{R}^n$, and $b \in \mathbb{R}$ the solution set of the equation $a^T x = b$ is a hyperplane in $\mathbb{R}^n$.

The solution sets of the inequalities $a^T x \leq b$ and $a^T x \geq b$ are __half spaces__ (in $\mathbb{R}^n$, where $x \in \mathbb{R}^n$)

__Remark__ The hyperplane $a^T x = b$ is the intersection of the half-spaces $a^T x \leq b$ and $a^T x \geq b$.

Convexity

__Definitions__ : If x_1 and $x_2 \in \mathbb{R}^n$ and $x_1 \neq x_2$, the line joining x_1 and x_2 is

$\{x_1 + \lambda(x_2 - x_1) \in \mathbb{R}^n \text{ s.t. } \lambda \in \mathbb{R}\} = \{(1-\lambda)x_1 + \lambda x_2 \text{ . s.t. } \lambda \in \mathbb{R}\}$

If x_1 and $x_2 \in \mathbb{R}^n$ (and $x_1 = x_2$ is possible) the line segment joining x_1 and x_2 is $\{x_1+\lambda(x_2-x_1)\in\mathbb{R}^2 \text{ s.t. } 0\leq\lambda\leq 1\}$

Definition $[0,1]=\{\lambda\in\mathbb{R} \text{ s.t. } 0\leq\lambda\leq 1\}$

Remark: If $x_1=x_2$, the line segment joining x_1 and x_2 is actually a single point:
$x_1=x_2$