MAT335 HW5 Rui Qiu #999292509

Chapter 9. Symbolic dynamics

1. Solution.

$\sigma^4(\overline{0001}) = \sigma^3(\overline{0010}) = \sigma^2(\overline{0100}) = \sigma(\overline{1000}) = (\overline{0001})$

$\sigma^4(\overline{0011}) = \sigma^3(\overline{0110}) = \sigma^2(\overline{1100}) = \sigma(\overline{1001}) = (\overline{0011})$

$\sigma^4(\overline{1011}) = \sigma^3(\overline{0111}) = \sigma^2(\overline{1110}) = \sigma(\overline{1101}) = (\overline{1011})$

So there are 12 cycles of prime period 4.

3. Solution.

$s = (\overline{100})$

$t = (\overline{010})$

$$d[s,t] = d[(\overline{100}), (\overline{010})] = \left(1+\frac{1}{2}\right) + \left(\frac{1}{8}+\frac{1}{16}\right) + \left(\frac{1}{64}+\frac{1}{128}\right) + \cdots$$

$$= \left(\frac{3}{2}\right) + \left(\frac{3}{16}\right) + \left(\frac{3}{128}\right) + \cdots$$

$$= 3\left(\frac{1}{2} + \frac{1}{16} + \frac{1}{128} + \cdots\right)$$

$$= 3 \cdot \frac{\frac{1}{2}}{1-\frac{1}{8}}$$

$$= 3 \cdot \frac{1}{2} \times \frac{8}{7}$$

$$= \frac{12}{7}$$

5. Solution: Suppose such string is t.

Firstly, the first digit must be 0,

otherwise $d[(000\ldots), t]$ must be larger than 1

So if the second digit is 0 then

$t = (00\overline{1})$ such that

$d[(000\ldots), (00\overline{1})] = \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}$

if the the second digit is 1 then

$t = (01\overline{0})$ such that

$d[(000\ldots), (01\overline{0})] = \frac{1}{2}$

9. Proof: Since $s \in \Sigma$ is periodic under $S: \Sigma \to \Sigma$

Say $s = (\overline{s_0 s_1 \cdots s_i})$

then the shift map sends $S^{-1}(s)$ to s

Therefore $S^{-1}(s) = (\overline{s_i s_0 s_1 \cdots s_{i-1}})$

Hence $S^{-1}(s)$ is a periodic point for Q_c in Λ with the same period.

If s is eventually periodic

Then say $s = (s_0 s_1 \cdots s_{i-1} \overline{s_i s_{i+1} \cdots s_{i+r}})$

$S^{-1}(s) = (s_{i+r} s_0 s_1 \cdots s_{i-1} \overline{s_i \cdots s_{i+r}})$

And we notice that $S^{-1}(s)$ is an effective eventually periodic point

iff $s_{i+r} = s_i = s_{i+1} = \cdots = s_{i+r}$

and this is to say, the repeating cycle here is 1 cycle.

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Chapter 10 Chaos

2. Solution:

S_2 is dense in $[0,1]$.

Since for every element in the set S_2 of the form $p/2^n$ it has a finitely many numbers' digit binary expansion, then we say the number of digits is s. then we can always find a number which is just $\frac{1}{2^t}$ away from it. where $(s>t)$

e.g. say $\frac{3}{2^9}$, we can always find a $\frac{3}{2^9}+\frac{1}{2^{10}}$.

6. Solution.

$T_1 = \{(s_0 s_1 s_2 \cdots) \mid s_4 = 0\}$

Any point $t \notin T_1$ is of the form $(s_0 s_1 s_2 s_3 1 s_5 s_6 \cdots)$

The point in T_1 closest to t is $(s_0 s_1 s_2 s_3 0 s_5 s_6 \cdots)$

and $d[(s_0 s_1 s_2 s_3 1 s_5 s_6 \cdots), (s_0 s_1 s_2 s_3 0 s_5 s_6 \cdots)] = \frac{1}{2^4}$

Hence T_1 is not dense in Σ. 2

7. Solution:

$T_2 = \{(s_0 s_1 s_2 \cdots) \mid s_4 = 1\}$

It's almost the same as the previous problem.

Similarly, T_2 is not dense in Σ. 2

8. Solution.

$T_3 = \{(s_0 s_1 \cdots) \mid$ the sequence ends in all 0's$\}$

Then say $s \in T_3$

s is of the form

$s = (s_0 s_1 \cdots s_{n+1} \bar{0})$

note that we have $t = (s_0 s_1 s_2 \cdots)$ in Σ 2

then $s \to t$ as $n \to \infty$

Thus T_3 is dense in Σ.

15. Solution:

No, actually $(010010001000001\cdots)$ does not have two consecutive one's so it is not dense in Σ.

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17. Proof: Suppose $s=(s_0 s_1 \cdots) \in \Sigma$

$t=(s_0 s_1 \cdots s_n \hat{s}_{n+1} \hat{s}_{n+2} \cdots)$

By the Proximity Thm,

$$d[s,t] \leq \frac{1}{2^n}$$

however,

$d[\sigma^k(s), \sigma^k(t)] = 2$ for all sufficiently large n.

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21. Proof

Firstly define D as a doubling function.

Then we need to show

$$\begin{tikzcd} [0,1] \arrow[r, "D"] \arrow[d, "T"] & [0,1] \arrow[d, "T"] \\ [0,1] \arrow[r, "T"] & [0,1] \end{tikzcd}$$

commutes

By thm that D is chaotic, then we need to prove

$$T \circ D^{n-1} = T^n, \forall n > 0 \qquad (*)$$

Suppose $(*)$ is true for $n=k$.

Then

$$T \circ D^{k-1} = T^k$$

$$\Rightarrow T \circ T \circ D^{k-1} = T \circ T^k$$

since $T \circ T = T \circ D$ $\quad (**)$

then $T \circ D^k = T^{k+1}$

$(*)$ is proved by induction.

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Now we need to prove (**) that we used before

For $T \circ T$:

$0 \le x \le \frac{1}{4} \Rightarrow 0 \le T(x) \le \frac{1}{2} \Rightarrow T \circ T(x) = T(2x) = 2(2x) = 4x$

$\frac{1}{4} \le x \le \frac{1}{2} \Rightarrow \frac{1}{2} \le T(x) \le 1 \Rightarrow T \circ T(x) = T(2x) = 2 - 2(2x) = 2 - 4x$

$\frac{1}{2} \le x \le \frac{3}{4} \Rightarrow \frac{1}{2} \le T(x) \le 1 \Rightarrow T \circ T(x) = T(2-2x) = 2 - 2(2-2x) = 4x - 2$

$\frac{3}{4} \le x \le 1 \Rightarrow 0 \le T(x) \le \frac{1}{2} \Rightarrow T \circ T(x) = T(2-2x) = 2(2-2x) = 4 - 4x$

For $T \circ D$:

$0 \le x \le \frac{1}{4} \Rightarrow 0 \le D(x) \le \frac{1}{2} \Rightarrow T \circ D(x) = T(2x) = 2(2x) = 4x$

$\frac{1}{4} \le x \le \frac{1}{2} \Rightarrow \frac{1}{2} \le D(x) \le 1 \Rightarrow T \circ D(x) = T(2x) = 2 - 2(2x) = 2 - 4x$

$\frac{1}{2} \le x \le \frac{3}{4} \Rightarrow 0 \le D(x) \le \frac{1}{2} \Rightarrow T \circ D(x) = T(2x-1) = 2(2x-1) = 4x - 2$

$\frac{3}{4} \le x \le 1 \Rightarrow \frac{1}{2} \le D(x) \le 1 \Rightarrow T \circ D(x) = T(2x-1) = 2 - 2(2x-1) = 4 - 4x$

Hence Specially note that since $D(x)$ not continuous at $x = \frac{1}{2}$ and not defined at $x = 1$

but $T \circ D(\frac{1}{2}) = T \circ T(\frac{1}{2})$

$T \circ D(1) = T \circ T(1)$ still.

Hence $T \circ D = T \circ T$

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∎

22. Proof. want to show

$$\begin{tikzcd}
{[0,1]} \arrow[r, "T"] \arrow[d, "C"'] & {[0,1]} \arrow[d, "C"] \\
{[-1,1]} \arrow[r, "G"] & {[-1,1]}
\end{tikzcd}$$

commutes

where $C(x) = \cos(\pi x)$ and it's a homeomorphism

i.e. $C \circ T = G \circ C$

$$C \circ T(x) = \begin{cases} \cos(2\pi x) & , 0 \le x \le \frac{1}{2} \\ \cos(2\pi x - 2\pi) & \frac{1}{2} \le x \le 1 \end{cases}$$

$\frac{13.5}{16}$

However, when $\cos(2\pi x - 2\pi) = \cos(2\pi x)$

then we conclude $C \circ T(x) = \cos(2\pi x)$

And $G \circ C(x) = 2\cos^2(\pi x) - 1 = \cos(2\pi x)$

Therefore $C \circ T(x) = G \circ C(x)$.

Hence G is chaotic since T is chaotic