University of Toronto
FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, APRIL 2011

MAT 301H1S - GROUPS AND SYMMETRY

Instructor: F. Murnaghan
Duration - 3 hours

Total marks: 100

No calculators or other aids allowed.

Notation: If n is an integer and $n \geq 2$, $\mathbb{Z}_n$ is the group of integers modulo n, S_n is the symmetric group of degree n, A_n is the alternating group of degree n, and, if $n \geq 3$, D_n is the dihedral group of order $2n$. Let $GL(2, \mathbb{R})$ be the group of 2×2 invertible matrices with real entries. Let $\mathbb{R}^{\times}$ be the group of nonzero real numbers (under multiplication). If G_1 and G_2 are groups, $G_1 \oplus G_2$ is the direct product of G_1 and G_2.

[18] 1. In each case below, determine whether there exists an element a in the group G such that $|a| = \ell$. If such an element exists, find one and explain why $|a| = \ell$. If no such element exists, explain why not.

a) Let $G = A_{10}$ and $\ell = 12$.

b) Let $G = \mathbb{Z}_{36}/\langle 24 \rangle$ and $\ell = 9$.

c) Let $G = S_5 \oplus D_{15}$ and $\ell = 30$.

[12] 2. Suppose that n is an odd integer and $n \geq 3$. Suppose that H is a subgroup of D_n that contains at least two reflections.

a) Prove that H is nonabelian. Indicate where your proof uses the fact that n is odd.

b) Prove that H contains at least three rotations and at least three reflections. (*Note*: e is a rotation.)

[11] 3. a) Find an element $\beta \in S_8$ such that $\beta^2 = (3\,8)(1\,7)(2\,5)(4\,6)$. Explain your answer.

b) Suppose that $n \geq 4$, $\alpha \in A_n$, and $|\alpha| = 2$. Prove that there exists $\beta \in S_n$ such $\beta^2 = \alpha$.

[12] 4. In each case below, prove or disprove that the subgroup H is normal in G. (*Note*: You do not need to prove that H is a subgroup of G.)

a) Let $G = D_9 \oplus \mathbb{Z}_{21}$ and let $H = \{\, a \in G \mid a^3 = e_G \,\}$. (Here, e_G is the identity element in G.)

b) Let ϕ be a homomorphism from a group G to the group $\mathbb{Z}$ of integers (under addition). Let $H = \{\, a \in G \mid \phi(a) \in \langle 4 \rangle \,\}$.

[12] 5. Let $G = GL(2,\mathbb{R}) \oplus \mathbb{R}^\times$ and let $H = \{\, (A, x) \in G \mid \det A = x^2 \,\}$.

a) Prove that H is a normal subgroup of G.

b) Prove that G/H is isomorphic to $\mathbb{R}^\times$.

[9] 6. Suppose that G and G' are groups and $\phi : G \to G'$ is a function. Prove that ϕ is a homomorphism if and only if $H = \{\, (a,\, \phi(a)) \mid a \in G \,\}$ is a subgroup of $G \oplus G'$.

[14] 7. Let $\alpha_1 = (1\,4\,2\,3)(2\,7\,5\,4\,8)$ and $\alpha_2 = (1\,4\,8\,5)(2\,6\,3) \in S_9$.

a) Determine whether α_1 and α_2 are conjugate in S_9. If α_1 and α_2 are not conjugate in S_9, give reasons. Otherwise, find an element $\beta \in S_9$ such that $\alpha_2 = \beta\alpha_1\beta^{-1}$.

b) Find a positive integer k such that $\alpha_2^k \neq \alpha_2$, $\alpha_2^k \neq \alpha_2^{-1}$ and α_2^k belongs to the conjugacy class of α_2 in S_9. Explain your answer.

[12] 8. Suppose that a and b are elements of a group G.

a) Prove that if a and b are conjugate in G, then a^n is conjugate to b^n in G for every integer n.

b) Suppose that a has finite order and $|a|$ is odd. Prove that if a^2 and b^2 are conjugate in G, then a and b are conjugate in G.