INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

20 April 2015 (pm)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 12 questions, beginning your answer to each question on a new page.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Explain why the running yield from property investments tends to be greater than that from equity investments. [3]

2 Calculate the time in days for £3,000 to accumulate to £3,800 at:

(a) a simple rate of interest of 4% per annum.
(b) a compound rate of interest of 4% per annum effective.

[4]

3 A 182-day treasury bill, redeemable at \$100, was purchased for \$96.50 at the time of issue and later sold to another investor for \$98 who held the bill to maturity. The rate of return received by the initial purchaser was 4% per annum effective.

(i) Calculate the length of time in days for which the initial purchaser held the bill. [2]

(ii) Calculate the annual simple rate of return achieved by the second investor. [2]

(iii) Calculate the annual effective rate of return achieved by the second investor. [2]

[Total 6]

4 (i) Describe what is meant by the "no arbitrage" assumption in financial mathematics. [2]

A 9-month forward contract is issued on 1 April 2015 on a stock with a price of £6 per share on that date. Dividends are assumed to be received continuously and the dividend yield is 3.5% per annum.

(ii) Calculate the theoretical forward price per share of the contract, assuming no arbitrage and a risk-free force of interest of 9% per annum. [2]

The actual forward price per share of the contract is £6.30 and the risk-free force of interest is as in part (ii).

(iii) Outline how an investor could make an arbitrage profit. [2]

[Total 6]

5 An investor pays £120 per annum into a savings account for 12 years. In the first four years, the payments are made annually in advance. In the second four years, the payments are made quarterly in advance. In the final four years, the payments are made monthly in advance.

The investor achieves a yield of 6% per annum convertible half-yearly on the investment.

Calculate the accumulated amount in the savings account at the end of 12 years. [7]

6 An ordinary share pays annual dividends. The next dividend is expected to be 6p per share and is due in exactly six months' time. It is expected that subsequent dividends will grow at a rate of 6% per annum compound and that inflation will be 4% per annum. The price of the share is 175p and dividends are expected to continue in perpetuity.

Calculate the expected effective real rate of return per annum for an investor who purchases the share. [6]

7 In a particular country, insurance companies are required by regulation to value their liabilities using spot rates of interest derived from the government bond yield curve.

Over time t (measured in years), the spot rate of interest is equal to:

$$i = 0.02t \text{ for } t \leq 5$$

An insurance company in this country has a group of annuity policies which involve making payments of £1m per annum for four years and £2m per annum in the fifth year. All payments are assumed to be paid halfway through the year.

(i) Calculate the value of the insurance company's liabilities. [3]

(ii) Outline two reasons why the spot yield curve might rise with term to redemption. [3]

(iii) Calculate the forward rate of interest from time $t = 3.5$ to time $t = 4.5$. [2]
[Total 8]

8 A fixed-interest security, redeemable at par in 10 years, pays annual coupons of 9% in arrear and has just been issued at a price to give an investor who does not pay tax a rate of return of 7% per annum effective.

(i) Calculate the price of the security at issue. [2]

(ii) Calculate the discounted mean term (duration) of the security at issue. [3]

(iii) Explain how your answer to part (ii) would differ if the annual coupons on the security were 3% instead of 9%. [2]

(iv) (a) Calculate the effective duration (volatility) of the security at the time of issue.

(b) Explain the usefulness of effective duration for an investor who expects to sell the security over the next few months. [3]
[Total 10]

9 A property development company has just purchased a retail outlet for \$4,000,000. A further \$900,000 will be spent refurbishing the outlet in six months' time.

An agreement has been made with a prospective tenant who will occupy the outlet beginning one year after the purchase date. The tenant will pay rent to the owner for five years and will then immediately purchase the outlet from the property development company for \$6,800,000. The initial rent will be \$360,000 per annum and this will be increased by the same percentage compound rate at the beginning of each successive year. The rental income is received quarterly in advance.

Calculate the compound percentage increase in the annual rent required to earn the company an internal rate of return of 12% per annum effective. [9]

10 The force of interest, $\delta(t)$, is a function of time and at any time t (measured in years) is given by

$$\delta(t) = \begin{cases} 0.08 & \text{for } 0 \le t \le 4 \\ 0.12 - 0.01t & \text{for } 4 < t \le 9 \\ 0.05 & \text{for } t > 9 \end{cases}$$

(i) Determine the discount factor, $v(t)$, that applies at time t for all $t \ge 0$. [5]

(ii) Calculate the present value at $t = 0$ of a payment stream, paid continuously from $t = 10$ to $t = 12$, under which the rate of payment at time t is $100e^{0.03t}$. [4]

(iii) Calculate the present value of an annuity of £1,000 paid at the end of each year for the first three years. [3]

[Total 12]

11 On 1 January 2016, a student plans to take out a five-year bank loan for £30,000 that will be repayable by instalments at the end of each month. Under this repayment schedule, the instalment at the end of January 2016 will be X, the instalment at the end of February 2016 will be $2X$ and so on, until the final instalment at the end of December 2020 will be $60X$. The bank charges a rate of interest of 15% per annum convertible monthly.

(i) Prove that $(Ia)_{\overline{n|}} = \dfrac{\ddot{a}_{\overline{n|}} - nv^n}{i}$. [3]

(ii) Show that X = £26.62. [4]

The student is concerned that she will not be able to afford the later repayments and so she suggests a revised repayment schedule. The student would borrow £30,000 on 1 January 2016 as before. She would now repay the loan by 60 level monthly instalments of $36X$ = £958.32 but the first repayment would not be made until the end of January 2019 and hence the final instalment is paid at the end of December 2023.

(iii) Calculate the APR on the revised loan schedule and hence determine whether you believe the bank should accept the student's suggestion. [5]

(iv) Explain the difference in the total repayments made under the two arrangements. [2]

[Total 14]

12 In any year, the yield on investments with an insurance company has mean j and standard deviation s and is independent of the yields in all previous years.

(i) Derive formulae for the mean and variance of the accumulated value after n years of a single investment of 1 at time 0 with the insurance company. [5]

Each year the value of $(1+i_t)$, where i_t is the rate of interest earned in the t^{th} year, is lognormally distributed. The rate of interest has a mean value of 0.04 and standard deviation of 0.12 in all years.

(ii) (a) Calculate the parameters μ and σ^2 for the lognormal distribution of $(1+i_t)$.

(b) Calculate the probability that an investor receives a rate of return between 6% and 8% in any year.

[8]

(iii) Explain whether your answer to part (ii) (b) looks reasonable. [2]

[Total 15]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2015 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

June 2015

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2015 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates. In general, the non-numerical questions were answered poorly by marginal candidates.

1 Dividends usually increase annually whereas rents are reviewed less often.
Property is less marketable.
Expenses associated with property investment are higher.
Large, indivisible units of property are less flexible.
On average, dividends will tend to rise more rapidly than rents because dividends benefit from retention and reinvestment of profits in earlier years.

The worst answered question on the paper with over one-third of candidates scoring no marks.

2 (a) Let the answer be t days

$$3{,}000\left(1+0.04\times\frac{t}{365}\right)=3{,}800$$

$t = 2{,}433.33$ days

(b) Let the answer be t days:

$$3{,}000\ (1.04)^{\frac{t}{365}} = 3{,}800$$

$$\therefore (1.04)^{\frac{t}{365}} = \frac{3{,}800}{3{,}000}$$

$$\frac{t}{365}\ln 1.04 = \ln\left(\frac{3{,}800}{3{,}000}\right)$$

$\therefore t = 2{,}199.91$ days.

3 (i) $96.5(1.04)^t = 98$

$t \times \ln(1.04) = \ln(98/96.5)$

Therefore, $t = 0.3933$ years $= 143.54$ days (144 days)

(ii) The second investor held the bill for 182–144 = 38 days

Therefore $98\left(1+\frac{38}{365}i\right) = 100$

$$i=\left(\frac{100}{98}-1\right)\times\frac{365}{38} = 0.19603 \text{ or } 19.603\%$$

(iii) The actual rate of interest over 38 days was (100/98) – 1 = 0.020408

Annual effective rate over 1 year would be:

$$(1 + 0.020408)^{365/38} - 1 = 0.21416 \text{ or } 21.416\%$$

4 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The theoretical price per share of the forward contract is $£6e^{(0.09-0.035)\times\frac{9}{12}}$ $= £6.2527$

(iii) In this case the actual forward price is too expensive in relation to the stock.

The investor should borrow $£6e^{-0.035\times\frac{9}{12}}$ and use this to buy $e^{-0.035\times\frac{9}{12}}$ units of the stock. The investor will also go short in one forward contract. The continuous dividends are reinvested in the stock. (Mark given for general strategy, exact amounts not required).

[After nine months, the investor will have $e^{0.035\times\frac{9}{12}} \times e^{-0.035\times\frac{9}{12}} = 1$ unit of stock that can be sold under the terms of the forward contract for £6.30. The investor will also have to repay cash of $£6e^{-0.035\times\frac{9}{12}}e^{0.09\times\frac{9}{12}} = £6.2527$.]

Whilst it was not required for candidates to give a full mathematical explanation for part (iii), they were expected to recognise that the forward was overpriced and to determine the arbitrage strategy accordingly.

5 We will use the ½-year as the time unit because the interest rate is convertible half yearly. The effective rate of interest is 3% per half year.

Accumulated amount = $\frac{120}{\ddot{a}_{\overline{2}|}}\ddot{s}_{\overline{8}|} \times (1.03)^{16} + 60\ddot{s}^{(2)}_{\overline{8}|} \times (1.03)^{8} + 60\ddot{s}^{(6)}_{\overline{8}|}$ at 3%

We need $d^{(6)}$ from $\left(1 - \frac{d^{(6)}}{6}\right)^6 = 1 - d = \frac{1}{1+i} = \frac{1}{1.03}$

$$\Rightarrow 1 - \frac{d^{(6)}}{6} = \left(\frac{1}{1.03}\right)^{1/6} \Rightarrow \quad d^{(6)} = \left(1 - \left(\frac{1}{1.03}\right)^{1/6}\right) \times 6$$

$$= 0\ .029486111$$

Thus accumulated amount =

$$\frac{120}{a_{\overline{2}|}} s_{\overline{8}|} \times (1.03)^{16} + 60\frac{i}{d^{(2)}} s_{\overline{8}|} \times (1.03)^{8} + 60\frac{i}{d^{(6)}} s_{\overline{8}|} \text{ at } 3\%$$

$$= \frac{120}{1.9135} * 8.8923 * 1.60471 + 60 \times 1.022445 * 8.8923 * 1.26677 + 60 * \frac{0.03}{0.029486111} * 8.8923$$

= 894.877 + 691.040 + 542.837

= £2,128.75

(above uses factors in formulae and tables book; if book not used then exact answer is £2,128.7**7)**.

Generally well-answered but marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity.

6 Let:

$i =$ nominal yield
$e =$ inflation rate
$i' =$ real rate

Then

$$1+i=(1+i')(1+e)$$

We first find i and then use above equation to find i':

$$175 = \frac{6}{(1+i)^{1/2}}+\frac{6\times 1.06}{(1+i)^{1\frac{1}{2}}}+\frac{6\times(1.06)^2}{(1+i)^{2\frac{1}{2}}}+\cdots$$

$$=\frac{6}{(1+i)^{1/2}}\left(1+\frac{1.06}{1+i}+\frac{(1.06)^2}{(1+i)^2}+\cdots\right)$$

$$=\frac{6}{(1+i)^{1/2}}\left(\frac{1}{1-\frac{1.06}{1+i}}\right)$$

$$=\frac{6(1+i)^{1/2}}{(1+i)\left(1-\frac{1.06}{1+i}\right)}$$

$$=\frac{6(1+i)^{1/2}}{i-0.06}$$

Let $i=10\%$, RHS = 157.32
$i=9\%$, RHS = 208.81

Hence

$$i\approx 0.09+0.01\times\left(\frac{208.81-175}{208.81-157.32}\right)$$

$= 0.09657$ (answer to nearest 0.1% is 9.6%)

Then we have

$$1.09657 = (1 + i')1.04$$

$\Rightarrow \quad i' = 5.44\%$ p.a. real return (answer to nearest 0.1% is 5.4%)

An alternative method of formulating the equation in real terms to find i' directly was perfectly valid.

7 (i)

Time	*Spot rate of interest*
½	0.01
1½	0.03
2½	0.05
3½	0.07
4½	0.09

Value of liabilities (£m)

$$V = 1\left(v_{1\%}^{0.5} + v_{3\%}^{1.5} + v_{5\%}^{2.5} + v_{7\%}^{3.5}\right) + 2v_{9\%}^{4.5}$$

$_{1\%}v^{½} = 0.99504$
$_{3\%}v^{1.5} = 0.95663$
$_{5\%}v^{2.5} = 0.88517$
$_{7\%}v^{3.5} = 0.78914$
$_{9\%}v^{4.5} = 0.67855$
$V = 4.98308$

(ii) Because expectations of short-term interest rates rise with term and the yield curve is determined by expectations theory.

Because investors have a preference for liquidity which puts an upwards bias on the yield curve (e.g. because long-term bonds are more volatile). A rising curve would be compatible, for example, with constant expectations of interest rates.

Because the market segmentation theory holds and investors short-term bonds might be in demand by investors such as banks (or there is an undersupply of short-term bonds or less demand/more supply for long-term bonds).

(iii) Spot rate to time 4.5 is 9%. Spot rate to time 3.5 is 7%. Therefore:

$$1.09^{4.5}/(1.07)^{3.5} = \text{forward rate from 3.5 to 4.5} = 16.3\%$$

Common errors in part (i) were to assume payments at the end of the year and/or to assume that the payments should be valued with the end of year spot rate (2%, 4%, 6% etc.)

8 (i) $P = 9a_{\overline{10}|7\%} + 100v_{7\%}^{10}$

$v^{10} = 0.50835; \quad a_{\overline{10}|} = 7.02358$

$P = 9 \times 7.02358 + 100 \times 0.50835$

$= 114.047$

(ii) Discounted mean term

$$= \frac{\sum tC_t v^t}{\sum C_t v^t}$$

$$= \frac{9(Ia)_{\overline{10}|} + 10 \times 100v^{10}}{114.047}$$

$$= \frac{9 \times 34.7391 + 10 \times 100 \times 0.50835}{114.047}$$

= 7.199 years

(iii) Duration will be higher because the payments will be more weighted towards the end of the term.

(iv) (a) Effective duration = duration $/(1+i)$

$= 7.199/1.07 = 6.728$ years

(b) Effective duration would indicate the extent to which the value of the bond would change if there were a uniform change in interest rates. It is therefore an indication of the risk to which the investor is exposed if interest rates rise and the price of the security falls before it is sold.

Many of the explanations from candidates in part (iii) were very unclear.

9 Present value of outgoings = 4,000,000 + 900,000$v^{½}$

$$@12\% = 4,850,420$$

Present value of income =

$$360,000v\ddot{a}_{\overline{1|}}^{(4)} + 360,000(1+k)v^2\ \ddot{a}_{\overline{1|}}^{(4)}$$

$$+\ldots+360,000v^5(1+k)^4\ \ddot{a}_{\overline{1|}}^{(4)} + 6,800,000\ v^6$$

$$= 360,000\ \ddot{a}_{\overline{1|}}^{(4)}\ v\left(1+v_j+v_j^2+v_j^3+v_j^4\right) + 6,800,000\ v^6$$

where $j = \dfrac{1.12}{1+k} - 1$

$$\ddot{a}_{\overline{1|}}^{(4)} = 0.95887@12\%$$

So, present value of income $= 360,000 \times 0.95887 \times \dfrac{1}{1.12} \times \ddot{a}_{\overline{5|}}^{j} + 6,800,000\ v^6$

$$= 308,209\ddot{a}_{\overline{5|}}^{j} + 3,445,092$$

Hence, for IRR = 12%, 4,850,420 = $308,209\ \ddot{a}_{\overline{5|}}^{j} + 3,445,092$

so $\ddot{a}_{\overline{5|}}^{j} = 4.55966$

At 4% $\ddot{a}_{\overline{5|}} = 4.62990$

5% $\ddot{a}_{\overline{5|}} = 4.54595$

$$j \approx 4 + \frac{4.62990 - 4.55966}{4.62990 - 4.54595} = 4.837\%$$

$$j = \frac{1.12}{1+k} - 1$$

$$0.04837 = \frac{1.12}{1+k} - 1$$

$$\therefore 1.04837(1+k) = 1.12$$

$$k = \frac{1.12}{1.04837} - 1 = 0.0683 = 6.83\% \text{ (exact answer is 6.84\%)}$$

Marginal candidates again would have benefited from showing more intermediate working. In project appraisal questions, it is good exam technique to show working and answers for each component of income and outgo separately so that partial marks can be given if any errors are made within a component.

10 (i) For $0 \le t \le 4$:

$$v(t) = \exp\left(-\int_0^t \delta(s)\,ds\right) = \exp\left(-\int_0^t 0.08ds\right)$$

$$= e^{-0.08t} \text{ and } v(4) = e^{-0.08\times 4} = e^{-0.32}$$

For $4 < t \le 9$:

$$v(t) = \exp\left(-\int_0^4 \delta(s)\,ds - \int_4^t \delta(s)\,ds\right) = e^{-0.32}\exp\left(-\int_4^t 0.12 - 0.01s ds\right)$$

$$= e^{-0.32}\exp\left[-0.12s + 0.005s^2\right]_4^t$$

$$= e^{-0.32}\exp\left(0.48 - 0.08 - 0.12t + 0.005t^2\right)$$

$$= e^{0.08-0.12t+0.005t^2}$$

and $v(9) = e^{0.08-0.12\times 9+0.005\times 81} = e^{-0.595}$

For $t > 9$:

$$v(t) = \exp\left(-\int_0^9 \delta(s)\,ds - \int_9^t \delta(s)\,ds\right) = e^{-0.595}\exp\left(-\int_9^t 0.05ds\right)$$

$$= e^{-0.595}\exp\left[-0.05s\right]_9^t = e^{-0.595}\exp\left(0.45 - 0.05t\right) = e^{-0.145-0.05t}$$

(ii) Present value is

$$\int_{10}^{12} 100e^{0.03t}\left(\exp\left(-\int_0^t \delta(s)\,ds\right)\right)dt$$

$$=\int_{10}^{12} 100e^{0.03t}e^{-0.145-0.05t}\,dt$$

$$=\int_{10}^{12} 100e^{-0.145-0.02t}\,dt=\left[\frac{100e^{-0.145-0.02t}}{-0.02}\right]_{10}^{12}$$

$$=\left[-5{,}000e^{-0.145-0.02t}\right]_{10}^{12}=5{,}000\left(e^{-0.345}-e^{-0.385}\right)$$

$$=138.85$$

(iii) Present value $=1{,}000a_{\overline{3}|\delta=8\%}=1{,}000\dfrac{1-e^{-0.08\times 3}}{e^{0.08}-1}=£2{,}561.89$

11 (i) $(Ia)_{\overline{n}|}=v+2v^2+3v^3+\ldots+nv^n$ (1)

$(1+i)(Ia)_{\overline{n}|}=1+2v+3v^2+\ldots+nv^{n-1}$ (2)

$(2)-(1)\Rightarrow i(Ia)_{\overline{n}|}=1+v+v^2+\ldots+v^{n-1}-nv^n$

$$\Rightarrow (Ia)_{\overline{n}|}=\frac{\left(1+v+v^2+\ldots+v^{n-1}\right)-nv^n}{i}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{i}$$

(ii) Work in months i.e. use a monthly interest rate of 1.25% per month effective:

$$30{,}000=Xv+2Xv^2+\ldots+60Xv^{60}=X\,(Ia)_{\overline{60}|@1.25\%}=X\left(\frac{\ddot{a}_{\overline{60}|}-60v^{60}}{i}\right)$$

$$=X\left(\frac{\frac{1-v^{60}}{d}-60v^{60}}{i}\right)=X\left(\frac{\frac{1-1.0125^{-60}}{0.0125/1.0125}-60\times 1.0125^{-60}}{0.0125}\right)$$

$$=1126.8774X\Rightarrow X=£26.62$$

(iii) Equation of value:

$$30,000 = v^{36} 958.32 a_{\overline{60|}}$$

$$\Rightarrow v^{36} a_{\overline{60|}} = 31.3048$$

Try i = 1%: LHS = 31.4202

Try i = 1.1%: LHS = 29.5098

Interpolate: $i = 1\% + 0.1\% \left(\dfrac{31.3048 - 31.4202}{29.5098 - 31.4202} \right) = 1.0060\%$

APR is $(1 + 0.010060)^{12} - 1 = 12.8\%$ to 1 d.p.

The bank is unlikely to be happy to accept the suggestion as it will be earning a lower rate of return compared with the original proposal of 15% per annum convertible monthly (=16.1% per annum effective).

(iv) The student's arrangement will lead to a greater total of payments (60 payments of $36X$) when compared to the original arrangement (60 payments of $30.5X$ on average) but will incur a lower rate of interest. This is because under the student's arrangement no capital or interest will be paid for three years. The extra total of payments will not be sufficient to cover the deferred interest at the bank's preferred rate.

In part (i), candidates were expected to show the first and last terms of each series used to derive the result so that the proof is absolutely clear. In part (ii), candidates should show enough steps to demonstrate that they have performed the calculations required to actually prove the answer (e.g. show the numerical values for the factors used) . In part (iv), if interpolating on a monthly interest rate (as in the above solution) the guesses most be close enough together to ensure the estimated annual rate is close enough to the correct answer.

12 (i) Let S_n = Accumulated value at time n of £1 invested at time 0

$$S_n = (1+i_1)(1+i_2)....(1+i_n)$$

$$\Rightarrow E[S_n] = E[(1+i_1)(1+i_2)....(1+i_n)]$$

$$= E(1+i_1).\, E(1+i_2)....\, .E(1+i_n) \text{ by independence}$$

and $E(1+i_t) = 1 + E(i_t) = 1 + j$

Hence

$$E\left(S_n\right)=\left(1+j\right)^n$$

Now

$$\operatorname{Var}\left[S_n\right] = E\left[S_n^2\right]-\left(E\left[S_n\right]\right)^2$$

$$E\left[S_n^2\right] = E\left[\left(1+i_1\right)^2\left(1+i_2\right)^2 \ldots .\left(1+i_n\right)^2\right]$$

$$= E\left[\left(1+i_1\right)^2\right].E\left[\left(1+i_2\right)^2\right]\ldots . \quad .E\left[\left(1+i_n\right)^2\right]$$

by independence

and

$$E\left[\left(1+i_t\right)^2\right] = E\left[\left(1+2i_t+i_t^2\right)\right]$$

$$= 1+2\,E\left(i_t\right)+E\left(i_t^2\right)$$

and

$$\operatorname{Var}\left[i_t\right]=s^2 = E\left(i_t^2\right)-\left[E\left(i_t\right)\right]^2$$

$$= E\left(i_t^2\right)-j^2$$

$$\Rightarrow E\left(i_t^2\right) = s^2+j^2$$

Hence

$$E\left[S_n^2\right] = \left(1+2j+j^2+s^2\right)^n$$

and

$$\operatorname{Var}\left[S_n\right]=\left(1+2j+j^2+s^2\right)^n-\left(1+j\right)^{2n}$$

(ii) (a) $E(1+i_t) = 1 + E(i_t) = 1 + j = 1.04 = e^{\left(\mu + \sigma^2/2\right)}$

$\text{Var}(1+i_t) = \text{Var}(i_t) = s^2 = 0.12^2 = e^{\left(2\mu+\sigma^2\right)} \times \left(e^{\sigma^2} - 1\right)$

$$\Rightarrow \frac{0.12^2}{(1.04)^2} = e^{\sigma^2} - 1$$

$$\Rightarrow \sigma^2 = Ln\left[1 + \left(\frac{0.12}{1.04}\right)^2\right]$$

$$\Rightarrow \sigma^2 = 0.013226$$

$$1.04 = e^{\left(\mu + \frac{0.013226}{2}\right)}$$

$$\Rightarrow \mu = Ln\,1.04 - \frac{0.013226}{2}$$
$$= 0.032608$$

(b) $Ln(1+i_t) \sim N(0.032608, 0.013226)$

and we require probability $0.06 < i_t < 0.08$

$$= \Pr(1.06 < 1 + i_t < 1.08)$$

$$= \Pr\left(Ln\,1.06 < Ln(1+i_t) < Ln\,1.08\right)$$

$$= \Pr\left(\frac{Ln\,1.06 - 0.032608}{\sqrt{0.013226}} < \frac{Ln(1+i_t) - \mu}{\sigma} < \frac{Ln\,1.08 - 0.032608}{\sqrt{0.013226}}\right)$$

$= \Pr(0.22 < Z < 0.39)$ where $Z \sim N(0,1)$

$$= \Phi(0.39) - \Phi(0.22)$$

$$= 0.65173 - 0.58706 = 0.0647$$

i.e. 6% probability (using exact Φ function gives probability of 6.2%)

(iii) The probability in (ii) (b) is small. This is reasonable since the expected return in any year is 4%, and we are being asked to calculate the probability that the return is between 6%, and 8% (i.e. a range which does not include the expected value).

In part (i), it is important to note when the assumption of independence is required for both proofs. Common mistakes in the calculation of μ and σ were to assume that s^2 was 0.12 (rather than s) and 1 + j was 0.04 (rather than 1.04).

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

30 September 2015 (pm)

Subject CT1 – Financial Mathematics Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all nine questions, beginning your answer to each question on a new page.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 An investor wishes to obtain a rate of interest of 3% per annum effective from a 91-day treasury bill.

Calculate:

(a) the price that the investor must pay per £100 nominal.

(b) the annual simple rate of discount from the treasury bill.

[3]

2 The nominal rate of discount per annum convertible monthly is 5.5%.

(i) Calculate, giving all your answers as a percentage to three decimal places:

(a) the equivalent force of interest.

(b) the equivalent effective rate of interest per annum.

(c) the equivalent nominal rate of interest per annum convertible monthly.

[3]

(ii) Explain why the nominal rate of interest per annum convertible monthly calculated in part (i)(c) is less than the equivalent annual effective rate of interest calculated in part (i)(b) [1]

(iii) Calculate, as a percentage to three decimal places, the effective annual rate of discount offered by an investment that pays £159 in eight years' time in return for £100 invested now. [1]

(iv) Calculate, as a percentage to three decimal places, the effective annual rate of interest from an investment that pays 12% interest at the end of each two-year period. [1]

[Total 6]

3 An insurance company has sold a pension product to an individual. Under the arrangement, the individual is to receive an immediate annuity of £500 per year annually in arrear for 12 years. The insurance company has invested the premium it has received in a fixed-interest bond that pays coupons annually in arrear at the rate of 5% per annum and which is redeemable at par in exactly eight years.

(i) Calculate the duration of the annuity at an interest rate of 4% per annum effective. [2]

(ii) Calculate the duration of the bond at an interest rate of 4% per annum effective. [3]

(iii) State with reasons whether the insurance company will make a profit or a loss if there is a small increase in interest rates at all terms. [2]

[Total 7]

4 A nine-month forward contract was issued on 1 October 2015 on a share with a price at that date of £10. Dividends of 50 pence per share are expected on 1 November 2015 and 1 May 2016. The risk-free force of interest is 5% per annum.

(i) Calculate the forward price at issue, stating any further assumptions made and showing all workings. [4]

(ii) Explain why the expected price of the share nine months after issue does not have to be taken into account when pricing the forward. [2]
[Total 6]

5 An individual can obtain a force of interest per annum at time t, measured in years, as given by the formula:

$$\delta(t) = \begin{cases} 0.03 + 0.005t & 0 \le t \le 3 \\ 0.005 & t > 3 \end{cases}$$

(i) Determine the amount the individual would need to invest at time $t = 0$ in order to receive a continuous payment stream of £5,000 per annum from time $t = 3$ to time $t = 6$. [5]

(ii) Determine the equivalent constant annual effective rate of interest earned by the individual in part (i). [3]

(iii) Determine the amount an individual would accumulate from the investment of £300 from time $t = 0$ to time $t = 50$. [2]
[Total 10]

6 Three bonds, each paying annual coupons in arrear of 3% and redeemable at £100 per £100 nominal, reach their redemption dates in exactly one, two and three years' time, respectively.

The price of each bond is £101 per £100 nominal.

(i) Determine the gross redemption yield of the three-year bond. [3]

(ii) Calculate the one-year, two-year and three-year spot rates of interest implied by the information given. [5]

(iii) Calculate the one-year forward rate starting from the end of the second year, $f_{2,1}$. [2]

The pattern of spot rates is upward sloping throughout the yield curve.

(iv) Explain, with reference to the various theories of the yield curve, why the yield curve might be upward sloping. [4]
[Total 14]

7 A special type of loan is to be issued by a company. The loan is made up of 100,000 bonds, each of nominal value €100. Coupons will be paid semi-annually in arrear at a rate of 4% per annum. The bonds are to be issued on 1 October 2015 at a price of €100 per €100 nominal. Income tax will be paid by the bond holders at a rate of 25% on all coupon payments.

Exactly half the bonds will be redeemed after ten years at €100 per €100 nominal. The bonds that are redeemed will be determined by lot (i.e. the bonds will be numbered and half the numbered bonds will be chosen randomly for redemption). Coupon payments on the remaining bonds will be increased to 7% per annum and these bonds will be redeemed 20 years after issue at €130 per €100 nominal.

An individual buys a single bond.

Calculate, as an effective rate of return per annum:

(i) the maximum rate of return the individual can obtain from the bond. [5]

(ii) the minimum rate of return the individual can obtain from the bond. [2]

(iii) the expected rate of return the individual will obtain from the bond [2]

An investor is considering buying the whole loan.

(iv) Show that the rate of return that the investor will obtain is greater than the expected rate of return that the above individual who buys a single bond will receive. [5]

[Total 14]

8 (i) State the characteristics of an equity. [4]

An investor was considering investing in the shares of a particular company on 1 August 2014. The investor assumed that the next dividend would be payable in exactly one year and would be equal to 6 pence per share.

Thereafter, dividends will grow at a constant rate of 1% per annum and are assumed to be paid in perpetuity. All dividends will be taxed at a rate of 20%. The investor requires a net rate of return from the shares of 6% per annum effective.

(ii) Derive and simplify as far as possible a general formula which will allow you to determine the value of a share for different values of:

- the next expected dividend.
- the dividend growth rate.
- the required rate of return.
- the tax rate.

(iii) Calculate the value of one share to the investor. [5]

The company announces some news that makes the shares more risky.

(iv) Explain what would happen to the value of the share, using the formula derived in part (ii). [2]

The investor bought 1,000 shares on 1 August 2014 for the price calculated in part (iii). He received the dividend of 6 pence on 1 August 2015 and paid the tax due on the dividend. The investor then sold the share immediately for 120 pence. Capital gains tax was charged on all gains of at a rate of 25%. On 1 August 2014, the index of retail prices was 123. On 1 August 2015, the index of retail prices was 126.

(v) Determine the net real return earned by the investor. [3]
[Total 14]

9 A student has inherited £1m and is considering investing the money in two projects, A and B.

Project A requires the investment of the whole sum in properties that are to be let out to tenants. The details are:

- The student expects to receive an income from rents at an annual rate of £60,000 a year for four years after an initial period of one year in which no income will be received.
- Rents are expected to rise thereafter at the start of each year at a rate of 0.5% per annum.
- The income will be received monthly in advance.
- The project involves costs of £10,000 per annum in the first year, rising at a constant rate of 0.5% per annum.
- The costs will be incurred at the beginning of each year.
- At the end of 20 years, the student expects to be able to sell the properties for £2m after which there will be no further revenue or costs.

Project B involves the investment of the whole sum in an investment fund.

- The fund is expected to pay an income of £60,000 per annum annually in advance and return the whole invested sum at the end of 20 years.

.

(i) (a) Calculate the payback period for project B.

(b) Show, by general reasoning or otherwise, that the payback period from project A is longer than that from project B. [5]

(ii) (a) Define the discounted payback period.

(b) Determine the discounted payback period from project B at a rate of interest of 1% per annum effective.

(c) Show, by general reasoning or otherwise, that the discounted payback period from project A is longer than that from project B. [5]

(iii) Determine the internal rate of return from project B expressed as an annual effective return. [3]

(iv) Show that the internal rate of return from project A is higher than that from project B. [10]

(v) Discuss which project is the better project given your answers to parts (i)–(iv) above. [3]

[Total 26]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2015

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners
December 2015

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

2. Student performance was poorer than in previous years. The performance across all questions was of a lower standard which indicated that the lower performance was not as the result of some particularly difficult individual questions.

3. There were some ambiguities in the wording and mark scheme for Q8 but the marking process was adjusted to ensure that candidates were not disadvantaged.

4. There were also elements of non-numerical explanation and analysis required in several questions and, as in previous papers, students performed relatively badly on these sections.

5. Finally, it appeared that many students left themselves short of time for the last question, Q9, which was worth 26 marks.

C. Comparative pass rates for the past 3 years for this diet of examination

Year	*%*
September 2015	44
April 2015	55
September 2014	57
April 2014	60
September 2013	57
April 2013	60

Reasons for any significant change in pass rates in current diet to those in the past:

Historically, the papers have been set by experienced examiners and this has led to very stable pass marks and relatively stable pass rates. This paper was set by the same examining team.

As highlighted in Section B, the performance by candidates on this paper has been significantly poorer than for past exams in this subject. The examiners do not believe that the paper was significantly different this session and any potential ambiguities in the paper were fully allowed for within the marking process.

We have no definite cause for the lower marks. An analysis on a question by question basis shows that candidates on average were only scoring above 60% on average on two questions out of nine, one of which was the 3-mark Q1. This all suggests a relatively weak cohort.

Solutions

Q1 $P = 100(1.03)^{-91/365} = 0.99266$ or £99.266

Let annual simple rate of discount = d

$$\frac{99.266}{100} = 1 - \frac{91}{365}d$$

Therefore $\frac{91}{365}d = 0.00734$; $d = 0.02945$

Q2 (i) (a) $\delta = \ln(1+i) = \ln\left[\left(1-\frac{d^{(12)}}{12}\right)^{-12}\right] = \ln\left[\left(1-\frac{0.055}{12}\right)^{-12}\right] = 0.055126$

$= 5.513\%$

(b) $1+i = \left(1-\frac{d^{(12)}}{12}\right)^{-12} \Rightarrow i = \left(1-\frac{0.055}{12}\right)^{-12} - 1 = 0.056674 = 5.667\%$

(c) $1+\frac{i^{(12)}}{12} = \left(1-\frac{d^{(12)}}{12}\right)^{-1} \Rightarrow i^{(12)} = 12\left[\left(1-\frac{0.055}{12}\right)^{-1} - 1\right] = 0.055253$

$= 5.525\%$

(ii) When interest is paid monthly, the interest that is paid in earlier months itself earns interest. This means that to achieve the same effective rate over the year, the nominal rate must be lower.

(iii) $100 = 159(1-d)^8 \Rightarrow d = 1 - \left(\frac{100}{159}\right)^{1/8} = 0.056319 = 5.632\%$

(iv) $1 + i = 1.12^{1/2} \Rightarrow i = 0.058301 = 5.830\%$

Parts (i) and (iii) were generally calculated well although many candidates chose not to give their final answer to the requested accuracy. Some candidates attempted to use their answers to parts (i)((a) and (i)(b) to find an answer to part (i)(c) – this was not accepted by the examiners . Part (iv) proved more challenging to many candidates. Amongst the marginal candidates, there were very few who gave a clear explanation for part (ii). It is a matter of concern that so many candidates were unable to articulate the relationship between nominal and effective interest rates.

Q3 (i) Duration of the annuity payment is $\frac{(Ia)_{\overline{12}|}}{a_{\overline{12}|}} = \frac{56.6328}{9.3851} =$ 6.0343 years

(ii) Duration of bond is:

$$\frac{5(Ia)_{\overline{8}|} + 800v^8}{5a_{\overline{8}|} + 100v^8}$$

$$= \frac{5 \times 28.9133 + 800 \times 0.73069}{5 \times 6.7327 + 100 \times 0.73069}$$

$$= \frac{729.119}{106.733} = 6.8313 \text{ years}$$

(iii) The duration of the assets (the bond) is greater than the duration of the liabilities (pension payments). If there is a rise in interest rates, the present value of the assets will fall by more than the present value of the liabilities and the insurance company will make a loss.

Parts (i) and (ii) were answered well. Where a term is calculated, it is particularly important to include the units in the final answer. Part (iii) was very poorly answered with many candidates stating that the company must make a loss because the durations were not equal.

Q4 (i) Assuming no arbitrage

The present value of the dividends (in £), I, is:

$I = 0.5(e^{-0.05\times(1/12)} + e^{-0.05\times(7/12)}) = 0.5 \times (0.995842 + 0.971255) = 0.98355$

Hence, forward price $F = (10 - 0.98355)e^{0.05(9/12)} = £9.3610$

(ii) The expected price of the share does not have to be taken into account because, using the no-arbitrage assumption, the purchaser of the forward is simply able to use the current price of the share (and the value of the dividends) given that the forward is simply an alternative way of exposing the investor to the same set of cash flows.

[The expected future price of the share will be taken into account by investors when determining the price they wish to pay for the share and therefore the current share price.]

> Part (i) was often answered well although some candidates miscalculated the timing of the dividends and the statement of the arbitrage assumption was often missed. Part (ii) was poorly answered despite being similar to previous exam questions.

Q5 (i) Present value is

$$\int_3^6 \rho(t)v(t)dt = \int_3^6 5000v(t)dt$$

For $t \geq 3$

$$v(t) = \exp\left(-\int_0^t \delta(t)\,dt\right) = \exp\left(-\int_0^3 0.03 + 0.005t\,dt - \int_3^t 0.005\,dt\right)$$

$$= \exp\left[-0.03t - 0.0025t^2\right]_0^3 \exp\left[-0.005t\right]_3^t$$

$$= \exp\left[-0.1125\right]\exp\left[0.015 - 0.005t\right]$$

$$= \exp(-0.005t - 0.0975)$$

Hence present value is

$$\int_3^6 5,000\exp(-0.005t - 0.0975)dt$$

$$= 5,000e^{-0.0975}\int_3^6 e^{-0.005t}dt$$

$$= \frac{5,000e^{-0.0975}}{-0.005}\left[e^{-0.005t}\right]_3^6 = -1,000,000e^{-0.0975}(e^{-0.03} - e^{-0.015})$$

$$= -880.293.42 + 893,597.35 = £13,303.93$$

(ii) $5,000(\bar{a}_{\overline{6}|} - \bar{a}_{\overline{3}|}) = 13,303.93$

i = 2%: LHS = 13,723
i = 3%: LHS = 13,136

Interpolating

$$i \approx 0.02 + 0.01 \times \frac{13,304 - 13,723}{13,136 - 13,723} = 2.714\% \text{ say } 2.7\%$$

(iii) Accumulation =

$$= 300A(50) = 300\exp(0.005 \times 50 + 0.0975)$$

$$= 300e^{0.3475} = £424.66$$

The discount factor was usually calculated correctly although some candidates just calculated this factor for t = 6 and assumed that the value of a single payment at this time was required. Part (ii) was poorly answered. The important point is that the rate of interest is obtained by equating the amount initially invested as calculated in part (i) with the present value of the annuity.

Q6 (i) $101 = 3a_{\overline{3}|} + 100v^3$

i = 3%: RHS = 100
i = 2.5%: RHS = 101.428

Interpolating

$$i \approx 0.025 + 0.005 \times \frac{101 - 101.428}{100 - 101.428} = 2.65\%$$

(ii) Let i_n = spot rate for term n

One year bond gives

$$101 = 103v_{i_1}$$

$$v_{i_1} = \frac{101}{103} = 0.98058$$

$$\Rightarrow i_1 = \frac{103}{101} - 1 = 1.980\%$$

Two year bond gives

$$101 = 3v_{i_1} + 103v_{i_2}^2$$

$$\Rightarrow v_{i_2}^2 = \frac{101 - 3\frac{101}{103}}{103} = 0.95202$$

$$\Rightarrow i_2 = 2.489\%$$

Three year bond gives

$$101 = 3v_{i_1} + 3v_{i_2}^2 + 103v_{i_3}^3$$

$$\Rightarrow v_{i_3}^3 = \frac{101 - 3\times 0.98058 - 3\times 0.95202}{103} = 0.92429$$

$$\Rightarrow i_3 = 2.659\%$$

(iii) Forward rate is $f_{2,1}$ where

$$1 + f_{2,1} = \frac{(1+i_3)^3}{(1+i_2)^2} = \frac{1.02659^3}{1.02489^2} = 1.03000 \Rightarrow f_{2,1} = 3.000\%$$

(iv) Reasons could include:

Expectations theory suggests that if short-term interest rates are expected to rise then if yields are the same on both long- and short-term bonds, short-term bonds will be more attractive and longer term bonds less attractive and so the yields on short-term bonds will fall relative to those on long-term bonds.

[Expected higher inflation could be a reason for this but could be allowed as a distinct point]

Liquidity preference theory suggests that investors demand higher rates of return for less liquid/longer term-to-maturity investments which are more sensitive to interest rate movements.

Market segmentation with the supply of bonds being restricted at shorter terms or some factor that leads to the demand for bonds of longer terms to be lower

> Common errors on this question included assuming the price of each bond was 100 and that the gross redemption yield of the 3-year bond was equal to the three-year spot rate. In part (iv) many candidates just gave the names of theories of the yield curve without explaining how this applied in this particular scenario. Otherwise, this was the best answered question on the paper apart from Q1.

Q7 (i) Maximum rate of return after 20 years

$$100 = 0.75\left(7a_{\overline{20}|}^{(2)} - 3a_{\overline{10}|}^{(2)}\right) + 130v^{20}$$

Try i = 5%:

$$\text{RHS} = 0.75\left(\frac{7\times(1-1.05^{-20}) - 3\times(1-1.05^{-10})}{2(1.05^{1/2}-1)}\right) + 130\times 1.05^{-20}$$

$$= 48.6460 + 48.9956 = 97.6417$$

Try i = 4%:

$$\text{RHS} = 0.75\left(\frac{7\times(1-1.04^{-20}) - 3\times(1-1.04^{-10})}{2(1.04^{1/2}-1)}\right) + 130\times 1.04^{-20}$$

$$= 53.6255 + 59.3303 = 112.9558$$

Interpolating

$$i \approx 0.04 + 0.01\times\frac{100-112.9558}{97.6417-112.9558} = 4.846\% = 4.8\%$$

(Exact answer is 4.8338%.)

(ii) Minimum rate of return after 10 years

In this case, the investor invests 100, receives 100 back and receives a net income at a rate of 1.5 per half-year. The rate of return per half-year effective is therefore 1.5 per cent.

The annual effective rate of return is $1.015^2 - 1 = 3.0225\%$.

(iii) There is a 0.5 probability of both early redemption and of late redemption. The expected return is therefore $0.5(4.8338\% + 3.0225\%) \approx 3.928\%$

(iv) If the investor buys the whole loan, the present value of the cash flows from the loan is as follows (per €100 nominal):

$$= 0.75 \times 4a_{\overline{10}|}^{(2)} + 0.5 \times 100v^{10} + 0.5\left(0.75 \times 7a_{\overline{10}|}^{(2)}v^{10} + 130v^{20}\right)$$

At i = 3.928% this is

$$= 3\frac{1-1.03928^{-10}}{2(1.03928^{1/2}-1)} + 50 \times 1.03928^{-10}$$

$$+0.5\left(5.25\frac{1-1.03928^{-10}}{2(1.03928^{1/2}-1)}1.03928^{-10} + 130 \times 1.03928^{-20}\right)$$

$$= 24.6575 + 34.0125 + 0.5(29.3538 + 60.1562)$$

$$= 103.4264$$

This is greater than 100 and so the rate of return will be greater than 3.928% (exact return is 4.212%).

This was the worst answered question on the paper with many candidates not recognising that the cases where the bond is redeemed after 10 years and after 20 years have to be calculated separately for parts (i) and (ii). If candidates obtained answers for parts (i) and (ii) then part (iii) was usually done well. However, few candidates recognised that substituting the return from part (iii) into the required equation for part (iv) would lead to the required answer.

The question did not state specifically that the coupons in the second 10 years were semi-annual although most students assumed this. Candidates who assumed a different coupon frequency were given full credit.

Q8 (i)

- Generally issued by commercial undertakings and other bodies.
- Shares are held by the owners of a company who receive a share in the company's profits in the form of dividends
- Potential for high returns relative to other asset classes…
- …but high risk particularly risk of capital losses
- Dividends are not fixed or known in advance and…
- …the proportion of profits paid out as dividends will vary from time-to-time
- No fixed redemption date
- Lowest ranking finance issued by companies.
- Return made up of income return and capital gains.

- Initial running yield low but has potential to increase with dividend growth…
- …in line with inflation and real growth in company earnings.
- Marketability depends on the size of the issue.
- Ordinary shareholders receive voting rights in proportion to their holding.

(ii) Let

P = price investor is willing to pay
d = next expected dividend
g = expected annual dividend growth rate
r = annual required return
t = tax rate

Then

$$P = \frac{d(1-t)}{1+r} + \frac{d(1-t)(1+g)}{(1+r)^2} + \frac{d(1-t)(1+g)^2}{(1+r)^3} + \dots$$

$$= \frac{d(1-t)}{1+r}\left[1 + \frac{1+g}{1+r} + \left(\frac{1+g}{1+r}\right)^2 + \left(\frac{1+g}{1+r}\right)^3 + \dots\right]$$

$$= \frac{d(1-t)}{1+r}\frac{1}{1-\frac{1+g}{1+r}} = \frac{d(1-t)}{r-g}$$

(iii) $d = 6p$
$g = 0.01$
$r = 0.06$
$t = 0.2$

$$P = \frac{d(1-t)}{r-g} = \frac{6(1-0.2)}{0.06-0.01} = 96p$$

(iv) If the share were regarded as more risky, then the required return, *r*, would increase. If *r* were to increase, this would reduce the value of the share as *r* is in the denominator (and is positive).

(v) Equation of value would be (working in money terms):

$$960 = 0.8 \times 60v + 1200v - 0.25(1200 - 960)v$$

$$\Rightarrow v = \frac{96}{118.8}$$

Therefore net money rate of return, *i*, is $\frac{118.8}{96} - 1 = 23.75\%$

Net real rate of return is $\frac{1.2375}{126/123} - 1 = 20.80\%$

> There was an error in the paragraph prior to part (v) of this question where the calculation was designed to be based on the purchase/sale of 1,000 shares but the question referred to the sale of "the share". Nearly all candidates based their calculation on the purchase/sale of the same number of shares (whether it be 1 share or 1,000) but candidates who made a different assumption were not penalised.
>
> There was no split of the marks between parts (ii) and part (iii) given on the paper. Candidates who just performed the calculation without the derivation of the formula were give appropriate credit but a formula derivation was required to obtain the full five marks for these parts. The question did not state specifically that the dividends were paid annually although almost all candidates assumed this. Candidates who assumed other payment frequencies were given full credit.

Q9 (i) (a) The payback period is the first time at which the total incoming cash flows are equal or greater in amount than the total outgoing cash flows.

Total incoming cash flows at the beginning of year t = 60,000t.

Determine t for which $60{,}000t \geq 1{,}000{,}000 \Rightarrow t \geq 16.67$

Therefore the payback period is 16 years.

(b) The net total income received in any year from project A is never greater than £60,000. As the costs are incurred at the beginning of the year, there is no point at which the total income from project A is greater than the total income from project B until the very end of the project when the properties are sold. The payback period for B must therefore be less than that for A.

(ii) (a) The discounted payback period occurs where the present value (or accumulated value) of incoming cash flows is equal to or greater than that of outgoing cash flows for the first time.

(b) Equation of value for project B is (in £000):

$$60\ddot{a}_{\overline{t|}} \geq 1,000$$

$$\Rightarrow a_{\overline{t|}} \geq \frac{1,000}{60(1+i)} = \frac{1,000}{60.6} = 16.5016$$

Need to solve for t. From inspection of tables, t = 19 and so the discounted payback period is 18 years.

(c) Again, given that the net income from project A is never greater in an individual year, than that from project B, at no rate of interest can the discounted value of the net income from project A be greater than that for the income from project B.

(iii) Internal rate of return from project B is the solution to the following equation of value (all figures in 000s):

$$1,000 = 60\ddot{a}_{\overline{20|}} + 1,000v^{20}$$

This can be solved by general reasoning.

As the investor invests 1,000 and receives an annual income of 60 in advance and receives his capital back at the end, the total rate of return, d, expressed as an effective rate of discount per annum is 6 per cent.

Internal rate of return is $i = \frac{d}{1-d} = \frac{0.06}{0.94} = 6.383\%$

(iv) If the IRR from project A is higher then it must have a net present value > zero at a rate of interest of 6.383 per cent.

Note that v = 0.94 at i = 6.383%

Present value of costs for project A:

$$= 1,000 + 10(1 + 1.005v + 1.005^2 v^2 + \ldots + 1.005^{19} v^{19})$$

$$= 1,000 + 10\left(\frac{1 - 1.005^{20} v^{20}}{1 - 1.005v}\right)$$

$$= 1,000 + 10\left(\frac{1 - (1.005 \times 0.94)^{20}}{1 - 1.005 \times 0.94}\right)$$

$$=1,000+10\frac{0.67946}{0.0553}=1,122.869$$

Present value of revenue for project A:

$$= 2,000v^{20}+60\ddot{a}^{(12)}_{\overline{4}|}v+60\ddot{a}^{(12)}_{\overline{1}|}v^5(1.005+1.005^2v+\ldots+1.005^{15}v^{14})$$

$$= 2,000v^{20}+60\ddot{a}^{(12)}_{\overline{4}|}v+60\ddot{a}^{(12)}_{\overline{1}|}1.005v^5(1+1.005v+\ldots+1.005^{14}v^{14})$$

$$= 2,000v^{20}+60\ddot{a}^{(12)}_{\overline{4}|}v+60\ddot{a}^{(12)}_{\overline{1}|}1.005v^5\left(\frac{1-1.005^{15}v^{15}}{1-1.005\ v}\right)$$

$$= 2,000\times 0.94^{20}+60\times 0.94\left(\frac{1-0.94^4}{12(1-(1-0.06)^{1/12})}\right)$$

$$+60\times 1.005\times 0.94^5\left(\frac{1-0.94}{12(1-(1-0.06)^{1/12})}\right)\left(\frac{1-(1.005\times 0.94)^{15}}{1-1.005\times 0.94}\right)$$

$$=580.212+200.365+446.577=1,227.154$$

NPV of project at IRR from project A is: 1,227.154 – 1,122.869 = 104.285 (= £104,285)

This is clearly positive so project A has a higher IRR.

(v) Project B would be preferred on the basis of both payback period and discounted payback period.

However, both these measures have shortcomings. The first does not take into account interest at all and the second does not take into account cash flows after the discounted payback period [or in the case of project A the occurrence of one large cash flow at the time of the discounted payback period]

Project A would be preferred on the basis of internal rate of return.

The internal rate of return measures the total return on the project and therefore is a better decision criterion than payback period or discounted payback period.

There may be other factors (comparison of NPVs at a particular rate of interest or the risk of the two projects) that should be taken into account.

Other factors could include, for example:

student's need for return of original investment
reliability of estimates of future cashflows

It appeared that many candidates were under time pressure when attempting this question. Nearly all candidates failed to recognise how the payments being at the start of the year for project B would impact on the payback period and the discounted payback period. Many candidates' general reasoning arguments for parts (i)(b) and (ii)(c) were unclear. In part (c), a common mistake was to miss out the return of the original investment in the calculation of the IRR. In part (iv), common errors were to miscount the number of terms (for both costs and revenue). As for similar long questions in previous years, marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity. There were a wide range of points that could be made to score marks in part (v) but few candidates scored well on this part.

END OF EXAMINERS' REPORT