Term test 2008

Problem 1.

Prove that in any non-isosceles triangle ABC the three points of intersection of the bisectors of its external angles with the opposite sides belong to one line.

Hint: If P is the point of intersection of the bisector of the external angle A with the extension of the side BC, then PC:PB= AC:AB. Prove it (using similar arguments to our proof of a similar statement for the interior bisector).

Problem 2.

1) Prove Radon's Theorem in $\mathbb{R}^d$: Assume that we have a collection of points $A \subset \mathbb{R}^d$ such that their number is $d+2$. Then there is a subset $B \subset A$ such that the convex hull of (B) and convex hull of $(A \setminus B)$ have a non empty intersection.

2) Prove Radon's theorem for $d = 2$.

Problem 3.

Take a couple of parallel segments AD and BC of lengths a and b respectively, where $a > b$. Consider a trapezoid ABCD. Let P be the point of intersection of the lines containing the sides AB and DC. Let Q be the point of intersection of the diagonals of the trapezoid. Prove that the line PQ intersects the sides BC and AD at their midpoints.

Hint: put masses at the points A, P and D in such a way that the center of masses would be located at the point Q.

Problem 4.

Consider a triangle ABC. Assume that angles at the vertexes A, B are smaller than 45 degrees. Take any point P inside the triangle. Find points $C' \in CB$, $B' \in BA$ and $A' \in AC$ for which the sum $PC' + C'B' + B'A' + A'P'$ will be the smallest.

Problem 5.

Take a regular triangle inscribed into a circle. Describe the image of the triangle (including its interior) under inversion with respect to the circle.

Term test 2009

Problem 1.

Consider a regular triangle ABC. Find all points O in the triangle for which the sum $2O_{AB} + 2O_{BC} + O_{CA}$ is the biggest possible. Here O_{AB}, O_{BC} and O_{CA} are distances from point O to the sides AB, BC and CA respectively.

Problem 2.

Consider a tetrahedron in $\mathbb{R}^3$. Mark a point at the middle of each side of the tetrahedron. Join by a segment the marked points belonging to the opposite sides. Prove that the three segments in $\mathbb{R}^3$ we constructed pass through one point. In what proportion the intersection point divides each segment?

Hint: put appropriate masses at the vertices of the tetrahedron and use uniqueness of the center of masses.

Problem 3. Consider plane $\mathbb{R}^2$ as complex plane. Consider a transformation $z \to \frac{z+1}{z-1}$. Which circles under this transformation will become lines?

Problem 4.

Take two circles S_1 and S_2 intersecting at points A and B. Consider all circles S orthogonal to S_1 and to S_2. Find the locus of centers of all such circles S.

Hint: How the points A and B are located with respect to a circle S?

Problem 5.

Prove the Menelaus's theorem:

Take three lines l_1, l_2, l_3 and consider three points L, M, N on them: $L \in l_1$, $M \in l_2$, $N \in l_3$. Assume that A, B, C are points of intersections of these lines: $A = l_1 \cap l_2$, $B = l_2 \cap l_3$, and $C = l_3 \cap l_1$.

Points L, M, N belong to one line, if and only if

$$\frac{AL}{CL} \cdot \frac{BM}{AM} \cdot \frac{CN}{BN} = 1.$$

Term test 2010

Problem 1.

Consider a triangle ABC. Assume that $\cos\alpha = \cos\beta = 1/4$, where α and β are angles at A and B. Find all points O in the triangle for which the sum $2O_{AB}+2O_{BC}+O_{CA}$ is the biggest possible. Here O_{AB}, O_{BC} and O_{CA} are distances from point O to the sides AB, BC and CA respectively.

Problem 2.

Consider triangle ABC. Let D be a point on the side AB such that $AD : DB = 10$. Let E be a point on the segment CD such that $CE : ED = 11$. Let F be the point of intersection of the line L passing through AE and the side CB. Find $CF : FB$.

Hint: put appropriate masses at the vertices of the triangle in such a way that the point E becomes the center of masses.

Problem 3.

Let T_1 and T_2 be the inversions in the circles $x^2+y^2 = 16$ and $(x-8)^2+y^2 = 1$. Consider the composition W of these inversions: $W = T_2 \circ T_1$. Which lines under the transformation W will become lines?

Problem 4.

Consider a simple convex polyhedron Δ in R^3 with 2010 edges.

1) How many vertices are there in Δ?

2) How many faces are there in Δ?

Hint: Try do 1) by hands and then use the Euler characteristic formula for 2).

Problem 5.

Prove the following theorem:

Let L be a line tangent at the point A to an ellipse with foci O_1 and O_2. Then the rays AO_1 and AO_2 make equal angles with the line L.

Term test 2011

Problem 1.
Consider a tetrahedron $ABCD$. Let E be the middle of the segment joining the vertex D and the point of intersection of the medians in the triangle ABC. In what proportion the plane containing the point E and the edge AB will cut the edge CD?

Problem 2.
Consider an angle $AOB = 2\pi/n$ as a billiard (with two infinite sides). Take billiard trajectory such that its first piece is a segment going inside the angle parallel to the side AO towards the side OB and intersecting it at the point $C \in OB$. Find the shortest distance from the trajectory to the point O assuming that the length of the segment OC is c.

Problem 3.
Take a prism having a convex planar 2011-gon as a base. Find F-polynomial and H-polynomial for this polyhedron. How many of its vertices have index one with respect to a linear function (not equal to a constant on each edge of the prism)?

Problem 4.
Is there an inversion mapping the points $(2, 0)$; $(-2, 0)$; $(0, 2)$; $(0, -1)$ into vertices of a square? (Show an example of such an inversion or prove that it does not exist).

Problem 5.
Prove the separation theorem: Let P be a point located outside of a compact convex set Δ. Then there exists a hyperplane that separates them, i.e. a hyperplane with the property that the point P and the set Δ lie on different sides of it.

Term test 2012

Problem 1. Every vertex of a triangle was connected by two lines to the points that divide the opposite side to three equal parts. Prove that in the hexagon these six lines form, the lines connecting opposite vertices are concurrent.

Problem 2.
Consider an angle $AOB = 30^o$ as a billiard (with two infinite sides). Take billiard trajectory such that its first piece is a segment going inside the angle parallel to the side AO towards the side OB and intersecting it at the point $C \in OB$. Find the shortest distance from the trajectory to the point O assuming that the length of the segment OC is c.

Problem 3.
Assume that Δ is a simple convex polyhedron in R^3 with 2012 vertices.
1) How many edges are there in Δ?
2) How many faces are there in Δ?
Present an example of such convex polyhedron in R^3.

Problem 4.
Describe the image under inversion of the family of circles passing through the points $(0, 1)$ and $(2, 0)$ after inversion in the unit circle centered at the origin.

Problem 5.
Prove the following theorem: Let R be a point located outside of an ellipse S with foci A and B. Let l_1, l_2 be the lines passing through the point R and tangent

to the ellipse S at the points A' and B'. Then the angles $A'RA$ and $B'RB$ are equal.

Term test 2013

Problem 1.

Consider triangle ABC. Let D be the point on the side AB such that $AD : DB = 10$. Let E be the point on the segment CD such that $CE : ED = 11$. Let F be the point of intersection of the line L passing through A and E and the side CB. Find $CF : FB$.

Problem 2.

Let S be a circle of radius $R = 5$ centered at $O = (0, 0)$. Consider the function $F(C) = |AC - CB|$ on the circle S (i.e. $C \in S$), where $A = (6, 1)$ and $B = (7, 0)$. Find M satisfying

$$M = \max_{C \in S} F(C).$$

Find all points $C \in S$ such that $F(C) = M$.

Problem 3. For three given lines in the plane find the point O such that after an inversion centered at O these three lines become three equal circles.

Problem 4.

Consider a simple convex polyhedron Δ in R^3 with 2013 faces.

1) How many vertices are there in Δ?

2) How many of the vertices have index one with respect to a linear function (not equal to a constant on each edge of Δ)?

Problem 5.

Prove the following theorem:

Let R be a point located outside of an ellipse S with foci A and B. Let l_1, l_2 be the lines passing through the point R and tangent to the ellipse S at the points A' and B'. Then the angles $A'RA$ and $B'RB$ are equal.