STAT2032/6046 - Solutions to Final Examination - First Semester 2013

1 Question 1 (6 marks)

Deposits of \$1000 are made into an investment fund at $t = 0$ and $t = 1$. The fund balance is \$1200 just before the deposit at $t = 1$, and \$2200 at $t = 2$.

(a) [2 marks] Calculate the money-weighted rate of return (MWRR) over the two year period. Express your answer as an effective rate per annum.

$$1000(1 + i_{MWRR})^2 + 1000(1 + i_{MWRR}) = 2200$$

$$(1 + i_{MWRR})^2 + (1 + i_{MWRR}) = 2.2$$

$$1 + i_{MWRR} = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times (-2.2)}}{2} = 1.06524 \text{ taking positive square root}$$

$$i_{MWRR} = 6.524\%$$

- (1 mark) - correct equation
- (1 mark) - final answer

(b) [2 marks] Calculate the time-weighted rate of return (TWRR) over the two year period. Express your answer as an effective rate per annum.

$$(1 + i_{TWRR})^2 = \frac{1200}{1000} \times \frac{2200}{2200} = 1.2$$

$$i_{TWRR} = \sqrt{1.2} - 1 = 9.5445\%$$

- (1 mark) - Growth factor period 1 & 2
- (1 mark) - final answer

(c) [2 marks] Compare your answers in parts (a) and (b). Is your MWRR greater than, less than, or approximately equal to your TWRR? Provide a written explanation for the relationship between the MWRR and TWRR values you calculated.

The MWRR is 3% less than the TWRR. Growth due to investment earnings was 0% in the second year. However, the MWRR applies equal weight to the deposits at $t = 0$ and $t = 1$. Hence, the MWRR underestimates the growth due to investment earnings only.

- (1 mark) - MWRR < TWRR
- (1 mark) - Explanation

2 Question 2 (14 marks)

(a) A private equity company is considering investing in a motorway project. The expenses consist of

EXPENSE AMOUNT ($m)	ITEM	TIME EXPENSE IS INCURRED
20	Initial capital expenditure	Now
3 (per annum)	Operations	Continuously
3.5	Motorway upgrade	5 years from now

Motorway revenue is estimated to be $4m per annum received continuously. After the first year, the revenue per annum increases by 5% per annum at the beginning of each year.

After ten years, the company plans to sell its investment in the motorway. The estimated sale price is $30m.

Assume the company can borrow and lend money at an effective rate of 10% per annum

(i) [6 marks] Calculate the Net Present Value (NPV) of the motorway investment for the private equity company.

- PV(initial cap. exp) = $-\$20$
- PV(operating exp) = $-3\bar{a}_{\overline{10}|} = -\$19.3407$
- PV(capital upgrade) = $-3.5v_{0.10}^{5} = -\$2.1732$
- PV(revenue)= $4\bar{a}_{\overline{1}|}(1 + (1.05)v + (1.05)^2v^2 + ... + (1.05)^9v^9) = \$31.2236$
- PV(sale price)=$30v^{10} = \$11.5663$

$$NPV = -20 - 19.3407 - 2.1732 + 31.2236 + 11.5663 = \$1.2759\text{m}$$

- (1 mark) each PV(operating exp.), PV (capital upgrade), PV(sale price)
- (2 marks) PV(revenue)
- (1 mark) NPV answer

(a) (ii) [4 marks] Write down the equation(s) you would solve to calculate the discounted payback period in years, of the motorway investment project for the private equity company.

(Note: there may be more than one equation).

Let t be discounted payback period in whole years.

(i) If $t < 5$, solve for t such that:

$$-20s_{\overline{t}|} - 3\bar{s}_{\overline{t}|} + 4\bar{s}_{\overline{1}|}\sum_{k=0}^{t-1}(1.05)^{t-(k+1)}(1.10)^k \geq 0$$

(ii) If $5 \leq t < 10$, solve for t such that:

$$-20s_{\overline{t}|} - 3\bar{s}_{\overline{t}|} - 3.5(1.10)^{t-5} + 4\bar{s}_{\overline{1}|}\sum_{k=0}^{t-1}(1.05)^{t-(k+1)}(1.10)^k \geq 0$$

(iii) Otherwise the discounted payback period = 10 years

- (1 mark) recognize equation to solve depends on value of t.
- (1 mark) equation for $t < 5$
- (1 mark) equation for $5 \leq t < 10$
- (1 mark) state that discounted payback period may be equal to ten years

(b) [4 marks] Lucy borrows \$400,000 to start a business. Her expected costs over three years are \$35,000 per month. The expected revenue is \$36,000 per month over three years. After three years, the business is sold for \$450,000.

Find the effective annual internal rate of return of Lucy's business to the nearest 0.5%. You can assume monthly expenses and revenue are paid or received at the end of each month.

Note: You may find the following values useful
$a_{\overline{3}|5\%} = 2.7232; a_{\overline{3}|6\%} = 2.6730; a_{\overline{3}|7\%} = 2.6243; a_{\overline{3}|8\%} = 2.5771$
$a_{\overline{3}|9\%} = 2.5313; a_{\overline{3}|10\%} = 2.4869$

Net cashflow per month is +\$1000. Let j be the monthly effective yield. Let i be the annual effective yield. Working in units of '000s.

$$-400 + a_{\overline{36}|j} + 450v_j^{36} = 0$$

$$-400 + 12\frac{i}{i^{(12)}}a_{\overline{3}|} + 450v^3 = 0$$

The exact solution is $i = 6.9836\%$. To the nearest 0.5%, the IRR = 7.00%.

- (1 mark) equation of value to solve
- (2 marks) interpolation
- (1 mark) Final answer, effective rate per annum, rounded to nearest 0.5%

3 Question 3 (20 marks)

(a) John purchases a \$1000 face value 10-year bond with coupons of 8% per annum half-yearly. The bond will be redeemed at C. The purchase price is \$800 and the present value of the redemption amount C is \$301.51.

(i) [1 mark] Is the bond sold at premium, par or discount? Bond is sold at a discount ($P < F$)

(ii) [1 mark] Given your answer to part (a)(i), provide a bound for the effective half-yearly yield j on the bond purchase.

$j > 4\%$.

(iii) [3 marks] Calculate the redemption amount C.

$$P = Fra_{\overline{n}|j} + Cv^n$$

$$800 = 40a_{\overline{20}|j} + 301.51$$

$$a_{\overline{20}|j} = 12.46225$$

$$j = 5\%$$

$$C = 301.51 \times (1.05)^{20}$$

$$C = 800$$

.

- (1 mark) Bond pricing equation
- (1 mark) $j = 5\%$
- (1 mark) Final answer

(b) (i) [5 marks] Tony is considering to purchase a \$1000 face value callable bond. The bond pays coupons of 10% per annum half-yearly and matures at the end of 10 years. The bond is callable at the following times for the following redemption amounts:

- **C=1050 ; ends of years 4 - 6.**
- **C=900 ; ends of years 8 - 10.**

Find the maximum price that Tony can offer and still be certain of a yield rate of 10% per annum convertible half-yearly.

$$F = 1000, r = 0.05, i = 0.05$$

$$C = 1050, n = 8, 10, 12 \rightarrow P = (1000)(0.05)a_{\overline{n|}} + 1050v^n$$

$$C = 900, n = 16, 18, 20 \rightarrow P = (1000)(0.05)a_{\overline{n|}} + 900v^n$$

$g_1 = \frac{1000\times0.05}{1050} = 4.76\%$; $g_2 = \frac{1000\times0.05}{900} = 5.55\%$

The bond is selling at a discount in interval 1 $\rightarrow$ price the bond at the latest date possible.

$$P = 1027.84 \ n = 12 \text{ (during the first interval)}$$

The bond is selling at a premium in interval 2 $\rightarrow$ price the bond at the earliest date possible

$$P = 954.19 \ n = 16 \text{ (during the second interval)}$$

Tony should offer the minimum price = \$954.19

- (1 mark) modified coupon rates
- (1 mark) recognize bond is selling at a discount interval 1, premium interval 2
- (1 mark each) Price in each interval evaluated at appropriate redemption date
- (1 mark) Final answer

(b) (ii) [4 marks]] Using the price you calculated in part (b)(i), find the yield to Tony if the bond is redeemed at the end of year 4. Explain in words why your answer is greater than the minimum required yield rate of 10% per annum convertible half-yearly.

Solve for i such that

$$954.19 = 1000(0.05)a_{\overline{8}|i} + 1050v_i^8$$

The exact solution is 6.2456% which exceeds the minimum yield.

The price offered is \$954.19. This is less than the breakeven price if the bond is redeemed at the end of year 4 to yield 10% per annum convertiable semiannually. Tony has paid less than the breakeven price, therefore his yield must be greater than the minimum required yield.

- (2 marks) interpolation
- (1 mark) final answer
- (1 mark) written explanation

(c)[6 marks] Simon purchases a $1000 face value 10-year bond with 8% per annum half-yearly coupons. He can reinvest coupons at a nominal rate of 6% per annum convertible half-yearly. The bond is redeemed at face value.

Coupon payments are taxed at 20%. Capital gains are taxed at 30%. There is no additional tax on reinvestment proceeds.

Find the price Simon should pay for the bond to earn a yield of 8% per annum convertible half-yearly.

Value of reinvested coupon payments after ten years

$$1000(0.04)(0.8)s_{\overline{20}|0.03} = \$859.85$$

Price before capital gains tax

$$P = 859.85v_i^{10} + 1000v_i^{10} \text{ where } i = 1.04^2 - 1$$

$$P = \$848.81216$$

$P < C$, therefore there is a capital gain. Calculate new price P' such that

$$P' = P - 0.3(1000 - P')v_i^{10}$$

$$P' = \frac{848.81216 - 0.3 \times 1000 \times v_i^{10}}{1 - 0.3v_i^{10}}$$

$$P' = \$824.83$$

- (1 mark each) Accumulated value reinvested proceeds
- (2 mark) Correct price for P
- (1 mark) Recognize capital gain
- (1 mark) Equation to calculate P'
- (1 mark) Final answer

4 Question 4 (18 marks)

(a) (i) [3 marks] Stock A pays dividends at a continuous rate q per annum. Use the replicating portfolio method and no-arbitrage principle to show that the forward price K on a unit of Stock A, for a forward contract with maturity date T is

$$\mathbf{K = S_0 e^{(\delta - q)T}}$$

where S_0 is the current stock price, and δ is the force of interest per annum.

HINT: Consider two portfolios I and II. Let Portfolio I consist of long one forward contract on Stock A with forward price K, plus $Ke^{-\delta T}$ units of cash. Let Portfolio II consist of e^{-qT} units of stock. Assume all dividends are reinvested immediately to purchase more stock.

The values of the portfolios at $t = 0$ and $t = T$ are given in the table below:

Portfolio	t=0	t=T
I	$Ke^{-\delta T}$	$S_T - K + Ke^{-\delta T}e^{\delta T} = S_T$
II	$S_0 e^{-qT}$	$S_T e^{-qT} e^{+qT} = S_T$

I and II have identical payoffs at t=T. Therefore, by the no-arbitrage principle, they must have the same value at t=0. That is $Ke^{-\delta T} = S_0 e^{-qT}$. Rearranging, the forward price is

$$K = S_0 e^{(\delta - q)T} \text{ as required}$$

- (1 mark each) Payoff at T for portfolio I and II
- (1 mark) Equate portfolio values at t=0

(a) (ii) [2 marks] Provide a written interpretation for the forward price formula $\mathbf{K = S_0 e^{(\delta - q)T}}$.

The forward price is the current stock price S_0 accumulated to maturity date T at the risk free rate of interest. However, S_0 includes the value of dividends received between now and T. These dividends are not available to the holder of the long position. Hence, the forward price is reduced for dividends not received during the life of the forward contract, but included in the price S_0.

- (1 mark) S_0 includes present value of future dividends
- (1 mark) dividends not available to long position of forward contract, therefore need to reduce price

(a) (iii) [2 marks] Given the forward price on a unit of Stock A is $K = S_0 e^{(\delta - q)T}$, what are the implications if $q > \delta$? Provide a written explanation.

If $q > \delta$, the $K < S_0$. That is, the forward price is less than the current stock price. Investors earn a higher rate of return by receiving dividends on Stock A than investing in cash, thus, investors prefer to hold stock over cash, and $S_0 > K$.

- (1 mark) $S_0 > K$
- (1 mark) dividend yield is higher than return on cash, investors prefer to hold stock over cash

(b) (i) [1 mark] On 30 June 2013, Kim enters into a forward contract to sell 5000 units of Stock A at 30 June 2018. The current share price is \$5.50. Dividends are paid out at a continuous rate of 2% per annum.

If the risk-free rate of interest is 5% per annum effective, calculate the forward price of the contract, using the formula in part (a)(i).

$$K = S_0 e^{(\delta - q)T} = 5.50 e^{(\ln(1.05) - 0.02)5} = \$6.35$$

(b) (ii) [3 marks] At 30 June 2014, the price of the stock, dividend and interest rates are unchanged. What is the value of Kim's forward contract on 30 June 2014? Provide a written explanation for why the value is positive or negative.

$$K_r = 5.50e^{(\ln(1.05)-0.02)4} = \$6.17$$

$$\text{Value} = ((6.35 - 6.17)e^{-\ln(1.05)\times 4}) \times 5000 = \$740.43$$

The value is positive because the forward price has dropped as at 30 June 2014. The short position locked in a higher selling price on 30 June 2013.

- (1 mark) K_r
- (1 mark) Numerical answer for value
- (1 mark) Explanation

(b) (iii)[1 mark] Suppose the stock price on 30 June 2018 is \$6.50. Calculate the value of Kim's forward contract on 30 June 2018.

$$\text{Value} = 5000 \times (6.35 - 6.50) = -\$750$$

(c) (i) [3 marks] Current spot rates in the market are as follows:

Time to maturity (years)	Spot rate (% per annum effective)
1	7.00
2	8.00
3	8.75
4	9.25
5	9.50

The current two-year deferred three year forward rate is $f_{2,5}$=11%.

Is there an arbitrage opportunity in the current market? If not, explain why not. If yes, what is the size of the arbitrage profit?

The no-arbitrage 2-year deferred 3-year forward rate is

$$f_{2,5} = \left(\frac{(1+0.0950)^5}{(1+0.08)^2}\right)^{1/3} - 1 = 10.51\%$$

Yes, there is an arbitrage opportunity. If I borrow \$100 now at 10.51% for five years, I will need to repay back $100(1.095)^5 = \$157.423$ at $t = 5$.

At t=0, I can invest the \$100 I borrowed at the two year spot rate $s_2 = 8.00\%$, then invest for a further 3 years at $f_{2,5}$=11%. The accumulated value of my investment $= 100(1.08)^2(1.11)^3 = \$159.52$

For zero initial outlay, I am guaranteed a profit of \$2.10 at $t = 5$.

- (1 mark) Identify arbitrage opportunity
- (1 mark) Numerical answer for arbitrage profit
- (1 mark) Explanation to borrow at low rate, lend at higher rate.

(c) (ii) [3 marks] Using the spot rates in part (c)(i), calculate the accumulated value at t=5 of a decreasing annuity of term 5 years, where payments are made at the beginning of each year. The first payment is $25, and each subsequent payment is $5 smaller than the previous one.

$$\text{Accum. Val} = \left(25 + 20(1.07)^{-1} + 15(1.08)^{-2} + 10(1.0875)^{-3} + 5(1.0925)^{-4}\right)(1.095)^{5} = \$106.79$$

- (1 mark) Correct cashflows
- (1 mark) Application of correct spot rates
- (1 mark) Numerical answer

5 Question 5 (18 marks)

(a) [6 marks] An investor buys large quantities of a fixed interest security when she expects interest rates to fall, with the intention of selling in a short time to realise a profit. At present, she is choosing between two securities:

- **Security 1 pays coupons at 5% per annum payable annually in arrears and is redeemed at face value in 5 years time.**
- **Security 2 pays coupons at 11% per annum payable annually in arrears and is redeemed at face value in 6 years time.**

Both securities have face value \$100 and can be bought at prices which should give the investor an effective yield of 10% per annum if held to maturity. Which security should she buy in order to achieve the larger capital gain on a small fall in interest rates? Provide numerical justification for your answer.

The % change in price for a small change in interests can be approximated by:

$$\frac{P(i_0+\epsilon)-P(i_0)}{P(i_0)} \approx \epsilon\frac{P'(i_0)}{P(i_0)}$$

where $\epsilon\frac{P'(i_0)}{P(i_0)} = -\epsilon\upsilon$. For $\epsilon < 0$, we want the maximum value of υ to achieve the larger capital gain.

$$P_1 = 5a_{\overline{5|}} + 100v^5 = 5(3.79079) + 100(0.62092) = 81.04607$$

$$\upsilon_1 = \frac{5v(Ia)_{\overline{5|}} + 100(5)v^6}{P_1} = \frac{5(0.90909)(10.6528) + 500(0.56447)}{P_1} = 4.07986$$

$$P_2 = 11a_{\overline{6|}} + 100v^6 = 11(4.35526) + 100(0.56447) = 104.35526$$

$$\upsilon_2 = \frac{11v(Ia)_{\overline{6|}} + 100(6)v^7}{P_2} = \frac{11(0.90909)(14.03964) + 600(0.51316)}{P_2} = 4.29581$$

Hence, select Security 2. [(1 mark each) P_1 P_2 υ_1 υ_2 ; (1 mark) identify criteria to select which security; (1 mark) select Security 2.]

(b) (i) [4 marks] Company X wants to immunise its portfolio of assets and liabilities against small changes in the interest rate. The present value of liabilities is \$250,000. The liabilities have duration (in years) $\tau_L = 4$ and convexity $c_L = 6.125$.

Two assets are available to construct an immunised portfolio. The duration and convexity of each asset is given in the table below.

Asset	Duration (years) (τ)	Convexity (c)
A	3.5	5.4
B	5.5	8.818

Let A and B denote the purchase price of Asset A and Asset B respectively. Calculate A and B such that the portfolio is immunised against small changes in the interest rate. Using convexity, show that these amounts will immunise your portfolio.

$$A + B = 250000 - (1)$$

$$\frac{3.5A + 5.5B}{250000} = 4 - (2)$$

Solve (1) and (2) get A=187500, B=62500. Check convexity

$$\frac{5.4 \times 187500 + 62500 \times 8.818}{250000} = 6.2545 > 6.125$$

- (1 mark) Equation (1)
- (1 mark) Equation (2)
- (1 mark) Answer for A and B.
- (1 mark) convexity check

(b)(ii) [2 marks] Suppose current interest rates are 5% effective per annum. Let ϵ denote a small change in the interest rate.

Calculate ϵ such that the present value of liabilities increases by no more than 3%.

$$0.03 \approx \epsilon \times \frac{4}{1.05} + \frac{\epsilon^2}{2} \times 6.125$$
$$6.43125\epsilon^2 + 8\epsilon - 0.063 = 0$$
$$\epsilon = \frac{-8 \pm \sqrt{64 - 4 \times 6.43125 \times (-0.063)}}{2 \times 6.43125}$$
$$\epsilon \approx 0.78\%$$

That is, at most a 0.78% drop in interest rates.

- (1 mark) Use Taylor series approximation to price change
- (1 mark) numerical answer for ϵ, must note a drop in interest rates of at most 0.78%

(c) [6 marks] For a 30-year home mortgage with level payments and an interest rate of 12% per annum convertible monthly, find the volatility and convexity of the mortgage payments.

Note: You may find the following result useful $\boldsymbol{\sum_{t=1}^{360} t^2 (1.01)^{-t} = 1410842}$.

The monthly interest rate is 1% effective per month. Per \$1 of level monthly payment, the price is

$$P(0.01) = \sum_{t=1}^{360} (1.01)^{-t} = \frac{1 - (1.01)^{-360}}{0.01} = \$97.22$$

and

$$P'(0.01) = -\sum_{t=1}^{360} t(1.01)^{-t-1} = -(1.01)^{-1} (Ia)_{\overline{360}|0.01} = -8730.347156$$

Thus, the volatility is

$$v = \frac{8730.347156}{97.22} = 89.80$$

Now

$$\begin{aligned} P''(0.01) &= \sum_{t=1}^{360} t(t+1)(1.01)^{-t-2} \\ &= v^2 \sum_{t=1}^{360} (t^2 + t) v^t \\ &= v^2 \left[\sum_{t=1}^{360} (t^2 v^t) + \sum_{t=1}^{360} (t v^t) \right] \\ &= v^2 \left[1410842 + (Ia)_{\overline{360}|0.01} \right] \end{aligned}$$

Therefore

$$c(0.01) = \left(1.01^{-2}\right) \frac{1410842 + 8817.650627}{97.22} = 14602$$

(1 mark) P(0.01); (1 mark) $P'(0.01)$; (1 mark) volatility answer; (2 marks) $P''(0.01)$; (1 mark) convexity calculation

6 Question 6 (24 marks)

(a) Nominal interest rates compounded monthly are assumed to be independent and identically distributed with mean $E[\tilde{i}^{(12)}] = 9\%$ and standard deviation $SD[\tilde{i}^{(12)}]$=2%.

Effective monthly interest rates are also assumed to be independent and identically distributed.

On 1 January 2013, Jenny invests $500 into an account that pays interest at the end of every month.

(a) (i) [2 marks] What is the expected value of Jenny's account on 1 January 2023?

$$
\begin{aligned}
E[500\tilde{S}(10)] &= 500E[\tilde{S}(10)] \\
&= 500E\left[\left(\left(1+\frac{\tilde{i}^{(12)}}{12}\right)^{12}\right)^{10}\right] \\
&= 500E\left[\left(1+\frac{\tilde{i}^{(12)}}{12}\right)^{120}\right] \\
&= 500\left[E\left[1+\frac{\tilde{i}^{(12)}}{12}\right]^{120}\right] \\
&= 500\left[\left(1+\frac{E[\tilde{i}^{(12)}]}{12}\right)^{120}\right] \\
&= 500\left[\left(1+\frac{0.09}{12}\right)^{120}\right] \\
&= 500\left[(1.0075)^{120}\right] \\
&= \$1225.68
\end{aligned}
$$

- (1 mark) working
- (1 mark) final answer

(a) (ii)[4 marks] What is the standard deviation of Jennys account value on 1 January 2023?

$$\begin{aligned} Var[500\tilde{S}(10)] &= 500^2 Var[\tilde{S}(10)] \\ &= 500^2\left[E[\tilde{S}(10)^2] - (E[\tilde{S}(10)])^2\right] \end{aligned}$$

$$\boxed{\begin{aligned} E[\tilde{S}(10)^2] &= E\left[\left(\left(1+\frac{\tilde{i}^{(12)}}{12}\right)^2\right)^{120}\right] \\ &= \left(E\left[\left(1+\frac{\tilde{i}^{(12)}}{12}\right)^2\right]\right)^{120} \end{aligned}}$$

$$\boxed{\begin{aligned} E\left[\left(1+\frac{\tilde{i}^{(12)}}{12}\right)^2\right] &= E\left[1+2\left(\frac{i^{(12)}}{12}\right)+\left(\frac{i^{(12)}}{12}\right)^2\right] \\ &= \left[1+2E\left[\frac{i^{(12)}}{12}\right]+E\left[\left(\frac{i^{(12)}}{12}\right)^2\right]\right] \\ &= 1+2\times 0.0075+\frac{1}{144}(0.02^2+0.09^2) \\ &= 1.01506 \end{aligned}}$$

$$E[\tilde{S}(10)^2] = 1.01506^{120}$$

$$\boxed{\begin{aligned} Var[500\tilde{S}(10)] &= 500^2(1.01506^{120} - ((1.0075)^{120})^2) \\ &= 666.15 \end{aligned}}$$

$$\boxed{SD[500\tilde{S}(10)] = \sqrt{666.15} = \$25.81}$$

- (1 mark) $E\left[\left(1+\frac{\tilde{i}^{(12)}}{12}\right)^2\right]$
- (1 mark) $E[\tilde{S}(10)^2]$
- (1 mark) $Var[500\tilde{S}(10)]$
- (1 mark) $SD[500\tilde{S}(10)]$

(a) (iii) [4 marks] Instead of an initial single deposit, Jenny decides to invest \$10 at the beginning of each month for ten years. The first deposit is made on 1 January 2013. After the first year, the deposit amount is increased at the beginning of each calendar year by 10%. What is the expected value of Jenny's account on 1 January 2023?

Let n=120 (months). Let $a = (1 + E[\tilde{i}])$. Let $j = E[\tilde{i}]$

$$= 10E[(1.10)^9((1+\tilde{i}_n) + (1+\tilde{i}_n)(1+\tilde{i}_{n-1}) + ... + (1+\tilde{i}_n)..(1+\tilde{i}_{n-12})) + ...$$

$$.......... + (1.10)^0((1+\tilde{i}_n)...(1+\tilde{i}_{12}) + ... + (1+\tilde{i}_n)..(1+\tilde{i}_1))]$$

$$= 10\left[(1.10)^9(a + a^2 + ... + a^{12}) + ... + (1.10)^0(a^{109} + a^{110} + ... + a^{120})\right] \text{ (because iid)}$$

$$= 10(1.10)^9(a + a^2 + ... + a^{12})(1 + a^{12\times 1}1.10^{-1} + ... + a^{12\times 11}1.10^{-11})$$

$$= 10(1.10)^9 \ddot{s}_{\overline{12}|j}\left(\frac{1 - \left(\frac{1.0075^{12}}{1.10}\right)^{10}}{1 - \left(\frac{1.0075^{12}}{1.10}\right)}\right)$$

$$= 10 \times 12.60139 \times 9.7504$$

$$= \$2897.18$$

- (1 mark) first line expression
- (1 mark) justification to evaluate at $E[\tilde{i}]$
- (1 mark) numerical working
- (1 mark) final answer

(b) (i) [2 marks] Find the expected present value of \$1000 paid in 5 years time, if effective annual discount rates $\tilde{d}_t$ (for t=1,2,3,4,5) are independent and identically distributed, and in each year $\tilde{d}_t$ has a uniform distribution on the interval $[0.01, 0.10]$.

$$\begin{aligned}
\text{Expected Present Value} &= E[1000(1-d_5)(1-d_4)(1-d_3)(1-d_2)(1-d_1)] \\
&= 1000E[(1-d_5)(1-d_4)(1-d_3)(1-d_2)(1-d_1)] \\
&= 1000E[(1-d_t)^5] \\
&= 1000(1-0.055)^5 \\
&= \$753.63
\end{aligned}$$

(1 mark) numerical working; (1 mark) final answer

(b) (ii) [4 marks] Find the standard deviation of the present value of $1000 paid in 5 years time, if effective annual discount rates $\tilde{d}_t$ follow the same distribution as in part (b)(i).

$$\boxed{\begin{aligned} \text{Variance} &= Var[1000(1-d_5)(1-d_4)(1-d_3)(1-d_2)(1-d_1)] \\ &= E[(1000(1-d_5)(1-d_4)(1-d_3)(1-d_2)(1-d_1))^2] - \$753.63^2 \\ &= 1000^2 E[(1-d_5)^2(1-d_4)^2(1-d_3)^2(1-d_2)^2(1-d_1)^2] - \$753.63^2 \\ &= 1000^2 (E[(1-d_t)^2])^5 - \$753.63^2 \end{aligned}}$$

$$\boxed{\begin{aligned} E[(1-d_t)^2] &= Var[1-d_t] + (E[1-d_t])^2 \\ &= Var(d_t) + (1-0.055)^2 \\ &= \frac{(0.10-0.01)^2}{12} + 0.945^2 \\ &= 0.000675 + 0.945^2 \\ &= 0.8937 \end{aligned}} \quad 2$$

$$\boxed{\begin{aligned} \text{Variance} &= 1000^2 \times 0.8937^5 - \$753.63^2 \\ &= 2149.734245 \end{aligned}}$$

Standard deviation $= \sqrt{2149.73245} = \$46.37$.

- (2 marks) Calculation $E[(1-d_t)^2]$
- (2 marks) Variance calculation

(c)(i) [4 marks] Effective annual interest rates for the next 5 years are independently and identically distributed with the following distribution:

$$\tilde{i} = \begin{cases} 0.08 & \text{prob} = 0.3 \\ 0.10 & \text{prob} = 0.5 \\ 0.15 & \text{prob} = 0.2 \end{cases}$$

A new fund has 1000 investors. Each investor has deposited \$5000. Assuming no new investors, deposits or withdrawals, and using the lognormal distribution, find the probability that the funds value exceeds \$8 million after 5 years.

We want $Pr(5\tilde{S}(5) > 8) = Pr(\ln(\tilde{S}(5)) > \ln(8/5))$. Using the lognormal distribution approximation, we assume $\ln(\tilde{S}(5)) \sim Norm(E[\ln(\tilde{S}(5))], Var[\ln(\tilde{S}(5))])$.

$$\boxed{E[\ln(\tilde{S}(5))] = 5 \times (0.3 \times \ln(1.08) + 0.5 \times \ln(1.10) + 0.2 \times \ln(1.15)) = 0.49348}$$

$$E[\delta^2] = (0.3 \times (\ln(1.08))^2 + 0.5 \times (\ln(1.10))^2 + 0.2 \times (\ln(1.15))^2) = 0.010226$$

$$\boxed{Var[\ln(\tilde{S}(5))] = 5 \times (0.010226 - 0.098696^2) = 0.002424}$$

$$\boxed{\begin{aligned} Pr(\ln(\tilde{S}(5)) > \ln(8/5)) &= Pr\left(Z > \frac{\ln(8/5) - 0.49348}{\sqrt{0.002424}}\right) \\ &= Pr(Z > -0.47684) \\ &= 1 - 0.317 \\ &= 0.683 \end{aligned}}$$

2

(nb: $Pr(Z > -0.47) = 0.6808$ and $Pr(Z > -0.48) = 0.6844$ from tables provided in exam. These answers are acceptable too).

(1 mark) $E[\ln(\tilde{S}(5))]$; (1 mark) $Var[\ln(\tilde{S}(5))]$; (2 marks) final answer and working

(c)(ii)[4 marks] The fund owners want to be 90% confident that the fund value will be at least $8 million in 5 years time. Assuming each investor deposits $5000, how many additional investors are required at t=0? Assume there are no new investors, deposits or withdrawals after t=0.

From standard normal tables, $\boxed{Pr(Z > -1.28) = 0.90.}$ Let n be the total number of investors required. Solve for n such that

$$\boxed{\begin{gathered} \frac{\ln\left(\frac{8000}{5 \times n}\right) - 0.49348}{\sqrt{0.002424}} = -1.28 \\ n = 1040.332 \\ n = 1041 \end{gathered}}$$

2

That is, $\boxed{\text{41 additional}}$ investors are required.

(1 mark) $Pr(Z > -1.28) = 0.90$; (2 marks) numerical working; (1 mark) 41 additional investors (not 1041)