Term average 70%

Solve for x_1, x_2

(E.g. $\begin{matrix} 3x_1 + 4x_2 = 2 \\ 5x_1 - 7x_2 = 0 \end{matrix}$)

In matrix form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

has a unique sol'n iff

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is non-zero

In this case, the solution is

$$x_1 = \frac{\begin{vmatrix} y_1 & b \\ y_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a & y_1 \\ b & y_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

or, in terms of inverse matrix

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Consider 2nd order linear hom. eq'n $L[y] = 0$ where

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

This is linear: $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$

$\Rightarrow$ Superposition principle: y_1, y_2 are solutions.

then $y = c_1 y_1 + c_2 y_2$ is again a solution

Suppose we have the IVP.

$L[y] = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$ can we fix c_1, c_2 to satisfy the condition?

Let's check:

$$y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \overset{!}{=} y_0$$
$$y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) \overset{!}{=} y_0'$$

Two equations for unknowns c_1, c_2

Matrix form $\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$

Has unique solution if

$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$ is non-zero

Definition: The Wronskian of two functions f, g is the function

$W(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$

Thus, the constant c_1, c_2 in $y = c_1 y_1 + c_2 y_2$ can be arranged to satisfy $L[y]=0$, $y(t_0)=y_0$, $y'(t_0)=y_0'$ provided the Wronskian of y_1, y_2 is non-zero at t_0.

In this case, y_1, y_2 are a fundamental set of solutions.

Eg 2: $L[y] = y'' - 4y' + 4y = 0$

char. eqn: $r^2 - 4r + 4 = 0$

repeated root: $r = 2$

$y_1(t) = e^{2t}$

$y_2(t) = te^{2t}$

Thus Wronskian is:

$$\begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}(1+2t) \end{vmatrix} = e^{4t}(1+2t) - 2te^{4t} = e^{4t} \text{ is non-zero}$$

$\Rightarrow y_1, y_2$ are fundamental set).

Example: $L[y] = ay'' + by' + cy = 0$

Case 1: $b^2 > 4ac$, two roots, $r_1 \neq r_2$

$y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$

$$\begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

Case 2: $b^2 = 4ac$, repeated roots $r_1 = r_2 = r$

$y_1(t) = e^{rt}$, $y_2(t) = re^{rt}$

Wronskian: $W(t) = e^{2rt} \neq 0$

Case 3: $b^2 < 4ac$, two complex roots, $r_1, r_2 = \lambda \pm i\mu$

$y_1(t) = e^{\lambda t}\cos(\mu t)$ $y_2(t) = e^{\lambda t}\sin(\mu t)$

Their Wronskian is $W(t) = e^{2\lambda t}\mu \neq 0$

Tomorrow: we'll see :

If y_1 and y_2 are solutions of $L[y]=0$ and $W(t_0)\neq 0$ for some t_0,

then $W(t)\neq 0$ for all t.