March 18th

Def'n: one-to-one (injective)
$f(x)=f(y) \Rightarrow x=y$

bijective = onto + one-to-one

onto (surjective)
$\forall y \in Y, \exists x \in X, f(x)=y$

$$\begin{array}{l} A=\{a,\ b,\ c\} \\ \quad\ \ \downarrow\ \ \downarrow\ \ \downarrow \\ N=\{1,\ 2,\ 3\} \end{array}$$

$f: A \to N$
$f(a)=1,\ f(b)=2,\ f(c)=3$
s.t. f is 1-1 and onto
1-1 $\quad f(x)=f(y)=1 \Rightarrow x=y=a$

onto $\quad$ If $y=1$ then $f(a)=1$

Produce a bijection from the even integers E to the odd integers O.

Let $f(2k)=2k+1$

Claim: f is one-to-one
If $f(2k)=f(2l)$

$2k+1=2l+1$
$\Rightarrow 2k=2l$
$\Rightarrow k=l$

Claim: f is onto
Let $n \in \mathbb{Z}$ be odd, then $n=2k+1$ for some $k \in \mathbb{Z}$
so $f(2k)=2k+1=n$

note: $2k$ is even

HOW TO PROVE BIJECTION

Def'n: The cardinality of a set S is the equivalence class of S under the relation $S \sim T$ if $\exists$ a bijection $f: S \to T$.

The cardinality $\{1,2,3\}=\{a,b,c\}=\{0,1,2\}$; - -

Def'n $|A| \leq |B|$ if there is a one-to-one map $f: A \to B$

$\{1\} \subseteq \{1,2,3\} \Rightarrow |\{1\}| \leq |\{1,2,3\}|$

Say $|A|=|B|$ if $\exists$ a bijection $f: A \to B \quad |\{a,b,c\}|=|\{1,2,3\}|$

Thm (Cantor-Bernstein):
If $|A| \leq |B|$, $|B| \leq |A|$ then $|A|=|B|$
$(|\mathbb{Z}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Z}| \Rightarrow |\mathbb{N}|=|\mathbb{Z}|)$

Defn For a set S define $P(S)=\{T : T\subseteq S\}$

Thm: $|S|\leq |P(S)|$ and $|S|\neq |P(S)|$

Step 1: Construct a 1–1 map

$f: S \longrightarrow P(S)$

consider $f(x)=\{x\}\in P(S)$

If $f(x)=f(y)$ then $\{x\}=\{y\}$ so $x=y$

Step 2: No bijection from S to $P(S)$

Suppose $g: S\to P(S)$ is bijective then g is onto

consider $T=\{x\in S : x\notin g(x)\}$

Claim: There's no x such that $g(x)=T$

For any x either $x\in T$ or $x\notin T$

Suppose that $\underbrace{g(x)=T}_{\Rightarrow\, x\notin T}$ and $\underbrace{x\in T}_{\text{This is a contradiction}} \Rightarrow x\notin g(x)=T$

Suppose $g(x)=T$ and $x\notin T$

$\Rightarrow x\in g(x)=T$

So there is no such x.

so — There is no bijection $S\to P(S)$

$|\mathbb{N}| \lneq |P(\mathbb{N})| \lneq |P(P(\mathbb{N}))|$

Claim: $|\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$

$\mathbb{N}\times\mathbb{N} = \{(x,y) : x,y\in\mathbb{N}\}$

Claim. $|\mathbb{N}|\leq|\mathbb{N}\times\mathbb{N}|$

$f(n)=(n,1)$

Check: f 1–1 $f(n)=f(l)$

$(n,1)=(l,1)$

$\Rightarrow n=l$

Claim: $|\mathbb{N}\times\mathbb{N}|\leq|\mathbb{N}|$

$f(x,y)=2^x3^y$

If $f(x,y)=f(n,m)$

$2^x3^y=2^n3^m$

FTA so $x=n, y=m$

$|P(\mathbb{N})| = |P_2| = |[0,1]|$

Consider $f: P(\mathbb{N})\to[0,1]$ given by $f(S)=\sum_{k\in S}\frac{1}{2^k}$

Note $f(S) = \sum_{k \in S} \frac{1}{2^k} \leq \sum_{k \in N}^{\infty} \frac{1}{2^k} = 1$

$f(S) \geq \sum_{k \in \emptyset} \frac{1}{2^k} = 0$