§10 - Partial Orders, Linear Orders and Well-Orders

1 Motivation

Partial ordering is a very natural relation in mathematics, and in the real world. We often want to sort out a bunch of data and order it somehow. A partial ordering is a basic relation that helps us to put various things in some order. These orderings are interesting on their own, and we will be using them to state an important axiom of mathematics called "the Axiom of Choice".

We will also take this time to introduce a rich class of topologies given by linear orders (a special type of partial order). Linear orders are basically just a collection of elements ordered into a line; given any two elements we will know "which one is in front of the other". Finally we will use a very special type of linear order (a "well-order") to give a very important example of a topological space, ω_1, which is the least *uncountable* well-ordered set.

2 The Definition

Definition. *Let $\mathbb{P}$ be a set, and let $\leq$ be a (partially defined) relation on $\mathbb{P}$. We say that $(\mathbb{P}, \leq)$ is a* ***partial ordering*** *(or a "partially ordered set" or "poset") if it has the following three properties:*

(Transitive) : *$a \leq b$ and $b \leq c$ implies $a \leq c$ $(\forall a, b, c \in \mathbb{P})$;*

(Reflexive) : *$a \leq a$ $(\forall a \in \mathbb{P})$;*

(Antisymmetric) : *$a \leq b$ and $b \leq a$ implies $a = b$ $(\forall a, b \in \mathbb{P})$.*

Notation: For a poset $(\mathbb{P}, \leq)$ we say $a < b$ iff $a \leq b$ and $a \neq b$. We also say $a \geq b$ iff $b \leq a$. Sometimes we use the symbol $\preceq$ and $\prec$ instead of $\leq$ and $<$. We also say $a \not\leq b$ if it is not true that $a \leq b$, (and note that this *does not* imply that $b < a$.)

This definition is just codifying a way of ordering elements of a set.

3 Examples

We already know a bunch of examples that we have known since elementary school:

Example. $(\mathbb{R}, \leq), (\mathbb{Q}, \leq), (\mathbb{Z}, \leq)$ *and* $(\mathbb{N}, \leq)$ *all with their usual orderings are partial orders.*

Example. *For any set X, $(\mathcal{P}(X), \leq)$ is a poset, where $A \leq B$ iff $A \subseteq B$. In particular, $(\mathcal{P}(\mathbb{N}), \subseteq)$ is a (very interesting!) poset.*

The example of $(\mathcal{P}(\mathbb{N}), \subseteq)$ has the property that for $\mathbb{E} := \{ 2n : n \in \mathbb{N} \}$ and $\mathbb{I} := \{ 2n+1 : n \in \mathbb{N} \}$ both

$$\mathbb{E} \not\leq \mathbb{I} \text{ and } \mathbb{I} \not\leq \mathbb{E}$$

and so we say that $\mathbb{E}$ and $\mathbb{I}$ are **not comparable**.

Example. *Orderings need not be defined on "mathy things". For X, the set of people who bought one iPad in 2012, we define the ordering $\leq$ by Person $A \leq$ Person B iff (1) both people bought their iPad in the same store and (2) Person A bought their iPad before Person B did.*

Here we see that this is a partial order, and we see that there are many incomparable elements of the set X. For example, each person who bought their iPad in Toronto is not comparable to anyone who bought their iPad in Edmonton. However, any two people who bought their iPad in the same store are comparable (that is you can say who bought theirs first).

Example. *Even though $\mathbb{N}$ has its usual ordering, we can ignore it and define a different ordering. Define $a \preceq b$ iff $a|b$, that is a divides b.*

Some high-school level manipulations tell us that this is indeed a partial order, (any number divides itself, $n|m$ and $m|n$ means that $n = m$, and $a|b$ with $b|c$ implies $a|c$). We notice that $3 \not\preceq 7$ and $7 \not\preceq 3$, so the two numbers are not comparable. Moreover we see that $P := \{ p \in \mathbb{N} : p \text{ is prime} \}$ has the property that any two elements are not comparable (in this ordering). So we call P an **antichain**.

We also see that

$$S := \{ 7^n : n \in \mathbb{N} \} = \{ 7, 49, 343, \dots \}$$

has the property that any two elements *are* comparable (in this ordering), so we say that S is a **chain**.

Example. *Let $X = \mathbb{N} \cup \{ \omega \}$, where ω is just a thing that isn't a natural number. Extend the (natural) ordering on $\mathbb{N}$ so that $n \leq \omega$ for all $n \in \mathbb{N}$. (Basically we have defined an element that "acts like ∞", which is above all of the natural numbers.) This is a handy partial order that we will refer to often, so we will call the order "$\omega + 1$" (read as "omega plus 1").*

Where's Waldo Exercise: Find a copy of the previous partial order inside $\mathbb{Q}$ (with the usual ordering). Is there a copy of this partial order inside $\mathbb{Z}$ (with the usual ordering)?

The following example and exercise are just for your own curiosity. **Don't worry too much about them!**

Example. *Finally, we present a neat example of a partial order, extracted from a topology. Let* $(X, \mathcal{T})$ *be a topological space, and define a partial order* $(\mathcal{T} \setminus \{\emptyset\}, \leq)$ *by* $U \leq V$ *iff* $U \subseteq V$*. (So we are comparing non-empty open sets.) Here we define a* ***topological antichain*** *to be a subset* $\mathcal{A} \subseteq \mathcal{T} \setminus \{\emptyset\}$ *such that any two elements of* $\mathcal{A}$ *are* ***not compatible****. We say that* $a, b \in \mathcal{T} \setminus \{\emptyset\}$ *are* ***compatible*** *if there is an* $r \in \mathcal{T} \setminus \{\emptyset\}$ *such that* $r \leq a$ *and* $r \leq b$*.*

ccc in posets Exercise: Show that the partial order we get from the topological space $\mathbb{R}_{\text{usual}}$ has the the property that all of its topological antichains are countable. Your proof might use the fact that $\mathbb{R}_{\text{usual}}$ is a ccc topological space, and you might be tempted to deduce that that is the origin of the name of the topological property "the countable chain condition". You would be right! If you want to know about the (deep) connections between topological spaces and posets come talk to me during my office hours. To tease you a bit, Paul Cohen won a Fields medal for his work involving the deep connections of posets, topology and statements unprovable from the usual axioms of mathematics.

4 Some Words

Let's collect some words that we will find useful. Throughout, we will let $(\mathbb{P}, \leq)$ be a poset.

- $a, b \in \mathbb{P}$ are **comparable** if $a \leq b$ or $b \leq a$;
- $a, b \in \mathbb{P}$ are **incomparable** if $a \not\leq b$ and $b \not\leq a$;
- $C \subseteq \mathbb{P}$ is a **chain** if a and b are comparable, for all $a, b \in C$.
- $A \subseteq \mathbb{P}$ is an **antichain** if a and b are incomparable, for all $a, b \in A$.

5 Linear Orders

A rather nice additional property that a partial order could have is that "there are no incomparable elements"; that is, any two elements are comparable. Intuitively, these partial orders look like lines, and are called "linear orders" or "total orders".

Definition. *A partial order* $(L, \leq)$ *is said to be a* ***linear order*** *(or a "linearly ordered set" or a "total order") if any two elements in* L *are comparable (i.e. it is a chain).*

Some examples:

- $(\mathbb{R}, \leq), (\mathbb{Q}, \leq), (\mathbb{Z}, \leq)$ and $(\mathbb{N}, \leq)$ all with their usual orderings are linear orders.
- On the set of letters $\{A, B, C, D, F\}$ there is the "academic order"

$$F < D < C < B < A.$$

- The example we gave before of $\omega + 1$ is a linear order.

Linear orders naturally give rise to topologies. Take a look at assignment 2 (A.1) where we described a subbasis for $\mathbb{R}_{\text{usual}}$. The subbasis was described completely in terms of the (linear) ordering on $\mathbb{R}$; we didn't use anything about addition, subtraction, multiplication or the distance function on $\mathbb{R}$. Let us use this idea to get a topological space from a linear order.

Definition. *If $(L, \leq)$ is a linear order (with at least 2 elements), then the **order topology on** L is given by the subbasis*

$$\mathcal{S} := \{ (-\infty, b) : b \in L \} \cup \{ (a, +\infty) : a \in L \},$$

where

$$(a, +\infty) := \{ x \in L : a < x \}$$

is the "half-open ray above a" and

$$(-\infty, b) := \{ x \in L : x < b \}$$

is the "half-open ray below b".

This *basically* tells us that the open sets in an order topology are unions of open intervals.

Subbasis/Basis Exercise: Why did we describe a *subbasis* for the order topology instead of a basis? What happens if our linear order has an end-point? (A right endpoint is an element $r \in L$ such that $x \leq r$ for all $x \in L$. A left endpoint is an element $l \in L$ such that $l \leq x$ for all $x \in L$.)

Example. *The order topology on $\mathbb{N}$ (with the usual ordering) is the discrete topology. (In what follows, all intervals are assumed to only contain elements of $\mathbb{N}$. For example, $(4, 8) = \{ 5, 6, 7 \}$.) First, $\{ 1 \} = (-\infty, 2)$ and then for $n > 1$ we have*

$$\{ n \} = (n - 1, n + 1) = (-\infty, n + 1) \cap (n - 1, +\infty).$$

So each singleton is open.

Example. *The order topology on $\mathbb{R}$ is identical to the usual topology on $\mathbb{R}$. Similarly, the order topology on $\mathbb{Q}$ is identical to its subspace topology in the usual topology.*

This leads to a boring, but handy fact (whose proof is obvious). First we recall a notion:

Definition. *Let $(L, \leq)$ be a linear order and $A \subseteq L$. We say that A is a **convex** subset of L provided that $\forall a, b \in A$ with $a < b$, then if $x \in L$ is such that $a < x < b$, then $x \in A$.*

Lemma. *If $(L, \leq)$ is a linear order, and $A \subseteq L$ is a* ***convex*** *set, then the subspace topology on A (as a subspace of L with the order topology) is identical to the order topology given by $(A, \leq_A)$. (Where $a \leq_A b$ iff $a, b \in A$ and $a \leq b$. It is the natural restriction!)*

Of course, our example with $\mathbb{Q}$ shows that the above lemma is not an if and only if.

Paul's Exercise: Find a subset $A \subseteq \mathbb{R}$ such that the subspace topology inherited from $\mathbb{R}_{\text{usual}}$ differs from the linear order topology on A.

The next example is a very important one. It is easy to trick yourself into thinking that something is true about order topologies when it isn't true. This next example is the baby sister of a serious example we will look at shortly.

Example. *The space $\omega + 1$ with the order topology has each of its points open except for the point ω. Any open set containing ω contains a basic open set of the form $(n, \omega]$ for some $n < \omega$. The only possibility is that $n \in \mathbb{N}$, and we see that $(n, \omega]$ contains lots of points other than ω itself. In fact, we have just proved that $\{ n : n \in \mathbb{N} \}$ converges to ω in this topology.*

Peek-a-$\mathbb{Q}$ Exercise: In a previous exercise you found a "copy" of $\omega + 1$ in $\mathbb{Q}$; is this copy you found homeomorphic to $\omega + 1$?

Before we leave, we remark about the following topological facts about order topologies.

Proposition. *If $(X, \mathcal{T})$ is a topological space generated by a linear order, then it is T_2, T_3 and T_4.*

Proof. Let $(X, \leq)$ be a linear order that generates $\mathcal{T}$.

$[T_2]$ Let $a, b \in X$ be distinct. Without loss of generality, assume $a < b$. Case 1: If there is a $c \in X$ that is between a and b (i.e. $a < c < b$), then $(-\infty, c)$ and (c, ∞) are the desired disjoint open sets. Case 2: If there are no elements between a and b, then $a \in (-\infty, b)$ and $b \in (a, \infty)$, but the two open sets are disjoint. (Do you see why?)

$[T_3]$ Let C be closed in X, and let $p \in X \setminus C$. There is an interval such that $p \in (a, b) \subseteq X \setminus C$. If there is a $c \in X$ such that $p < c < b$ and a $d \in X$ such that $a < d < p$, i.e.

$$a < d < p < c < b$$

then our desired open sets are $(-\infty, d) \cup (c, \infty) \supseteq C$ and $(d, c) \ni p$. In the case where there is no such c, then we adjust the open sets to $(-\infty, d) \cup [b, \infty) \supseteq C$ and $(d, p] \ni p$. A similar adjustment can be made if there is no such d.

$[T_4]$ This proof is a tedious exercise. We will not need it, so we won't look into it. $\square$

6 Well-orders

We now move on to a subclass of linear orders that is even nicer: well-orders. Since kindergarten we have known that $\mathbb{N}$ is well-ordered. That is, there is a smallest element (1) and a next-smallest element (2) and a next-smallest element (3) and so on... It is convienient to code this property is such a way that makes sense for "large" linear orders. For example, our linear order $\omega + 1$ has a similar property to $\mathbb{N}$ that it has a first element, a second element, a third element, etc. It also has an element (ω) that is greater than all of those. In that sense ω is the "next biggest element after $\mathbb{N}$". This should give you a sense of why we need to come up with a better way of discussing these things.

Definition. *A partial order $(W, \leq)$ is said to be a* ***well-order*** *if every (non-empty) subset $S \subseteq W$ has a $\leq$-least element of S, denoted by* $\min(S)$. *(An element $l \in S$ is a $\leq$-least element of S if $\forall s \in S$, $l \leq s$.)*

Some Examples:

- $(\mathbb{R}, \leq)$ with the usual order is *not* a well-order. For example $(0, \infty)$ is a non-empty set without a minimal element. ($\mathbb{Q}$ and $\mathbb{Z}$ with their usual orders also fail to be well-orders for similar reasons.)
- $\mathbb{N}$ with the usual order is a beautiful well-order that we have known since kindergarten.
- $\omega + 1$ is a well-order. (If $S \subseteq \omega + 1$ is a non-empty set that intersects $\mathbb{N}$, then use the well-orderedness of $\mathbb{N}$ to find your least element. If $S \cap \mathbb{N} = \emptyset$, then the only element of S is ω, so *it* is the minimal element of S.)

Lets unwrap the definition by looking at some immediate facts for a well-order $(W, \leq)$:

- For $\emptyset \neq S \subseteq W$, $\min(S)$ is unique. (If x, y are both least elements of S, then $x \leq y$ and $y \leq x$, so $x = y$.)
- $(W, \leq)$ is a linear order. (If $x, y \in W$, then $S := \{ x, y \}$ is a non-empty subset of W. If $x = \min(S)$, then $x \leq y$, and if $y = \min(S)$, then $y \leq x$.)
- W does not contain infinite *decreasing* chains. (Prove this!)
- W contains a least/smallest element (provided that $W \neq \emptyset$). (Take $\min(W)$.)
- W contains a "next-smallest element" (provided that W contains at least two elements). (Take the least element of $W \setminus \{ \min(W) \}$, which is non-empty provided that W contains at least two elements).
- Every infinite well-order contains a "copy" of $\mathbb{N}$ as an initial segment. (Just proceed inductively as in the previous line.) In this sense we call $\mathbb{N}$ the "least countable well-order".

- Every uncountable well-order contains a "copy" of $\omega + 1$ as an initial segment.

Wait, wait, wait... What did I just say? "Every uncountable well-order contains a "copy" of $\omega + 1$ as an initial segment." Oh, ok. Wait, what?! "Every **uncountable well-order** ...". That's weird, do uncountable well-orders even exist? What do they look like? (Can I touch it?)

Let's tackle that. And, in the eternal words of Samuel L. Jackson's character Ray Arnold from Jurassic Park, "Hold on to your butts."

7 ω_1, the least uncountable well-order

The most difficult part of this section will be that we are investigating a space that we have not "constructed", *per se.* We will be told that a nice space exists, then we will investigate its properties. One analogy might be that we have discovered some alien vessel from another galaxy, that has crashed on our planet; we don't know how it got here, we can't read the language, we don't *really* understand what all the buttons do, but if we press enough buttons we might just be able to figure out how to fly it. Another analogy is Descartes' ontological "proof" of the existence of God; he doesn't try to "fully understand" everything about God (whatever that means), but he does use two or three properties about God (Omniscience, creator) to deduce other properties (namely, existence). So as we investigate ω_1, focus on things we do understand about ω_1, rather than what we don't understand. There is an old saying "What you don't know could fill a library."

We start with the part that is just given to us (i.e. the crashed alien vessel). The following axiom is something we can take for granted.

Theorem (The Well-ordering Principle)**.** *If X is a set, then there is a well-order $\preceq$ such that $(X, \preceq)$ is a well-ordered set.*

The sound bite here is "**Every set can be well-ordered**".

Note that this does *not* say "Every order is a well-order", which is absurd (we know that the usual order on $\mathbb{R}$ is not a well-order). **The well-ordering principle does not say anything about the usual orders we know**.

Since we know that there is an uncountable set (obviously), there is a very useful corollary to the Well-ordering principle:

Corollary. *There is an uncountable well-ordered set $(W, \leq)$.*

From here we create (*cue stormy night, lightning and maniacal laughter*) $\ldots \omega_1$.

Definition. *Let* $(W, \leq)$ *be an uncountable well-order from the corollary. Define*

$$\omega_1 := \{\, x \in W : \textit{ there are only countably many elements } y \textit{ of } W \textit{ such that } y \leq x \,\}$$

Ok, so ω_1 is just the set of all elements of W with countable initial segments. Let's write down the things we know so far:

Facts about ω_1

1. $(\omega_1, \leq)$ is a well-order;
2. If $\alpha \in \omega_1$ then the initial segment $\{x \in \omega_1 : x \leq \alpha\}$ is countable.
3. ω_1 is uncountable;
4. "Countable subsets of ω_1 are bounded above", which means if $S \subseteq \omega_1$ is countable, then there is an $\alpha \in \omega_1$ such that $s < \alpha$ for all $s \in S$.

Proof. The first two of these properties follow immediately from our construction.

[3.] Suppose for the sake of contradiction that ω_1 was countable. Notice that since W (from our construction of ω_1) is uncountable, it must be that $W \setminus \omega_1$ is non-empty. Since W is well-ordered, let $M := \min(W \setminus \omega_1)$, which must exist. Notice that by definition of ω_1, the intitial segment

$$\{\, y \in W : y \leq M \,\}$$

is uncountable (because $M \notin \omega_1$), and we notice that since M was a least element,

$$\{\, y \in W : y \leq M \,\} = \omega_1 \cup \{\, M \,\}$$

which surely is countable! ("And don't call me Shirley!") A contradiction.

[4.] Suppose that $S \subseteq \omega_1$ is a (non-empty) countable set. Define the initial segments as $I_s := \{\, x \in \omega_1 : x \leq s \,\}$ for $s \in S$. We note that each I_s is a countable set, so $\bigcup_{s \in S} I_s$ is a countable set, and the union contains S. Thus

$$\omega_1 \setminus (\bigcup_{s \in S} I_s) \neq \emptyset$$

since ω_1 is uncountable, so take α to be in this difference.

<u>Claim</u>: $s < \alpha$ for all $s \in S$.

Suppose not. Let $s_0 \in S$ be such that $\alpha \leq s_0$, (recall that this really *is* the negation since well-orders are linear orders). Thus $\alpha \in I_{s_0}$. Oops! That's a contradiction! $\square$

So those are the basic properties of ω_1. Regardless of where we found ω_1 we know that it has those 4 properties. (i.e. Regardless of what the other buttons on the alien ship do, we have learned to fly it . . . a little.) Taking the analogy a little bit further, let's fly it by our old neighbourhood and show it to our friends. (What can I say, I like analogies a lot.)

Let's look at what happens when we give ω_1 the order topology:
Topological Facts about ω_1:

1. ω_1 is a Hausdorff, T_4 space.
2. ω_1 is first countable;
3. ω_1 is not separable.

Proof. Since ω_1 has an order topology, [1] is true.

[2.] For each $\alpha \in \omega_1$, we know that $\{ x \in \omega_1 : x \leq \alpha \}$ is a countable set. If $\{ \alpha \}$ is an open set (like $\{ \min(\omega_1) \}$ is!), then that is a countable local basis for α. Otherwise, $\{ (x, \alpha] : x < \alpha \}$ is the desired local basis.

[3.] Let $S \subseteq \omega_1$ be countable. Since S is bounded, take $\alpha \in \omega_1$ above all of S. Thus $(\alpha, +\infty)$ is an open set in ω_1 that is disjoint from S. Hence S is not dense in ω_1. □

8 ω_1's big brother: $\omega_1 + 1$

Before we leave, let us remark that we can get another really interesting space "$\omega_1 + 1$" in the same way that we defined "$\omega + 1$". This space turns out to answer a question we asked a long time ago.

Definition. *We define the set $\omega_1 + 1 := \omega_1 \cup \{ \Omega \}$, (where Ω is just some other point not in ω_1), and we extend the ordering on ω_1 by adding the relation $\alpha < \Omega$ for all $\alpha \in \omega_1$.*

Giving this the order topology (since it *is* a linear order) we get the following:

1. $\omega_1 + 1$ is a Hausdorff, T_4 space.
2. $\omega_1 + 1$ is not separable.
3. $\omega_1 + 1$ is *not*(!) first countable (and therefore not second countable);

Proof. We get [1] since it is an order topology.

[2] The same proof works as with ω_1, but instead of taking the open set $(\alpha, +\infty)$ we need to take the open set (α, Ω). This is because the countable set we take might include the

element Ω.

[3] Let $\mathcal{B}$ be a countable collection of basic open sets containing Ω. So the elements of $\mathcal{B}$ look like $(\alpha_n, \Omega]$, for $n \in \mathbb{N}$ (where $\alpha_n \in \omega_1$). Since countable sets are bounded above in ω_1, let $\beta \in \omega_1$ be such a bound. Thus $(\beta, \Omega]$ is an open set containing Ω, but each element $(\alpha_n, \Omega]$ of $\mathcal{B}$ is such that $(\alpha_n, \Omega] \nsubseteq (\beta, \Omega]$. So $\mathcal{B}$ is not a local basis at Ω. $\square$

Now we have an example of a Hausdorff, non-first countable space. Some of you investigated the Arens-Fort space, which is also an example of a Hausdorff, non-first countable space, but that example seemed somewhat contrived. It seemed designed to precisely break first-countability. This example, of $\omega_1 + 1$ is much more "naturally not first countable". Later on in the course, when we investigate compactness, we will see that this is an extremely handy example of a compact space.

These next exercises will help you understand more about ω_1, (you will understand more of the foreign buttons in the alien vessel).

ccc and ω_1 Exercise: Prove that ω_1 is not ccc. (This will also show that $\omega_1 + 1$ is not ccc!) This might seem a bit challeging, but start like this: "For each $\alpha \in \omega_1$ note that $\{\min(\omega_1 \setminus I_\alpha)\}$ is an open set" where $I_\alpha := \{ x \in \omega_1 : x \leq \alpha \}$. Unwrap what that says, and then complete the proof.

Discreteness Exercise: The previous exercise might get you mistakenly thinking that somehow ω_1 is discrete; this is very much not true! Show that ω_1 has infinitely many points that are not open (hint: look at what we did for $\omega + 1$). Then show that ω_1 has uncountably many points which are not open. (Is this true for $\mathbb{R}_{\text{usual}}$?)

Boundedness Exercise: Prove that $A \subseteq \omega_1$ is bounded iff it is countable. (Notice that we already proved one of these directions). Conclude that $C \subseteq \omega_1$ is uncountable iff it is unbounded. (Is this true for $\mathbb{R}$?)

Closedness Exercise: Prove that $\{ x \in \omega_1 : \{ x \} \text{ is not open } \}$ is a closed subset of ω_1. Conclude from a previous exercise that it is a closed and unbounded set.

Club Exercise: A subset $C \subseteq \omega_1$ is said to be a **club** subset of ω_1 if it is closed and unbounded. Note that final segments (i.e. everything to the *right* of a point α) are club. Prove that if C_1 and C_2 are club subsets then $C_1 \cap C_2$ is club. (Is this true for $\mathbb{R}_{\text{usual}}$?) If you are really hungry show that the intersection of countably many club sets is again club.

9 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Where's Waldo : Find a copy of $\omega+1$ inside $\mathbb{Q}$ (with the usual ordering). Is there a copy of this partial order inside $\mathbb{Z}$ (with the usual ordering)?

ccc posets : Show that the partial order we get from the topological space $\mathbb{R}_{\text{usual}}$ has the the property that all of its topological antichains are countable.

Paul : Find a subset $A \subseteq \mathbb{R}$ such that the subspace topology inherited from $\mathbb{R}_{\text{usual}}$ differs from the linear order topology on A.

Basis/Subbasis : Why did we describe a *subbasis* for the order topology instead of a basis? What happens if our linear order has an end-point? (A right endpoint is an element $r \in L$ such that $x \leq r$ for all $x \in L$. A left endpoint is an element $l \in L$ such that $l \leq x$ for all $x \in L$.)

Peek-a-$\mathbb{Q}$: If you give the subspace topology to the copy of $\omega+1$ inside $\mathbb{Q}$ you found earlier, is it homeomorphic to $\omega+1$ with its order topology?

Infinite Decr. : Prove that a linear order is a well-order iff it does not contain any infinite decreasing chains.

ccc + ω_1 : Prove that ω_1 is not ccc. (This will also show that ω_1+1 is not ccc!) This might seem a bit challeging, but start like this: "For each $\alpha \in \omega_1$ note that $\{\min(\omega_1 \setminus I_\alpha)\}$ is an open set" where $I_\alpha := \{\, x \in \omega_1 : x \leq \alpha \,\}$. Unwrap what that says, and then complete the proof.

Discreteness : The previous exercise might get you mistakenly thinking that somehow ω_1 is discrete; this is very much not true! Show that ω_1 has infinitely many points that are not open (hint: look at what we did for $\omega+1$). Then show that ω_1 has uncountably many points which are not open. (Is this true for $\mathbb{R}_{\text{usual}}$?)

Boundedness : Prove that $A \subseteq \omega_1$ is bounded iff it is countable. (Notice that we already proved one of these directions). Conclude that $C \subseteq \omega_1$ is uncountable iff it is unbounded. (Is this true for $\mathbb{R}$?)

Closedness : Prove that $\{\, x \in \omega_1 : \{\, x \,\}$ is not open $\}$ is a closed subset of ω_1. Conclude from a previous exercise that it is a closed and unbounded set.

Club : A subset $C \subseteq \omega_1$ is said to be a **club** subset of ω_1 if it is closed and unbounded. Note that final segments (i.e. everything to the *right* of a point α) are club. Prove that if C_1 and C_2 are club subsets then $C_1 \cap C_2$ is club. (Is this true for $\mathbb{R}_{\text{usual}}$?) If you are really hungry show that the intersection of countably many club sets is again club.