Selected practice questions from past exams

DEFINITIONS

1) Discuss a method taught in class for removing (or modeling) seasonality of time series data.
2) Describe the Dickey Fuller unit root test.
3) Define strictly and weakly stationary time series.

Answer:

a. *Strictly stationary time series: The joint probability distribution of* $(X_t, X_{t+1}, \dots, X_{t+n})$ *is is exactly the same as that of* $(X_{t+k}, X_{t+k+1}, \dots, X_{t+k+n})$*, for any* t*,* k *and* $n \in Z$ *(integers).*

b. *Weak stationary time series: (i)* $E(X_t) = m < \infty$ *is a constant; (ii)*$var(X_t) = \sigma_X^2 < \infty$ *is a constant; (iii) the auto-covariance function* $\gamma(t, t+h)$ *is independent of time and a function of the distance between two time points, for all* $t, h \in Z$ *.*

4) State Wold decomposition and how this theorem supports the use of ARMA model. (5%)

Answer:

a. *A stochastic process* $\{V_t\}$ *is called deterministic if the values of* $V_{t+j}, j \geq 1$ *are perfectly predictable in terms of the span of its past observations, or* $sp\{X_t, -\infty < t \leq n\}$*. Thus, a deterministic process has the form of an AR(*∞*) process.*

b. *Wold decomposition: any zero-mean process* $\{X_t\}$ *which is not deterministic can be expressed a sum of* $X_t = U_t + V_t$*, where* $\{V_t\}$ *is a deterministic process,* $\{U_t\}$ *denotes an* $MA(\infty)$ *process, and* $\{U_t\}$ *and* $\{V_t\}$ *are uncorrelated.*

c. *That is, Wold decomposition implies that any zero-mean process can be expressed as the form of an* $ARMA(\infty, \infty)$ *process so we may be able to approximate any data/process by an higher order* $ARMA(p, q)$ *model.*

5) Define an invertible ARMA(p,q) process and state the reason why we discuss invertible processes in class.

Answer:

a. *Consider an* $ARMA(p, q)$ *process as follows:*

$$\phi(B)X_t = \theta(B)a_t, \quad a_t \sim WN(0, \sigma^2),$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ *and* $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$*. The above process is invertible if all zeros of* $\theta(B)$ *(the roots satisfies* $\theta(B) = 0$*) are outside unit circle.*

b. *Studying only the invertible process, we ensure that we are able to match any set of the autocorrelation functions to a unique ARMA model.*

6) Define partial autocorrelation functions (PACFs) and describe how to use PACFs for model identification.

Answer:

a. *Two definitions are discussed in class. First,*

Partial autocorrelation function (PACF)

The correlation between X_t and X_{t+k} after mutual linear dependency on the intervening variables, X_{t+1}, X_{t+2}, ..., and X_{t+k-1} has been removed.

- The conditional correlation $\phi_{kk} = corr(X_t, X_{t+k} | X_{t+1}, \dots, X_{t+k-1})$ may be used to define the partial autocorrelation functions in time series analysis.
- In this case, PACF between X_t and X_{t+k} can be obtained as the regression coefficient associated with X_t when regressing X_{t+k} on its k lagged variables X_{t+k-1}, X_{t+k-2}, ..., and X_t.

Second definition is as follows (also the definition given in the textbook):

The partial autocorrelation function of a stationary process at lag h, denoted as ϕ_{hh}, can be defined as the correlation between two prediction errors as follows:

- $\phi_{11} = Corr(X_{t+1}, X_t) = \rho(1)$
- $\phi_{hh} = Corr\left(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t\right), h \geq 2$, where
  - $\hat{X}_{t+h} = \beta_1 X_{t+h-1} + \beta_2 X_{t+h-2} + \cdots + \beta_{h-1} X_{t+1}$
  - $\hat{X}_t = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \cdots + \beta_{h-1} X_{t+h-1}$
  - $\beta_1, \dots, \beta_{h-1}$ are obtained by minimized mean squared error forecast

Note that $\phi_{hh} = 0 \; \forall \; h \geq p$ for $AR(p)$ processes.

b. *The theoretical value of PACF functions at lag h of an AR(p) process is zero if $h > p$. Using this fact, we may guess that the data comes from an AR(P) process preliminarily if the sample PACF plot cut off at lag p.*

BOX-JENKINS APPROACH (ARMA MODELS)

1. Autoregressive model of order 2

Answer the questions using the following AR(2) process

$$(1-0.5B)(1-0.1B)X_t = a_t, \quad a_t \sim NID(0,1), \qquad (2)$$

1) Is the AR(2) process in eqn. (2) stationary? Explain why? (3%)

Answer: *Consider the polynomial* $\phi(B) = (1-.5B)(1-.1B) = 0$*. The roots* $\frac{1}{.5} = 2$ *and* $\frac{1}{.1} = 10$ *are both outside unit circle so the AR(2) process in eqn. (2) is stationary.*

2) Find the autocorrelation function at lag 1 and 2 for the above AR(2) process (4%)

Answer: *We can rewrite eq. (2) as*

$$X_t - \underbrace{0.6}_{\phi_1} X_{t-1} \underbrace{+0.05}_{-\phi_2} X_{t-2} = a_t, \qquad a_t \sim NID(0,1)$$

Using the result,

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2), k \geq 1$$

and the above equation, we can solve for

$$\rho(1) = \frac{\phi_1}{1-\phi_2} = \frac{0.6}{1.05} = 0.5714286$$

$$\rho(2) = \frac{\phi_1^2}{1-\phi_2} + \phi_2 = \frac{0.6^2}{1.05} - 0.05 = 0.2928571$$

3) Find the partial autocorrelation functions at lag 1, 2 and 3 (3%)

Answer: *Using Yule=Walker equation, we have*

$$\phi_{11} = \rho(1) = \frac{\phi_1}{1-\phi_2} = 0.5714286$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \phi_2 = -0.05$$

$$\phi_{33} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}} = 0$$

2. Method of moment estimation

Consider a time series $\{x_t\}, t = 1, \dots, 100$ with sample autocovariances $\hat{\gamma}(0) = 1800$, $\hat{\gamma}(1) = 1200, \hat{\gamma}(2) = 600$. Suppose that we decide to fit $\{x_t\}$ using an AR(2) process as follows:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t, \quad a_t \sim NID(0, \sigma^2). \quad (3)$$

1) Find the Yule-Walker estimates of ϕ_1, ϕ_2 and σ^2.(10%)

Answer: *The Yule-Walker equations for the AR(2) model in eqn. (3) are given by*

$$\phi_1\gamma(1) + \phi_2\gamma(2) = \gamma(0) - \sigma^2$$
$$\phi_1\gamma(0) + \phi_2\gamma(1) = \gamma(1)$$
$$\phi_1\gamma(1) + \phi_2\gamma(0) = \gamma(2)$$

Rewrite the above equations as the matrix form

$$\Gamma_p \Phi = \gamma_p$$

and

$$\sigma^2 = \gamma(0) - \Phi'\gamma_p,$$

where $\Gamma_p = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}$, $\Phi = [\phi_1, \phi_2]'$ *and* $\gamma_p = [\gamma(1), \gamma(2)]'$. *Using these results and the sample autocovariances, we can solve for*

$$\hat{\Phi} = \hat{\Gamma}_p^{-1}\hat{\gamma}_p = \begin{bmatrix} 0.001 & -0.0006666667 \\ -0.0006666667 & 0.001 \end{bmatrix} \begin{bmatrix} 1200 \\ 600 \end{bmatrix} = \begin{bmatrix} 0.8 \\ -0.2 \end{bmatrix}$$

$$\widehat{\sigma^2} = \hat{\gamma}(0) - \hat{\phi}_1\hat{\gamma}(1) - \hat{\phi}_2\hat{\gamma}(2) = 1800 - 0.8 * 1200 + 0.2 * 600 = 960$$

2) Find the 95% confidence intervals for ϕ_1 and ϕ_2.(5% bonus question)

Answer: *The Yule-Walker estimator are asymptotically distributed as*

$$\hat{\Phi} - \Phi \to N\left(\mathbf{0}, \frac{\sigma^2 \Gamma_p^{-1}}{n}\right).$$

Replacing the sample estimates into the above result, we have

$$\hat{\Phi} - \Phi \to N\left(\mathbf{0}, \begin{bmatrix} 0.0096 & -0.0064 \\ -0.0064 & 0.0096 \end{bmatrix}\right).$$

Therefore, the 95% confidence interval for ϕ_1 *and* ϕ_2 *are respectively*

$$0.8 \pm 1.96 * \sqrt{0.0096} \text{ and } -0.2 \pm 1.96 * \sqrt{0.0096}$$

FORECAST

1. Consider an ARIMA(1,1,0) model,

$$(1-0.5B)(1-B)=a_t,\;\; a_t \sim NID(0,1).$$

   a) Write down the forecast function for origin t.
   b) What is the variance of the 1-step-ahead forecast error.

2. Consider the AR(1) model as follows:

$$(1-0.6B)(X_t-9)=a_t,\quad a_t \sim NID(0,1),\qquad (1)$$

   Suppose that we observe $(X_{97}, X_{98}, X_{99}, X_{100}) = (9.6,9,9,8.9)$.

   c) Is the process in eqn. (1) stationary? Why?
   d) Forecast $\{X_t\}$, t=101, 102, 103 and 104 and their associated 95% forecast limits.
   e) Suppose now that the observation at $t=101$ turns out to be $X_{101}=8.8$. Calculate $\hat{X}_{101}(l)$ for lead time, $l=1,2,3$, using "updating forecast".

3. Forecast an ARMA(1,1) model

$$X_t - 0.5X_{t-1} = a_t + 0.25a_{t-1},\quad a_t \sim NID(0,1),\qquad (*)$$

   Suppose that $(X_{97}, X_{98}, X_{99}, X_{100}) = (-0.7,-1,-0.8,-0.4)$. Answer the following questions:

   a) Write down the forecasting function for eqn. (*).
   b) Calculate the best linear forecast of $\mathrm{X}_{101}+\mathrm{X}_{102}+\mathrm{X}_{103}$.
   c) Calculate the 95% forecast (confidence) interval of the forecast in question 5b). For simplicity, use $Z_{0.975}=1$ in your calculation.

4. Consider the ARIMA(1,1,0) model

$$(1-B)(1+0.9B)X_t = a_t,\quad a_t \sim NID(0,\sigma^2).$$

   The most recent 8 observations for 1989 to 1996 were
   $(X_{89},\ldots,X_{96}) = (0,-0.1,-1.5,-2.2,-4.3,-4.9,-7.2,-6.3)$.

   a) Write out the recursive formula for forecasting X_{t+l} at original t. Consider $l=1,2,3$ and m.
   b) Is this process stationary? Why or why not?
   c) Is this process invertible? Why or why not?

d) Derive the formulas for predictions for 1997 to 1999 in terms of previously observed values. (These may be expressed in terms of other predictions, as long as you describe how to calculate each term before you use it another formula.)

e) Let $\bar{X}$ be the average of X_{97}, X_{98} and X_{99}. Use the values above and your formulas to calculate the estimate $\hat{\bar{X}}$ of $\bar{X}$. (You should give this estimate both as a formula and numerically.) Assume that all earlier values of the series are zero if you need them in your predictions.

f) Calculate the variance of the forecast error $\bar{X} - \hat{\bar{X}}$ in terms of σ^2.

- ***Remarks: Skip answers for question 1,2, and 4.***

  a) *Write down the forecasting function for eqn. (*).*

  Answer:

  Lead time=1: $\hat{X}_t(1) = 0.5X_t + 0.25\left(X_t - \hat{X}_{t-1}(1)\right)$

  Lead time h, h>1: $\hat{X}_t(h) = 0.5 \cdot \hat{X}_t(h-1)$

  b) *Calculate the best linear forecast of* $X_{101} + X_{102} + X_{103}$*. For simplicity, assume that* $\hat{X}_{99}(1) = 0$*.*

  Answer:

  The best linear forecast of $X_{101} + X_{102} + X_{103}$ *at origin* $t = 100$ *is* $\hat{X}_{100}(1) + \hat{X}_{100}(2) + \hat{X}_{100}(3)$*. Using the general forecast formula in 3 a). We have*

$$\hat{X}_{100}(1) = 0.5 \times -0.4 + 0.25(-0.4 - 0) = -0.3$$
$$\hat{X}_{100}(2) = 0.5 \times -0.3 = -0.15$$
$$\hat{X}_{100}(3) = 0.5 \times -0.15 = -0.075$$

  Therefore, our best linear forecast is -0.525.

  c) *Calculate the 95% forecast (confidence) interval of the forecast in question 5b). For simplicity, use* $Z_{0.975} = 1.96$ *in your calculation.*

  Answer:

  Consider a stationary ARMA(1) process

$$(1 - \phi B)X_t = (1 - \theta B)a_t, \quad a_t \sim NID(0, \sigma^2).$$

  The corresponding ψ_j *coefficients (that expresses the above process as a causal process) are given by*

$$\psi_j = \phi^{j-1}(\phi - \theta), \qquad j \geq 1.$$

  The forecast error at original t *and lead time* l *can be express as*

$$e_t(l) = \sum_{j=0}^{l-1} \psi_j a_{t+l-j}.$$

Students need to calculate confidence level using $var(e_t(1) + e_t(2) + e_t(3))$, where $e_t(h)$ is the forecast error at time t with lead time h.

PS. The variance of corresponding forecast errors is given by

$$var\big(e_t(l)\big) = \sigma^2 \left\{ 1 + \sum_{j=1}^{l-1} \left[\phi^{j-1}(\phi - \theta)\right]^2 \right\}.$$

5. Consider and ARMA(1,1) model*

$$(1 - 0.5B)(X_t - 4) = (1 + 0.5B)a_t, \quad a_t \sim NID(0,1).$$

Its one-step forecast at origin $t = 99$ is $\widehat{X}_{99}(1) = 2.09$, and

$$\{X_{99}, X_{100}, X_{101}, X_{102}, X_{103}, X_{104}, X_{105}\} = \{2.11, 1.39, 2.57, 4.11, 6.28, 4.89, 5.94\}.$$

We shall refer the above $ARMA(1,1)$ model as Model A. Use the above information to answer the following question.

a) [6%] Calculate the l step ahead forecast $\widehat{X}_{100}(l)$ for $l = 1,2,3$ under Model A.
b) [4%] Calculate the forecast error variance with origin $t = 100$ and lead time $l = 2,3$ under Model A.

- ***Remark: Forecast evaluation is not yet taught in class and won't be tested in the midterm test. That said, questions c) and d) are skipped.***

a) Calculate the l step ahead forecast $\widehat{X}_{100}(l)$ for $l = 1,2,3$.

<u>Answer</u>: *The difference equation form the above $ARMA(1,1)$ process at time $n + l$is given by*

$$X_{n+l} = 4 + 0.5(X_{n+l-1} - \mu) + a_{n+l} + 0.5a_{n+l-1}.$$

Using the conditional expectation given filtration at time n, we have

$$\hat{X}_n(1) = 4 + 0.5(X_n - 4) + 0.5\hat{a}_n, \quad \hat{a}_n = X_n - \hat{X}_{n-1}(1), \quad (1)$$

and

$$\hat{X}_n(l) = 4 + 0.5\left(\hat{X}_n(l-1) - 4\right), l \geq 2, \qquad (2)$$

Use the above results and let $n = 100$. We have

$$\hat{a}_{100} = X_{100} - \hat{X}_{99}(1) = 1.39 - 2.09 = -0.7,$$

i. $\hat{X}_{100}(1) = 4 + 0.5(1.39 - 4) - 0.5 \cdot 0.7 = 2.345,$

ii. $\hat{X}_{100}(2) = 4 + 0.5(2.345 - 4) = 3.1725,$

iii. $\hat{X}_{100}(3) = 4 + 0.5(3.175 - 4) = 3.58625.$

b) Calculate the forecast error variance for $l = 1,2,3$.

Answer: *Since* $\phi = 0.5 < 1$ *we can calculate the* ψ *weights as follows:*

$$(1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1 - \theta B.$$

Equating the coefficients of B^j *on both sides give*

$$\psi_j = 0.5^{j-1}(0.5 + 0.5) = 0.5^{j-1}, \quad j \geq 1. \qquad (3)$$

Since the forecast error variance is given by $var\big(e_n(l)\big) = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2$, *we have*

i. $var\big(e_{100}(1)\big) = 1$

ii. $var\big(e_{100}(2)\big) = 1 + \sum_{j=1}^{2-1} \left[0.5^{j-1}\right]^2 = 1 + 0.5^0 = 2$

iii. $var\big(e_{100}(3)\big) = 1 + \sum_{j=1}^{3-1} \left[0.5^{j-1}\right]^2 = 1 + 0.5^0 + 0.5^2 = 2.25$