from $\mathbb{R}$ to $\mathbb{R}^n$

Cartesian product

$\mathbb{R}^2$ is $\mathbb{R}\times\mathbb{R}$ together with algebraic structure of Vector space

We are designing properties for $\mathbb{R}^n$ so that we can extend our knowledge and intuition of one variable calculus to multivariate i.e. to $\underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n \text{ times}}$

Properties of $\mathbb{R}$

is ordered: $\forall a,b\in\mathbb{R}$ $a<b$ or $a>b$ or $a=b$

$\Downarrow$

Completeness axiom

lub, glb, bounded above

has algebraic Structure known as field

$a+b$ $\quad ka$ $\quad a/b$

$|a-b|$ is interpreted as distance between a and b

$|x-c|<\epsilon$ means $c-\epsilon<x<c+\epsilon$

or

$c-\epsilon$ $\quad c$ $\quad c+\epsilon$

Properties of $\mathbb{R}^n$

parallel to properties of $\mathbb{R}$

order

$\mathbb{R}^n$ is NOT ordered so we cannot have lub or glb so Completeness of $\mathbb{R}^n$ is defined differently

Through Cauchy Sequences See bottom of page 27 & Thm 1.20

Algebraic Structure

$\mathbb{R}^n$ has vectorspace properties $\mathbf{x}+\mathbf{y}$, $k\mathbf{x}$ but NOT $\mathbf{x}/\mathbf{y}$!!

multiplications on $\mathbb{R}^n$

Scalar

$k\mathbf{x}$ only extends $\mathbf{x}$ in magnitude

$\mathbf{x}\parallel\mathbf{y}$ if $\mathbf{y}=k\mathbf{x}$

in $\mathbb{R}^3$ only Cross product result is a vector

$\boxed{\mathbf{a}\times\mathbf{b}}$

$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$\mathbf{a}\perp\mathbf{a}\times\mathbf{b}$

$\mathbf{b}\perp\mathbf{a}\times\mathbf{b}$

$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$=$ area of parallelogram of $\mathbf{a},\mathbf{b}$

dot product

$\mathbf{x}\circ\mathbf{y}$

result is a number

$\mathbf{a}\circ\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ projection

orthogonality

$\mathbf{x}\perp\mathbf{y}$ if $\mathbf{x}\circ\mathbf{y}=0$

Euclidean norm

$|\mathbf{x}| = \sqrt{\mathbf{x}\circ\mathbf{x}}$

distance $d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$

Cauchy inequality

$|\mathbf{a}\circ\mathbf{b}|\le|\mathbf{a}||\mathbf{b}|$

or $-|\mathbf{a}||\mathbf{b}|\le\mathbf{a}\circ\mathbf{b}\le|\mathbf{a}||\mathbf{b}|$

or

$-1\le\dfrac{\mathbf{a}\circ\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\le 1$

angle

$\cos\theta =$

triangle $\le$

$|\mathbf{a}+\mathbf{b}|\le|\mathbf{a}|+|\mathbf{b}|$

or

$||\mathbf{a}|-|\mathbf{b}||\le|\mathbf{a}-\mathbf{b}|$