PROBLEM-SOLVING AND PROOFS: ASSIGNMENT 1
DUE FRIDAY, MARCH 3, 4PM.

Warm-up problems. These are completely optional.

1.14

(1) Let $a < b < c < d$ be real numbers. Express $[a, b] \cup [c, d]$ as a difference of sets.

(2) For what conditions on sets A and B does $A - B = B - A$. 1.15

(3) Suppose you play the coin game repeatedly, starting with a single pile of 5 coins. What happens? What if you start with a single pile of 6 coins? 1.16

Problems to be handed in. Solve three of the following four problems. One of the three must be Problem (4).

(1) Prove that $\sqrt{11}$ is irrational. You may use the fact that every integer can be uniquely decomposed as a product of primes.

(2) Let S denote the set of all prime numbers of the form $4k + 3$ with $k \in \mathbb{N}$. (So $3 \in S$, $7 \in S$, but $5 \notin S$.) Prove that S is infinite.

(3) Let $f : S \to T$ be a function, and let A and B be subsets of S. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. Give an example to show that the reverse inclusion need not hold.

(4) Let f and g denote functions from $\mathbb{R}$ to $\mathbb{R}$. Recall that such a function is *bounded* if there exists a real number M such that $|f(x)| < M$ for all $x \in \mathbb{R}$. Determine whether each of the following statements are true. If true, provide a proof. If false, provide a counterexample.
   - If f and g are bounded, then $f + g$ is bounded.
   - If f and g are bounded, then fg is bounded. 1.49
   - If $f + g$ is bounded, then f and g are bounded.
   - If fg is bounded, then f and g are bounded.
   - If $f + g$ and fg are bounded, then f and g are bounded.

   You may use the *triangle inequality* which states that for all $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|.$$