PLEASE HAND IN

FACULTY OF ARTS AND SCIENCE
UNIVERSITY OF TORONTO

PLEASE HAND IN

Final Examinations
MAT301H1F – Groups and Symmetry
Tuesday, December 14, 2011

Instructor: Prof. J. W. Lorimer
Duration – 3 hours
No Examination Aids Allowed

LAST NAME: ________________________________

FISRT NAME: ________________________________

STUDENT NUMBER: ________________________________

INSTRUCTIONS:

1. DO **THREE** questions out of **FOUR** from **PART A** and the **TWO** questions from **PART B.**
2. Write the final solutions in the pages provided.
3. There are 25 pages and 6 questions in this examination paper.

FOR EXAMINER ONLY		
Question	**Value**	**Mark**
PART A		
1.	20	
2.	20	
3.	20	
4.	20	
PART B		
5.	20	
6.	20	
TOTAL	**100**	

PART A

PART A

Do any **THREE QUESTIONS**.

1. *[20 marks]* DEFINE or EXPLAIN the following notions: (6 parts)

1. (a) *[3 marks]* The Klein-4- group.

1. (b) *[3 marks]* The canonical (natural) homomorphism of a normal subgroup of a group G.

1. (c) *[3 marks]* The kernel of a homomorphism.

1. (d) *[3 marks]* A quotient group.

1. (e) *[4 marks]* The internal direct product of n subgroups of a group G.

1. (f) *[4 marks]* The BASIS THEOREM and the FUNDAMENTAL THEOREM of finitely generated abelian group.

2. *[20 marks]* In S_3, let $H = <(12)>$ and $K = <(123)>$ (8 parts)

2. (a) *[2 marks]* Determine the order of (12) and (123).

2. (b) *[2 marks]* Show that $K = A_3$.

2. (c) *[2 marks]* Determine $(123)^{-1}$.

2. (d) *[2 marks]* Calculate $(123)(12)(123)^{-1}$.

2. (e) *[2 marks]* Show that H is NOT NORMAL in S_3.

2. (f) *[3 marks]* Show that $S_3 =< (12), (123) >$.

2. (g) *[3 marks]* Show that $S_3 = HK$.

2. (h) *[4 marks]* Show that S_3/A_3 is isomorphic to $\mathbb{Z}_2$.

3. *[20 marks]* Let G be a group with subgroups H and K. Define a relation $\sim$ on G by $a \sim b$ if and only if $a = hbk$ where $h \in H$ and $k \in K$.

3. (a) *[6 marks]* Show that $\sim$ is an equivalence relation on G.

3. (b) *[3 marks]* Show that the equivalence classes of $\sim$ are the sets HxK where $x \in G$.

3. (c) *[5 marks]* For each $x \in G$, prove that $x^{-1}Hx$ is a subgroup of G.

3. (d) *[6 marks]* Prove that $|HxK| = \dfrac{\circ(H) \circ (K)}{\circ(x^{-1}Hx \cap K)}$ if G is a finite group.

3. (d) (Continued)

4. *[20 marks]* Let $\varphi : G_1 \to G_2$ be a group homomorphism and H_i a normal subgroup of $G_i (i = 1, 2)$ so that $\varphi[H_1] \subseteq H_2$.

Define, $\varphi^* : G_1/H_1 \to G_2/H_2$ by $\varphi^*(H_1 g_1) = H_2 \varphi(g_1)$.

4. (a) *[8 marks]* Show that φ^* is a well defined homorphism.

4. (b) *[2 marks]* If φ is an epimorphism, show that φ^* is an epimorphism.

4. (c) *[3 marks]* If φ_{H_i} is the canonical (natural) homomorphism of $H_i (i = 1, 2)$ show that $\varphi^* \circ \varphi_{H_1} = \varphi_{H_2} \circ \varphi$.

4. (d) *[7 marks]* If φ is a monomorphism and $\varphi[H_1] = H_2$, prove that φ^* is a monomorphism.

PART B

Do **BOTH QUESTIONS**.

5. *[20 marks]* Let G be a group and Auto (G) its automorphism group. For each $g \in G, \varphi_g : G \to G$ defined by $\varphi_g(x) = gxg^{-1}$ for each $x \in G$, is an inner automorphism of G, and Inn (G) is the set of all inner automorphisms of G.

5. (a) *[4 marks]* For each $\psi \in$ Auto (G), prove that $\psi \circ \varphi_g \circ \psi^{-1} = \varphi_{\psi(g)}$ for each $g \in G$.

5. (b) *[4 marks]* Prove that Inn (G) is a normal subgroup of Auto (G).

5. (c) *[3 marks]* For each $g \in G$, prove that $\varphi_g = 1$ if and only if $g \in \mathbb{Z}(G)$, the centre of G.

5. (d) *[3 marks]* Define the map $\varphi : G \to Auto(G)$ by $\varphi(g) = \varphi_g$ for every $g \in G$. Show that φ is a homomorphism.

5. (e) *[6 marks]* Prove that $G/\mathbb{Z}(G)$ is isomorphic to Inn (G).

6 *[20 marks]* Let G be a finite group of order p^2q for distinct primes p and q.

6. (a) *[7 marks]* Prove that there exists subgroups $\mathbb{P}$ and $\mathbb{Q}$ so that

(i) $\circ(\mathbb{P}) = p^2$ and $\circ(\mathbb{Q}) = q$.

(ii) $\mathbb{P} \cap \mathbb{Q} = \{e\}$.

(iii) $G = \mathbb{P}\mathbb{Q}$.

6. (a) (Continued)

6. (b) *[3 marks]* Give an example to show that a group of order p^2q for distinct primes p and q is not necessarily abelian.

6. (c) *[6 marks]* Using the subgroups $\mathbb{P}$ and $\mathbb{Q}$ from (a) prove that G is abelian if and only if $\mathbb{P}$ and $\mathbb{Q}$ are normal.

6. (c) (Continued)

6. (d) *[4 marks]* If G is abelian, prove that $G \cong \mathbb{Z}_{p^2q}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_P \times \mathbb{Z}_q$.