**UNIVERSITY OF TORONTO**
**FACULTY OF ARTS AND SCIENCES**
**AUGUST 2013 EXAMINATIONS**
**MAT301H1Y - GROUPS & SYMMETRIES**

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DURATION: 3 HOURS
NO AIDS ALLOWED

Total: 90 points, 6 questions + 1 Bonus, 2 pages

(1) [7 points] Consider the groups $G = (\mathbb{R}, +)$, the real numbers under addition, and $H = (\mathbb{R}^+, \times)$, the *positive* real numbers under multiplication. Prove that $G$ and $H$ are isomorphic.
*Hint*: Think back to previous math courses, and maps you know taking addition to multiplication, or vice versa.

(2)(a) [7 points] Let $G = \{x \in \mathbb{R} \mid x \neq (-1)\}$. For any $x, y \in G$, define $x \star y = x + y + xy$, using regular addition and multiplication of real numbers. Prove or disprove that $(G, \star)$ is a group.

(b) [5 points] Let $H, K$ be any groups such that $H$ is simple (i.e.: the only normal subgroups of $H$ are $\{e\}$ and $H$). Let $\varphi : H \to K$ be a nontrivial homomorphism of groups. Prove that $\varphi$ is injective.

(3) Consider $\mathbb{R}^n$ as the set of $n \times 1$ column vectors of real numbers. Let $\mathrm{GL}(n, \mathbb{R})$ act on $\mathbb{R}^n$ via left matrix multiplication.

(a) [8 points] Determine the orbits of the action.

(b) [10 points] Pick a point in each orbit, and determine what its stabiliser is, up to isomorphism. (note: try to pick the points cleverly to make your analysis much easier)

(c) [4 points] Is this group action free? Is it transitive? Justify.

(4) Let $G = \mathbb{R}^* \oplus \mathbb{R}^*$, where $\mathbb{R}^*$ is the nonzero real numbers as a group under multiplication. Fix two integers $n, m \in \mathbb{Z}$, and define a map $\varphi : G \to \mathbb{R}^*$ by $\varphi(x, y) = x^m y^n$

(a) [8 points ] Prove that $\varphi$ is an homomorphism.

(b) [12 points] Let $H = \{(x, x^2) \mid x \in \mathbb{R}^*\} \subset G$. Prove $H$ is a subgroup of $G$, and prove $G/H$ is isomorphic to $\mathbb{R}^*$. (Hint: the first isomorphism theorem is supremely useful here)

(5)(a) [6 points] Find an element of order 20 in $A_{11}$

(b) [7 points] Let $\alpha, \beta \in S_{10}$ be $\alpha = (1,3,5,7,9)(2,4,6)(8,10)$ (we use commas to separate the elements since we have some with 2 digits), $\beta = (1,5,6)(2,5,4)(3,5,9)$. Compute $|\alpha^2\beta^2|$, and $|\beta\alpha\beta^{-1}|$

(c) [8 points] Using the same $\alpha$ as in part (*b*), determine all the $m$ such that $\alpha^m$ is a 5-cycle. (Hint: don't use brute force)

(6) [8 points] For $\langle 8 \rangle \leqslant \mathbb{Z}$, $\langle 48 \rangle \leqslant \mathbb{Z}$, prove $\langle 8 \rangle / \langle 48 \rangle$ is isomorphic to $\mathbb{Z}_6$.

BONUS QUESTION: [7 marks]

**Definition 1.** A *groupoid* is an object comprised of two sets, and a number of maps. The first set $X$ is called the *object space*; the second set $\Gamma$, which is called the *set of arrows*. A way to visualise this is to imagine the objects $X$ are a bunch of points sitting in the plane, and $\Gamma$ is a set of arrows between them (note: not necessarily all possible arrows). We have the following structure maps:

(1) Two maps, $s : \Gamma \to X$, $t : \Gamma \to X$, called the *source* and *target* maps, respectively. If we're imagining arrows between points, then for some arrow $g \in \Gamma$ with $s(g) = x$, $t(g) = y$ for $x, y \in X$, then $g$ is an arrow from $x$ to $y$:
$$\bullet_x \xrightarrow{g} \bullet_y$$
(2) A *product map*, defined for elements $g, h \in G$ such that $s(g) = t(h)$ (if two elements satisfy this, we say they are *composable*). The product takes the elements $(g, h)$ and assigns another element $g \circ h \in \Gamma$, satisfying the conditions
- $s(g \circ h) = s(h)$, $t(g \circ h) = t(g)$
- The map is associative: if $g, h, k \in \Gamma$ such that $s(g) = t(h)$, and $s(h) = t(k)$, we have $g \circ (h \circ k) = (g \circ h) \circ k$.

A way to visualise this product is $\bullet_x \xrightarrow{h} \bullet_y \xrightarrow{g} \bullet_z$, so $g \circ h$ is the arrow you get from $x$ to $z$ by composing these arrows.

We also have two important kinds of arrows: for every $x \in X$, there is an element of $\Gamma$ called the *identity at x*, denoted $1_x$ or $\mathrm{id}_x$ which satisfies $s(1_x) = t(1_x) = x$, and $g \circ 1_x = g$, $1_x \circ h = h$ for all $g, h \in \Gamma$ for which these products are defined. For any arrow $g \in G$ with $s(g) = x$, $t(g) = y$, we also have the *inverse* arrow $g^{-1}$ which goes from $y$ to $x$, such that $g^{-1} \circ g = 1_x$, $g \circ g^{-1} = 1_y$.
An important thing to note is that if two arrows share the same sources and targets, it does not mean they are the same arrow.

QUESTION: Let $\Gamma$ be a groupoid over $X$. Fix some point $x \in X$. Let
$$G_x = \{g \in \Gamma \mid s(g) = t(g) = x\}$$
all of the arrows which start at $x$ and end at $x$. Prove that $G_x$ is a group.
(note: not many part marks will be given for incomplete solutions)