Natural Logarithms

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
3. $\ln(x^n) = n \times \ln(x)$
4. $\ln(e^x) = e^{\ln(x)} = x$

Exponents

1. $x^m x^n = x^{m+n}$
2. $\frac{x^m}{x^n} = x^{m-n}$
3. $x^{-n} = \frac{1}{x^n}$
4. $x^0 = 1$
5. $(x^m)^n = x^{mn}$
6. $(x^m)(y^m) = (xy)^m$

The Derivative and Differentiation

Consider a continuous smooth function $y = f(x)$ and two points A and B on the graph of the function, where $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$.

The slope of the line joining *A* and *B* is $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$.

As Δx gets shorter, the slope of the line joining *A* and *B* approaches the slope of the tangent line at point x_0.

We say that the derivative of $y=f(x)$ at x_0 is the slope of the tangent line at the point x_0:

$$\left.\frac{dy}{dx}\right|_{x_0}=f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$f'(x_0)$ is the derivative of $y=f(x)$ at $x=x_0$.

Some rules of differentiation

In the following, *a*, *b* and *c* are constants.

1. If $f(x)=ax+b$, $f'(x)=a$
2. If $f(x)=ax^2+bx+c$, $f'(x)=2ax+b$
3. If $f(x)=x^n$, $f'(x)=nx^{n-1}$
4. If $h(x)=\sum_{i=1}^{n}g_i(x)$, $h'(x)=\sum_{i=1}^{n}g'_i(x)$
5. Product Rule: If $h(x)=f(x)g(x)$, then $h'(x)=f'(x)g(x)+f(x)g'(x)$
6. Quotient Rule: If $h(x)=\frac{f(x)}{g(x)}$ and $g'(x)\neq 0$, $h'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$
7. If $f(x)=e^x$, $f'(x)=e^x$
8. If $f(x)=e^{g(x)}$, $f'(x)=g'(x)e^{g(x)}$
9. If $f(x)=\ln(x)$, $f'(x)=\frac{1}{x}$
10. If $f(x)=\ln(g(x))$, $f'(x)=\frac{g'(x)}{g(x)}$
11. L'Hopitals Rule: Suppose that as $x\to a$ both $f(x)$ and $g(x)$ either both tend to 0, both tend to $+\infty$ or both tend to $-\infty$. Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

Higher order derivatives

If $y=f(x)$,

The first derivative is $\frac{dy}{dx}=f'(x)$

The second derivative is $\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d^2y}{dx^2}=f''(x)$

The first derivative is $\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]=\frac{d^3y}{dx^3}=f'''(x)$

Taylor Series Formula

The Taylor Series Formula will be used when we cover duration and convexity of cash flow sequences and Redington immunisation.

Consider the function $y=f(x)$ is differentiable as many times as required. If we know $f(x_0)$ and the associated derivative values, the value of the function at the point x_1 can be approximated using the n^{th} order Taylor series approximation:

$$f(x_1)\cong f(x_0)+(x_1-x_0)f'(x_0)+\frac{(x_1-x_0)^2}{2!}f''(x_0)+\ldots+\frac{(x_1-x_0)^n}{n!}f^{(n)}(x_0)$$

where $f^{(n)}$ is the n^{th} derivative of $y=f(x)$, and $n!=n(n-1)\ldots 1$ i.e., $5!=5.4.3.2.1=120$

For example, consider the exponential function e^x. Let $y=f(x)=e^x$ and set $x_0=0$. Using the Taylor series approximation, this can be written as:

$$e^{x_1} = f(x_1) \cong 1 + x_1 + \frac{(x_1)^2}{2!} + \ldots$$

Integration

If $y = f(x) = \frac{d}{dx}F(x)$, then $F(x) + c = \int f(x)dx$

Fundamental theorem of Integral Calculus

If the function $f(x)$ is continuous on the closed interval $[a,b]$ and if $F(x)$is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Some rules of integration

In the following, *a* and *b* are constants.

1. $\int_a^b x^n\, dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$
2. $\int_a^b e^x\, dx = \left. e^x \right|_a^b = e^b - e^a$

For example, $500\int_3^8 x^4\, dx = \left.\frac{500x^5}{5}\right|_3^8 = 100\left(8^5 - 3^5\right)$

Probability and Statistics

The section on stochastic interest rate models will assume a basic knowledge of statistics. The main results that we will be using are summarised below:

For a **discrete random variable** $\tilde{X}$, with probability function $p(x)=\Pr[\tilde{X}=x]$, the mean is:

$E[\tilde{X}]=\sum xp(x)$ and the variance is: $Var[\tilde{X}]=E[\tilde{X}^2]-(E[\tilde{X}])^2=\sum x^2.p(x)-\left(\sum xp(x)\right)^2$

For a **continuous random variable** $\tilde{X}$, with probability density function $f(x)$, the

probability $\mathrm{P}[a<\tilde{X}<b]=\int_a^b f(x)dx$

$\tilde{X}$ has mean: $E[\tilde{X}]=\int_{-\infty}^{+\infty} xf(x)dx$

and variance: $Var[\tilde{X}]=E[\tilde{X}^2]-(E[\tilde{X}])^2=\int_{-\infty}^{+\infty} x^2.f(x)dx-\left(\int_{-\infty}^{+\infty} xf(x)dx\right)^2$

If *a* and *b* are constants then $Var[a\tilde{X}+b]=a^2\times Var[\tilde{X}]$

The standard deviation of $\tilde{X}$ is $\sqrt{Var[\tilde{X}]}$

For a function $h(\bullet)$: $E[h(\tilde{X})]=\int_{-\infty}^{+\infty} h(x)f(x)dx$

If $\tilde{X}$ and $\tilde{Y}$ are independent random variables then $Var[\tilde{X}+\tilde{Y}]=Var[\tilde{X}]+Var[\tilde{Y}]$

We will also be using a number of continuous distributions:

Uniform distribution

$$f(x)=\frac{1}{b-a} \qquad \text{for } a<x<b$$

$$E\left[\tilde{X}\right]=\frac{a+b}{2}$$

$$Var\left[\tilde{X}\right]=\frac{(b-a)^2}{12}$$

Normal distribution

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad \text{for } -\infty<x<+\infty$$

$$E\left[\tilde{X}\right]=\mu$$

$$Var\left[\tilde{X}\right]=\sigma^2$$

Recall that if $\tilde{X}$ is normally distributed with mean and variance as above, then

$$P\left[a<\tilde{X}<b\right]=P\left[\frac{a-\mu}{\sigma}<\frac{\tilde{X}-\mu}{\sigma}<\frac{b-\mu}{\sigma}\right]=P\left[\frac{a-\mu}{\sigma}<\tilde{Z}<\frac{b-\mu}{\sigma}\right]$$

where $\tilde{Z}$ has a standard normal distribution (i.e. normal distribution with mean 0 and variance 1).

Statistical tables can be used with a standard normal variable to find probabilities.