- The final exam is **7pm → 10pm** on Tuesday, August 19, 2014. Please arrive about 10 minutes early. You can find all relevant information about the exam time/place/etc by following the link at the bottom of the following webpage. Please read the reminder on this page concerning rules for exams. **http://www.artsci.utoronto.ca/current/exams/reminder**
- The exam has 2 parts: One part is short answer (T/F and fill in the blanks), the second is long answer. For the long answer, you are expected to show your work and provided full solutions to earn full credit.
- The exam is cumulative and covers all material, up to and including the material covered in class August 12. More specifically it covers the material from class corresponding to 1.1→1.6, 2.1→2.6, 3.1→3.4 in the textbook.
- Please use the discussion board for posting questions while studying.
- Extra office hours will be as follows:
  Th Aug 14 @ 4-6pm (Lishak), M Aug 18 @ 3-5pm (Thind), Tu Aug 19 @ 2-4pm (Dranovski)
  Office hours will be in BA 6283, unless otherwise posted on the course page.
- There are no calculators, notes, books or other aids allowed.
- Remember to bring your student ID to the exam.
- Below is a list of some things that you should know. I make no claim that there is nothing missing from this list. I will promise that nothing on the test is independent from the course. (I'm not trying to trick you with this list, of course. I just do not want 100 people yelling at me after the test. So use the list below as a guide, but make sure you are good with the assignment and practice problems.)

Things to know for the test

- **Basics of Complex Numbers** - You should be familiar with operations on complex numbers, and how to represent complex numbers in regular and polar form. You should be able to know when to use $x + iy$ or $re^{i\theta}$ form, and how to convert between them. You should be familiar with modulus, Re, Im, and basic inequalities. You should be able to solve various equations over complex numbers.
- **Geometry and Topology of the Plane** - You should know the basics of topology of $\mathbb{C}$. In particular, you should be able to determine if a given set is open, closed, connected, convex. You should be able to determine the boundary of a given set. You should know how to write the equation of circles and lines in complex form, and be able to describe simple sets, such as disks, annuli and half planes.
- **Complex Functions and Limits** - You should know the basics of complex functions. You should know how to take limits, and determine continuity. You should be familiar with exponential, trig and logarithmic functions, and their properties. You should understand the subtlety of complex logarithms, and how to deal with some of the issues (e.g. non-continuity, multi-valuedness) of complex logarithm. You should be familiar with the various "branches" of logarithm, obtained by deleting a ray from 0.
- **Line Integrals and Green's Theorem** - You should be able to parametrize line segments, circle arcs, and piecewise curves made up of line segments and circle arcs. You should be able to compute line integrals. You should know the complex form of Green's Theorem, and how/when to use it. You should know the estimate

$$\left| \int_\gamma f(z)dz \right| \leq \max_{z\in\gamma} \left| f(z) \right| \cdot \text{length}(\gamma)$$

- **Analytic and Harmonic Functions, Cauchy-Riemann Equations** - You should know how the derivative of a complex function is defined. You should understand that this is a very strong condition, and there are many functions which are not analytic. You should know the Cauchy-Riemann (C-R) equations, and be able to use them to test/prove analyticity of a given function. You should know what a harmonic function is, and that the real and imaginary parts of an analytic function are harmonic. You should know how to find the "conjugate" harmonic function, to a harmonic function. You should know when a given function is, or is not, the real or imaginary part of an analytic function.

- **Sequences, Series and Power Series** - You should be able to determine convergence of sequences. You should know how to determine convergence of complex series, and be able to use tests such as ratio, comparison, root, etc., to determine absolute convergence. You should know how to determine the radius of convergence of a power series, and that a convergent power series is an analytic function on its domain of convergence. You should know the power series expansions for $e^z, \sin(z), \cos(z), \dfrac{1}{1-z}$, and you should be able to find power series for easy compositions or products of functions given by power series (e.g. find the power series for e^{z^2} using the power series for e^z.)
- **Cauchy Theorem and Cauchy Formula** - You should know the statement of Cauchy's Theorem, as well as the important Theorems in Ch. 2.3 that follow from Cauchy's Theorem. You should be able to use these Theorems to answer problems, compute line integrals. You should know Cauchy's Formula. You should be able to use it to compute line integrals, or values of a given function. Hint: If you're asked to compute a line integral over a curve you can't parametrize (e.g. an arbitrary curve joining two points) you should know to try to apply Cauchy's Theorem. You should know that the line integral of an analytic function depends only on the endpoints of the curve, and not that actual path taken. This is *very* useful.
- **Application of Cauchy Theorem and Formula** - You should be able to use these as tools to solve other problems, including computing real integrals by realizing them as line integrals, then using Cauchy's Formula to evaluate them. You should be able to recognize when/how to apply Cauchy's Formula to compute line integrals. You should know the formula for the power series of an analytic function centred at z_0 as well as the coefficients of the series. You should know how to write the kth derivative of a function in terms of a line integral.
- **Zeroes** - You should know what it means for a function f to have a zero of order n at a point z_0, and what that means for the power series of f centered at z_0. You should be able to find zeroes of simple functions and determine their order. Please be mindful that for rational functions $\frac{P}{Q}$ you need to fully factor both the top and bottom when trying to find zeroes and their orders. If a function $\frac{P}{Q}$ is not defined at z_0, it can't have a zero there. It is often useful to know that if f has a zero of order n at z_0 and g has a zero of order m at z_0, then $f \cdot g$ has a zero of order $n+m$ at z_0.
- **Isolated Singularities** - You should know what an isolated singularity is, and how they are classified. You should know that $e^{\frac{1}{z-z_0}}$ has an essential singularity at z_0. You should know how to find and classify all the isolated singularities of a function. For rational functions, you should factor top and bottom, and be mindful of factors that appear in both the numerator and denominator. You should be able to find the order of a pole, and to determine the Laurent series for a function around a pole.
- **Residues** - You should know how a residue of a pole is defined (as an integral), and you should know the various ways to compute it (by coefficient in Laurent expansion, by $\text{Res}(\frac{P}{Q} : z_0) = \frac{P(z_0)}{Q'(z_0)}$ when it applies...be warned, it does not always apply!!!!). You should know the Residue Theorem, and how to use it to compute line integrals.
- **Real Integrals** You should be able to use the Residue Theorem to compute real integrals. For this test, you should know how to compute real integrals of the form $\int \frac{P}{Q}$ where $\deg P + 2 \leq \deg Q$. You should also know how to compute real integrals of the form $\int \frac{P(x)}{Q(x)} \sin x dx$ or $\int \frac{P(x)}{Q(x)} \cos x dx$. For these types of questions, you will need to know how to apply Residue Theorem. I.e. what complex function to use, what curve to use, how to split up the resulting line integral into the integral you're after plus some leftover terms, as well as how to make estimates for the left over terms so that you can take a limit.

  You should know the following estimate for a polynomial. If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots +$

$a_1 z + a_0$, then we get the estimates for $|z| = R$ with R very large:

$$\frac{1}{2}|a_n|R^n \leq |p(z)| \leq 2|a_n|R^n$$

- **Argument Principle** - You should know the statement of the Argument Principle, and how to use it to compute line integrals. You should also be able to use it to determine the number of zeroes a polynomial can have in a given region, and how many poles a rational function can have in a given region. You should be familiar with the technique of approximating a polynomial with its leading term for $|z| = R$ very large, and how this can be used to determine the change in argument of $p(z)$ as z goes along a curve.
- **Rouché's Theorem** - You should be able to use Rouché's Theorem to find the number of zeros of a given function in a given region. You should be able to figure out what function to use as a comparison, in order to apply the theorem. Remember that zeros are counted with multiplicity. You should remember the Fundamental Theorem of Algebra.
- **Maximum Modulus Principle** - You should know what the Maximum Modulus Principle says, and how you can use this to verify whether or not a given function can have a maximum on a given region. (E.g. Can $u(x, y) = x^2 - y^2 = Re(z^2)$ have a maximum on the open unit disc?) You should also be able to use it to test whether a given function can be the real part of an analytic function.
- **Fractional Linear Transformations** - You should know what a FLT is, and the basic facts about FLT's. You should be able to relate a FLT with a matrix, and use this to compute composition and inverses of FLT's. You should know the formula for the FLT that takes a given triple (z_1, z_2, z_3) to $(0, 1, \infty)$ and how to use this to find the FLT that takes the triple (z_1, z_2, z_3) to (w_1, w_2, w_3). You should know that a FLT takes circles and lines to circles and lines and be able to figure out the image of a circle/line (or disk/halfplane) under a FLT.
- **Conformal Mappings** - You should know the definition of a conformal map. You should know that an injective (one-to-one) analytic function is conformal. You should know that a FLT is conformal. You should know that a composition of conformal maps is conformal. You should know that if $f'(z_0) \neq 0$, then f is conformal at z_0. You should be able to find conformal maps between simple regions, and explain why the map you constructed is conformal.
- **Sections & topics not covered** - 1.1.1, 2.1.1, 2.3.1, in 2.6 fractional powers and logarithms, 3.1.1, in 3.2 Schwarz Lemma, in 3.3 circulation by FLT, in 3.4 level curves, 3.4.1.

Extra Review Problems

Here are some extra review problems. Do as many as you need to feel comfortable and confident about the material. Don't forget to look back at the problems already assigned, either as HW, or as practice. (There is likely to be some repetition between these problems and the previous assignment and practice problems.)

1.1 Do whatever problems you need, if you think you need review on the basics.
1.2 Do whatever problems you need, if you think you need review on the basics.
1.3 # 1→8, 11
1.4 # 1, 9, 15, 39, 40
1.5 # 3, 5, 9, 23
1.6 # 1, 2, 3
2.1 # 18, 20(a,b)
2.2 # 1, 3, 5, 11
2.3 # 1, 3, 5, 9, 10
2.4 # 1, 5, 9, 13, 17
2.5 # 1, 3, 8, 9, 22(a,c) 23(a) Hint: don't use their approach, use partial fractions.
Also find a Laurent series for $f(z) = \dfrac{4z-5}{z^2-3z+2}$ centered at $z_0 = 1$.
2.6 # 3, 5, 23(a). Also, find $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}$ using residues.
3.1 # 3, 4, 11, 13
3.2 # 1 Also: Suppose that u satisfies $|u(x,y)| \leq \cos(x^2+y^2)$ for all $z = x+iy$. Is u the real part of an entire function?
3.3 # 3, 4(a,b,c), 5(b), 7(a) - Also, find the image of the unit disk $|z| < 1$ under the map $T(z) = \dfrac{2z+1}{z-i}$
3.4 # 1 - Also, find a bijective conformal map $f : S \to D$, where D is the disc of radius π centred at i, and $S = \{z \in \mathbb{C} \mid z - 3 = re^{i\theta} \text{ with } r > 0, \frac{-\pi}{3} < \theta < \frac{\pi}{6}\}$. Explain why the map you've constructed is conformal and bijective. (Hint: Sketch the region S first!)

Tips for Studying

Please (please, please, please, ...) don't wait until the night before the test to start studying. If you find out then that you don't know something, it is doubtful that you'll have anybody to help you.
Math is like everything else: you need to practice to succeed. Do as many problems as you can. The more you do, the easier they become.
Aim to get to the point where you can answer problems without looking things up in the book. Practice until you can't make a mistake, rather than until you get it right once.
For many people it can be helpful to write things out. Take your notes, or the book, and make a full list of all the definitions and theorems that we've talked about. The simple act of writing something explicitly yourself can help you internalize it.
Many of the things in this course (definitions, theorems, proofs) has a picture that go with it. For more visual thinkers, make a list of each definition and theorem together with the appropriate picture, if applicable.