

Tutorial 1

P40 (1.6)

(a). $X_t = \underbrace{\beta_1 + \beta_2 t}_{\text{Signal}} + \underbrace{w_t}_{\text{noise}}, w_t \sim WN(0, \sigma_w^2)$

$$E(X_t) = E(\beta_1 + \beta_2 t + w_t) = \beta_1 + \beta_2 t + E(w_t) = \beta_1 + \beta_2 t$$

if $\beta_2 \neq 0$, then non-stationary
if $\beta_2 = 0$, stationary

(b). $Y_t = X_t - X_{t-1}$ is stationary
 $= (\beta_1 + \beta_2 t + w_t) - (\beta_1 + \beta_2(t-1) + w_{t-1})$
 $= \beta_2 + w_t - w_{t-1}$

$$E(Y_t) = \beta_2$$

$$\gamma(0) = E[(Y_t - \beta_2)(Y_t - \beta_2)] = E[(w_t - w_{t-1})^2] = E(w_t^2 - 2w_t w_{t-1} + w_{t-1}^2) = 2\sigma_w^2$$

||

$$\gamma(t, t+0)$$

$$\gamma(1) = E[(w_{t+1} - w_t)(w_t - w_{t-1})]$$

$$= E(w_{t+1}w_t - w_{t+1}w_{t-1} - w_t^2 + w_t w_{t-1}) = -\sigma_w^2$$

$$\gamma(2) = E[(w_{t+2} - w_{t+1})(w_t - w_{t-1})] = 0$$

$$\gamma(h) = \begin{cases} 2\sigma_w^2 & h=0 \\ -\sigma_w^2 & h=1 \\ 0 & \text{o.w.} \end{cases}$$

(c). $V_t = \frac{1}{2q+1} \sum_{j=-q}^q X_{t-j}$ to show $E(V_t) = \beta_1 + \beta_2 t$

$$E(V_t) = E\left(\frac{1}{2q+1} \sum_{j=-q}^q X_{t-j}\right)$$

$$= \frac{1}{2q+1} \sum_{j=-q}^q E(X_{t-j})$$

$$= \frac{1}{2q+1} \sum_{j=-q}^q (\beta_1 + \beta_2 t)$$

$$= \beta_1 + \beta_2 t$$

$$E(aX + bY) = aE(X) + bE(Y)$$

Linearity of expectation

Fact: X_t & $X_t - f(t)$ have the same $\gamma(s, t)$

$$v_t' = v_t - \beta_1 - \beta_2 t = \left(\frac{1}{2q+1} \sum_{j=-q}^q [\beta_1 + \beta_2(t-j) + w_{t-j}]\right) - \beta_1 - \beta_2 t$$

$$= \frac{1}{2q+1} \sum_{j=-q}^q w_{t-j}$$

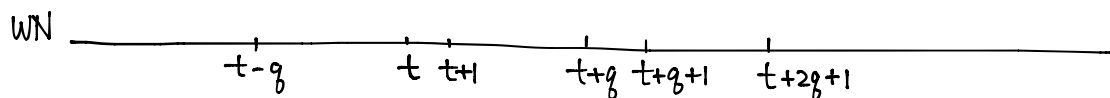
$$\gamma(0) = E\left[\left(\frac{1}{2q+1} \sum_{j=-q}^q w_{t-j}\right)^2\right]$$

$$= \left(\frac{1}{2q+1}\right)^2 E\left[\left(\sum_{j=-q}^q w_{t-j}\right)^2\right]$$

$$= \left(\frac{1}{2q+1}\right)^2 (2q+1) \sigma_w^2$$

$$= \sigma_w^2 / (2q+1)$$

$$E(w_t w_s) = 0 \text{ if } t \neq s$$



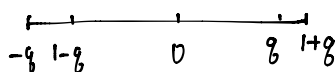
$$V'_t \quad V'_{t+1} \quad V_{t+2g+1} = \frac{1}{2g+1} \sum_{j=-g}^g X_{t+2g+1-j}$$

$$V_t \quad t+g+1$$

$$\gamma(2g+1) = \text{Cov}(V'_t, V'_{t+2g+1}) = 0$$

$$\gamma(h) = 0 \text{ if } h \geq 2g+1$$

$$\gamma(1) = \left(\frac{1}{2g+1}\right)^2 E\left[\left(\sum_{j=-g}^g w_{-j}\right)\left(\sum_{j=-g}^g w_j\right)\right] = \left(\frac{1}{2g+1}\right)^2 \cdot 2g \cdot \sigma_w^2$$



$$\gamma(h) = \begin{cases} \left(\frac{1}{2g+1}\right)^2 (2g+1-h) \sigma_w^2 & \text{for } 0 \leq h \leq 2g \\ 0 & \text{o.w.} \end{cases}$$

1.8

$$X_t = \delta + X_{t-1} + w_t \quad \text{with } X_0 = 0$$

(a). Show that $X_t = \delta + \sum_{k=1}^t w_k$

$$X_t = \delta + (\delta + X_{t-2} + w_{t-1})$$

$$= \dots = \delta + \sum_{k=1}^t w_k$$

(b). $\mu_X(t) = E(X_t) = E(\delta + \sum_{k=1}^t w_k) = \delta + \sum_{k=1}^t E(w_k) = \delta$

$$\gamma(s, t) = E[(X_t - \delta_t)(X_s - \delta_s)]$$

$$= E\left[\left(\sum_{k=1}^t w_k\right)\left(\sum_{j=1}^s w_j\right)\right]$$

$$= \min(s, t) \sigma^2 \text{ depends on } t$$

(d). Show $\rho(t-1, t) = \sqrt{\frac{t-1}{t}} \rightarrow 1$ as $t \rightarrow \infty$

$$\rho(t-1, t) = \frac{\gamma(t-1, t)}{\sqrt{\gamma(t-1, t-1)} \cdot \sqrt{\gamma(t, t)}} = \frac{\sqrt{t-1} \sigma^2 \sqrt{t-1}}{\sqrt{(t-1) \sigma^2} \sqrt{t \sigma^2}} = \sqrt{\frac{t-1}{t}}$$

(e). $X_t = \delta + X_{t-1} + w_t$

$$E(X_t) = \delta \text{ suggest to take 1st order diff op.}$$

$$y_t = X_t - X_{t-1} = X_{t-1} + w_t - X_{t-1} = w_t$$

$$E(y_t) = 0 \quad -\delta \quad \gamma(0) = \sigma^2, \gamma(h) = 0 \text{ if } h > 0.$$