University of Toronto

STA 447/2006: Stochastic Processes (Winter 2015)

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Handout to Chapter 3

Additional Examples of Compound Distributions

1) This example will be employed later when studying the *Rao damage process*.

Assume that compound Poisson r.v.

$$U = \sum_{i=1}^{N} B_i,$$

where r.v. N is Poisson-distributed with mean η, and $\{B_n, \ n \geq 1\}$ are i.i.d.r.v.'s with common Bernoulli 0/1 distribution with probability of success p, such that $\mathbf{P}\{B_i = 1\} = p$. Suppose that the r.v.'s $\{B_n, \ n \geq 1\}$ do not depend on the Poisson counting r.v. N.

Then the compound Poisson r.v. U is also Poisson-distributed, but with mean $\eta \cdot p$.

This important result can be derived by using the following identity between the moment generating functions $m_U(t)$, $m_N(t)$ and $m_B(t)$ of the r.v.'s U, N, and B_i, respectively:

$$m_U(t) = m_N(\ln \ m_B(t)). \tag{1}$$

Indeed,

$$m_N(t) = \exp\{\eta \cdot (e^t - 1)\} \tag{2}$$

and $m_B(t) = p \cdot (e^t - 1) + 1$. Hence,

$$\ln \ m_N(\ln m_B(t)) = \eta(e^{\ln \ (m_B(t))} - 1)$$

$$= \eta(m_B(t) - 1) = \eta(p \cdot (e^t - 1) + 1 - 1).$$

In view of (1),

$$\ln \ m_U(t) = \ln \ m_N(\ln \ m_B(t)) = \eta \cdot p \cdot (e^t - 1).$$

By (2), this coincides with the natural logarithm of the m.g.f. of a Poisson r.v. with mean $\eta \cdot p$.

2) This example will be employed later when studying the discrete-time *Galton-Watson* and the continuous-time *birth-and-death* branching stochastic processes. Members of the class of zero-modified geometric distributions introduced by formulas (3)–(4) below also emerge in some models of Queueing Theory.

Definition. A non-negative integer-valued r.v. $Y_{\gamma,r}$ is said to have zero-modified geometric law with real-valued parameters $\gamma \in (0,1)$ and
$r \in (max\{-1, -(1-\gamma)/\gamma\},\ 1)$ if

$$\mathbf{P}\{Y_{\gamma,r} = 0\} = \gamma, \tag{3}$$

and for each $k \in \mathbf{N}$,

$$\mathbf{P}\{Y_{\gamma,r} = k\} = \gamma \cdot (1-\gamma) \cdot (1-r) \cdot \{1 - \gamma + \gamma \cdot r\}^{k-1}. \tag{4}$$

A problem to practice.

Prove that a Bernoulli 0/1 sum of a *shifted* geometric variable can be represented as a particular zero-modified geometric r.v.

Namely, consider independent *Bernoulli* r.v. $D_p = \mathbf{B}(1,p)$ with probability of success $p \in (0,1)$ and *(shifted) geometric* r.v. $H_\gamma = Y_{\gamma,0} + 1$ such that $\gamma \in (0,\ min\{1, 2\cdot(1-p)\})$. Then

$$\sum_{k=1}^{D_p} H_\gamma = Y_{1-p,\ (1-p-\gamma)/(1-p)}. \tag{5}$$

Some useful comments:

Please, note that for the purposes of this course, the (shifted) geometric r.v. is the same r.v. which is usually called the (standard) geometric r.v. in most calculus-based courses of Probability Theory.

In order to establish (5), you should use an analogue to (1).