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Theorem 3.1, the implicit function theorem (IFT) presents the sufficient conditions for solving a functional equation $F(\boldsymbol{x}, y) = 0$ for one of the variables, in this case $y = f(\boldsymbol{x})$. That is, a equation determined by the function of $n+1$ variables F, can be reorganized so that one of the variables, in this case y is isolated and is written in terms of other n variables $\boldsymbol{x} = (x_1, x_2, \ldots x_n)$, and the function f is the formulas that expresses y. Note that

- The IFT can guarantee the existence of a solution such as $y = f(\boldsymbol{x})$ only on a neighborhood of a given point $(\boldsymbol{x}_0, y_0)$ which satisfies the equation $F(\boldsymbol{x}_0, y_0) = 0$ and this solution may not even make sense outside this neighborhood.
- The implicit function theorem is non constructive; that is, it does not present a formulas for f, nor could it specify a particular neighborhood on which this solution is possible. It only suggests that there is a neighborhood and on this neighborhood the solution is possible!
- However the IFT locally specifies a formula for $\frac{\partial y}{\partial x_j}$. This partial derivative can be used in the context of Calculus for any practical purposes such as linear approximation of the values of the function f near the point $\boldsymbol{x}_0$.
- A property of the continuous functions is that if it is positive at a particualr point then the values if the function will be all positive on a neighborhood that point. The choice of this neighborhood is not constructive; it is merely a consequence of the continuity where $\forall \epsilon \exists \delta \ldots$ See how they used this property of continuous functions in line 1 of the proof.
- Make sure to carefully inspect the applications of the Intermediate Value Theorem in line 7 from top of page 116. IVT is not constructive either. This is another source of non-constructivity of the IFT.
- See how they use the Mean Value Theorem in the second part of the proof, in the middle of page 116. Similarly the MVT is not constructive.
- Try to read the first two paragraphs of the proof to understand why f is continuous. See the how r_1 and r_0 play the role of ϵ and δ in the proof of continuity of f. You may have to read the two paragraphs a few times and carefully.
- Corollary 3.3 is trying to tell us that y is not more important than any other one of the x_j. Indeed any one of the variables can be solved and isolated and written in terms of the others (provided the conditions of the theorem hold true for that particular variable.) So, we can just ignore the special role of the variable y and assume that $\partial_j F(\boldsymbol{x}_0) \neq 0$ for some j. But this really means that $nablaF(\boldsymbol{x}_0) \neq \mathbf{0}$. Read the statement and the proof of 3.3.
- pay attention to see how the condition "F is C^1" is used in the first two lines of the proof. In this application F needs only be C^1 with respect to the variable y. Later, when they prove that the j^{th} partial derivative is the ratio of the two partial derivatives they use the fact that F is C^1 with respect to the variable x_j also. So if F is not completely C^1 then the conditions of the IFT are partially satisfied and a lot of the conclusion may still be valid.

- Theorem 3.9 is the system's version of the IFT. From Linear Algebra we know that for example a system of two equations and three unknowns x, y_1 and y_2 can be solved parametrically: one of the unknowns, say x becomes the parameter $x = t$ and the other two unknowns can be solved in terms of this parameter: $y_1 = g_1(t)$ and $y_2 = g_2(t)$. This example can be interpreted as: intersection of two planes is a line; or in general, the intersection of two surfaces is a curve.

- To understand the condition for the IFT (the general version) read the first half of page 118 to see that this condition in the case of a linear system is given in formula 3.8 (the invertibility of some matrix B, the matrix of partial derivatives given in formula 3.7.) It is interesting that this condition comes directly from the simplest case (linear case) but it is actually the condition in most general case.

- Note that theorem 3.9 does not give any formula for the partial derivatives, but in part (b) it suggests that we have to compute those partial derivatives by solving a system of partial derivatives.