FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 5

EQUATIONS OF VALUE

Cash flow models were briefly introduced in week 1. Recapping, in a cash flow model payments received are income and are positive cash flows. Payments made are outgo and are negative cash flows. The net cash flow equals income minus outgo.

The transactions in a cash flow model can be represented as a mathematical equation that balances the income and outgo according to the dates of the transactions and the time value of money (ie. present and accumulated values). An **equation of value** equates the present value of money received (income) to the present value of money paid out (outgo):

PV income = PV outgo

Analysing financial transactions involves constructing and solving equations of value.

For example,

- the 'fair price' to pay for an investment such as a fixed interest security is the present value of the proceeds from the investment discounted at the rate of interest required by the investor.

- the premium for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and expenses paid out by the insurer (not allowing for profit).

All of the equations of value examples and questions that you will encounter in this course will involve compound interest and not simple interest.

In order to solve an equation of value all income transactions and all outgo transactions must be condensed into two equations that are then equated to one another.

An equation of value can be expressed mathematically as follows: Assume that there are n income payments denoted by I_{t_1}, I_{t_2}, I_{t_3}, $\ldots, I_{t_n}$ and outgo payments denoted by O_{t_1}, O_{t_2}, O_{t_3}, $\ldots, O_{t_n}$, which occur at times t_1, $t_2, t_3, \ldots, t_n$. If we assume that interest is paid or due at the effective compound rate i, then the equation of value can be written:

$$\sum_{k=1}^{n} I_{t_k} \cdot (1+i)^{-t_k} = \sum_{k=1}^{n} O_{t_k} \cdot (1+i)^{-t_k}$$

or,

$$\sum_{k=1}^{n} I_{t_k} \cdot v^{t_k} = \sum_{k=1}^{n} O_{t_k} \cdot v^{t_k}$$

or,

$$\sum_{k=1}^{n} \left(I_{t_k} - O_{t_k}\right) \cdot v^{t_k} = 0$$

In relation to continuous payments, if we let $\rho_I(t)$ and $\rho_O(t)$ be the rates of receiving and paying money at time t respectively, then the equation of value is:

$$\int_0^n \rho_I(t) \cdot (1+i)^{-t}\, dt = \int_0^n \rho_O(t) \cdot (1+i)^{-t}\, dt$$

If the continuous payments above are constant or changing at a linear rate, they can be valued using the techniques for continuous annuities as previously defined. If the payments are a more complex function of time, then the above integral(s) will need to be completed.

When both discrete and continuous cash flows are present, the equation of value is:

$$\sum_{k=1}^{n} I_{t_k} \cdot (1+i)^{-t_k} + \int_0^n \rho_I(t) \cdot (1+i)^{-t}\, dt = \sum_{k=1}^{n} O_{t_k} \cdot (1+i)^{-t_k} + \int_0^n \rho_O(t) \cdot (1+i)^{-t}\, dt$$

In the notation above, the present value of all income transactions is equated to the present value of all outgo transactions.

For transactions involving compound interest, when solving for an unknown in an equation of value, we need not equate income and outgo at time 0. We can choose more than one reference time point and achieve the same solution. If the reference time point selected is such that some transactions occur prior to the reference time point, then those transactions would have to be accumulated to the reference time point, while others may have to be discounted to a present value at the reference time point.

An example is given below to illustrate cash flow models under the conditions of compound interest.

EXAMPLE
At the beginning of each week for four weeks Walt places a $1,000 bet, on credit, with his bookie. Walt agrees to repay the debt during the four weeks following the last bet. He pays $1,100 on the first three repayment dates. The transactions can be written out in a table as below.

Time t (weeks)	Income ($)	Outgo ($)
0	1,000	
1	1,000	
2	1,000	
3	1,000	
4		1,100
5		1,100
6		1,100
7		X

Find the amount that must be paid at time $t = 7$ in order to repay the debt if the bookie charges an effective weekly interest rate of 8% on all credit extended.

First choose a reference time point at which to formulate the equation of value. We will solve this equation using three different reference time points to illustrate that any reference time point can be used with compound interest transactions.

Solution
(i) Choose the initial date (t=0) as the reference time point.

We need to equate the present values of all income transactions (PV_I) and all outgo transactions (PV_O). If $v = (1.08)^{-1}$, then,

$$PV_I = \$1000(1 + v^1 + v^2 + v^3)$$
$$PV_O = \$1100(v^4 + v^5 + v^6) + Xv^7$$

Equating the present values, and solving for X,

$$\$1100(v^4 + v^5 + v^6) + Xv^7 = \$1000(1 + v^1 + v^2 + v^3)$$

$$X = \frac{\$1000(1 + v^1 + v^2 + v^3) - \$1100(v^4 + v^5 + v^6)}{v^7} = \$2{,}273.79$$

(ii) Choose the final date (t=7) as the reference time point.

We need to equate the accumulated values of all income transactions (AV_I) and all outgo transactions (AV_O).

$$AV_I = \$1000((1+i)^7 + (1+i)^6 + (1+i)^5 + (1+i)^4)$$
$$AV_O = \$1100((1+i)^3 + (1+i)^2 + (1+i)) + X$$

Equating the accumulated values, and solving for X,

$$\$1000((1+i)^7 + (1+i)^6 + (1+i)^5 + (1+i)^4) = \$1100((1+i)^3 + (1+i)^2 + (1+i)) + X$$

$$X = \$1000((1+i)^7 + (1+i)^6 + (1+i)^5 + (1+i)^4) - \$1100((1+i)^3 + (1+i)^2 + (1+i)) = \$2{,}273.79$$

(iii) Choose the date t=4 as the reference time point.

We need to equate the accumulated values of all income transactions (AV_I) and the present value of all outgo transactions (PV_O) at the reference time point.

$$AV_I = \$1000((1+i)^4 + (1+i)^3 + (1+i)^2 + (1+i))$$
$$PV_O = \$1100(1 + v^1 + v^2) + Xv^3$$

Equating the values, and solving for X,

$$\$1000((1+i)^4 + (1+i)^3 + (1+i)^2 + (1+i)) = \$1100(1 + v^1 + v^2) + Xv^3$$

$$X = \frac{\$1000((1+i)^4 + (1+i)^3 + (1+i)^2 + (1+i)) - \$1100(1 + v^1 + v^2)}{v^3} = \$2{,}273.79$$

<u>SOLVING FOR UNKNOWN TIME</u>

We will show how to solve for n for situations such as:

$$A = Bs_{\overline{n}|i} \text{ or } A = Ba_{\overline{n}|i}$$

when A, B and i are known.

Consider the problem: $A = Ba_{\overline{n}|i} = B\left(\dfrac{1-(1+i)^{-n}}{i}\right)$

eg. How long can I continue to withdraw an amount of *B* each year, if my initial bank account value is *A* and the effective annual rate of interest is *i*?

If we know A, B and i, then we can use logarithms to find n:

$$A = B\left(\frac{1-(1+i)^{-n}}{i}\right) \Rightarrow \left(1-\frac{Ai}{B}\right) = (1+i)^{-n} \Rightarrow \ln\left(1-\frac{Ai}{B}\right) = -n\ln(1+i)$$

$$\Rightarrow n = \frac{-\ln\left(1-\frac{Ai}{B}\right)}{\ln(1+i)} = \frac{-\ln\left(1-\frac{Ai}{B}\right)}{\delta}$$

Consider the problem: $A = Bs_{\overline{n}|i} = B\left(\frac{(1+i)^n - 1}{i}\right)$

If we know A, B and i, then we can use logarithms to find n:

$$A = B\left(\frac{(1+i)^n - 1}{i}\right) \Rightarrow \left(\frac{Ai}{B}+1\right) = (1+i)^n \Rightarrow \ln\left(\frac{Ai}{B}+1\right) = n\ln(1+i)$$

$$\Rightarrow n = \frac{\ln\left(\frac{Ai}{B}+1\right)}{\ln(1+i)} = \frac{\ln\left(\frac{Ai}{B}+1\right)}{\delta}$$

In some cases, we cannot find an analytic solution to n. Approximation methods have to be used in these circumstances.

EXAMPLE

Smith contributes a gross amount of 100 per month to a fund earning an effective monthly rate of 1%, with interest credited at the end of each month. After expenses and administration fees are deducted, this leaves a net amount of 90 invested per month. How many payments are required such that the accumulated value of the fund is larger than the total gross contributions immediately after the payment is made?

Solution

We need to find the smallest integer n such that:

$$90s_{\overline{n}|0.01} > 100n \qquad \Rightarrow 90\frac{(1.01)^n - 1}{0.01} > 100n \qquad \Rightarrow 9000(1.01)^n - 9000 - 100n > 0$$

This cannot be solved analytically, so we need an alternative method. Two possibilities include:

(i) trial and error - ie. select values of n until the problem is satisfied

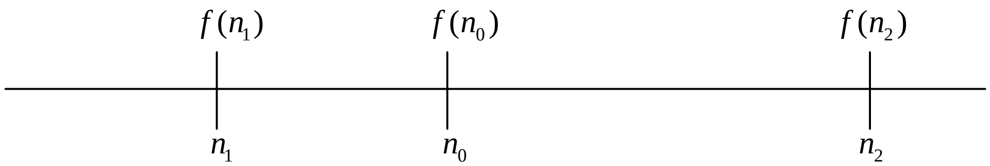
(ii) linear interpolation

LINEAR INTERPOLATION

Briefly, given n_1, n_2, $f(n_1)$ and $f(n_2)$, an estimate of $f(n_0)$ or n_0 on the basis of linear interpolation can be found from:

$$\frac{f(n_0)-f(n_1)}{f(n_2)-f(n_1)} \cong \frac{n_0-n_1}{n_2-n_1}$$

$$\Rightarrow n_0 \cong n_1 + \frac{f(n_0)-f(n_1)}{f(n_2)-f(n_1)} \cdot (n_2-n_1)$$



In order to use linear interpolation, two values n_1 and n_2 that lie close to the true time period must be selected and corresponding functions $f(n_1)$ and $f(n_2)$ must be determined.

Note: This approximation works best if the trial values are close to the true value.

Because the approximation works best with values close to the true value, interpolation may have to be carried out a number of times in order to find a solution to an equation of value.

Trial and error is generally used to identify values that are reasonably close to the true value. Trial and error can then be continued to find the true value, or linear interpolation can be used to identify an approximate answer.

In the previous example, you would generally use trial and error only, as the answer is an integer.

Solving by trial and error:

Test n = 20 $\Rightarrow 9000(1.01)^{20} - 9000 - 100\times 20 = -18.28$
Test n = 25 $\Rightarrow 9000(1.01)^{25} - 9000 - 100\times 25 = 41.89$
Test n = 22 $\Rightarrow 9000(1.01)^{22} - 9000 - 100\times 22 = 2.44$
Test n = 21 $\Rightarrow 9000(1.01)^{21} - 9000 - 100\times 21 = -8.47$

Therefore, it takes 22 months for the accumulated value of the fund to be greater than the total value of the contributions.

However, in the case of solving for an unknown interest rate, or an exact time value, you would generally use linear interpolation to generate an approximate answer. This is considered in the next section.

SOLVING FOR UNKNOWN INTEREST

For any given transaction, the equation of value may have no solutions, a unique solution, or several solutions.

The rate of interest i_0 that solves an equation of value is called the "yield", the "internal rate of return" or the "money-weighted rate of return" for the transaction. We will explore these terms in more detail in a later section of the course. In some cases it is possible to solve for the interest rate analytically from an equation of value.

For example, if we can reduce an equation of value to a quadratic formula:

$$a(1+i)^2 + b(1+i) + c = 0$$

then we can use the well-known quadratic solution $(1+i) = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the equation. This will often result in both a negative and positive solution. Most questions specify that a positive interest rate is required, in which case the negative root can be discarded.

Note that if an equation can be expressed in the quadratic form:

$$a\left((1+i)^n\right)^2 + b(1+i)^n + c = 0$$

for any value of n, then $(1+i)^n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and solutions for i can be found when n is known.

When an analytical solution cannot be found, we need to resort to approximation methods. The example below uses linear interpolation to identify the approximate interest rate.

EXAMPLE

Use linear interpolation to find an interest rate i_0 that satisfies the following equation:

$2s_{\overline{20}|i} + s_{\overline{8}|i} = 182.1938$

Solution

We first need to choose two interest rates i_1 and i_2 that produce results close to $f(i_0) = 182.1938$.

With i_1=15%, $2s_{\overline{20|}0.15} + s_{\overline{8|}0.15} = 218.6140 = f(i_1)$

This is greater than the value of 182.1938. We now need to choose another value to use in the interpolation.

With i_2=14%, $2s_{\overline{20|}0.14} + s_{\overline{8|}0.14} = 195.2826 = f(i_2)$

$$\frac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cong \frac{i_0 - i_1}{i_2 - i_1} \Rightarrow \frac{182.1938 - 218.614}{195.2826 - 218.614} \cong \frac{i_0 - 0.15}{0.14 - 0.15} \Rightarrow i_0 \cong 0.134$$

Ideally for linear interpolation, you would look to use figures i_k that give values $f(i_k)$ that lie on either side of the value ($f(i_0)$) that you are trying to find. However in the case above, 13.4% does give a satisfactory answer in this case.

Also if you needed a more accurate answer, a closer approximation could be obtained by using a smaller interval of approximation. For example, if we use: 13.2% and 13.6% as the two interest rates, then:

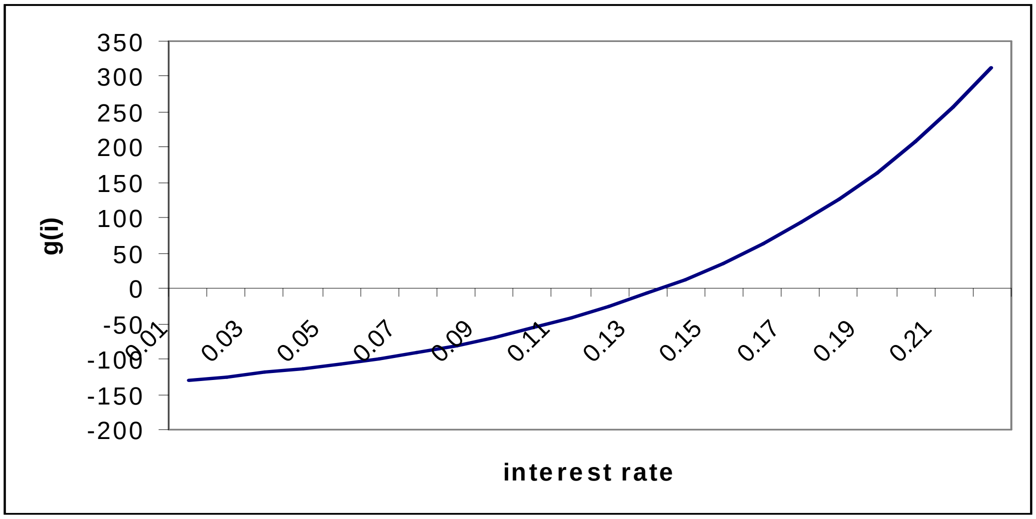
With i_1=13.2%, $2s_{\overline{20|}0.132} + s_{\overline{8|}0.132} = 178.5769 = f(i_1)$

With i_2=13.6%, $2s_{\overline{20|}0.136} + s_{\overline{8|}0.136} = 186.7244 = f(i_2)$

Use interpolation to find a value for i_0:

$$\frac{f(i_0) - f(i_1)}{f(i_2) - f(i_1)} \cong \frac{i_0 - i_1}{i_2 - i_1} \Rightarrow \frac{182.1938 - 178.5769}{186.7244 - 178.5769} \cong \frac{i_0 - 0.132}{0.136 - 0.132} \Rightarrow i_0 \cong 0.1338$$

The chart below shows the value of $g(i) = 2s_{\overline{20|}i} + s_{\overline{8|}i} - 182.1938$ as i changes. The solution to this equation is i_0 where $g(i_0) = 0$.



In practice, computer packages such as Excel can be used to quickly find exact solutions.

ACTUARIAL TABLES

Interest rate tables are available as part of the Formulae and Tables that you can download from WebCT, and will have available for exam purposes. The pages given are for a selection of effective rates of interest.

Interest rate tables are not required to be used in this course, but it is recommended that you get acquainted with the tables, and try to use them when answering questions. Using the tables will enable you to answer some questions more quickly than otherwise.

Interest rate tables will be provided during the mid-semester and final exams.

For the moment look at the 5% table. On the right of the page, equivalent nominal interest rates and discount rates convertible half-yearly, quarterly and monthly are given. The page also gives the effective discount rate and force of interest that is equivalent to $i = 5\%$, as well as a number of other useful constants.

On the left of the page, $(1+i)^n$, v^n, $s_{\overline{n|}}$, $a_{\overline{n|}}$, $(Ia)_{\overline{n|}}$ and $(Da)_{\overline{n|}}$ are given for a range of values of n.

A number of examples are given that show how the tables can be used in practice.

All of the examples are based on a hypothetical financial security which operates as follows:

A price P is paid by an investor in return for a series of interest payments of D payable at the end of each of the next n years and a final redemption payment of R payable at the end of the n years.

The equation of value for this investment is:

$$P = Da_{\overline{n}|i} + Rv_i^n$$

EXAMPLES

Solving for the present value (P)

Find P, if $D = 5$, $R = 125$, $i = 5\%$ and $n = 10$

Solution

$P = 5a_{\overline{10}|0.05} + 125v_{0.05}^{10}$

Using the Tables for 5%, $a_{\overline{10}|0.05}$ = 7.7217 and $v_{0.05}^{10}$ =0.61391

$\Rightarrow P = 5(7.7217) + 125(0.61391) = 115.3473$

That is, if a price of $P \cong \$115.35$ is invested at time 0 in an account returning 5% per annum effective, this will be enough to generate a payment of \$5 at the end of each of the next 10 years, and a repayment of \$125 at the end of 10 years.

Solving for the present value (P) **when** $D = iR$

Find P, if $D = 6.25$, $R = 125$, $i = 5\%$ and $n = 20$

Solution

$P = 6.25a_{\overline{20}|0.05} + 125v_{0.05}^{20}$

Using the Tables for 5%, $a_{\overline{20}|0.05}$ = 12.4622 and $v_{0.05}^{20}$ =0.37689

$\Rightarrow P = 6.25(12.4622) + 125(0.37689) = 125$

In fact, for any value of n, if $D = iR$ then $P = R$. We can prove this algebraically:

$$P = iRa_{\overline{n}|i} + Rv_i^n = iR\left(\frac{1 - v_i^n}{i}\right) + Rv_i^n = R\left(1 - v_i^n\right) + Rv_i^n = R$$

This can also be proved by general reasoning: Suppose that I have a sum of money R and deposit this in a bank account that pays an effective annual interest rate i. If the interest is paid out to me at the end of each year, I will receive interest payments of iR at the end of each year. At the end of n years after receiving the last interest payment, my initial capital of R will be repaid. Therefore, since the amount invested to generate this stream of cash flows is R, the value P of a security that generates equivalent cash flows must be the same, and $P = R$.

We will be discussing this result, and these types of cash flows in detail later in the course when we cover fixed interest securities.

Solving for the redemption amount (R)

Find R, if $D = 3$, $P = 100$, $i = 5\%$ and $n = 15$

Solution

$$100 = 3a_{\overline{15}|0.05} + Rv_{0.05}^{15}$$

Using the Tables for 5%, $a_{\overline{15}|0.05}$ = 10.3797 and $v_{0.05}^{15}$ =0.48102

$$\Rightarrow 100 = 3(10.3797) + R(0.48102)$$

$$\Rightarrow R = \frac{100 - 3(10.3797)}{0.48102} = 143.156$$

Solving for the timing of a payment (n)

Find n, if $D = 2.8883$, $P = 100$, $i = 5\%$ and $R = 130$

Solution

$$100 = 2.8883a_{\overline{n}|0.05} + 130v_{0.05}^{n}$$

$$100 = 2.8883\left(\frac{1 - v_{0.05}^{n}}{0.05}\right) + 130v_{0.05}^{n} = 57.766\left(1 - v_{0.05}^{n}\right) + 130v_{0.05}^{n} = 57.766 + 72.234v_{0.05}^{n}$$

$$\Rightarrow v_{0.05}^{n} = 0.58468$$

Using the tables we can see that $v_{0.05}^{11} = 0.58468$.

Therefore, $n = 11$.

Annuities payable m-thly or continuously.

The tables can also be useful for finding the value of annuities payable m-thly or continuously.

For example, to find $a^{(12)}_{\overline{10}|0.05}$ we can use the fact that: $a^{(12)}_{\overline{10}|0.05} = \frac{i}{i^{(12)}} a_{\overline{10}|0.05}$

Using the Tables, $\frac{i}{i^{(12)}}$=1.022715, so $a^{(12)}_{\overline{10}|0.05} = \frac{i}{i^{(12)}} a_{\overline{10}|0.05} = 1.022715 \cdot (7.7217) \cong 7.8971$

As another example, to find $\bar{s}_{\overline{10}|0.05}$ we can use the fact that: $\bar{s}_{\overline{10}|0.05} = \frac{i}{\delta} s_{\overline{10}|0.05}$

Using the Tables, $\frac{i}{\delta}$=1.024800, so $\bar{s}_{\overline{10}|0.05} = \frac{i}{\delta} s_{\overline{10}|0.05} = 1.024800 \cdot (12.5779) \cong 12.8898$

INFLATION

It is often important to factor inflation into measuring investment returns since inflation erodes the purchasing power of money over time.

A measurement of the return on an investment after taking inflation into account is known as the "real return", the "real interest rate", or the "inflation-adjusted return".

The real rate of interest is the rate interest after allowing for the effect of inflation.

EXAMPLE
\$100 is invested at time 0 in return for \$120 at time 1. This implies a 20% per annum effective rate of return. However, if inflation has been 5% per annum over the year, then \$120 at time 1 would only have the purchasing power of $\frac{120}{1.05} = \$114.29$ in terms of the value of money at time 0. The inflation-adjusted or real rate of interest is then:

$\frac{14.29}{100} = 14.29\%$.

We can instead work with inflation-adjusted year-end dollars. In this instance, 5% (\$5) of the initial investment of \$100 is needed to maintain dollar value against inflation. The remainder of the investment growth (\$20-\$5=\$15) is the real amount of growth in year-end dollars. \$105 is the value of the initial investment in year-end dollars. The real rate of interest is then:

$\frac{15}{105} = 14.29\%$.

The real rate of interest is often approximated by $i - r$, where r is the rate of inflation and i is the effective interest rate. However, $i - r = 0.20 - 0.05 = 0.15$ is only an approximation and is ***not*** the exact real rate of interest:

$i-r$ ***is <u>not</u> the real rate of interest***

The key point in calculations when finding real rates of interest, is that we need to express all payments in units of purchasing power at the same date.

If we use year-end dollars as the reference date then, in general, an investment of 1 at the start of the year will grow to $1+i$ by the end of the year. $1+r$ of this amount is needed to maintain dollar value against inflation. The remainder $(1+i)-(1+r)=i-r$ is the real amount of growth in yr-end dollars. The investment of 1 has an inflation-adjusted value of $1+r$ in yr-end dollars, so, therefore,

$$i_{real} = \frac{\textit{value of amount of real return (yr-end dollars)}}{\textit{initial value of invested amount (yr-end dollars)}}$$

$$\boxed{i_{real} = \frac{i-r}{1+r}}$$

In the above example,

$$i_{real} = \frac{0.20-0.05}{1+0.05} = \frac{0.15}{1.05} = 14.29\%$$

It follows from this formula that:

$$\boxed{1+i_{real} = \frac{1+i}{1+r}}$$

Rather than a fixed level of inflation, real rates of return often have to be found when inflation varies according to an index such as the CPI. An example is given below which illustrates how calculations should be performed in a simple situation involving inflation indices:

EXAMPLE

$100 is invested at time 0 in return for $8 at time 1 and $108 at time 2. What is the real rate of return if the inflation index at time 0 was $Q_0 = 150$; at time 1, $Q_1 = 156$; and at time 2, $Q_2 = 170$?

Solution
We can solve this question by expressing the payments in units of purchasing power at time 0. Alternatively, the problem could be solved by first expressing the payments in units of purchasing power at time 1 or 2. The key point is that all payments should be expressed in units of purchasing power at the same date.

Expressing all payments in terms of purchasing power at time = 0:

\$8 at time 1 would have the purchasing power of $8\left(\frac{Q_0}{Q_1}\right) = 8\left(\frac{150}{156}\right) = \$7.69$ in dollar-value at time 0.

\$108 at time 2 would have the purchasing power of $108\left(\frac{Q_0}{Q_2}\right) = 108\left(\frac{150}{170}\right) = \$95.29$ in dollar-value at time 0.

The real rate of return in this instance could be found by solving the equation:

$$100 = 7.69v^1 + 95.29v^2$$

where $v = (1 + i_{real})^{-1}$

$$95.29v^2 + 7.69v \ -100 = 0$$

$$v \ = \frac{-7.69 \pm \sqrt{7.69^2 - 4 \times 95.29 \times -100}}{2 \times 95.29} = \frac{-7.69 + \sqrt{38175.14}}{190.58} = 0.98486 \quad \text{(ignore negative root)}$$

$$(1 + i_{real})^{-1} = 0.98486$$

$$i_{real} = \frac{1}{0.98486} - 1 = 0.0154$$

Longer equations need to be solved by using numerical methods such as linear interpolation. Alternatively, for an exact answer, you can use “Goal Seek” in Excel.