Question 1. [7 MARKS]

Consider the statement:

(S1) $A \Rightarrow (B \vee C)$.

Assuming that statement (S1) is true, give the best answer for each of the following questions:

Part (a) [1 MARK]

What can be concluded from (S1), if A is true?

$\boldsymbol{(B \vee C)}$ **is true.**

Part (b) [1 MARK]

What can be concluded from (S1), if B is true?

Nothing. (Also accepted, the consequent is true.)

Part (c) [1 MARK]

What is the converse of (S1)?

$\boldsymbol{(B \vee C) \Rightarrow A}$**.**

Part (d) [2 MARKS]

What is the contrapositive of (S1)? (Work the negation(s) all the way in.)

$$\begin{aligned} \neg(B \vee C) \Rightarrow \neg A &\iff (\neg B \wedge \neg C) \Rightarrow \neg A \\ &\iff B \vee C \vee \neg A \end{aligned}$$

Part (e) [2 MARKS]

What is the negation of (S1)? (Work the negation(s) all the way in.)

$$\begin{aligned} \neg(A \Rightarrow (B \vee C)) &\iff \neg(\neg A \vee (B \vee C)) \\ &\iff \neg\neg A \wedge \neg(B \vee C)) \\ &\iff A \wedge \neg B \wedge \neg C \end{aligned}$$

Question 2. [11 MARKS]

Consider the domain $D = \{$all CSC courses and all MAT courses$\}$, and the predicate symbols $C(x)$: "x is a CSC course", $M(x)$: "x is a MAT course", and $P(x, y)$: "course x is a prerequisite for course y".

Using only these symbols (in addition to appropriate connectives and quantifiers), give a clear symbolic statement that corresponds to each given English sentence. Quantifiers may **only** be over the domain D.

Part (a) [1 MARK]

CSC108 is a prerequisite for CSC148.

$$\boldsymbol{P(\mathbf{CSC108},\mathbf{CSC148})}$$

Part (b) [2 MARKS]

There is no prerequisite for CSC104.

$$\neg(\exists x \in D, P(x, \mathrm{CSC104})) \textbf{ or } \forall x \in D, \neg P(x, \mathrm{CSC104})$$

Part (c) [2 MARKS]

Every course has a prerequisite.

$$\forall x \in D, \exists y \in D, P(y, x)$$

Part (d) [2 MARKS]

No course is a prerequisite for itself.

$$\forall x \in D, \neg P(x, x) \textbf{ or } \neg\exists x \in D, P(x, x)$$

Part (e) [2 MARKS]

Some CSC course has a prerequisite.

$$\exists x \in D, C(x) \wedge (\exists y \in D, P(y, x))$$

Part (f) [2 MARKS]

Every MAT course has a prerequisite.

$$\forall x \in D, M(x) \Rightarrow (\exists y \in D, P(y, x))$$

Question 3. [12 MARKS]

The following terms are used frequently to describe logical statements in CSC165. For each of them:

(i) Write a **definition** of the term, in English.

(ii) Write a **statement in English** that is true and is an example of a statement that meets the definition of the term.

(iii) After defining suitable domain(s) and/or predicate(s), write a **statement in logic** that is true and is an example of a statement that meets the definition of the term.

Part (a) [4 MARKS]

A universally quantified statement.

(i) definition: **A universally quantified statement is a statement that makes a claim about all objects in a domain.**

(ii) statement in English: **All CSC165 lectures are under an hour in length.**

(iii) domains, predicates and statement in logic:
Let $D = \{\text{CSC165 lectures}\}$, $P(x)$: "lecture x was under an hour". $\forall x \in D, P(x)$.

Part (b) [4 MARKS]

A tautology.

(i) definition: **A tautology is a logical statement that is always true, independent of the domains or predicates involved.**

(ii) statement in English:
Either this statement is a tautology or it isn't.

(iii) domains, predicates and statement in logic:
$P \vee \neg P$ or $\forall D \in \mathcal{D}, \forall P \in \mathcal{P}, \forall x \in D, P(x) \vee \neg P(x)$, where $\mathcal{D}$ is the set of all possible domains and $\mathcal{P}$ is the set of all possible predicates on $\mathcal{D}$.

Part (c) [4 MARKS]

A vacuous truth.

(i) definition: **A vacuous truth is an implication that is always true because its antecedent/assumption is always false.**

(ii) statement in English:
All unicorns are pink and purple.

(iii) domains, predicates and statement in logic:
$\forall x \in \mathbb{R}, x^2 < 0 \Rightarrow x = 42$.

Question 4. [6 MARKS]

Show that $\exists x \in D, (P(x) \Rightarrow Q(x))$ is equivalent to $(\forall x \in D, P(x)) \Rightarrow (\exists x \in D, Q(x))$. Justify each step of your derivation. You may use the list of standard equivalences given below.

$$
\begin{array}{lll}
 & \boldsymbol{\exists x \in D, (P(x) \Rightarrow Q(x))} & \\
\iff & \boldsymbol{\exists x \in D, (\neg P(x) \vee Q(x))} & \textbf{(implicatation)} \\
\iff & \boldsymbol{(\exists x \in D, \neg P(x)) \vee (\exists x \in D, Q(x))} & \textbf{(quantifier distributivity)} \\
\iff & \boldsymbol{\neg(\forall x \in D, P(x)) \vee (\exists x \in D, Q(x))} & \textbf{(quantifier negation)} \\
\iff & \boldsymbol{(\forall x \in D, P(x)) \Rightarrow (\exists x \in D, Q(x))} & \textbf{(implication)}
\end{array}
$$

Standard Equivalences (where P, Q, $P(x)$, $Q(x)$, *etc.* are arbitrary sentences. All quantifications are over a domain D.)

- *Commutativity*
  $P \wedge Q \iff Q \wedge P$
  $P \vee Q \iff Q \vee P$
  $P \Leftrightarrow Q \iff Q \Leftrightarrow P$
- *Associativity*
  $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R$
  $P \vee (Q \vee R) \iff (P \vee Q) \vee R$
- *Identity*
  $P \wedge (Q \vee \neg Q) \iff P$
  $P \vee (Q \wedge \neg Q) \iff P$
- *Absorption*
  $P \wedge (Q \wedge \neg Q) \iff Q \wedge \neg Q$
  $P \vee (Q \vee \neg Q) \iff Q \vee \neg Q$
- *Idempotency*
  $P \wedge P \iff P$
  $P \vee P \iff P$
- *Double Negation*
  $\neg\neg P \iff P$
- *DeMorgan's Laws*
  $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
  $\neg(P \vee Q) \iff \neg P \wedge \neg Q$
- *Distributivity*
  $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- *Implication*
  $P \Rightarrow Q \iff \neg P \vee Q$
- *Biconditional*
  $P \Leftrightarrow Q \iff (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- *Renaming* (where $P(x)$ does not contain variable y)
  $\forall x, P(x) \iff \forall y, P(y)$
  $\exists x, P(x) \iff \exists y, P(y)$
- *Quantifier Negation*
  $\neg\forall x, P(x) \iff \exists x, \neg P(x)$
  $\neg\exists x, P(x) \iff \forall x, \neg P(x)$
- *Quantifier Commutativity*
  $\forall x, \forall y, S(x, y) \iff \forall y, \forall x, S(x, y)$
  $\exists x, \exists y, S(x, y) \iff \exists y, \exists x, S(x, y)$
- *Quantifier Distributivity* (where S does not contain variable x)
  $S \wedge \forall x, Q(x) \iff \forall x, S \wedge Q(x)$
  $S \vee \forall x, Q(x) \iff \forall x, S \vee Q(x)$
  $S \wedge \exists x, Q(x) \iff \exists x, S \wedge Q(x)$
  $S \vee \exists x, Q(x) \iff \exists x, S \vee Q(x)$
  $(\forall x, P(x)) \wedge (\forall x, Q(x)) \iff \forall x, (P(x) \wedge Q(x))$
  $(\exists x, P(x)) \vee (\exists x, Q(x)) \iff \exists x, (P(x) \vee Q(x))$

Question 5. [6 MARKS]

At a murder trial, four witnesses give the following testimony.

Alice: If either Bob or Carol is innocent, then so am I.

Bob: Alice is guilty, and either Carol or Dan is guilty.

Carol: If Bob is innocent, then Dan is guilty.

Dan: If Bob is guilty, then Carol is innocent. In addition, Bob is innocent.

Is it possible that everyone is telling the truth? Justify your response.

Let A represent the statement "Alice is innocent.", B represent the statement "Bob is innocent.", C represent the statement "Carol is innocent.", and D represent the statement "Dan is innocent.".

We are given the 4 statements:

(1) $(B \vee C) \Rightarrow A$

(2) $\neg A \wedge (\neg C \vee \neg D)$

(3) $B \Rightarrow \neg D$

(4) $(\neg B \Rightarrow C) \wedge B$

If all statements are true,

- **(4) tells us B.**
- **Then (1) tells us A.**
- **While (2) tells us $\neg A$.**

But we cannot have both A and $\neg A$. We have a contradiction and so it is not possible that everyone is telling the truth.

(There are many other arguments that lead to the same conclusion.)