Method of undetermined coefficients

Consider ODE

$ay'' + by' + cy = g$

Depending on form of $g(t)$, are following trial for particular solution $Y(t)$

$g(t)$	$Y(t)$
$e^{\lambda t}$	$Ae^{\lambda t}$
$\cos(\mu t)$ $\sin(\mu t)$	$a\cos(\mu t) + b\sin(\mu t)$
t^k	$a_0 t^k + a_1 t^{k-1} + \cdots + a_k$
Sum of product of these	sums of products of these

This works, unless this $Y(t)$ solves the homogeneous equation $L[Y]=0$
In this case, multiply by t.

Example: $y'' + 2y - 4y = e^t(\cos(2t))$

Trial: $Y(t) = e^t(a\cos(2t) + b\sin(2t))$

Plug in, compare coeff's of $e^t\cos(2t)$, $e^t\sin(2t)$ to find a and b.

Another method: Use $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = Re(e^{i\theta})$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Consider $y'' + 2y' - 4y = e^t e^{2it} = e^{(1+2i)t}$

Trial: $Z(t) = Ce^{(1+2i)t}$

$Z = Ce^{(1+2i)t}$

$Z' = C(1+2i)e^{(1+2i)t}$

$Z'' = C\underbrace{(1+2i)^2}_{-3+4i}e^{(1+2i)t}$

$L[Z] = Ce^{(1+2i)t}((-3+4i) + 2(1+2i) - 4)$

$= Ce^{(1+2i)t}(-5+8i) \overset{!}{=} e^{(1+2i)t}$

$\Rightarrow C = \dfrac{1}{-5+8i} = \dfrac{-5-8i}{(-5+8i)(-5-8i)} = \dfrac{-5-8i}{25+64} = \dfrac{-5-8i}{89}$

$\Rightarrow Z(t) = \left(\dfrac{-5}{89} - \dfrac{8}{89}i\right)e^{(1+2i)t}$

$Z(t) = \dfrac{-5}{89}e^t(\cos 2t) + \dfrac{8}{89}e^t\sin(2t) + i(\cdots)$

$\Rightarrow Y(t) = Re(Z(t))$ solves the original problem.

$y'' + 2y - 4y = e^t\cos(2t) = Re(e^{2it})e^t$

Variation of problem

Consider general linear 2nd order ODE

$$\underbrace{y''+py'+qy}_{L[y]}=g$$

Sps y_1, y_2 are a fund. set of solutions of $L[y]=0$. Thus, general solution of $L[y]=0$ is $y=c_1y_1+c_2y_2$

Idea: Try solution of $L[Y]=g$ by replacing c_1, c_2 with functions v_1, v_2.

$$Y=v_1y_1+v_2y_2$$
$$Y'=v_1'y_1+v_2'y_2+v_1y_1'+v_2y_2'$$

Big Trick: Impose condition $v_1'y_1+v_2'y_2=0$

$$Y'=v_1y_1'+v_2y_2'$$
$$Y''=v_1'y_1'+v_2'y_2'+v_1y_1''+v_2y_2''$$

$$\Rightarrow L[Y]=Y''+pY'+qY$$
$$=\underbrace{v_1\ L[y_1]}_{0}+\underbrace{v_2\ L[y_2]}_{0}+v_1'\ y_1'\ +v_2'\ y_2'$$
$$\overset{!}{=} g(t)$$

$\Rightarrow$ get two conditions for two unknowns v_1', v_2'.

$$v_1'y_1+v_2'y_2=0$$
$$v_1'y_1'+v_2'y_2'=g$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}=\begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}=\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ g \end{pmatrix}=\frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ g \end{pmatrix}=\frac{1}{W}\begin{pmatrix} -y_2g \\ y_1g \end{pmatrix}$$

$$\Rightarrow v_1'=\frac{1}{W}(-y_2g),\ v_2'=\frac{1}{W}(y_1g)$$

The upshot is: Put $v_1=\int\frac{1}{W}(-y_2g)dt$

$v_2=\int\frac{1}{W}(y_1g)dt$

Then $Y=v_1y_1+v_2y_2$ solves $L[Y]=g$

Example: $t^2y''-2y=3t^2-1$

hom. equation $L[y]=0$ has solution $y_1(t)=t^2, y_2(t)=t^{-1}$.

$$W=W[y_1,y_2]=\begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix}=-1-2=-3$$

$$v_1=-\frac{1}{3}\int(-t^{-1})(3t^2-1)=\frac{1}{3}\int(3t-t^{-1})=\frac{t^2}{2}-\frac{1}{3}\ln(t)$$
$$v_2=-\frac{1}{3}\int t^2(3t-1)=-\int t^4+\frac{1}{3}\int t^2=-\frac{t^5}{5}+\frac{t^3}{9}$$

$$\Rightarrow Y(t) = \left(\frac{t^2}{2} - \frac{t}{3}\ln(t)\, t^2 + \left(-\frac{t^5}{5} + \frac{t^3}{9}\right) t^{-1} = \cdots\right.$$