CSC165H1 S - Exercise 4
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Question 1:

(a) The contrapositive of the original statement is:

If $m,n\in Z$, with $(m+n)^2$ even, then m^2-n^2 is even.

Proof:

Assume $m,n\in Z$ # m and n are generic integers
  Assume $(m+n)^2$ is even
    Assume m+n is odd # proof by contradiction
      Then $\exists k\in Z$ such that $m+n = 2k+1$
      Then $(m+n)^2 = (2k+1)^2 = m^2+4k+1 = 2(2k^2+2k)+1$
      Then $(m+n)^2$ is odd. # Contradiction with assumption that $(m+n)^2$ is even
    Then m+n is even.
    Then $\exists k\in Z$ such that $m+n = 2k$
    Then $n = 2k-m$
    Then $m^2-n^2 = m^2-(2k-m)^2 = m^2-4k^2+4km-m^2 = -4k^2+4km = 2(-2k^2+2km)$
    Then m^2-n^2 is even.
  Then $(m+n)^2$ is even => m^2-n^2 is even.
Then $\forall m,n\in Z$, $(m+n)^2$ is even $\Rightarrow$ m^2-n^2 is even.
Then $\forall m,n\in Z$, m^2-n^2 is odd $\Rightarrow (m+n)^2$ is odd. # the contrapositive of the statement above.

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(b) Proof:

Assume $m,n\in Z$. # m and n are generic integers
  Assume $(m+n)^2$ is odd.
    Assume m+n is even # proof by contradiction
    Then $\exists k\in Z$ such that $m+n = 2k$
    Consider $j\in Z$ such that $m+n = 2j$
    Then $(m+n)^2 = (2j)^2 = 4j^2 = 2(2j^2)$
    Then $(m+n)^2$ is even. # contradiction with the assumption that $(m+n)^2$ is odd
    Then m+n is odd.
    Then $\exists k\in Z$ such that $m+n = 2k+1$
    Consider $i\in Z$ such that $m+n = 2i+1$
    Then $n = 2i-m$
    Then $m^2-n^2 = m^2-(2i+1-m)^2 = m^2-(2i+1)^2+2m(2i+1)-m^2$
    $= -(2i+1)^2+2m(2i+1) = -4i^2-4i-1+4mi+2m$
    $= 2(-2i^2-2i+2mi+m)-1$ #some algebra
    Then m^2-n^2 is odd.
  Then $(m+n)^2$ is odd $\Rightarrow$ m^2-n^2 is odd.
Then $\forall m,n\in Z,(m+n)^2$ is odd $\Rightarrow$ m^2-n^2 is odd. ■

(c)

Conclusion: $\forall m,n\in Z$, $(m+n)^2$ is odd if and only if m^2-n^2 is odd.

Question 2:

Proof:

Assume x is a real number. # x is a typical real number

Assume $x<=0$ or $x>=1$. # negation of the consequent

Case1: $x<=0$

Then $(x^2+1)^2>=1$ and $1+2x<=1$. # some algebra

Then $(x^2+1)^2>=1+2x$. # since $(x^2+1)^2>=1$ and $1+2x<=1$

Then $x^4+2x^2+1>=1+2x$. #some algebra

Then $x^4+2x^2-2x>=0$. # minus 1 on both sides and move 2x to the LHS

Case2: $x>=1$

Then $x^4+2x^2-2x=x^4+2x(x-1)$. # some algebra

Let a, b, c be three real numbers such that $a=x^4$, $b=x-1$, $c=2bx$. # name them a, b, c

Then $a>=1$, $b>=0$, $c>=0$. # since $x=\sqrt[4]{a}=b+1=c/2b>=1$

Then $x^4+2x^2-2x=a+b+c>=1>0$. #some algebra

Then $x^4+2x^2-2x>=0$. # since when $x<=0$ or $x>=1$, the statement is True

Then $x<=0$ or $x>=1 \Rightarrow x^4+2x^2-2x>=0$. # assuming $x<=0$ or $x>=1$ leads to $x^4+2x^2-2x>=0$

Then $x^4+2x^2-2x<0 \Rightarrow 0<x<1$. # implication is equivalent to contrapositive

Then if x is a real number such that $x^4+2x^2-2x<0$, then $0<x<1$.

$\blacksquare$

Question 3:

(a) The statement is true.
Proof:
Assume $x \in R$.

Let a be the integer part of x and b be the fraction part of x such that x = a+b and a,b has the same positive or negative sign with x. # divide a real number x into two parts

Then $\lfloor x \rfloor = \lfloor a+b \rfloor = a$ when x>=0 and $\lfloor x \rfloor = \lfloor a+b \rfloor$=a-1 when x<0.
$\lceil x \rceil = \lceil a+b \rceil = a$+1 when x>=0 and $\lceil x \rceil = \lceil a+b \rceil$=a when x<0.
by definitions of $\lfloor x \rfloor$ and $\lceil x \rceil$ functions

Then when x>0, -x<0, so $\lceil -x \rceil = \lceil -a-b \rceil$=-a and $-\lfloor x \rfloor = -\lfloor a+b \rfloor$=-a.
when x<0, -x>0, so $\lceil -x \rceil = \lceil -a-b \rceil$=-a+1 and $-\lfloor x \rfloor = -\lfloor a+b \rfloor$=-a+1.
When x=0, -x=0, so $\lceil -x \rceil$= 0 and $-\lfloor x \rfloor$=0
solve the functions

Then $\lceil -x \rceil = -\lfloor x \rfloor$. # since in all three cases the equation holds
Then $\forall x \in R, \lceil -x \rceil = -\lfloor x \rfloor$. # assume x was a typical real number

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(b) The statement is false.

In order to disprove the statement, we prove its negation:
$\exists$ x∈R, $\exists$ n∈N, $\lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor$.

Proof:
Let x=-1.5. # choose a particular element that will work
Then x∈R. # verify that the element x is in the domain
Let n=2. # choose a particular element n that will work
Then n∈N. # verify that the element is in the domain
Then $n \cdot \lfloor x \rfloor = 2\lfloor -1.5 \rfloor$=2(-2)=-4
substitute -1.5 for x and substitute 2 for n
But $\lfloor n \cdot x \rfloor = \lfloor 2 \cdot (-1.5) \rfloor = \lfloor -3 \rfloor$=-3.
substitute -1.5 for x and substitute 2 for n
Then $\lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor$. # definition of inequality
Then n=2, $\lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor$. # introduce existential of n
Then x=-1.5, n=2, $\lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor$. # introduce existential of x
Then $\exists$x∈R, $\exists$n∈N, $\lfloor n \cdot x \rfloor \neq n \cdot \lfloor x \rfloor$.
Then $\neg(\forall x \in R, \forall n \in N, \lfloor n \cdot x \rfloor = n \cdot \lfloor x \rfloor)$.
equivalent form of the above statement

Therefore the original statement is false.

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