§14 - Arbitrary Products

1 Motivation

We have seen that taking the product of two topological spaces gives rise to new topological spaces. We have studied how some topological invariants behave under products. Generally things were very well understood and the proofs were straightforward; for example our proof that the product of two separable spaces is again separable required almost no thought. Moving forward we will investigate infinite products of topological spaces which will give us access to some very interesting topological spaces. We will see that the proofs sometimes generalize in a straightforward way, but often the proofs will be more difficult, or there might not even be a proof in the general case!

The first thing to investigate is how to make sense of "$\mathbb{R}^{\mathbb{N}}$" as an infinite product. It turns out that there will be many different topologies we could give it.

2 Introducing $\mathbb{R}^{\mathbb{N}}$

How do we think about the *set* $\mathbb{R}^2$? Since kindergarten we have thought of this set as the collection of all pairs of real numbers. That is fine, but then how do we think about $\mathbb{R}^{\mathbb{N}}$? Saying it is the collection of all "$\mathbb{N}$-tuples" of real numbers is the right idea, but we should make sure that this makes mathematical sense. That leads to this definition:

Definition. *We define the set* $\mathbb{R}^{\mathbb{N}} := \{f : \mathbb{N} \longrightarrow \mathbb{R}\}$*, the collection of all functions from* $\mathbb{N}$ *to* $\mathbb{R}$*. This is also the collection of all sequences of real numbers.*

It Works! Exercise: Show that there is a natural correspondence between $\mathbb{R}^2$ and $\{ f : \{0,1\} \longrightarrow \mathbb{R} \}$, the collection of all functions from $\{0,1\}$ to $\mathbb{R}$.

The important question is "how do we visualize $\mathbb{R}^{\mathbb{N}}$?". This is important because as we know, visualizing $\mathbb{R}^3$ can be challenging and visualizing $\mathbb{R}^{10}$ is near impossible. So here's one way to do it: Write down an un-ending list of "$\mathbb{N}$-many" columns of $\mathbb{R}$, written from the center to the right. Then we write a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ on this picture by writing the real number $f(1)$ in the first column, $f(2)$ in the second column, $f(3)$ in the third column, etc..

[Include a picture here for yourself]

Now what does it mean for two functions $f, g \in \mathbb{R}^{\mathbb{N}}$ to "be close" or to "be within $\epsilon > 0$ of each other"? There are many ways for us to say that two functions are close. Here are some potential options:

1. Say that f, g are within ϵ of each other if $|f(n) - g(n)| < \epsilon$ for all $n \in \mathbb{N}$;
2. We could also say that f, g are within ϵ of each other if $|f(n) - g(n)| < \epsilon$ for $n \leq 10$;
3. Or, we could say that f, g "are close" if whenever you have a sequence of $\{\epsilon_n : n \in \mathbb{N}\}$ then $|f(n) - g(n)| < \epsilon_n$ for all $n \in \mathbb{N}$.

3 Bounded Metrics

It turns out that each of these ideas will give rise to a different topology. To make these notions precise we have to introduce the notion of a bounded metric. This seems arbitrary, but it is necessary to avoid difficulties like having "infinite distance" between two functions. For example, what is a good suggestion for the distance between the constant function 0 and the identity function $f(n) = n$? We can't say that they have "infinite distance" (whatever that means) but we should agree that they are fairly far apart.

Definition. *Let (X, d) be a metric space. The metric $\overline{d}(x, y) = \min\{\, d(x, y), 1 \,\}$ is called the* ***bounded metric corresponding to*** *d. A metric with bounded range is called a* ***bounded metric****.*

Strictly speaking we should check that $\overline{d}$ really is a metric, but the proof is straightforward. The next proposition says that we can always use bounded metrics to generate a metrizable space.

Proposition. *Let $(X, \mathcal{T})$ be a metrizable space. $\mathcal{T}$ is generated by a bounded metric.*

Proof. Let (X, d) be a metric space, and let $\overline{d}$ be the corresponding bounded metric. Let $B_\epsilon(x)$ be a basic open set in (X, d) and let $p \in B_\epsilon(x)$. Let $0 < \delta < \min\{\frac{1}{2}, \frac{\epsilon - d(x,p)}{2}\}$. By the triangle inequality we see that $B_\delta(p) \subseteq B_\epsilon(x)$. We also see that for any $q \in B_\delta(p)$, that $B_\delta(p)$ is a basic open set that contains q. Together these tell us that the metrics generate the same topology. $\square$

Now we are in a position to give two different metrics on $\mathbb{R}^{\mathbb{N}}$. Here let $\overline{d}$ be the bounded metric corresponding to the usual metric on $\mathbb{R}$.

1. Let $\rho_{\text{unif}}(f, g) := \sup_{n \in \mathbb{N}}\{\overline{d}(f(n), g(n))\}$, where $f, g \in \mathbb{R}^{\mathbb{N}}$, which is called the **Uniform Metric** on $\mathbb{R}^{\mathbb{N}}$.
2. Let $\rho_{\text{prod}}(f, g) := \sup_{n \in \mathbb{N}}\{\frac{\overline{d}(f(n), g(n))}{n}\}$, where $f, g \in \mathbb{R}^{\mathbb{N}}$ which is called the **standard product metric** on $\mathbb{R}^{\mathbb{N}}$.

We know that we can generate topologies from these metrics, and those will give rise to topologies on $\mathbb{R}^{\mathbb{N}}$. We will investigate them in a moment. For the uniform metric topology, the metric is quite natural, so we often work directly with the metric. For the standard product metric topology, the metric is a bit awkward, so we tend to use a more topological description of the topology.

4 Topologies on Products

We now describe what an arbitrary cartesian product of sets is, so that we may define topologies on it. It looks quite abstract, but keep in mind the example of $\mathbb{R}^{\mathbb{N}}$.

Definition. *Let X, I be non-empty sets. Define X^I to be the collection of all functions $f : I \longrightarrow X$.*

Definition. *Let I be a non-empty set and let $\{ A_\alpha : \alpha \in I \}$ be a collection of non-empty sets. We define*

$$\prod_{\alpha \in I} A_\alpha$$

to be the collection of functions

$$f : I \longrightarrow \bigcup_{\alpha \in I} A_\alpha$$

such that $f(\alpha) \in A_\alpha, \forall \alpha \in I$.

Now this second defintion looks like a mess, but it's just saying that f sends α to something in A_α. The language of functions is because it is messy to talk about "I-tuples".

Example. *Let $I = \mathbb{N}$, and let $A_n := [0, 2n] \subseteq \mathbb{R}$. Then*

$$\prod_{\alpha \in I} A_\alpha = \prod_{n \in \mathbb{N}} [0, 2n] = [0, 2] \times [0, 4] \times [0, 6] \times \dots$$

and the constant 0 function is in this product, but the constant 3 function is not (because $3 \notin A_1$).

Now we are ready to describe topologies on products. The first topology we present is the most naive, and turns out to be very misbehaved!

Definition. *Let $X = \prod_{\alpha \in I} A_\alpha$, with $(A_\alpha, \mathcal{T}_\alpha)$ each topological spaces. Define a basis for $\mathcal{T}_{box}$, the box topology, as*

$$\mathcal{B}_{box} := \{ \prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \}$$

Here the basis for the box topology consists of arbitrary products of open sets. For example, in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$, the set

$$\mathbb{R} \times \mathbb{R} \times (0,1) \times (0, \frac{1}{2}) \times (0, \frac{1}{3}) \times \dots$$

is open.

Constrast that with this next topology:

Definition. *Let $X = \prod_{\alpha \in I} A_\alpha$, with $(A_\alpha, \mathcal{T}_\alpha)$ each topological spaces. Define a basis for $\mathcal{T}_{prod}$, the product topology, as*

$$\mathcal{B}_{prod} := \{ \prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \text{ and at most finitely many } U_\alpha \neq A_\alpha \}$$

So again we take products of open sets, but this time we make sure that most (i.e. all but finitely many) of the coordinates contain just the open set A_α. For example, in $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ the set

$$\mathbb{R} \times (0,1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$$

is open.

First, let's notice how these topologies compare:

Proposition. *Let $\mathcal{T}_{prod}, \mathcal{T}_{unif}$ and $\mathcal{T}_{box}$ be the product topology, uniform metric topology and box topology on $\mathbb{R}^{\mathbb{N}}$. Then*

$$\mathcal{T}_{prod} \subseteq \mathcal{T}_{unif} \subseteq \mathcal{T}_{box}$$

5 Projection Maps

Not only does $\mathbb{R}^{\mathbb{N}}$ have nice properties, it is also quite natural as is made evident from the next theorem which is a direct generalization of the finite product case.

Theorem. *X^I_{prod} is the weakest topology where each projection map is continuous.*

Generalize! Exercize!: Prove the previous theorem by finding the corresponding theorem from §8, and copying down the proof but replacing all references to "1 or 2" with "$\forall \alpha \in I$".

The previous theorem actually tells us that we could take "the weakest topology where each projection map is continuous" as our definition of the product topology. This has a certain appeal for some mathematicians. It also allows us to define the "weak topologies" on a product. This won't be particularly interesting for us in this course, but it is a neat way to get topologies.

Definition. *Let X be a set, and $\{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \}$ be a collection of topological space. For each $\alpha \in I$, let $f_\alpha : X \longrightarrow X_\alpha$ be a function. The* ***weak topology on X induced by*** *$\{ f_\alpha : \alpha \in I \}$ is the weakest topology on X for which each f_α is continuous.*

Now just as in the finite product case we have a theorem about when a function on the product is continuous.

Proposition. *Let $(X, \mathcal{T})$ and $(Y_\alpha, \mathcal{U}_\alpha)$ (for $\alpha \in I$) be topological spaces and let $f : X \longrightarrow \prod_{\alpha \in I} Y_\alpha$ be a function, where the range has the product topology. TFAE:*

- *f is continuous;*
- *$\pi_\alpha \circ f$ is continuous (for each $\alpha \in I$).*

Generalize 2! Exercize!: Prove the previous theorem by finding the corresponding theorem from §8, and copying down the proof but replacing all references to "1 or 2" with "$\forall \alpha \in I$".

Weak Exercise: State and prove a theorem for when a function $f : X \longrightarrow \prod_{\alpha \in I} Y_\alpha$ is continuous, when the range is given the weak topology induced by some family of functions.

6 Topological Properties

A recurring theme is that the box topology is misbehaved while the product topology is nicely behaved. Here is one piece of evidence towards that:

Proposition. *$\mathbb{R}^{\mathbb{N}}_{box}$ is not metrizable because it is not first countable.*

Proof. This proof really comes down to a Cantor-style Diagonalization argument.

Let $\{ V_n : n \in \mathbb{N} \}$ be a collection of open sets in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ that all contain the constant 0 function $g(n) = 0$, for all $n \in \mathbb{N}$. We know that V_n contains a product of ϵ balls around g, so without loss of generality we may assume

$$V_n = B_{\epsilon_1^n}(0) \times B_{\epsilon_2^n}(0) \times B_{\epsilon_3^n}(0) \times B_{\epsilon_4^n}(0) \times \dots$$

Now we can see (by writing!) that the "diagonal open set"

$$U := B_{\frac{\epsilon_1^1}{2}}(0) \times B_{\frac{\epsilon_2^2}{2}}(0) \times B_{\frac{\epsilon_3^3}{2}}(0) \times B_{\frac{\epsilon_4^4}{2}}(0) \times \dots$$

does not contain any V_n. Thus $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is not first countable, hence not metrizable. $\square$

Contrast that proposition with the following fact:

Proposition. $\mathbb{R}^{\mathbb{N}}_{prod}$ *is metrizable. Moreover* ρ_{prod} *generates the product topology.*

Just like we have investigated how various topological properties are preserved under subspaces and finite products we are also concerned about how these properties behave under arbitrary products.

Definition. *A property ϕ is said to be* ***countably productive*** *if whenever* $(X_n, \mathcal{T}_n)$ *(for* $n \in \mathbb{N}$*) are each a topological space with property ϕ then the space* $\prod_{n\in\mathbb{N}} X_n$*, with the product topology, has property ϕ.*

Definition. *A property ϕ is said to be* ***productive*** *if whenever* $(X_\alpha, \mathcal{T}_\alpha)$ *(for $\alpha \in I$) are each a topological space with property ϕ then the space* $\prod_{\alpha\in I} X_\alpha$*, with the product topology, has property ϕ.*

We already know that normality is *not* a productive property (it isn't even finitely productive!) but the other separation axioms are productive.

Proposition. *The Hausdorff property is a productive property.*

Proof. You proved that the Hausdorff property is finitely productive on Assignment 3, C.3. This proof will be shorter (surprisingly!) than most of the proofs that students presented.

Let $(X_\alpha, \mathcal{T}_\alpha)$ (for $\alpha \in I$) be Hausdorff spaces, and let $X = \prod_{\alpha\in I} X_\alpha$, with the product topology. Let $f, g \in X$ be distinct functions. Therefore they disagree on some coordinate $\alpha_0 \in I$; that is $f(\alpha_0) \neq g(\alpha_0)$. Since $f(\alpha_0), g(\alpha_0)$ are distinct elements of X_{α_0}, a Hausdorff space, there are disjoint open sets $U, V \subseteq X_{\alpha_0}$ such that $f(\alpha_0) \in U$ and $g(\alpha_0) \in V$. Clearly, $\pi_{\alpha_0}^{-1}[U]$ and $\pi_{\alpha_0}^{-1}[V]$ are the desired disjoint open sets in X that contain f and g respectively. $\square$

Note that by Assignment 3, C.4 we now know that a product of Hausdorff spaces with the box topology is also a Hausdorff space.

Other Productive Properties:

- X is regular;
- X is T_1;

This is a pretty meager list! It doesn't contain any of the interesting countability properties that we discussed in this course! The moral of this story is that countability properties say (in some sense) that a space is small, but uncountable products are quite large! This is why we often only concern ourselves with countable products of spaces (like $\mathbb{R}^{\mathbb{N}}$!).

Let us record some of the **countably productive properties**:

- X is first countable;
- X is separable;
- X is second countable;
- X is metrizable.

The sound bite here is "**A small product of small spaces is small**", (where ccc is the outlier that does not conform to this).

These proofs are straightforward, but they require an idea that is related to the product topology (as we would expect since, for example, a countable box product of first countable spaces need not be first countable). The proof that first countability, separability and second countability are countably productive all include the same basic idea: "take finite intersections of things you want".

Theorem. *Second Countability is a countably productive property.*

Proof. Let $(X_n, \mathcal{T}_n)$ for $n \in \mathbb{N}$ be second countable spaces. For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a countable basis for X_n. Note that if infinitely many $\mathcal{B}_n$ have at least two elements, then $\prod_{n\in\mathbb{N}} \mathcal{B}_n$ would be uncountable, so it would not be useful for us here. (Also, most of those sets probably are not open in the product topology.) Instead, we look at $\mathcal{B}$, the collection all finite intersections of things from

$$\mathcal{S} := \{\pi_n^{-1}(B) : B \in \mathcal{B}_n\}$$

The keener among you will notice that this $\mathcal{S}$ a subbasis! Also observe that $\mathcal{S}$ is a countable set, since each $\mathcal{B}_n$ is countable, then taking finite intersections keeps it countable (see Assignment 3, C.1).

It is now an easy exercise to show that $\mathcal{B}$ is a basis for $\prod_{n\in\mathbb{N}} X_n$, with the product topology. $\square$

7 Convergence in Different Topologies on $\mathbb{R}^{\mathbb{N}}$

We have already seen that the box product behaves differently from the product topology; for example, $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is metrizable, but $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is not. Now we will look at a much more "real" difference between them. We will look at what types of sequences converge in these topologies.

First recall the obvious lemmata:

Lemma. *Let $(X, \mathcal{T})$ and $(X, \mathcal{U})$ be topological spaces, and let $\langle x_n \rangle$ be a sequence in X. If $\mathcal{T} \subseteq \mathcal{U}$ and $\langle x_n \rangle$ converges in $\mathcal{U}$, then $\langle x_n \rangle$ converges in $\mathcal{T}$. Moreover, if $\langle x_n \rangle \to x$ in $(X, \mathcal{U})$, then $\langle x_n \rangle \to x$ in $(X, \mathcal{T})$.*

Lemma. *In a Hausdorff space, a sequence converges to at most one point.*

Example: The following sequence converges in $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$, but not $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$ (or $\mathbb{R}^{\mathbb{N}}_{\text{box}}$):

$$\begin{aligned} f_1 &: \langle 10, 0, 0, 0, \ldots \rangle \,; \\ f_2 &: \langle 0, 100, 0, 0, \ldots \rangle \,; \\ f_3 &: \langle 0, 0, 1000, 0, \ldots \rangle \,; \\ &\vdots \end{aligned}$$

Proof. Clearly, the only possible choice is that these functions converge to $\langle \overline{0} \rangle$, the constant 0 function.

[$\mathbb{R}^{\mathbb{N}}_{\text{prod}}$] Let $U \ni \langle \overline{0} \rangle$ be open in $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$. Since $U = \prod_{n \in \mathbb{N}} U_n$ is open in the product topology, it means that there is a finite set $F \subseteq \mathbb{N}$ such that for $n \in \mathbb{N} \setminus F$ we have $U_n = \mathbb{R}$. Thus, for $N = \max(F) + 1$, and $n \geq N$ we see that the first N coordinates of f_N are 0. That is

$$f_n(i) = 0, \forall 1 \leq i \leq N$$

So we see that $f_n \in U$ for all $n \geq N$. Hence this sequence converges to $\langle \overline{0} \rangle$ in the product topology.

(This previous paragraph really needs a picture to go with it, otherwise it is impossible to follow. Draw a picture- do you remember how to visualize $\mathbb{R}^{\mathbb{N}}$?- and follow the proof).

[$\mathbb{R}^{\mathbb{N}}_{\text{unif}}$] Now we will show that this sequence does not converge in $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$. Consider the open set

$$U := \prod_{n \in \mathbb{N}} (-1, 1)$$

which is open in the uniform topology (it is in fact the open unit ball in the uniform metric), which contains $\langle \overline{0} \rangle$ but does not contain any f_n. (The reason being that for $n \in \mathbb{N}$ we have $f_n(n) = 10^n > 1$.) $\square$

8 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

It Works : Show that there is a natural correspondence between $\mathbb{R}^2$ and $\{\, f : \{0, 1\} \longrightarrow \mathbb{R} \,\}$, the collection of all functions from $\{0, 1\}$ to $\mathbb{R}$.

Generalize! : Prove the theorem about projections by looking for the similar statement in §8, and copying down the proof but replacing all references to "1 or 2" with "$\forall \alpha \in I$".

Generalize 2! : Prove the other projection theorem by finding the corresponding theorem from §8, and copying down the proof but replacing all references to "1 or 2" with "$\forall \alpha \in I$".

Weak : State and prove a theorem for when a function $f : X \longrightarrow \prod_{\alpha \in I} Y_\alpha$ is continuous, when the range is given the weak topology induced by some family of functions.