AST121 Tutorial on Kepler's Laws

AST121 TAs

January 26, 2013

OUTLINE

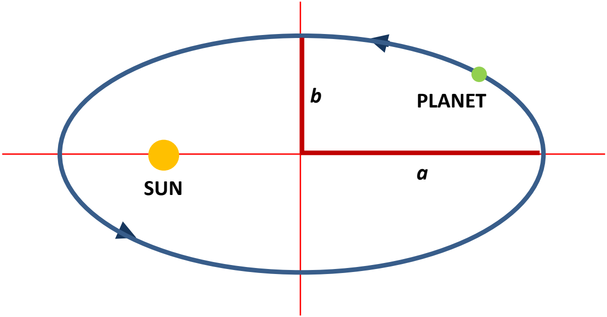
- Discovery of the laws
- Kepler's laws stated
- Newtonian derivation
- Applications and worked problems

Discovery of the laws

- Kepler published his first two laws in 1609, based off Tycho Brahe's work. His third law was published ten years later.
- Kepler's laws not only supported Copernicus' heliocentric model of the Solar System, but stated planets' orbits were elliptical rather than perfectly circular.
- Almost a century later Issac Newton proved that Kepler's laws would apply for any system of two bodies orbiting one another.
- Voltaire's *Eléments de la philosophie de Newton* was the first publication to call Kepler's laws "laws".

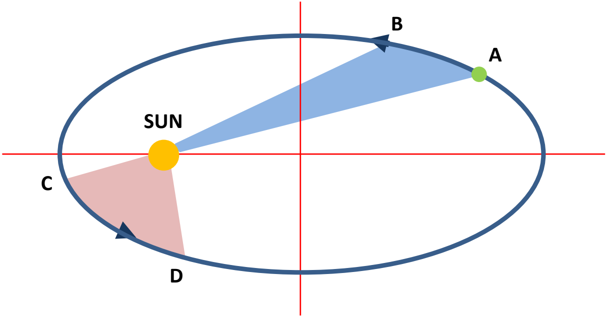
Johannes Kepler (December 27, 1571 November 15, 1630). From Wikipedia.

Kepler's 1st Law



The orbit of a planet forms an ellipse (semi-major axis a) with the Sun at one focus.

Kepler's 2nd Law



The Sun-planet radius vector sweeps out equal areas in equal times.

Kepler's 3rd Law

The square of the period of revolution of a planet is proportional to the cube of the semimajor axis of its elliptical orbit. In other words:

$$P^2 \propto a^3 \tag{1}$$

If we were to use units of years for the period, and AU for the semimajor axis, then:

$$P_{\mathrm{yr}}^2 = a_{\mathrm{AU}}^3 \tag{2}$$

This can only be used for objects orbiting the Sun. Using it for any other purpose will give incorrect answers.

Newtonian derivation

- Kepler determined his laws specifically for planets moving around the Sun. Newton, using his laws of motion and the universal law of gravitation[1]

$$\vec{F}_g = \frac{GMm}{r^2}\hat{r} \tag{3}$$

  showed that Kepler's laws hold for any body with mass m that is orbiting a much more massive body M, in the absence of any other objects [2].
- Newton's derivation (beyond the scope of this video) also leads to the general form of Kepler's 3rd Law:

$$P^2 = \frac{4\pi^2 a^3}{G(M+m)} \tag{4}$$

  where a is the semimajor axis of the object's orbit. Note how **both** m and M are included in the equation. In most cases where we use Kepler's laws, $m << M$, so $M + m \approx M$.

[1] $\hat{r}$ is a unit vector parallel to the line connecting the two bodies.

[2] To be exact, both m and M orbit the centre of mass of the system, but if $m << M$ the centre of mass is inside M.

- We can derive Kepler's 3rd Law for circular orbits (where $a = r$, the radius of the circle). Take an object of mass m orbiting a much more massive object of mass M in a circle. For uniform circular motion[3] the centripetal force $F_c = mv^2/r = m4\pi^2 r/P^2$. Equating this to the force of gravity (Eqn. 3):

$$\begin{aligned} \frac{m4\pi^2 r}{P^2} &= \frac{GMm}{r^2} \\ \frac{1}{P^2} &= \frac{GM}{4\pi^2 r^3} \\ P^2 &= \frac{4\pi^2 r^3}{GM} \end{aligned} \tag{5}$$

  We have M rather than $M + m$ because we have assumed m orbits M, which requires that $m << M$.
- Kepler's laws no longer hold when there are more than two objects in the system.

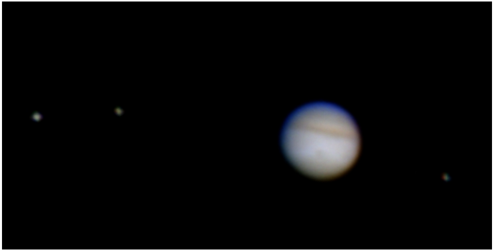
[3] Recall that $v = 2\pi r/P$.

Example 1

Uranus and Neptune have periods of 84 and 165 years. How far are they from the Sun?

This is the Solar System, so use Eqn. 2. For Uranus, $84^{2/3} = 19.2$ and for Neptune, $165^{2/3} = 30.0$. So Uranus is 19.2 AU away from the Sun, and Neptune 30.0 AU.

EXAMPLE 2



From Wikipedia.

Orbiting Jupiter (5.20 AU, or 7.78×10^{11} *m from the Sun, from observations and Kepler's 3rd Law) are the Galilean moons, Io, Europa, Ganymede and Callisto. Telescope observations of Callisto from Earth show that Callisto orbits Jupiter once every 16.7 days in a nearly circular orbit, and its maximum angular separation from Jupiter is 498″. What is the mass of Jupiter?*

We can determine the semimajor axis of Callisto's orbit using using the small angle formula, $\theta = l/d$. 498″ is about 0.138 degrees, or 0.00242 radians. We take the Earth-Jupiter distance to be the same as the Sun-Jupiter distance, which is 7.78×10^{11} m. Plugging in our numbers, we find Callisto's orbital radius is 1.88×10^{9} m.

We can then rearrange the general form of Kepler's 3rd Law, Eqn. 4, to obtain the mass:

$$M_J + M_C = \frac{4\pi^2 a^3}{GP^2} \tag{6}$$

Plugging numbers in (remembering to convert Callisto's orbital period into seconds), and noticing that Callisto's mass is much, much less than that of Jupiter's, we obtain $M_J = 1.89 \times 10^{27}$ kg, or 317 Earth masses.

Conclusion

- Planets orbiting the Sun (or, more generally, objects orbiting bodies much more massive than themselves) trace out an ellipse with the Sun at one focus.
- The planet-Sun (object-massive body) vector sweeps out equal areas in equal times.
- $P^2 = \frac{4\pi^2 a^3}{G(M+m)}$, with $P_{\text{yr}}^2 = a_{\text{AU}}^3$ for our Solar System.

Further Reading

- Kepler's Laws on Wikipedia
- Newton's law of universal gravitation on Wikipedia
- Shu, *The Physical Universe: An Introduction to Astronomy*, pg. 33 - 38 (covers Newtonian mechanics) and pg. 463 - 466 (Kepler's Laws)
- Carroll & Ostlie, *An Introduction to Modern Astrophysics*, Ch. 2