Sta347 Probability I

Selected Practice Problems for Midterm

Oct. 12, 2013

(1) Problem 2 on Page 24 of the Textbook.

The loss can be written as

$$L = a(y-z)\,I\{y > z\} + b\,I\{y < z\}$$

$$= a(\beta x+\varepsilon-z)\,I\{\beta x+\varepsilon > z\} + b\,I\{\beta x+\varepsilon < z\}$$

$$\Rightarrow EZ = a\,E[(\beta x+\varepsilon-z)\,I\{\varepsilon > z-\beta x\}] + b\,p\{\varepsilon \leq z-\beta x\}$$

$$= a\int_{z-\beta x}^{\infty}(\beta x+u-z)f(u)\,du + b\int_{-\infty}^{z-\beta x} f(u)\,du .$$

Take derivative w.r.t. x and we have

$$b f(z-\beta x) = a\int_{z-\beta x}^{\infty} f(\varepsilon)\,d\varepsilon$$

(2) Problem 4 on Page 36 of the Textbook.

Since $H(x)$ is an increasing function.

$$\therefore\quad X \geq a \Longleftrightarrow H(x) \geq H(a)$$

Note that $H(a) \geq 0$

Hence by Markov's Inequality

$$p(X \geq a) = p(H(x) \geq H(a)) \leq \frac{E(H(X))}{H(a)}$$

(3) Show that if A and B are events, then

$$P(\bar{A}\bar{B}) = P(\bar{A}) - P(B) + P(AB).$$

By Inclusion – Exclusion formula

$$P(A\cup B) = P(A) + P(B) - P(AB) \quad \circledast$$

Meanwhile, $\bar{A}\bar{B} = \overline{(A\cup B)}$. $\Rightarrow P(\bar{A}\bar{B}) = 1 - P(A\cup B)$.

$$\Rightarrow P(\bar{A}\bar{B}) = 1 - P(A) - P(B) + P(AB)$$

$$= P(\bar{A}) - P(B) + P(AB).$$

(4) (a) Show that if A and B are events and $P(A) = P(B) = 0$, then $P(A \bigcup B) = 0$.

(b) Show that if $A_1, A_2, \cdots, A_n$ are events and $P(A_i) = 0$ $i = 1, 2, \cdots, n$, then $P(\bigcup_{i=1}^{n} A_i) = 0$.

(c) Show that if infinitely many events $A_1, A_2, \cdots,$ all have probability 0, then $P(\bigcup_{i=1}^{\infty} A_i) = 0$.

(a): Note that $A\cap B \subseteq A \Rightarrow P(A\cap B) \leq P(A) = 0 \Rightarrow P(A\cap B) = 0$.

$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B) = 0 + 0 - 0 = 0$

(b). Use induction. If $n=2$. (b) holds by (a). Suppose $n=k$, (b) holds. Now for $n=k+1$.

Let $B = \bigcup_{i=1}^{k} A_i \Rightarrow P(\bigcup_{i=1}^{k+1} A_i) = P(B\cup A_{k+1}) = P(B) + P(A_{k+1}) - P(A_{k+1}B)$

$= 0 + 0 - 0 = 0$

(Note $P(A_{k+1}B) \leq P(A_{k+1}) = 0 \Rightarrow P(A_{k+1}B) = 0$)

(c). Let $C_k = \bigcup_{i=1}^{k} A_i$. Then $P(C_k) = 0$ for any k by (b).

Further note that C_k is an non-decreasing sequence of events.

So $P(\bigcup_{k=1}^{\infty} C_k) = \lim_{k\to\infty} P(C_k) = 0$. (Axiom 5)

On the other hand, note $\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} A_k$. $\therefore$ (c) holds.

(5) Show that for any random variable X such that $E(|X|^3) < \infty$, we have

$$(E|X^2|)^2 \leq E|X| \times E(|X|^3).$$

By Cauchy's inequality

$$(E|X^2|)^2 = \left(E\left[|X|^{0.5}|X|^{1.5}\right]\right)^2$$

$$\leq E|X| \, E\left[|X|^3\right].$$

(6) Let X be a Binomial(25, 0.6) random variable. Find $P(X \geq 2)$, $P(X = 10)$, $E(X)$ and $V(X)$.

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - \binom{25}{0} 0.4^{25} - \binom{25}{1} 0.6 \; 0.4^{24} \approx 1$$

$$P(X=10) = 0.021$$

$$E(X) = 25 \times 0.6 = 15$$

$$V(X) = 25 \times 0.6 \times 0.4 = 6$$

(7) A recruiting firm finds that 20% of the applications are fluent in both English and French. Applicants are selected randomly from a pool and interviewed sequentially. Find the probability that at least five applicants are interviewed before finding the first applicant who is fluent in both English and French.

This prob. is the same as the prob. that the first 5 applicants are not fluent for both languages.

Hence the desired prob. $= (0.8)^5 = 0.328$.

(8) A swimming pool repair person has three check valves in stock. Then percent of the service calls require a check valve. What is the expected number and standard deviation of the number of service calls she will make before running out of check valves?

Let X be the # of service calls she will make before running out of check values.

Then $X = Y+3$, where Y is negative binomial with $r=3$ & $p=0.1$

$$\Rightarrow EX = EY + 3 = \frac{rq}{p} + 3 = \frac{3 \times 0.9}{0.1} + 3$$

$$= 30$$

$$V(Y) = V(X) = \frac{rq}{p^2} = \frac{3 \times 0.9}{(0.1)^2} = 270$$

(9) Let X be a negative binomial random variable with parameters r and p, and let Y be a binomial random variable with parameters n and p. Show that

$$P(X > n-r) = P(Y < r).$$ [Note the question is changed to $P(X>n-r) = P(Y<r)$]

Conduct a series of independent Bernoulli trials with success prob. p.

Let Y be the # of successes in the first n trials.

Note that Y is a Binomial(n,p) random variable.

$Y<r \iff$ there are less than r successes in the first n trials $\iff$ the # of trials until the rth success $> n \iff$ # of fails until the rth success $> n-r \Rightarrow P(Y<r) = P(X>n-r)$

(10) The number of car accidents in a city in a certain week follows a Poisson distribution with a mean of two accidents per square kilometer.

(a) If four one-square-kilometer regions from the city are selected independently, find the probability that at least one region will contain at least one car accident in a particular week.

(b) How many one-square-kilometer regions should be selected in order to have probability of approximately 95% of containing at least one car accident in a particular week?

(a) Let ~~Xi be the~~ $X_i = \begin{cases} 0 & \text{o.w.} \\ 1 & \text{if region } i \text{ has at least one accident} \end{cases}$

$\Rightarrow P(X_i = 1) = 1 - \frac{2^0}{0!} e^{-2} = 0.865$.

Desired prob. $= 1 - P(X_1 = X_2 = X_3 = X_4 = 0)$
$= 0.9997$.

(b). Suppose k region are required.

Then $1 - (e^{-2})^k \approx 0.95$

$\Rightarrow k = 2$ ($k=2$ yields the nearest prob. to 0.95).