PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATIONS, DECEMBER 2010
MAT 244 F - INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

EXAMINERS: P. CARRASCO AND F. RECIO

INSTRUCTIONS:

1. ATTEMPT ALL QUESTIONS.
2. SHOW AND EXPLAIN YOUR WORK IN ALL QUESTIONS.
3. GIVE YOUR ANSWERS IN THE SPACES PROVIDED. USE BOTH SIDES OF THE PAPERS, IF NECESSARY.
4. DO NOT TEAR OUT ANY PAGES.
5. NON-PROGRAMMABLE POCKET CALCULATORS, BUT NO OTHER AIDS ARE PERMITTED.
6. THIS EXAM CONSISTS OF 9 QUESTIONS IN 15 PAGES. THE VALUE OF EACH QUESTION IS INDICATED (IN BRACKETS) BY THE QUESTION NUMBER.
7. TIME ALLOWED: 3 HOURS.
8. PLEASE WRITE YOUR NAME, YOUR STUDENT NUMBER, AND YOUR SIGNATURE IN THE SPACES PROVIDED AT THE BOTTOM OF THIS PAGE.

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QUESTION NUMBER	QUESTION VALUE	GRADE
1	10	
2	13	
3	10	
4	13	
5	10	
6	13	
7	8	
8	13	
9	10	
TOTAL:	100	

NAME: ______________________ , ______________________

(FAMILY NAME. PLEASE PRINT.) (GIVEN NAME.)

STUDENT No.: ______________________ SIGNATURE: ______________________

1. (10 marks) Solve the differential equation

$$(1+\sec t)\,y' + (\sin t + \tan t)\,y = e^{\cos t}\sin t\tan t \quad \text{,where } -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

2. a) (5 marks) Show that the equation $x^2 y'' - 2xy' + f(x)\,y = 0$, where $x > 0$, can be transformed into $u'' + g(x)\,u = 0$ by letting $y = u(x)\,v(x)$ and choosing $v(x)$ appropriately.

b) (8 marks) Solve the differential equation $x^2 y'' - 2xy' + 2(1 + 2x^2)\,y = 0$, where $x > 0$.

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3. (10 marks) Solve the initial value problem

$$y'' - 4\,y' + 4\,y = 4\,e^{2x}\,,\ y(0) = 1\,,\ y'(0) = 1\,.$$

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4. a) (5 marks) Find all the values of m for which $y = x^m$ is a solution of the differential equation

$$x^3 y''' - 4x^2 y'' + 8x y' - 8y = 0 .$$

b) (8 marks) Solve the differential equation

$$x^3 y''' - 4x^2 y'' + 8x y' - 8y = \frac{6}{1+x^2} .$$

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5. (10 marks) Find the first five nonzero terms in the series solution in powers of x of the initial value problem

$$y'' - x\,y' - 3\,y = 0\,,\ y(0) = 2\,,\ y'(0) = 0\,.$$

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6. Consider the first order system

$$x' = x\,(10 - 2y)$$

$$y' = -y\,(1 - 2x)$$

where $x > 0$ and $y > 0$.

a) (3 marks) Find the point (x_0, y_0) for which $x' = 0$ and $y' = 0$, introduce the new variables

$$z = x - x_0$$

$$w = y - y_0$$

and find the resulting equations for these new variables.

b) (5 marks) Neglect the terms involving the product zw and solve the resulting linear system.

c) (5 marks) Solve the original system.

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7. a) (4 marks) Suppose that the vector functions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$, are solutions of the linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Show that if the vectors $\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \ldots, \mathbf{x}^{(n)}(t_0)$ are linearly independent for some real number t_0 then the vectors $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are also linearly independent for every real value of t.

b) (4 marks) Show that the functions $y_1(t) = t^3$ and $y_2(t) = |\,t\,|^3$, where $-\infty < t < \infty$, are linearly independent both they do not form a fundamental set of solutions of any second order linear differential equation.

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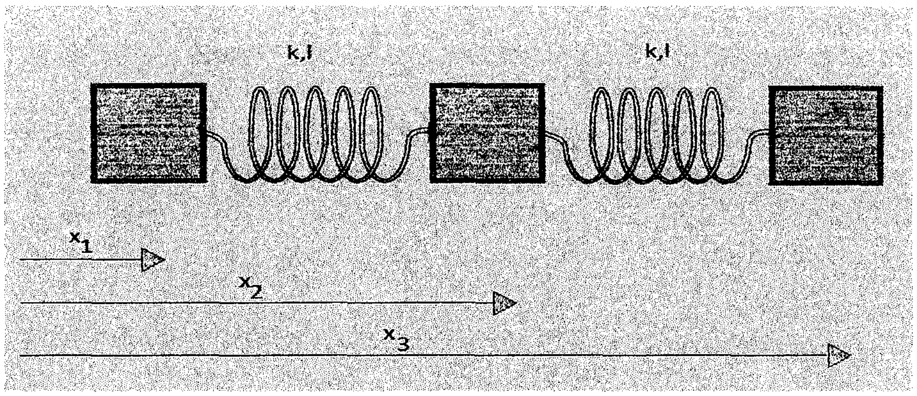
8. a) (8 marks) Let k be a positive constant. Find the general solution of the system:

$$\mathbf{y}'' = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \mathbf{y}$$

b) (5 marks) Three identical particles of unit mass are attached by ideal springs of elasticity constant k and relaxed length l as depicted below



Find the equations of motion of the system and introduce the new coordinates

$$y_1 = x_1 - x_1^0, \; y_2 = x_2 - x_2^0 \text{ and } y_3 = x_3 - x_3^0, \text{ where } l = x_2^0 - x_1^0 = x_3^0 - x_2^0 .$$

Use part (a) to determine the natural frequencies of the system.

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9. Let $P_n(x)$ denote the nth Legendre polynomial and consider its generating relation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

where $|x| \leq 1$ and $|t| < 1$.

a) (4 marks) Show that $P_0(x) = 1$ for all values of x and that for $n = 1, 2, 3, \ldots$

$$\text{a)}\quad P_n(1) = 1 \text{ and } P_{2n}(0) = (-1)^n \frac{1\cdot 3 \cdots (2n-1)}{2^n\, n!}.$$

Hint: Recall that if $|t| < 1$ then

$$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2!}t^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}t^n + \cdots$$

b) (6 marks) Differentiate the generating relation to obtain a recurrence relation among $P_{n+1}(x)$, $P_n(x)$ and $P_{n-1}(x)$. Hint: After differentiating, use the given generating relation and express both sides of the equation as power series.