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1. Completeness axiom for the real numbers $\mathbb{R}$ states that if a set S in non empty and it is bounded above then it is guaranteed that there exists a least upper bound for the set S. The meaning of this statement can be analyzed as follows:
   a) A set S can be identified with a property, expressed in the language of Algebra,
   b) $S \neq \emptyset$ means that the property expressed by S is consistent,
   c) S is bounded above means that the property is specific enough to exclude some of the numbers, in other words it is not too general to include anything, suggesting an edge or a point beyond which the set fails to describe.
   d) The existence of the lub is a magical phenomenon, it is a guarantee that the property described is about to give birth to a fantasy. And the lub is the goal of defining S and with an application of the magic word, completeness axiom, it becomes real. As such, real numbers are the ones that are described in such manners by an application of this axiom. Try some obvious examples: $S = \{1\}$ has a lub, that is 1.

2. Algebra can be used to describe some numbers:
   a) for example the rational number 3/7 is, of course, the repeated decimal $0.\overline{428571}$, a never ending decimal description, can be realized or described as the solutions to the equation $7x = 3$. Now why should such a solutions exist? Clearly the meaning and the purpose of 3/7 as an act of dividing a pie in seven pieces and selecting three out of seven can be achieved, but this action is really not a number; what is then 3/7 as a number? We know that using our calculator we can never reach such a number. So, to accept 3/7 as a number is just a belief.
   b) some irrational numbers can be described as solutions to more complex equations: for example the solution (or a solution) to $x^3 - 2 = 0$ is the irrational number known as the cube root of 2. we know that this number is never to be completely known to us through its decimal representation. Why such as number should exist? In other words why should we believe there is a solutions to the equation $x^3 - 2 = 0$? Don't we believe that equation like $x^2 + 1 = 0$ has no real root? So why should the equation $x^3 - 2 = 0$ must have a solution? The solutions to polynomial equations are known as algebraic numbers.

3. However algebra is not capable of describing numbers such as e or π; such numbers are called transcendental numbers. Remember the title of the first year calculus is usually involves something like 'early transcendentals', which point at transcendental numbers or the functions such as trig functions who deal with such numbers: for example the $Arcsin$ function presents us with $\pi/2$ upon the input of 1. Methods of Calculus such as limit and infinite series are used to define such transcendental numbers.

4. the completeness axiom guarantees the existence of limits of sequences etc. Indeed so far we have learned how to prove $\lim\limits_{n\to\infty} x_n = a$, but we don't know how to prove that a given sequence actually converges. The completeness axiom helps with this task. The first theorem in this regard is the monotone sequence theorem 1.16, which uses completeness axiom to guarantee the existence of the limit for a very convenient sequence: a monotone increasing bounded sequence.

5. Completeness axiom for $\mathbb{R}$ depends on the ordering of the real numbers. This ordering is very special because it is used to define the topology of the reals. However the space $\mathbb{R}^n$ does not have a natural ordering that is related to its topology, so the completeness axiom does not obviously extend to $\mathbb{R}^n$. To Try to extend the main idea of completeness axiom we will try to translate it to some other language which is understandable by the space $\mathbb{R}^n$. This is done through a series of theorems due to Bolzano and Weierstrass (1.18, 1.19, 1.20, 1.21) Briefly these theorems furnish a version of completeness that can be applied to $\mathbb{R}^n$. This version is capturing the idea that a sequence that behaves as though converging will actually converge to a point (which of course must exist.)

   a) A non empty bounded set is replaced by a bounded sequence, and then the technique of nested intervals (1.17) is used to show that a bounded sequence must have at least a subsequence that converges. This requirement is the version of completeness axiom that we can extend to $\mathbb{R}^n$ (in 1.19).

   b) Another version of this requirement is that if a sequence looks converging then it must be converging. A sequence that looks converging is called a Cauchy sequence. In other words, in a complete space the sequences cannot fake convergence: if they behave like converging then they must converge to something (and obviously that limit must exist.) This is theorem 1.20.

   c) Finally, $\mathbb{R}^n$ is complete, but how about an arbitrary subset S of $\mathbb{R}^n$? Obviously the subset $\mathbb{Q}$ of $\mathbb{R}$ is not complete as it is seen in MAT137. What condition on a subset S of $\mathbb{R}^n$, makes S complete? Bolzano-Weierestrass theorem (1.21) suggests that a compact subset of $\mathbb{R}^n$ is complete. A compact subset is a set that is closed and bounded, and as such it contains all the limit points (recall that the limit points are in the closure ...)