70

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Term Exam: Tuesday October 18

i) **Exam is 6 pages, duration 1 hour and 50 minutes.**

ii) **Answer on question sheet. Hand in only question sheet.**

iii) **No materials allowed except question sheet and rough paper.**

iv) **Write name and student number on every page.**

v) **Take more rough paper from the front if you need it. Raise your hand only to visit bathroom, if there is an error in the test, or to hand in your test early.**

1	21
2	15
3	20
4	4
5	10

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Question 1 (30 points). Write T(rue) or F(alse).

i) It is not possible for two linear subspaces to have an empty intersection. T

ii) If V is spanned by a list (v_1, v_2, v_3, v_4) and none of the v_i are zero, then $\dim V = 4$. F

(1,0,0) (0,1,0) (0,0,0)

iii) If (v_1, v_2, v_3) is linearly dependent, then v_1 must be a linear combination of v_2 and v_3. F

iv) If $v + v + v = 0$ for a vector v, then v must be the zero vector. T X

21

v) If U_1, U_2 are subspaces of a finite dimensional vector space V, then $\dim(U_1 + U_2) = \dim U_1 + \dim U_2$. F

vi) $((1, 2, -3), (0, 1, -1), (0, 0, 2))$ is a basis for $\mathbb{R}^3$. T

vii) Every vector space over the field $\mathbb{F}_2 = \{0, 1\}$ has a finite total number of vectors. T X

viii) If (v_1, v_2, v_3, v_4) is linearly independent, then the sum of subspaces $\text{Span}(v_1, v_2) + \text{Span}(v_3, v_4)$ is direct. T

ix) If $\text{Span}(v_1, v_2) + \text{Span}(v_3, v_4)$ is direct, then (v_1, v_2, v_3, v_4) is linearly independent. T X

x) If $(v_1, \ldots, v_n)$ is a basis, then $(v_1 + w, \ldots, v_n + w)$ is a basis for any vector w. F

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Question 2 (20 points). Short answers, be precise:

i) What is the definition of linear dependence of a list $(v_1, \ldots, v_n)$ of vectors?

5

Solution: Except for $a_1 = a_2 = a_3 = \cdots = a_n = 0$, $a_1, a_2, \cdots a_n \in \mathbb{F}$, there exist other choices for $a_1, \cdots a_n$ that make $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ equal to 0. We can say this list $(v_1, \cdots, v_n)$ is linearly dependent.

ii) Let X be a set, $\mathbb{F}$ be a field, and $\mathbb{F}^X$ be the set of functions from X to $\mathbb{F}$. Define the vector addition and scalar multiplication operations making $\mathbb{F}^X$ into a vector space over $\mathbb{F}$. (Only give the operations, don't verify axioms)

0

Solution. Suppose x, y are two vectors in $\mathbb{F}^X$, $\lambda \in X$.

If $x \in \mathbb{F}^X, y \in \mathbb{F}^X$ then $x+y \in \mathbb{F}^X$. (vector addition)

If $x \in \mathbb{F}^X, \lambda \in X$, then $\lambda x \in \mathbb{F}^X$ (scalar multiplication)

Those are two operations.

iii) Is the vector $(6, 7, 4, -8) \in \mathbb{R}^4$ contained in $\mathrm{Span}((2, -1, 0, 3), (3, 1, 1, 0), (-1, -2, -1, 2))$? (Hint: you may want to replace the list with a more convenient one)

5

Solution: Suppose $(6,7,4,-8)$ is contained in $\mathrm{Span}((2,-1,0,3), (3,1,1,0), (-1,-2,-1,2))$ such that $\exists\ a, b, c \in \mathbb{R}$, then

$$a(2,-1,0,3) + b(3,1,1,0) + c(-1,-2,-1,2) = (6,7,4,-8)$$

That is:
$$\begin{cases} 2a + 3b + (-c) = 6 \\ -a + b + (-2c) = 7 \\ b - c = 4 \\ 3a + 2c = -8 \end{cases}$$

So $a = -2, b = 3, c = -1$. Therefore vector $(6,7,4,-8)$ is contained in the span.

iv) If U_1 and U_2 are five-dimensional subspaces of $\mathbb{R}^7$, what are the possible dimensions for $U_1 \cap U_2$?

5

Solution: By theorem $\dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$, we can get $\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2)$

As U_1 and U_2 are five-dimensional subspaces of $\mathbb{R}^7$

so $\dim U_1 = \dim U_2 = 5$. $5 \leq \dim(U_1 + U_2) \leq 7$

Therefore
$$5 \leq \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \leq 7$$
$$5 \leq 5 + 5 - \dim(U_1 \cap U_2) \leq 7$$
$$3 \leq \dim(U_1 \cap U_2) \leq 5$$

That is, the possible dimensions for $U_1 \cap U_2$ are $3, 4$ or 5.

20

Name: Qiu Rui Student number: 999292509

Question 3 (20 points).

i) Let V be a vector space over $\mathbb{C}$, and let (u, v, w) be a linearly independent list of vectors in V. Prove that $(u+v, v+w, w+u)$ is linearly independent.

Proof: Suppose not $(u+v, v+w, w+u)$ is linearly independent.
such that $a_1(u+v)+a_2(v+w)+a_3(w+u)=0$, a_1, a_2, a_3 are not all zeroes.

Thus $(a_1+a_3)u+(a_1+a_2)v+(a_3+a_2)w=0$

Let (*) $a_1+a_3=b_1$, $a_1+a_2=b_2$, $a_3+a_2=b_3$, we can get
$b_1u+b_2v+b_3w=0$

As (u, v, w) is linearly Independent $\Rightarrow c_1u+c_2v+c_3w=0$ only for $c_1=c_2=c_3=0$

so $b_1=b_2=b_3=0$

Then $a_2=-a_1$, $a_3=-a_1$
$a_3+a_2=-2a_1=b_3=0$

so $a_1=0$, similarly we can get $a_2=a_3=0$

As we mentioned before a_1, a_2, a_3 are not all zeroes.

Contradiction. Thus $(u+v, v+w, w+u)$ is linearly independent. ∎

10

ii) What is the set of $(a, b, c) \in \mathbb{R}^3$ such that the list of vectors

$$((1, 1, a, b), (c, 0, -1, 1), (2, 1, 0, 1))$$

is linearly dependent in $\mathbb{R}^4$?

Solution: Using Row Echelon Operation we can get.

$$\begin{pmatrix} 1, & 1, & a, & b \\ c, & 0, & -1, & 1 \\ 2, & 1, & 0, & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1, & \frac{1}{2}, & 0, & \frac{1}{2} \\ 0, & 1, & 2a, & 2b-1 \\ 0, & 0, & 1-ac, & -bc+c-1 \end{pmatrix}$$

As the list is linearly dependent in $\mathbb{R}^4$, the last row could be 0.

So $\begin{cases} 1-ac=0 & \Rightarrow c=\frac{1}{a}, (a\neq 0) \\ -bc+c-1=0 & \Rightarrow a+b=1 \end{cases}$

10

Therefore, any number $a, b, c \in \mathbb{R}$ that satisfy these two equations $c=\frac{1}{a}$, $a+b=1$ $(a\neq 0, c\neq 0)$ would form a set (a, b, c) such that the list is linearly dependent. e.g. $a=1, b=0, c=1$, $(a,b,c)=(1,0,1)$

4

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Question 4 (40 points – four parts). Let $X = \mathbb{F}_5$, and let V be the vector space of functions from X to $\mathbb{F}_5$. Recall that $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ is the field with five elements.

i) Give a basis for V and state the dimension of V (no justification required).

Solution: $((1,0),(0,1))$ is a basis for V. (as the only choice of making $a_1(1,0)+a_2(0,1)=0$ is that $a_1=a_2=0$, which means, $((1,0),(0,1))$ is linearly independent, and $\operatorname{span}((0,1),(1,0))=V$)

So the dimension of V, $\dim V = 2$.

ii) Define the following subsets

$$V_e = \{f \in V \ : \ f(x) = f(-x), \quad \text{for all } x \in X\}$$
$$V_o = \{f \in V \ : \ f(x) = -f(-x), \quad \text{for all } x \in X\}$$

Prove that V_e and V_o are subspaces and determine their dimensions, justifying your answer.

Proof: Suppose that f_1, f_2 are two functions in V_e. f_3, f_4 are two functions in V_o.

Let $g_1(x) = f_1(x) + f_2(x)$, $g_2(x) = \lambda_1 f_1(x)$

$h_1(x) = f_3(x) + f_4(x)$ $h_2(x) = \lambda_2 f_3(x)$, $\lambda_1, \lambda_2 \in \mathbb{F}_5$

So we can get

$g_1(x) = f_1(x) + f_2(x) = f_1(-x) + f_2(-x) = g_1(-x) \Rightarrow g_1(x) \in V_e$

$g_2(x) = \lambda_1 f_1(x) = \lambda_1 f_1(-x) = g_2(-x) \Rightarrow g_2(x) \in V_e$

$h_1(x) = f_3(x) + f_4(x) = -f(-x_3) - f_4(-x_4) = -h_1(-x) \Rightarrow h_1(x) \in V_o$

$h_2(x) = \lambda_2 f_3(x) = -\lambda_2 f_3(-x) = -h_2(-x) \Rightarrow h_2(x) \in V_o$

Therefore, V_e and V_o are both closed under addition and closed under scalar multiplication.

4

Besides we can verify that 0 is also in V_e and V_o (because $f(0) = f(-0)$, $f(0) = -f(-0)$)

Thus V_e and V_o are subspaces.

As $V_e \cap V_o = \{0\}$, because $f(x) = f(-x) = -f(-x) \Rightarrow x = 0$ and V_e, V_o are two subspaces of V.

Continued....

By theorem we can get $V_e \oplus V_o = V$

Thus $\dim V_o + \dim V_e = \dim V_5 = 2$

From above we know that neither V_o nor V_e is empty, thus $\dim V_o = \dim V_e = 1$.

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iii) Find a nonzero even function $f \in V_e$ such that $f(0) + f(2) + f(4) = 0$ and $f(1) + f(3) = 0$. (No justification required)

Solution: There exists an even function $f(x) = x^2$
such that $f(0) + f(2) + f(4) = 0 + 4 + 1 = 0$
and. $f(1) + f(3) = 1 + 4 = 0$

10

iv) Prove that any function $f \in V$ can be written uniquely as a polynomial in one variable with coefficients in $\mathbb{F}_5$, of degree ≤ 4. State clearly any results you may need to use, including any relevant properties of the Lagrange interpolating polynomials $(f_0, \ldots, f_n)$, defined for fixed distinct numbers $c_0, \ldots, c_n$ by

$$f_i(x) = \prod_{\substack{0 \leq k \leq n \\ k \neq i}} \frac{x - c_k}{c_i - c_k}.$$

Proof: