Solution to excercise 12 (from exercise set 3)

Exercise 12: Prove that if G is connected, then its block graph is a tree.

Let G be a connected graph, and let B be its block graph. So for every cut vertex x of G, x is a vertex of B, and for every block H of G, there is a vertex v_H in B (a *block vertex*). Vertices v_H and x are adjacent in B if the block H contains the cut vertex x.
We will show that B is a tree, by showing that it is connected and contains no cycles.

Claim: B is connected.
Proof: If G contains only one block H, then $G = H$, so G contains no cut vertices. Then B consists of the single vertex v_H, and thus is connected. Now we may assume that G contains at least two blocks.

First we will show that for any pair x, y of cut vertices of G, B contains an (x, y)-walk. Let x, y be two cut vertices of G, and consider an (x, y)-path $P = u_0, u_1, \ldots, u_k$ in G (this exists since G is connected). We view P as a vertex sequence. A *segment* of P is a maximal subpath that has no cut vertices as internal vertices. We map P to an (x, y)-path in B by replacing every segment as follows:
A segment $S = u_i, \ldots, u_j$ is replaced by $S' = u_i, v_H, u_j$, where H is the block that contains the edge $u_i u_{i+1}$ (by Observation 4.7, H exists and is unique).
We argue that S' is a path in B. By definition, v_H and u_i are adjacent in B. Suppose that the edge $u_{j-1}u_j$ is not part of H. Then we may consider the first edge $u_\ell u_{\ell+1}$ on the path S that is not part of H. So $i < \ell < j$. The vertex u_ℓ is then part of two blocks, and thus a cut vertex (Proposition 4.8). This contradicts that S is a segment. We conclude that u_j is part of the block H as well, and thus S' is a path in B.
By replacing every segment of P this way, we obtain an (x, y)-walk in B. (Not necessarily a path!)
We conclude that all cut vertices of G lie in the same component C of B.

Secondly, we show that for every block H of G, v_H lies in the component C as well. Consider a block H of G. Since we may assume that there are at least two blocks, we may choose a different block H' of G. Choose $u \in V(H) \setminus V(H')$ and $v \in V(H') \setminus V(H)$. (Such vertices exist by Observation 4.4.) Let P be a (u, v)-path in G. Similar to above, by considering the first edge of P that is not part of H (such an edge exists by choice of u and v), we find a cut vertex x that is part of H. Hence v_H is adjacent to x in B, and thus v_H is part of the component C of B as well.
We conclude that all vertices of B lie in the same component, so B is connected.

Claim: B contains no cycles.
Suppose to the contrary that B does contain a cycle. Let C be a cycle in B of *minimum length.* Since B is simple and bipartite, C has length at least 4 and thus contains at least two block vertices.
The minimum length of C guarantees the following property:
Property 1: Let H and H' be distinct blocks of G that share a vertex x, such that v_H and $v_{H'}$ are both part of C. Then in the cycle C, v_H and x are adjacent, and $v_{H'}$ and x are adjacent.
Indeed, suppose that Property 1 does not hold: Note that the common vertex x is a cut vertex (Proposition 4.8), so $x \in V(B)$. Either we can shorten C by replacing a subwalk

from v_H to $v_{H'}$ by the walk $v_H, x, v_{H'}$ (which contradicts the choice of C), or H and H' share another vertex $y \neq x$ (which contradicts Proposition 4.5).

Now we will map C to a closed walk W in G, as follows: Replace every subpath x, v_H, y of C by an (x, y)-path P in the block H (which exists since H is connected). If P has length at least two, then some vertices are added. These are called the *new vertices for the block H*.

Replacing subsequences of C this way clearly gives a closed walk W in G. W contains edges of at least two blocks, since C has length at least 4. We now argue that W is a cycle.

If W is not a cycle, then it contains a vertex w at least twice. Since C is a cycle, w must be a new vertex for some block H. Let x and y be the cut vertices of G that are adjacent to v_H in C, so $w \notin \{x, y\}$ (since w is a new vertex). Since w occurs twice in W, there is a different block H' with $v_{H'} \in V(C)$ and $w \in V(H')$. This, together with $w \notin \{x, y\}$, contradicts Property 1.

We conclude that W is a cycle that contains edges of at least two blocks of G. Since a cycle is 2-connected, this contradicts the maximality of blocks. $\square$