STAT2001/6039 Mid-Semester Exam 2014 Solutions

Solution to Problem 1

First, $\mu = EX = \sum_x xf(x) = (c+1)\frac{1}{k} + ... + (c+k)\frac{1}{k} = \frac{1}{k}\{kc + (1+2+...+k)\}$

$$= c + \frac{1}{k} \times \frac{k(k+1)}{2} = c + \frac{k+1}{2}.$$

Next, $\sigma^2 = VX = VY$, where $f(y) = 1/k,\ y = 1, 2, ..., k$ (i.e. $Y = X$ with $c = 0$)

$$= EY^2 - (EY)^2$$

$$= \frac{1}{k}(1^2 + 2^2 + ... + k^2) - \left(\frac{k+1}{2}\right)^2 = \frac{1}{k} \times \frac{k(k+1)(2k+1)}{6} - \left(\frac{k^2+2k+1}{4}\right)$$

$$= 2\left(\frac{2k^2+3k+1}{12}\right) - 3\left(\frac{k^2+2k+1}{12}\right) = \frac{1}{12}(k^2-1) = \frac{(k-1)(k+1)}{12}.$$

Therefore, $\lambda = \frac{\mu}{\sigma} = \boxed{\frac{c+(k+1)/2}{\sqrt{(k-1)(k+1)/12}}}$ (simple formula).

For the case $c = 5$ and $k = 7$, we see that $\lambda = \frac{5+(7+1)/2}{\sqrt{(7-1)(7+1)/12}} = \frac{9}{2} = \boxed{4.500}$.

Observe that $\lambda \to \frac{1/2}{\sqrt{1/12}} = \sqrt{3}$ as $k \to \infty$.

So if $c = 5$ and $k = 10^{50}$ then $\lambda = \sqrt{3} = \boxed{1.732}$.

Solution to Problem 2

There are $20 - 2 - 5 = 13$ balls remaining in the box if neither of the two balls initially selected are red. Also, the number of balls in the box is 14 if exactly one of those two balls is red, and that number is 15 if both are red. There are no other possibilities.

With this in mind, let: A_k = "Exactly k of the two balls initially selected are red"

B_k = "Exactly k of the 5 balls subsequently selected are red ".

Then the required probability is $P(\overline{A}_0 \mid B_0) = 1 - P(A_0 \mid B_0) = 1 - \dfrac{P(A_0 B_0)}{P(B_0)}$.

Now, $P(A_0 B_0) = P(A_0)P(B_0 \mid A_0) = \dfrac{\binom{8}{0}\binom{12}{2}}{\binom{20}{2}} \times \dfrac{\binom{8}{0}\binom{10}{5}}{\binom{18}{5}} = \dfrac{33}{3230}$ = 0.0102 1672.

Also, $P(B_0) = P(A_0 B_0) + P(A_1 B_0) + P(A_2 B_0)$,

where: $P(A_1 B_0) = P(A_1)P(B_0 \mid A_1) = \dfrac{\binom{8}{1}\binom{12}{1}}{\binom{20}{2}} \times \dfrac{\binom{8}{0}\binom{11}{5}}{\binom{19}{5}} = \dfrac{616}{30685}$ = 0.0200 7496

$P(A_2 B_0) = P(A_2)P(B_0 \mid A_2) = \dfrac{\binom{8}{2}\binom{12}{0}}{\binom{20}{2}} \times \dfrac{\binom{8}{0}\binom{12}{5}}{\binom{20}{5}} = \dfrac{231}{30685}$ = 0.0075 2811.

Thus $P(B_0)$ = 0.0102 1672 + 0.0200 7496 + 0.0075 2811 = 0.0378 1979

(= 2321/61370).

It follows that $P(\overline{A}_0 \mid B_0) = 1 - \dfrac{0.0102\ 1672}{0.0378\ 1979} = 1 - 0.2701$ = $\boxed{0.7299}$ (= 154/211).

Solution to Problem 3

Observe that: $P(X = 2) = P(AB)$

$P(X = 0) = P(\overline{A}\overline{B})$

$P(X = 1) = P(A\overline{B}) + P(\overline{A}B)$.

Therefore $\mu = EX = 0 \times P(X=0) + 1 \times P(X=1) + 2 \times P(X=2)$

$$= P(X=1) + 2P(X=2) = P(A\bar{B}) + P(\bar{A}B) + 2P(AB)$$

$$= \{P(A\bar{B}) + P(AB)\} + \{P(\bar{A}B) + P(AB)\} = P(A) + P(B).$$

Alternatively, and more simply, let $U = I(A)$ and $W = I(B)$.

Then $X = U + W$, where $U \sim Bern(P(A))$ and $W \sim Bern(P(B))$.

It follows that $EX = EU + EW = P(A) + P(B)$.

Now observe that: $\mu = P(A) + P(B) \le \frac{1}{2}P(B) + P(B) = 1.5P(B) \le 1.5 \times 0.8 = 1.2$

$$\mu = P(A) + P(B) \ge P(A) + 2P(A) = 3P(A) = 3 \times 0.2 = 0.6.$$

Thus, μ is not known exactly but must be in the interval $\boxed{[0.6, 1.2]}$.

We may also report the value of μ as $\boxed{[0.9 \pm 0.3]}$.

Next, observe that

$$\mu_2' = EX^2 = 0^2 \times P(X=0) + 1^2 \times P(X=1) + 2^2 \times P(X=2)$$

$$= 0 + P(A\bar{B}) + P(\bar{A}B) + 4P(AB)$$

$$= \{P(A\bar{B}) + P(AB)\} + \{P(\bar{A}B) + P(AB)\} + 2P(AB)$$

$$= P(A) + P(B) + 2P(A) \quad \text{(since } A \subseteq B \text{ and so } P(AB) = P(A)\text{)}$$

$$= 3P(A) + P(B).$$

Hence: $\mu_2' = 3P(A) + P(B) \le 3 \times \frac{1}{2}P(B) + P(B) = \frac{5}{2}P(B) \le \frac{5}{2} \times \frac{4}{5} = 2$

$$\mu_2' = 3P(A) + P(B) \ge 3P(A) + 2P(A) = 5P(A) \ge 5 \times \frac{1}{5} = 1.$$

Thus, μ_2' is not known exactly but must be in the interval $\boxed{[1, 2]}$.

We may also report the value of μ_2' as $\boxed{[1.5 \pm 0.5]}$.

Solution to Problem 4

We will apply a first step analysis. First, define A = "Ann wins" and B = "Bob wins".

Then consider the first roll as resulting in 1, 6 or 0, where 0 denotes a number which is 2, 3, 4 or 5 (i.e., a number other than 1 or 6). Then, by the law of total probability,

$$P(A) = P(1)P(A|1) + P(0)P(A|0) + P(6)P(A|6)$$

$$= \frac{1}{6}P(A|1) + \frac{4}{6}P(B) + \frac{1}{6}P(B). \tag{1}$$

Note: If any number except 1 comes up on the first roll, Ann will be in exactly the same situation as Bob was before the first roll. Thereby, $P(A|0) = P(A|6) = P(B)$.

Next consider each of the three numbers 1, 0 and 6 as coming up on the second roll, conditional on the first roll coming up 1. Then, again by the law of total probability,

$$P(A|1) = P(11|1)P(A|1,11) + P(10|1)P(A|1,10) + P(16|1)P(A|1,16)$$

$$= \frac{1}{6}P(B|1) + \frac{4}{6}P(A) + \frac{1}{6} \times 0. \tag{2}$$

Note: If two 1s come up on the first two rolls, then Ann will be in exactly the same situation as Bob after the first roll results in a 1. Also, if the first roll comes up 1 and the second roll comes up 2, 3, 4 or 5, then Ann will be in the same situation as she was initially, before any rolls. Also, if 1 and 6 come up in that order, then Ann loses.

Now observe that $P(A) + P(B) = 1$ and $P(A|1) + P(B|1) = 1$.

Then, with $p = P(A)$ and $q = P(A|1)$, equations (1) and (2) imply, respectively:

$$p = \frac{1}{6}q + \frac{4}{6}(1-p) + \frac{1}{6}(1-p) \quad , \qquad q = \frac{1}{6}(1-q) + \frac{4}{6}p + 0 .$$

Solving these two equation in two unknowns, we obtain the required probability

$p = P(A)$ = 36/73 = $\boxed{0.49315}$.

Note: Thus Bob has a slight advantage. Also, $q = P(A|1)$ = 217/511 = 0.42466. This means that if Ann first rolls 1, Bob's initial advantage becomes even greater.