Exercise 2 - Beta Binomial Poisson example.

$y \sim \text{Binomial}(n, \theta)$.
$n \sim \text{Pois}(\lambda)$
$\theta \sim \text{Beta}(a, b)$

$$p(y, n, \theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1} \frac{e^{-\lambda}\lambda^{n}}{n!} \binom{n}{y} \theta^{y}(1-\theta)^{n-y}.$$

$$\prod_{i=1}^{n} \binom{n}{y} \theta^{y}(1-\theta)^{n-y}$$

$y = 0, 1, 2, \ldots n$

$0 < \theta < 1$

$n = 0, 1, 2, 3, \ldots$

Steps of Gibbs sampling algorithm

Full conditional distributions

$$f(\theta \mid y, n) \propto \theta^{a+y-1} (1-\theta)^{b+n-y-1}$$

$$\propto \text{Beta}(a+y, b+n-y)$$

$$f(n \mid \theta, y) \propto \frac{\lambda^{n}}{n!} \binom{n}{y} (1-\theta)^{n-y}.$$

$$\prod \frac{n!}{y_i!(n-y_i)!}.$$

$$\propto \frac{[\lambda(1-\theta)]^{n-y}}{(n-y)!}.$$

$$\propto \frac{\prod_{i=1}^{n-1} n!}{\prod_{i=1}^{n} (n-y_i)!} [\lambda(1-\theta)]^{\sum n-y_i}$$

Let $z = n-y$. So $z \mid \theta, y \sim \text{Pois}(\lambda(1-\theta))$.

Iteration (t) of Gibbs sampler.

1. Sample $\theta^{(t)}$ from $\text{Beta}(a+y, b+n^{(t-1)}-y)$.
2. Sample $z^{(t)}$ from $z \sim \text{Pois}(\lambda(1-\theta^{(t)}))$

   Set $n^{(t)} = y + z^{(t)}$.