MAT 237, Quiz 6

NAME: **STUDENT ID NUMBER :**

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Part A: (4 marks) Present the definition of integrability of a bounded function f on an interval $[a, b]$. Also give the ϵ characterization of integrability (Lemma 4.5).

2 f is integrable on $[a,b]$ if the least upper bound of all the lower Riemann Sums = the greatest lower bound of all the upper sums of f on $[a,b]$.

2 Lemma 4.5: for a bdd function f on $[a,b]$

f is integrable $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists P$ st. $\bar{S}_P f - \underline{S}_P f < \varepsilon$

Part B: (2 marks) Use Lemma 4.5 as in part A to show that the function $f(x) = 2$ for all $x \in [0, 2]$ except for $f(1) = 3$, is integrable, and calculate the integral. (Please define your partition)

Given $\varepsilon > 0$, define $P = \{0, 1-\frac{\varepsilon}{3}, 1+\frac{\varepsilon}{3}, 2\}$ ← 0.5

$\Rightarrow \bar{S}_P f = 2(1-\frac{\varepsilon}{3}) + 3(1+\frac{\varepsilon}{3}-1+\frac{\varepsilon}{3}) + 2(2-1-\frac{\varepsilon}{3}) = 4+\varepsilon-\frac{2\varepsilon}{3}$

1 $\underline{S}_P f = 2(1-\frac{\varepsilon}{3}) + 2(1+\frac{\varepsilon}{3}-1+\frac{\varepsilon}{3}) + 2(2-1-\frac{\varepsilon}{3}) = 4+\frac{2\varepsilon}{3}-\frac{2\varepsilon}{3}$

$\Rightarrow \bar{S}_P f - \underline{S}_P f = \frac{\varepsilon}{3} < \varepsilon$

$\Rightarrow f$ is integrable.

Let $\varepsilon \to 0$, $\lim_{\varepsilon\to 0} \bar{S}_P f = \lim_{\varepsilon\to 0} \underline{S}_P f = 4 \Rightarrow I_a^b f = 4$ ← 0.5

Part C: (4 marks) Prove Lemma 4.5 only the direction f is integrable, therefor $\forall\epsilon\ldots$

Assume $\underline{I}_a^b f = \bar{I}_a^b f = I_a^b f$, then

Given $\varepsilon > 0$, $\exists P$ and Q st.

$$\underline{I}_a^b f - \frac{\varepsilon}{2} < \underline{S}_P f \le \underline{I}_a^b f = \bar{I}_a^b f \le \bar{S}_p f < \bar{I}_a^b f + \frac{\varepsilon}{2}$$ 2

Let $R = P \cup Q$, then R is a refinement of both P and Q. ← 1

$$\Rightarrow I_a^b f - \frac{\varepsilon}{2} = \underline{I}_a^b f - \frac{\varepsilon}{2} < \underline{S}_P f \le \underline{S}_R f \le \bar{S}_R f \le \bar{S}_Q f < \bar{I}_a^b f + \frac{\varepsilon}{2} = I_a^b f + \frac{\varepsilon}{2}$$

$$\Rightarrow \bar{S}_R f - \underline{S}_R f < I_a^b f + \frac{\varepsilon}{2} - (I_a^b f - \frac{\varepsilon}{2}) = \varepsilon$$ ← 1