MAT135H1S Calculus I(A)

Solution to even-numbered problems in Section 3.1, 3.2, 3.3 and 3.4

(Section 3.1, Q10)

$$h(x) = (x-2)(2x+3) = 2x^2 - x - 6$$

$$\text{Therefore, } h'(x) = 4x - 1.$$

(Section 3.1, Q28)

$$y = ae^v + \frac{b}{v} + \frac{c}{v^2} = ae^v + bv^{-1} + cv^{-2}$$

$$\text{Therefore, } \frac{dy}{dv} = ae^v - bv^{-2} - 2cv^{-3}.$$

(Section 3.1, Q30)

$$v = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2 = \left(\sqrt{x}\right)^2 + 2\sqrt{x}\cdot\frac{1}{\sqrt[3]{x}} + \left(\frac{1}{\sqrt[3]{x}}\right)^2$$

$$= x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}$$

$$\text{Therefore, } \frac{dv}{dx} = 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}.$$

(Section 3.2, Q8)

$$G'(x) = \frac{(2x+1)(2x) - (x^2-2)(2)}{(2x+1)^2} = \frac{2x^2+2x+4}{(2x+1)^2}$$

(Section 3.2, Q34)

$$\frac{dy}{dx} = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

At the point $(1,1)$, $\dfrac{dy}{dx} = \dfrac{2-2(1)^2}{\left((1)^2+1\right)^2} = 0$ and so the equation of the tangent line is $y - 1 = 0(x-1)$, or $y = 1$. The slope of the normal line is undefined, and therefore, the equation of the normal line is $x = 1$.

(Section 3.2, Q42)

$$g(x) = \frac{x}{e^x}$$
$$g'(x) = \frac{e^x(1) - x(e^x)}{(e^x)^2} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}$$
$$g''(x) = \frac{e^x(-1) - (1-x)(e^x)}{(e^x)^2} = \frac{e^x(x-2)}{e^{2x}} = \frac{x-2}{e^x}$$
$$g'''(x) = \frac{e^x(1) - (x-2)(e^x)}{(e^x)^2} = \frac{e^x(3-x)}{e^{2x}} = \frac{3-x}{e^x}$$
$$g^{(4)}(x) = \frac{e^x(-1) - (3-x)(e^x)}{(e^x)^2} = \frac{e^x(x-4)}{e^{2x}} = \frac{x-4}{e^x}$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(-1)^n(x-n)}{e^x}$.

(Section 3.2, Q46)

$$\frac{d}{dx}\Big(\frac{h(x)}{x}\Big) = \frac{xh'(x) - h(x)}{x^2}$$
$$\text{Therefore, } \frac{d}{dx}\Big(\frac{h(x)}{x}\Big)\Big|_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - 4}{4} = -\frac{5}{2}.$$

(Section 3.3, Q8)

$$f(t) = \frac{\cot t}{e^t}$$
$$f'(t) = \frac{e^t(-\csc^2 t) - (\cot t)(e^t)}{(e^t)^2} = \frac{-e^t(\csc^2 t + \cot t)}{e^{2t}} = -\frac{\csc^2 t + \cot t}{e^t}$$

(Section 3.3, Q12)

$$y = \frac{\cos x}{1 - \sin x}$$
$$\frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1}{1-\sin x}$$

Note that we have made use of the identity $\sin^2 x + \cos^2 x = 1$.

(Section 3.3, Q40)
Rewriting, we have

$$\frac{\sin 4x}{\sin 6x} = \frac{4}{6}\Big(\frac{\sin 4x}{4x}\Big)\Big(\frac{6x}{\sin 6x}\Big)$$

Taking limits, we have

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x}=\lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{x}{\sin 6x}\right)=\lim_{x\to 0}\left(\frac{4\sin 4x}{4x}\cdot\frac{6x}{6\sin 6x}\right)$$
$$=4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x}=4(1)\cdot\frac{1}{6}(1)=\frac{2}{3}$$

(Section 3.3, Q48)

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}=\frac{1}{3}\cdot 1=\frac{1}{3}$$

(Section 3.4, Q12)

$$\begin{aligned}f(t)&=\sin(e^t)+e^{\sin t}\\ f'(t)&=\cos(e^t)\cdot(e^t)+e^{\sin t}\cdot\cos t\\ &=e^t\cos(e^t)+e^{\sin t}\cos t\end{aligned}$$

(Section 3.4, Q30)

$$\begin{aligned}F(v)\ &=\left(\frac{v}{v^3+1}\right)^6\\ F'(v)\ &=6\left(\frac{v}{v^3+1}\right)^5\cdot\frac{(v^3+1)(1)-v(3v^2)}{(v^3+1)^2}\\ &=\frac{6v^5}{(v^3+1)^5}\cdot\frac{1-2v^3}{(v^3+1)^2}\\ &=\frac{6v^5(1-2v^3)}{(v^3+1)^7}\end{aligned}$$

(Section 3.4, Q78)

$$\begin{aligned}f(x)&=xe^{-x}\\ f'(x)&=e^{-x}-xe^{-x}=(1-x)e^{-x}\\ f''(x)&=-e^{-x}+(1-x)(-e^{-x})=(x-2)e^{-x}\\ f'''(x)&=e^{-x}+(x-2)(-e^{-x})=(3-x)e^{-x}\\ f''''(x)&=-e^{-x}+(3-x)(-e^{-x})=(x-4)e^{-x}\end{aligned}$$

The pattern suggests that $f^{(1000)}(x)=(x-1000)e^{-x}$.