Assignment 2 - MAT 327 - Summer 2014

Due June 2nd, 2013 at 4:10 PM

Comprehension

For this section please complete these questions independently without consulting other students.

[**C.1**] Let $A \subseteq X$, a topological space. Prove that

$$\overline{A} = \bigcap \{ \, C \subseteq X : C \text{ is closed and } A \subseteq C \, \}$$

Going in a similar direction, define **the interior of** A as

$$\text{int}(A) := \bigcup \{ \, U \subseteq X : U \text{ is open and } U \subseteq A \, \}$$

Convince yourself, on your own, that this is always an open set.
[**C.2**] Find the interior and closure of the following sets (no proof needed):

1. $(3, 4]$ in $\mathbb{R}$ with the usual topology;
2. $(3, 4]$ in $\mathbb{R}$ with the Sorgenfrey Line;
3. $(3, 4]$ in $\mathbb{R}$ with the discrete topology;
4. $B_{77}(0, 0, 0) \setminus B_{13}(0, 0, 0)$ in $\mathbb{R}^3$ with the usual topology;
5. $\mathbb{Q}$ (as a subset of $\mathbb{R}$ with the usual topology).

[**C.3**] Let $A \subseteq X$ a topological space. Prove that A is open iff $A = \text{int}(A)$. (Notice the similar statement we made in lecture about closed sets!) Conclude that

$$\text{int}(\text{int}(A)) = \text{int}(A)$$

[**C.4**] Is it true $\text{int}(\overline{A}) = \overline{(\text{int}(A))}$? Is $\text{int}(\overline{A}) \subseteq \overline{(\text{int}(A))}$? What about $\text{int}(\overline{A}) \supseteq \overline{(\text{int}(A))}$?

[**C.5**] Is a dense subset of the $\mathbb{R}_{\text{cofinite}}$ always dense in $\mathbb{R}_{\text{usual}}$? Is a dense subset of $\mathbb{R}_{\text{usual}}$ always dense in $\mathbb{R}_{\text{cofinite}}$? State your own proposition that captures the general idea here and provide a short proof.

Application

For this section you may consult other students in the course as well as your notes and textbook, but please avoid consulting the internet. See the course Syllabus for more information.

For the next exercise you will need to know about the boundary of a set:

Definition. *For $A \subseteq X$ a topological space, define $\partial(A) := \overline{A} \cap \overline{X \setminus A}$, the **boundary of** A.*

(Notice that the boundary of a set is always closed.)

[**A.1**] For $A \subseteq X$ a topological space, prove that $X = \mathrm{int}(A) \sqcup \partial(A) \sqcup \mathrm{int}(X \setminus A)$, (where $D = B \sqcup C$' means that $D = B \cup C$ and $B \cap C = \emptyset$. Give an example of a set A where both $\mathrm{int}(A) \neq \emptyset, \mathrm{int}(X \setminus A) \neq \emptyset$ but $\partial(A) = \emptyset$. What properties must such an A have?

[**A.2**] *Let's think about dense sets...* Is it true that the intersection of two dense sets is again dense? What about the intersection of a dense set with an open set? What if both sets are dense and open? What about the intersection of finitely many such sets? Let's now shift our focus to $\mathbb{R}$ (with the usual topology). Is the only open set that contains $\mathbb{Q}$ all of $\mathbb{R}$? Is it possible to make such an open set that has an infinite complement? What about an uncountable complement? What about a dense complement? (This is one way that mathematicians think about problems; they ask a reasonable question, get an answer and then keep pushing their answer.)

New Ideas

*For this section please work on and sumbit **at least one** of the following problems. You may consult other students, texts, online resources or other professors, but you must cite all sources used. See the course Syllabus for more information.*

[**NI.1**] (Let's go further than A.2.) Prove that the intersection of countably many sets that are dense and open in $\mathbb{R}$ is again dense. (This is a famous theorem, called the "Baire Category Theorem". Don't be scared that I'm asking you to prove a named theorem! Try it on your own; I can give you a hint if you need one.) Does the intersection have to be open? What if we allow uncountable intersections? Write a couple sentences explaining your thoughts on the following assertion: "There is an uncountable collection of mutually different dense and open sets whose intersection is dense." *After* you've thought about this on your own, read the Wikipedia article on Martin's Axiom and explain in your own words (in a way that a student in this course would understand) what Martin's Axiom says about that assertion.

It is a useful skill for a mathematician to be able to read technical writing (in this case the Wikipedia article) and extract a little bit of useful information from a lot of irrelevant information.

[**NI.2**] Write a program that:

1. Allows a user to input a collection of subsets of $X = \{0, 1, 2, 3, 4\}$;
2. Generates the smallest topology $\mathcal{T}$ that contains those sets (always include the full space X);
3. Allows the user to input a subset of $A \subseteq X$, and your program returns the interior, closure and boundary of A (in the topology $\mathcal{T}$).

Feel free to do more if you like. Customize it and make it your own. It would be nice if you could get it to the point where we could post it on the course website for others to play with it. Also feel free to work in small groups (up to 5) and submit a single program. If you do decide to work in a group of 5, please make sure it is 5 times more awesome than if only one person did it on their own. I'm kind of imagining something that looks like this:

`http://humpheh.com/magic/c/`

[**NI.3**] This is a very famous (and fun!) problem called the **Kuratowski 14-set problem**. Let $A \subseteq X$ a topological space.

1. Prove that there are at most 14 different subsets of X that can be obtained from A by applying the operations of closures and complements successively. (There is a tutorial problem that you will probably find useful!)
2. Find a subset A of $\mathbb{R}$ (with the usual topology) such that the 14 subsets of $\mathbb{R}$ can be obtained from A by applying the operations of closure and complements successively.

Since this is such a famous problem, please avoid consulting the internet. The point of this exercise is for you to *discover* the magic of the problem, and looking up a solution side-steps that. Please *do* consult other students and discuss this with others though.
There is also a solution of this problem contained in Counterexamples in Topology. Please don't consult it until after you've written up your own solution.