1. For each equivalence below, either provide a derivation from one side of the equivalence to the other (justify each step of your derivation with a brief explanation — for example, by naming one of the equivalences (see over for a list), or show that the equivalence does not hold (warning: you cannot use a derivation to show non-equivalence — instead, think carefully about what an equivalence means, and how you can disprove it).

   (a) $(P \Rightarrow R) \wedge (Q \Rightarrow R) \iff (P \vee Q) \Rightarrow R$

   **Sample Solution:**

$$\begin{aligned}
(P \Rightarrow R) \wedge (Q \Rightarrow R) &\iff (\neg P \vee R) \wedge (\neg Q \vee R) && \text{(implication)}\\
&\iff (\neg P \wedge \neg Q) \vee R && \text{(distributivity)}\\
&\iff \neg(P \vee Q) \vee R && \text{(DeMorgan's law)}\\
&\iff (P \vee Q) \Rightarrow R && \text{(implication)}
\end{aligned}$$

   (b) $P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow R$

   **Sample Solution:**

   The equivalence does not always hold:
   Let $P = \textsf{False}$, $Q = \textsf{True}$, $R = \textsf{False}$.
   Then $P \Rightarrow (Q \Rightarrow R) = \textsf{False} \Rightarrow (\textsf{True} \Rightarrow \textsf{False}) = \textsf{False} \Rightarrow \textsf{False} = \textsf{True}$ and $(P \Rightarrow Q) \Rightarrow R = (\textsf{False} \Rightarrow \textsf{True}) \Rightarrow \textsf{False} = \textsf{True} \Rightarrow \textsf{False} = \textsf{False}$.

   (c) $P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$

   **Sample Solution:**

$$\begin{aligned}
(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) &\iff \neg(P \Rightarrow Q) \vee (\neg P \vee R) && \text{(implication)}\\
&\iff (P \wedge \neg Q) \vee (\neg P \vee R) && \text{(implication negation)}\\
&\iff \big((P \vee \neg P) \wedge (\neg Q \vee \neg P)\big) \vee R && \text{(distributivity)}\\
&\iff (\neg P \vee \neg Q) \vee R && \text{(identity and commutativity)}\\
&\iff \neg P \vee (\neg Q \vee R) && \text{(associativity)}\\
&\iff P \Rightarrow (Q \Rightarrow R) && \text{(implication)}
\end{aligned}$$

**Standard Equivalences** (where $P$, $Q$, $P(x)$, $Q(x)$, *etc.* are arbitrary sentences)

- *Commutativity*
  $P \wedge Q \iff Q \wedge P$
  $P \vee Q \iff Q \vee P$
  $P \Leftrightarrow Q \iff Q \Leftrightarrow P$
- *Associativity*
  $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R$
  $P \vee (Q \vee R) \iff (P \vee Q) \vee R$
- *Identity*
  $P \wedge (Q \vee \neg Q) \iff P$
  $P \vee (Q \wedge \neg Q) \iff P$
- *Absorption*
  $P \wedge (Q \wedge \neg Q) \iff Q \wedge \neg Q$
  $P \vee (Q \vee \neg Q) \iff Q \vee \neg Q$
- *Idempotency*
  $P \wedge P \iff P$
  $P \vee P \iff P$
- *Double Negation*
  $\neg\neg P \iff P$
- *DeMorgan's Laws*
  $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
  $\neg(P \vee Q) \iff \neg P \wedge \neg Q$
- *Distributivity*
  $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- *Implication*
  $P \Rightarrow Q \iff \neg P \vee Q$
- *Biconditional*
  $P \Leftrightarrow Q \iff (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- *Renaming* (where $P(x)$ does not contain variable $y$)
  $\forall x, P(x) \iff \forall y, P(y)$
  $\exists x, P(x) \iff \exists y, P(y)$
- *Quantifier Negation*
  $\neg\forall x, P(x) \iff \exists x, \neg P(x)$
  $\neg\exists x, P(x) \iff \forall x, \neg P(x)$
- *Quantifier Commutativity*
  $\forall x, \forall y, S(x,y) \iff \forall y, \forall x, S(x,y)$
  $\exists x, \exists y, S(x,y) \iff \exists y, \exists x, S(x,y)$
- *Quantifier Distributivity* (where $S$ does not contain variable $x$)
  $S \wedge \forall x, Q(x) \iff \forall x, S \wedge Q(x)$
  $S \vee \forall x, Q(x) \iff \forall x, S \vee Q(x)$
  $S \wedge \exists x, Q(x) \iff \exists x, S \wedge Q(x)$
  $S \vee \exists x, Q(x) \iff \exists x, S \vee Q(x)$

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2. An "interpretation" for a logical statement consists of a domain $D$ (any non-empty set of elements) and a meaning for each predicate symbol. For example, $D = \{1, 2\}$ and $P(x)$: "$x > 0$" is an interpretation for the statement $\forall x \in D, P(x)$ (in this case, one that happens to make the statement True). For each statement below, provide one interpretation under which the statement is true and another interpretation under which the statement is false — if either case is not possible, explain why clearly and concisely.

   (a) $\forall x \in D, P(x) \iff \exists y \in D, P(y)$

   **Sample Solution:**

   Let $D = \mathbb{N}$ and $P(x)$ mean "$x$ is nonnegative". Then $\forall x \in D, P(x)$ means "every natural number is nonnegative", which is clearly true. And $\exists y, P(y)$ means "some natural number is nonnegative", which is also clearly true. This interpretation makes the statement True.
   Let $D = \mathbb{N}$ and $P(x)$ mean "$x$ is even". Then $\forall x, P(x)$ means "every natural number is even", which is clearly false. But $\exists y, P(y)$ means "some natural number is even", which is clearly true. This interpretation makes the statement False.

   (b) $\forall x \in D, \exists y \in D, P(x,y) \wedge \forall z \in D, P(z,y) \Rightarrow z = x$

   **Sample Solution:**

   When $D = \mathbb{R}^* = \mathbb{R} - \{0\}$ and $P(x,y)$ means "$xy = 1$", the entire statement means "Every number has a multiplicative inverse that is the multiplicative inverse of no other number."
   This makes the statement True because it is a known property of the real numbers that every non-zero number has a unique multiplicative inverse.
   When $D = \mathbb{N}$ and $P(x,y)$ means "$y$ is a multiple of $x$", the entire statement means "Every number has some multiples that are multiples of no other number."
   This makes the statement False because when $x = 2$, every multiple of 2 (of the form $y = 2k$ for some $k \in \mathbb{N}$) is also a multiple of $z = 1 \neq 2 = x$ ($y = 1 \times (2k)$).