Exercise 9

Mixture of conjugate priors

$$g(p) = \gamma g_1(p) + (1-\gamma) g_2 p.$$

Let. $\gamma = 0.5$

(1.) $g_1(p) = \dfrac{p^{15-1}(1-p)^{5-1}}{Beta(15,5)}$; $g_2(p) = \dfrac{p^{9-1}(1-p)^{11-1}}{Beta(9,11)}$.

(Beta(a,b) = $\int_0^1 t^{a-1}(1-t)^{b-1}dt$

(2). $n = 168$ $y = 91$.

$$g(p|y) \propto g(p)\, p(y|p).$$

$$= \gamma g_1(p) p(y|p_1) + (1-\gamma) g_2(p)(p(y|p_2))$$

$$= \frac{\gamma}{Beta(15,5)} p^{15+91-1} (1-p)^{5+77-1} + \frac{(1-\gamma)}{Beta(9,11)} p^{9+91-1} (1-p)^{11+77-1}$$

$$= \frac{\gamma}{Beta(15,5)} p^{106-1}(1-p)^{82-1} + \frac{(1-\gamma) p^{100-1}(1-p)^{88-1}}{Beta(9,11)}.$$

We require $\int_0^1 g(p|y=91)\, dp = 1$.

So solve for k such that.

$$k \int_0^1 \frac{\gamma\, p^{106-1}(1-p)^{82-1}}{Beta(15,5)} + \frac{(1-\gamma) p^{100-1}(1-p)^{88-1}}{Beta(9,11)}\, dp = 1$$

$$1 = K\left[\frac{\gamma\,\mathrm{Beta}(106,82)}{\mathrm{Beta}(15,5)}\int_0^1 \frac{p^{106-1}(1-p)^{82-1}}{\mathrm{Beta}(106,82)}\,dp\right.$$

$$\left.+\frac{(1-\gamma)\,\mathrm{Beta}(100,88)}{\mathrm{Beta}(9,11)}\int_0^1 \frac{p^{100-1}(1-p)^{88-1}}{\mathrm{Beta}(100,88)}\,dp\right]$$

$$= K\left[\gamma\,\frac{\mathrm{Beta}(106,82)}{\mathrm{Beta}(15,5)} + (1-\gamma)\,\frac{\mathrm{Beta}(100,88)}{\mathrm{Beta}(9,11)}\right]$$

$$K = \frac{1}{\dfrac{\gamma\,\mathrm{Beta}(106,82)}{\mathrm{Beta}(15,5)} + \dfrac{(1-\gamma)\,\mathrm{Beta}(100,88)}{\mathrm{Beta}(9,11)}}$$

So $g(p \mid y=91) = \gamma' g_1^*(p) + (1-\gamma')\, g_2^*(p)$

$$\gamma' = \frac{\gamma\,\mathrm{Beta}(106,82)/\mathrm{Beta}(15,5)}{\dfrac{\gamma\,\mathrm{Beta}(106,82)}{\mathrm{Beta}(15,5)} + \dfrac{(1-\gamma)\,\mathrm{Beta}(100,85)}{\mathrm{Beta}(9,11)}} = \cancel{\mathrm{Beta}(106,82)}$$

$$g_1^*(p) = \frac{p^{106-1}(1-p)^{82-1}}{\mathrm{Beta}(106,82)}\,; \quad g_2^*(p) = \frac{p^{100-1}(1-p)^{88-1}}{\mathrm{Beta}(100,88)}.$$

(3) Posterior weights.

$\gamma' = 0.18 \quad (1-\gamma') = 0.82.$

$$\left.\begin{array}{l} E_{g_1}(p|y) = 0.56 \\ \\ E_{g_2}(p|y) = 0.53 \end{array}\right\} \text{very similar.}$$

Prior distribution is bimodal

Posterior distribution is unimodal

(4) Frequentist

$H_0 : p = 0.5 \qquad H_A = p > 0.5$

$$z = \frac{\frac{91}{168} - 0.5}{\sqrt{\frac{0.5(1-0.5)}{168}}} = 1.08$$

p-value $= \Pr(Z \geq 1.08) = 0.1401$

(not significant)

Bayesian

$\Pr(p > 0.5 | y) = 0.804$. (Monte Carlo simulation)

Indicates strong evidence that auction clearance rate is greater than 0.5.