1. **Predicate:** $P(n)$: "$\forall x \in \mathbb{R}, (\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$."

   We prove $\forall n \in \mathbb{Z}^+, P(n)$, by simple induction on $\mathbb{Z}^+$.

   Base Case: Let $x \in \mathbb{R}$ be arbitrary. Then $(\cos(x) + i\sin(x))^1 = \cos(x) + i\sin(x) = \cos(1 \cdot x) + i\sin(1 \cdot x)$. Hence, $P(1)$.

   Ind. Hyp.: Suppose $n \in \mathbb{Z}^+$ and $P(n)$.

   Ind. Step: Let $x \in \mathbb{R}$ be arbitrary.

   Then
   $$\begin{aligned}(\cos(x) + i\sin(x))^{n+1} &= (\cos(x) + i\sin(x)) \cdot (\cos(x) + i\sin(x))^n \\ &= (\cos(x) + i\sin(x)) \cdot (\cos(nx) + i\sin(nx)) \quad \text{(by I.H.)} \\ &= \cos(x)\cos(nx) + i^2\sin(x)\sin(nx) + i\sin(x)\cos(nx) + i\cos(x)\sin(nx) \\ &= (\cos(x)\cos(nx) - \sin(x)\sin(nx)) + i(\sin(x)\cos(nx) + \cos(x)\sin(nx)) \\ &= \cos(x + nx) + i\sin(x + nx) \\ &= \cos((n+1)x) + i\sin((n+1)x).\end{aligned}$$

   Hence, $P(n+1)$.

   Conclusion: $\forall n \in \mathbb{Z}^+, \forall x \in \mathbb{R}, (\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$.

2. **Predicate:** $P(n)$: "$f(n) = 3^n - 2^n$."

   We prove $\forall n \in \mathbb{Z}^+, P(n)$, by complete induction on $\mathbb{Z}^+$.

   Base Cases: $f(1) = 1 = 3 - 2 = 3^1 - 2^1$, so $P(1)$.

   $f(2) = 5 = 9 - 4 = 3^2 - 2^2$, so $P(2)$.

   Ind. Hyp.: Suppose $n \in \mathbb{Z}^+$ and $n > 2$ and $\forall k \in \mathbb{Z}^+, k < n \Rightarrow P(k)$.

   Ind. Step:
   $$\begin{aligned}f(n) &= 5f(n-1) - 6f(n-2) \quad \text{(by definition and since } n > 2\text{)} \\ &= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) \quad \text{(by I.H.)} \\ &= 15 \cdot 3^{n-2} - 10 \cdot 2^{n-2} - 6 \cdot 3^{n-2} + 6 \cdot 2^{n-2} \\ &= 9 \cdot 3^{n-2} - 4 \cdot 2^{n-2} \\ &= 3^n - 2^n.\end{aligned}$$

   Conclusion: $\forall n \in \mathbb{Z}^+, f(n) = 3^n - 2^n$.

3. **Predicate:** $P(T)$: "T contains some monochromatic complete binary subtree."

   We prove $P(T)$ for all coloured complete ternary trees T, by structural induction.

   Base Case: If T is a single coloured node, then T is already a monochromatic complete binary tree.

   Ind. Hyp.: Suppose T_1, T_2, and T_3 are coloured complete ternary trees with the same height, and $P(T_1)$, $P(T_2)$, $P(T_3)$.

   Ind. Step: Let T be the coloured complete ternary tree constructed from T_1, T_2, T_3 by placing them under a new root node.

   Let T_1', T_2', T_3' be the monochromatic complete binary subtrees of T_1, T_2, T_3 (respectively) that we know exist by the I.H.

   Then two of T_1', T_2', T_3' must have leaves of the same colour (because there are only two possible colours). Without loss of generality, assume that T_1' and T_2' have leaves of the same colour.[1] Then the tree constructed by placing both T_1' and T_2' under T's root node is a monochromatic complete binary subtree of T.

   Conclusion: By structural induction, every coloured complete ternary tree contains some monochromatic complete binary subtree.

[1] The phrase "without loss of generality" means that we are picking only one out of many possible cases, but where the argument would be exactly the same for every case (up to the names of the subtrees, in this situation), so we don't waste time repeating the argument.

4. **Predicate:** $S(P)$: "$\exists P' \in \mathcal{F}$, P' contain no negation symbol except next to variables and $P' \Leftrightarrow P$."

   We prove that $\forall P \in \mathcal{F}, S(P) \wedge S(\neg P)$, by structural induction on $\mathcal{F}$.

   Base Case: Let $i \in \mathbb{N}$ be arbitrary.

   Then p_i and $\neg p_i$ are equivalent to themselves and both contain no negation except next to variables.

   Hence, $\forall i \in \mathbb{N}, S(p_i) \wedge S(\neg p_i)$.

   Ind. Hyp. 1: Suppose $P \in \mathcal{F}$ and $S(P) \wedge S(\neg P)$ — with $P \Leftrightarrow P'$ and $\neg P \Leftrightarrow P''$.

   Ind. Step 1: Then $\neg P \Leftrightarrow P''$ and $\neg\neg P \Leftrightarrow P \Leftrightarrow P'$, by the I.H.

   Hence, $S(\neg P) \wedge S(\neg\neg P)$.

   Ind. Hyp. 2: Suppose $P_1, P_2 \in \mathcal{F}$ and $S(P_1) \wedge S(\neg P_1)$ and $S(P_2) \wedge S(\neg P_2)$ — with $P_1 \Leftrightarrow P_1'$ and $\neg P_1 \Leftrightarrow P_1''$ and $P_2 \Leftrightarrow P_2'$ and $\neg P_2 \Leftrightarrow P_2''$.

   Ind. Step 2: Then,

$$
\begin{aligned}
(P_1 \wedge P_2) &\Leftrightarrow (P_1' \wedge P_2') \\
\neg(P_1 \wedge P_2) &\Leftrightarrow (\neg P_1 \vee \neg P_2) \Leftrightarrow (P_1'' \vee P_2'') \\
(P_1 \vee P_2) &\Leftrightarrow (P_1' \vee P_2') \\
\neg(P_1 \vee P_2) &\Leftrightarrow (\neg P_1 \wedge \neg P_2) \Leftrightarrow (P_1'' \wedge P_2'') \\
(P_1 \Rightarrow P_2) &\Leftrightarrow (P_1' \Rightarrow P_2') \\
\neg(P_1 \Rightarrow P_2) &\Leftrightarrow (P_1 \wedge \neg P_2) \Leftrightarrow (P_1' \wedge P_2'') \\
(P_1 \Leftrightarrow P_2) &\Leftrightarrow (P_1' \Leftrightarrow P_2') \\
\neg(P_1 \Leftrightarrow P_2) &\Leftrightarrow (P_1 \Leftrightarrow \neg P_2) \Leftrightarrow (P_1' \Leftrightarrow P_2'').
\end{aligned}
$$

   Hence, $S((P_1 \wedge P_2)) \wedge S(\neg(P_1 \wedge P_2))$ and $S((P_1 \vee P_2)) \wedge S(\neg(P_1 \vee P_2))$ and $S((P_1 \Rightarrow P_2)) \wedge S(\neg(P_1 \Rightarrow P_2))$ and $S((P_1 \Leftrightarrow P_2)) \wedge S(\neg(P_1 \Leftrightarrow P_2))$.

   Conclusion: For all $P \in \mathcal{F}$, there is an equivalent $P' \in \mathcal{F}$ such that P' contains no negation except next to variables.

   (There is also a $P'' \in \mathcal{F}$ such that $P'' \Leftrightarrow \neg P$ and P'' contains no negation except next to variables, but that was required for the proof only.)