STAT2001 Tutorial 7 Solutions

Problem 1

(a) $1 = c\int_0^2 (2-y)dy = c\left[2y - \frac{y^2}{2}\Big|_0^2\right] = c\left[2(2) - \frac{2^2}{2}\right] = 2c \Rightarrow c = \frac{1}{2}$.

$\mu = \frac{1}{2}\int_0^2 y(2-y)dy = \frac{1}{2}\left[y^2 - \frac{y^3}{3}\Big|_0^2\right] = \frac{1}{2}\left[2^2 - \frac{2^3}{3}\right] = \frac{2}{3}$ (mean).

$\mu_2' = \frac{1}{2}\int_0^2 y^2(2-y)dy = \frac{1}{2}\left[\frac{2y^3}{3} - \frac{y^4}{4}\Big|_0^2\right] = \frac{1}{2}\left[\frac{2(2)^3}{3} - \frac{2^4}{4}\right] = \frac{2}{3}$ (2nd raw moment).

$\sigma^2 = \mu_2' - \mu^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$ (variance).

$\sigma = \frac{\sqrt{2}}{3} = 0.4714$ (standard deviation).

(b) $E(7Y^2 - 2Y + 6) = 7EY^2 - 2EY + 6 = 7(2/3) - 2(2/3) + 6 = 28/3$.

Problem 2

(a) $m(t) = Ee^{Zt} = \int_{-\infty}^{\infty} e^{zt}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Q}dz$,

where $Q = z^2 - 2zt = z^2 - 2zt + t^2 - t^2 = (z-t)^2 - t^2$

(we have completed the square in the exponent).

Therefore $m(t) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\{(z-t)^2 - t^2\}}dz = e^{\frac{1}{2}t^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-t)^2}dz = e^{\frac{1}{2}t^2}$.

(b) Now $m'(t) = te^{\frac{1}{2}t^2}$. So $\mu = m'(0) = 0$ (mean).

We can also write $m'(t) = tm(t)$.

Therefore $m''(t) = tm'(t) + m(t)$. So $\mu_2' = m''(0) =$ 0 + 1 = 1.

Hence $\sigma^2 = \mu_2' - \mu^2 = 1 - 0^2 = 1$ (variance).

$m'''(t) = tm''(t) + m'(t) + m'(t)$. So $\mu_3' = m'''(0) = 0 + 0 + 0$ = 0.

So $\mu_3 = E(Z-\mu)^3 = E(Z-0)^3 = \mu_3' = 0$ (third central moment).

(NB: μ_3 = 0 corresponds to the fact that the dsn of *Z* is *symmetric*.
In general, a distribution is *right-skewed* if μ_3 > 0 (eg the gamma dsn),
and *left-skewed* if μ_3 < 0. We may call μ_3 the *skewness* or *skewness parameter*.)

(c) $EY = a + bEZ = a + b(0) = a$.

$VarY = b^2 VarZ = b^2$.

$E(Y - EY)^3 = E(a + bZ - a)^3 = b^3 EZ^3 = 0$.

(NB: We have proved that $Y = a + bZ$ has a symmetric dsn with mean a and variance b^2. We have *not* proved that Y has a normal dsn.
This will be proved elsewhere.)

Problem 3

(a) Let Y be the time to failure of a randomly chosen printer, in 100's of hours.
Then $Y \sim \mathrm{N}(15, 4)$.

So $P(Y < 10) = P\left(\frac{Y - 15}{2} < \frac{10 - 15}{2}\right) = P(Z < -2.5) = P(Z > 2.5) = 0.0062$.

So 0.62% of printers will fail before 1000 hours.

(b) We wish to find the value of c such that $P(Y < c) = 0.05$.

Thus: $0.05 = P\left(\frac{Y - 15}{2} < \frac{c - 15}{2}\right) = P\left(Z < \frac{c - 15}{2}\right)$.

But $0.05 = P(Z < -1.645)$ from tables.

Therefore $(c - 15)/2 = -1.645$, which implies that $c = 11.71$.

So the guarantee time should be 1171 hours.

(c) $P(Y < 5) = \frac{1}{2} P(Y < 5 \text{ or } Y > 25)$ by symmetry about the mean $\mu = 15$

$= \frac{1}{2} P(|Y - 15| > 10)$

$= \frac{1}{2} P(|Y - \mu| \geq k\sigma)$ where $\sigma = 2$ and $k = 5$

$\leq \frac{1}{2}\frac{1}{k^2}$ by Chebyshev's theorem

$= \frac{1}{2}\frac{1}{5^2}$

$= 0.02$.

Thus no more that 2% of printers will fail before 500 hours.

(Using tables, we find that the exact proportion that will fail before 500 hours is

$$P(Y < 5) = P\left(\frac{Y - 15}{2} < \frac{5 - 15}{2}\right) = P(Z < -5) = P(Z > 5) = 0.000\,000\,287,$$

which is indeed no greater than the upper bound of 0.02.)

Problem 4

(a) Let R be the radius of the crater (in metres).
Then R ~ Expo(3), and the area of the crater is $A = \pi R^2$.

So $EA = \pi ER^2$.
Now ER = 3 and $VarR$ = 3^2 = 9. So ER^2 = $VarR$ + $(ER)^2$ = $9 + 3^2$ = 18.
Hence $EA = 18\pi$ = 56.55 m^2.

Also, $VarA = \pi^2 VarR^2 = \pi^2\{E(R^2)^2 - (ER^2)^2\}$.

$$\text{Now } E(R^2)^2 = ER^4 = \int_0^\infty r^4 \frac{1}{3} e^{-r/3} dr$$

$$= \frac{3^5\Gamma(5)}{3}\int_0^\infty \frac{r^{5-1}e^{-r/3}}{3^5\Gamma(5)} dr$$

$$= 3^4 4!$$

$$= 1944.$$

Hence $VarA = \pi^2(1944 - 18^2) = 1620\pi^2 = 15989$ m^4 (ie, m^2-squared).

(b) $P(A > 12) = P(\pi R^2 > 12) = P(R > \sqrt{12/\pi})$

$$= \int_{\sqrt{12/\pi}}^\infty \frac{1}{3} e^{-r/3} dr = e^{-\frac{\sqrt{12/\pi}}{3}} = 0.5213.$$