Mathematics for STAT 2032

Natural Logarithms

1. $\ln(xy) = \ln(x) + \ln(y)$

2. $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$

3. $\ln(x^n) = n\ln(x)$

4. $\ln(e^x) = e^{\ln(x)} = x$

Exponents

1. $\left(x^m\right)\left(x^n\right) = x^{m+n}$

2. $x^m / x^n = x^{m-n}$

3. $x^{-n} = \dfrac{1}{x^n}$

4. $x^0 = 1$

5. $\left(x^m\right)^n = x^{mn}$

6. $x^m y^m = (xy)^m$

Series

If a_t, $t = 1, 2, 3, \ldots$ is a sequence then $s_n = \sum_{t=1}^{n} a_t$ is a series.

The standard arithmetic series is

$$s_n = a + \left(a + d\right) + \left(a + 2d\right) + \ldots + \left(a + (n-1)d\right) = \sum_{t=1}^{n} a + (t-1)d$$

The summation formula for the n^{th} term of an arithmetic series is:

$$s_n = a + \left(a + d\right) + \left(a + 2d\right) + \ldots + \left(a + (n-1)d\right) = \frac{n}{2}\left(2a + (n-1)d\right)$$

Proof:

$$
\begin{aligned}
s_n &= a+\left(a+d\right)+\left(a+2d\right)+\ldots+\left(a+(n-1)d\right) \\
&= \left[a+\left(a+(n-1)d\right)\right]+\left[(a+d)+\left(a+(n-2)d\right)\right]+\left[(a+2d)+\left(a+(n-3)d\right)\right]+\ldots.. \\
&= \frac{n}{2}\left(2a+(n-1)d\right)
\end{aligned}
$$

The standard geometric series is $s_n = a+ar+ar^2+\ldots+ar^{n-1} = \sum_{t=1}^{n} ar^{t-1}$

The summation formula for the n^{th} term of a geometric series is:

$$s_n = a+ar+ar^2+\ldots+ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad \text{where } r \neq 1$$

Proof:

$$
\begin{aligned}
s_n &= a\left(1+r+r^2+\ldots+r^{n-1}\right) \\
s_n(1-r) &= a\left(1+r+r^2+\ldots+r^{n-1}\right)-a\left(r+r^2+\ldots+r^{n-1}+r^n\right) \\
&= a-ar^n \\
&= a\left(1-r^n\right) \\
s_n &= \frac{a\left(1-r^n\right)}{1-r}
\end{aligned}
$$

<u>Quadratic Formula</u>

To find the roots of the equation:

$$Ax^2+Bx+C=0$$

solve the following:

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

<u>The Derivative and Differentiation</u>

Consider a continuous smooth function $y=f(x)$ and two points A and B on the graph of the function, where $A=\left(x_0, f(x_0)\right)$ and $B=\left(x_1, f(x_1)\right)$.

The slope of the line joining A and B is $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}=\dfrac{\Delta y}{\Delta x}$.

As Δx gets shorter and shorter, the slope of the line joining A and B approaches the slope of the tangent line at point x_0.

We say that the derivative of $y=f(x)$ at x_0 is the slope of the tangent line at the point x_0:

$$\left.\frac{dy}{dx}\right|_{x_0} = f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$f'(x_0)$ is the derivative of $y=f(x)$ at $x=x_0$.

Some rules of differentiation

In the following, a,b and c are constants.

1. If $f(x)=ax+b$, $f'(x)=a$
2. If $f(x)=ax^2+bx+c$, $f'(x)=2ax+b$
3. If $f(x)=x^n$, $f'(x)=nx^{n-1}$
4. If $h(x)=\sum_{i=1}^{n} g_i(x)$, $h'(x)=\sum_{i=1}^{n} g_i{}'(x)$
5. Product Rule: If $h(x)=f(x)g(x)$, $h'(x)=f'(x)g(x)+f(x)g'(x)$
6. Quotient Rule: If $h(x)=\dfrac{f(x)}{g(x)}$ and $g'(x)\neq 0$, $h'(x)=\dfrac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$
7. If $f(x)=e^x$, $f'(x)=e^x$
8. If $f(x)=e^{g(x)}$, $f'(x)=g'(x)e^{g(x)}$
9. If $f(x)=\ln(x)$, $f'(x)=\dfrac{1}{x}$
10. If $f(x)=\ln\left(g(x)\right)$, $f'(x)=\dfrac{g'(x)}{g(x)}$

11. L'Hopitals Rule: Suppose that as $x \to a$ both $f(x)$ and $g(x)$ either both tend to 0, both tend to $+\infty$ or both tend to $-\infty$. Then: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Higher-order derivatives

If $y = f(x)$,

The first derivative is $\frac{dy}{dx} = f'(x)$

The second derivative is $\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2} = f''(x)$

The third derivative is $\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3} = f'''(x)$

Taylor Series Formula

The Taylor Series Formula will be used when we cover duration and convexity of cash flow sequences and Redington immunisation.

Consider the function $y = f(x)$ is differentiable as many times as required. If we know $f(x_0)$ and the associated derivative values, the value of the function at the point x_1 can be approximated using the n^{th} order Taylor series approximation:

$$f(x_1) \cong f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) + ... + \frac{(x_1 - x_0)^n}{n!} f^{(n)}(x_0)$$

where $f^{(n)}$ is the n^{th} derivative of $y = f(x)$, and

$n! = n(n-1)(n-2)....$
eg. $5! = 5(4)(3)(2)(1) = 120$

For example, consider the exponential function e^x. Let $y = f(x) = e^x$ and set $x_0 = 0$. Using the Taylor series approximation, this can be written as:
$e^{x_1} = f(x_1) \cong 1 + x_1 + \frac{(x_1)^2}{2!} + ...$

Integration

If $f(x) = \frac{d}{dx}F(x)$, then

$F(x) + c = \int f(x)dx$

Fundamental theorem of Integral Calculus

If the function $f(x)$ is continuous on the closed interval $[a,b]$ and if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Some rules of integration

In the following, a and b are constants.

1. $\int_a^b x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$

2. $\int_a^b e^x dx = \left. e^x \right|_a^b = e^b - e^a$

eg. $500\int_3^8 x^4 dx = \left.\frac{500x^5}{5}\right|_3^8 = 100\left(8^5 - 3^5\right)$

Probability and Statistics

The section on stochastic interest rate models will assume a basic knowledge of statistics. The main results that we will be using are summarised below:

For a **discrete random variable** $\tilde{X}$, with probability function $p(x) = \Pr\left[\tilde{X} = x\right]$, the mean is: $E\left[\tilde{X}\right] = \sum_x x \cdot p(x)$

and the variance is: $Var\left[\tilde{X}\right] = E\left[\tilde{X}^2\right] - \left(E\left[\tilde{X}\right]\right)^2 = \sum_x x^2 \cdot p(x) - \left(\sum_x x \cdot p(x)\right)^2$

For a **continuous random variable** $\tilde{X}$, with probability density function $f(x)$, the probability $P\left[a < \tilde{X} < b\right] = \int_a^b f(x)dx$.

$\tilde{X}$ has mean: $E\left[\tilde{X}\right] = \int_{-\infty}^{\infty} x \cdot f(x)dx$

and variance: $Var\left[\tilde{X}\right] = E\left[\tilde{X}^2\right] - \left(E\left[\tilde{X}\right]\right)^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - \left(\int_{-\infty}^{\infty} x \cdot f(x)dx\right)^2$

If a and b are constants then $Var\left[a\tilde{X} + b\right] = a^2 Var\left[\tilde{X}\right]$

The standard deviation of $\tilde{X}$ is $\sqrt{Var\left[\tilde{X}\right]}$.

For a function $h(\cdot)$: $E\left[h(\tilde{X})\right] = \int_{-\infty}^{\infty} h(x) \cdot f(x)dx$

If $\tilde{X}$ and $\tilde{Y}$ are independent random variables then $Var\left[\tilde{X} + \tilde{Y}\right] = Var\left[\tilde{X}\right] + Var\left[\tilde{Y}\right]$

We will also be using a number of continuous distributions:

Uniform distribution

$$f(x) = \frac{1}{b-a} \text{ for } a < x < b$$
$$E\left[\tilde{X}\right] = \frac{a+b}{2}$$
$$Var\left[\tilde{X}\right] = \frac{(b-a)^2}{12}$$

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \text{ for } -\infty < x < \infty$$
$$E\left[\tilde{X}\right] = \mu$$
$$Var\left[\tilde{X}\right] = \sigma^2$$

Recall that if $\tilde{X}$ is normally distributed with mean and variance as above, then

$$P\left[a < \tilde{X} < b\right] = P\left[\frac{a-\mu}{\sigma} < \frac{\tilde{X}-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right] = P\left[\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right]$$

where Z has a standard normal distribution (ie. normal distribution with mean 0 and variance 1).

Statistical tables can be used with a standard normal variable to find probabilities.