UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE
AUGUST 2013 FINAL EXAMINATION

MAT334H1Y - Complex Variables

Instructor - Jaimal Thind

Duration: 3 hours

Aids: None

NAME (PRINT): ______________________________
Last / Surname   First / Given Name

STUDENT #: ____________________   SIGNATURE: ____________________

INSTRUCTIONS

(1) There are two parts to this examination:
PART I **(30 marks)**: Ten (10) short answer questions. Each question is worth 3 marks.
PART II **(70 marks)**: Seven (7) written questions. Each question is worth 10 marks.

(2) This examination has 11 different pages including this page. Make sure your copy of the examination has 11 different pages and **sign** at the top of this page. You can use pages 3, 11, and its back for rough work.

(3) The last page contains some formulas and information you might find useful.

Good Luck!

Part I	Part II Q#1	Part II Q#2	Part II Q#3	Part II Q#4	Part II Q#5	Part II Q#6	Part II Q#7	TOTAL
/30	/10	/10	/10	/10	/10	/10	/10	/100

PART I (30 marks)

Answer the following true/false questions. Each question is worth **3 marks**. **No explanation is needed.**

1. The function $f(z) = |z|$ is analytic. True False

2. The set $S = \{z \in \mathbb{C} \mid |z-2| = 2\}$ is closed. True False

3. If D is simply connected, then a continuous function $f : D \to \mathbb{C}$ is analytic if and only if $\int_\gamma f(z)dz = 0$ for all simple closed curves γ in D. True False

4. The function $f(z) = \dfrac{(z^4-1)^2(1-z)^3}{(z^3-1)^4}$ has a pole of order 2 at $z_0 = 1$. True False

5. If $f(z)$ is entire, and satisfies $|f(z)| \leq e^{-|z|^2}$, then f must be a constant. True False

6. The function $u(x,y) = e^{-(x^2+y^2)}$ is the real part of an analytic function. True False

Fill in the blanks. Each question is worth **3 marks**. **No explanation is needed.**

7. Let γ be the unit circle, oriented positively. Then $\displaystyle\int_\gamma \frac{e^z}{z^5} =$ __________

8. The power series $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k^k} z^k$ converges for __________.

9. $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{\sin z}{z} =$ __________, where γ is the unit circle, oriented positively.

10. Let $f(z) = \dfrac{z^2}{z^2+1}$. Then $\mathrm{Res}(f : i) =$ __________.

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Rough Work – This page will not be marked

PART II (70 marks)

Answer the following questions in the space provided. Each question is worth **10 marks.**

Provide **complete solutions and justify your answers.**

1. Let $f(z) = \dfrac{2z-2}{z^2-2z}$.

(a) Find a Laurent series for f, centred at $z_0 = 0$, that converges in the punctured disc $0 < |z| < 2$.

(b) Find $\operatorname{Res}(f : 0)$.

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2. Let $f(z) = e^z$.

(a) Let γ_1 be the line segment joining $\frac{-i\pi}{2}$ to $\frac{i\pi}{2}$. Parametrize the curve γ_1, and compute $\int_{\gamma_1} f(z)dz$ using the definition of line integral.

(b) Let γ be any curve joining $\frac{-i\pi}{2}$ to $\frac{i\pi}{2}$. Find $\int_{\gamma} f(z)dz$.

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3. Compute $\displaystyle\int_\gamma \frac{\cos(\pi z)}{z^3 - 1}\,dz$, where γ is the circle $|z-2| = 2$, oriented positively.

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4. How many zeroes does the function $p(z) = \frac{1}{5}z^5 + \frac{1}{3}z^3 + \frac{1}{4}z^2 + z$ have inside the annulus $1 < |z| < 2$? (Zeroes are counted with multiplicities.)

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5. Find a fractional linear transformation that takes $0, 1, 2$ to $1, 2, -1$.

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6. Let $S = \{z \in \mathbb{C} \mid 0 < Arg(z) < \frac{\pi}{2}\}$, and let D be the unit disc. Find a conformal map $f : S \to D$. (Hint: The fractional linear transformation $T(z) = i\dfrac{z+1}{1-z}$ is a conformal map from the unit disc D to the upper half plane H.)

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7. Evaluate the real integral $\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 + 5x^2 + 4} dx$.

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Some formulae:

$$\int_{\partial\Omega} f(z)dz = i \iint_{\Omega} (\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})dxdy \quad \text{(Complex Version of Green's Theorem)}$$

$$\cos(\theta) = \frac{1}{2}(z + \frac{1}{z}) \quad \text{(under the substitution } z = e^{i\theta}\text{)}$$

$$\sin(\theta) = \frac{1}{2i}(z - \frac{1}{z}) \quad \text{(under the substitution } z = e^{i\theta}\text{)}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Some useful estimates:

If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, then we get the following estimates for $|z| = R$ with R very large:

$$\frac{1}{2}|a_n|R^n \leq |p(z)| \leq 2|a_n|R^n$$

For a continuous function f and continuous curve γ, we get:

$$\left| \int_{\gamma} f(z)dz \right| \leq \text{length}(\gamma) \cdot \max_{z\in\gamma}|f(z)|$$

Rough Work – This page will not be marked

The End.