9.12.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x|\theta) = \theta \exp(-\theta x)$

- Derive the LRT for:

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

$$\lambda = \frac{\sup_{\Theta_0} L(\theta|\underset{\sim}{x})}{\sup_{\Theta} L(\theta|\underset{\sim}{x})}$$

$$L(\theta|\underset{\sim}{x}) = \prod_{i=1}^{n} \theta \exp(-\theta x_i)$$

$$= \theta^n \exp(-\theta \Sigma x_i)$$

$$\ell(\theta|\underset{\sim}{x}) = n \log(\theta) - \theta \Sigma x_i$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \Sigma x_i = 0 \Rightarrow \hat{\theta} = 1/\bar{X}$$

$$\therefore \lambda = \frac{\theta_0^n \exp(-\theta_0 \Sigma x_i)}{\hat{\theta}^n \exp(-\hat{\theta} \Sigma x_i)}$$

$$\lambda = \frac{\theta_0^{\,n} \exp(-\theta_0 \Sigma x_i)}{\left(\frac{1}{\bar{x}}\right)^n \exp\left(-\frac{n}{\Sigma x_i} \Sigma x_i\right)}$$

$$= \frac{\theta_0^{\,n} \bar{x}^n \exp(-\theta_0 \Sigma x_i)}{\exp(-n)}$$

$$= \frac{\theta_0^{\,n} \bar{x}^n \exp(-\theta \bar{x} n)}{\exp(-n)}$$

$$R = \{ \lambda < c \}$$

$$= \left\{ \frac{\theta_0^{\,n} \bar{x}^n \exp(-\theta \bar{x} n)}{\exp(-n)} < c \right\}$$

$$= \left\{ \left[\theta_0 \bar{x} \exp(-\theta \bar{x})\right]^n < c^* \right\}$$

$$= \left\{ \bar{x} \exp(-\bar{x}) < c^{**} \right\}$$

9.13)

$H_0: \theta = 1$ $\quad n = 10, \ \alpha = 0.05.$

$H_1: \theta \neq 1$

$$R = \{ \bar{X} \exp(-\bar{X}) < c^{**} \}$$

a.) We want to show that Rejection region can be written in the following form:

$$R = \{ \bar{X} \leq x_0 \} \cup \{ \bar{X} \geq x_1 \}$$

- Let's differentiate the following:

$$g(a) = a \exp(-a)$$

$$g'(a) = \exp(-a)(1-a)$$

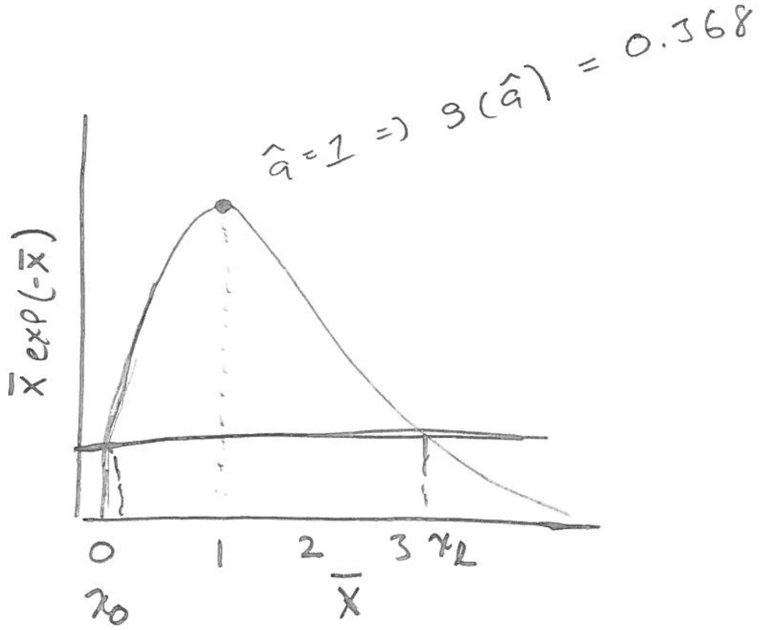
$g'(a)$ is positive for $a \in (0,1)$

$g'(a)$ is negative for $a > 1$

$\therefore$ $g(a)$ is strictly increasing from $(0,1)$ and strictly decreasing from $(1, \infty)$.

$\therefore$ for $g(a)$ we can find two cut-points :

$\bar{X} = a$



$\therefore$ There are exactly two solutions for $g(\bar{x}) = C$.

b.) Based on N-P theory for this LRT

we determine c based on

$$R = \{ \bar{x} \exp(-\bar{x}) \le c \}$$

$$P_{H_0}(R) = \alpha = 0.05$$

$$P_{H_0}(\bar{x} \exp(-\bar{x}) \le c) = 0.05$$

$\rightarrow$ under $H_0: \theta = 1$.

c.) Note $X \sim \exp(\theta)$

$$X \sim \text{gamma}(1, \theta) \overset{\text{under } H_0}{\Rightarrow} X \sim \text{gamma}(1, 1)$$

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha - 1} \exp(-x\beta)$$

$$= \frac{\beta^{1}}{\Gamma(1)} x^{1-1} \exp(-x\beta) = \beta \exp(-x\beta)$$

$$\Rightarrow MGF_x(t) = \left(\frac{1}{1 - t/\beta} \right)^{\alpha} = \left(\frac{\beta}{\beta - t} \right)^{\alpha}$$

↑ worked out in lecture.

$$\therefore \quad Y = \sum_{i=1}^{n} X_i$$

$$MGF_{\Sigma X_i}(t) = \left(\frac{1}{1 - t/\beta}\right)^{\alpha_1} \times \cdots \times \left(\frac{1}{1 - t/\beta}\right)^{\alpha_n}$$

$$= \left(\frac{1}{1 - t/\beta}\right)^{\alpha_1 + \cdots + \alpha}$$

$$\therefore \quad Y \sim \text{gamma}(\alpha_1 + \cdots + \alpha_n, \beta)$$

under Ho

$$\Rightarrow \quad \text{gamma}(n, 1)$$

$$\therefore \quad \bar{X} \sim \text{gamma}(n, n)$$

$$\Rightarrow \quad P_{H_0}\left(\bar{X} \exp(-\bar{x}) \leq c\right) = \alpha$$

$$P_{H_0}\left(\bar{X} \in [0, x_0(c)] \cup [x_1(c), \infty)\right)$$

$$= F(x_0(c)) + 1 - F(x_1(c))$$

Solve computationally:

1.) try a value c

2.) Calculate x_0, x_1

3.) Calculate $F(x_0(c)) + 1 - F(x_1(c))$

and get this close to $\alpha = 0.05$

d.) How can we solve via simulation?

$$R = \{ \bar{x} \exp(-\bar{x}) < c \}$$

1.) $X_1, \ldots, X_{10} \sim \exp(\theta_0 = 1)$

2.) Calculate $\bar{X}$

3.) Calculate $\bar{x} \exp(-\bar{x})$

4.) We reject for small values:

$$P_{H_0}(\bar{X} \exp(-\bar{x}) < c) = 0.05$$

$\therefore$ We use the empirical quantile for 0.05.

$\rightarrow$ See the R portion for the solution.

9.24.) $X \sim \text{binomial}(n, p)$

a.) $H_0: p = 0.5$

$H_1: p \neq 0.5$

$$\lambda = \frac{\sup_{p_0} L(p \mid \underset{\sim}{x})}{\sup_{p} L(p \mid \underset{\sim}{x})} = \frac{\binom{n}{x} p_0^{x} (1-p_0)^{n-x}}{\binom{n}{x} \hat{p}^{x} (1-\hat{p})^{n-x}}$$

- We can work out $\hat{p} = \bar{x}$

$$\therefore \lambda = \frac{(1/2)^{x} (1-1/2)^{n-x}}{\left(\frac{x}{n}\right)^{x} (1-x/n)^{n-x}} = \frac{(1/2)^{n}}{(x/n)^{x} (1-x/n)^{n-x}}$$

$$\frac{n^n}{n^n} = \frac{n^n}{n^x \, n^{n-x}}$$

$$\Rightarrow \lambda = \frac{n^n (1/2)^n}{n^x (x/n)^x \, n^{n-x} (1-x/n)^{n-x}} = \frac{(n/2)^n}{x^x (n-x)^{n-x}}$$

$$\therefore R = \left\{ \frac{(n/2)^n}{x^x (n-x)^{n-x}} < c \right\}$$

b.) We want to show that the LRT

rejects for

$$R = \{ |x - n/2| > k \}$$

- We have $\lambda = \dfrac{(n/2)^n}{x^x (n-x)^{n-x}}$

Let $h(x) = \dfrac{1}{x^x (n-x)^{n-x}}$

Also notice $h(n-x) = \dfrac{1}{(n-x)^{n-x} (n-(n-x))^{n-(n-x)}}$

$$= \frac{1}{(n-x)^{n-x} x^x}$$

$\therefore \quad h(x) = h(n-x)$

- So consider $y = x - n/2 \Rightarrow x = n/2 + y$

$$\Rightarrow \quad h(n/2 + y) = h(n/2 - y)$$

- Consider $h(n/2 + y)$ for $y \geq 0$.

- $h(n/2+y) = \left(\frac{n}{2}-y\right)^{-(n/2-y)} \left(\frac{n}{2}+y\right)^{-(\frac{n}{2}+y)}$

- Let $h^*(y) = \log(h(n/2+y))$

  - We want to show that $h^*(y)$ is a non-decreasing function of y

$\Rightarrow$ $y\uparrow \Rightarrow x-n/2 \uparrow \Rightarrow h^*(y)\downarrow \Rightarrow \lambda \Downarrow$

$\downarrow$

due to symmetry

$X+n/2 \uparrow \Rightarrow \lambda\Downarrow$

$$h^*(y) = -(n/2-y)\log(n/2-y) - (n/2+y)\log(n/2+y)$$

$$\frac{d\,h^*(y)}{dy} = \log(n/2-y) - \log(n/2+y) \leq 0$$

as $y \geq 0$

$\therefore$ $R = \{\,|x-n/2| > K\,\}$ is an equivalent rejection region for some K.

c.) $R = \{ |X - n/2| > K \}$

$$P_{H_0}\left(|X - n/2| > K \right) = P_{H_0}\left(X - n/2 > K \right)$$

$$+ P\left(X - n/2 < -K \right) = \alpha$$

$$= P_{H_0}\left(X > K + n/2 \right) + P_{H_0}\left(X < -K + n/2 \right)$$

$$= 1 - P_{H_0}\left(X \leq K + n/2 \right) + P_{H_0}\left(X < -K + n/2 \right) = \alpha$$

d.) If $n = 10$, $K = 2$ what is α:

$$\alpha = 1 - P_{H_0}\left(X \leq 2 + 10/2 \right) + P_{H_0}\left(X < 10/2 - 2 \right)$$

$$= 1 - P_{H_0}\left(X \leq 7 \right) + P_{H_0}\left(X < 3 \right)$$

$$= 1 - \text{Pbinom}(10, 7, 1/2) + \text{Pbinom}(10, 2, 1/2)$$

$$\approx 0.11.$$

$$\text{e.)} \quad X \underset{}{\overset{\text{under } H_0}{\sim}} \text{Binomial}(n=100,\ p=1/2)$$

$$\alpha = 1 - P_{H_0}(X \le k + n/2) + P_{H_0}(X < -k + n/2)$$

$$= 1 - P_{H_0}(X \le 60) + P_{H_0}(X < 40)$$

Using CLT: $\quad E(x) = 100(1/2) = 50;\ V(x) = 100(1/2)^2$

$= 25$

$$= 1 - P_{H_0}\left(\frac{X-50}{5} \le \frac{60-50}{5}\right) + P_{H_0}\left(\frac{X-50}{5} \le \frac{40-50}{5}\right)$$

$$= 1 - P_{H_0}(Z \le 2) + P_{H_0}(Z \le -2)$$

$$= 1 - \text{Pnorm}(2) + \text{Pnorm}(-2)$$

$$= 0.0455$$

32.) If $A \Rightarrow x \sim N(\mu = 100, \sigma = 25)$

$B \Rightarrow x \sim N(\mu = 125, \sigma = 25)$

A datapoint is drawn: $x = 120$.

a.) $H_0 : \mu = 100$ | Simple vs Simple

$H_1 : \mu = 125$

$$\lambda = \frac{\exp\left(-\frac{1}{2(25^2)}(x-100)^2\right)}{\exp\left(-\frac{1}{2(25^2)}(x-125)^2\right)}$$

$$= \exp\left(-\frac{1}{2(25^2)}\left[(x-100)^2 - (x-125)^2\right]\right)$$

- when $x = 120 \Rightarrow \lambda = 0.7408$

b.) $$\frac{P(A|x)}{P(B|x)} = \frac{P(x|A)\,P(A)}{P(x|B)\;P(B)} = \frac{P(x|A)}{P(x|B)} = 0.7408$$

$$= \frac{1 - P(B|x)}{P(B|x)} = 0.7408$$

$\therefore \; P(B|x) = 0.574$

c.) $R = \{ X > 125 \}$

$$P_{H_0}(R) = P_{H_0}\left(\frac{X-100}{25} > \frac{125-100}{25}\right)$$

$$= P_{H_0}(Z > 1) = 1 - P_{H_0}(Z \leq 1)$$

$$= 0.1587$$

d.) Power $= P_{H_1}(R) = P_{H_1}\left(Z > \frac{125-125}{25}\right)$

$$= P_{H}(Z > 0) = 1 - P_{H_0}(Z \leq 0)$$

$$= 1/2$$

e.) Let's calculate the p-value:

$$P_{H_0}(X > 120) = P_{H_0}\left(Z > \frac{120-100}{25}\right)$$

$$= P_{H_0}(Z > 0.8) = 1 - P_{H_0}(Z \leq 0.8)$$

$$= 1 - \text{Pnorm}(0.8)$$

$$\approx 0.2119.$$