APM462H1S, Winter 2014 , Assignment 2,

due: Monday February 24, at the beginning of the lecture.

Exercise 1. Assume that Q is a symmetric $n \times n$ matrix, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and with an *orthonormal basis* of eigenvectors $w_1, \ldots, w_n$.

Since $w_1, \ldots, w_n$ is a basis, any vector $v \in E^n$ can be written in the form

$$v = a_1 w_1 + \cdots + a_n w_n. \tag{1}$$

(In fact, $a_i = w_i^T v$ for every i — this follows by multiplying equation (1) by w_i^T on the left and using the fact that the vectors $w_1, \ldots, w_n$ are orthonormal.)

a. Show that if $v = a_1 w_1 + \cdots + a_n w_n$ and at least one a_i is nonzero, then

$$\frac{v^T Q v}{v^T v} = \theta_1 \lambda_1 + \ldots + \theta_n \lambda_n, \qquad \text{where } \theta_i = \frac{a_i^2}{a_1^2 + \cdots + a_n^2}.$$

b. Using part **a** (if you like), prove that

$$\lambda_n = \text{largest eigenvalue of } Q = \max_{v \neq 0} \frac{v^T Q v}{v^T v}. \tag{2}$$

hints: it may be convenient to break this into two parts: first, that $\frac{v^T Q v}{v^T v} \leq \lambda_n$ for every nonzero vector v, and second, that there is some choice of a nonzero vector v such that $\frac{v^T Q v}{v^T v} = \lambda_n$.

remark. By almost the same argument, one can also show that

$$\lambda_1 = \text{smallest eigenvalue of } Q = \min_{v \neq 0} \frac{v^T Q v}{v^T v}. \tag{3}$$

Exercise 2. Assume that Q is a symmetric $n \times n$ matrix, $c \in E^n$ is a nonzero (column) vector, and μ is a positive number.

Consider the symmetric matrix $R = Q + \mu c c^T$.

Let $\lambda_i(Q)$ denote the ith eigenvalue of Q, and similarly and $\lambda_i(R)$ the ith eigenvalue of R, where they are arranged so that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, for both Q and R.

a. Using formula (2) from exercise 1 above, prove that

$$\lambda_n(R) \geq \mu |c|^2 + \lambda_1(Q).$$

b. Using formula (3) above, prove that if $n \geq 2$, then

$$\lambda_1(R) \leq \lambda_n(Q).$$

c. Conclude that if Q is positive semidefinite, then the condition number of R satisfies

$$\text{condition number of } R = \frac{\lambda_n(R)}{\lambda_1(R)} \geq \frac{\mu |c|^2}{\lambda_n(Q)}.$$

Thus, the condition number is very large if μ is large compared to $\lambda_n(Q)$.

Exercise 3. Luenberger and Ye, problem 21 on page 260. In the definition of f in the book, s^2 should be replaced by x^2.

Exercise 4. Luenberger and Ye, problem 24 on page 260, parts **a - c**.

Extra marks will be awarded for a correct solution of part **d**, which is harder.