CSC336 Tutorial 5 – Norms and condition numbers of matrices

QUESTION 1 *Prove that* $\max_{x\neq 0}\left\{\frac{\|Ax\|}{\|x\|}\right\} = \max\limits_{\|x\|=1}\{\|Ax\|\}$

PROOF:

1. Let $S_1 = \{x : \|x\| = 1\}$, $S_2 = \{x : x \neq 0\}$. Clearly, $S_1 \subset S_2$. Then, we have $\max\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|} \geq \max\limits_{\|x\|=1}\frac{\|Ax\|}{\|x\|} = \max\limits_{\|x\|=1}\|Ax\|$.

2. Note that $\max\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \max\limits_{x\neq 0}\left\|\frac{Ax}{\|x\|}\right\| = \max\limits_{x\neq 0}\left\|A\left(\frac{x}{\|x\|}\right)\right\|$
   because $\|x\|$ is just a positive scalar and can be pushed into the norm (or pulled out of it).

   Let $S_3 = \left\{x' : x' = \frac{x}{\|x\|}, x \neq 0\right\}$. We have $x' \in S_3 \Rightarrow x' = \frac{x}{\|x\|}, x \neq 0 \Rightarrow \|x'\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|}, x \neq 0 \Rightarrow \|x'\| = 1 \Rightarrow x' \in S_1 \Rightarrow S_3 \subseteq S_1$

   So $\max\limits_{x\neq 0}\left\|A\left(\frac{x}{\|x\|}\right)\right\| \leq \max\limits_{\|x'\|=1}\|Ax'\| = \max\limits_{\|x\|=1}\|Ax\|$, where the (first) inequality is due to the fact that $S_3 \subseteq S_1$, and the (second) equality is just a variable renaming. To summarize, $\max\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \max\limits_{x\neq 0}\left\|A\left(\frac{x}{\|x\|}\right)\right\| \leq \max\limits_{\|x\|=1}\|Ax\|$.

Combining the results of 1 and 2 gives us the desired result.

Note: We can avoid showing 1 above, if we show equality in 2.
In 2, we have already shown that $S_3 \subseteq S_1$. Below, we show that $S_1 \subseteq S_3$, thus $S_1 = S_3$.

1. $x \in S_1 \Rightarrow \|x\| = 1 \Rightarrow x = \frac{x}{\|x\|}, x \neq 0 \Rightarrow x \in S_3 \Rightarrow S_1 \subseteq S_3$

   So, $\max\limits_{\|x\|=1}\|Ax\| \leq \max\limits_{x\neq 0}\left\|A\left(\frac{x}{\|x\|}\right)\right\|$.

Combining the results of this new 1 with the results of the previous 2, we have $S_1 = S_3$, and $\max\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \max\limits_{x\neq 0}\left\|A\left(\frac{x}{\|x\|}\right)\right\| = \max\limits_{\|x\|=1}\|Ax\|$

Note: By definition, the induced matrix norm is $||A|| \equiv \max_{x\neq 0}\left\{\frac{\|Ax\|}{\|x\|}\right\}$. So we essentially proved that there is an equivalent definition of the induced matrix norm, namely, $||A|| \equiv \max_{\|x\|=1}\{\|Ax\|\}$.

QUESTION 2 *Let* $A = \begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix}$ *for some* $\delta \sim 0$ *(but* $\delta \neq 0$*) and let* $D = \begin{bmatrix} 1/\delta & 0 \\ 0 & 1 \end{bmatrix}$. *Find* $\kappa_\infty(A)$ *and* $\kappa_\infty(DA)$.

ANSWER: We have $A^{-1} = \begin{bmatrix} 1/\delta & 0 \\ 0 & 1 \end{bmatrix}$, $DA = \begin{bmatrix} 1/\delta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix} = I$.

Also,
$\kappa_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty = 1 \cdot \frac{1}{\delta} = \frac{1}{\delta}$ (large),
$\kappa_\infty(DA) = \|DA\|_\infty\|(DA)^{-1}\|_\infty = \|I\|_\infty\|I\|_\infty = 1$ (as small as can be).

Thus, scaling may affect the condition number, and sometimes may improve (decrease) it substantially. This matrix A has a large condition number due to bad scaling.

QUESTION 3 *Let* $A = \begin{bmatrix} \delta & \delta \\ 1-\delta & 1 \end{bmatrix}$ *for some* $\delta \sim 0$ *(but* $\delta > 0$*) and let* $D = \begin{bmatrix} 1/\delta & 0 \\ 0 & 1 \end{bmatrix}$. *Find* $\kappa_\infty(A)$ *and* $\kappa_\infty(DA)$.

ANSWER: We have
$A^{-1} = \frac{1}{\delta^2}\begin{bmatrix} 1 & -\delta \\ \delta-1 & \delta \end{bmatrix}$, $DA = \begin{bmatrix} 1/\delta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta & \delta \\ 1-\delta & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1-\delta & 1 \end{bmatrix}$, and $(DA)^{-1} =$

$\frac{1}{\delta}\begin{bmatrix} 1 & -1 \\ \delta-1 & 1 \end{bmatrix}$.

Also,
$\kappa_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty = (1-\delta+1)\cdot\frac{1}{\delta^2}(1+\delta) = \frac{(2-\delta)(1+\delta)}{\delta^2} \approx \frac{2}{\delta^2}$ (huge),
$\kappa_\infty(DA) = \|DA\|_\infty\|(DA)^{-1}\|_\infty = (1+1)\frac{1+1}{\delta} = \frac{4}{\delta}$ (large).

In this case, A still has a large condition number, even after scaling by D. This matrix A has a large condition number because it is nearly singular.

QUESTION 4 *Let* $A = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}$ *for some* $\delta \sim 0$ *(but* $\delta \neq 0$*). Find* $\kappa_\infty(A)$ *and* $\det(A)$.

ANSWER: We have
$\kappa_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty = \delta \cdot \frac{1}{\delta} = 1$ (small),
$\det(A) = \delta^2$ (small)

Note: In general, there is no relation between $\kappa(A)$ and $\det(A)$. A singular matrix A has $\kappa(A) = \infty$ and $\det(A) = 0$. However, a matrix A with small determinant is not necessarily nearly singular, and a matrix with large condition number is not necessarily nearly singular. If a matrix has large condition number after scaling, then it is nearly singular.

QUESTION 5 *Consider* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *for some* θ*. Find the condition number of* A *in the Euclidian norm.*

ANSWER: It is easy to see that $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Note that $Ax = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$ and

$$\begin{aligned} \|Ax\|_2 &= \left[(x_1\cos\theta - x_2\sin\theta)^2 + (x_1\sin\theta + x_2\cos\theta)^2\right]^{1/2} \\ &= [x_1^2\cos^2\theta + x_2^2\sin^2\theta - 2x_1x_2\cos\theta\sin\theta \\ &\qquad + x_1^2\sin^2\theta + x_2^2\cos^2\theta + 2x_1x_2\cos\theta\sin\theta]^{1/2} \\ &= [x_1^2 + x_2^2]^{1/2} = \|x\|_2 \end{aligned}$$

Thus $\|A\|_2 = \max_{x\neq 0} \dfrac{\|Ax\|_2}{\|x\|_2} = \max_{x\neq 0} 1 = 1$.
Similarly, $\|A^{-1}\|_2 = 1$.
Then $\kappa_2(A) = 1$.

A geometric way of viewing this:
First, note that the condition number of a matrix denotes the ratio of the maximal stretching over the minimal stretching (or maximal shrinking) that the matrix gives rise to, when applied to any non-zero vector: $\kappa_a(A) = ||A||_a||A^{-1}||_a =$

$$= \max_{x\neq 0} \frac{||Ax||_a}{||x||_a} \max_{x\neq 0} \frac{||A^{-1}x||_a}{||x||_a} = \max_{x\neq 0} \frac{||Ax||_a}{||x||_a} \max_{y\neq 0} \frac{||y||_a}{||Ay||_a} = \frac{\max_{x\neq 0} \frac{||Ax||_a}{||x||_a}}{\min_{y\neq 0} \frac{||Ay||_a}{||y||_a}}$$

Then, notice that A represents a counter-clockwise rotation of x by θ radians. We can see that the Euclidian norm (length) of x does not change if A is applied to it. Since A produces neither stretching nor shrinking when applied to any vector, it has condition number 1 with respect to Euclidian norm.

Another way of viewing this:
It is easy to see that A is orthogonal, i.e., $A^TA = \mathbf{I}$. Then $||A||_2 = \max_{x\neq 0} \dfrac{||Ax||_2}{||x||_2} =$ $\max_{x\neq 0} \dfrac{\sqrt{(Ax)^TAx}}{\sqrt{x^Tx}} = \max_{x\neq 0} \dfrac{\sqrt{x^TA^TAx}}{\sqrt{x^Tx}} = \max_{x\neq 0} \dfrac{\sqrt{x^Tx}}{\sqrt{x^Tx}} = 1$. Also, since A is square (and orthogonal), its inverse is its transpose, i.e., $A^{-1} = A^T$. So we have $AA^T = \mathbf{I}$, i.e., A^T is also orthogonal (and square). Thus $||A^T||_2 = 1$, and thus $\kappa_2(A) = 1$.

Note: For the above matrix A, it is easy to find $\kappa_2(A)$. For arbitrary matrices, it is not straightforward. Even if we have the inverse explicitly, it is not always easy to calculate $\|A\|_2$ and $\|A^{-1}\|_2$.

QUESTION 6 *With the same* A *as before, find the condition number of* A *in the one-norm (possibly in terms of* θ*).*

ANSWER: We have
$\|A\|_1 = \max\{|\cos\theta| + |\sin\theta|, |-\sin\theta| + |\cos\theta|\} = |\cos\theta| + |\sin\theta|$,
$\|A^{-1}\|_1 = \max\{|\cos\theta| + |-\sin\theta|, |\sin\theta| + |\cos\theta|\} = |\cos\theta| + |\sin\theta|$.
Then

$$\begin{aligned} \kappa_1(A) &= \|A\|_1\|A^{-1}\|_1 = (|\cos\theta| + |\sin\theta|)^2 \\ &= \cos^2\theta + \sin^2\theta + 2|\sin\theta||\cos\theta| = 1 + 2|\sin\theta\cos\theta| \end{aligned}$$

Note: For this A, we have $\|A\|_2 \leq \|A\|_1$, and $\kappa_2(A) \leq \kappa_1(A)$, but for a general matrix, we cannot tell which norm or condition number is larger.

QUESTION 7 *Let* $A = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$. *Find the condition number of A in the infinity norm.*

ANSWER: It is easy to see that $A^{-1} = \begin{bmatrix} 1/8 & 0 \\ 0 & 1/2 \end{bmatrix}$.
$\|A\|_\infty = \max\{8, 2\} = 8$, $\|A^{-1}\|_\infty = \max\{1/8, 1/2\} = 1/2$.
$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 8 \cdot \frac{1}{2} = 4$.

Note: The condition number of a matrix denotes the ratio of the maximal stretching over the minimal stretching (or maximal shrinking) that the matrix gives rise to, when applied to any non-zero vector. We can use this interpretation of the condition number to find the condition number of A.

First, as an example for this A, take $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for which we have $\|x\|_\infty = 1$.

Then $Ax = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$. Thus $\|Ax\|_\infty = 8$, which means that A stretches x by a factor or 8.

Now take $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for which we also have $\|x\|_\infty = 1$.

Then $Ax = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Thus $\|Ax\|_\infty = 2$, which means that A stretches x by a factor or 2.

We can show that, for this A, 8 and 2 are the maximal and minimal stretching A produces, when applied to any $x \neq 0$:
Consider $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $Ax = \begin{bmatrix} 8x_1 \\ 2x_2 \end{bmatrix}$, $\|Ax\|_\infty = \max\{8|x_1|, 2|x_2|\}$ and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.

Case $\frac{1}{4}|x_2| \le |x_1| < |x_2|$
$\|x\|_\infty = |x_2|$, $\|Ax\|_\infty = 8|x_1|$. Then $\frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{8|x_1|}{|x_2|}$ is ≥ 2 and < 8.

Case $|x_2| \le |x_1|$
$\|x\|_\infty = |x_1|$, $\|Ax\|_\infty = 8|x_1|$. Then $\frac{\|Ax\|_\infty}{\|x\|_\infty} = 8$.

Case $|x_1| \le \frac{1}{4}|x_2|$
$\|x\|_\infty = |x_2|$, $\|Ax\|_\infty = 2|x_2|$. Then $\frac{\|Ax\|_\infty}{\|x\|_\infty} = 2$.

Thus, for any $x \neq 0$, we have $2 \le \frac{\|Ax\|_\infty}{\|x\|_\infty} \le 8$, and there exist vectors for which the maximal and minimal stretchings 8 and 2 are obtained (see above $x = (1, 0)^T$ and $x = (0, 1)^T$), so $\kappa_\infty(A) = \frac{8}{2} = 4$.

QUESTION 8 *With the same A as above, find the condition number of A^{-1} in the infinity norm.*
ANSWER: Note that $\kappa(A) = \|A\| \|A^{-1}\| = \kappa(A^{-1})$.
So $\kappa(A^{-1}) = 4$, and we don't need to do any further calculations.
Notice also that $A^{-1} = \begin{bmatrix} 1/8 & 0 \\ 0 & 1/2 \end{bmatrix}$ gives rise to maximal stretching (minimal shrinking) of $1/2$ and minimal stretching (maximal shrinking) of $1/8$.

General note:
For diagonal (and some other special) matrices, it is easy to calculate $\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ and $\min_{x \neq 0} \frac{\|Ax\|}{\|x\|}$, so it is easy to calculate the condition number through the ratio.
For arbitrary matrices, it is not straightforward (and it may not be possible).

QUESTION 9 *Let* $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$. *Find the condition number of* A *in the infinity and one norms.*

ANSWER: It is easy to find that $A^{-1} = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$. Thus

$\|A\|_\infty = \max\{|1| + |-2|, |3| + |-4|\} = \max\{3, 7\} = 7$,

$\|A^{-1}\|_\infty = \frac{1}{2}\max\{|-4| + |2|, |-3| + |1|\} = \frac{1}{2}\max\{6, 4\} = 3$, and

$\kappa_\infty(A) = 7 \cdot 3 = 21$.

Also,

$\|A\|_1 = \max\{|1| + |3|, |-2| + |-4|\} = \max\{4, 6\} = 6$,

$\|A^{-1}\|_1 = \frac{1}{2}\max\{|-4| + |-3|, |2| + |1|\} = \frac{1}{2}\max\{7, 3\} = 3.5$, and

$\kappa_1(A) = 6 \cdot 3.5 = 21$.

Note: In this question, $\kappa_\infty(A)$ and $\kappa_1(A)$ turn out to be the same value, but this is not necessarily the general case.

