Department of Mathematics	Problem Set #2
University of Toronto	**Deadline: Tuesday October 18, 3:00 p.m.**
MAT332F, 2011	Assignment Posted/Revised: October 3, 2011

Read the following instructions carefully! In contains guidelines on handing in assignments for this course.

For solutions to all problem sets, please remember that when you are asked to find or calculate something you must justify that what you have found is correct and complete. You may use results from your lecture to help in your justification. You are learning to present your results in a clear and convincing manner. Thus, you will be graded on your presentation and justification; we are not simply verifying whether you know the answer.

Present your solutions with complete sentences. Pretend the marker does **not** know how to solve the question.

Required Information. The front page must include your name and student number. *Failure to put your name and/or your student number on your problem set will result in a zero on your assignment.* A cover page is not required as long as the necessary information is on the top of the first page.

Submitting your assignment. You must hand your assignment to your instructor before the beginning of lecture, or deposit the instructor's personal mailbox on the 6th floor of the Bahen Centre.

If you are unable to complete homework or if you miss a term test due to illness or other circumstances outside of your control, please contact your instructor immediately in order to receive special consideration. Note that special consideration will be given on an individual basis and will not be given automatically. In other words, you risk getting a mark of zero for missed work unless you contact the instructor promptly.

In the case of illness, medical documentation must be supplied on the standard University of Toronto Student Medical Certificate. You can also obtain a paper copy of this certificate from your college registrar or in your registration handbook. (A simple "note" from your doctor is not acceptable.)

Late submission. Late assignments will be accepted up to 25 hours after their deadlines with the following penalties.

Submission time	Penalty
by 3pm on Tuesday	none
by 10am on Wednesday	-10%
by 4pm on Wednesday	-25%

Note that lateness penalties will be computed as a percentage of the total marks on the assignment, not of the mark you obtain. Late assignments must be submitted directly into the instructor's personal mailbox on the 6th floor of Bahen Center (in the Math Department office), unless you require special consideration (see the section above for details). Please write the *exact* submission time on your assignment if you are submitting late.

Policy on Plagiarism on Assignments. Plagiarism is a form of academic fraud and is treated very seriously by the Faculty. *The assignments you hand in must not contain anyone else's work or ideas without proper attribution.* A working definition of plagiarism suitable for this course may be found at `http://www.northwestern.edu/provost/students/integrity/plagiarism.html`.

In science, collaboration is the norm, and in this course student collaboration is permitted to an extent. Namely, you <u>are</u> permitted to abstractly discuss possible solutions to a problem with other students. However, a student is forbidden from guiding another student through a solution step by step.

You are permitted to submit a joint answer to a problem set question. If two students have contributed to the solution of the problem, please write both names and student numbers near the problem, and you may share the marks.

Core Problems.

(1) Two digraphs $D_{1,2} = (V_{1,2}, E_{1,2}, \psi_{1,2})$ are said to be *isomorphic* if there exist bijections $\theta\colon V_1 \to V_2$ and $\phi\colon E_1 \to E_2$ satisfying $\psi_1(e) = (u, v)$ if and only if $\psi_2(\phi(e)) = (\theta(u), \theta(v))$.

(a) List all tournaments with four vertices, up to isomorphism. Prove the list is complete. (*5pt*)

(b) List all tournaments with five vertices, up to isomorphism. Prove the list is complete. (*10pt*)

(2)

(a) Prove that a k-regular graph with girth 5 must have at least $k^2 + 1$ vertices. (*10pt*)

(b) Find a cubic graph with girth 6 which has the smallest possible number n of vertices. Prove that n is indeed minimal. (*8pt*)

(3) Prove or disprove: Every connected graph contains a walk which transverses each of its edges precisely once in each direction. (*17pt*)

(4) The *Wiener index* of a graph G with n vertices is $W(G) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$ where d_{ij} denotes the distance (number of edges) from vertex i to vertex j. Wiener indexes are important in chemistry, where they are used to predict physical parameters such as boiling points and molar volumes.

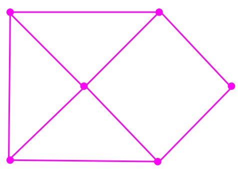
If $T = (E, V)$ is a non-trivial tree, deleting an edge $e \in E$ gives a graph with two connected components $T - \{e\} = T_1^e \cup T_2^e$. Let $n_1(e)$ and $n_2(e)$ denote the numbers of vertices in T_1^e and in T_2^e respectively. Prove that $W(T) = \sum_{e \in E} n_1(e) \cdot n_2(e)$. (*20pt*)

(5) Let T be a non-trivial tree, and let n_i denote the number of vertices in T of degree i. Prove that T has $2 + \sum_{i=3}^{\infty}(i-2)n_i$ leaves. (*15pt*)

(6) A graph may have many spanning trees. Find and prove necessary and sufficient conditions (on the graph itself) for a graph G such that if T is a spanning tree of G, then T is unique. (*15pt*)

Bonus Problems.

(1) How many spanning trees are in the graph below? Prove your answer. (*9 bonus points*)



(2) In how many ways can 30 points on a circle be joined so that the corresponding chords to not intersect? (one chord per pair of points, see illustration below for the case of 6 points). To put it another way, in how many ways can you fold a piece of paper 15 times so as never to fold over another fold-line? Prove your answer. (*6 bonus points*)