Assignment 8 SOLUTIONS- MAT 327 - Summer 2014

Comprehension

[**C.1**] On Assignment 5, A.4 you proved that in ω_1 the intersection of a finite number of closed unbounded sets was again closed unbounded, and in particular, nonempty. Does this prove that ω_1 is compact?

Solution to C.1. The theorem is that a space is compact if and only if every collection of closed sets with the finite intersection property has a point in the intersection of the collection. You proved the weaker fact that every (finite) collection of closed *and unbounded* sets has a point in its intersection. This is not enough to show compactness, and indeed, ω_1 is not a compact space so it's a good thing that we didn't prove that it is compact. $\square$

[**C.2**] Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Prove that $(X, \mathcal{T})$ is compact if and only if every cover of the space by *basic* open sets has a finite subcover.

Solution to C.2. Certainly the $\Rightarrow$ direction is obvious, so let us prove the $\Leftarrow$ direction. Suppose that every open cover of X by basic open sets has a finite subcover. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$ and $U \in \mathcal{U}$ that contains x, choose a basic open set $B_{x,U} \subseteq U$ that contains x. Notice that $\{B_{x,U} : x \in U, U \in \mathcal{U}\}$ is an open cover of X consisting of basic open sets. Let F be a finite subcover of this cover. Notice that $\{U : B_{x,U} \in F\}$ is a finite subcollection of $\mathcal{U}$ that covers X, since the corresponding $B_{x,U}$ cover X and $B_{x,U} \subseteq U$. $\square$

[**C.3**] Here's a really cute and useful fact: Let $(X, \mathcal{T})$ be a compact space, let $(Y, \mathcal{U})$ be a Hausdorff space and let $f : X \longrightarrow Y$ be a continuous function. Prove that f is a closed map. Conclude that, if additionally f is a bijection, then f is a homeomorphism.

Solution to C.3. Let us use C.5! Let $C \subseteq X$ be a closed set. Since X is compact, we know that C is also compact. Since f is continuous, $f[C]$ is compact. Since Y is Hausdorff, we know that $f[C]$ must be closed, as desired.

If we assume that f is a bijection and continuous then it is also closed (as we just showed), so f is a homeomorphism by Assignment 4, C.5. $\square$

[**C.4**] Consider $\mathbb{R}^n$ with the usual metric d, and define

$$\rho(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

for $A, B \subseteq \mathbb{R}^n$. Assume that $C \subseteq \mathbb{R}^n$ is closed, and $K \subseteq \mathbb{R}^n$ is compact. Show that they are disjoint if and only if $\rho(C, K) > 0$. Find an example where this fails if both sets are closed, but not compact.

Solution to C.4. The $\Leftarrow$ direction is fairly obvious. The $\Rightarrow$ direction requires an argument. Use the fact that the distance function $d(C, x)$ is a continuous, real-valued function defined on K, so it achieves its minimum value m at some value $k \in K$. If $m > 0$ then we are finished, so assume for the sake of contradiction that $m = 0$. This says that $d(C, k) = 0$. This means that there are points $c_n \in C$ such that $d(c_n, k) < \frac{1}{n}$. In particular, $\langle c_n \rangle$ converges to k, and since C is closed, it must contain k. This contradicts the fact that C and K are disjoint.

This can fail in $\mathbb{R}$ if $A = \{n : n \in \mathbb{N}\}$ and $B = \{n + \frac{1}{n+1} : n \in \mathbb{N}\}$ which are both closed.

Note that there is a tempting, but *false* argument for the $\Rightarrow$ direction:

Let $C, K \subseteq \mathbb{R}^n$ be disjoint with K compact and C closed. For each point $x \in K$, since $K \subseteq \mathbb{R}^n \setminus C$ is open, find an $\epsilon(x) > 0$ such that $B_{\epsilon(x)}(x) \subseteq \mathbb{R}^n \setminus C$. We observe that $\{B_{\epsilon(x)}(x) : x \in K\}$ is an open cover of K, so let $F \subseteq K$ be a finite set such that $\{B_{\epsilon(x)}(x) : x \in F\}$ is a finite subcover. We observe that $\rho(C, K) \geq \min_{x \in F}\{\epsilon(x)\} > 0$ which exists since F is finite. (*You should check that the previous line does not have to be true!*)

$\square$

[**C.5**] Prove that the continuous image of a compact set is compact.

Solution to C.5. Let $f : X \longrightarrow Y$ be a continuous surjection with X compact. Let $\mathcal{V}$ be an open cover of Y. Since f is a surjection we have that $\mathcal{U} := \{f^{-1}(V) : V \in \mathcal{V}\}$ is a cover of X, and since f is continuous, it is an open cover. Since X is compact, let $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$ be a finite subcover of $\mathcal{U}$. Since f is a surjection we have that $f(f^{-1}(V)) = V$, thus $\{V_1, \ldots, V_n\}$ is a finite open subcover of $\mathcal{V}$. $\square$

Application

[**A.1**] Let $(X, \leq)$ be a linear order, and let $(X, \mathcal{T})$ be its order topology. Prove that X is compact if and only if every non-empty set in X has a least upper bound (supremum) and a greatest lower bound (infimum).

Solution to A.1. This is very similar to the creeping along proof of the Heine-Borel Theorem in the notes (§15). A careful reading of the proof of that theorem together with the proof technique used to prove that ω_1 is compact (in Assignment 5, A.2 or section 2 of §15 in the notes) should yield a full solution without much trouble. $\square$

[**A.2**] Let $2 := \{0, 1\}$ be given the discrete topology, and let $\mathbb{N}$ be given the discrete topology, prove that $2^{\mathbb{N}}$, with the product topology is a compact, Hausdorff, metrizable space. You may wish to observe that $2^{\mathbb{N}}$ is a metrizable space. Do not use Tychonoff's theorem.

Solution to A.2. We already know that $2^{\mathbb{N}}$ is Hausdorff and metrizable, since these are countably productive properties. All that we need to show is that this is a compact space. (Of course, if on assignment 7 you showed that this space is homeomorphic to the Cantor set there is nothing to show, since the Cantor set is compact!)

To show that $2^{\mathbb{N}}$ is compact let us show that every infinite set has an accumulation point, which is equivalent to compactness since $2^{\mathbb{N}}$ is a metric space. Let $\mathcal{F}$ be an infinite subset of $2^{\mathbb{N}}$. We note that in the first coordinate their is a value $g(1)$ (which is 0 or 1) such that infinitely many functions in $\mathcal{F}$ take that value. Let $\mathcal{F}_1 := \{f \in \mathcal{F} : f(1) = g(1)\}$. Recursively choose the $\mathcal{F}_n$ and $g(n)$ so that $\mathcal{F}_n = \{f \in \mathcal{F}_{n-1} : f(n) = g(n)\}$, and each $\mathcal{F}_n$ is infinite.

Now it is easy to check that $g : \mathbb{N} \longrightarrow 2$ is an accumulation point of $\mathcal{F}$. $\square$

[**A.3**] Let $(X, \mathcal{T})$ is a compact subpace of $\mathbb{R}^n$, and let $f : X \longrightarrow \mathbb{R}$ be continuous. Prove that f is uniformly continuous.

Solution to A.3. Let $\epsilon > 0$. For each $x \in X$ there is a $\delta(x) > 0$ such that if $y \in B_{\delta(x)}(x)$ then $d(f(x), f(y)) < \epsilon$. Notice that $\{B_{\delta(x)}(x) : x \in X\}$ is an open cover of X, so let $F \subseteq X$ be a finite set such that $\{B_{\delta(x)}(x) : x \in F\}$

is a finite cover of X. Since F is finite $\delta := \min_{x \in F}\{\delta(x)\} > 0$ exists, and it is easy to check that if $d(x,y) < \frac{\delta}{2}$ then $d(f(x), f(y)) < \epsilon$. $\square$