Mat 337 Assignment 1 solutions

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Problems from Davidson and Donsig, *Real Analysis*. I have tried to write these solutions so that you could actually read them. They are not sketches, they are complete solutions, so although they look long they should not take hours to understand. I therefore recommend that you read them to see how someone with a lot of experience with analysis writes a solution to a problem. These solutions are also written with more detail than you need to get full marks on your solutions.

Finally, for this problem set the goal was to get practice writing precise statements with sequences and limits and being able to correctly use quantifiers ($\forall$ and $\exists$) in the right order. If you want to say that the sequence x_n converges to a, the following is wrong:

$$\forall \epsilon > 0 \exists n \geq N \, |x_n - a| < \epsilon.$$

The following is correct:

$$\forall \epsilon > 0 \exists N \forall n \geq N \, |x_n - a| < \epsilon.$$

And since the purpose of these problems was to become expert in writing statements like this, even if you made what might seem like a small mistake it needs to be punished harshly so that when we come to hard material you can use notation correctly unconsciously. If you take a course in mountain climbing and the first lesson is tying your shoes, since that's such a basic part to a much harder goal you should expect to be marked very low if you make mistakes tying those shoes. But on the other hand, no one is born knowing how to tie shoes, so if you have trouble tying your shoes you may be able to use willpower and have enoguh energy to master tying your shoes and climb the mountain.

p. 18, Problem B. Suppose that $\lim_{n\to\infty} \sin \frac{n\pi}{2}$ had a limit L. This means that for all $\epsilon > 0$ there is some N such that $n \geq N$ implies that

$$\left| \sin \frac{n\pi}{2} - L \right| < \epsilon.$$

This is true for all $\epsilon > 0$ so it's true in particular for $\epsilon = \frac{1}{2}$, and there is some N such that $n \geq N$ implies that

$$\left| \sin \frac{n\pi}{2} - L \right| < \frac{1}{2}.$$

But there is certainly some $n \geq N$ with $\sin \frac{n\pi}{2} = 0$, so $|L| < \frac{1}{2}$. And there is certainly some $n \geq N$ with $\sin \frac{n\pi}{2} = 1$, so $|1 - L| < \frac{1}{2}$. But $|1 - L| \geq 1 - |L|$, and combining this with $|1 - L| < \frac{1}{2}$ we obtain $1 - |L| < \frac{1}{2}$, so $\frac{1}{2} < |L|$. But we also had $|L| < \frac{1}{2}$, which is a contradiction. Therefore there is no limit L of the sequence.

p. 22, Problem B. For a problem like this we first prove that the sequence is bounded and monotone. This will imply that the sequence converges, and once we know this we can find the limit of the sequence. We have $0 \leq a_1 < a_2 = \sqrt{5} < 5$. (The reason I include the ≥ 0 is to ensure that we are not taking square roots of negative numbers.) Assume that $0 \leq a_n < a_{n+1} < 5$. Then (here I am using that $\sqrt{\cdot}$ is strictly increasing, i.e. that if $x > y$ then $\sqrt{x} > \sqrt{y}$)

$$a_{n+2} = \sqrt{5 + 2a_{n+1}} > \sqrt{5 + 2a_n} = a_{n+1}.$$

This shows that $0 \leq a_{n+1} < a_{n+2}$. On the other hand,

$$a_{n+2} = \sqrt{5 + 2a_{n+1}} < \sqrt{5 + 2 \cdot 5} = \sqrt{15} < \sqrt{25} = 5,$$

showing that $a_{n+2} < 5$. Therefore we have shown by induction that for all n we have

$$0 \leq a_n < a_{n+1} < 5,$$

i.e. that the sequence a_n is strictly increasing and bounded.[1] Therefore by the monotone convergence theorem the sequence a_n converges to some limit L.

If f is a continuous function then

$$f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n),$$

i.e.

$$f(L) = \lim_{n\to\infty} f(a_n).$$

But if $f(a_n) = a_{n+1}$, then

$$f(L) = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L,$$

i.e. the limit L is a fixed point of the function f. Here the function f is $f(x) = \sqrt{5 + 2x}$, which is a continuous function $(0, \infty) \to (0, \infty)$, so the limit satisfies $\sqrt{5 + 2L} = L$, hence $5 + 2L = L^2$, hence $L^2 - 2L - 5 = 0$, hence

$$L = \frac{2 \pm \sqrt{4 + 20}}{2} = \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 2\sqrt{6}}{2} = 1 \pm \sqrt{6}.$$

What we have shown is that the limit must be one of these two numbers. The terms in the sequence are all positive, so the limit is $1 + \sqrt{6}$.

[1] By the way, you don't need to say "monotonic increasing". "monotonic" means either increasing or decreasing, so if you say increasing you don't need to add that it's monotonic; that's like saying a "nonzero positive number". I know the book uses the term monotone increasing but I assure you that both are correct ways of speaking. Also, in case you're curious the word "monotonic" means the same thing as "monotone".

p. 26, Problem A. As $|\cos^n(n)| \leq 1$ for all n (in fact it is strictly less than 1, because π is irrational, but we don't need to use this)

$$|a_n| \leq \frac{n}{\sqrt{n^2+2n}} < \frac{n}{\sqrt{n^2}} = \frac{n}{n} = 1,$$

showing that the sequence a_n is bounded. Therefore by the Bolzano-Weierstrass theorem it has a convergent subsequence.

p. 31, Problem A. Suppose that x_n is a Cauchy sequence with a subsequence $x_{f(n)}$ that converges to some a.[2] Let's write out what we know: for all $\epsilon > 0$, there is some N_1 such that $n, m \geq N$ implies

$$|x_n - x_m| < \epsilon$$

and there is some N_2 such that $n \geq N_2$ implies

$$|x_{f(n)} - a| < \epsilon.$$

The first was the Cauchy sequence fact and the second was the fact that the subsequence converges to a. Our goal is to prove that $x_n \to a$, i.e., that for all $\epsilon > 0$ there is some N such that $n \geq N$ implies that $|x_n - a| < \epsilon$.

Let $\epsilon > 0$. There is some N_1 such that $n, m \geq N_1$ implies

$$|x_n - x_m| < \frac{\epsilon}{2},$$

and there is some N_2 such that $n \geq N_2$ implies

$$|x_{f(n)} - a| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|x_n - a| = |x_n - x_{f(n)} + x_{f(n)} - a| \leq |x_n - x_{f(n)}| + |x_{f(n)} - a|.$$

We have $f(n) \geq n$ because $x_{f(n)}$ is a subsequence of x_n, i.e. because f is strictly increasing. So $n, f(n) \geq N \geq N_1$, so the above inequality becomes

$$|x_n - a| \leq \frac{\epsilon}{2} + |x_{f(n)} - a|.$$

And $n \geq N \geq N_2$, so this inequality now becomes

$$|x_n - a| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

[2] If I want to talk about a subsequence of a sequence a_n, I write $a_{f(n)}$ where $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. If you do complicated arguments with the a_{n_k} notation it becomes hard to figure out whether you want to talk about $n_k \to \infty$ or $k \to \infty$, and becomes impossible to use if you have subsequences within subsequences. This notation I am recommending is not standard, but any mathematician would understand what I am writing so it would never cause confusion.

We have shown that there is some N such that $n \geq N$ implies $|x_n - a| < \epsilon$, and therefore $x_n \to a$.

p. 43, H. To prove that the series $\sum_{n=1}^{\infty} a_n$ converges it suffices to prove that it converges absolutely, i.e., that $\sum_{n=1}^{\infty} |a_n|$ converges, i.e. that $\sum_{n=1}^{\infty} |a_n| < \infty$.

Furthermore, using limsups can be confusing, so let's write out the definition of what $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$ means. Define

$$A_N = \sup\left\{\frac{|a_N|}{b_N}, \frac{|a_{N+1}|}{b_{N+1}}, \dots\right\}.$$

Then $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$ means that there is some $0 \leq A < \infty$ such that $A_N \to A$. (A limsup is literally a limit of suprema.)

Because $A_n \to A$, there is some M such that[3] $N \geq M$ implies that $|A_N - A| \leq 1$, which is the same as $A_N - A \leq 1$ because A_N is decreasing to A (the fewer terms over which you take a supremum the smaller the supremum becomes). So, there is some M such that $N \geq M$ implies that $A_N \leq A + 1$. In particular, $A_M \leq A + 1$, hence for all $n \geq M$ we have (using the definition of A_M)

$$\frac{|a_n|}{b_n} \leq A + 1,$$

i.e. for all $n \geq M$ we have

$$|a_n| \leq (A+1)b_n.$$

Therefore,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{M-1} |a_n| + \sum_{n=M}^{\infty} |a_n| \leq \sum_{n=1}^{M-1} |a_n| + \sum_{n=M}^{\infty} (A+1)b_n \leq \sum_{n=1}^{M-1} |a_n| + (A+1)\sum_{n=1}^{\infty} b_n.$$

But $\sum_{n=1}^{M-1} |a_n|$ is a finite sum so it is finite, and we are given that $\sum_{n=1}^{\infty} b_n < \infty$, and therefore $(A+1)\sum_{n=1}^{\infty} b_n < \infty$, as $A+1$ is some finite number. Therefore

$$\sum_{n=1}^{\infty} |a_n| < \infty,$$

showing that $\sum_{n=1}^{\infty} a_n$ converges absolutely, and therefore converges.

p. 43, I. We are told that $a_n > 0$ for all n and that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. Define

$$A_N = \sup\left\{\frac{a_{N+1}}{a_N}, \frac{a_{N+2}}{a_{N+1}}, \dots\right\}.$$

Then there is some A, $0 \leq A < 1$, such that $A_N \to A$, i.e., for all $\epsilon > 0$ there is some M such that $N \geq M$ implies that $|A_N - A| < \epsilon$, equivalently $A_N - A < \epsilon$ because A_N is decreasing. Using $\epsilon = \frac{1-A}{2}$, there is some M such that $N \geq M$ implies that $A_N - A < \frac{1-A}{2}$, i.e., there is some M such that $N \geq M$ implies

[3] We could have chosen any $\epsilon > 0$ rather than 1 but it turns out we just need to use some specific number, so we use 1.

that $A_N < \frac{1+A}{2}$. Define $r = \frac{1+A}{2}$, which satisfies $0 < r < 1$ (because $A < 1$), so that for all $N \geq M$ we have $A_N < r$. In particular, $A_M < r$, so by the definition of A_M for all $n \geq M$ we have

$$\frac{a_{n+1}}{a_n} < r,$$

i.e. for all $n \geq M$ we have

$$a_{n+1} < ra_n.$$

Therefore, $a_{M+1} < ra_M$, $a_{M+2} < ra_{M+1} < r^2 a_M$, and generally $a_{M+k} < r^k a_M$. Hence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{M-1} a_n + \sum_{n=M}^{\infty} a_n = \sum_{n=1}^{M-1} a_n + \sum_{k=0}^{\infty} a_{M+k} < \sum_{n=1}^{M-1} a_n + \sum_{k=0}^{\infty} r^k a_M.$$

Now, $\sum_{n=1}^{M-1} a_n$ is a finite sum so it is finite, and using the formula for a geometric series because $0 < r < 1$,

$$\sum_{k=0}^{\infty} r^k a_M = a_M \sum_{k=0}^{\infty} r^k = a_M \frac{1}{1-r} < \infty,$$

and therefore

$$\sum_{n=1}^{\infty} a_n < \infty.$$

a_n is a sequence of positive terms so this is equivalent to saying it converge.

Now suppose that $a_n > 0$ for all n and that

$$\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} > 1.$$

Write out what this means: Define

$$A_N = \inf \left\{ \frac{a_{N+1}}{a_N}, \frac{a_{N+2}}{a_{N+1}}, \ldots \right\}.$$

There are two possibilities. Either $A_N \to \infty$, or there is some $A > 1$ such that $A_N \to A$. (liminf might indeed be ∞) In the first case, since A_N becomes arbitrarily large there is in particular some M such that $N \geq M$ implies that $A_N \geq 2$. Then, for all $n \geq M$ we have

$$\frac{a_{n+1}}{a_n} \geq 2.$$

Then $a_{M+1} \geq a_M$, and $a_{M+2} \geq 2a_{M+1} \geq 2^2 a_M$, and generally $a_{M+k} \geq 2^k a_M$. Thus

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{M-1} a_n + \sum_{n=M}^{\infty} a_n = \sum_{n=1}^{M-1} a_n + \sum_{k=0}^{\infty} a_{M+k} \geq \sum_{n=1}^{M-1} a_n + \sum_{k=0}^{\infty} 2^k a_M.$$

But $\sum_{k=0}^{\infty} 2^k = \infty$, and $a_M > 0$, so the right-hand side is infinite. Therefore

$$\sum_{n=1}^{\infty} a_n$$

diverges. Now we do the second case, where there is some $A > 1$ such that $A_N \to A$. This means that for all $\epsilon > 0$ there is some M such that $N \geq M$ implies that $|A_N - A| < \epsilon$, which is the same as $A - A_N < \epsilon$ because A_N is increasing. So, for all $\epsilon > 0$ there is some M such that $N \geq M$ implies that $A_N > A - \epsilon$. Use $\epsilon = \frac{A-1}{2}$. So there is some M such that $N \geq M$ implies that $A_N > A - \frac{A-1}{2} = \frac{A+1}{2}$. Let $r = \frac{A+1}{2}$, and $r > 1$ because $A > 1$. So, there is some M such that $N \geq M$ implies that $A_N > r$, and in particular $A_M > r$. This means that for all $n \geq M$ we have

$$\frac{a_{n+1}}{a_n} > r,$$

i.e.,

$$a_{n+1} > ra_n.$$

So $a_{M+1} > ra_M$, and $a_{M+2} > ra_{M+1} > r^2 a_M$, and generally $a_{M+k} > r^k a_M$. Therefore we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{M-1} a_n + \sum_{n=M}^{\infty} a_n = \sum_{n=1}^{M-1} a_n + \sum_{k=0}^{\infty} a_{M_k} > \sum_{n=1}^{M-1} a_n + \sum_{k=0}^{\infty} r^k a_M.$$

And because $r > 1$ and $a_M > 0$, we have $\sum_{k=0}^{\infty} r_k a_M = \infty$. Therefore

$$\sum_{n=1}^{\infty} a_n = \infty,$$

i.e.

$$\sum_{n=1}^{\infty} a_n$$

diverges.