Lecture 15

§7.2 $c < -2$

- Not all orbits escape to infinity.

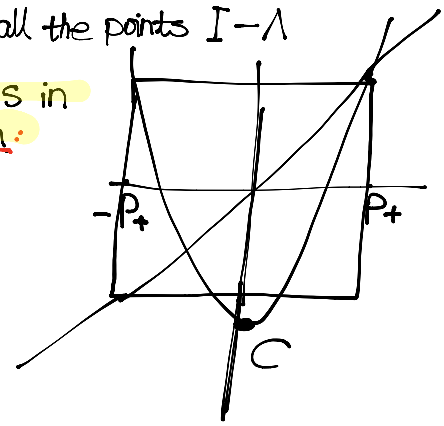
We define a set

$$\Lambda = \{x \in I,\ Q_c^n(x) \in I,\ \forall n\}$$

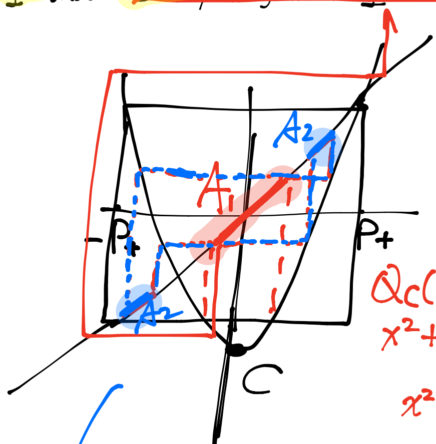
'lambda'

Q: how do we describe Λ?

We will describe Λ by describing all the points $I - \Lambda$



- Let us find the set A_1 of points in I that escape after 1 iteration:



$$Q_c(x) < -P_+$$

$$x^2 + c < -\frac{1+\sqrt{1-4c}}{2}$$

$$x^2 < -\left(c + \frac{1+\sqrt{1-4c}}{2}\right)$$

$$< -(c + P_+)$$

$$-\sqrt{-\left(c+\frac{1+\sqrt{1-4c}}{2}\right)} < x < \sqrt{-\left(c+\frac{1+\sqrt{1-4c}}{2}\right)} \implies -\sqrt{-(c+P_+)} < x < \sqrt{-(c+P_+)}$$

So $A_1 = \left(-\sqrt{-\left(c+\frac{1+\sqrt{1-4c}}{2}\right)},\ \sqrt{-\left(c+\frac{1+\sqrt{1-4c}}{2}\right)}\right)$

for $c = -2.5$, $A_1 = \left(-\sqrt{\frac{4-\sqrt{11}}{2}},\ \sqrt{\frac{4-\sqrt{11}}{2}}\right)$

- More generally, we define $A_n = \{x \in I : Q_c^n(x) \notin I,\ Q_c^i(x) \in I \text{ for } 0 < i \leq n-1\}$

Orbits escape I in n iterations and not before

A_2 gets double from A_1

A_3 doubles from A_2 ...

Q: what do we know about these sets?

① : A_n is the union of 2^{n-1} disjoint open intervals (why not closed? cuz if close then still in I → eventually fixed)

② $Q_c(A_n) = A_{n-1}$

③ $\bigcup_{n=1}^{\infty} A_n$ is the set of all pts in I whose orbits escape to infinities

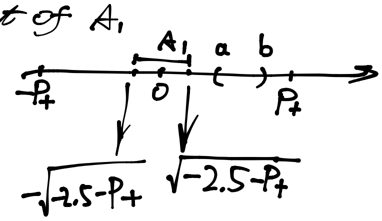
④ $\Lambda = I - \bigcup_{n=1}^{\infty} A_n$ is the set of all seeds whose orbits remain in I.

⑤ Λ must be a close set

⑥ Λ can be expressed as I by removing the centre A_1 then removing the centres (A_2) of the 2 remaining intervals
Then removing the centres A_3 of the 4 remaining intervals
...
This type of sets is called CANTOR SET

⑦ Λ contains no open intervals, it is a collection of isolated points

Proof of ⑦:
Assume $(a,b) \subset \Lambda$
So (a,b) is either to the left or to the right of A_1
Assume (a,b) is to the right of A_1



$\sqrt{-2.5-P_+} < a < b$

we know that Q_c is increasing for $x>0$
so $Q_c(a,b) = (Q_c(a), Q_c(b)) = (a^2-2.5, b^2-2.5)$

The centre of $Q_c(a,b)$ is ...