Reliable Communications and Expander Graphs

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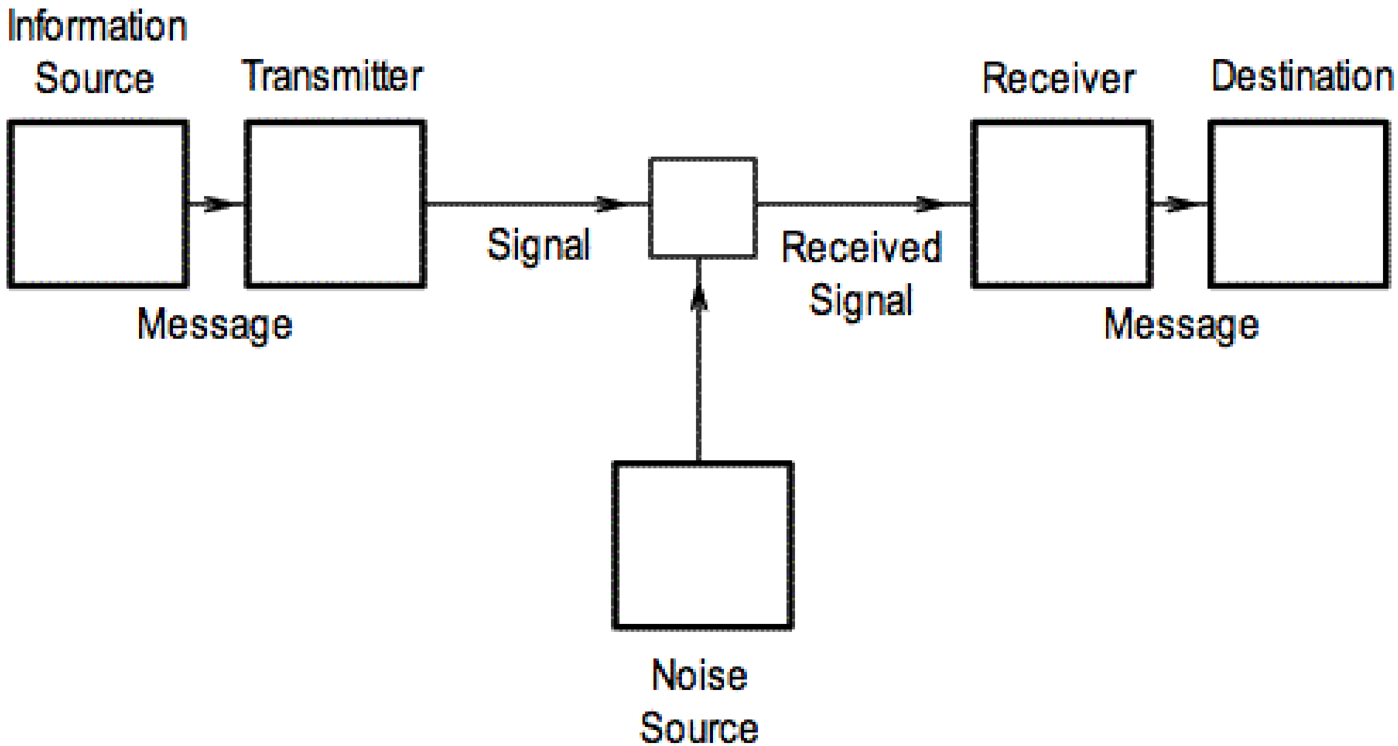
Overview of the Talk

- Background in Communication Theory
- Transition to Graph Theory
- Expander Graphs
  - Definitions and Properties
- Construction of Expander Graphs
  - Graph Squaring
  - Zig-Zag Product

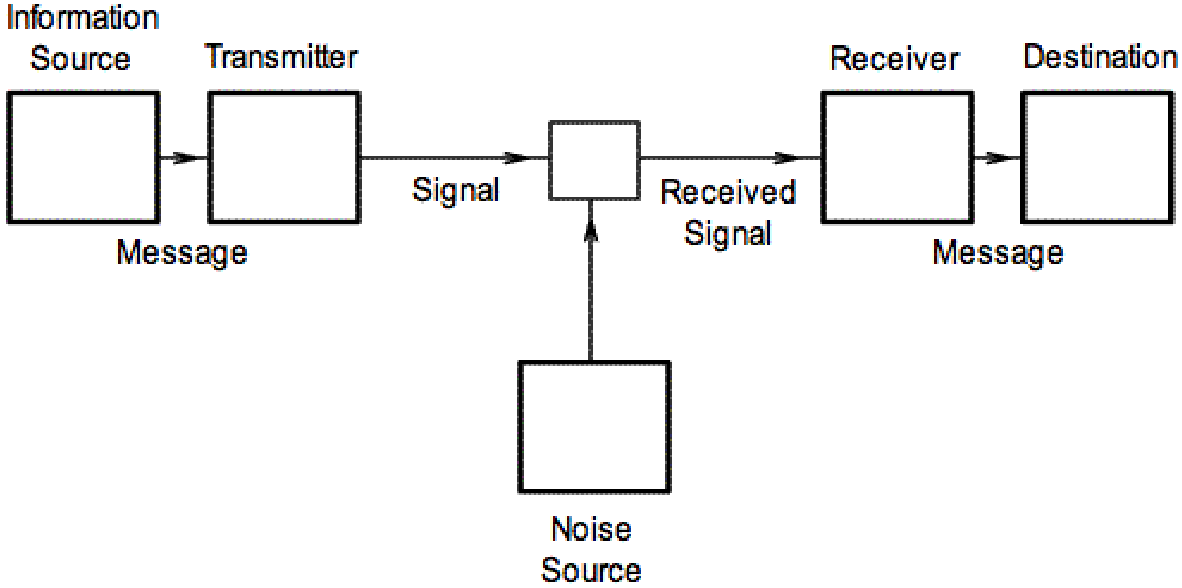
Physical Layer Communication Theory



Shannon, “A Mathematical Theory of Communication” (1948)

Communication over noisy channel

- Alice and Bob can communicate over a noisy channel that might corrupt a proportion p of the bits sent through it. How can Alice send Bob a message of k bits ?



Error Detection *and* Correction

- Build a dictionary (or code) $\mathcal{C} \subseteq \{0,1\}^n$

  such that $|\mathcal{C}_k| = 2^k$

- Every *k*-bits message is encoded by a code word in *C* and transmitted.

- Bob receives *n* (corrupted) bits and finds the closest code word that matches in *C.*

A Good Dictionary (code)

- Key idea is to construct a good dictionary (code).

- A good dictionary is the one
  - That is big ($|C|$ is big)
  - Length of the words in C are small

Communication Theory to Graph Theory

- Is it possible to design a series of dictionaries

$\{\mathcal{C}_k\}$ such that $|\mathcal{C}_k| = 2^k$?

Recent trends in designing good codes are based on …….

Communication Theory to Graph Theory

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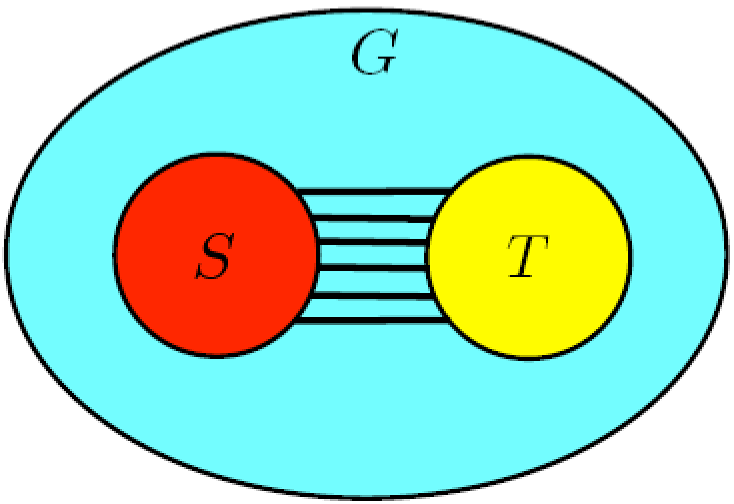
Expander Graphs !

Expander Graphs

- Combinatorially, Expander graphs are ***strongly connected*** graphs, and one has to remove many edges to disconnect a large part of the graph.

- Geometric, Algebraic, Probabilistic definitions omitted here.

Terminology



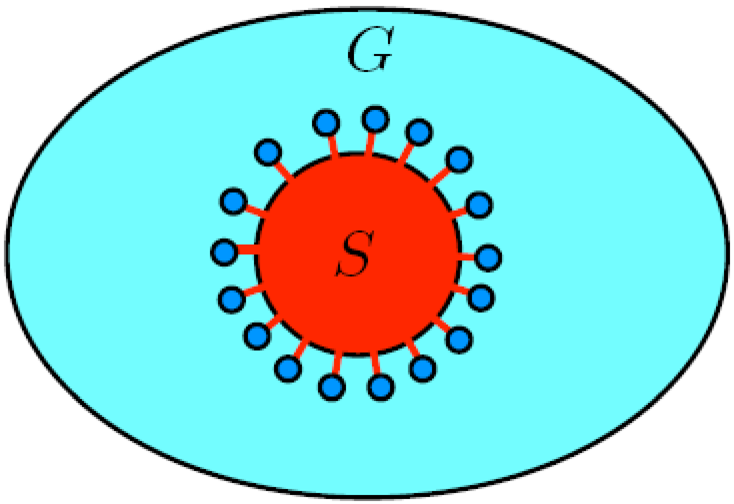
$$E(S,T) = \{(u,v) \in E \mid u \in S, v \in T \text{ or } u \in T, v \in S\}$$

Terminology

Let $G = (V, E)$ be an undirected graph. For any set $S \subseteq V$, let

$$\partial S := E(S, \overline{S})$$

be the edge boundary of S.



$$\begin{aligned} \partial S &= \{(u, v) \in E \mid u \in S, v \notin S\} \\ &= E(S, \overline{S}) \end{aligned}$$

Terminology

- Let $G = (V, E)$ be an undirected graph. For any set, $S \subseteq V$ let

$$\partial_{\text{out}}(S) = \{v \in V \mid \exists u \in S \colon (u, v) \in E\}$$

be the <u>neighbouring set</u> of *S.*

Vertex Expansion

Definition: Vertex Expansion Factor

$$h_{\mathrm{out}}(G) = \min_{0<|S|\leq \frac{n}{2}} \frac{|\partial_{\mathrm{out}}(S)|}{|S|}$$

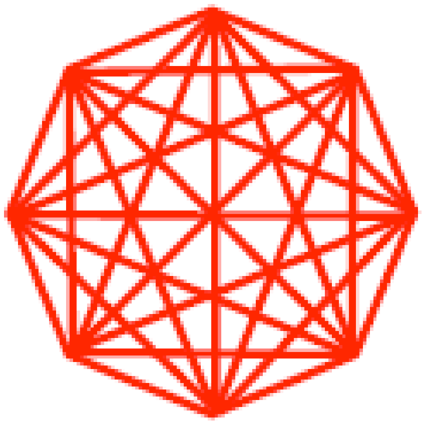
Edge Expansion

The edge expansion of a graph $G = (V, E)$ is

$$h(G) = \min_{1 \leq |S| \leq |V|/2} \frac{|\partial S|}{|S|}$$

Examples: Edge Expansion

$$G = \mathcal{K}_n$$



$$h(G) = \frac{n}{2}$$

Examples:

- If G is a complete graph, then $h(G) = \lceil |V|/2 \rceil$.
- If G is not connected, then $h(G) = 0$.

Expander Graphs

Definition

Let $d \in \mathbb{N}$. A sequence of d-regular graphs $\{G_i\}_{i \in \mathbb{N}}$ of size increasing with i is a family of expanders if there is $\epsilon > 0$ such that $h(G_i) \geq \epsilon$ for all i.

Lemma

Any expander graph is a connected graph.

Construction of Expander Graph: Graph Squaring

- The *square* $G^2 = (V, E')$ is a graph that has the same number of vertices and $(u, w) \in E'$ iff there is path of length 2 in *G* from *u to v.*

Increased expansion factor !

Construction of Expander Graph: Zig-Zag Product

Let G be d-regular graph on n vertices

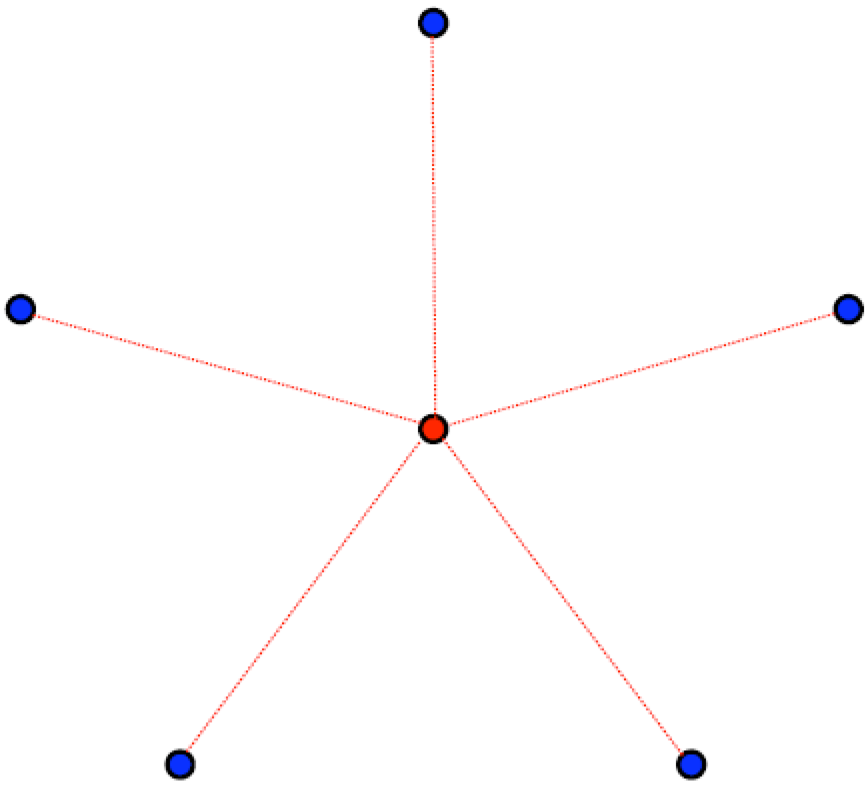
Let H be r-regular graph on d vertices

Then, the zig-zag product of G and H, $G \textcircled{z} H$ is a r^2-regular graph on *nd* vertices.

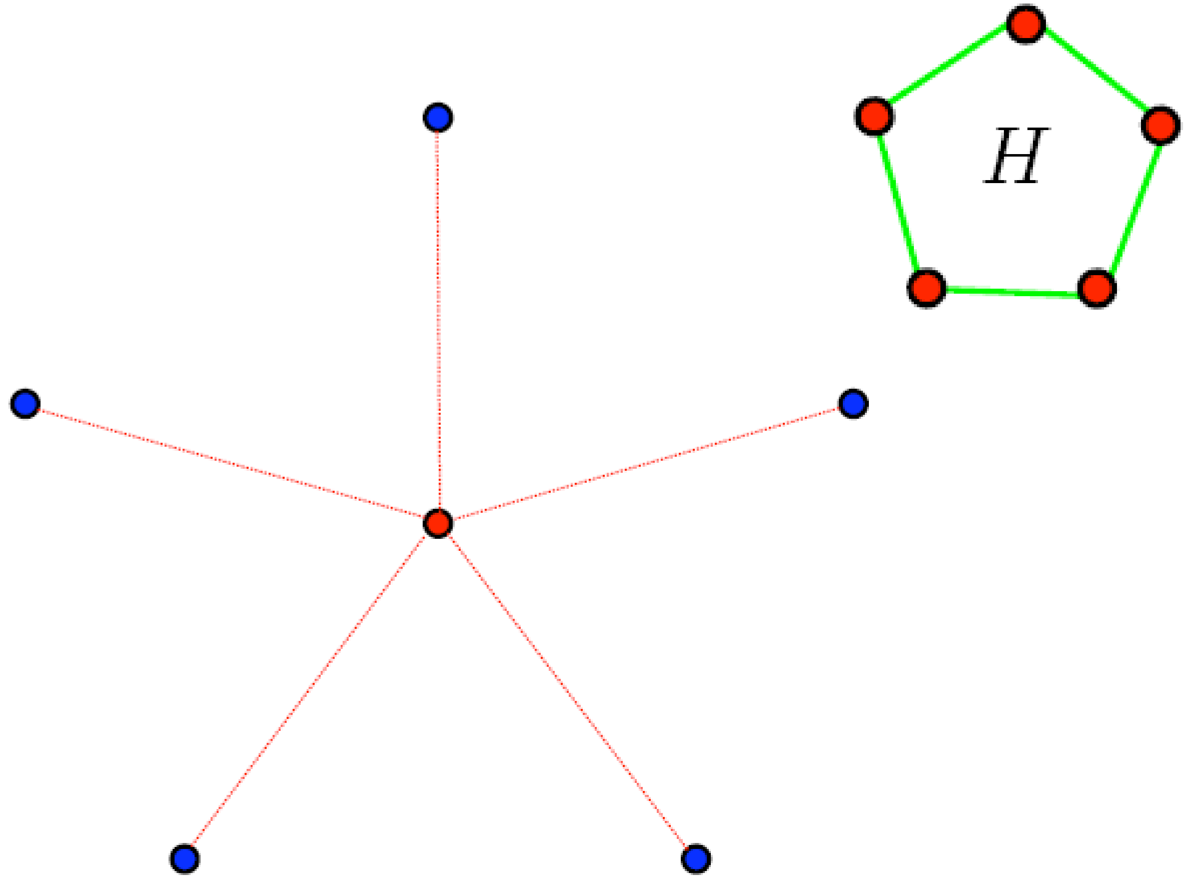
- Inherits the size of larger graph, but the degree of the smaller one.

Zig – Zag Product

- Neighbourhood of a vertex in G



Zig – Zag Product



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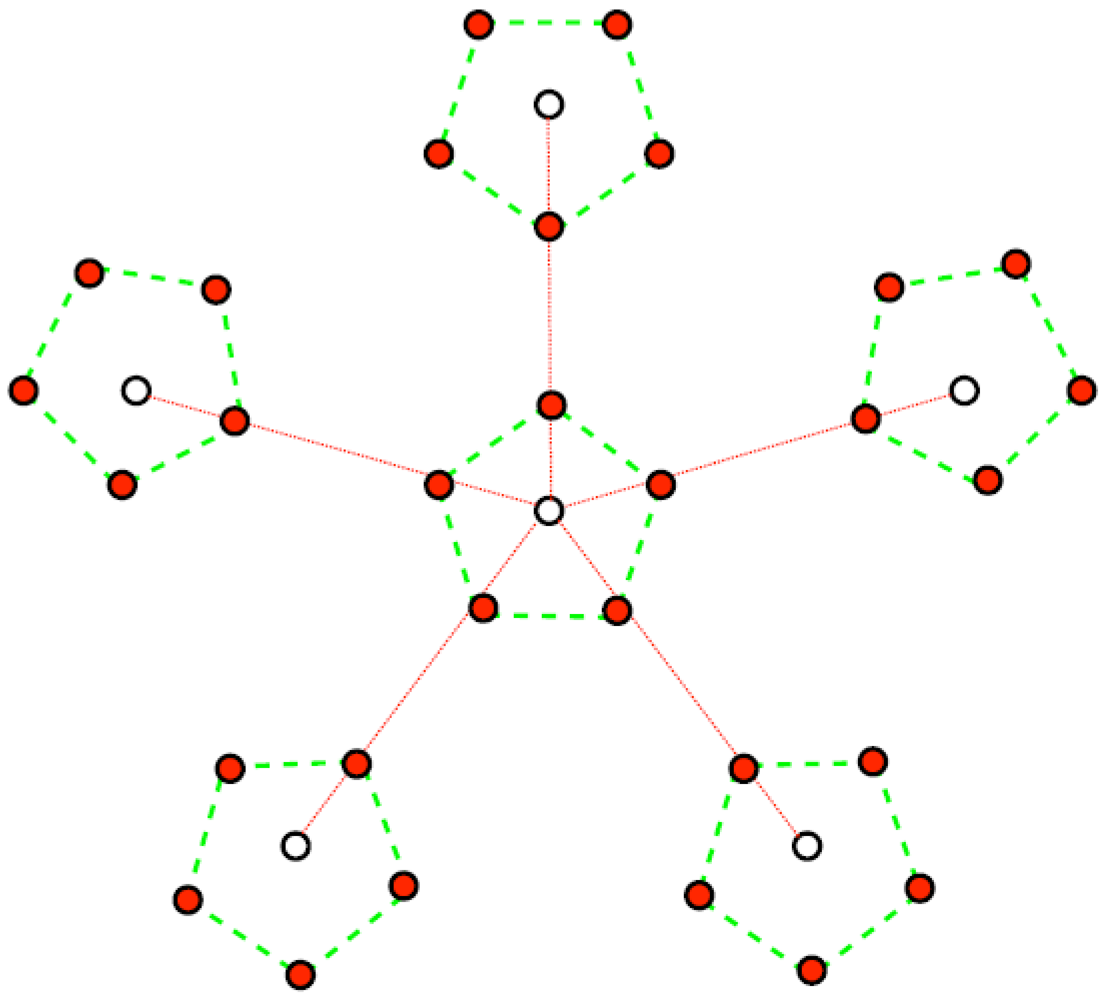

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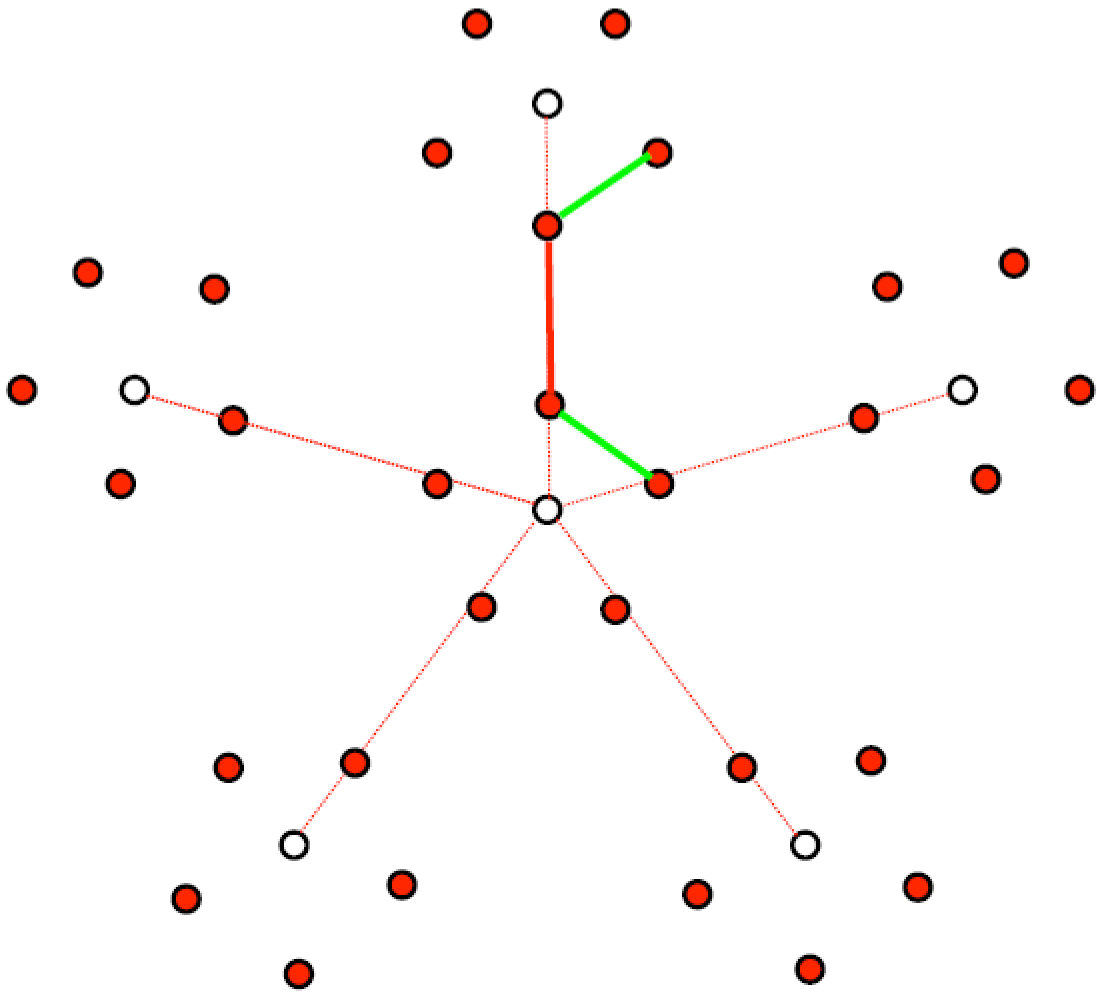

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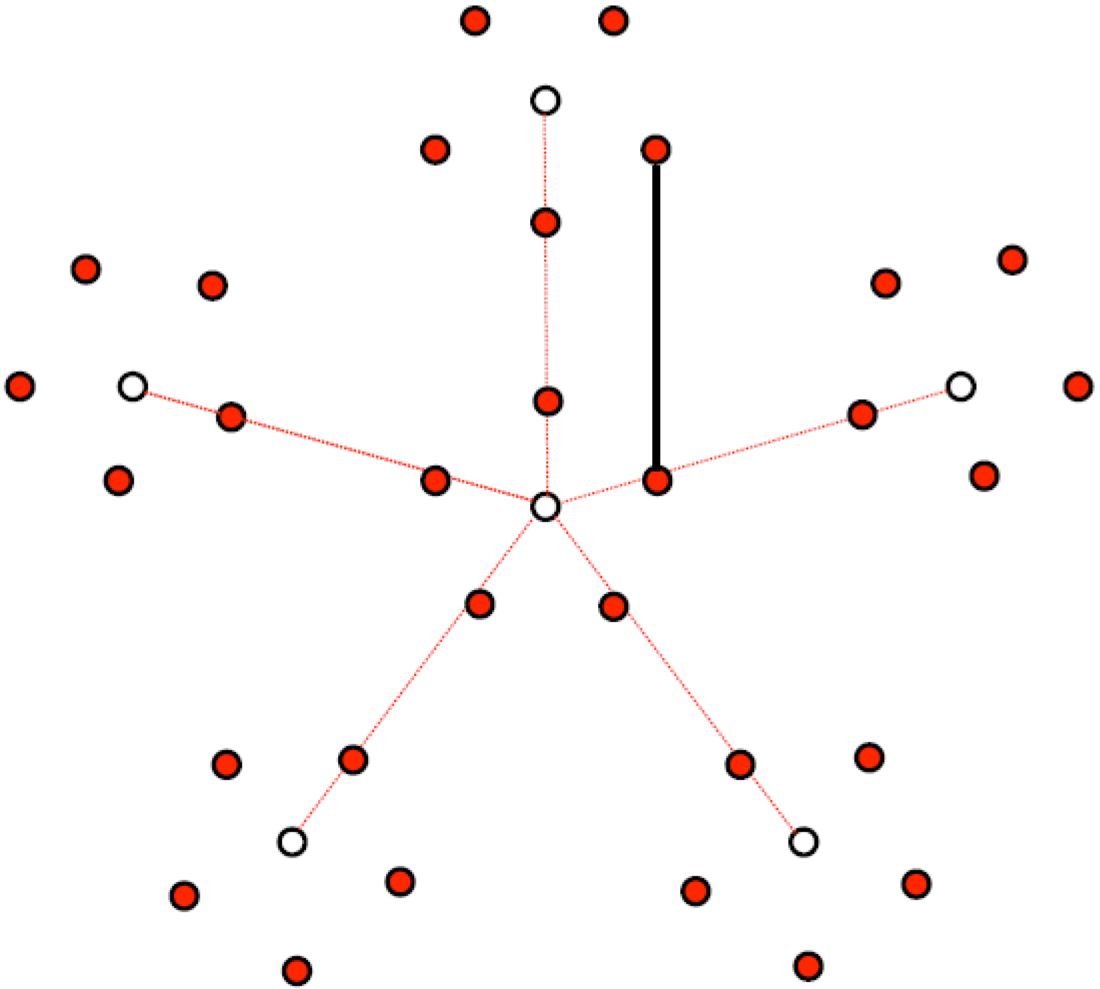

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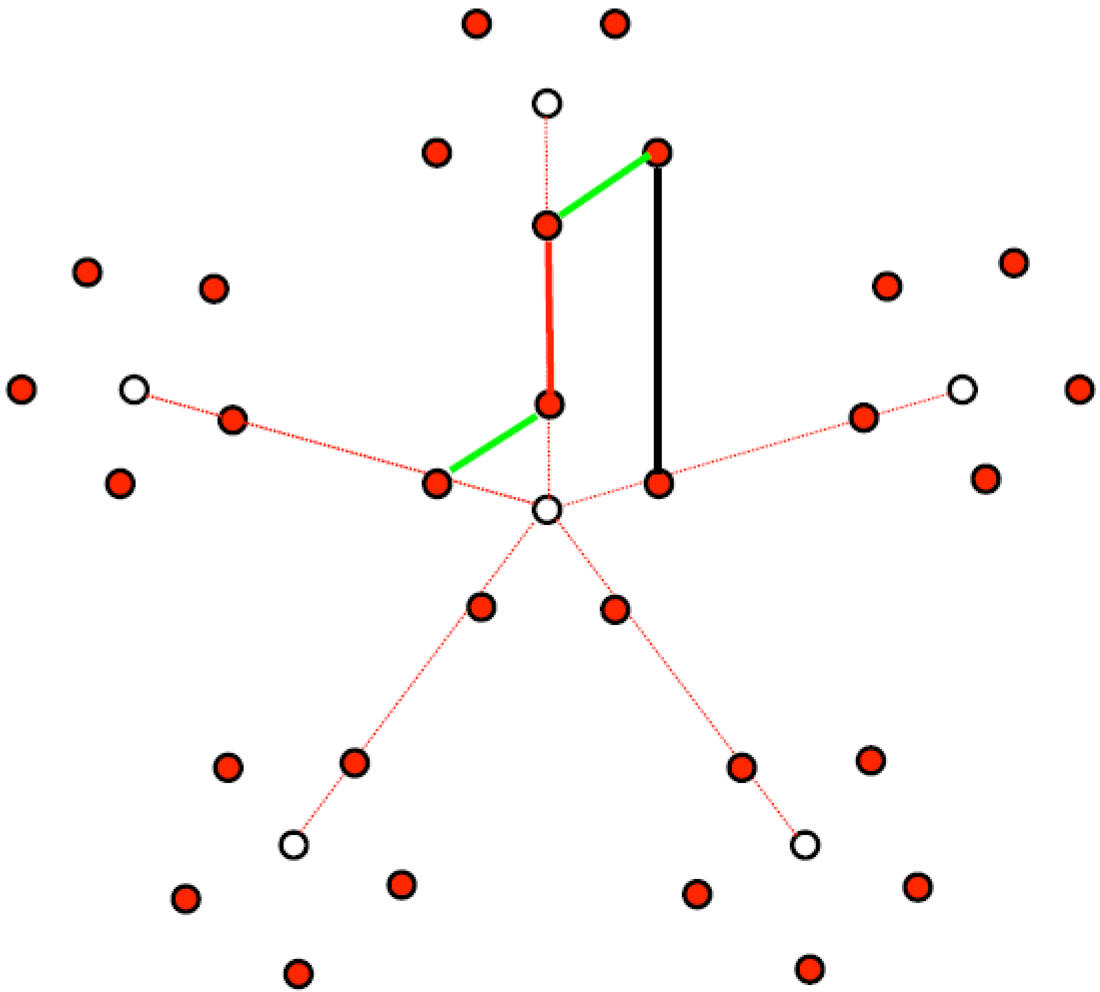

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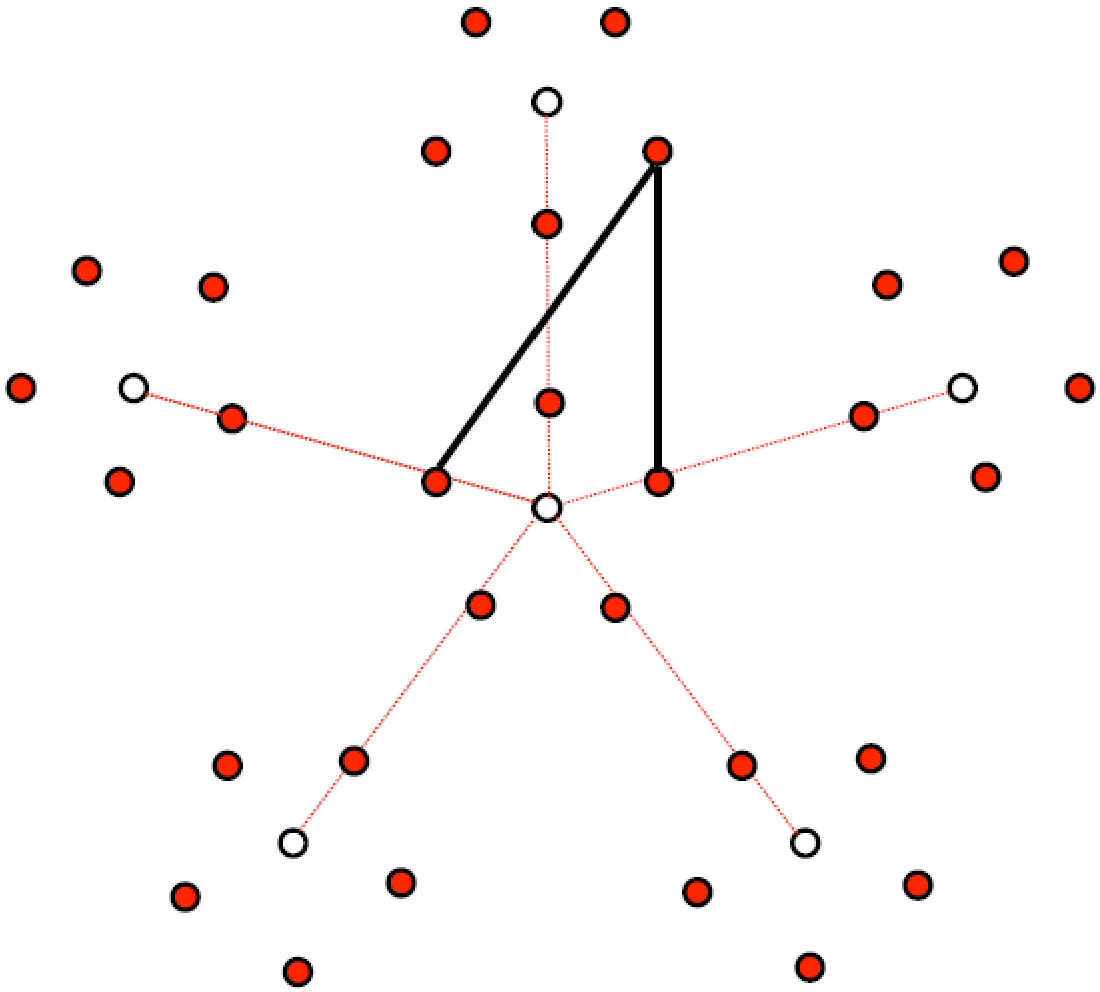

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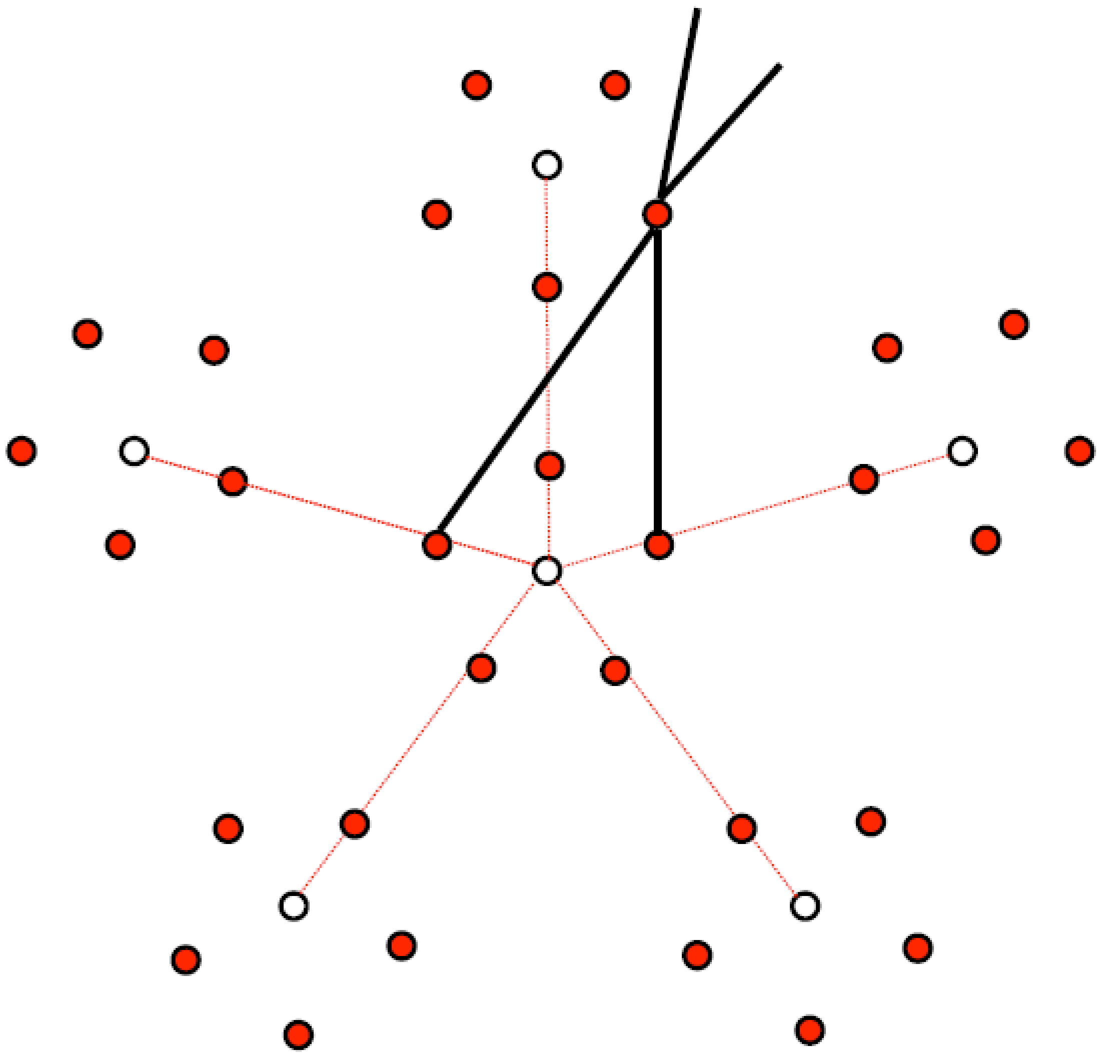

Applications of Expanders

- In Computer Science
  - Derandomization
  - Circuit complexity
  - Error correcting codes
  - Communication networks
  - Approximation algorithms

- In Mathematics
  - Graph theory
  - Group theory
  - Number theory
  - Information theory

References

1. Amin Shokrollahi's talk on “*Expander Graphs*” at EPFL, Switzerland.

2. He Sun's lecture notes on “*Introduction to Expander Graphs*”, Max Planck Institute for Informatics

3. Nati Linial and Avi Wigderson's lecture notes on “*Expander Graphs and their Applications*” at the Hebrew University, Israel, 2003.