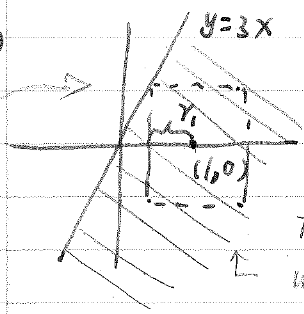
1. $F(x,y) = -x + y + \frac{1}{x-\frac{y}{3}}$ is C' on all the plane except at $x-\frac{y}{3}=0$ or $y=3x$

a) 

The half plane under $y=3x$

b) $F(1,0) = -1 + 0\cdot\frac{1}{1-0} = 0$

$\frac{\partial F}{\partial y}(1,0) = 1 + \frac{\frac{1}{3}}{(x-\frac{y}{3})^2} > 0$

as long as $(x-\frac{y}{3})^2 > 0$

c) let $f(\vec{a}) > 0$, choose $\varepsilon = \frac{f(\vec{a})}{2}$ and by continuity

$\exists\ \delta > 0$, s.t. $\forall\ x$, $|x-a| < \delta \Rightarrow |f(\vec{x}) - f(\vec{a})| < \varepsilon = \frac{f(\vec{a})}{2}$

$\Rightarrow -\frac{f(\vec{a})}{2} < f(\vec{x}) - f(\vec{a}) < \frac{f(\vec{a})}{2}$

$\Rightarrow \frac{f(\vec{a})}{2} < f(\vec{x}) < \frac{3}{2} f(\vec{a})$

$\Rightarrow f(x) > 0$

d) largest r_1 must satisfy $x = 1 - r_1$, $y = r_1$

The only restriction is the upper right corner is on the line $y = 3x$

$\Rightarrow r_1 = y = 3x = 3(1-r_1)$

$\Rightarrow 4r_1 = 3 \Rightarrow r_1 = \frac{3}{4}$

e) $F(1, \frac{3}{4}) = -\frac{1}{4} + \frac{4}{3} > 0$, $F(1, -\frac{3}{4}) = -\frac{7}{4} + \frac{4}{5} < 0$

to find r_0 (largest r_0) s.t. $|x-1| < r_0 \Rightarrow F(x, \frac{3}{4}) > 0$

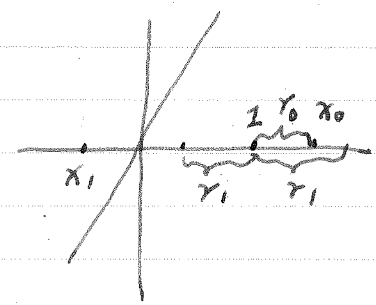
we solve $F(x, \frac{3}{4}) = 0$ to find possible change of sign locations.

$$F(x, \tfrac{3}{4}) = 0 \Rightarrow -x + \frac{3}{4} + \frac{1}{x - \frac{3/4}{4}} = 0$$

$$\Rightarrow 16x^2 - 16x - 13 = 0$$

$$\Rightarrow x = \frac{2 \pm \sqrt{17}}{4} \begin{cases} x_0 \approx \frac{2+4}{4} = \frac{3}{2} \\ x_1 \approx \frac{2-4}{4} = -\frac{1}{2} \end{cases}$$

$$\Rightarrow |x_0 - 1| \approx \tfrac{1}{2} < r_1 < |x_1 - 1| \approx \tfrac{3}{2}$$



(*) $\cdots \Rightarrow r_0 = x_0 - 1 = \frac{-2 + \sqrt{17}}{4} \approx 0.53$

repeat for $F(x, -\frac{3}{4}) < 0$

$F(x, -\frac{3}{4}) = 0$ implies

$$-x - \frac{3}{4} + \frac{1}{x + \frac{1}{4}} = 0$$

$$\Rightarrow x + \frac{3}{4} = \frac{1}{x + \frac{1}{4}} \Rightarrow x^2 + x - \frac{13}{16} = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{17}/2}{2} \begin{cases} x_0 \approx \frac{1}{2} \\ x_1 \approx -\frac{3}{2} \end{cases}$$

$$\Rightarrow r_0 = 1 - x_0 = \frac{6 - \sqrt{17}}{4} \approx 0.47$$

compare with (*).

$$\Rightarrow \boxed{r_0 = \frac{6 - \sqrt{17}}{4}}$$

f) $$f'(1) = \frac{-\partial_x F(1, f(1))}{\partial_y F(1, f(1))} = \frac{-1 - \frac{1}{(x - \frac{y}{3})^2} \Big|_{\text{at } x=1,\, y=0}}{1 + \frac{1/3}{(x - \frac{y}{3})^2} \Big|_{\text{at } x=1,\, y=0}}$$

$$= -\frac{3}{2}$$

at $(1, 1, e, 0, -1)$

2. $$B = \begin{bmatrix} \partial_u F_1 & \partial_w F_1 \\ \partial_u F_2 & \partial_w F_2 \end{bmatrix} = \begin{bmatrix} \frac{x}{w} & -\frac{xy}{w^2} \\ 0 & \frac{1}{w} \end{bmatrix}\Bigg|_{\substack{\text{when} \\ u=1,\, v=1,\, w=e \\ x=0,\, y=-1}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{e} \end{bmatrix}$$

$\det B = 0$

$\Rightarrow$ The IFT does not guarantee the soln.

at $(0, 1, e^2, -1, -2)$

$$B = \begin{bmatrix} -\frac{1}{e^2} & 0 \\ 0 & \frac{1}{e^2} \end{bmatrix} \Rightarrow \det B = -\frac{1}{e^4} \neq 0$$

$\Rightarrow$ by IFT, it is possible to compute $\partial_x u$ and $\partial_x w$.

we differentiate both eqns. w.r.t to x

$$\begin{cases} \frac{u}{w} + \frac{x\,\partial_x u}{w} - \frac{xu}{w^2}\partial_x w = 0 \\ 2xe^{v-1} + 1 + \frac{\partial_x w}{w} = 0 \end{cases}$$

at $(0, 1, e^2, -1, -2) = (u, v, w, x, y)$

$$\begin{cases} 0 - \frac{\partial_x u}{e^2} - 0 = 0 \\ -2e^0 + 1 + \frac{\partial_x w}{w} = 0 \end{cases} \Rightarrow \boxed{\begin{matrix} \partial_x u = 0 \\ \partial_x w = e^2 \end{matrix}}$$

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3. Def of a smooth curve: $S \subset \mathbb{R}^2$ is a <u>smooth curve</u> near the point (a, b) if $\exists$ N (nbd) of (a, b) s.t. $S \cap N$ is the graph of the C_1 function $y = f(x)$ or $x = g(y)$.

But near $(0,0)$, the locus $F(x, y) = x^2 - y^2 = 0$ is

X and any nbd N of the $(0,0)$ still keeps the cross (X). In fact, for the cross, both horizonal & vertical line test fail.

so X is not the graph of $y=f(x)$ or $x=g(y)$

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4. a) S is the part of the sphere in the positive octant and U is the upper right quarter of a disc.

b) $G_3(u,v) = \sqrt{1-u^2-v^2}$

c) $H_1(u,v) = \sqrt{1-u^2-v^2}$

d) $h \circ G(u,v) = 10 - 3u^2 - 3v^2 + 2uv + 4(u+v)\sqrt{1-u^2-v^2}$

$\Rightarrow D(h\circ G) = 0$ gives

$$\left.\begin{aligned} -6u + 2v + 4\sqrt{1-u^2-v^2} - \frac{4u(u+v)}{\sqrt{1-u^2-v^2}} &= 0 \\ -6v + 2u + 4\sqrt{1-u^2-v^2} - \frac{4v(u+v)}{\sqrt{1-u^2-v^2}} &= 0 \end{aligned}\right\} \quad \cdots (1)$$

Subtracting the top equation from the bottom gives

$$4(u-v)\left(-1 - \frac{4(u+v)}{\sqrt{1-u^2-v^2}}\right) = 0$$

$$\Rightarrow \begin{cases} u = v & \cdots (3) \\ \text{or} & \\ \sqrt{1-u^2-v^2} = -4(u+v) & \cdots (4) \end{cases}$$

claim: (4) gives a value of u, v on the boundary of U.
plug (4) into (1) gives $u=0=v$, which is on the boundary.

when $u = v$, plug into (1)

$$-u\sqrt{1-2u^2} + 1 - 4u^2 = 0 \quad \cdots \quad (5)$$

$\Rightarrow (1-4u^2)^2 = u^2(1-2u^2)$

let $a = u^2$

$\Rightarrow (1-4a)^2 = a(1-2a)$

$\Rightarrow a = 1/3$ or $1/6$

$\Rightarrow u = \pm \frac{1}{\sqrt{3}}$ or $\pm \frac{1}{\sqrt{6}}$

We are only considering $u > 0$ and only $\frac{1}{\sqrt{6}}$ solves (5).

$\Rightarrow$ the only critical point is

$(u_0, v_0) = (1/\sqrt{6}, \frac{1}{\sqrt{6}})$ and $h \circ G(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) = 12$

Now, we need to compare this value, 12, to the values of $h \circ G$ on the boundary.

Since $h \circ G$ is continuous on the closure of U, if we show that the maximum value of $h \circ G$ on the boundary is less than 12, then in fact 12 is the maximum of $h \circ G$ on both U and its closure.

The three boundary components of U are.

① $v = 0$, $0 \le u \le 1$

② $u = 0$, $0 \le v \le 1$

③ $v = \sqrt{1-u^2}$, $0 \le u \le 1$.

For ②. $u = 0$, $\Rightarrow h \circ G(0, v, \sqrt{1-v^2}) = 7v^2 + 10(1-v^2) + 4v\sqrt{1-v^2}$

$\Rightarrow \frac{\partial}{\partial v} h \circ G(0, v, \sqrt{1-v^2}) = -3v\sqrt{1-v^2} + 2 - 4v^2 = 0 \quad \cdots \quad (6)$

$\Rightarrow v = 1/\sqrt{5}$, $2/\sqrt{5}$. and only $1/\sqrt{5}$ solves (6).

$\Rightarrow h \circ G(0, 1/\sqrt{5}, 2/\sqrt{5}) = 11$.

By symmetry: for ①. max of $h \circ G = 11$

for ②: $h\circ G = 7 + 2v\sqrt{1-v^2} \le 9$ for $0 \le v \le 1$

Thus: $\boxed{\text{max of } h\circ G \text{ on the closure of } U \text{ is } 12}$

e). Same rigmarole:

$$h\circ G = 7 + 3v^2 + 4uv + 2(u+2v)\sqrt{1-u^2-v^2}$$

$\Rightarrow \nabla h\circ G$:

$$4v + 2\sqrt{1-u^2-v^2} - \frac{2u(u+2v)}{\sqrt{1-u^2-v^2}} = 0 \quad \text{--- (7a)}$$

$$3v + 2u + 2\sqrt{1-u^2-v^2} - \frac{v(u+2v)}{\sqrt{1-u^2-v^2}} = 0 \quad \text{--- (7b)}$$

$$\xrightarrow{(7a)-(7b)} (v-2u)\left(u+2v+\sqrt{1-u^2-v^2}\right) = 0$$

$$\Rightarrow \begin{cases} v = 2u & \text{--- (8a)} \\ \text{or} & \\ u + 2v + \sqrt{1-u^2-v^2} = 0 & \text{--- (8b)} \end{cases}$$

From (8b): $u = v = 0$, which is on the boundary.

From (8a): $v = 2u$ and the critical point is

$$(u_1, v_1) = \left(1/\sqrt{6}, \frac{2}{\sqrt{6}}\right) \text{ and } h\circ H\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = 12.$$

On the boundry, all the values of $h\circ H$ are all less than 12.

f) $G(u_0, v_0) = G\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = H\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = H(u_1, v_1)$.

5(a) $\vec{f}(s,t) = \vec{p} + s\vec{u} + t\vec{v} = \begin{pmatrix} p_1 + su_1 + tv_1 \\ p_2 + su_2 + tv_2 \\ p_3 + su_3 + tv_3 \end{pmatrix},$

where $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

(b) $\vec{f}_s = \begin{pmatrix} u_1 \\ u_2 \\ v_3 \end{pmatrix} = \vec{u}, \quad \vec{f}_t = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$

(c) From (b): $\partial_s \vec{f} \times \partial_t \vec{f} = \vec{u} \times \vec{v}$

Since $\vec{u}, \vec{v}$ are direction vector for the plane

$\vec{n} = \vec{u} \times \vec{v}$ is perpendicula to the plane, but not ~~necessary~~ necessary of unit length.

i.e. it is a normal to the plane.

(d) The point (x,y,z) is in the plane

$\Leftrightarrow \left((x,y,z) - \vec{p} \right) \cdot \vec{n} = 0 \quad \cdots \quad (*)$

If $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad D = \vec{p} \cdot \vec{n}$

Then define $F(x,y,z) = Ax + By + Cz - D$

by $(*)$, the equation of the plane is

$F = 0$

and

$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = (A, B, C) = \partial_s \vec{f} \times \partial_t \vec{f}$

e). $\nabla F = \vec{u} \times \vec{v} = (0, -2, 2)$

- Since $F_z = 2 \neq 0$ we can solve $z = \varphi(x, y)$
  s.t. $F(x, y, \varphi(x,y)) = 0$

- Since $\dfrac{\partial(x,y)}{\partial(s,t)} = F_z = 2 \neq 0$
  we can solve $\begin{cases} x = \vec{f}_1(s,t) \\ y = \vec{f}_2(s,t) \end{cases}$
  for s, t: $s = g(x,y)$
  $t = h(x,y)$
  and set $z = f_3(s,t) = f_3(g(x,y), h(x,y)) = \varphi(x,y)$

- Since $F(x,y,z) = -2y + 2z - D$, then $z = y + \frac{D}{2}$

Same reasoning using $F_y = -2 \neq 0$ shows that the plane is transverse to the y axis and we can solve $y = z - \frac{D}{2}$

However, $F_x = 0$, so $\nabla F \perp x$-axis
so the plane is parallel to the x-axis
we can not solve for x.