PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

APRIL 2011 EXAMINATIONS

FINAL EXAM

CSC 165H1 S
DURATION — 3 HOURS

NO AIDS ALLOWED

PLEASE HAND IN

LAST NAME: ______________________________

FIRST NAME: ______________________________

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 11 questions on 17 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question,". You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-17 of this test.

Marking breakdown (Total = 136 marks).

Question 1	10 marks	Question 6	15 marks
Question 2	16 marks	Question 7	15 marks
Question 3	22 marks	Question 8	8 marks
Question 4	12 marks	Question 9	8 marks
Question 5	10 marks	Question 10	10 marks
Question 11	10 marks		

Good Luck!

Here are some definitions and results that will be useful throughout the exam. (note: you may also use any result from X = {notes, assignments, tutorials, lectures} by saying "from the x", where $x \in X$)

1. Let $\mathbb{N}$= the set of natural numbers (i.e $\{0, 1, 2, 3, ...\}$)
2. Let $\mathbb{R}$= the set of real numbers and $\mathbb{R}^+$ = the set of positive real numbers
3. Let $\mathbb{F}$= $\{\mathbb{N} \to \mathbb{R}^{\geq 0}\}$
4. $\forall f, g \in \mathbb{F}$: $f \in O(g) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c * g(n)$
5. $\forall f, g \in \mathbb{F}$: $f \in \Omega(g) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq c * g(n)$
6. $\forall f, g \in \mathbb{F}$: $f \in \Theta(g) \Leftrightarrow \exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 * g(n) \leq f(n) \leq c_2 * g(n)$
7. $\forall f, g \in \mathbb{F}, \forall z \in \mathbb{R}^+, f = z * g \Rightarrow f \in \Theta(g)$
8. $\forall m, n, r \in \mathbb{N}, r = m\%n \Leftrightarrow (0 \leq r < n) \wedge (\exists q \in \mathbb{N}, m = q * n + r)$
9. $W_P(n) \in \Omega(f(n)) \Leftrightarrow \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, \exists x \in I, size(x) = n \wedge n \geq B \Rightarrow t_P(x) \geq c * f(n)$
10. $y = log_b(x) \Leftrightarrow b^y = x$
11. $log_b(xy) = log_b(x) + log_b(y)$
12. $log_b(x/y) = log_b(x) - log_b(y)$

commutative laws	$P \wedge Q$	$\Leftrightarrow$	$Q \wedge P$
	$P \vee Q$	$\Leftrightarrow$	$Q \vee P$
	$(P \Leftrightarrow Q)$	$\Leftrightarrow$	$(Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R$	$\Leftrightarrow$	$P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R$	$\Leftrightarrow$	$P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R)$	$\Leftrightarrow$	$(P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R)$	$\Leftrightarrow$	$(P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q$	$\Leftrightarrow$	$\neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q$	$\Leftrightarrow$	$\neg P \vee Q$
equivalence	$(P \Leftrightarrow Q)$	$\Leftrightarrow$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
double negation	$\neg(\neg P)$	$\Leftrightarrow$	P
DeMorgan's laws	$\neg(P \wedge Q)$	$\Leftrightarrow$	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	$\Leftrightarrow$	$\neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q)$	$\Leftrightarrow$	$P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q)$	$\Leftrightarrow$	$\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x))$	$\Leftrightarrow$	$\exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x))$	$\Leftrightarrow$	$\forall x \in D, \neg P(x)$
identity	$P \vee (Q \wedge \neg Q)$	$\Leftrightarrow$	P
	$P \wedge (Q \vee \neg Q)$	$\Leftrightarrow$	P
idempotence	$P \vee P$	$\Leftrightarrow$	P
	$P \wedge P$	$\Leftrightarrow$	P
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x)$	$\Leftrightarrow$	$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
	$\exists x \in D, P(x) \vee Q(x)$	$\Leftrightarrow$	$(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

QUESTION 1. [10 MARKS]

Symbolic Representations of Ideas.

PART (A) [5 MARKS]

Consider the following predicates:

FP(x) : x is a Fermat prime
Prime(x) : x is prime.

Using the above predicates, provide an equivalent symbolic statement for the statement below:

(S1A) A natural number n is a Fermat prime if and only if n is a prime number that can be written in the form $2^{2^k} + 1$ for some non-negative integer k.

PART (B) [5 MARKS]

Provide an equivalent symbolic statement for the following statement:

(S1B) Conjecture: The only Fermat primes are those for which $k \leq 4$.

QUESTION 2. [16 MARKS]

Distributive laws for quantifiers: For each of the following determine if they are true in both directions, one direction, or false in both directions. Briefly explain your answers.

PART (A) [4 MARKS]

$\forall x \in D, P(x) \wedge Q(x) \Leftrightarrow (\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$

PART (B) [4 MARKS]

$\exists x \in D, P(x) \wedge Q(x) \Leftrightarrow (\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$

PART (C) [4 MARKS]

$\forall x \in D, P(x) \vee Q(x) \Leftrightarrow (\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$

PART (D) [4 MARKS]

$\exists x \in D, P(x) \vee Q(x) \Leftrightarrow (\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

QUESTION 3. [22 MARKS]

PART (A) [6 MARKS]

Show that if $x^2 - 2x + 2 \leq 0$, then $x^3 \geq 8$

PART (B) [6 MARKS]

Find all primes that are one less than a perfect cube.

PART (C) [10 MARKS]

Let $a_1 = 1, a_2 = 4, a_3 = 9, a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3), n \geq 4$. Prove that $a_n = n^2$ for all positive integers n.

QUESTION 4. [12 MARKS]

Show that the following pair(s) of statements are logically equivalent using any method:

PART (A) [3 MARKS]

$(P \wedge Q) \Leftrightarrow P$ and $P \Rightarrow Q$

PART (B) [3 MARKS]

$P \Rightarrow (Q \vee R)$ and $\neg Q \Rightarrow (\neg P \vee R)$

PART (C) [3 MARKS]

$\neg Q \Rightarrow (P \wedge \neg P)$ and Q

PART (D) [3 MARKS]

$(P \wedge Q) \Rightarrow R$ and $(P \wedge \neg R) \Rightarrow \neg Q$

QUESTION 5. [10 MARKS]

Recall: $\mathbb{F}= \{\mathbb{N} \to \mathbb{R}^{\geq 0}\}$. Prove the following statement:

THEOREM: For any functions $f_1, f_2, g_1, g_2 \in \mathbb{F}$, if $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then $f_1(n) + f_2(n) \in O(max(g_1(n), g_2(n)))$.

QUESTION 6. [15 MARKS]

Show that for non-zero x, if $x + \frac{1}{x} < 2$, then $x < 0$ using each of the following proof methods.

PART (A) [5 MARKS]

A direct proof.

PART (B) [5 MARKS]

A proof of contrapositive.

PART (C) [5 MARKS]

A proof by contradiction.

QUESTION 7. [15 MARKS]

Prove that for any $a_n, a_{n-1}, ..., a_1, a_0 \in \mathbb{R}^+$ and $n \in \mathbb{N}$, $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \in \Theta(x^n)$.

QUESTION 8. [8 MARKS]

Define n^{th} triangular number T_n and n^{th} pyramidal number P_n as follows:

$$T_n = 1 + 2 + ... + n$$
$$P_n = T_1 + T_2 + ... + T_n$$

Prove termination and correctness of the following function.

```
#pre-condition: n ∈ ℕ, n ≥ 1
#post-condition: Returns the n^th pyramidal number P_n.
DEF calc(n):
    T := 0
    P := 0
    k := 1
    WHILE k<= n do
        T := T+k
        P := P+T
        k := k+1
    end WHILE
    RETURN P
```

QUESTION 9. [8 MARKS]

PART (A) [4 MARKS]

Suppose f(x) = $ln(x)$ (the natural log, i.e. log_e, of x). Explain how the condition number of f is related to the relative error of f's input versus the relative error of f's output. Explain what this tells you about implementing f for x $\in$ (1, 3)?

PART (B) [4 MARKS]

Suppose you have a floating-point number system with base $\beta = 4$, one sign bit, emin = -3 and emax = 3, t = 4 digits in the mantissa (significand), with the convention that the significand for non-zero values begins with a left-most non-zero digit (i.e. normalized), which is the unit value to the left of the radix point. Round-to-nearest is used for numbers that cannot be represented exactly. How many distinct numbers are there in this system in the range (-20, 20)?

QUESTION 10. [10 MARKS]

```
# Pre-condition: A is an array of constant time comparable objects
""" insertionSort(A) sorts the elements of A in non-decreasing order """
DEF insertionSort(A):
    n = len(A)
    j = 1
    WHILE j <= n
        key = A[j]
        i = j - 1
        WHILE (i >= 0) AND (A[i] > key):
            A[i+1] = A[i]
            i = i - 1
        A[i+1]=key

  # post-condition: A is sorted in non-decreasing order
  RETURN A
```

Let $t(A)$ be the number of lines executed by insertionSort on the Array A and $W(n)$ be the worst-case number of lines executed over all arrays of length n. Prove that $W(n) \in \Omega(n^2)$. (i.e. prove $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \exists A \in I, length(A) = n \wedge t(A) \geq cn^2$)

QUESTION 11. [10 MARKS]

Lemma: Given $m, n \in \mathbb{N}$, if there exist values $q \geq 0, 0 \leq r < m$ such that $n = mq + r$, then GCD (m,n) = GCD(m,r).

Use the above lemma to prove the termination and correctness of the following:

```
# precondition: m,n ∈ ℕ, m ∤ n.
# postcondition: returns GCD(m,n).
DEF gcd(m,n) :
 a = n
 b = m
 WHILE b > 0 do
        c = 0
        WHILE c ≤ a do
                c = c + b
        end WHILE
        r = a + b − c   # Loop Invariant: r = a mod b
        a = b
        b = r
 end WHILE
 RETURN a
```

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# 1:	______/ 10
# 2:	______/ 16
# 3:	______/ 22
# 4:	______/ 12
# 5:	______/ 10
# 6:	______/ 15
# 7:	______/ 15
# 8:	______/ 8
# 9:	______/ 8
# 10:	______/ 10
# 11:	______/ 10
TOTAL:	______/136