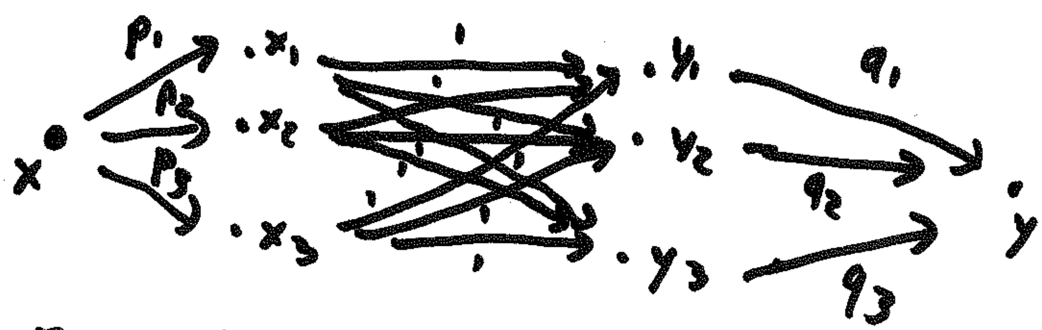
Problem Set 3 Solutions

Question 1 solution

a) Consider the Network $N(x,y)$ with source x, sink y, intermediate vertices $(\{x_1,\ldots,x_m\}, \{y_1,\ldots,y_n\})$, and edges:

- $e_{x\to x_i}$ with capacity p_i $\quad 1\le i\le m$;
- $e_{x_i\to y_j}$ with capacity 1 $\quad 1\le i\le m$; $1\le j\le n$;
- $e_{y_j\to y}$ with capacity q_j $\quad 1\le j\le n$.

Then G is realizable iff $N(x,y)$ has max flow $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j$.



Example: $m=n=3$

b) ($\Rightarrow$) $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_i$ is the condition that if flow $\sum_{i=1}^{m} p_i$ leaves the source, $\sum_{j=1}^{n} q_i$ enters the sink.

Let u_i denote the number of neighbours of x_i in $Y_k := \{y_1, \dots, y_k\}$.
Because the underlying graph of $N(x,y)$ is simple, $u_i \le \min(p_i, k)$.

$$\Rightarrow \sum_{i=1}^{m} \min(p_i, k) \ge \sum_{i=1}^{m} u_i = \sum_{j=1}^{k} q_j. \quad (*)$$

So we're double-counting neighbours of Y_k.

($\Leftarrow$) If $\{e_{x \to x_i}\}$ is a min cut, we are finished.
Assume there is a smaller cut. This cut will consist of $e_{y_j \to y}$ for $j \ge k+1$ for some $1 \le k \le n$, and for each x_i choose either $e_{x \to x_i}$ or $\{e_{x_i \to y_j}\}_{1 \le j \le k}$ to be in the cut - whichever is smaller.
We get: $\sum_{j=k+1}^{m} q_j + \sum_{i=1}^{m} \min(p_i, k) < \sum_{j=1}^{m} q_j$. Contradicts $(*)$.

Question 2 solution

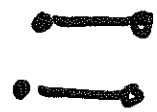
Consider the network $N'(x,y)$ obtained from $N(x,y)$ by deleting all edges pointing to x or from y.

$N'(x,y)$ has the same max flow as $N(x,y)$ because it has the same min cut (edges to source or from sink can't be in a cut).

A flow function on $N'(x,y)$ induces a flow function on $N(x,y)$ by setting flow to zero on deleted edges.

Extend a maximal flow from $N'(x,y)$ to $N(x,y)$.

Question 3 Solution

If G is disconnected, False: 

If G is connected, true.

If G has cycle C, then C contains an edge e not in its perfect matching M. $G-\{e\}$ has a perfect matching induced by that of G.

Delete edges not in M to break cycles until we obtain a tree. This tree is a spanning tree for G with perfect matching induced by M.

Question 4 solution

($\Rightarrow$) Let M be a perfect matching of T, and let u be matched to v in M.

Let $T_0, T_1, \ldots T_k$ be connected components of $T - \{v\}$, with $u \in T_0$.

$T_1, \ldots T_k$ have perfect matchings $\Rightarrow$ they are of even order.

$T - (T_1 \cup \ldots \cup T_k)$ has a perfect matching $\Rightarrow$ it has even order $\Rightarrow$ T_0 has odd order.

($\Leftarrow$) For each v, match v with its neighbour u in the odd order component of $T - \{v\}$.

With notation as before, $T_1, \ldots, T_k$ have even order $\Rightarrow$ $T - T_0$ has odd order

$\Rightarrow$ u gets matched with v also in $T - \{u\}$

$\Rightarrow$ the matching procedure given above is well-defined.

Question 5 solution

Consider the graph with vertices

$$\Big(\underbrace{\{b_{ij}\}_{\substack{1\le i\le k\\ 1\le j\le n_i}}}_{X},\ \underbrace{\{g_l\}_{1\le l\le n}}_{Y} \Big)$$

and an edge $e_{b_{ij}\to g_l}$ iff b_i fancies g_l. (clone b_i n_i times).

The original marriage problem has a solution iff this marriage problem satisfies the marriage condition

$$|N(S)| \ge |S| \text{ for all } S \subseteq X.$$

Question 6 solution

Induction on d, strong induction on $|X|$.

Assume the claim holds for $d=m$, and for all $|X|$ up to $|X|=n$. Let $d=m+1$.

If for any $S\subset X$, $S\neq X$ we have $|S|<|N(S)|$, match vertex $x_1\in X$ to any of its $\geq m+1$ neighbours, and conclude by induction (d becomes $d-1$).

Otherwise, there is a set $S\subset X$, $S\neq X$ with $|S|=|N(S)|$. Match all vertices in $X\setminus S$ (*) (such a matching exists by the marriage theorem), and conclude by strong induction on $|X|$ (note that $|N(S)|\geq d$ because each vertex has degree at least d.

We obtain $d!$ perfect matchings if $d\leq |X|$, and at least $d(d-1)\cdots(d-|X|+1)$ if $d>|X|$, because we have proven that the marriage condition continues to hold after each step.

(*) To vertices outside $N(S)$. Such a matching exists, as in the proof of the marriage theorem.