PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

DECEMBER 2009 EXAMINATIONS

CSC 165H1F
DURATION — 3 HOURS

PLEASE HAND IN

AIDS ALLOWED: HANDWRITTEN AID SHEET, 8.5"x11", BOTH SIDES

STUDENT NUMBER: |_|_|_|_|_|_|_|_|_|

LAST NAME: ____________________

FIRST NAME: ____________________

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This exam consists of 9 questions on 14 pages (including this one). *When you receive the signal to start, please make sure that your copy of the exam is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Write your student number at the bottom of pages 2-14 of this exam.

1: ______/10

2: ______/10

3: ______/10

4: ______/10

5: ______/10

6: ______/10

7: ______/10

8: ______/10

9: ______/10

TOTAL: ______/90

Good Luck!

QUESTION 1. [10 MARKS]

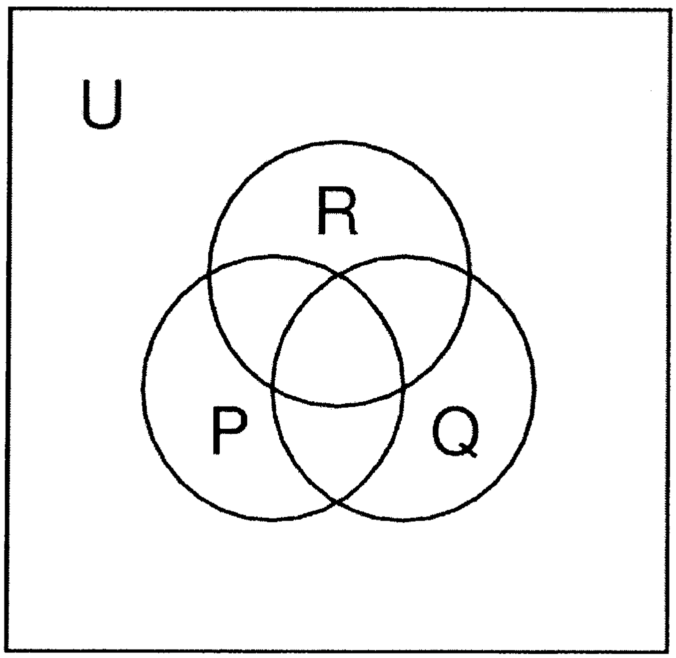
In the Venn diagram below, U represents ungulates, P represents persistent ungulates, R represents reticent ungulates, and Q represents queasy ungulates.

PART (A) [8 MARKS]

Write an F in each region of the diagram where, if it were non-empty, it would provide a counter-example to the statement $S1$:

$$S1: \qquad \forall u \in U, u \in R \vee u \in P \Leftrightarrow u \in Q$$

You will receive a mark for each correctly-placed F, and for each correctly-omitted F.



PART (B) [2 MARKS]

Write a statement equivalent to $S1$ in natural English.

QUESTION 2. [10 MARKS]

Use the definition:

$$\forall m \in \mathbb{N}, m \bmod 7 = r \in \{0,1,2,3,4,5,6\} \Leftrightarrow \exists q \in \mathbb{N}, m = 7q + r$$

...and the proof structure from this course to prove:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (m \bmod 7 = 3 \wedge n \bmod 7 = 4) \Rightarrow mn \bmod 7 = 5$$

QUESTION 3. [10 MARKS]

Use the definition:

$$\forall m \in \mathbb{N}, m \bmod 7 = r \in \{0,1,2,3,4,5,6\} \Leftrightarrow \exists q \in \mathbb{N}, m = 7q + r$$

... and the proof structure from this course to DISPROVE:

$$\forall m \in \mathbb{N}, m^2 \bmod 7 = 4 \Rightarrow m \bmod 7 = 2$$

QUESTION 4. [10 MARKS]

The interval $[0, 1] \subset \mathbb{R}$ represents the real numbers from 0 to 1, inclusive. Use the proof structure from this course to prove:

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [0,1], \forall y \in [0,1], |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$$

QUESTION 5. [10 MARKS]

The interval $[0,1] \subset \mathbb{R}$ represents the real numbers from 0 to 1, inclusive. Use the proof structure from this course to DISPROVE:

$$\exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, \forall x \in [0,1], \forall y \in [0,1], |x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$$

QUESTION 6. [10 MARKS]

Recall the definition of the ceiling function:

$$\forall x \in \mathbb{R}, y = \lceil x \rceil \Leftrightarrow y \in \mathbb{Z} \wedge y \geq x \wedge (\forall z \in \mathbb{Z}, z \geq x \Rightarrow z \geq y)$$

Use the definition, and the proof structure from this course, to prove:

$$\forall x \in \mathbb{R}, 2\lceil x \rceil - 2 < 2x$$

QUESTION 7. [10 MARKS]

Recall the definition of big-Oh:

$$\mathcal{O}(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)\}$$

Define $f(n) = n^4 - 3n^3 + n^2 - 2n + 5$, and $g(n) = n^3 + n^2 - n + 7$. Use the proof structure from this course to prove that $f \notin \mathcal{O}(g)$.

QUESTION 8. [10 MARKS]

Recall the definition of big-Oh:

$$\mathcal{O}(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)\}$$

Define $f(n) = n^3 + 7$, and $g(n) = 3^n$. Use the proof structure from this course to prove that $f \in \mathcal{O}(g)$.

QUESTION 9. [10 MARKS]

Suppose you have a floating-point representation with base $\beta = 7$, exponents from the set $\{-4, \ldots, 4\}$, $t = 4$ digits in the significand, a single sign symbol $+$ or $-$, a radix point following the first digit, and the convention that the digit preceding the radix point is non-zero unless we are representing zero itself.

PART (A) [2 MARKS]

What is the largest positive number you can represent in this number system? Explain.

PART (B) [2 MARKS]

What is the smallest positive number you can represent in this number system? Explain.

PART (C) [3 MARKS]

How many different positive numbers can you represent exactly with this number system? Explain.

PART (D) [3 MARKS]

What is the bound on relative error in this number system, assuming round-to-nearest? Explain.

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Total Marks = 90