STA447/STA2006 Stochastic Processes

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Note

This lecture note is prepared for the course STA447/STA2006 Stochastic Processes. This lecture note may contain flaws. Please consult text book or reference books for confidence.

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* indicates graduate level. So you may skip those parts.

5.3 Optional Sampling Theorem

Theorem 68. Let X_n be a submartingale and T is a stopping time with $P(T \le k) = 1$. Then $\mathbb{E}X_0 \le \mathbb{E}X_T \le \mathbb{E}X_k$.

Proof. Since $X_{T\wedge n}$ is also a submartingale, $\mathbb{E}X_0 = \mathbb{E}X_{T\wedge 0} \le \mathbb{E}X_{T\wedge k} = \mathbb{E}X_T$. Let $K_n = 1(T < n) = 1(T \le n-1) \in \mathcal{F}_{n-1}$ so that it is predictable. $(K \cdot X)_n = X_n - X_{N\wedge n}$ is also a submartingale and $\mathbb{E}X_k - \mathbb{E}X_N = \mathbb{E}(K \cdot X)_k \ge \mathbb{E}(K \cdot X)_0 = 0$. □

Example 63. Let X_n be a submartingale and S, T be stopping times with $S \le T$ a.s. Further suppose $P(T \le k) = 1$. Let $K_n = 1(M < n \le N) = 1(M \le n-1) - 1(M \le n-1, N \le n-1) = 1(M \le n-1) - 1(N \le n-1) \in \mathcal{F}_{n-1}$. Then $(K \cdot X)_n = (X_n - X_{S\wedge n}) - (X_n - X_{T\wedge n}) = X_{T\wedge n} - X_{S\wedge n}$ and $\mathbb{E}X_T - \mathbb{E}X_S = \mathbb{E}(K \cdot X)_k \ge \mathbb{E}(K \cdot X)_0 = 0$. Hence $\mathbb{E}X_S \le \mathbb{E}X_T$.

Theorem 69 (Doob's inequality)**.** Let X_n be a submartingale. Then, for any $\lambda > 0$,

$$\lambda P(\max_{0\le m\le n} X_m^+) \le \mathbb{E}X_n^+.$$

Proof. Let $T = n \wedge \inf\{m : X_m \ge \lambda\}$ so that $X_N \ge \lambda$ on $A = \{\max_{0\le m\le n} X_m^+ \ge \lambda\}$. Hence,

$$\lambda P(A) \le \mathbb{E}X_T 1_A = \mathbb{E}X_T - \mathbb{E}X_T 1_{A^c} \le \mathbb{E}X_n - \mathbb{E}X_n 1_{A^c} = \mathbb{E}X_n 1_A \le \mathbb{E}X_n^+.$$

In the last inequality, $X_n 1_A \le X_n^+ 1_A \le X_n^+$ is used. □

Example 64. Let $X_1, X_2, \dots$ be independent with $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 < \infty$ and $S_n = X_1 + \cdots + X_n$. Then, S_n is a martingale w.r.t. $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and S_n^2 is a submartingale. The Kolmogorov's maximal inequality, for $\alpha > 0$,

$$P(\max_{1\le i\le n} |S_n| > \alpha) \le \alpha^{-2}\mathbb{V}\mathrm{ar}(S_n)$$

is a special case of Doob's inequality, that is, $P(\max_{1\le i\le n} S_n^2 > \alpha^2) \le \alpha^{-2}\mathbb{E}\max(0, S_n^2) = \alpha^{-2}\mathbb{E}S_n^2$.

Theorem 70. Let X_n be a submartingale and T be a stopping time. Suppose there exists $c > 0$ such that $|X_{T\wedge n}| < c$ a.s. for all n. Then $\mathbb{E}X_T \ge \mathbb{E}X_0$.

Proof. Note that $\sup_n \mathbb{E}X^+_{T\wedge n} \le c$. By the martingale convergence theorem $X_{T\wedge n}$ converges to X_T almost surely. By the bounded convergence theorem, $\mathbb{E}X_T = \lim_{n\to\infty} \mathbb{E}X_{T\wedge n} \ge \lim_{n\to\infty} \mathbb{E}X_{T\wedge 0} = \mathbb{E}X_0$. $\square$

Example 65 (Branching process)**.** Let $X, X_{i,j}$ be i.i.d. F which having positive probability only on non-negative integers. Let $Z_0 = 1$ and $Z_n = X_{n,1} + \cdots + X_{n,Z_{n-1}}$ if $Z_{n-1} > 0$ and $Z_n = 0$ otherwise. Then Z_n is a homogeneous Markov chain having transition probability p given by $p(0,0) = 1, p(0,j) = 0$ for $j > 0$ and

$$p(i,j) = P(Y_1 + \cdots + Y_i = j)$$

for $i > 0$ and $j \ge 0$ where $Y_1, Y_2, \ldots$ are i.i.d. F.

Let $W_n = Z_n/\mu^n$ for $\mu = \mathbb{E}X$. Then

$$\mathbb{E}[W_{n+1} \,|\, \mathcal{F}_n] = \mathbb{E}[\mu^{-n-1} \sum_{i=1}^{Z_n} X_{n+1,i} \,|\, \mathcal{F}_n] = \mu^{-n-1} Z_n \mu = \mu^{-n} Z_n = W_n.$$

implies W_n is a martingale.
Subcritical: If $\mu < 1$, then $P(Z_n > 0) \le \mu^n Z_0 \to 0$ as $n \to \infty$.
Note $\mathbb{E}Z_n = \mu^n \mathbb{E}W_n = \mu^n \mathbb{E}W_0 = \mu^n \mathbb{E}Z_0$. Using Markov's inequality, $P(Z_n \ge 1) \le \mathbb{E}Z_n = \mu^n \mathbb{E}Z_0 \to 0$.
Critical: If $\mu = 1$ and $P(X = 1) < 1$, then $Z_n \to 0$ almost surely as $n \to \infty$.
The martingale convergence theorem implies $Z_n \to Z$ almost surely. If $P(X = 0) = 0$, then $\mathbb{E}X \ge 1$ with the equality only when $P(X = 1) = 0$. Hence $p_0 = P(X = 0) > 0$. Note that the state 0 is absorbing and i is transient for $i > 0$ because $p(i,0) = p_0^i > 0$ but $\rho_{0,i} = 0$. Hence, for $i > 0$, $P(Z_n = i) = p^{(n)}(1,i) \to 0$. Which implies $P(Z_n > 0) \to 0$.
Supercritical: If $\mu > 1$, then $Z_n/\mu^n \to W$ almost surely as $n \to \infty$.
Since W_n is a nonnegative supermartingale, it converges to W with $\mathbb{E}W < \infty$.
Nontrivial limit: $P(W > 0) > 0$ if and only if $\mathbb{E}[1(X > 0) X \log X] < \infty$.
A proof is in "A Limit Theorem for Multidimensional Galton-Watson Processes by H. Kesten and B.P. Stigum in Annals of Mathematical Statitics, vol 37."

6 Brownian Motion

Brownian motion was introduced to describe the movement of particles. Nowadays Brownian motion is one of the most popular stochastic processes.

Definition 44. A stochastic process B_t is called a *standard Brownian motion* if it satisfies
(a) $B_0 = 0$
(b) [independent increment] For $0 \le t_1 < t_2 \le t_3 < t_4$, $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent
(c) [stationary increment] The distribution of $B_t - B_s$ only depends on $t - s$ for $0 \le s < t$.
(d) [normal distribution] $B_t \sim N(0,t)$ for all $t \ge 0$.
(e) [continuity] The map $t \mapsto B_t$ is continuous.

The existence of Brownian motion should be proved. A heuristic construction is given below. Let $X_1, X_2, \ldots$ be i.i.d. with mean 0 and variance 1 and $S_n = X_1 + \cdots + X_n$. Consider $B_{t,n} = n^{-1/2} S_{\lfloor tn \rfloor}$. For any fixed $t \in (0,1)$, $B_{t,n} = (\lfloor tn \rfloor/n)^{1/2} \times \lfloor tn \rfloor^{-1/2} S_{\lfloor tn \rfloor} \to tN(0,1) \sim N(0,t)$ in distribution by the central limit theorem. Independent increment is each to check because $X_{\lfloor t_1 n \rfloor + 1}, \ldots, X_{\lfloor t_2 n \rfloor}$ and $X_{\lfloor t_3 n \rfloor + 1}, \ldots, X_{\lfloor t_4 n \rfloor}$ are independent. Similarly stationary increment is satisfied. Also the continuity can be satisfied but it requires higher level of probability theory. Hence there must exist a standard Brownian motion.

For a Brownian motion, we consider a Brownian filtration which is the natural filtration $\mathcal{F}_t = \sigma(B_s : s \le t)$.

Theorem 71. A standard Brownian motion B_t is a Markov chain and a martingale.

Exercise 38. Prove the above theorem.

Theorem 72. Let B_t be a standard Brownian motion.
(a) $\mathbb{E}B_t = 0$, $\mathbb{V}\mathrm{ar}(B_t) = t$.
(b) $\mathbb{E}(B_t - B_s) = 0$, $\mathbb{V}\mathrm{ar}(B_t - B_s) = t - s$.
(c) $\mathbb{E}(B_t B_s) = \min(s, t)$ for $s, t \geq 0$.
(d) $B_t^2 - t$ is a martingale.

Proof. (c) if $s \leq t$, then $B_t - B_s$ and B_s are independent. Thus $\mathbb{E}B_tB_s = \mathbb{E}(B_t - B_s)B_s + \mathbb{E}B_s^2 = \mathbb{E}(B_t - B_s)\mathbb{E}B_s + \mathbb{E}B_s^2 = 0 + s = s$.
(d) for $s < t$, $\mathbb{E}[B_t^2 - t \,|\, \mathcal{F}_s] = -t + \mathbb{E}[B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 \,|\, \mathcal{F}_s] = -t + B_s^2 + 0 + t - s = B_s^2 - s$. □

Exercise 39. Show that B_t^{2k-1} is a Martingale for any positive integer k.

Definition 45. A stochastic process X_t is called a *Lévy process* if it satisfies
(a) $X_0 = 0$
(b) [independent increment] For $0 \leq t_1 < t_2 \leq t_3 < t_4$, $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent
(c) [stationary increment] For $0 \leq s < t$, $X_t - X_s$ and X_{t-s} have the same distribution.
(e) [continuity in probability] The map $t \mapsto X_t$ is continuous, that is, for any $\epsilon > 0$ and $t \geq 0$, $\lim_{h\to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$.

Example 66. A homogeneous Poisson process with parameter λ is a Lévy process. A standard Brownian motion is a Lévy process.

Definition 46. A process X_t satisfying $X_t = \mu t + \sigma B_t$ is called a *Brownian motion with drift* μ.

Exercise 40. Show that a Brownian motion with drift is a Lévy process.

Theorem 73 (Reflection principle)**.** Let B_t be a standard Brownian motion. Then $P(\sup_{0\leq s\leq t} B_s \geq x) = 2P(B_t \geq x)$ for $x > 0$.

Proof. Let $T = \inf\{t \geq 0 : B_t = x\}$. It is known that $P(T < \infty) = 1$. Using the strong Markov property, $X_t = B_{T+t} - x$ and B_t have the same distribution and are independent given B_T. Then $P(\sup_{0\leq s\leq t} B_s \geq x) = P(\sup_{0\leq s\leq t} B_s \geq x, W_t \geq x) + P(\sup_{0\leq s\leq t} B_s \geq x, B_t < x) = P(W_t \geq x) + P(\sup_{0\leq s\leq t} B_s \geq x, X_{t-T} < 0) = P(W_t \geq x) + \mathbb{E}P(T \leq t, X_{t-T} < 0 \,|\, B_T, T) = P(W_t \geq x) + \mathbb{E}1(T \geq t)P(X_{t-T} < 0 \,|\, B_T, T) = P(W_t \geq x) + \mathbb{E}1(T \geq t)(1/2) = P(W_t \geq x) + (1/2)P(\sup_{0\leq s\leq t} B_s \geq x)$. Hence $P(\sup_{0\leq s\leq t} B_s \geq x) = 2P(B_t \geq x)$. □

Example 67. The water level of a reservoir follows a stochastic process $X_t = a + B_t$ where $a > 0$ and B_t is the standard Brownian motion. What is the probability the reservoir dries up within a 4 unit time when $a = 5$?
The probability is $P(\inf_{0\leq t\leq 4} X_t \leq 0) = P(\inf_{0\leq t\leq 4} B_t \leq -5) = P(\sup_{0\leq t\leq 4} -B_t \geq 5) = P(\sup_{0\leq t\leq t} B_t \geq 5) = 2P(B_4 \geq 5) = 2(1 - \Phi(5/\sqrt{4})) = 0.0124$