APM462H1S, Winter 2014 , Assignment 5 (optional).

due: Tuesday April 15 by 5pm, either at the instructor's office (room 1001B, 215 Huron) or in his mailbox in the Math Department (6th floor of Bahen).

Exercise 1. This is a linear algebra problem needed to answer a control question below.

a. Let A by the $n \times n$ matrix defined by

$$\begin{aligned} &a_{11} = a_{nn} = 1, \quad a_{ii} = 2 \text{ for } 2 \le i \le n-1 \\ &a_{ij} = -1 \quad \text{if } |i-j| = 1 \\ &a_{ij} = 0 \quad \text{if } |i-j| \ge 2 \end{aligned}$$

(Write it out to see what it looks like.)

Prove that A is positive semidefinite. *Hint:* It suffices to show that $x^T Ax \ge 0$ for all $x \in \mathbb{R}^n$. Try to do this first for $n = 2$ and $n = 3$, by expanding and just trying to figure it out. If so, you may be able to see a pattern that will allow you to prove it in the general case.

b. Let M be the $2n \times 2n$ matrix that can be written in block form as

$$M = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$$

where 0 denotes a $n \times n$ block of zeros, I is the $n \times n$ identity matrix, and B is a symmetric $n \times n$ matrix.

Prove that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of B, then the eigenvalues of M are $\pm\sqrt{\lambda_1}, \dots, \pm\sqrt{\lambda_n}$. *Hint:* For each j, attempt to build two eigenvectors of M out of the jth eigenvector of B, by trial and error if necessary.

Exercise 2.

A train with n cars connected by springs, powered by a rocket on the last car, is governed by the system of n second-order equations:

$$(\frac{d}{dt})^2 x(t) = -A\,x(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \alpha(t), \qquad x(0) = x^0, \quad \frac{d}{dt}x(0) = v^0,$$

with $\alpha(t) \in [-1, 1]$ for all t. Here A is the same matrix as in Exercise **1a**, and $x(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$, where $x_j(t)$ is the displacement along the railroad track between the jth car and its desired rest position at the station.

Prove that this system is controllable for every value of n. (In other words, for any initial data x^0, v^0, it is possible to find a control that brings the train to the state $x(T) = \frac{dx}{dt}(T) = 0$ at some finite time T.)