Lecture 18
March 19th, 2015

$$\overbrace{\underline{x_0}\quad\underline{x_1}}^{n \text{ digits}}$$
$$\underline{y_0}\quad\underline{y_1}$$

$$x = 10^{\frac{n}{2}} x_0 + x_1$$
$$y = 10^{\frac{n}{2}} y_0 + y_1$$

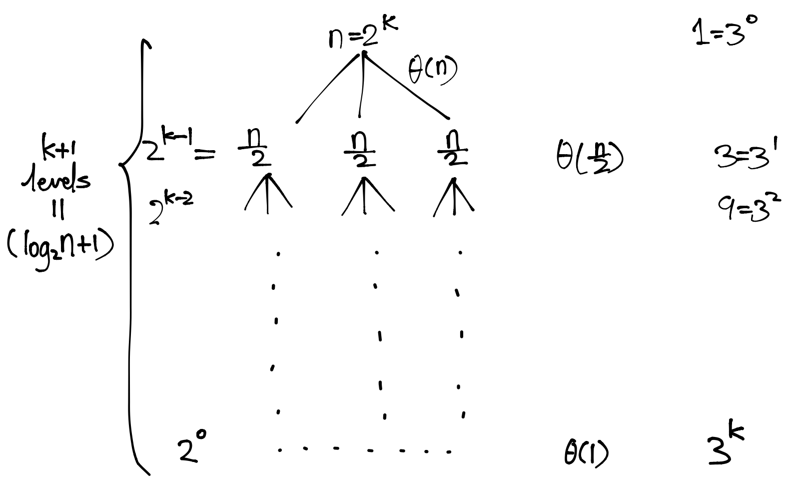
e.g.

$$\overbrace{\underline{1234}\ \ \underline{5678}}^{8} = 10^4 \cdot 1234 + 5678$$
$$\underline{3141}\ \ \underline{5926} = 10^4 \cdot 3141 + 5926$$

\# multiples of size $\Theta(n)$: 3

$$xy = 10^n x_0 y_0 + 10^{\frac{n}{2}}(x_0 y_1 + x_1 y_0) + x_1 y_1$$
$$= 10^n x_0 y_0 + 10^{\frac{n}{2}}\left[(x_0 + x_1)(y_0 + y_1) - x_0 y_0 - x_1 y_1\right] + x_1 y_1$$

3 multiplications

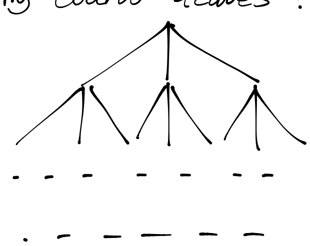


arity of tree of calls
$a = 3$
division of problems of
$\frac{1}{b} = \frac{1}{2}$ the size

use these 2 ☆ $\log_2 3 < \log_2 4 = 2$
☆ $x^{\log_y z} = z^{\log_y x}$

$$\Longrightarrow 3^k = 3^{(\log_2 n)} = (2^{\log_2 3})^{\log_2 n}$$
$$= 2^{(\log_2 3)(\log_2 n)} = n^{\log_2 3} \in O(n^2)$$

Why count leaves?



$$3^0 + 3^1 + \cdots + 3^{k-1} > 3^k - 1$$

For binary tree:

$$2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$$

3^k leaves at bottom

For n a power of 2

$$T(n) = \begin{cases} 3T(\frac{n}{2}) + n, & n > 1 \\ 1, & n = 1 \end{cases}$$

$$T(n) = 3T(\tfrac{n}{2}) + n = 3[3T(\tfrac{n}{4}) + \tfrac{n}{2}] + n$$
$$= 3[3[3T(\tfrac{n}{8}) + \tfrac{n}{4}] + \tfrac{n}{2}] + n$$
$$= n + \frac{3}{2}n + \frac{3^2}{2^2}n + \frac{3^3}{2^3}n + \cdots + \frac{3^k}{2^k}n$$

If $n = 2^k$, the last term is $3^k = 3^{\log_2 n}$

$$\underbrace{n + \frac{3}{2}n + \cdots + 3^{\log_2 n}}_{(1+\log_2 n) \text{ terms}}$$

$$= n\left[1 + 3\cdot\frac{1}{2} + 3^2\cdot\frac{1}{2^2} + 3^3\cdot\frac{1}{2^3} + \cdots + \frac{3^{\log_2 n}}{2^{\log_2 n}}\right]$$

$$= n\left[1 + 3\cdot\frac{1}{2} + 3^2\cdot\frac{1}{2^2} + 3^3\cdot\frac{1}{2^3} + \cdots + \frac{3^{\log_2 n}}{2^{\log_2 n}}\right]$$

$$= \Theta\left(n\left(\frac{3}{2}\right)^{\log_2 n}\right)$$
$$= \Theta(n \cdot n^{\log_2 \frac{3}{2}})$$
$$= \Theta\left(\frac{n(n^{\log_2 3})}{n^{\log_2 2}}\right)$$
$$= \Theta(n^{\log_2 3})$$

So doing work at each division
= doing work at these leaves

$$S = 1 + r^1 + r^2 + \cdots + r^n$$
$$rS = r + r^2 + r^3 + \cdots + r^{n+1}$$
$$rS - S = r^{n+1} - 1$$
$$S = \frac{r^{n+1}-1}{r-1} = \Theta(r^n) \text{ if } r > 1$$

comparing with fixed r

For n a power of b.

$$T(n) = \begin{cases} aT(\frac{n}{b}) + n^k, & n > 1 \\ 1, & n = 1 \end{cases}$$

$$\sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^k = \left[\sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i\right] n^k$$

$\boxed{\frac{a}{b^k} > 1}$ $\qquad \frac{a}{b^k} = 1 \qquad$ $\boxed{\frac{a}{b^k} < 1}$

To be continued.

Tutorial

For n a power of b

$$T(n) = \begin{cases} aT(\frac{n}{b}) + n^k, & n>1 \\ 1 & , n=1 \end{cases}$$

- $f \in \Theta(n^k)$

  $\exists B, \exists c_1, c_2 . \forall n \geq B, c_1 \cdot n^k \leq f(n) \leq c_2 n^k$

  $\downarrow$ n sufficiently large, $f(n)$ is bounded (sandwiched)

$$\left.\begin{aligned} T(n) &\leq c_2 \cdot n^k \sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i \\ T(n) &\geq c_1 \cdot n^k \sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i \end{aligned}\right\} \Rightarrow \text{Ok}$$, just take $f(n) = n^k$

The assignment asks you to compute this part:

$$\left(\frac{a}{b^k}\right)^i$$

$L = \{0^n 1^n \mid n \in \mathbb{N}\}$

X defined by:

- $\varepsilon \in X$
- $\forall s,t \in \{0,1\}^*$ · $s \in X \Rightarrow 00s11 \in X$
- $s01t \in X \Rightarrow st \in X$

$X \subseteq L$ Intuition for $L \subseteq X$

0, 1 Apply rule 3 in reverse: 0011

Then apply rule 2 in reverse to get ε.

Proof of $X \subseteq L$

By structure induction on def'n of X

Let $s \in X$ be arbitrary

WTS $s \in L$

① s, $\varepsilon \in X$ by def'n

② s is generated by 2nd Rule.

So $s = 00s'11$ for some $s' \in X$

s' generated before, $s \Rightarrow$ IH applies $s' \in L$

$\exists n \in \mathbb{N}, s' = 0^n 1^n$

Then $s = 0^{n+2} 1^{n+2} \in L$ ✓

③ s is generated by 3rd rule, so

$\exists s', t' \in \{0,1\}^*$ s.t. $s = s'01t'$

IH applies to $s'01t'$

$s'01t' \in L$ so $\exists n \in \mathbb{N}$ $s'01t' = 0^n 1^n$

Then $s't' = 0^{n-1} 1^{n-1} \in L$ (and $n \geq 1$ since $|\underbrace{s'01t'}_{\text{length}}| \geq 2$)