Today – conclusion of last day's example
– discuss the extreme point theorem
– §1.5

Eg (concluded) To show that $\begin{bmatrix}0\\0\end{bmatrix}$ is the only extreme point of $\{\begin{bmatrix}x\\y\end{bmatrix}$ s.t. $x\geq 0, y\geq 0\} = S$

For $\begin{bmatrix}x\\y\end{bmatrix}$ in S so that either $x>0$ or $y>0$

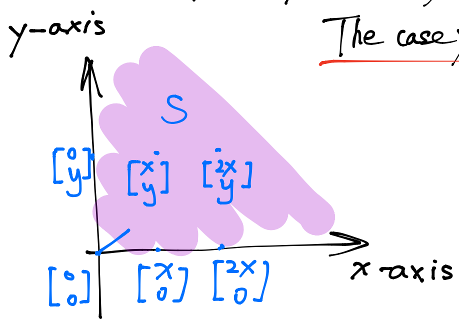
In case $x>0$:

Let $\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$ and let $\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}2x\\y\end{bmatrix}$

Both are in S and $\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{2}\begin{bmatrix}x_1\\y_1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}x_2\\y_2\end{bmatrix}$

Yet $\begin{bmatrix}x\\y\end{bmatrix}\neq\begin{bmatrix}x_1\\y_1\end{bmatrix}$ and $\begin{bmatrix}x\\y\end{bmatrix}\neq\begin{bmatrix}x_2\\y_2\end{bmatrix}$ (because $x>0$).

The case $y>0$ is similar



The extreme point theorem (Theorem 1.7)

Let S be the feasible region of a linear programming problem.

① If S is non-empty and bounded, the problem has an optimal solution at an extreme point of S.

This is the contrapositive of ③ If the problem has an optimal solution then either S is empty or S is unbounded.

② (correction): If S has an extreme point, and S is unbounded but the problem has an optimal solution, then it has an optimal solution at an extreme point.

§1.5 Basic Solutions

Consider the system $Ax=b$ where A is an $m\times n$ matrix having columns $A_1,\cdots,A_n$, $x\in\mathbb{R}^n$, and $b\in\mathbb{R}^m$

There is a basic solution having basic variables $x_{i_1},\cdots,x_{i_m}$ if and only if $A_{i_1},\cdots A_{i_m}$ are linearly independent.

Defn: The basic solution having basic variables $x_{i_1},\cdots,x_{i_m}$ is (provided $A_{i_1},\cdots,A_{i_m}$ are linearly independent). the unique solution of $Ax=b$ where $x_j=0$ when $j\neq i_1,\cdots,j\neq i_m$.

Defn: The x_j, when $j\neq i_1,\cdots,j\neq i_m$ are non-basic variable.

Remark: Non-basic variables are always 0.

Basic variables can be positive, negative, or 0.

Definition: Consider the canonical constraints $\begin{array}{c}Ax=b\\ x\geq 0\end{array}$

A basic, positive solution of this system is a basic solution of $Ax=b$, where $x\geq 0\in\mathbb{R}^n$

Theorem: Let S be the solution set of $\begin{array}{c}Ax=b\\ x\geq 0\end{array}$

x is extreme in S if and only if x is a basic feasible positive solution for the constraints.

Note: Theorem 1.8: "if"

Theorem 1.9: "only if"