APM462H1S, Winter 2014, About the Final
second installment

The Final Exam for APM462 will take place on Tuesday April 22, 9am-noon, in EX 310.

A list of topics to be covered on the Exam can be found in the earlier document, **About the Final, first installment**, together with a good deal of other material.

What kind of questions to expect?

to repeat: in general, it is the instructor's job to design test questions in such a way that a well-prepared student can solve them in the available time. So if the instructor does his job correctly, you will not be asked to find the eigenvalues and eigenvectors of a 6 by 6 matrix or to minimize a complicated function of 10 variables.

A large portion of the test will consist of routine questions designed to test your grasp of basic material. Some practice questions appear below, on topics not covered in the first installment.

What *other* kinds of questions to expect?

There will also certainly be some questions that are not as straightforward as most of the ones above, and should require a deeper understanding of the material.

The best way to prepare for these is to understand the material as well as possible.

There will also probably be some proofs. Some of these may be quite easy, unless you are a person who finds *all* proofs difficult or impossible.

practice questions

Calculus of variations:

- Find a function $x(\cdot)$ that minimizes
$$I[x(\cdot)] = \frac{1}{2}\int_0^1 (x'(t) - t^2)^2$$
in the set
$$\mathcal{A} := \{x(\cdot) \in C^2([0,1]) \ : \ x(0) = 0 = x(1) = 0\}.$$

- Let
$$I[x(\cdot)] = \int_0^1 \frac{1}{2}(x'(t)^2 + x(t)^2)\,dt - \int_0^1 f(t)x(t)\,dt.$$
and define
$$\mathcal{A} := \{x(\cdot) \in C^2([0,1]) \ : \ x(0) = x^0 x(1) = x^1\},$$
Assume that $x(\cdot)$ minimizes I in $\mathcal{A}$. (Thus, $x(\cdot) \in \mathcal{A}$ and $I[x(\cdot)] \le I[y(\cdot)]$ for all $y(\cdot) \in \mathcal{A}$.)
a. Write down the equation that is satisfied by $x(\cdot)$.
b. Assume that $x^0 = 1$, $x^1 = e^2$ and $f(t) = -3e^{2t}$. Find an explicit expression for the minimizer $x(t)$. (*Hint: compute* $-f'' + f$ *to get inspired.*)

- Define

$$I[x(\cdot)] := \int_{-\infty}^{\infty} x'(t)^2 \ dt$$
$$J[x(\cdot)] := \int_{-\infty}^{\infty} x(t)^2 \ dt$$

(where the integrals are actually improper integrals, so that for example $\int_{-\infty}^{\infty} x'(t)^2 \ dt$ means $\lim_{M\to\infty} \int_{-M}^{M} x'(t)^2 \ dt \in [0, +\infty]$.)

a. Given any function $x(\cdot)$ and any $\lambda > 0$, let $x_\lambda(\cdot)$ denote the function defined by

$$x_\lambda(t) = x(\lambda t).$$

Use a change of variables to show that

$$I[x_\lambda(\cdot)] = \lambda^p I[x(\cdot)], \qquad J[x_\lambda(\cdot)] = \lambda^q J[x(\cdot)]$$

for some constants p and q that do not depend on $x(\cdot)$ or on λ.

b. Prove that a minimizer does not exist for the problem:

$$\text{minimize} \ \ I[x(\cdot)] \qquad \text{subject to } J[x(\cdot)] = 1.$$

Systems of ODEs and matrix exponentials. As mentioned earlier, this is not a topic that we are particularly interested in for its own sake, in this class, but it is an important tool in linear control problems.

In particular, it is a good idea to remember

$$\text{if } M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{then } e^{tM} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

$$\text{if } M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \text{then } e^{tM} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}.$$

These formulas are both easy to derive *if* you remember formulas for the power series of $\sin t, \cos t, e^t$, or if you remember certain aspects of ODEs that we have not reviewed in this class. If you do not remember these things, the formulas are not so easy to derive. (But if you remember the formula for a matrix exponential, you should certainly also remember the power series for e^t, since it is exactly the same formula, for a "1×1 matrix" *i.e.* a number.)

Some matrix exponentials are straightforward to compute from the definition. This is the case for the examples in the problems that appear below.

Note that you are not so likely to see a problem that says, like the ones below, "find e^{tM}" or "use the formula for e^{tM} to write down the solution of a system of equations." Instead, you may have to do things like this in order to solve other problems, such as "Solve the control problem ..."

- Let

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

a compute e^{tM}.

b. Write down a formula for the solution of

$$\frac{d}{dt} x(t) = Mx(t) + \begin{pmatrix} 0 \\ \alpha(t) \\ 0 \end{pmatrix}.$$

- Similar questions with

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

- A somewhat harder example is

$$M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

controllability

- Consider the equation

$$\frac{d}{dt}x(t) = Mx(t) + N\alpha(t), \qquad \alpha(t) \in [-1,1] \text{ for all } t$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Is this system controllable?

- Same question with different M and N, for example

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

or

$$M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

or

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$M = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 4 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{or } N = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Or instead, same system as one of the above, or a similar one, but different question: is the origin in the interior of the "reachable set" $\mathcal{C}$, where

$$\mathcal{C} := \{x^0 \in \mathbb{R}^n : \text{ there exists a control steering the system from } x^0 \text{ to the origin in finite time}\} \text{ ?}$$

Pontryagin's Maximum Principle

- For any of the examples about controllability above, one could write down the same system of equations and ask: Suppose $\alpha^*(\cdot)$ is an optimal control, steering the system from x^0 to the origin in the minimum possible time. What can we deduce about $\alpha^*(t)$ from the Pontryagin maximum principle?

- For example: consider the equation

$$\frac{d}{dt}x(t) = Mx(t) + N\alpha(t), \qquad \alpha(t) \in [-1,1] \text{ for all } t$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Suppose $\alpha^*(\cdot)$ is an optimal control, steering the system from x^0 to the origin in the minimum possible time. What can we deduce about $\alpha^*(t)$ from the Pontryagin maximum principle?

- Or: consider the equation

$$\frac{d}{dt}x_1(t) = x_2(t) + \alpha_1(t),$$
$$\frac{d}{dt}x_2(t) = -x_1(t) + \alpha_2(t),$$

with

$$\alpha_1^2(t) + \alpha_2^2(t) \leq 1 \quad \text{for all } t. \tag{1}$$

(Note that this is different from the usual constraint, which would be $\alpha(t) \in [-1,1]^2$ for all t.)

Suppose $\alpha^*(\cdot)$ is an optimal control, steering the system from x^0 to the origin in the minimum possible time.

a. What can we deduce about $\alpha^*(t)$ from the Pontryagin maximum principle?

(It is a fact that the Pontryagin maximum principle still holds with the constraint (1), although it has to be modified a little bit. There's only one reasonable choice for the correct modification, and you can probably figure it out.)

b. What is the optimal control? Could you have figured this out without using the Pontryagin maximum principle?