Lecture 2

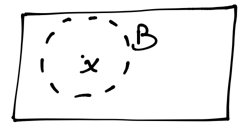
(X, τ)

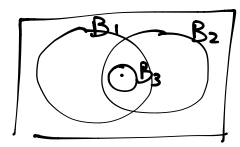
X: set; τ: open subsets of X

Def'n: Let X be a set, a collection $\mathcal{B} \subseteq P(X)$ is called a basis (for some top. on X) if:

i). $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$, and
($\mathcal{B}$ cover X)

ii). $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}$
Such that $x \in B_3 \subseteq B_1 \cap B_2$
($\mathcal{B}$ is directed)





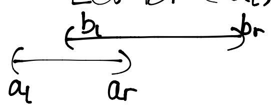
e.g.1 Let $\mathcal{B} = \{(a,b) : a, b \in \mathbb{R}\}$ be the collection of open intervals in $\mathbb{R}$.

Check that this is a basis:

i) Fix $x \in \mathbb{R}$. Note $x \in (x-\pi, x+\pi)$ (also $x \in (x-1, x+1)$)

ii) Fix $B_1, B_2 \in \mathcal{B}$. We'll show their intersection is actually an element of $\mathcal{B}$

Let $B_1 = (a_l, a_r)$, $B_2 = (b_l, b_r)$

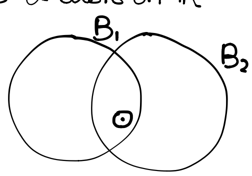


Note $B_1 \cap B_2 = (\max\{a_l, b_l\}, \min\{a_r, b_r\})$

If $x \in B_1 \cap B_2$, then this open interval is not empty, and in $\mathcal{B}$.

e.g.2 Let $\mathcal{B} = \{B_\varepsilon(x) : x \in \mathbb{R}^2\}$

This is a basis on $\mathbb{R}^2$

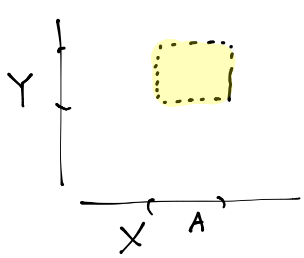


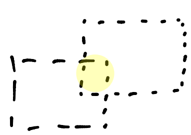
e.g.3 Let X be a set. $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for some top on X.

eg4. Let (X, τ) and $(Y, \mathcal{U})$ be top spaces.

Let $\mathcal{B} = \tau \times \mathcal{U} = \{A \times B : A \in \tau, B \in \mathcal{U}\}$.

This is a basis on $X \times Y$





Idea: A basis is almost a topology, we're just missing unions.

<u>Def'n</u>: Let $\mathcal{B}$ be a basis on a set X.
Let $\mathcal{T}_{\mathcal{B}} = \{\bigcup \mathcal{C} : \mathcal{C} \subseteq \mathcal{B}\}$
(be the collection of all unions of elements of $\mathcal{B}$)
$\mathcal{T}_{\mathcal{B}}$ is called the topology generated by $\mathcal{B}$.

e.g. if $X = \mathbb{N}$. $\mathcal{C} = \{\{1,2,3\}, \{3,100\}, \{9,3\}\}$, then $\bigcup \mathcal{C} = \{1,2,3\} \cup \{3,100\} \cup \{9,3\}$
$= \{1,2,3,9,100\}$

so $\mathcal{C} \subseteq P(\mathbb{N})$

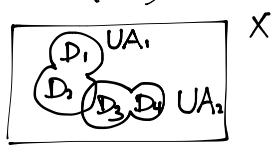
<u>Proof that $\mathcal{T}_{\mathcal{B}}$ is a topology</u>

1) $\emptyset \in \mathcal{T}_{\mathcal{B}}$, because we included it. since $\mathcal{B}$ covers X, $\bigcup \mathcal{B} = X$
2) [closed under unions]
Let $\{\bigcup A_\alpha : \alpha \in I\}$ be a collection of elements of $\mathcal{T}_{\mathcal{B}}$.

$$\bigcup_{\alpha \in I} (\bigcup A_\alpha) = \bigcup (\bigcup_{\alpha \in I} A_\alpha)$$

each $A_\alpha \subseteq \mathcal{B}$, so $\bigcup_{\alpha \in I} A_\alpha \subseteq \mathcal{B}$, so $\bigcup(\bigcup_{\alpha \in I} A_\alpha) \in \mathcal{T}_{\mathcal{B}}$

Pic: Let $I = \{1,2\}$



$A_1 = \{D_1, D_2\}$
$A_2 = \{D_3, D_4\}$

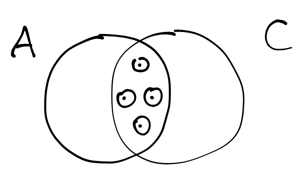
$$\bigcup_{\alpha \in I} A_\alpha = \{D_1, D_2, D_3, D_4\}$$

$$\bigcup(\bigcup_{\alpha \in I} A_\alpha) = D_1 \cup D_2 \cup D_3 \cup D_4$$

3). [$\mathcal{T}_{\mathcal{B}}$ is closed under finite intersections]

By induction, we'll only need to show $\mathcal{T}_{\mathcal{B}}$ is closed under intersection of two things.
Let $\bigcup \mathcal{A}$ and $\bigcup \mathcal{C}$ be elements of $\mathcal{T}_{\mathcal{B}}$.

Note $(\bigcup \mathcal{A}) \cap (\bigcup \mathcal{C}) = \bigcup \{A \cap C, A \in \mathcal{A}, C \in \mathcal{C}\}$
We know $\mathcal{T}_{\mathcal{B}}$ is closed under unions so it is enough to show that $A \cap C \in \mathcal{T}_{\mathcal{B}}$ (Both A & C are in $\mathcal{B}$)



For each $x \in A \cap C$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq A \cap C$

Note that $\bigcup_{x \in A \cap C} B_x \subseteq A \cap C$ and $A \cap C \subseteq \bigcup_{x \in A \cap C} B_x$

So $A \cap C = \bigcup_{x \in A \cap C} B_x \in \mathcal{T}_{\mathcal{B}}$

$\blacksquare$

e.g.1 If $\mathcal{B} = \{(a,b) : a, b \in \mathbb{R}\}$ then $\tau_{\mathcal{B}}$ is the usual topology on $\mathbb{R}$

e.g.2 If $\mathcal{B} = \{\{x\} : x \in X\}$, then $\tau_{\mathcal{B}}$ is the discrete topology.

e.g.3 $\mathcal{B} = \mathbb{R}_{usual} \times \mathbb{R}_{usual}$. What $\tau_{\mathcal{B}}$ on $\mathbb{R}^2$?

This is the usual topology on $\mathbb{R}^2$!

Pic