FINANCIAL MATHEMATICS
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LECTURE NOTES WEEK 7

MEASURING INVESTMENT PERFORMANCE

It is often necessary to be able to measure the investment performance of a fund over a period.

Fund value will go up or down as a result of changes in:

- Income generated by the fund – interest payments, dividends, rental payments earned by the fund's assets.
- Changes in market value (capital gains/losses) – the prices investors are prepared to pay for assets in the fund will vary from day to day.
- "New money"- extra money paid into the fund that was not generated by the fund itself (withdrawals are negative "new money").

In investment performance calculations it is important to distinguish between money generated by the fund and "new money". Otherwise, there is a danger of double counting or under counting.

MONEY-WEIGHTED RATE OF RETURN (MWRR)

The internal rate of return is sometimes referred to as the money-weighted rate of return for the transaction. This terminology is more likely to be used in the context of a transaction of one year duration, when measuring the performance of an investment fund.

"The money-weighted rate of return is the interest rate satisfying the equation of value incorporating the initial and final fund values and the intermediate cash flows."

Note that the intermediate cash flows in the equation of value used in calculating the MWRR only takes account of new money. Any cash flows generated as investment income by the fund itself must be ignored. This is because the equation of value we are setting up is to value the rate of return that generated the income and changes in market value.

EXAMPLE

The market value of a small pension fund's assets was \$2.7m on 1 January 2012 and \$3.1 m on 31 December 2012. During 2012 the only cash flows were:

- Bank interest and dividends totaling \$125,000 received on 30 June
- A lump sum retirement benefit of \$75,000 paid on 1 May
- A contribution of \$50,000 paid to the fund on 31 December

Show that the MWRR is 16%.

Solution
The only "new money" is the lump sum retirement benefit of $75,000 and the contribution of $50,000.

The bank interest and dividends lead to the growth in the value of the fund, which is what the MWRR i is measuring. These payments **are already "absorbed" in the value of i.** Including them as cash flows in the equation would result in double counting. Cash flows in respect of new money, on the other hand, are not reflected in the value of i, and so these must be included as extra terms in the equation of value.

The equation of value using compound interest (in $'000) is:

$$3{,}100 = 2{,}700(1+i) - 75(1+i)^{8/12} + 50$$

Evaluating the LHS at interest rates either side of 16%, we find:
$i = 15.95\% \Rightarrow LHS = 3{,}097.9 < 3{,}100$
$i = 16.05\% \Rightarrow LHS = 3{,}100.5 > 3{,}100$

Therefore, the MWRR is 16.0% (to the nearest 0.1%).

TIME-WEIGHTED RATE OF RETURN (TWRR)

The MWRR is sensitive to the amounts and timing of the net cash flows. If, for example, we are assessing the skill of a fund manager, the MWRR is not ideal, as the fund manager does not control the timing or amount of the cash flows – he or she is merely responsible for investing the positive net cash flows and realising cash from the sale of assets to meet the negative net cash flows. For example, if a large amount of money was invested just after a market boom, the MWRR is likely to be lower than if the money was invested just before the boom. This is not a reflection of the skill of the fund manager, simply a result of the timing of the cash flows.

An alternative to the MWRR for measuring investment fund performance, which tries to eliminate the sensitivity to the amounts and timing of the net cash flows, is the **time-weighted rate of return (TWRR).**

The TWRR is found by calculating "growth factors" reflecting the change in the value of the fund between the times of consecutive cash flows, then by combining these growth factors to come up with an overall rate of return for the whole period.

"The time-weighted rate of return is found from the product of the growth factors between consecutive cash flows"

The time-weighted rate of return for a fund over a one-year period is found by first identifying the time points at which cash flows occur in the fund. These are the time points at which new contributions are added, or withdrawals are made.

Assume that these time points are: $0 < t_1 < t_2 < ... < t_n = 1$. For each successive time interval $[t_{k-1}, t_k]$, a periodic rate of return i_k, and corresponding growth factor $G_k = 1 + i_k = \frac{B_k}{A_k}$ is calculated where, A_k is the fund value at time t_{k-1} after all transactions are completed, and B_k is the fund value at time t_k after interest is credited but just before the cash flow due at time t_k occurs. The aim of this is to identify the change in fund value due to interest credited and capital value change only, before any new money is added.

The **growth factors are compounded over the full-year** to produce the annual growth factor
$G = G_1 G_2 ....G_n = (1+i_1)(1+i_2)...(1+i_n) = (1+TWRR)$

A disadvantage of the TWRR is that it requires the fund values at all the cash flow dates.

EXAMPLE
Using the same example above, as that used for the MWRR, find the TWRR. Use the additional information that the fund value at 30 April was \$3m.

Solution
First we need to identify the time points at which cash flows occur in the fund. In this example we know the final fund value is \$3.1m. The timing of "new money" is:

1 May	\$75,000 lump sum benefit paid
31 December	\$50,000 contribution paid

The progress of the fund (in \$'000) was as follows:

1 January to 30 April	Fund value increased from 2,700 to 3,000
1 May	Cash flow of –75
1 May to 30 December	Fund value increased from 3,000-75 = 2,925 to 3,100-50 = 3,050
31 December	Cash flow of +50, taking fund value to 3,100

So, during the period from 1 January to 30 April the fund grew by a factor of:
$$\frac{3,000}{2,700} = 1.111$$
During the period from 1 May to 30 December the fund grew by a factor of
$$\frac{3,100-50}{3,000-75} = 1.043$$

Therefore, the TWRR for the year is $(1.111)(1.043) - 1 = 15.9\%$

FIXED INTEREST SECURITIES

Governments and corporations will often wish to raise or borrow money from investors to fund their spending plans. They may wish to repay the money in the near future by issuing short-term loans, or at a later date by issuing medium-term or long-term loans.

BILLS

Government **bills** are **short-dated** securities issued by governments to fund their short-term spending requirements. They are issued at a discount price P and redeemed at par (the face value of the bill) with no coupons (interest payments). Three months is a typical term for a government bill.

The yield on government bills is typically quoted as a **simple annual rate of discount** for the term of the bill.

EXAMPLE

A 91-day bill with a par-value of $100 is quoted as being offered at a discount of 12% per annum.

Assuming that the discount rate quoted is a simple annual rate of discount, this means that the initial investment required to buy the bill would be approximately:
$\frac{91}{365}d = \frac{91}{365}0.12 \cong 3\%$ less than the payment received 91 days later:

$$P = \$100(1 - \frac{91}{365}d) = \$100(0.97) = \$97$$

We will not be considering bills further in this course. Instead, we will spend some time developing methods for determining the price and yield of bonds.

BONDS

To borrow large amounts over terms generally greater than one year, governments and corporations can issue bonds.

A **bond** is a security in which the issuer (borrower or debtor) promises to repay to the investor (lender) the amount borrowed plus interest over some specified period of time.

Bonds generally have a fixed term with return of principal plus the last interest payment at maturity of the bond. Interest payments are also called **coupons** and bonds may be split into two categories based on the regularity of these payments, **zero-coupon bonds** or **coupon-paying bonds**.

The initial purchaser of a bond might not retain ownership for the full term to maturity, but might sell the bond to another party. There is a very active and liquid bond secondary market in which bonds are bought and sold.

There are various parameters associated with a bond:

P - the price of a bond.

F - the face amount (also known as the amount borrowed, par value, principal or nominal amount).

r - the effective interest rate per coupon period used in determining the amount of coupon. The most common frequency for bond coupons is half-yearly. Coupon rates are usually quoted as nominal rates of interest convertible half-yearly. eg. An 8% bond with half-yearly coupons means the effective half-yearly coupon rate is $r = \frac{8\%}{2} = 4\%$. The coupon amount is always a percentage of the face amount, that is $F \times r$.

C - the redemption amount. ie. the amount of money paid at a redemption date to the holder of the bond. Often $C = F$ (that is, "redeemed at par").

n - the number of coupon payments until maturity.

i - the annual redemption yield of a bond (see below)

Note the coupon rate and the redemption rate are two different parameters.

You should be aware that there are different "yields" associated with a bond.

1. The annual "**redemption yield**" is the internal rate of return or the effective annual rate of interest. The half-yearly redemption yield is the 6-month effective interest rate.
2. The "**nominal yield**" is the annual redemption yield expressed as a nominal rate of interest per annum usually convertible half-yearly ($i^{(2)}$). Yields are sometimes expressed in this way since coupons are usually payable half-yearly.
3. The "**running yield**" (or flat yield) is the ratio of the coupon rate per annum to the original price of the bond per unit nominal. For example if a $100 par value bond with coupons of $9 per annum is selling for $90, then the current yield on the bond is $\frac{0.09}{0.9}$ =10% per annum.

EXAMPLE

An 8% three-year bond with par-value $100,000, has coupons paid half-yearly, where the amount of interest paid at every coupon payment date is:

$$Fr = \$100,000\left(\frac{0.08}{2}\right) = \$4,000$$

Standard naming conventions dictate that 8% is a nominal rate convertible half-yearly, so the effective half-yearly coupon rate is $r = \frac{8\%}{2} = 4\%$.

For this bond, the investor will receive $4,000 at the end of each half-year for three years. Also assume the bond is redeemable at par, so the investor receives the return of principal (=$100,000) in three years' time.

The cashflows received by the investor are given below:

t	Principal received	Coupon payments	Total received
0.5		$4,000	$4,000
1		$4,000	$4,000
1.5		$4,000	$4,000
2		$4,000	$4,000
2.5		$4,000	$4,000
3	$100,000	$4,000	$104,000

If fixed interest bonds are held until redemption, the monetary amounts of coupons and principal are known and fixed. To this extent, the cash flows and returns are known at the outset.

The return achieved over the period up to the maturity date of the bond will, however, be uncertain for various reasons:

(a) The investor may need to reinvest the coupon payments. The terms that will be available for reinvestment are not known at the outset.

(b) For an investor who plans to sell before redemption, the ultimate sale price is not known at the outset.

(c) The real return (ie in excess of inflation) is uncertain. If inflation turns out to be higher than expected at outset, the real returns from fixed interest bonds will be lower than originally anticipated. Some governments therefore issue index-linked bonds that provide interest and redemption payments that are linked to an inflation index.

Bonds issued by financially and politically stable governments are virtually risk-free and are a safe investment option. There are agencies that rate the risk of default on interest and principal payment associated with a bond issuer. The purchaser of a bond will take into account the level of risk associated with the bond when determining its value. For example, if two bonds have exactly the same cash flows, the bond that has the highest default risk will generally sell for a lower price.

The main differences between corporate and government bonds are:

- Corporate bonds are usually less secure than government bonds.
- Corporate bonds are usually less marketable (liquid) than government bonds.

Investors require a greater yield on corporate bonds than on corresponding government bonds to compensate for the lower security and marketability.

DETERMINATION OF BOND PRICES

The price, P, to be paid to achieve a nominal yield of $i^{(2)}$ per annum convertible half-yearly, is equal to the present value of the interest and principal payments:

$$\boxed{P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n}$$

where $j = \dfrac{i^{(2)}}{2}$ is the effective half-yearly yield (assuming coupons are to be paid half-yearly) and other parameters are as discussed previously:

If an annual redemption yield i is given (instead of the nominal yield convertible half-yearly), then the equivalent half-yearly redemption yield can be found by: $j = (1+i)^{1/2} - 1$.

Alternatively, the equation for the price can be written using an interest conversion period of one year. If coupons are payable half-yearly, then the price is:

$$P = 2Fr \cdot a^{(2)}_{\overline{n/2}|i} + C \cdot v_i^{n/2}$$

Bonds with the same redemption price as face value are “redeemable at par”. In this case $C = F$ and the price is:

$$P = Fr \cdot a_{\overline{n}|j} + F \cdot v_j^n$$

If you are not told otherwise, you should assume the bond is redeemable at par.

EXAMPLE
What price should an investor who requires an effective redemption yield of 10% per annum pay for \$10,000 nominal of a six-year bond with half-yearly coupon payments of 13% per annum? The first coupon is due in six months' time and the bond is redeemable at par.

Solution
Since the bond is redeemable at par, $C = \$10,000$.

Since the coupon payments are 13% per annum, the half-yearly coupon payments are $\frac{13\%}{2} = 6.5\%$ of 10,000. There are $n = 6 \times 2 = 12$ coupon payments.

We are given an effective annual yield, so we can find the half-yearly effective rate of interest: $j = (1.1)^{1/2} - 1$

The price P is:

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n = 10{,}000(0.065)a_{\overline{12}|j} + 10{,}000 v_j^{12} = \$11{,}445$$

Alternatively,

$$P = 2Fr \cdot a^{(2)}_{\overline{n/2}|i} + C \cdot v_i^{n/2} = 20{,}000(0.065)a^{(2)}_{\overline{6}|i} + 10{,}000 v_i^6 = \$11{,}445 .$$

BOND PRICES BETWEEN COUPON DATES

The formula previously introduced can be used to a price bond at its issue date or some time later immediately after a coupon is paid. In practice bonds are traded daily, so we need to consider the valuation of a bond at a time between coupon dates.

Suppose that we wish to find the price P_t of a bond at time t, where $0 \le t \le 1$, with t measured as the time since the last coupon payment as a portion of the total coupon period.

If P_0 is the value of the bond just after the last coupon, then the value of the bond at time t is:

$$P_t = P_0(1 + j)^t$$

where j is the effective yield over the coupon period.

EXAMPLE
What price should an investor who requires a redemption yield of 10% per annum pay for $10,000 nominal of a six-year bond with half-yearly coupon payments of 13% pa? The first coupon is due in *one* months' time.

Solution
The price at the issue date was found in the previous example: $P = \$11{,}445$
The value of the bond one month before the first coupon is equivalent to the bond price five months after the issue date. t in this example is, therefore, equal to 5/6. The price is:

$$P_t = \$11{,}445(1+j)^{5/6} = \$11{,}445(1.1^{1/2})^{5/6} = \$11{,}909$$

(since $j = (1.1)^{1/2} - 1$)

ALLOWANCE FOR INCOME TAX ON COUPONS

Since tax payments represent negative cashflows for an investor, the price an investor should pay to obtain a given yield can be found by:

Price = PV income – PV tax

If the investor is liable to income tax at rate t_I on coupon payments, the price to be paid for a net redemption yield of i per annum is:

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n - \left(Frt_I\right) a_{\overline{n}|j}$$

which can be written as:

$$\boxed{P = Fr(1-t_I) \cdot a_{\overline{n}|j} + C \cdot v_j^n}$$

This formula assumes that tax is payable at the same time as income is incurred.

EXAMPLE
What price should an investor who requires a net redemption yield of 10% per annum pay for $10,000 nominal of a six-year bond with half-yearly coupon payments of 13% pa? The next coupon is due in six months' time. Assume that the investor pays income tax at 33% at the same time that payments are received, and that the bond is redeemable at par.

Solution
The amount of tax payable on each coupon will be 33% x 10,000 x 0.065. This means that each net coupon payment received will be 67% x 10,000 x 0.065.

$$P = Fr(1-t_1)\cdot a_{\overline{n|}j} + C\cdot v_j^n$$
$$= 10{,}000(0.065)(0.67)a_{\overline{12|}j} + 10{,}000v_j^{12} = \$9{,}531$$

Comparing this with the earlier example without income tax, an investor subject to tax must buy at a lower price in order to obtain the same yield.

ALLOWANCE FOR CAPITAL GAINS TAX

We just showed that if the investor is liable to income tax at rate t_I, but is not liable to capital gains tax, the price to be paid for a redemption yield of i per annum is

$$P = Fr(1-t_1)\cdot a_{\overline{n|}j} + C\cdot v_j^n$$

If the investor is also subject to tax at rate t_C on the capital gains, then let the new price payable be P'.

If there is no capital gain - that is, if the redemption amount C is less than the purchase price, then there is no additional tax liability, and the price paid is simply P.

If there is a capital gain, then at the redemption date of the contract there is an additional liability of $t_C(C-P')$. This additional liability will have the effect of reducing the price in order to maintain the same net yield.

One way to determine whether or not there is a capital gain is to calculate P and compare it to C.

If $P \ge C$ then there is no capital gain, so the price is $P' = P = Fr(1-t_I)\cdot a_{\overline{n|}j} + C\cdot v_j^n$

If $P < C$ then there is a capital gain of $(C-P')$ taxed at t_C and the price is:

$$P' = Fr(1-t_I)\cdot a_{\overline{n|}j} + C\cdot v_j^n - t_C(C-P')v_j^n = P - t_C(C-P')v_j^n$$

EXAMPLE
What price should an investor who requires a redemption yield of 10% per annum and pays 33% tax on income and capital gains pay for $10,000 nominal of a six-year stock with half-yearly coupon payments of 13% pa.

Solution
The coupon payments will be subject to 33% income tax.
The half-yearly redemption yield is $j = (1.1)^{1/2} - 1$

From the previous example, we know that when capital gains tax is not payable, the price will be:

$$P = Fr(1-t_I)\cdot a_{\overline{n}|j} + C\cdot v_j^n = 10{,}000(0.065)(0.67)a_{\overline{12}|j} + 10{,}000v_j^{12} = \$9{,}531$$

If capital gains tax is payable, then the price paid will be lower than \$9,531, since the additional tax liability will have the effect of reducing the price in order to maintain the same yield.

Since \$9,531 < \$10,000 $(P < C)$ there will be a capital gain equal to $10{,}000 - P'$, which will be taxed at 33%, payable when the capital is repaid. The price P' is

$$\begin{aligned} P' &= P - t_C(C - P')v_j^n \\ &= 9{,}531 - 0.33(10{,}000 - P')0.56447 \\ &= 7{,}668 + 0.18628P' \\ &= \$9{,}423 \end{aligned}$$

EXAMPLE
Find the price, as at 30 April 2004, of a \$100,000 bond which matures at 30 June 2015. The bond pays coupons half yearly at 11% p.a., with the next coupon due on 30 June 2004. Tax is payable on coupons and capital gains at 30% and the investor requires a net yield of 9% p.a. convertible half yearly.

Solution
First we must calculate the price at 31 December 2003 (just after the coupon paid at that date), allowing for income tax but not capital gains tax:

$$P_{12/03} = Fr(1-t_I)\cdot a_{\overline{n}|j} + C\cdot v_j^n = 100{,}000(0.055)(1-0.3)a_{\overline{23}|0.045} + 100{,}000v_{0.045}^{23}$$
$$= \$90{,}803.95$$

Next we accumulate this forward to 30 April 2004:

$$P_{04/04} = P_{12/03} \times 1.045^{0.6667} = \$93{,}508.03$$

We can now observe that as $P < C$ there is a capital gain which must be allowed for.

$$P'_{04/04} = P_{04/04} - t_C(C - P'_{04/04})v_j^n$$
$$= 93{,}508.03 - 0.3(100{,}000 - P'_{04/04})v_{0.045}^{22.333}$$
$$P'_{04/04} = \$92{,}687.14$$