LN 11.1.

Ex: Sol: ① A: $PV_{(A)} = 100 \cdot (1+i)^{-11}$

$$C_A(i) = \frac{PV_A''(i)}{PV_A(i)} = \frac{\cancel{100} \cdot (-11) \cdot (-12) \cdot (1+i)^{\cancel{-13}\ -2}}{\cancel{100}\,\cancel{(1+i)^{-11}}} = \frac{132}{(1+i)^2}$$

$$\boxed{\begin{aligned} PV_A'(i) &= 100 \cdot (-11) \cdot (1+i)^{-12}. \\ PV_A''(i) &= 100 \cdot (-11) \cdot (-12) \cdot (1+i)^{-13}. \end{aligned}}$$

$$i = 10\% \Rightarrow C_A(i) = \frac{132}{(1+10\%)^2} = \boxed{109.1}$$

② B: $PV_B(i) = 9663 \cdot (1+i)^{-5} + 26910 \cdot (1+i)^{-20}.$

$$C_B(i) = \frac{PV_B''(i)}{PV_B(i)} = \frac{9663 \cdot (-5)(-6) \cdot (1+i)^{-7} + 26910(-20)(-21)(1+i)^{-22}}{9663(1+i)^{-5} + 26910(1+i)^{-20}}.$$

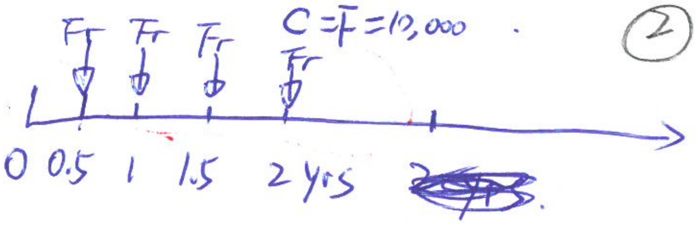
$$i = 10\% \Rightarrow C_B(0.1) = \boxed{153.7}.$$

③ C: $\boxed{P_C(i) = \frac{1}{i}} = (i)^{-1}$

$$C_C(i) = \frac{PV_C''(i)}{PV_C(i)} = \frac{(-1) \cdot (-2) \cdot (i)^{-3}}{(i)^{-1}} = \frac{2}{i^2}.$$

$$i = 10\% \Rightarrow C_C(0.1) = \boxed{200}.$$

Ex: $F = \$10{,}000$



$n = 2 \times 2 = 4$ coupon payments

$r = \frac{13\%}{2} = 6.5\%$.

$i^{(2)} = 6\% \Rightarrow j = 3\%$ per half year. $\Rightarrow \boxed{i = (1+j)^2 - 1 = ..}$

Sol: $\boxed{C(i)} = \dfrac{\sum_{k=1}^{n} C_{t_k} \cdot t_k \cdot (t_k+1) \cdot v_i^{(t_k+2)}}{PV}$

$$\boxed{PV(i) = \sum_{k=1}^{n} C_{t_k} \cdot v_i^{t_k}} \Rightarrow \boxed{PV''(i) = \sum_{k=1}^{n} C_{t_k} \cdot t_k \cdot (t_k+1) \cdot v^{t_k+2}}$$

$$\boxed{= \sum_{k=1}^{n} C_{t_k} \cdot (1+i)^{-t_k}} = \sum_{k=1}^{n} C_{t_k} (-t_k)(-t_k+1) \cdot (1+i)^{-t_k+2}$$

$$v_i^{0.5+2} = v_{0.03}^{(0.5+2)\times 2}$$

Fr

$$C(i) = \frac{\boxed{10{,}000 \times 6.5\%} \times \left(0.5 \times 1.5 \cdot v_{0.03}^{(0.5+2)\times 2} + 1 \times 2 \cdot v_{0.03}^{(1+2)\times 2} + 1.5 \times 2.5 \cdot v_{0.03}^{(1.5+2)\times 2} + 2 \times 3 \cdot v_{0.03}^{(2+2)\times 2}\right) + 10{,}000 \times 2 \times 3 \cdot v_{0.03}^{(2+2)\times 2}}{\left(10{,}000 \times 6.5\% \times a_{\overline{4}|0.03} + 10{,}000 \cdot v_{0.03}^{4}\right)}$$

$C(i) = 4.77254.$

Taylor's series.

Approximation.

$$\boxed{\frac{PV(i_0+\varepsilon)-PV(i_0)}{PV(i_0)} \simeq \varepsilon \cdot \boxed{\frac{PV'(i_0)}{PV(i_0)}} + \frac{\varepsilon^2}{2} \cdot \boxed{\frac{PV''(i_0)}{PV(i_0)}}}$$

$$= -\varepsilon \frac{\tau}{(1+i_0)} + \frac{\varepsilon^2}{2} \cdot C.$$

$$\Rightarrow \boxed{PV(i_0+\varepsilon) = PV(i_0) \times \left(1 - \varepsilon \frac{\tau}{(1+i_0)} + \frac{\varepsilon^2}{2} \cdot C\right)}.$$

per. half year

Ex: $P(j_0 = 10\%) = \$21.77$, $\tau = 6.56$ yrs, $C = 66$.

$P(j = 11\%) \cong$? (two-term Taylor series)

Sol: ~~PV~~ $i_0 = (1+j_0)^2 - 1 = 0.21.$

$PV(i_0) = \$21.77$, $i = (1+11\%)^2 - 1$

$\varepsilon = i - i_0 = 0.0221$

$$\Rightarrow PV(i_0+\varepsilon) = 21.77 \times \left(1 - 0.0221 \cdot \frac{6.56}{1+0.21} + \frac{0.0221^2}{2} \cdot 66\right)$$

$$= \$19.44.$$

Immunisation

$$PV_A - PV_L \geq 0 \quad \text{surplus.}$$

$$V_A > V_L$$

$$S(i) = V_A(i) - V_L(i) = PV_A(i) - PV_L(i).$$

$\boxed{\text{surplus}}$ →
① $S(i_0) = 0$
② $S(i_0+\varepsilon) \geq 0$
$\Rightarrow \boxed{\text{Immunised}}$

$\Updownarrow$

$$\boxed{\begin{array}{l} ① \ V_A(i_0) = V_L(i_0) \\ ② \ V_A(i_0+\varepsilon) \geq V_L(i_0+\varepsilon) \end{array}}$$

* $\boxed{\text{Redington Immunisation}}$

$$S(i_0+\varepsilon) = \boxed{S(i_0)} + \boxed{\varepsilon \cdot S'(i_0)} + \boxed{\frac{\varepsilon^2}{2} \cdot S''(i_0)} + \cdots$$

① $S(i_0) = 0. \iff V_A(i_0) = V_L(i_0).$

② $\varepsilon S'(i_0) = 0$

③ $\frac{\varepsilon^2}{2} S''(i_0) \geq 0.$

2 : $\Leftrightarrow V_A'(i_0) = V_L'(i_0)$

$$\overset{①}{\Longrightarrow} -\frac{V_A'(i_0)}{V_A(i_0)} = -\frac{V_L'(i_0)}{V_L(i_0)}$$

$$\boxed{\begin{array}{c} \nu_A = \nu_B \\ \hline \tau_A = \tau_B \end{array}}$$

$$\boxed{\tau = (1+i)\nu}$$

3 : $\underline{S''(i_0) \geq 0}$.

$V_A''(i_0) \geq V_L''(i_0)$.

$$\overset{①}{\Longrightarrow} \frac{V_A''(i_0)}{V_A(i_0)} \geq \frac{V_L''(i_0)}{V_L(i_0)}.$$

$$\Longrightarrow \boxed{C_A(i_0) \geq C_L(i_0)}.$$

Ex: Sol: \$P : 5-year zero coupon bond.

\$Q : 10-year zero coupon bond.

$$V_A(0.07) = P \cdot \nu_{0.07}^5 + Q\nu_{0.07}^{10} = 0.71299P + 0.50835Q$$

$$V_L(0.07) = 50{,}000 \cdot (\nu_{0.07}^6 + \nu_{0.07}^8) = 62{,}418$$

① . $V_A = V_L$

$$\Longleftrightarrow \quad 0.71299 \cdot P + 0.50835 \cdot Q = 62{,}418 \quad ①.$$

② $\upsilon_A = \upsilon_L$

$$\upsilon_A(0.07) = -\frac{V_A'}{V_A} = -\frac{\left(-5 \cdot P\, v_{0.07}^{6} - 10Q\, v_{0.07}^{11}\right)}{0.71299P + 0.50835Q}.$$

$$= \frac{3.3317P + 4.7509Q}{62{,}418}$$

$$\upsilon_L(0.07) = -\frac{\upsilon_L'}{V_L} = -\frac{50{,}000\left(-6 \cdot v_{0.07}^{7} - 8 \cdot v_{0.07}^{9}\right)}{62{,}418}$$

$$= \frac{404{,}398}{62{,}418}.$$

$$\upsilon_A = \upsilon_L \quad \Longrightarrow \quad 3.3317P + 4.7509Q = 404{,}398 \quad ②$$

$$① \,\&\, ② \Longrightarrow \begin{cases} P = \$83{,}710 \\ Q = \$47{,}454. \end{cases}$$

③ $\cdot\ C_A(0.07) = \dfrac{V_A''}{V_A} = \dfrac{(-5)\cdot(-6)\cdot P\cdot v_{0.07}^{7} + (-10)(-11)\,Q\cdot v_{0.07}^{12}}{62{,}418}$

$C_L(0.07) = \dfrac{V_L''}{V_L} = \dfrac{50{,}000\cdot\left[(-6)\cdot(-7)\cdot v_{0.07}^{8} + (-8)\cdot(-9)\cdot v_{0.07}^{10}\right]}{62{,}418}.$

$C_A(0.07) = 53.21$ ✓

$C_L(0.07) = 48.90.$

Stochastic Interest Rate Models

$\boxed{T_0,\ j}$

$\tilde{X}$, $\quad p(x) = \Pr[\tilde{X} = x]$

↓

discrete

$E[\tilde{X}] = \sum_x x\cdot p(x)$

$Var[\tilde{X}] = E\left[(\tilde{X} - E[\tilde{X}])^2\right]$

$\boxed{= E[\tilde{X}^2] - (E[\tilde{X}])^2}$

$= \sum_x x^2\cdot p(x) - \left[\sum_x x\cdot p(x)\right]^2$

$\tilde{X}$ $\downarrow$ cont.

$$P[a<\tilde{X}<b] = \int_a^b f(x)\,dx.$$

$$E[\tilde{X}] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx.$$

$$\begin{aligned} Var[\tilde{X}] &= E[\tilde{X}^2] - (E[\tilde{X}])^2 \\ &= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left[\int_{-\infty}^{\infty} x \cdot f(x)\,dx\right]^2. \end{aligned}$$

---

a, b. constants.

$$Var[a\tilde{X}+b] = a^2 \cdot Var[\tilde{X}]$$

$$SD[\tilde{X}] = \sqrt{Var[\tilde{X}]}$$

<u>$h(x)$.</u>

$$E[h(\tilde{X})] = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx.$$

$$\tilde{X} \perp\!\!\!\perp \tilde{Y} \Rightarrow Var[\tilde{X}+\tilde{Y}] = Var[\tilde{X}] + Var[\tilde{Y}]$$

---

* Uniform distributions:

$$f(\tilde{x}) = \frac{1}{b-a} \quad \text{for } a < x < b.$$

$$E[\tilde{x}] = \frac{a+b}{2}$$

$$Var[\tilde{x}] = \frac{(b-a)^2}{12}.$$

* Normal Distribution. $\tilde{X} \sim N(\mu, \sigma^2)$.

$$f(\tilde{x}) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right); (-\infty < X < \infty).$$

$$E[\tilde{x}] = \mu$$

$$Var[\tilde{x}] = \sigma^2$$

Standard Normal : $\tilde{X} \sim N(0, 1)$.

Assume $\tilde{X} \sim N(\mu, \sigma^2)$

$$P(a < \tilde{X} < b) = P\left(\frac{a-\mu}{\sigma} < \frac{\tilde{X}-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} < \boxed{Z} < \frac{b-\mu}{\sigma}\right).$$

$\sim N(0,1)$.

$\tilde{i}$: random variable

$\hookrightarrow$ Single Cash flow < P.V. / A.V.

Annuity < P.V. / A.V.

① Single C.F.

A.V.

$$\tilde{i} = \begin{cases} i_a & prob = a \\ i_b & prob = b \end{cases}$$

$$E[\tilde{i}] = \sum_{i} i \cdot p(i) = i_a \cdot a + i_b \cdot b$$

$$Var[\tilde{i}] = \boxed{E[\tilde{i}^2]} - E[\tilde{i}]^2 = \tilde{i}_a^2 \cdot a + \tilde{i}_b^2 \cdot b - (i_a \cdot a + i_b \cdot b)^2$$

$$E[\tilde{i}^2] = \sum_{i} h(\tilde{i}) \cdot p(\tilde{i}) = \tilde{i}_a^2 \cdot a + \tilde{i}_b^2 \cdot b$$

Ex: $\tilde{i} = \begin{cases} 0.1 & , prob = 0.5 \\ 0.15 & , prob = 0.5 \end{cases}$

Sol:

$$E[\tilde{i}] = 0.1 \cdot 0.5 + 0.15 \cdot 0.5 = 0.125$$

$$\mathrm{Var}[\tilde{i}] = E[\hat{i}^2] - E[\hat{i}]^2 = 0.01625 - 0.125^2$$

$$= 0.000625$$

$$E[\hat{i}^2] = 0.1^2 \cdot 0.5 + 0.15^2 \cdot 0.5 = 0.01625$$