SPECTRAL ANALYSIS

Frequency-domain or spectral analysis starts with describing the value of Y_t as a weighted sum of periodic functions of the form $cos(\omega t)$ and $sin(\omega t)$, where ω denote a particular frequency

$$Y_t = \mu + \int_0^\pi \alpha(\omega) \cdot \cos(\omega t)\, d\omega + \int_0^\pi \delta(\omega) \cdot \sin(\omega t)\, d\omega.$$

Its goal is to determine how important cycles of different frequencies are in accounting for the behavior of Y_t.

THE SPECTRAL REPRESENTATION AND SPECTRAL DISTRIBUTION

Consider a time series represented as

$$Y_t = \sum_{j=1}^{m} \left[A_j \cos(2\pi f_j t) + B_j \sin(2\pi f_j t)\right], \quad (1)$$

where the frequencies $0 < f_1 < f_2 < \cdots < f_m < ½$ are fixed and A_j and B_j are independent normal random variables with zero means and $var(A_j) = var(B_j) = \sigma_j^2$. Then we could show that $\{Y_t\}$ is stationary with mean zero and

$$\gamma_k = \sum_{j=1}^{m} \sigma_j^2 \cos(2\pi k f_j). \quad (2)$$

In particular, the process variance, γ_0, is a sum of the variances due to each component at the various fixed frequencies:

plug in 0
each cos = 1

$$\gamma_0 = \sum_{j=1}^{m} \sigma_j^2. \quad (3)$$

If for $0 < f < 1/2$ we define two random step functions by

$$a(f) = \sum_{\{j|f_j \le f\}} A_j$$

and

$$b(f) = \sum_{\{j|f_j \le f\}} B_j$$

then we can write eqn. (1) as

$$Y_t = \int_0^{1/2} \cos(2\pi f t) da(f) + \int_0^{1/2} \sin(2\pi f t) db(f). \qquad (4)$$

It turns out that *any* zero-mean stationary process may be represented as in eqn. (4)[1]. It shows how stationary processes may be represented as linear combinations of infinitely many cosine-sine pairs over a continuous frequency band. In general, $a(f)$ and $b(f)$ are zero-mean stochastic processes indexed by frequency on $0 \le f \le 1/2$, each with orthogonal increments, and the increments of $a(f)$ are uncorrelated with the increments of $b(f)$. Furthermore, we have

$$var\left(\int_{f_1}^{f_2} da(f)\right) = var\left(\int_{f_1}^{f_2} db(f)\right) = F(f_2) - F(f_1).$$

Eqn. (4) is called the *spectral representation* of the process. The nondecreasing function $F(f)$ defined on $0 < f < 1/2$ is called the *spectral distribution function* of the process.

We say that the special process defined by eqn. (1) has a *purely discrete*(or *line*) *spectrum* and, for $0 \le f \le 1/2$,

$$F(f) = \sum_{\{j | f_j \le f\}} \sigma_j^2.$$

Here the heights of the jumps in the spectral distribution give the variances associated with the various periodic components, and the positions of the jumps indicate the frequencies of the periodic components. In general, a spectral distribution function has the properties

1. F is nondecreasing
2. F is right continuous
3. $F(f) \ge 0$ for all f
4. $\lim_{f \to \frac{1}{2}} F(f) = var(Y_t) = \gamma_0$

If we consider the scaled spectral distribution function $F(f)/\gamma_0$, we have a function with the same mathematical properties as a cumulative distribution function (CDF) for a random variable on the interval 0 to ½ since now $F(½)/\gamma_0 = 1$. Hence, we interpret the spectral distribution by saying that, for $0 \le f_1 < f_2 \le ½$, the integral $\int_{f_1}^{f_2} dF(f)$ gives the portion of the (total) process variance $F(½) = \gamma 0$ that is attributable to frequencies in the range f_1 to f_2.

[1] The proof is beyond the scope of this course.

POPULATION SPECTRUM/SPECTRAL DENSITY FUNCTION

Let Y_t be a causal/stationary process and γ_j denote its autocovariance of lag j. As a causal process, we can express Y_t as

$$Y_t = \sum_{i=0}^{\infty} \psi_i a_{t-i} = \psi(B)a_t, \quad a_t \sim WN(0, \sigma^2), \quad (5)$$

where

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots$$

and its autocovariance of lag j equals

$$\gamma_k = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}. \qquad (6)$$

For a given sequence of autocovarinces $\gamma_k = 0, \pm 1, \pm 2, \ldots$, the *autocovariance generating function* is defined as

$$g_Y(B) = \sum_{k=-\infty}^{\infty} \gamma_k B^k, \quad (7)$$

where the variance of the process, γ_0, is the coefficient of B^0 and the autocovariance of lag k, γ_k, is the coefficient of both B^k and B^{-k}. Substituting eqn. (6) into eqn. (7), we have

$$\begin{aligned} g_Y(B) &= \sigma^2 \sum_{k=-\infty}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_{i+k} B^k \\ &= \sigma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j B^{j-i} \\ &= \sigma^2 \sum_{i=0}^{\infty} \psi_i B^{-i} \sum_{j=0}^{\infty} \psi_j B^j \\ &= \sigma^2 \psi(B^{-1}) \psi(B), \qquad (8) \end{aligned}$$

where we let $j = i + k$ and note that $\psi_j = 0$ for $j < 0$.

If eqn. (7) is divided by 2π and evaluated at some B represented by $B = e^{-i\omega}$ for $i = \sqrt{-1}$ and ω a real scalar, the result is called the *population spectrum* or *theoretical spectral density function*[2] of Y_t:

[2] $F(f) = \int_0^f s_Y(x)dx$, $0 \le f \le 1/2$, and $\omega = 2\pi f$.

$$s_Y(\omega) = \frac{1}{2\pi} g_Y\left(e^{-i\omega}\right) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\omega k}. \quad (9)$$

Note that the spectrum is a function of ω: given any particular value of ω and a sequence of autocovariances $\{\gamma_k\}$, we could calculate the value of $s_Y(\omega)$.

De Moivre's theorem allows us to write $e^{-i\omega k}$ as

$$e^{-i\omega k} = \cos(\omega k) - i \cdot \sin(\omega k). \quad (10)$$

Substituting eqn. (10) into (9), we have

$$s_Y(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k [\cos(\omega k) - i \cdot \sin(\omega k)]. \quad (11)$$

Note that autocovariance is an even function so $\gamma_k = \gamma_{-k}$. Hence, eqn. (11) implies

$$\begin{aligned} s_Y(\omega) = &\frac{1}{2\pi} \gamma_0 [\cos(0) - i \cdot \sin(0)] \\ &+ \frac{1}{2\pi} \left\{ \sum_{k=1}^{\infty} \gamma_k \left[\cos(\omega k) + \cos(-\omega k) - i \cdot \sin(\omega k) - i \cdot \sin(-\omega k)\right] \right\}. \quad (12) \end{aligned}$$

Next, make use of the following trigonometry:

$\cos(0) = 1$, $\sin(0) = 0$, $\sin(-\theta) = -\sin(\theta)$, and $\cos(-\theta) = \cos(\theta)$.

Eqn. (8) can be simplified as

$$s_Y(\omega) = \frac{1}{2\pi} \left\{ \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\omega k) \right\}. \quad (13)$$

Properties of population spectrum:

1. The spectrum is symmetric around $\omega = 0$ since $\cos(\omega j) = \cos(-\omega j)$ for any ω
2. Since $\cos[(\omega + 2\pi k) \cdot j] = \cos(\omega j)$ for any integer k and j, it follows from eqn. (9) that $s_Y(\omega + 2\pi k) = s_Y(\omega)$ for any integer k. Hence, the spectrum is a periodic function of ω. If we know the value of $s_Y(\omega)$ for all ω between 0 and π, we can infer the value of $s_Y(\omega)$ for any ω.

CALCULATING THE POPULATION SPECTRUM

Use eqn. (8) and (9), the population spectrum for an MA(∞) process is given by

$$s_Y(\omega) = \frac{1}{2\pi}\sigma^2\psi(e^{-i\omega})\psi(e^{i\omega}). \qquad (14)$$

<u>Example 1 (White noise process)</u>

1. $\psi(B) = 1$
2. $s_Y(\omega) = \sigma^2/2\pi$

<u>Example 2 (MA(1) process)</u>

1. $\psi(B) = 1 + \theta B$
2. $s_Y(B) = (2\pi)^{-1} \cdot \sigma^2[1 + \theta^2 + 2\theta \cdot \cos(\omega)]$

   Hint: Use the fact that $e^{-i\omega} + e^{i\omega} = 2 \cdot \cos(\omega)$.

<u>Example 3 (causal/stationary AR(1) process)</u>

1. $\psi(B) = (1 - \phi B)^{-1}$
2. $s_Y(\omega) = \frac{1}{2\pi}\frac{\sigma^2}{(1-\phi e^{-i\omega})(1-\phi e^{i\omega})}$

$$= \frac{1}{2\pi}\frac{\sigma^2}{(1 - \phi e^{-i\omega} - \phi e^{i\omega} + \phi^2)}$$

$$= \frac{1}{2\pi}\frac{\sigma^2}{1 + \phi^2 - 2\phi\, cos(\omega)}$$

<u>Example 4 (causal/stationary and invertible ARMA(p,q) process)?? Practice</u>