Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 2

- Due Tues. Feb 5, 6:10pm sharp . Late assignments will not be accepted - even if it's one minute late!
- You may hand in your problem set either to your instructor in class on Tuesday, during S. Uppal's office hours Tuesdays 3-4pm, or in the drop boxes for MAT224 in the Sidney Smith Math Aid Center (SS 1071), arranged according to tutorial sections. Note: If you are in the T6-9 evening class, the problem set is due at 6:10pm **before** lecture begins.
- Be sure to clearly write your name, student number, and your tutorial section on the top right-hand corner of your assignment. Your assignment must be written up clearly on standard size paper, stapled, and cannot consist of torn pages otherwise it will not be graded.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly and will result at the minimum of a grade of zero for the problem set. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ that has the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 & -1 \\ 2 & 6 & 4 & 6 & 4 \\ 1 & 3 & 2 & 2 & 1 \end{bmatrix}$$

relative to the bases $\{(1,1,1,1,1),(1,1,1,1,0),(1,1,0,0,0),(1,0,0,0,0),(0,0,0,0,1)\}$ of $\mathbb{R}^5$ and $\{(1,1,1),(0,1,0),(1,0,0)\}$ of $\mathbb{R}^3$.

(a) Find a basis for the kernel of T.

(b) Find a basis for the image of T.

Replace (c-d) by (d-c) and follow the solutions posted on Blackboard---suggestion by Mohammad El Smaily to keep the posted solutions valid

2. Let $T\colon M_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-b) + (a-d)x + (b-c)x^2 + (c-d)x^3.$$

Consider the bases $\alpha = \left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$ of $M_{2\times 2}(\mathbb{R})$, and $\beta = \{x, x-x^2, x-x^3, x-1\}$ of $P_3(\mathbb{R})$.

(a) Find $[T]_{\beta\alpha}$.

(b) Use $[T]_{\beta\alpha}$ to find a basis for the kernel of T.

(c) Use $[T]_{\beta\alpha}$ to find a basis for the image of T.

(d) State the nullity and rank of T. Is T injective? surjective?

3. Textbook, Section 2.3, **12**.

4. Let $T\colon \mathbb{Z}_3^3 \to \mathbb{Z}_3^2$ be defined by $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

(a) Is T surjective? If not, find $Im(T)$.

(a) Is T injective? If not, find $Ker(T)$.

5. Let V be a vector space over a field F and let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Let $T\colon V \to F^n$ defined by

$$T(v) = [v]_\alpha$$

for every $v \in V$.

(a) Show that T is a linear transformation.

(b) Show that T is bijective.

6. Let $T\colon V \to W$ be a bijective linear transformation. Prove that if $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

7. Let V and W be vector spaces over a field F. Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, and $\beta = \{w_1, w_2, \ldots, w_m\}$ a basis for W. Let $T\colon V \to W$ be a linear transformation.

(a) Prove that T is surjective if and only if the columns of $[T]_{\beta\alpha}$ span F^m.

(b) Prove that T is injective if and only if the columns of $[T]_{\beta\alpha}$ are linearly independent in F^m.

Suggested Extra Problems (not to be handed in):

Note: the questions marked with a * are all related.

- Textbook, Section 2.1, **12***
- Textbook, Section 2.2, **11***
- Textbook, Section 2.3 **1, 2, 3, 4, 9*, 13**
- Textbook, Section 2.4 **1, 2, 5, 6, 10, 12**
- Let $T\colon \mathbb{Z}_3^2 \to \mathbb{Z}_3^3$ be defined by $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(i) Is T surjective? If not, find $Im(T)$.

(ii) Is T injective? If not, find $Ker(T)$.