FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 4

PERPETUITIES

An annuity where payments continue forever is called a ***perpetuity***.

The present value of a perpetuity where 1 is payable at the end of each year (in arrears or immediate perpetuity) is:

$$\boxed{a_{\overline{\infty|}} = \lim_{n\to\infty} a_{\overline{n|}} = \frac{1}{i}}$$

Proof
This can be proved by general reasoning or by finding the limit of $a_{\overline{n|}}$ as $n \to \infty$. Below is a proof by general reasoning:

Let X denote the present value of the perpetuity $a_{\overline{\infty|}}$. Now suppose that we invest a single payment of amount X in a bank account. By definition, this amount is just sufficient to provide a payment of 1 unit at the end of each future year.

At the end of the first year the initial investment will have accumulated to $X(1+i)$. After we have made the payment of 1 due at the end of the first year we will have $X(1+i)-1$ left. This amount is required to pay 1 unit at the end of each future year, and since we still have ∞ future years to go, the amount required is equal to X. Therefore,

$$(1+i)X - 1 = X \;\Rightarrow X = a_{\overline{\infty|}} = \frac{1}{i}$$

The present value of a perpetuity where 1 is payable at the beginning of each year (in advance or perpetuity-due) is:

$$\boxed{\ddot{a}_{\overline{\infty|}} = \frac{1}{d}}$$

The present value of a perpetuity where $\frac{1}{m}$ is payable at the end of each period of length $\frac{1}{m}$ is:

$$\boxed{a_{\overline{\infty}|}^{(m)} = \frac{1}{i^{(m)}}}$$

The present value of a perpetuity where $\frac{1}{m}$ is payable at the beginning of each period of length $\frac{1}{m}$ is:

$$\boxed{\ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}}}$$

CONTINUOUS ANNUITIES

We have just introduced annuities payable every $\frac{1}{m}^{th}$ of a year. As m increases, the frequency of payments becomes larger.

The limit as $m \to \infty$ of an annuity payable every $\frac{1}{m}^{th}$ of a year is an ***annuity payable continuously***.

We derive the accumulated value of a continuously payable annuity below.

Consider an annuity that has a rate of continuous payment of 1 per period, and an effective rate of interest of i per period.

The amount paid from time 0 to time n is n (1 is paid per period and there are n periods), and the amount paid from time t to time $t + dt$ is equal to dt (since the payments are continuous).

The accumulated value from time t to time n of an amount dt is $(1+i)^{n-t}\,dt$. The sum of all accumulated amounts can be found by integrating between 0 and n :

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^{n-t}dt = \int_0^n \exp\left(\ln(1+i)^{n-t}\right)dt = \int_0^n \exp\left((n-t)\ln(1+i)\right)dt$$

Since $\int \exp(at)dt = \frac{\exp(at)}{a}$, it follows that,

$$\bar{s}_{\overline{n|}} = \int_0^n \exp\left((n-t)\ln(1+i)\right)dt = \left.\frac{\exp\left((n-t)\ln(1+i)\right)}{-\ln(1+i)}\right|_0^n$$

$$= \frac{-\exp(0)}{\ln(1+i)} + \frac{\exp(n\ln(1+i))}{\ln(1+i)} = \frac{(1+i)^n - 1}{\ln(1+i)}$$

In the lecture notes on force of interest we showed that when we have a constant force of interest δ,

$$e^{\delta} = (1+i) \Rightarrow \delta = \ln(1+i)$$

Therefore,

$$\bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\ln(1+i)} = \frac{(1+i)^n - 1}{\delta}$$

Since $s_{\overline{n|}} = \frac{(1+i)^n - 1}{i} \Rightarrow \bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} \cdot s_{\overline{n|}}$

$$\boxed{\bar{s}_{\overline{n|}} = \int_0^n (1+i)^{n-t}dt = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} \cdot s_{\overline{n|}}}$$

The present value of an annuity payable continuously at the rate of 1 per annum for n periods is:

$$\bar{a}_{\overline{n|}} = \int_0^n v^t dt = \int_0^n \exp(t \cdot \ln(v))dt = \left.\frac{v^t}{\ln(v)}\right|_0^n = \frac{v^n - 1}{\ln(v)} = \frac{v^n - 1}{-\ln(1+i)} = \frac{1 - v^n}{\ln(1+i)} = \frac{1 - v^n}{\delta}$$

$$\boxed{\bar{a}_{\overline{n|}} = \int_0^n v^t dt = \frac{1 - v^n}{\delta} = \frac{i}{\delta} \cdot a_{\overline{n|}}}$$

The annuities introduced above are based on effective rates of interest i. If accumulation is based on a force of interest δ_r, then,

$$\bar{s}_{\overline{n}|\delta_r} = \int_0^n \exp\left(\int_t^n \delta_r \, dr\right) dt$$

$$\bar{a}_{\overline{n}|\delta_r} = \int_0^n \exp\left(-\int_o^t \delta_r \, dr\right) dt$$

EXAMPLE
Smith deposits \$12 every day in 2011 and 2012 (730 days), and deposits \$15 every day in 2013 (365 days). Interest accrues on the deposits at an effective annual rate of 9% in 2011 and 2012, and 12% in 2013.

Find the amount in the account at the end of 2013 (after interest is credited):
(a) exactly
(b) using an approximation that deposits are made continuously.

Solution
(a) Let j be the daily effective rate of interest based on 9%:

$$j = (1.09)^{1/365} - 1$$

Let k be the daily effective rate of interest based on 12%:

$$k = (1.12)^{1/365} - 1$$

The accumulated amount of the payments of \$12 per day made in 2011 and 2012 at the end of 2012 is:

$$12 \cdot s_{\overline{730}|j}$$

Since the effective annual rate of 12% applies over 2013, the amount at the end of 2012 accumulated to the end of 2013 is:

$$12 \cdot s_{\overline{730}|j}(1.12)$$

The accumulated amount of the payments of \$15 per day made in 2013 at the end of 2013 is:

$$15 \cdot s_{\overline{365}|k}$$

Therefore, the total accumulated amount at the end of 2013 is:

$$\begin{aligned} 12 \cdot s_{\overline{730}|j}(1.12) + 15 \cdot s_{\overline{365}|k} &= 12 \cdot \left[\frac{(1+j)^{730} - 1}{j}\right](1.12) + 15 \cdot \left[\frac{(1+k)^{365} - 1}{k}\right] \\ &= 12 \cdot \left[\frac{(1.09^{1/365})^{730} - 1}{1.09^{1/365} - 1}\right](1.12) + 15 \cdot \left[\frac{(1.12^{1/365})^{365} - 1}{1.12^{1/365} - 1}\right] \end{aligned}$$

$$= 12 \cdot \left[\frac{(1.09)^2 - 1}{1.09^{1/365} - 1} \right](1.12) + 15 \cdot \left[\frac{1.12 - 1}{1.12^{1/365} - 1} \right]$$
$$= 10,706.19 + 5,796.40 = 16,502.59$$

(b) We have deposits of $12 \times 365 = \$4,380$ per annum in 2011 and 2012, and $15 \times 365 = \$5,475$ in 2013.

If we let δ_1 be the force of interest equivalent to $i = 0.09$, and let δ_2 be the force of interest equivalent to $i = 0.12$, then the accumulated value at the end of 2013 is:

$$4380 \cdot \bar{s}_{\overline{2|}0.09}(1.12) + 5475 \cdot \bar{s}_{\overline{1|}0.12} = 4380 \cdot \left[\frac{(1.09)^2 - 1}{\delta_1} \right](1.12) + 5475 \cdot \left[\frac{(1.12) - 1}{\delta_2} \right]$$
$$= 4380 \cdot \left[\frac{(1.09)^2 - 1}{\ln(1.09)} \right](1.12) + 5475 \cdot \left[\frac{(1.12) - 1}{\ln(1.12)} \right]$$
$$= 10,707.45 + 5,797.30 = 16,504.75$$

INCREASING OR DECREASING ANNUITIES

Increasing annuities

Consider a series of n payments where the t^{th} payment has the amount t, and payments are made at the end of each period

0	1	2	3	4	…	n-1	n	time
	1	2	3	4	…	n-1	n	amount

If we assume an effective interest rate of i for each period, then the accumulated value of an ***increasing annuity*** at time n is:

$$\boxed{(Is)_{\overline{n|}i} = \frac{\ddot{s}_{\overline{n|}i} - n}{i}}$$

and the present value at time 0 of this stream of payments is:

$$\boxed{(Ia)_{\overline{n|}i} = \sum_{t=1}^{n} t v^t = \frac{\ddot{a}_{\overline{n|}i} - n v^n}{i}}$$

Proof

The accumulated value at time n is the accumulated value of each of the series of payments.

$$S(n) = 1\cdot(1+i)^{n-1} + 2\cdot(1+i)^{n-2} + 3\cdot(1+i)^{n-3} + \ldots + (n-1)\cdot(1+i) + n$$

Multiply this by $(1+i)$,

$$(1+i)\cdot S(n) = 1\cdot(1+i)^{n} + 2\cdot(1+i)^{n-1} + 3\cdot(1+i)^{n-2} + \ldots + (n-1)\cdot(1+i)^2 + n\cdot(1+i)$$

Now, subtract $S(n)$ from $(1+i)\cdot S(n)$,

$$\begin{aligned}(1+i)\cdot S(n) - S(n) = S(n)\cdot i &= (1+i)^{n} + (1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^2 + (1+i) - n \\ &= \ddot{s}_{\overline{n}|} - n\end{aligned}$$

Therefore,

$$S(n) = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

The present value of this stream of payments is just the accumulated value discounted back to time zero:

$$S(0) = S(n)\cdot v^n = \frac{\ddot{s}_{\overline{n}|} - n}{i}\cdot v^n = \frac{\ddot{s}_{\overline{n}|}v^n - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

We can derive a similar function for the present value of an increasing annuity-due:

$$\boxed{(I\ddot{a})_{\overline{n}|i} = \sum_{t=0}^{n-1}(t+1)v^t = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{d}}$$

<u>Decreasing annuities</u>

Now consider a series of n payments where the 1st payment has the amount n, the second payment has the amount $n-1,\ldots$, and the last payment is of amount 1 (ie. payments are decreasing from n to 1 in units of 1. As before, assume that payments are made in arrears (at the end of each period).

0	1	2	3	4	…	n-1	n	time
	n	n-1	n-2	n-3	…	2	1	amount

If we assume an effective interest rate of i for each period, then the accumulated value of a ***decreasing annuity*** at time n is:

$$\boxed{(Ds)_{\overline{n|}i} = \frac{n \cdot (1+i)^n - s_{\overline{n|}i}}{i}}$$

and the present value at time 0 of this stream of payments is:

$$\boxed{(Da)_{\overline{n|}i} = \sum_{t=1}^{n} (n-t+1)v^t = \frac{n - a_{\overline{n|}i}}{i}}$$

The present value at the time of the first payment is:

$$\boxed{(D\ddot{a})_{\overline{n|}i} = \sum_{t=0}^{n-1} (n-t)v^t = \frac{n - a_{\overline{n|}i}}{d}}$$

These results can be proved using a proof similar to that used above for increasing annuities. ***Try and prove these yourself.***

<u>General formula for increasing annuity</u>

In general we consider an n-payment annuity with first payment A and subsequent payment B larger (or smaller) than the previous one.

0	1	2	3	4	…	n-1	n	time
Increasing	A	A+B	A+2B	A+3B	…	A+(n-2)B	A+(n-1)B	amount

The series can be decomposed into two parts:

(i) a level series of n payments of amount $A-B$ each

(ii) a series of n payments starting at amount B and increasing by amount B each period.

For example, the accumulated value of this series can be written:

$$S(n) = A \cdot (1+i)^{n-1} + (A+B) \cdot (1+i)^{n-2} + \ldots + (A+(n-2)B) \cdot (1+i) + (A+(n-1)B)$$

$$\Rightarrow S(n) = (A-B) \cdot \left[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i) + 1 \right]$$
$$+B \cdot \left[(1+i)^{n-1} + 2 \cdot (1+i)^{n-2} \ldots + (n-1) \cdot (1+i) + n \right]$$

$$\Rightarrow S(n) = (A-B) \cdot s_{\overline{n|}i} + B \cdot \left[(1+i)^{n-1} + 2 \cdot (1+i)^{n-2} \ldots + (n-1) \cdot (1+i) + n \right]$$

If we use the method above, we can find a formula for the second increasing annuity:

$$B \cdot \left[(1+i)^{n-1} + 2 \cdot (1+i)^{n-2} \ldots + (n-1) \cdot (1+i) + n \right] = (Is)_{\overline{n|}i}$$

$$\boxed{S(n) = (A-B)s_{\overline{n|}i} + B\left(Is\right)_{\overline{n|}i}}$$

EXAMPLE
Find the accumulated value of an increasing annuity, where there are 30 payments in arrears, the first payment being 100 and each subsequent payment being 20 more than the last. Assume that the annuity earns interest at an annual effective rate of 9%.

Solution

0	1	2	3	4	…	29	30	time
	100	120	140	160	…	660	680	amount

This is equivalent to an increasing annuity with two parts. The first part is a level annuity of amount 80 payable for n periods. The second part is an increasing annuity of n payments, where the first payment is 20 and each subsequent payment increases by 20.

We can, therefore, use the formula just derived:

$$S(n) = (A-B)s_{\overline{n|}i} + B\left(Is\right)_{\overline{n|}i} = (A-B)\left(\frac{(1+i)^n - 1}{i}\right) + B\left(\frac{\ddot{s}_{\overline{n|}i} - n}{i}\right)$$

$$= (A-B)\left(\frac{(1+i)^n - 1}{i}\right) + B\left(\frac{\left(\frac{(1+i)^n - 1}{d}\right) - n}{i}\right)$$

$$= 80\left(\frac{(1.09)^{30} - 1}{0.09}\right) + 20\left(\frac{\left(\frac{(1.09)^{30} - 1}{0.09/1.09}\right) - 30}{0.09}\right)$$

$= 37,254.65$

Alternatively, we could decompose this into a level annuity of amount 100 payable for n periods and an increasing annuity of $n-1$ payments, where the first payment is 20 and each subsequent payment increases by 20.

$$S(n) = 100 s_{\overline{n}|i} + 20\left(Is\right)_{\overline{n-1}|i} = 100\left(\frac{(1+i)^n - 1}{i}\right) + 20\left(\frac{\ddot{s}_{\overline{n-1}|i} - (n-1)}{i}\right)$$

$$= 100\left(\frac{(1+i)^n - 1}{i}\right) + 20\left(\frac{\left(\frac{(1+i)^{n-1} - 1}{d}\right) - (n-1)}{i}\right)$$

$$= 100\left(\frac{(1.09)^{30} - 1}{0.09}\right) + 20\left(\frac{\left(\frac{(1.09)^{29} - 1}{0.09/1.09}\right) - 29}{0.09}\right)$$

= 37,254.65

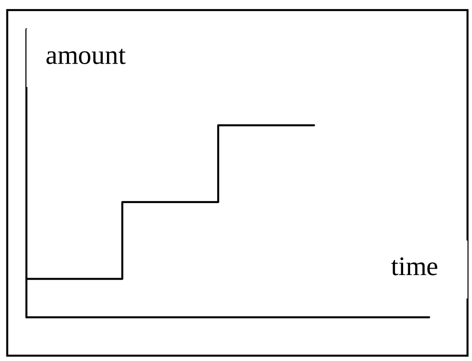
A general formula for a decreasing annuity could also be derived but is not included here.

<u>Continuous payments</u>

The symbols for continuously payable increasing and decreasing annuities have a bar over the annuity symbol (to show that payments are continuous).

eg.

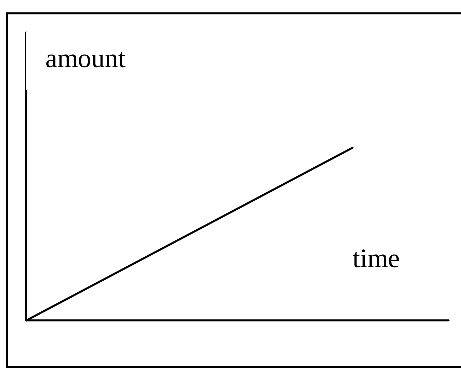
$$(I\bar{a})_{\overline{n}|i} = \int_0^n \lceil t \rceil v^t dt = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{\delta} = \frac{i}{\delta}(Ia)_{\overline{n}|i}$$



A bar over the I indicates that increases occur continuously, rather than at the end of the year.

eg.

$$(\bar{I}\bar{a})_{\overline{n}|i} = \int_0^n t v^t dt = \frac{\bar{a}_{\overline{n}|i} - nv^n}{\delta}$$



EXAMPLE
Calculate the present value of a series of continuous payments starting immediately at a rate of $500 per annum and increasing linearly to a rate of $1,000 per annum when the payments cease in 5 years time. Use an effective interest rate of 4% p.a.

Solution
This can be split into two separate annuities of $500 per annum payable continuously for 5 years and a continuously increasing, continuously payable annuity of $100 per annum.

We can, therefore, use the formula just derived:

$$S(0) = 500\bar{a}_{\overline{5}|0.04} + 100\left(\overline{Ia}\right)_{\overline{5}|0.04}$$

$$= 500\left(\frac{1-v^5}{\ln(1.04)}\right) + 100\left(\frac{\bar{a}_{\overline{5}|0.04} - 5v^5}{\ln(1.04)}\right)$$

$$= \$2,270.14 + \$1,097.99 = \$3,368.13$$

ANNUITIES WITH INDEXATION

If payments are linked to a specific rate of inflation (r), then it is necessary to modify the standard annuity formulae introduced previously.

Annuities in arrears

Consider a series of n payments of 1 where payments are indexed at an effective rate r, and payments are made at the end of each period.

0	1	2	3	4	…	n-1	n	time
	1	$(1+r)$	$(1+r)^2$	$(1+r)^3$	…	$(1+r)^{n-2}$	$(1+r)^{n-1}$	amount

If we assume an effective interest rate of i for each period, the accumulated value of this series of payments at time n is:

$$S(n) = (1+i)^{n-1} + (1+r)\cdot(1+i)^{n-2} + (1+r)^2\cdot(1+i)^{n-3} + ... + (1+r)^{n-2}\cdot(1+i) + (1+r)^{n-1}$$

In general, in order to find a formula that reduces the series, first try and combine the terms in $(1+r)$ and the terms in $(1+i)$.

In this example, take out $(1+i)^{n-1}$ as a factor:

$$S(n) = (1+i)^{n-1}\left[1 + (1+r)\cdot(1+i)^{-1} + (1+r)^2\cdot(1+i)^{-2} + ... + (1+r)^{n-1}(1+i)^{-(n-1)}\right]$$

$$S(n) = (1+i)^{n-1}\left[1 + \left(\frac{1+r}{1+i}\right) + \left(\frac{1+r}{1+i}\right)^2 + \ldots + \left(\frac{1+r}{1+i}\right)^{n-1}\right]$$

Now, we can use the geometric series expansion formula from week 3:

$$1 + x + x^2 + x^3 + \ldots + x^k = \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1}-1}{x-1}$$

Therefore,

$$S(n) = (1+i)^{n-1}\left[1 + \left(\frac{1+r}{1+i}\right) + \left(\frac{1+r}{1+i}\right)^2 + \ldots + \left(\frac{1+r}{1+i}\right)^{n-1}\right]$$

$$S(n) = (1+i)^{n-1}\left[\frac{\left(\frac{1+r}{1+i}\right)^n - 1}{\left(\frac{1+r}{1+i}\right) - 1}\right] = (1+i)^n\left(\frac{\left(\frac{1+r}{1+i}\right)^n - 1}{r-i}\right)$$

$$S(n) = \left(\frac{(1+r)^n - (1+i)^n}{r-i}\right)$$

This is equivalent to:

$$S(n) = \left(\frac{(1+i)^n - (1+r)^n}{i-r}\right)$$

The present value at time 0 of this series of payments is:

$$S(0) = S(n) \cdot v_i^n = (1+i)^{-n} \cdot \frac{(1+r)^n - (1+i)^n}{r-i} = \frac{\left(\frac{1+r}{1+i}\right)^n - 1}{r-i}$$

Annuities-due

Consider a series of n payments of 1 (in advance) where payments are indexed at r per period. This is like the previous example, but payments are in advance.

0	1	2	3	4	…	n-1	n	time
1	$(1+r)$	$(1+r)^2$	$(1+r)^3$	$(1+r)^4$	…	$(1+r)^{n-1}$		amount

The accumulated value in this case is:

$$S(n) = (1+i)^n + (1+r)\cdot(1+i)^{n-1} + (1+r)^2\cdot(1+i)^{n-2} + \ldots + (1+r)^{n-1}\cdot(1+i)$$
$$= (1+i)\cdot\left[\frac{(1+i)^n - (1+r)^n}{i-r}\right]$$

Sometimes it may be possible to further simplify an expression by introducing a new rate of interest that is a function of i and r. For example, the present value in this case can be written as

$$S(0) = S(n)\cdot v_i^n = (1+i)^{-n}\cdot(1+i)\cdot\left[\frac{(1+i)^n - (1+r)^n}{i-r}\right] = \left[\frac{1-\left(\frac{1+r}{1+i}\right)^n}{\frac{i-r}{1+i}}\right] = \left[\frac{1-\left(\frac{1+r}{1+i}\right)^n}{1-\left(\frac{1+r}{1+i}\right)}\right]$$

If we define a new rate of interest j such that,

$v_j^n = \left(\frac{1+r}{1+i}\right)^n = (1+j)^{-n}$, then $j = \frac{i-r}{1+r}$ and

$$S(0) = \left[\frac{1-\left(\frac{1+r}{1+i}\right)^n}{1-\left(\frac{1+r}{1+i}\right)}\right] = \frac{1-v_j^n}{1-v_j} = \frac{1-v_j^n}{d_j} = \ddot{a}_{\overline{n}|j}$$

Rather than memorising the formula derived here, it is better to understand the steps taken in reducing a series of payments to a simple formula. A standard approach is to:

1) write out the series of payments either as a present value or an accumulated value depending on what the question is asking (use a timeline first if you find it helpful)

2) try and combine terms in $(1+r)$ and $(1+i)$.

3) try and reduce the final expression to ones involving standard annuities (due or immediate) with modified interest rates or by using the geometric series expansion formula directly.

EXAMPLE

Smith receives monthly family allowance payments on the last day of each month from January 31, 1995 to December 31, 2012 inclusive. The payments are deposited in an account earning 1% effective per month. The first payment of $25 is made on January 31, 1995. The payments are then increased by 2% each month to meet cost-of-living increases. Find the accumulated value of the payments on December 31, 2012.

Solution
There are 18 years of payments, which is equal to 216 monthly payments. Working in time units of one month:

The first payment of 25 accumulates to $25(1.01)^{215}$
The second payment is $25(1.02)$ and accumulates to $25(1.02)(1.01)^{214}$
The third payment is $25(1.02)^2$ and accumulates to $25(1.02)^2(1.01)^{213}$
…
The last payment is $25(1.02)^{215}$ and is made on the date of valuation.

$$\Rightarrow S(n) = 25\left[(1.01)^{215} + (1.02)(1.01)^{214} + \ldots + (1.02)^{214}(1.01) + (1.02)^{215}\right]$$

We can reduce this by taking out $(1.02)^{215}$:

$$\Rightarrow S(n) = 25(1.02)^{215}\left[1 + \left(\frac{1.01}{1.02}\right) + \ldots + \left(\frac{1.01}{1.02}\right)^{214} + \left(\frac{1.01}{1.02}\right)^{215}\right]$$
$$\Rightarrow S(n) = 25(1.02)^{215}\ddot{a}_{\overline{216}|j}$$

where $v_j = \dfrac{1.01}{1.02}$

Therefore, $S(n) = 25(1.02)^{215}\left(\dfrac{1-v_j^{216}}{1-v_j}\right) = \$158,679.78$

Another way of looking at the above expression would be to reduce it by taking out $(1.01)^{215}$ instead:

$$\Rightarrow S(n) = 25(1.01)^{215}\left[1 + \left(\frac{1.02}{1.01}\right) + \ldots + \left(\frac{1.02}{1.01}\right)^{214} + \left(\frac{1.02}{1.01}\right)^{215}\right]$$
$$\Rightarrow S(n) = 25(1.01)^{215}s_{\overline{216}|j}$$

where $j = \dfrac{1.02}{1.01} - 1$

Therefore, $S(n) = 25(1.01)^{215}\left(\dfrac{(1+j)^{216}-1}{j}\right) = \$158,679.78$

When payment period and index period don't coincide

Deriving the formula above is relatively simple since the timing of indexation of payments coincides with the actual payments themselves.

In the situation when they don't coincide, the payment period should be modified to coincide with the index period.

EXAMPLE

Smith receives monthly family allowance payments on the last day of each month from January 31, 1995 to December 31, 2012 inclusive. The payments are deposited in an account earning 1% effective per month. The first payment of $25 is made on January 31, 1995.

Assume that payment increases of 12% per annum occur each calendar year. In other words, the payment period is monthly, but the indexation is annual. Monthly payments are constant during each calendar year:

In 1995 - monthly payments are 25
In 1996 - monthly payments are $25(1.12) = 28$
In 1997 - monthly payments are $25(1.12)^2 = 31.36$
etc….

Find the accumulated value of the payments on December 31, 2012.

Solution

There are 18 years of payments, which is equal to 216 monthly payments. Working in time units of one month:

The first payment of 25 accumulates to $25(1.01)^{215}$
The second payment is 25 and accumulates to $25(1.01)^{214}$
…
The thirteenth payment is $25(1.12)$ and accumulates to $25(1.12)(1.01)^{203}$
…
The twenty-fifth payment is $25(1.12)^2$ and accumulates to $25(1.12)^2(1.01)^{191}$
…
The second-last payment is $25(1.12)^{17}$ and accumulates to $25(1.12)^{17}(1.01)$.
The last payment is $25(1.12)^{17}$ and is made on the date of valuation.

We can write the accumulated value out as below:

$$S(n) = 25\left((1.01)^{215} + ... + (1.01)^{204}\right) + 25(1.12)\left((1.01)^{203} + ... + (1.01)^{192}\right) + ...$$
$$... + 25(1.12)^{17}\left((1.01)^{11} + ... + 1\right)$$

We can take out the common factor $25\left((1.01)^{11} + ... + 1\right)$:

$$S(n) = 25\left((1.01)^{11} + ... + 1\right)\left[(1.01)^{12x17} + (1.12)^{1}(1.01)^{12x16} + ... + (1.12)^{17}(1.01)^{12x0}\right]$$

Take $(1.01)^{12x17}$ out of the right hand series:

$$\Rightarrow S(n) = 25(1.01)^{12x17} s_{\overline{12}|0.01}\left[1 + \left(\frac{1.12}{1.01^{12}}\right) + \left(\frac{1.12}{1.01^{12}}\right)^{2} ...... + \left(\frac{1.12}{1.01^{12}}\right)^{17}\right]$$

The steps above have reduced the 216 monthly payments into 18 annual payments at the end of each year, where the timing of the new payments is consistent with the timing of indexation.

We can write the series as the present value of an annuity-due with interest rate w:

$$\ddot{a}_{\overline{18}|w} \text{ where } v_w = \frac{1.12}{1.01^{12}} \Rightarrow w = \frac{1.01^{12}}{1.12} - 1.$$

Therefore, the accumulated value in this example can be expressed by the following formula:

$$S(n) = (1.01)^{204} 25 s_{\overline{12}|0.01} \ddot{a}_{\overline{18}|w} = \$41,282.55$$