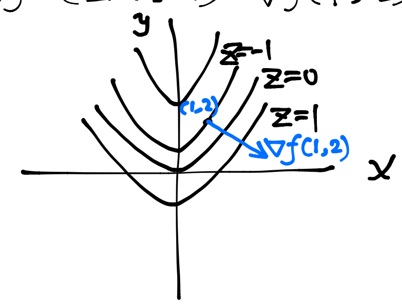
June 12th

1. The discussion at the bottom of page 60 suggests that $\nabla f(\mathbf{a})$ is the vector of $\mathbb{R}^n$ along which the values of $z = f(x_1, ..., x_n)$ changes fastest. Consider the function $z = f(x,y) = x^2 - y$. In this case the function f is defined on $\mathbb{R}^2$. Try to draw a few level curves in the x-y plane (ie z=0, z=1, z=2, etc). Use these level curves to see how the graph of the function in $\mathbb{R}^3$ looks. Which level curve is the point (1,2) on? Calculate $\nabla f(1,2)$ and try to draw it on the x-y plane together with the level curves in the plane. Analytically, show that this vector is perpendicular to the corresponding level curve at the point (1,2). Show that the perpendicular direction to the level curve is the direction of fastest change in the value of the function (of course it is a general fact, but try to analytically prove it in this special case). Show that along the direction tangent to the level curve at the point (1,2) there is only insignificant (order $h^2$) change in the value of z.

$z = f(x, y) = x^2 - y$. Consider $(1, 2) = (x, y) \in \mathbb{R}^2$

$z = f(1,2) = -1 \Rightarrow$ The point $(1,2)$ is on the $z=-1$ level set of $f$.

$\nabla f = (2x, -1)$ $\quad \nabla f(1,2) = (2, -1)$




tangent line to the level set $z=-1$

The eqn of this curve is

$-1 = x^2 - y$

$y = x^2 + 1$

$\frac{dy}{dx} = 2x \qquad \left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{1}$

$\Rightarrow$ The vector tangent to $z=-1$ is $(1,2)$

Notice $(1,2)\cdot(2,-1) = 0 \Rightarrow \nabla f \perp$ (level set $z=-1$)

Check that $\nabla f(1,2)$ is the direction that $f$ is changing most quickly

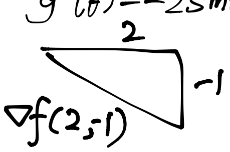
Let $u = (u_1, u_2)$ be a unit vector

$$\partial_u f = \nabla f \cdot u = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} \Rightarrow \partial_u f(1,2) = 2u_1 - u_2$$

I want to find the direction $u$ in which $\partial_u f$ is maximized. Since $u = (u_1, u_2)$ is a unit vector, let $u_1 = \cos\theta$, $u_2 = \sin\theta \Rightarrow \partial_u f = 2\cos\theta - \sin\theta = g(\theta)$

Maximize $g(\theta)$

$g'(\theta) = -2\sin\theta - \cos\theta = 0 \Rightarrow \tan\theta = \frac{-1}{2}$



2. Now consider the function $w = F(x, y, z) = x^2 - y - z$. What is the relationship between this function and the function f of the previous problem? Determine the level surfaces of this function corresponding to w=0 and w=1. Which level surface does the point (1,2,-1) belong to? Calculate $\nabla F(1, 2, -1)$. This is a vector in $\mathbb{R}^3$, how is this related to $\nabla f(1, 2)$ in the previous problem? Analytically show that this vector is perpendicular to the level surface w = 0 at the point (1,2,-1).

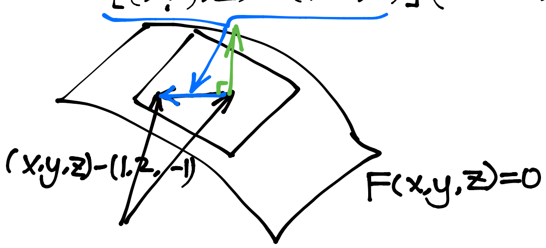
The 0-level set of F is the function f from previous problem.

$\nabla F = (2x, -1, -1)$ $\quad\quad$ $\nabla F(1,2,-1) = (2, -1, -1)$

Next, find the tangent plane to the w=0 level set, at the point (1,2,-1)

To find tangent plane, we use the fact that $\nabla F(1,2,-1)$ is $\perp$ to the tangent plane.

$[(x, y, z) - (1, 2, -1)] \cdot (2, -1, -1) = 0$ $\}$ see also the textbook $\boxed{\nabla f(a) \cdot (x - a) = 0}$



3. Carefully read through Theorems 2.19, 2.26, and 2.42. Then create a brief summary of the big ideas of each proof, perhaps in bullet point form. You should feel that after reading your summary of the proof that you understand the larger structure of the proof and would be able to fill in the details of your summary if asked.

The point of this question is to practice the skill of reading a complicated proof and being able to take away from it the major structural ideas and key points of the proof. The flip side of this is going the other direction: taking these summaries of the proofs and being able to use that to reproduce the proof in full with all details included. I would suggest that letting several days or a week go by after coming up with your summaries, you go back and read your summary and then try to reproduce the proof in full. Putting these two skills together, this should help us distill an otherwise long list of complicated theorems that I have asked you to be able to reproduce, into a list with much smaller summaries, from which you will be able to fill in the details on demand, as well, of course, as being able to help you prove new theorems yourselves.

I'll give you an example of what I mean. Consider the theorem 1.26 asked for on the midterm that the image of a connected set under a continuous map is connected. A proof outline might go as follows:
-Proof by contrapositive
-Start with a disconnection on f(S).

-Pull it back through inverse images to form a candidate for a disconnection on S
-It easily satisfies all property of a disconnection except we need to check the intersection property
- Assume intersection non trivial, then get limit points (ie points in the closure) in S1 or S2 become limit points in f(S1) or f(S2) through assumption of continuity.
-Would thus get that the intersections in f(S) are not empy, a contradiction

Reading this summary, can you fill in all the details for the proof? (on your own, we won't do this in the problem solving session). If not, feel free to adjust my summary to one you prefer, and from which you would be able to fill in all the details.

Thm 2.19 : $f: \mathbb{R}^n \to \mathbb{R}$, sps $\partial_i f$ are continuous $\Rightarrow f$ is diff.

Need to show that $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)-c\cdot h}{|h|} = 0$, $c = \nabla f(a)$

① Look at $f(a+h)-f(a)$ Re-write this by adding & substracting terms as a sum of pairs where each pair only involves one variable changing.

$$f(a_1+h_1, a_2+h_2) - f(a_1, a_2)$$
$$= \underbrace{[f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2)]}_{A} + [f(a_1+h_1, a_2) - f(a_1, a_2)]$$

② Apply MVT to each pair
e.g. $A = \partial_2 f(a_1+h_1, a_2+c_2)\cdot h_2$, where $c_2$ is between $0$ & $h_2$

③ Result is a bunch of terms that look like $\underbrace{(\cdots)}_{\text{junk}}\cdot h_i$

④ $\dfrac{f(a+h)-f(a)-\nabla f(a)\cdot h}{|h|} = \text{sum}(\cdots)\dfrac{h_i}{|h|}$

⑤ Use fact that $\left|\dfrac{h_i}{|h|}\right| \leq 1 \Rightarrow \lim\limits_{h\to 0}\left|\dfrac{f(a+h)-f(a)-\nabla f(a)\cdot h}{|h|}\right| \leq \lim\limits_{h\to 0}\left|\text{Sum (junk)}\cdot 1\right|$

⑥ $\lim_{h\to 0} |\text{Sum}(\cdots)| = 0$ by continuity of partials.

Thm 2.26 (Chain Rule)