March 13th

When do Gram-Schmidt for vectors in a Hermitian vector space ...

$\{u_1, \cdots, u_n\}$ basis of V

$v_1 = u_1$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$ <u>not</u> $u_2 - \frac{\langle v_1, u_2 \rangle}{\langle v_1, v_1 \rangle} v_1$ on book (it's for $\mathbb{R}$)

<u>Recall</u>: $N: V \to V$ is nilpotent if there is a k s.t $N^k = 0$, or equivalently its only eigenvalues are zero.

$0 \neq v \in V$ $v, N(v), N^2(v), \cdots, N^{j-1}(v) \neq 0$, but $N^j(v) = 0$

We want to study $C(v) = sp\{N^{j-1}(v), N^{j-2}(v), \cdots, v\}$

<u>Claim:</u> $N: V \to V$, $v \in V$, $N^{j-1}(v) \neq 0$, $N^j(v) = 0$

Then $\{N^{j-1}(v), N^{j-2}(v), \cdots, v\}$ is independent

In particular, $\dim C(v) = j$ = "length of cycle"

<u>Proof</u>: $a_1 N^{j-1}(v) + a_2 N^{j-2}(v) + \cdots + a_j v = 0$

Apply N to both sides: $a_2 N^{j-1}(v) + a_3 N^{j-2}(v) + \cdots + a_j N(v) = 0$

Apply N^{j-1} to both sides: $a_j N^{j-1}(v) = 0$

Since $N^{j-1}(v) \neq 0$ this implies $a_j = 0$

So we have $a_1 N^{j-1}(v) + a_2 N^{j-2}(v) + \cdots + a_{j-1} N(v) = 0$

Apply N^{j-2} to both sides, get $a_{j-1} N^{j-1}(v) = 0 \Rightarrow a_{j-1} = 0$

Continuing in this manner shows $a_i = 0$ for all i. ■

Properties of $C(v)$

1. $\dim C(v)$ = length of the cycle

2. $C(v)$ is invariant under N

Why? $C(v) = span\{N^{j-1}(v), \cdots, v\}$

Suffices to show that $N(N^i(v)) \in C(v)$, where $1 \leq i \leq j$.

$N(N^i(v)) = N^{i+1}(v) \in C(v)$

So we have that $N(C(v)) \subset C(v)$

3. $\alpha = \{N^{j-1}(v), \cdots, v\}$ is a basis of $C(v)$.

$N|_{C(v)} : C(v) \to C(v)$

$$[N|_{C(v)}]_\alpha = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & 0 & 1 \\ 0 & \cdots & & 0 \end{bmatrix}$$

4. $N^{j-1}(v)$ is an eigenvector of N with eigenvalue 0.

EX: $N = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$

check: $N^3 = 0$

$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ cycle is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$N(v) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, N^2(v) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, N^3(v) = 0$

$C(v) = \mathbb{C}^3$

$[N]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This is the first non-triv example of JCF

Ex: $N = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ check $N^2 = 0$

$v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $N(v) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $N^2(v) = 0$

$C(v) = sp\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is 2-dim $\left(\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \right)$

$[N|_{C(v)}]_\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$N(w) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $C(w) = sp\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$[N|_{C(w)}]_\beta = [0]$

Putting this together: $\delta = \alpha \cup \beta$ is a basis of $\mathbb{C}^3$ (need to check)

$\delta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ $[N]_\delta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ This is our second ex of JCF.

Prop: Let $\{v_1, \cdots, v_r\}$ be some vectors in V
$N: V \longrightarrow V$ nilpotent operator

Let $\alpha_1 = \{N^{j_1-1}(v_1), \cdots, v_1\}$ be the cycle of v_1

$\alpha_2 = \{N^{j_2-1}(v_2), \cdots, v_2\}$ be the cycle of v_2

$\alpha_3 = \{N^{j_3-1}(v_3), \cdots, v_3\}$ be the cycle of v_3

$\vdots$

$\alpha_r = \{N^{j_r-1}(v_r), \cdots, v_r\}$ be the cycle of v_r

If $\{N^{j_1-1}(v), \cdots, N^{j_r-1}(v_r)\}$ are linearly indpt. then $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_r$ is lin. ind.

In the second example $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\alpha_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\alpha_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ lin ind $\Rightarrow \alpha_1 \cup \alpha_2$ is a basis