Generalizations of IFT

IFT:

$F: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^1 \quad n \geq 1$

$F(\mathbf{x}_0, y_0) = 0$

$\partial_y F(\mathbf{x}_0, y_0) \neq 0$

Then

$\exists r_1, r_0 < r_1$ s.t.

$\forall \mathbf{x} \quad |\mathbf{x} - \mathbf{x}_0| < r_0 \Rightarrow$

$\exists! y$ st $|y - y_0| < r_1$ & $y = f(x)$

$F(\mathbf{x}, y) = 0$

$\frac{\partial f}{\partial x_j} = -\frac{\partial_{x_j} F(x,y)}{\partial_y F(x,y)}$

Generalization #1 3.3

$F: \mathbb{R}^n \longrightarrow \mathbb{R}^1 \quad n \geq 2$

$F(\mathbf{x}_0) = 0$

$\nabla F(\mathbf{x}_0) \neq 0$

so $\begin{bmatrix} \partial_1 F(\mathbf{x}_0) \\ \vdots \\ \partial_n F(\mathbf{x}_0) \end{bmatrix} \neq 0$ so

for some i $\partial_{x_j} F(\mathbf{x}_0) \neq 0$

let that x_j be the y

so $\exists r_1, r_0 \ldots$

so $x_j = f(\text{rest of the } x_i)$

Generalization #2 Thm 3.9

$\mathbb{F}: \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k \quad n \geq 1 \quad k \geq 2$

at least in $\mathbb{R}^2$ so we have a system

$\mathbb{F}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$, $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{y}_0 \in \mathbb{R}^k$

$\mathbb{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}$

$\left[\partial_{y_j} F_i\right](\mathbf{x}_0, \mathbf{y}_0) = B$

$\det B \neq 0$ or B is invertible

Then exists r_0, r_1 — r_1 → nbd of $\mathbf{y}_0$, r_0 → nbd of $\mathbf{x}_0$

St. $\forall \mathbf{x} \ \exists! \mathbf{y} \quad \mathbb{F}(\mathbf{x}, f(\mathbf{x})) = \mathbf{0}$

$f(\mathbf{x})$ is C^1

Generalization #3

Generalization of 3.9

$\mathbb{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^k \quad n \geq k+1$

$\mathbb{F}(\mathbf{x}_0) = \mathbf{0}$

rank $D\mathbb{F}(\mathbf{x}_0) = k$

$D\mathbb{F}(\mathbf{x}_0) = \left[\partial_{x_j} F_i\right]$

This means there is at least one $k \times k$ submatrix of $D\mathbb{F}(\mathbf{x}_0)$ with $\det \neq 0$. That submatrix corresponds to a set of k variables say $x_1 \ldots x_k$. Then

$\exists r_0, r_1$ — r_1 → nbd of $(x_1 \ldots x_k)$ st $\mathbf{y}$, r_0 → nbd $(x_{k+1} \ldots x_n)$ $\mathbf{x}$

st. $\forall \mathbf{x} \ldots$

$\exists! \mathbf{y} = f(\mathbf{x})$

i.e. $x_1 \ldots x_k$ can be written in terms of the rest.

See also IMT Thm 3.18

See also example 2, 2.10