Map of 1.6

1.21 Bolzano-Weierstrass Theorem (Characterization of Compactness)

S is compact $\iff$ ANY sequence $\{x_k\} \subset S$ has a subsequence $\{x_{k_j}\}$ that converges to a point $x \in S$.

Pf:

$\Rightarrow$) assume S is cpt & let $\{x_k\}$ be ANY sequence in S.

S is closed

$S = \overline{S}$

$x \in S$

S is bounded

$\{x_k\} \subset S$

by 1.19 (B-W2)

$\exists \{x_{k_j}\} \to$ Some x

Subsequence of $\{x_k\}$

$x \in \overline{S}$

1.14

$x \in \overline{S}$ iff a seq of S converg. to x

$\therefore \exists$ Subseq $\{x_{k_j}\} \to x \in S$

$\Leftarrow$) Assumption 1: ANY sequence $\{x_k\} \subset S$ will have a subsequence converging to some $x \in S$.

Assumption 2: (for the sake of a contradiction) S is NOT compact!!!

Either S is not closed OR S is not bounded

S is not closed:

$\exists a \in \overline{S} \setminus S$ or $\exists a \in \partial S$, $a \notin S$

$\forall r > 0 \ \exists x \in B(r, a) \cap S$

$r = 1 \quad \exists x_1$

$r = \frac{1}{2} \quad \exists x_2$

$\vdots$

$r = \frac{1}{k} \quad \exists x_k \in B(\frac{1}{k}, a) \cap S$

$\Rightarrow \{x_k\} \to a \notin S$

S is not bounded:

$\sim(\exists C : \forall x \in S \ |x| < C)$

not ($\underbrace{\qquad}_{\text{S is bounded}}$)

$\forall C \ \exists x \in S \ |x| \geq C$

let $C = 1$, $x_1 \in S$ st. $|x_1| \geq 1$

$\vdots$

$x_{k+1} \in S$ st $|x_{k+1}| \geq \max_{j \leq k}\{1, |x_j| + 1\}$

$\{x_k\}$ unbounded, not conv.

$\uparrow$ Both contradict the assumption

$\therefore$ S cannot be not cpt

1.22

Continuous image of a compact set is compact.

i.e. if S is compact and f is continuous then

$f(S) = \{f(x) : \underset{\text{all}}{x} \in S\}$ is also compact.

Pf

use 1.21, Bolzano-Weierstrass characterization of compactness:

S is cpt $\iff$ any seq $\{x_k\} \subset S$ has $\{x_{k_j}\} \xrightarrow{} \overset{\text{some}}{x} \in S$

or $f(S)$ is cpt $\iff$ any " $\{y_k\} \subset f(S)$ has $\{y_{k_j}\} \longrightarrow \underset{\text{some}}{y} \in f(S)$

pick any $\{y_k\} \subset f(S)$ $\longrightarrow$ $\exists \{x_k\} \subset S$ s.t. $f(x_k) = y_k$ for all k

$\downarrow$ S is cpt, so by 1.21

$\exists x \in S$ & $\{x_{k_j}\} \longrightarrow x$

1.15 $\leftarrow$ $f(x_{k_j}) \longrightarrow f(x)$ as f is cont.

$\{y_{k_j}\} \longrightarrow y = f(x) \in f(S)$

by 1.21

$f(S)$ is cpt

1.23 Extrem Value Theorem: if f is cont $f: S \to \mathbb{R}$ & S is cpt $\exists\, a, b \in S : \forall x \in S \; f(a) \le f(x) \le f(b)$

proof:

assumptions: S is compact & $f: S \to \mathbb{R}$ is continuous

$f(S)$ is compact by 1.22

$f(S) \subset \mathbb{R}$ is bounded

$f(S)$ is closed

by completeness of $\mathbb{R}$ $f(S)$ has lub & glb

$f(S) = \overline{f(S)}$

ϵ-characterisation of lub & glb of a set S:

$\forall \epsilon > 0 \; \exists x \in S \quad glb(S) \le x < glb(S) + \epsilon$

$\therefore \quad B(\epsilon, glb(S)) \cap S \ne \emptyset$

$\therefore \quad glb(S) \in \overline{S}$

$\therefore$ both $glb(f(S))$ and $lub(f(S))$ belong to $f(S)$

$\exists\, a, b \in S$ Such That

$f(a) = glb(f(S))$ & $f(b) = lub(f(S))$

definition of lub, glb

$\forall x \in S$

$f(a) \le f(x) \le f(b)$