Solutions to Midterm

Bonus: *(4 marks)* Suppose $X_1, \cdots, X_n$ are iid and follow the Normal distribution $N(\mu, 1)$ with unknown μ. Show the distributions of $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\sum_{i=1}^{n}(X_i - \bar{X})^2$.

The dist. of $\bar{X}$ is normal (1 mark) $N(\mu, \frac{1}{n})$ (1 mark)

The dist. of $\sum_{i=1}^{n}(X_i - \bar{X})^2$ is chi square (1 mark) $\chi^2(n-1)$ (1 mark)

Question 1: *(5 marks)* Consider $X_1, \cdots, X_n$ are iid Gamma distribution $G(\alpha/2, 1/2)$:

$$f(x|\alpha, \lambda) = \frac{1}{\Gamma(\alpha/2)} \frac{1}{2^{\alpha/2}} x^{\alpha/2-1} e^{-x/2}, \quad 0 \le x < \infty.$$

a. *(2 marks)* Find the moment estimate of α.

$$E(X) = \frac{\alpha/2}{1/2} = \alpha \qquad (1 \text{ mark})$$

$$\hat{\alpha} = \bar{X} \qquad (1 \text{ mark})$$

b. *(3 marks)* Find the sample distribution of the moment estimate by central limit theorem.

$$Var(X) = 2\alpha \qquad (1 \text{ mark})$$

$$\sqrt{n} \cdot \frac{\hat{\alpha} - \alpha}{\sqrt{2\alpha}} \longrightarrow N(0, 1) \qquad (2 \text{ marks})$$

Question 2: *(5 marks)* Consider $X_1, \cdots, X_n$ are iid Normal distribution $N(0, \sigma^2)$.

a. *(1 mark)* Compute the MLE estimate of σ^2.

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{X_i^2}{2\sigma^2}\right\}$$

$$\ell(\sigma^2) = \log(L(\sigma^2))$$

(0.5 mark) (Detail needed)

$$\frac{\partial \ell(\sigma^2)}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i^2)$$

(0.5 mark)

b. *(2 marks)* What's the distribution of the MLE estimate by the Exact method?

chi square distribution (1 mark)

$$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n)$$

(1 mark)

c. *(2 marks)* What's the distribution of the MLE estimate by the Large sample theory?

Fisher information for σ^2: $I(\sigma^2) = \frac{1}{2\sigma^4}$ (1 mark)

(Detail needed)

$$\sqrt{nI(\hat{\sigma}^2)}\,(\hat{\sigma}^2 - \sigma^2) \rightarrow N(0, 1)$$

(1 mark)

Question 3: *(5 marks)* Consider the model

$$\begin{aligned} X_1, \cdots, X_n | \theta &\sim \text{iid Normal}, N(\theta, 1), \\ \Theta &\sim N(2, 2) \end{aligned}$$

a. *(3 marks)* Find the posterior pdf of Θ. Justify your answer.

Normal distribution (1 mark)

Posterior dist. $N\left(\underbrace{\bar{X} \cdot \frac{2}{2 + 1/n} + 2 \cdot \frac{1/n}{2 + 1/n}}_{\text{(1 mark)}}, \underbrace{\frac{2/n}{2 + 1/n}}_{\text{(1 mark)}}\right).$

(Detail needed)

b. *(2 marks)* Suppose a sample size $n = 2$ results in the observations $X_1 = 0$ and $X_2 = 2$. Given these data and square loss, find the Bayes' estimate of θ.

Posterior mean (1 mark)

$$\hat{\theta} = \frac{0+2}{2} \frac{2}{2 + 1/2} + 2 \frac{1/2}{2 + 1/2}$$

$$= 1 \times \frac{4}{5} + 2 \times \frac{1}{5}$$

$$= \frac{6}{5} = 1.2$$

(1 mark)

Question 4: *(5 marks)* Consider $X_1, \cdots, X_n$ are iid Poisson distribution $Pois(\lambda)$ with pdf

$$P(X = x|\lambda) = \lambda^x e^{-\lambda}/x!.$$

a. *(3 marks)* Find the Fisher information. Justify your answer.

$l(\lambda) = \log L(\lambda) = x \log \lambda - \lambda - \log x!$ (1 mark)

$l''(\lambda) = -\frac{x}{\lambda^2}$ (1 mark)

Fisher infor $I(\lambda) = -E(l''(\lambda)) = \frac{E(X)}{\lambda^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$ (1 mark)

(Detail needed)

b. *(2 marks)* Find the efficient estimate of λ.

$E(\bar{X}) = \lambda$ 0.5 mark

$Var(\bar{X}) = \frac{Var(X)}{n} = \frac{\lambda}{n} = \frac{1}{nI(\lambda)}$ 0.5 mark

$\Rightarrow \bar{X}$ is efficient 1 mark

Question 5: *(5 marks)* Consider $X_1, \cdots, X_n$ are iid Exponential distribution $Exp(\lambda)$ with pdf

$$f(x|\lambda) = \lambda e^{-\lambda x} \text{ if } x \geq 0, \text{ otherwise } 0.$$

a. *(2 marks)* Determine the MLE of λ.

$$l(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} X_i \qquad (0.5 \text{ mark})$$

$$l'(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0 \qquad (0.5 \text{ mark})$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}} \qquad (1 \text{ mark})$$

(Detail needed)

b. *(3 marks)* Is the MLE a consistent estimate of λ? Justify your answer.

MLE is a consistent estimate of λ by large sample theory. (3 marks).

or: $\bar{X}$ is a consistent estimate of $E(X) = 1/\lambda$ by the law of large number. (1 mark)

So $\bar{X} \to 1/\lambda$ in prob. We prove $\frac{1}{\bar{X}} \to \frac{1}{1/\lambda}$ in prob.

$\forall \varepsilon > 0$

$$P\left(\left|\frac{1}{\bar{X}} - \frac{1}{1/\lambda}\right| > \varepsilon\right) = P\left(\left|\frac{\bar{X} - 1/\lambda}{\bar{X}/\lambda}\right| > \varepsilon\right)$$

$$= P(|\bar{X} - 1/\lambda| > \varepsilon|\bar{X}|/\lambda) = P(|\bar{X} - 1/\lambda| > \varepsilon|\bar{X}|/\lambda, |\bar{X} - 1/\lambda| < \tfrac{1}{2\lambda})$$

$$+ P(|\bar{X} - 1/\lambda| > \varepsilon|\bar{X}|/\lambda, |\bar{X} - 1/\lambda| \geq \tfrac{1}{2\lambda})$$

$$\leq P(|\bar{X} - 1/\lambda| > \varepsilon \cdot \tfrac{1}{2\lambda^2}) + P(|\bar{X} - 1/\lambda| \geq \tfrac{1}{2\lambda})$$

$\to 0$ by $\bar{X} \to \frac{1}{\lambda}$ in prob. (1 mark) (Detail needed)

MLE $\hat{\lambda}$ is a consistent estimate of λ. (1 mark)

Question 6: *(5 marks)* Consider $X_1, \cdots, X_n$ are iid Uniform distribution $U(-\theta, 0)$ with pdf

$$f(x|\theta) = 1/\theta \text{ if } -\theta \le x \le 0, \text{ otherwise } 0, \text{ where } 0 < \theta < \infty.$$

a. *(2 marks)* Determine the sufficient statistic for θ.

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}(-\theta, X_i) \quad (0.5 \text{ mark})$$

where $1(a,b) = 1$ if $a \le b$, otherwise 0.

$$\Rightarrow L(\theta) = \frac{1}{\theta^n} \mathbb{1}(-\theta, \min\{X_1, X_2, \cdots, X_n\}) \quad (0.5 \text{ mark})$$

(Detail needed)

$\Rightarrow$ $\min\{X_1, \cdots, X_n\}$ is the sufficient statistis for θ. (1 mark)

b. *(3 marks)* Find an unbiased estimate of θ based on the sufficient statistic.

Denote $X_{(1)} = \min\{X_1, \cdots, X_n\}$

$$P(X_{(1)} \le x) = 1 - P(X_{(1)} \ge x) = 1 - P(X_1 \ge x, \cdots, X_n \ge x)$$

$$= 1 - \prod_{i=1}^{n} P(X_i \ge x) = 1 - \left(\frac{-x}{\theta}\right)^n$$

$\Rightarrow$ pdf of $X_{(1)}$ is $\dfrac{n(-x)^{n-1}}{\theta^n}$ (1 mark)

$$E(X_{(1)}) = \int_{-\theta}^{0} x \cdot \frac{n(-x)^{n-1}}{\theta^n} dx = \int_{-\theta}^{0} (-x) \frac{n(-x)^{n-1}}{\theta^n} d(-x)$$

$t = -x$

$$= \int_{\theta}^{0} t \frac{n t^{n-1}}{\theta^n} dt = -\int_{0}^{\theta} \frac{n t^n}{\theta^n} dt = -\frac{n}{\theta^n} \frac{t^{n+1}}{n+1}\Big|_0^{\theta} = -\frac{n\theta}{n+1}$$

(1 mark)
(Detail needed)

$\Rightarrow$ The unbiased estimate of θ based on the sufficient stat. is

$$-\frac{(n+1)}{n} \min\{X_1, \cdots, X_n\} \quad (1 \text{ mark})$$