University of Toronto
Department of Mathematics

MAT224H1S
Linear Algebra II

Midterm Examination
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Duration: 1 hour 50 minutes

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

Tutorial Group: ______________________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

[10] **1.** Let T: $\mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Find the matrix of T with respect to the basis $\alpha = \{(1,0,0), (1,1,0), (1,1,1))\}$.

(b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution:

(a) We have (keeping in mind that we're working over $\mathbb{Z}_2$, where $2 = 0$ and $-1 = 1$)

$$\begin{aligned} T(1,0,0) &= (1,1,1) = [(0,0,1)]_\alpha \\ T(1,1,0) &= (2,1,2) = (0,1,0) = [(1,1,0)]_\alpha \\ T(1,1,1) &= (2,2,2) = (0,0,0) = [(0,0,0)]_\alpha. \end{aligned}$$

Thus

$$[T]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) We know that $\mathrm{Ker}(T) = \mathrm{null}([T]_\alpha)$ and $\mathrm{Im}(T) = \mathrm{col}([T]_\alpha)$, so let's row reduce $[T]_\alpha$:

$$[T]_\alpha \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\mathrm{null}([T]_\alpha) = \{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}_\alpha \} = \{(t,t,t)\} = \{(1,1,1)t\}.$$

So a basis for $\mathrm{Ker}(T)$ is $\{(1,1,1)\}$.

Next, as the leading 1s in $\mathrm{rref}([T]_\alpha)$ occur in columns 1 and 2, we conclude that the corresponding columns of $[T]_\alpha$ form a basis for its column space. Thus

$$\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_\alpha, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_\alpha \} = \{(1,1,1), (0,1,0)\}$$

is a basis for $\mathrm{Im}(T)$.

[10] **2.** Let $T\colon \mathbb{R}^4 \to P_2(\mathbb{R})$ be the linear transformation that is represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

relative to the standard bases of $\mathbb{R}^4$ and $P_2(\mathbb{R})$. Find the matrix of T with respect to the bases $\alpha = \{(1,0,0,0), (0,0,1,0), (1,-1,0,0), (0,-1,1,1)\}$ and $\beta = \{x^2+1, x, 1\}$.

Solution #1:

We have:

$$T(1,0,0,0) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\text{std}} = 1 + x^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\beta}.$$

Similarly,

$$\begin{aligned} T(0,0,1,0) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\text{std}} = x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\beta} \\ T(1,-1,0,0) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\text{std}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\beta} \\ T(0,-1,1,1) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\text{std}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\beta} \end{aligned}$$

Thus our desired matrix is

$$[T]_{\beta\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution #2:

Let $S_{\beta,\text{std}}$ and $S_{\text{std},\alpha}$ denote the change of bases matrices from the standard basis for $P_2(\mathbb{R})$ to β and from the α to the standard basis for $\mathbb{R}^4$, respectively. Then the matrix we want is

$$[T]_{\beta\alpha} = S_{\beta,\text{std}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} S_{\text{std},\alpha}.$$

So let's determine these change of basis matrices. Let's start with $S_{\beta,\text{std}}$:

$$[1]_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [x]_{\beta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [x^2]_{\beta} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Thus

$$S_{\beta,\text{std}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Next, let's find $S_{\text{std},\alpha}$:

$$[(1,0,0,0)]_{\text{std}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and similarly for the other vectors in α. Thus

$$S_{\text{std},\alpha} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally,

$$\begin{aligned} [T]_{\beta\alpha} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

[10] **3.** Let $W = \{p(x) \in P_2(\mathbb{R}) \mid p(0) = 0\}$. Show that W and $\mathbb{R}^2$ are isomorphic and find an isomorphism $T\colon W \to \mathbb{R}^2$.

Solution:

A polynomial $p(x) = a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R})$ is in W iff $p(0) = 0$ iff $a_0 = 0$. That is,

$$p(x) \in W \iff p(x) = a_1 x + a_2 x^2.$$

This shows that

$$W = \{a_1 x + a_2 x^2 \in P_2(\mathbb{R}) \mid a_1, a_2 \in \mathbb{R}\} = \operatorname{span}\{x, x^2\}.$$

Since the set $\{x, x^2\}$ is clearly linearly independent, we conclude that $\dim W = 2 = \dim \mathbb{R}^2$. Hence W and $\mathbb{R}^2$ are isomorphic, as they have the same dimension.

An isomorphism $T\colon W \to \mathbb{R}^2$ is given by

$$T(a_1 x + a_2 x^2) = (a_1, a_2).$$

Indeed, T is linear:

$$\begin{aligned} T((a_1 x + a_2 x^2) + \lambda(b_1 x + b_2 x^2)) &= T((a_1 + \lambda b_1)x + (a_2 + \lambda b_2)x^2) \\ &= (a_1 + \lambda b_1, a_2 + \lambda b_2) \\ &= (a_1, a_2) + \lambda(b_1, b_2) \\ &= T(a_1 x + a_2 x^2) + \lambda T(b_1 x + b_2 x^2); \end{aligned}$$

T is injective:

$$T(a_1 x + a_2 x^2) = (0,0) \iff (a_1, a_2) = (0,0) \iff a_1 x + a_2 x^2 = 0;$$

hence T is also surjective because $\dim W = \dim \mathbb{R}^2$. (We can also check surjectivity directly: given $(a,b) \in \mathbb{R}^2$, we have $ax + bx^2 \in W$ and $T(ax + bx^2) = (a,b)$, so $(a,b) \in \operatorname{Im}(T)$.)

[10] **4.** Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the linear transformation defined by

$$T(z_1, z_2, z_3) = ((1+i)z_1, -2iz_1 + (1+i)z_2 + 2iz_3, iz_1 + z_3),$$

where $\mathbb{C}^3$ is seen as a vector space over the field of complex numbers. Find the eigenvalues of T and bases for each of the corresponding eigenspaces.

Solution:

Let's find the standard matrix of T:

$$\begin{aligned} T(1,0,0) &= (1+i, -2i, i) \\ T(0,1,0) &= (0, 1+i, 0) \\ T(0,0,1) &= (0, 2i, 1). \end{aligned}$$

Thus

$$[T]_{\text{std}} = \begin{bmatrix} 1+i & 0 & 0 \\ -2i & 1+i & 2i \\ i & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\det([T]_{\text{std}} - \lambda I) = \det \begin{bmatrix} (1+i)-\lambda & 0 & 0 \\ -2i & (1+i)-\lambda & 2i \\ i & 0 & 1-\lambda \end{bmatrix} = ((1+i)-\lambda)((1+i)-\lambda)(1-\lambda),$$

where the last equality was obtained by cofactor expansion along the first row. So the eigenvalues of A are

$$\lambda = 1 \quad \text{and} \quad \lambda = 1+i$$

(with multiplicities 1 and 2, respectively).

To find the eigenvectors of $\lambda = 1$, we need to determine $\ker(T - \lambda I) = \ker(T - I)$. For this we row reduce $[T]_{\text{std}} - I$:

$$\begin{bmatrix} i & 0 & 0 \\ -2i & i & 2i \\ i & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + 2iR_1, R_3 - R_1} \begin{bmatrix} i & 0 & 0 \\ 0 & i & 2i \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \times \frac{1}{i}, R_2 \times \frac{1}{i}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\ker(T - I) = \{ \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} \}$$

and $\{(0, -2, 1)\}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

Now we do the same thing for $\lambda = 1+i$. Here we row reduce $[T]_{\text{std}} - (1+i)I$:

$$\begin{bmatrix} 0 & 0 & 0 \\ -2i & 0 & 2i \\ i & 0 & -i \end{bmatrix} \xrightarrow{R_2 + (2i)R_3, R_3 \times \frac{1}{i}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\ker(T-(i+1)I) = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and $\{(1,0,1),(0,1,0)\}$ is a basis for the eigenspace corresponding to $\lambda = 1+i$.

[10] **5.** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (ax_1 + bx_2, bx_1 + ax_2 + bx_3, bx_2 + ax_3).$$

Show that T is diagonalizable for all values of $a, b \in \mathbb{R}$.

Solution:

Let's find the standard matrix of T:

$$\begin{aligned} T(1,0,0) &= (a,b,0) \\ T(0,1,0) &= (b,a,b) \\ T(0,0,1) &= (0,b,a). \end{aligned}$$

Thus

$$[T]_{\text{std}} = \begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{aligned} \det \begin{bmatrix} a-\lambda & b & 0 \\ b & a-\lambda & b \\ 0 & b & a-\lambda \end{bmatrix} &= (a-\lambda)((a-\lambda)^2 - b^2) - b(b(a-\lambda)) \\ &= (a-\lambda)((a-\lambda)^2 - 2b^2) \\ &= (a-\lambda)(a-\lambda-b\sqrt{2})(a-\lambda+b\sqrt{2}b). \end{aligned}$$

Hence the eigenvalues of T are

$$\lambda = a, a \pm b\sqrt{2}.$$

So if $b \neq 0$ then we have 3 *distinct* eigenvalues, and since $[T]_{\text{std}}$ is 3×3, this means that T is diagonalizable in this case. On the other hand, if $b = 0$ then we see that $[T]_{\text{std}} = \text{diag}(a,a,a)$ is already diagonal. So in either case T is diagonalizable.

[**Side note:** A very quick proof is possible if we simply observe that the matrix $[T]_{\text{std}}$ is symmetric hence diagonalizable, by a theorem to be covered later in the course.]

[10] **6.** Let $T\colon V \to W$ be an injective linear transformation. Prove that if $T(v_4)$ is dependent on $\{T(v_1), T(v_2), T(v_3)\}$, then v_4 is dependent on $\{v_1, v_2, v_3\}$.

Solution:

If $T(v_4)$ is dependent on $\{T(v_1), T(v_2), T(v_3)\}$ then we can find scalars a_1, a_2, a_3 such that

$$\begin{aligned} T(v_4) &= a_1T(v_1) + a_2T(v_2) + a_3T(v_4) \\ &= T(a_1v_1 + a_2v_2 + a_3v_3). \end{aligned}$$

Hence

$$0 = T(v_4) - T(a_1v_1 + a_2v_2 + a_3v_3) = T(v_4 - (a_1v_1 + a_2v_2 + a_3v_3)),$$

i.e., $v_4 - (a_1v_1 + a_2v_2 + a_3v_3)$ is in the kernel of T. But T is injective, so its kernel is $\{0\}$, and consequently

$$v_4 - (a_1v_1 + a_2v_2 + a_3v_3) = 0,$$

or, in other words,

$$v_4 = a_1v_1 + a_2v_2 + a_3v_3.$$

This shows that v_4 is dependent on $\{v_1, v_2, v_3\}$, as desired.