Tutorial Problems - Sections 6 to 7 - MAT 327 - Summer 2014

6 Continuous Functions and Homeomorphisms

1. Give examples of topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and a function $f : X \longrightarrow Y$ such that
   (a) f is open but not closed.
   (b) f is closed but not open.
2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and define $g : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $g(x) = (x, f(x))$. Prove that g is continuous.
3. Prove that the function $\times : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\times(a, b) = a \cdot b$ is a continuous function.
4. Assuming the previous problem and all the material proved in class, convince us that the determinant function $\det : M_n \longrightarrow \mathbb{R}$ is continuous, (where M_n is the collection of all $n \times n$ matrices with real entries).
5. Give an example of a homeomorphism $f : (X, \mathcal{T}) \longrightarrow (X, \mathcal{U})$, where $\mathcal{T}$ does not refine $\mathcal{U}$ and $\mathcal{U}$ does not refine $\mathcal{T}$ (hint: we've mentioned this in one of the first lectures).
6. Give an example of a homeomorphism $f : (X, \mathcal{T}) \longrightarrow (X, \mathcal{U})$, where $\mathcal{T} \subsetneq \mathcal{U}$.
7. Recall that $A \subseteq \mathbb{R}$ is **convex** if whenever $a < b < c$ and $a, c \in A$ then $b \in A$. Prove that a convex subset of $\mathbb{R}_{\text{usual}}$ is never homeomorphic to a non-convex subset of $\mathbb{R}_{\text{usual}}$. (Hint: You may wish to use the intermediate value theorem.)
8. Remind us of the defintion of convex in $\mathbb{R}^2$ and convince us that the previous problem is false in $\mathbb{R}^2$.
9. Using only material from 1st year calculus, convince us that $[0, 1]$ is not homeomorphic to $\mathbb{R}$ (both with their usual topology).

7 Subspaces

1. Give an example of a topological space $(X, \mathcal{T})$, a subspace $(A, \mathcal{T}_A)$ of $(X, \mathcal{T})$, and a closed set in $(A, \mathcal{T}_A)$ that is not closed in $(X, \mathcal{T})$. (Is there more than one way to "break" this question?)

2. Let $(X, \mathcal{T}), (Y, \mathcal{U})$, and $(Z, \mathcal{V})$ be topological spaces such that there is an embedding of X in Y and an embedding of Y in Z. Prove that there is an embedding of X in Z. (X is embedded in Y if X is homeomorphic to a subspace of Y.)

3. Let $(X, \mathcal{T})$ be a topological space, and let $B \subseteq A \subseteq X$. Show that the boundary of B, considered as a subset of A, is a subset of the boundary of B, considered as a subset of X, intersected with A.

4. Let A be an open subset of a separable space $(X, \mathcal{T})$. Prove that $(A, \mathcal{T}_A)$ is separable.

5. Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(X, \mathcal{T})$. Prove that the inclusion map $i : A \longrightarrow X$ defined by $i(a) = a$ for each $a \in A$ is continuous.

6. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $(A, \mathcal{T}_A)$ be a subspace of $(X, \mathcal{T})$, and let $f : X \longrightarrow Y$ be a continuous function. Prove that $f|_A : A \longrightarrow Y$ is continuous.

7. Let $\mathcal{B}'$ be the collection of all open disks in $\mathbb{R}^2$ with a finite number of straight lines through the center removed, and let

$$\mathcal{B} = \{\, B \cup \{c\} : B \in \mathcal{B}' \text{ and } c \text{ is the center of } B \,\}$$

   (a) Show that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2$.
   (b) Compare $\mathcal{T}$ with the usual topology $\mathcal{U}$ on $\mathbb{R}^2$.
   (c) Let A denote a striaght line in $\mathbb{R}^2$. Describe $\mathcal{T}_A$.
   (d) Let A denote a circle in $\mathbb{R}^2$. Compare $\mathcal{T}_A$ and $\mathcal{U}_A$.

8. Let $\mathcal{T}$ denote the subspace topology on $[0, 1)$ determined by the usual topology on $\mathbb{R}$, and let $\mathcal{U}$ denote the subspace topology on $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ determined by the usual topology on $\mathbb{R}^2$. Define $f : [0, 1) \longrightarrow (S^1, \mathcal{U})$ by

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

   (a) Prove that f is a bijection.
   (b) Prove that f is continuous.
   (c) Prove that f^{-1} is not continuous.