UNIT 3: DERIVATIONS FOR SENTENTIAL LOGIC
NATURAL DEDUCTION
Part 1

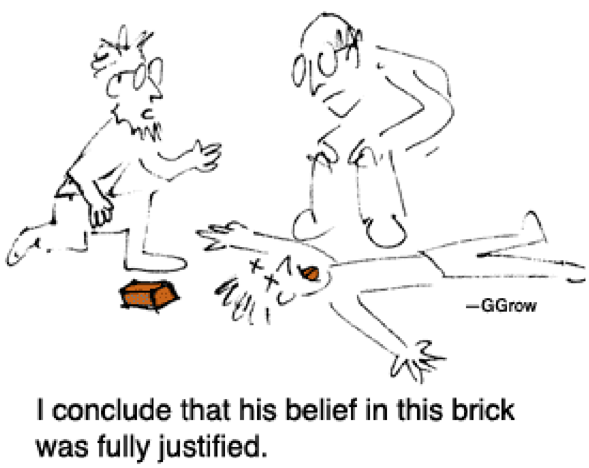
3.1 What is a derivation?

A derivation is a proof or demonstration that shows how a sentence or sentences can be derived (obtained by making valid inferences) from a set of sentences. A derivation can be used to demonstrate that an argument is valid; that a sentence is a tautology or that a set of sentences is inconsistent. In a derivation, you are proving that the conclusion logically follows from the premises.

We'll be using a natural deduction system for first-order logic that uses the symbolic language that we have learned. Our system is based on that presented by Kalish and Montague in their text, *Techniques of Formal Reasoning*.[1] All of our derivation rules are truth-preserving, so that if we follow the rules and the premises are true, we can only derive true conclusions.

Every sentence that is logically entailed by a set of sentences can be derived from that set of sentences using our derivation system – the system is complete. Every one of the infinite number of valid theorems and valid arguments (within the scope of first-order sentential logic) can be proven.

Every argument arrived at through our sentential derivation system will be deductively valid – our system is consistent. Thus, true premises will always lead to a true conclusion.



Our natural deduction system should

be a little less painful than this!

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[1] Kalish, Donald, and Montague, Richard, 1964. *Logic: Techniques of Formal Reasoning*. Harcourt, Brace, and Jovanovich.

Our system is based on Kalish and Montague's natural deduction system for first-order logic, an elegant logical system that treats negation and material conditional as primary. It uses three types of derivation which we will be learning in this unit: direct derivation, indirect derivation and conditional derivation. Except for a single rule of inference (*modus ponens)* no other logical symbol or rule of inference is necessary to express any sentence (for any possible truth-value assignment) or to complete a derivation in sentential logic. However, the other logical operators and rules of inference will make things a little easier and more intuitive!

[2] This illustration is the property of Gerald Grow, Professor of Journalism, Florida A&M University. http://www.longleaf.net/ggrow/CartoonPhil.html

Three Types of Derivation for Sentential Logic:

Direct Derivation

Using the premises, you derive the sentence that you want to prove through the application of the derivation rules.

Conditional Derivation

You can derive a conditional sentence by assuming the antecedent and deriving the consequent from it using the derivation rules.

Indirect Derivation

You can derive a sentence of any form by assuming its negation and deriving a contradiction from it. If you can show that one sentence leads to a contradiction, then you can infer that that sentence is false, and hence, you can infer its negation. To derive a contradiction, you derive a sentence and its negation using the derivation rules.

Derivation Rules:

The derivation rules determine what sentences you can derive from the sentences that you already have (either premises or other sentences that you have derived) and provides the justification for each step in the derivation. Each rule is itself a valid argument form; or a pattern of valid reasoning.

Basic Rules: There are ten basic rules in sentential logic. The rules for conjunction ($\wedge$), disjunction ($\vee$) and biconditional ($\leftrightarrow$) come in pairs: we need rules that allow us to move from sentences without these connectives to sentences that have connectives as the main logical operators (introduction rules); and we need rules that allow us to move from sentences for which these are the main logical operators to sentences that do not (elimination rules). The basic rules for negation ($\sim$) and conditional ($\rightarrow$), the primary logical operators, are a little different because conditional and indirect derivations are the primary methods for introducing the conditional sign and both introducing and eliminating the negation sign.

Derived Rules: In addition to the basic rules, there are a potentially endless number of derived rules. A derived rule is a rule that is developed out of a pattern of valid reasoning – once you've shown that the pattern of reasoning is valid (given certain starting conditions), you can validly deduce the end product without going through the whole reasoning process. It's a short-cut. We will be introducing five derived rules, each of which you will prove with the basic rules.

Theorems as Rules: Finally, the derivation system allows us to derive an infinite number of theorems – sentences that can be validly derived from the empty set (from no premises at all). These are tautologies, such as "$P \vee \sim P$" or "It will rain tomorrow or it won't." It is not logically possible for such sentences to be false. And every tautology can also be used as a derived derivation rule. Once you've proved it, you can use the theorem as a short cut!

3.2 The Basic Rules for ~ and →

The basic rules concern the first two logical operators we discussed – negation and conditional. The Greek letters ϕ (phi) and ψ (psi) can represent any sentence, whether atomic or molecular.

Modus Ponens (mp or MP)

$(\phi \rightarrow \psi)$

ϕ

———————

ψ

This rule allows us to infer the consequent of a conditional from a conditional sentence and the antecedent.

'Modus ponens' is a Latin term. It's short for 'Modus ponendo ponens', which means, 'The mode of argument that asserts by asserting.' By asserting the antecedent, you can assert the consequent as your conclusion.

This makes sense – with a conditional sentence if the antecedent is true so is the consequent. We just follow the direction of the arrow.

Modus Tollens (mt or MT)

$(\phi \rightarrow \psi)$

$\sim\psi$

———————

$\sim\phi$

This rule allows us to infer the negation of the antecedent from a conditional sentence and the negation of the consequent.

'Modus tollens' is short for 'Modus tollendo tollens' which means 'The mode of argument that denies by denying.' By denying the consequent, you can conclude the denial of the antecedent.

This makes sense – with a conditional sentence the antecedent cannot be true and the consequent false – thus if the consequent is false, the antecedent must be false as well.

Double Negation (dn or DN)

ϕ	$\sim\sim\phi$
———————	———————
$\sim\sim \phi$	ϕ

This rule allows us to infer from a sentence the same sentence with two negations in front of it, or to infer the unnegated sentence from the sentence with two negations in front of it.

This makes sense – a doubly negated sentence is logically equivalent to the unnegated sentence.

Repetition (r or R)

ϕ

———————

ϕ

This rule allows us to infer a sentence from itself. This may look too trivial to be a rule, but it we will need it!

Rules of inference are valid no matter how simple or complex the sentences being inferred are. What matters is the logical form of the inference. No matter how complex the components are, as long as the argument fits the pattern, the rule applies.

The following are all valid instances of the rule:

Modus Ponens:	$P \to Q$ P $\therefore Q$	$(\sim P \to R) \to S$ $(\sim P \to R)$ $\therefore S$	$(P \to \sim T) \to ((T \to S) \to P)$ $(P \to \sim T)$ $\therefore (T \to S) \to P$
Modus Tollens:	$P \to Q$ $\sim Q$ $\therefore \sim P$	$(\sim P \to R) \to S$ $\sim S$ $\therefore \sim (\sim P \to R)$	$(P \to \sim T) \to ((T \to S) \to P)$ $\sim ((T \to S) \to P)$ $\therefore \sim (P \to \sim T)$
Double Negation:	$P \to Q$ $\therefore \sim\sim(P \to Q)$	$(\sim P \to R) \to S$ $\therefore \sim\sim((\sim P \to R) \to S)$	$(P \to \sim T) \to ((T \to S) \to P)$ $\therefore \sim\sim[(P \to \sim T) \to ((T \to S) \to P)]$
Repetition	$P \to Q$ $\therefore P \to Q$	$(\sim P \to R) \to S$ $\therefore (\sim P \to R) \to S$	$(P \to \sim T) \to ((T \to S) \to P)$ $\therefore (P \to \sim T) \to ((T \to S) \to P)$

Fallacies (Errors in Reasoning):

Be careful … especially with MP and MT.

The following are INVALID applications of the rules! They are errors in reasoning.

Don't do this …

$P \to Q$
Q
$\therefore P$

FALLACY!

This error in reasoning is called the fallacy of affirming the consequent. Modus Ponens goes in the direction of the arrow… move from the antecedent to the consequent!

Don't do this …

$P \to Q$
$\sim P$
$\therefore \sim Q$

FALLACY!

This error in reasoning is called the fallacy of denying the antecedent.

The rules of inference act on the entire sentence … not on sentential components!

Don't do this …

$(P \to Q) \to R$
P
$\therefore Q \to R$ MP

ERROR!

MP can only be used if the conditional being acted on is the full sentence (in this case it is NOT – rather it is in the antecedent of another conditional.

Make sure you can identify the main connective of a sentence. That is the logical operator that the rule pertains to.

Here is another case: Remember rules of inference can't work for sentential components!

Don't do this … $P \rightarrow {\sim}{\sim}Q$ $\therefore P \rightarrow Q$ dn

ERROR!

The full sentence must be doubly negated. Here DN is incorrectly used to remove the double negation on the consequent.

In the case of DN, it is merely a formal error but not a logical error. (We could introduce a rule of replacement that allows us to replace a doubly negated sentential component with its unnegated counterpart.)

However, it is still an error in our system!

3.2 E1

Which inference rule justifies the following arguments? (mp, mt, dn or none)

a) $\sim R \rightarrow P$, $\sim R$, $\therefore P$ — mp

b) $\sim\sim S \rightarrow T$, $\therefore S \rightarrow T$ — None! dn cannot be used on a sentential part

c) $P \rightarrow \sim Q$, Q, $\therefore \sim P$ — None! First you have to use dn then you can use mt

d) $(P \rightarrow \sim R) \rightarrow \sim S$, $\sim\sim S$, $\therefore \sim(P \rightarrow \sim R)$ — mt

e) $\sim(\sim P \rightarrow Q)$, $\therefore P \rightarrow Q$ — none

f) $P \rightarrow (P \rightarrow \sim P)$, P, $\therefore P \rightarrow \sim P$ — mp

g) $S \rightarrow R$, $\sim P$, $\therefore \sim S$ — none

h) $Q \rightarrow (S \rightarrow P)$, $\sim (S \rightarrow P)$, $\therefore \sim Q$ — mt

3.2 E2

What can you infer (if anything) in one step from the following? What rule of inference are you using? (mp, mt, dn)

a) $P \rightarrow R$, $\sim P$, $\therefore ?$ — Nothing with mp/mt

b) $\sim\sim(V \rightarrow W)$, $\sim W$, $\therefore ?$ — Nothing in one step with mp/mt. After dn on the first premise, mt yields ~V

c) $\sim S \rightarrow \sim\sim T$, $\sim S$, $\therefore ?$ — ~~T mp

d) $\sim Y \rightarrow \sim Z$, $\sim Z$, $\therefore ?$ — Nothing with mp/mt

e) $P \rightarrow (Q \rightarrow R)$, $\sim Q \rightarrow R$, $\therefore ?$ — Nothing with mp/mt

f) $P \rightarrow (Q \rightarrow R)$, $\sim(Q \rightarrow R)$, $\therefore ?$ — ~P mt

g) $\sim\sim(\sim P \rightarrow \sim\sim\sim Q)$, $\therefore ?$ — (~P->~~~Q) dn

h) $\sim Z \rightarrow \sim X$, $\therefore ?$ — ~~(~Z->~X) dn

i) $(P \rightarrow Q) \rightarrow R$, $P \rightarrow Q$, $\therefore ?$ — R mp

j) $X \rightarrow \sim Y$, Y, $\therefore ?$ — Nothing in one step. After dn on the second premise, MT gets you ~X.

k) $\sim W \rightarrow (Z \rightarrow \sim X)$, $\sim\sim X$, $\therefore ?$ — Nothing

l) $(\sim P \rightarrow R) \rightarrow \sim Q$, $\sim\sim Q$, $\therefore ?$ — ~(~P->R) mt

3.3 The Direct Derivation

What does a derivation look like? A Simple Direct Derivation

Line #'s	**Symbolic Sentences**	**Justifications and Annotations**
1	~~Show~~ ϕ	Show Sentence. It states the goal, what sentence you want to prove.
2	Premise	Premise
3	Premise	Premise
4	Derived Sentence	Line number(s), Rule
5	Derived Sentence	Line number(s), Rule
6	Derived Sentence	Line number(s), Rule
7	ϕ	Line number(s), Rule
8		The last line, 7, was what you wanted to show, and now the derivation has shown that the sentence, ϕ, can be inferred from your premises. The derivation is complete, showing that the argument is valid.

Line Numbers:

Line numbers, written on the far left, keep your derivation organized. Justifications refer to the line numbers.

Symbolic Sentences:

The symbolic sentences form your derivation or proof. The proof or derivation shows that the sentence is true, given the premises.

Main Show Line (line 1 above): The first line states what is being proved. When first written (before it is shown or proved) "show" will not be crossed out. Later, when it is proven, "show" is crossed out and a box drawn around the proof (as shown here).

After a "show line", indent. This makes it clear that the subsequent lines are the proof for the sentence you are showing.

Justifications and Annotations:

Every time you write down a symbolic sentence as part of your proof, you have to give a justification to explain where the sentence comes from (perhaps it's a premise or it follows from other sentences by some logical rule, etc.) Normally the justification includes a line number or two, as well as the rule used to derive the sentence. You may want to use further annotations to make your reasoning clearer.

The Box

The 'box' is drawn around the completed proof for a sentence (from the first line after the show line to the last line, with the final justification). It can be drawn as a closed box or a box that is open on the right hand side (as shown above.)

Direct Derivation: Step by Step

Consider the following argument:

> Mary will pass only if the course isn't difficult. If the professor is mean then the course will be difficult. So Mary will not pass! The professor is as mean as they come.

We might reason as follows:

> We were given the following three facts: If Mary passes then the course isn't difficult. If the professor is mean then the course is difficult. And the professor is mean. From these last two facts we can conclude that the course is difficult. So we can infer that it is not the case that the course isn't difficult. But Mary passes only if it isn't difficult. Therefore, Mary won't pass.

If we symbolize the argument we can use the inference rules to derive the conclusion from the premises, showing that it is a valid argument.

P: Mary will pass.

Q: The course is difficult.

R: The professor is mean.

The argument can be symbolized: $P \rightarrow \sim Q$. $R \rightarrow Q$. R. $\therefore \sim P$

Here the argument is put on one line, with a period after each premise (rather than in standard form).

Now the first line of the derivation:

1	Show ~P	The first thing we want to do is state what our goal is … to show that ~ P, the conclusion. Every derivation begins with a 'show line'. If ϕ is the conclusion of the argument, the first line is "show ϕ". This line needs no justification. It is what we are trying to achieve. However, you can put in an annotation that states what you are showing – for the first line of the derivation this would be "show conc" or "show conclusion".

Now we want to put in the premises of the argument.

After any show line you must indent all the lines, showing that the indented lines are the proof for the sentence on the show line.

$P \rightarrow \sim Q$. $R \rightarrow Q$. R. $\therefore \sim P$

1	Show ~P	Show conc	
2	$P \rightarrow \sim Q$	pr1	This is the first premise
3	$R \rightarrow Q$	pr2	This is the second premise
4	R	pr3	This is the third premise

Next we can start symbolizing the reasoning …

P→~Q.   R →Q.   R. ∴ ~P

1	Show ~P	Show conc.	
2	P → ~Q	pr1	
3	R → Q	pr2	
4	R	pr3	
5	Q	3 4 mp	Modus ponens allows us to infer the consequent, Q, from a conditional, R→Q, and the antecedent, R – lines 3 and 4. The justification states the two relevant line numbers and the rule, mp.
6	~~Q	5 dn	Double negation allows us to infer a doubly negated sentence, ~~ Q, from the unnegated sentence, Q – line 5. The justification states the relevant line number and the rule, dn.
7	~P	2 6 mt	Modus tollens allows us to infer the negation of the antecedent, ~P, from a conditional, P→~Q, and the negation of the consequent, ~~Q – lines 2 and 6. The justification states the relevant line number and the rule, mt.

Line 7 is what we were trying to show. Through a direct derivation we have shown that the conclusion follows from the premises. Now we can box the proof by drawing a box from immediately below the show line to just after the last line of the proof. We then cancel the show line by crossing out the word 'show' (leaving us just with the shown sentence) and claim victory.

P→~Q.   R →Q.   R. ∴ ~P

1	~~Show~~ ~P		
2	P → ~Q	pr1	
3	R → Q	pr2	
4	R	pr3	
5	Q	3,4 mp	
6	~~Q	5 dn	
7	~P	2, 6 mt	
8		7 dd	Line 7 is what we wanted to show. The justification states that line 7 shows that we have succeeded in the direct derivation.

In this derivation, line 8 is empty. It really just says that line 7 completes the direct derivation – we've successfully shown that line 7 (the conclusion) follows from the premises.

A few things to shorten the proof:

We don't have to repeat the premises. We can just refer to them in the justifications.

We could just write 'dd' after the justification on line 7. (Not use a new line for that.)

P→~Q. R →Q. R. ∴ ~P

1	~~Show~~ ~P	
2	Q	pr2 pr3 mp
3	~~Q	2 dn
4	~P	pr1 3 mt dd

Also, in the above version of the proof, the 'box' is open on the right side. This is just another representational style and is equally acceptable.

Either the closed or the open box makes it clear that all the lines within the box function as a proof of the line immediately above the box (the canceled show line.)

The 'show conc.' or 'show conclusion' annotation is also left off the first line. It is not part of the proof, but such annotations can help you keep track of what you are doing.

Available lines:

Some lines are 'available' in that the sentences written on the line can be used to infer other sentences from. The available lines are the ones that you use in your argument. But not all lines are available.

Unavailable lines:

- Uncanceled show lines. The show line states what you are trying to prove – it would be a poor argument indeed if you used the conclusion as part of your reasoning.
- Boxed lines. Lines that are already boxed cannot be used. If the line is in a box, then it may follow from an assumption that you are no longer working under.

Available lines:

- Canceled show lines. These are sentences that you have already shown – so they are good! Once you have shown it, you can use it.
- Previously derived lines – provided they are the same assumption, under the same uncanceled show lines. This should be any line that is neither boxed off, nor an uncanceled show line.
- Premises

Completed Derivation:

A derivation is **complete** if and only if all the show lines are canceled and all the lines that aren't show lines are boxed. (And, of course, if and only if every line is legitimate!)

A complete derivation shows that the argument is deductively valid. The derivation validates the argument, or proves the conclusion follows from the premises.

A few things to remember for a direct derivation:

- The first step of any derivation is a 'show line' – you write the word 'show' and then the symbolic sentence that you are going to prove. The first line of a derivation is a show line for the conclusion of the argument. But, you can introduce a show line at any time, for any sentence (although, of course, not all sentences can be shown!) You don't have to give a justification for a show line, although an annotation may be helpful especially in longer derivations.
- Premises can be introduced on any line of the derivation. The justification is 'pr' followed by the premise number, (e.g. pr1 is the first premise). But you don't have to put them under the show line. You can just refer to them in your justification (pr1 or pr2).
- You can introduce a sentence on a line if and only if it can be inferred from previous available lines by a rule of inference. The justification states the lines from which it is legitimately inferred and the rule of inference.
- Once you have introduced a sentence that is identical to the sentence in the nearest, earlier uncanceled show line, then you have all but completed the direct derivation. You should write 'dd' in the justification column, draw a box around all the lines following the show line (including the one with dd written on it) and cross out the word 'show' on the show line by putting a line through it like this: ~~show~~. This is canceling the show line. Now that sentence is shown and your proof is complete!
- Note: it is okay either to write 'dd' after the justification for the shown sentence on the last line of your proof, or to write 'dd' on the next line (a blank line). Both are correct notations.
- Note: the box can be completely closed or open on the right hand side. Both are correct notations.

3.3 E1:

Check the work in the following derivations. Does each line follow from available lines using the rule cited?

(a) ~T → ~S. R→~~T. S. ∴R

1	~~Show~~ ~R		ERROR. Show line incorrect.
2	S	pr3	
3	~T → ~S	pr1	
4	T	2 3 mt	ERROR. You need the negated consequent to use mt.
5	R → ~~T	pr2	
6	R → T	5 dn	ERROR. dn cannot be used on a sentential component.
7	R	4 6 mp	ERROR. mp moves from a conditional and thr antecedent to the consequent.
8		7 dd	

This cannot be fixed. It is not valid, so the conclusion cannot be derived from the premises.

(b) ~(P → ~Q) → ~~S.   Q → ~S.   Q ∴~P

1	~~Show~~ ~P		
2	~S	pr2 pr3 mp	
3	~~~(P → ~Q) → S~~	pr1 dn	ERROR. dn cannot be used on a sentential component.
4	~~(P → ~Q) ~~~S	2 3 mt	change to dn 2
5	P → ~Q	4 dn	
6	~~Q	pr3 dn	
7	~P	5 6 mt	
8		7 dd	

(c) Z→ (X →~W).   ~Z→ ~X.   X.   ∴~W

1	~~Show~~ ~W		
2	X	pr3	
3	~~X	2 dn	
4	~Z→~X	pr2	ALL CORRECT!
5	~~Z	3 4 mt	
6	Z	5 dn	
7	Z→ (X →~W).	pr1	
8	X →~W	6 7 mp	
9	~W	2 8 mp dd	

(d) ~P → S.   ~S.   Q → (P → Q).   ~(~S → Q) → ~P.   ∴P → Q

1	~~Show~~ P → Q		
2	~P → S	pr1	
3	~S	pr2	
4	~~P	2 3 mt	
5	P	4 dn	
6	Q	1 5 mp	VERY BIG ERROR! The show line cannot be used in your proof. After all, it is what you are trying to show! You can use canceled show lines, but those would be in a previous subderivation (section 3.6)
7	Q → (P → Q)	pr3	
8	P → Q	6 7 mp dd	

3.3 E2

Construct direct derivations for the following, showing that the conclusion can be validly inferred from the premises.

(a) P → Q.   R → ~Q.   ~S → R.   P.   ∴S

(b) Y.   X → (Y → Z).   ~X → ~W.   W.   ∴ ~~Z

(c) P → (Q → (R → S)).   ~~P.   (R → S) → ~P.   ∴ ~Q

(d) (P → ~Q) → (~R → S).   ~S.   ~(P → ~Q) → T.   T → S.   ∴R

skipped

3.4 Conditional Derivations

One of the most powerful techniques for proving that an argument is valid is the 'conditional proof'. A conditional derivation or proof allows us to derive a conditional sentence by considering what follows from the assumption that the antecedent is true. By assuming that something, P, is the case, one can show that *if* P *then* Q (where Q follows from P.) To derive a conditional, assume the antecedent and derive the consequent from it.

This type of argument is sometimes called hypothetical reasoning or hypothetical inference.

The reasoning would go something like this … Suppose we know the following:

> If Poppy studies, then Poppy will pass.
>
> If Poppy passes, then Poppy will graduate.
>
> Poppy will lose her job only if she doesn't graduate.

We can't use these facts to determine whether or not Poppy will lose her job. However, we can draw a hypothetical conclusion – if Poppy studies then she will not lose her job.

The rules we have so far (mp, mt, dn and r) aren't powerful enough to show that this conclusion follows from the premises using a direct derivation. We need to be able to make the assumption that Poppy studies, and show that if that is the case then she won't lose her job. That's what a conditional derivation does.

Let's try it out:

> P: Poppy studies.
>
> Q: Poppy will pass.
>
> R: Poppy will graduate.
>
> S: Poppy will not lose her job.

The argument can be symbolized: $P \rightarrow Q$. $Q \rightarrow R$. $\sim S \rightarrow \sim R$. $\therefore$ $P \rightarrow S$

1	Show $P \rightarrow S$	show conc.	The first line is still a show line. We state what we are going to prove.
2	P	ass cd	Assumption for conditional derivation. This justification shows that we are making an assumption for the sake of a conditional derivation. We assume the antecedent of the conditional that we want to show. Now we have a new goal, S. We want to show that S follows from this assumption. When using cd, the show line is a conditional sentence, and the first line after the show line is the assumption of the antecedent.

Here is the rest of the proof:

$P\to Q$. $Q \to R$. $\sim S \to \sim R$. $\therefore$ $P\to S$

1	Show $P\to S$	show conc.	
2	P	ass cd	Remember, our new goal is to show that S follows from P.
3	$P\to Q$	pr1	We need to use our premises too! The first premise lets us use P to derive Q.
4	Q	2 3 mp	Using mp, we can infer Q from the first premise and line 2 – the assumption, P, that we are working under.
5	$Q\to R$	pr2	The second premise will let us derive R from Q.
6	R	4 5 mp	Using mp again, we can infer R from the second premise and line 4.
7	$\sim S\to\sim R$	pr3	Here's the last premise we have to work with. We could use mt with it to get ~~S (which is logically equivalent to S, which is what our goal is) if we had the negation of the consequent, ~~R.
8	$\sim\sim R$	6 dn	
9	$\sim\sim S$	7 8 mt	
10	S	9 dn	Now we have shown on lines 2-10 that if we assume P then S follows. That's what we wanted to show – if P then S.

$P\to Q$. $Q \to R$. $\sim S \to \sim R$. $\therefore$ $P\to S$

1	~~Show~~ $P\to S$	show conc.
2	P	ass cd
3	$P\to Q$	pr1
4	Q	2 3 mp
5	$Q\to R$	pr2
6	R	4 5 mp
7	$\sim S\to\sim R$	pr3
8	$\sim\sim R$	6 dn
9	$\sim\sim S$	7, 8 mt
10	S	9 dn
11		10 cd

.

We complete the derivation by canceling the show, boxing the derivation and writing 'cd'.

Now it is a compete conditional derivation

Here's another.

$\sim(R \rightarrow S) \rightarrow \sim P. \quad \therefore P \rightarrow (R \rightarrow S)$

1	~~Show~~ $P \rightarrow (R \rightarrow S)$		The show line is for a conditional sentence.
2	P	ass cd	The next line is the assumption of the antecedent of the show line.
3	$\sim(R \rightarrow S) \rightarrow \sim P$	pr1	
4	$\sim\sim P$	2 dn	
5	$\sim\sim(R \rightarrow S)$	3 4 mt	
6	$R \rightarrow S$	5 dn cd	Now we want to complete the derivation by canceling the show, boxing the derivation and writing 'cd'.

This time, we didn't add a separate line on which to write 'cd'. We can do that on the next line or the last line of the proof (on which we derive the consequent.) This is a complete conditional derivation.

3.4 E1 Now, let's try a few:

(a) $W \rightarrow (X \rightarrow \sim Y)$. $Z \rightarrow Y$. Z. $\therefore W \rightarrow \sim X$

1	~~Show~~ $W \rightarrow \sim X$	Show conclusion
2	W	ass cd
3	$X \rightarrow \sim Y$	2 pr1 mp
4	Y	pr2 pr3 mp
5	$\sim\sim Y$	4 dn
6	$\sim X$	3 5 mt
7		5 cd
8		
9		

(b) $P \rightarrow (R \rightarrow T)$. $P \rightarrow \sim T$. $W \rightarrow R$. $\therefore P \rightarrow \sim W$

1	~~Show~~ $P \rightarrow \sim W$	Show conclusion
2	P	Ass cd
3	$P \rightarrow (R \rightarrow T)$	pr1
4	$P \rightarrow \sim T$	pr2
5	$W \rightarrow R$	pr3
6	$R \rightarrow T$	2 3 mp
7	$\sim T$	2 4 mp
8	$\sim R$	6 7 mt
9	$\sim W$	8 5 mt cd
10		

3.5 Indirect Derivations

Another useful technique for proving that an argument is valid is the 'indirect proof'. An indirect derivation or proof allows us to derive a sentence 'indirectly' by showing that its negation would lead to a contradiction or absurdity. If we can derive a sentence and its negation (contradictory sentences) from an assumption, then there must be something wrong with the assumption! Since every sentence is either true or false (the Law of Excluded Middle), if a sentence can't be true then its opposite must be true. By assuming the opposite of what we want to prove, and showing that it leads to a contradiction, we can validly derive our desired sentence.

This is a form of reasoning that is sometimes called *reductio ad absurdum*. You can show an argument is invalid by showing it leads to a contradiction. Likewise, you can show that a particular assumption is faulty if making it leads to a contradiction – leads to absurdity.

The reasoning would go something like this … Suppose we know the following:

> If Poppy studies then so does George.
>
> If the course is exciting George doesn't study.
>
> Poppy doesn't study if the course isn't exciting.
>
> > And we want to show that Poppy doesn't study.

It isn't obvious we can do this with either direct or conditional proofs using the rules that we have (mp, mt, dn and r). But, we could reason as follows …

Just assume for the sake of argument that Poppy does study. If Poppy studies, then so does George. But if George studies then the course is not exciting, because if it is George doesn't study. If the course isn't exciting then Poppy doesn't study. But that is impossible! We were assuming that Poppy does study. The assumption must be wrong, since it leads to a contradiction!

To use an indirect derivation to prove that a sentence is true you need to make an assumption and then derive a sentence and the negation of that sentence, showing that the assumption leads to a contradiction. The sentence immediately after a show line, will be the faulty assumption – the negation (or un-negation) of the show sentence. The last line of the indirect derivation will consist of a sentence that is the negation (or un-negation) of another sentence under the same assumption. The contradiction can be any sentence and its negation, an atomic sentence, a molecular sentence, even the sentence being assumed.

The elements:

- The show line
- The assumption for ID – the 'opposite' of the show sentence. This is the negated show sentence, or the unnegated show sentence (if the main logical operator of the show sentence is negation, ~.)
- The contradictory sentences (any sentence and its negation). The assumption itself is an acceptable sentence for this. Both contradictory sentences must be under the show line (as the assumption is). Thus, you cannot just cite a premise, you must state the premise under the assumption for ID. Later, in indirect subderivations, you may have to reiterate/repeat (R) an available line in order to get it under your show line for ID.

Every valid argument can be proved by ID! (But it isn't always the most efficient or obvious method.)

Let's try it out:

P: Poppy studies.

Q: George studies.

R: The course is exciting.

The argument above can be symbolized:

$P \rightarrow Q$. $R \rightarrow \sim Q$. $\sim R \rightarrow \sim P$. $\therefore \sim P$

1	Show $\sim P$		The first line is still a show line. We state what we are going to prove.
2	P	ass id	Assumption for indirect derivation. This justification shows that we are making an assumption for the sake of an indirect derivation. We assume the opposite of the sentence on the show line. Now we have a new goal, showing that some sentence and its negation both follow from the assumption (and the premises of course.)
3	$P \rightarrow Q$	pr1	We need to use our premises too! The first premise will let us use P to derive Q.
4	Q	2 3 mp	Using mp, we can infer Q from the first premise and line 2 – the assumption, P, that we are working under.
5	$R \rightarrow \sim Q$	pr2	The second premise will let us derive ~R from a doubly negated Q.
6	$\sim\sim Q$	4 dn	Using dn, we can infer ~~Q from line 4.
7	$\sim R$	5 6 mt	Using mt, we can infer ~R from lines 5 and 6.
8	$\sim R \rightarrow \sim P$	pr3	Here's the last premise we have to work with.
9	$\sim P$	7 8 mp	Using mp, we can infer ~P from lines 7 and 8. But ~P contradicts the assumption we made! We have achieved the goal. Two contradictory sentences under the show line!
10		2 9 ID	Now we have shown on lines 2-9 that if we assume P then a logical contradiction follows – lines 2 and 9 (cited in the justification). This shows ~P is the case. The indirect derivation is all but complete.

Here's the completed proof:

$P \to Q$. $R \to \sim Q$. $\sim R \to \sim P$. $\therefore \sim P$

1	~~Show~~ $\sim P$		
2	P	ass id	
3	$P \to Q$	pr1	
4	Q	2 3 mp	
5	$R \to \sim Q$	pr2	
6	$\sim\sim Q$	4 dn	
7	$\sim R$	5 6 mt	
8	$\sim R \to \sim P$	pr3	
9	$\sim P$	7 8 mp	
10		2 9 id	To complete the derivation …

Write id and cite the contradiction (lines 2 and 9).

Box the indirect proof.

Cancel the show line by crossing out 'show'.

Let's do another: $P \to Q$. $\sim P \to Q$. $\therefore Q$

1	~~Show~~ Q		
2	$\sim Q$	ass id	The assumption is the opposite (negation) of the show sentence.
3	$P \to Q$	pr1	first premise
4	$\sim P \to Q$	pr2	second premise
5	$\sim P$	2 3 mt	
6	$\sim\sim P$	2 4 mt 5 id	Once I've inferred $\sim\sim P$ using mt, I can see that it is the negation of line 5.

I just write down '5' (to show that this line is the negation of line 5) and 'id' to show that this completes the indirect proof. Then all I need to do to complete the derivation is to cancel the show line and box the proof.

Both of the above are complete indirect derivations.

3.5 E1 Now, try a few:

(a) P→R. Q →~R. P. ∴~ (P→Q)

1	~~Show~~ ~(P→Q)	show conc
2	P → Q	ass id
3	P → R	pr1
4	Q → ~R	pr2
5	P	pr3
6	Q	2 5 mp
7	R	3 5 mp
8	~~R	7 dn
9	~Q	8 4 mt
10		6 9 id

(b) ~(~Y → Y) → Y. ∴Y

1	~~Show~~ Y	Show conc.
2	~Y	ass id
3	~~(~Y→Y)	2 pr1 mt
4	~Y→Y	3 dn
5	Y	2 4 mp
6		2 5 id
7		

(c) P → R. R→S. P. S → ~P. ∴ Z

Note: here the conclusion is totally unrelated to the premises of the argument. This is a sure sign that you will need to use indirect derivation.

1	Show Z	show conc.
2	~Z	ass id
3	P→R	pr1
4	R→S	pr2
5	P	pr3
6	S→~P	pr4
7	R	3 5 mp
8	S	4 7 mp
9	~P	6 8 mp
10		5 9 id

3.6 Subderivations

Conditional Subderivations

Some arguments are complex – to show they are valid requires arguments within arguments.

For instance, consider the following argument:

> If Poppy doesn't study then she loses her scholarship. If Poppy is happy only if she succeeds then Poppy doesn't lose her scholarship. That's because Poppy succeeds if she wants to succeed; and she does want to.
>
> P: Poppy studies. U: Poppy loses her scholarship.
> S: Poppy succeeds. W: Poppy wants to succeed.
> T: Poppy is happy.
>
> $(\sim P \to U)$. $((T \to S) \to \sim U)$. $(W \to S)$. W. $\therefore$ P

The only way to derive P from the premises is to use modus tollens on the first premise (followed by dn) but in order to do that we need: $T \to S$, the antecedent of the second premise. If we had that we could finish the proof!

Here's an argument for $T \to S$:

> Assume T is true for the sake of argument. Now we need to show that S is true.
>
> If W is true then S is true. W is true. Thus, by modus ponens, S is true.

Now back to the main argument. We've shown that $(T \to S)$ is true. Thus, we can infer ~U. From that and the first premise we can get ~~P. After double negation this gives us P.

Now we can do it formally: $\sim P \to U$. $(T \to S) \to \sim U$. $W \to S$. W. $\therefore$ P

1	Show P		Here's the main show line. It sets the goal, P, which we will use dd to get.
2	$\sim P \to U$	pr1	Indent. We can list the premises here.
3	$(T \to S) \to \sim U$	pr2	
4	$(W \to S)$	pr3	
5	W	pr4	
6	Show $T \to S$	show ant 3	This is the show line for the conditional subderivation. It states the goal: $T \to S$ The annotation (show antecedent of line 3) is optional.
7	T	ass cd	Immediately after a show line for a conditional, assume the antecedent of the conditional. The conditional is $T \to S$ so assume T. Now we have a new goal, the consequent S. The assumption is for the conditional derivation of line 5. Indent again (after every show line)

$\sim P \to U$. $(T \to S) \to \sim U$. $W \to S$.   W. $\therefore$ P

1	Show P	show conc	
2	$\sim P \to U$	pr1	
3	$(T \to S) \to \sim U$	pr2	
4	$W \to S$	pr3	
5	W	pr4	
6	~~Show~~ $T \to S$	show ant 3	This annotation is optional. We want to show the antecedent of line 3 (or premise 2) so that we can use mp to derive ~U.
7	T	ass cd	This is the assumption for the conditional derivation in support of line 5. Now we have a new goal : S, the consequent of the show line we are working under.
8	S	4 5 mp	S can be derived using lines 4 and 5 with MP.
9		8 cd	Now the conditional sub-derivation is complete. We derived the consequent of show line 6! Box and cancel by drawing a box from line 7-9 and crossing out the show on line 6! Now we have proven line 6, T→S, and so that line is available for us to use.
10	~U	3 6 mp	From line 6 (now available to use since the 'show' is canceled) and line 3, we get ~U using MP.
11	~~P	2 10 mt	Using line 10 and line 2 we get ~~P using MT.
12	P	11 dn	Using DN on line 11 we get P – our main goal. The proof is now all but done. We need our final justification (DD) and we have to box and cancel.

Finally, the completed derivation!

$\sim P \to U$. $(T \to S) \to \sim U$. $W \to S$. W. $\therefore P$

1	~~Show~~ P	show conc
2	$\sim P \to U$	pr1
3	$(T \to S) \to \sim U$	pr2
4	$W \to S$	pr3
5	W	pr4
6	~~Show~~ $T \to S$	show ant 3
7	T	ass cd
8	S	4 5 mp
9		8 cd
10	$\sim U$	3 6 mp
11	$\sim\sim P$	2 10 mt
12	P	11 dn
13		12 dd

Let's try another one. It is similar, but it has a sub-derivation inside a subderivation:

$\sim P \to R$. $(T \to (S \to T)) \to \sim R$. $\therefore P$

1	Show P		Here's the main show line. It sets the goal, P, which we will use dd to get.
2	$\sim P \to R$	pr1	Indent. We can list the premises here.
3	$(T \to (S \to T)) \to \sim R$	pr2	
4	Show $T \to (S \to T)$	show ant 3	This is the show line for the conditional subderivation. It states the goal: $T \to (S \to T)$ The annotation (show antecedent 3) is optional. We want to show the antecedent of line 3/pr2.
5	T	ass cd	Immediately after a show line for a conditional, we need to assume the antecedent of the conditional. The conditional is $T \to (S \to T)$ so assume T. Now we have a new goal, the consequent, $S \to T$. The assumption is for conditional derivation. Indent again (after every show line)

$\sim P \rightarrow R$. $(T\rightarrow (S\rightarrow T))\rightarrow\sim R$. $\therefore$ P

1	Show P	show conc	
2	$\sim P \rightarrow R$	pr1	
3	$(T\rightarrow (S\rightarrow T))\rightarrow\sim R$	pr2	
4	Show $T \rightarrow (S\rightarrow T)$	show ant 3	This annotation is optional. We want to show the antecedent of premise 2 so that we can use mp to derive ~R.
5	T	ass cd	This is the assumption for the conditional derivation in support of line 4. Now we have a new goal : S →T, the consequent of the show line we are working under. We can either derive S→T directly OR, as we will do here, put in a show line for the consequent of the conditional on line 4 that we are trying to show. A new subderivation.
6	~~Show~~ $S\rightarrow T$	show cons	This is the show line for the next subderivation. The annotation is optional, it means 'show consequent'. We want to show the consequent of the conditional show line that the preceding line is an assumption for (the sentence on line 4). The new goal is T.
7	S	ass cd	After a show line for a conditional, we need to assume the antecedent of the conditional. The conditional is S→T, so assume S. Now we have a new goal, the consequent, T. Indent again (after every show line).
8	T	5 r	We can get T under the assumption S by reiterating line 5. Line 5 is available – it's an assumption we are working under.
9		8 cd	We've inferred the consequent of the conditional on line 6 that we are trying to show. So we write the line number of the consequent, 'cd', box it and cancel the show on line 6. This is the closest show line above. Now line 6 is available to use.
10			Now that we have shown that S→ T follows from the assumption T, we have all but completed the conditional derivation started on line 5. We need to box from 5-10 & cancel the 'show' on line 4! Line 4 will then be available to use, and we can finish the derivation.

Finally, the completed derivation!

~P → R.   (T→ (S→T))→~R. ∴ P

1	~~Show~~ P	show conc
2	~P → R	pr1
3	(T→ (S→T))→~R	pr2
4	~~Show~~ T → (S→T)	show ant 3
5	T	ass cd
6	~~Show~~ S→T	show cons
7	S	ass cd
8	T	5 r
9		8 cd
10		6 cd
11	~R	4 3 mp
12	~~P	2 11 mt
13	P	12 dn dd

Notice that each time a conditional subderivation is used it begins with a show line for a conditional sentence. The line after the show line is the assumption of the antecedent.

The next line after that will often be ‘show cons.’ (show consequent). That’s the consequent of the conditional you are trying to show for which the preceding line is the antecedent. It is a good idea to do a new show line for the consequent every time – unless you can see how to derive it directly from the available lines. It may not be necessary, but it never hurts. (The worst case scenario is that your derivation will be a couple lines longer than it needs to be, with an extra subderivation.)

Once the consequent is derived, write the line number of the consequent and ‘cd’, box the conditional derivation and cancel the show line by crossing out ‘show’.

3.6 EG1 Let’s try one:

(a) P → Q. S → ~Q. (P→~S) → T. ∴ P → T

1	~~Show~~ P → T	show conc.
2	P	ass cd
3	~~show~~ P → ~S	show ant. pr3
4	P	ass cd
5	Q	4 pr1 mp
6	~~Q	5 dn
7	~S	6 pr2 mt
8		7 cd
9	T	3 pr3 mp
10		9 cd
11		

Indirect Subderivations

We can use the indirect derivation within a larger derivation as well.

Consider the following argument:

Poppy studies. If Poppy studies, she passes. If Poppy passes then, she graduates only if she doesn't plagiarize. If Poppy doesn't plagiarize then she doesn't get into trouble. If Poppy doesn't plagiarize, Poppy doesn't succeed. If Poppy doesn't succeed then she gets into trouble. ∴ Poppy doesn't graduate.

We might reason as follows. Since Poppy studies, she passes. Thus she graduates only if she doesn't plagiarize. In order to show that she doesn't graduate we need to know that Poppy plagiarizes. If we could show that she plagiarized then we could finish the proof.

> Here's an argument that Poppy plagiarizes:
>
> Assume for the sake of argument that Poppy doesn't plagiarize. Then, she doesn't get into trouble. She also doesn't succeed. But if Poppy doesn't succeed she gets into trouble. That contradicts the inference that she doesn't get into trouble which followed from the assumption that she doesn't plagiarize. Thus the assumption must be wrong. Poppy does plagiarize.

Poppy plagiarizes (as shown above) Thus, returning to the main argument, Poppy doesn't graduate (for she graduates only if she doesn't plagiarize.)

Now we can do it formally:

P:	Poppy studies	R:	Poppy graduates.	T:	Poppy gets into trouble.
Q:	Poppy passes.	S:	Poppy plagiarizes.	U:	Poppy succeeds.

$P.\ \ P\rightarrow Q.\ \ Q\rightarrow(R\rightarrow\sim S).\ \ \sim S\rightarrow\sim T.\ \ \sim S\rightarrow\sim U.\ \ \sim U\rightarrow T.\ \ \therefore\ \sim R.$

1	Show ~R			We start with the show line. It states our goal: ~R.
2		P	pr1	We can get started with the proof. Indent.
3		P→Q	pr2	
4		Q	2 3 mp	This can be inferred from lines 2 and 3 using mp.
5		Q→(R→~S)	pr3	
6		R →~S	4 5 mp	This is as far as we can go until we've shown S.

P. P→Q. Q→(R→~S). ~S→~T. ~S →~U. ~U→T. ∴ ~R.

1	Show ~R		We start with the show line. It states our goal: ~R.
2	P	pr1	We can get started with the proof. Indent.
3	P→Q	pr2	
4	Q	2 3 mp	This can be inferred from lines 2 and 3 using mp.
5	Q→(R→~S)	pr3	
6	R →~S	4 5 mp	This is as far as we can go until we've shown S.
7	Show S		We want to show S using an indirect subderivation.
8	~S	Ass id	This is the assumption for the subderivation. Since we want to do an indirect proof of S, we need to assume the negation of S – the sentence on the show line immediately above. Indent again!
9	~S → ~T	pr4	
10	~S → ~U	pr5	We're going to need the rest of the premises.
11	~U → T	pr6	
12	~T	8 9 mp	
13	~U	8 10 mp	
14	T	11 13 mp	Now we have the contradiction.
15		12 14 id	This shows the two contradicting sentences. The assumption of line 8 that we are working under leads to absurdity. We can box it off. We cancel the nearest show line above. That's line 7. (The only other show line is line 1 - it's further away!)

P. P→Q. Q→(R→~S). ~S→~T. ~S →~U. ~U→T. ∴ ~R.

1	Show ~R			
2	P		pr1	
3	P→Q		pr2	
4	Q		2 3 mp	
5	Q→(R→~S)		pr3	
6	R →~S		4 5 mp	
7	~~Show~~ S			
8		~S	Ass id	
9		~S → ~T	pr4	
10		~S → ~U	pr5	
11		~U → T	pr6	
12		~T	8 9 mp	
13		~U	8 10 mp	
14		T	11 13 mp	
15			12 14 id	Now none of these lines 8-15 are available!
16	~~S		7 dn	Outdent- return to the main derivation. Line 7 is now available – the show has been struck out. Now it is just like any other sentence.
17	~R		6, 16 mt	We can infer this from lines 16 and 6 (which is still available since it is under the same show line – line 1.) We can write 'dd', box the derivation, cancel the show on line 1 and the derivation will be complete.

Here is the complete derivation:

P. P→Q. Q→(R→~S). ~S→~T. ~S →~U. ~U→T. ∴ ~R.

1	~~Show~~ ~R	
2	P	pr1
3	P→Q	pr2
4	Q	2 3 mp
5	Q→(R→~S)	pr3
6	R →~S	4 5 mp
7	~~Show~~ S	
8	~S	Ass id
9	~S → ~T	pr4
10	~S → ~U	pr5
11	~U → T	pr6
12	~T	8 9 mp
13	~U	8 10 mp
14	T	11 13 mp
15		12 14 id
16	~~S	7 dn
17	~R	6 16 mt dd

3.6 EG2 Let's try one:

P → ~Q. S → P. Q → S. ∴ Q → ~R

1	~~Show~~ Q → ~R	show conc.
2	Q	ass cd
3	~~Show~~ ~R	show cons.
4	R	ass id
5	S	pr3 2 mp
6	P	5 pr2 mp
7	~Q	6 pr1 mp
8	Q	2 r
9		7 8 id
10		3 cd
11		

3.6 E1 Here’s one based on the Trolley Problem

Suppose you are the driver of a trolley. The trolley rounds a bend, and you see five workmen on the track. But you are on a steep hill and the brakes don’t work. You see that you can turn the trolley onto a spur by pulling a lever, saving the five men from certain death. Unfortunately, there is one workman on that track and you will surely kill him.
(…The Trolley Problem is from Philippa Foot and Judith Jarvis Thomson… much is changed here to make it easy to symbolize with just negation and conditional!)[2]

> But what do you think? Given that you are the only one who can act in this situation, should you pull the lever and save the five men from certain death even though it means that you will kill somebody. Or should you do nothing (since it is wrong to kill somebody) and let five people die? Is it okay to kill an innocent person in these circumstances?
>
> Do you think that what you do is ***the right thing*** to do? Or do you think there is no morally right thing to do?

Here our conclusion will be that there is no right thing to do…

Symbolize and derive the conclusion to this argument:

> If you don’t pull the lever, then five people will die.
> If you pull the lever, then you kill a person.
> You are good only provided that you don’t kill anyone.
> If five people are saved only if it’s not the case that you don’t kill a person, then if you are good, you pull the lever.
> You are good.
>
> ---
>
> Therefore, there is no right or wrong thing to do.

P:	You pull the lever.
R:	Five people die (negated: five people are saved).
S:	You kill a person.
T:	You are good.
U:	There is no right or wrong thing to do.

---

2 Philippa Foot, *The Problem of Abortion and the Doctrine of the Double Effect* in *Virtues and Vices* (Oxford: Basil Blackwell, 1978)
Judith Jarvis Thomson, *Killing, Letting Die, and the Trolley Problem*, 59 The Monist 204-17 (1976)

Subderivation notes

1. Subderivations can be used in any proof. So if you are doing a direct derivation, a conditional derivation or an indirect derivation, you might want to use a subderivation. Even a subderivation can have a subderivation within it.

2. When you complete a subderivation, that line is available to use.

3. You begin a subderivation by writing a show line – the word ‘show’ followed by the sentence you want to prove. Your show lines should be sentences that you need and that you can show.

4. Boxes must be neatly nested within one another. That means that you must cancel the first previous uncanceled show line (the one with the largest line number) before you can cancel show lines higher up in the derivation (with smaller line numbers).

   - If your boxes are criss-crossing or there is an uncanceled show inside a box then you have not been canceling the correct show line. Big mistake! Try it again!

5. When working with subderivations you must be really careful about using only the available lines.

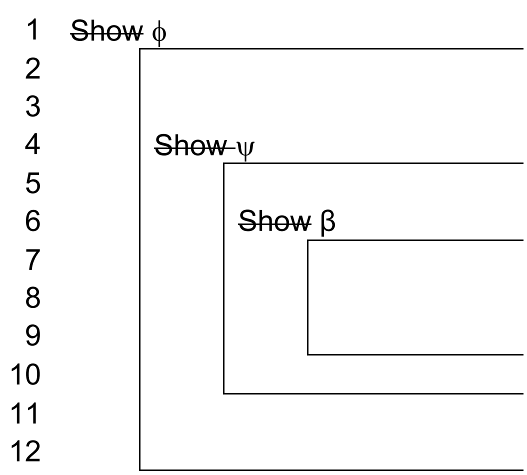
   What’s available?

   - Canceled show lines.
   - Premises
   - Assumptions that you are working under.
   - Sentences under the same assumptions (following the same uncanceled show lines).

   What’s not available?

   - Any sentence that has already been boxed.
   - Any uncanceled show line (canceled show lines will be followed by boxed proofs.)

6. At the end of your complete derivation, every show line should be canceled. Under every canceled show line there should be a box. The boxes should be neatly nested within one another. So the only unboxed line will be the very first line – and show will be crossed out on that line.



Neatly nested boxes.

3.6 E 2 Provide a derivation to prove that the following arguments are valid.

a) $S \to \sim P$. $P \to R$. $R \to (\sim S \to Q)$. $\therefore$ $P \to Q$

b) $(P \to Q) \to R$. $S \to \sim P$. $\sim S \to (T \to Q)$. $\therefore$ $T \to (P \to R)$

c) $P \to \sim W$. $S \to W$. $P \to (T \to R)$. $(\sim R \to \sim T) \to S$. $\therefore$ $\sim P$

d) $\sim(P \to S)$. $R \to \sim P$. $\sim R \to (T \to S)$. $P \to T$. $\therefore$ W

3.7 Shortcuts

Shortcut 1: Cite premises directly – don't rewrite them first!

Why rewrite a premise (risk copying it wrong) when they have already been written out?

Here is a proof we did earlier in which the premises are copied on lines 3, 5 and 8:

$P \to Q$. $R \to \sim Q$. $\sim R \to \sim P$. $\therefore$ $\sim P$

1	~~Show~~ ~P		
2	P	ass id	
3	P→Q	pr1	premise copied
4	Q	2 3 mp	
5	R→~Q	pr2	premise copied
6	~~Q	4 dn	
7	~R	5 6 mt	
8	~R→~P	pr3	premise copied
9	~P	7 8 mp	
10		2 9 id	

We can cut it quite a bit shorter just by referring directly to the premises in the justifications:

$P \to Q$. $R \to \sim Q$. $\sim R \to \sim P$. $\therefore$ $\sim P$

1	~~Show~~ ~P		
2	P	ass id	
3	Q	2 pr1 mp	Cite the first premise as if it was already a line. The premises are always available.
4	~~Q	3 dn	
5	~R	4 pr2 mt	Cite the second premise here. Use it in mt.
6	~P	5 pr3 mp id	Cite the third premise for mp. Then write id, box and cancel the show.

But, make sure you have written the question out accurately – the premises do have to be clear!

EXCEPTION: In ID you need your contradiction under your show line. So if the contradiction is one of the premises, you need to write it in – don't just refer to it!

Shortcut 2: Mixed Derivations

Sometimes you are working on a derivation and it turns out that you infer something that would have been more useful if only you had made a different assumption, or started a different type of derivation.

Of course, you can make a new assumption, use your sentence, and prove your point. However, it is shorter to just recognize that the hard work has already been done – the proof is already valid.

The derivation rules essentially tell us what is required to complete that type of derivation. For DD, you need to derive the sentence. For CD, you need to derive the consequent. For ID, you need to derive a contradiction. Although it's nice when you end up using the derivation rule that you plan on using (and for which you may have made an assumption), it is not necessary for a valid proof.

Let's see how it would work:

$P \rightarrow Z.\ \ P \rightarrow (Z \rightarrow {\sim}Z).\ \ V \rightarrow Z.\ \ \therefore\ P \rightarrow V$

1	Show $P \rightarrow V$		A conditional. So it makes sense to assume the antecedent.
2	P	Ass cd	
3	Z	2 pr1 mp	
4	$Z \rightarrow {\sim}Z$	2 pr2 mp	
5	${\sim}Z$	3 4 mp	NOTE! This contradicts line 3. Now I can get whatever I want – I just have to assume the negation of what I want and reiterate lines 3 and 5 for an indirect subderivation.
6	~~Show~~ V	Ass id	
7	Z	3 r	
8	${\sim}Z$	5 r id	It's done. Box and cancel.
9		6 cd	Now the main derivation is done. All we need to do is box and cancel and it's complete.

Lines 6-8 are superfluous. We knew that we were done at line 5 – from a contradiction you can prove anything!

It's shorter like this:

$P \rightarrow Z.\ \ P \rightarrow (Z \rightarrow {\sim}Z).\ \ V \rightarrow Z.\ \ \therefore\ P \rightarrow V$

1	~~Show~~ $P \rightarrow V$		
2	P	Ass cd	
3	Z	2 pr1 mp	
4	$Z \rightarrow {\sim}Z$	2 pr2 mp	
5	${\sim}Z$	3 4 mp 3 id	The proof is complete. Box and cancel.

On line 2, we made an assumption for CD. But, instead of sticking with the original plan (CD), we saw that we had satisfied the conditions for ID and finished off the derivation using ID.

Here's another example:

$P \to R$. $R \to (P \to Q)$. $\therefore P \to Q$

1	~~Show~~ $P \to Q$		
2	P	Ass cd	We need to show Q, the consequent.
3	R	2 pr1 mp	
4	$P \to Q$	3 pr2 mp dd	NOTE! That's actually what we wanted to show. Of course we could just go one more step and derive Q from P, but why bother? The proof is complete now. Just write dd, box and cancel!

On line 2, we made an assumption for CD. But, instead of sticking with the original plan (CD), we saw that we had satisfied the conditions for DD and finished off the derivation using DD.

R. $R \to P$. $P \to \sim R$. $\sim R \to S$. $S \to \sim T$. $\sim T \to Q$. $\therefore$ Q

1	~~Show~~ Q		It looks like we can do this directly. We just have to use mp in sequence.
2	P	pr1 pr2 mp	
3	~R	2 pr3 mp	Wait a second. This is the contradiction of the first premise. Why finish all those bothersome modus ponens? From a contradiction we can prove anything.
4	R	pr1 3 id	R. The proof is all but complete. Just write id, box and cancel.

On line 2, didn't make an assumption but proceeded as for DD. But, instead of sticking with the original plan (DD), we saw that we had satisfied the conditions for ID and finished off the derivation using ID.

We've seen three examples. In the first, we were all ready for a conditional derivation, but we ended up using id after deriving a contradiction. In the second, we were ready for a conditional derivation but ended up deriving the conditional directly. In the third, we were ready for a direct proof but ended up using id after deriving a contradiction. But, there are other types of mixed derivations as well.

The final justifications (dd, cd, and id) can be used to complete a derivation whenever you have derived the required sentence, whether or not you made an assumption, and regardless of the type of assumption you might have made.

Mixed derivations are just shortcuts. The rules already allowed for them. The derivation rules tell us what conditions must be met to complete a valid derivation – how you began doesn't actually matter. The chart on the next page gives all the different uniform and mixed derivation combinations.

The show sentence.	**The assumption immediately after the show sentence.**	**What you derive for the derivation (or subderivation) to be complete.**	**What is the justification? (NOTE: Line # never refers to the show line!)**	**Type of Derivation**
ϕ Any sentence.	No assumption (DD)	ϕ The show sentence.	line # for ϕ. DD	Uniform DD
$\phi \rightarrow \psi$ A conditional	ϕ The antecedent of the show sentence. ass CD	ψ The consequent of the show sentence.	line # for ψ CD	Uniform CD
ϕ Any sentence	$\sim\phi$ The negation of the show sentence. ass ID	ψ & $\sim\psi$ Any sentence and its negation.	line # for ψ, $\sim\psi$ ID	Uniform ID
ϕ Any sentence.	No assumption (DD)	ψ & $\sim\psi$ Any sentence and its negation.	line # for ψ, $\sim\psi$ ID	Mixed DD/ID
$\phi \rightarrow \psi$ A conditional	No assumption (DD)	ψ The consequent of the show sentence.	line # for ψ CD	Mixed DD/CD
$\phi \rightarrow \psi$ A conditional	ϕ The antecedent of the show sentence. ass CD	χ & $\sim\chi$ Any sentence and its negation.	line # for χ , $\sim\chi$ ID	Mixed CD/ID
$\phi \rightarrow \psi$ A conditional	ϕ The antecedent of the show sentence. ass CD	$\phi \rightarrow \psi$ The show sentence.	line # for $\phi \rightarrow \psi$ DD	Mixed CD/DD
$\phi \rightarrow \psi$ A conditional	$\sim(\phi \rightarrow \psi)$ The negation of the show sentence. ass ID	ψ The consequent of the show sentence.	line # for ψ CD	Mixed ID/CD
ϕ Any sentence	$\sim\phi$ The negation of the show sentence. ass ID	ϕ The show sentence.	line # for ϕ. DD	Mixed ID/DD

Note – if you don't go to a new line you may not need to cite all the line #s for justification.

3.8 Theorems

What is a theorem?

A theorem is a sentence that is true no matter what else is the case. It is a tautology – a sentence that is logically true or analytically true. A complete system of first order logic should be able to demonstrate that such sentences are true. Our system can do it – in our system we can demonstrate that a sentence is a theorem by deriving it from the empty set. This makes sense, since any argument with a tautology (or theorem) as a conclusion is valid. Thus, such a conclusion does not rely on any premises. So, we can indicate a sentence is a theorem by putting a therefore sign ($\therefore$) in front of it (after all, it is a conclusion from no premises!)

Both conditional derivations and indirect derivations can be used to prove theorems.

Let's give it a try:

T2 $\quad \therefore Q \rightarrow (P \rightarrow Q)$

1	~~Show~~ $Q \rightarrow (P \rightarrow Q)$		This is a conditional so it is obvious how to begin. We assume the antecedent and show the consequent follows.
2	Q	ass cd	Assume the antecedent. Now we have a goal – the consequent $P \rightarrow Q$.
3	~~Show~~ $P \rightarrow Q$		We need to prove another conditional. This requires a conditional subderivation.
4	P	ass cd	
5	Q	2 r cd	2 is available. The cd is complete. Write 'cd', box and cancel.
6		3 cd	We're done. Box and cancel.

3.8 EG1 Let's try another.

T4 $\quad \therefore (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

3.8 E 1 Prove the following theorems:

a) $\therefore ((P \rightarrow Q) \rightarrow (P \rightarrow R)) \rightarrow (P \rightarrow (Q \rightarrow R))$

b) $\therefore \sim(P \rightarrow Q) \rightarrow \sim Q$

c) $\therefore (P \rightarrow S) \rightarrow ((T \rightarrow P) \rightarrow (\sim S \rightarrow \sim T))$

You should be able to prove any of the theorems on the next page!

Some Theorems[3]

T1 $\therefore P \rightarrow P$

T2 $\therefore Q \rightarrow (P \rightarrow Q)$

T3 $\therefore P \rightarrow ((P \rightarrow Q) \rightarrow Q)$

T4 $\therefore (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

T5 $\therefore (Q \rightarrow R) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

T6 $\therefore (P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

T7 $\therefore ((P \rightarrow Q) \rightarrow (P \rightarrow R)) \rightarrow (P \rightarrow (Q \rightarrow R))$

T8 $\therefore P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))$

T9 $\therefore (P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$

T10 $\therefore ((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P)$

T11 $\therefore \sim\sim P \rightarrow P$

T12 $\therefore P \rightarrow \sim\sim P$

T13 $\therefore (P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$

T14 $\therefore (P \rightarrow \sim Q) \rightarrow (Q \rightarrow \sim P)$

T15 $\therefore (\sim P \rightarrow Q) \rightarrow (\sim Q \rightarrow P)$

T16 $\therefore (\sim P \rightarrow \sim Q) \rightarrow (Q \rightarrow P)$

T17 $\therefore P \rightarrow (\sim P \rightarrow Q)$

T18 $\therefore \sim P \rightarrow (P \rightarrow Q)$

T19 $\therefore (\sim P \rightarrow P) \rightarrow P$

T20 $\therefore (P \rightarrow \sim P) \rightarrow \sim P$

T21 $\therefore \sim(P \rightarrow Q) \rightarrow P$

T22 $\therefore \sim(P \rightarrow Q) \rightarrow \sim Q$

T23 $\therefore ((P \rightarrow Q) \rightarrow P) \rightarrow P$

THEOREMS

Theorems are general patterns of reasoning, just like rules of inference are. Indeed, any theorem that has a conditional as the main connective can be used as a rule of inference (Unit 3.12, in part 2.)

Think of P, Q and R as general terms that stand for any three sentences – molecular or atomic.

For instance, consider theorem 13.

$(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$

If we replace "P" with "R" and "Q" with "$T \vee S$" we would get the following sentence:

$(R \rightarrow T \vee S) \rightarrow (\sim(T \vee S) \rightarrow \sim R)$

This sentence is also a theorem.

Since these theorems do express regular patterns of reasoning, many of them have traditional names.

T4, T5: Hypothetical Syllogism

T11, T12: Double Negation

T13, T14, T15, T16: Transposition

T19, T20: Reductio ad absurdum

[3] The number of possible theorems is infinite. The numbering is arbitrary. This list of theorems is from: Kalish, Montague, and Mar. *Logic: Techniques of Formal Reasoning*. Oxford: 1980. Pg. 107
This is the list used in David Kaplan's Logic 2010.

3. 9 TERMS FOR DERIVATIONS EXPLAINED

Derivation: A derivation is a proof or demonstration showing how a sentence or sentences can be derived (obtained by making valid inferences) from a set of sentences. It consists of a numbered sequence of symbolic sentences (indented after the show line). Each line that is neither a show line nor a premise should follow from previous available lines according to the derivation rules (which are cited in the justification for that line.) When complete, it should be boxed and canceled (the word 'show' crossed off.)

Show line: The first line of *every* derivation or subderivation. Write the word "show" and then the symbolic sentence you are proving. A show line may be introduced on any line – it begins a subderivation when introduced after line 1. Show lines don't need justification. You may wish to annotate them (for instance, "show conclusion" on the main show line or "show consequence" after an assumption for CD.) Always indent immediately following a show line. When the derivation or subderivation is completed and a box drawn around it (boxed), then the sentence on the show line has been derived and is available to use. The word, 'show' is canceled (crossed out: ~~show~~).

Premises: These are the symbolic sentences from which the conclusion is to be derived. Premises are always available for use. You can introduce any premise any time by copying it into your derivation, and justifying it with the notation "pr" followed by the number of the premise (for example, 'pr1'). Alternatively, a justification can just refer to the premise, again with the notation "pr" followed by the number of the premise.

Rule: A symbolic sentence can be introduced on a line if it follows by a rule of inference from premises or sentences on previous available lines. The justification consists of the line or premise numbers and the name of the rule. A rule is a valid pattern of inference.

Subderivation: A subderivation is a derivation within the main derivation – under the main show line. It is used to prove, and make available, a sentence that cannot be directly derived from the premises. A subderivation begins with a show line and consists of a numbered sequence of symbolic sentences (indented after the show line). Each line that is neither a show line nor a premise should follow from previous available lines according to the derivation rules. When a subderivation is complete (boxed and canceled), the 'show line' sentence is available to use within the main derivation. At that point, outdent to the level of the previous uncanceled show line. Subderivations are usually indirect or conditional subderivations.

Direct derivation: One of the three types of derivation for SL. The show sentence can be any symbolic sentence. The sentence is derived directly, using the rules and available lines and premises. A direct derivation is complete when a sentence (not a show sentence) is derived that is the same as the sentence being shown (which should be the sentence on the nearest, earlier uncanceled show line.) Write "DD" following the rule justification for that line, cross out the word "show" on the relevant show line. Then draw a box around all the lines below the show line including the line with DD written on it.

Conditional derivation: One of the three types of derivation for SL. The show sentence must be a conditional sentence. On the line immediately following the show line, the antecedent of the conditional to be shown is assumed (justification is assumption for conditional derivation, "ass CD"). The consequent of the show sentence is then derived, using the rules, the assumption and any other available lines or premises. A conditional derivation or subderivation is complete when a sentence is derived that is the same as the consequent of the conditional being 'shown' (which should be on the nearest, earlier uncanceled show line). Write "CD" following the rule justification for that line, cross out the word "show" on the relevant show line, and draw a box around all the lines below the show line, including the last line with CD written on it.

Assumption for conditional derivation: When a show sentence is a conditional sentence, the next line is often an assumption for conditional derivation. If so, the line following the show line must be antecedent of the conditional – the justification is "ass CD" (assumption for conditional derivation.)

Indirect derivation: One of the three types of derivation for SL. The show sentence can be any sentence. Immediately following the show line, the 'opposite' of the show sentence is assumed (either negation of the show sentence or the unnegated sentence, if the show sentence is itself a negation). A pair of contradictory sentences (a sentence and its negation) are then derived, (using the rules, the assumption and any other available lines or premises.) An indirect (sub)derivation is complete when a sentence is derived that is the negated form of another sentence under the assumption for ID or under the show line. After the rule justification for that line, write the line number of the contradictory sentence and write "ID", cross out the word "show" on the relevant show line, and draw a box around all the lines below the show line, including the last line with ID written on it. Note: BOTH contradictory sentences must be physically under the show line (not just available). You may want to use "R" to get an available line under the show line.

Assumption for indirect derivation: For any show sentence, the next line may be an assumption for indirect derivation. If so, the line following the show line must be the negated show sentence (or unnegated show sentence if the show sentence is itself a negation.) The justification is "ass ID" (assumption for indirect derivation.)

Uniform derivation (or subderivation): in a uniform derivation or subderivation, the justification 'matches' any assumption made – the (sub)derivation is completed according to the original plan. If there was no assumption made (the derivation planned to be direct), then the justification would be "DD". If the assumption was the antecedent of the show line ("ass CD"), then the justification would be "CD". If the assumption was the opposite of the show line ("ass ID"), then the justification would be "ID".

Mixed derivation (or subderivation): in a mixed derivation or subderivation, the justification does not 'match' the assumption made (if one was made). If a sentence satisfying the conditions for a certain type of (sub)derivation is derived (given the nearest, earlier uncanceled show line), then the derivation is complete regardless of any assumption made – regardless of the original plan for the (sub)derivation.

Keep your derivations neat and orderly! Every line should be written immediately below previous lines, indenting after a show line and outdenting after boxing and canceling.

When a derivation/subderivation is complete, the justifications for direct, conditional or indirect derivation (DD, CD or ID) may be written on the last line of the (sub)derivation. If the justification is on the line on which the needed sentence is derived, then no line number is needed for DD and CD. In the case of ID, if the justification is written on the line of one of the two contradictory sentences, only the line number of the other sentence is necessary.

Alternatively, when boxing and canceling, the justification may be written a new line (the last line of the subderivation or derivation) which contains no sentence at all. In this case, the justification should be followed by the number of the line (or two lines, in the case of ID) that satisfies the conditions for the derivation.

3.10 DERIVATION FAQ's

How do I begin?

- Your first line is a show line. It shows the conclusion of the argument. Now you have a goal.
- Try to think through the argument. Can you explain why the conclusion follows from the premises? If so, that should suggest a strategy.
- Analyze the goal sentence (start with your primary goal – the show line)
  - What is the main connective? (If it's a conditional, consider a conditional derivation.)
  - What are the sentential components? Do any of those sentential components appear (negated or unnegated) in the premises? How could you 'free' that sentential component? (Look at the other premises – consider MP and MT.)
- Analyze the premises.
  - Check for a contradiction. If the premises are inconsistent then you can quickly derive a contradiction. If so, you can achieve your goal with an indirect derivation.
  - Do any sentential components (or their negations) occur in more than one premise? Can you think of a strategy to work with those premises together? (Think about what you can get with MP and MT).
- If the goal sentence is not at all related to the premises then you might want to show that the premises contain a contradiction (are they inconsistent?) Use ID to derive your goal sentence.
- If you are still stuck, just work with what you've got to make things a bit simpler.
  - Use modus ponens and modus tollens on whatever you can.

How do I know when to put a show line for a subderivation?

- Use a show line whenever you can't get what you need through a direct derivation. Most of the subderivations will be conditional or indirect derivations – after all, if you can derive it directly, just using the rules of inference, you don't need a subderivation.
- Whenever your show line is a conditional, consider putting in a show line for the consequent after your assumption for CD (assumption of the antecedent.)
- Look at your working goal (the show line you are working with). Thinking about all the other available lines. Ask yourself what you need to achieve that goal? (For instance, if you have $\phi \rightarrow \psi$ on an available line, and your main goal is ψ, then what you need to achieve it is ϕ.)
- Once you know what you need, make it a show line.

How do I know what sentence to assume?

- If you know ***why*** you are starting a subderivation then you will know what assumption to make.
- Once you have your show line, ask yourself how you want to show it? Then it should be obvious what assumption to make.
  - Is it a conditional? Then you probably want to do a conditional derivation. Assume the antecedent. Your new goal is the consequent.
  - If it isn't a conditional, then you want to do an indirect derivation. Assume the negated (or unnegated) sentence. Your new goal is a contradiction.
  - Occasionally when you want to show a conditional the best way is with an indirect derivation. Assume the negated conditional. Your new goal is a contradiction.

What you want to show in the main derivation or in a sub-derivation.	**Try this strategy:** (If you need to derive a particular sentence to follow the suggested strategy think about whether you can get it directly through the rules and available lines, or if you want to use ID or CD to derive it in a subderivation.)
A conditional: $\phi\to\psi$ (Do you already have ψ? If so, you are almost done if you use this strategy!)	Show $\phi\to\psi$. Assume ϕ. Derive ψ (don't forget to use R if ψ is on an earlier available line). 'CD', box and cancel. (Uniform derivation)
A conditional: $\phi\to\psi$ (Do you already have $\sim\phi$? If so, you are almost done if you use this strategy!)	Show $\phi\to\psi$. Assume ϕ. Derive $\sim\phi$ (don't forget to use R if $\sim\phi$ is on an earlier available line.) 'ID', box and cancel. (Mixed derivation, CD/ID)
The negation of a conditional: $\sim(\phi\to\psi)$	Show $\sim(\phi\to\psi)$. Assume $(\phi\to\psi)$ Derive a contradiction. If you have or can derive ϕ, use MP with your assumption to get ψ and see if that helps. If you have or can derive $\sim\psi$, use MT with your assumption to get $\sim\phi$ and see if that helps. 'ID', box and cancel. (Uniform derivation.)
Unnegated Sentence: ϕ	Show ϕ Assume $\sim\phi$ Derive a contradiction. Try to use MP and MT with $\sim\phi$ and any available lines. Do you see a contradiction? If you have or can derive the negation of a conditional $\sim(\chi\to\psi)$, you might be able use CD to derive $(\chi\to\psi)$. 'ID', box and cancel. (Uniform derivation.)
Negated Sentence: $\sim\phi$	Show $\sim\phi$ Assume ϕ (or $\sim\sim\phi$) Derive a contradiction. Try to use MP and MT with ϕ (you may need DN) and any available lines. Do you see a contradiction? If you have or can derive the negation of a conditional $\sim(\chi\to\psi)$, you might be able use CD to derive $(\chi\to\psi)$. 'ID', box and cancel. (Uniform derivation.)