Australian National University
Research School of Finance, Actuarial Studies and Applied Statistics

STAT2032/6046: Financial Mathematics

Review Questions (Week 4 – Week 6)

WEEK 4

Question 1

Evaluate the following at i = 15%

a. $(Ia)_{\overline{60|}}$
b. $(I\ddot{a})_{\overline{60|}}$
c. $(I\bar{a})_{\overline{60|}}$
d. $(Is)_{\overline{60|}}$

Solution

a. $(Ia)_{\overline{60|}} = \dfrac{\ddot{a}_{\overline{60|}} - 60v^{60}}{i} = 51.01821$

b. $(I\ddot{a})_{\overline{60|}} = \dfrac{\ddot{a}_{\overline{60|}} - 60v^{60}}{d} = \dfrac{i}{d}(Ia)_{\overline{60|}} = 58.65944$

c. $(I\bar{a})_{\overline{60|}} = \dfrac{\ddot{a}_{\overline{60|}} - 60v^{60}}{\delta} = \dfrac{i}{\delta}(Ia)_{\overline{60|}} = 54.74474$

d. $(Is)_{\overline{60|}} = \dfrac{\ddot{s}_{\overline{60|}} - 60}{i} = (Ia)_{\overline{60|}}(1.15)^{60}$=223,620

Question 2

i. Assuming a rate of interest of 6% pa, find the present value as at 1 January 2001 of the following annuities, each with a term of 25 years:
   a. an annuity payable annually with the first payment of \$5,000 being made on 1 January 2002 and payments increasing by \$100 pa on each subsequent 1 January.
   b. an annuity as in (i), but only 10 increases are to be made, the annuity then remaining level for the remainder of the term.

ii. An investor is to receive an annual annuity at the end of each year for a term of 15 years such that payments are increased by 5% compound each year. The first payment is to be \$1000. Find the accumulated value of the annuity payments after 20 years if the investor invests at an effective rate of interest of 6% per half year.

Solution

i. Part a. $PV = 4900a_{\overline{25}|} + 100(Ia)_{\overline{25}|} = 75{,}514.10$

   Part b. $PV = 4900a_{\overline{25}|} + 100(Ia)_{\overline{11}|} + 1100v^{11}a_{\overline{14}|} = 72{,}300.28$

ii.

$$AV = 1000(1.06)^{10}\left(1.06^{28} + 1.06^{26}1.05 + 1.06^{24}1.05^{2} + \ldots\ldots + 1.05^{14}\right)$$

$$= 1000(1.06^{38})\left(1 + \frac{1.05}{1.06^{2}} + \frac{1.05^{2}}{1.06^{4}} + \frac{1.05^{3}}{1.06^{6}} + \ldots\ldots + \frac{1.05^{14}}{1.06^{28}}\right) = 1000(1.06^{38})\ddot{a}_{\overline{15}|k}$$

$$= \$89{,}166.79$$

where $k = \dfrac{1.06^{2}}{1.05} - 1 = 0.0700952$ and $\ddot{a}_{\overline{15}|k} = 1.0700952\left(\dfrac{1-1.0700952^{-15}}{0.0700952}\right)$.

WEEK 5

Question 3

In return for an initial investment of \$20,000, a sum of \$12,000 is to be received after 3 years and a further sum of \$9,000 after 6 years. What is the real rate of return on the investment if inflation is 5% pa in the first two years and 4% pa thereafter?

Solution

Convert the payments to dollar amounts consistent with the initial investment.

The payment after 3 years in year 0 dollars is: $\dfrac{12000}{1.05^2 1.04}$

The payment after 6 years in year 0 dollars is: $\dfrac{9000}{1.05^2 1.04^4}$

The equation of value is:

$$20000 = \frac{12000}{1.05^2 1.04} v^3 + \frac{9000}{1.05^2 1.04^4} v^6 = (10465.72) v^3 + (6977.993) v^6$$

Solving the quadratic in v^3 gives:

$$v^3 = \frac{-10465.72 \pm \sqrt{10465.72^2 - 4(-20000)(6977.99)}}{2(6977.99)} = 1.1017,\ -2.602$$

The positive root is 1.1017. $v^3 = \left(\dfrac{1}{1+i}\right)^3 = 1.1017 \Rightarrow i = -0.031774$

Therefore the real rate of return is -3.18%

Question 4

One annuity pays 6 at the end of each year for 36 years. Another annuity pays 10 at the end of each year for 18 years. The present values of both annuities are equal at effective rate of interest i >0. If an amount of money invested at the same rate i will triple in n years, find n.

Solution

$$PV_A = 6a_{\overline{36|}}$$

$$PV_B = 10a_{\overline{18|}}$$

$$PV_A = PV_B \Leftrightarrow 6a_{\overline{36|}i} = 10a_{\overline{18|}i} \Rightarrow 6\frac{1-v^{36}}{i} = 10\frac{1-v^{18}}{i} \Rightarrow 6v^{36} - 10v^{18} + 4 = 0$$

Solve the quadratic for $v^{18} \Rightarrow \dfrac{10 \pm \sqrt{100-96}}{12} = \dfrac{10 \pm 2}{12} = 1,\ \dfrac{2}{3}$

$$\Rightarrow v^{18} = \frac{2}{3} \Rightarrow i = 2.278\%$$

$$1.02278^{n} = 3 \Rightarrow n = \frac{\ln(3)}{\ln(1.02278)} = 48.77$$

Question 5

An investor is making level payments at the beginning of each year for 15 years to accumulate $10,000 at the end of the 15 years in a bank which is paying 5% effective interest rate.

a. Find the level annual deposit

At the end of 5 years the bank drops its interest rate to 4% effective. You revise your level payments to ensure the total account will still accumulate to $10,000.

b. Find the annual deposit for the last 10 years

Solution

a. The annual deposits would have initially been calculated assuming interest rates of 5% for the 15 years:

$$10{,}000 = X\ddot{s}_{\overline{15}|0.05} = X\left(s_{\overline{16}|0.05} - 1\right) \Rightarrow X = \frac{10{,}000}{\left(s_{\overline{16}|0.05} - 1\right)} = \frac{10{,}000}{22.6575} = 441.35$$

b. After 5 years interest rates drop to 4%. The annual deposit for the remaining ten years is calculated from:

$$10{,}000 = 441.35\ddot{s}_{\overline{5}|0.05}(1.04)^{10} + X\ddot{s}_{\overline{10}|0.04} = 3{,}790.42 + X\ddot{s}_{\overline{10}|0.04}$$

$$\Rightarrow X = \frac{6{,}209.58}{\ddot{s}_{\overline{10}|0.04}} = \frac{6{,}209.58}{\left(s_{\overline{11}|0.04} - 1\right)} = 497.31$$

WEEK 6

Question 6

You deposit $1,000 then withdraw level annual payments starting one year after the initial deposit was made. If immediately after the 10th drawing, you have $200 left in the account, find the amount of each withdrawal if the annual effective rate of interest is 4%.

Solution

If the amount of the annual drawings is X , then we need to solve the equation:

$$1000(1+i)^{10} - Xs_{\overline{10|}0.04} = 200$$

This is equivalent to:

$$1000 = Xa_{\overline{10|}0.04} + 200v_{0.04}^{10} \Rightarrow X = \$106.63$$

Question 7

A loan is to be repaid with level installments at the end of each half-year for 5 years, at a nominal rate of interest 10% payable half-yearly. After the third payment the outstanding loan balance is $8,000. Find the initial amount of the loan.

Solution

Work with periods of half-years:

$n = 10$

$$OB_3 = 8{,}000 = Ka_{\overline{n-t|}} = Ka_{\overline{7|}0.05} = K(5.7864) \Rightarrow K = 1382.55$$

$$OB_0 = Ka_{\overline{10|}0.05} = (1382.55)(7.7217) = 10{,}675.64$$

Question 8

A loan of $50,000 is being repaid with 20 installments at the end of each year at 5% effective. Show that the amount of interest in the 11th installment is $\frac{2500}{1+v^{10}}$.

Solution

OB_0 = 50,000

n = 20

i = 5%

$$OB_0 = Ka_{\overline{n}|} \Rightarrow K = \frac{50{,}000}{a_{\overline{20}|}}$$

$$OB_t = Ka_{\overline{n-t}|}$$

$$I_t = OB_{t-1} \cdot i = Ka_{\overline{n-t+1}|} i = \frac{2500}{a_{\overline{20}|}} a_{\overline{n-t+1}|}$$

$$I_{11} = \frac{2500}{a_{\overline{20}|}} a_{\overline{10}|} = 2500 \frac{1-v^{10}}{1-v^{20}} = \frac{2500}{1+v^{10}}$$

since $\left(1-v^{10}\right)\left(1+v^{10}\right) = \left(1-v^{20}\right)$.

Question 9

A loan is being repaid with series of payments at the end of each quarter for five years. If the amount of principal in the fourth payment is $200, find the total amount of principal in the last four payments. Interest is at the rate of 10% convertible quarterly.

Solution

Work with periods of one quarter.

n = 20

i = 2.5% (quarterly effective interest rate)

$$PR_t = K_t - I_t = K - OB_{t-1} i = K - iKa_{\overline{n-t+1}|} = K(1-(1-v^{n-t+1})) = Kv^{n-t+1}$$
$$\Rightarrow PR_4 = 200 = Kv^{17} \Rightarrow K = 200(1+i)^{17}$$

The principal in the last four payments is:

$$PR_{20} + PR_{19} + PR_{18} + PR_{17} = K\left(v^1 + v^2 + v^3 + v^4\right) = Ka_{\overline{4}|} = 200(1+i)^{17} a_{\overline{4}|} = 1{,}144.86$$

Question 10

A loan of $2,000 is to be repaid by level monthly installments over 15 years using an interest rate of 10% per annum.

a. What is the monthly installment amount?
b. What is the capital repaid in the sixth year?

Solution

a. The level monthly instalment K is found from:

$$2000 = 12Ka_{\overline{15|}}^{(12)} \Rightarrow K = \frac{2000}{12a_{\overline{15|}}^{(12)}} = \frac{2000}{12\left(a_{\overline{15|}}\frac{i}{i^{(12)}}\right)} = 20.97$$

b. The capital outstanding at the beginning of the sixth year is:

$$OB_5 = 12Ka_{\overline{10|}}^{(12)} = 251.64a_{\overline{10|}}^{(12)} = 251.64\left(a_{\overline{10|}}\frac{i}{i^{(12)}}\right) = 1{,}615.87$$

The capital outstanding at the end of the sixth year is:

$$OB_6 = 12Ka_{\overline{9|}}^{(12)} = 251.64a_{\overline{9|}}^{(12)} = 251.64\left(a_{\overline{9|}}\frac{i}{i^{(12)}}\right) = 1{,}514.47$$

The capital repaid during the sixth year is, therefore: 1,615.87 – 1,514.47 = $101.40

Question 11

A project is expected to provide in return for an initial outlay of $120,000, payments of $3,000 per month in year 1, $42,000 throughout year 2 and a lump sum of $50,000 after 3 years. Find the net present value of the project using a risk discount rate of 8% per annum.

Solution

$$\begin{aligned} NPV &= -120{,}000 + 12(3000)a_{\overline{1|}}^{(12)} + 42{,}000\bar{a}_{\overline{1|}}v + 50{,}000v^3 \\ &= -120{,}000 + 12(3000)\left(a_{\overline{1|}}\frac{i}{i^{(12)}}\right) + 42{,}000\left(a_{\overline{1|}}\frac{i}{\delta}\right)v + 50{,}000v^3 \\ &= -120{,}000 + 34{,}538.57 + 37{,}430.09 + 39{,}691.61 \\ &= \$8{,}339.73 \end{aligned}$$

Question 12

An investment project requires an outlay of $100,000 at time t_1 years and produces income of $50,000 at time t_2 years and $70,000 at time t_3 years. Assuming that these are the only cashflows, calculate the internal rate of return if t_1 = 2, t_2 = 4 and t_3 = 5.

Solution

The equation of value is: $-100{,}000v^2 + 50{,}000v^4 + 70{,}000v^5 = 0$ or $-10v^2 + 5v^4 + 7v^5 = 0$

At i = 8% $\quad -10v^2+5v^4+7v^5=-0.13416$

At i = 7% $\quad -10v^2+5v^4+7v^5=0.07099$

$$\frac{0.08-i}{0.08-0.07}=\frac{-0.1342-0}{-0.1342-0.07099}=0.654028\Rightarrow i=0.08-0.00654028=0.07346$$

$IRR\approx 7.35\%$

Question 13

An investor can raise money by two options:

a. by lending \$800 and receiving \$900 at the end of the period
b. by lending \$1000 and receiving \$1120 at the end of the period.

i. Find the IRR for both options.

Using NPV as the criterion for selecting projects, which option should be chosen if

ii. the interest preference rate for the period is 10%
iii. the interest preference rate for the period is 8%
iv. the interest preference rate for the period is 12.25%

Solution

a. $NPV=-800+900v=0\Rightarrow v=\frac{8}{9}\Rightarrow i_A=12.5\%$

$NPV=-1000+1120v=0\Rightarrow v=\frac{1}{1.12}\Rightarrow i_B=12\%$

b. $NPV_A(0.10)=-800+900v=18.18$

$NPV_B(0.10)=-1000+1120v=18.18$

Both options have an equivalent net present value, so either could be chosen. Since both options have an internal rate of return greater than the interest preference rate, both options are profitable.

c. $NPV_A(0.08)=-800+900v=33.33$

$NPV_B(0.08)=-1000+1120v=37.04$

Option B is preferred if the interest preference rate is 8% since it has a higher NPV.

d. Since the internal rate of return for option A (12.5%) is greater than the interest preference rate, option A will produce a profit. The internal rate of return for option B is 12%. Since this is less than the interest preference rate, option B will produce a loss. Therefore, option A is preferred.

Question 14

An investor borrows at an effective rate of interest of 6% per annum and invests spare funds at 5% per annum. The investor borrows $4500 to be repaid after 3 years (with no early repayment option) and uses the money to invest in a project that will provide income of $150 at the end of each month for 36 months. Interest on the money borrowed is paid at the end of each month. What is the borrower's accumulated profit at the end of 3 years?

Solution

The amount of monthly interest payable at the end of each month is:

$$4500\left(1.06^{1/12}-1\right)=21.904$$

The investor's net monthly income is 150 – 21.904 = $128.096

The accumulated profit at the end of 3 years will be:

$$AV=12(128.096)s_{\overline{3}|0.05}^{(12)}-4500=12(128.096)\left(s_{\overline{3}|0.05}\frac{i}{i^{(12)}}\right)-4500=\$455.95.$$