5.1 Line integral

in $\mathbb{R}^n$

$f:\mathbb{R}^n \longrightarrow \mathbb{R}$ is integrable

$$\int_C f(\mathbf{x})\, ds := \int_a^b f(\mathbf{g}(t))\, |\mathbf{g}'(t)|\, dt$$

C' ← Smooth curve in $\mathbb{R}^n$

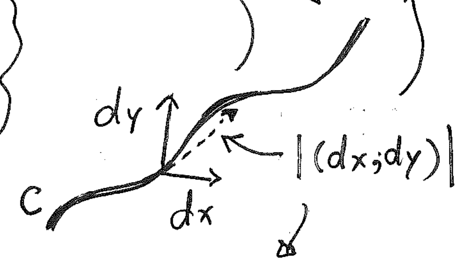
$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \mathbf{g}(t) \qquad \mathbf{g} : [a\ b] \longrightarrow \mathbb{R}^n$$

$$\mathbf{g}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

arc length of C is

$$= \int_C ds = \int_a^b |\mathbf{g}'(t)|\, dt \qquad = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$\sqrt{(dx)^2 + (dy)^2}$$



line integral of v.f.

$$\int_C \mathbb{F}(\mathbf{x}) \bullet d\mathbf{x} := \int_a^b \mathbb{F}(\mathbf{g}(t)) \bullet \mathbf{g}'(t)\, dt$$

unit vector tangent to C

$$\int_C \underbrace{\mathbb{F} \bullet \mathbb{t}}\, ds = \int \mathbb{F} \bullet \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}\, |\mathbf{g}'(t)|\, dt \qquad = \int_C F_1\, dx + F_2\, dy$$

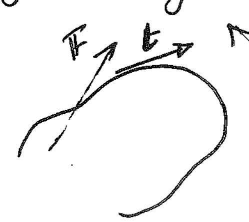
Scalar valued function measures effect of $\mathbb{F}$ along C

$$\mathbb{t} = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$$

but

$$\int_C d\mathbf{x} = \int_a^b \mathbf{g}'(t)\, dt = \mathbf{g}(b) - \mathbf{g}(a)$$

so $\int_C \mathbb{F} \cdot d\mathbf{x} =$ work of $\mathbb{F}$ along C



$\mathbb{F} \cdot \mathbb{t} = F_{tang}$

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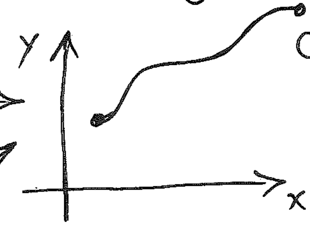
line integral is independent of parametrization $\mathbf{g}(t)$



$t = \varphi(u)$

$dt = \varphi'(u)\, du$

$\underset{a}{[} \overset{t}{\rule{2em}{0.4pt}} \underset{b}{]} \xrightarrow[\mathbf{g}]{①}$

$\frac{d}{dt}\mathbf{g}(t)$

$$① \int_C ds = \int_a^b |\mathbf{g}'(t)|\, dt = \int_{u=c}^{d} |\mathbf{g}'(\varphi(u))|\, \varphi'(u)\, du$$

$$② \int_C ds = \int_c^d \left| (\mathbf{g} \circ \varphi)'(u) \right| du = \int_c^d |\mathbf{g}'(\varphi(u))|\, \varphi'(u)\, du$$

$\frac{d}{du}$ $\qquad$ $\frac{d}{dt}$

$\underset{c}{[} \overset{u}{\rule{2em}{0.4pt}} \underset{d}{]} \xrightarrow{\varphi}$

$\xrightarrow{②}$ $\mathbf{g} \circ \varphi$ ← a different parametrization