§18 - Locally Compact Spaces and Compactifications

1 Motivation

Metrizable spaces and Compact Hausdorff spaces are two of the nicest, most common topological spaces. We have seen that they both have nice separation properties, and they both behave fairly well with respect to countability properties. We have also seen that compact spaces have some very nice properties like "Every continuous real-valued function with compact domain is bounded" and "Every sequence in a compact space has a convergent subsequence". However, we often don't need the full strength of compactness *of an entire space*, often it is enough to have just compactness locally. For example, if we think of the universe as $\mathbb{R}^3$, then a real-valued function on it might be unbounded, but if we restrict our attention to that function defined on the earth (a compact ball) then it will be bounded there. This will give rise to the notion of local compactness, which is a very common topological property.

From there we will look at the notion of a compactification of a space. We do this for three reasons: (1) it is fun; (2) it is natural; and (3) it is intellectually and morally satisfying. The idea is partly inspired by our use of the "extended reals" (which contain the elements $+\infty$ and $-\infty$), which was defined in analysis so that the supremum of a set was always defined. This also helped us distinguish which types of sequences don't converge (in $\mathbb{R}$) because in the extended reals there are sequences that converge to $\pm\infty$ and those that diverge because they are "all over the place" (think about a sequence that is an enumeration of $\mathbb{Q}$). In a general topological space we will often be able to "extend" the space so that we can make sense of "converging to the boundary" or "converging to ∞". In practice we have already seen most of these "extensions".

2 Some Examples

Before we get into definitions, let us try to answer the question: "Does $I = (0, 1)$, the open unit interval (with the usual topology), embed densely into a compact space?". In other words, is there a compact space E such that $I \cong X$ is a dense subset of E? Of course!

Some Options:

1. We could just take the closed unit interval $[0, 1]$. (Which is called the **two point compactification of** I for obvious reasons.)
2. We could also embed I into the circle S^1 by the map $f(x) = (1, 2\pi x)$ written in polar coordinates. This is the **one point compactification of** I.

3. Getting a bit wacky we could also map I into the topologist's sine curve T, by sending I to the sine portion of T.

A natural question is: "Is there a 'largest' compactification of I?" to which I would respond: "Why are you thinking about such horrible things?" and "Yes." We won't look into this question now, but you should be able to answer it on your own by the end of these notes.

3 The Definitions

First we introduce the notion of local compactness which facilitates our discussion of compactifications.

Definition. *A topological space $(X, \mathcal{T})$ is **locally compact** provided that for each point $p \in X$ there is a compact set $C \subseteq X$ that contains p and contains an open neighbourhood U of p (i.e. $p \in U \subseteq C \subseteq X$).*

Clearly every compact space is locally compact. We also see that $\mathbb{R}^n_{\text{usual}}$ is locally compact because if $p \in \mathbb{R}^n$, then $\{x \in \mathbb{R}^n : d(x, p) \leq 10\}$ is a compact set containing p and the open neighbourhood $B_{10}(p)$. $\mathbb{Q}_{\text{usual}}$ is not locally compact (as is easy to see but annoying to write down).

Here is one reason to care about local compactness: it characterizes when a space is an open subspace of a compact Hausdorff Space.

Theorem. *A topological space $(X, \mathcal{T})$ is homeomorphic to an open subspace of a compact Hausdorff space iff $(X, \mathcal{T})$ is a locally compact Hausdorff space.*

We will be mostly interested in the $\Leftarrow$ direction of this proof, which involves creating a nice compact space. The other direction will be omitted, but a motivated student should be able to come up with the proof (and a lazier, less motivated student could just look it up in Munkres, and an even more torpid student would just take my word for it.)

Where will this compact space come from? The most naive thing possible would be to add a single point p to a space X and then put a compact topology on $X \cup \{p\}$. This will turn out to work!

Definition. *Let $(X, \mathcal{T})$ be a locally compact space and let p be a point that is not in X. Define the **one point compactification of** X on the set $X \cup \{p\}$ by declaring open (in $X \cup \{p\}$) all sets $U \in \mathcal{T}$ and $(X \cup \{p\}) \setminus C$ where $C \subseteq X$ is closed and compact. Here $X \cup \{p\}$ is called the **one point compactification of** X.*

If we insist that our topological space X is in fact *Hausdorff*, then recall that compact in X implies closed in X.

Details! Exercise: Check that (1) the one point compactification of a topological space really forms a topology; (2) the point p we choose doesn't matter, any two different p yield homeomorphic compactifications.

Theorem. *If X is a locally compact Hausdorff space (which is not compact) and $Y = X \cup \{p\}$ is its one point compactification then:*

1. *Y is a compact Hausdorff space;*

2. *X is dense in Y;*

3. *X is an open subspace of Y.*

Proof. Showing that Y is Hausdorff is straightforward. If x, y are distinct points in X, then since that space is Hausdorff, there are disjoint open (in X and hence Y) sets $U \ni x$ and $V \ni y$. For $x \in X$ and p (the additional point) by local compactness of x there is a compact set C and an open (in X and so Y) set U such that $x \in U \subseteq C$. So then $p \in Y \setminus C$, which is an open set in Y, and this set is disjoint from U.

To show compactness, let $\mathcal{U}$ be an open cover of Y. There is an open set $U_p \in \mathcal{U}$ that contains p. By definition, there is a compact set $C \subseteq X$ such that $U_p = Y \setminus C$. Now it only takes a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ to cover C. Thus $\mathcal{V} \cup \{U_p\}$ is the desired finite subcover.

To show that X is dense, note that since X is not compact, any open set containing p cannot be just the singleton $\{p\}$, it must contain elements of X. Hence $p \in \overline{X}$, so we have $\overline{X} = Y$.

For the open subspace part of the proof, we note that Y is Hausdorff, and so points are closed. Hence $Y \setminus \{p\} = X$ is open. Also, any open (in Y) set U is either already open in X, or $U \cap X$ is open in X. Also, any set open in X is open in Y. □

This tells us that the one point compactification of a locally compact Hausdorff space is exactly the answer to our (modified) question: "Does every locally compact Hausdorff space embed densely into a compact space?".

4 Examples

Here is the fun part, we look at examples of one point compactifications. (For convenience here we will denote the one point compactification of a space X by $\sigma(X)$.)

1. $\sigma(I) \cong S^1$, the circle (where I is the open unit interval).

2. If $X = (0, 1) \cup (3, 4)$, then $\sigma(X)$ is a "figure 8".

3. For X an open disc in $\mathbb{R}^2$, $\sigma(X) = S^2$ the sphere.

4. $\sigma(\mathbb{N}) \cong \omega + 1$, and in fact, if X is any countable, discrete space, then $\sigma(X) \cong \omega + 1$. Note that these spaces are *not* homeomorphic to the cofinite topology on a countable set.

5. $\sigma(\omega_1) = \omega_1 + 1$. Cool!

6. $\sigma(\mathbb{R}^n) \cong S^n \subseteq \mathbb{R}^{n+1}$.

There are a lot of interesting things going on here. Notice that the first fact tells us that $\sigma(\mathbb{R}) \cong S^1$ since $\mathbb{R} \cong I$. The third fact also tells us that $\sigma(\mathbb{R}^2) \cong S^2$. To show in general that $\sigma(\mathbb{R}^n) \cong S^n$ we use what's called "stereographic projection". We will illustrate this with the two dimensional case, but there is a clear generalization to higher dimensions.

Theorem. *$\sigma(\mathbb{R}^2) \cong S^2$.*

Proof. Consider the map $s : S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$ defined by:

$$s(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z})$$

You can check that this is a continuous bijection with (continuous) inverse given by

$$s^{-1}(a,b) = (\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{-1+a^2+b^2}{1+a^2+b^2})$$

The actual computations aren't so interesting, but the picture is quite instructive. □

5 The Big Compactification

Finally, let's leave this section with a much, much larger compactification than the one point compactification. It turns out that there is a maximal (in some sense) compactification of a locally compact Hausdorff space. It turns out to be a very interesting object that has many different descriptions. Since we have seen ultrafilters I will present that one. There is *a lot* to say about this space, but I will leave you with some teasers.

Definition. *Let $\beta(\mathbb{N}) := \{\mathcal{U} : \mathcal{U}$ is an ultrafilter on $\mathbb{N}\}$ and define a basis for this topology by $B_A := \{\mathcal{U} \in \beta(\mathbb{N}) : A \in \mathcal{U}\}$, where $A \subseteq \mathbb{N}$. This is called the Čech-Stone Compactifiaction of $\mathbb{N}$.*

Now this is big object, but you can easily check the following facts (possibly using some previous assignment questions):

1. There is an open, dense homeomorphic copy of $\mathbb{N}$ in $\beta(\mathbb{N})$ (namely the prinipal ultrafilters).

2. The basis described for $\beta(\mathbb{N})$ contains only clopen sets (so the space is zero dimensional).

3. $\beta(\mathbb{N})$ is a compact Hausdorff space.

Here is one of the main reasons we care about $\beta(\mathbb{N})$:

Theorem. *Let $(X, \mathcal{T})$ be a compact Hausdorff space and let $f : \mathbb{N} \to X$ be a continuous function. Then there is a continuous function $F : \beta(\mathbb{N}) \to X$ such that F restricted to $\mathbb{N}$ is the function f.*

This has the following neat application:

Theorem. *$\beta(\mathbb{N})$ is a compact, Hausdorff semigroup.*

The idea is that since "Addition on the left by n" is a continuous map from $\mathbb{N}$ to $\mathbb{N} \subseteq \beta(\mathbb{N})$ a compact Hausdorff space, we can extend that function to a continuous function whose domain is all of $\beta(\mathbb{N})$. That's right! We can define a way to add together any two ultrafilters! (Of course there are some details missing, but this is the main idea).

You can find a full write-up of this (by Leo Goldmakher) where he gives a survey of an ultrafilter proof of Hindman's Theorem:

`http://boolesrings.org/mpawliuk/2012/09/13/hindmans-theorem-write-up/`