Problem Set 4 Solutions

Question 1 solution

Proof by contradiction. Let G be a graph with the minimal number of vertices that cannot be embedded in $\mathbb{R}^3$. Either G is empty (so it can be embedded in $\mathbb{R}^3$), or G has an edge e. $\psi(e)=\{x,y\}$. Contract e to obtain $H := G/\{e\}$, and embed H in $\mathbb{R}^3$ (possible by assumption). $\{x,y\}$ contract to a vertex z. Choose a straight line l through z. There are a finite number of points on l such that a straight line from a neighbour of z to that point intersects the graph. l can be chosen not to intersect the graph except at z. So there is a point w on l so that no straight line from a neighbour of z to w intersects H. Form a graph from the induced embedding of $H\backslash\{e\}$ by adding points z and w, the line segment between them in l, connecting z and w to the neighbours of x and of y in G correspondingly.

This is an embedding of G in $\mathbb{R}^3$. Contradiction.

"3"

Question 2 solution

There are a number of ways to prove this - we present Alex Edmonds's nice solution, which uses:

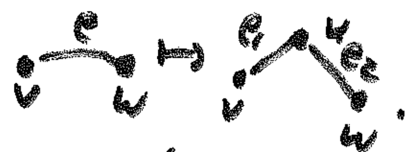
(*) G is a forest $\iff G^*$ has one vertex (and loops).

Let G be a simple connected planar graph, and choose a spanning tree T^* for G^*. Then $G^* \setminus T^*$ has one vertex, so its dual, $G - T$ (deletion-contraction duality) is a forest.

Next, partition $T^{**} =: T \subseteq G$ into two forests. G is simple, so G^* is bridgeless. For each cut set of G^* whose edges are all in T^*, add one edge to a set H^* and the others to a set F^*. Add all remaining edges in T^* both to F^* and to H^*. Both $G^* - H^*$ and $G^* - F^*$ are connected spanning subgraphs of G^* by construction, therefore both $G^* \setminus (G^* - H^*)$ and $G^* \setminus (G^* - F^*)$ have one vertex, and their duals H and F are both forests. $T = H \cup F$ so we are finished.

$G = (G - T) \cup H \cup F$.

Question 3 solution

Let H be obtained from G by a single edge subdivision, $e \mapsto \{e_1, e_2\}$ with $\psi_G(e) = \{v, w\}$ and $\psi_{G'}(e_1) = \{v, u\}$, $\psi_{G'}(e_2) = \{u, w\}$. .

It suffices to prove that G is planar iff H is planar.

($\Rightarrow$) Choose a point in the interior of l, the embedding of e, to be u. This divides l into two paths l_1 and l_2 in $\mathbb{R}^2$, which we take to be embeddings of e_1 and of e_2 correspondingly. The rest of the graph H is embedded as induced by G.

($\Leftarrow$) Let l_1 and l_2 be embeddings of e_1 and of e_2. "Forget" the point $l_1 \cap l_2$, and set $l_1 \cup l_2$ to be the embedding of e. The rest of G is embedded as induced from H.

Question 4 solution

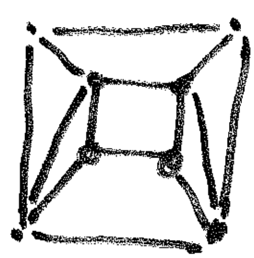
a) Let G be a simple planar graph on $n \geq 11$ vertices. Then $2e \geq 3f$ and from Euler's formula $n-e+f=2 \Rightarrow n+\frac{2e}{3}-e \geq 2 \Rightarrow$
$\Rightarrow \boxed{3n-6 \geq e}$ (*)

$$|E(\bar{G})| = |E(K_n)| - |E(G)| \underset{(*)}{\geq} \frac{n(n-1)}{2} - 3n + 6 =$$
$$= \frac{n^2-7n+12}{2}$$

By (*), if $\bar{G}$ is planar, then $\frac{n^2-7n+12}{2} \leq 3n-6 \Rightarrow$
$\Rightarrow n^2-13n+24 \leq 0$

By elementary algebra this has no solution for $n \geq 11$.

b)



is self-complementary.

Question 5 solution

Let G be a d-regular graph with colouring C, with $|C| = \chi(G)$.
If vertex $v \in V(G)$ is coloured $c \in C$, then none of its d neighbours can be coloured c. So there are at most $n-d$ vertices coloured c. Summing over all colours in C:

$$n = \sum_{c \in C} \left| \begin{matrix}\text{vertices}\\ \text{coloured } c\end{matrix} \right| \leq (n-d)|C| = (n-d)\chi(G)$$

So

$$\chi(G) \geq \frac{n}{n-d}.$$

Q.E.D.

Question 6 Solution

By the question, $\deg(f) \geq 4$ for all faces.

Handshake Lemma: $\sum \deg(f) = 2e$

$\Rightarrow 4f \leq 2e \leq \boxed{f \leq \frac{e}{2}}$ $(*)$

If $\deg(v) \geq 4$ for all vertices, by the Handshake Lemma: $\sum \deg(v) = 2e \Rightarrow 4v \leq 2e \Rightarrow \boxed{v \leq \frac{e}{2}}$ $(**)$

Substitute $(*)$ and $(**)$ into Euler's formula:

$v - e + f = 2 \Rightarrow \frac{e}{2} - e + \frac{e}{2} \geq 2 \Rightarrow 0 \geq 2$ contradiction.

Thus, $\deg(v) \leq 3$ for some $v \in V(G)$ ✓

Assume by induction that any graph with n vertices is 4-colourable if it is simple, planar, girth ≥ 4. (base cage of $n = 1, 2, 3$ are trivial). Let G be a graph with $n+1$ vertices, satisfying the conditions. As we have shown, $\exists v \in V(G)$ s.t. $\deg(v) \leq 3$. $G - \{v\}$ is 4-colourable by the induction hypothesis. Colouring v be a colour different from that of its ≤ 3 neighbours induces a 4-colouring of G.

Q.E.D.

Question 7 solution

Let G be a planar graph with $\deg(v)$ even for all $v \in V$.

Any cut set of G has even order:

If C cuts $V(G)$ into $V(G_1)$ and $V(G_2)$

then $C = \{\psi(e) = \{u,v\} \mid u \in V(G_1), v \in V(G_2)\}$

$$\sum_{u \in V(G_1)} \deg(u) = \sum_{u \in V(G_1)} \#\{\psi(e) = \{u,v\} \mid v \in V(G)\} =$$

$$= \sum_{u \in V(G_1)} \#\{\psi(e) = \{u,v\} \mid v \in V(G_1)\} + \#\{\psi(e) = \{u,v\} \mid v \in V(G_2)\}$$

$$= 2\,|E(G_1)| + |C|.$$

$\deg u$ is even for all $u \Rightarrow |C|$ is even.

Each cut corresponds to a cycle in G^*

$\Rightarrow$ cycles of G^* are all of even length

$\overset{\text{Thm}}{\Rightarrow}$ G^* is bipartite $\Rightarrow$ G^* is 2-colourable

$\Rightarrow$ G is 2-face-colourable.

Q.E.D (Solution of Yan Li)