Lecture 22

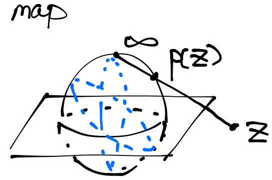
FRACTIONAL LINEAR TRANSFORMATION

Note: The class notes are different from the book.
(e.g. the book doesn't mention any thing about matrices)

Remainder: A F.L.T is a function $T(z)=\dfrac{az+b}{cz+d}$ where $ad-bc\neq 0$

- F.L.T are invertible & their inverses are also FLT's
- We can extend T to a map

$$T:\ S \to S,\quad S = \mathbb{C}\cup\{\infty\}$$

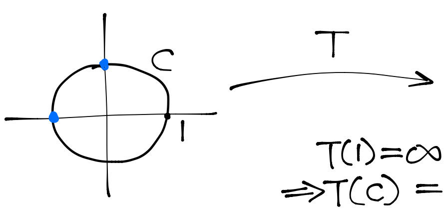


- F.L.T's take circles/lines to circles/lines.

Q: Given T & a circle (or line) how can find out whether T maps it to a circle or a line & which one it is?

Ex: $T(z)=\dfrac{z}{z-1}$, $C=\{z \mid |z|=1\}$

Q: What is $T(C)$?



$T(1)=\infty$
$\Rightarrow T(C)=\text{line}$

pick 2 pts

Compute $T(i)$, $T(-1)$, then $T(C)$ is the line passing through $T(i)$ & $T(-1)$

$T(i)=\dfrac{i}{i-1}=\dfrac{i(-1-i)}{2}=\dfrac{1-i}{2}=\dfrac{1}{2}-\dfrac{i}{2}$

$T(-1)=\dfrac{-1}{-1-1}=\dfrac{1}{2}$

So $T(C)$ is the vertical line $\mathrm{Re}(z)=\frac{1}{2}$.

(b) Let $C=\{z \mid |z|=2\}$
Find $T(C)$

It's circle this time, cuz $z\in C$, then $T(z)\neq\infty$ ($T(z)=\infty$ only if $z=1$)

$T(C)=\{z=x+iy \mid (x-x_0)^2+(y-y_0)^2=r\}$

Find x_0, y_0, r. to do this, pick 3 distinct pts, calculate the image & solve for the pts.

pts z_1, z_2, z_3

Find $T(z_1), T(z_2), T(z_3)$

$T(z_1)=x_1+iy_1 \leftarrow$ satisfies $(x_1-x_0)^2+(y_1-y_0)^2=r^2$

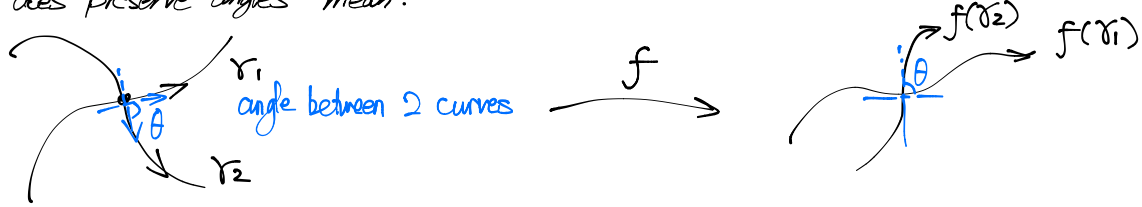
$T(z_2)=x_2+iy_2 \leftarrow$ satisfies $(x_2-x_0)^2+(y_2-y_0)^2=r^2$

$T(z_3)=x_3+iy_3 \leftarrow \cdots$ $(x_3-x_0)^2+(y_3-y_0)^2=r^2$

CONFORMAL MAPS

Def'n: A map $f: \mathbb{C} \to \mathbb{C}$ is called conformal at $z_0 \in \mathbb{C}$ if it's analytic at z_0 and preserves angles.

What does "preserve angles" mean?



Angle b/w γ_1 & γ_2 = angle b/w $f(\gamma_1)$ & $f(\gamma_2)$

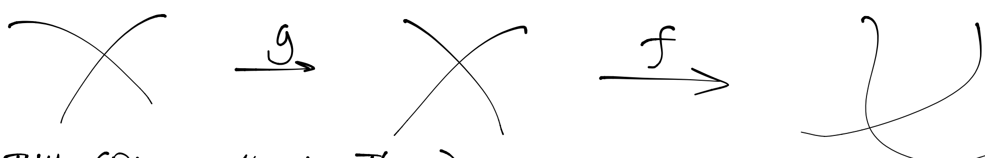
Def'n We say f is conformal on a domain D, if it is conformal at all $z_0 \in D$.

Thm: If f is analytic on D and $f'(z_0) \neq 0$, then f is conformal at z_0

Ex: ① $f(z) = z^2$ is conformal for all $z \neq 0$. ($f'(z) = 2z$)
② $f(z) = z^n$ is conformal for all $z \neq 0$
③ $f(z) = e^z$ is conformal everywhere ($f'(z) = e^z \neq 0$)

THM: If f is analytic in D & one-to-one (injective) then it's conformal)

THM The composition of conformal maps is conformal.
"Pf": (If f, g conformal, $f \circ g$ is conformal)



THM: (Riemann Mapping Theorem)
If S_1, S_2 are two simply-connected domains, then $\exists$ a conformal bijection $f: S_1 \to S_2$.
(proof hard)