Australian National University

STAT 6046 Tutorial Week 11

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Today's plan

- Brief review of course material
- Go through selective tutorial questions

Effective duration & Duration

$$PV = \sum_{k=1}^{n} C_{t_k} v_i^{t_k} = \sum_{k=1}^{n} C_{t_k} (1+i)^{-t_k}$$

- Effective duration/volatility:

$$\upsilon = -\frac{1}{PV}\frac{d}{di}PV = \frac{\sum_{k=1}^{n} C_{t_k} t_k (1+i)^{-t_k-1}}{\sum_{k=1}^{n} C_{t_k} (1+i)^{-t_k}} = \frac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k+1}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} = \frac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k+1}}{PV}$$

- Duration/discounted mean term/Macaulay's duration:

$$\text{DMT} = \tau = \frac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} = \frac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k}}{PV} = (1+i)\upsilon$$

Duration of a bond

- Coupon-paying Bond price:

$$P = Fr \cdot a_{\overline{n}|j} + C \cdot v_j^n = \sum_{t=1}^{n} Fr \cdot v_j^t + C \cdot v_j^n$$

- Bond duration (coupon paid at the end of each half year):

$$\tau = \frac{\sum_{t=1}^{n} t \cdot Fr \cdot v_j^t + n \cdot C \cdot v_j^n}{\sum_{t=1}^{n} Fr \cdot v_j^t + C \cdot v_j^n} = \frac{Fr \cdot (Ia)_{\overline{n}|j} + n \cdot C \cdot v_j^n}{P}$$

- Zero coupon bond: $\tau = \dfrac{n \cdot C \cdot v_i^n}{C \cdot v_i^n} = n$

- The larger the duration (or volatility), the larger the sensitivity of a series of cash flows to an interest rate movement.

Taylor's approximation

- For small ε, the function $f(x+\varepsilon)$ can be written as:

$$f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2!} f''(x) + \ldots$$

- Our focus will be on the first two orders, the impact of higher orders will be small when ε is small.
- First order derivative $f'(x)$ → Duration τ

$$\varepsilon \frac{PV'(i_0)}{PV(i_0)} = -\varepsilon \upsilon = -\varepsilon \frac{\tau}{(1+i_0)}.$$

- Second order derivative $f''(x)$ → Convexity c

$$c(i) = \frac{1}{PV} \frac{d^2}{di^2} PV = \frac{1}{PV} \frac{d}{di} \left(-\sum_{k=1}^{n} C_{t_k} t_k (1+i)^{-t_k-1} \right) = \frac{\sum_{k=1}^{n} C_{t_k} t_k (t_k+1) v_i^{t_k+2}}{PV}$$

- For bond:

$$\frac{PV(i_0+\varepsilon) - PV(i_0)}{PV(i_0)} \cong \varepsilon \frac{PV'(i_0)}{PV(i_0)} + \frac{\varepsilon^2}{2} \frac{PV''(i_0)}{PV(i_0)} = -\varepsilon \frac{\tau}{(1+i_0)} + \frac{\varepsilon^2}{2} c$$

Immunisation

- Immunisation is the process of selecting a portfolio of assets that will protect a fund's surplus against small changes in interest rates.

$$S(i) = V_A(i) - V_L(i)$$

- A fund is said to be immunised against small changes in the interest rate if:
  - The surplus in the fund at the current interest rate is zero and
  - Any small change in the interest rate (in either direction) would lead to a positive surplus.

Thus, at rate of interest i_0 the fund is immunised against small movements in the rate of interest of ε if and only if $V_A(i_0) = V_L(i_0)$ and $V_A(i_0 + \varepsilon) \geq V_L(i_0 + \varepsilon)$

Immunisation: 3 conditions

The first condition is that the surplus at the current interest rate is zero. That is, $S(i_0) = 0$.

The second term $\varepsilon S'(i_0)$ will be equal to zero if and only if $S'(i_0) = 0$. This is satisfied if $V_A'(i_0) = V_L'(i_0)$.

- The second condition is that the assets and liabilities must have the same volatility.

$$\tau_A(i_0) = \tau_L(i_0)$$

The third condition is that $\frac{\varepsilon^2}{2} S''(i_0) \geq 0$. ε^2 is always positive, so we need to ensure that $S''(i_0) \geq 0$ or equivalently that $V_A''(i_0) \geq V_L''(i_0)$.