FUNCTIONS OF RANDOM VARIABLES (Chapter 6)

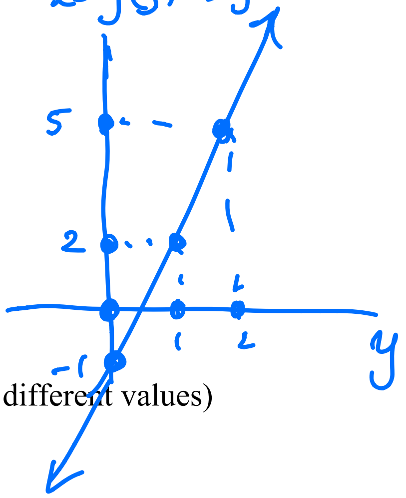
1. The discrete case

Example 1 A coin is tossed twice. Let Y be the number heads that come up. Find the dsn of $X = 3Y - 1$.

$x = g(Y)$ where $g(t) = 3t - 1$

$x = g(y) = 3y - 1$

Here, $Y \sim \text{Bin}(2,1/2)$. So $p(y) = \begin{cases} 1/4, & y=0 \\ 1/2, & y=1 \\ 1/4, & y=2 \end{cases}$



If $y = 0$ then $x = 3(0) - 1 = -1$.
If $y = 1$ then $x = 3(1) - 1 = 2$.
If $y = 2$ then $x = 3(2) - 1 = 5$.

Therefore $p(x) = \begin{cases} 1/4, & x=-1 \\ 1/2, & x=2 \\ 1/4, & x=5 \end{cases}$ (same probabilities but different values)

Note that there is a *one-to-one correspondence* here between x and y values. This made the solution easy.

In general, if Y is a discrete random variable, then $X = g(Y)$ has pdf
$$p(x) = \sum_{y:g(y)=x} p(y).$$

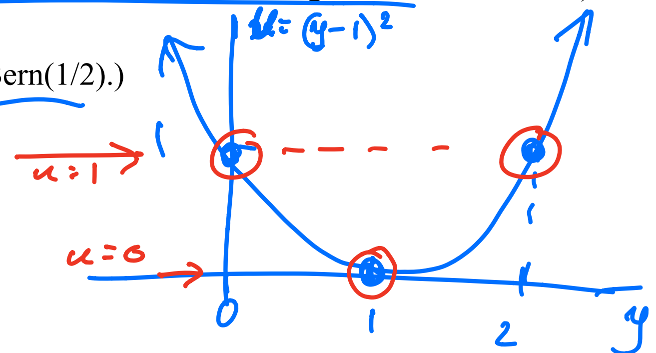
Example 2 $Y \sim \text{Bin}(2,1/2)$. Find the dsn of $U = (Y-1)^2$.

In this case there are two possible values of u: 0 (if $y = 1$), and 1 (if $y = 0$ or 2).

$$p_U(0) = \sum_{y:(y-1)^2=0} p(y) = p(1) = 1/2.$$
$$p_U(1) = \sum_{y:(y-1)^2=1} p(y) = p(0) + p(2) = 1/4 + 1/4 = 1/2. \quad (= 1 - \tfrac{1}{2})$$

(Note: The second 1/2 could have been obtained by subtracting the first 1/2 from 1.)

Thus $\boxed{p(u) = 1/2, \quad u = 0,1.}$ (Ie, $U \sim \text{Bern}(1/2)$.)



What if we want to find the dsn of a function of *two* rv's?
Then we use the same formula as above, interpreting y as a vector quantity.

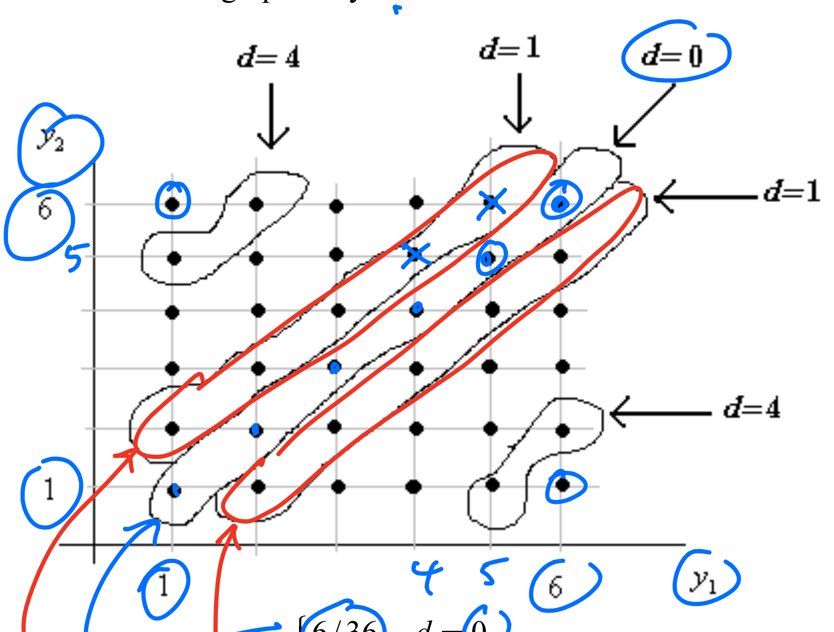
Example 3 If we roll two dice, what is the expected difference between the two numbers that come up?

Let Y_i be the number on the ith die.
We wish to find the expected value of $\boxed{D = |Y_1 - Y_2|}$.

?
$ED = \sum_d d\, f(d) = 0$
$y = (y_1, y_2)$

We will first obtain the pdf of D, according to $p(d) = \sum\sum_{y_1, y_2 : |y_1 - y_2| = d} p(y_1, y_2)$.

This is best done graphically.



We see that: $p(d) = \begin{cases} 6/36, & d = 0 \\ 10/36, & d = 1 \\ 8/36, & d = 2 \\ 6/36, & d = 3 \\ 4/36, & d = 4 \\ 2/36, & d = 5 \end{cases}$ (note that these pr's sum to 1)

It follows that $ED = \sum_{d=0}^{5} dp(d) = 0 \times \frac{6}{36} + 1 \times \frac{10}{36} + \ldots + 5 \times \frac{2}{36} = \frac{35}{18}$. ($\approx 2$, < 2)

Alternatively,

$ED = \sum_{y_1, y_2} |y_1 - y_2| f(y_1, y_2) = |1-1|\frac{1}{36} + |1-2|\frac{1}{36} + \ldots + |6-6|\frac{1}{36} = \frac{35}{18}$.

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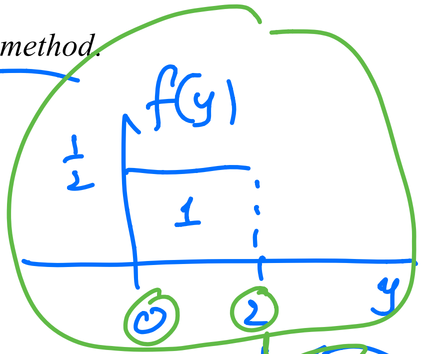
(Don't need $f(d)$)

2. The continuous case

There are three main strategies we'll look at:

the *cdf method*, the *transformation method (or rule)*, the *mgf method*.



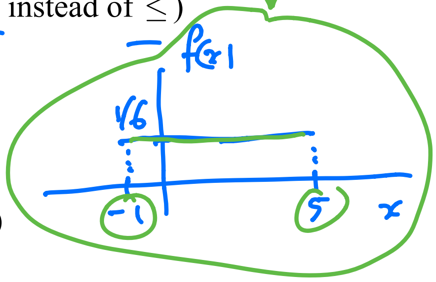
1. The cdf method

This consists of two steps:

1. Find the cdf of the rv of interest.
2. Differentiate this cdf to obtain the required pdf.

Example 4 Suppose that $Y \sim U(0,2)$. Find the pdf of $X = 3Y - 1$.

$X = 3Y - 1$ "transformation"



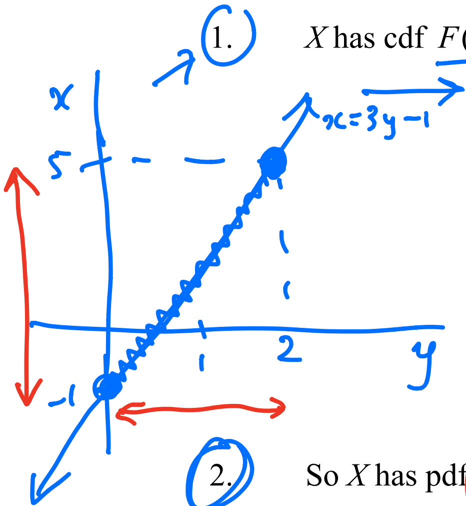
1. X has cdf $F(x) = P(X < x)$ (since X is cts, we may write $<$ instead of $\le$)

$$= P(3Y - 1 < x)$$

$$= P\left(Y < \frac{x+1}{3}\right)$$

$$= \int_0^{(x+1)/3} \frac{1}{2}\, dy \quad \text{(since } f(y) = 1/2,\ 0 < y < 2\text{)}$$

$$= \frac{x+1}{6}, \quad -1 < x < 5 \quad \text{(since } 3(0) - 1 = -1 \text{ and } 3(2) - 1 = 5\text{).}$$



2. So X has pdf $f(x) = F'(x) = \frac{1}{6}$, $-1 < x < 5$. (Ie, $X \sim U(-1, 5)$.)

Example 5 Suppose that $X, Y \sim \text{iid } U(0,1)$. Find the pdf of $U = X + Y$.

$X \perp Y \Rightarrow f(x,y) = f(x) f(y) = 1 \times 1$

$f(u) = ?$

$U \in (0,2)$

First observe that $f(x,y) = 1$, $0 < x < 1$, $0 < y < 1$.

1. So U has cdf $F(u) = P(U < u)$

$$= P(X + Y < u)$$

$$= P(Y < u - X)$$

$$= \frac{1}{2}u^2 \quad \text{(area of shaded region below).}$$

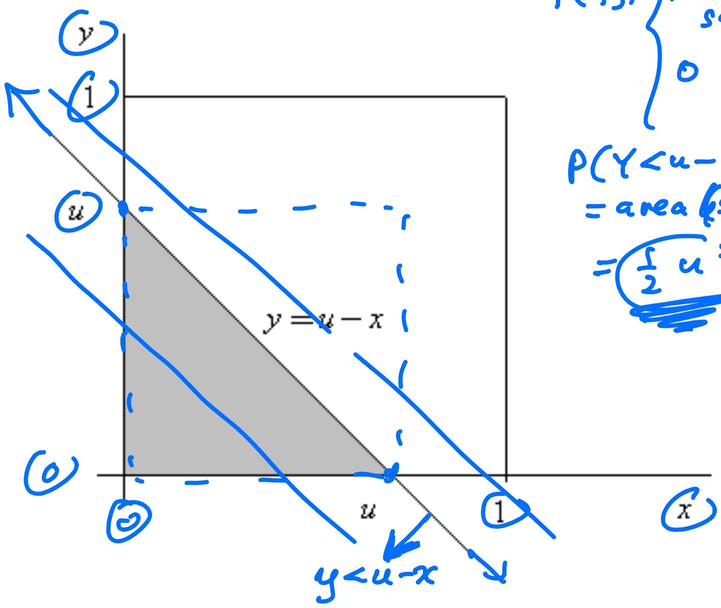
$f(x) = 1,\ 0 < x < 1$
$f(y) = 1,\ 0 < y < 1$

$$\int_{x=0}^{1} \int_{\substack{y=0 \\ y < u-x}}^{1} f(x,y)\, dx\, dy = ???$$

= volume under $f(x,y)$ over the region $y < u - x$

= area of the region × height of $f(x,y)$

1

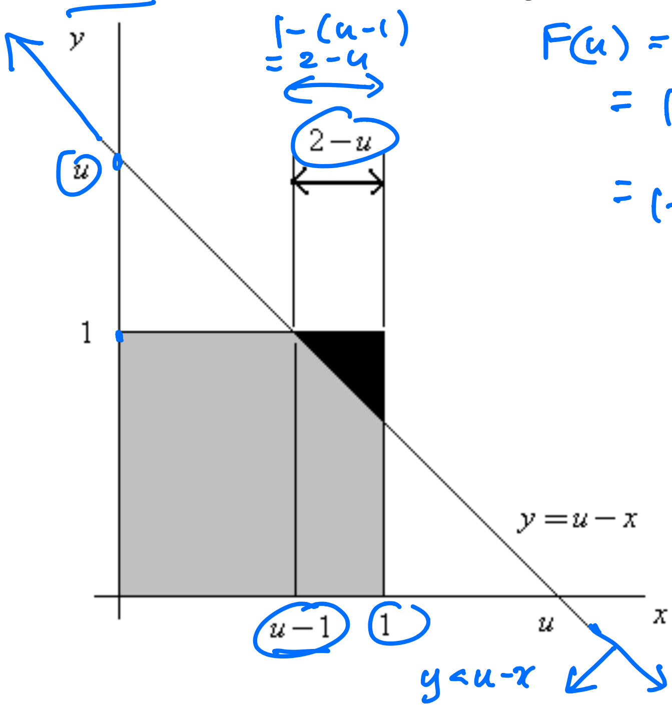


$f(x,y) = \begin{cases} 1 & \text{in the square} \\ 0 & \text{o/w} \end{cases}$

$P(Y<u-X)$
$= \text{area of shaded}$
$= \frac{1}{2}u^2$

But this is true only if $u < 1$.

For $u > 1$, we need to draw another diagram.



$F(u) = \text{area(grey)}$
$= 1 - \text{area(black)}$
$= 1 - \frac{1}{2}(2-u)^2$

We see that $F(u) = P(Y < u - X)$ (area of grey region)

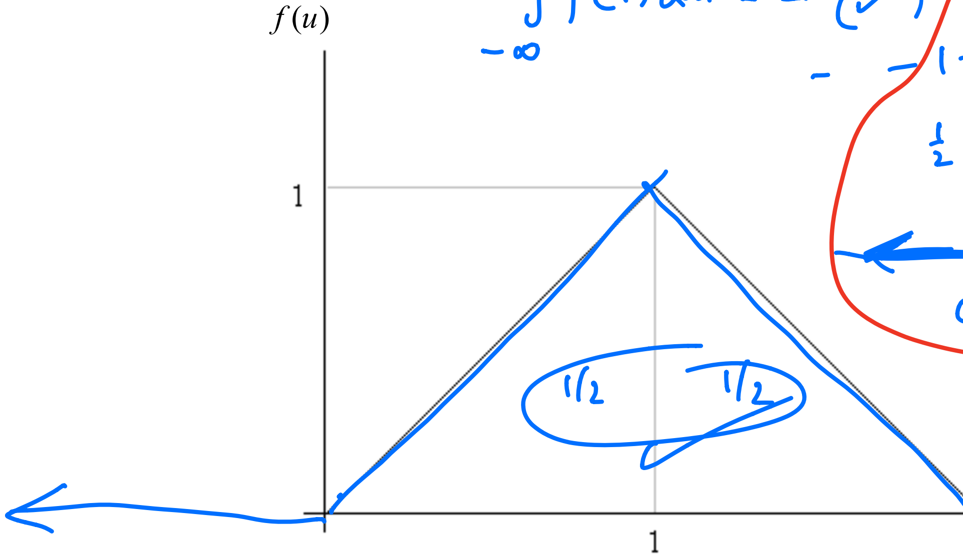
$= 1 - P(Y > u - X)$ (1 minus area of black region)

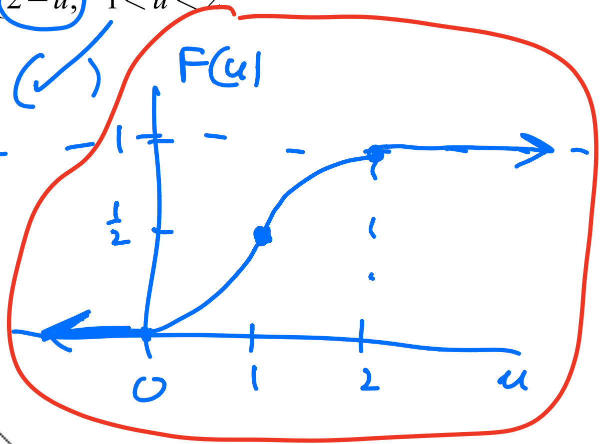
$= 1 - \frac{1}{2}(2-u)^2$.

In summary, $U = X + Y$ has cdf $F(u) = \begin{cases} 0, & u \le 0 \\ \frac{1}{2}u^2, & 0 < u \le 1 \\ 1 - \frac{1}{2}(2-u)^2, & 1 < u < 2 \\ 1, & u \ge 2 \end{cases}$

2. Therefore U has pdf $f(u) = F'(u) = \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 < u < 2 \end{cases}$

$\int_{-\infty}^{\infty} f(u)\,du = 1$ (✓)





2. The transformation method

This is a shortcut version of the cdf method.

Suppose that Y is a cts rv with pdf $f(y)$, and $x = g(y)$ is a function which is either

(a) strictly increasing

or **(b)** strictly decreasing,

for all possible values y of Y.

Then $X = g(Y)$ has pdf

$$\boxed{f(x) = f(y)\left|\frac{dy}{dx}\right|},$$

where $y = g^{-1}(x)$. (This is the inverse function of g.)

x

x

Example 6 Suppose that $Y \sim U(0,2)$.
Find the pdf of $X = 3Y - 1$. (This is the same as Example 4.)

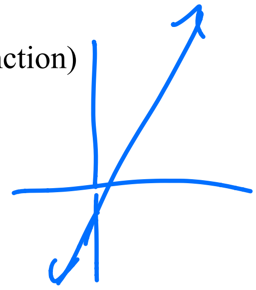
$g(y)$

Here: $x = 3y - 1$ ($x = g(y)$ is a strictly increasing function)

$g^{-1}(x)$ $y = \dfrac{x+1}{3}$ (the inverse function of g)

$\dfrac{dy}{dx} = \dfrac{1}{3}$

$f(y) = 1/2,\ 0 < y < 2.$



So: $f(x) = f(y)\left|\dfrac{dy}{dx}\right| = \dfrac{1}{2}\left|\dfrac{1}{3}\right| = \dfrac{1}{6},\ -1 < x < 5$ (as before).

Example 7 $Y \sim N(a, b^2)$. Find the dsn of $Z = \dfrac{Y-a}{b}$.

Here: $z = \dfrac{y-a}{b} = g(y)$ (a strictly increasing function of y)

$y = a + bz = g^{-1}(z)$

$\dfrac{dy}{dz} = b$

$f(y) = \dfrac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(y-a)^2},\ -\infty < y < \infty$

So $f(z) = f(y)\left|\dfrac{dy}{dz}\right| = \dfrac{1}{\cancel{b}\sqrt{2\pi}} e^{-\frac{1}{2b^2}((a+bz)-a)^2}\cancel{b} = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2},\ -\infty < z < \infty.$

Thus $\boxed{Z \sim N(0,1)}$.

Exercise: $Z \sim N(0,1)$. Find the dsn of $Y = a + bZ$ (very similar to above).
$Y \sim N(a, b^2)$

Example 8 $Z \sim N(0,1)$. Find the dsn of $X = Z^2$.

In this case, $x = z^2$ is neither strictly increasing nor strictly decreasing.
So the transformation method cannot be used (at least not directly).

We could find the pdf of X using the cdf method (do this as an exercise).

Another way to proceed is via the mgf method.