2017-03-06
MATH6222 week 3 lecture 7

Principle of Induction

Suppose we $P(1), P(2), P(3), \ldots$ is a sequence of mathematical statements, i.e. we have a statement $P(k)$ for each natural number $k \in \mathbb{N}$.

[Correction: The textbook does not include zero as a natural number.]

Suppose we can prove:

1. $P(1)$ [Base step]
2. $P(k) \implies P(k+1)$ for each $k \in \mathbb{N}$ [Induction step]

Then $P(1), P(2), P(3), \ldots$ are all true.

Proof: Suppose, for a contradiction, that some of the $P(i)$ are false.
Consider the "First" which is false, i.e., we have $P(k)$ false but $P(k-1), P(k-2), \ldots$ true.
Note: This statement cannot be $P(1)$.
Therefore, $P(k-1)$ exists and is true.
But we also know by (2) that $P(k-1) \implies P(k)$ is true.
This is a contradiction.
We conclude none of the statements can be false.

Example: Find a pattern for the sum of the first n odd, positive integers.

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

$$P(n) := 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

We want to show this is true for all n.

1. Check $P(1).1 = 1$. It's true.
2. Check $P(k) \implies P(k+1)$.

Assume $P(k)$ (induction hypothesis), which is $1+3+5+\cdots+(2k-1) = k^2$.
What about $P(k+1)$? It is $1+3+5+\cdots+(2k-1)+(2k+1) = (k+1)^2$.

$1{+}3{+}5{+}\cdots{+}2k{-}1{+}2k{+}1 = k^2{+}2k{+}1 = (k{+}1)^2$.(Done with the induction step)

Question: How many squares are contained in an $n \times n$ chessboard?

The answer is $1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$.........$P(n)$.

1. $P(1), 1 = \frac{1\cdot 2\cdot 3}{6}$. (Base step)
2. $P(k-1) \implies P(k), 1^2 + 2^2 + \cdots + (k-1)^2 = \frac{(k-1)k(2k-1)}{6}$.

$$
\begin{aligned}
P(k-1) \implies P(k), 1^2 + 2^2 + \cdots + (k-1)^2 &= \frac{(k-1)k(2k-1)}{6}. \\
1^2 + 2^2 + \cdots + (k-1)^2 + k^2 &= \frac{(k-1)k(2k-1)}{6} + k^2 \\
&= \frac{(k-1)k(2k-1) + 6k^2}{6} \\
&= \frac{(k^2-k)(2k-1) + 6k^2}{6} \\
&= \frac{2k^3 - k^2 - 2k^2 + k + 6k^2}{6} \\
&= \frac{2k^3 + 3k^2 + k}{6} \\
&= \frac{k(k+1)(2k+1)}{6}
\end{aligned}
$$

Problem: For which natural numbers n is it true that $3^n \geq 2^{n+1}$.

For $n = 1$, false. $n = 2$, true. $n = 3$, true...
Let $P(n) := 3^n \geq 2^{n+1}$. We claim $P(n)$ is true for $n \geq 2$.
Need

1. $P(2)$ is true.
2. $P(k) \implies P(k+1)$ for all $k = 2, 3, \ldots$

$P(2) : 9 > 8$ True.
$P(k) : 3^{k+1} > 3 \cdot 2^{k+1} > 2 \cdot 2^{k+1} = 2^{k+2}$. This completes the induction.

$$\int_0^1 x dx = ?$$