PROBLEM-SOLVING AND PROOFS: LECTURE DIARY

1. Weeks 1-6: A Summary

We covered Chapters 1-6 of the textbook. Within each chapter, there were a few topics that we did not cover in lecture. While it would not hurt you to read about these, I will not expect you to know them for examination purposes.

In Chapter 1, we did not cover the sections on the quadratic formula or elementary inequalities (pp. 2-6).

In Chapter 2, we covered most everything, but not Theorem 2.2 (p. 26).

In Chapter 3, we covered most everything, but *not* the discussion of polynomials nor the handshake problem (pp. 59-60).

In Chapter 4, we did not discuss the weights problem or base q representations of natural numbers (76-80). On the other hand, we proved a few important facts that are not in the book, e.g. $\mathbb{Q}$ is countable, but $\mathbb{R}$ is uncountable. We discussed the statement, but not the proof of the Cantor-Schroder-Bernstein theorem (p.91).

In Chapter 5, we did not discuss permutations (pp. 111-115).

In Chapter 6, we did not discuss the dart board problem or the optional material on polynomials (pp.129-134).

2. Week 6

2.1. **Monday, March 27.** We discussed the "walls and balls" method for proving that the number of ways to select n objects from k types is $\binom{n+k-1}{k-1}$. See Theorem 5.23 on page 107.

We discussed what distances can be measured if one has only one rope of length 15 and one rope of length 6 to work with, and concluded that the answer is all multiples of 3. Make sure you check out the definitions of divisibility and greatest common divisor in Chapter 6 before the next lecture.

2.2. **Thursday, March 30.** After much wailing and gnashing of teeth, we proved that if a, b, c are integers, then the equation

$$ax + by = c$$

has a solution in integers if and only if $\gcd(a, b)|c$. The hardest part of the proof was the Euclidean algorithm, which gives a way of writing the greatest common divisor of two numbers as an integer linear combination of this two numbers. This algorithm is very important. The relevant pages of the textbook are 126-128.

2.3. **Friday, March 31.** We proved the fundamental theorem of arithmetic, which states that every positive integer admits a *unique* decomposition as a product of primes. We observed that in a world where the only integers are $\{2, 4, 6, \ldots\}$, we can define primality, but the fundamental theorem does not hold. Relevant pages of the textbook are 124-126.

3. Week 7

3.1. **Thursday, April 20.** We introduced the notion of two integers being congruent modulo n, and the congruence class of an integer modulo n. We proved two key facts: First, that the there are exactly n distinct congruence classes modulo n, and each integer falls into exactly one of these classes (Definition 7.17 and Remark 7.18 in the textbook). Second, the "key lemma of modular arithmetic" (Lemma 7.19 in the textbook).

3.2. **Friday, April 21.** I spent sometime reviewing the two key facts from last time. We used modular arithmetic to do various fun problems: finding the last digit of $971^{216} + 523^{121}$, finding the reading on a clock face 47^{37} hours from now, Proving that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. Finally, I tried to explain how the key lemma of modular arithmetic implies that addition and multiplication are well-defined on the set of congruent classes themselves, so that (for each n) we can think of a $\mathbb{Z}_n$ as a finite number system, which satisfies all the usual laws of arithmetic. I will discuss this perspective in greater depth next week. This week's material is basically covered in pages 140-145 of the textbook though our approach was a little different.

4. Week 8

4.1. **Monday, April 24.** We went over the midsemester exam.

4.2. **Thursday, April 26.** We showed that a congruence class $\bar{a} \in \mathbb{Z}_n$ has a multiplicative inverse if $(a, n) = 1$. We used this to prove two classical theorems from number theory: Fermat's Little Theorem and Wilson's Theorem. See 147-151 in the textbook.

4.3. **Friday, April 27.** We discussed the functional digraph associated to a permutation, and used it to show that the minimum number of swaps needed to sort of an out-of-order list of numbers is the number of cycles in the functional digraph of the associated permutation (pp. 111-115). We also discussed the concept of an equivalence relation (pp. 140-41).

5. Week 9

5.1. **Monday, May 1.** We discussed the definition of a finite *probability space*. We also discussed the definition of *conditional probability*, considering in detail the following problem: there are three jars, one of which has two white marbles, one has one white and one black marble, and one has two marbles. Suppose you pick a jar at random, and then pick a marble from that jar and see that it is black. What is the probability that you're holding the third jar.

5.2. **Thursday, May 4.** We discussed *Bayes' Rule*, and described its application to the problem of medical testing. We also defined *random variables* and the *expectation* of random variable.

5.3. **Friday, May 5.** We used *linearity of expectation* to compute the expected number of heads in n coin tosses, as well as the expected number of complete paris of socks to be returned by an evil washing machine. We discussed the so-called "finger game," proving that A can guarantee a positive expectation with a certain probabilisitic strategy.

This week's material is all contained in Chapter 9 of the textbook. The things we did not cover are: Bertrand's Ballot Problem (9.10-9.11), the Coupon Collector Problem (9.28-9.30), and multinomial coefficients (pp. 182-184).

6. Week 10

6.1. **Monday, May 8.** We did several problems involving the Pigeonhole principle.

6.2. **Thursday, May 12.** We proved the inclusion-exclusion principle and used to to compute a formula for the Euler ϕ-function, and to count the number of permutations of $[n]$ without fixed points.

6.3. **Friday, May 13.** We did a couple more examples of Pigeonhole principle and inclusion-exclusion. In total, we have covered nearly all the examples in Chapter 10 of the textbook. We also spent some time discussing Newcomb's Paradox.

7. Week 11

7.1. **Monday, May 15.** We discussed all the definitions on pages 203-205 of the text: graph, simple graph, trails, closed trails, degree, etc.

7.2. **Thursday, May 18.** We proved that a connected graph has an Eulerian trail if and only if every vertex degree is even. (Theorem 11.13)

8. Week 12

8.1. **Monday, May 22.** We proved that graph is bipartite if and only if it has no odd cycles. (Pages 211-216 of the text.)

8.2. **Thursday, May 25.** We proved that the graphs K_5 and $K_{3,3}$ are planar. (Pages 223-226)

8.3. **Friday, May 27.** We proved that any simple planar graph has chromatic number less than or equal to 6 (Exercise 11.46 on Page 29).

The only topics we did not cover (and are therefore not examinable) from Chapter 11 are perfect matchings (p. 218) and the chromatic polynomial (pp. 221-222).