Quiz: ① $2y'' + 10y' + 17y = 0$, $y(0) = 3$, $y'(0) = -3$.
② $y'' + 6y' + 9y = 0$ fundamental set & Calculate $W$

# Reduction of Order

Consider: $L[y] = y'' + py' + qy = 0$

Method: Sps $y_1$ is a soln of $L[y] = 0$. Put $y(t) = v(t)y_1(t)$. Then the equation $L[y] = 0$ becomes a first order separable eqn for $w$, where $w(t) = v'(t)$

Indeed: $y = vy_1$

$$y' = vy_1' + v'y_1$$

$$y'' = vy_1'' + 2v'y_1' + v''y_1$$

$$0 = y'' + py' + qy$$

$$= v(y_1'' + py_1' + qy_1) + v'(2y_1' + py_1) + v''y_1$$

$$= v'(2y_1' + py_1) + v''y_1 .$$

So, $w = v'$ satisfies $0 = w(2y_1' + py_1) + w'y_1$

Example: $y'' + 6y' + 9y = 0$

$y_1(t) = e^{-3t}$

Write $y(t) = v(t)e^{-3t}$

$$y'(t) = v(t)\cdot(-3)e^{-3t} + v'e^{-3t}$$

$$y''(t) = v(t)\cdot 9e^{-3t} - 6v'e^{-3t} + v''e^{-3t}$$

$$\Rightarrow y'' + 6y' + 9y = ve^{-3t}(\cancel{9 + (-3)6} + 9) + v'e^{-3t}(\cancel{-6+6}) + v''e^{-3t}$$

$$\Rightarrow v'' = 0$$

Example: Solve $t^3y'' - ty' + y = 0$

Note: $y_1(t) = t$ is a solution

Put $y(t) = v(t)\, t$

$$y'(t) = v'(t)\,t + v(t)$$

$$y''(t) = v''(t)\,t + 2v'(t)$$

$$\Rightarrow 0 = t^3y'' - ty' + y$$

$$= t^3(tv'' + 2v') - t(tv' + v) + tv$$

$$= v''t^4 + v'(2t^3 - t^2)$$

$$\Rightarrow 0 = v''t^4 + v'(2t^3 - t^2)$$

$$0 = v''t^2 + v'(2t - 1)$$

$$\frac{v''}{v'} = -\frac{2t-1}{t^2} = -\frac{2}{t} + \frac{1}{t^2}$$

$$\|$$

$$\frac{d}{dt}\ln(v')$$

$\ln(v') = -2\ln(t) - \frac{1}{t}$ (just need one solution, needn't worry about constant)

$\Rightarrow v' = \frac{1}{t^2}\exp(-\frac{1}{t}) = \frac{d}{dt}\exp(-\frac{1}{t})$

$\Rightarrow v(t) = \exp(-\frac{1}{t})$

$\Rightarrow y_2(t) = v(t)y_1(t) = \exp(-\frac{1}{t})\, t$ is a second solution.