University of Toronto
Department of Mathematics

MAT224H1F
Linear Algebra II

Midterm Examination
October 25, 2011

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Duration: 1 hour 50 minutes

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

Tutorial Group: ______________________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

[10] **1.** Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation defined by

$$T(A) = \frac{A + A^T}{2}.$$

Find the matrix of T relative to the basis $\alpha = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}(\mathbb{R})$.

SOLUTION: Let $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$T(v_1) = \frac{1}{2}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix} = \frac{1}{2}\left(3v_1 + 3v_2 - v_3 - 2v_4\right),$$

$$T(v_2) = \frac{1}{2}\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2}\left(-v_1 - v_2 + v_3 + 2v_4\right),$$

$$T(v_3) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = v_3,$$

$$T(v_4) = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = v_4.$$

Therefore,

$$[T]_{\alpha\alpha} = \begin{bmatrix} 3/2 & -1/2 & 0 & 0 \\ 3/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

EXTRA PAGE FOR QUESTION 1 - do not remove.

[10] **2.** Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$T(a + bx + cx^2) = (-2b + 11c) + (-2a + c)x + (3a - b + 4c)x^2.$$

Find bases for the kernel and image of T.

SOLUTION: Let $\alpha = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$. Then

$$[T]_{\alpha\alpha} = \begin{bmatrix} 0 & -2 & 11 \\ -2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}.$$

Perform row operations to obtain:

$$\begin{bmatrix} 0 & -2 & 11 \\ -2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} -6 & 0 & 3 \\ -2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} -6 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} -6 & 0 & 3 \\ 0 & -2 & 11 \\ 0 & 0 & 0 \end{bmatrix}.$$

So a basis for the null space of $[T]_{\alpha\alpha}$ is given by the vector $\begin{bmatrix} 1 \\ 11 \\ 2 \end{bmatrix}$; translating this back to $P_2(\mathbb{R})$ via the basis α, we have that a basis for $\ker(T)$ is given by the polynomial $1+11x+2x^2$.

As for the image, note that the leading ones of the r.r.e.f of $[T]_{\alpha\alpha}$ are in columns 1 and 2. Hence the first two columns of $[T]_{\alpha\alpha}$ give a basis for the range of that matrix, which means that a basis for the image of T is

$$\{T(1), T(x)\} = \{-2x + 3x^2, -2 - x^2\}.$$

EXTRA PAGE FOR QUESTION 2 - do not remove.

[10] **3.** Let $V = P_4(\mathbb{R})$ and $W = \{p(x) \in P_5(\mathbb{R}) \mid p(1) = 0\}$. Show that V and W are isomorphic and find an isomorphism $T\colon V \to W$.

SOLUTION: There are many possible approaches to this problem. This is probably the most straightforward solution; we define a natural map, and show that it is an isomorphism. Let $T : V \to P_5(\mathbb{R})$ be defined by the formula

$$T(p)(x) = (x-1)p(x).$$

Note $T(p+q) = T(p) + T(q)$, and $T(c \cdot p) = c \cdot T(p)$, for any polynomials $p, q \in P_4(\mathbb{R})$ and scalar $c \in \mathbb{R}$; in other words, T is linear. Moreover,

$$T(p)(1) = (1-1)p(1) = 0,$$

so T is in fact a linear map $T : V \to W$.

Next, we note that T is injective, since if $T(p) = 0$, then $(x-1)p(x)$ is the 0 polynomial, which means $p(x) = 0$. In particular, $dim(ker(T)) = 0$.

Now, it's clear that $W \neq P_5(\mathbb{R})$, the latter of which has dimension 6. Therefore $\dim(W) \leq 5$. On the other hand, $\dim V = 5$. Hence, by the dimension theorem, we know that

$$5 = \dim(V) = \dim(ker(T)) + \dim(im(T)) = \dim(im(T)) \leq \dim(W) \leq 5.$$

So $5 = \dim(W) = \dim(im(T))$, and hence T is also surjective.

Another possible approach would be to simply find bases for V, and W, and notice that they are the same dimension; then you can define a isomorphism by sending one basis to the other.

[10] **4.** Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation whose matrix with respect to the standard basis of $\mathbb{C}^2$ is

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

Find a basis α for $\mathbb{C}^2$ consisting of eigenvectors of T and find $[T]_{\alpha\alpha}$.

SOLUTION: Let $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$. First, we find the characteristic polynomial:

$$\det(A - \lambda \cdot \mathbb{I}) = \det\left(\begin{bmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - 1 = \lambda(\lambda - 2)$$

Hence the eigenvalues are 0 and 2; note that as they all have multiplicity one, A is in fact diagonalizable.

To find E_0, i.e. the null space of $A - 0 = A$: By Gaussian elimination:

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix},$$

so a basis for E_0 is given by the vector $\begin{bmatrix} i \\ -1 \end{bmatrix}$.

To find E_2, we compute

$$A - 2\mathbb{I} = \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \to \begin{bmatrix} -1 & i \\ 0 & 0 \end{bmatrix},$$

so a basis for E_2 is given by the vector $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

Therefore, $\alpha = \{\begin{bmatrix} i \\ -1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}\}$ is a basis of eigenvectors, and

$$[T]_{\alpha\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

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[10]**5.** Let T: $P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be the linear transformation defined by

$$T(a + bx + cx^2) = (a - 3b + c) + (2a - 6b + 3c)x.$$

Find bases α' for $P_2(\mathbb{R})$, and β' for $P_1(\mathbb{R})$ such that $[T]_{\beta'\alpha'}$ is the reduced row echelon form of $[T]_{\beta\alpha}$ where α and β are the standard bases for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively.

SOLUTION: Let $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x\}$ be the standard bases of $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively. Then

$$[T]_{\beta\alpha} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 3 \end{bmatrix}.$$

Perfoming Gaussian elimination:

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

the first step involved adding -2 times the first row to the second, and the second step involved subtracting the second row from the first. In terms of elementary matrices, we get

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we're looking for bases α', β' such that $[T]_{\beta'\alpha'} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. On the other hand, we know in general that

$$[T]_{\beta'\alpha'} = [\mathbb{I}]_{\beta'\beta}[T]_{\beta\alpha}[\mathbb{I}]_{\alpha\alpha'}.$$

So we can look for bases α' and β' such that $[\mathbb{I}]_{\alpha\alpha'} = Id$, and

$$[\mathbb{I}]_{\beta'\beta} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

The first is easy: just take $\alpha' = \alpha$.

For the second, note that

$$[\mathbb{I}]_{\beta\beta'} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix},$$

which means we can read off the basis β' by the column of this matrix, (relative to the basis β):

$$\beta' = \{1 + 2x, 1 + 3x\}.$$

EXTRA PAGE FOR QUESTION 5 - do not remove.

6. Let V and W be vector spaces over a field F. Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, and $\beta = \{w_1, w_2, \ldots, w_m\}$ a basis for W. Let $T\colon V \to W$ be a linear transformation.

[5]**(a)** Prove that T is surjective if and only if the columns of $[T]_{\beta\alpha}$ span F^m.

[5]**(b)** Prove that T is injective if and only if the columns of $[T]_{\beta\alpha}$ are linearly independent in F^m.

SOLUTION: Let $\Phi : W \to F^m$ denote the map defined by $\Phi(w) = [w]_\beta$; in your problem sets, you've shown this map is an isomorphism. Essentially by definition, the j'th column of $[T]_{\beta\alpha}$ is equal to $\Phi(T(v_j))$.

Since Φ is an isomorphism, we get

$$\begin{aligned} span\{\text{ columns of } [T]_{\beta\alpha}\} = F^m &\iff span\{\Phi(T(v_1)), \Phi(T(v_2)), \ldots \Phi(T(v_n))\} = F^m \\ &\iff span\{T(v_1), T(v_2), \ldots T(v_n)\} = W \\ &\iff T \text{ is surjective,} \end{aligned}$$

which proves part (a).

Similarly,

$$\begin{aligned} \text{The columns of } [T]_{\beta\alpha} \text{ are lin. indep.} &\iff \{\Phi(T(v_1)), \ldots, \Phi(T(v_n))\} \text{ is lin. indep.} \\ &\iff \{T(v_1), \ldots T(v_n)\} \text{ is lin. indep.} \\ &\iff T \text{ is injective,} \end{aligned}$$

which proves part (b).

In case you haven't seen it before, the last equivalence can be proved as follows: suppose

$$a_1 T(v_1) + \ldots a_n T(v_n) = 0,$$

for some scalars $a_1, \ldots a_n \in F$. Then $T(a_1 v_1 + \ldots a_n v_n) = 0$, by linearity, and hence

$$a_1 v_1 + \ldots a_n v_n \in ker(T).$$

In other words, if the original linear combination is non-trivial, we get a non-zero vector in $ker(T)$. This proves that $ker(T) \neq 0$ if and only if the set $\{T(v_1), \cdots T(v_n)\}$ is linearly dependent, which then implies the statement we want.