MATH6222: Homework #1

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Problem 1

Prove that $\sqrt{11}$ is irrational. You may use the fact that every integer can be uniquely decomposed as a product of primes.

Proof:
The idea is very similar to the one we used to prove the irrationality of $\sqrt{2}$ in class.

Suppose $\sqrt{11}$ is a rational number, i.e., for two co prime integers p, q, i.e.p, q have no common factors, it can be written as

$$\sqrt{11} = \frac{p}{q}$$

Square the both sides and multiple by q^2 we have

$$11q^2 = p^2$$

Now we recall that some fact proved in class:

- The product of two odd numbers is odd.
- The product of two even numbers is even.
- The product of an even and an odd is even.

and consider this:

- If p is odd, q is even. Then the right hand side (RHS) is odd, the left hand side (LHS) is even. Contradiction.
- If p is even, q is odd. Then RHS is even, LHS is odd. Contradiction.
- If p, q both even, contradicts the fact that p, q are co primes.

Therefore, p, q can only be two odd co primes.

$$\begin{aligned}
p &= 2n + 1 \\
q &= 2m + 1 \\
11(2m + 1)^2 &= (2n + 1)^2 \\
11(4m^2 + 4m + 1) &= 4n^2 + 4n + 1 \\
44m^2 + 44m + 11 &= 4n^2 + 4n + 1 \\
44m^2 + 44m + 10 &= 4n^2 + 4n \\
22m^2 + 22m + 5 &= 2n^2 + 2n
\end{aligned}$$

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$22m^2, 22m, 2n^2, 2n$ are even numbers. So the RHS is even. But an even number $22m^2 + 22m$ plus an odd number 5 equals an odd number (LHS). So we have a contradiction here.

Hence, the original hypothesis that $\sqrt{11}$ is rational fails. So we proved that $\sqrt{11}$ is irrational.

Problem 2

Let S denote the set of all prime numbers of the form $4k+3$ with $k \in \mathbb{N}$. (So $3 \in S, 7 \in S$, but $5 \notin S$). Prove that S is infinite.

Proof

Suppose there are only finitely many primes $p_1, \ldots p_k$ in the set S. Consider the number $N = 4p_1 \cdot p_2 \cdots p_k - 1 = 4(\Pi_{i=1}^k p_i - 1) + 3$ which is also of the form $4n+3$.

Since it is greater than any p_i, so consider it not a prime. Then N is divisible by a prime.

Note that all integers should be one of the form $4n, 4n+1, 4n+2, 4n+3$. The factors of N cannot be of the form $4n, 4n+2$ since N is odd. On the other hand, none of the elements of S divides N. So the only possible form of factors of N is $4n+1$.

However

$$\forall a, b \in \mathbb{Z}, (4a+1)(4b+1) = 16ab + 4a + 4b + 1 = 4(4ab + a + b) + 1.$$

So the product of any two primes of the form $4n+1$ is still $4n+1$. It's like an infinite loop. But remember that N itself is of the form $4n+3$ in the end. Contradiction!

Hence the original hypothesis is incorrect, i.e. S is infinite.

Problem 4

Let f and g denote functions from $\mathbb{R}$ to $\mathbb{R}$. Recall that such a function is *bounded* if there exists a real number M such that $|f(x)| < M$ for all $x \in \mathbb{R}$. Determine whether each of the following statements are true. If true, provide a proof. If false, provide a counterexample.

- If f and g are bounded, then $f+g$ is bounded.
- If f and g are bounded, then fg is bounded.
- If $f+g$ is bounded, then f and g are bounded.
- If fg is bounded, then f and g are bounded.

Problem 4 continued on next page. . .

- If $f + g$ and fg are bounded, then f and g are bounded.

You may use the *triangle inequality* which states that for all $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|.$$

Solution

Statement 1

True. According to the definition of *boundedness*, $\forall x \in \mathbb{R}, \exists M_1, M_2 \in \mathbb{R}$ such that

$$\begin{aligned} |f(x)| &< M_1 \\ |g(x)| &< M_2 \\ |(f+g)(x)| = |f(x) + g(x)| \leq |f(X)| + |g(x)| &< M_1 + M_2 = M \end{aligned}$$

Hence $f + g$ is bounded by $M = M_1 + M_2$.

Statement 2

True. Similarly,

$$|fg(x)| = |f(x)g(x)| = |f(x)||g(x)| < M_1 \cdot M_2 = M'$$

Hence fg is bounded by $M' = M_1 \cdot M_2$.

Statement 3

False. Suppose $f(x) = \pi \cdot x, g(x) = -\pi \cdot x$, then $(fg)(x) = 0$ which is bounded since $0 <= 0$ all the time. But neither of $f(x), g(x)$ is bounded.

Statement 4

False. The counterexample is similar to the one above. Suppose

$$f(x) = \begin{cases} \dfrac{1}{x}, \, \forall x \in \mathbb{R} - \{0\} \\ 0, \, x \in \{0\} \end{cases}$$
$$g(x) = x, \ \forall x \in \mathbb{R}.$$

The product of them, $fg(x) = 1$ is bounded, but neither f nor g is bounded.

Statement 5

True. Since $f+g$ and fg are bounded, $\exists M_1, M_2 \in \mathbb{R}$, such that

$$\begin{aligned}
|f(x)+g(x)| &< M_1 \\
|f(x)\cdot g(x)| &< M_2 \\
|f(x)^2+g(x)^2| &= |\left(f(x)+g(x)\right)^2 - 2f(x)g(x)| \\
&\leq |\left(f(x)+g(x)\right)^2| + 2|f(x)g(x)| \\
&< M_1^2 + 2M_2
\end{aligned}$$

This is to say, the sum of two squares f^2+g^2 is bounded. As we know, the magic of a square number is that it is always nonnegative. So

$$\begin{aligned}
f(x)^2 \leq f(x)^2+g(x)^2 = M_1^2+2M_2 \\
g(x)^2 \leq f(x)^2+g(x)^2 = M_1^2+2M_2
\end{aligned}$$

Hence $f(x) \leq \sqrt{M_1^2+2M_2}$, $g(x) \leq \sqrt{M_1^2+2M_2}$, i.e., f and g are both bounded.