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LOGIC, SETS and TOPOLOGY

Set notations and operations (see optional reading postings) can be thought of as a transliation of symbolic logic into the language of sets. Symbolic logic is about properties like P which is a statment about a variable x. To say that the property P holds true for the variable x is to just say $P(x)$. Now if we define $S = \{x : P(x)\}$ then to say P is true of x is to say $x \in S$.

To say that P is not true of x is to say $x \notin S$, which mean $x \in S^c$ (S^c stands for the complement of S.) So, the negation of logic is represented by the complement operation on sets.

Consider two properties P and Q, and let two sets A and B to correspond to the two properties, that is $P(x)$ whenever $x \in A$, and $Q(x)$ whenever $x \in B$. To say that $P(x)\&Q(x)$ is true is to say that $x \in A \cap B$. So intersection corresponds to the logical AND. Similarly the logical OR corresponds to the union of two sets. Finally the property $P \implies Q$, which means if P is true for x then Q must also be true for x, can be translated as if $x \in A$ then $x \in B$, which in set notations translates to $A \subseteq B$.

Any set S divides/partitions the universe (all the range of the variable x) into two groups, those who belong to S and those who do not belong to S. That is the entire universe is $S \cup S^c$, while $S \cap S^c = \emptyset$. This should remind us of the two valued logic, idea of good and bad, black and white, in and out, pass and fail. Under the pass-fail arrangement once you pass it is not clear nor is it important to know how comfortably you passed or how marginal you were. The two valued logic is not sensitive enough to capture such details, nor is the theory of sets capable of capturing more details beyond the idea of 'in and out'.

Topology is a theory invented as a refinement to the theory of sets: it takes advantage of the concept of distance and to capture the idea of marginality also. In topology any set S partitions the space into three pieces: $S^{int}, \partial S$, and $(S^c)^{int}$. Interior of S stands for all the points which are comfortably inside S. Similarly interior of the complement includes all the points that are comfortably outside S. But the boundary of S is the collection of all the points which are marginal (no matter if they are in or out.) Of course English language is sensitive enough to express this idea, but symbolic logic is not capable of formally expressing this refinement because the notion of distance is not defined there. Topology, however takes advantage of the notion of distance/norm that was induced by dot product, to capture the idea of marginality. Now imagine if Topology could be translated back to the language of symbolic logic then we could capture this idea of marginality or 'comfortably in', within our formal logic. This would've given us some kind of logic that we would've named 'three valued, borderline logic': a statement is true, or false or borderline! However we are learning mathematics so that we become capable of understanding such refinements in a different mathematical realm which might be more naturally suitable for understanding some extensions and some refinements without having to translate them back to the language of formal symbolic logic.