## 1. Natural logarithms

$\delta = \ln(1+i)$.

① $\ln(x \cdot y) = \ln(x) + \ln(y)$.

② $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$.

③ $\ln x^n = n \cdot \ln x$ → Notes. 5.1, P8.

④ $\ln \cdot e^x = e^{\ln x} = x$. → Notes 4.1. P3

Ex: $\bar{s}_{\overline{n}|i} = \int_0^n (1+i)^{n-t} \, dt$

$= \int_0^n \exp\{\ln(1+i)^{n-t}\} \, dt$

$= \int_0^n \exp[(n-t)\ln(1+i)] \, dt$.

## 2. Exponents.

① $x^m \cdot x^n = x^{m+n}$.

② $x^m / x^n = x^{m-n}$

③ $x^{-n} = \frac{1}{x^n}$

④ $x^0 = 1$

⑤ $(x^m)^n = x^{m \cdot n}$

⑥ $x^m \cdot y^m = (x \cdot y)^m$.

# 3. Series.

$\{a_t\}$ $t = 1, 2, 3, \cdots$

$a_1, a_2, \cdots$

$S_n = \sum_{t=1}^{n} a_t$ is a series.

## ① Arithmetic Series.

$a, \; a+d, \; a+2d, \; a+3d, \cdots \quad a+(n-1)d.$

$$S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] \quad ①$$

$$= \sum_{t=1}^{n} a + (t-1)d$$

$$S_n = [a+(n-1)d] + [a+(n-2)\cdot d] + \cdots + a+d+a \quad ②$$

$$\xRightarrow{①+②} 2S_n = [2a+(n-1)d] \times n.$$

$$\Rightarrow S_n = \frac{n \cdot [2a+(n-1)d]}{2}.$$

## ② Geometric Series.

$a, \; a \cdot r, \; a \cdot r^2, \cdots \; a \cdot r^{n-1}.$

$$S_n = a + \cancel{a \cdot r} + \cancel{a \cdot r^2} + \cdots + \cancel{a \cdot r^{n-1}} \quad ①$$

$$= \sum_{t=1}^{n} a \cdot r^{t-1}$$

$$r S_n = \cancel{r \cdot a} + \cancel{a \cdot r^2} + \cancel{a r^3} \cdots + a \cdot r^n \quad ②$$

$$\xrightarrow{① - ②} (1-r) S_n = a - a \cdot r^n$$

$$\Rightarrow S_n = \frac{a - a \cdot r^n}{1-r}$$

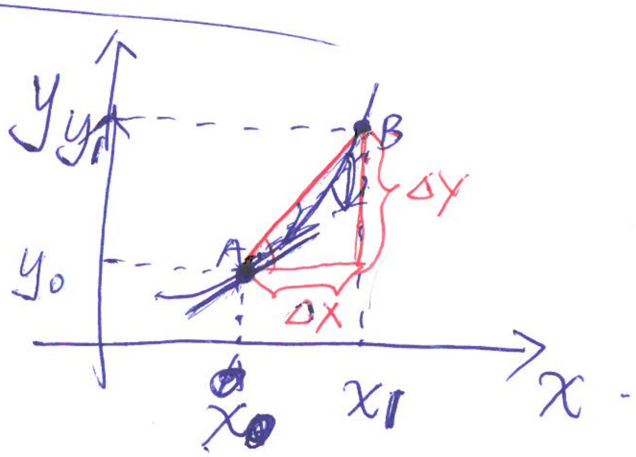
4. Quadratic Formula.

$$A x^2 + B x + C = 0$$

$$\Rightarrow \quad X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

5. Derivatives.

Cont. Smooth. $\boxed{y = f(x)}$




A. B.

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

$f(x_1)$

$$\left.\frac{dy}{dx}\right|_{x_0} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

1. Common functions.

| function | derivative |
|---|---|
| $c$ | $0$ |
| $ax+c$ | $a$ |
| $x^n$ | $n \cdot x^{n-1}$ |
| $e^x$ | $e^x$ |
| $a^x$ | $a^x \cdot \ln a$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\log_a(x)$ | $\frac{1}{x \ln(a)}$ |

2. Rules. $f\ g,\ f'\ g'$

$cf$ $\quad c \cdot f'$

$f+g$ $\quad f'+g'$

$f \cdot g$ $\quad f'g + g'f$

$\frac{f}{g}$ $\quad \frac{f'g - g'f}{g^2}$

$f(g(x))$ $\quad f'(g(x)) \cdot g'(x)$

L'Hopitals Rules:

$$x \to a \qquad \begin{cases} f(x) \\ g(x) \end{cases} \longrightarrow \begin{cases} 0 \\ +\infty \\ -\infty \end{cases}$$

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}.$$

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Ex: $\delta_t = \dfrac{S'(t)}{S(t)} \overset{?}{=} \dfrac{d}{dt} \ln[S(t)]$

$$\begin{cases} \ln x = \dfrac{1}{x} \\ [f(g(x))]' = f'(g(x)) \cdot g'(x). \end{cases}$$

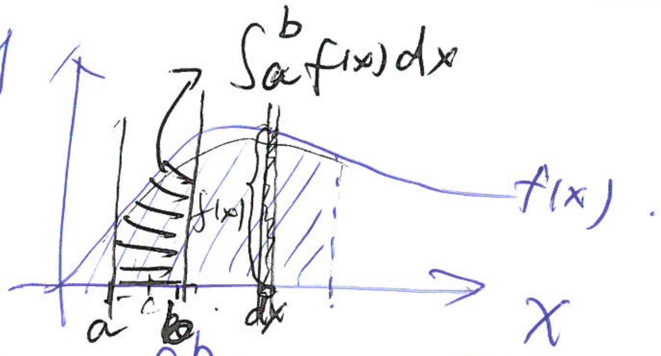
Pf. $\left(\ln[S(x)]\right)' = \dfrac{S'(x)}{S(x)}.$

$f(x) \triangleq \ln x, \quad g(x) \triangleq S(x).$

$f(g(x)) = \ln[S(x)]$

$\left(f[g(x)]\right)' = \left\{\ln[S(x)]\right\}' = \dfrac{1}{S(x)} \cdot S'(x)$

Integration:




- definite integral: $\int_a^b f(x)dx = \underline{F(b) - F(a)}$
- indefinite integral: $\boxed{\int f(x)\, dx} = F(x) + C$

symbol

① Integral of common functions.

| function | indefinite integral |
|---|---|
| $a$ | $\int a\,dx = a \cdot x + C$ |
| $x^n$ | $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ |
| $\frac{1}{x}$ | $\int \frac{1}{x} dx = \ln\lvert x\rvert + C$ |
| $e^x$ | $\int e^x dx = e^x + C$ |
| $a^x$ | $\int a^x dx = \frac{a^x}{\ln a} + C$ |

→ Notes 4.1 P3. P4.

Ex: $\int \ln(x)\,dx = \boxed{x\ln(x) - x + C}$

first der.

$1 \cdot \ln(x) + \frac{1}{x} \cdot x - 1 = \ln(x)$

② **Rules**:

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx.$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

* Integration by parts. → Notes 4.2 . Page 1.

$$\int uv\,dx = u \cdot \int v\,dx + \int u' \cdot \left(\int v\,dx\right) dx.$$

$$\Leftrightarrow \int u \cdot v'\,dx = u \cdot v - \int u' \cdot v\,dx.$$

$$\Leftrightarrow \boxed{\int u \cdot dv = u \cdot v - \int v\,du}$$

* Substitution rule: → Notes 4.2. Page 2

$$\int f(g(x))\,\cancel{dx}\,g'(x)\,dx = \int f(u)\,du.$$

$$\|$$

$$\int_a^b f(g(x))\,dg(x) = \int_{a'}^{b'} f(u)\,du.$$

Ex: $\bar{s}_{\overline{n}|i} = \int_0^n e^{(n-t)\cdot \ln(1+i)} dt$

$$= -\frac{e^{(n-t)\cdot\ln(1+i)}}{\ln(1+i)}\Big|_0^n$$

$$= \frac{-(1+i)^{n-t}}{\ln(1+i)}\Big|_0^n$$

Ex: $(\bar{I}\bar{a})_{\overline{n}|i} = \int_0^t \boxed{s}\cdot \boxed{e^{-\delta s}} ds$

$$= \frac{-1}{\delta}\int_0^t \underbrace{s}_{=u} \, d\underbrace{e^{-\delta\cdot s}}_{=v}$$

Integration by parts

$$= \frac{-1}{\delta}\left[ s\cdot e^{-\delta s}\Big|_0^t - \int_0^t e^{-\delta\cdot s}\cdot ds\right]$$

$$= \frac{-s\cdot e^{-\delta s}}{\delta}\Big|_0^t + \int_0^t \frac{e^{-\delta s}}{\delta} ds$$

$$= \frac{-t\cdot e^{-\delta t}}{\delta} - \frac{e^{-\delta s}}{\delta^2}\Big|_0^t$$

$$= \frac{\bar{a}_{\overline{t}|i} - t\cdot v^t}{\delta}$$

Ex: $(I\bar{a})_{\overline{n}|i} = \sum_{t=1}^{n} \int_{t-1}^{t} t\, v^s ds$.

Substitution Rule.

$$= \sum_{t=1}^{n} t \boxed{\int_{t-1}^{t} v^s ds}$$

$s = \boxed{q} + t - 1$

$\Downarrow$

$$\begin{cases} s = t-1, \Rightarrow q = 0 \\ s = t, \Rightarrow q = 1 \end{cases}$$

$$= \sum_{t=1}^{n} \cdot t \cdot \int_0^1 v^{q+t-1} \cdot \overset{dq}{\overset{\parallel}{d(q+t-1)}}$$

$$= \sum_{t=1}^{n} t \cdot \int_0^1 v^q \cdot v^{t-1} dq$$

$g(x) \triangleq q+t-1$

$u \triangleq s$

$$= \sum_{t=1}^{n} t\, v^{t-1} \cdot \int_0^1 v^q dq$$

$$= (I\ddot{a})_{\overline{n}|} \cdot \bar{a}_{\overline{1}|}$$

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① $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

② $\int_a^a f(x)dx = 0$.

③ $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

# Higher-order derivatives.

$$y = f(x) = x^2$$

$$\frac{dy}{dx} = f'(x) = 2x$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2} = f''(x) = 2$$

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

$n$

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# Taylor Series Formula.

$$y = f(x). \quad \boxed{f(x_0),\ f'(x_0),\ \cdots f^{(n)}(x_0)}$$

$$f(x) \boxed{\cong} f(x_0) + (x - x_0)\cdot f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0)$$

$$+ \cdots \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \cdots$$

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$$f(x) = f(x_0) + \quad \cdots \quad + \cdots$$

$$(n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1)$$

Ex: $f(x) = e^x$ $\quad f^{(n)}(x) = e^x$

$x_0 = 0$

$$f(x) = e^x \cong e^0 + (x-0) \cdot e^0 + \frac{(x-0)^2}{2!} \cdot e^0 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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Probability & Statistics.

$\tilde{X}$ discrete $\quad Pr(\tilde{X} = x) = p(x)$

$$E[\tilde{X}] = \sum_x x \cdot p(x)$$

$$Var[\tilde{X}] = E\left[(\tilde{X} - E[\tilde{X}])^2\right]$$

$$\overset{☆}{=} E[\tilde{X}^2] - (E[\tilde{X}])^2.$$

X continuous: density of X.

$$P[a<X<b] = \int_a^b f(x)\,dx.$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left[\int_{-\infty}^{\infty} x \cdot f(x)\,dx\right]^2 \end{aligned}$$

$$Var[aX+b] = a^2\,Var[X]$$

$$SD[X] = \sqrt{Var[X]}$$

$$E[h(x)] = \int_{-\infty}^{\infty} h(x) \cdot f(x) \cdot dx.$$

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$X \perp Y$: $\quad Var[X+Y] = Var[X] + Var[Y]$