Statistical Inference

Lecture 02a

ANU - RSFAS

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A Bit of Revision

Theorem R1: If Z is a standard normal random variable, the $U = Z^2$ is a χ^2 distribution with 1 degree of freedom.

Theorem R2: If $U_1, \ldots, U_n$ are independent and $\overset{U_i}{\cancel{X_1}} \sim \chi_1^2$ then

$$\sum_{i=1}^{n} U_i \sim \chi_n^2$$

Proof of (1): Let's first consider the sums of independent gamma distributions.

Question: Suppose $X \sim \mathrm{gamma}(\alpha_1, \lambda)$ and $Y \sim \mathrm{gamma}(\alpha_2, \lambda)$, what is the distribution of $X + Y$?

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x)$$

- Let's get the moment generating function for X:

move sth. out to observe the kernel & make some adjustments.

$$
\begin{aligned}
M_X(t) &= E[exp(xt)] = \int_0^\infty exp(xt) \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x) dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty exp(xt) x^{\alpha-1} \exp(-\lambda x) dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \exp(-(\lambda - t)x) dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda - t)^\alpha} \underbrace{\int_0^\infty \frac{(\lambda - t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-(\lambda - t)x) dx}_{1} \\
&= \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\lambda - t)^\alpha} \\
&= \left(\frac{\lambda}{\lambda - t} \right)^\alpha
\end{aligned}
$$

- Back to our question:

$$
\begin{aligned}
M_{X+Y}(t) &= M_X(t)M_Y(t) \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1}\left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2} \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1+\alpha_2}
\end{aligned}
$$

$$W = X + Y \sim \mathrm{gamma}(\alpha_1 + \alpha_2, \lambda)$$

- The MGF for a χ^2 distribution is:

$$M(t) = (1-2t)^{-n/2}$$

- If we take our MGF for a single gamma distribution and set $\alpha = n/2$ and $\lambda = 1/2$ we have:

χ^2 distribution is a gamma distribution with $\alpha = \frac{n}{2}$ & $\lambda = \frac{1}{2}$

$$\begin{aligned} M_X(t) &= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \\ &= \left(\frac{1/2}{1/2 - t}\right)^{n/2} \\ &= \left(\frac{1}{1-2t}\right)^{n/2} \end{aligned}$$

So a χ^2 distribution with n degrees of freedom.

- Now let's determine the sum of two χ^2 random variables. $U_1 \sim \chi^2_n$ and $U_2 \sim \chi^2_m$ then:

$$M_{U_1+U2}(t) = M_{U_1}(t)M_{U_2}(t) = (1-2t)^{-n/2}(1-2t)^{-m/2} = (1-2t)^{-(n+m)/2}$$

$$U_1 + U_2 \sim \chi^2_{n+m}$$

Theorem R3: If

- $Z \sim \text{normal}(0, 1)$
- $U \sim \chi^2_n$
- Z and U are independent, then:

$$T = Z/\sqrt{U/n} \text{ is a t distrbution with } n \text{ degrees of freedom}$$

$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

- We have a transformation based on two independent random variables. We will use the standard transformation method.

$$t = z/\sqrt{u/n} \qquad v = u$$

- Now let's solve for the inverse of these solve for z and u in terms of t and v.

$$z = \frac{t\sqrt{v}}{\sqrt{n}} \qquad u = v$$

- Now let's get the determinant of the Jacobian:

$$|J| = \begin{vmatrix} \frac{\partial z}{\partial v} & \frac{\partial u}{\partial v} \\ \\ \frac{\partial z}{\partial t} & \frac{\partial u}{\partial t} \end{vmatrix} = \sqrt{v}/\sqrt{n}$$

$$f_{TV}(t,v) \;=\; f_{ZU}\left(\frac{t\sqrt{v}}{\sqrt{n}}, v\right)|J|$$

Note: the joint distribution of of Z and U is (remember they are independent):

joint of Z & U is density of Z times density of U.

$$f_{ZU}(z,u) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}z^2\right)\frac{1}{2^{n/2}\Gamma(n/2)}u^{n/2-1}exp(-u/2)$$

b/c independence

- So now we plug in for z and u.

$$\begin{aligned} f_{ZU}(z,u) &= \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{t\sqrt{v}}{\sqrt{n}}\right)^2\right)\frac{1}{2^{n/2}\Gamma(n/2)}v^{n/2-1}exp(-v/2) \\ &= \frac{v^{n/2-1}}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} exp\left(-\frac{1}{2}\left(\frac{t\sqrt{v}}{\sqrt{n}}\right)^2\right) exp(-v/2) \\ &= \frac{v^{n/2-1}}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right) \end{aligned}$$

$$
\begin{aligned}
f_{TV}(t,v) &= f_{ZU}\left(\frac{t\sqrt{v}}{\sqrt{n}}, v\right)|J| \\
&= \frac{v^{n/2-1}}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right) \\
&= \frac{v^{(n+1)/2-1}}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right)
\end{aligned}
$$

- Now we integrate out v to get t:

$$
\begin{aligned}
f_T(t) &= \int f_{TV}(t, v) dv \\
&= \int_0^{\infty} \frac{v^{(n+1)/2-1}}{\sqrt{2\pi n} 2^{n/2} \Gamma(n/2)} exp\left(-\frac{v}{2}(1+t^2/n)\right) dv \\
&= \frac{1}{\sqrt{2\pi n} 2^{n/2} \Gamma(n/2)} \int_0^{\infty} v^{(n+1)/2-1} exp\left(-\frac{v}{2}(1+t^2/n)\right) dv
\end{aligned}
$$

- So the integrand is a kernel of a gamma distribution with $a = (n+1)/2$ and $b = (1+t^2/n)/2$.

$$
\begin{aligned}
f_T(t) &= \frac{\Gamma\left((n+1)/2\right)}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} \frac{1}{[(1+t^2/n)/2]^{(n+1)/2}} \\
&= \frac{\Gamma\left((n+1)/2\right)}{\sqrt{\pi n}\Gamma(n/2)} \left[(1+t^2/n)\right]^{-(n+1)/2}
\end{aligned}
$$

Theorem R4: If

- $U \sim \chi^2_m$
- $V \sim \chi^2_n$
- U and V are independent then:

$$W = \frac{U/m}{V/n} \sim \mathrm{F}(m, n)$$

Proof: Through a similar approach we can show the result.

Sampling from the Normal Distribution

Theorem R5: If $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2)$, then

1. $\bar{X} \sim \text{normal}(\mu, \sigma^2/n)$
2. $\bar{X}$ and S^2 are independent
3. $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$

Proof of (1):

- If $X \sim n(\mu, \sigma^2)$ then the mgf of X is:

$\begin{cases} E(\bar{X}) = \mu \\ V(\bar{X}) = \frac{\sigma^2}{n} \end{cases}$

$$M_X(t) = E\left[e^{tX}\right] = e^{\mu t + \sigma^2 t^2/2}$$

So For $\bar{X}$ we have: $\bar{X} = \frac{1}{n}\sum X_i$

$$\begin{aligned} M_{\bar{X}}(t) &= \left[exp\left(\mu t/n + \sigma^2 (t/n)^2/2\right)\right]^n \\ &= exp\left(n\left(\mu t/n + \sigma^2 (t/n)^2/2\right)\right) \\ &= exp\left(\mu t + (\sigma^2/n)t^2/2\right) \end{aligned}$$

$$\bar{X} \sim n(\mu, \sigma^2/n)$$

Proof of (2): $\bar{X}$ and S^2 are independent.

- All we need to do is show that $\bar{X}$ and $Y_j = X_j - \bar{X}$ are independent for all j.
- Note: If $Z = \sum_{j=1}^{n} a_j X_j$ and $W = \sum_{j=1}^{n} b_j X_j$ are any two distinct linear combinations of iid normals, then Z and W have a joint bivariate normal distribution.
- For jointly bivariate normal quantities, independence is equivalent to being uncorrelated.

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$
independent

$$
\begin{aligned}
Cov(\bar{X}, X_j - \bar{X}) &= Cov(\bar{X}, X_j) - Cov(\bar{X}, \bar{X}) \\
&= Cov(\bar{X}, X_j) - V(\bar{X}) \\
&= Cov(\bar{X}, X_j) - \sigma^2/n \\
&= Cov(\frac{1}{n}(X_1 + \cdots + X_j + \cdots + X_n), X_j) - \sigma^2/n \\
&= Cov\left(\frac{1}{n}X_1, X_j\right) + \cdots + Cov\left(\frac{1}{n}X_j, X_j\right) + \cdots - \sigma^2/n \\
&= 0 + \cdots + Cov\left(\frac{1}{n}X_j, X_j\right) + \cdots - \sigma^2/n \\
&= \frac{1}{n}Cov(X_j, X_j) - \sigma^2/n \\
&= \frac{1}{n}V(X_j) - \sigma^2/n \\
&= \sigma^2/n - \sigma^2/n = 0
\end{aligned}
$$

uncorrelated
= independent

- Now examine the functions $\bar{X}$ and $S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2$
- As S^2 is a function of $X_1 - \bar{X}, \ldots, X_n - \bar{X}$ then $\bar{X}$ and S^2 are independent.
- For fun let's do this again in terms of matrices:

$$\boldsymbol{W} = \begin{bmatrix} \bar{X} \\ \boldsymbol{Y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \boldsymbol{X}$$

- Where:

$$\boldsymbol{Y} = (X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})'$$

$$\boldsymbol{v}' = (1/n, 1/n, \ldots, 1/n) = (1/n)\mathbf{1}'$$

$$\mathbf{1}' = (1, 1, \ldots, 1)$$

- As $\boldsymbol{W}$ is a linear transformation of multivariate normal random vector, it is a multivariate normal distirbution.

MNRV.

$$
\begin{aligned}
E[\boldsymbol{W}] &= E\left[\begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \boldsymbol{X}\right] \\
&= \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} E[\boldsymbol{X}] \\
&= \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \mu\mathbf{1} \\
&= \begin{bmatrix} \mu \\ \mathbf{0}_n \end{bmatrix}
\end{aligned}
$$

→ is ~~the~~ only random value.

$X_i \sim N(\mu, \sigma^2)$
$V(a X_i)$
$= a^2 V(X_i) = a^2 \sigma^2$

$$
\begin{aligned}
V[\boldsymbol{W}] &= V\left[\begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \boldsymbol{X}\right] \\
&= \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} V[\boldsymbol{X}] \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix}' \\
&= \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \sigma^2 \mathrm{I} \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix}' \\
&= \sigma^2 \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix}' \\
&= \sigma^2 \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \begin{bmatrix} \boldsymbol{v} & (\mathrm{I} - \mathbf{1}\boldsymbol{v})' \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \begin{bmatrix} \boldsymbol{v} & (\mathrm{I} - \mathbf{1}\boldsymbol{v})' \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{bmatrix} \begin{bmatrix} \boldsymbol{v} & \mathrm{I} - \boldsymbol{v}\mathbf{1}' \end{bmatrix}
\end{aligned}
$$

$V(AX)$
$= A V(X) A'$
$V(X) = ?$
$= \sigma^2 I$

$$
\begin{aligned}
V[\boldsymbol{W}] =& \sigma^2 \left[\begin{array}{c} \boldsymbol{v}' \\ \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{array} \right] \left[\begin{array}{cc} \boldsymbol{v} & \mathrm{I} - \boldsymbol{v}\mathbf{1}' \end{array} \right] \\
=& \sigma^2 \left[\begin{array}{cc} 1/n & \mathbf{0}_n' \\ \mathbf{0}_n & \mathrm{I} - \mathbf{1}\boldsymbol{v}' \end{array} \right]
\end{aligned}
$$

- As we saw before, $\bar{X}$ and $\boldsymbol{Y}$ are independent. Now as $S^2 = (n-1)^{-1}\boldsymbol{Y}'\boldsymbol{Y}$, so a function of $\boldsymbol{Y}$, then $\bar{X}$ and S^2 are independent.

Proof of R5 (3)

$$\sum (X_i - \mu)^2 = (n-1)S^2 + n(\bar{X} - \mu)^2$$

$$\sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2$$

$$\sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2$$

$$W = U + V$$

$$\left(\frac{X_i - \mu}{\sigma}\right)^2 = Z^2 \sim \chi_1^2$$

$$W = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

$$V = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 = Z^2 \sim \chi_1^2$$

Based on $\bar{X}$ and S^2 being independent then U and V are independent.

The MGF for a $\chi_p^2 = (1-2t)^{-p/2}$.

$$W = U + V$$
$$M_W(t) = M_U(t)M_V(t)$$
$$(1-2t)^{-n/2} = M_U(t)(1-2t)^{-1/2}$$
$$M_U(t) = \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}}$$
$$M_U(t) = (1-2t)^{-(n-1)/2}$$

$$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{(n-1)}^2$$

Theorem R6: t-statistic

- Consider the following statistic:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Proof: All we need to do is rewrite the statistic in the form of a *t*-distribution:

$$\begin{aligned}
\frac{\bar{X} - \mu}{S/\sqrt{n}} &= \frac{\bar{X} - \mu}{S/\sqrt{n}} \left(\frac{\sigma}{\sigma}\right) \\
&= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{Z}{\sqrt{U/(n-1)}}
\end{aligned}$$