Worth: 3% **Due:** By 12 noon on Tuesday 31 January.

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1. (a) $\forall x \in D, T(x) \Rightarrow \neg L(x)$

   (b) There is a long exam question. (Alternate: Some exam question is long.)

   (c) $\exists x \in D, T(x) \wedge (\forall y \in D, E(y) \Rightarrow H(x, y))$ (The parentheses are not required, but are added to aid comprehension.)

   (d) No exam question is harder than every test question.

2. (a) Every prime number other than 2 is odd.

   (b) There is a largest prime number.

3. (a) These two statements are equivalent. Suppose $\forall x \in D, (P(x) \wedge Q(x))$ is true. Then it follows that $\forall x \in D$, both $P(x)$ and $Q(x)$ are true. Hence, $\forall x \in D, P(x)$ is true and $\forall x \in D, Q(x)$ is true. Therefore, $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$ is true.

   Now suppose $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$ is true. If x is any element in the domain D, then $P(x)$ is true and $Q(x)$ is true. Hence, for any element x in the domain, $P(x) \wedge Q(x)$ is true. Since this is true for any x in domain, $\forall x \in D, (P(x) \wedge Q(x))$ is true.

   Hence whenever one of the two statements is true, the other one is true. The two given statements are equivalent.

   (b) Let the domain D be the set $\mathbb{R}$ of real numbers, the predicate $P(x)$ be the expression $x \geqslant 0$ and the predicate $Q(x)$ be the expression $x < 0$.

   With these definitions, it follows that $\forall x \in D, (P(x) \vee Q(x))$, since every real number is either $\geqslant 0$ or < 0.

   However, the statement $\forall x \in D, P(x)$ is not true, because $P(x)$ is false for negative numbers. And the statement $\forall x \in D, Q(x)$ is also not true, because $Q(x)$ is false for nonnegative numbers. Hence, the statement $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$ is false. The two given statements are not equivalent.

   (c) These two statements are equivalent. Suppose $\exists x \in D, (P(x) \vee Q(x))$ is true. Then it follows that there is an $x_0 \in D$ such that $(P(x) \vee Q(x))$ is true. We could have $P(x_0)$ true, $Q(x_0)$ true or both $P(x_0)$ and $Q(x_0)$ true. Consider the statements $(\exists x \in D, P(x))$ and $(\exists x \in D, Q(x))$. They cannot both be false, since at least one of $P(x_0)$, $Q(x_0)$ is true. Hence $(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$ is true.

   Now suppose $(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$ is true. Then there must be an $x_0 \in D$ which is such that at least one of $P(x_0)$, $Q(x_0)$ is true. Then it follows that $(P(x_0) \vee Q(x_0))$ is true, and then $\exists x \in D, (P(x) \vee Q(x))$ is true.

   Hence whenever one of the two statements is true, the other one is true. The two given statements are equivalent.

   (d) If we can construct a domain D and predicate P for which $P(x)$ is sometimes true and sometimes false, then $(\forall x \in D, P(x))$ will be false. This will make the second statement vaccuously true.

   If we can further construct a Q which is always false, then the statement $P(x) \Rightarrow Q(x)$ will be false for the $x \in D$ for which $P(x)$ is true. This will make the first statement false, and will show that the two statements are not equivalent.

   Let D represent the set of real numbers, P be the predicate "$x = |x|$" and Q be the predicate "$|x| < 0$." Then $(\forall x \in D, Q(x))$ is false. The value of $P(42)$ is true. While the value of $P(-42)$ is false. The statement $P(42) \Rightarrow Q(42)$ is false, and so the first statement is false. We know that $(\forall x \in D, P(x))$ is false, which makes the second statement true. The two statements are not equivalent.

4.

$$\neg(\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, (\forall x \in \mathbb{R}, 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon)) \quad \text{(given)}$$

$$\iff \exists \epsilon \in \mathbb{R}^+, \neg(\exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon) \quad \text{(negation of } \forall\text{)}$$

$$\iff \exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \neg(\forall x \in \mathbb{R}, 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon \quad \text{(negation of } \exists\text{)}$$

$$\iff \exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, 0 < |x-a| < \delta \wedge \neg(|f(x) - L| < \epsilon) \quad \text{(negation of } \Rightarrow\text{)}$$

$$\iff \exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, 0 < |x-a| < \delta \wedge |f(x) - L| \geqslant \epsilon \quad \text{(negation of } <\text{)}$$