Example: (we continue last class's reduction of order)

Solve:

$$\overbrace{t^2y'' - t(t+2)y' + (t+2)y}^{L[y]} = 0.$$

One solution is $y_1(t) = t$

Write $y(t) = v(t)y_1(t)$

$$y = vt$$
$$y' = v + v't$$
$$y'' = 2v' + v''t$$

$\Rightarrow$

$$L[y] = t^2(2v' + v''t) - t(t+2)(v + v't) + (t+2)vt$$
$$= \cancel{v(-t(t+2)+t(t+2))} + v'(2t^2 - t(t+2)t) + v''t^3$$
$$= v'(2t^2 - 2t^2 - t^3) + v''t^3$$
$$= t^3(v'' - v')$$

$\Rightarrow L[y] = 0,$ if $v'' - v' = 0$

Solution: $v' = e^t$, $v = e^t$

(just used <u>one</u> solution)

$\Rightarrow y_2(t) = v(t)y_1(t) = e^t \cdot t$

2nd order linear inhomogeneous equation

Recall notation:

$$L[y] = y'' + py' + gy$$

Inhom. eqn: $L[y] = g$.

i.e. $y''(t) + p(t)y'(t) + g(t)y(t) = g(t)$

Sps Y_1, Y_2 are two solutions of $L[y] = g$.

$$L[Y_1] = g$$
$$L[Y_2] = g$$

Then $Y_1 - Y_2$ solves the homogeneous equation:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g - g = 0.$$

Conversely, if Y solves $L[Y] = g$ and y solves $L[y] = 0$ then $Y + y$ solves

$$L[Y + y] = L[Y] + L[y] = g$$

Theorem: The general solution of $L[y]=g$ has the form

$y = Y + c_1 y_1 + c_2 y_2$ where y_1, y_2 are fundamental set of solutions of $L[y]=0$, and Y is a particular solution of $L[Y]=g$.

Thus, assuming we've solved $L[y]=0$, need to find one particular soln Y.

Consider first constant coefficient equation.

Example:

$$5y'' + y' + 6y = t^3$$

To find particular solution, try

$$Y(t) = At^3 + Bt^2 + Ct + D$$

$$Y'(t) = 3At^2 + 2Bt + C$$

$$Y''(t) = 6At + 2B$$

$$\Rightarrow L[Y] = 5(6At + 2B) + (3At^2 + 2Bt + C) + 6(At^3 + Bt^2 + Ct + D)$$
$$= t^3$$

Compare coefficients of t^3, t^2, t, t^0:

t^3: $6A = 1 \quad \Rightarrow A = \frac{1}{6}$

t^2: $3A + 6B = 0 \quad \Rightarrow B = -\frac{1}{2}A = -\frac{1}{12}$

t: $30A + 2B + 6C = 0 \Rightarrow C = \cdots$

$t^0 = 1$, $10B + C + 6D = 0 \Rightarrow D = \cdots$

Example: $y'' + 2y' - 2y = e^{-2t}$

Try $Y(t) = Ae^{-2t}$

$$Y'(t) = -2Ae^{-2t}$$

$$Y''(t) = 4Ae^{-2t}$$

$L[y] = 4Ae^{-2t} + 2(-2Ae^{-2t}) - 2Ae^{-2t}$

$= Ae^{-2t}\underbrace{(4-2-2)}_{0} \overset{!}{=} e^{-2t}$

$0 \cdot Ae^{-2t} = e^{-2t}$

so here it didn't work

Example: $y'' + y' - 2y = e^{-2t}$

Try $Y = Ae^{-2t}$. Doesn't work since $L[Y] = 0$.

Problem: e^{-2t} solves the homogeneous equation.

Try instead $Y = Ate^{-2t}$

$Y' = Ae^{-2t} - 2Ate^{-2t}$

$Y'' = 4Ae^{-2t} + 4At^2e^{-2t}$

$L[Y] = (-4Ae^{-2t} + 4Ate^{-2t}) + (Ae^{-2t} - 2Ate^{-2t}) - 2(Ate^{-2t})$

$= Ae^{-2t}(-4+1) + Ate^{-2t} \cdot 0$

$\overset{!}{=} e^{-2t}$

$\Rightarrow -3Ae^{-2t} = e^{-2t}$

$\Rightarrow A = \frac{-1}{3}$

$\Rightarrow Y(t) = -\frac{1}{3}te^{-2t}$ is a solution

Example: $y'' - 4y' + 4y = 3e^{2t}$

Char. eqn. for $L[y] = 0$ is

$r^2 - 4r + 4 = 0$

has $r = 2$ as repeated root.

$Y(t) = Ae^{2t}$ doesn't work, Ate^{2t} also doesn't work.

Try instead $Y = At^2e^{2t}$

You'll find $A = \frac{3}{2}$