FACULTY OF ARTS AND SCIENCE

DEPARTMENT OF MATHEMATICS

University of Toronto

MAT135Y Final Examination - Aug. 19 2010

Time allowed: 3 hours

NO AIDS ALLOWED

PLEASE HAND IN

Name: ______________________________

Student Number: ______________________

Tutorial Section: ______________________

Signature: ___________________________

Are you writing this exam as a deferred exam? Please circle: YES NO

Please read the following instructions closely:

1. This exam has 23 pages (including this cover page). Do not remove any pages from the exam.
2. This exam consists of 24 questions, in two parts; you must answer all the questions.
   - PART A: 16 short answer questions. (Right answer receives full credit. You may get partial marks even if your answer is incorrect if you choose to show your work. Circle your answer.)
   - PART B: 8 long answer questions. (You have to justify your answer/show your work; the answer by itself is worth little or no credit, even if correct.)
3. No aids of any kind allowed. NO CALCULATORS.
4. Answer each question in the space provided. You can use the back of each page, or the last page of the exam, for rough work. If you need extra space for a solution, use the back of the page, and make sure this is **clearly indicated** on the front of the page.

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Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
7	3	
8	3	
9	3	
10	3	
11	3	
12	3	
13	3	
14	3	
15	3	
16	3	
17	8	
18	10	
19	10	
20	10	
21	8	
22	8	
23	8	
24	5	
Total:	115	

PART A: SHORT ANSWER QUESTIONS: Read the questions carefully, and answer all the questions in the space provided. The correct answer receives full marks. You are not required to justify your answer; however part marks may be awarded to an incorrect answer, should you wish to show your work. Please circle your final answer.

1. (3 points) Find the limit:

$$\lim_{x \to 0} \frac{tan(7x)}{tan(4x)}$$

2. (3 points) If

$$\int_{x}^{x^2} f(t)\, dt = \sqrt{x}$$

for all x, find $f(1)$.

3. (3 points) Find the integral:

$$\int_{2}^{\infty} \frac{1}{1+e^x}\,dx$$

4. (3 points) Let R be the region bounded by the curves

$y = 1/x^3,\ x = 1,\ x = 3$ and the x-axis.

Find the number a in the interval $(1, 3)$ such that the line $x = a$ divides R into two regions (not necessarily the same shape) of equal area.

5. (3 points) Evaluate the following sum:

$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{2n-1}}$$

6. (3 points) Find the integral:

$$\int ln(x^2 - 1)dx$$

7. (3 points) Find the equation of the line that is tangent to the curve

$$y^2 = x^3 + x^2 - x + 3$$

at the point $(1, 2)$

8. (3 points) Find the integral

$$\int \frac{cos^2(1 + ln(x))}{x} dx$$

9. (3 points) Find the radius of convergence and the interval of convergence of the following series:

$$\sum_{n=4}^{\infty} \frac{x^n}{n\ ln(n)}$$

10. (3 points) Find the integral:

$$\int \frac{1}{\sqrt{9-x^2}}\, dx$$

11. (3 points) Find the limit:

$$\lim_{t\to 1} \sqrt{ln(t) + \cos(t\pi/2)}\ \sin\left(\frac{\pi}{t-1}\right)$$

Hint: Use the squeeze theorem.

12. (3 points) Where on the curve

$$y = x^4 - 24x^2 + 27x - 11$$

is the slope of the tangent at a local maximum?

13. (3 points) Find the volume of the solid obtained by rotating the region bounded by the curves

$$y^2 - y^3 - x = 0, \ x = 0$$

about the line $x = 0$.

14. (3 points) If $f(x) = \sin\left(\sqrt{ln(x)}\right)$, find $f'(x)$.

15. (3 points) Solve the initial value problem:

$$y\frac{dy}{dx} - x^3 = 0, \quad y(0) = -1.$$

16. (3 points) Of all the pairs of positive numbers whose product is 100, what is the smallest possible value of their sum?

PART B: LONG ANSWER QUESTIONS: Read the questions carefully, and answer all questions in the space provided. You are required to fully justify your answers; a correct answer with no justification is worth little or no credit.

17. This question has two parts:

    (a) (2 points) State the formal definition of the derivative of a function f at a point x.

    (b) (6 points) Find the derivative of the function $f(x) = x^2 - x$ **using only the formal definition**. Do not use any differentiation rules you may know.

18. (10 points) Find the integral

$$\int \frac{x\ ln(x)}{\sqrt{x^2-1}}\,dx$$

19. For the function $f(x) = e^{2x-x^2}$:

(a) (2 points) Find the domain of the function, $x-$ and $y-$ intercepts, and horizontal and vertical asymptotes, if applicable.

(b) (1 point) Does the function possess odd or even symmetry?

(This question continues on the following page)

(c) (4 points) Find the critical points, and points of inflection, if any. Find the intervals on which the function is increasing and decreasing. Find the intervals on which the curve is concave up and concave down.

(This question continues on the following page)

(d) (3 points) Draw a neat sketch of the graph of f; your sketch should include labels indicating the critical points, points of inflection, and regions of increase/decrease and concavity.

20. Researchers are interested in studying a certain strain of bacteria. The bacteria are assumed to have birth and death rates that are constant proportions of the population. The bacteria are grown in a very large nutrient-rich environment, which can support many more bacteria than the researchers will ever need to use. In addition, the researchers need to remove a fixed number of bacteria each hour for testing.

    (a) (3 points) Explain briefly why

    $$\frac{dP}{dt} = kP - m$$

    is a reasonable model for this population; your answer should include an explanation of the significance of the constants k and m. You may answer in point-form. *Suggestion: This problem is worth only 3 marks; keep your answer brief.*

(This question continues on the following page)

(b) (6 points) Given an initial population $P(0) = P_0$, solve the initial value problem for the population.

(This question continues on the following page)

(c) (1 point) Suppose $k = 0.02$ and $P_0 = 1000$. How many bacteria should the researchers collect each hour so that the population remains constant?

21. (8 points) You are driving through the remote Ontario wilderness, and notice that you are low on fuel. According to your map, which has a grid measuring a 1km scale drawn on it (ie the distance from one grid-line to the next represents 1km in the real world), you are at the point $(0, 1)$, and the road you are on appears to be described by the curve

$$y = arcsin(x) + \sqrt{1 - x^2}$$

You have enough gas to drive for exactly half a kilometre along this road, and there is a gas station at the end of the road, at the point $(1, \pi/2)$. Do you make it to the gas station? The following approximation may be useful: $\frac{1}{\sqrt{2}} \simeq 0.707...$

22. For each of the following series, decide whether it converges or diverges. If it converges, does it converge conditionally and/or absolutely? Justify your answers.

(a) (2 points) $\sum_{n=1}^{\infty} n \sin(1/n)$

(b) (2 points) $\sum_{n=3}^{\infty} \frac{n^6}{n^7+2n+1}$ *Hint: consider a comparison*

(c) (2 points) $\sum_{k=1}^{\infty} \frac{k(2k+1)}{(-2)^k}$

(d) (2 points) $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$

23. This problem has two parts.

(a) (2 points) State the mean value theorem.

(b) (6 points) Prove the following inequality:

$$\sin(ln(x+1)) \leq x \quad \text{for all } x \geq 0$$

24. (5 points) Find the limit, if it exists; if it does not exist, explain why.

$$\lim_{n\to\infty} (n!)^{1/n}$$

Recall that $n!$ is defined to be the product of all the integers between 1 and n inclusive.

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