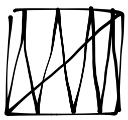
Lecture 28

$V: [-2,2] \to [-2,2]$
$V(x) = 2|x| - 2$ is a chaotic dynamic system

observe that the graph of the nth iterate of V consists of 2^n lines with slope 2^n and maps an interval J_n^i of length $\frac{1}{2^n}$ to $[-2,2]$

Proof: Density. Let $x \in [-2,2]$. For any $n \in \mathbb{N}$, there is an interval J_n^i s.t. $x \in J_n^i$



$\widetilde{J_n^i}$

Then $V^n(J_n^i) = [-2,2]$ so there is a fixed point x_n of V^n in J_n^i
so x_n is a periodic point of V and $|x - x_n| \leq \frac{1}{2^{n-2}}$
so $|x - x_n| \xrightarrow{n \to \infty} 0$

Transitivity. Let $x, y \in [-2,2]$ and $\varepsilon > 0$. choose n such that $\frac{1}{2^{n-2}} < \varepsilon$
There is an interval s.t. $x \in J_n^i$
Since $V^n(J_n^i) = [-2,2]$, there is $z \in J_n^i$ s.t. $V^n(z) = y$
so $|x - z| \leq \frac{1}{2^{n-2}} < \varepsilon$

$|y - V^n(z)| = 0 < \varepsilon$

Sensitivity. Let $\beta = 2$
Let $x \in [-2,2]$ and $\varepsilon > 0$. choose n st. $\frac{1}{2^{n-2}} < \varepsilon$
There is J_n^i s.t. $x \in J_n^i$.
Take $y \in J_n^i$ s.t. $|x - y| \geq \frac{1}{2}\,\text{length}(J_n^i) = \frac{1}{2^{n-1}}$

Then $|x - y| \leq \frac{1}{2^{n-2}} < \varepsilon$ and

$\frac{V^n(x) - V^n(y)}{x - y} = (V^n)'(c)$ for some c between x and y
$\Rightarrow c \in J_n^i$ so $|(V^n)'(c)| = 2^n$
Thus $|V^n(x) - V^n(y)| = 2^n |x - y| \geq 2^n \cdot \frac{1}{2^{n-1}} = 2$

This proves that V is chaotic. ■

We now use V to prove that Q_{-2} is chaotic. Define $C(x) = -2\cos(\frac{\pi}{2}x)$
$C(V(x)) = -2\cos(\frac{\pi}{2}(2|x| - 2)) = -2\cos(\pi|x| - \pi) = 2\cos(\pi x)$
$Q_{-2}(C(x)) = (-2\cos(\frac{\pi}{2}(x))^2 - 2 = 4\cos^2(\frac{\pi}{2}x) - 2$
$= 2(1 + \cos(\pi x)) - 2$
$= 2\cos(\pi x)$

Note:
$\cos(\alpha - \pi) = -\cos(\alpha)$
$\cos^2(x) = \frac{1 + \cos(2x)}{2}$

$$\begin{tikzcd} {[-2,2]} \arrow[r, "V"] \arrow[d, "C"] & {[-2,2]} \arrow[d, "C"] \\ {[-2,2]} \arrow[r, "Q_{-2}"'] & {[-2,2]} \end{tikzcd}$$

C seems to be a conjugacy, but it is not one-to-one
it is two-to-one
it is also onto and continuous.

so the density proposition still applies to C and C still takes periodic pts of V to per. pts of Q_{-2}. So C can still be used to prove that Q_{-2} is chaotic.