STA 347, Probability I
Homework 2 Solutions
Note: All questions are marked out of 5 points for a total of 35 points.

Problem 1)

Solution 1: Let Ω be the sample space. Notice that for any $\omega \in \Omega$ we can write the random variable X as

$$X(\omega) = \sum_{j=0}^{\infty} jI(X(\omega) = j)$$

since the random variable X only takes non-negative values. Now, we have

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}[\sum_{j=0}^{\infty} jI(X = j)] = \sum_{j=0}^{\infty} j\mathrm{E}[I(X = j)] \\
&= \sum_{j=0}^{\infty} jP(X = j) = \sum_{j=0}^{\infty}\sum_{n=1}^{j} P(X = j) \\
&= \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} P(X = j) = \sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=0}^{\infty} P(X > n)
\end{aligned}$$

Note that we were able to exchange the expectation and the infinite sum as a consequence of axiom (5) of the expectation operator.

Solution 2) We have

$$\begin{aligned}
\sum_{n=0}^{\infty} P(X > n) &= \sum_{n=0}^{\infty} \mathrm{E}[I(X > n)] = \mathrm{E}[\sum_{n=0}^{\infty} I(X > n)] \\
&= \mathrm{E}[\sum_{n=0}^{X-1} 1 + \sum_{n=X}^{\infty} 0] = \mathrm{E}[X]
\end{aligned}$$

Again note that the exchangeability of the expectation and the infinite sum is justified by the axiom (5) of the expectation operator.

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Problem 2)

Since S_1 has a geometric distribution then

$$P(S_1 = s) = pq^{s-1}I(s = 1, 2, 3, \dots)$$

Thus

$$\pi(z) = \mathrm{E}[z^{S_1}] = \sum_{s=1}^{\infty} P(S_1 = s) z^s = \sum_{s=1}^{\infty} pq^{s-1} z^s$$
$$= pz \sum_{s=0}^{\infty} q^s z^s = \frac{pz}{1-qz} \qquad \text{if } |qz| < 1$$

Now

$$\mathrm{E}[S_1] = \pi'(1) = \Big(\frac{pz}{1-qz}\Big)'\Big|_{z=1}$$
$$= \frac{(1-q)p + pq}{(1-q)^2} = \frac{p}{p^2} = p^{-1}$$

Next

$$\mathrm{E}[S_1(S_1-1)] = \pi''(1) = \Big(\frac{pz}{1-qz}\Big)''\Big|_{z=1}$$
$$= \frac{2pq}{(1-q)^3} = \frac{2pq}{(p)^3} = \frac{2q}{p^2}$$

This leads to

$$\mathrm{Var}(S_1) = E[S_1^2] - (E[S_1]))^2 = E[S_1(S_1-1)] + E(S_1) - (E[S_1])^2$$
$$= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = qp^{-2}$$

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Problem 3)

Since the random variables N_j are independent we have

$$\pi_X(z) = \prod_{j=1}^{\infty} \pi_{jN_j}(z)$$

Since the N_j have a Poisson distribution, we have

$$\pi_{jN_j}(z) = \mathrm{E}[z^{jN_j}] = \sum_{n=0}^{\infty} P(N_j = n) z^{nj} = \sum_{n=0}^{\infty} \frac{\lambda_j^n e^{-\lambda_j}}{n!} z^{nj}$$
$$= e^{-\lambda_j} \sum_{n=0}^{\infty} \frac{(\lambda_j z^j)^n}{n!} = e^{-\lambda_j} e^{\lambda_j z^j} = e^{\lambda_j (z^j - 1)}$$

Thus

$$\pi_X(z) = \prod_{j=1}^{\infty} e^{\lambda_j (z^j - 1)} = e^{\sum_{j=1}^{\infty} \lambda_j (z^j - 1)}$$

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Problem 4)

This is a Multinomial distribution problem. Let X_a, X_b and X_c be the number of customers that exit through gates A, B and C respectively and let p_a, p_b and p_c be the corresponding probabilities. We have $X_a + X_b + X_c = n$ and $p_a + p_b + p_c = 1$. Hence

$$\begin{pmatrix} X_a \\ X_b \\ X_c \end{pmatrix} \sim Multinomial(n, p_a, p_b, p_c)$$

In this problem $n = 4$ and since exiting through any of the three gates is equally likely we have $p_a = p_b = p_c = \frac{1}{3}$.

(a)

$$P(X_a = 2, X_b = 1, X_c = 1) = \binom{4}{2,1,1}(\frac{1}{3})^2(\frac{1}{3})(\frac{1}{3}) = \frac{4}{27}$$

(b) The events of all four customers selecting the same gate are mutually exclusive, and hence

$$\begin{aligned} &P(\text{All four select the same gate}) \\ &= P(X_a = 4, X_b = 0, X_c = 0) + P(X_a = 0, X_b = 4, X_c = 0) + P(X_a = 0, X_b = 0, X_c = 4) \\ &= \binom{4}{4,0,0}(\frac{1}{3})^4 + \binom{4}{0,4,0}(\frac{1}{3})^4 + \binom{4}{0,0,4}(\frac{1}{3})^4 = 3(\frac{1}{3})^4 = \frac{1}{27} \end{aligned}$$

(c) Since there are 4 customers and 3 gates, if we require that all three gates are used then one and only one gate must have 2 customers exiting through. There are three possible situations

$$\begin{aligned} &P(X_a = 2, X_b = 1, X_c = 1) + P(X_a = 1, X_b = 2, X_c = 1) + P(X_a = 1, X_b = 1, X_c = 2) \\ &= 3P(X_a = 2, X_b = 1, X_c = 1) = 3\binom{4}{2,1,1}(\frac{1}{3})^2(\frac{1}{3})(\frac{1}{3}) = \frac{4}{9} \end{aligned}$$

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Problem 5)

This is a multinomial distribution problem. Let X_0, X_1 and X_2 be the number of manufactured items with zero, one, and at least two defects respectively and let p_0, p_1 and p_2 be the corresponding probabilities. We have

$$X_0 + X_1 + X_2 = n = 10 \qquad \text{and } p_0 = 0.85, \qquad p_1 = 0.1, \qquad p_2 = 0.05$$

Hence

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} \sim Multinomial(10, 0.85, 0.1, 0.05)$$

In a Multinomial distribution, the marginals also have a multinomial distribution. In particular the one-dimensional marginals X_0, X_1 and X_2 have a binomial distribution of size n and probabilities p_0, p_1 and p_2 respectively. Also, each two marginals have a negative covariance. For example $\mathrm{Cov}(X_1, X_2) = -np_1p_2$. These facts help us solve the problem:

$$\mathrm{E}[X_1 + 4X_2] = \mathrm{E}[X_1] + 4\mathrm{E}[X_2] = np_1 + 4np_2 = 10 * 0.1 + 4 * 10 * 0.05 = 3$$

and

$$\begin{aligned} \mathrm{Var}[X_1 + 4X_2] &= \mathrm{Var}[X_1] + 4^2\mathrm{Var}[X_2] + 2 * 4\mathrm{Cov}(X_1, X_2) \\ &= np_1(1 - p_1) + 16np_2(1 - p_2) - 8 * np_1p_2 \\ &= 10 * 0.1 * 0.9 + 16 * 10 * 0.05 * 0.95 - 8 * 10 * 0.1 * 0.05 = 8.1 \end{aligned}$$

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Problem 6)

Suppose $Y \sim Bin(n, p)$. We can either calculate the MGF directly by using the definition of MGF and the density of Y or alternatively by noting that a binomial distribution of size n is the sum of n independent and identically distributed Bernoulli random variables; i.e. $Y = X_1 + \ldots, X_n$ where $X_1, \ldots, X_n \overset{i.i.d.}{\sim} Ber(p)$. We have

$$P(X_i = x) = p^x(1 - p)^{1-x} \qquad \text{where} \qquad x = 0, 1$$

So

$$M_{X_i}(t) = \mathrm{E}[e^{tX_i}] = pe^t + (1 - p)e^0 = pe^t + q$$

and

$$\begin{aligned} M_Y(t) &= \mathrm{E}[e^{tY}] = \mathrm{E}[e^{t\sum_{i=1}^{n} X_i}] = \mathrm{E}[\prod_{i=1}^{n} e^{tX_i}] \\ &= \prod_{i=1}^{n} \mathrm{E}[e^{tX_i}] = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} pe^t + q = (pe^t + q)^n \end{aligned}$$

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Problem 7)

Similar to previous question we can either compute the MGF using brute force or make life easier and use the fact that a negative binomial random variable with parameters p and r is the sum of r independent geometric random variables with parameter p. So let

$$X = \sum_{i=1}^{r} Y_i$$

where $Y_i \overset{i.i.d.}{\sim} Geometric(p)$. Then

$$\begin{aligned} M_Y(t) &= \mathrm{E}[e^{tY}] = \sum_{y=0}^{\infty} P(Y=y)e^{ty} = \sum_{y=0}^{\infty} pq^y e^{ty} = p\sum_{y=0}^{\infty}(qe^t)^y \\ &= \frac{p}{1-qe^t} \qquad \text{provided } |qe^t| < 1 \end{aligned}$$

Thus

$$\begin{aligned} M_X(t) &= \mathrm{E}[e^{t\sum_{i=1}^{r} Y_i}] = \mathrm{E}[\prod_{i=1}^{r} e^{tY_i}] = \prod_{i=1}^{r} \mathrm{E}[e^{tY_i}] \\ &= \prod_{i=1}^{r}\left(\frac{p}{1-qe^t}\right) = \left(\frac{p}{1-qe^t}\right)^r \end{aligned}$$

Next compute

$$\begin{aligned} E[X] = M_X'(0) &= \frac{d}{dt}\left(\frac{p}{1-qe^t}\right)^r\Big|_{t=0} = rqp^r e^t(1-qe^t)^{-r-1}\Big|_{t=0} = \frac{rq}{p} \\ E[X^2] = M_X''(0) &= \frac{d^2}{dt^2}\left(\frac{p}{1-qe^t}\right)^r\Big|_{t=0} = \frac{rq(1+rq)}{p^2} \end{aligned}$$

Finally

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{rq(1+rq)}{p^2} - \left(\frac{rq}{p}\right)^2 = \frac{rq}{p^2}$$