Problem 4

Solution:

The core of efficient computation of the problem is to avoid calculating B^{-1} directly. Instead, we try to treat some parts including B^{-1} as a whole, and solve that part using LU factorization which genuinely saves steps.

First, we need to expand z:

$$\begin{aligned} z &= B^{-1}(2A+I)(B^{-1}+A)b \\ &= (2B^{-1}A+B^{-1})(B^{-1}+A)b \\ &= 2B^{-1}B^{-1}Ab + B^{-1}B^{-1}b + 2B^{-1}A^2b + B^{-1}Ab \end{aligned}$$

Now we assume $B^{-1}B^{-1}A = x, B^{-1}A = y, B^{-1}B^{-1}b = m$, then

$$\begin{aligned} A &= B^2x \\ A &= By \\ b &= B^2m \end{aligned}$$

Note that x, y are two $n \times n$ matrices, m is a $n \times 1$ vector.
Thereafter we can solve for x, y, m by LU factorization.

- We need to calculate B^2, which takes $2n-1$ flops for each entry. And the result of B^2 is an $n \times n$ matrix which has n^2 entries. So in total, the calculation of B^2 takes $(2n-1)\cdot n^2 = 2n^3 - n^2$ flops.
    - Also note that all $n \times n$ matrix multiplication takes $2n^3 - n^2$ flops.
- To calculate x, apply LU factorization on B^2 ($n^3/3$ flops), then do forward and backward substitutions (n^3 flops). So in total, $\frac{4}{3}n^3$ flops.
- Similarly for calculating y, also $\frac{4}{3}n^3$ flops.
- But for m, apply LU factorization on B^2 ($n^3/3$ flops), then do forward and backward substitutions ($n \cdot 1 \cdot n \cdot 1 = n^2$ flops). So this time in total $n^3/3 + n^2$ flops.
    - **Not sure whether we can save this calculation of B^2, i.e. saving $n^3/3$ flops since we already have that previously.**

So the original equation can be rewritten as:

$$z = 2xb + m + 2yAb + yb$$

Now we are able to know the flop counts of z:

- xb is a $n \times n$ matrix times $n \times 1$ vector, $(2n-1)n = 2n^2 - n$ flops. Scalar takes extra n flops. So in total $2n^2 + \frac{4}{3}n^3$ flops for $2xb$ (including calculating x).
- m takes $\frac{n^3}{3} + n^2$ flops.
- Scalar 2 takes n flops, y times A takes $2n^3 - n^2$ flops, times b takes another $2n^2 - n$ flops. So in total $n + 2n^3 - n^2 + 2n^2 - n + \frac{4}{3}n^3 = \frac{10}{3}n^3 + n^2$ flops (including calculating y).
- yb takes another $2n^2 - n$ flops.
- Each addition takes n flops. Three additions take $3n$ flops.

Hence, we have:

$$2n^2 + \frac{4}{3}n^3 + \frac{1}{3}n^3 + n^2 + \frac{10}{3}n^3 + n^2 + 2n^2 - n + 3n = 5n^3 + 6n^2 + 2n$$

flops for calculating z.