FINANCIAL MATHEMATICS
STAT 2032 / STAT 6046

LECTURE NOTES WEEK 3

<u>SUMMARY OF FORMULAE</u>

The important relationships between the variables introduced so far are given below. These also appear on page 1 of the formula sheet that you will have available for the mid semester and final exams.

<u>The accumulated value of 1 from time 0 to time t under compound interest:</u>

$$S(t)=\left(1+\frac{i^{(m)}}{m}\right)^{mt}=(1+i)^{t}=v^{-t}=(1-d)^{-t}=\left(1-\frac{d^{(m)}}{m}\right)^{-mt}=e^{\delta t}$$

<u>The present value at time 0 of 1 payable at time t under compound interest is:</u>

$$S(0)=\left(1+\frac{i^{(m)}}{m}\right)^{-mt}=(1+i)^{-t}=v^{t}=(1-d)^{t}=\left(1-\frac{d^{(m)}}{m}\right)^{mt}=e^{-\delta t}$$

These relationships enable you to translate an interest rate that is given in a particular form (e.g. $i^{(12)}$, d), into any other form as required.

EXAMPLE

If $i^{(12)}=0.12$, then

$$i=\left(1+\frac{i^{(12)}}{m}\right)^{12}-1=0.126825$$

$$d=1-\left(1+\frac{i^{(12)}}{m}\right)^{-12}=0.112551$$

$$\delta=12\cdot\ln\left(1+\frac{i^{(12)}}{m}\right)=0.119404$$

THE VALUATION OF PERIODIC PAYMENTS – ANNUITIES

Many financial transactions involve a series of payments. It is often possible to simplify a stream of payments by using algebraic methods. For example in the tutorial in week 2 we showed the equivalency of income payments under compound interest with a simple algebraic expression, thereby deriving the following geometric series expansion:

$$1+(1+i)+(1+i)^2+(1+i)^3+...+(1+i)^{n-1} = \frac{(1+i)^n-1}{i}$$

Many of the calculations in this section involve the familiar geometric series summation formula:

$$\boxed{1+x+x^2+x^3+...+x^k = \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1}-1}{x-1}}$$

The equivalency of these two formula can be seen by substituting in $(1+i)$ for x.

EXAMPLE

John receives a payment of $50 every month. Each payment is deposited in a bank account at the end of each month. The account earns interest at an annual rate of 12% compounded monthly and payable on the last day of each month. If the first deposit is on June 30, 2006, what is the balance, including the payment just made, on December 31, 2013?

Solution

We want to find the accumulated value at December 31, 2013 of all monthly payments made. Since the rate is a nominal rate of interest convertible monthly, we will use months as our unit of time. The one-month effective (compound) interest rate is

$$j = \frac{0.12}{12} = 0.01.$$

We need to write out the equation for the accumulated values. Since the first deposit is on June 30, 2006, the number of months that this deposit will accumulate is 6 months until the end of 2006, and 84 additional months from the end of 2006 until the end of 2013 (12 months x 7 years). Therefore, the first payment of 50 accumulates for 90 months to: $50(1+j)^{90}$.

The second deposit is one month after the first, so this accumulates for 89 months: $50(1+j)^{89}$.

Since there is a payment of 50 made every month, including on the final date of December 31, 2013, the total accumulated value is the sum of the accumulated amounts from each month:

$$50\left[(1+j)^{90}+(1+j)^{89}+(1+j)^{88}...+(1+j)^{1}+1\right]$$

Using the geometric series summation formula from above, where $x=(1+j)$ and $k=90$, this can be simplified to:

$$50\left[\frac{(1+j)^{91}-1}{(1+j)-1}\right]=50\left[\frac{(1.01)^{91}-1}{0.01}\right]=\$7,365.60$$

ACCUMULATED VALUE OF AN IMMEDIATE ANNUITY

In the previous example we found the accumulated value of a series of periodic payments.

The term used to describe a series of periodic payments is ***annuity***.

All of the annuities that we will deal with in this course are of type ***annuity-certain***, which means that the payments are *not* contingent on the occurrence of a specified event such as the survival of an individual. When we refer to annuities in this course, we mean annuities-certain. Annuities that are *not* certain, but are contingent on the life of an individual (or multiple lives), are covered in STAT3037 (Life contingencies).

We have already found the accumulated value of an annuity in the previous example. We now introduce the formal definition:

Consider a series of n payments of 1 unit made at the ***end*** of equally spaced time intervals, where each payment is invested at an effective interest rate of i per time interval, and where interest is credited on payment dates.

The accumulated value of these payments at time n, where the final payment is made at time n, can be found by noting the following:

The first payment accumulates from time 1 to time n, ie. $n-1$ periods of time, or:

$$(1+i)^{n-1}$$

The second payment accumulates from time 2 to time n, ie. $n-2$ periods of time, or:

$$(1+i)^{n-2}$$

$$\ldots$$

The second-last payment accumulates from time $n-1$ to time n, ie. 1 period of time, or:

$$(1+i)$$

The last payment of 1 is made at time n.

Therefore, using the geometric series expansion, the summation of these accumulated payments is:

$$(1+i)^{n-1}+(1+i)^{n-2}+(1+i)^{n-3}+...+(1+i)+1=\frac{(1+i)^n-1}{(1+i)-1}=\frac{(1+i)^n-1}{i}$$

In standard notation, this is given the symbol $s_{\overline{n|}}$, or $s_{\overline{n|}i}$ to indicate that an interest rate of i is associated with the accumulation.

In summary, the accumulated value at the end of n periods of an immediate annuity of 1 unit per period payable at the end of each period for a total of n periods is:

$$\boxed{s_{\overline{n|}}=\sum_{t=0}^{n-1}(1+i)^t=\frac{(1+i)^n-1}{i}}$$

Since the payments are made at the end of each period this is also referred to as the accumulated value of an annuity certain payable in ***arrears***.

In the case of accumulated value, when the annuity is valued at the time of the final payment this is referred to as an ***immediate*** annuity.

PRESENT VALUE OF AN IMMEDIATE ANNUITY

The present value at time 0 of an immediate annuity of 1 unit per period payable at the end of each period for n periods is:

$$\boxed{a_{\overline{n|}}=s_{\overline{n|}}\cdot v^n=\frac{1-v^n}{i}}$$

In the case of present value, an ***immediate*** annuity refers to an annuity valued one payment period before the first payment.

A proof of this result from first principles is given below:

$a_{\overline{n|}}$ is the present value at t = 0 of a series of payments of 1 unit payable at times 1,2,3,…,n:

$$a_{\overline{n|}}=v+v^2+...+v^{n-1}+v^n$$

Multiplying both sides through by $(1+i)$,

$$(1+i)\cdot a_{\overline{n}|} = (1+i)\cdot\left(v+v^2+\ldots+v^{n-1}+v^n\right) = 1+v+v^2+\ldots+v^{n-1}$$

Subtracting the first equation from the second,

$$(1+i)\cdot a_{\overline{n}|} - a_{\overline{n}|} = i\cdot a_{\overline{n}|} = \left(1+v+\ldots+v^{n-1}\right)-\left(v+v^2+\ldots+v^{n-1}+v^n\right) = 1-v^n$$

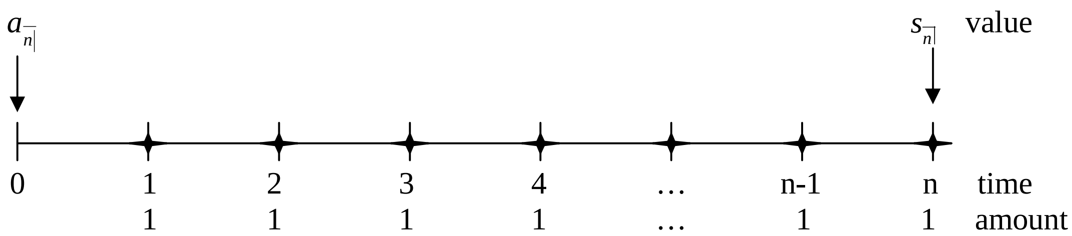
Therefore,

$$a_{\overline{n}|} = \frac{1-v^n}{i}.$$

If you are asked to prove an annuity formula (of which there will be several of relevance to this course) from first principles, you should follow the general methodology above. That is, you should derive the result without having to directly reference any other result without proof (such as the formula for the summation of a geometric series, or any other annuity formula).

An alternative derivation of this result involves using the formula for geometric series as we did previously for the accumulated value of an immediate annuity.

Often it may help to visualise payments on a timeline:



indicates that a payment is made

EXAMPLE
Find the present value at 1 January 2013 of a series of payments of $1,000 payable on the first day of each month from February 2013 to December 2013 inclusive, assuming a nominal rate of interest of 6% per annum convertible monthly. Next, find the accumulated value at 1 December 2013 of this series of payments.

Solution
We have an annuity of $1,000 payable each month for 11 months, starting in one month's time. Since payments are monthly, work with time units of a month. First convert the nominal interest rate into an effective monthly interest rate.

6% per annum convertible monthly is equivalent to an effective monthly interest rate of

$$i = \frac{i^{(m)}}{m} = \frac{0.06}{12} = 0.005$$

The present value (using units of time of one month) is:

$$\$1{,}000 \cdot a_{\overline{11}|0.005} = \$1{,}000 \cdot \left(\frac{1 - v_{0.005}^{11}}{i} \right) = \$1{,}000 \cdot \left(\frac{1 - 1.005^{-11}}{0.005} \right) = \$10{,}677$$

In other words, $10,677 invested at 1 January 2013, will be enough to return $1,000 per month for 11 months if the compound return on the investment is 0.5% per month.

Note that we are taking the present value of the annuity one month before the first payment is due (ie. valuing the annuity at 1 January when the first payment is at 1 February).

The accumulated value of this annuity on 1 December 2013 is:

This can simply be calculated by accumulating the present value by 11 months; ie.

$$\$10{,}677 \cdot (1.005)^{11} = \$11{,}279$$

This can also be written: $\$1{,}000 \cdot s_{\overline{11}|0.005} = \$11{,}279$

ANNUITIES DUE

An annuity payable in advance (ie. payments at the beginning of each period) is called an ***annuity due***.

The accumulated value at the end of n periods of an annuity of 1 unit per period payable at the ***beginning*** of each period for n periods is:

$$\boxed{\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \frac{i}{d} s_{\overline{n}|}}$$

Since this is the accumulated value at the <u>end</u> of n periods, the payment stream is valued <u>one payment period after</u> the final annuity payment.

$$\ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + ... + (1+i)^n$$

$$= (1+i)\left[1 + (1+i)^1 + ... + (1+i)^{n-1}\right] = (1+i)s_{\overline{n}|}$$

$$= (1+i) \cdot \left[\frac{(1+i)^n - 1}{i}\right] = \frac{(1+i)^n - 1}{i/(1+i)} = \frac{(1+i)^n - 1}{d}$$

Therefore, $\ddot{s}_{\overline{n}|} = (1+i) \cdot s_{\overline{n}|} = \frac{i}{d} s_{\overline{n}|}$

The present value of 1 unit per period payable at the ***beginning*** of each period for n periods is:

$$\boxed{\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d} = \frac{i}{d} a_{\overline{n}|}}$$

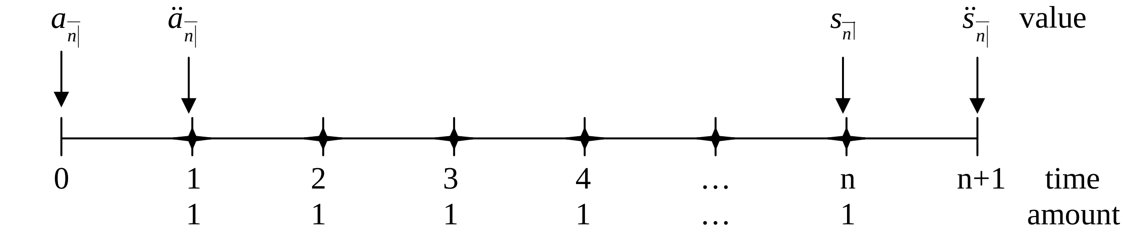
In this case, the payment stream is valued at the <u>time of the first payment</u>.

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + ... + v^{n-1} = \frac{v^n - 1}{(v-1)} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}$$

Therefore, $\ddot{a}_{\overline{n}|} = v^{-1}\left(v + v^2 + ... + v^n\right) = (1+i) \cdot a_{\overline{n}|} = \frac{i}{d} a_{\overline{n}|}$

You should also be able to derive this result from <u>first principles</u>.

The payments for $\ddot{a}_{\overline{n}|}$ correspond exactly with those for $a_{\overline{n}|}$, except that each payment is made one year earlier. i.e. each payment has a value that is greater by a factor of $(1+i)$.



✦ indicates that a payment is made

DEFERRED ANNUITIES

The present values considered so far have assumed that the annuities commence payment at the date of valuation, or one unit of time after the date of valuation.

If an annuity is to be valued more than 1 unit of time before commencement of the stream of payments, we call this a ***deferred annuity***.

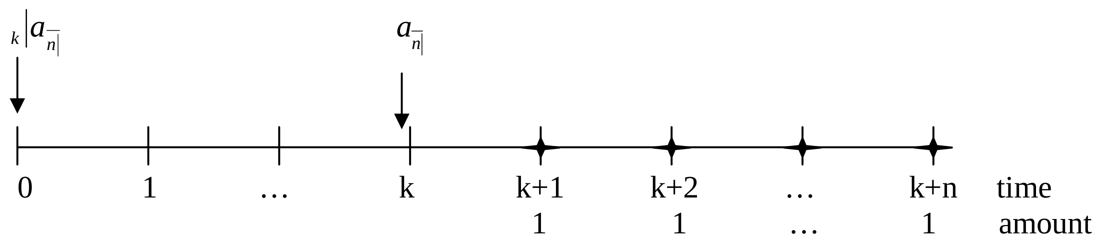
For example, suppose that k and n are non-negative integers. The value at time 0 of a series of n payments, each of amount 1, commencing at time $k+1$, is denoted by ${}_{k|}a_{\overline{n}|}$.

$$ {}_{k|}a_{\overline{n}|} = v^{k+1} + v^{k+2} + ... + v^{k+n} = v^k \left[v^1 + v^2 + ... + v^n\right] = v^k \cdot a_{\overline{n}|} $$

It can also be shown that:

$$ \boxed{{}_{k|}a_{\overline{n}|} = v^k \cdot a_{\overline{n}|} = a_{\overline{n+k}|} - a_{\overline{k}|}} $$

This is called an n-payment immediate annuity deferred for k payment periods.



✦ indicates that a payment is made

The equivalent n-payment annuity-due deferred for k payment periods is:

$$ \boxed{{}_{k|}\ddot{a}_{\overline{n}|} = v^k \cdot \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n+k}|} - \ddot{a}_{\overline{k}|}} $$

If we want to find the accumulated value of an annuity k periods after the last payment date, then we simply multiply the accumulated annuity $s_{\overline{n}|}$ by $(1+i)^k$.

$$\boxed{s_{\overline{n}|}(1+i)^k = s_{\overline{n+k}|} - s_{\overline{k}|}}$$

EXAMPLE

Find the present value at 15 May 2006 of a series of payments of $200 starting on 15 June 2009 and payable quarterly thereafter until the last payment on 15 June 2013, assuming an effective interest rate of 2.5% per quarter. Next, find the accumulated value at 15 August 2015 of this series of payments.

Solution

First we calculate the present value at the date of the first payment, 15 June 2009:

$$= 200\ddot{a}_{\overline{17}|0.025} = 200\left(\frac{1-v_{0.025}^{17}}{d}\right) = 200\left(\frac{1-1.025^{-17}}{0.025/1.025}\right) = \$2,811.00$$

We then discount the value calculated at 15 June 2009 to 15 May 2006:

$$= 200 \times {}_{12.333}\big|\ddot{a}_{\overline{17}|0.025} = \$2,811.00 \times 1.025^{-12.333} = \$2,073.00$$

Alternatively, we could have used annuities in arrears:

$$= 200 \times {}_{11.333}\big|a_{\overline{17}|0.025} = \$2,742.44 \times 1.025^{-11.333} = \$2,073.00$$

The accumulated value of this annuity on 15 August 2015 can simply be calculated by accumulating the present value by 9.25 years (37 quarters); ie.

$$2,073 \times (1.025)^{37} = \$5,168.71$$

This can also be written as $200s_{\overline{17}|0.025} \times (1.025)^{8.667} = \$5,168.71$ or

$200\ddot{s}_{\overline{17}|0.025} \times (1.025)^{7.667} = \$5,168.71$

VALUING ANNUITIES WITH MORE THAN ONE INTEREST RATE

Consider a $n+k$-payment annuity with an interest rate of i per payment period up to the time of the n^{th} payment, followed by an effective interest rate of j per payment period from the time of the n^{th} payment onward.

The accumulated value of an annuity at the time of the final payment can be found by:

(a) Finding the accumulated value of the first n payments at the time of the n^{th} payment:

$$s_{\overline{n}|i}$$

(b) Accumulating the result of part (a) for an additional k periods at compound rate j:

$$s_{\overline{n}|i}(1+j)^k$$

(c) Finding the accumulated value of the final k payments at compound rate j:

$$s_{\overline{k}|j}$$

(d) Adding the results of (b) and (c) together to get the final accumulated amount:

$$s_{\overline{n}|i}(1+j)^k + s_{\overline{k}|j}$$

When finding the present value of an annuity with different interest rates a similar approach can be used, as shown in the example below.

Alternatively, we can find the present value one period before the first payment (ie. annuity-immediate) by discounting the accumulated value above:

$$\left(s_{\overline{n}|i}(1+j)^k + s_{\overline{k}|j}\right)v_j^k v_i^n = s_{\overline{n}|i}v_i^n + s_{\overline{k}|j}v_j^k v_i^n = a_{\overline{n}|i} + v_i^n \cdot a_{\overline{k}|j}$$

ie. this is the accumulated value discounted k periods at the rate j, and then discounted to the present at the rate i for an additional n periods.

EXAMPLE

Find the present value at time 0 of payments of \$200 at time 0,1,2,…..,10 at a periodic interest rate of 5% for time 0-5 and 6% at time 5-10.

Solution

First we calculate the present value of the first six payments:

$$= 200\ddot{a}_{\overline{6}|0.05} = \$1,065.90$$

We then find the present value of the last five payments:

$$= 200 \times a_{\overline{5}|0.06} v_{0.05}^5 = 200 \times 4.21236 \times 0.78353 = \$660.10$$

The total of these gives \$1,726.00.

ANNUITIES PAYABLE MORE FREQUENTLY THAN ANNUALLY

In many situations annuities will be payable more frequently than interest is convertible. For example, we may have an annuity that is payable monthly, but we are told that interest is payable annually rather than monthly.

In these situations, we need to modify the calculations so that the points at which interest is paid are consistent with the points at which payments are made.

We can do this by working in units of time consistent with the timing of annuity payments. This involves converting the interest rate quoted to an equivalent effective interest rate consistent with the timing of the annuity payments. We then can use the annuity formulae introduced above (where n represents the number of payments).

EXAMPLE

Find the present value as at 1 January 2013 of a series of payments of $100 payable on the first day of each month during 2014, 2015 and 2016 (ie. first payment made on 1 January 2014 and last payment made on 1 December 2016), assuming an effective rate of interest of 8% per annum.

Solution

The first thing to note is that the annuity is payable monthly, but our interest conversion period is yearly. i.e. the annuity is payable more frequently than interest is convertible.

One way to solve this problem is to work directly in units of 1-month, and convert the effective annual interest rate into an effective monthly interest rate, and then use the annuity formulae already introduced.

The effective monthly interest rate (periodic rate) is:

$$i = (1.08)^{1/12} - 1 = 0.006434$$

First find the present value of this annuity at 1 January 2014. Working in units of time of 1-month, the present value of $1 per month as at 1 January 2014 is:

$\ddot{a}_{\overline{36|}0.006434}$. (Note: the first payment is on 1 January 2014 so we use $\ddot{a}_{\overline{36|}}$, not $a_{\overline{36|}}$).

The present value at 1 January 2014 of $100 per month is:

$100 \cdot \ddot{a}_{\overline{36|}0.006434}$, where: $d = \dfrac{i}{1+i} = \dfrac{0.006434}{1.006434} = 0.006393$

Therefore, the present value as at 1 January 2013 is found by discounting this value back one year at 8%:

$$PV = 100 \cdot v_{0.08} \ddot{a}_{\overline{36}|0.006434} = 100 \cdot v_{0.08} \frac{1 - v_{0.006434}^{36}}{d} = \frac{100}{1.08}\left(\frac{1-(1.006434)^{-36}}{0.006393}\right) = \$2,986$$

Another way to solve this type of question is to formulate the solution in terms of annuities payable m times per annum. Below, we introduce algebraic expressions for these types of annuities, and then we use the new notation as an alternative way of solving the above example.

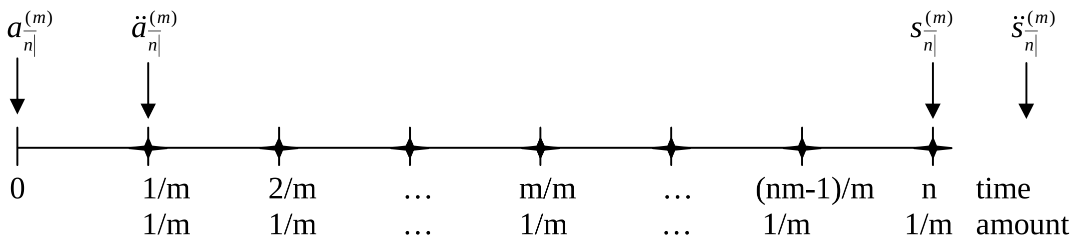
Consider a series of payments of $\frac{1}{m}$ each $\frac{1}{m}^{th}$ of a year for n years (total number of payments is nm).

The present value $\frac{1}{m}^{th}$ of a year before the first payment of this series of payments is:

$$\boxed{a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|}}$$

Where annuity payments are made m times per year, a superscript (m) is added in the top right hand corner of the symbol.

Since the annuity consists of m payments per annum of $\frac{1}{m}$, **the total annual amount paid is still 1 unit.**



indicates that a payment is made

Proof

$a_{\overline{n}|}^{(m)}$ is the present value of nm payments of $\frac{1}{m}$ units payable at times

$\frac{1}{m}, \frac{2}{m}, \ldots, \frac{nm-1}{m}, n$

This is equal to the sum of the present values of the individual payments:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}v^{1/m} + \frac{1}{m}v^{2/m} + \ldots + \frac{1}{m}v^{(nm-1)/m} + \frac{1}{m}v^{n}$$

Multiplying by m, then multiplying through by $(1+i)^{1/m}$ and subtracting gives:

$$ma_{\overline{n}|}^{(m)} = v^{1/m} + v^{2/m} + \ldots + v^{(nm-1)/m} + v^{n}$$
$$(1+i)^{1/m}\, ma_{\overline{n}|}^{(m)} = 1 + v^{1/m} + \ldots + v^{(nm-1)/m}$$
$$\Rightarrow ma_{\overline{n}|}^{(m)}\left[(1+i)^{1/m} - 1\right] = 1 - v^{n}$$

Therefore,

$$a_{\overline{n}|}^{(m)} = \frac{1-v^{n}}{m\left[(1+i)^{1/m} - 1\right]} = \frac{1-v^{n}}{i^{(m)}}$$

Other results (without proof) are given below.

The present value at the time of the first payment of a series of payments of $\frac{1}{m}$ each $\frac{1}{m}^{th}$ of a year for n years (total number of payments is nm) is:

$$\boxed{\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^{n}}{d^{(m)}} = \frac{i}{d^{(m)}}a_{\overline{n}|}}$$

The accumulated value at the date of the final payment of a series of payments of $\frac{1}{m}$ each $\frac{1}{m}^{th}$ of a year for n years (total number of payments is nm) is:

$$\boxed{s_{\overline{n}|}^{(m)} = \frac{(1+i)^{n} - 1}{i^{(m)}} = \frac{i}{i^{(m)}}s_{\overline{n}|}}$$

The accumulated value $\frac{1}{m}^{th}$ of a year after the date of the final payment of a series of payments of $\frac{1}{m}$ each $\frac{1}{m}^{th}$ of a year for n years (total number of payments is nm) is:

$$\boxed{\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^{n} - 1}{d^{(m)}} = \frac{i}{d^{(m)}}s_{\overline{n}|}}$$

EXAMPLE
This is the same as the previous example. Instead of solving this as we had done above, we can formulate the solution in terms of the new notation.

Find the present value as at 1 January 2013 of a series of payments of \$100 payable on the first day of each month during 2014, 2015 and 2016, assuming an effective rate of interest of 8% per annum.

Solution
Since payments are monthly, use the algebraic formulae just introduced where the number of payments per annum $m = 12$.

The present value of $\$\frac{1}{12}$ per month (\$1 per annum) in advance for three years as at 1 January 2014 is $\ddot{a}_{\overline{3}|}^{(12)} = \frac{1-v^3}{d^{(12)}}$.

Therefore, the present value of \$100 per month (\$1,200 per annum) is $1{,}200 \cdot \ddot{a}_{\overline{3}|}^{(12)}$, where

$$d^{(12)} = m \cdot \left[1-(1+i)^{-1/m}\right] = 12 \cdot \left[1-(1.08)^{-1/12}\right] = 0.076715$$

The present value as at 1 January 2013 is $1{,}200 \cdot v_{0.08} \ddot{a}_{\overline{3}|0.08}^{(12)}$

$$1{,}200 \cdot v_{0.08} \ddot{a}_{\overline{3}|0.08}^{(12)} = 1{,}200 \cdot v \frac{1-v^3}{d^{(12)}} = \frac{1{,}200}{1.08}\left(\frac{1-(1.08)^{-3}}{0.076715}\right) = \$2{,}986$$

EXAMPLE
On the last day of every March, June, September, and December, Smith makes a deposit of \$1,000 into a savings account. The first deposit is March 31, 1998 and the final one is December 31, 2013.

Find the balance in the account on the day of the final payment if the effective annual interest rate is 10%.

Solution
We will solve this using both methods introduced.

a) Since payments are quarterly, work in units of quarters (3-months). There are 4 quarters per year, and 16 years of payments between March 31, 1998 and the end of 2013. Therefore, there are 16 x 4 = 64 deposits made.

The three-month effective rate j is: $(1+j)^4 = 1.10 \Rightarrow j = 1.10^{1/4} - 1 = 0.02411369$.

The balance on 31 December 2013 (after the final payment) is:

$$1{,}000 \cdot s_{\overline{64|}j} = 1{,}000 \cdot \frac{(1+j)^{64} - 1}{j} = 1{,}000 \cdot \frac{(1.10^{1/4})^{64} - 1}{1.10^{1/4} - 1} = \$149{,}084$$

b) As an alternative to the method above we can work with nominal rates of interest and use $s_{\overline{n|}}^{(m)} = \dfrac{(1+i)^n - 1}{i^{(m)}}$.

The number of years in which deposits are made is $n = 16$, and \$4,000 is paid per annum. We need to find the annual nominal rate of interest convertible quarterly $i^{(4)}$:

$$i^{(4)} = m\left[(1+i)^{1/m} - 1\right] = 4\left[(1.1)^{1/4} - 1\right] = 0.096455$$

The accumulated value is then: $4{,}000 \cdot s_{\overline{16|}}^{(4)} = 4{,}000 \cdot \dfrac{(1.1)^{16} - 1}{4\left[(1.1)^{1/4} - 1\right]} = \$149{,}084$