Lecture 6

$$\left.\begin{array}{l} P(0) \\ \wedge (P(0) \to P(1)) \\ \wedge ([P(0) \wedge P(1)] \to P(2)) \\ \wedge ([P(0) \wedge P(1) \wedge P(2)] \to P(3)) \\ \wedge \cdots \end{array}\right\} \forall n \in \mathbb{N}, P(n)$$

$\Downarrow$

$\forall n \in \mathbb{N}, (\forall k \in \mathbb{N}, k < n \to P(k)) \to P(n)$

$\boxed{\{\forall n \in \mathbb{N}, n \geq 2 \to [(\forall k \in \mathbb{N}, 2 \leq k < n \to P(k)) \to P_n]\}}$ (*) $\to \forall n \in \mathbb{N}, n \geq 2 \to P(n)$

$$\left.\begin{array}{l} (T) \to P(2) \\ \wedge P(2) \to P(3) \\ \wedge (P(2) \wedge P(3)) \to P(4) \\ \wedge (P(2) \wedge P(3) \wedge P(4)) \to P(4) \end{array}\right\} \to P(2) \wedge P(3) \wedge P(4) \wedge \cdots$$

For $n \in \mathbb{N}$, let $P(n)$ be: n is a product of one or more primes.

Prove $\forall n \in \mathbb{N}, n \geq 2 \to P(n)$ by proving (*)

Let $n \in \mathbb{N}$

Assume $n \geq 2$

Assume k is a product primes, for each $k \in \mathbb{N}$ s.t. $2 \leq k < n$ (IH)

Case n is prime: $n = n$, which is a product of the one prime n.

Case n is not prime: let $a, b \in \mathbb{N}$, $a, b > 1$, s.t. $n = ab$

i.e. $a, b \geq 2$

$a = n/b < n$ since $b \geq 2$

$b = n/a < n$ since $a \geq 2$

$a, b \in \mathbb{N}$, $2 \leq a, b < n$, so by (IH) a is a product of primes, b is a product of primes

So $n = a \cdot b$ is a product of primes

Note: better to check each assumptions when doing proofs in cases.

Back to the dollar problem

For $n \in \mathbb{N}$, let $P(n)$ be: $\exists k, l \in \mathbb{N}, n = 3k + 5l$

We proved:

$\forall n \in \mathbb{N}, n \geq 8 \to P(n)$ by $P(8) \wedge P(9) \wedge P(10) \wedge \forall n \in \mathbb{N} [(n \geq 8 \wedge P(n)) \to P(n+3)]$

Proof by Complete/Strong Induction

Let $n \in \mathbb{N}$ $\to$ Inductive Step

Assume $n \geq 8$

Assume $P(i)$ for each $i \in \mathbb{N}$ s.t. $8 \leq i < n$ (IH)

Base Case:

- Case $n = 8$: $8 = 3 + 5$
- Case $n = 9$: $9 = 3 \times 3$
- Case $n = 10$: $10 = 5 \times 2$

Case $n \geq 11$: Then $n - 3 < n$ # descend

Also $n - 3 \geq 11 - 3$ (by Case)

$= 8$

And $n - 3 \in \mathbb{Z}$, in fact $n - 3 \in \mathbb{N}$, since $n \geq 8$

So by IH for $i = n-3$

let $k, l \in \mathbb{N}$ s.t. $n-3 = 3k+5l$

Then $n = (n-3)+3 < 3(k+1)+5l$

where $k+1, l \in \mathbb{N}$, since $k, l \in \mathbb{N}$