After 4 pm today, see the course website to find:
(1) whether "Portal" is made available, and has your term test marks
(2) what to bring to upcoming classes (examples).

Today Complementary slackness (end of §3.2)

Remark: If $a_1 \geq 0, \cdots, a_n \geq 0, b_1, \cdots, \geq b$ but $a_1b_1 + \cdots + a_nb_n \leq 0$, then $a_ib_i = 0$, for $i = 1, \cdots, n$ and $a_i = 0$ or $b_i = 0$ for $i = 1, \cdots, n$.

Complementary Slackness Theorem

Consider the following dual problem

Maximize $z = c^Tx$ s.t.

$$Ax \leq b$$
$$x \geq 0 \in \mathbb{R}^n$$

Minimize $z' = b^Tw$ s.t.

$$A^Tw \geq c$$
$$w \geq 0 \in \mathbb{R}^n$$

(A is $m \times n$)

and suppose x_0 & w_0 are optimal for the respective problems.

Then, for $i = 1, \cdots, m$, the product of of the slack, at x_0, in the ith primal constraint, and w_{i0} (the ith component of w_0) is 0.

Proof: After putting slack variables
$x' = \begin{bmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$ is the primal problem to put it in canonical form, its constraints read

$$Ax + x' = b, x \geq 0 \in \mathbb{R}^n, x' \geq 0 \in \mathbb{R}^n.$$

If $w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \in \mathbb{R}^m$, then

($\underbrace{\text{P}}_{\text{misprint}}$ 179)

$$w^TAx + w^Tx' = w^Tb$$

Taking transpose, $x^TA^Tw + (x')^Tw = b^Tw$

At the new respective optimal solutions, x_0 & w_0, $c^Tx_0 = b^Tw_0$ ($\underbrace{\text{strong}}_{\text{duality}}$).

So $c^Tx_0 = x_0^TA^Tw_0 + (x_0')^Tw_0$ (x_0' has the values of the primal slack variables at $x = x_0$).

By the feasibility of w_0 for the dual problem $A^T w_0 \geq c$ so $x_0^T(A^T w_0) \geq x_0^T c$

($x_0 \geq 0$, so $x_0^T(A^T w_0 - c) \geq 0 \in \mathbb{R}$).

So $x_0^T(A^T w_0) \geq c^T x_0$ ($x_0^T c$ is a 1×1 matrix)

Putting the circled expressions together

$c^T x_0 \geq c^T x_0 + (x_0')^T w_0$, so that $0 \geq (x_0')^T w_0$

so each of the m terms $(x_0')^T w_0$,

$$x_{i0} \cdot w_{i0} = 0$$

Eg. Consider the primal problem :

Maximize $z = -x_1 - 5x_2 - 7x_3 - 6x_4$ s.t.

$$-2x_1 + 2x_2 + x_3 + x_4 \geq 9$$

$$x_1 + x_2 - x_3 - 3x_4 = 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

Its dual is

(using Maximize "z" $= x_1 + 5x_2 + 7x_3 + 6x_4$)

Maximize $z' = 9w_1 + 2w_2$ s.t.

$$-2w_1 + w_2 \leq 1$$

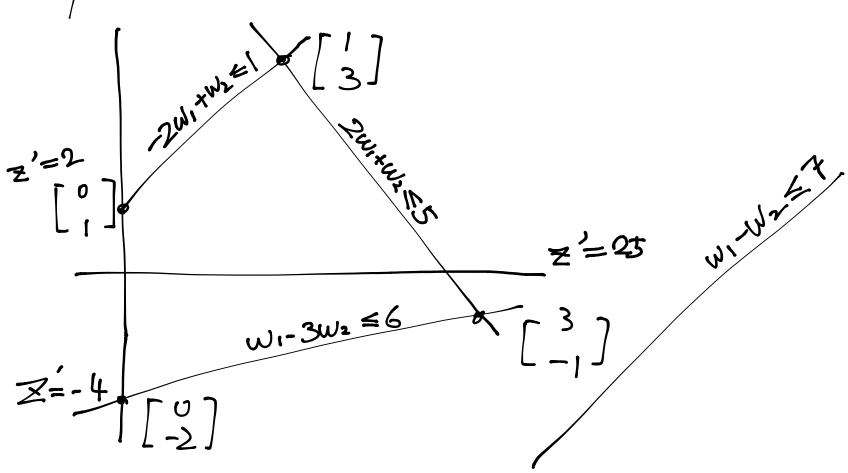
$$2w_1 + w_2 \leq 5$$

$$w_1 - w_2 \leq 7$$

$$w_1 - 3w_2 \leq 6$$

$w_1 \geq 0$, w_2 unrestricted

Graphical solution



To find primal optimality:

At dual optimality neither dual variable, so both primal constraints are tight (have no slack) at optimality

At dual optimality the 1st & 3rd dual constraints have slack, so $x_1=0$, $x_3=0$ at primal optimality.

So $\left.\begin{aligned} 2x_2+4x_4&=9 \\ x_2-3x_4&=2 \end{aligned}\right\}$ at primal optimal

Solving, get $x_2=\frac{29}{7}$, $x_4=\frac{5}{7}$ at primal optimality