UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2009 EXAMINATIONS

PLEASE HAND IN

MAT246H1F

Duration - 3 hours

No calculators, scrap paper, or other aids permitted

Examiner: Brooke Feigon

LAST NAME:_____

FIRST NAME:_____

STUDENT NUMBER:_____

NOTES

- There are ten questions, each of which is worth 10 marks.
- Before you start, check that this test has 12 pages, including this cover page.
- Explain and justify your work. If you need more space use the back of the page.
- DO NOT tear any pages from this test.

FOR MARKER ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
TOTAL	/100

1. If the numbers a_n are defined by $a_1 = 11$, $a_2 = 21$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$, prove that $a_n = 5 \cdot 2^n + 1$ for all natural numbers n.

2. What is the last digit in the ordinary decimal representation of

$$1 + 7 + 7^2 + 7^3 + \cdots + 7^{213}?$$

3. Suppose that a, b and m are natural numbers and that the greatest common divisor of a and m divides b (that is, $\gcd(a, m) \mid b$). Prove that there is a natural number x such that $ax \equiv b \bmod m$.

4. Find *all* integers x such that $0 \leq x \leq 96$ and

$$x^{483} \equiv x \bmod 97.$$

Prove that your answer is correct.

5. Suppose

$$a\sqrt{6} + b\sqrt{7} = c\sqrt{6} + d\sqrt{7}$$

with a, b, c and d rational. Show that $a = c$ and $b = d$.

6. Let $\mathbb{R}$ denote the set of all real numbers. What is the cardinality $(\aleph_0, \mathfrak{c}, 2^{\mathfrak{c}}, \dots)$ of the set of all functions from $\mathbb{R}$ to $\mathbb{R}$? Justify your answer.

7. Prove that if S is an infinite set and $\{a, b\} \subset S$, then $|S| = |S \backslash \{a, b\}|$, where $S \backslash \{a, b\} = \{s \in S : s \notin \{a, b\}\}$.

8. Let t be a transcendental number, that is, a real number that is *not* algebraic. Prove that t cannot be a root of any equation of the form $ax^2 + bx + c = 0$, where a, b and c are constructible numbers.

9. For each of the following numbers, state whether or not it is constructible and justify your answer.

   (a) $\cos 1°$

   (b) $\frac{\sqrt{2}-\sqrt[6]{7}}{5\sqrt{8}+1}$

   (c) a root of $2x^2 - \sqrt{99}x + \sqrt{7} = 0$

   (d) a root of $x^3 - x^2 + x + 1 = 0$

10. Does $2x^3 - x + \sqrt{2} = 0$ have a constructible root? Justify your answer.

(This page is intentionally left blank.)