SOLUTIONS TO PS4

CHEN

Problem 1. Solution. a) Denote $u_1 = (0, -i, 1)$ and $u_2 = (1+i, 2, 1)$. Then, let

$$\begin{aligned} v_1 &= u_1 \\ &= (0, -i, 1) \\ v_2 &= u_2 - \frac{< u_2, u_1 >}{< u_1, u_1 >} u_1 \\ &= (1+i, 2, 1) - \frac{< (1+i, 2, 1), (0, -i, 1) >}{< (0, -i, 1), (0, i, 1) >} (0, -i, 1) \\ &= (1+i, 1+\frac{i}{2}, \frac{1}{2} - i), \end{aligned}$$

so an orthonormal basis for W is $\{w_1, w_2\} = \{\frac{v_1}{|v_1|}, \frac{v_2}{|v_2|}\} = \{\frac{1}{\sqrt{2}}(0, -i, 1), \frac{\sqrt{2}}{3}(1 + i, 1 + \frac{i}{2}, \frac{1}{2} - i)\}$.

b)We will try to find the matrix of the projection with respect to the standard basis of C^3 and the above orthonormal basis of W. Compute
$\mathrm{Proj}_W(1,0,0) =< (1,0,0), w_1 > w_1 + < (1,0,0), w_2 > w_2 = \frac{\sqrt{2}}{3}(1-i)w_2$;
$\mathrm{Proj}_W(0,1,0) =< (0,1,0), w_1 > w_1 + < (0,1,0)), w_2 > w_2 = \frac{1}{\sqrt{2}} i w_1 + \frac{\sqrt{2}}{3}(1 - \frac{i}{2})w_2$;
$\mathrm{Proj}_W(0,0,1) =< (0,0,1)), w_1 > w_1 + < (0,0,1), w_2 > w_2 = \frac{1}{\sqrt{2}} w_1 + \frac{\sqrt{2}}{3}(\frac{1}{2} + i)w_2$;

So the matrix is $\begin{bmatrix} \frac{\sqrt{2}}{3}(1-i)w_2 & \frac{1}{\sqrt{2}} i w_1 + \frac{\sqrt{2}}{3}(1 - \frac{i}{2})w_2 & \frac{1}{\sqrt{2}} w_1 + \frac{\sqrt{2}}{3}(\frac{1}{2} + i)w_2 \end{bmatrix}$
Note that $\frac{\sqrt{2}}{3}(1-i)w_2$, $\frac{1}{\sqrt{2}} i w_1 + \frac{\sqrt{2}}{3}(1 - \frac{i}{2})w_2$ and $\frac{1}{\sqrt{2}} w_1 + \frac{\sqrt{2}}{3}(\frac{1}{2} + i)w_2$ are column vectors.

Problem 2. Solution.

Proof. For any $p(x), r(x), q(x) \in \mathbf{P}_n(\mathbb{C})$, $a, b \in \mathbb{C}$

$$\begin{aligned} 1) < (ap(x) + br(x), q(x) > &= \sum_{i=0}^{n} (ap(x_i) + br(x_i))\overline{q(x_i)} \\ &= \sum_{i=0}^{n} ap(x_i)\overline{q(x_i)} + \sum_{i=0}^{n} br(x_i)\overline{q(x_i)} = a < p(x), q(x) > + b < r(x), q(x) > \end{aligned}$$

2)

$$< p(x), q(x) > = \sum_{i=0}^{n} p(x_i)\overline{q(x_i)} = \overline{\sum_{i=0}^{n} \overline{p(x_i)} q(x_i)} = \overline{< q(x), p(x) >}.$$

3)

$$< p(x), p(x) > = \sum_{i=0}^{n} p(x_i)\overline{p(x_i)} = \sum_{i=0}^{n} |p(x_i)|^2 \geq 0,$$

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and the equality is achieved when $|p(x_i)|^2 = 0$ for $i = 0, \cdots, n$, which is equivalent to $p(x_i) = 0$ for $i = 0, \cdots, n$, namely, $p(x)$ has $n + 1$ distinct roots, then by the fundamental theorem of algebra, we have that $p(x) = 0$.

Problem 3. The question is still wrong.

The reason is as follows.

$< 3x^2-2x-1, cx^2+x-1 >= 0$ implies $c = 0$, while $< 3x^2-2x-1, sx^2+cx-9 >= 0$ implies $c = -7$.

Problem 4. Solution.

Take the standard basis $\{1, x, x^2\}$. Now let

$$\begin{aligned}
v_1 &= 1 \\
v_2 &= x - \frac{< x, 1 >}{< 1, 1 >} = x - i \\
v_3 &= x^2 - \frac{< x^2, 1 >}{< 1, 1 >} - \frac{< x^2, x - i >}{< x - i, x - i >}(x - i) = x^2 - 2ix - \frac{1}{3},
\end{aligned}$$

so an orthonormal basis for $\mathbf{P}_n(\mathbb{C})$ is

$$\{\frac{v_1}{|v_1|}, \frac{v_2}{|v_2|}, \frac{v_3}{|v_3|}\} = \{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}(x - i), \frac{\sqrt{3}}{\sqrt{2}}x^2 - 2ix - \frac{1}{3}\}.$$

Problem 5. Solution.

$(i) \Rightarrow (ii)$ by definition.

$(ii) \Rightarrow (iii)$ Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$, by the uniqueness in (ii), we see $w_1 = w_2 = \mathbf{0}$.

$(iii) \Rightarrow (iv)$ Denote $\alpha_1 = \{e_1, \cdots, e_k\}$, and $\alpha_2 = \{f_1, \cdots, f_l\}$, all we need to prove is that the vectors $e_1, \cdots, e_k, f_1, \cdots, f_l$ are linear independent. If not, we can find numbers $a_i, i = 1, \cdots, k$ and $b_j, j = 1, \cdots, l$ (among a_i and b_j at least one of the numbers doesnt equal to 0), such that

$$\sum_{i=1}^{k} a_i e_i + \sum_{j=1}^{k} b_j f_j = \mathbf{0},$$

by (iii)we have

$$\sum_{i=1}^{k} a_i e_i = \sum_{j=1}^{k} b_j f_j = \mathbf{0}.$$

Since $\alpha_1 = \{e_1, \cdots, e_k\}$ ($\alpha_2 = \{f_1, \cdots, f_l\}$) is a basis of W_1 (W_2), so $a_i = 0, i = 1, \cdots, k$, and $b_j = 0, j = 1, \cdots, l$ which is a contradiction. So $e_1, \cdots, e_k, f_1, \cdots, f_l$ are linear independent.

$(iv) \Rightarrow (i)$ We use the same notations as above. Since $\alpha_1 \cup \alpha_2$ is a basis for V, so for any vector $v \in V$, we can find numbers $a_i, i = 1, \cdots, k$ and $b_j, j = 1, \cdots, l$ so that $v = \sum_{i=1}^{k} a_i e_i + \sum_{j=1}^{k} b_j f_j$, with $\sum_{i=1}^{k} a_i e_i \in W_1$ and $\sum_{j=1}^{k} b_j f_j \in W_2$.

Now suppose $u \in W_1 \cap W_2$, we can find numbers $\tilde{a}_i, i = 1, \cdots, k$ and $\tilde{b}_j, j = 1, \cdots, l$ so that

$$u = \sum_{i=1}^{k} \tilde{a}_i e_i = \sum_{j=1}^{k} \tilde{b}_j f_j,$$

the second equality implies

$$\sum_{i=1}^{k}\tilde{a}_i e_i - \sum_{j=1}^{k}\tilde{b}_j f_j = 0.$$

Now by the linear independence of the vectors $e_1, \cdots, e_k, f_1, \cdots, f_l$, we have $\tilde{a}_i = 0, i = 1, \cdots, k$, and $\tilde{b}_j = 0, j = 1, \cdots, l$, namely, $u = \mathbf{0}$. So $W_1 \cap W_2 = \{\mathbf{0}\}$.

Problem 6. Solution.

Proof. For any $A \in M_{n\times n}(R)$, let $A_1 = \frac{A+A^T}{2}$, $A_2 = \frac{A-A^T}{2}$. Notice that $A_1^T = \frac{(A+A^T)^T}{2} = A_1$, so $A_1 \in W_1$. Similarly one can check that $A_2 \in W_2$. Now we need only to prove $W_1 \cap W_2 = \{\mathbf{0}\}$.

Suppose $B \in W_1 \cap W_2$, we have $B = B^T = -B^T$, which means $B^T = \mathbf{0}$.

Problem 7. Solution.

Proof. $\Rightarrow$ Since T is diagonalizable, by definition, we have a basis $\{e_1, \cdots, e_n\}$ so that each of them is an eigenvector of T. Since T has only two distinct eigenvalues λ_1 and λ_2, we have that e_i either belongs to E_{λ_1} or belongs to E_{λ_2}, for all i. This means $E_{\lambda_1} + E_{\lambda_2} = V$. Then one needs only to show that $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$.

Assume $\mathbf{0} \neq v \in E_{\lambda_1} \cap E_{\lambda_2}$, we have $Tv = \lambda_1 v = \lambda_2 v$, since $\lambda_1 \neq \lambda_2$, this implies $v = \mathbf{0}$, which is a contradiction. So $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$.

$\Leftarrow$ Denote $\{e_1, \cdots, e_k\}$ as a basis of E_{λ_1} and $\{f_1, \cdots, f_l\}$ as a basis of E_{λ_2}. All we need to check is that $\{e_1, \cdots, e_k, f_1, \cdots, f_l\}$ forms a basis of V. And the only thing needs to be checked is that they are linearly independent. If not, we can find numbers $a_i, i = 1, \cdots, k$ and $b_j, j = 1, \cdots, l$ (among a_i and b_j at least one of the numbers doesnt equal to 0), so that

$$\sum_{i=1}^{k} a_i e_i + \sum_{j=1}^{k} b_j f_j = \mathbf{0}. \tag{0.1}$$

Then,

$$\mathbf{0} = T\left(\sum_{i=1}^{k} a_i e_i + \sum_{j=1}^{k} b_j f_j\right) \tag{0.2}$$

$$= \lambda_1 \sum_{i=1}^{k} a_i e_i + \lambda_2 \sum_{j=1}^{k} b_j f_j. \tag{0.3}$$

Since λ_1 and λ_2 are distinct, by (0.1) and (0.3), we can solve

$$\sum_{i=1}^{k} a_i e_i = \sum_{j=1}^{k} b_j f_j = \mathbf{0},$$

which implies $a_i = 0, i = 1, \cdots, k$, and $b_j = 0, j = 1, \cdots, l$. This is a contradiction, so $e_1, \cdots, e_k, f_1, \cdots, f_l$ are linearly independent.

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