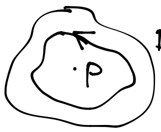
Lecture 10

MORE ABOUT CAUCHY FORMULA

THM: If $f: D \to \mathbb{C}$ is analytic, $\gamma$ simple closed curve so that inside of $\gamma$ is in D.




$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}\,dz = f(p) \text{ if } p \text{ inside } \gamma.$$

## Trig Integrals

$\int_0^{2\pi}(\text{trig function})\,d\theta \longrightarrow$ is actually a line integral

If $z = e^{i\theta}$ then $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$

$$\sin\theta = \frac{1}{2i}(z-\frac{1}{z})$$

$$d\theta = \frac{1}{i}\frac{dz}{z}$$

(How to get this?)

$$z = e^{i\theta}$$
$$\frac{dz}{d\theta} = ie^{i\theta}$$
$$dz = ie^{i\theta}d\theta$$
$$\frac{dz}{iz} = \frac{dz}{ie^{i\theta}d\theta} = d\theta \quad ✓$$

Ex: Find $\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}$ $\qquad \gamma$ = unit circle

$$= \int_\gamma \frac{\frac{1}{i}\frac{dz}{z}}{2+\frac{1}{2i}(z-\frac{1}{z})}$$

$$= \int_\gamma \frac{1}{i}\frac{1}{z}\frac{2i\,dz}{4i+(z-\frac{1}{z})}$$

$$= \int_\gamma \frac{2\,dz}{4iz+z^2-1}$$

$$= 2\int \frac{dz}{z^2+4iz-1}$$

recall: Cauchy Formula up there

$$= 2\int \frac{dz}{\underbrace{(z-(-2+\sqrt{3})i)}_{z-p}\underbrace{(z+(2+\sqrt{3})i)}_{f}}$$

$z^2+4iz-1=0$

$$z = \frac{-4i \pm\sqrt{-16+4}}{2} = \frac{-4i\pm 2\sqrt{3}i}{2}$$
$$= -2i \pm\sqrt{3}i$$
$$= (-2\pm\sqrt{3})i$$

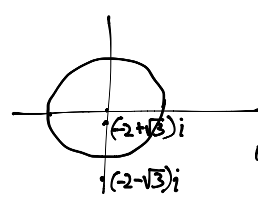
$$= 2\int f(p)\cdot 2\pi i$$

where $f(z) = \frac{1}{z+(2+\sqrt{3})i}$

$$= 2\frac{1}{(-2+\sqrt{3})i+(2+\sqrt{3})i}\cdot 2\pi i$$

$$= \frac{4\pi}{2\sqrt{3}}$$

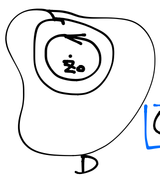
$$= \frac{2\pi}{\sqrt{3}}$$




one inside $\gamma$, one outside

THM: Suppose $f$ is analytic in $D$, $z_0 \in D$. Then we can find a disk $D_R(z_0)$ & a power series representation of $f$. $f(z)=\sum a_k(z-z_0)^k$ which cvges absolutely in $D_R(z_0)$.

Moreover, $a_k$ is given by $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}dz$ where $\gamma$ is a simple closed curve around $z_0$




(k=0 is usual Cauchy Formula)

Ex: Find $\int_\gamma \frac{\cos z}{z^{31}}dz$, $\gamma$=unit circle

$(z_0=0, k+1=31, k=30) \Rightarrow \int_\gamma \frac{\cos z}{z^{31}}dz = 2\pi i\, a_{30} = \frac{-2\pi i}{30!}$

$\cos z = \sum \frac{(-1)^n z^{2n}}{(2n)!}$

$\hookrightarrow a_{30} = \frac{(-1)^{15}}{30!} = \frac{-1}{30!}$

Pf: (Easy Part) Suppose $f(z)=\sum_{k=0}^\infty a_k(z-z_0)^n$ converges in $D_R(z_0)$:

$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^k}dz = \frac{1}{2\pi i}\int_\gamma \left(\sum_{n=0}^\infty \frac{a_n(z-z_0)^n}{(z-z_0)^k}\right)dz = \frac{1}{2\pi i}\int_\gamma \sum \frac{a_n}{(z-z_0)^{k-n+1}}dz$

$= \frac{1}{2\pi i}\sum_{n=0}^\infty \int_\gamma \frac{a_n}{(z-z_0)^{k-n+1}}dz \Rightarrow = 0$ unless $k-n+1=1$

$= \frac{1}{2\pi i} a_k \cdot 2\pi i \quad (n=k)$

$= a_k$

$$\boxed{\int_\gamma \frac{1}{(z-z_0)^m}dz = \begin{cases} 2\pi i, & m=1 \\ 0 & \text{o.w.}\end{cases}}$$

(Hard Part). Show such series converges. (skip)

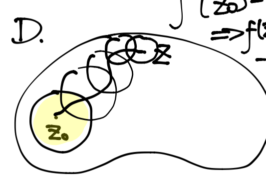
COR: If $f$ is analytic, then so is $f'$. Hence $f$ is infinitely diff.

Ex: Not true for real diff. functions.

$f(x) = \begin{cases} -x^2 & x<0 \\ x^2 & x\geq 0\end{cases}$

$f'(x) = 2|x|$

Pic IDEA $\Rightarrow$




$f^{(k)}(z_0)=0 \Rightarrow a_k=0$
$\Rightarrow f(z)=0$ in $D_R(z_0)$
Then do this trick

COR: Suppose $f$ is analytic in $D$ & for some point $z_0 \in D$ we have $f^{(k)}(z_0)=0$ for all $k$. Then $f(z)=0$ for all $z \in D$.

$\boxed{a_k = \frac{f^{(k)}(z_0)}{k!}}$