TUTORIAL 3

(1) Consider the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x-(1/2)}{x} \text{ if } x \leq 1/2 \\ \frac{x-(1/2)}{1-x} \text{ if } x \geq 1/2 \end{cases}$$

(a) Sketch a graph of f
(b) Prove that f gives a bijection from $(0, 1)$ to $\mathbb{R}$.

(2) Let $\mathbb{I} := \mathbb{R}/\mathbb{Q}$ denote the set of irrational numbers. Prove that $\mathbb{I}$ is uncountable. (Hint: You may find it easier to prove the more general statement that if A is an uncountable set, and $B \subset A$ is a countable subset, then $A\backslash B$ is uncountable.)

(3) (a) Prove that $\mathbb{R}^n$ has the same cardinality as $\mathbb{R}$ for any $n \geq 1$.
(b) Prove that $\mathbb{R}^\infty$ has the same cardinality as $\mathbb{R}$. Here we define $\mathbb{R}^\infty$ to be the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{R}\}.$$

Just for fun.

(1) Show that it is possible to decompose $\mathbb{R}$ into a disjoint union of uncountably many uncountable subsets.