MVT for integrals $\boxed{4.24}$

average value of $f(x)$ weighted by a density function $g(x)$ is achieved at some $a \in S$

$$\frac{\iint_S f(x) g(x)\,dx}{\iint_S g(x)\,dx} = f(a)$$

Let $S \subset \mathbb{R}^n$ be <u>Compact</u>, <u>Connected</u>, <u>measurable</u>, f and g be <u>Cont</u> <u>on S</u>

$g(x) \geq 0$ on S. Then

$\exists\, a \in S$ st. $\iint_S f(x) g(x)\,dA = f(a) \iint g(x)\,dA$

in $\mathbb{R}^1$:

$$\frac{\int_a^b f(x)\,dx}{b-a} = f(c) \quad \text{some } c \in [a,b]$$

or

$$\frac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx} = f(c) \quad c$$

MVT was:

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$\frac{\text{rise}}{\text{run}}$ = average rise

f is Cont, S is cpt $\Rightarrow$ $\exists\, d_1 \in S$ $\quad m = \min$ of $f(x)$ on S $= f(d_1)$

$\exists\, d_2 \in S$ $\quad M = \max$ " $= f(d_2)$

<u>Compact</u> EVT 1.23

or $m \leq f(x) \leq M$ or $f(d_1) \leq f(x) \leq f(d_2)$ for all $x \in S$

$g(x) \geq 0$ on S $\Rightarrow$ $m\, g(x) \leq f(x) g(x) \leq M\, g(x)$

<u>S is measurable</u>

f, g are Cont $\therefore$ integrable

4.17 (c) $\quad \iint_S m\, g(x)\,dA \leq \iint_S f(x) g(x)\,dA \leq \iint_S M\, g(x)\,dA$

4.17 (a) $\quad m \iint_S g(x)\,dA \leq \iint_S f g\,dA \leq M \iint g\,dA$

So assume $\iint g\,dA \neq 0$

$$f(d_1) = m \leq \frac{\iint_S fg\,dA}{\iint_S g\,dA} \leq M = f(d_2)$$

intermediate value

by IVT 1.27 <u>Connected</u> $\Rightarrow \exists\, a \in S$ st. $f(a) =$

if $g(x) = 0$ $\forall x \in S$ Then $\iint_S g\,dA = 0$ so equality in MVT holds.

assume $g(x_0) > 0$ for some $x_0 \in S$, (A)

Then $\exists\, \delta > 0$ st. $\forall x \in S$ $\quad x \in B(\delta, x_0) \Rightarrow$

$|g(x) - g(x_0)| < \frac{A}{3}$ (ϵ)

so $\iint_{B(\delta, x_0)} g\,dA > \frac{A}{3}$ volume of $B(\delta, x_0) > 0$

Applications: 1. if $g(x)$ is a probability density function on S, Then $\iint_S g\,dA = 1$ ($g \geq 0$) so $\iint_S f(x) g(x)\,dA = f(a)$ $= E[f] = f(a)$

2. proof of 5.60

$$\frac{1}{h}\int_0^h G_1(x+t, y)\,dt = G_1(x+c, y) \quad 0 \leq c \leq h$$

& as $h \to 0$ $\quad G_1(x+c, y) \to G_1(x, y)$