§1 - Introduction to Topologies

1 Some Motivating Questions

Here are some things to chew on:

- What does the "$\longrightarrow$" mean in the sentence "$x_n \longrightarrow x$"?
- What does it mean to say that two sets of points are close to each other? Is $(-\infty, 0)$ close to $(0, +\infty)$? Is π close to $\mathbb{Q}$?
- When are two shapes or figures fundamentally the same? How can we distinguish two different spaces? Are $\mathbb{R}$ and $\mathbb{R}^2$ really the same space? *Why?*
- How can we describe mathematically the (obvious) fact that a torus (a doughnut) has a hole, but a sphere doesn't?
- What are some useful mathematical notions of "largeness" or "smallness"? Is $\mathbb{R}$ small? Is $\mathbb{Q}$ small? Is $\mathbb{N}$ small? Is $[0, 1]$ small?

Throughout this course we will answer all of those questions (and more!). Most of these questions involve some notion of distance or convergence. A topology will be a structure on a set that allows us to describe various convergence properties. In many ways a topology will act as an abstract version of distance. It will help us to determine when points and sets are "very close" to each other. We will make this precise in a moment. First let's remember some things we learned about $\mathbb{R}^2$ in analysis.

2 Open Sets in $\mathbb{R}^2$

In analysis we learned about open sets in $\mathbb{R}^2$, the set of all points in the plane. It had to do with ϵ-balls $B_\epsilon(x) := \{\, y \in \mathbb{R}^2 : d(x, y) < \epsilon \,\}$, where $\epsilon > 0$. Recall that in $\mathbb{R}^2$,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \text{where } x = (x_1, x_2), y = (y_1, y_2).$$

An ϵ-ball is just the set of points within distance ϵ of x.

From there we describe what an open set in $\mathbb{R}^2$ is:

Definition. *A set $U \subseteq \mathbb{R}^2$ is open if for every point $x \in U$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.*

(Naively: Once we wrapped our heads around this, we realized that it *basically* just amounts to "a blob is open if it doesn't contain its boundary".)

From here we realized that open sets "play nicely" with each other, and we proved the following proposition:

Proposition. *i. The empty set, $\emptyset$, and $\mathbb{R}^2$ are each open.*

ii. The intersection of finitely many open sets ($U_1 \cap \cdots \cap U_N$) is open.

iii. The union of arbitrarily many open sets is open.

If you go through the proof of these facts you will see that [i] and [iii] don't really use anything special about $\mathbb{R}^2$. For part [ii] we needed to use something about being able to take the minimum of two real numbers. This suggests that these properties don't *heavily* rely on the distance function from $\mathbb{R}^2$. If we step back and just look at the proposition we see that it doesn't mention *anything* about the Euclidean distance function on $\mathbb{R}^2$.

It turns out that many theorems about $\mathbb{R}^2$ that are about open sets, closed sets or connected sets are still true if we don't use the fact that $\mathbb{R}^2$ has a distance function and instead only use the fact that the previous proposition is true. Assuming that a space has a distance function is often an unneeded assumption, and often it will be difficult (or even impossible!) to define such a distance. (For example, is there a natural distance function on the sphere? What about on the Möebius band? What about the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?)

This course will be devoted to proving general facts about spaces that follow from that proposition. In fact, this proposition leads to the main definition of the course, and surprisingly provides enough structure to describe meaningful phenomena.

3 Topologies!

Definition. *Let X be a set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is* ***a topology on*** *X if the following three properties are all satisfied:*

i. Both $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

ii. If $U_1, U_2, \ldots, U_N$ are finitely many sets all in $\mathcal{T}$, then the intersection $U_1 \cap \cdots \cap U_N \in \mathcal{T}$. (We say that a topology is ***closed under finite intersections****.)*

iii. If $\{ U_\alpha : \alpha \in I \} \subseteq \mathcal{T}$ is a family of subsets of $\mathcal{T}$ (indexed by the index set I), then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$. (We say that a topology is ***closed under arbitrary unions****.)*

Here we will say that any set $U \in \mathcal{T}$ is an ***open*** *set. We also say that $(X, \mathcal{T})$ is a* ***topological space****.*

In this course we will see many examples of topological spaces, and in fact, most of our work will be in expressing various geometrical objects and phenomena in the language of open sets. We will see that we can express the following things by only refering to open sets:

- Continuous functions;

- Compact spaces;
- Connected Spaces.

We should look at some examples. Some of these are natural examples that you will see "in the wild", and some are contrived to help you understand what a topology is. (It's like the difference between a program that generates random numbers and a "Hello World!" program.)

(CiT stands for Counterexamples in Topology.)

Example 1 (Example 28 in CiT)**.** *Let $X = \mathbb{R}$ and let*

$$\mathcal{T}_{usual} := \{\, U \subseteq \mathbb{R} : \forall x \in U, \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U \,\}.$$

*We refer to $(X, \mathcal{T}_{usual})$ as **the reals with the usual topology** or **the reals with the Euclidean topology**. We will often write $\mathbb{R}_{usual}$ to represent this space.*

It isn't surprising that this is an example of a topological space, as this space is what we modeled our definition after.

Example 2 (Example 1 in CiT)**.** *Let X be any non-empty set, and let*

$$\mathcal{T}_{discrete} := \mathcal{P}(X), \text{ the collection of all subsets of } X.$$

Then $\mathcal{T}_{discrete}$ is called the discrete topology on X.

Boring Exercise: Check (in your head at least) that $\mathcal{T}_{\text{discrete}}$ is a topology on X.

The discrete topology looks kind of silly, but it turns out to be extremely useful. We will often see this space showing up naturally in our discussions.

Thought Exercise: Try to justify the name "Discrete Topology" by explaining how $(\,\mathbb{N}, \mathcal{T}_{\text{discrete}}\,)$ relates to $(\,\mathbb{R}, \mathcal{T}_{\text{usual}}\,)$. (Thinking about this will help you with the concept of a **subspace** that shows up soon.)

Prior Knowledge Exercise: If you have taken a course that involved metric spaces, try to justify how the Discrete Metric on a set X is related to $(X, \mathcal{T}_{\text{discrete}})$. (Hint: Look at how we defined $(\mathbb{R}, \mathcal{T}_{\text{usual}})$. This exericse will help you with the concept of **a topology generated by a metric** which shows up later in the course.)

Example 3 (Example 4 in CiT)**.** *Let X be any set (with at least 2 points), let*

$$\mathcal{T}_{indiscrete} := \{\, \emptyset, X \,\}.$$

*This topology is called the **indiscrete topology on** X.*

This topology is *very* silly, and is basically just used to show students an example of a topology. It *never* shows up in "real-life" mathematics and it is very badly behaved. Some people think of it a little like a swear word. If someone says: "Can you think of a topological space that has 'property X' but not 'property Y'?" and you reply with "How about $\mathbb{R}$ with the indiscrete topology?!" they might just squint their eyes and grimace at you. If at all possible, try to avoid using this space as a counterexample! (In this course, if you really need to use it then go ahead, this course is a safe place!)

Example 4 (Example 18 in CiT, Exercise C.3 on Assignment 1)**.** *Let X be a set and let*

$$\mathcal{T}_{\textit{co-finite}} := \{\, A \subseteq X : A = \emptyset \textit{ or } X \setminus A \textit{ is finite} \,\}.$$

This is called the ***co-finite topology on*** *X.*

The co-finite topology is only *somewhat useful*, but it does hint at something we will look at later, (the co-compact topology). On the first assignment you will need to show that this is indeed a topology, (to do that you will need to check the three defining properties of a topology).

The next topology is also *fairly useless*, but it is being presented as an extension of the previous example.

Example 5 (Example 20 in CiT)**.** *Let X be a set and let*

$$\mathcal{T}_{\textit{co-countable}} := \{\, A \subseteq X : A = \emptyset \textit{ or } X \setminus A \textit{ is countable} \,\}.$$

This is called the ***co-countable topology on*** *X.*

Your proof that the co-finite topology on a set is really a topology will also basically show that the co-countable topology is really a topology.

Co-property Exercise: For fun, let us define a bizarre topology based on some fixed property ϕ that a set could have. Here ϕ could be the property "finite" or "countable" or "is an interval" or something else, but we will fix it for now. Let

$$\mathcal{T}_{\text{co-}\phi} := \{\, A \subseteq X : A = \emptyset \text{ or } X \setminus A \text{ has property } \phi \,\}.$$

When is this a topology?

4 Comparing Topologies

If we have two topologies $\mathcal{T}$ and Γ on the same space X it is sometimes useful to determine how $\mathcal{T}$ and Γ relate to each other. We sometimes want to know: "Does $\mathcal{T}$ refine Γ?".

Definition 6. *Let $\mathcal{T}, \Gamma$ both be topologies on a set X. We say that $\mathcal{T}$* ***refines*** *Γ if $\mathcal{T} \supseteq \Gamma$, i.e. $\mathcal{T}$ contains more open sets than Γ. Here we might also say that Γ* ***is refined by*** *$\mathcal{T}$.*

Example 7. *1.* $(\mathbb{R}, \mathcal{T}_{discrete})$ *refines* $(\mathbb{R}, \mathcal{T}_{usual})$.

2. $(X, \mathcal{T}_{co\text{-}finite})$ *is refined by* $(X, \mathcal{T}_{co\text{-}countable})$.

Refining Exercise: Which topology that we have discussed refines *every* other topology on the same space X? Which topology is refined by every other topology?

Directional Exercise: Is what I said above about the co-countable and co-finite topologies correct? That is, did I get the implication right or should it go in the opposite direction? Convince yourself on at least two different days that you know which way the implication goes.

5 More Topologies

We have seen some examples of topologies, but you should be unsatisfied. I gave you two "interesting" topologies ($\mathbb{R}$ with the usual topology, and discrete topological spaces) and a slew of contrived examples. The reason for this is that it is often difficult (or impossible!) to list out all of the open sets in a topology. Even for $\mathbb{R}$ with the usual topology our description of the open sets is a bit ... weird.

Our next task is then to provide a method for describing topologies. We will actually only describe *some* of the ("basic") open sets of a given topology and the rest of the open sets will come from unions of "basic" open sets.

Prepatory Exercise: Let $\mathcal{B} := \{\{x\} : x \in \mathbb{R}\}$ be the set of singletons in $\mathbb{R}$. If $\mathcal{T}$ is a topology on $\mathbb{R}$ such that $\mathcal{B} \subseteq \mathcal{T}$, then prove that $\mathcal{T}$ must be the discrete topology. (Use the fact that topologies are closed under arbitrary unions. This exercise will help you understand what a **basis for a topology** is. It also might help with NI.2 on assignment 1.)

6 Summary of Exercises

These exercises aren't for submission, but are useful for understanding.

Boring : Check (in your head at least) that $\mathcal{T}_{\text{discrete}}$ is a topology on X.

Thought Exercise : Try to justify the name "Discrete Topology" by explaining how $(\mathbb{N}, \mathcal{T}_{\text{discrete}})$ relates to $(\mathbb{R}, \mathcal{T}_{\text{usual}})$.

Knowledge Exercise : If you have taken a course that involved metric spaces, try to justify how the Discrete Metric on a set X is related to $(X, \mathcal{T}_{\text{discrete}})$.

Co-property : Fix a property ϕ that a set could have (like "is finite" or "is countable" or "is an interval"). Let $\mathcal{T}_{\text{co-}\phi} := \{\, A \subseteq X : A = \emptyset \text{ or } X \setminus A \text{ has property } \phi \,\}$. What conditions must ϕ have so that this is a topology?

Refining : Which topology that we have discussed refines *every* other topology on the same space X? Which topology is refined by every other topology?

Directional : Is what I said about the co-countable and co-finite topologies correct? Did I get the implication right or should it go in the opposite direction? Convince yourself on at least two different days that you know which way the implication goes.

Prepatory : Let $\mathcal{B} := \{\, \{x\} : x \in \mathbb{R} \,\}$ be the set of singletons in $\mathbb{R}$. If $\mathcal{T}$ is a topology on $\mathbb{R}$ such that $\mathcal{B} \subseteq \mathcal{T}$, then prove that $\mathcal{T}$ must be the discrete topology.