/52 Problem Set 2 Solutions

1a) For $f(x) = \sqrt{x}$, $x \in [0,1]$

$$|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}| \quad \text{via triangle}$$

$$\Rightarrow |\sqrt{x}-\sqrt{y}|\,|\sqrt{x}-\sqrt{y}| \leq |\sqrt{x}+\sqrt{y}|\,|\sqrt{x}-\sqrt{y}|$$

$$\Rightarrow |\sqrt{x}-\sqrt{y}|^2 \leq |\sqrt{x}^2 - \sqrt{y}^2| = |x-y|$$

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$$\Rightarrow |\sqrt{x}-\sqrt{y}| \leq 1 \cdot |x-y|^{1/2}$$

$\Rightarrow$ f(x) is Hölder Continuous with $\lambda = 1/2$, $C = 1$

Exercise 1 $\Rightarrow$ $\sqrt{x}$ is uniformly continuous
$\Rightarrow$ $\sqrt{x}$ is continuous ∎

b) Consider $f(x) = \begin{cases} \frac{1}{\ln(x)} & 0 < x \leq 1/2 \\ 0 & 0 \end{cases}$

Suppose f(x) was Hölder Continuous for some $\lambda > 0$ at $x = 0$.

ie $\exists C > 0$ so $\left|0 - \frac{1}{\ln x}\right| \leq C|x|^\lambda$

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$\Rightarrow 1 \leq |x|^\lambda |\ln x|$. But $\lim_{x \to 0^+} |x|^\lambda |\ln x| = 0 \Rightarrow \in$

$\therefore$ Not Hölder Continuous on $[0, 1/2]$ as property doesn't hold at $x = 0$

However, f(x) is continuous at 0 as $\lim_{x \to 0^+} \frac{1}{\ln x} = 0$

and f(x) is uniformly continuous as it is continuous on a compact interval via 1.33 ∎

d) Let $f : [0, \infty) \to \mathbb{R}$ be continuous & $\lim_{x \to \infty} f(x) = L$ exists.

Let $\varepsilon > 0$

That is, $\exists\, M \geq 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \text{ when } x \geq M.$$ (If you prefer $x > M$, then use $x > M+1$ instead)

Then $[0, \infty) = [0, M] \cup (M, \infty)$

As $[0, M]$ is compact, Theorem 1.33 $\Rightarrow$ $f(x)$ is uniformly continuous on $[0, M]$.

ie $\forall \varepsilon > 0\ \exists\, \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{2}$ when $|x - y| < \delta$.

Choose $x, y \in [0, \infty)$. Clearly if $x, y \in [0, M]$ or $x, y \in (M, \infty)$

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we get $|f(x) - f(y)| < \varepsilon$ when $|x - y| < \delta$.

Suppose $x \in [0, M]$, $y \in (M, \infty)$, $|x - y| < \delta$.

Then $|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)|$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

As $|x - M| \leq |x - y|$ (as $x, y, M > 0$, $x \leq M \leq y$)

& $y \geq M$

Hence f is uniformly continuous on $[0, \infty)$

3) Basis: $n=1$: for $f(x)=x^1$, $f(x+h)=x+h$

$= f(x) + 1\cdot h + E(h)$

and $E(h)=0$ hence $\lim_{h\to 0} \frac{E(h)}{h} = 0$

so $f'(x)=1$ ✓

Assume $(x^n)' = nx^{n-1}$. Let $f(x) = x^{n+1}$

$$f(x+h) = (x+h)^{n+1} = (x+h)(x+h)^n$$
$$= (x+h)\left[x^n + nx^{n-1}h + E(h)\right]$$

s.t. $\lim_{h\to 0} \frac{E(h)}{h} = 0$, via induction hypothesis.

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so $$f(x+h) = x^{n+1} + hx^n + nx^n h + nx^{n-1}h^2 + (x+h)E(h)$$

$$= x^{n+1} + (n+1)x^n h + E_2(h)$$

where $E_2(h) = nx^{n-1}h^2 + (x+h)E(h)$

has $\lim_{h\to 0} \frac{E_2(h)}{h} = 0$.

$\therefore$ $(x^{n+1})' = (n+1)x^n$. The result follows via induction ∎

Hilroy

4a) Consider $f(x) = x\sin\frac{1}{x}$ if $x \neq 0$, $f(0) = 0$

- For $x \neq 0$, clearly continuous (composition of cont. fcns)

- For $x = 0$, $0 \leq |x\sin\frac{1}{x}| \leq |x|$ as $|\sin x| \leq 1$

and $\lim_{x\to 0} |x| = 0$ thus $\lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$

so f is continuous on $\mathbb{R}$.

- For $x \neq 0$, $f'(x) = \sin\frac{1}{x} + x\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

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which is continuous at $x \neq 0 \Rightarrow f$ is diff at $x \neq 0$

- At $x = 0$ $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ which does not exist.

So f not diff at 0.

b) For $g(x) = xf(x) = x^2\sin\left(\frac{1}{x}\right)$ if $x \neq 0$, $g(0) = 0$

- $g'(x) = 2x\sin\frac{1}{x} + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

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Continuous for $x \neq 0 \Rightarrow$ diff for $x \neq 0$.

- If $x = 0$,

$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

$\Rightarrow g'(0) = 0$ so g diff $\forall x \in \mathbb{R}$.

4c) $h(x) = x^2 \sin \frac{1}{x} + \frac{x}{2}$, $x \neq 0$, $h(0) = 0$

$h'(0) = g'(0) + \frac{1}{2} = \frac{1}{2} > 0$

But $\forall x \neq 0$ $h'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} + 1/2$

4 as $x \to 0$, $2x \sin \frac{1}{x} \to 0$ so $h'(x) \to \frac{1}{2} - \cos \frac{1}{x}$

which oscillates between $-1/2$ & $3/2$

so not increasing on any neighborhood of 0.

5a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ have $f(x,y) = \frac{xy^2}{x^2+y^2}$ $(x,y) \neq (0,0)$

$f(0,0) = 0$

- Clearly cont for $(x,y) \neq (0,0)$
- Method A: $|xy| \leq \frac{1}{2}|x^2+y^2|$

$$|f(x,y)| \leq \frac{|y|}{2} \frac{|\cancel{x^2+y^2}|}{|\cancel{x^2+y^2}|} = \frac{|y|}{2} \to 0 \text{ as } (x,y) \to 0$$

5 Method B: using $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r^3 \cos\theta \sin^2\theta}{r^2} = \lim_{r \to 0} r \cos\theta \sin\theta = 0 = f(0,0)$$

Either way, f is continuous at 0 so continuous $\forall x \in \mathbb{R}$.

b) For $(x,y) \neq (0,0)$, $\partial_x f(x,y) = \frac{y^2(x^2+y^2) - 2x(xy^2)}{(x^2+y^2)^2} = \frac{y^4 - x^2y^2}{(x^2+y^2)^2}$

$$\partial_y f(x,y) = \frac{2xy(x^2+y^2) - 2y(xy^2)}{(x^2+y^2)} = \frac{2x^3y}{x^2+y^2}$$

Both continuous at $(x,y) \neq (0,0)$ so f diff at $(x,y) \neq (0,0)$ via 2.19.

At $(x,y) = (0,0)$,

$$\partial_x f(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\partial_y f(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

$\therefore \nabla f(0,0) = (0,0)$

Now $\displaystyle\lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0) - \nabla f(0,0) \cdot (x,y)}{\sqrt{x^2+y^2}}$

$$= \lim_{(x,y) \to (0,0)} \frac{xy^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \lim_{r \to 0} \frac{r^3 \cos\theta \sin^2\theta}{r^3} = \lim_{r \to 0} \cos\theta \sin^2\theta$$

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which does not exist as θ not determined.

$\therefore$ f not differentiable at $(0,0)$

c) Let $u = (u_1, u_2)$ be a unit vector i.e. $|u| = \sqrt{u_1^2 + u_2^2} = 1$

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t \to 0} \frac{t^3 u_1 u_2^2}{t^3(u_1^2+u_2^2)} = \frac{u_1 u_2^2}{u_1^2+u_2^2}$$

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$$= u_1 u_2^2$$

$\Rightarrow$ all directional derivatives exist at $(0,0)$.