Notes on Wall-crossing Formula

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Contents

Abstract

This short note aims to write down all the details of the wall-crossing formula as explicitly as possible. If you find any errors or typos in this article, please tell me in time to correct them.

1 Graded Lie Algebra, Central Charge and Strict Cones

Let $\Gamma := \mathbb{Z}^n$ be a free abelian group with rank $n \in \mathbb{N}$, we say a Γ-graded $\mathbb{Q}$-vector space $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ is a Γ**-graded Lie alegbra** over $\mathbb{Q}$ if there exists is a bilinear function $[-,-]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, such that for every element $\alpha, \beta \in \Gamma$, we have:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \tag{1}$$

and for every vector x, y and z in $\mathfrak{g}$, the followings hold:

$$[x,y] = [y,z], \tag{2}$$

$$[x,[y,z]] + [y,[z,x]] + [z,[y,x]] = 0. \tag{3}$$

We often refer the equality (3) a **Jacobi identity**.

A **central charge** $Z\colon \Gamma \to \mathbb{C}$, is a group homomorphism from Γ to $\mathbb{C}$. We may regard a central chage as an elements in $\mathbb{C}^n$. Let $\{e_1,\dots,e_n\}$ be the set of generators of $\Gamma \simeq \mathbb{Z}^n$, a group homomorphism from $\mathbb{Z}^n$ to $\mathbb{C}$ is uniquely determined by the image of $e_1,\dots,e_n$ in $\mathbb{C}$. Therefore we may treat the central charge Z as a vector $(z_1,\dots,z_n) \in \mathbb{C}^n$ if there is no ambiguity.

Let $\Gamma_\mathbb{R} := \Gamma\otimes\mathbb{R} \simeq \mathbb{R}^n$ be the n-dimensional real vector space associated to the free ableian group Γ.

2 Nilpotent and Pro-nilpotent Lie Alegbra