Categories and Morphisms

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1 Categories, Functor and Natural Transformation

Definition 1 (Quick Review)**.** 我们稱$\mathscr{C}$ 是一**category** 范疇, 如果$\mathscr{C}$, 有**objects** 對象$X, Y, Z, \cdots$, **morphisms** 態射$f, g, h, \cdots$, 每个對象x, 有**identity** 單位態射 id_x, 以及對態射 $f\colon X \to Y, g\colon Y \to Z$ 有 **composition** 複合$g \circ f\colon X \to Z$, 这个複合是結合的, 且態射于單位的複合仍然是原態射.

我们稱$F\colon \mathscr{C} \to \mathscr{D}$ 是一 **functor** 函子, 若F 映對象至對象, 映態射至態射, 并且F 保持複合及單位.

我们稱$\tau\colon F \to G$ 是一 **natural transformation** 自然變換, 若對每个X, 有$\tau_X\colon FX \to GX$ 滿足: 對任意的$f\colon X \to Y$

$$Gf \circ \tau_X = \tau_Y \circ Ff\colon FX \to GY$$

Note 2. 事實上在任意范疇中, 如果給定了相同的複合, 則單位的選取是唯一的. 这是因爲若有$1_X, 1'_X$ 是單位, 則有

$$1_X = 1_X \circ 1'_X = 1'_X.$$

Example 3. 在之前我们已經給過范疇的一些例子, 我们在这邊一口氣多給一些.

- Set 是一个對象爲集合, 態射爲集合間函數的范疇.
- Top 是一个對象爲拓撲空間, 態射爲連續函數的范疇.
- Grp 是一个對象爲羣, 態射爲羣同態的范疇.
- 對任意范疇$\mathscr{C}$, $\mathscr{C}^{\mathrm{op}}$ 是一个對象爲$\mathscr{C}$ 中對象, 態射爲$\mathscr{C}$ 態射, 但態射的方向和複合均相反的范疇.
- $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ 是一个對象爲$\mathscr{C}^{\mathrm{op}}$ 到 Set 間函子, 態射爲自然變換的范疇. 我们一般稱这个范疇爲**presheaf category**.
- Δ 是一个對象爲非空有限偏序集$[n] := \{0 < 1 < \cdots < n\}$, 態射爲保序映射的范疇.

Definition 4. A $\mathscr{C}$ is connected if $\pi_0(\mathscr{C})$ is a connected space, i.e. for each $x \to y$ there is a finite zig-zag

$$x = x_0 \leftarrow x_1 \rightarrow\leftarrow \cdots \leftarrow\rightarrow x_{n-1} \rightarrow x_n = y.$$

2 Yoneda Lemma

Example 5 (Hom-Set/Hom-Functor)**.** 給定局部小 **(locally small)** 范疇$\mathscr{C}$, 即, 對于每對對象$X, Y \in \mathscr{C}$, 所有從X 到Y 的態射恰組成一个集合, 记爲$\mathscr{C}(X, Y)$, 并稱爲$\mathscr{C}$ 中的**Hom-set.**

給定$f\colon X \to Y$, 我们可以定义映射 $f^*\colon \mathscr{C}(W, X) \to \mathscr{C}(W, Y), (g\colon W \to X) \mapsto (f \circ g\colon W \to Y)$, 和 $f_*\colon \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z), (h\colon Y \to Z) \mapsto (h \circ f\colon X \to Z)$

若定义$\mathscr{C}(W, f) := f^*\colon \mathscr{C}(W, X) \to \mathscr{C}(W, Y), \mathscr{C}(f, Z) := f_*\colon \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$ 則容易驗證對任意的對象$X \in \mathscr{C}$ 有:

$$\mathscr{C}(X, -)\colon \mathscr{C} \to \mathrm{Set} \quad \mathscr{C}(-, X)\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$$

是兩个函子.

Definition 6. A functor F in $\mathrm{Set}^{\mathscr{C}}$ is called representable iff $F \simeq \mathrm{Hom}(-, X)$ for some $X \in \mathscr{C}$. 同時我们也說F 被X 表示.

Theorem 7 (Yoneda Lemma)**.** 令$\mathscr{C}$ 是一 locally small category, 對于任一$P \in \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ 以及$C \in \mathscr{C}$, 我们有

$$\mathrm{Nat}(\mathrm{Hom}(-, C), P) \simeq P(C)$$

Proof. 我们直接給出这个映射:

$$\tau \in \mathrm{Nat}(\mathrm{Hom}(-,C),P) \to P(C) \ni \tau_C(\mathrm{id}_C)$$

我们來證明它是雙射. 首先我们驗證这是一个滿射, 對于每个$x \in P(C)$, 我们定义映射τ^x 滿足

$$f \in \mathrm{Hom}(C',C) \xrightarrow{\tau^x_{C'}} P(C') \ni P(f)(x).$$

我们驗證τ^x 是一个自然變換, 这是因為對于任意$f\colon X \to Y \in C$, 對于任意的$g \in \mathrm{Hom}(Y,C)$, 有

$$\tau^x_X \circ f_*(g) = \tau^x_X(gf) = P(gf)(x) = P(f)P(g)(x) = P(f) \circ \tau^x_X(g).$$

于是由于$\tau^x_C(\mathrm{id}_C) = P(\mathrm{id}_C)(x) = x$, 我们說这是一个滿射. 同样的, 對于任一$f \in \mathrm{Hom}(C',C)$ 注意到

$$\tau^{\tau_C(\mathrm{id}_C)}_{C'}(f) = P(f)(\tau_C(\mathrm{id}_C)) = \tau_{C'} f_*(\mathrm{id}_C) = \tau_{C'},$$

这样就有$\tau^{\tau_C(id_C)} = \tau$. 这使我们知道上述映射是一个雙射. 對偶地, 我们也有:

$$\mathrm{Nat}(\mathrm{Hom}(C,-),P) \simeq P(C)$$

式中$P\colon \mathscr{C} \to \mathrm{Set}$. □

雖然这一定理的證明是初等的, 但其在肉眼可見的未來有着豐富的應用.

3 Morphisms in Categories

Definition 8 (Monic and epi)**.** 我们稱態射$f\colon X \to Y$ 是**monic** 單的, 如果對于任意$x,y\colon \cdot \to X$ 若$fx = fy$ 就有$x = y$ 成立. 對偶地, 若f^{op} 在$\mathscr{C}^{\mathrm{op}}$ 中是單的, 則稱f 是**epi** 滿的.

Note 9. If f is monic, then for every $hg = f$, g is monic.And the composition of monic morphisms is also monic. Therefore all the monic morphism defined a subcategory.

Definition 10 (Retraction, Section and Isomorphism)**.** 若$rs = \mathrm{id}_\bullet$ 成立, 則我们稱$r\colon \star \to \bullet$ 是一个**retraction**, $s\colon \bullet \to \star$ 是一个**section**. 若額外的我们還有$sr = \mathrm{id}_\star$, 則我们稱r 是一个**isomorphism**, 并稱$\star, \bullet$ 是**isomorphic** 同構的.

Note 11. Obviously every section is monic and every retraction is epic, and every epic section is an isosmorphism.

Note 12. In the categories of sets, the injections are monic and the surjection are epi. But in other category, monic/epi morphisms act differently.For example, $\mathbb{Z} \to \mathbb{Q}$ in the category Rng of rings is monic and epic, but not an isomorphism.

Definition 13. 我们稱$0 \in \mathscr{C}$ 是一个**initial objects**, 若對任意的$c \in \mathscr{C}$, 對于任意$f,g\colon 0 \to c$, $f = g$. 對偶地, 我们稱$1 \in \mathscr{C}$ 是一个**terminal objects**. 若對象0 不僅是initial, 而且terminal 則稱其為**zero objects**, 并把$0_{x,y}\colon x \to 0 \to y$ 稱為**zero morphisms**.

Note 14. 顯然任意兩个initial/ternimal object 彼此同構.

Note 15. $f : X \to Y$ is monic iff $X \times_Y X \simeq X$.Dually $f : X \to Y$ is epic iff $Y \coprod_X Y \simeq Y$.

4 Morphisms Between Categories

Note 16. Any functor preserve retraction and section. Therefore functor preserve isomorphism.

Definition 17. $F\colon \mathscr{C} \to \mathscr{D}$ 誘導了$F^\flat\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX,FY), f \mapsto Ff$ 的一个映射. 如果这一映射是單的, 則稱F 是**faithful functor** 的, 如果这一映射是滿的, 則稱F 是**fully functor**.

Note 18. $\mathrm{Hom}(X,-)$ 是faithful 的.

Note 19. faithful functor creates monic/epi, i.e. $F\colon \mathscr{C} \to \mathscr{D}$, $Ff\colon FX \to FY$ is monic/epic so is f.Moreover, faithful functor creates commute diagram.

Note 20. In fact the functor can be viewed as the morphism in the category of categories, but there is a difference bewteen faithful functor and mono functor. Consider the **free category of complete graph** K_n, the functor $K_n \to \mathbb{1}$ is faithful, but not mono apparently.

Definition 21. $F\colon \mathscr{C} \to \mathscr{D}$ is said to be essential surjective is for every object $d \in \mathscr{D}$ there exists an object $c \in \mathscr{C}$ that $FC \simeq D$.

5 Morphisms Between Morphisms

Definition 22. An equivalence of categories is a functor pair $F\colon \mathscr{C} \leftrightarrow \mathscr{D} : G$ that $FG \simeq 1_D$ and $1_C \simeq GF$. 容易地, 我们可以通過范疇等價定义一个等價關系.

Definition 23. A category $\mathscr{C}$ is skeletal if for each $x, y \in \mathscr{C}$, $x \simeq y$ implies $x = y$.

Similarly we can defined the skeleton category $\mathrm{sk}\mathscr{C}$ be the category which objects are the equivalence class of objects in $\mathscr{C}$ with the relation of isomorhism, and morphisms are the equivalence class of arrows $\mathscr{C}$ with the same relation but in $\mathscr{C}^{\rightarrow}$

Theorem 24 (equivalence criterion)**.** The following are equivalent:

- $\mathscr{C} \simeq \mathscr{D}$
- There exists a functor from $F\colon \mathscr{C} \to \mathscr{D}$ which is fully faitful and essential surjective.
- $\mathrm{sk}\mathscr{C}$ is isomorphic to $\mathrm{sk}\mathscr{D}$.