Adv. Macroeconomics II Problem Set 1 Solution

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September 26, 2020

1 Define a competitive equilibrium for this economy

A Competitive Equilibrium consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ solves:

$$\pi = \max_{\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t n_t^d)$$
$$s.t. \quad y_t = F(k_t^d, n_t^d) \quad \forall t \geq 0$$
$$y_t, k_t^d, n_t^d \geq 0 \quad \forall t \geq 0$$

2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ solves:

$$\max_{\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} B_t U(c_t)$$
$$s.t. \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t^s + w_t n_t^s) + \pi$$
$$x_{t+1} = (1-\delta)x_t + i_t \quad \forall t \geq 0$$
$$0 \leq n_t^s \leq 1, \quad 0 \leq k_t^s \leq x_t \quad \forall t \geq 0$$
$$c_t, x_{t+1} \geq 0 \quad \forall t \geq 0$$
$$k_0 \quad given$$

3. Markets Clear

$$y_t = c_t + i_t \quad (Goods\ Market) \tag{1}$$
$$n_t^d = n_t^s \quad (Labor\ Market) \tag{2}$$
$$k_t^d = k_t^d \quad (Capital\ Market) \tag{3}$$

2 Define the social planner's problem for this economy

The social planner's problem is:

$$\begin{aligned} w(k_0) = &\max_{\{k_t, n_t, y_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t. \quad &F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t \quad \forall t \geq 0 \\ &c_t \geq 0, k_t \geq 0, 0 \leq n_t \leq 1 \quad \forall t \geq 0 \\ &k_0 \; given \end{aligned} \tag{4}$$

3 Show that the equilibrium allocation of consumption, capital, and labor coincides with those of the planner's.

The Euler equation and the transversality condition are jointly sufficient for the maximizing sequence of capital stocks, so if we prove that the competitive equilibrium problem and the planner's problem share the same Euler equation and TVC, then their allocations must be the same.

Planner's Problem. With $\delta = 1$ and no allocation for leisure, $n_t = 1$, and market clearing $c_t = f(k_t) - k_{t+1}$, rewrite the problem.

$$\begin{aligned} w(k_0) = \max_{\{k_t\}_{t=1}^{\infty}} &\sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ s.t. \quad &0 \leq k_{t+1} \leq f(k_t) \quad \forall t \geq 0 \\ &k_0 \; given \end{aligned} \tag{5}$$

Take FOC to derive the Euler equation:

$$U'(f(k_t) - k_{t+1}) = \beta f'(k_{t+1}) U'(f(k_{t+1}) - k_{t+2}) \quad \forall t \geq 0 \tag{6}$$

TVC makes sure the value of the capital stock the social planner chooses converge to zero in the limit:

$$\lim_{t \to \infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0 \tag{7}$$

Competitive equilibrium. Solving household's problem with market clearing in firm:

$$
\begin{aligned}
\beta^t U'(c_t) &= \mu p_t \\
\beta^{t+1} U'(c_{t+1}) &= \mu p_{t+1} r_{t+1} \\
\mu p_t &= \mu p_{t+1} r_{t+1} \\
c_t &= f(k_t) - k_{t+1} \\
r_t &= f'(k_t) \\
\rightarrow U'(f(k_t) - k_{t+1}) &= \beta f'(k_{t+1}) U'(f(k_{t+1}) - k_{t+2})
\end{aligned} \tag{8}
$$

We also need to make sure the value of the capital stock carried forward by the household converges to zero:

$$
\begin{aligned}
\lim_{t\to\infty} p_t k_{t+1} = 0 & \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} u'(c_{t-1}) k_t \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} \beta u'(c_{t-1}) r_t k_t \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} \beta u'(f(k_t) - k_{t+1}) k_t
\end{aligned} \tag{9}
$$

With μ being positive, both the Euler equation and TVC from the competitive equilibrium are same as for the planner's, therefore they have he same capital and consumption allocation.

4 Pose the planner's dynamic programming problem. Write down the appropriate Bellman equation.

$$
v(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\} \tag{10}
$$

5 Solve the planner's dynamic programming problem (find the value and policy functions).

First, plug in the given utility function and production function into the Bellman equation:

$$
v(k) = \max_{0 \leq k' \leq f(k)} \{log(zk^\alpha - k') + \beta v(k')\} \tag{11}
$$

Then, use the guess and verify method, set $v(k) = Alog(k) + B$. Plug it in the Bellman equation and take FOC.

$$\begin{aligned} -\frac{1}{zk^\alpha - k'} + \frac{\beta A}{k'} &= 0 \\ k' &= \frac{\beta A z k^\alpha}{1+\beta A} \quad (Policy\, Func.) \end{aligned} \tag{12}$$

Next, plug k' and the expression for v into the bellman equation, and solve for A and B:

$$\begin{aligned} A &= \frac{\alpha}{1-\alpha\beta}, \\ B &= \frac{1}{(1-\beta)(1-\alpha\beta)}[(\alpha\beta)log(\alpha\beta) + (1-\alpha\beta)log(1-\alpha\beta) + log(z)] \end{aligned} \tag{13}$$

*Proof for contraction mapping is omitted.

6 Use the solution to the planner's problem to obtain the steady state value of $\{c, k, r, w, y\}$.

Set $\overline{k} = \frac{\beta A z \overline{k}^\alpha}{1+\beta A}$ and solve for $\overline{k}$ and rest of the steady state values.

$$\begin{aligned} \overline{k} &= (\alpha\beta z)^{\frac{1}{1-\alpha}}; \\ \overline{c} &= f(\overline{k}) - \frac{\beta A z \overline{k}^\alpha}{1+\beta A} = z\overline{k}^\alpha - \alpha\beta z \overline{k}^\alpha = (1-\alpha\beta) z (\alpha\beta z)^{\frac{\alpha}{1-\alpha}}; \\ \overline{r} &= f_k(\overline{k}) = \alpha z \overline{k}^{\alpha-1} = \frac{1}{\beta}; \\ \overline{w} &= f_l(\overline{k}) = (1-\alpha) z \overline{k}^\alpha = (1-\alpha) z (\alpha\beta z)^{\frac{\alpha}{1-\alpha}}; \\ \overline{y} &= f(\overline{k}) = z\overline{k}^\alpha = z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}. \end{aligned} \tag{14}$$

7 Coding

7.1 Capital decreases to 80% of its steady state value

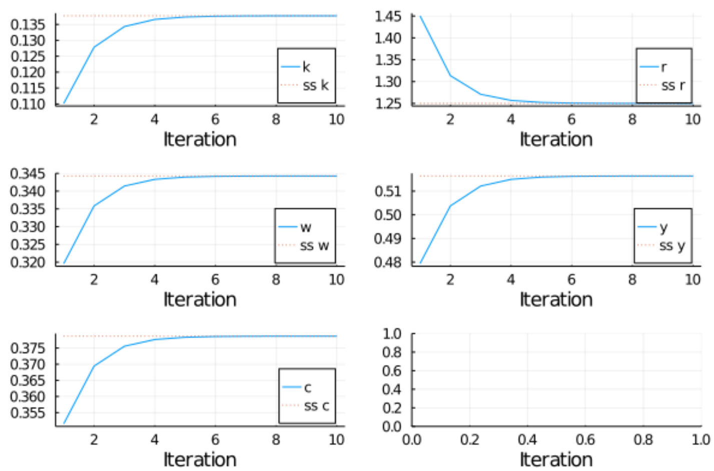

Figure 1: Capital decreases to 80% with β=0.8

7.2 Productivity increases permanently by 5%

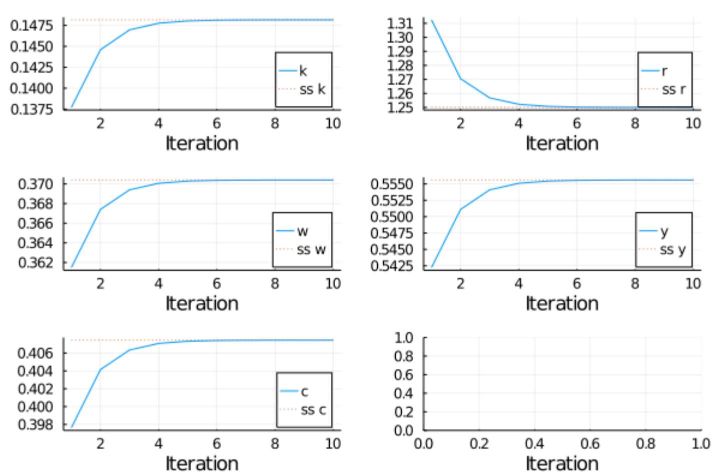

Figure 2: Productivity increases permanently by 5% with β=0.8