Adv. Macroeconomics II Problem Set 2 Solution

Chong Wang

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1 Define a competitive equilibrium for this economy

A Competitive Equilibrium consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves:

$$\begin{aligned}
\pi = \max_{\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t l_t^d) \\
s.t. \quad & y_t = z(k_t^d)^{\alpha}(l_t^d)^{1-\alpha} \quad \forall t \geq 0 \\
& y_t, k_t^d, l_t^d \geq 0 \quad \forall t \geq 0
\end{aligned}$$

2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves:

$$\begin{aligned}
\max_{\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta_t \left(\frac{(c_t)^{1-\sigma}}{1-\sigma} - \chi \frac{(l_t^s)^{1+\eta}}{1+\eta}\right) \\
s.t. \quad & \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t^s + w_t l_t^s) + \pi \\
& x_{t+1} = (1-\delta)x_t + i_t \quad \forall t \geq 0 \\
& 0 \leq l_t^s \leq 1, \quad 0 \leq k_t^s \leq x_t \quad \forall t \geq 0 \\
& c_t, x_{t+1} \geq 0 \quad \forall t \geq 0 \\
& k_0 \quad given
\end{aligned}$$

3. Markets Clear

$$y_t = c_t + i_t \quad (Goods\ Market) \tag{1}$$
$$l_t^d = l_t^s = l_t \quad (Labor\ Market) \tag{2}$$
$$k_t^d = k_t^s = k_t \quad (Capital\ Market) \tag{3}$$

2 Find the steady state value for $\{c, l, k, r, w, y\}$

Firm's problem: take FOC w.r.t k_t and l_t

$$
\begin{aligned}
r_t &= z\alpha(k_t)^{\alpha-1}(l_t)^{1-\alpha} \\
w_t &= z(1-\alpha)(k_t)^{\alpha}(l_t)^{-\alpha} \\
\pi &= \sum_{t=0}^{\infty} p_t(z(k_t)^{\alpha}(l_t)^{1-\alpha} - r_t k_t - w_t l_t) = 0
\end{aligned}
\tag{4}
$$

Household's problem: there is no gains from not fully using capital on k_t, so $k_t = x_t \quad \forall t \geq 0$. Then, solve the Lagrange function.

$$
\begin{aligned}
\mathcal{L}(\{c_t, k_t, l_t\}_{t=0}^{\infty}, \lambda_t) = &\sum_{t=0}^{\infty} \beta^t (\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}) \\
&+ \lambda_t (\sum_{t=0}^{\infty} p_t (z k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta)k_t) \\
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t c_t^{-\sigma} - \lambda_t p_t = 0 \\
\frac{\partial \mathcal{L}}{\partial l_t} &= -\beta^t \chi l_t^{\eta} + \lambda_t p_t z(1-\alpha) k_t^{\alpha} l_t^{-\alpha} = 0 \\
\frac{\partial \mathcal{L}}{\partial k_t} &= \lambda_t p_t (z\alpha k_t^{\alpha-1} l_t^{1-\alpha} + 1 - \delta) - \lambda_{t-1} p_{t-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta)k_t - c_t = 0
\end{aligned}
\tag{5}
$$

In steady state, $c_t = c, k_t = k, l_t = l$. When we normalize $p_0 = 1$, we get:

$$
\begin{aligned}
\lambda_t &= c \\
p_t &= \beta^t \\
c^{\sigma} l^{\eta} &= z(1-\alpha)k^{\alpha} l^{-\alpha}/\chi \\
z\alpha k^{\alpha-1} l^{1-\alpha} &= 1/\beta - 1 + \delta \\
c &= z k^{\alpha} l^{1-\alpha} - \delta k
\end{aligned}
\tag{6}
$$

Now, solve for $\frac{k}{l}$ and $\frac{c}{l}$, then use them to solve the steady state variables:

$$\begin{aligned}
Let\ M &= \frac{k}{l} = \left(\frac{z\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} \\
Let\ N &= \frac{c}{l} = z(\frac{k}{l})^{\alpha} - \delta\frac{k}{l} = zM^{\alpha} - \delta M \\
c^{\sigma}l^{\eta} &= (lN)^{\sigma}l^{\eta} \\
&= z(1-\alpha)(\frac{k}{l})^{\alpha}/\chi \\
&\Rightarrow l^{\sigma+\eta} = \frac{z(1-\alpha)M^{\alpha}}{\chi N^{\sigma}} \\
&\Rightarrow l = \left(\frac{z(1-\alpha)M^{\alpha}}{\chi N^{\sigma}}\right)^{\frac{1}{\sigma+\eta}} \quad (*)
\end{aligned} \tag{7}$$

With l in steady state, we can solve for the other steady state variables:

$$\begin{aligned}
k &= Ml \quad (*) \\
c &= Nl \quad (*) \\
y &= zk^{\alpha}l^{1-\alpha} = zM^{\alpha}l \quad (*) \\
r &= \alpha zk^{\alpha-1}l^{1-\alpha} = \alpha zM^{\alpha-1} \quad (*) \\
w &= (1-\alpha)zk^{\alpha}l^{-\alpha} = (1-\alpha)zM^{\alpha} \quad (*)
\end{aligned} \tag{8}$$

3 Pose the planner's dynamic programming problem. Write down the appropriate Bellman equation.

The social planner's problem is:

$$\begin{aligned}
w(k_0) &= \max_{\{k_t,c_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\frac{(c_t)^{1-\sigma}}{1-\sigma} - \chi\frac{(l_t^s)^{1+\eta}}{1+\eta}) \\
s.t. \quad z(k_t)^{\alpha}(l_t)^{1-\alpha} &= c_t + k_{t+1} - (1-\delta)k_t \quad \forall t \geq 0 \\
c_t &\geq 0, k_t \geq 0, 0 \leq l_t \leq 1 \quad \forall t \geq 0 \\
k_0\ given &
\end{aligned} \tag{9}$$

Then, substitute c_t using k_t and turn it into Bellman equation:

$$v(k_0) = \max_{\substack{0\leq k'\leq z(k)^{\alpha}(l)^{1-\alpha}+(1-\delta)k \\ 0\leq l\leq 1}} \{\frac{(z(k)^{\alpha}(l)^{1-\alpha}+(1-\delta)k-k')^{1-\sigma}}{1-\sigma} - \chi\frac{(l)^{1+\eta}}{1+\eta} + \beta v(k')\} \tag{10}$$

4 Find χ such that $l_{ss} = 0.4$

Rewrite the equation l_{ss} for χ

$$\begin{aligned} l_{ss}^{\sigma+\eta} &= \frac{z(1-\alpha)M^{\alpha}}{\chi N^{\sigma}} \\ \chi &= \frac{z(1-\alpha)M^{\alpha}}{l_{ss}^{\sigma+\eta} N^{\sigma}} \end{aligned} \tag{11}$$

Plug in $\alpha = 1/3, z = 1, \delta = 2, \eta = 1, l_{ss} = 0.4$

$$\chi = 57.12 \tag{12}$$

5 Solve the planner' problem numerically using value function iteration. You must do it using:

5.1 Plain VFI

n_k = 100, Converging Iteration: 266, Time: 176s

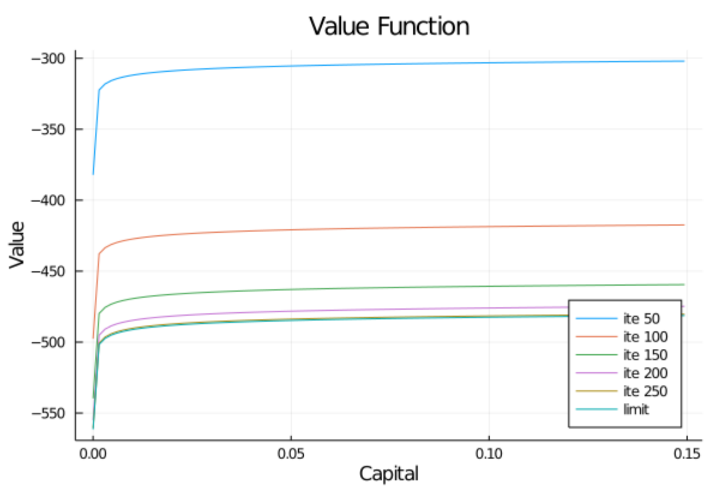

Figure 1: Approximated Value Function

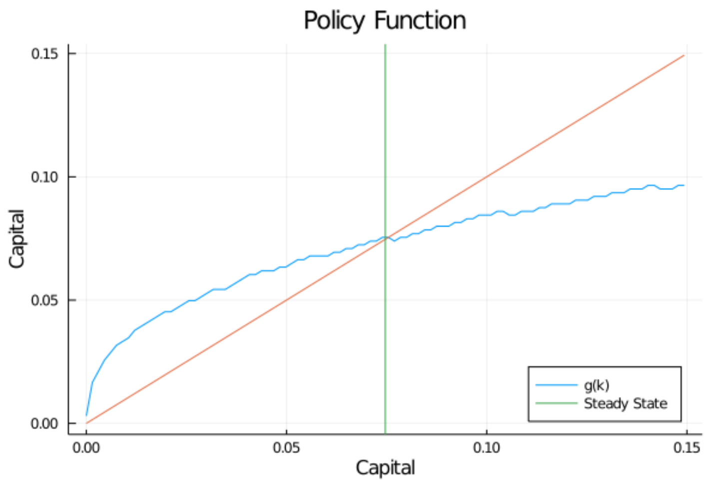

Figure 2: Approximated Policy Function

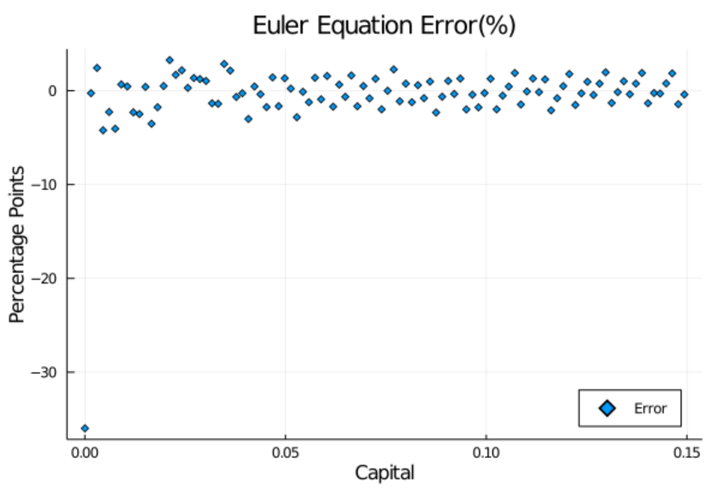

Figure 3: Approximated Error(%)

5.2 Modified Howard's Policy Iteration (you must choose the number of policy iterations)

n_k = 500, Converging Iteration: 19, Time: 33s

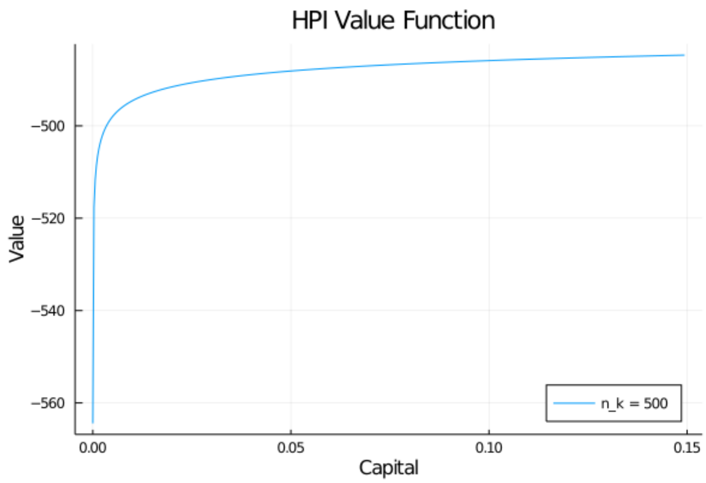

Figure 4: Approximated Value Function

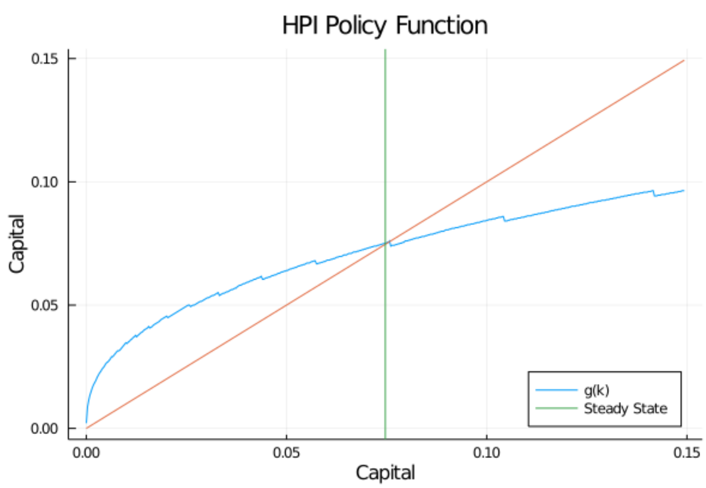

Figure 5: Approximated Policy Function

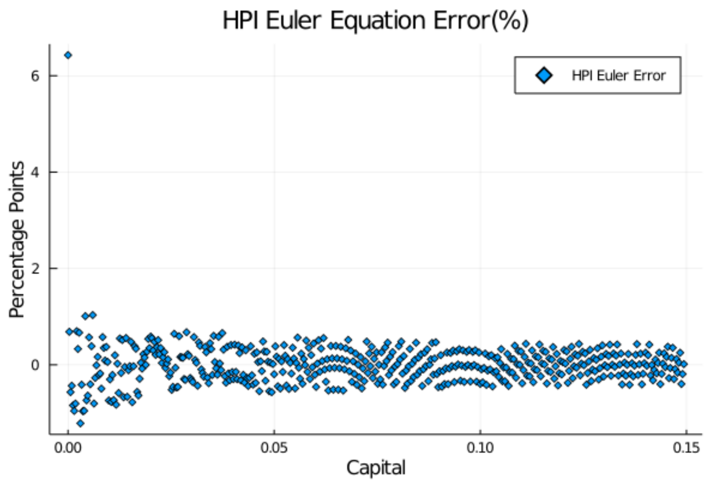

Figure 6: Approximated Error(%)

5.3 MacQueen-Porteus Bounds

n_k = 600, Converging Iteration: 17, Time: 20s

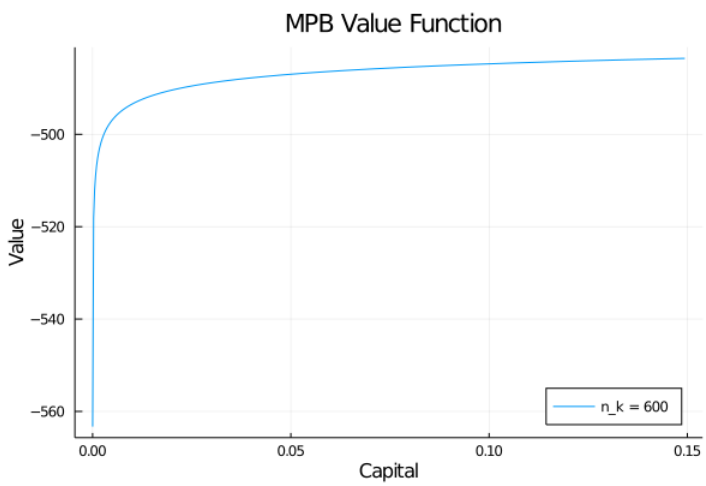

Figure 7: Approximated Value Function

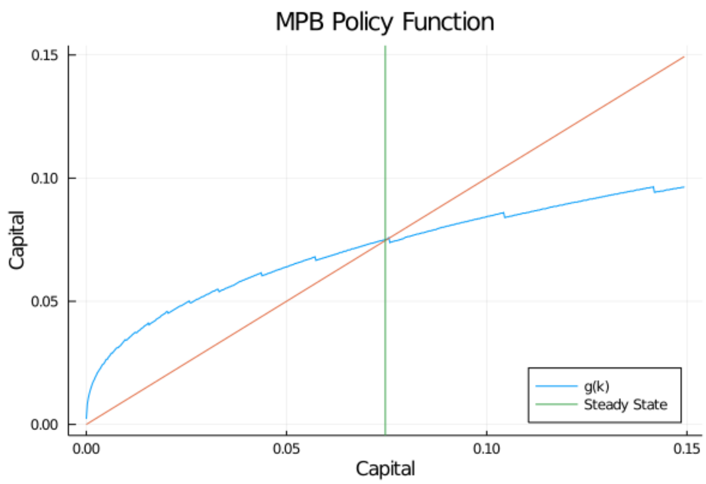

Figure 8: Approximated Policy Function

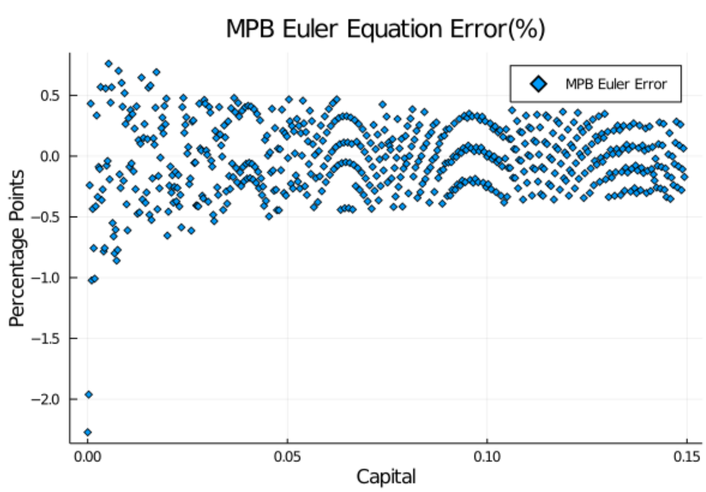

Figure 9: Approximated Error(%)

6 Use the solution to the planner's problem to obtain the steady state value of $\{c, k, r, l, w, y\}$. (Compare estimation to solved steady state)

6.1 Capital decreases to 80% of its steady state value

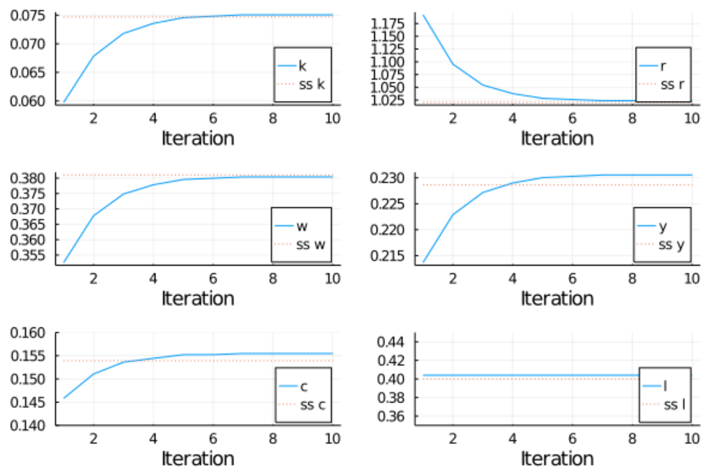

Figure 10: SS capital decreases to 80%

6.2 Productivity increases permanently by 5%

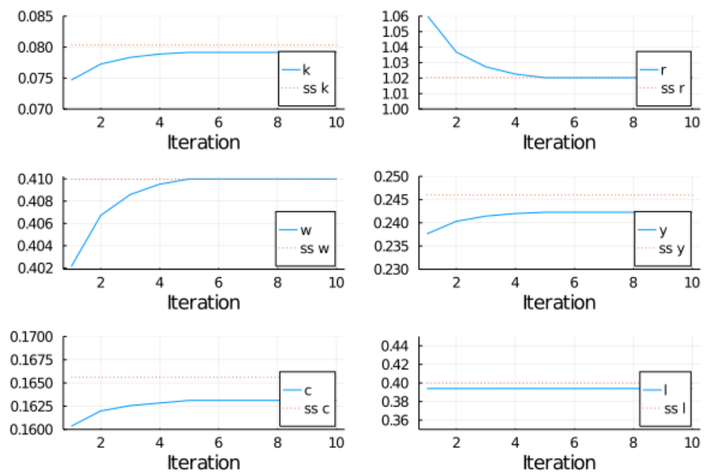

Figure 11: Productivity increases permanently by 5%

7 Prove that the mapping used in Howard's policy iteration algorithm is a contraction.

I will prove the contraction by Blackwell's theorem.

$$Tv^n(k) = \max_{\substack{0 \le k' \le z(k)^\alpha (l)^{1-\alpha} + (1-\delta)k \\ 0 \le l \le 1}} \{ \frac{(z(k)^\alpha (l)^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi \frac{(l)^{1+\eta}}{1+\eta} + \beta v^n(k') \} \tag{13}$$

- First, we assume $\frac{(z(k)^\alpha (l)^{1-\alpha}+(1-\delta)k-k')^{1-\sigma}}{1-\sigma} - \chi \frac{(l)^{1+\eta}}{1+\eta}$ is bounded, so Tv is also bounded.
- Next, consider monotonicity. Suppose $v(k) \le w(k)$ for all k. Let $\overline{g}(k)$be Howard's fixed policy function.

$$\begin{aligned} T^H v(k) &= U(\overline{g}(k)) + \beta v(\overline{g}(k)) \\ &\le U(\overline{g}(k)) + \beta w(\overline{g}(k)) \\ &= T^H w(k) \end{aligned} \tag{14}$$

- Lastly, consider discounting.

$$\begin{aligned} T^H(v+a)(k) &= U(\overline{g}(k)) + \beta(v(\overline{g}(k)) + a) \\ &= U(\overline{g}(k)) + \beta v(\overline{g}(k)) + \beta a \\ &= T^H v(k) + \beta a \end{aligned} \tag{15}$$

The mapping used in Howard's fixed policy function satisfies Blackwell's theorem, therefore it is a contraction mapping.