2nd Order homogeneous differential equation (context first)

a non-homogeneous diffential equation is a second order differential equation where the end product $\neq 0$ eg $ay'' + by' + cy = x^3$

first Recap 2nd order homogeneous Equations (1st case)

given

$$ay'' + by' + cy = 0$$

suggest a derivative where its a constant multiplied by its original

meaning

$$f'(t) = Cf(t)$$

a derivative that will fulfill this requirement is the exponential derivative e^{Cx}

where $a \neq 0$ as if $C = 0$

$e^{0x} = 1$ which is constant

lets say C

c is all +ve integer values to make our lives simpler

$f(t) = e^{Ct}$

$f'(t) = Ce^{Ct}$

$f''(t) = C^2e^{ct}$ //

first Understanding

observe that

this fulfils the sugestion $f'(t) = Cf(t)$

as $e^{Ct} = f(t)$

lets now rewrite C as m so as to not confuse it with constant C

$C \equiv M$

$$am^2e^{mt} + bme^{mt} + Ce^{mt} = 0$$

observe common factor $= e^{mt}$

Rewrite

$$e^{mt}(am^2 + bm + c) = 0$$

as $e^{mt} \neq 0$ as the asymptote of $e^{mt} = 0$

to satisfy the equation as homogeneous

$$(am^2 + bm + c) = 0$$

hence we can think of this as quadratic Eqn

Given some constants

$$1y'' + 5y' + 6y = 0$$

$$1m^2 + 5m + 6 = 0$$

$$(m+3)(m+2) = 0$$

$m = -3$ or $m = -2$

Remembering from our first understanding that

$$\boxed{f'(t) = Cf(t)}$$

e^{-3t} or e^{-2t}

understanding that in 2nd Order differential Equations there will be 2 constants as you will differentiate a function twice hence

$$y = C_1e^{-3t} + C_2e^{-2t}$$

Proving for e^{-2t}

$$1 \cdot -2^2 e^{-2t} + 5 \cdot -2e^{-2t} + 6e^{-2t} = 0$$

$$4e^{-2t} - 10e^{-2t} + 6e^{-2t} = 0$$

Proving that this is allowed

$$1 \cdot -2^2 Ce^{-2t} + 5 \cdot -2Ce^{-2t} + 6Ce^{-2t} = 0$$

$$4Ce^{-2t} - 10Ce^{-2t} + 6Ce^{-2t} = 0$$

Next Page for NHDE

non-homogeneous Eqn

$$\boxed{y'' + 5y' + 6y = x^2}$$

homogeneous Eqn

non-homogeneous Solving

$$ay'' + by' + cy = G(x)$$

$$ay'' + by' + cy = 0$$

$$\boxed{\begin{array}{l}\underline{\text{Gen Soln}} \\ y(x) = y_p(x) + y_c(x)\end{array}}$$

first solve homogeneous equation where $G(x) = 0$

$$y'' + 5y' + 6y = 0$$

into auxiliary eqn

$$am^2 + bm + c = 0$$

$$m^2 + 5m + 6$$

$$(m+2)(m+3) = 0$$

$$m_1 = -2 \quad m_2 = -3$$

Plug into Equation

$$C_1 e^{m_1 x} + C_2 e^{m_2 x} = y$$

$$\boxed{C_1 e^{-2x} + C_2 e^{-3x} = y_c}$$ General Solution 1

$$\boxed{y'' + 5y' + 6y = x^2}$$

Observe $G(x)$ is x^2 hence y must be a polynomial of degree 2

$$y_p(x) = Ax^2 + Bx + C$$

$$y'_p(x) = 2Ax + B$$

$$y''_p(x) = 2A$$

sub into equation of x^2

hence : $2A + 5(2Ax+B) + 6(Ax^2+Bx+C) = x^2$

Distribute : $2A + 10Ax + 5B + 6Ax^2 + 6Bx + C = x^2$

(to show symmetry)

Rearrange: $(6A)x^2 + (10A+6B)x + (2A+5B+6C) = x^2 + 0x + 0$

$x^2 + 0x + 0$

So we can find coefficients

first coefficient	second coefficient	third coefficient
$6A = 1$	$10A + 6B = 0$	$2(\frac{1}{6}) + 5(-\frac{5}{18}) + 6C = 0$
$A = \frac{1}{6}$	$\frac{10}{6} = -6B$	$2(\frac{1}{6}) + 5(-\frac{5}{18}) = -6C$
	$-\frac{10}{36} = B$	$\frac{2(\frac{1}{6}) + 5(-\frac{5}{18})}{-6} = C$
	$-\frac{5}{18} = B$	$\frac{19}{108} = C$

hence $\boxed{Y_p(x) = \frac{1}{6}x^2 - \frac{5}{18}x + \frac{19}{108}}$ General Solution 2

Gen Soln

$\boxed{y(x) = y_p(x) + y_c(x)}$

sub $y_p(x) + y_c(x)$

$\boxed{y_c(x)\ C_1e^{-2x} + C_2e^{-3x}}$ General Solution 1

$$\boxed{Y_p(x) = \frac{1}{6}x^2 - \frac{5}{18}x + \frac{19}{108}}$$ General Solution 2

$$y(x) = C_1e^{-2x} + C_2e^{-3x} + \frac{1}{6}x^2 - \frac{5}{18}x + \frac{19}{108} \quad //$$

to summarise:
the general solution of 2nd Order non-homogeneous Eqn

$$Y(x) = Y_c(x) + Y_p(x)$$

where $Y_c(x)$ is the General solution where $G(x) = 0$

&

where $Y_p(x)$ is the General Solution where $G(x) = x^2$
but solved via finding the undetermined coefficients

A, B, C there are many different methods used to solve SONHDEs such as methods of variation of Parameters but to keep it within 3 pages I have only done it for the first case & via method of undetermined coefficients.