Appendix E.2

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Proof. Using the expression of ∇F_δ, the inequality $\nabla_{j_0} F_\delta(\mu) < 0$ is equivalent to

$$\sum_{i\in[n]} B_i \left(\frac{1}{\sum_{j\in[m]} e^{\frac{\mu_j - \mu_{j_0} + \log(d_{ij_0}) - \log(d_{ij})}{\delta}}} \right) < q_{j_0}(\mu) = \sum_{i\in[n]} B_i \left(\frac{1}{\sum_{j\in[m]} e^{\mu_j - \mu_{j_0}}} \right).$$

Let $a = \max_{i,j}\{\log(d_{ij})\} - \min_{i,j}\{\log(d_{ij})\}$. It suffices to show that

$$\sum_{j\in[m]} e^{\frac{\mu_j - \mu_{j_0} - a}{\delta}} > \sum_{j\in[m]} e^{\mu_j - \mu_{j_0}}. \tag{1}$$

We partition $[m]$ into two sets:

$$J_0 \coloneqq \{j : \mu_j - \mu_{j_0} \le a/(1-\delta) + \log(2)\}; \qquad J_1 = [m] \setminus J_0.$$

We then show that $J_1 \neq \emptyset$. Since $\sum_{j\in[m]} q_j(\mu) = \sum_{i\in[n]} B_i$ and $q_j(\mu) > 0$, there exists an index $j_1 \in [m]$ such that $q_{j_1}(\mu) \ge \sum_{i\in[n]} B_i/m$, i.e.,

$$\mu_{j_1} \ge \log\Big(\sum_{j\in[m]} e^{\mu_j}\Big) - \log(m).$$

On the other hand, since $q_{j_0}(\mu) < e^{\underline{\mu_\delta}}$, we have

$$\mu_{j_0} < \log\Big(\sum_{j\in[m]} e^{\mu_j}\Big) - \log\Big(\sum_{i\in[n]} B_i\Big) + \underline{\mu_\delta}.$$

Combining the two inequalities and using the definition of $\underline{\mu_\delta}$, we see that

$$\mu_{j_1} - \mu_{j_0} \ge \log\left(\frac{\sum_{i\in[n]} B_i}{m}\right) - \underline{\mu_\delta} > \frac{a}{(1-\delta)} + \log(2), \tag{2}$$

which implies that $j_1 \in J_1$. Therefore, J_1 is non-empty.

The definition of J_1, together with $\delta \le 1/(2 + \log(m-1)) < \frac{1}{2}$, implies that

$$\frac{\mu_j - \mu_{j_0} - a}{\delta} + \log\left(\frac{1}{2}\right) \ge \mu_j - \mu_{j_0} \qquad \text{for all } j \in J_1.$$

It follows that

$$\frac{1}{2} e^{\frac{(\mu_j - \mu_{j_0} - a)}{\delta}} > e^{\mu_j - \mu_{j_0}}, \qquad \forall\, j \in J_1. \tag{3}$$

We then estimate

$$\Phi(\mu) := \sum_{j\in[m]} e^{\frac{\mu_j-\mu_{j_0}-a}{\delta}} - \sum_{j\in[m]} e^{\mu_j-\mu_{j_0}}$$

to prove (1):

$$\begin{aligned}\Phi(\mu) &= \sum_{j\in J_0}\left(e^{\frac{\mu_j-\mu_{j_0}-a}{\delta}} - e^{\mu_j-\mu_{j_0}}\right) + \sum_{j\in J_1}\left(e^{\frac{\mu_j-\mu_{j_0}-a}{\delta}} - e^{\mu_j-\mu_{j_0}}\right)\\ &\geq \sum_{j\in J_0} -e^{\frac{a}{1-\delta}+\log(2)} + \sum_{j\in J_1}\frac{1}{2}e^{\frac{\mu_j-\mu_{j_0}-a}{\delta}}.\end{aligned}$$

Here the inequality is due to $e^{\frac{\mu_j-\mu_{j_0}-a}{\delta}} \geq 0$, $\mu_j-\mu_{j_0} \leq a/(1-\delta)+\log(2)$ for $j\in J_0$, and (3). Recall $j_1\in J_1$. We further have

$$\Phi(\mu) \geq -2me^{\frac{a}{1-\delta}} + \frac{1}{2}e^{\frac{\mu_{j_1}-\mu_{j_0}-a}{\delta}} \geq -2me^{\frac{a}{1-\delta}} + \frac{1}{2}e^{\frac{-\underline{\mu_\delta}+\log(\sum_{i\in[n]}B_i/m)-a}{\delta}} \geq 0,$$

where the second inequality is due to (2); the last one follows from the definition of $\underline{\mu_\delta}$. We complete the proof. $\square$